ON CHARACTER SUMS OF LEE-WEINTRAUB, ARAKAWA, AND IBUKIYAMA, AND RELATED SUMS

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Abstract

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In this dissertation, we prove a dual version of an identity first conjectured by Lee and Weintraub and later proved by Arakawa and Ibukiyama. Our method follows a similar line of investigation. As a corollary, we prove a character sum identity conjectured by Ibukiyama.

To my wife Juliana, my universe.

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Contents

1 Introduction

1.1 History and Main Result

The beauty of character sums can be intoxicating, and their theory unquestionably rich. Letting χ be a Dirichlet character modulo k, one may think, for example, of the Gaussian sum

$$\tau(\chi) = \sum_{n=0}^{k-1} \chi(n) e^{2\pi i n/k},$$

about which there exists a whole literature inaugurated by Gauss himself.

In this dissertation, we obtain formulas expressing various character sums in terms of generalized Bernoulli numbers. We often encounter such formulas when we compare the dimension formulas for the vector space of certain modular forms in two different ways: by either the holomorphic Lefshetz fixed-point theorem or by the Selberg trace formula. Often, the character sums appear in the first method and the generalized Bernoulli numbers appear in the second. A famous instance of the these types of formulas is the Lee-Weintraub conjecture[12] stated in 1982, and proved by Ibukiyama[9] in 1994. To state it, we fix an odd prime p (the most interesting case being $p \equiv 3(4)$). We denote by ψ the Legendre symbol mod p: $\psi(a) = \left(\frac{a}{p}\right)$. We put $T = \{(a, b, c) \in (\mathbb{F}_p^\times)^3 \mid ab + bc + ca = 0\}$ and $\zeta = exp(2\pi i/p)$. Then, we have

$$\sum_{(a,b,c)\in T} \frac{\psi(abc)}{(\zeta^a - 1)(\zeta^b - 1)(\zeta^c - 1)} = \sqrt{\psi(-1)p} \left(\frac{1}{6} B_{3,\psi} - \frac{3}{2} B_{2,\psi} + \frac{p+1}{4} B_{1,\psi} \right),$$

where the $B_{k,\psi}$ are generalized Bernoulli numbers defined by the generating function (3.1.3). Several people considered the challenge of finding an elementary proof, but no one succeeded as of yet. We briefly describe Ibukyama's proof. Hashimoto[8] introduced an L-function $L_2^*(s, \psi_{H,p})$ attached to the ternary zero form $4xy - z^2$ (for the precise definition, see page 117). Arakawa[1] proved that the Lee-Weintraub conjecture is equivalent to the identity $L_2^*(0, \psi_{H,p}) = \frac{1}{24} B_{1,\psi}$. Ibukiyama[9] then proved the relation $L_2^*(s,\psi_{H,p}) = -\frac{2^{2s-1}B_{1,\psi}}{p^s}\zeta(2s-1)$ with the Riemann zeta function $\zeta(s)$, which together with Arakawa's result immediately implies the Lee-Weintraub conjecture.

Not only is the work of Arakawa[1] an essential part of Ibukiyama's proof of the Lee-Weintraub conjecture, but it is also very interesting in its own right because of the techniques used to express the special values of $L_2^*(s,\psi_{H,p})$ at non-positive integer values of s by finite sums of products of periodic Bernoulli polynomials. We describe how this was done. Shintani[19] established an ingenious method of evaluating the special values at non-positive integers of partial zeta functions for totally real number fields, giving remarkable expressions of partial zeta functions by integrals taken over complex contour paths. Following Shintani's method, Satake[16] introduced zeta functions of self-dual homogeneous cones and studied a general method of obtaining nice expressions of the zeta functions by integrals over contour paths. Kurihara[11], also following Shintani, evaluated the special values at non-positive integers of Siegel zeta functions of $\mathbb{Q}$-anisotropic quadratic forms (non-zero forms) with signature $(1, n-1)\,(n = 3, 4)$. However, their methods are not applicable to the zeta functions of cones having the property that some of the associated edge vectors are contained in the boundary of the underlying self-dual homogeneous cone. We now turn our attention to the Hashimoto L-function $L_2^*(s,\psi_{H,p})$ attached to the ternary zero form $4xy - z^2$. Following the method of Satake-Kurihara, Arakawa expressed $L_2^*(s,\psi_{H,p})$ as a finite linear combination of partial zeta functions of cones. However, since the quadratic form $4xy - z^2$ is a zero form (which represents zero non-trivially), one needs to deal with partial zeta functions of cones whose edge vectors are not necessarily in the interior of $\mathscr{P}_2$, the self-dual homogeneous cone of positive definite symmetric matrices of size two. Because of this, Satake-Kurihara's method cannot be applied to this situation. New ideas were needed to obtain useful integral representations of these partial zeta functions, and Tsuneo Arakawa did indeed overcome this obstruction. By skillfully evaluating these integral representations via the residue theorem, Arakawa was able to express the special values of $L_2^*(s,\psi_{H,p})$ at non-positive integer values of s by finite sums of products of periodic Bernoulli polynomials.

In our work (Chapter 5), we consider the dual Hashimoto L-function $L_2(s,\psi_{H,p})$ attached to the ternary zero form $xy-z^2$, whose definition will be given now. Let L_2 denote the lattice formed by 2×2 integral symmetric matrices and let $L_{2,+}$ be the subset consisting of all positive definite matrices of L_2. We fix an odd prime integer p. We denote by ψ the Legendre symbol mod p: $\psi(a)=\left(\frac{a}{p}\right)$. For each $T\in L_2$, we define $\psi_{H,p}(T)$ as follows: We put $\psi_{H,p}(T)=0$, if $\det(T)\not\equiv 0$ mod p. When $\det(T)\equiv 0(p)$, we have ${}^tgTg\equiv\left(\begin{smallmatrix}a&0\\0&0\end{smallmatrix}\right)$ mod p for some $g\in SL_2(\mathbb{F}_p)$ and $a\in\mathbb{F}_p$, and we put $\psi_{H,p}(T)=\psi(a)$. Then $L_2(s,\psi_{H,p})$ is defined by

$$L_2(s,\psi_{H,p})=\sum_{T\in L_{2,+}/SL_2(\mathbb{Z})}\frac{\psi_{H,p}(T)}{\epsilon(T)(\det(T))^s}\qquad(\mathrm{Re}(s)>3/2),$$

where $L_{2,+}/SL_2(\mathbb{Z})$ denotes the representatives of $SL_2(\mathbb{Z})$-equivalence classes in $L_{2,+}$ and $\epsilon(T)=\#\{g\in SL_2(\mathbb{Z})\mid {}^tgTg=T\}$.

Following Arakawa's method, with the help of Carlitz's reciprocity theorem for generalized Dedekind-Rademacher sums[5] (needed to overcome additional difficulties not occurring in Arakawa's work), we express the special values of $L_2(s,\psi_{H,p})$ at non-positive integer values of s by finite sums of products of periodic Bernoulli polynomials (Theorem 5.2.36). Combining this result with Ibukiyama's identity[9] $L_2(s,\psi_{H,p})=-\frac{B_{1,\psi}}{p^s}\zeta(2s-1)$, we obtain as a corollary, for each $s=0,-1,-2,\cdots$, formulas (to be called Arakawa Identities) expressing finite sums of products of periodic Bernoulli polynomials in terms of generalized Bernoulli numbers. These formulas are of great importance, interest, and significance, since I strongly believe that they cannot be easily obtained by using elementary techniques from algebra and number theory.[1] For this reason, I call these formulas Arakawa Identities (Theorem 5.5). This infinite sequence of Arakawa Identities is the main theorem of this dissertation. From the Arakawa Identity for $s = 0$, we obtain the dual version of the Lee-Weintraub identity,

Theorem 5:

$$\sum_{(a,b,c)\in S} \frac{\psi(abc)}{(\zeta^a - 1)(\zeta^b - 1)(\zeta^c - 1)} = \sqrt{\psi(-1)p}\left(\frac{1}{6}B_{3,\psi} - \frac{3}{4}B_{2,\psi} + (p-2)B_{1,\psi}\right),$$

where $S = \{(a,b,c) \in (\mathbb{F}_p^\times)^3 \mid 4ab - (c-a-b)^2 = 0\}$.

As a corollary to the above identity, we prove the following conjecture of Ibukiyama (see Example 8.19 and Remark 8.20 in [2]).

Corollary 5.4.1:

$$\sum_{(a,b,c)\in T} abc\,\psi(abc) = -\frac{p^2}{6}B_{3,\psi} - \frac{3p^3}{4}B_{2,\psi} + \frac{p^2(p+1)}{2}B_{1,\psi},$$

where $T = \{(a,b,c) \in \mathbb{Z}^3 \mid 1 \le a,b,c \le p-1,\ ab + bc + ca \equiv 0(p)\}$.

The above results follow from the simplest case of an infinite sequence of Arakawa Identities for each non-positive integer value of s, which we prove in Theorem 5.5. In Section 5.5, we also explicitly state the Arakawa Identities for $s = 0, -1, -2$.

1.2 Further Results

We fix an odd prime p. We denote by ψ the Legendre symbol mod p: $\psi(a) = \left(\frac{a}{p}\right)$. Let $P_k(x)$ denote the kth periodic Bernoulli polynomial (see (3.1.4)). We define the

[1] The situation is reminiscent of the classical unsolved problem to give a proof of Dirichlet's class number formula by elementary means and methods; that is, to give an elementary proof that $-B_{1,\psi} = \#$ of reduced quadratic irrationalities in the complex upper half-plane of discriminant $-p$, $p \equiv 3(4)$, $p \ge 7$.

Mordell-Tornheim L-function of depth 2 by

$$\begin{aligned} &L_{MT,2}(s_1, s_2, s_3; \chi_1, \chi_2, \chi_3) \\ &= \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \frac{\chi_1(m_1)\chi_2(m_2)\chi_3(m_1+m_2)}{m_1^{s_1} m_2^{s_2} (m_1+m_2)^{s_3}} \qquad (\mathrm{Re}(s_j) \geq 1,\ 1 \leq j \leq 3) \end{aligned}$$

for complex variables s_1, s_2, s_3 and primitive Dirichlet characters χ_1, χ_2, χ_3. We also put $\zeta = exp(2\pi i/p)$.

In Section 3.3, we define the sum

$$L_p = \sum_{k,t(p)} P_1\left(\frac{2kt}{p}\right) P_1\left(\frac{k^2+2kt}{p}\right) P_1\left(\frac{(k+t)^2}{p}\right).$$

In a completely elementary manner, we prove

Theorem 3.3:

$$L_p = -\frac{1}{16p} B_{3,\psi} - \frac{4p-7+3\psi(2)}{16p} B_{1,\psi}.$$

In Section 3.4, we define the sum

$$J_p = \sum_{(a,b,c)\in S} \frac{\psi(abc)}{(\zeta^a-1)(\zeta^b-1)(\zeta^c-1)},$$

where $S = \{(a,b,c) \in (\mathbb{F}_p^\times)^3 \mid a+b+c=0\}$. For $p \equiv 3(4)$, we prove

Theorem 3.4:

$$J_p = \frac{3p^3 i}{4\pi^3} L_{MT,2}(1,1,1;\psi,\psi,\psi).$$

As a corollary, for $p \equiv 3(4)$, we obtain the rationality statement

Corollary 3.4.3:

$$L_{MT,2}(1,1,1;\psi,\psi,\psi) \in \frac{\pi^3}{\sqrt{p}}\mathbb{Q},$$

a result for which we are not able to find any reference in the literature.

In Section 3.5, we make our contribution towards an elementary proof of the Lee-Weintraub identity. We prove, in a completely elementary manner,

Theorem 3.5: *The Lee-Weintraub identity is equivalent to any of the following three identities:*

$$(i)\qquad \sum_{k,t(2p)} P_1\left(\frac{t^2}{2p}\right) P_1\left(\frac{k^2+2kt}{2p}\right) P_1\left(\frac{k^2-2kt}{2p}\right) = 0,$$

$$(ii)\qquad \sum_{k,t(p)} P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{k^2+2kt}{p}\right) = \sum_{k,t(p)} P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{2kt}{p}\right) P_1\left(\frac{k^2+2kt}{p}\right),$$

$$(iii)\qquad \lim_{t\to\infty} \sideset{}{'}\sum_{\substack{x,y,z\in\mathbb{Z}\\ xy-z^2\equiv 0(p)\\ |x|,|y|,|z|<t}} \frac{\psi(x)}{x(y^2-z^2)} = 0.$$

Thus proving any one of these identities in an elementary manner would complete an elementary proof of the Lee-Weintraub identity.

Let χ be a primitive Dirichlet character with conductor $f > 1$. Let l be any positive integer which is prime to f. Let $n \in \mathbb{Z}$, $n \geq 0$. We also put $\zeta = exp(2\pi i/f)$. In Chapter 4, we obtain a formula expressing the following character sum

$$M_n(l,\chi) = \sum_{a=1}^{f-1} \frac{\chi(a)}{(\zeta^{la}-1)^n(\zeta^a-1)}$$

in terms of generalized Bernoulli numbers using only elementary methods from algebra and number theory (see Theorem 4.2.6). In Section 4.1, we evaluate the following very special character sum

$$Z_k(l,c,\chi) = f^{k-1}\sum_{a(f)} P_k\left(\frac{la+c}{fl}\right)\chi(la+c),$$

see Theorem 4.1. A special case of this theorem was established earlier by Ibukiyama[10]. In Section 4.2, we relate the sums $M_n(l,\chi)$ and $Z_k(l,c,\chi)$, and in Section 4.3, we work out some examples.

2 Notation

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of natural numbers, the ring of rational integers, the rational number field, the real number field, and the complex number field. For any communitive ring S, let $M_n(S)$, $GL_n(S)$, and $SL_n(S)$ denote the ring of matrices of size n with entries in S, the group of invertible elements in $M_n(S)$, and the group of elements in $M_n(S)$ whose determinants are one, respectively. For any element A of $M_n(S)$, let ${}^tA$, $\mathrm{tr}(A)$, and $\det(A)$ denote the transposed matrix of A, the trace of A, and the determinant of A, respectively. Additionally, we have

B_k : the kth Bernoulli number

$B_k(x)$: the kth Bernoulli polynomial

$P_k(x)$: the kth periodic Bernoulli polynomial

$c_k(x)$: given by (3.1.5)

$\delta_{i,j}$: the Kronecker delta function

$\delta_{\{x\in A\}}$: the indicator function $\mathbf{1}_A(x)$

$\delta(x,y)$: given by (3.1.8)

$\delta_1(x,y,z)$: given by (3.1.9)

$\delta_2(x,y,z)$: given by (3.1.10)

$\phi_{r,s}(h,k;x,y)$: the Carlitz Phi function given by (3.1.14)

$\psi_{r,s}(h,k;x,y)$: the Carlitz Psi function given by (3.1.14)

$\Gamma(s)$: the Gamma function

$\zeta(s)$: the Riemann zeta function

$e(s)$: the abbreviation for $exp(2\pi is)$

$\sum'$: means the meaningless terms are to be excluded

3 Background and Elementary Results

3.1 Background Information

In this section, we discuss all of the relevant background information. We start by introducing the Bernoulli polynomials. The Bernoulli polynomials $\{B_k(x)\}_{k=0}^{\infty}$ are defined recursively as follows:

$$\begin{aligned} B_0(x) &= 1, \\ B_k'(x) &= kB_{k-1}(x) \quad (k \geq 1), \\ \int_0^1 B_k(x)dx &= 0 \quad (k \geq 1). \end{aligned} \tag{3.1.1}$$

Thus, from the penultimate equation, $B_k(x)$ is obtained by integrating $kB_{k-1}(x)$, and the constant of integration is determined from the last condition. From this, one can easily prove the formula for the generating function of the Bernoulli polynomials:

$$\frac{te^{tx}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} t^k \quad (|t| < 2\pi). \tag{3.1.2}$$

This is a particularly useful formula. For example, we have the multiplication formula,

Proposition 3.1.1 [2]. *For any $k \in \mathbb{Z}$, $k \geq 0$, $n \in \mathbb{N}$, $x \in \mathbb{R}$, we have*

$$\sum_{a=0}^{n-1} B_k\left(x + \frac{a}{n}\right) = \frac{1}{n^{k-1}} B_k(nx).$$

Proof. From (3.1.2), we get

$$\sum_{a=0}^{n-1} \sum_{k=0}^{\infty} \frac{B_k\left(x + \frac{a}{n}\right)}{k!} t^k = \sum_{a=0}^{n-1} \frac{te^{t\left(x + \frac{a}{n}\right)}}{e^t - 1} = \frac{te^{tx}}{e^{t/n} - 1} = n \sum_{k=0}^{\infty} \frac{B_k(nx)}{k!} \left(\frac{t}{n}\right)^k,$$

from which the assertion follows from equating the coefficients. □

Proposition 3.1.2 [2]. *For $k \in \mathbb{Z}$, $k \geq 0$, $x \in \mathbb{R}$, we have*

$$B_k(1-x) = (-1)^k B_k(x).$$

Proof. From (3.1.2), we get

$$\sum_{k=0}^{\infty} \frac{B_k(1-x)}{k!} t^k = \frac{te^{t(1-x)}}{e^t - 1} = \frac{(-t)e^{(-t)x}}{e^{(-t)} - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!}(-t)^k.$$

Thus the assertion follows from equating the coefficients. □

Proposition 3.1.3 [2]. Let $k \in \mathbb{Z}$, $k \geq 0$, $x, y \in \mathbb{R}$. Then, we have

$$B_k(x+y) = \sum_{j=0}^{k} \binom{k}{j} B_{k-j}(y) x^j.$$

Proof. Taking the Taylor series expansion of the kth Bernoulli polynomial centered at $x = y$, we get

$$B_k(x+y) = \sum_{j=0}^{k} \frac{B_k^{(j)}(y)}{j!} x^j.$$

From (3.1.1), we see that $B_k^{(j)}(y) = k(k-1)\cdots(k-j+1)B_{k-j}(y)$ for $0 < j \leq k$. Thus the assertion follows. □

We now introduce the Bernoulli numbers. The sequence of Bernoulli numbers $\{B_k\}_{k=0}^{\infty}$ is defined by

$$B_k = B_k(0) \quad \text{for } k \geq 0.$$

The following proposition expresses Bernoulli polynomials in terms of Bernoulli numbers.

Proposition 3.1.4 [2]. *For $k \in \mathbb{Z}$, $k \geq 0$, we have*

$$B_k(x) = \sum_{j=0}^{k} \binom{k}{j} B_{k-j} x^j.$$

Proof. This follows from letting $y = 0$ in Proposition 3.1.3. □

Proposition 3.1.5 [2]. *We have*

$$B_{2k+1} = 0 \text{ for } k \geq 1.$$

Proof. From (3.1.2), we have

$$\frac{t}{2} + \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k = \frac{t}{e^t - 1} + \frac{t}{2} = \frac{t(e^t + 1)}{2(e^t - 1)}.$$

Since the right-hand side is an even function, we get $B_k = 0$ for k odd, $k \geq 3$. □

We would be remiss if we did not include the following proposition.[1]

Proposition 3.1.6 [2]. *Let $k, N \in \mathbb{N}$. Then, we have*

$$\sum_{n=1}^{N} n^k = \frac{B_{k+1}(N+1) - B_{k+1}}{k+1}.$$

Proof. From (3.1.2), we get

$$\sum_{k=0}^{\infty} \frac{B_k(n+1)}{k!} t^k = \frac{te^{t(n+1)}}{e^t - 1} = te^{tn}\left\{1 + \frac{1}{e^t - 1}\right\} = \sum_{k=0}^{\infty} \frac{n^k}{k!} t^{k+1} + \sum_{k=0}^{\infty} \frac{B_k(n)}{k!} t^k.$$

Equating the coefficient for t^{k+1}, we obtain $B_{k+1}(n+1) = (k+1)n^k + B_{k+1}(n)$. Hence,

$$\sum_{n=0}^{N} n^k = \sum_{n=0}^{N} \frac{B_{k+1}(n+1) - B_{k+1}(n)}{k+1} = \frac{B_{k+1}(N+1) - B_{k+1}}{k+1}.$$

□

We next introduce the generalized Bernoulli numbers. Given a Dirichlet character χ modulo f, the generalized Bernoulli numbers $\{B_{k,\chi}\}_{k=0}^{\infty}$ are defined by using the

[1] Jacob Bernoulli, who introduced the Bernoulli numbers, claims that he did not take "a half of a quarter of an hour" to compute the sum of tenth powers of 1 to 1,000, which he computed correctly as 91409924241424243424241924242500.

generating function

$$\sum_{a=1}^{f} \frac{\chi(a)te^{at}}{e^{ft}-1} = \sum_{k=0}^{\infty} \frac{B_{k,\chi}}{k!}t^k. \tag{3.1.3}$$

We have seen that all the Bernoulli numbers B_k with k odd, $k \geq 3$, are 0. For generalized Bernoulli numbers, we have the following.

Proposition 3.1.7 [2]. *Let χ be a nontrivial Dirichlet character modulo f. Then, for any k with $\chi(-1) \neq (-1)^k$, we have $B_{k,\chi} = 0$. In other words, if χ is an even character, then $B_{k,\chi}$ with odd indices are 0; if χ is an odd character, then $B_{k,\chi}$ with even indices are 0.*

Proof. We note that $\chi(f) = 0$ since χ is nontrivial. Replacing a by $f-a$ in the generating function (3.1.3), we get

$$\begin{aligned}\sum_{a=1}^{f-1} \frac{\chi(a)te^{at}}{e^{ft}-1} &= \sum_{a=1}^{f-1} \frac{\chi(f-a)te^{(f-a)t}}{e^{ft}-1} \\ &= \chi(-1)\sum_{a=1}^{f-1} \frac{\chi(a)te^{-at}}{1-e^{ft}} \\ &= \chi(-1)\sum_{a=1}^{f-1} \frac{\chi(a)(-t)e^{a(-t)}}{e^{f(-t)}-1}.\end{aligned}$$

Thus the generating function is an even function if $\chi(-1) = 1$ and an odd function if $\chi(-1) = -1$, from which the assertion follows. □

The following proposition expresses generalized Bernoulli numbers in terms of Bernoulli polynomials.

Proposition 3.1.8 [2]. *Let χ be a Dirichlet character modulo f. Then for any $k \in \mathbb{Z}$, $k \geq 0$, we have*

$$B_{k,\chi} = f^{k-1}\sum_{a=1}^{f} \chi(a)B_k(a/f).$$

Proof. Replacing a by a/f in (3.3.3), we get

$$\frac{te^{at}}{e^{ft}-1} = \frac{1}{f}\sum_{k=0}^{\infty}\frac{B_k(a/f)}{k!}(ft)^k.$$

Thus, we have

$$\sum_{k=0}^{\infty}\frac{B_{k,\chi}}{k!}t^k = \sum_{a=1}^{f}\frac{\chi(a)te^{at}}{e^{ft}-1} = \sum_{a=1}^{f}\chi(a)\frac{1}{f}\sum_{k=0}^{\infty}\frac{B_k(a/f)}{k!}(ft)^k$$
$$= \sum_{k=0}^{\infty}\frac{f^{k-1}\sum_{a=1}^{f}\chi(a)B_k(a/f)}{k!}t^k.$$

Thus the assertion follows from equating the coefficients. □

Next we define the periodic Bernoulli polynomials. They are used everywhere in this dissertation. Their usefulness goes well beyond providing the remainder term in the Euler-Maclaurin summation formula (which relates sums to integrals). For any $k \in \mathbb{Z}$, $k \geq 0$, we define

$$P_k(x) = \begin{cases} 0, & \text{if } k = 1,\ x \in \mathbb{Z}, \\ B_k(\{x\}), & \text{otherwise,} \end{cases} \tag{3.1.4}$$

where $\{x\}$ denotes the fractional part of x. We remark that nearly all of the important properties of the Bernoulli polynomials carry over to their periodic counterparts.

Proposition 3.1.9. *Let $k \in \mathbb{Z}$, $k \geq 0$, $n \in \mathbb{N}$, $x \in \mathbb{R}$, and let χ be a nontrivial Dirichlet character modulo f. Then, we have*

(i) $\displaystyle\sum_{a(n)} P_k\left(x+\frac{a}{n}\right) = \frac{1}{n^{k-1}}P_k(nx).$

(ii) $P_k(-x) = (-1)^k P_k(x).$

(iii) $\displaystyle B_{k,\chi} = f^{k-1}\sum_{a(f)} P_k\left(\frac{a}{f}\right)\chi(a).$

Proof. The assertions (i), (ii), (iii) follow immediately from Proposition 3.1.1, Proposition 3.1.2, Proposition 3.1.8, respectively. □

We will henceforth refer to (i) in Proposition 3.1.9 as "the multiplication formula". From the Fourier series expansions of the P_k's, we get the following proposition.

Proposition 3.1.10 [2]. *Let $k \in \mathbb{N}$, $x \in \mathbb{R}$. We put $e(x) = e^{2\pi i x}$. Then,*

$$P_k(x) = -\frac{k!}{(2\pi i)^k} \sideset{}{'}\sum_{n \in \mathbb{Z}} \frac{e(nx)}{n^k},$$

where the prime on the summation sign means that the meaningless terms (i.e. when $n = 0$) are to be excluded. If $k = 1$, then the infinite sum on the right-hand side is understood to be

$$\lim_{N \to \infty} \sideset{}{'}\sum_{n=-N}^{N} \frac{e(nx)}{n}.$$

Proof. Let $0 < x < 1$. Consider the function

$$f(z) = \frac{e^{zx}}{e^z - 1}.$$

Observe that $f(z)$ has poles at $z = 2\pi i n \, (n \in \mathbb{Z})$ and all of them are of order 1. The residue at $z = 2\pi i n$ is given by

$$\lim_{z \to 2\pi i n} (z - 2\pi i n) \frac{e^{zx}}{e^z - 1} = e^{2\pi i n x} = e(nx).$$

Let N be a natural number and put $R = 2\pi(N + 1/2)$. Let C_N be a square path in the complex plane with vertices at $R + iR$, $-R + iR$, $-R - iR$, $R - iR$. If t is a point inside C_N such that $t \neq 2\pi i n$ $(n \in \mathbb{Z})$, then it follows from the residue theorem that

$$\int_{C_N} \frac{f(z)}{z - t} \, dz = 2\pi i \left(f(t) + \sum_{n=-N}^{N} \frac{e(nx)}{2\pi i n - t} \right).$$

As $N \to \infty$, one can show that

$$\int_{C_N} \frac{f(z)}{z-t}\,dz \to 0.$$

Hence, if $0 < |t| < 2\pi$, we get

$$\begin{aligned}
f(t) &= -\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{e(nx)}{2\pi in - t} \\
&= \frac{1}{t} - \lim_{N\to\infty} \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{e(nx)}{2\pi in - t} \\
&= \frac{1}{t} - \lim_{N\to\infty} \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{e(nx)}{2\pi in\left(1 - \frac{t}{2\pi in}\right)} \\
&= \frac{1}{t} - \lim_{N\to\infty} \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{e(nx)}{2\pi in} \sum_{k=1}^{\infty} \left(\frac{t}{2\pi in}\right)^{k-1} \\
&= \frac{1}{t} - \lim_{N\to\infty} \sum_{\substack{n=-N \\ n\neq 0}}^{N} \sum_{k=1}^{\infty} \frac{e(nx)}{(2\pi in)^k} \cdot t^{k-1}
\end{aligned}$$

Since the sum

$$\sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \sum_{k=2}^{\infty} \frac{e(nx)}{(2\pi in)^{k+1}} \cdot t^k$$

converges absolutely, we get

$$f(t) = \frac{1}{t} - \lim_{N\to\infty} \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{e(nx)}{2\pi in} - \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \sum_{k=2}^{\infty} \frac{e(nx)}{(2\pi in)^k} \cdot t^{k-1}.$$

From (3.1.2), we see that

$$f(t) = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} t^{k-1} \quad (|t| < 2\pi).$$

Thus, comparing the coefficients, we obtain, for $0 < x < 1$,

$$(i) \quad P_1(x) = -\frac{1}{2\pi i} \lim_{N\to\infty} \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{e(nx)}{n},$$
$$(ii) \quad P_k(x) = -\frac{k!}{(2\pi i)^k} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e(nx)}{n^k} \qquad (k \geq 2).$$

Since both sides of the above formulas are periodic with respect to x with period 1, it suffices to show that both formulas hold for $x = 0$. The formula (i) clearly holds for $x = 0$. Since the sum on the right-hand side of (ii) converges uniformly on $[0, 1)$ and $P_k(x)$ $(k \geq 2)$ is continuous in the same interval, the formula (ii) also holds for $x = 0$. Thus the assertion follows. □

We remark in passing that from this proposition, we get the values of the Riemann zeta function at positive even integers.

Proposition 3.1.11 [2]. *Let $k \in \mathbb{N}$. Then, we have*

$$\zeta(2k) = -\frac{(2\pi i)^{2k}}{2(2k)!} B_{2k}.$$

Proof. This follows from Proposition 3.1.10 where we let $x = 0$. □

We now discuss finite Fourier series expansions. They can often be helpful in evaluating finite sums.

Proposition 3.1.12 [3]. *Let $f(n)$ be a any k-periodic function on $\mathbb{Z}$ and put $\zeta = e^{2\pi i/k}$. Then we have the following finite Fourier series expansion:*

$$f(n) = \frac{1}{k} \sum_{j=0}^{k-1} \hat{f}(j)\zeta^{nj}, \text{ where the Fourier coefficients are } \hat{f}(j) = \sum_{i=0}^{k-1} f(i)\zeta^{-ji}.$$

Proof. By direct substitution, we see that

$$f(n) = \frac{1}{k}\sum_{j=0}^{k-1} \hat{f}(j)\zeta^{nj} = \frac{1}{k}\sum_{j=0}^{k-1}\sum_{i=0}^{k-1} f(i)\zeta^{(n-i)j} = \frac{1}{k}\sum_{i=0}^{k-1} f(i)\sum_{j=0}^{k-1}\zeta^{(n-i)j} = f(n).$$

□

Proposition 3.1.13. *Let $f(n)$ be a k-periodic function on the integers so that we have $f(n) = \frac{1}{k}\sum_{j=0}^{k-1}\hat{f}(j)\zeta^{nj}$ by Proposition 3.1.12. Then $\hat{\hat{f}}(n) = kf(-n)$.*

Proof. From Proposition 3.1.12, we get, by definition,

$$\hat{\hat{f}}(n) = \sum_{j=0}^{k-1} \hat{f}(j)\zeta^{-jn} = kf(-n).$$

□

Proposition 3.1.14 [3]. *Let $f(n)$ be a k-periodic function on the integers so that we have $f(n) = \frac{1}{k}\sum_{j=0}^{k-1}\hat{f}(j)\zeta^{nj}$ by Proposition 3.1.12. Then $f(n)$ is odd if and only if $\hat{f}(j)$ is odd, and $f(n)$ is even if and only if $\hat{f}(j)$ is even.*

Proof. $f(-n) = \frac{1}{k}\sum_{j=0}^{k-1}\hat{f}(j)\zeta^{-nj} = \frac{1}{k}\sum_{j=0}^{k-1}\hat{f}(-j)\zeta^{nj}$. Thus, $f(-n) = -f(n)$ if and only if $\hat{f}(-j) = -\hat{f}(j)$, and $f(-n) = f(n)$ if and only if $\hat{f}(-j) = \hat{f}(j)$. □

We next consider the sequence of functions $\{c_k(x)\}_{k=0}^{\infty}$ defined in Sczech[17]. They are given as follows: $c_0(x) = 1$, and

$$c_k(x) = \sum_{n\in\mathbb{Z}}{}' \frac{1}{(n+x)^k} \qquad (k \geq 1), \tag{3.1.5}$$

where x is a complex number and the prime on the summation sign indicates that meaningless terms are to be excluded. If $k = 1$, then the infinite sum on the right-hand side is understood to be

$$\lim_{t\to\infty} \sum_{\substack{n\in\mathbb{Z} \\ |n+x|<t}}{}' \frac{1}{n+x}.$$

From Sczech[17], we see that $c_1(x) = \pi\cot(\pi x)$ if $x \notin \mathbb{Z}$ and 0 otherwise, and that if $x \notin \mathbb{Z}$, then $c_k(x) = V_k(c_1(x))$ with a polynomial $V_k(t)$ of degree k given recursively by

$$V_0(t) = 1, \quad V_1(t) = t, \quad kV_{k+1}(t) = (t^2 + \pi^2)V_k'(t) \text{ for } k \geq 1.$$

From this, we get the following useful facts that will be used in Section 3.2.

$$c_2(x) = \begin{cases} c_1^2(x) + \pi^2, & \text{if } x \notin \mathbb{Z}, \\ \pi^2/3, & \text{if } x \in \mathbb{Z}, \end{cases} \tag{3.1.6}$$

$$c_3(x) = c_1^3(x) + \pi^2 c_1(x) = c_1(x)c_2(x).$$

We now turn our attention to some of the various properties of the c_k's. From the definition (3.1.5), it is clear that $c_k(x)$ is periodic with period 1. Moreover, it is clear that $c_k(-x) = (-1)^k c_k(x)$. The c_k's also possess a multiplication formula.

Proposition 3.1.15. *Let $k, r \in \mathbb{N}$. Then, we have*

$$\sum_{a(r)} c_k\left(x + \frac{a}{r}\right) = r^k c_k(nx).$$

Proof. From (3.1.5), we get

$$\begin{aligned} \sum_{a(r)} c_k\left(x + \frac{a}{r}\right) &= \sum_{a(r)} \sum_{n\in\mathbb{N}}{}' \frac{1}{\left(n + x + \frac{a}{r}\right)^k} \\ &= r^k \sum_{a(r)} \sum_{n\in\mathbb{N}}{}' \frac{1}{\left((nr + a) + rx\right)^k} \\ &= r^k \sum_{m\in\mathbb{N}}{}' \frac{1}{(m + rx)^k} \\ &= r^k c_k(rx). \end{aligned}$$

□

We have the following finite Fourier expansions of the P_k's and c_k's.

Proposition 3.1.16. *Let $k, f \in \mathbb{N}$, $x \in \mathbb{Z}$, and put $\zeta = e^{2\pi i/f}$. Then, we have*

$$(i) \quad P_k\left(\frac{x}{f}\right) = -\frac{k!}{(2\pi i f)^k}\sum_{a(f)} c_k\left(\frac{a}{f}\right)\zeta^{ax},$$

$$(ii) \quad c_k\left(\frac{x}{f}\right) = -\frac{(-1)^k(2\pi i f)^k}{fk!}\sum_{a(f)} P_k\left(\frac{a}{f}\right)\zeta^{ax}.$$

Proof. By Proposition 3.1.12, we have

$$P_k\left(\frac{x}{f}\right) = \frac{1}{f}\sum_{a(f)} \hat{P}_k\left(\frac{a}{f}\right)\zeta^{ax} \quad \text{where} \quad \hat{P}_k\left(\frac{a}{f}\right) = \sum_{b(f)} P_k\left(\frac{b}{f}\right)\zeta^{-ba}. \tag{3.1.7}$$

Employing the Fourier series of the P_k's (Proposition 3.1.10), we get

$$\begin{aligned}
\hat{P}_k\left(\frac{a}{f}\right) = \sum_{b(f)} P_k\left(\frac{b}{f}\right)\zeta^{-ba} &= \sum_{b(f)}\left\{-\frac{k!}{(2\pi i)^k}{\sum_{n\in\mathbb{Z}}}' \frac{e\left(\frac{nb}{f}\right)}{n^k}\right\}\zeta^{-ba} \\
&= -\frac{k!}{(2\pi i)^k}{\sum_{n\in\mathbb{Z}}}' \frac{1}{n^k}\sum_{b(f)}\zeta^{b(n-a)} \\
&= -\frac{fk!}{(2\pi i)^k}{\sum_{\substack{n\in\mathbb{Z}\\ n\equiv a(f)}}}' \frac{1}{n^k} \\
&= -\frac{fk!}{(2\pi i)^k}{\sum_{m\in\mathbb{Z}}}' \frac{1}{(mf+a)^k} \\
&= -\frac{fk!}{(2\pi i f)^k}{\sum_{m\in\mathbb{Z}}}' \frac{1}{\left(m+\frac{a}{f}\right)^k} \\
&= -\frac{fk!}{(2\pi i f)^k}c_k\left(\frac{a}{f}\right).
\end{aligned}$$

Thus the assertion (i) follows from (3.1.7). In view of the above, it follows from Proposition 3.1.13 that

$$\hat{c}_k\left(\frac{x}{f}\right) = -\frac{(2\pi i f)^k}{fk!}\hat{\hat{P}}_k\left(\frac{x}{f}\right) = -\frac{(2\pi i f)^k}{k!}P_k\left(\frac{-x}{f}\right) = -\frac{(-1)^k(2\pi i f)^k}{k!}P_k\left(\frac{x}{f}\right).$$

Hence, from Proposition 3.1.12, we get

$$c_k\left(\frac{x}{f}\right) = -\frac{(-1)^k(2\pi i f)^k}{fk!}\sum_{a(f)} P_k\left(\frac{a}{f}\right)\zeta^{ax}.$$

□

Proposition 3.1.16 shows that the P_k's and c_k's are dual to each other. One may wonder which is easier to work with, the P_k's or c_k's. The answer is: both are easy to work with, but both have their advantages. Since the P_k's are rational expressions, often times it is easier to deal with the P_k's rather than the c_k's, and is why the majority of the sums in this dissertation are converted into sums involving P_k's. However, the c_k's have their advantages as well. There is a wealth of well-known trig identities that can be used when dealing with c_k's. In addition, a sum involving c_k's can provide an alternative viewpoint to the equivalent sum involving P_k's. In mathematics, it is always a good idea to look at a problem from different angles/viewpoints, for perhaps some insight will reveal itself. This was certainly the case with me and the c_k's in Section 4.2.

We now discuss the fundamental property of the P_k's, the addition formulas. We first prove the "2-term addition formula".

Proposition 3.1.17. *Let $v_1, v_2 \in \mathbb{R}$. Then, we have*

$$\begin{aligned} &P_1(v_1)P_1(v_2) - P_1(v_1)P_1(v_1+v_2) - P_1(v_2)P_1(v_1+v_2) \\ &\quad = -\frac{1}{2}\left(P_2(v_1) + P_2(v_2) + P_2(v_1+v_2)\right) + \frac{1}{4}\delta(v_1,v_2), \end{aligned}$$

where

$$\delta(v_1,v_2) = \begin{cases} 1, & \text{if } v_1, v_2 \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1.8}$$

Proof. The assertion is clear when $v_1, v_2 \in \mathbb{Z}$. If exactly one of the arguments is an integer, say v_2, then the assertion becomes $-P_1^2(v_1) = -P_2(v_1) - \frac{1}{12}$, which is clearly true for $v_1 \notin \mathbb{Z}$. Thus we assume that $v_1, v_2 \notin \mathbb{Z}$. Since P_1, P_2 are 1-periodic, we may assume without a loss of generality that $0 < v_1, v_2 < 1$. Then $0 < v_1 + v_2 < 2$. If $v_1 + v_2 \in \mathbb{Z}$, then $v_2 \equiv -v_1 \bmod 1$, and the assertion becomes $-P_1^2(v_1) = -P_2(v_1) - \frac{1}{12}$, which again is clearly true for $v_1 \notin \mathbb{Z}$. Thus it remains to verify the assertion for $v_1, v_2, v_1 + v_2 \notin \mathbb{Z}$ with $0 < v_1 + v_2 < 2$. We have two cases to consider, the case where $0 < v_1 + v_2 < 1$, and where $1 < v_1 + v_2 < 2$. If $0 < v_1 + v_2 < 1$, the assertion becomes

$$\left(v_1 - \frac{1}{2}\right)\left(v_2 - \frac{1}{2}\right) - \left(v_1 - \frac{1}{2}\right)\left(v_1 + v_2 - \frac{1}{2}\right) - \left(v_2 - \frac{1}{2}\right)\left(v_1 + v_2 - \frac{1}{2}\right)$$
$$= \frac{1}{2}\left\{\left(v_1^2 - v_1 + \frac{1}{6}\right) + \left(v_2^2 - v_2 + \frac{1}{6}\right) + \left((v_1 + v_2)^2 - (v_1 + v_2) + \frac{1}{6}\right)\right\},$$

which is easily verified to be true. Similarly, if $1 < v_1 + v_2 < 2$, the assertion becomes

$$\left(v_1 - \frac{1}{2}\right)\left(v_2 - \frac{1}{2}\right) - \left(v_1 - \frac{1}{2}\right)\left(v_1 + v_2 - \frac{3}{2}\right) - \left(v_2 - \frac{1}{2}\right)\left(v_1 + v_2 - \frac{3}{2}\right)$$
$$= \frac{1}{2}\left\{\left(v_1^2 - v_1 + \frac{1}{6}\right) + \left(v_2^2 - v_2 + \frac{1}{6}\right) + \left((v_1 + v_2 - 1)^2 - (v_1 + v_2 - 1) + \frac{1}{6}\right)\right\},$$

which is also easily verified to be true. □

We called the above addition formula the 2-term addition formula because it contained products of two P_1's. One might wonder, is there an addition formula containing products of three P_1's? Or even more generally, does there exist an addition formula containing products of an arbitrary number of, say n, P_1's? Thanks to the revolutionary work of Gunnells and Sczech[7], the answer is yes. We will now prove a 3-term addition formula which will henceforth be referred to as the "Sczech 3-term addition formula".

Proposition 3.1.18. *Let $v_1, v_2, v_3 \in \mathbb{R}$. Then, we have*

$$\begin{aligned}
&- P_1(v_1)P_1(v_2)P_1(v_3)\\
&\quad + P_1(v_1)P_1(v_2 - v_1)P_1(v_3 - v_1)\\
&\quad + P_1(v_1 - v_2)P_1(v_2)P_1(v_3 - v_2)\\
&\quad + P_1(v_1 - v_3)P_1(v_2 - v_3)P_1(v_3)\\
&\quad = \frac{1}{2}\Big\{ P_1(v_2 - v_1)(P_2(v_2 - v_3) - P_2(v_1 - v_3))\\
&\quad + P_1(v_2 - v_1)(P_2(v_2) - P_2(v_1))\\
&\quad + P_1(v_3 - v_1)(P_2(v_3) - P_2(v_1))\\
&\quad + P_1(v_3 - v_2)(P_2(v_3) - P_2(v_2))\Big\}\\
&\quad - \frac{1}{6}\Big\{ P_3(v_1 - v_3) + P_3(v_2 - v_3) + 2P_3(v_1) + 2P_3(v_2) + 2P_3(v_3)\Big\}\\
&\quad - \frac{1}{4}\delta_1\left(v_1, v_2, v_3\right) + \frac{1}{4}\delta_2\left(v_1, v_2, v_3\right),
\end{aligned}$$

where

$$\delta_1(v_1, v_2, v_3) = \begin{cases} P_1(v_1), & \text{if } v_1 \equiv v_2 \equiv v_3 \bmod 1, \\ 0, & \text{otherwise,} \end{cases} \tag{3.1.9}$$

and

$$\delta_2(v_1, v_2, v_3) = \begin{cases} P_1(v_1), & \text{if } v_2, v_3 \text{ are integers,} \\ P_1(v_2), & \text{if } v_1, v_3 \text{ are integers,} \\ P_1(v_3), & \text{if } v_1, v_2 \text{ are integers,} \\ 0, & \text{otherwise.} \end{cases} \tag{3.1.10}$$

Proof. We shall apply Theorem 3.3 in [7] with lattice $L = \mathbb{Z}^3$ and vectors

$$\sigma_0 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

Let Q be a product of real-valued linear forms on $L_{\mathbb{R}}$ that do not vanish on $L_{\mathbb{Q}} \backslash \{0\}$. Let $\langle x, v \rangle$ be the inner product of x and v, and put $e(x) = exp(2\pi i x)$. Then, we get, from Theorem 3.3 in [7],

$$\sum_{j=0}^{3} S(L, \sigma^j, v)|_Q = \sum_{j=0}^{3} S(L \cap \sigma_j^{\perp}, \sigma^j, v)|_Q,$$

where

$$S(L, \sigma, v)|_Q = \lim_{t \to \infty} \left((2\pi i)^{-3} \sum_{\substack{x \in L \\ |Q(x)| < t}}{}' \frac{e(\langle x, v \rangle)}{\langle x, \sigma_1 \rangle \langle x, \sigma_2 \rangle \langle x, \sigma_3 \rangle} \right).$$

This limit always exists (see Sczech[18], Theorem 2), and the value depends on Q in a rather simple way. Thus, by making transformations of x that change Q into some other product of real-valued linear forms on $L_{\mathbb{R}}$ that do not vanish on $L_{\mathbb{Q}} \backslash \{0\}$, say Q', we can easily determine the difference $S(L, \sigma, v)|_Q - S(L, \sigma, v)|_{Q'}$. In essence, we can rearrange the terms of a conditionally convergent series and keep track of the error that any such rearrangement creates. What I have just described is what is called the "Q-limit process". For the details, we refer the interested reader to Gunnells and Sczech[7]. We first treat the case where at least two of the v_j's ($j = 1, 2, 3$) are not integers and that it is not the case that $v_1 \equiv v_2 \equiv v_3 \bmod 1$. Then, by the Q-limit process, all of the rearrangements that we make below will be justified. Therefore, to keep notation to a minimum, we assume that everything converges absolutely. That is,

$$\sum_{j=0}^{3} S(L, \sigma^j, v) = \sum_{j=0}^{3} S(L \cap \sigma_j^{\perp}, \sigma^j, v), \tag{3.1.11}$$

where

$$S(L, \sigma, v) = (2\pi i)^{-3} \sum_{x \in L}{}' \frac{e(\langle x, v \rangle)}{\langle x, \sigma_1 \rangle \langle x, \sigma_2 \rangle \langle x, \sigma_3 \rangle}.$$

is assumed to converge absolutely. By Proposition 3.1.10, it follows that

$$S(L, \sigma^0, v) = (2\pi i)^3 \sum_{x,y,z \in \mathbb{Z}}{}' \frac{e(xv_1 + yv_2 + zv_3)}{xyz} = -P_1(v_1)P_1(v_2)P_1(v_3).$$

Next, let $(a, b, c) = (x + y + z, y, z)$. Then, again by Proposition 3.1.10, we get

$$\begin{aligned} S(L, \sigma^1, v) &= (2\pi i)^3 \sum_{x,y,z \in \mathbb{Z}}{}' \frac{e(xv_1 + yv_2 + zv_3)}{-(x + y + z)yz} \\ &= -(2\pi i)^3 \sum_{a,b,c \in \mathbb{Z}}{}' \frac{e(av_1 + b(v_2 - v_1) + c(v_3 - v_1))}{abc} \\ &= P_1(v_1)P_1(v_2 - v_1)P_1(v_3 - v_1). \end{aligned}$$

Similar reasoning reveals that

$$\begin{aligned} S(L, \sigma^2, v) &= P_1(v_1 - v_2)P_1(v_2)P_1(v_3 - v_1), \\ S(L, \sigma^3, v) &= P_1(v_1 - v_3)P_1(v_2 - v_3)P_1(v_3). \end{aligned}$$

We now turn our attention to the right-hand side of (3.1.11). As $\sigma_0^{\perp} = \{y \in \mathbb{R}^3 : \langle y, \sigma^0 \rangle = -y_1 - y_2 - y_3 = 0\} = \{(x, y, -x-y) \in \mathbb{R}^3\}$, we have $L \cap \sigma_0^{\perp} = \{(q, r, -q-r) \in \mathbb{Z}^3\}$. Hence,

$$S(L \cap \sigma_0^{\perp}, \sigma^0, v) = (2\pi i)^3 \sum_{q,r \in \mathbb{Z}}{}' \frac{e(q(v_1 - v_3) + r(v_2 - v_3))}{-qr(q + r)}.$$

By the partial fraction decomposition

$$\text{(3.1.12)} \qquad \frac{1}{r(q+r)} = \frac{1}{qr} - \frac{1}{q(q+r)} \quad (q, r, q+r \neq 0),$$

we get

$$\begin{aligned}
&S(L\cap\sigma_0^{\perp}, \sigma^0, v)\\
&\quad = -(2\pi i)^3 \left\{ \sideset{}{'}\sum_{\substack{q,r\in\mathbb{Z}\\ r+q\neq 0}} \frac{e(q(v_1-v_3)+r(v_2-v_3))}{q^2 r} - \sideset{}{'}\sum_{\substack{q,r\in\mathbb{Z}\\ r\neq 0}} \frac{e(q(v_1-v_3)+r(v_2-v_3))}{q^2(q+r)} \right\}\\
&\quad = -(2\pi i)^3 \left\{ \sideset{}{'}\sum_{q,r\in\mathbb{Z}} \frac{e(q(v_1-v_3)+r(v_2-v_3))}{q^2 r} - \sideset{}{'}\sum_{q,r\in\mathbb{Z}} \frac{e(q(v_1-v_3)+r(v_2-v_3))}{q^2(q+r)} \right.\\
&\qquad \left. - \left[\sideset{}{'}\sum_{q\in\mathbb{Z}} \frac{e(q(v_1-v_2))}{-q^3} - \sideset{}{'}\sum_{q\in\mathbb{Z}} \frac{e(q(v_1-v_3))}{q^3} \right] \right\}.
\end{aligned}$$

Letting $(a,b) = (q, q+r)$, we get, by Proposition 3.1.10,

$$\begin{aligned}
&S(L\cap\sigma_0^{\perp}, \sigma^0, v)\\
&\quad = -(2\pi i)^3 \left\{ \sideset{}{'}\sum_{q,r\in\mathbb{Z}} \frac{e(q(v_1-v_3)+r(v_2-v_3))}{q^2 r} - \sideset{}{'}\sum_{a,b\in\mathbb{Z}} \frac{e(a(v_1-v_2)+b(v_2-v_3))}{a^2 b} \right.\\
&\qquad \left. - \left[\sideset{}{'}\sum_{q\in\mathbb{Z}} \frac{e(q(v_1-v_2))}{-q^3} - \sideset{}{'}\sum_{q\in\mathbb{Z}} \frac{e(q(v_1-v_3))}{q^3} \right] \right\}\\
&\quad = \frac{1}{2} P_1(v_2-v_3)\Big(P_2(v_1-v_2) - P_2(v_1-v_3)\Big) + \frac{1}{6}\Big(P_3(v_1-v_2) + P_3(v_1-v_3)\Big).
\end{aligned}$$

As $\sigma_1^{\perp} = \{y\in\mathbb{R}^3 : \langle y, \sigma^1\rangle = y_1 = 0\} = \{(0,y,z)\in\mathbb{R}^3\}$, we have $L\cap\sigma_1^{\perp} = \{(0,q,r)\in\mathbb{Z}^3\}$. Hence,

$$S(L\cap\sigma_1^{\perp}, \sigma^1, v) = (2\pi i)^3 \sideset{}{'}\sum_{q,r\in\mathbb{Z}} \frac{e(qv_2+rv_3)}{-(q+r)qr}.$$

From the partial fraction decomposition (3.1.12), we get

$$\begin{aligned}
&S(L\cap\sigma_1^{\perp},\sigma^1,v)\\
&\quad=-(2\pi i)^3\left\{\sideset{}{'}\sum_{\substack{q,r\in\mathbb{Z}\\ r+q\neq 0}}\frac{e(qv_2+rv_3)}{q^2r}-\sideset{}{'}\sum_{\substack{q,r\in\mathbb{Z}\\ r\neq 0}}\frac{e(qv_2+rv_3)}{q^2(q+r)}\right\}\\
&\quad=-(2\pi i)^3\left\{\sideset{}{'}\sum_{q,r\in\mathbb{Z}}\frac{e(qv_2+rv_3)}{q^2r}-\sideset{}{'}\sum_{q,r\in\mathbb{Z}}\frac{e(qv_2+rv_3)}{q^2(q+r)}\right.\\
&\qquad\left.-\left[\sideset{}{'}\sum_{q\in\mathbb{Z}}\frac{e(q(v_2-v_3)}{-q^3}-\sideset{}{'}\sum_{q\in\mathbb{Z}}\frac{e(qv_2}{q^3}\right]\right\}.
\end{aligned}$$

Letting $(a,b)=(q,q+r)$, we get, by Proposition 3.1.10,

$$\begin{aligned}
&S(L\cap\sigma_1^{\perp},\sigma^1,v)\\
&\quad=-(2\pi i)^3\left\{\sideset{}{'}\sum_{q,r\in\mathbb{Z}}\frac{e(qv_2+rv_3)}{q^2r}-\sideset{}{'}\sum_{a,b\in\mathbb{Z}}\frac{e(a(v_2-v_2)+bv_3)}{a^2b}\right.\\
&\qquad\left.-\left[\sideset{}{'}\sum_{q\in\mathbb{Z}}\frac{e(q(v_2-v_3)}{-q^3}-\sideset{}{'}\sum_{q\in\mathbb{Z}}\frac{e(qv_2}{q^3}\right]\right\}\\
&\quad=\frac{1}{2}P_1(v_3)\Big(P_2(v_2-v_3)-P_2(v_2)\Big)+\frac{1}{6}\Big(P_3(v_2-v_3)+P_3(v_2)\Big).
\end{aligned}$$

Similar reasoning reveals that

$$\begin{aligned}
S(L\cap\sigma_2^{\perp},\sigma^2,v)&=\frac{1}{2}P_1(v_3)\Big(P_2(v_1-v_3)-P_2(v_1)\Big)+\frac{1}{6}\Big(P_3(v_1-v_3)+P_3(v_1)\Big),\\
S(L\cap\sigma_3^{\perp},\sigma^3,v)&=\frac{1}{2}P_1(v_2)\Big(P_2(v_1-v_2)-P_2(v_1)\Big)+\frac{1}{6}\Big(P_3(v_1-v_2)+P_3(v_1)\Big).
\end{aligned}$$

Summing up the results above, we have, by (3.1.11),

$$\begin{aligned}
&- P_1(v_1)P_1(v_2)P_1(v_3)\\
&\quad + P_1(v_1)P_1(v_2-v_1)P_1(v_3-v_1)\\
&\quad + P_1(v_1-v_2)P_1(v_2)P_1(v_3-v_2)\\
&\quad + P_1(v_1-v_3)P_1(v_2-v_3)P_1(v_3)\\
&\quad = \frac{1}{2}\Big\{P_1(v_2-v_3)\Big(P_2(v_1-v_2)-P_2(v_1-v_3)\Big)\\
&\quad + P_1(v_3)\Big(P_2(v_1-v_3)+P_2(v_2-v_3)-P_2(v_1)-P_2(v_2)\Big)\\
&\quad + P_1(v_2)\Big(P_2(v_1-v_2)-P_2(v_1)\Big)\Big\}\\
&\quad \frac{1}{6}\Big\{(2P_3(v_1-v_2)+2P_3(v_1-v_3)+P_3(v_2-v_3)+2P_3(v_1)+P_3(v_2)\Big\}.
\end{aligned}$$

Noting that the condition that {at least two of the v_j's ($j = 1, 2, 3$) are not integers}∩{it is not the case that $v_1 \equiv v_2 \equiv v_3 \bmod 1$} is equivalent to the condition that {at least two of $\{v_1, v_1 - v_3, v_1 - v_2\}$ are not integers}∩{it is not the case that $v_1 \equiv v_1 - v_3 \equiv v_1 - v_2 \bmod 1$}, we obtain the assertion by replacing (v_1, v_2, v_3) with $(v_1, v_1 - v_3, v_1 - v_2)$ and multiplying both sides by (-1). We now treat the remaining cases where at least two of the v_j's ($j = 1, 2, 3$) are integers, or, where it is the case that $v_1 \equiv v_2 \equiv v_3 \bmod 1$. Suppose at least two of the v_j's ($j = 1, 2, 3$) are integers, say $v_2, v_3 \in \mathbb{Z}$. Then the assertion becomes $P_1(v_1)^3 = \frac{3}{2}P_1(v_1)P_2(v_1) - \frac{1}{2}P_3(v_1) - \frac{1}{4}\delta_{\{v_1\in\mathbb{Z}\}}$, which clearly holds for all $v_1 \in \mathbb{R}$. Next, suppose that $v_1 \equiv v_2 \equiv v_3 \bmod 1$. Then the assertion becomes $-P_1(v_1)^3 = -P_3(v_1) - \frac{1}{4}P_1(v_1) + \frac{1}{4}\delta_{\{v_1\in\mathbb{Z}\}}$, which also clearly holds for all $v_1 \in \mathbb{R}$. This completes the proof. □

As we shall see, the Sczech 3-term addition formula is a very useful resource in evaluating sums containing products of three P_1's. In general, when confronted with a product of n P_1's, it is a good start to look at an n-term addition formula via the work of Gunnells and Sczech[7]. We now introduce a less general family of addition formulas which do not contain products of P_1's. These formulas are much easier to prove since we can avoid the complications of rearranging the terms of a conditionally convergent series.

Proposition 3.1.19. Let $n \in \mathbb{N}$ with $n \geq 3$, and let $x, y \in \mathbb{R}$. Then, we have

$$P_1(x)P_{n-1}(x+y) + P_1(y)P_{n-1}(x+y) - \frac{1}{n}\sum_{j=1}^{n-1}\binom{n}{j}P_j(x)P_{n-j}(y)$$
$$= \frac{1}{n}\Big(P_n(x) + (n-1)P_n(x+y) + P_n(y)\Big).$$

Proof. Let $p, q, r \in \mathbb{R}$ such that $\frac{1}{pq} + \frac{1}{qr} + \frac{1}{rp} = 0$, $pqr \neq 0$, and $p + q + r = 0$. Moreover, we fix q so that $\frac{\partial q}{\partial p} = 0$, $\frac{\partial r}{\partial p} = -\frac{\partial p}{\partial p} = -1$. Then, it follows that

$$\frac{1}{(n-2)!}\partial_p^{(n-2)}\left(\frac{1}{pq} + \frac{1}{qr} + \frac{1}{rp}\right) = \frac{(-1)^n}{p^{n-1}q} + \frac{1}{qr^{n-1}} + \sum_{j=1}^{n-1}(-1)^{j-1}\frac{1}{p^j r^{n-j}} = 0.$$

Multiplying both sides by $e(p(x+y)+qy) = e(-qx - r(x+y)) = e(px - ry)$, and summing symmetrically (so as to give meaning to the conditionally convergent series) over all $p, q, r \in \mathbb{Z}$, we get

$$(-1)^n \sum_{\substack{p,q\in\mathbb{Z}\\ r\neq 0}}{}' \frac{e(p(x+y)+qy)}{p^{n-1}q} + \sum_{\substack{q,r\in\mathbb{Z}\\ p\neq 0}}{}' \frac{e(-qx-r(x+y))}{qr^{n-1}} + \sum_{j=1}^{n-1}(-1)^{j-1}\sum_{\substack{r,p\in\mathbb{Z}\\ q\neq 0}}{}' \frac{e(px-ry)}{p^j r^{n-j}}$$
$$= (-1)^n \sum_{p,q\in\mathbb{Z}}{}' \frac{e(p(x+y)+qy)}{p^{n-1}q} + \sum_{q,r\in\mathbb{Z}}{}' \frac{e(-qx-r(x+y)}{qr^{n-1}} + \sum_{j=1}^{n-1}(-1)^{j-1}\sum_{r,p\in\mathbb{Z}}{}' \frac{e(px-ry)}{p^j r^{n-j}}$$
$$-\left\{(-1)^{n-1}\sum_{p\in\mathbb{Z}}{}' \frac{e(px)}{p^n} + (-1)^{n-1}\sum_{q\in\mathbb{Z}}{}' \frac{e(qy)}{q^n} + (-1)^{n-1}\sum_{j=1}^{n-1}\sum_{p\in\mathbb{Z}}{}' \frac{e(p(x+y))}{p^n}\right\}$$
$$= (-1)^n\left\{\sum_{p,q\in\mathbb{Z}}{}' \frac{e(px-qy)}{p^{n-1}q} + \sum_{q,r\in\mathbb{Z}}{}' \frac{e(q(x+y)+rx)}{qr^{n-1}} - \sum_{j=1}^{n-1}\sum_{r,p\in\mathbb{Z}}{}' \frac{e(px+ry)}{p^j r^{n-j}}\right.$$
$$\left. + \sum_{p\in\mathbb{Z}}{}' \frac{e(px)}{p^n} + \sum_{r\in\mathbb{Z}}{}' \frac{e(ry)}{r^n} + (n-1)\sum_{p\in\mathbb{Z}}{}' \frac{e(p(x+y))}{p^n}\right\} = 0.$$

Thus the assertion follows from multiplying both sides by $\frac{(-1)^n(n-1)!}{(2\pi i)^n}$ and applying

Proposition 3.1.10. □

We remark that the addition formulas of Proposition 3.1.19 with parameter n can easily be obtained from n-term addition formulas of Gunnells and Sczech, which tell the complete story. For example, the addition formula of Proposition 3.1.19 with parameter $n = 3$ follows from the Sczech 3-term addition formula with $(v_1, v_2, v_3) = (x, y, x+y)$. Despite this, the addition formulas of Proposition 3.1.19 can still prove useful (see Proposition 3.2.34).

We now state a powerful reciprocity theorem given by Carlitz. Let $r, s \in \mathbb{Z}$ with $r, s \geq 0$, $h, k \in \mathbb{Z}$ with $k \geq 1$, and $x, y \in \mathbb{R}$. Letting $\bar{B}_k(x) = B_k(x - [x])$, Carlitz[5] introduced the functions

$$\begin{cases} \tilde{\phi}_{r,s}(h,k;x,y) = \displaystyle\sum_{a(k)} \bar{B}_r\left(h\frac{a+y}{k} + x\right)\bar{B}_s\left(\frac{a+y}{k}\right), \\ \tilde{\psi}_{r,s}(h,k;x,y) = \displaystyle\sum_{j=0}^{r} (-1)^{r-j}\binom{r}{j} h^{r-j}\tilde{\phi}_{j,r+s-j}(h,k;x,y), \end{cases}$$

and proved the following reciprocity formula,

$$\begin{aligned} &(s+1)k^s\tilde{\psi}_{r+1,s}(h,k;x,y) - (r+1)h^r\tilde{\psi}_{s+1,r}(k,h;y,x) \\ &\quad = (s+1)k\bar{B}_{r+1}(x)\bar{B}_s(y) - (r+1)h\bar{B}_r(x)\bar{B}_{s+1}(y). \end{aligned} \tag{3.1.13}$$

Since I prefer to work with the periodic Bernoulli polynomials P_k rather than the Bernoulli functions $\bar{B}_k$, we consider the functions

$$\begin{cases} \phi_{r,s}(h,k;x,y) = \displaystyle\sum_{a(k)} P_r\left(h\frac{a+y}{k} + x\right)P_s\left(\frac{a+y}{k}\right), \\ \psi_{r,s}(h,k;x,y) = \displaystyle\sum_{j=0}^{r} (-1)^{r-j}\binom{r}{j} h^{r-j}\phi_{j,r+s-j}(h,k;x,y). \end{cases} \tag{3.1.14}$$

We note that $\phi_{1,1}(h,k;0,0)$ are Dedekind sums and $\phi_{1,1}(h,k;x,y)$ are Rademacher sums. We now prove the following reciprocity theorem which will also be referred to as "Carlitz reciprocity".

Proposition 3.1.20. *Let $r, s \in \mathbb{Z}$ with $r, s \geq 0$, $h, k \in \mathbb{N}$ with $(h, k) = 1$, and $x, y \in \mathbb{R}$. Then, we have*

$$\begin{aligned}&(s+1)k^s\psi_{r+1,s}(h,k;x,y) - (r+1)h^r\psi_{s+1,r}(k,h;y,x)\\ &\quad = (s+1)kP_{r+1}(x)P_s(y) - (r+1)hP_r(x)P_{s+1}(y) - 2(-1)^r k^s h^r B_1^2 \delta_{r+s,1}\delta_{x,0}\delta_{y,0},\end{aligned}$$

where ψ is given by (3.1.14).

Proof. In view of the reciprocity law given in (3.1.13), the assertion is equivalent to

$$\begin{aligned}&(s+1)k^s\Big(\tilde{\psi}_{r+1,s}(h,k;x,y) - \psi_{r+1,s}(h,k;x,y)\Big)\\ &\quad - (r+1)h^r\Big(\tilde{\psi}_{s+1,r}(k,h;y,x) - \psi_{s+1,r}(k,h;y,x)\Big)\\ &\quad = (s+1)k\Big(\bar{B}_{r+1}(x)\bar{B}_s(y) - P_{r+1}(x)P_s(y)\Big)\\ &\quad - (r+1)h\Big(\bar{B}_r(x)\bar{B}_{s+1}(y) - P_r(x)P_{s+1}(y)\Big)\\ &\quad + 2(-1)^r k^s h^r B_1^2\delta_{r+s,1}\delta_{x,0}\delta_{y,0}.\end{aligned} \tag{3.1.15}$$

Without a loss of generality, we may assume that $0 \leq x, y < 1$. Since $\bar{B}_k(x) = P_k(x) + \delta_{k,1}\delta_{\{x\in\mathbb{Z}\}}B_1$, we have, by definition,

$$\begin{aligned}&\tilde{\phi}_{r,s}(h,k;x,y) - \phi_{r,s}(h,k;x,y)\\ &\quad = \delta_{r,1}\sum_{a(k)}\delta_{\left\{\frac{h(a+y)}{k}+x\in\mathbb{Z}\right\}}B_1P_s\left(\frac{a+y}{k}\right) + \delta_{s,1}\delta_{y,0}B_1P_r(x) + \delta_{r,1}\delta_{s,1}\delta_{x,0}\delta_{y,0}B_1^2.\end{aligned}$$

Hence, we get

$$\begin{aligned}
&\tilde{\psi}_{r+1,s}(h,k;x,y) - \psi_{r+1,s}(h,k;x,y) \\
&\quad = \sum_{j=0}^{r+1} (-1)^{r+1-j} \binom{r+1}{j} h^{r+1-j} \Big(\tilde{\phi}_{j,r+s+1-j}(h,k;x,y) - \phi_{j,r+s+1-j}(h,k;x,y) \Big) \\
&\quad = \sum_{j=0}^{r+1} (-1)^{r+1-j} \binom{r+1}{j} h^{r+1-j} \Bigg\{ \delta_{j,1} \sum_{a(k)} \delta_{\left\{\frac{h(a+y)}{k}+x\in\mathbb{Z}\right\}} B_1 P_{r+s+1-j}\left(\frac{a+y}{k}\right) \\
&\qquad + \delta_{r+s+1-j,1}\delta_{y,0} B_1 P_j(x) + \delta_{j,1}\delta_{r+s+1-j,1}\delta_{x,0}\delta_{y,0} B_1^2 \Bigg\} \\
&\quad = (-1)^r (r+1) h^r \sum_{a(k)} \delta_{\left\{\frac{h(a+y)}{k}+x\in\mathbb{Z}\right\}} B_1 P_{r+s}\left(\frac{a+y}{h}\right) \\
&\qquad + (-1)^{1-s} \binom{r+1}{1-s} h^{1-s} \delta_{\{s\le 1\}} \delta_{y,0} B_1 P_{r+s}(x) + (-1)^r (r+1) h^r B_1^2 \delta_{r+s,1}\delta_{x,0}\delta_{y,0}.
\end{aligned}$$

And similarly,

$$\begin{aligned}
&\tilde{\psi}_{s+1,r}(k,h;y,x) - \psi_{s+1,r}(k,h;y,x) \\
&\quad = (-1)^s (s+1) k^s \sum_{b(k)} \delta_{\left\{\frac{k(b+x)}{h}+y\in\mathbb{Z}\right\}} B_1 P_{r+s}\left(\frac{b+x}{h}\right) \\
&\qquad + (-1)^{1-r} \binom{s+1}{1-r} k^{1-r} \delta_{\{r\le 1\}} \delta_{x,0} B_1 P_{r+s}(y) + (-1)^s (s+1) k^s B_1^2 \delta_{r+s,1}\delta_{x,0}\delta_{y,0}.
\end{aligned}$$

We now show that

$$\sum_{a(k)} \delta_{\left\{\frac{h(a+y)}{k}+x\in\mathbb{Z}\right\}} P_{r+s}\left(\frac{a+y}{h}\right) = (-1)^{r+s} \sum_{b(k)} \delta_{\left\{\frac{k(b+x)}{h}+y\in\mathbb{Z}\right\}} P_{r+s}\left(\frac{b+x}{h}\right). \tag{3.1.16}$$

Observe that the existence of an $\tilde{a} \in \{0,\cdots,k-1\}$ such that $\frac{h(\tilde{a}+y)}{k}+x \in \mathbb{Z}$ is equivalent to the existence of a $\tilde{b} \in \{0,\cdots,h-1\}$ such that $\frac{k(\tilde{b}+x)}{h}+y \in \mathbb{Z}$. Thus, if there is not an $\tilde{a} \in \{0,\cdots,k-1\}$ such that $\frac{h(\tilde{a}+y)}{k}+x \in \mathbb{Z}$, then (3.1.16) clearly holds since both sides vanish. Otherwise, there is an $\tilde{a} \in \{0,\cdots,k-1\}$, $\tilde{b} \in \{0,\cdots,h-1\}$ such that $\frac{h(\tilde{a}+y)}{k}+x$, $\frac{k(\tilde{b}+x)}{h}+y \in \mathbb{Z}$, or equivalently, $\frac{\tilde{a}+y}{k}+\frac{\tilde{b}+x}{h} \in \{0,1\}$. Hence, we get

$$\sum_{a(k)} \delta_{\left\{\frac{h(a+y)}{k}+x\in\mathbb{Z}\right\}} P_{r+s}\left(\frac{a+y}{h}\right) = P_{r+s}\left(\frac{\tilde{a}+y}{k}\right) = P_{r+s}\left(-\frac{\tilde{b}+x}{h}\right)$$
$$= (-1)^{r+s} P_{r+s}\left(\frac{\tilde{b}+x}{h}\right) = (-1)^{r+s} \sum_{b(k)} \delta_{\left\{\frac{k(b+x)}{h}+y\in\mathbb{Z}\right\}} P_{r+s}\left(\frac{b+x}{h}\right).$$

Thus (3.1.16) is established. Computing the left-hand side of (3.1.15) where we take (3.1.16) into account, we get

$$\begin{aligned}
&(s+1)k^s\Big(\tilde{\psi}_{r+1,s}(h,k;x,y) - \psi_{r+1,s}(h,k;x,y)\Big)\\
&\quad - (r+1)h^r\Big(\tilde{\psi}_{s+1,r}(k,h;y,x) - \psi_{s+1,r}(k,h;y,x)\Big)\\
&\quad = (s+1)(r+1)k^s h^r B_1\Bigg\{(-1)^r \sum_{a(k)} \delta_{\left\{\frac{h(a+y)}{k}+x\in\mathbb{Z}\right\}} P_{r+s}\left(\frac{a+y}{h}\right)\\
&\qquad - (-1)^s \sum_{b(k)} \delta_{\left\{\frac{k(b+x)}{h}+y\in\mathbb{Z}\right\}} P_{r+s}\left(\frac{b+x}{h}\right)\Bigg\}\\
&\qquad + (-1)^{1-s}(s+1)\binom{r+1}{1-s} k^s h^{1-s} \delta_{\{s\le 1\}}\delta_{y,0} B_1 P_{r+s}(x)\\
&\qquad - (-1)^{1-r}(r+1)\binom{s+1}{1-r} h^r k^{1-r} \delta_{\{r\le 1\}}\delta_{x,0} B_1 P_{r+s}(y)\\
&\qquad + (s+1)(r+1)k^s h^r B_1^2 \delta_{r+s,1}\delta_{x,0}\delta_{y,0}\Big((-1)^r - (-1)^s\Big)\\
&\quad = (-1)^{1-s}(s+1)\binom{r+1}{1-s} k^s h^{1-s} \delta_{\{s\le 1\}}\delta_{y,0} B_1 P_{r+s}(x)\\
&\qquad - (-1)^{1-r}(r+1)\binom{s+1}{1-r} h^r k^{1-r} \delta_{\{r\le 1\}}\delta_{x,0} B_1 P_{r+s}(y)\\
&\qquad + 4(-1)^r k^s h^r B_1^2 \delta_{r+s,1}\delta_{x,0}\delta_{y,0}.
\end{aligned}$$

Computing the right-hand side of (3.1.15), we get

$$\begin{aligned}
&(s+1)k\Big(\bar{B}_{r+1}(x)\bar{B}_s(y) - P_{r+1}(x)P_s(y)\Big)\\
&\quad - (r+1)h\Big(\bar{B}_r(x)\bar{B}_{s+1}(y) - P_r(x)P_{s+1}(y)\Big)\\
&\quad + 2(-1)^r k^s h^r B_1^2 \delta_{r+s,1}\delta_{x,0}\delta_{y,0}\\
&\quad = (s+1)k\Big(\delta_{s,1}\delta_{y,0}B_1P_{r+1}(x) + \delta_{r,0}\delta_{x,0}B_1P_s(y) + \delta_{r,0}\delta_{s,1}\delta_{x,0}\delta_{y,0}B_1^2\Big)\\
&\quad - (r+1)h\Big(\delta_{s,0}\delta_{y,0}B_1P_r(x) + \delta_{r,1}\delta_{x,0}B_1P_{s+1}(y) + \delta_{r,1}\delta_{s,0}\delta_{x,0}\delta_{y,0}B_1^2\Big)\\
&\quad + 2(-1)^r k^s h^r B_1^2 \delta_{r+s,1}\delta_{x,0}\delta_{y,0}.
\end{aligned}$$

Thus (3.1.15) is equivalent to

$$\begin{aligned}
&(-1)^{1-s}(s+1)\binom{r+1}{1-s}k^s h^{1-s}\delta_{\{s\le 1\}}\delta_{y=0}B_1P_{r+s}(x)\\
&\quad - (-1)^{1-r}(r+1)\binom{s+1}{1-r}h^r k^{1-r}\delta_{\{r\le 1\}}\delta_{x=0}B_1P_{r+s}(y)\\
&\quad + 2(-1)^r k^s h^r B_1^2 \delta_{r+s,1}\delta_{x,0}\delta_{y,0}\\
&\quad = (s+1)k\Big(\delta_{s,1}\delta_{y,0}B_1P_{r+1}(x) + \delta_{r,0}\delta_{x,0}B_1P_s(y) + \delta_{r,0}\delta_{s,1}\delta_{x,0}\delta_{y,0}B_1^2\Big)\\
&\quad - (r+1)h\Big(\delta_{s,0}\delta_{y,0}B_1P_r(x) + \delta_{r,1}\delta_{x,0}B_1P_{s+1}(y) + \delta_{r,1}\delta_{s,0}\delta_{x,0}\delta_{y,0}B_1^2\Big).
\end{aligned} \tag{3.1.17}$$

We shall establish (3.1.17) by verifying all of the various possibilities. If $x, y \neq 0$, then (3.1.17) clearly holds since everything vanishes. Moreover, if $r, s \geq 2$, then (3.1.17) also clearly holds since everything vanishes. Next, we assume that $x = 0$, $y \neq 0$. From (3.1.17), we get

$$\begin{aligned}
&- (-1)^{1-r}(r+1)\binom{s+1}{1-r}h^r k^{1-r}\delta_{\{r\le 1\}}B_1P_{r+s}(y)\\
&\quad = (s+1)k\delta_{r,0}B_1P_s(y) - (r+1)h\delta_{r,1}B_1P_{s+1}(y),
\end{aligned}$$

which is easily verified for $r = 0$, $r = 1$, and $r \geq 2$. Next, we assume that $x \neq 0$, $y = 0$.

From (3.1.17), we get

$$(-1)^{1-s}(s+1)\binom{r+1}{1-s}k^s h^{1-s}\delta_{\{s\le 1\}}B_1P_{r+s}(x)$$
$$= (s+1)k\delta_{s,1}B_1P_{r+1}(x) - (r+1)h\delta_{s,0}B_1P_r(x),$$

which is also easily verified for $s = 0$, $s = 1$, and $s \geq 2$. Lastly, we assume that $x = y = 0$. From (3.1.17), we get

$$\begin{aligned}
&(-1)^{1-s}(s+1)\binom{r+1}{1-s}k^s h^{1-s}\delta_{\{s\le 1\}}B_1P_{r+s}(0)\\
&- (-1)^{1-r}(r+1)\binom{s+1}{1-r}h^r k^{1-r}\delta_{\{r\le 1\}}B_1P_{r+s}(0)\\
&+ 2(-1)^r k^s h^r B_1^2\delta_{r+s,1}\\
&= (s+1)k\Big(\delta_{s,1}B_1P_{r+1}(0) + \delta_{r,0}B_1P_s(0) + \delta_{r,0}\delta_{s,1}B_1^2\Big)\\
&- (r+1)h\Big(\delta_{s,0}B_1P_r(0) + \delta_{r,1}B_1P_{s+1}(0) + \delta_{r,1}\delta_{s,0}B_1^2\Big),
\end{aligned}$$

from which the cases $(r, s) = (0, 0), (0, 1), (1, 0), (1, 1), (0, l), (1, l), (l, 0), (l, 1)$ for $l \geq 2$ are all easily verified. This establishes (3.1.17), and consequently, completes the proof. □

3.2 Auxiliary Sums

In this section, we obtain formulas for all of the auxiliary sums needed in Chapters 4 and 5. In many cases, they are interesting in their own right. To fix our standpoint, we assume here that we are satisfied if an exponential sum or a character sum can be expressed in terms of generalized Bernoulli numbers. We remark that all of the results in this section will be obtained using only elementary methods from algebra and number theory.

We fix an odd prime p. We denote by ψ the Legendre symbol mod p: $\psi(a) = \left(\frac{a}{p}\right)$. We also put $\zeta = exp(2\pi i/p)$.

Proposition 3.2.1. *Let $n \in \mathbb{Z}$, $n \geq 0$, and let l be any integer. Then, we have*

$$\sum_{k(p)} P_{2n+1}\left(\frac{lk}{p}\right)\psi(k) = \sum_{k(p)} P_{2n+1}\left(\frac{lk^2}{p}\right) = \frac{\psi(l)}{p^{2n}} B_{2n+1,\psi}.$$

Proof. The assertion is clear when $l \equiv 0(p)$, so we assume $l \not\equiv 0(p)$. Since P_{2n+1} is an odd function, we have $\sum_{k(p)} P_{2n+1}\left(\frac{lk}{p}\right) = 0$. Hence,

$$\begin{aligned}\sum_{k(p)} P_{2n+1}\left(\frac{lk}{p}\right)\psi(k) &= \sum_{k(p)} P_{2n+1}\left(\frac{lk}{p}\right)\psi(k) + \sum_{k(p)} P_{2n+1}\left(\frac{lk}{p}\right) \\ &= 2\sum_{\substack{k(p)\\ \psi(k)=1}} P_{2n+1}\left(\frac{lk}{p}\right) \\ &= \sum_{k(p)} P_{2n+1}\left(\frac{lk^2}{p}\right).\end{aligned}$$

Thus we get the first equality. Replacing k by $l^{-1}k$ in the first sum where we note that $\psi(l^{-1}) = \psi(l)$, we get

$$\sum_{k(p)} P_{2n+1}\left(\frac{lk}{p}\right)\psi(k) = \psi(l^{-1})\sum_{k(p)} P_{2n+1}\left(\frac{k}{p}\right)\psi(k) = \frac{\psi(l)}{p^{2n}} B_{2n+1,\psi}.$$

Thus we obtain the second equality. $\square$

Proposition 3.2.2. *Let l be any integer prime to p. Then, we have*

$$(i) \quad \sum_{k(p)} P_2\left(\frac{lk}{p}\right) = \frac{1}{6p},$$

$$(ii) \quad \text{If } p \equiv 3(4), \text{ then } \sum_{k(p)} P_2\left(\frac{lk^2}{p}\right) = \frac{1}{6p}.$$

Proof. Replacing k by $l^{-1}k$ in (i), we get

$$\sum_{k(p)} P_2\left(\frac{lk}{p}\right) = \sum_{k(p)} P_2\left(\frac{k}{p}\right) = \sum_{k=0}^{p-1}\left(\left(\frac{k}{p}\right)^2 - \left(\frac{k}{p}\right) + \frac{1}{6}\right) = \frac{1}{6p}.$$

Thus the assertion (i) is proved. If $p \equiv 3(4)$, then since P_2 is even and ψ odd, we get

$$\sum_{k(p)} P_2\left(\frac{lk}{p}\right)\psi(k) = 0.$$

Hence,

$$\begin{aligned}\sum_{k(p)} P_2\left(\frac{lk}{p}\right) &= \sum_{k(p)} P_2\left(\frac{lk}{p}\right) + \sum_{k(p)} P_2\left(\frac{lk}{p}\right)\psi(k) \\ &= 2\sum_{\substack{k(p)\\ \psi(k)=1}} P_2\left(\frac{lk}{p}\right) \\ &= \sum_{k(p)} P_2\left(\frac{lk^2}{p}\right).\end{aligned}$$

Thus the assertion (ii) follows from (i). □

Proposition 3.2.3. *Let l be any integer. We have*

$$\sum_{k(p)} P_1\left(\frac{lk}{p}\right) P_1\left(\frac{k^2}{p}\right) = \sum_{k(p)} P_1\left(\frac{lk}{p}\right) P_2\left(\frac{k^2}{p}\right) = 0.$$

Proof. This follows from replacing k by $-k$ in the two sums above and noting that P_1 is odd. □

Proposition 3.2.4. *We have*

$$\sum_{k,t(p)} P_1\left(\frac{t^2}{p}\right) P_2\left(\frac{k^2-t^2}{p}\right) = -\frac{1}{3p}B_{3,\psi} + \frac{1}{6p}B_{1,\psi}.$$

Proof. If $p \equiv 1(4)$, replacing (k,t) by $(k\sqrt{-1}, t\sqrt{-1})$, noting that P_1 is odd and P_2 is even, reveals that

$$\sum_{k,t(p)} P_1\left(\frac{t^2}{p}\right) P_2\left(\frac{k^2-t^2}{p}\right) = 0.$$

Since $B_{3,\psi}$ and $B_{1,\psi}$ also vanish (by Proposition 3.1.7), the assertion is clear in the case $p \equiv 1(4)$. Thus we assume that $p \equiv 3(4)$. Applying the finite Fourier transform (Proposition 3.1.16) to $P_1\left(\frac{t^2}{p}\right)$ and $P_2\left(\frac{k^2-t^2}{p}\right)$, we find that

$$\begin{aligned}
\sum_{k,t(p)} P_1\left(\frac{t^2}{p}\right) P_2\left(\frac{k^2-t^2}{p}\right) &= \sum_{k,t(p)} \frac{2}{(2\pi i p)^3} \sum_{a,b(p)} c_1\left(\frac{a}{p}\right) c_2\left(\frac{b}{p}\right) \sum_{k,t(p)} \zeta^{(a-b)t^2+bk^2} \\
&= \frac{2}{(2\pi i p)^3}\left(\sum_{\substack{a,b(p)\\ b\equiv 0(p)}} + \sum_{\substack{a,b(p)\\ b\not\equiv 0(p)\\ a-b\equiv 0(p)}} + \sum_{\substack{a,b(p)\\ b\not\equiv 0(p)\\ a-b\not\equiv 0(p)}}\right) c_1\left(\frac{a}{p}\right) c_2\left(\frac{b}{p}\right) \sum_{k,t(p)} \zeta^{(a-b)t^2+bk^2} \\
&= \frac{2}{(2\pi i p)^3}\Bigg\{p\,\tau(\psi)\,c_2(0)\sum_{a(p)} c_1\left(\frac{a}{p}\right)\psi(a) + p\,\tau(\psi) \sum_{\substack{a(p)\\ b\not\equiv 0(p)\\ a-b\equiv 0(p)}} c_1\left(\frac{a}{p}\right) c_2\left(\frac{b}{p}\right)\psi(b) \\
&\quad + \tau^2(\psi) \sum_{\substack{a,b(p)\\ b\not\equiv 0(p)\\ a-b\not\equiv 0(p)}} c_1\left(\frac{a}{p}\right) c_2\left(\frac{b}{p}\right)\psi(b)\psi(a-b)\Bigg\} \\
&= \frac{2}{(2\pi i p)^3}\Bigg\{p\,\tau(\psi)\,c_2(0)\sum_{a(p)} c_1\left(\frac{a}{p}\right)\psi(a) + p\,\tau(\psi)\sum_{a(p)} c_1\left(\frac{a}{p}\right) c_2\left(\frac{a}{p}\right)\psi(a) \\
&\quad + \tau^2(\psi)\sum_{a,b(p)} c_1\left(\frac{a}{p}\right) c_2\left(\frac{b}{p}\right)\psi(b)\psi(a-b)\Bigg\}.
\end{aligned}$$

Replacing (a,b) by $(-a,-b)$ in the third sum, noting that c_1, ψ are odd and c_2 is even, we see that the third sum vanishes. Hence, by (3.1.6), we get

$$\sum_{k,t(p)} P_1\left(\frac{t^2}{p}\right) P_2\left(\frac{k^2-t^2}{p}\right) = \frac{2p\,\tau(\psi)}{(2\pi i p)^3}\left\{ c_2(0)\sum_{a(p)} c_1\left(\frac{a}{p}\right)\psi(a) + \sum_{a(p)} c_3\left(\frac{a}{p}\right)\psi(a)\right\}.$$

From Proposition 3.1.16 and (iii) in Proposition 3.1.9, we get

$$\begin{aligned}\sum_{a(p)} c_k\left(\frac{a}{p}\right)\psi(a) &= -\frac{(-1)^k(2\pi i p)^k}{pk!}\sum_{b(p)} P_k\left(\frac{b}{p}\right)\sum_{a(p)}\zeta^{ba}\psi(a)\\ &= -\frac{(-1)^k\tau(\psi)(2\pi i)^k}{k!}p^{k-1}\sum_{b(p)} P_k\left(\frac{b}{p}\right)\psi(b) \qquad (3.2.1)\\ &= -\frac{(-1)^k\tau(\psi)(2\pi i)^k}{k!}B_{k,\psi}.\end{aligned}$$

Noting that $c_2(0) = \pi^2/3$ and applying (3.2.1), we obtain the assertion of the proposition. □

Proposition 3.2.5. *We have*

$$(i)\quad \sum_{k,t(p)} P_1\left(\frac{t^2}{p}\right) P_2\left(\frac{k^2 \pm 2kt}{p}\right) = -\frac{1}{3p}B_{3,\psi} + \frac{1}{6p}B_{1,\psi},$$

$$(ii)\quad \sum_{k,t(p)} P_1\left(\frac{(k\pm t)^2}{p}\right) P_2\left(\frac{k^2 \pm 2kt}{p}\right) = -\frac{1}{3p}B_{3,\psi} + \frac{1}{6p}B_{1,\psi}.$$

Proof. The assertion (i) follows from replacing (k,t) by $(k \mp t, t)$ and applying Proposition 3.2.4. The assertion (ii) follows from replacing (k,t) by $(t \mp k, k)$, noting that P_2 is even, and applying Proposition 3.2.4. □

Proposition 3.2.6. *We have*

$$\sum_{k,t(p)} P_1\left(\frac{2kt}{p}\right) P_2\left(\frac{k^2 + 2kt}{p}\right) = \frac{1}{3p}B_{3,\psi}.$$

Proof. If $p \equiv 1(4)$, replacing (k,t) by $(k\sqrt{-1}, t\sqrt{-1})$, noting that P_1 is odd and

P_2 is even, reveals that

$$\sum_{k,t(p)} P_1\left(\frac{2kt}{p}\right) P_2\left(\frac{k^2+2kt}{p}\right) = 0.$$

Since $B_{3,\psi}$ also vanishes (by Proposition 3.1.7), the assertion is clear in the case $p \equiv 1(4)$. Thus we assume that $p \equiv 3(4)$. Replacing (k,t) by $(k,(2k)^{-1}t)$ where $k \not\equiv 0(p)$, we get

$$\sum_{k,t(p)} P_1\left(\frac{2kt}{p}\right) P_2\left(\frac{k^2+2kt}{p}\right) = \sum_{\substack{k,t(p)\\ k\not\equiv 0(p)}} P_1\left(\frac{t}{p}\right) P_2\left(\frac{k^2+t}{p}\right).$$

Since

$$\sum_{t(p)} P_1\left(\frac{t}{p}\right) P_2\left(\frac{t}{p}\right) = 0$$

follows from replacing t by $-t$, we have

$$\sum_{k,t(p)} P_1\left(\frac{2kt}{p}\right) P_2\left(\frac{k^2+2kt}{p}\right) = \sum_{k,t(p)} P_1\left(\frac{t}{p}\right) P_2\left(\frac{k^2+t}{p}\right).$$

Applying the finite Fourier transform (Proposition 3.1.16) to $P_1\left(\frac{t}{p}\right)$ and $P_2\left(\frac{k^2-t}{p}\right)$, we find that

$$\begin{aligned}
\sum_{k,t(p)} P_1\left(\frac{t}{p}\right) P_2\left(\frac{k^2+t}{p}\right) &= \sum_{k,t(p)} \frac{2}{(2\pi i p)^3} \sum_{a,b(p)} c_1\left(\frac{a}{p}\right) c_2\left(\frac{b}{p}\right) \sum_{k,t(p)} \zeta^{(a+b)t+bk^2} \\
&= \frac{2p}{(2\pi i p)^3} \sum_{a(p)} c_1\left(\frac{a}{p}\right) c_2\left(\frac{-a}{p}\right) \sum_{k(p)} \zeta^{-ak^2} \\
&= -\frac{2p\,\tau(\psi)}{(2\pi i p)^3} \sum_{a(p)} c_1\left(\frac{a}{p}\right) c_2\left(\frac{a}{p}\right) \psi(a).
\end{aligned}$$

By (3.1.6), (3.2.1), we get

$$\sum_{k,t(p)} P_1\left(\frac{t}{p}\right) P_2\left(\frac{k^2+t}{p}\right) = -\frac{2p\,\tau(\psi)}{(2\pi i p)^3}\sum_{a(p)} c_3\left(\frac{a}{p}\right)\psi(a) = \frac{1}{3p}B_{3,\psi}.$$

□

Proposition 3.2.7. We have

$$(i) \qquad \sum_{k,t(2p)} P_1\left(\frac{t^2}{2p}\right) P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{k^2 \pm 2kt}{2p}\right) = \sum_{k,t(p)} P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{k^2 \pm 2kt}{p}\right),$$

$$(ii) \qquad \sum_{k,t(2p)} P_1\left(\frac{t^2}{2p}\right) P_1\left(\frac{2kt}{p}\right) P_1\left(\frac{(k\pm t)^2}{2p}\right) = \sum_{k,t(p)} P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{2kt}{p}\right) P_1\left(\frac{(k\pm t)^2}{p}\right),$$

$$(iii) \qquad \sum_{k,t(2p)} P_1\left(\frac{(k\pm t)^2}{2p}\right) P_1\left(\frac{2kt}{p}\right) P_1\left(\frac{k^2 \pm 2kt}{2p}\right) = \sum_{k,t(p)} P_1\left(\frac{(k\pm t)^2}{p}\right) P_1\left(\frac{2kt}{p}\right) P_1\left(\frac{k^2 \pm 2kt}{p}\right),$$

$$(iv) \qquad \sum_{k,t(2p)} P_1\left(\frac{(k+t)^2}{2p}\right) P_2\left(\frac{2kt}{p}\right) = 2\sum_{k,t(2p)} P_1\left(\frac{(k+t)^2}{p}\right) P_2\left(\frac{2kt}{p}\right).$$

Proof. We first prove the assertion (i). We have

$$\begin{aligned}
\sum_{k(2p)} P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{k^2 \pm 2kt}{2p}\right) &= \sum_{k=0}^{p-1} P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{k^2 \pm 2kt}{2p}\right) \\
&\quad + \sum_{k=0}^{p-1} P_1\left(\frac{(k+p)^2}{p}\right) P_1\left(\frac{(k+p)^2 \pm 2(k+p)t}{2p}\right) \\
&= \sum_{k=0}^{p-1} P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{k^2 \pm 2kt}{2p}\right) + \sum_{k=0}^{p-1} P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{k^2 \pm 2kt}{2p} + \frac{1}{2}\right) \\
&= \sum_{k=0}^{p-1} P_1\left(\frac{k^2}{p}\right) \left\{ P_1\left(\frac{k^2 \pm 2kt}{2p}\right) + P_1\left(\frac{k^2 \pm 2kt}{2p} + \frac{1}{2}\right) \right\}.
\end{aligned}$$

Applying the multiplication formula (Proposition 3.1.8), we get

$$\sum_{k(2p)} P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{k^2 \pm 2kt}{2p}\right) = \sum_{k(p)} P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{k^2 \pm 2kt}{p}\right).$$

Hence, we have

$$\begin{aligned}(3.2.2)\qquad &\sum_{k,t(2p)} P_1\left(\frac{t^2}{2p}\right) P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{k^2 \pm 2kt}{2p}\right)\\ &\qquad = \sum_{k(p)} P_1\left(\frac{k^2}{p}\right) \sum_{t(2p)} P_1\left(\frac{t^2}{2p}\right) P_1\left(\frac{k^2 \pm 2kt}{p}\right).\end{aligned}$$

Then,

$$\begin{aligned}\sum_{t(2p)} P_1\left(\frac{t^2}{2p}\right) P_1\left(\frac{k^2 \pm 2kt}{p}\right) &= \sum_{t=0}^{p-1} P_1\left(\frac{t^2}{2p}\right) P_1\left(\frac{k^2 \pm 2kt}{p}\right)\\ &\quad + \sum_{t=0}^{p-1} P_1\left(\frac{(t+p)^2}{2p}\right) P_1\left(\frac{k^2 \pm 2k(t+p)}{p}\right)\\ &= \sum_{t=0}^{p-1} P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{k^2 \pm 2kt}{p}\right) + \sum_{t=0}^{p-1} P_1\left(\frac{t^2}{p}+\frac{1}{2}\right) P_1\left(\frac{k^2 \pm 2kt}{p}\right)\\ &= \sum_{t=0}^{p-1}\left\{P_1\left(\frac{t^2}{2p}\right) + P_1\left(\frac{t^2}{2p}+\frac{1}{2}\right)\right\} P_1\left(\frac{k^2 \pm 2kt}{p}\right).\end{aligned}$$

Again applying the multiplication formula, we get

$$\sum_{t(2p)} P_1\left(\frac{t^2}{2p}\right) P_1\left(\frac{k^2 \pm 2kt}{p}\right) = \sum_{t(p)} P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{k^2 \pm 2kt}{p}\right).$$

Thus, from (3.2.2), we get the assertion (i). The other identities are similarly verified, so the proofs are omitted. □

Proposition 3.2.8. We have

$$(i)\quad \sum_{k,t(2p)} P_1\left(\frac{t^2}{2p}\right) P_2\left(\frac{k^2 \pm 2kt}{2p}\right) = -\frac{1}{6p}B_{3,\psi} + \frac{1}{12p}B_{1,\psi},$$

$$(ii)\quad \sum_{k,t(2p)} P_1\left(\frac{2kt}{p}\right) P_2\left(\frac{k^2 \pm 2kt}{2p}\right) = \pm\frac{1}{3p}B_{3,\psi},$$

$$(iii)\quad \sum_{k,t(2p)} P_1\left(\frac{t^2}{2p}\right) P_2\left(\frac{k^2}{p}\right) = \frac{1}{3p}B_{1,\psi},$$

$$(iv)\quad \sum_{k,t(2p)} P_1\left(\frac{t^2}{2p}\right) P_2\left(\frac{(k \pm t)^2}{p}\right) = \frac{1}{12p}B_{1,\psi},$$

$$(v)\quad \sum_{k,t(2p)} P_1\left(\frac{t^2}{2p}\right) P_2\left(\frac{2kt}{p}\right) = \frac{1}{3p}B_{1,\psi},$$

$$(vi)\quad \sum_{k,t(2p)} P_1\left(\frac{(k \pm t)^2}{2p}\right) P_2\left(\frac{t^2}{2p}\right) = \frac{1}{12p}B_{1,\psi},$$

(vii) *For $n \in \mathbb{Z}$, $n \geq 0$, we get*

$$\sum_{k(2p)} P_{2n+1}\left(\frac{k^2}{2p}\right) = \sum_{k(2p)} P_{2n+1}\left(\frac{k}{2p}\right)\psi(k) = \frac{1}{(2p)^{2n}}B_{2n+1,\psi}.$$

Proof. We apply the same reasoning as in the proof of (i) in Proposition 3.2.7. Then, the assertion (i) follows from Proposition 3.2.4, the assertion (ii) follows from Proposition 3.2.6, and the assertions (iii)-(vii) follow from Proposition 3.2.1 and Proposition 3.2.2. □

Proposition 3.2.9. We have

$$\sum_{k,t(2p)} P_1\left(\frac{(k+t)^2}{2p}\right) P_2\left(\frac{(k-t)^2}{2p}\right) = \frac{3\psi(2)-1}{6p}B_{1,\psi}.$$

Proof. Suppose $p \equiv 1(4)$. If $\sqrt{-1} \bmod p$ is relatively prime to $2p$, we replace (k,t) by $(k\sqrt{-1}, t\sqrt{-1})$. Otherwise $(\sqrt{-1}+p)$ is relatively prime to $2p$ and we replace

(k,t) by $(k(\sqrt{-1}+p), t(\sqrt{-1}+p))$. Since P_1 is odd and P_2 is even, we find that

$$\sum_{k,t(2p)} P_1\left(\frac{(k+t)^2}{2p}\right) P_2\left(\frac{(k-t)^2}{2p}\right) = 0.$$

Since $B_{1,\psi}$ also vanishes (by Proposition 3.1.7), the assertion is clear in the case $p \equiv 1(4)$. Thus we assume that $p \equiv 3(4)$. Replacing (k,t) by $(k-t,t)$, we get

$$\sum_{k,t(2p)} P_1\left(\frac{(k+t)^2}{2p}\right) P_2\left(\frac{(k-t)^2}{2p}\right) = \sum_{k(2p)} P_1\left(\frac{k^2}{2p}\right) \sum_{t(2p)} P_2\left(\frac{(k-2t)^2}{2p}\right).$$

We have two cases to consider. Suppose k is even. Then $k=2r$ for some r, and we get

$$\sum_{t(2p)} P_2\left(\frac{(k-2t)^2}{2p}\right) = 2\sum_{t(p)} P_2\left(\frac{2(r-t)^2}{p}\right).$$

Replacing t by $t+r$ and applying (ii) in Proposition 3.2.2, we get

$$\sum_{t(2p)} P_2\left(\frac{(k-2t)^2}{2p}\right) = \frac{1}{3p}. \tag{3.2.3}$$

If k is odd, then $k=2r+p$ for some r, and we get

$$\sum_{t(2p)} P_2\left(\frac{(k-2t)^2}{2p}\right) = 2\sum_{t(p)} P_2\left(\frac{2(r-t)^2}{p}+\frac{1}{2}\right).$$

By the multiplication formula, we have $P_2\left(\frac{2(r-t)^2}{p}+\frac{1}{2}\right) = \frac{1}{2}P_2\left(\frac{4(r-t)^2}{p}\right) - P_2\left(\frac{2(r-t)^2}{p}\right)$. Replacing t by $t+r$ and applying (ii) in Proposition 3.2.2, we get

$$\sum_{t(2p)} P_2\left(\frac{(k-2t)^2}{2p}\right) = -\frac{1}{6p}. \tag{3.2.4}$$

From (3.2.3), (3.2.4), we have

$$\sum_{t(2p)} P_2\left(\frac{(k-2t)^2}{2p}\right) = \begin{cases} \dfrac{1}{3p}, & \text{if } k \text{ is even,} \\ -\dfrac{1}{6p}, & \text{if } k \text{ is odd.} \end{cases}$$

Hence,

$$\begin{aligned} \sum_{k,t(2p)} P_1\left(\frac{k^2}{2p}\right) P_2\left(\frac{(k-2t)^2}{2p}\right) &= \frac{1}{3p} \sum_{\substack{k(2p) \\ k \text{ even}}} P_1\left(\frac{k^2}{2p}\right) - \frac{1}{6p} \sum_{\substack{k(2p) \\ k \text{ odd}}} P_1\left(\frac{k^2}{2p}\right) \\ &= \frac{1}{3p} \sum_{r(p)} P_1\left(\frac{(2r)^2}{2p}\right) - \frac{1}{6p} \sum_{r(p)} P_1\left(\frac{(2r+p)^2}{2p}\right) \\ &= \frac{1}{3p} \sum_{r(p)} P_1\left(\frac{2r^2}{p}\right) - \frac{1}{6p} \sum_{r(p)} P_1\left(\frac{2r^2}{p} + \frac{1}{2}\right). \end{aligned}$$

By the multiplication formula, we have $P_1\left(\frac{2r^2}{p} + \frac{1}{2}\right) = P_1\left(\frac{4r^2}{p}\right) - P_1\left(\frac{2r^2}{p}\right)$. Then, by Proposition 3.2.1, we obtain

$$\begin{aligned} \sum_{k,t(2p)} P_1\left(\frac{k^2}{2p}\right) P_2\left(\frac{(k-2t)^2}{2p}\right) &= \frac{1}{2p} \sum_{r(p)} P_1\left(\frac{2r^2}{p}\right) - \frac{1}{6p} \sum_{r(p)} P_1\left(\frac{4r^2}{p}\right) \\ &= \frac{3\psi(2)-1}{6p} B_{1,\psi}. \end{aligned}$$

□

Proposition 3.2.10. We have

$$\begin{aligned} (i) \quad & \sum_{k,t(p)} P_1\left(\frac{2kt}{p}\right) P_2\left(\frac{k^2+t^2}{p}\right) = \sum_{k,t(p)} P_1\left(\frac{kt}{p}\right) P_2\left(\frac{t^2-k^2}{p}\right) \\ & \qquad = \sum_{k,t(p)} P_3\left(\frac{2kt}{p}\right) = 0, \\ (ii) \quad & \sum_{k,t(p)} P_3\left(\frac{(k+t)^2}{p}\right) = \frac{1}{p} B_{3,\psi}, \\ (iii) \quad & \sum_{k,t(p)} P_3\left(\frac{k^2+t^2}{p}\right) = 0. \end{aligned}$$

Proof. The assertion (i) follows from replacing k by $-k$ in each of the sums and noting that P_1, P_3 are odd. The assertion (ii) follows from replacing k by $k-t$ and applying Proposition 3.2.1. Applying the finite Fourier transform (Proposition 3.1.16) to $P_3\left(\frac{k^2+t^2}{p}\right)$, we get

$$\begin{aligned}\sum_{k,t(p)} P_3\left(\frac{k^2+t^2}{p}\right) &= -\frac{3!}{(2\pi i p)^3}\sum_{a(p)} c_3\left(\frac{a}{p}\right)\sum_{k,t(p)}\zeta^{a(k^2+t^2)}\\ &= -\frac{3!\,\tau^2(\psi)}{(2\pi i p)^3}\sum_{a(p)} c_3\left(\frac{a}{p}\right).\end{aligned}$$

Since c_3 is odd, the above sum vanishes. This completes the proof of the assertion (iii). □

Proposition 3.2.11. We have

$$\sum_{x(p)}\sum_{i,j,k=1}^{p-1} P_1\left(\frac{i}{p}\right)\psi(i+x)\psi(j+x)\psi(k+x) = -B_{1,\psi}.$$

Proof. Clearly, we have

$$\begin{aligned}\sum_{x(p)}\sum_{i,j,k=1}^{p-1} P_1\left(\frac{i}{p}\right)\psi(i+x)\psi(j+x)\psi(k+x) &= \sum_{x(p)}\sum_{i=1}^{p-1} P_1\left(\frac{i}{p}\right)\psi(i+x)\psi(x^2)\\ &= -\sum_{i=1}^{p-1} P_1\left(\frac{i}{p}\right)\psi(i) = -B_{1,\psi}.\end{aligned}$$

□

Proposition 3.2.12. We have

(*i*) *For any integer* k, *we have*

$$\sum_{x(p)} P_1\left(\frac{x+k}{p}\right) = 0.$$

(*ii*) $$\sum_{i,k(p)} P_1\left(\frac{k-i}{p}\right)\psi(i^2)\psi(k) = -B_{1,\psi}.$$

Proof. The assertion (i) follows from replacing x by $x-k$ and noting that P_1 is odd. Next, we have

$$\sum_{i,k(p)} P_1\left(\frac{k-i}{p}\right)\psi(i^2)\psi(k) = \sum_{i,k(p)} P_1\left(\frac{k-i}{p}\right)\psi(k) - \sum_{k(p)} P_1\left(\frac{k}{p}\right)\psi(k).$$

Thus the assertion (ii) follows from (i) and Proposition 3.2.1. □

Proposition 3.2.13 [2]. *Let χ be a nontrivial primitive Dirichlet character modulo f, and let a be an integer. Then, we have*

$$(i) \quad \sum_{n(f)} \chi(n) = 0,$$

$$(ii) \quad \sum_{n(f)} \zeta^{an}\chi(n) = \bar{\chi}(a)\tau(\chi),$$

$$(iii) \quad \sum_{n(f)} \zeta^{an} = \begin{cases} f, & \text{if } a \equiv 0(f), \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since χ is nontrivial, there is a b such that $(b,f)=1$ and $\chi(b)\neq 1$. Replacing n by bn in (i), we obtain

$$\sum_{n(f)} \chi(n) = \chi(b)\sum_{n(f)} \chi(n).$$

Thus the assertion (i) follows. If $a \equiv 0(f)$ in (ii), then the assertion (ii) follows from (i). Thus we assume that $a \not\equiv 0(f)$. Replacing n by $a^{-1}n$, we obtain

$$\sum_{n(f)} \zeta^{an}\chi(n) = \chi(a^{-1})\sum_{n(f)} \zeta^{n}\chi(n) = \chi(a^{-1})\tau(\chi).$$

Thus the assertion (ii) follows from the fact that $\chi(a^{-1}) = \bar{\chi}(a)$. The assertion (iii) is clear. □

Proposition 3.2.14 [2]. *For any a with $(a,p)=1$, we have*

$$\frac{1}{\zeta^a - 1} = \frac{1}{p}\sum_{n=1}^{p-1} n\zeta^{an}.$$

Proof. Observe that $(x-1)(1+x+\cdots+x^{p-1}) = x^p - 1$, which yields after differentiation,

$$(x-1)(1+2x+\cdots+(p-1)x^{p-2}) + (1+x+\cdots+x^{p-1}) = px^{p-1}.$$

Multiplying by x and replacing x by ζ^a, we get

$$(\zeta^a - 1)\sum_{n=1}^{p-1} n\zeta^{an} + \sum_{n(p)} \zeta^{an} = p.$$

Thus the assertion follows from (iii) in Proposition 3.2.13. □

Lemma 3.2.15 [2]. *We have*

$$\sum_{x(p)} \psi(x^2+e) = \begin{cases} -1, & \text{if } e \not\equiv 0(p), \\ p-1, & \text{if } e \equiv 0(p). \end{cases}$$

Proof. From (ii) in Proposition 3.2.13, for $\chi = \psi$, we see that

$$\psi(a) = \frac{1}{\tau(\psi)}\sum_{n(p)} \zeta^{an}\psi(n).$$

Hence,

$$\begin{aligned}\sum_{x(p)} \psi(x^2+e) &= \sum_{x(p)} \frac{1}{\tau(\psi)} \sum_{n(p)} \zeta^{(x^2+e)n}\psi(n) \\ &= \frac{1}{\tau(\psi)} \sum_{n(p)} \zeta^{en}\psi(n) \sum_{x(p)} \zeta^{nx^2} \\ &= \sum_{n(p)} \zeta^{en}\psi^2(n).\end{aligned}$$

Thus the assertion follows from (iii) in Propostion 3.2.13. □

Proposition 3.2.16 [2]. *Let $a, b, c \in \mathbb{Z}$, $a \not\equiv 0(p)$. Then, we have*

$$\sum_{x(p)} \psi(ax^2+bx+c) = \begin{cases} -\psi(a), & \text{if } b^2-4ac \not\equiv 0(p), \\ (p-1)\psi(a), & \text{if } b^2-4ac \equiv 0(p). \end{cases}$$

Proof. Completing the square yields

$$\sum_{x(p)} \psi(ax^2+bx+c) = \psi(a) \sum_{x(p)} \psi((x+(2a)^{-1}b)^2 - (4a^2)^{-1}(b^2-4ac)).$$

Replacing x by $x-(2a)^{-1}b$, we get

$$\sum_{x(p)} \psi(ax^2+bx+c) = \psi(a) \sum_{x(p)} \psi(x^2 - (4a^2)^{-1}(b-4ac)).$$

Thus the assertion of the proposition follows from Lemma 3.2.15. □

Proposition 3.2.17 [4]. *For any integer a prime to p, we have*

$$\sum_{n(p)} \zeta^{an}\psi(n) = \sum_{n(p)} \zeta^{an^2} = \psi(a)\tau(\psi).$$

Proof. From (ii), (iii) in Proposition 3.2.13, we get

$$\psi(a)\tau(\psi) = \sum_{n(p)} \zeta^{an}\psi(n) = \sum_{n(p)} \zeta^{an}(1+\psi(n)) = \sum_{n(p)} \zeta^{an^2}.$$

□

Here, we state the following beautiful result of Gauss.

$$\tau(\psi) = \sqrt{\psi(-1)p}. \tag{3.2.5}$$

While it is very easy to see that $\tau^2(\psi) = \psi(-1)p$, it is considerably more difficult to determine that the + sign of the square root in $\tau(\psi)$ is correct in all cases.[2] For the proof, we refer the interested reader to Berndt, Evans, and Williams[4].

Proposition 3.2.18. *Let $a, b \in \mathbb{Z}$. Then, we have*

$$\sum_{x(p)} \zeta^{ax^2+bx} = \begin{cases} p, & \text{if } a, b \equiv 0(p), \\ 0, & \text{if } a \equiv 0(p),\ b \not\equiv 0(p), \\ \tau(\psi)\psi(a)\zeta^{-(4a)^{-1}b^2}, & \text{if } a \not\equiv 0(p). \end{cases}$$

Proof. The case where $a \equiv 0(p)$ was already handled in (iii) of Proposition 3.2.13. Thus we assume that $a \not\equiv 0(p)$. Completing the square and replacing x by $x-(2a)^{-1}b$, we obtain

$$\sum_{x(p)} \zeta^{ax^2+bx} = \sum_{x(p)} \zeta^{ax^2-(4a)^{-1}b^2},$$

from which the assertion follows from Proposition 3.2.17. □

Proposition 3.2.19. *We have*

$$\sum_{k,m=1}^{p-1} km\,\psi(k+m) = -\frac{p^2}{2}(B_{2,\psi} + 2B_{1,\psi}).$$

[2]On August 30, 1805, Gauss wrote in his diary that he devoted some time to this problem every week for more than four years before he was able to prove his conjecture on the sign of these sums.

Proof. Expressing everything in terms of periodic Bernoulli polynomials, we get

$$\begin{aligned}\sum_{k,m=1}^{p-1} km\,\psi(k+m) &= p^2 \sum_{k,m=1}^{p-1} \left(P_1\left(\frac{k}{p}\right)+\frac{1}{2}\right)\left(P_1\left(\frac{m}{p}\right)+\frac{1}{2}\right)\psi(k+m)\\ &= p^2\left\{\sum_{k,m=1}^{p-1} P_1\left(\frac{k}{p}\right)P_1\left(\frac{m}{p}\right)\psi(k+m)\right.\\ &\qquad +\frac{1}{2}\sum_{k,m=1}^{p-1}\left(P_1\left(\frac{k}{p}\right)+P_1\left(\frac{m}{p}\right)\right)\psi(k+m)\\ &\qquad \left.+\frac{1}{4}\sum_{k,m=1}^{p-1}\psi(k+m)\right\}.\end{aligned}$$

From (i) in Proposition 3.2.13, we get

$$\sum_{k,m=1}^{p-1} km\,\psi(k+m) = p^2\left\{\sum_{k,m(p)} P_1\left(\frac{k}{p}\right)P_1\left(\frac{m}{p}\right)\psi(k+m) - B_{1,\psi}\right\}. \tag{3.2.6}$$

Replacing k by $k-m$ in the sum on the right side, we get

$$\sum_{k,m(p)} P_1\left(\frac{k}{p}\right)P_1\left(\frac{m}{p}\right)\psi(k+m) = \sum_{k,m(p)} P_1\left(\frac{k-m}{p}\right)P_1\left(\frac{m}{p}\right)\psi(k) \tag{3.2.7}$$

Applying the two-term addition formula (Proposition 3.1.17) with the arguments $\frac{k-m}{p}$, $\frac{m}{p}$, we get

$$\begin{aligned}&P_1\left(\frac{k-m}{p}\right)P_1\left(\frac{m}{p}\right) - P_1\left(\frac{k-m}{p}\right)P_1\left(\frac{k}{p}\right) - P_1\left(\frac{m}{p}\right)P_1\left(\frac{k}{p}\right)\\ &\quad = -\frac{1}{2}\left(P_2\left(\frac{k-m}{p}\right)+P_2\left(\frac{m}{p}\right)+P_2\left(\frac{k}{p}\right)\right)+\frac{1}{4}\delta\left(\frac{k-m}{p},\frac{m}{p}\right).\end{aligned}$$

Multiplying throughout by $\psi(k)$, carefully summing over $k, m(p)$, applying (i) in Proposition 3.2.12, (i) in Proposition 3.2.2, and (i) in Proposition 3.2.13, we obtain

$$\sum_{k,m(p)} P_1\left(\frac{k-m}{p}\right)P_1\left(\frac{m}{p}\right)\psi(k) = -\frac{1}{2}B_{2,\psi}. \tag{3.2.8}$$

Thus the assertion follows from (3.2.6), (3.2.7), and (3.2.8). □

Proposition 3.2.20. *We have*

$$\sum_{l,m(p)} P_1\left(\frac{m^2l+ml}{p}\right) P_1\left(\frac{l+ml}{p}\right)\psi(l) = -\sum_{l,m(p)} P_1\left(\frac{m^2l+ml}{p}\right) P_1\left(\frac{ml}{p}\right)\psi(l)$$
$$= -\sum_{l,m(p)} P_1\left(\frac{l+ml}{p}\right) P_1\left(\frac{ml}{p}\right)\psi(l) = -\frac{1}{2}B_{2,\psi}.$$

Proof. Replacing (l,m) by $(l,-m-1)$ in the first sum and noting that P_1 is odd, we get the first equality. Replacing (l,m) by (l^2m^{-1}, ml^{-1}) in the second sum and by $(m, m^{-1}l)$ in the third sum, where $l, m \not\equiv 0(p)$, we obtain the second equality. Replacing (l,m) by $(l,-ml^{-1})$ in the third sum where $l \not\equiv 0(p)$, we get

$$-\sum_{l,m(p)} P_1\left(\frac{l+ml}{p}\right) P_1\left(\frac{ml}{p}\right)\psi(l) = \sum_{l,m(p)} P_1\left(\frac{l-m}{p}\right) P_1\left(\frac{m}{p}\right)\psi(l).$$

Thus the third equality follows from (3.2.8). □

Proposition 3.2.21. *We have*

$$(i)\qquad \sum_{l,m(p)} P_1\left(\frac{l+ml}{p}\right)\psi(l) = \sum_{l,m(p)} P_1\left(\frac{ml}{p}\right)\psi(l) = 0,$$
$$(ii)\qquad \sum_{l,m(p)} P_1\left(\frac{m^2l+ml}{p}\right)\psi(l) = -B_{1,\psi}.$$

Proof. We first prove the assertion (i). Replacing (l,m) by $(l,m-1)$ in the first sum, we get the first equality. The second equality follows from replacing (l,m) by $(l,-m)$ in the second sum and noting that P_1 is odd. We now prove the assertion (ii). Replacing (l,m) by (l^2m^{-1}, ml^{-1}) where $l, m \not\equiv 0(p)$ and then applying (i) in Proposition 3.2.12, we get

$$\sum_{l,m(p)} P_1\left(\frac{m^2l+ml}{p}\right)\psi(l) = \sum_{l,m(p)} P_1\left(\frac{m+l}{p}\right)\psi(l^2m) = -B_{1,\psi}.$$

□

Proposition 3.2.22. *We have*

$$(i)\quad \sum_{k,t(p)} P_1\left(\frac{kt}{p}\right) P_1\left(\frac{k^2+kt}{p}\right) = \frac{1}{3p}B_{3,\psi},$$

$$(ii)\quad \sum_{k,t(p)} P_1\left(\frac{kt}{p}\right) P_1\left(\frac{k^2+2kt}{p}\right) = \frac{1}{6p}B_{3,\psi} + \frac{1-\psi(2)}{2p}B_{2p}.$$

Proof. The assertion (i) follows immediately from Proposition 3.2.6 by replacing t by $2t$. Following the same reasoning as in Proposition 3.2.6, we see that the assertion (ii) is clear for $p \equiv 1(4)$, and for $p \equiv 3(4)$, we have

$$\begin{aligned}\sum_{k,t(p)} P_1\left(\frac{kt}{p}\right) P_2\left(\frac{k^2+2kt}{p}\right) &= \sum_{k,t(p)} P_1\left(\frac{t}{p}\right) P_2\left(\frac{k^2+2t}{p}\right)\\ &= \sum_{k,t(p)} \frac{2}{(2\pi ip)^3} \sum_{a,b(p)} c_1\left(\frac{a}{p}\right) c_2\left(\frac{b}{p}\right) \sum_{k,t(p)} \zeta^{(a+2b)t+bk^2}\\ &= \frac{2p}{(2\pi ip)^3} \sum_{a(p)} c_1\left(\frac{a}{p}\right) c_2\left(\frac{-2^{-1}a}{p}\right) \sum_{k(p)} \zeta^{-2^{-1}ak^2}\\ &= -\frac{2p\,\tau(\psi)}{(2\pi ip)^3} \sum_{a(p)} c_1\left(\frac{2a}{p}\right) c_2\left(\frac{a}{p}\right) \psi(a).\end{aligned}$$

We define following trig function,

$$t_1(x) = \begin{cases} \pi\tan(\pi x), & \text{if } x \notin \mathbb{Z},\\ 0, & \text{otherwise.}\end{cases}$$

From the trig identity $c_1(2x) = \frac{1}{2}(c_1(x) - t_1(x))$ together with the fact that $c_2(x) =$

$c_1^2(x)+\pi^2$ for $x \notin \mathbb{Z}$, we get

$$\begin{aligned}
&\sum_{a(p)} c_1\left(\frac{2a}{p}\right) c_2\left(\frac{a}{p}\right)\psi(a)\\
&\qquad = \frac{1}{2}\sum_{a(p)}\left\{c_1\left(\frac{a}{p}\right) - t_1\left(\frac{a}{p}\right)\right\} c_2\left(\frac{a}{p}\right)\psi(a)\\
&\qquad = \frac{1}{2}\sum_{a(p)} c_1\left(\frac{a}{p}\right) c_2\left(\frac{a}{p}\right)\psi(a) - \frac{\pi^2}{2}\sum_{a(p)} c_1\left(\frac{a}{p}\right)\psi(a) - \frac{\pi^2}{2}\sum_{a(p)} t_1\left(\frac{a}{p}\right)\psi(a).
\end{aligned}$$

Observe that $t_1(x) = -c_1(x+1/2)$. Therefore, by the multiplication formula for the c_k's (Proposition 3.1.15), we get

$$\begin{aligned}
\sum_{a(p)} t_1\left(\frac{a}{p}\right)\psi(a) &= -\sum_{a(p)} c_1\left(\frac{a}{p}+\frac{1}{2}\right)\psi(a)\\
&= -2\sum_{a(p)} c_1\left(\frac{2a}{p}\right)\psi(a) + \sum_{a(p)} c_1\left(\frac{a}{p}\right)\psi(a)\\
&= (1-2\psi(2))\sum_{a(p)} c_1\left(\frac{a}{p}\right)\psi(a).
\end{aligned}$$

Hence, by (3.1.6), (3.2.1), we get

$$\begin{aligned}
\sum_{a(p)} c_1\left(\frac{2a}{p}\right) c_2\left(\frac{a}{p}\right)\psi(a) &= \frac{1}{2}\sum_{a(p)} c_3\left(\frac{a}{p}\right)\psi(a) - \pi^2(1-\psi(2))\sum_{a(p)} c_1\left(\frac{a}{p}\right)\psi(a)\\
&= \frac{2\pi^3 ip}{\tau(\psi)}\left(\frac{1}{3}B_{3,\psi} + (1-\psi(2))B_{1,\psi}\right).
\end{aligned}$$

Thus the assertion (ii) follows. □

Proposition 3.2.23. *We have*

$$(i) \quad \sum_{k,t(p)} P_1\left(\frac{k^2+kt}{p}\right) P_2\left(\frac{t^2-k^2}{p}\right) = \sum_{k,t(p)} P_1\left(\frac{k^2+kt}{p}\right) P_2\left(\frac{k^2+2kt}{p}\right)$$
$$= \sum_{k,t(p)} P_1\left(\frac{t^2+kt}{p}\right) P_2\left(\frac{t^2+2kt}{p}\right) = -\frac{1}{6p}B_{3,\psi} - \frac{1-\psi(2)}{2p}B_{1,\psi},$$

$$(ii) \quad \sum_{k,t(p)} P_1\left(\frac{k^2+kt}{p}\right) P_2\left(\frac{t^2+kt}{p}\right) = \sum_{k,t(p)} P_1\left(\frac{k^2+kt}{p}\right) P_2\left(\frac{kt}{p}\right)$$
$$= \sum_{k,t(p)} P_1\left(\frac{t^2+kt}{p}\right) P_2\left(\frac{kt}{p}\right) = -\frac{1}{3p}B_{3,\psi},$$

$$(iii) \quad \sum_{k,t(p)} P_1\left(\frac{t^2-k^2}{p}\right) P_2\left(\frac{t^2+2kt}{p}\right) = -\sum_{k,t(p)} P_1\left(\frac{t^2-k^2}{p}\right) P_2\left(\frac{k^2+2kt}{p}\right)$$
$$= \sum_{k,t(p)} P_1\left(\frac{k^2+2kt}{p}\right) P_2\left(\frac{t^2-k^2}{p}\right),$$

$$(iv) \quad \sum_{k,t(p)} P_3\left(\frac{t^2+kt}{p}\right) = \sum_{k,t(p)} P_3\left(\frac{kt}{p}\right) = 0,$$

$$(v) \quad \sum_{k,t(p)} P_3\left(\frac{t^2-k^2}{p}\right) = \sum_{k,t(p)} P_3\left(\frac{t^2+2kt}{p}\right) = \sum_{k,t(p)} P_3\left(\frac{k^2+2kt}{p}\right) = 0.$$

Proof. We first prove (i). Replacing (k,t) by $(k+t,-t)$ in the first sum yields the first equality. Replacing (k,t) by (t,k) in the second sum gives the second equality. Replacing (k,t) by $(-k-t,k)$ in the third sum, noting that P_1 is odd and P_2 is even, we get

$$\sum_{k,t(p)} P_1\left(\frac{t^2+kt}{p}\right) P_2\left(\frac{t^2+2kt}{p}\right) = -\sum_{k,t(p)} P_2\left(\frac{kt}{p}\right) P_1\left(\frac{k^2+2kt}{p}\right).$$

Thus the third equality follows from (ii) in Proposition 3.2.22. We next prove (ii). Replacing (k,t) by $(k+t,-t)$ in the first sum yields the first equality. Replacing (k,t) by (t,k) in the second sum gives the second equality. Replacing (k,t) by $(-k-t,k)$

in the third sum, noting that P_1 is odd and P_2 is even, we get

$$\sum_{k,t(p)} P_1\left(\frac{t^2+kt}{p}\right) P_2\left(\frac{kt}{p}\right) = -\sum_{k,t(p)} P_1\left(\frac{kt}{p}\right) P_2\left(\frac{k^2+kt}{p}\right).$$

Thus the third equality follows from (i) in Proposition 3.2.22. We next prove (iii). Replacing (k,t) by (t,k) in the first sum yields the first equality. Replacing (k,t) by $(k+t,-t)$ in the second sum gives the second equality. We next prove (iv). Replacing (k,t) by $(k-t,t)$ in the first sum yields the first equality. Replacing (k,t) by $(-k,t)$ in the second sum and noting that P_3 is odd, yields the second equality. Lastly we prove (v). Replacing (k,t) by $(k,t+k)$ in the first sum yields the first equality. Replacing (k,t) by (t,k) in the second sum yields the second equality. The third equality follows from replacing (k,t) by (t,k) in the first sum and noting that P_3 is odd. □

Proposition 3.2.24. *We have*

$$(i) \quad \sum_{\substack{x(2p)\\ x \text{ even}}} P_j\left(\frac{x}{2p}\right)\psi(x) = \frac{\psi(2)}{p^{j-1}} B_{j,\psi},$$

$$(ii) \quad \sum_{\substack{x(2p)\\ x \text{ odd}}} P_j\left(\frac{x}{2p}\right)\psi(x) = \frac{1-2^{j-1}\psi(2)}{(2p)^{j-1}} B_{j,\psi}.$$

Proof. If x is even, then $x = 2k$ for some k. Hence,

$$\sum_{\substack{x(2p)\\ x \text{ even}}} P_j\left(\frac{x}{2p}\right)\psi(x) = \psi(2)\sum_{k(p)} P_j\left(\frac{k}{p}\right)\psi(k).$$

Thus the assertion (i) follows from Proposition 3.2.1. Next, observe that

$$\sum_{\substack{x(2p)\\ x \text{ odd}}} P_j\left(\frac{x}{2p}\right)\psi(x) = \sum_{x(2p)} P_j\left(\frac{x}{2p}\right)\psi(x) - \sum_{\substack{x(2p)\\ x \text{ even}}} P_j\left(\frac{x}{2p}\right)\psi(x).$$

Thus the assertion (ii) follows from (vii) in Proposition 3.2.8 and the assertion (i). □

Proposition 3.2.25 [6]. *Let $n \in \mathbb{Z}$, $n \geq 0$. Then, we have*

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} = \delta_{n,0}.$$

Proof. For $n = 0$, the assertion is clear. Thus we assume that $n \geq 1$. From the Binomial Theorem, we have $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$. Putting $x = 1$, $y = -1$, we get

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} = (1-1)^N = 0.$$

□

I am particularly fond of this little, yet very important identity.

Proposition 3.2.26 [6]. *Let $n \in \mathbb{N}$, and let $k \in \mathbb{Z}$ with $0 \leq k \leq n-1$. Then, we have*

$$\sum_{j=0}^{k} (-1)^j \binom{n}{j} = (-1)^k \binom{n-1}{k}.$$

Proof. We proceed by induction on k. If $k = 0$, the assertion is clear. Assume the assertion holds for $k = K \leq n-2$. Then,

$$\begin{aligned}
\sum_{j=0}^{K+1} (-1)^j \binom{n}{j} &= \sum_{j=0}^{K} (-1)^j \binom{n}{j} + (-1)^{K+1} \binom{n}{K+1} \\
&= (-1)^K \binom{n-1}{K} + (-1)^{K+1} \binom{n}{K+1} \\
&= (-1)^{K+1} \left\{ \binom{n}{K+1} - \binom{n-1}{K} \right\} = (-1)^{K+1} \binom{n-1}{K+1}.
\end{aligned}$$

Since n is arbitrary, the assertion follows. □

Proposition 3.2.27. *Let $n \in \mathbb{N}$, and let $k \in \mathbb{Z}$ with $0 \leq k \leq n-1$. Then,*

$$\sum_{j=2k+1}^{2n-1} (-1)^{j-1} \binom{2n-1}{j-1} \binom{j-1}{2k} = \binom{2n-1}{2k}.$$

Proof. We have

$$\begin{aligned}
&\sum_{j=2k+1}^{2n-1} (-1)^{j-1} \binom{2n-1}{j-1} \binom{j-1}{2k} \\
&\quad = \sum_{j=0}^{2n-2-2k} (-1)^{j} \binom{2n-1}{j+2k} \binom{j+2k}{2k} \\
&\quad = \binom{2n-1}{2k} \sum_{j=0}^{2n-2-2k} (-1)^{j} \binom{2n-1-2k}{j} \\
&\quad = \binom{2n-1}{2k} \left\{ \sum_{j=0}^{2n-1-2k} (-1)^{j} \binom{2n-1-2k}{j} - (-1)^{2n-1-2k} \binom{2n-1-2k}{2n-1-2k} \right\}.
\end{aligned}$$

Thus the assertion follows from Proposition 3.2.25. □

Proposition 3.2.28. *Let ϕ and ψ denote the Carlitz Phi and Psi functions given by (3.1.14). Let $N, M \in \mathbb{N}$, $h, k \in \mathbb{N}$ with $(h,k) = 1$, and $x, y \in \mathbb{R}$. Then, we have*

$$\begin{aligned}
&\sum_{j=0}^{N} \frac{(-1)^{N-j} \binom{N}{j} h^{N-j}}{N+M-j} \cdot \phi_{j,N+M-j}(h,k;x,y) \\
&\quad = \frac{1}{(N+M)\binom{N+M-1}{N}} \sum_{j=0}^{N} \binom{N+M}{j} h^{N-j}\, \psi_{j,N+M-j}(h,k;x,y).
\end{aligned}$$

Proof. By definition of ψ, we get

$$\begin{aligned}
&\sum_{j=0}^{N} \binom{N+M}{j} h^{N-j}\, \psi_{j,N+M-j}(h,k;x,y) \\
&\quad = \sum_{j=0}^{N} \sum_{i=0}^{j} (-1)^{j-i} \binom{N+M}{j} \binom{j}{i} h^{N-i}\, \phi_{i,N+M-i}(h,k;x,y) \\
&\quad = \sum_{i=0}^{N} \sum_{j=i}^{N} (-1)^{j-i} \binom{N+M}{j} \binom{j}{i} h^{N-i}\, \phi_{i,N+M-i}(h,k;x,y) \\
&\quad = \sum_{i=0}^{N} \sum_{j=0}^{N-i} (-1)^{j} \binom{N+M}{j+i} \binom{j+i}{i} h^{N-i}\, \phi_{i,N+M-i}(h,k;x,y) \\
&\quad = \sum_{i=0}^{N} \binom{N+M}{i} \left\{ \sum_{j=0}^{N-i} (-1)^{j} \binom{N+M-i}{j} \right\} h^{N-i}\, \phi_{i,N+M-i}(h,k;x,y).
\end{aligned}$$

From Proposition 3.2.26, we get

$$\begin{aligned}
&\sum_{j=0}^{N} \binom{N+M}{j} h^{N-j}\, \psi_{j,N+M-j}(h,k;x,y) \\
&\quad = \sum_{i=0}^{N} (-1)^{N-i} \binom{N+M}{i} \binom{N+M-i-1}{N-i} h^{N-i}\, \phi_{i,N+M-i}(h,k;x,y).
\end{aligned}$$

Thus the assertion follows from the fact that

$$\frac{1}{(N+M)\binom{N+M-1}{N}} \binom{N+M}{i} \binom{N+M-i-1}{N-i} = \frac{\binom{N}{i}}{N+M-i}.$$

$\square$

Proposition 3.2.29. *Let ϕ and ψ denote the Carlitz Phi and Psi functions given by (3.1.14). Let $N, M \in \mathbb{N}$ with $M \geq 3$ and odd. Let $h, k \in \mathbb{N}$ with $(h,k) = 1$, and*

$x, y \in \mathbb{R}$. Then, we have

$$(i)\quad \sum_{j=0}^{N} \binom{N+M}{j} h^{1-j}\, \psi_{j,N+M-j}(h,k;x,0) = \sum_{j=M+1}^{N+M} \binom{N+M}{j} k^{1-j}\, \psi_{j,N+M-j}(k,h;0,x),$$

$$(ii)\quad \sum_{j=0}^{M} \binom{N+M}{j} h^{1-j}\, \psi_{j,N+M-j}(h,k;0,y) = \sum_{j=N+1}^{N+M} \binom{N+M}{j} k^{1-j}\, \psi_{j,N+M-j}(k,h;y,0).$$

Proof. We have

$$\begin{aligned} &\sum_{j=0}^{N} \binom{N+M}{j} h^{1-j}\, \psi_{j,N+M-j}(h,k;x,0) \\ &= h\, \psi_{0,N+M}(h,k;x,0) + \sum_{j=0}^{N-1} \binom{N+M}{j+1} h^{-j}\, \psi_{j+1,N+M-1-j}(h,k;x,0). \end{aligned} \tag{3.2.9}$$

From Proposition 3.2.1, we have

$$\psi_{0,N+M}(h,k;x,0) = \phi_{0,N+M}(h,k;x,0) = \frac{1}{k^{N+M-1}} B_{N+M}.$$

By Carlitz reciprocity (Proposition 3.1.20), we have

$$\begin{aligned} \psi_{j+1,N+M-1-j}(h,k;x,0) = \frac{1}{(N+M-j)k^{N+M-1-j}} \Big\{ &(j+1)h^{j}\, \psi_{N+M-j,j}(k,h,0,x) \\ &+ (N+M)kB_{j+1}(x)B_{N+M-1-j} - (j+1)hB_j(x)B_{N+M-j} \Big\}. \end{aligned}$$

Hence, plugging in the above results back into (3.2.9), we get

$$\sum_{j=0}^{N}\binom{N+M}{j}h^{1-j}\,\psi_{j,N+M-j}(h,k;x,0)$$
$$=\frac{1}{k^{N+M-1}}\sum_{j=0}^{N-1}\binom{N+M}{j}k^{j}\,\psi_{N+M-j,j}(k,h;0,x)+\frac{1}{h^{N-1}k^{M-1}}\binom{N+M}{N}P_N(x)B_M.$$

Since $M\geq 3$ is odd, $B_M=0$, and we get

$$\sum_{j=0}^{N}\binom{N+M}{j}h^{1-j}\,\psi_{j,N+M-j}(h,k;x,0)$$
$$=\frac{1}{k^{N+M-1}}\sum_{j=0}^{N-1}\binom{N+M}{j}k^{j}\,\psi_{N+M-j,j}(k,h;0,x).$$

Thus the assertion (i) follows from replacing j by $N+M-j$. The assertion (ii) is similarly verified, so the proof is omitted. □

Proposition 3.2.30. *Let ϕ and ψ denote the Carlitz Phi and Psi functions given by (3.1.14). Let $N\in\mathbb{N}$, $h,k\in\mathbb{N}$ with $(h,k)=1$, and $x,y\in\mathbb{Z}$. Then, we have*

$$\sum_{j=0}^{N}\binom{N}{j}h^{1-j}\,\psi_{j,N-j}\left(h,k;\frac{x}{k},\frac{y}{h}\right)=\frac{B_N}{(hk)^{N-1}}-B_1\delta_{N,1}.$$

Proof. From the definition of ψ, we have

$$\sum_{j=0}^{N}\binom{N}{j}h^{1-j}\,\psi_{j,N-j}\left(h,k;\frac{x}{k},\frac{y}{h}\right)$$
$$=\sum_{j=0}^{N}\sum_{i=0}^{j}(-1)^{j-i}\binom{N}{j}\binom{j}{i}h^{1-i}\,\phi_{i,N-i}\left(h,k;\frac{x}{k},\frac{y}{h}\right)$$
$$=\sum_{i=0}^{N}\sum_{j=i}^{N}(-1)^{j-i}\binom{N}{j}\binom{j}{i}h^{1-i}\,\phi_{i,N-i}\left(h,k;\frac{x}{k},\frac{y}{h}\right)$$
$$=\sum_{i=0}^{N}\binom{N}{i}h^{1-i}\,\phi_{i,N-i}\left(h,k;\frac{x}{k},\frac{y}{h}\right)\left\{\sum_{j=0}^{N-i}(-1)^{j}\binom{N-i}{j}\right\}.$$

By Proposition 3.2.25, we get

$$\sum_{j=0}^{N}\binom{N}{j}h^{1-j}\,\psi_{j,N-j}\left(h,k;\frac{x}{k},\frac{y}{h}\right)=\sum_{i=0}^{N}\binom{N}{i}h^{1-i}\,\phi_{i,N-i}\left(h,k;\frac{x}{k},\frac{y}{h}\right)\delta_{N-i,0} \tag{3.2.10}$$
$$=h^{1-N}\,\phi_{N,0}\left(h,k;\frac{x}{k},\frac{y}{h}\right).$$

From the definition of ϕ, we have

$$\phi_{N,0}\left(h,k;\frac{x}{k},\frac{y}{h}\right)=\sum_{a(k)}P_N\left(h\left(\frac{a+\frac{y}{h}}{k}\right)+\frac{x}{k}\right)=\sum_{a(k)}P_N\left(\frac{ha+x+y}{k}\right).$$

Replacing a by $h^{-1}(a-x-y)$ (which is permissible since $(h,k)=1$ and $x,y\in\mathbb{Z}$), we get

$$\phi_{N,0}\left(h,k;\frac{x}{k},\frac{y}{h}\right)=\sum_{a(k)}P_N\left(\frac{a}{k}\right).$$

From the multiplication formula, we get

$$\phi_{N,0}\left(h,k;\frac{x}{k},\frac{y}{h}\right)=\frac{1}{k^{N-1}}B_N-B_1\delta_{N,1}.$$

Thus the assertion follows from plugging back into (3.2.10). □

Proposition 3.2.31. *Let ϕ and ψ denote the Carlitz Phi and Psi functions given by (3.1.14). Let $M\in\mathbb{N}$ with $M\geq 3$, $N\in\mathbb{Z}$ with $0\leq N\leq M-1$, and $x\in\mathbb{Z}$. Then, we have*

$$\begin{aligned}
&\psi_{M-N,N}\left(2,p;\frac{x}{p},0\right)\\
&=\frac{(M-N)2^{M-1-N}}{(N+1)p^N}\sum_{k=0}^{N+1}(-1)^{N+1-k}\binom{N+1}{k}2^{N+1-k}\,\phi_{k,M-k}\left(2,p;\frac{x}{p},0\right)\\
&+\frac{B_N}{p^{N-1}}\cdot P_{M-N}\left(\frac{x}{p}\right)-\frac{2(M-N)B_{N+1}}{(N+1)p^N}\cdot P_{M-1-N}\left(\frac{x}{p}\right)\\
&+\frac{\delta_{N,1}}{2}\cdot P_{M-1}\left(\frac{x}{p}\right)-M\delta_{N,0}\cdot P_{M-1}\left(\frac{x}{p}\right).
\end{aligned}$$

Proof. From Carlitz reciprocity (Proposition 3.1.20), we get

$$\begin{aligned}\psi_{M-N,N}\left(2,p;\frac{x}{p},0\right) &= \frac{(M-N)2^{M-1-N}}{(N+1)p^N}\cdot\psi_{N+1,M-1-N}\left(p,2;0,\frac{x}{p}\right)\\ &\quad+\frac{B_N}{p^{N-1}}\cdot P_{M-N}\left(\frac{x}{p}\right)-\frac{2(M-N)B_{N+1}}{(N+1)p^N}\cdot P_{M-1-N}\left(\frac{x}{p}\right)\\ &\quad+\frac{\delta_{N,1}}{2}\cdot P_{M-1}\left(\frac{x}{p}\right)-M\delta_{N,0}\cdot P_{M-1}\left(\frac{x}{p}\right).\end{aligned}$$

Thus the assertion follows from the definition of $\psi_{N+1,M-1-N}\left(p,2;0,\frac{x}{p}\right)$. □

Proposition 3.2.32. *Let ϕ denote the Carlitz Phi function given by (3.1.14). Let $M\in\mathbb{N}$, $k\in\mathbb{Z}$ with $0\le k\le M$. Then, we have*

$$\begin{aligned}&\sum_{x(p)}\psi(x)\,\phi_{k,M-k}\left(p,2;0,\frac{x}{p}\right)\\ &\quad=\frac{(B_k-B_k(1/2))2^{M-1-k}\psi(2)+B_k(1/2)}{(2p)^{M-1-k}}\cdot B_{M-k,\psi}+\frac{\psi(2)\delta_{k,1}}{2p^{M-2}}\cdot B_{M-1,\psi}.\end{aligned}$$

Proof. From the definition of ϕ, we have

$$\begin{aligned}\phi_{k,M-k}\left(p,2;0,\frac{x}{p}\right) &= \sum_{a(2)}P_k\left(\frac{p(a+\frac{x}{p})}{2}\right)P_{M-k}\left(\frac{a+\frac{x}{p}}{2}\right)\\ &= P_k\left(\frac{x}{2}\right)P_{M-k}\left(\frac{x}{2p}\right)+P_k\left(\frac{x+1}{2}\right)P_{M-k}\left(\frac{x}{2p}+\frac{1}{2}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\phi_{k,M-k}\left(p,2;0,\frac{x}{p}\right)\\ &\quad=\begin{cases}\left(B_k+\frac{\delta_{k,1}}{2}\right)P_{M-k}\left(\frac{x}{2p}\right)+B_k(1/2)P_{M-k}\left(\frac{x}{2p}+\frac{1}{2}\right), & \text{if } x \text{ is even,}\\ B_k(1/2)P_{M-k}\left(\frac{x}{2p}\right)+\left(B_k+\frac{\delta_{k,1}}{2}\right)P_{M-k}\left(\frac{x}{2p}+\frac{1}{2}\right), & \text{if } x \text{ is odd.}\end{cases}\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{x(p)} \psi(x)\phi_{k,M-k}\left(p,2;0,\frac{x}{p}\right) \\
&\quad = \left(B_k + \frac{\delta_{k,1}}{2}\right) \sum_{\substack{x=1 \\ x \text{ even}}}^{p-1} P_{M-k}\left(\frac{x}{2p}\right)\psi(x) + B_k(1/2) \sum_{\substack{x=1 \\ x \text{ even}}}^{p-1} P_{M-k}\left(\frac{x}{2p}+\frac{1}{2}\right)\psi(x) \\
&\qquad + B_k(1/2) \sum_{\substack{x=1 \\ x \text{ odd}}}^{p-1} P_{M-k}\left(\frac{x}{2p}\right)\psi(x) + \left(B_k + \frac{\delta_{k,1}}{2}\right) \sum_{\substack{x=1 \\ x \text{ odd}}}^{p-1} P_{M-k}\left(\frac{x}{2p}+\frac{1}{2}\right)\psi(x) \\
&\quad = \left(B_k + \frac{\delta_{k,1}}{2}\right) \sum_{\substack{x=1 \\ x \text{ even}}}^{2p-1} P_{M-k}\left(\frac{x}{2p}\right)\psi(x) + B_k(1/2) \sum_{\substack{x=1 \\ x \text{ odd}}}^{2p-1} P_{M-k}\left(\frac{x}{2p}\right)\psi(x).
\end{aligned}$$

Thus the assertion follows from Propostion 3.2.24. □

Proposition 3.2.33. *Let ϕ and ψ denote the Carlitz Phi and Psi functions given by (3.1.14). Let $M \in \mathbb{N}$ with $M \geq 3$ and odd. Let $h, k \in \mathbb{N}$ with $(h,k) = 1$, and $x \in \mathbb{R}$. Let $\{b_n\}_{n=1}^{M-1}$ be any set of numbers. Then, we have*

$$\sum_{n=1}^{M-1} b_n\, \phi_{M-n,n}\,(h,k;x,0) = \sum_{n=1}^{M-1} c_n\, \psi_{M-n,n}\,(h,k;x,0)\,,$$

where

$$c_n = \sum_{j=1}^{n} \binom{M-j}{n-j} h^{n-j} b_j \quad (n = 1, \cdots, M-1). \tag{3.2.11}$$

Proof. We note that $\phi_{0,M} = \frac{1}{k^{M-1}} B_M = 0$ since $M \geq 3$ and odd. With c_n given by (3.2.11) together with the definition of ψ, we have

$$\begin{aligned}
&\sum_{n=1}^{M-1} c_n\, \psi_{M-n,n}(h,k,x,0) \\
&\quad = \sum_{n=1}^{M-1}\sum_{j=1}^{n} \binom{M-j}{n-j} h^{n-j} b_j \sum_{k=1}^{M-n} (-1)^{M-n-k} \binom{M-n}{k} h^{M-j-k}\, \phi_{k,M-k}(h,k,x,0) \\
&\quad = \sum_{k=1}^{M-1}\sum_{n=1}^{k-1}\sum_{j=1}^{n} (-1)^{M-n-k} \binom{M-j}{n-j}\binom{M-n}{k} h^{M-j-k} b_j\, \phi_{k,M-k}(h,k,x,0) \\
&\quad = \sum_{k=1}^{M-1}\sum_{j=1}^{M-k}\sum_{n=j}^{M-k} (-1)^{M-n-k} \binom{M-j}{n-j}\binom{M-n}{k} h^{M-j-k} b_j\, \phi_{k,M-k}(h,k,x,0).
\end{aligned}$$

Replacing k by $M-k$ and n by $n+j$, we get

$$\begin{aligned}
&\sum_{n=1}^{M-1} c_n\, \psi_{M-n,n}(h,k,x,0) \\
&\quad = \sum_{k=1}^{M-1} (-1)^k h^k\, \phi_{M-k,k}(h,k,x,0) \sum_{j=1}^{k} (-1)^j h^{-j} b_j \sum_{n=0}^{k-j} (-1)^n \binom{M-j}{n}\binom{M-j-n}{M-k}.
\end{aligned}$$

Evaluating the innermost sum, we get, by Proposition 3.2.25,

$$\sum_{n=0}^{k-j} (-1)^n \binom{M-j}{n}\binom{M-j-n}{M-k} = \binom{M-j}{k-j} \sum_{n=0}^{k-j} (-1)^n \binom{k-j}{n} = \delta_{k-j,0}.$$

Thus the assertion of the proposition follows. □

Proposition 3.2.34. *We put*

$$S_p(n,m) = \sum_{k,t(p)} P_n\left(\frac{k^2-2kt}{p}\right) P_m\left(\frac{t^2-k^2}{p}\right).$$

Then, we have

$$S_p(1,4) = -S_p(2,3).$$

Proof. From Proposition 3.1.19, for $n = 4$, we get

$$\begin{aligned} P_1(x)P_3(x+y) + P_1(y)P_1(x+y) - P_1(x)P_3(y) - \frac{3}{2}P_2(x)P_2(y) \\ - P_3(x)P_1(y) = \frac{1}{4}\left(P_4(x) + 3P_4(x+y) + P_4(y)\right). \end{aligned} \tag{3.2.12}$$

Noting that

$$\begin{aligned} P_1^2(y) &= P_2(y) + \frac{1}{12} - \frac{1}{4}\delta_{\{y\in\mathbb{Z}\}}, \\ P_1(y)P_2(y) &= P_3(y) + \frac{1}{6}P_1(y), \\ P_1(y)P_3(y) &= P_4(y) + \frac{1}{4}P_2(y) - \frac{1}{120}, \\ P_1(y)P_4(y) &= P_5(y) + \frac{1}{3}P_3(y) - \frac{1}{30}P_1(y), \end{aligned}$$

multiplying (3.2.12) throughout by $P_1(y)$ yields

$$\begin{aligned} &P_1(x)P_1(y)P_3(x+y) + P_3(x+y)P_2(y) - P_1(x)P_4(y) - \frac{1}{4}P_1(x)P_2(y) - \frac{3}{2}P_2(x)P_3(y) \\ &\quad - \frac{1}{4}P_2(x)P_1(y) - P_3(x)P_2(y) + \frac{1-3\delta_{\{y\in\mathbb{Z}\}}}{12}P_3(x+y) - \frac{1-3\delta_{\{y\in\mathbb{Z}\}}}{12}P_3(x) + \frac{1}{120}P_1(x) \\ &\quad = \frac{1}{4}\left(P_4(x)P_1(y) + 3P_4(x+y)P_1(y) + P_5(y) + \frac{1}{3}P_3(y) - \frac{1}{30}P_1(y)\right). \end{aligned}$$

Letting $(x, y, x+y) = \left(\frac{k^2+2kt}{p}, \frac{t^2-k^2}{p}, \frac{t^2+2kt}{p}\right)$, carefully summing over $k, t(p)$, and simplifying, we get

$$- S_p(1,4) - \frac{1}{4}S_p(1,2) - \frac{3}{2}S_p(2,3) - \frac{1}{4}S_p(2,1) = -\frac{1}{2}S_p(4,1). \tag{3.2.13}$$

Replacing (k, t) by $(k+t, t)$ in the definition of S_p, we find that

$$S_p(n,m) = (-1)^{n+m}S_p(m,n) \quad (n, m \in \mathbb{N}).$$

Hence $S_p(4,1) = -S_p(1,4)$ and $S_p(2,1) = -S_p(1,2)$. Thus the proposition follows from (3.2.13). □

Lemma 3.2.35 [15]. *Let f be multiplicative. Suppose that*

$$n = \prod_{p^\alpha || n} p^\alpha$$

is the unique factorization of n into powers of distinct primes. Then, we have

$$\sum_{d|n} f(d) = \prod_{p^\alpha || n} (1 + f(p) + f(p^2) + \cdots + f(p^\alpha)).$$

The notation $p^\alpha || n$ means that p^α is the exact power dividing n.

Proof. A typical divisor d of n is of the form $d = \prod_{p|n} p^{\beta(p)}$, where $\beta(p) \leq \alpha$ and $p^\alpha || n$. Thus $f(d) = \prod_{p|n} f(p^{\beta(p)})$, which is a typical term on the right-hand side. □

Proposition 3.2.36. *Let $l \in \mathbb{N}$. Let χ, δ be Dirichlet characters and let μ denote the Möbius function. Then, we have*

$$\sum_{m|l} \mu(m)\chi(m)\delta(m)m^k = \prod_{\substack{q|l \\ q \text{ prime}}} (1 - q^k \chi(q)\delta(q)).$$

Proof. This follows from letting $f(m) = \mu(m)\chi(m)\delta(m)m^k$ and applying Lemma 3.2.35. □

3.3 The Evaluation of the Sum L_p

We fix an odd prime p. The following sums will play a major role througout this section:

$$L_p = \sum_{k,t(p)} P_1\left(\frac{2kt}{p}\right) P_1\left(\frac{k^2+2kt}{p}\right) P_1\left(\frac{(k+t)^2}{p}\right),$$

$$F_p = \sum_{k,t(p)} P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{k^2+2kt}{p}\right),$$

$$G_p = \sum_{k,t(p)} P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{2kt}{p}\right) P_1\left(\frac{k^2+2kt}{p}\right),$$

$$T_p = \sum_{k,t(p)} P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{2kt}{p}\right) P_1\left(\frac{(k+t)^2}{p}\right),$$

$$R_p = \sum_{k,t(2p)} P_1\left(\frac{t^2}{2p}\right) P_1\left(\frac{k^2+2kt}{2p}\right) P_1\left(\frac{k^2-2kt}{2p}\right),$$

$$Z_p = \sum_{k,t(2p)} P_1\left(\frac{t^2}{2p}\right) P_1\left(\frac{(k+t)^2}{2p}\right) P_1\left(\frac{(k-t)^2}{2p}\right),$$

$$a_p = \sum_{k,t(p)} P_1\left(\frac{(k+t)^2}{p}\right) P_2\left(\frac{k^2+t^2}{p}\right),$$

$$b_p = \sum_{k,t(p)} P_1\left(\frac{2kt}{p}\right) P_2\left(\frac{(k+t)^2}{p}\right),$$

$$c_p = \sum_{k,t(p)} P_1\left(\frac{(k+t)^2}{p}\right) P_2\left(\frac{2kt}{p}\right).$$

Theorem 3.3. With notation and assumptions as above, we have

$$L_p = -\frac{1}{16p}B_{3,\psi} - \frac{4p-7+3\psi(2)}{16p}B_{1,\psi}.$$

Before proving this theorem, we must first establish several results.

Proposition 3.3.1. With notation and assumptions as above, we have

$$L_p = -\frac{1}{12p}B_{3,\psi} - \frac{6p-5}{24p}B_{1,\psi} + \frac{1}{4}\left(b_p + c_p\right).$$

Proof. Applying the 2-term addition formula (Proposition 3.1.17) to the arguments $\frac{k^2}{p}$, $\frac{2kt}{p}$, we get

$$
\begin{aligned}
&P_1\left(\frac{k^2}{p}\right)P_1\left(\frac{2kt}{p}\right)-P_1\left(\frac{k^2}{p}\right)P_1\left(\frac{k^2+2kt}{p}\right)-P_1\left(\frac{2kt}{p}\right)P_1\left(\frac{k^2+2kt}{p}\right)\\
&\quad=-\frac{1}{2}\left(P_2\left(\frac{k^2}{p}\right)+P_2\left(\frac{2kt}{p}\right)+P_2\left(\frac{k^2+2kt}{p}\right)\right)+\frac{1}{4}\delta\left(\frac{k^2}{p},\frac{2kt}{p}\right),
\end{aligned}
\tag{3.3.1}
$$

where δ is given by (3.1.8). Multiplying throughout by $P_1\left(\frac{t^2}{p}\right)$, carefully summing over $k,t(p)$, and applying Proposition 3.2.3, Proposition 3.2.2, Proposition 3.2.1, and Proposition 3.2.4, yields

$$F_p+G_p=-\frac{1}{6p}B_{3,\psi}-\frac{p-1}{4p}B_{1,\psi}. \tag{3.3.2}$$

Multiplying (3.3.1) throughout by $P_1\left(\frac{(k+t)^2}{p}\right)$, carefully summing over $k,t(p)$, and applying Proposition 3.2.2, Proposition 3.2.1, and Proposition 3.2.5, yields

$$L_p=T_p+F_p+\frac{1}{2}c_p-\frac{1}{6p}B_{3,\psi}-\frac{3p-2}{12p}B_{1,\psi}. \tag{3.3.3}$$

Applying the two-term addition formula (Proposition 3.1.17) to the arguments $-\frac{t^2}{p}$, $\frac{(k+t)^2}{p}$, we get

$$
\begin{aligned}
&-P_1\left(\frac{t^2}{p}\right)P_1\left(\frac{(k+t)^2}{p}\right)+P_1\left(\frac{t^2}{p}\right)P_1\left(\frac{k^2+2kt}{p}\right)-P_1\left(\frac{k^2+2kt}{p}\right)P_1\left(\frac{(k+t)^2}{p}\right)\\
&\quad=-\frac{1}{2}\left(P_2\left(\frac{t^2}{p}\right)+P_2\left(\frac{k^2+2kt}{p}\right)+P_2\left(\frac{(k+t)^2}{p}\right)\right)+\frac{1}{4}\delta\left(-\frac{t^2}{p},\frac{(k+t)^2}{p}\right).
\end{aligned}
$$

Multiplying throughout by $P_1\left(\frac{2kt}{p}\right)$, carefully summing over $k,t(p)$, and applying Proposition 3.2.3, Proposition 3.2.6, yields

$$L_p=G_p-T_p+\frac{1}{2}b_p+\frac{1}{6p}B_{3,\psi}. \tag{3.3.4}$$

From (3.3.3) and (3.3.4), we get

$$2L_p = F_p + G_p + \frac{1}{2}(b_p + c_p) - \frac{3p-2}{12p} B_{1,\psi},$$

and from (3.3.2), we obtain

$$L_p = -\frac{1}{12p} B_{3,\psi} - \frac{6p-5}{24p} B_{1,\psi} + \frac{1}{4}(b_p + c_p).$$

This completes the proof of the Proposition 3.3.1. □

To complete the proof of Theorem 3.3, we must evaluate b_p and c_p. To this end, we will find three independent relations involving a_p, b_p, and c_p. We note that $e(x) = exp(2\pi ix)$.

Proposition 3.3.2. With notation and assumptions as above, we have

$$a_p + c_p = \frac{1}{3p} B_{1,\psi}.$$

Proof. First we note that the assertion is obvious in the case of $p \equiv 1(4)$ (since $a_p = c_p = B_{1,\psi} = 0$). Thus, we will assume that $p \equiv 3(4)$ for the remainder of the proof. Replacing (k,t) by $(k+t,-t)$ in the definition of a_p, we get

$$a_p = \sum_{k,t(p)} P_1\left(\frac{k^2}{p}\right) P_2\left(\frac{(k+t)^2+t^2}{p}\right).$$

Similarly, replacing (k,t) by $(-k-t,t)$ in the definition of c_p, noting that P_2 is an even function, we get

$$c_p = \sum_{k,t(p)} P_1\left(\frac{k^2}{p}\right) P_2\left(\frac{(k+t)^2+t^2-k^2}{p}\right).$$

Applying the finite Fourier transform (Proposition 3.1.16) to $P_2\left(\frac{(k+t)^2+t^2}{p}\right)$,

$P_2\left(\frac{(k+t)^2+t^2-k^2}{p}\right)$ and summing over $t(p)$, we obtain

$$\sum_{t(p)} P_2\left(\frac{(k+t)^2+t^2}{p}\right) = -\frac{2!}{(2\pi i p)^2}\sum_{b(p)} c_2\left(\frac{b}{p}\right)\sum_{t(p)} e\left(\frac{b((k+t)^2+t^2)}{p}\right)$$
$$= -\frac{2!}{(2\pi i p)^2}\sum_{b(p)} c_2\left(\frac{b}{p}\right) e\left(\frac{bk^2}{p}\right)\sum_{t(p)} e\left(\frac{2bt^2+2bkt}{p}\right),$$

$$\sum_{t(p)} P_2\left(\frac{(k+t)^2+t^2-k^2}{p}\right) = -\frac{2!}{(2\pi i p)^2}\sum_{b(p)} c_2\left(\frac{b}{p}\right)\sum_{t(p)} e\left(\frac{b((k+t)^2+t^2-k^2)}{p}\right)$$
$$= -\frac{2!}{(2\pi i p)^2}\sum_{b(p)} c_2\left(\frac{b}{p}\right)\sum_{t(p)} e\left(\frac{2bt^2+2bkt}{p}\right).$$

By Proposition 3.2.18, it follows that

$$\sum_{t(p)} e\left(\frac{2bt^2+2bkt}{p}\right) = \begin{cases} p, & b \equiv 0(p), \\ \tau(\psi)\,\psi(2b)(-1)^{-bk^2} e\left(\dfrac{-bk^2}{2p}\right), & b \not\equiv 0(p). \end{cases}$$

Hence, we get

$$\sum_{t(p)} P_2\left(\frac{(k+t)^2+t^2}{p}\right)$$
$$= -\frac{2!}{(2\pi i p)^2}\left\{p\,c_2(0) + \tau(\psi)\psi(2)\sum_{b=1}^{p-1} c_2\left(\frac{b}{p}\right)(-1)^{bk^2}\psi(b) e\left(\frac{bk^2}{2p}\right)\right\},$$

$$\sum_{t(p)} P_2\left(\frac{(k+t)^2+t^2-k^2}{p}\right)$$
$$= -\frac{2!}{(2\pi i p)^2}\left\{p\,c_2(0) + \tau(\psi)\psi(2)\sum_{b=1}^{p-1} c_2\left(\frac{b}{p}\right)(-1)^{bk^2}\psi(b) e\left(\frac{-bk^2}{2p}\right)\right\}.$$

Replacing b by $p-b$ in the sum above, noting that ψ is odd and c_2 is even, we get

$$\sum_{t(p)} P_2\left(\frac{(k+t)^2+t^2-k^2}{p}\right)$$
$$= -\frac{2!}{(2\pi i p)^2}\left\{p\, c_2(0) - \tau(\psi)\psi(2)\sum_{b=1}^{p-1} c_2\left(\frac{b}{p}\right)(-1)^{-bk^2}\psi(b)e\left(\frac{bk^2}{2p}\right)\right\}.$$

Therefore,

$$\sum_{t(p)}\left\{P_2\left(\frac{(k+t)^2+t^2}{p}\right)+P_2\left(\frac{(k+t)^2+t^2-k^2}{p}\right)\right\} = -\frac{4p\, c_2(0)}{(2\pi i p)^2} = \frac{1}{3p}.$$

Multiplying the above equation by $P_1\left(\frac{k^2}{p}\right)$ and summing over $k(p)$, we get

$$a_p + c_p = \sum_{k(p)} P_1\left(\frac{k^2}{p}\right)\sum_{t(p)}\left\{P_2\left(\frac{(k+t)^2+t^2}{p}\right)+P_2\left(\frac{(k+t)^2+t^2-k^2}{p}\right)\right\}$$
$$= \frac{1}{3p}\sum_{k(p)} P_1\left(\frac{k^2}{p}\right) = \frac{1}{3p}B_{1,\psi}.$$

□

Proposition 3.3.3. With notation and assumptions as above,

$$R_p = -2T_p + \frac{1}{3p}B_{3,\psi} + \frac{3p-2}{12p}B_{1,\psi}.$$

Proof. Applying the two-term addition formula (Proposition 3.1.17) to the arguments $\frac{k^2+2kt}{2p}$, $\frac{k^2-2kt}{2p}$, we get

$$P_1\left(\frac{k^2+2kt}{2p}\right)P_1\left(\frac{k^2-2kt}{2p}\right) - P_1\left(\frac{k^2+2kt}{2p}\right)P_1\left(\frac{k^2}{p}\right) - P_1\left(\frac{k^2-2kt}{2p}\right)P_1\left(\frac{k^2}{p}\right)$$
$$= -\frac{1}{2}\left(P_2\left(\frac{k^2+2kt}{2p}\right)+P_2\left(\frac{k^2-2kt}{2p}\right)+P_2\left(\frac{k^2}{p}\right)\right)$$
$$+\frac{1}{4}\delta\left(\frac{k^2+2kt}{2p},\frac{k^2-2kt}{2p}\right).$$

Multiplying throughout by $P_1\left(\frac{t^2}{2p}\right)$, carefully summing over $k, t(2p)$, and applying Proposition 3.2.7, Proposition 3.2.8, and Proposition 3.2.1, yields

$$R_p = 2F_p + \frac{1}{6p}B_{3,\psi} + \frac{p-1}{4p}B_{1,\psi}.$$

From (3.3.2), we see that

$$R_p = F_p - G_p, \tag{3.3.5}$$

and from (3.3.3), (3.3.4), we obtain

$$R_p = -2T_p + \frac{1}{3p}B_{3,\psi} + \frac{3p-2}{12p}B_{1,\psi}.$$

□

Proposition 3.3.4. With notation and assumptions as above,

$$Z_p = -2T_p + \frac{1}{4p}B_{1,\psi}.$$

Proof. Applying the two-term addition formula (Proposition 3.1.17) to the arguments $\frac{(k+t)^2}{2p}, -\frac{(k-t)^2}{2p}$, we get

$$\begin{aligned}
&- P_1\left(\frac{(k+t)^2}{2p}\right) P_1\left(\frac{(k-t)^2}{2p}\right) - P_1\left(\frac{(k+t)^2}{2p}\right) P_1\left(\frac{2kt}{p}\right) \\
&\quad + P_1\left(\frac{(k-t)^2}{2p}\right) P_1\left(\frac{2kt}{p}\right) = -\frac{1}{2}\left(P_2\left(\frac{(k+t)^2}{2p}\right) + P_2\left(\frac{(k-t)^2}{2p}\right) + P_2\left(\frac{2kt}{p}\right)\right) \\
&\quad + \frac{1}{4}\delta\left(\frac{(k+t)^2}{2p}, \frac{-(k-t)^2}{2p}\right).
\end{aligned}$$

Multiplying throughout by $P_1\left(\frac{t^2}{2p}\right)$, carefully summing over $k, t(2p)$, and applying

Proposition 3.2.7, Proposition 3.2.8, yields

$$Z_p = -2T_p + \frac{1}{4p} B_{1,\psi}.$$

□

Proposition 3.3.5. With notation and assumptions as above, we have

$$b_p - c_p = \frac{1}{4p} B_{3,\psi} + \frac{1 - 3\psi(2)}{12p} B_{1,\psi}.$$

Proof. Applying Sczech's 3-term addition formula (Proposition 3.1.18) to the arguments

$$v_1 = -\frac{t^2}{2p}, \; v_2 = \frac{k^2 + 2kt}{2p}, \; v_3 = \frac{k^2 - 2kt}{2p},$$

we obtain

$$
\begin{aligned}
&P_1\left(\frac{t^2}{2p}\right)P_1\left(\frac{k^2+2kt}{2p}\right)P_1\left(\frac{k^2-2kt}{2p}\right)\\
&\quad -P_1\left(\frac{t^2}{2p}\right)P_1\left(\frac{(k+t)^2}{2p}\right)P_1\left(\frac{(k-t)^2}{2p}\right)\\
&\quad +P_1\left(\frac{(k+t)^2}{2p}\right)P_1\left(\frac{k^2+2kt}{2p}\right)P_1\left(\frac{2kt}{p}\right)\\
&\quad -P_1\left(\frac{(k-t)^2}{2p}\right)P_1\left(\frac{k^2-2kt}{2p}\right)P_1\left(\frac{2kt}{p}\right)\\
&\quad =\frac{1}{2}P_1\left(\frac{(k+t)^2}{2p}\right)\left(P_2\left(\frac{2kt}{p}\right)-P_2\left(\frac{(k-t)^2}{2p}\right)\right)\\
&\quad +\frac{1}{2}P_1\left(\frac{(k+t)^2}{2p}\right)\left(P_2\left(\frac{k^2+2kt}{2p}\right)-P_2\left(\frac{t^2}{2p}\right)\right)\\
&\quad +\frac{1}{2}P_1\left(\frac{(k-t)^2}{2p}\right)\left(P_2\left(\frac{k^2-2kt}{2p}\right)-P_2\left(\frac{t^2}{2p}\right)\right)\\
&\quad -\frac{1}{2}P_1\left(\frac{2kt}{p}\right)\left(P_2\left(\frac{k^2-2kt}{2p}\right)-P_2\left(\frac{k^2+2kt}{2p}\right)\right)\\
&\quad -\frac{1}{6}\left(-P_3\left(\frac{(k-t)^2}{2p}\right)+P_3\left(\frac{2kt}{p}\right)\right)\\
&\quad -\frac{1}{6}\left(-2P_3\left(\frac{t^2}{2p}\right)+2P_3\left(\frac{k^2+2kt}{2p}\right)+2P_3\left(\frac{k^2-2kt}{p}\right)\right)\\
&\quad -\frac{1}{4}\delta_1\left(-\frac{t^2}{2p},\frac{k^2+2kt}{2p},\frac{k^2-2kt}{2p}\right)+\frac{1}{4}\delta_2\left(-\frac{t^2}{2p},\frac{k^2+2kt}{2p},\frac{k^2-2kt}{2p}\right),
\end{aligned}
$$

where

$$
\delta_1(x,y,z)=\begin{cases} P_1(x), & \text{if } x\equiv y\equiv z \bmod 1,\\ 0, & \text{otherwise,}\end{cases}
$$

and

$$
\delta_2(x,y,z)=\begin{cases} P_1(x), & \text{if } y,z \text{ are integers},\\ P_1(y), & \text{if } x,z \text{ are integers},\\ P_1(z), & \text{if } x,y \text{ are integers},\\ 0, & \text{otherwise.}\end{cases}
$$

Carefully summing over $k, t(2p)$ and applying Proposition 3.2.7, Proposition 3.2.9, Proposition 3.2.5, Proposition 3.2.8, and Proposition 3.2.1, we get

$$R_p - Z_p + 2L_p = c_p + \frac{5}{12p}B_{3,\psi} + \frac{-3p+1-3\psi(2)}{12p}B_{1,\psi}.$$

Applying Proposition 3.3.1, Proposition 3.3.3, and Proposition 3.3.4, we obtain

$$\frac{1}{6p}B_{3,\psi} - \frac{1}{4}B_{1,\psi} + b_p = c_p + \frac{5}{12p}B_{3,\psi} + \frac{-3p+1-3\psi(2)}{12p}B_{1,\psi},$$

from which the assertion of the proposition immediately follows. □

Proposition 3.3.6. With notation and assumptions as above, we have

$$a_p + b_p - c_p = \frac{1}{3p}B_{3,\psi}.$$

Proof. Applying Proposition 3.1.19 with $n = 3$ and the arguments $x = \frac{(k+t)^2}{p}$, $y = -\frac{2kt}{p}$, we obtain

$$\begin{aligned} &P_1\left(\frac{(k+t)^2}{p}\right)P_2\left(\frac{k^2+t^2}{p}\right) - P_1\left(\frac{2kt}{p}\right)P_2\left(\frac{k^2+t^2}{p}\right) - P_1\left(\frac{(k+t)^2}{p}\right)P_2\left(\frac{2kt}{p}\right) \\ &\quad + P_1\left(\frac{2kt}{p}\right)P_2\left(\frac{(k+t)^2}{p}\right) = \frac{1}{3}\left(P_3\left(\frac{(k+t)^2}{p}\right) + 2P_3\left(\frac{k^2+t^2}{p}\right) - P_3\left(\frac{2kt}{p}\right)\right). \end{aligned}$$

Carefully summing over $k, t(p)$ and applying Proposition 3.2.10, we obtain

$$a_p + b_p - c_p = \frac{1}{3p}B_{3,\psi},$$

which was to be demonstrated. □

Theorem 3.3.7. With notation and assumptions as above, we have

$$\begin{aligned} a_p &= \frac{1}{12p}B_{3,\psi} - \frac{1-3\psi(2)}{12p}B_{1,\psi}, \\ b_p &= \frac{1}{6p}B_{3,\psi} + \frac{1-\psi(2)}{2p}B_{1,\psi}, \\ c_p &= -\frac{1}{12p}B_{3,\psi} + \frac{5-3\psi(2)}{12p}B_{1,\psi}. \end{aligned}$$

Proof. This follows immediately from Propostion 3.3.2, Proposition 3.3.5, and Proposition 3.3.6. □

We are now in position to prove Theorem 3.3.

Proof of Theorem 3.3. This follows immediately from Proposition 3.3.1 and Theorem 3.3.7. □

3.4 The Sum J_p and the Mordell-Tornheim L-Function

In this section, we discuss a very interesting exponential sum and its connection with a particular Mordell-Tornheim L-function. Matsumoto et al.[14] define the Mordell-Tornheim L-functions of depth k by

$$\begin{aligned} & L_{MT,k}(s_1, \cdots, s_{k+1}; \chi_1, \cdots, \chi_{k+1}) \\ & \quad = \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} \frac{\chi_1(m_1) \cdots \chi_k(m_k) \chi_{k+1}(m_1 + \cdots + m_k)}{m_1^{s_1} \cdots m_k^{s_k} (m_1 + \cdots + m_k)^{s_{k+1}}} \\ & \quad (\mathrm{Re}(s_j) \geq 1,\ 1 \leq j \leq k+1) \end{aligned}$$

for complex variables $s_1, \cdots, s_{k+1}$ and primitive Dirichlet characters $\chi_1, \cdots, \chi_{k+1}$.

We fix an odd prime $p \equiv 3(4)$. We denote by ψ the Legendre symbol mod p: $\psi(a) = \left(\frac{a}{p}\right)$. We also put $\zeta = exp(2\pi i/p)$.

Theorem 3.4. *With notation and assumptions being the same as above, put*

$$J_p = \sum_{(a,b,c) \in S} \frac{\psi(abc)}{(\zeta^a - 1)(\zeta^b - 1)(\zeta^c - 1)},$$

where $S = \{(a, b, c) \in (\mathbb{F}_p^\times)^3 \mid a + b + c = 0\}$. *Then, we get*

$$J_p = \frac{3p^3 i}{4\pi^3} L_{MT,2}(1, 1, 1; \psi, \psi, \psi).$$

Incidentally, when $p \equiv 1(4)$, the sum J_p is not so interesting since replacing (a, b, c) by $(-a, -b, -c)$ reveals that $J_p = 0$. Before proving this theorem, we first establish two propositions.

Proposition 3.4.1. *With notation and assumptions being the same as above, we have*

$$J_p = p\sqrt{-p}\, j_p,$$

where

$$j_p = \sum_{i,j,k(p)} P_1\left(\frac{j+i}{p}\right) P_2\left(\frac{j+k}{p}\right) \psi(ijk). \tag{3.4.1}$$

Proof. We have

$$\begin{aligned} J_p &= \sum_{(a,b,c)\in S} \frac{\psi(abc)}{(\zeta^a-1)(\zeta^b-1)(\zeta^c-1)} \\ &= \frac{1}{p}\sum_{x(p)}\sum_{a,b,c=1}^{p-1} \frac{\psi(abc)\zeta^{x(a+b+c)}}{(\zeta^a-1)(\zeta^b-1)(\zeta^c-1)}. \end{aligned}$$

Applying Proposition 3.2.14, we get

$$\begin{aligned} J_p &= \frac{1}{p^4}\sum_{x(p)}\sum_{a,b,c=1}^{p-1}\sum_{i,j,k=1}^{p-1} ijk\,\zeta^{(i+x)a+(j+x)b+(k+x)c}\,\psi(abc) \\ &= \frac{\tau(\psi)^3}{p^4}\sum_{x(p)}\sum_{i,j,k=1}^{p-1} ijk\,\psi(i+x)\psi(j+x)\psi(k+x), \end{aligned}$$

where $\tau(\psi)$ is the Gaussian sum $\tau(\psi)=\sum_{a=1}^{p-1}\psi(a)\zeta^a$. We will now express everything in terms of periodic Bernoulli polynomials and take full advantage of their periodicity. Noting that P_1 and ψ are odd functions and applying Proposition 3.2.11, we obtain

$$\begin{aligned} J_p &= \frac{\tau(\psi)^3}{p}\sum_{x(p)}\sum_{i,j,k=1}^{p-1}\left(P_1\left(\frac{i}{p}\right)+\frac{1}{2}\right)\left(P_1\left(\frac{j}{p}\right)+\frac{1}{2}\right)\left(P_1\left(\frac{k}{p}\right)+\frac{1}{2}\right) \\ &\quad\times \psi(i+x)\psi(j+x)\psi(k+x) \\ &= \frac{\tau(\psi)^3}{p}\Bigg\{\sum_{x(p)}\sum_{i,j,k=1}^{p-1} P_1\left(\frac{i}{p}\right)P_1\left(\frac{j}{p}\right)P_1\left(\frac{k}{p}\right)\ \psi(i+x)\psi(j+x)\psi(k+x) \\ &\quad+\frac{1}{4}\sum_{x(p)}\sum_{i,j,k=1}^{p-1}\left(P_1\left(\frac{i}{p}\right)+P_1\left(\frac{j}{p}\right)+P_1\left(\frac{k}{p}\right)\right)\psi(i+x)\psi(j+x)\psi(k+x)\Bigg\} \\ &= \frac{\tau(\psi)^3}{p}\Bigg\{\sum_{i,j,k,x(p)} P_1\left(\frac{i+x}{p}\right)P_1\left(\frac{j+x}{p}\right)P_1\left(\frac{k+x}{p}\right)\ \psi(ijk)-\frac{3}{4}B_{1,\psi}\Bigg\}. \end{aligned}$$

Applying the Sczech 3-term addition formula (Proposition 3.1.18) with the arguments

$$v_1 = \frac{i+x}{p},\ v_2 = \frac{j+x}{p},\ v_3 = \frac{k+x}{p},$$

we obtain

$$\begin{aligned}
&- P_1\left(\frac{i+x}{p}\right) P_1\left(\frac{j+x}{p}\right) P_1\left(\frac{k+x}{p}\right)\\
&\quad + P_1\left(\frac{i+x}{p}\right) P_1\left(\frac{j-i}{p}\right) P_1\left(\frac{k-i}{p}\right)\\
&\quad + P_1\left(\frac{i-j}{p}\right) P_1\left(\frac{j+x}{p}\right) P_1\left(\frac{k-j}{p}\right)\\
&\quad + P_1\left(\frac{i-k}{p}\right) P_1\left(\frac{j-k}{p}\right) P_1\left(\frac{k+x}{p}\right)\\
&\quad = \frac{1}{2}P_1\left(\frac{j-i}{p}\right)\left(P_2\left(\frac{k-j}{p}\right) - P_2\left(\frac{k-i}{p}\right)\right)\\
&\quad + \frac{1}{2}P_1\left(\frac{j-i}{p}\right)\left(P_2\left(\frac{j+x}{p}\right) - P_2\left(\frac{i+x}{p}\right)\right)\\
&\quad + \frac{1}{2}P_1\left(\frac{k-i}{p}\right)\left(P_2\left(\frac{k+x}{p}\right) - P_2\left(\frac{i+x}{p}\right)\right)\\
&\quad + \frac{1}{2}P_1\left(\frac{k-j}{p}\right)\left(P_2\left(\frac{k+x}{p}\right) - P_2\left(\frac{j+x}{p}\right)\right)\\
&\quad - \frac{1}{6}\left(-P_3\left(\frac{k-i}{p}\right) - P_3\left(\frac{k-j}{p}\right)\right)\\
&\quad - \frac{1}{6}\left(2P_3\left(\frac{i+x}{p}\right) + 2P_3\left(\frac{j+x}{p}\right) + 2P_3\left(\frac{k+x}{p}\right)\right)\\
&\quad - \frac{1}{4}\delta_1\left(\frac{i+x}{p}, \frac{j+x}{p}, \frac{k+x}{p}\right) + \frac{1}{4}\delta_2\left(\frac{i+x}{p}, \frac{j+x}{p}, \frac{k+x}{p}\right).
\end{aligned}$$

Multiplying throughout by $\psi(ijk)$, carefully summing over $i, j, k, x(p)$, and applying Proposition 3.2.12, Proposition 3.2.2, we find that

$$\sum_{i,j,k,x(p)} P_1\left(\frac{i+x}{p}\right) P_1\left(\frac{j+x}{p}\right) P_1\left(\frac{k+x}{p}\right) \psi(ijk) = -p\, j_p + \frac{3}{4}B_{1,\psi},$$

where j_p is given by (3.4.1). Hence, plugging this result back into the above expression for J_p, we obtain

$$J_p = -\tau(\psi)^3 j_p.$$

As $p \equiv 3(4)$, $\tau(\psi) = \sqrt{-p}$, and the assertion of the proposition immediately follows.

□

Proposition 3.4.2. With notation and assumptions as above, we have

$$j_p = \frac{3p\sqrt{p}}{4\pi^3}\, L_{MT,2}(1,1,1;\psi,\psi,\psi).$$

Proof. Employing the Fourier series for $P_1\left(\frac{j+i}{p}\right)$ and $P_2\left(\frac{j+k}{p}\right)$ (Proposition 3.1.10), we get

$$\begin{aligned} j_p &= \frac{2!}{(2\pi i)^3} \sum_{i,j,k(p)} \sideset{}{'}\sum_{x,y\in\mathbb{Z}} \frac{e\left(\frac{x(j+i)+y(j+k)}{p}\right)\psi(ijk)}{xy^2} \\ &= \frac{2\tau(\psi)^3}{(2\pi i)^3} \sideset{}{'}\sum_{x,y\in\mathbb{Z}} \frac{\psi(xy(x+y))}{xy^2}. \end{aligned}$$

From the partial fraction expansion $\frac{1}{xy} = \frac{1}{x(x+y)} + \frac{1}{y(x+y)}$, together with $\psi(-x) = -\psi(x)$, it follows that

$$\begin{aligned} j_p &= \frac{\tau(\psi)^3}{(2\pi i)^3} \sideset{}{'}\sum_{x,y\in\mathbb{Z}} \frac{\psi(xy(x+y))}{xy(x+y)} \\ &= \frac{2\tau(\psi)^3}{(2\pi i)^3} \sideset{}{'}\sum_{\substack{x,y\in\mathbb{Z}\\ y>0}} \frac{\psi(xy(x+y))}{xy(x+y)} \\ &= \frac{2\tau(\psi)^3}{(2\pi i)^3} \left\{ \sideset{}{'}\sum_{\substack{x,y\in\mathbb{Z}\\ x,y>0}} \frac{\psi(xy(x+y))}{xy(x+y)} + \sideset{}{'}\sum_{\substack{x,y\in\mathbb{Z}\\ y>x>0}} \frac{\psi(xy(y-x))}{xy(y-x)} + \sideset{}{'}\sum_{\substack{x,y\in\mathbb{Z}\\ x>y>0}} \frac{\psi(xy(y-x))}{xy(y-x)} \right\} \\ &= \frac{6\tau(\psi)^3}{(2\pi i)^3} \sideset{}{'}\sum_{\substack{x,y\in\mathbb{Z}\\ x,y>0}} \frac{\psi(xy(x+y))}{xy(x+y)} \\ &= \frac{6\tau(\psi)^3}{(2\pi i)^3}\, L_{MT,2}(1,1,1;\psi,\psi,\psi). \end{aligned}$$

As $p \equiv 3(4)$, $\tau(\psi) = \sqrt{-p}$, and the assertion of the proposition immediately follows.

□

We are now in position to prove Theorem 3.4.

Proof of Theorem 3.4. This follows immediately from Proposition 3.4.1 and Proposition 3.4.2. □

We now obtain the following interesting corollary of Proposition 3.4.2, a result for which we are not able to find any reference in the literature.

Corollary 3.4.3. *With notation and assumptions being the same as above, we have*

$$L_{MT,2}(1,1,1;\psi,\psi,\psi) \in \frac{\pi^3}{\sqrt{p}}\mathbb{Q}.$$

Proof. This follows immediately from Proposition 3.4.2 and the fact that $j_p \in \mathbb{Q}$.

□

3.5 A Contribution Towards an Elementary Proof of the Lee-Weintraub Identity

In this section, we make our contribution towards an elementary proof of the Lee-Weintraub identity. We prove, in a completely elementary manner, that the Lee-Weintraub identity is equivalent to any of the three identities given in Theorem 3.5. Thus proving any one of these identities in an elementary manner would complete an elementary proof of the Lee-Weintraub identity.

We fix an odd prime p. We denote by ψ the Legendre symbol mod p: $\psi(a) = \left(\frac{a}{p}\right)$. We also put $\zeta = exp(2\pi i/p)$. We recall the sums F_p, G_p, R_p, and introduce the sum S_p:

$$F_p = \sum_{k,t(p)} P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{k^2+2kt}{p}\right)$$
$$G_p = \sum_{k,t(p)} P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{2kt}{p}\right) P_1\left(\frac{k^2+2kt}{p}\right)$$
$$R_p = \sum_{k,t(2p)} P_1\left(\frac{t^2}{2p}\right) P_1\left(\frac{k^2+2kt}{2p}\right) P_1\left(\frac{k^2-2kt}{2p}\right)$$
$$S_p = \sum_{k,t(p)} P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{(k+t)^2}{p}\right)$$

We also recall the Lee-Weintraub identity, proved by Ibukiyama.

The Lee-Weintraub Identity [9]. *With notation and assumptions being the same as above, put*

$$LW_p = \sum_{(a,b,c)\in T} \frac{\psi(abc)}{(\zeta^a-1)(\zeta^b-1)(\zeta^c-1)},$$

where $T = \{(a,b,c) \in (\mathbb{F}_p^\times)^3 \mid ab+bc+ca=0\}$. *Then, we get*

$$LW_p = \sqrt{\psi(-1)p}\left(\frac{1}{6}B_{3,\psi} - \frac{3}{2}B_{2,\psi} + \frac{p+1}{4}B_{1,\psi}\right).$$

Theorem 3.5 *With notation and assumptions being the same as above, the Lee-Weintraub identity is equivalent any of the following three identities:*

$$(i) \quad R_p = 0,$$

$$(ii) \quad F_p = G_p,$$

$$(iii) \quad \lim_{t\to\infty} \sum_{\substack{x,y,z\in\mathbb{Z} \\ xy-z^2\equiv 0(p) \\ |x|,|y|,|z|<t}}{}' \frac{\psi(x)}{x(y^2-z^2)} = 0.$$

Proof. By applying the same reasoning as in Section 5.1, in a completely elementary manner, one easily shows that

$$LW_p = \sqrt{\psi(-1)p}\left(-pS_p - \frac{3}{2}B_{2,\psi} + \frac{3}{4}B_{1,\psi}\right).$$

Thus the Lee-Weintraub identity is equivalent to the identity

$$S_p = -\frac{1}{6p}B_{3,\psi} - \frac{p-2}{4p}B_{1,\psi}. \tag{3.5.1}$$

Applying the 2-term addition formula (Proposition 3.1.17) to the arguments

$$v_1 = -\frac{t^2}{p},\ v_2 = \frac{(k+t)^2}{p},$$

we obtain

$$\begin{aligned}
&-P_1\left(\frac{t^2}{p}\right)P_1\left(\frac{(k+t)^2}{p}\right) + P_1\left(\frac{t^2}{p}\right)P_1\left(\frac{k^2+2kt}{p}\right) - P_1\left(\frac{(k+t)^2}{p}\right)P_1\left(\frac{k^2+2kt}{p}\right) \\
&= -\frac{1}{2}\left(P_2\left(\frac{t^2}{p}\right) + P_2\left(\frac{(k+t)^2}{p}\right) + P_2\left(\frac{k^2+2kt}{p}\right)\right) + \frac{1}{4}\delta\left(\frac{t^2}{p}, \frac{(k+t)^2}{p}\right),
\end{aligned}$$

where

$$\delta(x,y) = \begin{cases} 1, & \text{if } x, y \text{ are integers,} \\ 0, & \text{otherwise.} \end{cases}$$

Multiplying both sides of the above equation by $P_1\left(\frac{k^2}{p}\right)$, carefully summing over $k, t(p)$, and applying Proposition 3.2.1, Proposition 3.2.2, we get

$$S_p = 2F_p + \frac{1}{4p}B_{1,\psi}.$$

From (3.3.2), (3.3.5), we see that

$$S_p = R_p - \frac{1}{6p}B_{3,\psi} - \frac{p-2}{4p}B_{1,\psi}.$$

Thus the assertion (i) follows from (3.5.1). The assertion (ii) follows immediately from (i) and (3.3.5). If $p \equiv 1(4)$, then the sum in the assertion (iii) vanishes by the usual even/odd argument. Thus we assume that $p \equiv 3(4)$ in what follows. By the multiplication formula, we find that

$$\begin{aligned} R_p &= \sum_{k,t(2p)} P_1\left(\frac{t^2}{2p}\right) P_1\left(\frac{k^2+2kt}{2p}\right) P_1\left(\frac{k^2-2kt}{2p}\right) \\ &= \sum_{\substack{k(2p) \\ 0\le t\le p-1}} P_1\left(\frac{t^2}{2p}\right) P_1\left(\frac{k^2+2kt}{2p}\right) P_1\left(\frac{k^2-2kt}{2p}\right) \\ &\quad + \sum_{\substack{k(2p) \\ 0\le t\le p-1}} P_1\left(\frac{t^2}{2p}+\frac{1}{2}\right) P_1\left(\frac{k^2+2kt}{2p}\right) P_1\left(\frac{k^2-2kt}{2p}\right) \\ &= \sum_{\substack{k(2p) \\ t(p)}} P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{k^2+2kt}{2p}\right) P_1\left(\frac{k^2-2kt}{2p}\right). \end{aligned}$$

Applying the Fourier series for the P_1's (Proposition 3.1.10), we get

$$\begin{aligned} R_p &= \sum_{\substack{k(2p) \\ t(p)}} P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{k^2+2kt}{2p}\right) P_1\left(\frac{k^2-2kt}{2p}\right) \\ &= -\frac{1}{(2\pi i)^3} \sum_{\substack{k(2p) \\ t(p)}} \sum_{x,y,z\in\mathbb{Z}}{}' \frac{e\left(\frac{2xt^2+y(k^2+2kt)+z(k^2-2kt)}{2p}\right)}{xyz}. \end{aligned}$$

Recall that we are summing symmetrically with respect to x, y, z. Applying the Q-limit formula (see Gunnells and Sczech[7]), we can replace (x, y, z) by $(x, y+z, y-z)$ while still summing symmetrically with respect to x, y, z. This together with Proposition 3.2.18 yields

$$\begin{aligned}
R_p &= -\frac{1}{(2\pi i)^3} \sideset{}{'}\sum_{x,y,z\in\mathbb{Z}} \frac{1}{x(y^2-z^2)} \sum_{t(p)} e\left(\frac{xt^2}{p}\right) \sum_{k(2p)} e\left(\frac{yk^2+2ztk}{p}\right) \\
&= -\frac{2\,\tau(\psi)}{(2\pi i)^3} \sideset{}{'}\sum_{x,y,z\in\mathbb{Z}} \frac{\psi(y)}{x(y^2-z^2)} \sum_{t(p)} e\left(\frac{\{x-y^{-1}z^2\}t^2}{p}\right) \\
&= -\frac{2p\,\tau(\psi)}{(2\pi i)^3} \sideset{}{'}\sum_{\substack{x,y,z\in\mathbb{Z} \\ xy-z^2\equiv 0(p)}} \frac{\psi(y)}{x(y^2-z^2)}.
\end{aligned}$$

Thus the assertion (iii) follows from (i). □

4 A Special Family of Character Sums

Let χ be a primitive Dirichlet character with conductor $f > 1$. Let l be any positive integer which is prime to f. Let $n \in \mathbb{Z}$, $n \geq 0$. We also put $\zeta = exp(2\pi i/f)$.

Consider the Dedekind sum

$$\begin{aligned} s(l,f) &= \sum_{a(f)} P_1\left(\frac{la}{f}\right) P_1\left(\frac{a}{f}\right) \\ &= -\frac{1}{f}\sum_{a=1}^{f-1} \frac{1}{(\zeta^{la}-1)(\zeta^a-1)} + \frac{f-1}{4f}. \end{aligned}$$

Observe that the mystery of the Dedekind sum $s(l,f)$ lies in the exponential sum

$$\sum_{a=1}^{f-1} \frac{1}{(\zeta^{la}-1)(\zeta^a-1)}.$$

Thus, the only natural (and responsible) thing to do is to twist this exponential sum with an arbitrary primitive Dirichlet character with conductor f. This is precisely what we did, and in fact, we generalized it even further.

The aim of this chapter is to obtain a formula expressing the following character sum

$$M_n(l,\chi) = \sum_{a=1}^{f-1} \frac{\chi(a)}{(\zeta^{la}-1)^n(\zeta^a-1)}$$

in terms of generalized Bernoulli numbers using only elementary methods from algebra and number theory. The following plays the starring role in accomplishing this.

4.1 The Star Character Sum $Z_k(l,c,\chi)$

Fix a primitive Dirichlet character χ with conductor $f_\chi > 1$. We write f for f_χ when there is no fear of confusion. For any natural number l prime to f and any integer c, we define a character sum $Z_k(l, c, \chi)$ by

$$Z_k(l, c, \chi) = f^{k-1} \sum_{a(f)} P_k\left(\frac{la + c}{fl}\right) \chi(la + c). \tag{4.1.1}$$

For any Dirichlet character δ, we denote by f_δ the conductor of δ. For any natural number m, we denote by $X(m)$ the set of primitive Dirichlet characters δ such that m is divisible by f_δ, and by $Y(m)$ the set of primitive Dirichlet characters with conductor m. The following theorem is a generalization of Ibukiyama's Theorem 2 in [10].

Theorem 4.1. *Let l be a natural number prime to f and c be a natural number prime to l with $1 \leq c \leq l-1$. For any integer u with $u \mid l$, denote by l_u the u-primary part of l, that is, the maximum integer which divides l and is prime to u. We get*

$$Z_k(l, c, \chi) = \frac{1}{\phi(l) l^{k-1}} \sum_{u|l} \sum_{\delta \in Y(u)} \left(\delta(c^{-1}) B_{k,\delta\chi} \prod_{\substack{q|l_u \\ q \text{ prime}}} (1 - q^{k-1}\chi(q)\delta(q)) \right),$$

where ϕ is the Euler function.

To prove Theorem 4.1, we prepare several lemmas. In preparation for Lemma 1, we recall the multiplication formula for periodic Bernoulli polynomials:

$$P_k(nx) = n^{k-1} \sum_{a=0}^{n-1} P_k\left(x + \frac{a}{n}\right).$$

Lemma 4.1.1. *For a natural number l prime to f_χ and any $\delta \in X(l)$, we get*

$$l^{k-1} \sum_{c=0}^{l-1} \delta(c) Z_k(l, c, \chi) = B_{k,\delta\chi}.$$

Proof. Since $\delta(c) = \delta(ln + c)$, we get

$$
\begin{aligned}
l^{k-1}\sum_{c=0}^{l-1}\delta(c)Z_k(l,c,\chi) &= (f_\chi l)^{k-1}\sum_{c=0}^{l-1}\sum_{n=0}^{f_\chi-1}P_k\left(\frac{ln+c}{f_\chi l}\right)\delta(ln+c)\chi(ln+c)\\
&= (f_\chi l)^{k-1}\sum_{m=0}^{f_\chi l-1}P_k\left(\frac{m}{f_\chi l}\right)\delta(m)\chi(m).
\end{aligned}
$$

Since $(l, f_\chi) = 1$, the Dirichlet character $\delta\chi$ is primitive with conductor $f_{\delta\chi} = f_\delta f_\chi$. Hence, we get

$$
\begin{aligned}
l^{k-1}\sum_{c=0}^{l-1}\delta(c)Z_k(l,c,\chi) &= (f_\chi l)^{k-1}\sum_{m=0}^{f_\chi l-1}P_k\left(\frac{m}{f_\chi l}\right)\delta\chi(m)\\
&= (f_\chi l)^{k-1}\sum_{a=0}^{f_{\delta\chi}-1}\sum_{b=0}^{lf_\delta^{-1}-1}P_k\left(\frac{f_{\delta\chi}b+a}{f_\chi l}\right)\delta\chi(f_{\delta\chi}b+a)\\
&= (f_\chi l)^{k-1}\sum_{a=0}^{f_{\delta\chi}-1}\left(\sum_{b=0}^{lf_\delta^{-1}-1}P_k\left(\frac{a}{f_\chi l}+\frac{b}{lf_\delta^{-1}}\right)\right)\delta\chi(a).
\end{aligned}
$$

Applying the multiplication formula, we obtain

$$
\begin{aligned}
l^{k-1}\sum_{c=0}^{l-1}\delta(c)Z_k(l,c,\chi) &= (f_\chi l)^{k-1}\left(\frac{f_\delta}{l}\right)^{k-1}\sum_{a=0}^{f_{\delta\chi}-1}B_k\left(\frac{a}{f_{\delta\chi}}\right)\delta\chi(a)\\
&= B_{k,\delta\chi}.
\end{aligned}
$$

□

From this formula, we shall extract a kind of inversion formula. We fix a natural number l which is prime to f_χ and put $L = \prod_{q|l} q$, where q runs over primes. For any $m \mid l$, denote by l_m the m-primary part of l.

Lemma 4.1.2. *For any fixed number $d \in (\mathbb{Z}/l\mathbb{Z})^*$, we get*

$$
\sum_{m|L}\phi(l_m)\chi(m)\left(l_m m\right)^{k-1}Z_k(l_m,e,\chi) = \sum_{\delta\in X(l)}\delta(d^{-1})B_{k,\delta\chi},
$$

where e is the unique integer such that $me \equiv d \bmod l_m$ *with* $0 \leq e \leq l_m - 1$.

Proof. We shall show this lemma by taking the sum over $\delta \in X(l)$ of both sides of the formula in Lemma 4.1.1. For an integer c with $0 \leq c \leq l-1$, there exists a unique $m \mid L$ such that $m \mid c$ and $(c, L/m) = 1$. Denote by $A(m)$ the following set of integers:

$$A(m) = \{c \in \mathbb{Z} : 0 \leq c \leq l-1,\ m \mid c,\ (c, L/m) = 1\}.$$

Then for any $c \in A(m)$, we have $\sum_{\delta \in X(l)} \delta(c) = \sum_{\delta \in X(l_m)} \delta(c)$, since $\delta(c) = 0$ whenever $(f_\delta, m) > 1$. Hence, applying Lemma 4.1.1, we get

$$\begin{aligned}
\sum_{\delta \in X(l)} \delta(d^{-1}) B_{k,\delta\chi} &= l^{k-1} \sum_{\delta \in X(l)} \sum_{c=0}^{l-1} \delta(d^{-1}c) Z_k(l, c, \chi) \\
&= l^{k-1} \sum_{m \mid L} \sum_{c \in A(m)} \sum_{\delta \in X(l)} \delta(d^{-1}c) Z_k(l, c, \chi) \qquad (4.1.2)\\
&= l^{k-1} \sum_{m \mid L} \sum_{c \in A(m)} \sum_{\delta \in X(l_m)} \delta(d^{-1}c) Z_k(l, c, \chi).
\end{aligned}$$

We note that

$$\sum_{\delta \in X(l_m)} \delta(d^{-1}c) = \begin{cases} \phi(l_m), & \text{if } d \equiv c \bmod l_m, \\ 0, & \text{otherwise,} \end{cases}$$

and denote by $C(m)$ the following set of integers:

$$C(m) = \{c \in \mathbb{Z} : 0 \leq c \leq l-1,\ m \mid c,\ (c, L/m) = 1, \text{ and } d \equiv c \bmod l_m\}.$$

Then, from (4.1.2), we get

$$\begin{aligned}
\sum_{\delta \in X(l)} \delta(d^{-1}) B_{k,\delta\chi} &= l^{k-1} \sum_{m \mid L} \sum_{c \in A(m)} \sum_{\delta \in X(l_m)} \delta(d^{-1}c) Z_k(l, c, \chi) \qquad (4.1.3)\\
&= l^{k-1} \sum_{m \mid L} \phi(l_m) \sum_{c \in C(m)} Z_k(l, c, \chi).
\end{aligned}$$

If we take the unique integer e such that $me \equiv d \bmod l_m$ with $0 \leq e \leq l_m - 1$, then $(e, l_m) = 1$, since $(d, l) = 1$. Hence we get

$$C(m) = \{m(e + l_m a) : a \in \mathbb{Z},\, 0 \leq a \leq l(l_m m)^{-1} - 1\},$$

and

$$\begin{aligned}
\sum_{c \in C(m)} Z_k(l, c, \chi) &= \sum_{a=0}^{l(l_m m)^{-1}-1} Z_k(l, m(e + l_m a), \chi) \\
&= f_\chi^{k-1} \sum_{a=0}^{l(l_m m)^{-1}-1} \sum_{n=0}^{f_\chi - 1} P_k\left(\frac{ln + m(e + l_m a)}{f_\chi l}\right) \chi(ln + m(e + l_m a)) \\
&= \chi(m) f_\chi^{k-1} \sum_{a=0}^{l(l_m m)^{-1}-1} \sum_{n=0}^{f_\chi - 1} P_k\left(\frac{l_m\left(l(l_m m)^{-1} n + a\right) + e}{f_\chi l m^{-1}}\right) \\
&\quad \times \chi\left(l_m\left(l(l_m m)^{-1} n + a\right) + e\right) \\
&= \chi(m) f_\chi^{k-1} \sum_{b=0}^{f_\chi l(l_m m)^{-1}-1} P_k\left(\frac{l_m b + e}{f_\chi l m^{-1}}\right) \chi\left(l_m b + e\right) \\
&= \chi(m) f_\chi^{k-1} \sum_{n=0}^{f_\chi - 1} \sum_{a=0}^{l(l_m m)^{-1}-1} P_k\left(\frac{l_m\left(f_\chi a + n\right) + e}{f_\chi l m^{-1}}\right) \chi\left(l_m\left(f_\chi a + n\right) + e\right) \\
&= \chi(m) f_\chi^{k-1} \sum_{n=0}^{f_\chi - 1} \left(\sum_{a=0}^{l(l_m m)^{-1}-1} P_k\left(\frac{l_m n + e}{f_\chi l m^{-1}} + \frac{a}{l(l_m m)^{-1}}\right)\right) \chi\left(l_m n + e\right).
\end{aligned}$$

Applying the multiplication formula, we get

$$\begin{aligned}
\sum_{c \in C(m)} Z_k(l, c, \chi) &= \chi(m) f_\chi^{k-1} \left(\frac{l_m m}{l}\right)^{k-1} \sum_{n=0}^{f_\chi - 1} P_k\left(\frac{l_m n + e}{f_\chi l_m}\right) \chi\left(l_m n + e\right) \\
&= \chi(m) \left(\frac{l_m m}{l}\right)^{k-1} Z_k(l_m, e, \chi).
\end{aligned}$$

Thus, from (4.1.3), we obtain

$$\sum_{\delta \in X(l)} \delta(d^{-1}) B_{k, \delta\chi} = \sum_{m \mid L} \phi(l_m) \chi(m) \left(l_m m\right)^{k-1} Z_{k, l_m, e}.$$

□

Lemma 4.1.3. *We fix natural numbers l prime to f_χ and c prime to l with $1 \le c \le l-1$. We define L and l_m for $m \mid l$ in the same way as in Lemma 4.1.2. Then, we get*

$$\phi(l)l^{k-1}Z_{k,l,c} = \sum_{m \mid L} \mu(m)\chi(m)m^{k-1} \sum_{\delta \in X(l_m)} \delta(mc^{-1})B_{k,\delta\chi},$$

where μ is the Möbius function.

Proof. For $u \mid v \mid L$ and any $d \in (\mathbb{Z}/l\mathbb{Z})^*$, we put

$$g(u,v,d) = \phi\left(l/l_u\right)\chi\left(u^{-1}v\right)\left(u^{-1}vl/l_u\right)^{k-1} Z_k(l/l_u, w, \chi),$$

where w is defined as the unique integer such that $(u^{-1}v)w \equiv d \bmod (l/l_u)$ with $1 \le w \le l/l_u - 1$. Also, we put

$$f(v,d) = \sum_{\delta \in X(l/l_v)} \delta(d^{-1})B_{k,\delta\chi}.$$

Next, we apply Lemma 4.1.2 for $(v, l/l_v)$ instead of (L, l). Noting that $(l/l_v)_m = l_m/l_v$ for any $m \mid v$, we get

$$\sum_{m \mid v} \phi(l_m/l_v)\chi(m)(l_m m)^{k-1} Z_k(l_m/l_v, e, \chi) = \sum_{\delta \in X(l/l_v)} \delta(d^{-1})B_{k,\delta\chi},$$

where e is determined by $me \equiv d \bmod (l_m/l_u)$ with $1 \le e \le l_m/l_v - 1$. For each $m \mid v$, we define u by $mu = v$. Then $l_m/l_v = l/l_u$, and we get

$$\sum_{u \mid v} g(u,v,d) = f(v,d).$$

For any $u \mid v \mid L$, we put $G(u) = g(u, L, c)$ and $F(v) = (L/v)^{k-1}\chi(v^{-1}L)f(v, L^{-1}vc)$.

Observe that $g(u,L,c) = (L/v)^{k-1}\chi(v^{-1}L)g(u,v,L^{-1}vc)$, where $L^{-1}vc$ is regarded as an element of $(\mathbb{Z}/l_u\mathbb{Z})^*$. Hence, we get

$$\begin{aligned}\sum_{u|v} G(u) &= (L/v)^{k-1}\chi(v^{-1}L)\sum_{u|v} g(u,v,L^{-1}vc)\\ &= (L/v)^{k-1}\chi(v^{-1}L)f(v,L^{-1}vc)\\ &= F(v).\end{aligned}$$

Applying the Möbius inversion formula for $v = L$, we get

$$G(L) = \sum_{m|L}\mu(m)F\left(\frac{L}{m}\right).$$

Thus, we obtain

$$\phi(l)l^{k-1}Z_k(l,c,\chi) = \sum_{m|L}\mu(m)\chi(m)m^{k-1}\sum_{\delta\in X(l_m)}\delta(mc^{-1})B_{k,\delta\chi}.$$

□

We are now in position to prove Theorem 4.1.

Proof of Theorem 4.1. We define L and l_m for $m \mid l$ in the same way as in Lemma 4.1.2. From Lemma 4.1.3, we get

$$\begin{aligned}Z_k(l,c,\chi) &= \frac{1}{\phi(l)l^{k-1}}\sum_{m|L}\sum_{\delta\in X(l_m)}\mu(m)\chi(m)\delta(m)m^{k-1}\delta(mc^{-1})B_{k,\delta\chi}\\ &= \frac{1}{\phi(l)l^{k-1}}\sum_{m|l}\sum_{\delta\in X(l_m)}\mu(m)\chi(m)\delta(m)m^{k-1}\delta(mc^{-1})B_{k,\delta\chi}\\ &= \frac{1}{\phi(l)l^{k-1}}\sum_{m|l}\sum_{u|l_m}\sum_{\delta\in Y(u)}\mu(m)\chi(m)\delta(m)m^{k-1}\delta(mc^{-1})B_{k,\delta\chi}.\end{aligned}$$

Observe that $u \mid l_m$ for $m \mid l$ is equivalent to $m \mid l_u$ for $u \mid l$. Thus, taking Proposition

3.2.36 into account, we get

$$\begin{aligned}
Z_k(l,c,\chi) &= \frac{1}{\phi(l)l^{k-1}} \sum_{u|l} \sum_{m|l_u} \sum_{\delta\in Y(u)} \mu(m)\chi(m)\delta(m)m^{k-1}\delta(mc^{-1})B_{k,\delta\chi} \\
&= \frac{1}{\phi(l)l^{k-1}} \sum_{u|l} \sum_{\delta\in Y(u)} \delta(c^{-1})B_{k,\delta\chi} \sum_{m|l_u} \mu(m)\chi(m)\delta(m)m^{k-1} \\
&= \frac{1}{\phi(l)l^{k-1}} \sum_{u|l} \sum_{\delta\in Y(u)} \delta(c^{-1})B_{k,\delta\chi} \prod_{\substack{q|l_u \\ q\ prime}} (1-q^{k-1}\chi(q)\delta(q)).
\end{aligned}$$

Hence, we get Theorem 4.1. □

As an immediate corollary, we get Ibukiyama's Theorem 2 in [10].

Corollary 4.2.4 [10]. With notation and assumptions being the same as in Theorem 4.1, we get

$$\sum_{n=0}^{f-1} \chi(ln+c)n = \frac{f}{\phi(l)} \sum_{u|l} \sum_{\delta\in Y(u)} \left(\delta(c^{-1})B_{1,\delta\chi} \prod_{\substack{q|l_u \\ q\ prime}} (1-\chi(q)\delta(q)) \right),$$

Proof. As $fZ_1(l,c,\chi) = \sum_{n=0}^{f-1}\chi(ln+c)n$, the assertion follows immediately from Theorem 4.1. □

4.2 The Evaluation of $M_n(l,\chi)$

Let us recall the definition of $M_n(l,\chi)$ introduced at the beginning of Chapter 4. Let χ be a primitive Dirichlet character with conductor $f > 1$. Let l by any positive integer which is prime to f. We also put $\zeta = exp(2\pi i/f)$. Then,

$$M_n(l,\chi) = \sum_{a=1}^{f-1} \frac{\chi(a)}{(\zeta^{la}-1)^n(\zeta^a-1)}.$$

The aim of this section is to express $M_n(l,\chi)$ in terms of the character sums $Z_j(l,c,\chi)$ given by (4.1.1). This is given in Theorem 4.2.6.

We put

$$(4.2.1) \qquad T_j(l,c,\chi) = \begin{cases} \chi(c), & \textit{if } j=0, \\ \displaystyle\sum_{a_1,\cdots,a_j=0}^{f-1} \left\{ \prod_{i=1}^{j} P_1\left(\frac{a_i}{p}\right) \right\} \chi\left(l \sum_{i=1}^{j} a_i + c \right), & \textit{if } j>0. \end{cases}$$

Proposition 4.2.1. *With notation and assumptions being the same as above, we have*

$$M_n(l,\chi) = \frac{(-1)^{n+1}\tau(\chi)}{2^{n+1}} \sum_{j=0}^{n+1} (-1)^j 2^j \binom{n+1}{j} \sum_{c=0}^{l-1} T_j(l,c,\bar{\chi}).$$

Proof. Since

$$\frac{\zeta^{la}-1}{\zeta^a-1} = \sum_{c=0}^{l-1} \zeta^{ca},$$

we get

$$\begin{aligned} M_n(l,\chi) &= \sum_{a=1}^{f-1} \frac{\chi(a)}{(\zeta^{la}-1)^n(\zeta^a-1)} \\ &= \sum_{a=1}^{f-1} \frac{\chi(a)(\zeta^{la}-1)}{(\zeta^{la}-1)^{n+1}(\zeta^a-1)} \\ &= \sum_{c=0}^{l-1} \sum_{a=1}^{f-1} \frac{\chi(a)\zeta^{ca}}{(\zeta^{la}-1)^{n+1}}. \end{aligned}$$

Applying Proposition 3.2.14, (ii) in Proposition 3.2.13, we get

$$\begin{aligned} M_n(l,\chi) &= \frac{1}{f^{n+1}} \sum_{c=0}^{l-1} \sum_{a_1,\cdots,a_{n+1}=1}^{f-1} \sum_{a=1}^{f-1} a_1 \cdots a_{n+1} \zeta^{la(a_1+\cdots+a_{n+1})+ca} \chi(a) \\ &= \frac{1}{f^{n+1}} \sum_{c=0}^{l-1} \sum_{a_1,\cdots,a_{n+1}=1}^{f-1} a_1 \cdots a_{n+1} \sum_{a=1}^{f-1} \zeta^{a\{l(a_1+\cdots+a_{n+1}+c\}} \chi(a) \\ &= \frac{\tau(\chi)}{f^{n+1}} \sum_{c=0}^{l-1} \sum_{a_1,\cdots,a_{n+1}=1}^{f-1} a_1 \cdots a_{n+1} \, \bar{\chi}(l(a_1+\cdots+a_{n+1})+c). \end{aligned}$$

Next, we shall express everything in terms of periodic Bernoulli polynomials and take full advantage of their periodicity. Continuing along, taking (i) in Proposition 3.2.13 into account, we get

$$\begin{aligned} M_n(l,\chi) &= \tau(\chi) \sum_{c=0}^{l-1} \sum_{a_1,\cdots,a_{n+1}=1}^{f-1} \left\{ \prod_{i=1}^{n+1} \left(P_1\left(\frac{a_i}{f}\right) + \frac{1}{2} \right) \right\} \bar{\chi}\left(l \sum_{i=1}^{n+1} a_i + c \right) \\ &= \tau(\chi) \sum_{c=0}^{l-1} \sum_{a_1,\cdots,a_{n+1}=1}^{f-1} \left\{ \sum_{j=0}^{n+1} \frac{1}{2^j} \binom{n+1}{j} \prod_{i=1}^{n+1-j} P_1\left(\frac{a_i}{f}\right) \right\} \bar{\chi}\left(l \sum_{i=1}^{n+1} a_i + c \right) \\ &= \tau(\chi) \sum_{c=0}^{l-1} \sum_{j=0}^{n+1} \sum_{a_1,\cdots,a_{n+1-j}=1}^{f-1} \frac{(-1)^j}{2^j} \binom{n+1}{j} \left\{ \prod_{i=1}^{n+1-j} P_1\left(\frac{a_i}{f}\right) \right\} \bar{\chi}\left(l \sum_{i=1}^{n+1-j} a_i + c \right) \\ &= \tau(\chi) \sum_{c=0}^{l-1} \sum_{j=0}^{n+1} \frac{(-1)^j}{2^j} \binom{n+1}{j} T_{n+1-j}(l,c,\bar{\chi}), \end{aligned}$$

from which the assertion of the proposition immediately follows. □

Proposition 4.2.2. *With notation and assumptions being the same as above, we have*

$$T_j(l,c,\chi) = \left(\frac{i}{2\pi}\right)^j \frac{1}{f} \sum_{a,b=0}^{f-1} c_1^j\left(\frac{b}{f}\right) \zeta^{ab} \chi(la+c),$$

where $c_1(x)$ is given by (3.1.5).

Proof. For $j = 0$, the assertion follows from (iii) in Proposition 3.2.13. We now assume that $j > 0$. Replacing a_1 by $a_1 - \sum_{i=2}^{j} a_i$ in the definition of $T_j(l,c,\chi)$ given by (4.2.1), we get

$$T_j(l,c,\chi) = \sum_{a_1,\cdots,a_j=0}^{f-1} P_1\left(\frac{a_1 - \sum_{i=2}^{j} a_i}{f}\right) \left\{ \prod_{i=2}^{j} P_1\left(\frac{a_i}{f}\right) \right\} \chi(la_1 + c).$$

Employing the finite Fourier transform to all of the P_1's (Proposition 3.1.16), we get

$$\begin{aligned} T_j(l,c,\chi) &= \left(\frac{-1}{2\pi i f}\right)^j \sum_{\substack{b_1,\cdots,b_j=0 \\ a_1,\cdots,a_j=0}}^{f-1} c_1\left(\frac{b_1}{f}\right)\cdots c_1\left(\frac{b_j}{f}\right) \zeta^{b_1\left(a_1-\sum_{i=2}^j a_i\right)+b_2a_2+\cdots+b_ja_j}\,\chi(la_1+c) \\ &= \left(\frac{-1}{2\pi i f}\right)^j \sum_{\substack{b_1,\cdots,b_j=0 \\ a_1,\cdots,a_j=0}}^{f-1} c_1\left(\frac{b_1}{f}\right)\cdots c_1\left(\frac{b_j}{f}\right) \zeta^{a_1b_1+a_2(b_2-b_1)+\cdots+a_j(b_j-b_1)}\,\chi(la_1+c) \\ &= \left(\frac{-1}{2\pi i f}\right)^j \cdot f^{j-1} \sum_{b_1,a_1=0}^{f-1} c_1^j\left(\frac{b_1}{f}\right) \zeta^{a_1b_1}\,\chi(la_1+c), \end{aligned}$$

from which the assertion of the proposition immediately follows. □

Recall from Sczech[17], for $x \notin \mathbb{Z}$, we have $c_k(x) = V_k(c_1(x))$ with a polynomial $V_k(t)$ of degree k given recursively by

$$V_0(t) = 1, \quad V_1(t) = t, \quad kV_{k+1}(t) = (t^2+\pi^2)V_k'(t) \text{ for } k \geq 1. \tag{4.2.2}$$

Then, $V_2(t) = t^2+\pi^2$, $V_3(t) = t^3+\pi^2 t$, $V_4(t) = t^4+\frac{4\pi^2}{3}t^2+\frac{\pi^4}{3}$, $V_5(t) = t^5+\frac{5\pi^2}{3}t^3+\frac{2\pi^4}{3}t$, and so on. Accordingly, $c_0(x) = 1$, $c_1(x) = c_1(x)$, $c_2(x) = c_1^2(x)+\pi^2$, $c_3(x) = c_1^3+\pi^2 c_1(x)$, $c_4(x) = c_1^4(x)+\frac{4\pi^2}{3}c_1^2(x)+\frac{\pi^4}{3}$, $c_5(x) = c_1^5(x)+\frac{5\pi^2}{3}c_1^3(x)+\frac{2\pi^4}{3}c_1(x)$ etc. From this, it easily follows that for $x \notin \mathbb{Z}$, there is a polynomial $U_k(t)$ of degree k such that $c_1^k(x) = U_k(c(x))$ where we use the symbolic power notation $(c(x))^j$ to denote $c_j(x)$. Consequently, the $U_k(t)$'s must satisfy the following condition:

$$V_k(U(t)) = t^k \quad (k = 0, 1, \cdots), \tag{4.2.3}$$

where we use the symbolic power notation $(U(t))^j$ to denote $U_j(t)$. Thus the $U_k(t)$'s can easily be determined recursively by (4.2.3) via (4.2.2). We find that $U_0(t) = t^0$, $U_1(t) = t$, $U_2(t) = t^2-\pi^2 t^0$, $U_3(t) = t^3-\pi^2 t$, $U_4(t) = t^4-\frac{4\pi^2}{3}t^2+\pi^4 t^0$, $U_5(t) = t^5-\frac{5\pi^2}{3}t^3+\pi^4 t$, and so on. Accordingly, $c_1^0(x) = c_0(x)$, $c_1(x) = c_1(x)$, $c_1^2(x) = c_2(x)-\pi^2 c_0(x)$, $c_1^3(x) = c_3(x)-\pi^2 c_1(x)$, $c_1^4(x) = c_4(x)-\frac{4\pi^2}{3}c_2(x)+\pi^4 c_0(x)$, $c_1^5(x) =$

$c_5(x) - \frac{5\pi^2}{3}c_3(x) + \pi^4 c_1(x)$, etc.

Proposition 4.2.3. *With notation and assumptions being the same as above, we have*

$$T_j(l,c,\chi) = \left(\frac{i}{2\pi}\right)^j \frac{1}{f} \sum_{a,b=0}^{f-1} U_j\left(c\left(\frac{b}{f}\right)\right) \zeta^{ab}\,\chi(la+c),$$

where $U_j(t)$ denotes the polynomial of degree j determined by (4.2.3) and $\left(c\left(\frac{b}{f}\right)\right)^k$ denotes $c_k\left(\frac{b}{f}\right)$.

Proof. We will show that this is the same expression for $T_j(l,c,\chi)$ given in Proposition 4.2.2. For $j=0$, the assertion is clear. We now assume that $j>0$. Since $c_1^j\left(\frac{b}{f}\right) = U_j\left(c\left(\frac{b}{f}\right)\right)$ for $b = 1, \cdots, f-1$, we only need to verify the equality in the case of $b=0$. Since $c_1(0)^j = 0$, we must show that

$$\sum_{a=0}^{f-1} U_j(c(0))\,\chi(la+c) = 0.$$

This follows immediately from (i) in Proposition 3.2.13, thus completing the proof.

□

In view of the finite Fourier transform of the P_k's ((i) in Proposition 3.1.16), we see that

$$\sum_{b=0}^{f-1} U_j\left(c\left(\frac{b}{f}\right)\right)\zeta^{ab} = U_j\left(P^*\left(\frac{a}{f}\right)\right),$$

where we use the symbolic power notation $\left(P^*\left(\frac{a}{f}\right)\right)^k$ to denote $P_k^*\left(\frac{a}{f}\right)$,

$$P_k^*\left(\frac{a}{f}\right) = \begin{cases} f, & \text{if } k=0,\ a=0,\\ 0, & \text{if } k=0,\ a\neq 0,\\ -\dfrac{(2\pi i f)^k}{k!}P_k\left(\dfrac{a}{f}\right), & \text{otherwise.}\end{cases}$$

Letting

$$(4.2.4)\qquad \begin{aligned} Q_k^*\left(l,c,\chi\right) &= \frac{1}{f}\sum_{a=0}^{f-1} P_k^*\left(\frac{a}{f}\right)\chi(la+c) \\ &= \delta_{k,0}\,\chi(c) - \frac{(2\pi i)^k f^{k-1}}{k!}\sum_{a=0}^{f-1} P_k\left(\frac{a}{f}\right)\chi(la+c), \end{aligned}$$

we get

$$(4.2.5)\qquad \frac{1}{f}\sum_{a,b=0}^{f-1} U_j\left(c\left(\frac{b}{f}\right)\right)\zeta^{ab}\,\chi(la+c) = U_j\left(Q^*\left(l,c,\chi\right)\right),$$

where we use the symbolic power notation $\left(Q^*\left(l,c,\chi\right)\right)^k$ to denote $Q_k^*\left(l,c,\chi\right)$. Hence from Proposition 4.2.3 and (4.2.5), we have

$$(4.2.6)\qquad T_j(l,c,\chi) = \left(\frac{i}{2\pi}\right)^j U_j\left(Q^*\left(l,c,\chi\right)\right).$$

For any natural number l prime to f and any integer c, recall the character sum $Z_k(l,c,\chi)$ given by (4.1.1),

$$Z_k(l,c,\chi) = f^{k-1}\sum_{a(f)} P_k\left(\frac{la+c}{fl}\right)\chi(la+c).$$

Lemma 4.2.4. With notation and assumptions being the same as above, we have

$$\sum_{a=0}^{f-1} P_k\left(\frac{a}{f}\right)\chi(la+c) = \sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}\left(\frac{c}{fl}\right)^{k-j}\frac{1}{f^{j-1}}\cdot Z_j(l,c,\chi) + \frac{1}{2}\delta_{k,1}\,\chi(c).$$

Proof. From Proposition 3.1.2, we see that

$$B_k\left(\frac{a}{f}\right) = \sum_{j=0}^{k}\binom{k}{j}\left(\frac{-c}{fl}\right)^{k-j} B_j\left(\frac{a}{f}+\frac{c}{fl}\right),$$

from which it follows that

$$\sum_{a=0}^{f-1} B_k\left(\frac{a}{f}\right)\chi(la+c) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\left(\frac{c}{fl}\right)^{k-j}\sum_{a=0}^{f-1} B_k\left(\frac{la+c}{fl}\right)\chi(la+c).$$

Since $B_k(0) = P_k(0) - (1/2)\delta_{k,1}$ and $\chi(0) = 0$, we have, for $0 \le c \le l-1$,

$$\sum_{a=0}^{f-1} P_k\left(\frac{a}{f}\right)\chi(la+c) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\left(\frac{c}{fl}\right)^{k-j}\frac{1}{f^{j-1}}\cdot Z_j(l,c,\chi) + \frac{1}{2}\delta_{k,1}\,\chi(c).$$

Since $Z_0(l,c,\chi) = 0$, the assertion of the lemma immediately follows. □

We put

$$\begin{aligned} &Z_k^*(l,c,\chi) \\ &= -\frac{(2\pi i)^k}{k!}\left\{\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}\left(\frac{c}{l}\right)^{k-j} Z_j(l,c,\chi) + \left(\frac{1}{2}\delta_{k,1} - \delta_{k,0}\right)\chi(c)\right\} \end{aligned} \tag{4.2.7}$$

Proposition 4.2.5. With notation and assumptions being the same as above, we have

$$Q_k^*(l,c,\chi) = Z_k^*(l,c,\chi).$$

Proof. From (4.2.4) and Lemma 4.2.4, we have

$$\begin{aligned} Q_k^*(l,c,\chi) &= \delta_{k,0}\,\chi(c) - \frac{(2\pi i)^k f^{k-1}}{k!} \\ &\quad\times\left\{\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}\left(\frac{c}{fl}\right)^{k-j}\frac{1}{f^{j-1}}Z_j(l,c,\chi) + \frac{1}{2}\delta_{k,1}\,\chi(c)\right\} \\ &= \delta_{k,0}\,\chi(c) - \frac{(2\pi i)^k}{k!}\left\{\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}\left(\frac{c}{l}\right)^{k-j} Z_j(l,c,\chi) + \frac{1}{2}\delta_{k,1}\,\chi(c)\right\} \\ &= Z_k^*(l,c,\chi). \end{aligned}$$

□

We are now in position to state the main theorem of this section.

Theorem 4.2.6. *Let χ be a primitive Dirichlet character with conductor $f > 1$ and let l by any positive integer which is prime to f. Let $n \in \mathbb{Z}$, $n \geq 0$. Then, we have*

$$M_n(l, \chi) = \frac{(-1)^{n+1}\tau(\chi)}{2^{n+1}} \sum_{j=0}^{n+1} (-1)^j \left(\frac{i}{\pi}\right)^j \binom{n+1}{j} \sum_{c=0}^{l-1} U_j\left(Z^*\left(l, c, \bar{\chi}\right)\right),$$

where $U_j(t)$ denotes the polynomial of degree j determined by (4.2.3), $\left(Z^*\left(l, c, \bar{\chi}\right)\right)^k$ denotes $Z_k^*\left(l, c, \bar{\chi}\right)$ given by (4.2.7), and $Z_j(l, c, \bar{\chi})$ is given by (4.1.1).

Proof. This follows immediately from Proposition 4.2.1, (4.2.6), and Proposition 4.2.5. □

By Theorem 4.2.6, it is clear that we can express the sum $M_n(l, \chi)$ in terms of the character sums $\{Z_j(l, c, \chi)\}_{j=1}^{n+1}$. Since the $Z_j(l, c, \chi)$'s can be expressed in terms of generalized Bernoulli numbers by virtue of Theorem 4.1, so can the sum $M_n(l, \chi)$. This is precisely what we were trying to show. Next we work out some examples.

4.3 Some Examples

Let χ be a primitive Dirichlet character with conductor $f > 1$ and let l by any positive integer which is prime to f. Let $n \in \mathbb{Z}$, $n \geq 0$. We will also consider the following character sum that is closely related to $M_n(l, \chi)$.

$$S_n(l, \chi) = \sum_{a_1, \cdots, a_{n+1}=1}^{f-1} a_1 \cdots a_{n+1}\, \chi(l(a_1 + \cdots + a_n) + a_{n+1}).$$

Proposition 4.3.1. *With notation and assumptions being the same as above, we have*

$$S_n(l, \chi) = \frac{f^{n+1}}{\tau(\bar{\chi})} M_n(l, \bar{\chi}).$$

Proof. By Proposition 3.2.14 and (ii) in Proposition 3.2.13, we have

$$\begin{aligned}
M_n(l,\bar{\chi}) &= \frac{1}{f^{n+1}} \sum_{a_1,\cdots,a_{n+1}=1}^{f-1} \sum_{a=1}^{f-1} a_1 \cdots a_{n+1} \zeta^{la(a_1+\cdots+a_n)+aa_{n+1}} \bar{\chi}(a) \\
&= \frac{1}{f^{n+1}} \sum_{a_1,\cdots,a_{n+1}=1}^{f-1} a_1 \cdots a_{n+1} \sum_{a=1}^{f-1} \zeta^{a\{l(a_1+\cdots+a_n)+a_{n+1}\}} \bar{\chi}(a) \\
&= \frac{\tau(\bar{\chi})}{f^{n+1}} \sum_{a_1,\cdots,a_{n+1}=1}^{f-1} a_1 \cdots a_{n+1}\, \chi(l(a_1+\cdots+a_n)+a_{n+1}) \\
&= \frac{\tau(\bar{\chi})}{f^{n+1}} S_n(l,\chi).
\end{aligned}$$

□

Lemma 4.3.2. *With notation and assumptions being the same as above, with* $Z_k(l,c,\chi)$ *given by (4.1.1), we have*

$$\sum_{c=0}^{l-1} Z_k(l,c,\chi) = \frac{1}{l^{k-1}} B_{k,\chi}.$$

Proof. This follows immediately from Lemma 4.1.1 where we let $\delta \in X(l)$ be the trivial character. □

Proposition 4.3.3. *With notation and assumptions being the same as in Theo-*

rem 4.2.6, we have

$$(i)\quad M_1(l,\chi)=\frac{\tau(\chi)}{4}\Bigg\{-4B_{1,\bar\chi}-\frac{2}{l}B_{2,\bar\chi}+\frac{4}{l}\sum_{c=1}^{l-1}c\,Z_1(l,c,\bar\chi)\Bigg\},$$

$$(ii)\quad M_2(l,\chi)=-\frac{\tau(\chi)}{8}\Bigg\{-8B_{1,\bar\chi}-\frac{6}{l}B_{2,\bar\chi}-\frac{4}{3l^2}B_{3,\bar\chi}$$
$$+\sum_{c=1}^{l-1}\left(\frac{12c}{l}-\frac{4c^2}{l^2}\right)Z_1(l,c,\bar\chi)+\frac{4}{l}\sum_{c=1}^{l-1}c\,Z_2(l,c,\bar\chi)\Bigg\},$$

$$(iii)\quad M_3(l,\chi)=\frac{\tau(\chi)}{16}\Bigg\{-16B_{1,\bar\chi}-\frac{44}{3l}B_{2,\bar\chi}-\frac{16}{3l^2}B_{3,\bar\chi}-\frac{2}{3l^3}B_{4,\bar\chi}$$
$$+\sum_{c=1}^{l-1}\left(\frac{88c}{3l}-\frac{16c^2}{l^2}+\frac{8c^3}{3l^3}\right)Z_1(l,c,\bar\chi)$$
$$+\sum_{c=1}^{l-1}\left(\frac{16c}{l}-\frac{4c^2}{l^2}\right)Z_2(l,c,\bar\chi)+\frac{8}{3l}\sum_{c=1}^{l-1}c\,Z_3(l,c,\bar\chi)\Bigg\},$$

$$(iv)\quad M_4(l,\chi)=-\frac{\tau(\chi)}{32}\Bigg\{-32B_{1,\bar\chi}-\frac{100}{3l}B_{2,\bar\chi}-\frac{140}{9l^2}B_{3,\bar\chi}-\frac{10}{3l^3}B_{4,\bar\chi}-\frac{4}{15l^4}B_{5,\bar\chi}$$
$$+\sum_{c=1}^{l-1}\left(\frac{200c}{3l}-\frac{140c^2}{3l^2}+\frac{40c^3}{3l^3}-\frac{4c^4}{3l^4}\right)Z_1(l,c,\bar\chi)$$
$$+\sum_{c=1}^{l-1}\left(\frac{140c}{3l}-\frac{20c^2}{l^2}+\frac{8c^3}{3l^3}\right)Z_2(l,c,\bar\chi)$$
$$+\sum_{c=1}^{l-1}\left(\frac{40c}{3l}-\frac{8c^2}{3l^2}\right)Z_3(l,c,\bar\chi)+\frac{4}{3l}\sum_{c=1}^{l-1}c\,Z_4(l,c,\bar\chi)\Bigg\}.$$

Proof. We get, by Theorem 4.2.6,

$$M_1(l,\chi)=\frac{\tau(\chi)}{4}\sum_{j=0}^{2}(-1)^j\left(\frac{i}{\pi}\right)^j\binom{2}{j}\sum_{c=0}^{l-1}U_j(Z^*(l,c,\bar\chi))$$
$$=\frac{\tau(\chi)}{4}\sum_{c=0}^{l-1}\Bigg\{U_0(Z^*(l,c,\bar\chi))-\frac{2i}{\pi}U_1(Z^*(l,c,\bar\chi))-\frac{1}{\pi^2}U_2(Z^*(l,c,\bar\chi))\Bigg\}.$$

From (4.2.3), we have $U_0(t) = t^0$, $U_1(t) = t$, and $U_2(t) = t^2 - \pi^2 t^0$. Hence, we get

$$\begin{aligned} M_1(l,\chi) &= \frac{\tau(\chi)}{4}\sum_{c=0}^{l-1}\left\{ Z_0^*(l,c,\bar{\chi}) - \frac{2i}{\pi} Z_1^*(l,c,\bar{\chi}) - \frac{1}{\pi^2}\left(Z_2^*(l,c,\bar{\chi}) - \pi^2 Z_0^*(l,c,\bar{\chi})\right)\right\} \\ &= \frac{\tau(\chi)}{4}\sum_{c=0}^{l-1}\left\{ 2Z_0^*(l,c,\bar{\chi}) - \frac{2i}{\pi} Z_1^*(l,c,\bar{\chi}) - \frac{1}{\pi^2} Z_2^*(l,c,\bar{\chi})\right\}. \end{aligned}$$

From (4.2.7), we have $Z_0^*(l,c,\bar{\chi}) = \bar{\chi}(c)$, $Z_1^*(l,c,\bar{\chi}) = -2\pi i\left(Z_1(l,c,\bar{\chi}) + \frac{1}{2}\bar{\chi}(c)\right)$, and $Z_2^*(l,c,\bar{\chi}) = 2\pi^2\left(-\frac{2c}{l} Z_1(l,c,\bar{\chi}) + Z_2(l,c,\bar{\chi})\right)$. Hence, we get

$$M_1(l,\chi) = \frac{\tau(\chi)}{4}\sum_{c=0}^{l-1}\left\{ \left(-4 + \frac{4c}{l}\right) Z_1(l,c,\bar{\chi}) - 2Z_2(l,c,\bar{\chi})\right\}.$$

Thus the assertion (i) follows from Lemma 4.3.2. The other identities are similarly verified, so they are omitted. □

We shall now work out some examples.

Proposition 4.3.4 [2]. *Let $f > 1$ be an odd number and let χ be a primitive Dirichlet character with conductor f. We also put $\zeta = exp(2\pi i/f)$. Then, we get*

$$\begin{aligned} (i)\quad M_2(2,\chi) &= \sum_{a=1}^{f-1} \frac{\chi(a)}{(\zeta^{2a}-1)^2(\zeta^a - 1)} \\ &= \tau(\chi)\left\{ \frac{1}{24} B_{3,\bar{\chi}} + \frac{1+\bar{\chi}(2)}{4} B_{2,\bar{\chi}} + \frac{3+5\bar{\chi}(2)}{8} B_{1,\bar{\chi}}\right\}, \\ (ii)\quad S_2(2,\chi) &= \sum_{a,b,c=1}^{f-1} abc\,\chi(2(a+b)+c) = \\ &= f^2\left\{ \frac{1}{24} B_{3,\chi} + \frac{1+\chi(2)}{4} B_{2,\chi} + \frac{3+5\chi(2)}{8} B_{1,\chi}\right\}. \end{aligned}$$

Proof. From (ii) in Proposition 4.3.3, we get

$$M_2(2,\chi) = -\frac{\tau(\chi)}{8}\left\{ -8B_{1,\bar{\chi}} - 3B_{2,\bar{\chi}} - \frac{1}{3}B_{3,\bar{\chi}} + 5Z_1(2,1,\bar{\chi}) + 2Z_2(2,1,\bar{\chi})\right\}.$$

From Theorem 4.1, we have

$$Z_j(2,1,\bar{\chi}) = \frac{1}{2^{j-1}}\left\{B_{j,\bar{\chi}}(1-2^{j-1}\bar{\chi}(2))\right\}.$$

Thus the assertion (i) follows. From Proposition 4.3.1, we get

$$S_2(2,\chi) = \frac{f^2}{\tau(\bar{\chi})}\,M_2(2,\bar{\chi}).$$

Thus the assertion (ii) follows from (i). □

Proposition 4.3.5. *Let $f>1$ be a natural number prime to 5 and let χ be a primitive Dirichlet character with conductor f. We denote by δ_1, δ_2, δ_3, the primitive Dirichlet characters modulo 5 characterized by $\delta_1(2)=i$, $\delta_2(2)=-1$, $\delta_3(2)=-i$. We also put $\zeta = exp(2\pi i/f)$. Then, we get*

$$(i)\quad M_1(5,\chi) = \sum_{a=1}^{f-1}\frac{\chi(a)}{(\zeta^{5a}-1)(\zeta^a-1)}$$

$$= \tau(\chi)\left\{-\frac{1}{10}B_{2,\bar{\chi}} - \frac{1+\bar{\chi}(5)}{2}B_{1,\bar{\chi}} + \frac{1}{4}B_{1,\delta_3}B_{1,\delta_1\bar{\chi}} + \frac{1}{4}B_{1,\delta_1}B_{1,\delta_3\bar{\chi}}\right\},$$

$$(ii)\quad S_1(5,\chi) = \sum_{a,b=1}^{f-1} ab\,\chi(5a+b) =$$

$$= f^2\left\{-\frac{1}{10}B_{2,\chi} - \frac{1+\chi(5)}{2}B_{1,\chi} + \frac{1}{4}B_{1,\delta_3}B_{1,\delta_1\chi} + \frac{1}{4}B_{1,\delta_1}B_{1,\delta_3\chi}\right\}.$$

Proof. From (i) in Proposition 4.3.3, we get

$$M_1(5,\chi) = \frac{\tau(\chi)}{4}\left\{-4B_{1,\bar{\chi}} - \frac{2}{5}B_{2,\bar{\chi}} + \frac{4}{5}\sum_{c=1}^{4} c\,Z_1(5,c,\bar{\chi})\right\}. \tag{4.3.1}$$

From Theorem 4.1, for $1 \le c \le 4$, we have

$$Z_1(5,c,\bar{\chi}) = \frac{1}{4}\left\{B_{1,\bar{\chi}}(1-\bar{\chi}(5)) + \sum_{i=1}^{3}\delta_i(c^{-1})B_{1,\delta_i\bar{\chi}}\right\}.$$

Hence,

$$\sum_{c=1}^{4} c\, Z_1(5,c,\bar{\chi}) = \frac{5(1-\bar{\chi}(5))}{2} B_{1,\bar{\chi}} + \frac{5}{4}\sum_{i=1}^{3} B_{1,\bar{\delta}_i} B_{1,\delta_i\bar{\chi}}.$$

From (4.3.1), we get

$$M_1(5,\chi) = \frac{\tau(\chi)}{4}\left\{ -2(1+\bar{\chi}(5))B_{1,\bar{\chi}} - \frac{2}{5}B_{2,\bar{\chi}} + \sum_{i=1}^{3} B_{1,\bar{\delta}_i} B_{1,\delta_i\bar{\chi}} \right\}.$$

As δ_2 is an even character, we get, by Proposition 3.1.7, $B_{1,\delta_2} = 0$. Since $\bar{\delta_1} = \delta_3$, $\bar{\delta_3} = \delta_1$, we obtain the assertion (i). From Proposition 4.3.1, we get

$$S_1(5,\chi) = \frac{f^2}{\tau(\bar{\chi})} M_1(5,\bar{\chi}).$$

Thus the assertion (ii) follows from (i). □

Proposition 4.3.6. *Let $f > 1$ be a natural number prime to 4 and let χ be a primitive Dirichlet character with conductor f. We denote by δ the unique primitive*

Dirichlet character modulo 4. We also put $\zeta = exp(2\pi i/f)$*. Then, we get*

$$
\begin{aligned}
(i)\quad M_4(4,\chi) &= \sum_{a=1}^{f-1} \frac{\chi(a)}{(\zeta^{4a}-1)^4(\zeta^a-1)} \\
&= \tau(\chi)\bigg\{ \frac{1}{30720}B_{5,\bar{\chi}} + \frac{17}{768}B_{4,\bar{\chi}} + \frac{1}{6144}B_{4,\delta\bar{\chi}} + \frac{1903-12\bar{\chi}(2)}{9216}B_{3,\bar{\chi}} \\
&\quad + \frac{1}{192}B_{3,\delta\bar{\chi}} + \frac{271-6\bar{\chi}(2)}{384}B_{2,\bar{\chi}} + \frac{173}{3072}B_{2,\delta\bar{\chi}} + \frac{2163-115\bar{\chi}(2)}{2048}B_{1,\bar{\chi}} \\
&\quad + \frac{15}{64}B_{1,\delta\bar{\chi}} \bigg\},
\end{aligned}
$$

$$
\begin{aligned}
(ii)\quad S_4(4,\chi) &= \sum_{a,b,c,d,e=1}^{f-1} abcde\,\chi(4(a+b+c+d)+e) \\
&= f^5\bigg\{ \frac{1}{30720}B_{5,\chi} + \frac{17}{768}B_{4,\chi} + \frac{1}{6144}B_{4,\delta\chi} + \frac{1903-12\chi(2)}{9216}B_{3,\chi} \\
&\quad + \frac{1}{192}B_{3,\delta\chi} + \frac{271-6\chi(2)}{384}B_{2,\chi} + \frac{173}{3072}B_{2,\delta\chi} + \frac{2163-115\chi(2)}{2048}B_{1,\chi} \\
&\quad + \frac{15}{64}B_{1,\delta\chi} \bigg\}.
\end{aligned}
$$

Proof. From (iv) in Proposition 4.3.3, we get

$$
\begin{aligned}
M_4(4,\chi) = -\frac{\tau(\chi)}{32}\bigg\{ & -32B_{1,\bar{\chi}} - \frac{25}{3}B_{2,\bar{\chi}} - \frac{35}{36}B_{3,\bar{\chi}} - \frac{5}{96}B_{4,\bar{\chi}} - \frac{1}{960}B_{5,\bar{\chi}} \\
& + \sum_{c=1}^{3}\left(\frac{50}{3}c - \frac{35}{12}c^2 + \frac{5}{24}c^3 - \frac{1}{192}c^4\right) Z_1(4,c,\bar{\chi}) \\
& + \sum_{c=1}^{3}\left(\frac{35}{3}c - \frac{5}{4}c^2 + \frac{1}{24}c^3\right) Z_2(4,c,\bar{\chi}) \\
& + \sum_{c=1}^{3}\left(\frac{10}{3}c - \frac{1}{6}c^2\right) Z_3(4,c,\bar{\chi}) + \frac{1}{3}\sum_{c=1}^{3} c\,Z_4(4,c,\bar{\chi}) \bigg\}.
\end{aligned}
\tag{4.3.2}
$$

From Theorem 4.1, we find that

$$\begin{aligned}
Z_j(4,1,\bar{\chi}) &= \frac{1}{2\cdot 4^{j-1}}((1-2^{j-1}\bar{\chi}(2))B_{j,\bar{\chi}} + B_{j,\delta\bar{\chi}}),\\
Z_j(4,2,\bar{\chi}) &= \bar{\chi}(2)\, Z_j(2,1,\bar{\chi}) = \frac{\bar{\chi}(2)}{2^{j-1}}(1-2^{j-1}\bar{\chi}(2))B_{j,\bar{\chi}},\\
Z_j(4,3,\bar{\chi}) &= \frac{1}{2\cdot 4^{j-1}}((1-2^{j-1}\bar{\chi}(2))B_{j,\bar{\chi}} - B_{j,\delta\bar{\chi}}) \quad (j\in\mathbb{N}).
\end{aligned}$$

Hence,

$$\sum_{c=1}^{3} c^i\, Z_j(4,c,\bar{\chi}) = \frac{1+3^i+2^{i+j}\bar{\chi}(2)}{2\cdot 4^{j-1}}(1-2^{j-1}\bar{\chi}(2))B_{j,\bar{\chi}} + \frac{1-3^i}{2\cdot 4^{j-1}}B_{j,\delta\bar{\chi}}. \tag{4.3.3}$$

Applying (4.3.3) to all of the sums in (4.3.2), we obtain the assertion (i). From Proposition 4.3.1, we get

$$S_4(4,\chi) = \frac{f^5}{\tau(\bar{\chi})}\, M_4(4,\bar{\chi}).$$

Thus the assertion (ii) follows from (i). □

5 The Dual Lee-Weintraub Identity

We fix an odd prime p. We denote by ψ the Legendre symbol mod p: $\psi(a) = \left(\frac{a}{p}\right)$. We put $\zeta = exp(2\pi i/p)$, and set

$$S = \{(a,b,c) \in (\mathbb{F}_p^\times)^3 \mid 4ab - (c-a-b)^2 = 0\},$$
$$T = \{(a,b,c) \in (\mathbb{F}_p^\times)^3 \mid ab + bc + ca = 0\}.$$

We consider the sets S and T to be dual to each other since the quadratic forms in their respective congruence conditions are dual to each other. Indeed,

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 4ab - (c-a-b)^2,$$
$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = ab + bc + ca.$$

In this chapter, we prove the dual version of the Lee-Weintraub identity, where the summation set is S instead of T.

Theorem 5 (The Dual Lee-Weintraub Identity). *With notation and assumptions being the same as above, put*

$$I_p = \sum_{(a,b,c)\in S} \frac{\psi(abc)}{(\zeta^a - 1)(\zeta^b - 1)(\zeta^c - 1)}.$$

Then, we get

$$I_p = \sqrt{\psi(-1)p}\left(\frac{1}{6}B_{3,\psi} - \frac{3}{4}B_{2,\psi} + (p-2)B_{1,\psi}\right).$$

To prove Theorem 5, we must first establish several results.

5.1 Rewriting the Sum I_p

We shall rewrite the sum I_p in the following way.

Theorem 5.1. *With notation and assumptions being the same as in Theorem 5, we get*

$$I_p = \sqrt{\psi(-1)p}\left(-6p\left(Q_p + \frac{1}{6}h_p\right) - \frac{3}{4}B_{3,\psi} - \frac{3}{4}B_{2,\psi} - \frac{9-3\psi(2)-2p\delta_{p,3}}{4}B_{1,\psi}\right),$$

where

$$Q_p = \sum_{k,t(p)} P_1\left(\frac{k^2+2kt}{p}\right)P_1\left(\frac{kt}{p}\right)P_1\left(\frac{t^2-k^2}{p}\right), \tag{5.1.1}$$

and

$$h_p = \sum_{k,t(p)} P_1\left(\frac{k^2+2kt}{p}\right)P_2\left(\frac{t^2-k^2}{p}\right). \tag{5.1.2}$$

Proof of Theorem 5.1. We have

$$\begin{aligned} I_p &= \sum_{a,b,c\in S} \frac{\psi(abc)}{(\zeta^a-1)(\zeta^b-1)(\zeta^c-1)} \\ &= \sum_{\substack{a,b,c(p) \\ a,b,c\not\equiv 0(p) \\ 4ab-(c-a-b)^2\equiv 0(p)}} \frac{\psi(abc)}{(\zeta^a-1)(\zeta^b-1)(\zeta^c-1)} \\ &= \sum_{\substack{a,b,c(p) \\ a,b,a+b+c\not\equiv 0(p) \\ 4ab-c^2\equiv 0(p)}} \frac{\psi(ab(a+b+c))}{(\zeta^a-1)(\zeta^b-1)(\zeta^{a+b+c}-1)}. \end{aligned}$$

We notice that $4ab - c^2 \equiv 0(p)$ with $a, b \not\equiv 0(p)$ implies that ab is a nonzero square mod p. Therefore, $b \equiv at^2(p)$, $c \equiv \pm 2at(p)$, and $a+b+c \equiv a(t\pm 1)^2(p)$ for some

$t : 1 \le t \le \frac{p-1}{2}$. Hence, we get

$$I_p = \sum_{\substack{1\le a\le p-1 \\ 1\le t\le \frac{p-1}{2} \\ a,t,t+1\not\equiv 0}} \frac{\psi(a^3t^2(t+1)^2)}{(\zeta^a-1)(\zeta^{at^2}-1)(\zeta^{a(t+1)^2}-1)} + \sum_{\substack{1\le a\le p-1 \\ 1\le t\le \frac{p-1}{2} \\ a,t,t-1\not\equiv 0}} \frac{\psi(a^3t^2(t-1)^2)}{(\zeta^a-1)(\zeta^{at^2}-1)(\zeta^{a(t-1)^2}-1)}.$$

Substituting $p-t$ for t in the second sum, we get

$$\begin{aligned} I_p &= \sum_{\substack{1\le a\le p-1 \\ 1\le t\le \frac{p-1}{2} \\ a,t,t+1\not\equiv 0}} \frac{\psi(a^3t^2(t+1)^2)}{(\zeta^a-1)(\zeta^{at^2}-1)(\zeta^{a(t+1)^2}-1)} + \sum_{\substack{1\le a\le p-1 \\ \frac{p-1}{2}\le t\le p-1 \\ a,t,t+1\not\equiv 0}} \frac{\psi(a^3t^2(t+1)^2)}{(\zeta^a-1)(\zeta^{at^2}-1)(\zeta^{a(t+1)^2}-1)} \\ &= \sum_{a=1}^{p-1}\sum_{t=1}^{p-2} \frac{\psi(a)}{(\zeta^a-1)(\zeta^{at^2}-1)(\zeta^{a(t+1)^2}-1)}. \end{aligned}$$

As $\frac{1}{\zeta^a-1} = \frac{1}{p}\sum_{n=1}^{p-1} n\zeta^{an}$ for any a with $(a,p)=1$ (Proposition 3.2.14), we get

$$\begin{aligned} I_p &= \frac{1}{p^3}\sum_{a=1}^{p-1}\sum_{t=1}^{p-2}\psi(a)\sum_{k,l,m=1}^{p-1} klm\zeta^{ka+lat^2+ma(t+1)^2} \\ &= \frac{1}{p^3}\sum_{k,l,m=1}^{p-1} klm\sum_{t=1}^{p-2}\sum_{a(p)}\zeta^{a\{(l+m)t^2+2mt+(k+m)\}}\psi(a) \\ &= \frac{\tau(\psi)}{p^3}\sum_{k,l,m=1}^{p-1} klm\sum_{t=1}^{p-2}\psi((l+m)t^2+2mt+(k+m)) \\ &= \frac{\tau(\psi)}{p^3}\left(\sum_{k,l,m=1}^{p-1} klm\sum_{t(p)}\psi((l+m)t^2+2mt+(k+m)) - 2\sum_{k,l,m=1}^{p-1} klm\,\psi(k+m)\right), \end{aligned}$$

where $\tau(\psi)$ is the Gaussian sum $\tau(\psi) = \sum_{a=1}^{p-1}\psi(a)\zeta^a$. From Proposition 3.2.16, it follows that

$$I_p = \frac{\tau(\psi)}{p^3}\left(\psi(-1)p\sum_{(k,l,m)\in T} klm\,\psi(klm) - 3\sum_{k,l,m=1}^{p-1} klm\,\psi(k+m)\right),$$

where $T = \{(k,l,m) \in \mathbb{Z}^3 \mid 1 \le k,l,m \le p-1,\ kl+lm+mk \equiv 0(p)\}$. Letting

$$A_p = \sum_{(k,l,m)\in T} klm\,\psi(klm), \tag{5.1.3}$$

and applying (3.2.5), Proposition 3.2.19, we obtain

$$I_p = \sqrt{\psi(-1)p}\left(\frac{\psi(-1)}{p^2}A_p + \frac{3(p-1)}{4}(B_{2,\psi} + 2B_{1,\psi})\right). \tag{5.1.4}$$

Next we shall evaluate A_p. To this end, we will express everything in terms of periodic Bernoulli polynomials and take full advantage of their periodicity.

$$\begin{aligned}
\frac{1}{p^3}A_p &= \sum_{(k,l,m)\in T} \frac{k}{p}\frac{l}{p}\frac{m}{p}\,\psi(klm)\\
&= \sum_{(k,l,m)\in T}\left(P_1\left(\frac{k}{p}\right)+\frac{1}{2}\right)\left(P_1\left(\frac{l}{p}\right)+\frac{1}{2}\right)\left(P_1\left(\frac{m}{p}\right)+\frac{1}{2}\right)\psi(klm)\\
&= \sum_{(k,l,m)\in T} P_1\left(\frac{k}{p}\right)P_1\left(\frac{l}{p}\right)P_1\left(\frac{m}{p}\right)\psi(klm)\\
&+\frac{1}{2}\sum_{(k,l,m)\in T}\left(P_1\left(\frac{k}{p}\right)P_1\left(\frac{l}{p}\right)+P_1\left(\frac{l}{p}\right)P_1\left(\frac{m}{p}\right)\psi(klm)\right.\\
&\quad\left.+P_1\left(\frac{m}{p}\right)P_1\left(\frac{k}{p}\right)\right)\psi(klm)\\
&+\frac{1}{4}\sum_{(k,l,m)\in T}\left(P_1\left(\frac{k}{p}\right)+P_1\left(\frac{l}{p}\right)+P_1\left(\frac{m}{p}\right)\right)\psi(klm)\\
&+\frac{1}{8}\sum_{(k,l,m)\in T}\psi(klm).
\end{aligned}$$

We observe the 1-to-1 correspondence between the sets

$$T = \{(k,l,m) \in \mathbb{Z}^3 \mid 1 \le k,l,m \le p-1,\ kl+lm+mk \equiv 0(p)\}$$

and

$$\{(m^2l+ml, l+ml, -ml) \in (\mathbb{F}_p^{\times})^3 \mid 1 \le l,m \le p-1,\ m \ne p-1\}.$$

Hence, substituting $m^2l+ml, l+ml, -ml$ for k,l,m in the sums above, we get

$$\begin{aligned}
\frac{\psi(-1)}{p^3}A_p = &-\sum_{l=1}^{p-1}\sum_{m=1}^{p-2} P_1\left(\frac{m^2l+ml}{p}\right) P_1\left(\frac{l+ml}{p}\right) P_1\left(\frac{ml}{p}\right)\psi(l) \\
&+\frac{1}{2}\sum_{l=1}^{p-1}\sum_{m=1}^{p-2} P_1\left(\frac{m^2l+ml}{p}\right) P_1\left(\frac{l+ml}{p}\right)\psi(l) \\
&-\frac{1}{2}\sum_{l=1}^{p-1}\sum_{m=1}^{p-2} P_1\left(\frac{l+ml}{p}\right) P_1\left(\frac{ml}{p}\right)\psi(l) \\
&-\frac{1}{2}\sum_{l=1}^{p-1}\sum_{m=1}^{p-2} P_1\left(\frac{ml}{p}\right) P_1\left(\frac{m^2l+ml}{p}\right)\psi(l) \\
&+\frac{1}{4}\sum_{l=1}^{p-1}\sum_{m=1}^{p-2}\left(P_1\left(\frac{m^2l+ml}{p}\right) + P_1\left(\frac{l+ml}{p}\right) - P_1\left(\frac{ml}{p}\right)\right)\psi(l) \\
&+\frac{1}{8}\sum_{l=1}^{p-1}\sum_{m=1}^{p-2}\psi(l).
\end{aligned}$$

Simplifying where we note that $\sum_{l=1}^{p-1}\psi(l)=0$, $\sum_{l=1}^{p-1}P_1\left(\frac{l}{p}\right)\psi(l)=B_{1,\psi}$, we obtain

$$\begin{aligned}\frac{\psi(-1)}{p^3}A_p = &-\sum_{l,m(p)}P_1\left(\frac{m^2l+ml}{p}\right)P_1\left(\frac{l+ml}{p}\right)P_1\left(\frac{ml}{p}\right)\psi(l)\\ &+\frac{1}{2}\sum_{l,m(p)}P_1\left(\frac{m^2l+ml}{p}\right)P_1\left(\frac{l+ml}{p}\right)\psi(l)\\ &-\frac{1}{2}\sum_{l,m(p)}P_1\left(\frac{l+ml}{p}\right)P_1\left(\frac{ml}{p}\right)\psi(l)\\ &-\frac{1}{2}\sum_{l,m(p)}P_1\left(\frac{ml}{p}\right)P_1\left(\frac{m^2l+ml}{p}\right)\psi(l)\\ &+\frac{1}{4}\sum_{l,m(p)}\left(P_1\left(\frac{m^2l+ml}{p}\right)+P_1\left(\frac{l+ml}{p}\right)-P_1\left(\frac{ml}{p}\right)\right)\psi(l)\\ &-\frac{1}{2}B_{1,\psi}.\end{aligned}$$

By Proposition 3.2.20, Proposition 3.2.21, it follows that

$$\begin{aligned}\frac{\psi(-1)}{p^3}A_p = &-\sum_{l,m(p)}P_1\left(\frac{m^2l+ml}{p}\right)P_1\left(\frac{l+ml}{p}\right)P_1\left(\frac{ml}{p}\right)\psi(l)\\ &-\frac{3}{4}B_{2,\psi}-\frac{3}{4}B_{1,\psi}.\end{aligned}\tag{5.1.5}$$

Observe that the sum above without the character,

$$\sum_{l,m(p)}P_1\left(\frac{m^2l+ml}{p}\right)P_1\left(\frac{l+ml}{p}\right)P_1\left(\frac{ml}{p}\right),$$

is odd in l and hence vanishes. Therefore,

$$\begin{aligned}
&\sum_{l,m(p)} P_1\left(\frac{m^2l+ml}{p}\right) P_1\left(\frac{l+ml}{p}\right) P_1\left(\frac{ml}{p}\right)\psi(l) \\
&\quad= \sum_{l,m(p)} P_1\left(\frac{m^2l+ml}{p}\right) P_1\left(\frac{l+ml}{p}\right) P_1\left(\frac{ml}{p}\right)\psi(l) \\
&\quad\quad+ \sum_{l,m(p)} P_1\left(\frac{m^2l+ml}{p}\right) P_1\left(\frac{l+ml}{p}\right) P_1\left(\frac{ml}{p}\right) \\
&\quad= 2\sum_{\substack{l,m(p)\\ \psi(l)=1}} P_1\left(\frac{m^2l+ml}{p}\right) P_1\left(\frac{l+ml}{p}\right) P_1\left(\frac{ml}{p}\right)\psi(l) \\
&\quad= \sum_{l,m(p)} P_1\left(\frac{m^2l^2+ml^2}{p}\right) P_1\left(\frac{l^2+ml^2}{p}\right) P_1\left(\frac{ml^2}{p}\right) \\
&\quad= \sum_{l,m(p)} P_1\left(\frac{m^2+ml}{p}\right) P_1\left(\frac{l^2+ml}{p}\right) P_1\left(\frac{ml}{p}\right),
\end{aligned}$$

where we replaced m by ml^{-1} where $l \not\equiv 0(p)$ in the last step. Let

$$X_p = \sum_{l,m(p)} P_1\left(\frac{m^2+ml}{p}\right) P_1\left(\frac{l^2+ml}{p}\right) P_1\left(\frac{ml}{p}\right). \tag{5.1.6}$$

Then, we get, by (5.1.5),

$$A_p = \psi(-1)(-p^3)\left(X_p + \frac{3}{4}B_{2,\psi} + \frac{3}{4}B_{1,\psi}\right). \tag{5.1.7}$$

Let

$$Y_p = \sum_{k,t(p)} P_1\left(\frac{kt}{p}\right) P_1\left(\frac{k^2+2kt}{p}\right) P_1\left(\frac{t^2+2kt}{p}\right).$$

Applying Sczech's 3-term addition formula (Proposition 3.1.18) to the arguments

$$v1 = \frac{kt}{p},\ v2 = \frac{k^2+2kt}{p},\ v3 = \frac{t^2+2kt}{p},$$

we obtain

$$
\begin{aligned}
&-P_1\left(\frac{kt}{p}\right)P_1\left(\frac{k^2+2kt}{p}\right)P_1\left(\frac{t^2+2kt}{p}\right)\\
&+P_1\left(\frac{kt}{p}\right)P_1\left(\frac{k^2+kt}{p}\right)P_1\left(\frac{t^2+kt}{p}\right)\\
&-P_1\left(\frac{k^2+kt}{p}\right)P_1\left(\frac{k^2+2kt}{p}\right)P_1\left(\frac{t^2-k^2}{p}\right)\\
&-P_1\left(\frac{t^2+kt}{p}\right)P_1\left(\frac{k^2-t^2}{p}\right)P_1\left(\frac{t^2+2kt}{p}\right)\\
&=\frac{1}{2}P_1\left(\frac{k^2+kt}{p}\right)\left(P_2\left(\frac{t^2-k^2}{p}\right)-P_2\left(\frac{t^2+kt}{p}\right)\right)\\
&+\frac{1}{2}P_1\left(\frac{k^2+kt}{p}\right)\left(P_2\left(\frac{k^2+2kt}{p}\right)-P_2\left(\frac{kt}{p}\right)\right)\\
&+\frac{1}{2}P_1\left(\frac{t^2+kt}{p}\right)\left(P_2\left(\frac{t^2+2kt}{p}\right)-P_2\left(\frac{kt}{p}\right)\right)\\
&+\frac{1}{2}P_1\left(\frac{t^2-k^2}{p}\right)\left(P_2\left(\frac{t^2+2kt}{p}\right)-P_2\left(\frac{k^2+2kt}{p}\right)\right)\\
&-\frac{1}{6}\left(-P_3\left(\frac{t^2+kt}{p}\right)-P_3\left(\frac{t^2-k^2}{p}\right)\right)\\
&-\frac{1}{6}\left(2P_3\left(\frac{kt}{p}\right)+2P_3\left(\frac{k^2+2kt}{p}\right)+2P_3\left(\frac{t^2+2kt}{p}\right)\right)\\
&-\frac{1}{4}\delta_1\left(\frac{kt}{p},\frac{k^2+2kt}{p},\frac{t^2+2kt}{p}\right)+\frac{1}{4}\delta_2\left(\frac{kt}{p},\frac{k^2+2kt}{p},\frac{t^2+2kt}{p}\right),
\end{aligned}
$$

where

$$
\delta_1(x,y,z)=\begin{cases} P_1(x), & \text{if } x\equiv y\equiv z \bmod 1,\\ 0, & \text{otherwise,}\end{cases}
$$

and

$$
\delta_2(x,y,z)=\begin{cases} P_1(x), & \text{if } y,z \text{ are integers},\\ P_1(y), & \text{if } x,z \text{ are integers},\\ P_1(z), & \text{if } x,y \text{ are integers},\\ 0, & \text{otherwise.}\end{cases}
$$

Carefully summing both sides over $k,t(p)$ and applying Proposition 3.2.23, Proposition 3.2.1, we get

$$X_p = 3Y_p + h_p + \frac{1}{4p}B_{3,\psi} + \frac{3p-3+3\psi(2)+p\delta_{p,3}}{4p}B_{1,\psi}. \tag{5.1.8}$$

Next, applying the 2-term addition formula (Proposition 3.1.17) to the arguments

$$v_1 = \frac{k^2+2kt}{p},\ v_2 = -\frac{t^2+2kt}{p},$$

we obtain

$$\begin{aligned}
&P_1\left(\frac{k^2+2kt}{p}\right)P_1\left(\frac{t^2-k^2}{p}\right) - P_1\left(\frac{t^2+2kt}{p}\right)P_1\left(\frac{t^2-k^2}{p}\right)\\
&- P_1\left(\frac{t^2+2kt}{p}\right)P_1\left(\frac{k^2+2kt}{p}\right)\\
&= -\frac{1}{2}\left(P_2\left(\frac{k^2+2kt}{p}\right) + P_2\left(\frac{t^2+2kt}{p}\right) + P_2\left(\frac{t^2-k^2}{p}\right)\right)\\
&+ \frac{1}{4}\delta\left(\frac{k^2+2kt}{p}, -\frac{t^2+2kt}{p}\right),
\end{aligned}$$

where

$$\delta(x,y) = \begin{cases} 1, & \text{if } x, y \text{ are integers,} \\ 0, & \text{otherwise.} \end{cases}$$

Multiplying both sides of the above equation by $P_1\left(\frac{kt}{p}\right)$, carefully summing over $k, t(p)$, and applying Proposition 3.2.22, (i) in Proposition 3.2.10, Proposition 3.2.1, we get

$$Y_p = 2Q_p + \frac{1}{6p}B_{3,\psi} + \frac{2-2\psi(2)-p\delta_{p,3}}{4p}B_{1,\psi}.$$

Plugging Y_p back into (5.1.8), we get

$$X_p = 6Q_p + h_p + \frac{3}{4p}B_{3,\psi} + \frac{3p+3-3\psi(2)-2p\delta_{p,3}}{4p}B_{1,\psi},$$

and plugging X_p back into (5.1.7), we get

$$A_p = \psi(-1)(-p^3)\left(6Q_p + h_p + \frac{3}{4p}B_{3,\psi} + \frac{3}{4}B_{2,\psi} + \frac{6p+3-3\psi(2)-2p\delta_{p,3}}{4p}B_{1,\psi}\right).$$

Upon plugging A_p back into (5.1.4), we finally obtain

$$I_p = \sqrt{-p}\left(-6p\left(Q_p + \frac{1}{6}h_p\right) - \frac{3}{4}B_{3,\psi} - \frac{3}{4}B_{2,\psi} - \frac{9-3\psi(2)-2p\delta_{p,3}}{4}B_{1,\psi}\right),$$

and the proof of Theorem 5.1 is now complete. □

5.2 Special Values of $L_2(s, \psi_{H,p})$

Next, we evaluate special values at non-positive integers of a particular Hashimoto L-function. We now fix the notation. Take an odd prime p and fix it. Let L_n (resp. L_n^*) denote the lattice formed by integral symmetric (resp. half-integral symmetric) matrices of size n, and let $L_{n,+}$, $L_{n,+}^*$ be the subsets consisting of all positive definite matrices of L_n, L_n^*, respectively. Denote by $L_{n,+}/SL_n(\mathbb{Z})$ (resp. $L_{n,+}^*/SL_n(\mathbb{Z})$) the set of $SL_n(\mathbb{Z})$-equivalence classes in $L_{n,+}$ (resp. $L_{n,+}^*$). In the case of $n = 2$, Hashimoto introduced the following L-functions $L_2(s, \psi_{H,p})$, $L_2^*(s, \psi_{H,p})$. Let ψ be the unique non-trivial quadratic character mod p, and let $\psi_{H,p}$ be a mapping from L_2^* to $\mathbb{R}$ given as follows: We put $\psi_{H,p}(T) = 0$, if $\det(T) \not\equiv 0$ mod p. When $\det(T) \equiv 0(p)$, we have ${}^t gTg \equiv \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$ mod p for some $g \in SL_2(\mathbb{F}_p)$ and $a \in \mathbb{F}_p$, and we put $\psi_{H,p}(T) = \psi(a)$. Set

$$L_2(s, \psi_{H,p}) = \sum_{T \in L_{2,+}/SL_2(\mathbb{Z})} \frac{\psi_{H,p}(T)}{\epsilon(T)(\det(T))^s} \qquad (\mathrm{Re}(s) > 3/2),$$

$$L_2^*(s, \psi_{H,p}) = \sum_{T \in L_{2,+}^*/SL_2(\mathbb{Z})} \frac{\psi_{H,p}(T)}{\epsilon(T)(\det(T))^s} \qquad (\mathrm{Re}(s) > 3/2),$$

where $\epsilon(T) = \#\{g \in SL_2(\mathbb{Z}) \mid {}^t gTg = T\}$.

Arakawa[1] worked primarily with the L-function $L_2^*(s, \psi_{H,p})$. He showed that $L_2^*(s, \psi_{H,p})$ can be continued analytically to a meromorphic function of s in the whole complex plane which is holomorphic at non-positive integers. Moreover, he established the rationality of the special values of $L_2^*(s, \psi_{H,p})$ at non-positive integers and gave a complicated explicit formula for $L_2^*(0, \psi_{H,p})$. We will now do the the same for $L_2(s, \psi_{H,p})$. Additionally, a formula for all of the special values of $L_2(s, \psi_{H,p})$ and $L_2^*(s, \psi_{H,p})$ at non-positive integers will be given. While we follow Arakawa's method for the most part, we also have to overcome additional difficulties not occurring in Arakawa's work. These difficulties are resolved using Carlitz's reciprocity theorem for generalized Dedekind-Rademacher sums (see Lemma 5.2.27).

Theorem 5.2. *Let p be an odd prime.*

(i) *The special values $L_2(1-m, \psi_{H,p})\,(m=1,2,\cdots)$ are rational numbers.*

(ii) *If $p \equiv 1(4)$, then $L_2(1-m, \psi_{H,p}) = 0\,(m=1,2,\cdots)$.*

(iii) *If $p \equiv 3(4)$, then, in particular,*

$$L_2(0, \psi_{H,p}) = Q_p + \frac{1}{6}h_p + \frac{11}{72p}B_{3,\psi} + \frac{6p+1-3\psi(2)-2p\delta_{p,3}}{24p}B_{1,\psi},$$

where Q_p, h_p are the constants given by (5.1.1), (5.1.2) in Theorem 5.1.

We must establish several results before proving Theorem 5.2.

5.2.1 Expressing $L_2(s, \psi_{H,p})$ in Terms of Partial Zeta Functions

The aim of this section is to represent the L-function $L_2(s, \psi_{H,p})$ as a finite linear combination of partial zeta functions. We need more notation. Let $V_{\mathbb{R}}^{(n)}$ be the $\mathbb{R}$-vector space of real symmetric matrices of size n, $\mathscr{P}_n \subset V_{\mathbb{R}}^{(n)}$ be the set of positive definite symmetric matrices of size n, and let $\partial\mathscr{P}_n$ denote the boundary of the domain $\mathscr{P}_n$ in $V_{\mathbb{R}}^{(n)}$, that is, $\partial\mathscr{P}_n$ is the set of positive semi-definite symmetric matrices of size n. Let $\{W_1, W_2, \cdots, W_r\}$ be an r-tuple of elements in $\mathscr{P}_2 \cup \partial\mathscr{P}_2$ such that $W_1, W_2, \cdots, W_r$ are linearly independent over $\mathbb{R}$. Then, necessarily $r \leq 3$. For any r-tuple $\xi = (\xi_1, \cdots, \xi_r)$ of positive numbers, we define a partial zeta function $\zeta(s; \{W_1, \cdots, W_r\}, \xi)$ as follows:

$$\zeta(s; \{W_1, \cdots, W_r\}, \xi) = \sum_{m_1, \cdots, m_r = 0}^{\infty} \det\left(\sum_{j=1}^{r} (\xi_j + m_j) W_j\right)^{-s}. \tag{5.2.1}$$

Let $C = C(W_1, \cdots, W_r)$ be a simplicial cone spanned by $W_1, \cdots, W_r$:

$$C = C(W_1, \cdots, W_r) = \left\{ \sum_{j=1}^{r} \lambda_j W_j \mid \lambda_j > 0 \ (1 \leq j \leq r) \right\}.$$

We assume that the cone $C = C(W_1, \cdots, W_r)$ is contained in $\mathscr{P}_2$. Then it is easily shown that the zeta function $\zeta(s;\{W_1,\cdots,W_r\},\xi)$ is absolutely convergent for $\mathrm{Re}(s)>r/2$. For any subset M of $V_{\mathbb{R}}^{(2)}$, the zeta function $\zeta(s;C,M)$, if it converges absolutely, is defined by

$$\zeta(s;C,M)=\sum_{T\in C\cap M}\det(T)^{-s}. \tag{5.2.2}$$

It is well-known that, as a fundamental domain of $\mathscr{P}_n$ under the usual action of $GL_n(\mathbb{Z})$, one can take the so-called Minkowski domain $\mathscr{R}_n$ of reduced matrices (see Maass[13]). In the case of $n=2$, the domain $\mathscr{R}_2$ has a simple form:

$$\mathscr{R}_2=\left\{\begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix} : 0\le 2y_{12}\le y_1\le y_2,\ 0<y_1\right\}.$$

We fix three very special elements V_1,V_2,V_3 in $\mathscr{P}_2\cup\partial\mathscr{P}_2$; put

$$V_1=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\qquad V_2=\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},\qquad V_3=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Remark. It is this choice of V_1,V_2,V_3 that will eventually lead us back to our sum I_p.

For simplicity, we set

$$\begin{cases} C_{123}=C(V_1,V_2,V_3), \\ C_{ij}=C(V_i,V_j)\quad(1\le i<j\le 3), \\ C_j=C(V_j)\quad(j=1,2), \end{cases} \tag{5.2.3}$$

which are simplicial cones contained in $\mathscr{P}_2$. Then the domain $\mathscr{R}_2$ has the decomposition

$$\mathscr{R}_2 = C_{123} \cup C_{12} \cup C_{13} \cup C_{23} \cup C_1 \cup C_2 \qquad \text{(disjoint union)}. \tag{5.2.4}$$

For each cone C in (5.2.3) and any $Y \in C$, observe that the order $\epsilon^*(Y)$ of the group $\{U \in GL_2(\mathbb{Z}) \mid {}^tUYU = Y\}$ takes the same value independent of Y belonging to C, and one can put

$$\epsilon^*(C) = \epsilon^*(Y) \qquad (Y \in C).$$

It is easily verified that

$$\epsilon^*(C_{123}) = 2, \quad \epsilon^*(C_{ij}) = 4 \quad (1 \le i < j \le 3), \quad \epsilon^*(C_1) = 8, \quad \epsilon^*(C_2) = 12. \tag{5.2.5}$$

For any $x \in \mathbb{R}$, we denote by $\langle x \rangle$ the unique real number which satisfies $0 < \langle x \rangle \le 1$ and $x - \langle x \rangle \in \mathbb{Z}$. Let p be an odd prime. Let

$$\mathscr{M}(p) = \{(\alpha, \gamma) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \mid (\alpha, \gamma) \ne (0,0) \bmod p\}. \tag{5.2.6}$$

For any integer μ prime to p, let $L(\mu)$ be the set consisting of all elements $T \in L_2$ satisfying $\psi_{H,p}(T) = \psi(\mu)$. Then it immediately follows that

$$L(\mu) = \left\{ T \in L_2 \mid T \equiv \mu \begin{pmatrix} \alpha^2 & \alpha\gamma \\ \alpha\gamma & \gamma^2 \end{pmatrix} \bmod p \text{ for some } (\alpha, \gamma) \in \mathscr{M}(p) \right\}$$

and that $L(\mu l^2) = L(\mu)$ for any integer l prime to p. For each $(\alpha, \gamma) \in \mathscr{M}(p)$ and for each integer μ prime to p, we put

$$\xi_{\alpha,\gamma,\mu} = (\langle \mu(\alpha^2 - 2\alpha\gamma)/p \rangle, \langle \mu\alpha\gamma/p \rangle, \langle \mu(\gamma^2 - \alpha^2)/p \rangle). \tag{5.2.7}$$

Let $\Xi_{H,\mu}$ be the set of all triples $\xi_{\alpha,\gamma,\mu}$: $\Xi_{H,\mu} = \{\xi_{\alpha,\gamma,\mu} \mid (\alpha, \gamma) \in \mathscr{M}(p)\}$. Observe that

$\mathscr{M}(p)/\{\pm 1\}$ corresponds to $\Xi_{H,\mu}$ bijectively by $\pm(\alpha,\gamma) \to \xi_{\alpha,\gamma,\mu} (= \xi_{-\alpha,-\gamma,\mu})$. For any integers i, j with $1 \le i < j \le 3$, we set

$$\Xi_{H,\mu}^{(i,j)} = \{\xi = (\xi_1, \xi_2, \xi_3) \in \Xi_{H,\mu} \mid \xi_k = 1\},$$

where k is the unique integer of $1, 2, 3$ satisfying $\{i, j, k\} = \{1, 2, 3\}$. We notice that

$$\begin{cases} \Xi_{H,\mu}^{(1,2)} = \{\xi_{\alpha,\gamma,\mu} \mid (\alpha,\gamma) \in \mathscr{M}(p),\ \alpha^2 \equiv \gamma^2(p)\}, \\ \Xi_{H,\mu}^{(1,3)} = \{\xi_{\alpha,\gamma,\mu} \mid (\alpha,\gamma) \in \mathscr{M}(p),\ \alpha\gamma \equiv 0(p)\}, \\ \Xi_{H,\mu}^{(2,3)} = \{\xi_{\alpha,\gamma,\mu} \mid (\alpha,\gamma) \in \mathscr{M}(p),\ \alpha^2 \equiv 2\alpha\gamma(p)\}. \end{cases} \tag{5.2.8}$$

For each cone C of the form (5.2.3), the zeta function $\zeta(s; C, L(\mu))$ given by (5.2.2) is absolutely convergent at least for Re$(s) > 3/2$.

Proposition 5.2.1. *The following expressions for the zeta functions $\zeta(s; C, L(\mu))$ hold:*

$$\zeta(s; C_{123}, L(\mu)) = p^{-2s} \sum_{\xi \in \Xi_{H,\mu}} \zeta(s; \{V_1, V_2, V_3\}, \xi),$$

$$\zeta(s; C_{ij}, L(\mu)) = p^{-2s} \sum_{\xi \in \Xi_{H,\mu}^{(i,j)}} \zeta(s; \{V_i, V_j\}, (\xi_i, \xi_j)) \qquad (1 \le i < j \le 3),$$

$$\zeta(s; C_1, L(\mu)) = 0,$$

$$\zeta(s; C_2, L(\mu)) = \begin{cases} 0, \text{ if } p > 3, \\ p^{-2s}\zeta(s; \{V_2\}, \langle 2\mu/p \rangle), \text{ if } p = 3. \end{cases}$$

Proof. Take $T \in C_{123} \cap L(\mu)$ and write $T = \sum_{j=1}^{3} m_j V_j$ with all $m_j \in \mathbb{N}$. Then for any pair $(\alpha,\gamma)\in\mathscr{M}(p)$ such that

$$T\equiv\mu\begin{pmatrix}\alpha^2 & \alpha\gamma\\ \alpha\gamma & \gamma^2\end{pmatrix} \bmod p, \tag{5.2.9}$$

the m_j's must necessarily satisfy the following congruences:

$$\begin{cases} m_1\equiv\mu(\alpha^2-2\alpha\gamma)\,(p),\\ m_2\equiv\mu\alpha\gamma\,(p),\\ m_3\equiv\mu(\gamma^2-\alpha^2)\,(p).\end{cases}$$

Therefore, there exists a triple $l=(l_1,l_2,l_3)$, l_j being non-negative integers, such that $(m_1,m_2,m_3)=p(\xi_{\alpha,\gamma,\mu}+l)$. As each $T\in C_{123}\cap L(\mu)$ determines a triple l uniquely and also $(\alpha,\gamma)\in\mathscr{M}(p)$ uniquely up to (±1)-multiplication, the first identity in Proposition 5.2.1 follows. Next, let $T\in C_{12}\cap L(\mu)$ and write $T=\sum_{j=1}^{2}m_jV_j$ ($m_j\in\mathbb{N}$). Taking a pair $(\alpha,\gamma)\in\mathscr{M}(p)$ as in (5.2.9), the congruences $m_1\equiv\mu(\alpha^2-2\alpha\gamma)\,(p)$, $m_2\equiv\mu\alpha\gamma\,(p)$ follow, and necessarily, the relation $\alpha^2\equiv\gamma^2\,(p)$ has to hold. Hence the identity for $\zeta(s;C_{12},L(\mu))$ immediately follows. The identities for $\zeta(s;C_{23},L(\mu))$ and $\zeta(s;C_{13},L(\mu))$ are similarly verified, so the proofs are omitted. Next, let $T\in C_1\cap L(\mu)$ and write $T=mV_1$ for some $m\in\mathbb{N}$. Then there is no pair $(\alpha,\gamma)\in\mathscr{M}(p)$ as in (5.2.9) and the identity for $\zeta(s;C_1,L(\mu))$ is vacuously true. Lastly, let $T\in C_2\cap L(\mu)$ and write $T=mV_2$ for some $m\in\mathbb{N}$. Taking a pair $(\alpha,\gamma)\in\mathscr{M}(p)$ as in (5.2.9), then necessarily, the relations $\alpha\equiv2\gamma\,(p)$ and $\gamma\equiv2\alpha\,(p)$ have to hold. This can only happen if $p=3$, in which case $\xi_{\alpha,\gamma,\mu}=(1,\langle\mu\alpha\gamma/p\rangle,1)=(1,\langle2\mu\alpha^2/p\rangle,1)=(1,\langle2\mu/p\rangle,1)$. Thus the identity for $\zeta(s;C_2,L(\mu))$ follows, and the proof of Proposition 5.2.1 is now complete. □

Let κ be a non-quadratic residue mode p so that we can write $L_2=L(1)\cup L(\kappa)$ (disjoint union).

Proposition 5.2.2. *Let ψ be the unique non-trivial quadratic character mod p. Then we have*

$$L_2(s,\psi_{H,p}) = \sum_{\mu} \psi(\mu) \left\{ \zeta(s; C_{123}, L(\mu)) + \frac{1}{2} \sum_{i<j} \zeta(s; C_{ij}, L(\mu)) \right.$$
$$\left. + \frac{1}{6}\delta_{p,3}\zeta(s; C_2, L(\mu)) \right\},$$

where μ is taken over 1 and κ, and the summation $\sum_{i<j}$ indicates that i, j run over all integers with $1 \leq i < j \leq 3$.

Proof. Only in this proof do we introduce the L-function $M_2(s,\psi_{H,p})$ which is very similar to $L_2(s,\psi_{H,p})$. We set

$$M_2(s,\psi_{H,p}) = \sum_{T \in L_{2,+}/GL_2(\mathbb{Z})} \frac{\psi_{H,p}(T)}{\epsilon^*(T)(\det(T))^s}$$

where T is taken over $GL_2(\mathbb{Z})$-equivalence classes of positive definite integral symmetric matrices of size two, and $\epsilon^*(T)$ is the order of the unit group $\{U \in GL_2(\mathbb{Z}) \mid {}^tUTU = T\}$ of T. Then an elementary observation shows that $L_2(s,\psi_{H,p}) = 2M_2(s,\psi_{H,p})$. In view of the decomposition (5.2.4) of $\mathscr{R}_2$, we may take a disjoint union $\cup_C (C \cap L_2)$ with C varying over all the simplicial cones in (5.2.3), as a complete set of $GL_2(\mathbb{Z})$-equivalence classes of all elements in $L_{2,+}$. Thus we get, with the help of the decomposition $L_2 = L(1) \cup L(\kappa)$ (disjoint union),

$$L_2(s,\psi_{H,p}) = 2\sum_{C} \epsilon^*(C)^{-1} \sum_{\mu} \psi(\mu)\zeta(s; C, L(\mu)),$$

where C runs over all the simplicial cones in (5.2.3) and μ is taken over 1 and κ. This together with (5.2.5) and Proposition 5.2.1, completes the proof of Proposition 5.2.2.

□

5.2.2 Integral Representations of Partial Zeta Functions I

The aim of this section is to obtain convenient expressions of partial zeta functions as integrals over contour paths, and then to evaluate special values of them at non-positive integers. Several results in this section immediately follow from Arakawa's work on $L_2^*(s, \psi_{H,p})$. All of the details will be included here for completeness.

Let $\zeta(s; \{W_1, \cdots, W_r\}, \xi)$ be a partial zeta function as defined in (5.2.1). We assume that the cone $C(W_1, \cdots, W_r)$ is contained in $\mathscr{P}_2$. The following formula is well-known (see Maass[13] and Satake[16]):

$$(5.2.10) \qquad \det(T)^{-s} = \frac{1}{\Gamma_2(s)} \int_{\mathscr{P}_2} \det(Y)^s e^{\mathrm{tr}(TY)} dv(Y) \quad (T \in \mathscr{P}_2,\ \mathrm{Re}(s) > 1/2),$$

where we put

$$\Gamma_2(s) = \pi^{1/2} \Gamma(s) \Gamma(s - 1/2) \quad \text{and} \quad dv(Y) = \det(Y)^{-3/2} \prod_{1 \le i \le j \le 2} dY_{ij}.$$

We set, for $t \in \mathbb{C}, x \in \mathbb{R}$,

$$\phi(t; x) = \frac{e^{tx}}{e^t - 1},$$

which is the generating function of Bernoulli polynomials $B_k(x)$. Recall that the Laurent expansion at $t = 0$ of $\phi(t; x)$ is given by

$$(5.2.11) \qquad \phi(t; x) = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} t^{k-1} \qquad (|t| < 2\pi).$$

By a usual argument which uses the formula (5.2.10), we get an expression of $\zeta(s; \{W_1, \cdots, W_r\}, \xi)$ for $\mathrm{Re}(s) > r/2$ by the integral taken over $\mathscr{P}_2$:

$$\zeta(s; \{W_1, \cdots, W_r\}, \xi) = \frac{1}{\Gamma_2(s)} \int_{\mathscr{P}_2} \det(Y)^s \prod_{j=1}^{r} \phi(\mathrm{tr}(W_j Y); 1 - \xi_j) dv(Y).$$

We set, for $\theta \in \mathbb{R}$, $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

Following Satake[16], we make a change of variables $Y \to (t, u, \theta)$ with $Y = tk_\theta \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} {}^t k_\theta \quad (0 < t,\ 0 < u \leq 1,\ 0 \leq \theta \leq \pi)$. Then using the relation $dv(Y) = t^{-1}u^{-3/2}(1-u)\,dt\,du\,d\theta$, we get

$$
\begin{aligned}
&\zeta(s; \{W_1, \cdots, W_r\}, \xi) \\
&= \frac{1}{\Gamma_2(s)} \int_0^\infty dt \int_0^1 du \int_0^\pi d\theta \cdot t^{2s-1} u^{s-3/2}(1-u) \\
&\quad \times \Phi((t, u, \theta); \{W_1, \cdots, W_r\}, \xi),
\end{aligned} \tag{5.2.12}
$$

where we put

$$
\Phi((t, u, \theta); \{W_1, \cdots, W_r\}, \xi) = \prod_{j=1}^{r} \phi(t\lambda((u, \theta), W_j); 1 - \xi_j),
$$

and

$$
\lambda((u, \theta), W) = \operatorname{tr}\left(W k_\theta \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} {}^t k_\theta \right) \quad \text{for any } W \in \mathscr{P}_2 \cup \partial\mathscr{P}_2.
$$

If all the W_j's $(1 \leq j \leq r)$ are contained in $\mathscr{P}_2$, then the integral in (5.2.12) has been studied in full generality by Satake[16] and Kurihara[11]. Unfortunately, we have to deal with the case where one of the edge vectors coincides with the special vector V_3 in $\partial\mathscr{P}_2$. Because of this complication, Satake-Kurihara's method cannot be applied directly to our situation, and as a result, new ideas are needed. Arakawa[1] has developed these new ideas in his evaluation of $L_2^*(s, \psi_{H,p})$. In view of Proposition 5.2.1, we only have to consider the cases in which, with respect to an r-tuple $\{W_1, \cdots, W_r\}$, the vectors $W_1, \cdots, W_{r-1}$ are all in $\mathscr{P}_2$ and W_r coincides with the special vector V_3 in $\partial\mathscr{P}_2$.

Now we set

$$\psi(t;x) = \phi(t;x) - \frac{1}{t},$$

which is a holomorphic function of t in the region $|t| < 2\pi$. Let $\{W_1, \cdots, W_{r-1}, V_3\}$ ($r = 2$ or 3) be an r-tuple of vectors in $\mathscr{P}_2 \cup \partial\mathscr{P}_2$ such that $W_1, \cdots, W_{r-1}$ are all in $\mathscr{P}_2$. We set, for an r-tuple $\xi = (\xi_1, \cdots, \xi_r)$ of positive numbers,

$$\Phi_P((t,u,\theta); \{W_1, \cdots, W_{r-1}, V_3\}, \xi) = \prod_{j=1}^{r-1} \phi(t\lambda((u,\theta), W_j); 1-\xi_j)\, \psi(t\lambda((u,\theta), V_3); 1-\xi_3),$$

and, for an $(r-1)$-tuple $\xi' = (\xi_1, \cdots, \xi_{r-1})$,

$$\Phi_S((t,u,\theta); \{W_1, \cdots, W_{r-1}, V_3\}, \xi') = \frac{1}{t\lambda((u,\theta), V_3)} \prod_{j=1}^{r-1} \phi(t\lambda((u,\theta), W_j); 1-\xi_j).$$

Next we define the principal and singular parts of $\zeta(s; \{W_1, \cdots, W_{r-1}, V_3\}, \xi)$.

$$\begin{aligned} &\zeta_P(s; \{W_1, \cdots, W_{r-1}, V_3\}, \xi) \\ &\quad = \frac{1}{\Gamma_2(s)} \int_0^\infty dt \int_0^1 du \int_0^\pi d\theta \cdot t^{2s-1} u^{s-3/2} (1-u) \\ &\qquad \times \Phi_P((t,u,\theta); \{W_1, \cdots, W_{r-1}, V_3\}, \xi), \end{aligned} \tag{5.2.13}$$

$$\begin{aligned} &\zeta_S(s; \{W_1, \cdots, W_{r-1}, V_3\}, \xi') \\ &\quad = \frac{1}{\Gamma_2(s)} \int_0^\infty dt \int_0^1 du \int_0^\pi d\theta \cdot t^{2s-1} u^{s-3/2} (1-u) \\ &\qquad \times \Phi_S((t,u,\theta); \{W_1, \cdots, W_{r-1}, V_3\}, \xi'). \end{aligned} \tag{5.2.14}$$

The integrals in (5.2.13) and (5.2.14) are absolutely convergent for $\mathrm{Re}(s) > 3/2$, and obviously,

$$\zeta(s; \{W_1, \cdots, V_3\}, \xi) = \zeta_P(s; \{W_1, \cdots, V_3\}, \xi) + \zeta_S(s; \{W_1, \cdots, V_3\}, \xi').$$

We now prepare more notation. For a positive number ϵ, let $I_\epsilon(\infty)$ (resp. $I_\epsilon(1)$) be the contour path consisting of the oriented half line $(+\infty, \epsilon)$ (resp. $(+\infty, 1)$), a counterclockwise circle of radius ϵ around the origin, and the oriented half line $(\epsilon, +\infty)$ (resp. $(\epsilon, 1)$). We will express the integral in (5.2.12) as the integral taken over contour paths $I_\epsilon(\infty)$ and $I_\epsilon(1)$ (for a small ϵ) with respect to t and u, respectively. However, the function $\phi(t\lambda((u,\theta), V_3); 1-\xi_3)$ has serious singularities as a function of t and u on the paths $I_\epsilon(\infty)$ and $I_\epsilon(1)$, because of the form of $\lambda((u,\theta), V_3) = u\, sin^2(\theta) + cos^2(\theta)$, and therefore such an integral representation cannot be easily obtained. To avoid the difficulties derived from such singularities, we divide the zeta function $\zeta(s; \{W_1, \cdots, V_3\}, \xi)$ into two parts as above. In the rest of this section, we shall mainly discuss the function $\zeta_P(s; \{W_1, \cdots, W_{r-1}, V_3\}, \xi)$ and its expression by an integral over contour paths. The singular part $\zeta_S(s; \{W_1, \cdots, W_{r-1}, V_3\}, \xi')$ will be dealt with in the next section.

For a positive number δ, we denote by $D_\delta(\infty)$ and $D_\delta(1)$ the regions given as follows:

$$D_\delta(\infty) = \{z \in \mathbb{C} \mid |z| < \delta\} \cup \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0 \text{ and } |\mathrm{Im}(z)| < \delta\},$$
$$D_\delta(1) = D_\delta(\infty) \cap \{z \in \mathbb{C} \mid |z| \leq 1\}.$$

If W is in $\mathscr{P}_2$, we can take positive constants a, b satisfying

$$a1_2 < W < b1_2. \tag{5.2.15}$$

We may write $\lambda((u,\theta), W) = \alpha_1 u + \alpha_2$ with $a < \alpha_1, \alpha_2 < b$. It then follows that

$$\begin{aligned} &|\lambda((u,\theta), W)| < b(1 + |u|), \\ &\mathrm{Re}(\lambda((u,\theta), W)) > a - b\delta \quad \text{if } \mathrm{Re}(u) > -\delta \quad (\delta > 0). \end{aligned} \tag{5.2.16}$$

We need some analytic properties of the functions $\phi(t\lambda((u,\theta),W);1-\xi)$ and $\psi(t\lambda((u,\theta),W);1-\xi)$ $(\xi>0)$.

Lemma 5.2.3 [1]. *Suppose that $W \in \mathscr{P}_2$ satisfies the condition (5.2.15). Let $\xi>0$ and $0<\delta<a/b$.*

(i) *If $|t|<\pi/2b$, and $u\in D_\delta(1)$, then, $t\phi(t\lambda((u,\theta),W);1-\xi)$, which is a holomorphic function of (t,u) for each θ in that region of (t,u), has the power series expansion with respect to t:*

$$t\phi(t\lambda((u,\theta),W);1-\xi)=\frac{1}{\lambda((u,\theta),W)}+\sum_{k=1}^{\infty}\frac{B_k(1-\xi)}{k!}\{\lambda((u,\theta),W)\}^{k-1}t^k,$$

(ii) *If $t>0$ and $0\le u\le 1$, then,*

$$t\phi(t\lambda((u,\theta),W);1-\xi)<\frac{te^{-t\xi a}}{1-e^{-ta}}.$$

Proof. If $|t|<\pi/2b, u\in D_\delta(1)$, then, we get, by (5.2.16),

$$|t\lambda((u,\theta),W)|<\pi(1+|u|)/2<2\pi.$$

Thus, the Laurent expansion in (5.2.11) implies the assertion (i). The assertion (ii) follows immediately from the inequality $\lambda((u,\theta),W)>a\,(0\le u\le 1)$. □

Lemma 5.2.4 [1]. *Let $\xi>0$ and $0<\delta<1$. Then $\psi(t\lambda((u,\theta),V_3);1-\xi)$ is a holomorphic function of t,u for each θ in the region $\{(t,u)\mid |t|<2\pi, u\in D_\delta(1)\}$, and has a Taylor expansion with respect to t:*

$$\psi(t\lambda((u,\theta),W);1-\xi)=\sum_{k=1}^{\infty}\frac{B_k(1-\xi)}{k!}(u\sin^2\theta+\cos^2\theta)^{k-1}t^{k-1}.$$

Proof. Recalling that $\lambda((u,\theta),V_3)=u\sin^2\theta+\cos^2\theta$, we see that $|t\lambda((u,\theta),V_3)|<2\pi$ if $|t|<2\pi, u\in D_\delta(1)$. The assertion of Lemma 5.2.4 then immediately follows. □

The function $\psi(t; 1-\xi)$ has the following preferable property which will be used in the proof of Proposition 5.2.6.

Lemma 5.2.5 [1]. *Let $\xi > 0$. There exist positive constants $M_k\,(k = 1, 2, \cdots)$ independent of t such that, if $0 \le t < +\infty$,*

$$|\psi^{(k)}(t; 1-\xi)| < M_k,$$

where $\psi^{(k)}(t; 1-\xi)$ denotes the k-th derivative of $\psi(t; 1-\xi)$ as a function of t.

We omit the proof of Lemma 5.2.5, which is an easy exercise of differential calculus.

From (5.2.16) and Lemma 5.2.5, we easily see that if δ is taken sufficiently small, then, $t\phi(t\lambda((u,t), W); 1-\xi)\,(W \in \mathscr{P}_2, \xi > 0)$ is holomorphic as a function of t, u for each $\theta \in \mathbb{R}$ in the region $D_\delta(\infty) \times D_\delta(1)$. Moreover, taking (5.2.16), Lemma 5.2.3, and Lemma 5.2.4 into account, we see that the integral

$$t^2 \int_0^\pi \varPhi_P((t,u,\theta); \{W_1, \cdots, W_{r-1}, V_3\}, \xi) d\theta$$

is a holomorphic function of t, u in the region $D_\delta(1) \times D_\delta(1)$. We notice here that the range of t is the region $D_\delta(1)$ (not $D_\delta(\infty)$).

To define the function $t^s = e^{s\,log\,t}$, we take the branch of $\log t$ with $0 < \arg t < 2\pi$.

Proposition 5.2.6 [1]. *The function $\zeta_P(s; \{W_1, \cdots, W_{r-1}, V_3\}, \xi)$ is analytically continued to a meromorphic function in the whole complex plane which is holomorphic at $s = 1-m\,(m = 1, 2, \cdots)$. Moreover, the special value at $s = 1-m$ is given by*

$$\begin{aligned}
&\zeta_P(1-m; \{W_1, \cdots, W_{r-1}, V_3\}, \xi) \\
&\quad = C(m) \int_{\varGamma_\epsilon} dt \int_{I_\epsilon(1)} du \int_0^\pi d\theta \cdot t^{1-2m} u^{-m-1/2} (1-u) \\
&\qquad \times \varPhi_P((t,u,\theta); \{W_1, \cdots, W_{r-1}, V_3\}, \xi),
\end{aligned}$$

where $C(m) = (2m-1)!\,/(2^{2m+2}\pi^2 i)$ *and* Γ_ϵ *denotes a circle of radius* ϵ *around the origin oriented counterclockwise,* ϵ *being taken sufficiently small.*

Proof. We set, only in the proof of this proposition,

$$f(t,u,\theta) = \Phi_P((t,u,\theta); \{W_1, \cdots, W_{r-1}, V_3\}, \xi).$$

We divide the integral in (5.2.13) into two parts by the range of the variable t. We set

$$\zeta_P(s; \{W_1, \cdots, W_{r-1}, V_3\}, \xi) = \frac{1}{\Gamma_2(s)}(I_1(s) + I_2(s)),$$

where

$$I_1(s) = \int_0^1 dt \int_0^1 du \int_0^\pi d\theta \cdot t^{2s-1} u^{s-3/2}(1-u) f(t,u,\theta),$$
$$I_2(s) = \int_1^\infty dt \int_0^1 du \int_0^\pi d\theta \cdot t^{2s-1} u^{s-3/2}(1-u) f(t,u,\theta).$$

We put $e(w)\,(w \in \mathbb{C})$ to denote $exp(2\pi i w)$. From the remarks made just before the statement of Proposition 5.2.6, it is easy to see that $I_1(s)$ has the following expression by an integral over contour paths:

$$I_1(s) = \frac{1}{(e(2s)-1)(e(s-3/2)-1)} \int_{I_\epsilon(1)} dt \int_{I_\epsilon(1)} du \int_0^\pi d\theta \times t^{2s-1} u^{s-3/2}(1-u) f(t,u,\theta), \tag{5.2.17}$$

where ϵ is taken sufficiently small. Since the integral in (5.2.17) is an entire function of s, the function $I_1(s)$ can be continued analytically to a meromorphic function in the whole complex plane. Thus we easily obtain

$$\lim_{s \to 1-m} \frac{I_1(s)}{\Gamma_2(s)} = C(m) \int_{\Gamma_\epsilon} dt \int_{I_\epsilon(1)} du \int_0^\pi d\theta \cdot t^{1-2m} u^{-m-1/2}(1-u) f(t,u,\theta) \quad (m = 1, 2, \cdots). \tag{5.2.18}$$

From Lemma 5.2.5, we see that $f(t,u,\theta)$ is a C^∞-function of (t,u,θ) in the region $(0,+\infty)\times[0,1]\times[0,\pi]$, and especially that the partial derivatives $(\partial^k f/\partial u^k)(t,u,\theta)(k = 0,1,\cdots)$ are bounded in the region $[1,+\infty]\times[0,1]\times[0,\pi]$. We set, for $\mathrm{Re}(s)>0$,

$$F(s;(t,\theta)) = \int_0^1 u^{s-1} f(t,u,\theta)du.$$

Then we have

$$I_2(s) = \int_1^\infty dt \int_0^\pi d\theta \cdot t^{2s-1}\{F(s-1/2;(t,\theta)) - F(s+1/2;(t,\theta))\}. \tag{5.2.19}$$

By repeatedly integrating by parts, we obtain

$$\begin{aligned} F(s;(t,\theta)) = &\sum_{j=0}^{m-1} \frac{(-1)^j}{s(s+1)\cdots(s+j)} \frac{\partial^j f}{\partial u^j}(t,1,\theta) \\ &+ \frac{(-1)^m}{s(s+1)\cdots(s+m-1)} \int_0^1 u^{s+m-1} \frac{\partial^m f}{\partial u^m}(t,u,\theta)du \\ &(\mathrm{Re}(s) > -m). \end{aligned} \tag{5.2.20}$$

Since any $m\in\mathbb{N}$ can be taken, it follows from (5.2.19), (5.2.20) that $I_2(s)$ can be continued to an entire function of s. Thus we get

$$\left[\frac{I_2(s)}{\Gamma_2(s)}\right]_{s=1-m} = 0. \tag{5.2.21}$$

The analytic continuation of $\zeta_P(s;\{W_1,\cdots,W_{r-1},V_3\},\xi)$ immediately follows those of $I_1(s), I_2(s)$. The last assertion of Proposition 5.2.6 is derived from (5.2.18) and (5.2.21). □

We also need the following proposition obtained by Satake which deals with partial zeta functions whose edge vectors are all in $\mathscr{P}_2$.

Proposition 5.2.7 [16]. *Let all vectors $W_j(1\le j\le r)$, which are linearly inde-*

pendent over $\mathbb{R}$, *be in* $\mathscr{P}_2$. *Then the zeta function* $\zeta(s;\{W_1,\cdots,W_r\},\xi)$ *is analytically continued to a meromorphic function in the whole complex plane which is holomorphic at* $s=1-m\,(m=1,2,\cdots)$. *Moreover, the special value at* $s=1-m$ *is given by*

$$\begin{aligned}\zeta(1-m;\{W_1,\cdots,W_r\},\xi) =&C(m)\int_{\Gamma_\epsilon}dt\int_{I_\epsilon(1)}du\int_0^\pi d\theta\cdot t^{1-2m}u^{-m-1/2}(1-u)\\ &\times\Phi((t,u,\theta);\{W_1,\cdots,W_r,\},\xi).\end{aligned}$$

Proof. Lemma 5.2.3 implies that the integral

$$t^2\int_0^\pi \Phi((t,u,\theta);\{W_1,\cdots,W_r\},\xi)d\theta$$

is a holomorphic function of (t,u) in the region $D_\delta(\infty)\times D_\delta(1)$. Thus one obtains, for a sufficiently small ϵ,

$$\begin{aligned}\zeta(s;\{W_1,\cdots,W_r\},\xi) =&\frac{1}{\Gamma_2(s)(e(2s)-1)(e(s-3/2)-1)}\int_{I_\epsilon(\infty)}dt\int_{I_\epsilon(1)}du\int_0^\pi d\theta\\ &\times t^{2s-1}u^{s-3/2}(1-u)\Phi((t,u,\theta);\{W_1,\cdots,W_r\},\xi).\end{aligned}$$

Since the integral on the right side of the equality is absolutely convergent, this identity gives the analytic continuation of $\zeta(s;\{W_1,\cdots,W_r\},\xi)$ to a meromorphic function of s in the whole complex plane. Passing to the limit as $s\to 1-m$, we get the identity in Proposition 5.2.7. □

In view of Proposition 5.2.1 and Proposition 5.2.2, we need only partial zeta functions of the form

$$\begin{aligned}&\zeta(s;\{V_1,V_2,V_3\},(\xi_1,\xi_2,\xi_3)),\\ &\zeta(s;\{V_i,V_j\},(\xi_i,\xi_j))\quad(1\le i<j\le 3),\\ &\zeta(s;\{V_2\},\xi).\end{aligned}$$

Now we discuss the evaluation of

$$\zeta_P(1-m;\{V_1,V_2,V_3\},(\xi_1,\xi_2,\xi_3)),$$
$$\zeta_P(1-m;\{V_j,V_3\},(\xi_j,\xi_3)) \quad (j=1,2),$$
$$\zeta(1-m;\{V_1,V_2\},(\xi_1,\xi_2)),$$
$$\zeta(1-m;\{V_2\},\xi),$$

as a continuation of Proposition 5.2.6, Proposition 5.2.7. For each triple (k_1,k_2,k_3) of integers such that $k_1,k_2\geq 0, k_3\geq 1$, and $k_1+k_2+k_3=2m+1$, we define a number $\Lambda_{(k_1,k_2,k_3)}$ by putting

$$\Lambda_{(k_1,k_2,k_3)}=\frac{1}{2\pi}\int_{I_\epsilon(1)}du\int_0^\pi d\theta\cdot u^{-m-1/2}(1-u)\prod_{j=1}^{3}\lambda((u,\theta),V_j)^{k_j-1}, \tag{5.2.22}$$

where the integral in the right side is independent of the choice of a small positive number ϵ.

Proposition 5.2.8. *Let $\xi=(\xi_1,\xi_2,\xi_3)$ be a triple of positive numbers, and $m\in\mathbb{N}$. Then we have*

$$\zeta_P(1-m;\{V_1,V_2,V_3\},\xi)=-4\pi^2 iC(m)\sum_{k_1,k_2,k_3}{}' \left\{\prod_{j=1}^{3}\frac{B_{k_j}(\xi_j)}{k_j!}\right\}\Lambda_{(k_1,k_2,k_3)},$$

where k_1,k_2,k_3 run over all integers satisfying the conditions $k_1,k_2\geq 0,k_3\geq 1$, and $k_1+k_2+k_3=2m+1$.

Proof. We take δ sufficiently small so that Lemma 5.2.3 for V_1,V_2 and Lemma 5.2.4 for V_3 hold. Then we get the following power series expansion, if $|t|<\delta, u\in D_\delta(1)$,

$$\Phi_P((t,u,\theta);\{V_1,V_2,V_3\},\xi)=\sum_{k_1,k_2,k_3}\prod_{j=1}^{3}\left\{\frac{B_{k_j}(1-\xi_j)}{k_j!}\lambda((u,\theta),V_j)^{k_j-1}\right\}\cdot t^{k_1+k_2+k_3-3},$$

where k_1,k_2,k_3 run over all integers satisfying $k_1,k_2\geq 0,k_3\geq 1$. Applying Proposition 5.2.6, the residue theorem, and noting that $B_k(1-\xi) = (-1)^k B_k(\xi)\,(k \geq 0)$, we obtain the expression for $\zeta_P(1-m; \{V_1, V_2, V_3\}, \xi)$ in Proposition 5.2.8. □

In exactly the same manner as in the proof of Proposition 5.2.8, one can evaluate the special values

$$\begin{aligned} &\zeta_P(1-m; \{V_j, V_3\}, (\xi_j, \xi_3)) \quad (j = 1, 2), \\ &\zeta(1-m; \{V_1, V_2\}, (\xi_1, \xi_2)), \\ &\zeta(1-m; \{V_2\}, \xi), \end{aligned}$$

so we omit the proof of the following proposition.

Proposition 5.2.9. *Let $\xi, \xi_j (j = 1, 2, 3)$ be positive numbers and $m \in \mathbb{N}$. Then the following expressions hold.*

(a) $$\zeta_P(1-m; \{V_1, V_3\}, (\xi_1, \xi_3)) = 4\pi^2 i C(m) \sum_{k_1, k_3}{}' \frac{B_{k_1}(\xi_1) B_{k_3}(\xi_3)}{k_1!\, k_3!} \Lambda_{(k_1, 0, k_3)},$$

where k_1, k_3 run over all integers with $k_1 \geq 0, k_3 \geq 1, k_1 + k_3 = 2m+1$.

(b) $$\zeta_P(1-m; \{V_2, V_3\}, (\xi_2, \xi_3)) = 4\pi^2 i C(m) \sum_{k_2, k_3}{}' \frac{B_{k_2}(\xi_1) B_{k_3}(\xi_3)}{k_2!\, k_3!} \Lambda_{(0, k_2, k_3)},$$

where k_2, k_3 run over all integers with $k_2 \geq 0, k_3 \geq 1, k_2 + k_3 = 2m+1$.

(c) $$\zeta(1-m; \{V_1, V_2\}, (\xi_1, \xi_2)) = 4\pi^2 i C(m) \sum_{k_1, k_2}{}' \frac{B_{k_1}(\xi_1) B_{k_2}(\xi_3)}{k_1!\, k_2!} \Lambda_{(k_1, k_2, 0)},$$

where k_1, k_2 run over all integers with $k_1, k_2 \geq 0, k_1 + k_2 = 2m+1$.

(d) $$\zeta(1-m; \{V_2\}, \xi) = -4\pi^2 i C(m) \frac{B_{2m-1}(\xi)}{(2m-1)!} \Lambda_{(1, 2m-1, 1)}.$$

To complete the evaluation of the special values of zeta functions above, we have to study some properties of the numbers $\Lambda_{(k_1, k_2, k_3)}$.

Proposition 5.2.10. *Let k_1, k_2, k_3 be integers with $k_1, k_2 \geq 0, k_3 \geq 1$, and $k_1 + k_2 + k_3 = 2m+1\,(m \in \mathbb{N})$. If k_1, k_2, k_3 satisfy one of the following two conditions,* *then* $\Lambda_{(k_1,k_2,k_3)}$ *is a rational number.*

(i) k_1, k_2, k_3 *are positive integers,*

(ii) $k_2 = 0$ *and* k_1, k_3 *are positive integers.*

Proof. A straightforward calculation shows that

$$\begin{cases} \lambda((u,\theta),V_1) = 1+u, \\ \lambda((u,\theta),V_2) = 2\{1+u+(1-u)\sin\theta\cos\theta\}, \\ \lambda((u,\theta),V_3) = u\sin^2\theta + \cos^2\theta. \end{cases} \tag{5.2.23}$$

Changing the variable by $\cot\theta = x$ in (21), we obtain

$$\Lambda_{(k_1,k_2,k_3)} = \int_{I_\epsilon(1)} u^{-m-1/2}(1-u)P_{k_1,k_2,k_3}(u)du, \tag{5.2.24}$$

where ϵ is taken to be sufficiently small and

$$\begin{aligned} P_{k_1,k_2,k_3}(u) = {} & \frac{1}{2\pi}(1+u)^{k_1-1}2^{k_2-1} \\ & \times \int_{\mathbb{R}} \left\{1+u+(1-u)\frac{x}{1+x^2}\right\}^{k_2-1} \left(\frac{x^2+u}{1+x^2}\right)^{k_3-1} \frac{dx}{1+x^2}. \end{aligned} \tag{5.2.25}$$

If k_1, k_2, k_3 satisfy condition (i), then from the binomial formula, we get

$$\begin{aligned} P_{k_1,k_2,k_3}(u) = {} & \frac{1}{2\pi}(1+u)^{k_1-1}2^{k_2-1} \sum_{i=0}^{\lfloor \frac{k_2-1}{2} \rfloor} \sum_{j=0}^{k_3-1} (-1)^j \binom{k_2-1}{2i}\binom{k_3-1}{j} \\ & \times (1+u)^{k_2-1-2i}(1-u)^{2i+j} \cdot I_{2i,2i+j+1}, \end{aligned} \tag{5.2.26}$$

where

$$I_{n,m} = \int_{\mathbb{R}} \frac{x^n}{(1+x^2)^m} dx.$$

By the following recursive relations,

$$\begin{cases} I_{n,m} = I_{n-2,m-1} - I_{n-2,m}, \\ I_{0,m} = \dfrac{2m-3}{2m-2} I_{0,m-1} \quad (n, m \geq 2), \end{cases} \tag{5.2.27}$$

and noting that $I_{0,1} = \pi$, we easily see that if k_1, k_2, k_3 satisfy condition (i), then $P_{k_1,k_2,k_3}(u)$ is a polynomial of u with rational coefficients. Noting that

$$\int_{I_\epsilon(1)} u^{k-1/2} du = -\frac{4}{2k+1} \qquad \text{for any } k \in \mathbb{Z}, \tag{5.2.28}$$

it follows from (5.2.24) that the value of $\Lambda_{(k_1,k_2,k_3)}$ is a rational number whenever k_1, k_2, k_3 satisfy condition (i).

Suppose $k_2 = 0$ and $k_1, k_3 \geq 1$. Set

$$Q(u) = 3u^2 + 10u + 3,$$

and

$$\omega(u) = \frac{-(1-u) + i\sqrt{Q(u)}}{2(1+u)} \qquad (0 \leq u \leq 1).$$

Observe that $\omega(u), \overline{\omega(u)}$ are the distinct roots of the quadratic equation: $(1+u)x^2 + (1-u)x + 1 + u = 0$. We write ω for $\omega(u)$. Applying the residue theorem in computing the integral in (5.2.25), we get

$$P_{k_1,0,k_3} = \frac{i}{2} \cdot \frac{(1+u)^{k_1-1}}{(1+u)(\omega - \overline{\omega})} \left(1 - \frac{1-u}{1+\omega^2}\right)^{k_3-1} + R_{k_1,k_3}(u), \tag{5.2.29}$$

where we put

$$\begin{aligned} R_{k_1,k_3}(u) = \frac{i}{2}&(1+u)^{k_1-1} \\ &\times \operatorname*{Res}_{x=i} \{(1+u)x^2 + (1-u)x + (1+u)\}^{-1} \left(1 - \frac{1-u}{1+x^2}\right)^{k_3-1}. \end{aligned} \tag{5.2.30}$$

Since the residue at $x = i$ of the function

$$\{(1+u)x^2 + (1-u)x + (1+u)\}^{-1} \left(\frac{1-u}{1+x^2}\right)^l \quad (l \in \mathbb{Z},\, l \geq 0),$$

is a polynomial of u with coefficients in the Gaussian field $\mathbb{Q}(i)$, so is $R_{k_1,k_3}(u)$. Therefore, the real part of $R_{k_1,k_3}(u)$ is a polynomial of u with rational coefficients. An elementary calculation shows us that

$$\begin{aligned} &(1+u)(\omega - \overline{\omega}) = i\sqrt{Q(u)}, \quad 1 + \omega^2 = -\frac{1-u}{1+u}\omega, \quad \omega\overline{\omega} = 1, \text{and} \\ &1 - \frac{1-u}{1+\omega^2} = \frac{1+u-i\sqrt{Q(u)}}{2}. \end{aligned}$$

Then, we get, by (5.2.29),

$$P_{k_1,0,k_3} = 2^{-k_3}(1+u)^{k_1-1}\left\{1+u-i\sqrt{Q(u)}\right\}^{k_3-1} Q(u)^{-1/2} + R_{k_1,k_3}(u). \tag{5.2.31}$$

Observe that the function $u^{-m+1}(1+u)^{2m-2-2j}Q(u)^j$ $(0 \leq j \leq m-1)$ is invariant under the transformation $u \to 1/u$, and is therefore a polynomial of $(u + 1/u)$ of degree $m - 1$. Thus we can write

$$u^{-m+1}(1+u)^{2m-2-2j}Q(u)^j = \sum_{k=0}^{m-1} b_{j,m,k}(u^k + u^{-k}) \quad (0 \leq j \leq m-1) \tag{5.2.32}$$

with some $b_{j,m,k} \in \mathbb{Q}$. We set, for a sufficiently small $\epsilon > 0$,

$$g(s) = \int_{I_\epsilon(1)} u^{s-1/2}Q(u)^{-1/2},$$

where we take the branch of $Q(u)^{1/2}$ so that $Q(u)^{1/2} > 0$, if $u \in \mathbb{R}$. The integral in the right side is independent of the choice of ϵ and converges for arbitrary $s \in \mathbb{C}$.

Consequently, $g(s)$ is an entire function of s. We define a sequence $\{\alpha_n\}$ by putting

$$\alpha_n = g(n) - g(-n) \qquad (n = 0, 1, \cdots).$$

Lemma 5.2.11 [1]. *The sequence* $\{\alpha_n\}$ *satisfies the following recursive formula*

$$3(n-1/2)\alpha_n + 10(n-1)\alpha_{n-1} + 3(n-3/2)\alpha_{n-2} = -16 \quad (n \geq 2)$$

with $\alpha_0 = 0, \alpha_1 = -16/3$. *Consequently, all* α_n *are rational numbers.*

Proof. If $\mathrm{Re}(s) > -1/2$, we get

$$\begin{aligned} 3g(s+1) + 5g(s) &= \int_{I_\epsilon(1)} u^{s-1/2}(3u+5)Q(u)^{-1/2}du \\ &= (e(s-1/2)-1)\int_0^1 u^{s-1/2}\frac{d}{du}Q(u)^{1/2}du. \end{aligned}$$

Then integration by parts implies that, if $\mathrm{Re}(s) > 1/2$,

$$3g(s+1) + 5g(s) = -4(1+e(s)) - (s-1/2)\int_{I_\epsilon(1)} u^{s-3/2}Q(u)^{1/2}du.$$

As $Q(u)^{1/2} = Q(u) \cdot Q(u)^{-1/2}$, we get the following functional equation:

$$3(s+1/2)g(s+1) + 10sg(s) + 3(s-1/2)g(s-1) = -4(1+e(s)), \tag{5.2.33}$$

which is valid for all $s \in \mathbb{C}$ by analytic continuation. Putting $s = 0$, we get $\alpha_1 = g(1) - g(-1) = -16/3$. Moreover, if we substitute $s = m-1$ and $s = 1-m$, respectively in (5.2.33), and add both of the equalities obtained, we get the recursive formula in Lemma 5.2.11. □

We continue the proof of Proposition 5.2.10. From (5.2.25), we see that $P_{k_1,0,k_3}(u)$ is real-valued, if $0 \leq u \leq 1$. Hence, it follows from (5.2.31) that $P_{k_1,0,k_3}(u)$ is a polynomial of u with rational coefficients plus a $\mathbb{Q}$-linear sum of the functions $(1+u)^{2m-1-2j}Q(u)^{j-1/2}$ $(0 \leq j \leq m-1)$. Moreover, from (5.2.32), it follows that

$$\int_{I_\epsilon(1)} u^{-m-1/2}(1-u)(1+u)^{2m-1-2j}Q(u)^{j-1/2}du$$
$$= -2b_{j,m,0}\alpha_1 + \sum_{k=1}^{m-1} b_{j,m,k}(\alpha_{k-1} - \alpha_{k+1}) \quad (0 \leq j \leq m-1),$$

which is a rational number by virtue of Lemma 5.2.11. Thus, by (5.2.24), (5.2.31), we have

$$\begin{aligned} \varLambda_{(k_1,0,k_3)} = 2^{-k_3} \sum_{j=0}^{\lfloor \frac{k_3-1}{2} \rfloor} (-1)^j \binom{k_3-1}{2j} \\ \times \left(-2b_{j,m,0}\alpha_1 + \sum_{k=1}^{m-1} b_{j,m,k}(\alpha_{k-1} - \alpha_{k+1})\right) \\ + \int_{I_\epsilon(1)} u^{-m-1/2}(1-u)\,\mathrm{Re}(R_{k_1,k_3}(u))du. \end{aligned} \tag{5.2.34}$$

Taking (5.2.28) into account, it follows that $\varLambda_{(k_1,0,k_3)}$ $(k_1, k_3 \geq 1)$ is a rational number. □

Remark. For triples (k_1, k_2, k_3) not satisfying the conditions of Proposition 5.2.10, it will be hard to compute $\varLambda_{(k_1,k_2,k_3)}$ in an elementary manner. However, as we shall see in Section 5.2.4, we do not need the explicit values of them.

To evaluate the special value at $s = 0$ of $L_2(s, \psi_{H,p})$, we only need the following explicit values of $\varLambda_{(k_1,k_2,k_3)}$.

Proposition 5.2.12. *We have*

$$\varLambda_{(1,1,1)} = 4, \quad \varLambda_{(2,0,1)} = 8/3, \quad \varLambda_{(1,0,2)} = 4/3.$$

Proof. The first identity is straightforward. Then, by (5.2.34), we obtain $\varLambda_{(2,0,1)} =$

$2\mathit{\Lambda}_{(1,0,2)} = -b_{0,1,0}\alpha_1 = 8/3.$ □

5.2.3 Integral Representations of Partial Zeta Functions II

We keep the notation used in 5.2.2. We shall study the analytic continuations of the functions $\zeta_S(s, \{V_1, V_2, V_3\}, (\xi_1, \xi_2))$, $\zeta_S(s, \{V_j, V_3\}, \xi)\,(j = 1, 2)$, and determine the first and second terms of the Laurant expansions at $s = 1 - m\,(m \in \mathbb{Z})$ of them.

For simplicity, we write λ_j for $\lambda((u, \theta), V_j)\,(j = 1, 2, 3)$, if there is no fear of confusion. We easily see from (5.2.14) that, for positive numbers ξ_1, ξ_2, ξ, and for $\mathrm{Re}(s) > 3/2$,

(5.2.35)
$$\begin{aligned} \zeta_S(s, \{V_1, V_2, V_3\}, (\xi_1, \xi_2)) &= \frac{1}{\mathit{\Gamma}_2(s)(e(2s) - 1)} I(s; (\xi_1, \xi_2)) \\ \zeta_S(s, \{V_j, V_3\}, \xi) &= \frac{1}{\mathit{\Gamma}_2(s)(e(2s) - 1)} I(s; \xi) \qquad (j = 1, 2), \end{aligned}$$

where we put

$$I(s; (\xi_1, \xi_2)) = \int_{I_\epsilon(\infty)} dt \int_0^1 du \int_0^\pi d\theta \cdot t^{2s-2} u^{s-3/2} (1 - u) \frac{1}{\lambda_3} \prod_{j=1}^{2} \phi(t\lambda_j; 1 - \xi_j),$$

$$I(s; \xi) = \int_{I_\epsilon(\infty)} dt \int_0^1 du \int_0^\pi d\theta \cdot t^{2s-2} u^{s-3/2} (1 - u) \frac{1}{\lambda_3} \phi(t\lambda_j; 1 - \xi_j) \quad (j = 1, 2),$$

where ϵ is taken sufficiently small. The absolute convergence for $\mathrm{Re}(s) > 3/2$ of the integrals above is easily verified by Lemma 5.2.3. We shall first integrate with respect to θ. Changing the variable by $\cot\theta = x\,(-\infty < x < \infty)$, we get, by (5.2.23),

(5.2.36)
$$\lambda_1 = 1 + u, \quad \lambda_2 = 2\left\{1 + u + (1 - u)\frac{x}{1 + x^2}\right\}, \quad \lambda_3 = \frac{u + x^2}{1 + x^2}.$$

As is easily seen, for each positive number $\beta < 1$, there exists a positive number $\delta = \delta(\beta)$ such that $\phi(t(1 + z); 1 - \xi)\,(\xi > 0)$, as a function of t, z, is holomorphic in the region $\{(t, z) \in \mathbb{C}^2 \mid t \in D_\delta(\infty),\, t \neq 0,\, |z| \leq \beta\}$. Then, $\phi(t(1 + z); 1 - \xi)$, as a function of z, has the power series expansion

$$\phi(t(1+z);1-\xi) = \sum_{k=0}^{\infty} \frac{\phi^{(k)}(t;1-\xi)t^k}{k!} \cdot z^k \qquad (t \in D_\delta, t \neq 0, |z| \leq \beta). \tag{5.2.37}$$

It follows from (5.2.37) that, if $t \in D_\delta(\infty), t \neq 0$, and $0 \leq u \leq 1$,

$$\phi(t\lambda_2;1-\xi) = \sum_{k=0}^{\infty} \frac{\phi^{(k)}(2t(1+u);1-\xi)(2t)^k(1-u)^k}{k!} \cdot \left(\frac{x}{1+x^2}\right)^k, \tag{5.2.38}$$

where δ is taken sufficiently small. We then define a function $\mathscr{H}_k(u)$ for each non-negative integer k:

$$\mathscr{H}_k(u) = (1-u)^k \int_{\mathbb{R}} \frac{1}{x^2+u} \left(\frac{x}{1+x^2}\right)^k dx \qquad (u > 0). \tag{5.2.39}$$

Obviously, we have $\mathscr{H}_k(u) = 0$ for any odd k. Applying the residue theorem in calculating the integral in (2.3.5), we can divide $\mathscr{H}_k(u)\,(k \in \mathbb{Z},\ k \geq 0)$ into two parts as follows:

$$\mathscr{H}_{2k}(u) = \pi u(-u)^k + \pi \mathscr{A}_{2k}(u), \tag{5.2.40}$$

where

$$\mathscr{A}_{2k}(u) = 2i(1-u)^{2k} \operatorname*{Res}_{x\,=\,i} \left(\frac{1}{x^2+u} \left(\frac{x}{1+x^2}\right)^{2k} \right). \tag{5.2.41}$$

An elementary calculation shows that $\mathscr{A}_0(u) = 0, \mathscr{A}_2(u) = (1+u)/2$. Moreover, we observe that each $\mathscr{A}_{2k}(u)$ is a polynomial with rational coefficients. We set, for $\xi > 0$,

$$F_1(t,u;1-\xi) = \sum_{k=0}^{\infty} \frac{\phi^{(2k)}(2t(1+u);1-\xi)}{(2k)!} \cdot (2t)^{2k}(-u)^k,$$
$$F_2(t,u;1-\xi) = \sum_{k=0}^{\infty} \frac{\phi^{(2k)}(2t(1+u);1-\xi)}{(2k)!} \cdot (2t)^{2k}\mathscr{A}_{2k}(u).$$

We shall discuss the convergence and regularity of $F_j(t,u;1-\xi)$ $(j=1,2)$ as functions of t,u. For that purpose, some preparations will be needed. We put, for $a\in\mathbb{R}$, and $j\in\mathbb{N}$,

$$\phi_j(t;a)=\frac{e^{ta}}{(e^t-1)^j}.$$

Lemma 5.2.13 [1]. *Let $n\in\mathbb{N}$ and $a<1$. If we write, in a unique way,*

$$\phi^{(n)}(t;a)=\sum_{j=0}^{n}\lambda_{j,n}(a)\phi_{j+1}(t;a)\qquad \text{with some } \lambda_{j,n}(a)\in\mathbb{R}, \tag{5.2.42}$$

then, we have $(-1)^n\lambda_{j,n}(a)>0$ for each j $(0\le j\le n)$.

Proof. Differentiating the both sides of (5.2.42) with respect to t, and using the identity $\phi'_{j+1}(t;a)=(a-j-1)\phi_{j+1}(t;a)-(j+1)\phi_{j+2}(t;a)$, we get the recursive relations:

$$\begin{cases}\lambda_{0,n+1}(a)=(a-1)\lambda_{0,n}(a),\\ \lambda_{j,n+1}(a)=(a-j-1)\lambda_{j,n}(a)-j\lambda_{j-1,n}(a),\\ \lambda_{n+1,n+1}(a)=-(n+1)\lambda_{n,n}(a).\end{cases} \tag{5.2.43}$$

In the case of $n=1$, we have, trivially, $(-1)\lambda_{j,1}(a)>0\,(j=0,1)$. Thus the assertion follows by induction on n from (5.2.43). □

Taking the k-th derivative of (5.2.11), one gets, if $|t|<2\pi$,

$$\frac{\phi^{(k)}(t;a)}{k!}=(-1)^k t^{-k-1}+\sum_{n=k+1}^{\infty}\frac{B_n(a)}{n!}\binom{n-1}{k}t^{n-1-k}. \tag{5.2.44}$$

For $x\in\mathbb{R}$, $[x]$ denotes the largest integer less than or equal to x.

Lemma 5.2.14 [1]. *Let* $0<\beta<1$ *and* $a\in\mathbb{R}$. *If* $|t|\leq \pi/2, |w|\leq\beta$, *then,*

$$t\sum_{k=0}^{\infty}\frac{\phi^{2k}(t;a)t^{2k}}{(2k)!}\cdot w^k=\frac{1}{1-w}+\sum_{n=1}^{\infty}\frac{B_n(a)}{n!}\left(\sum_{k=0}^{\kappa_n}\binom{n-1}{2k}w^k\right)t^k, \tag{5.2.45}$$

where we put $\kappa_n=[(n-1)/2]$, *and the infinite series of both sides are absolutely convergent. Moreover, the function defined by the infinite series (5.2.45) is a holomorphic function of* t,w *in the region* $\{(t,w)\mid |t|\leq\pi/2,\ |w|\leq\beta\}$.

Proof. By virtue of the fact that $t\phi(t;a)=\sum_{k=0}^{\infty}(B_k(a)/k!\,)t^k$ is absolutely convergent for $|t|<2\pi$, there exists a positive constant C_1 independent of k which satisfies

$$|B_k(a)/k!\,|<C_1(3\pi/2)^{-k}\qquad(k=1,2,\cdots).$$

Then, from (5.2.44), we get

$$\begin{aligned}\sum_{k=0}^{\infty}\left|\frac{\phi^{(2k)}(t;a)t^{2k+1}}{(2k)!}\cdot w^k\right| &\leq \sum_{k=0}^{\infty}|w|^k+C_1\sum_{k=0}^{\infty}\sum_{n=2k+1}^{\infty}\binom{n-1}{2k}|w|^k\left(\frac{2|t|}{3\pi}\right)^n\\ &\leq\frac{1}{1-|w|}+C_1\sum_{n=1}^{\infty}\left(\sum_{k=0}^{\kappa_n}\binom{n-1}{2k}\beta^k\right)\left(\frac{2|t|}{3\pi}\right)^n\\ &\leq\frac{1}{1-|w|}+C_1\sum_{n=1}^{\infty}\left(\frac{4|t|}{3\pi}\right)^n,\end{aligned}$$

where the last infinite series is convergent for $|t|\leq\pi/2$. Thus, the infinite series in both sides of (5.2.45) are absolutely and uniformly convergent for $|t|\leq\pi/2$, $|w|\leq\beta$. In a similar manner, the identity (5.2.45) is easily shown to hold. □

Proposition 5.2.15. *If we take* δ *sufficiently small, then, the infinite series* $2tF_j(t,u;1-\xi)$ $(\xi>0,\ j=1,2)$ *are absolutely and uniformly convergent in the region* $D_\delta(\infty)\times D_\delta(1)$. *Consequently,* $2tF_j(t,u,1-\xi)$ $(j=1,2)$ *are holomorphic in the same region. Moreover,* $2tF_j(t,u,1-\xi)$, *as functions of* t, *have the following power series* *expansions; if* $|t| < \delta, u \in D_\delta(1)$, *then, we have*

$$(5.2.46) \qquad 2tF_1(t,u;1-\xi) = \frac{1+u}{1+3u+u^2} + \sum_{n=1}^{\infty} \frac{B_n(1-\xi)}{n!}\mu_n(u)2^n t^n,$$

$$(5.2.47) \qquad 2tF_2(t,u;1-\xi) = \sum_{n=0}^{\infty} \frac{\mathscr{A}_{2n}}{(1+u)^{2n+1}} + \sum_{n=1}^{\infty} \frac{B_n(1-\xi)}{n!}\nu_n(u)2^n t^n,$$

where we put

$$(5.2.48) \qquad \begin{cases} \mu_n(u) = \displaystyle\sum_{k=0}^{\kappa_n} \binom{n-1}{2k}(1+u)^{n-1-2k}(-u)^k, \\ \nu_n(u) = \displaystyle\sum_{k=0}^{\kappa_n} \binom{n-1}{2k}(1+u)^{n-1-2k}\mathscr{A}_{2k} \quad \left(\kappa_n = \left[\frac{n-1}{2}\right]\right). \end{cases}$$

Proof. First we consider the infinite series $2tF_1(t,u;1-\xi)$. We put $t' = 2t(1+u)$, $w = -u/(1+u)^2$. Then, we get

$$2tF_1(t,u;1-\xi) = \frac{t'}{1+u}\sum_{k=0}^{\infty} \frac{\phi^{(2k)}(t';1-\xi)(t')^{2k}}{(2k)!} \cdot w^k.$$

Let δ_1 be a small positive constant such that $|-u/(1+u)^2| \leq 1/2$ for $u \in D_{\delta_1}(1)$. If $|t| < \pi/8, u \in D_{\delta_1}(1)$, then we get $|t'| < \pi/2, |w| \leq 1/2$. It follows easily from Lemma 5.2.14 that $2tF_1(t,u;1-\xi)$ converges absolutely and is holomorphic in the region $\{(t,u) \mid |t| < \pi/8,\ u \in D_{\delta_1}\}$, and moreover that the power series expansion (5.2.46) holds in the same region. We take δ sufficiently small with $\delta < \delta_1$. Let $|t| \geq \pi/8$, $t \in D_\delta(\infty)$, and $u \in D_\delta$. Set $\tau' = \mathrm{Re}(t')$. Then we may have $\tau' > |t'|/\sqrt{2}$, δ being taken sufficiently small. An elementary observation shows that

$$|\phi_j(t';a)| \leq \phi_j(\tau';a) \qquad (a \in \mathbb{R},\ j = 0,1,\cdots),$$

from which, in addition to Lemma 5.2.13, we get

$$|\phi_j^{(k)}(t';1-\xi)|\leq (-1)^k\phi_j^{(k)}(\tau';1-\xi) \qquad (\xi>0, k=0,1,\cdots).$$

Hence, from Lemma 5.2.13 and the expansion (5.2.37), we see that

$$\begin{aligned}\sum_{k=0}^{\infty}\left|\frac{\phi^{(2k)}(t';1-\xi)(t')^{2k}}{(2k)!}\cdot w^k\right| &\leq \sum_{k=0}^{\infty}\left|\frac{\phi^{(k)}(t';1-\xi)(t')^{k}}{k!}\cdot w^{k/2}\right| \\ &\leq \sum_{k=0}^{\infty}\frac{\phi^{(k)}(\tau';1-\xi)(\tau')^{k}}{k!}\cdot\left(\frac{-|t'|}{\sqrt{2}\tau'}\right)^k \\ &= \phi\left(\tau'-\frac{|t'|}{\sqrt{2}};1-\xi\right).\end{aligned}$$

Thus, $F_1(t,u;1-\xi)$ is absolutely convergent in the region $\{(t,u) \mid |t|\geq \pi/8, t \in D_\delta(\infty), u\in D_\delta(1)\}$. Moreover, we see from the above calculations that $F_1(t,u;1-\xi)$ is uniformly convergent in some small neighborhood of (t,u) contained in the region above. Consequently, $F_1(t,u;1-\xi)$ is holomorphic in that region.

Next we consider the series $2tF_1(t,u;1-\xi)$. We have to estimate $\mathscr{A}_{2k}(u)$ from above. The definition of $\mathscr{A}_{2k}(u)$ implies that

$$\pi\mathscr{A}_{2k}(u) = (1-u)^{2k}\int_{\mathbb{R}}\frac{1}{x^2+u}\left\{\left(\frac{x}{1+x^2}\right)^{2k}-\left(\frac{-u}{(1-u)^2}\right)^k\right\}dx.$$

Letting $s(x)=(x/1+x^2)^2$ for simplicity, we get the expression

(5.2.49) $$\pi\mathscr{A}_{2k}(u) = \int_{\mathbb{R}}\frac{ux^2+1}{(1+x^2)^2}\sum_{j=0}^{k-1}\left\{s(x)(1-u)^2\right\}^{k-1-j}(-u)^j dx,$$

which holds for any $u\in\mathbb{C}$. We take a positive number δ_2 in such a manner that, if $|u|<\delta_2$, then, $|-u/(1-u)^2|<1/16$. Accordingly, $\delta_2\leq 9-4\sqrt{5}=0.0557\cdots$. Using the inequality $s(x) \leq 1/4$, we see easily from (5.2.49) that, if $|u| < \delta_2$,

$$\left|\frac{\mathscr{A}_{2k}(u)}{(1+u)^{2k}}\right| < (1/4)^{k-1}\frac{|1-u|^{2(k-1)}}{|1+u|^{2k}}\sum_{j=0}^{k-1}\left|\frac{-4u}{(1-u)^2}\right|^j$$
$$< (1/4)^{k-1}\frac{1}{(1-\delta_2)^2}\left(\frac{1+\delta_2}{1-\delta_2}\right)^{2(k-1)}\cdot\frac{4}{3}.$$

Observe that we have the inequality $(1+\delta_2)/(2(1-\delta_2)) < 3/5$. Thus it follows that there exists a constant C_2 independent of k such that

(5.2.50) $$\left|\frac{\mathscr{A}_{2k}(u)}{(1+u)^{2k}}\right| < C_2(3/5)^{2k} \quad \text{if } |u| < \delta_2 \quad (k = 0, 1, 2, \cdots).$$

We put $\delta_3 = \delta_2/2$. On the other hand, if $|u| \geq \delta_2$ and $u \in D_{\delta_3}(1)$, then,

$$|1+u| \geq 1 + \mathrm{Re}(u) > 1 + \delta_3, \quad |1-u| \leq 1, \quad \text{and} \quad |u| \leq 1.$$

Thus the inequalities above and (5.2.49) imply that, for any non-negative integer k,

(5.2.51) $$\left|\frac{\mathscr{A}_{2k}(u)}{(1+u)^{2k}}\right| < \frac{4}{3}\left(\frac{1}{1+\delta_3}\right)^{2k} \qquad \text{if } |u| \geq \delta_2 \text{ and } u \in D_{\delta_3}(1).$$

Putting $t' = 2t(1+u)$ as before, we get

(5.2.52) $$2tF_2(t, u; 1-\xi) = \frac{t'}{1+u}\sum_{k=0}^{\infty}\frac{\phi^{(2k)}(t'; 1-\xi)(t')^{2k}}{(2k)!}\cdot\frac{\mathscr{A}_{2k}}{(1+u)^{2k}}.$$

With the help of (5.2.50), (5.2.51), we can prove that the right side of (5.2.52) is absolutely convergent if $|t| < \pi/8$, $u \in D_{\delta_3}(1)$, and we obtain, similarly as in the proof of (5.2.46), the identity (5.2.47). Since $\mathscr{A}_{2k}$ is a polynomial of u, we easily see from the expression (5.2.47) that $2tF_2(t, u; 1-\xi)$ is holomorphic in the region $\{(t, u) \mid |t| < \pi/8,\ u \in D_{\delta_3}\}$. The rest of the assertions for $2tF_2(t, u; 1-\xi)$ can be verified in the same manner as in the case of $2tF_1(t, u; 1-\xi)$ by using the inequalities

(5.2.50), (5.2.51). □

We are now in position to integrate with respect to θ. We take δ sufficiently small so that the identity (5.2.38) and Proposition 5.2.15 simultaneously hold. Then, taking the identities (5.2.38), (5.2.39), and (5.2.40) into account, noting that $|x/(1+x^2)| \leq 1/2$, we obtain

$$\int_0^\pi \frac{1}{\lambda_3}\phi(t\lambda_2; 1-\xi)d\theta = \pi u^{-1/2} F_1(t,u;1-\xi) + \pi F_2(t,u;1-\xi) \qquad (t \in D_\delta(\infty),\ 0 < u \leq 1). \tag{5.2.53}$$

We set, for positive numbers ξ_1, ξ_2, ξ,

$$\begin{cases} \Phi^{(0)}(t,u;\xi_1,\xi_2) = \phi(t(1+u); 1-\xi_1) F_1(t,u;1-\xi_2), \\ \Phi^{(1)}(t,u;\xi) = \phi(t(1+u); 1-\xi_1), \\ \Phi^{(2)}(t,u;\xi) = F_1(t,u;1-\xi_2), \end{cases} \tag{5.2.54}$$

$$\begin{cases} \Psi^{(0)}(t,u;\xi_1,\xi_2) = \phi(t(1+u); 1-\xi_1) F_2(t,u;1-\xi_2), \\ \Psi^{(1)}(t,u;\xi) = 0, \\ \Psi^{(2)}(t,u;\xi) = F_2(t,u;1-\xi_2). \end{cases} \tag{5.2.55}$$

Let $\Phi(t,u)$ (resp. $\Psi(t,u)$) be one of the three functions in (5.2.54) (resp. in (5.2.55)). We set

$$\begin{aligned} J(s;\Phi) &= \int_{I_\epsilon(\infty)} dt \int_{I_\epsilon(1)} du \cdot t^{2s-2} u^{s-2} (1-u)\,\Phi(t,u), \\ K(s;\Psi) &= \int_{I_\epsilon(\infty)} dt \int_{I_\epsilon(1)} du \cdot t^{2s-2} u^{s-3/2} (1-u)\,\Psi(t,u), \end{aligned} \tag{5.2.56}$$

where ϵ is taken sufficiently small with $\epsilon < \delta$, δ being the same as in (5.2.53). Then, by virtue of Proposition 5.2.15 and its proof, the integrals $J(s;\Phi), K(s;\Phi)$ are independent of the choice of ϵ and absolutely convergent for arbitrary $s \in \mathbb{C}$. Moreover, they are entire functions of s. For convenience, we write

$$\text{(5.2.57)} \qquad \begin{cases} J(s;(\xi_1,\xi_2) = J(s;\Phi^{(0)}(t,u;\xi_1,\xi_2)), \\ J_j(s;(\xi) = J(s;\Phi^{(j)}(t,u;\xi)) & (j=1,2), \\ K(s;(\xi_1,\xi_2) = K(s;\Phi^{(0)}(t,u;\xi_1,\xi_2)), \\ K_j(s;(\xi) = K(s;\Phi^{(j)}(t,u;\xi)) & (j=1,2). \end{cases}$$

Trivially, $K_1(s;\xi) = 0$. Then, using (5.2.53), we obtain the following convenient expressions for $I(s;(\xi_1,\xi_2))$, $I_j(s;\xi)\,(j=1,2)$ by the integrals (5.2.57) over contour paths $I_\epsilon(\infty), I_\epsilon(1)$.

Proposition 5.2.16. *Let ξ_1, ξ_2, ξ be positive numbers. We have*

$$I(s;(\xi_1,\xi_2)) = \frac{\pi}{e(s)-1} J(s;(\xi_1,\xi_2)) + \frac{\pi}{e(s-3/2)-1} K(s;(\xi_1,\xi_2)),$$
$$I_j(s;\xi) = \frac{\pi}{e(s)-1} J_j(s;\xi) + \frac{\pi}{e(s-3/2)-1} K_j(s;\xi) \qquad (j=1,2),$$

which give the analytic continuation to meromorphic functions of s in the whole complex plane.

The following corollary is an immediate consequence of Proposition 5.2.16 and (5.2.35).

Corollary to Proposition 5.2.16. *The functions*

$$\zeta_S(s;\{V_1,V_2,V_3\},(\xi_1,\xi_2)),$$
$$\zeta_S(s;\{V_j,V_3\},\xi)\,(j=1,2)$$

can be continued analytically to meromorphic functions of s in the whole complex plane.

From Proposition 5.2.16, the Laurent expansions at $s = 1 - m\,(m \in \mathbb{N})$ of $I(s;(\xi_1,\xi_2))$, $I_j(s;\xi)\,(j=1,2)$ are given as follows:

(5.2.58)
$$\begin{aligned} I(s;(\xi_1,\xi_2)) = \frac{J(1-m;(\xi_1,\xi_2))}{2i}(s+m-1)^{-1} + \frac{1}{2i}\{J'(1-m;(\xi_1,\xi_2)) \\ - \pi i J(1-m;(\xi_1,\xi_2)) - \pi i K(1-m;(\xi_1,\xi_2))\} \\ + \text{higher terms of } (s+m-1), \end{aligned}$$

(5.2.59)
$$\begin{aligned} I_j(s;\xi) = \frac{J_j(1-m;\xi)}{2i}(s+m-1)^{-1} + \frac{1}{2i}\{J_j'(1-m;\xi) - \pi i J_j(1-m;\xi) \\ - \pi i K_j(1-m;\xi)\} + \text{higher terms of } (s+m-1) \qquad (j=1,2). \end{aligned}$$

Let $C(m)\,(m \in \mathbb{N})$ be the constant given in Proposition 5.2.6. Then, as a Taylor expansion at $s = 1-m\,(m \in \mathbb{N})$, we have

$$\frac{1}{\Gamma_2(s)(e(2s)-1)} = -2C(m) + \beta_m(s+m-1) + \text{higher terms of } (s+m-1)$$

with some constant $\beta_m \in \mathbb{C}$. Thus, by (5.2.35), (5.2.58), (5.2.59), we get the Laurent expansions at $s = 1-m$ of $\zeta_S(s;\{V_1,V_2,V_3\},(\xi_1,\xi_2)), \zeta_S(s;\{V_j,V_3\},\xi)\,(j=1,2)$:

(5.2.60)
$$\begin{aligned} &\zeta_S(s;\{V_1,V_2,V_3\},(\xi_1,\xi_2)) \\ &\quad = iC(m)J(1-m;(\xi_1,\xi_2))(s+m-1)^{-1} \\ &\quad + \frac{1}{2i}\{(\beta_m + 2\pi i C(m))J(1-m;(\xi_1,\xi_2)) - 2C(m)J'(1-m;(\xi_1,\xi_2)) \\ &\quad + 2\pi i C(m)K(1-m;(\xi_1,\xi_2))\} + \text{higher terms of } (s+m-1), \end{aligned}$$

(5.2.61)
$$\begin{aligned} &\zeta_S(s;\{V_j,V_3\},\xi) \\ &\quad = iC(m)J_j(1-m;\xi)(s+m-1)^{-1} + \frac{1}{2i}\{(\beta_m + 2\pi i C(m))J_j(1-m;\xi) \\ &\quad - 2C(m)J_j'(1-m;\xi) + 2\pi i C(m)K_j(1-m;\xi)\} \\ &\quad + \text{higher terms of } (s+m-1) \qquad (j=1,2). \end{aligned}$$

We shall now evaluate $J(1-m;(\xi_1,\xi_2))$, $J'(1-m;(\xi_1,\xi_2))$, $K(1-m;(\xi_1,\xi_2))$, and so on.

We consider the integral $J(s,\Phi)$ in (5.2.56). Putting $s=1-m$, we get

$$J(1-m;\Phi)=\int_{\Gamma_\epsilon}dt\int_{\Gamma_\epsilon}du\cdot t^{-2m}u^{-m-1}(1-u)\Phi(t,u) \tag{5.2.62}$$

(for the path Γ_ϵ, see Proposition 5.2.6).

Furthermore, differentiating the integrand of $J(s,\Phi)$ with respect to s, we obtain

$$\begin{aligned} &J'(1-m;\Phi)\\ &\quad=\int_{I_\epsilon(\infty)}dt\int_{I_\epsilon(1)}du\cdot t^{-2m}u^{-m-1}(1-u)(2\log t+\log u)\Phi(t,u), \end{aligned} \tag{5.2.63}$$

where the integral is absolutely convergent again by Proposition 5.2.15. For non-negative integers n, we define the functions $\Phi_n(t)$, according to the choice of $\Phi(t,u)$, as follows:

$$\begin{cases} \Phi_n(t)=\dfrac{\phi(t;1-\xi_1)\phi^{(2n)}(2t;1-\xi_2)(-4)^n}{(2n)!} & \text{if } \Phi(t,u)=\Phi^{(0)}(t,u;\xi_1,\xi_2),\\ \Phi_0(t)=\phi(t;1-\xi_1),\quad \Phi_n(t)=0\quad (n\geq 1) & \text{if } \Phi(t,u)=\Phi^{(1)}(t,u;\xi),\\ \Phi_n(t)=\dfrac{\phi^{(2n)}(2t;1-\xi_2)(-4)^n}{(2n)!} & \text{if } \Phi(t,u)=\Phi^{(2)}(t,u;\xi), \end{cases}$$

Moreover, we see from Proposition 5.2.15 that $\Phi(t,u)$ has a Laurent expansion with respect to t:

$$\Phi(t,u)=\sum_{n=-2}^{\infty}b_n(u;\Phi)t^n \qquad \text{if } |t|<\delta,\ u\in D_\delta(1). \tag{5.2.64}$$

Proposition 5.2.17. *Let $m \in \mathbb{N}$. For a sufficiently small ϵ, we have the following:*

$$(i) \quad J(1-m;\Phi) = 2\pi i \int_{\Gamma_\epsilon} \Phi_m(t)dt,$$

$$(ii) \quad J'(1-m;\Phi) = 4\pi i \int_{I_\epsilon(\infty)} \log t \cdot \Phi_m(t)dt - 4\pi i \sum_{j=0}^{m-1} \frac{(2m-2j-1)!}{\{(m-j)!\}^2} \times \int_{\Gamma_\epsilon} t^{2(j-m)} \Phi_j(t)dt + 2\pi i \int_{I_\epsilon(1)} u^{-m-1}(1-u)\log u \cdot b_{2m-1}(u;\Phi)du.$$

Proof. The function $\Phi(t,u)$ is holomorphic if $t \in D_\delta(\infty)$, $t \neq 0$, and $|u| < \delta$, and therefore be expanded in a power series of u as follows:

$$\Phi(t,u) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n \Phi}{\partial u^n}(t,0)u^n. \tag{5.2.65}$$

On the other hand, it is easy to see from the definition of $\Phi_n(t)$ that

$$t^2\Phi(t,u) = t^2 \sum_{n=0}^{\infty} \Phi_n(t(1+u))t^{2n}u^n \qquad (t \in D_\delta(\infty),\ u \in D_\delta(1)). \tag{5.2.66}$$

Since the infinite series in the right side of (5.2.66) is uniformly convergent in $D_\delta(\infty) \times D_\delta(1)$ by Proposition 5.2.15, and each term $t^2\Phi_n(t(1+u))t^{2n}u^n$ is a holomorphic function of t,u, we can differentiate it termwise. Thus, taking the k-th derivative of (5.2.66) with respect to u, we get

$$\frac{\partial^k \Phi}{\partial u^k}(t,u) = \sum_{j=0}^{k}\sum_{n=j}^{\infty} \binom{k}{j} n(n-1)\cdots(n-j+1)\Phi_n^{(k-j)}(t(1+u))t^{k-j+2n}u^{n-j}.$$

Therefore,

$$\frac{\partial^k \Phi}{\partial u^k}(t,0) = \sum_{j=0}^{k} \binom{k}{j} j!\, \Phi_n^{(k-j)}(t)t^{k+j}. \tag{5.2.67}$$

It follows from (5.2.65), (5.2.67) that

$$\begin{aligned}(5.2.68)\qquad &\int_{I_\epsilon(1)} u^{-m-1}(1-u)\,\Phi(t,u)du \\ &= 2\pi i\left\{\frac{1}{m!}\cdot\frac{\partial^m \Phi}{\partial u^m}(t,0)-\frac{1}{(m-1)!}\cdot\frac{\partial^{m-1}\Phi}{\partial u^{m-1}}(t,0)\right\} \\ &= 2\pi i\Bigg(\Phi_m(t)t^{2m} \\ &\quad+\sum_{j=0}^{m-1}\left\{\frac{t^{m+j}}{(m-j)!}\,\Phi_j^{(m-j)}(t)-\frac{t^{m-1+j}}{(m-1-j)!}\,\Phi_j^{(m-1+j)}(t)\right\}\Bigg).\end{aligned}$$

By integrating by parts, the identity

$$(5.2.69)\qquad -s\int_{I_\epsilon(\infty)} t^{s-1}\,\Phi_j^{(m-1-j)}(t)dt=\int_{I_\epsilon(\infty)} t^s\,\Phi_j^{(m-j)}(t)dt$$

holds for each $j\,(0\le j\le m-1)$. Putting $s=j-m$, we get

$$(5.2.70)\qquad (m-j)\int_{I_\epsilon(\infty)} t^{j-m-1}\,\Phi_j^{(m-1-j)}(t)dt=\int_{I_\epsilon(\infty)} t^{j-m}\,\Phi_j^{(m-j)}(t)dt.$$

Therefore, the identities (5.2.62), (5.2.68), and (5.2.70) imply assertion (i). Differentiating both sides of (5.2.69) with respect to s and then, putting $s=j-m$, we get

$$\begin{aligned}(5.2.71)\qquad &(m-j)\int_{I_\epsilon(\infty)} \log t\cdot t^{j-m-1}\,\Phi_j^{(m-1-j)}(t)dt-\int_{I_\epsilon(\infty)} t^{j-m-1}\,\Phi_j^{(m-1-j)}(t)dt \\ &=\int_{I_\epsilon(\infty)} \log t\cdot t^{j-m}\,\Phi_j^{(m-j)}(t)dt\qquad (0\le j\le m-1).\end{aligned}$$

By repeatedly integrating by parts, we see that

$$(5.2.72)\qquad \int_{I_\epsilon(\infty)} t^{j-m-1}\,\Phi_j^{(m-1-j)}(t)dt=\frac{(2m-2j-1)!}{(m-j)!}\int_{\Gamma_\epsilon} t^{2(j-m)}\,\Phi_j(t)dt.$$

Thus, from (5.2.68), (5.2.71), and (5.2.72), we get

$$\begin{aligned}(5.2.73)\qquad & 2\int_{I_\epsilon(\infty)} dt \int_{I_\epsilon(1)} du\cdot t^{-2m}u^{-m-1}(1-u)\log t\cdot \Phi(t,u) \\ &= 4\pi i\int_{I_\epsilon(\infty)} \log t\cdot \Phi_m(t)dt - 4\pi i\sum_{j=0}^{m-1}\frac{(2m-2j-1)!}{\{(m-j)!\}^2}\int_{\Gamma_\epsilon} t^{2(j-m)}\Phi_j(t)dt.\end{aligned}$$

Moreover, we get, using the expansion (5.2.64),

$$\begin{aligned}&\int_{I_\epsilon(\infty)} dt \int_{I_\epsilon(1)} du\cdot t^{-2m}u^{-m-1}(1-u)\log u\cdot \Phi(t,u) \\ &\qquad = 2\pi i\int_{I_\epsilon(1)} u^{-m-1}(1-u)\log u\cdot b_{2m-1}(u;\Phi)du,\end{aligned}$$

which, in addition to (5.2.63), (5.2.73), completes the proof.

□

Let $m\in\mathbb{N}$. For integers k,n with $k,n\geq 0$, $k+n=2m+1$, we define the numbers $\mathscr{M}_{(k-1,n-1)}$ by putting

$$(5.2.74)\qquad \mathscr{M}_{(k-1,n-1)} = \frac{1}{\pi i}\int_{I_\epsilon(1)} \log u\cdot u^{-m-1}(1-u)(1+u)^{k-1}2^{n-1}\mu_n(u)du,$$

where $\mu_n(u)\,(n\geq 1)$ is a polynomial of u given by (5.2.48), and

$$\mu_0(u) = \frac{1+u}{1+3u+u^2}.$$

The numbers $\mathscr{M}_{(k-1,n-1)}$ are independent of the choice of small ϵ.

Lemma 5.2.18. *If $k,n\geq 1$ with $k+n=2m+1$, then $\mathscr{M}_{(k-1,n-1)}$ are rational numbers.*

Proof. It follows from (5.2.48) that

$$(1-u)(1+u)^{k-1}\mu_n(u) = \sum_{j=0}^{\kappa_n}\binom{n-1}{j}(1-u)^{2(m-j)-1}(-u)^j.$$

By the conditions $k \geq 1$ and $k+n = 2m+1$, we have $m > j$ for each $j\,(0 \leq j \leq \kappa_n = [(n-1)/2])$. As is easily shown, the coefficient of the term u^m of the polynomial $(1-u)(1+u)^{2(m-j)-1}(-u)^j$ vanishes. Therefore, the assertion of Lemma 5.2.18 follows from the formula

$$(5.2.75) \qquad \frac{1}{\pi i}\int_{I_\epsilon(1)} \log u \cdot u^p du = \frac{2}{p+1} \qquad \text{for } p \in \mathbb{Z}, p \neq -1).$$

□

In the case of $\Phi(t,u) = \Phi^{(0)}(t,u;(\xi_1,\xi_2))$, Proposition 5.2.17 yields

Proposition 5.2.19. *Let ξ_1, ξ_2 be positive numbers and $m \in \mathbb{N}$. Then,*

$$(i) \quad J(1-m;(\xi_1,\xi_2)) = \frac{2\pi^2(-1)^m}{(2m+1)!}\{B_{2m+1}(\xi_1) + 2^{2m+1}B_{2m+1}(\xi_2)\},$$

$$(ii) \quad J'(1-m;(\xi_1,\xi_2)) = \frac{2^{2m+2}\pi i(-1)^m}{(2m)!}\int_{I_\epsilon(\infty)} \log t \cdot \phi(t;1-\xi_1)\phi^{(2m)}(2t;1-\xi_2)dt$$
$$- 4\pi i \sum_{j=0}^{m-1} \frac{(2m-2j-1)!\,2^{2j}(-1)^j}{\{(m-j)!\,\}^2(2j)!}\int_{\Gamma_\epsilon} t^{2(j-m)} \cdot \phi(t;1-\xi_1)\phi^{(2j)}(2t;1-\xi_2)dt$$
$$+ 2\pi^2 \sum_{n=0}^{2m+1} \frac{B_{2m+1-n}(\xi_1)B_n(\xi_2)}{(2m+1-n)!\,n!} \cdot \mathscr{M}_{(2m-n,n-1)}.$$

Proof. In the proof, we have $\Phi(t,u) = \Phi^{(0)}(t,u;(\xi_1,\xi_2))$ and

$$\Phi_n(t) = \frac{\phi(t;1-\xi_1)\phi^{(2n)}(2t;1-\xi_2)(-4)^n}{(2n)!} \qquad (n = 0,1,2,\cdots).$$

The expansions (5.2.11), (5.2.44) show that the coefficient of the term t^{-1} in the Laurent expansion at $t=0$ of $\Phi_m(t)$ is given by

$$\frac{(-1)^m}{2(2m+1)!}\{B_{2m+1}(1-\xi_1) + 2^{2m+1}B_{2m+1}(1-\xi_2)\}.$$

Thus, by (i) of Proposition 5.2.17, the assertion (i) follows. In view of the expansions

(5.2.11), (5.2.46) of Proposition 5.2.15, the coefficient $b_{2m-1}(u;\Phi)$ in (5.2.64) is given as follows:

$$b_{2m-1}(u;\Phi) = \frac{B_{2m+1}(1-\xi_1)(1+u)^{2m+1}}{2(2m+1)!\,(1+3u+u^2)} + \sum_{n=1}^{2m+1} \frac{B_{2m+1-n}(1-\xi_1)B_n(1-\xi_2)}{(2m+1-n)!\,n!} \times (1+u)^{2m-n}\mu_n(u)2^{n-1}.$$

Therefore, we see from (5.2.74) that

$$\int_{I_\epsilon(1)} u^{-m-1}(1-u)\log u \cdot b_{2m-1}(u;\Phi)du = -\pi i \sum_{n=0}^{2m+1} \frac{B_{2m+1-n}(\xi_1)B_n(\xi_2)}{(2m+1-n)!\,n!} \cdot \mathscr{M}_{(2m-n,n-1)}.$$

Hence, by (ii) of Proposition 5.2.17 together with the above result, we obtain the assertion (ii). □

In the other two cases of $\Phi(t,u)$, we obtain the following.

Proposition 5.2.20. *Let $m \in \mathbb{N}$ and $\xi > 0$. Then,*

(i) $J_j(1-m;1-\xi) = 0 \qquad (j=1,2),$

(ii) $$J_1'(1-m;1-\xi) = \frac{4\pi^2 B_{2m}(\xi)}{m(m!\,)^2} - 2\pi^2 \frac{B_{2m}(\xi)}{(2m)!} \cdot \mathscr{M}_{(2m-1,0)},$$

$$J_2'(1-m;1-\xi) = \frac{2^{2m+1}\pi^2(-1)^m B_{2m}(\xi)}{m(2m)!} + \frac{2^{2m+1}\pi^2 B_{2m}(\xi)}{m} \sum_{j=0}^{m-1} \frac{(-1)^j}{\{(m-j)!\,\}^2(2j)!} - 2\pi^2 \frac{B_{2m}(\xi)}{(2m)!} \cdot \mathscr{M}_{(0,2m-1)}.$$

(iii) *In particular,* $J_j'(1-m;\xi) \in \mathbb{Q} \quad (j=1,2)$.

Proof. If $\Phi(t,u) = \Phi^{(1)}(t,u;\xi) = \phi(t(1+u);1-\xi)$, then, we have $\Phi_0(t) = \phi(t;1-\xi)$, $\Phi_n(t) = 0\,(n \geq 1)$, and $b_{2m-1}(u;\Phi) = B_{2m}(1-\xi)(1+u)^{2m-1}/\{(2m)!\,\}$.

Hence, we immediately see from Proposition 5.2.17 that $J_1(1-m;\xi)=0$, and that

$$J_1'(1-m;\xi) = -\frac{4\pi i(2m-1)!}{(m!)^2}\int_{\Gamma_\epsilon} t^{-2m}\phi(t;1-\xi)dt + \frac{2\pi i B_{2m}(1-\xi)}{(2m)!}\int_{I_\epsilon(\infty)} \log u \cdot u^{-m-1}(1-u)(1+u)^{2m-1}du,$$

from which we get the expression for $J_1'(1-m;\xi)$ in the assertion (ii) (note that $\mu_1(u)=1$). In the case of $\Phi(t,u)=\Phi^{(2)}(t,u;\xi)=F_1(t,u;1-\xi)$, we have

$$\Phi_n(t) = \frac{\phi^{(2n)}(2t;1-\xi)(-4)^n}{(2n)!} \quad \text{and} \quad b_{2m-1}(u;\Phi) = \frac{B_{2m}(1-\xi)}{(2m)!}\cdot \mu_{2m}(u)2^{2m-1}.$$

The expansion (5.2.44) shows that the coefficient of the term t^{-1} in the Laurent expansion at $t=0$ of $\Phi_m(t)$ is equal to 0. Hence, we see from Proposition 5.2.17 that $J_2(1-m;\xi)=0$. Since integration by parts implies that

$$\int_{I_\epsilon(\infty)} t^s\phi^{(2m)}(2t;1-\xi)dt = -\frac{s}{2}\int_{I_\epsilon(\infty)} t^{s-1}\phi^{(2m-1)}(2t;1-\xi)dt,$$

differentiating both sides with respect to s and setting $s=0$, we obtain

$$\int_{I_\epsilon(\infty)} \log t\cdot\phi^{(2m)}(2t;1-\xi)dt = -\frac{1}{2}\int_{I_\epsilon(\infty)} t^{-1}\phi^{(2m-1)}(2t;1-\xi)dt = -\frac{\pi i B_{2m}(1-\xi)}{2m}. \tag{5.2.76}$$

Thus,

$$4\pi i\int_{I_\epsilon(\infty)} \log t\cdot \Phi_m(t)dt = \frac{2^{2m+1}\pi^2(-1)^m B_{2m}(1-\xi)}{m(2m)!}.$$

Moreover, we have, by a usual argument,

$$\int_{\Gamma_\epsilon} t^{2(j-m)}\,\Phi_j(t)dt = \frac{2^{2m}\pi i B_{2m}(1-\xi)}{(2m)!}\cdot\binom{2m-1}{2j}.$$

Hence, by (ii) of Proposition 5.2.17 together with the results above, we obtain the expression for $J_2'(1-m;\xi)$ in the assertion (ii). By assertion (ii) together with Lemma 5.2.18, we obtain assertion (iii). □

Finally, we evaluate the special values at $s = 1-m$ of $K(s;(\xi_1,\xi_2)), K_2(s;\xi)$ (for the definition, see (5.2.57)).

Let $\nu_n(u)\,(n \geq 1)$ be the polynomials defined by (5.2.48). Then, we have $\nu_1(u) = \nu_2(u) = 0, \nu_3(u) = (1+u)/2$, and so on. For convenience, we put

$$\nu_0(u) = \sum_{j=0}^{\infty} \frac{\mathscr{A}_{2j}}{(1+u)^{2j+1}} \qquad \text{for } |u| \text{ sufficiently small.}$$

For any pairs (k,n) of non-negative integers with $k+n = 2m+1\,(m \in \mathbb{N})$, we define the numbers $\mathscr{N}_{(k-1,n-1)}$ by putting

$$\mathscr{N}_{(k-1,n-1)} = \int_{I_\epsilon(1)} u^{-1/2-m}(1-u)(1+u)^{k-1}2^{n-1}\nu_n(u)du, \tag{5.2.77}$$

where ϵ is taken sufficiently small. Observe that the integral on the right side of (5.2.77) is independent of the choice of small ϵ. Then the identity (5.2.28) implies that

$$\mathscr{N}_{(k-1,n-1)} \in \mathbb{Q} \qquad \text{for } k,n \geq 1. \tag{5.2.78}$$

Proposition 5.2.21. *Let $\xi_1, \xi_2, \xi > 0$ and $m \in \mathbb{N}$. Then,*

(i) $$K(1-m;(\xi_1,\xi_2)) = -2\pi i \sum_{n=0}^{2m+1} \frac{B_{2m+1-n}(\xi_1)B_n(\xi_2)}{(2m+1-n)!\,n!} \cdot \mathscr{N}_{(2m-n,n-1)},$$

(ii) $$K_1(1-m;\xi) = 0,$$

(iii) $$K_2(1-m;\xi) = \frac{2\pi i B_{2m}(\xi)}{(2m)!} \cdot \mathscr{N}_{(0,2m-1)}.$$

Proof. Let $\Psi(t,u)$ be one of the functions given in (5.2.55). Recalling the definition

(5.2.56) of $K(s;\,\Psi)$, we have by the Fubini theorem,

$$K(1-m;\Psi) = \int_{I_\epsilon(1)} u^{-m-1/2}(1-u)du \int_{\Gamma_\epsilon} dt \cdot t^{-2m}\,\Psi(t,u).$$

If $\Psi(t,u) = \Psi^{(0)}(t,u;\xi_1,\xi_2)$, then, by the expansions (5.2.11), (5.2.48), we get

$$\int_{\Gamma_\epsilon} t^{-2m}\,\Psi(t,u)dt = 2\pi i \sum_{n=0}^{2m+1} \frac{B_{2m+1-n}(1-\xi_1)B_n(1-\xi_2)}{(2m+1-n)!\,n!} \cdot (1+u)^{2m-n}\nu_{2m}(u)2^{2m-1}.$$

Thus we obtain the assertion (i). The assertion (ii) is clear. If $\Psi(t,u) = \Psi^{(2)}(t,u;\xi)$, then, by the expansion (5.2.48), we get

$$\int_{\Gamma_\epsilon} t^{-2m}\,\Psi(t,u)dt = 2\pi i \frac{B_{2m}(1-\xi)}{(2m)!} \cdot \nu_{2m}(u)2^{2m-1},$$

from which the assertion (iii) immediately follows. □

5.2.4 Determination of Special Values of $L_2(s,\psi_{H,p})$

The aim of this next section is to obtain a formula for all of the special values at non-positive integers of $L_2(s,\psi_{H,p})$, and in particular, to obtain an explicit formula for the special value $L_2(0,\psi_{H,p})$. We keep the notation used in the previous sections.

Suppose that p is an odd prime. For any integer μ prime to p, let $L(\mu)$ be the same as in 5.2.1. Corresponding to $L(\mu)$, we shall define the principal part $\zeta_P(s;C,L(\mu))$ and the singular part $\zeta_S(s;C,L(\mu))$ of the zeta function $\zeta(s;C,L(\mu))$, C being the simplicial cones $C_{123}, C_{j3}\,(j=1,2)$. In view of Proposition 5.2.1, we set

$$
\begin{aligned}
&\zeta_P(s; C_{123}, L(\mu)) = p^{-2s} \sum_{\xi \in \Xi_{H,\mu}} \zeta_P(s; \{V_1, V_2, V_3\}, \xi), \\
&\zeta_S(s; C_{123}, L(\mu)) = p^{-2s} \sum_{\xi \in \Xi_{H,\mu}} \zeta_S(s; \{V_1, V_2, V_3\}, (\xi_1, \xi_2)), \\
&\zeta_P(s; C_{j3}, L(\mu)) = p^{-2s} \sum_{\xi \in \Xi^{(j,3)}_{H,\mu}} \zeta_P(s; \{V_j, V_3\}, (\xi_j, \xi_3)), \\
&\zeta_S(s; C_{j3}, L(\mu)) = p^{-2s} \sum_{\xi \in \Xi^{(j,3)}_{H,\mu}} \zeta_S(s; \{V_j, V_3\}, \xi_j) \qquad (j=1,2),
\end{aligned} \tag{5.2.79}
$$

$$(\xi \text{ being denoted by } (\xi_1, \xi_2, \xi_3)).$$

Proposition 5.2.2 then makes it possible to define the principal and singular parts of the L-function $L_2(s, \psi_{H,p})$. We set

$$
\begin{aligned}
L_{2,P}(s, \psi_{H,p}) &= \sum_{\mu} \psi(\mu) \left\{ \zeta_P(s; C_{123}, L(\mu)) + \frac{1}{2} \sum_{j=1}^{2} \zeta_P(s; C_{j3}, L(\mu)) \right. \\
&\qquad \left. + \frac{1}{2} \zeta(s; C_{12}, L(\mu)) + \frac{\delta_{p,3}}{6} \zeta(s; C_2, L(\mu)) \right\}, \\
L_{2,S}(s, \psi_{H,p}) &= \sum_{\mu} \psi(\mu) \left\{ \zeta_S(s; C_{123}, L(\mu)) + \frac{1}{2} \sum_{j=1}^{2} \zeta_S(s; C_{j3}, L(\mu)) \right\},
\end{aligned} \tag{5.2.80}
$$

where μ runs over 1 and κ, (κ being a non-quadratic residue mod p). Thus we have the obvious identity

$$
L_2(s, \psi_{H,p}) = L_{2,P}(s, \psi_{H,p}) + L_{2,S}(s, \psi_{H,p}). \tag{5.2.81}
$$

We see from Proposition 5.2.6, Proposition 5.2.7, and Corollary to Proposition 5.2.16 that $L_{2,P}(s, \psi_{H,p}), L_{2,S}(s, \psi_{H,p})$ can be continued analytically to meromorphic functions of s in the whole complex plane.

In the rest of this section, we shall discuss the evaluation of special values at

$s = 1 - m\,(m \in \mathbb{N})$ of $L_2(s, \psi_{H,p})$. We note that

$$B_j(\langle a \rangle) = \begin{cases} B_1(1), & \text{if } j = 1,\ a \in \mathbb{Z} \\ P_j(a), & \text{otherwise.} \end{cases} \tag{5.2.82}$$

We will need two lemmas related to the Bernoulli polynomials.

Lemma 5.2.22. *Let $m \in \mathbb{N}$. Then,*

$$\sum_{\xi \in \Xi_{H,\mu}} B_{2m+1}(\xi_j) = 0 \qquad (j = 1, 2, 3),$$

where $\xi_j = \xi^{(j)}_{\alpha,\gamma,\mu}$ is the j-component of $\xi_{\alpha,\gamma,\mu} \in \Xi_{H,p}$.

Proof. From (5.2.82) and the fact that $P_{2m+1}(0) = 0$, we have

$$\begin{aligned}
&(i) && \sum_{(\alpha,\gamma)\in \mathscr{M}(p)} B_{2m+1}(\xi^{(1)}_{\alpha,\gamma,\mu}) = \sum_{\alpha,\gamma(p)} P_{2m+1}\left(\frac{\mu(\alpha^2 - 2\alpha\gamma)}{p}\right),\\
&(ii) && \sum_{(\alpha,\gamma)\in \mathscr{M}(p)} B_{2m+1}(\xi^{(2)}_{\alpha,\gamma,\mu}) = \sum_{\alpha,\gamma(p)} P_{2m+1}\left(\frac{\mu\alpha\gamma}{p}\right),\\
&(iii) && \sum_{(\alpha,\gamma)\in \mathscr{M}(p)} B_{2m+1}(\xi^{(3)}_{\alpha,\gamma,\mu}) = \sum_{\alpha,\gamma(p)} P_{2m+1}\left(\frac{\mu(\gamma^2 - \alpha^2)}{p}\right).
\end{aligned}$$

As $P_{2m+1}(-x) = -P_{2m+1}(x)$ for any $x \in \mathbb{R}$, replacing (α, γ) by $(\alpha+\gamma, \gamma)$, $(-\alpha, \gamma)$, (γ, α) in (i), (ii), (iii) respectively yields the desired result. □

Lemma 5.2.23. *Let $m \in \mathbb{N}$ and let k_2, k_3 be positive integers with $k_2+k_3 = 2m+1$.*

Then,

(*i*) *If* $k_2, k_3 \geq 2$, *then,*

$$\sum_{\xi \in \Xi_{H,p}} B_{k_2}(\xi_2) B_{k_3}(\xi_3) = 0.$$

(*ii*) *If* $k_2 = 1$, *then,*

$$\sum_{\xi \in \Xi,\, \xi_2 \neq 1} B_1(\xi_2) B_{2m}(\xi_3) = 0,$$

(*iii*) *If* $k_3 = 1$, *then,*

$$\sum_{\xi \in \Xi,\, \xi_3 \neq 1} B_{2m}(\xi_2) B_1(\xi_3) = 0,$$

where $\xi_j = \xi^{(j)}_{\alpha,\gamma,\mu}$ *is the* j*-component of* $\xi_{\alpha,\gamma,\mu} \in \Xi_{H,p}$.

Proof. From (5.2.82) and the fact that $P_{2j-1}(0) = 0\,(j \in \mathbb{N})$, we have

$$\sum_{(\alpha,\gamma)\in \mathscr{M}(p)} B_{k_2}\left(\xi^{(2)}_{\alpha,\gamma,\mu}\right) B_{k_3}(\xi^{(3)}_{\alpha,\gamma,\mu}) = \sum_{\alpha,\gamma(p)} P_{k_2}\left(\frac{\mu\alpha\gamma}{p}\right) P_{k_3}\left(\frac{\mu(\gamma^2-\alpha^2)}{p}\right),$$
$$\sum_{(\alpha,\gamma)\in \mathscr{M}(p),\, \alpha\gamma\not\equiv 0} B_1(\xi^{(2)}_{\alpha,\gamma,\mu}) B_{2m}(\xi^{(3)}_{\alpha,\gamma,\mu}) = \sum_{\alpha,\gamma(p)} P_1\left(\frac{\mu\alpha\gamma}{p}\right) P_{2m}\left(\frac{\mu(\gamma^2-\alpha^2)}{p}\right),$$
$$\sum_{(\alpha,\gamma)\in \mathscr{M}(p),\, \gamma^2\not\equiv\alpha^2} B_1(\xi^{(2)}_{\alpha,\gamma,\mu}) B_{2m}(\xi^{(3)}_{\alpha,\gamma,\mu}) = \sum_{\alpha,\gamma(p)} P_{2m}\left(\frac{\mu\alpha\gamma}{p}\right) P_1\left(\frac{\mu(\gamma^2-\alpha^2)}{p}\right).$$

As $P_{2j-1}(-x) = -P_{2j-1}(x)$ for any $x \in \mathbb{R}$, $j \in \mathbb{N}$, replacing (α,γ) by $(-\gamma,\alpha)$ in the three sums above yields the desired result. □

Let $m \in \mathbb{N}$. In the below, let k_1, k_2, k_3 be integers satisfying $k_1, k_2 \geq 0$, $k_3 \geq 1$ and $k_1 + k_2 + k_3 = 2m+1$. For any triple $\xi = (\xi_1, \xi_2, \xi_3)$ of positive numbers, we write, for convenience,

$$B(k_1,k_2,k_3;\xi) = \prod_{j=1}^{3} \frac{B_{k_j}(\xi_j)}{k_j!}, \qquad P(k_1,k_2,k_3;\xi) = \prod_{j=1}^{3} \frac{P_{k_j}(\xi_j)}{k_j!}. \tag{5.2.83}$$

Let μ be any integer prime to p. Viewing Proposition 5.2.8, Proposition 5.2.9, and taking (5.2.82), $P_{2j-1}(0) = 0\,(j \in \mathbb{N})$, and Proposition 3.2.1 into account, we define the numbers $\mathscr{A}_{(k_1,k_2,k_3)}(\mu)$ as follows:

(i) If $k_1, k_2, k_3 \neq 1$, we set

$$\begin{aligned}\mathscr{A}_{(k_1,k_2,k_3)}(\mu) &= -4\pi^2 i C(m) \Lambda_{(k_1,k_2,k_3)} \sum_{\xi \in \Xi_{H,p}} B(k_1,k_2,k_3;\xi)\\ &= -2\pi^2 i C(m) \Lambda_{(k_1,k_2,k_3)} \sum_{\alpha,\gamma(p)} P(k_1,k_2,k_3;\xi).\end{aligned}$$

(ii) Let r be an integer with $1 \leq r \leq 3$. If $k_r = 1$ and the other k_j's $\neq 1$, we set

$$\begin{aligned}\mathscr{A}_{(k_1,k_2,k_3)}(\mu) &= -4\pi^2 i C(m) \Lambda_{(k_1,k_2,k_3)} \sum_{\xi \in \Xi_{H,p},\, \xi_r \neq 1} B(k_1,k_2,k_3;\xi)\\ &= -2\pi^2 i C(m) \Lambda_{(k_1,k_2,k_3)} \sum_{\alpha,\gamma(p)} P(k_1,k_2,k_3;\xi).\end{aligned}$$

(iii) Let r, n be integers with $1 \leq r < n \leq 3$. If $k_r = k_n = 1$, and the remaining $k_j \neq 1$ (then, necessarily, $m > 1$), we set

$$\begin{aligned}\mathscr{A}_{(k_1,k_2,k_3)}(\mu) &= -4\pi^2 i C(m) \Lambda_{(k_1,k_2,k_3)}\\ &\qquad \times \left\{ \sum_{\xi \in \Xi_{H,p},\, \xi_r, \xi_n \neq 1} B(k_1,k_2,k_3;\xi) - s(k_1,k_2,k_3;\mu) \right\},\\ &= -2\pi^2 i C(m) \Lambda_{(k_1,k_2,k_3)}\\ &\qquad \times \left\{ \sum_{\alpha,\gamma(p)} P(k_1,k_2,k_3;\xi) - 2s(k_1,k_2,k_3;\mu) \right\},\end{aligned}$$

where

$$s(k_1,k_2,k_3;\mu) = \begin{cases} \dfrac{1}{8} g_{2m-1}(\mu) & (k_1,k_2,k_3) = (1,1,2m-1),\\ \dfrac{1}{24} \delta_{p,3}\, g_{2m-1}(2\mu) & (k_1,k_2,k_3) = (1,2m-1,1),\\ 0 & (k_1,k_2,k_3) = (2m-1,1,1),\end{cases}$$

$$g_{2m-1}(\mu) = \frac{1}{(2m-1)!} \sum_{\substack{\alpha(p)\\ \alpha \not\equiv 0(p)}} B_{2m-1}(\langle \mu\alpha^2/p \rangle) = \frac{\psi(\mu)}{(2m-1)!\, p^{2m-2}} \cdot B_{2m-1,\psi}.$$

(iv) In the case of $(k_1, k_2, k_3) = (1, 1, 1)$, we set

$$\begin{aligned}\mathscr{A}_{(1,1,1)}(\mu) &= -4\pi^2 i C(1) \Lambda_{(1,1,1)} \\ &\quad\times \left\{ \sum_{\xi \in \Xi_{H,p},\, \xi_j \neq 1\,(j=1,2,3)} B(1,1,1;\xi) - \frac{1}{8} g_1(\mu) - \frac{1}{24}\delta_{p,3} g_1(2\mu) \right\} \\ &= -2\pi^2 i C(1) \Lambda_{(1,1,1)} \\ &\quad\times \left\{ \sum_{\alpha,\gamma(p)} P(1,1,1;\xi) - \frac{1}{4} g_1(\mu) - \frac{1}{12}\delta_{p,3} g_1(2\mu) \right\}.\end{aligned}$$

We note that $\mathscr{A}_{(k_1,k_2,k_3)}(d^2\mu) = \mathscr{A}_{(k_1,k_2,k_3)}(\mu)$ for any d prime to p. We put

$$\text{(5.2.84)} \qquad \begin{cases} c(k_1, k_2, k_3) = \dfrac{\Lambda_{(k_1,k_2,k_3)}}{k_1!\, k_2!\, k_3!}, \\ S_p(k_1, k_2, k_3) = \displaystyle\sum_{\alpha,\gamma(p)} P_{k_1}\left(\frac{\alpha^2 - 2\alpha\gamma}{p}\right) P_{k_2}\left(\frac{\alpha\gamma}{p}\right) P_{k_3}\left(\frac{\gamma^2 - \alpha^2}{p}\right). \end{cases}$$

The special values at $s = 1 - m\,(m \in \mathbb{N})$ of $L_{2,P}(s, \psi_{H,p})$ can now be evaluated with the use of the numbers defined above.

Proposition 5.2.24. *Let $m \in \mathbb{N}$. Then,*

$$(i) \qquad L_{2,P}(1-m, \psi_{H,p}) = -\frac{(2m-1)!\, p^{2(m-1)}}{2^{2m}} \sum_{k_1,k_2,k_3}'' c(k_1,k_2,k_3) S_p(k_1,k_2,k_3) + \frac{(2m-1)!\,\{3c(1,1,2m-1) - c(1,2m-1,1)\delta_{p,3}\}}{3 \cdot 2^{2m+2}} \cdot B_{2m-1,\psi},$$

where (k_1, k_2, k_3) runs over all triples of integers with $k_2 \geq 0$, $k_1, k_3 \geq 1$, and $k_1 + k_2 + k_3 = 2m + 1$.

(ii) *Accordingly, $L_{2,P}(1-m, \psi_{H,p}) \in \mathbb{Q}$.*

(iii) *If $p \equiv 1(4)$, then $L_{2,P}(1-m, \psi_{H,p}) = 0$,*

If $p \equiv 3(4)$, then, in particular,

$$L_{2,P}(0, \psi_{H,p}) = Q_p + \frac{1}{6} h_p + \frac{3 - \delta_{p,3}}{12} B_{1,\psi},$$

where Q_p, h_p are the constants given by (5.1.1), (5.1.2) in Theorem 5.1.

Proof. Let $\xi_{\alpha,\gamma,\mu}$ be the triple of $\Xi_{H,\mu}$ given by (5.2.7). We notice that $B_1(1) = 1/2$, and moreover that

$$\begin{cases} \Xi_{H,\mu}^{(2,3)} \cap \Xi_{H,\mu}^{(1,3)} = \{\xi_{\alpha,\gamma,\mu} \mid \alpha \equiv 0(p),\, \gamma \not\equiv 0(p)\}, \\ \Xi_{H,\mu}^{(2,3)} \cap \Xi_{H,\mu}^{(1,2)} = \begin{cases} \phi & (p > 3), \\ \{\xi_{\alpha,\gamma,\mu} \mid \gamma \equiv -\alpha(p),\, \alpha \not\equiv 0(p)\} & (p = 3), \end{cases} \\ \Xi_{H,\mu}^{(1,3)} \cap \Xi_{H,\mu}^{(1,2)} = \phi. \end{cases} \tag{5.2.85}$$

Taking very carefully (5.2.79), (5.2.80), Proposition 5.2.8, Proposition 5.2.9 and (5.2.85) into account, we get

$$L_{2,P}(1-m, \psi_{H,p}) = p^{2(m-1)} \sum_{\mu} \sum_{k_1,k_2,k_3}{}' \psi(\mu) \mathscr{A}_{(k_1,k_2,k_3)}(\mu),$$

where μ is over 1 and κ, and (k_1, k_2, k_3) runs over all triples of integers with $k_1, k_2 \geq 0$, $k_3 \geq 1$, and $k_1 + k_2 + k_3 = 2m + 1$. By Lemma 5.2.22 and Lemma 5.2.23, we see that $\mathscr{A}_{(0,k_2,k_3)}(\mu) = 0$ for $k_2 \geq 0, k_3 \geq 1$ with $k_2 + k_3 = 2m + 1$. Hence, we have

$$L_{2,P}(1-m, \psi_{H,p}) = p^{2(m-1)} \sum_{\mu} \sum_{k_1,k_2,k_3}{}'' \psi(\mu) \mathscr{A}_{(k_1,k_2,k_3)}(\mu), \tag{5.2.86}$$

where μ is over 1 and κ, and (k_1, k_2, k_3) runs over all triples of integers with $k_2 \geq 0$, $k_1, k_3 \geq 1$, and $k_1 + k_2 + k_3 = 2m + 1$. Noting that $P_k(-x) = (-1)^k P_k(x)$ for any $x \in \mathbb{R}$, $k \geq 0$, we easily see that, for any triple (k_1, k_2, k_3) with $k_2 \geq 0$, $k_1, k_3 \geq 1$, and $k_1 + k_2 + k_3 = 2m + 1$,

$$\mathscr{A}_{(k_1,k_2,k_3)}(-\mu) = -\mathscr{A}_{(k_1,k_2,k_3)}(\mu). \tag{5.2.87}$$

Suppose $p \equiv 1(4)$. Then -1 is a quadratic residue mod p, say, $d^2 \equiv -1(p)$. Hence, $\mathscr{A}_{(k_1,k_2,k_3)}(-\mu) = \mathscr{A}_{(k_1,k_2,k_3)}(d^2\mu) = \mathscr{A}_{(k_1,k_2,k_3)}(\mu)$, which implies that $\mathscr{A}_{(k_1,k_2,k_3)}(\mu) = 0$.

Thus the first assertion of (iii) follows. Moreover, replacing (α, γ) by $(d\alpha, d\gamma)$ in the sum S_p, we get $S_p(k_1, k_2, k_3) = (-1)^{k_1+k_2+k_3} S_p(k_1, k_2, k_3)$. If $k_1+k_2+k_3 = 2m+1$, then $S_p(k_1, k_2, k_3) = 0$, and as $B_{2m-1,\psi} = 0$, we obtain the expression (i) for $L_2(1-m, \psi_{H,p})$ in the case of $p \equiv 1(4)$.

Suppose $p \equiv 3(4)$. Then we may take -1 as κ. By (5.2.86), together with (5.2.87) and the property $\psi(-1) = -1$, we get

$$L_{2,P}(1-m, \psi_{H,p}) = 2p^{2(m-1)} \sum_{k_1,k_2,k_3}'' \mathscr{A}_{(k_1,k_2,k_3)}(1),$$

where (k_1, k_2, k_3) runs over all triples of integers with $k_2 \geq 0$, $k_1, k_3 \geq 1$, and $k_1 + k_2 + k_3 = 2m + 1$. Thus we obtain the expression (i) for $L_2(1 - m, \psi_{H,p})$ in the case of $p \equiv 3(4)$.

If a triple (k_1, k_2, k_3) satisfies either of the conditions (i), (ii) in Proposition 5.2.10, then $\Lambda_{(k_1,k_2,k_3)}$, and consequently, $\mathscr{A}_{(k_1,k_2,k_3)}(\mu)$, is a rational number. Therefore, the assertion (ii) follows from (i).

It remains to prove the second assertion of (iii). Suppose $p \equiv 3(4)$. By assertion (i) and Proposition 5.2.12, we get

$$L_{2,P}(0, \psi_{H,p}) = -\frac{1}{6} S_p(1, 0, 2) - S_p(1, 1, 1) - \frac{1}{3} S_p(2, 0, 1) + \frac{3 - \delta_{p,3}}{12} B_{1,\psi},$$

where

$$S_p(1, 0, 2) = \sum_{\alpha,\gamma(p)} P_1((\alpha^2 - 2\alpha\gamma)/p) P_2((\gamma^2 - \alpha^2)/p),$$

$$S_p(1, 1, 1) = \sum_{\alpha,\gamma(p)} P_1((\alpha^2 - 2\alpha\gamma)/p) P_1(\alpha\gamma/p) P_1((\gamma^2 - \alpha^2)/p),$$

$$S_p(2, 0, 1) = \sum_{\alpha,\gamma(p)} P_2((\alpha^2 - 2\alpha\gamma)/p) P_1((\gamma^2 - \alpha^2)/p).$$

Replacing (α, γ) by $(-\alpha, \gamma)$ in the first two sums and by $(\alpha + \gamma, \gamma)$ in the third sum, noting that $P_k(-x) = (-1)^k P_k(x)$, we get

$$S_p(1,0,2) = \sum_{\alpha,\gamma(p)} P_1((\alpha^2+2\alpha\gamma)/p)P_2((\gamma^2-\alpha^2)/p) = h_p,$$

$$S_p(1,1,1) = -\sum_{\alpha,\gamma(p)} P_1((\alpha^2+2\alpha\gamma)/p)P_1(\alpha\gamma/p)P_1((\gamma^2-\alpha^2)/p) = -Q_p,$$

$$S_p(2,0,1) = \sum_{\alpha,\gamma(p)} P_2((\alpha^2-\gamma^2)/p)P_1((-\alpha^2-2\alpha\gamma)/p) = -h_p.$$

Thus the second assertion of (iii) follows. □

Now we study the singular part $L_{2,S}(s, \psi_{H,p})$.

Proposition 5.2.25. *Let $m \in \mathbb{N}$ and let μ be an integer prime to p. The functions $\zeta_S(s; C_{123}, L(\mu))$, $\zeta_S(s; C_{j3}, L(\mu))$ $(j = 1, 2)$ are holomorphic at $s = 1-m$, and the special values at $s = 1-m$ are given by*

(i) $\zeta_S(1-m; C_{123}, L(\mu))$

$$= iC(m)p^{2(m-1)} \sum_{\xi \in \Xi_{H,\mu}} \{J'(1-m; (\xi_1,\xi_2)) - \pi i K(1-m; (\xi_1,\xi_2))\},$$

(ii) $\zeta_S(1-m; C_{j3}, L(\mu))$

$$= iC(m)p^{2(m-1)} \sum_{\xi \in \Xi^{(j,3)}_{H,\mu}} \{J'_j(1-m; (\xi_1,\xi_2)) - \pi i K_j(1-m; (\xi_1,\xi_2))\}$$

$(j = 1, 2)$.

Proof. Proposition 5.2.19, Lemma 5.2.22, and Proposition 5.2.20, show that

$$\sum_{\xi \in \Xi_{H,p}} J(1-m; (\xi_1,\xi_2)) = 0, \qquad \sum_{\xi \in \Xi^{(j,3)}_{H,p}} J_j(1-m; (\xi_1,\xi_2)) = 0 \quad (j = 1, 2).$$

Thus we immediately see from (5.2.79) and the expansions (5.2.60), (5.2.61) that $\zeta_S(s; C_{123}, L(\mu))$, $\zeta_S(s; C_{j3}, L(\mu))$ $(j = 1, 2)$ are holomorphic at $s = 1-m$, and that the special values at $s=1-m$ are expressed as in the proposition. □

It follows from Proposition 5.2.25 and (5.2.80) that $L_{2,S}(s,\psi_{H,p})$ is holomorphic at $s=1-m\,(m\in\mathbb{N})$.

Proposition 5.2.26. *Let $m\in\mathbb{N}$. Set $(\eta_1,\eta_2)=(\langle(x-2u)/p\rangle,\langle u/p\rangle)$. Then,*

$$
\begin{aligned}
(i)\quad & \sum_{\mu}\psi(\mu)\zeta_S(1-m;C_{123},L(\mu))=iC(m)p^{2(m-1)}\\
&\times\sum_{\substack{x(p)\\ x\not\equiv 0(p)}}\psi(x)\sum_{u(p)}\{J'(1-m;(\eta_1,\eta_2))-\pi iK(1-m;(\eta_1,\eta_2))\},\\
(ii)\quad & \sum_{\mu}\psi(\mu)\zeta_S(1-m;C_{13},L(\mu))=iC(m)p^{2(m-1)}\\
&\times\sum_{\substack{x(p)\\ x\not\equiv 0(p)}}\psi(x)\{J'_j(1-m;\langle x/p\rangle)-\pi iK_j(1-m;\langle x/p\rangle)\},\\
(iii)\quad & \sum_{\mu}\psi(\mu)\zeta_S(1-m;C_{23},L(\mu))=iC(m)p^{2(m-1)}\\
&\times\sum_{\substack{x(p)\\ x\not\equiv 0(p)}}\psi(x)\{J'_j(1-m;\langle 2^{-1}x/p\rangle)-\pi iK_j(1-m;\langle 2^{-1}x/p\rangle)\},
\end{aligned}
$$

where μ runs over 1 and κ (κ being a non-quadratic residue mod p).

Proof. Let $\xi^{(j)}_{\alpha,\gamma,\mu}\,(j=1,2,3)$ be the j-component of $\varXi_{H,\mu}$ (see (5.2.7)). We set $\mu\alpha^2=x$, $\mu\alpha\gamma=u$. If (α,γ) runs over all elements of $\mathscr{M}(p)$ with $\alpha\not\equiv 0(p)$, and μ is over 1 and κ, then, $(x,u)=(\mu\alpha^2,\mu\alpha\gamma)$ just doubly covers all elements of $\mathscr{M}(p)$ with $x\not\equiv 0(p)$. If $\alpha\equiv 0(p)$, then, $\xi^{(1)}_{\alpha,\gamma,\mu}=\xi^{(2)}_{\alpha,\gamma,\mu}=1$, and all of the sums over μ would vanish. Thus, by Proposition 5.2.25, we get

$$\sum_{\mu} \psi(\mu)\zeta_S(1-m; C_{123}, L(\mu)) = iC(m)p^{2(m-1)}$$

$$\times \sum_{\mu} \psi(\mu) \sum_{\xi \in \Xi_{H,\mu}} \{J'(1-m; (\xi_1, \xi_2)) - \pi i K(1-m; (\xi_1, \xi_2))\},$$

$$= iC(m)p^{2(m-1)}$$

$$\times \frac{1}{2} \sum_{\mu} \psi(\mu) \sum_{(\alpha,\gamma) \in \mathscr{M}(p)} \{J'(1-m; (\xi^{(1)}_{\alpha,\gamma,\mu}, \xi^{(2)}_{\alpha,\gamma,\mu})) - \pi i K(1-m; (\xi^{(1)}_{\alpha,\gamma,\mu}, \xi^{(2)}_{\alpha,\gamma,\mu}))\},$$

$$= iC(m)p^{2(m-1)}$$

$$\times \sum_{\substack{x(p) \\ x \not\equiv 0(p)}} \psi(x) \sum_{u(p)} \{J'(1-m; (\eta_1, \eta_2)) - \pi i K(1-m; ((\eta_1, \eta_2)))\},$$

which proves the assertion (i). Similarly, we get

$$\sum_{\mu} \psi(\mu)\zeta_S(1-m; C_{13}, L(\mu)) = iC(m)p^{2(m-1)}$$

$$\times \sum_{\mu} \psi(\mu) \sum_{\xi \in \Xi^{(1,3)}_{H,\mu}} \{J'(1-m; \xi_1) - \pi i K(1-m; \xi_1)\},$$

$$= iC(m)p^{2(m-1)}$$

$$\times \frac{1}{2} \sum_{\mu} \psi(\mu) \sum_{\substack{(\alpha,\gamma) \in \mathscr{M}(p) \\ \alpha\gamma \equiv 0(p)}} \{J'(1-m; \xi^{(1)}_{\alpha,\gamma,\mu}) - \pi i K(1-m; \xi^{(1)}_{\alpha,\gamma,\mu})\},$$

$$= iC(m)p^{2(m-1)}$$

$$\times \sum_{\substack{x(p) \\ x \not\equiv 0(p)}} \psi(x) \sum_{\substack{u(p) \\ u \equiv 0(p)}} \{J'(1-m; \eta_1) - \pi i K(1-m; \eta_1)\},$$

from which the assertion (ii) follows, and

$$\sum_{\mu} \psi(\mu)\zeta_S(1-m; C_{23}, L(\mu)) = iC(m)p^{2(m-1)}$$
$$\times \sum_{\mu} \psi(\mu) \sum_{\xi \in \Xi_{H,\mu}^{(2,3)}} \{J'(1-m;\xi_1) - \pi i K(1-m;\xi_1)\},$$
$$= iC(m)p^{2(m-1)}$$
$$\times \frac{1}{2} \sum_{\mu} \psi(\mu) \sum_{\substack{(\alpha,\gamma)\in\mathscr{M}(p) \\ \alpha^2 \equiv 2\alpha\gamma(p)}} \{J'(1-m;\xi^{(2)}_{\alpha,\gamma,\mu}) - \pi i K(1-m;\xi^{(2)}_{\alpha,\gamma,\mu})\},$$
$$= iC(m)p^{2(m-1)}$$
$$\times \sum_{\substack{x(p) \\ x\not\equiv 0(p)}} \psi(x) \sum_{\substack{u(p) \\ u\equiv 2^{-1}x(p)}} \{J'(1-m;\eta_2) - \pi i K(1-m;\eta_2\},$$

from which the assertion (iii) follows. □

The following lemma plays a key role in evaluating the special values of $L_{2,S}(s, \psi_{H,p})$ at $s = 1-m\,(m \in \mathbb{N})$.

Lemma 5.2.27. *Let $m \in \mathbb{N}$, $j \in \mathbb{Z}$ with $j \geq 0$, and x be any integer prime to p.*

Then,

$$(i)\quad \sum_{u(p)} \phi(t;1-\langle (x-2u)/p\rangle)\phi^{(2j)}(2t;1-\langle u/p\rangle)$$

$$= \sum_{k=0}^{2j+1} \left(\gamma^1_{k,j}(x)\phi^{(k)}\left(2t;\frac{1}{2}-\frac{x}{2p}\right)+\gamma^2_{k,j}(x)\phi^{(k)}\left(2t;1-\frac{x}{2p}\right)\right)$$

with

$$\gamma^1_{k,j}(x) = \delta_{k,0}\delta_{j,0}B_1 + \frac{(-1)^k\binom{2j+1}{k}}{(2j+1)p^{2j-k}}\cdot\begin{cases} B_{2j+1-k} & \text{if } x \text{ is odd,}\\ B_{2j+1-k}(1/2) & \text{if } x \text{ is even,}\end{cases}$$

$$\gamma^2_{k,j}(x) = \delta_{k,0}\delta_{j,0}B_1 + \frac{(-1)^k\binom{2j+1}{k}}{(2j+1)p^{2j-k}}\cdot\begin{cases} B_{2j+1-k}(1/2) & \text{if } x \text{ is odd,}\\ B_{2j+1-k} & \text{if } x \text{ is even,}\end{cases}$$

$$(0\le k\le 2j+1).$$

$$(ii)\quad \int_{I_\epsilon(\infty)} \log t\cdot\sum_{u(p)}\phi(t;1-\langle (x-2u)/p\rangle)\phi^{(2m)}(2t;1-\langle u/p\rangle)dt$$

$$= -\pi i\sum_{k=1}^{2m+1}\frac{(-1)^k}{k}\left(\gamma^1_{k,m}(x)B_k\left(\frac{x}{2p}+\frac{1}{2}\right)+\gamma^2_{k,m}(x)B_k\left(\frac{x}{2p}\right)\right).$$

(iii) *For* $0\le j\le m-1$*, we have*

$$\int_{\Gamma_\epsilon} t^{2(j-m)}\sum_{u(p)}\phi(t;1-\langle (x-2u)/p\rangle)\phi^{(2j)}(2t;1-\langle u/p\rangle)dt$$

$$= 2\pi i\sum_{k=0}^{2j+1}\frac{(-1)^k\binom{k+2m-2j-1}{k}k!\,2^{2m-2j-1}}{(k+2m-2j)!}$$

$$\times\left\{\gamma^1_{k,j}(x)B_{k+2m-2j}\left(\frac{x}{2p}+\frac{1}{2}\right)+\gamma^2_{k,j}(x)B_{k+2m-2j}\left(\frac{x}{2p}\right)\right\}.$$

Proof. We may take x so that $1\le x\le p-1$. Noting the obvious identity

$$\phi(t;1-a)=\phi\left(2t;1/2-a/2\right)+\phi\left(2t;1-a/2\right),$$

we get, with the help of Lemma 5.2.13,

$$
\begin{aligned}
&\sum_{u(p)} \phi(t; 1 - \langle (x-2u)/p \rangle) \phi^{(2j)}(2t; 1 - \langle u/p \rangle) \\
&\quad = \sum_{k=0}^{2j} \sum_{u(p)} \lambda_{k,2j}(1 - \langle u/p \rangle) \frac{e^{2t\left(\frac{3}{2} - \langle u/p \rangle - \frac{1}{2}\langle (x-2u)/p \rangle\right)}}{(e^{2t}-1)^{k+2}} \\
&\quad + \sum_{k=0}^{2j} \sum_{u(p)} \lambda_{k,2j}(1 - \langle u/p \rangle) \frac{e^{2t\left(2 - \langle u/p \rangle - \frac{1}{2}\langle (x-2u)/p \rangle\right)}}{(e^{2t}-1)^{k+2}} \\
&\quad = \sum_{k=0}^{2j} \frac{1}{(e^{2t}-1)^{k+2}} \left\{ \sum_{u=1}^{\lceil x/2 \rceil - 1} \lambda_{k,2j}(1-u/p) e^{2t\left(\frac{3}{2} - \frac{x}{2p}\right)} \right. \\
&\quad + \left. \sum_{u=\lceil x/2 \rceil}^{\lceil (x+p)/2 \rceil - 1} \lambda_{k,2j}(1-u/p) e^{2t\left(1 - \frac{x}{2p}\right)} + \sum_{u=\lceil (x+p)/2 \rceil}^{p} \lambda_{k,2j}(1-u/p) e^{2t\left(\frac{1}{2} - \frac{x}{2p}\right)} \right\} \\
&\quad + \sum_{k=0}^{2j} \frac{1}{(e^{2t}-1)^{k+2}} \left\{ \sum_{u=1}^{\lceil x/2 \rceil - 1} \lambda_{k,2j}(1-u/p) e^{2t\left(2 - \frac{x}{2p}\right)} \right. \\
&\quad + \left. \sum_{u=\lceil x/2 \rceil}^{\lceil (x+p)/2 \rceil - 1} \lambda_{k,2j}(1-u/p) e^{2t\left(\frac{3}{2} - \frac{x}{2p}\right)} + \sum_{u=\lceil (x+p)/2 \rceil}^{p} \lambda_{k,2j}(1-u/p) e^{2t\left(1 - \frac{x}{2p}\right)} \right\} \\
&\quad = \sum_{k=0}^{2j} \left\{ \sum_{u=1}^{\lceil (x+p)/2 \rceil - 1} \lambda_{k,2j}(1-u/p) \phi_{k+1}\left(2t; \frac{1}{2} - \frac{x}{2p}\right) \right. \\
&\qquad + \left. \sum_{u=\lceil (x+p)/2 \rceil}^{p} \lambda_{k,2j}(1-u/p) \phi_{k+2}\left(2t; \frac{1}{2} - \frac{x}{2p}\right) \right\} \\
&\quad + \sum_{k=0}^{2j} \left\{ \sum_{u=1}^{\lceil x/2 \rceil - 1} \lambda_{k,2j}(1-u/p) \phi_{k+1}\left(2t; 1 - \frac{x}{2p}\right) \right. \\
&\qquad + \left. \sum_{u=\lceil x/2 \rceil}^{p} \lambda_{k,2j}(1-u/p) \phi_{k+2}\left(2t; 1 - \frac{x}{2p}\right) \right\}.
\end{aligned}
$$

Then, repeatedly using the formula

$$\phi_{k+1}(2t; a) = -(1/(2k))\phi_k'(2t; a) + ((a-k)/k)\phi_k(2t; a) \quad (k \geq 1),$$

we obtain the expression

$$(5.2.88)\quad \begin{aligned}&\sum_{u(p)} \phi(t;1-\langle (x-2u)/p\rangle)\phi^{(2j)}(2t;1-\langle u/p\rangle)\\ &\qquad = \sum_{k=0}^{2j+1}\left(\gamma_{k,j}^{1}(x)\phi^{(k)}\left(2t;\frac{1}{2}-\frac{x}{2p}\right)+\gamma_{k,j}^{2}(x)\phi^{(k)}\left(2t;1-\frac{x}{2p}\right)\right),\end{aligned}$$

where $\gamma_{k,j}^{1}(x), \gamma_{k,j}^{2}(x)\,(0\le k\le 2j+1)$ are certain rational numbers depending on x. Recalling the Laurent expansion (5.2.44) of $\phi^{(k)}(t;a)$, we shall compare the coefficients in the Laurent expansions at $t=0$ of both sides of (5.2.88). For $|t|<\pi$, we have

$$\begin{aligned}&\phi(t;1-\langle (x-2u)/p\rangle)\phi^{(2j)}(2t;1-\langle u/p\rangle)\\ &\quad = (2j)!\,2^{-2j-1}\sum_{n=0}^{2j}\frac{B_n(1-\langle (x-2u)/p\rangle)}{n!}\cdot t^{n-2j-2}\\ &\quad + \frac{1}{2j+1}\left\{2^{-2j-1}B_{2j+1}(1-\langle (x-2u)/p\rangle)+B_{2j+1}(1-\langle u/p\rangle)\right\}t^{-1}\\ &\quad + \sum_{r=0}^{\infty}\left\{\frac{(2j)!\,2^{-2j-1}B_{r+2j+2}(1-\langle (x-2u)/p\rangle)}{(r+2j+2)!}+\frac{1}{(r+1)!}\right.\\ &\qquad \left.\times\sum_{k=0}^{r+1}\frac{\binom{r+1}{k}2^{r+1-k}}{r+2j+2-k}\cdot B_k(1-\langle (x-2u)/p\rangle)B_{r+2j+2-k}(1-\langle u/p\rangle)\right\}t^{r}.\end{aligned}$$

By (5.2.82) and Proposition 3.2.1, it follows that

$$\begin{aligned}
&\sum_{u(p)} \phi(t;1-\langle (x-2u)/p\rangle)\phi^{(2j)}(2t;1-\langle u/p\rangle) \\
&\quad = (2j)!\,2^{-2j-1}\sum_{n=0}^{2j}\frac{B_n}{n!\,p^{n-1}}\cdot t^{n-2j-2} \\
&\quad + \frac{(2^{-2j-1}+1)B_{2j+1}}{(2j+1)p^{2j}}\cdot t^{-1} \\
&\quad + \sum_{r=0}^{\infty}\Bigg\{\frac{(2j)!\,2^{-2j-1}B_{r+2j+2}}{(r+2j+2)!\,p^{r+2j+1}} \\
&\qquad + \frac{(-1)^{r+1}2^r B_1}{r!\,(r+2j+1)}\cdot P_{r+2j+1}\left(\frac{x}{2p}+\frac{x}{2}\right) + \frac{(-1)^{r+1}B_1\delta_{j,0}}{(r+1)!}\cdot P_{r+1}\left(\frac{x}{p}\right) \\
&\qquad + \frac{(-1)^{r+1}}{(r+1)!}\sum_{k=0}^{r+1}\frac{(-1)^{r+1-k}\binom{r+1}{k}2^{r+1-k}}{r+2j+2-k}\cdot \phi_{k,r+2j+2-k}\left(2,p;\frac{x}{p},0\right)\Bigg\}t^r,
\end{aligned} \tag{5.2.89}$$

where $\phi_{r,s}(h,k;x,y)$ denotes the Carlitz Phi function given by (3.1.14). Similarly, for $|t|< \pi$, we have

$$\begin{aligned}
&\sum_{k=0}^{2j+1}\left(\gamma^1_{k,j}(x)\phi^{(k)}\left(2t;\frac{1}{2}-\frac{x}{2p}\right)+\gamma^2_{k,j}(x)\phi^{(k)}\left(2t;1-\frac{x}{2p}\right)\right) \\
&\quad = \sum_{n=0}^{2j+1}(-1)^{2j+1-n}(2j+1-n)!\,2^{n-2j-2} \\
&\qquad\qquad \times\left(\gamma^1_{2j+1-n,j}(x)+\gamma^2_{2j+1-n,j}(x)\right)t^{n-2j-2} \\
&\quad + \sum_{r=0}^{\infty}\Bigg\{\sum_{k=0}^{2j+1}\frac{(-1)^{r+k+1}}{r!\,(r+k+1)} \\
&\qquad\qquad \times\left(\gamma^1_{k,j}(x)P_{r+k+1}\left(\frac{x}{2p}+\frac{1}{2}\right)+\gamma^2_{k,j}(x)P_{r+k+1}\left(\frac{x}{2p}\right)\right)2^r\Bigg\}t^r.
\end{aligned} \tag{5.2.90}$$

Suppose $\gamma^1_{k,j}(x),\gamma^2_{k,j}(x)$ are given as in the assertion (i). We will show that all of the coefficients in the Laurent expansions (5.2.89), (5.2.90) are the same; that is,

$$\begin{aligned}
&\text{(I)}\quad (-1)^{2j+1-n}(2j+1-n)!\,2^{n-2j-2}\left(\gamma^1_{2j+1-n,j}(x)+\gamma^2_{2j+1-n,j}(x)\right)\\
&\qquad = (2j)!\,2^{-2j-1}\frac{B_n}{n!\,p^{n-1}} \qquad (0\le n\le 2j),\\
&\text{(II)}\quad \frac{1}{2}\left(\gamma^1_{0,j}(x)+\gamma^2_{0,j}(x)\right)=\frac{(2^{-2j-1}+1)B_{2j+1}}{(2j+1)p^{2j}},\\
&\text{(III)}\quad \sum_{k=0}^{2j+1}\frac{(-1)^{r+k+1}\left(\gamma^1_{k,j}(x)P_{r+k+1}\left(\frac{x}{2p}+\frac{1}{2}\right)+\gamma^2_{k,j}(x)P_{r+k+1}\left(\frac{x}{2p}\right)\right)2^r}{r!\,(r+k+1)}\\
&\qquad = \frac{(2j)!\,2^{-2j-1}B_{r+2j+2}}{(r+2j+2)!\,p^{r+2j+1}}+\\
&\qquad + \frac{(-1)^{r+1}2^rB_1}{r!\,(r+2j+1)}\cdot P_{r+2j+1}\left(\frac{x}{2p}+\frac{x}{2}\right)+\frac{(-1)^{r+1}B_1\delta_{j,0}}{(r+1)!}\cdot P_{r+1}\left(\frac{x}{p}\right)\\
&\qquad + \frac{(-1)^{r+1}}{(r+1)!}\sum_{k=0}^{r+1}\frac{(-1)^{r+1-k}\binom{r+1}{k}2^{r+1-k}}{r+2j+2-k}\cdot\phi_{k,r+2j+2-k}\left(2,p;\frac{x}{p},0\right)\\
&\qquad (r=0,1,2,\cdots).
\end{aligned}$$

(I) follows immediately from the fact that $B_n(0)+B_n(1/2)=B_n/2^{(n-1)}$ $(0\le n\le 2j)$, and (II) is obvious. We observe that the left side of (III) is equal to

$$\begin{aligned}
&\frac{(-1)^{r+1}2^rB_1}{r!\,(r+2j+1)}\cdot P_{r+2j+1}\left(\frac{x}{2p}+\frac{x}{2}\right)+\frac{(-1)^{r+1}B_1\delta_{j,0}}{(r+1)!}\cdot P_{r+1}\left(\frac{x}{p}\right)\\
&\quad+\frac{(-1)^r2^r}{p^{2j}r!\,(2j+1)}\sum_{k=0}^{2j+1}\frac{(-1)^{2j+1-k}\binom{2j+1}{k}p^{2j+1-k}}{r+2j+2-k}\cdot\phi_{k,r+2j+2-k}\left(p,2;0,\frac{x}{p}\right).
\end{aligned}$$

Thus, (III) reduces to

$$\begin{aligned}
&\binom{r+2j+2}{r+1}\left\{\frac{2j+1}{2^r}\sum_{k=0}^{r+1}\frac{(-1)^{r+1-k}\binom{r+1}{k}2^{r+1-k}}{r+2j+2-k}\cdot\phi_{k,r+2j+2-k}\left(2,p;\frac{x}{p},0\right)\right.\\
&\qquad\left.+\frac{r+1}{p^{2j}}\sum_{k=0}^{2j+1}\frac{(-1)^{2j+1-k}\binom{2j+1}{k}p^{2j+1-k}}{r+2j+2-k}\cdot\phi_{k,r+2j+2-k}\left(p,2;0,\frac{x}{p}\right)\right\}\\
&=\frac{B_{r+2j+2}}{(2p)^{r+2j+1}}\qquad (r=0,1,2,\cdots).
\end{aligned}$$

Applying Proposition 3.2.28 to both of the sums above, we get

$$\begin{aligned}&\sum_{k=0}^{r+1}\binom{r+2j+2}{k}2^{1-k}\psi_{k,r+2j+2-k}\left(2,p;\frac{x}{p},0\right)\\&\quad+\sum_{k=0}^{2j+1}\binom{r+2j+2}{k}p^{1-k}\psi_{k,r+2j+2-k}\left(p,2;0,\frac{x}{p}\right)\\&=\frac{B_{r+2j+2}}{(2p)^{r+2j+1}}\qquad(r=0,1,2,\cdots),\end{aligned}$$

where $\psi_{r,s}(h,k;x,y)$ denotes the Carlitz Psi function given by (3.1.14). Applying Proposition 3.2.29 to the first sum, we get

$$\sum_{k=0}^{r+2j+2}\binom{r+2j+2}{k}p^{1-k}\psi_{k,r+2j+2-k}\left(p,2;0,\frac{x}{p}\right)=\frac{B_{r+2j+2}}{(2p)^{r+2j+1}}$$
$$(r=0,1,2,\cdots),$$

which follows immediately from Proposition 3.2.30, and hence establishes (III). The proof of the assertion (i) is now complete.

Similarly as in (5.2.76), we have

$$\int_{I_\epsilon(\infty)}\log t\cdot\phi^{(k)}\left(2t;1-a\right)dt=-\pi i\frac{(-1)^k B_k\left(a\right)}{k}\qquad(k\in\mathbb{N},\ a\notin\mathbb{Z}).$$

This together with (i), noting that $\gamma^1_{0,m}(x)=\gamma^2_{0,m}(x)=0$ for $m\in\mathbb{N}$, completes the proof of the assertion (ii).

From (5.2.44), the coefficient of the term $t^{2(m-j)-1}$ $(0\le j\le m-1)$ in the Laurent expansion at $t=0$ of $\phi^{(k)}(2t;1-a)$ for $t<|\pi|$, is

$$\frac{(-1)^k\binom{k+2m-2j-1}{k}k!\,2^{2m-2j-1}B_{k+2m-2j}(a)}{(k+2m-2j)!}\qquad(k\ge 0,\ a\in\mathbb{R}).$$

Thus the assertion (iii) readily follows from (i). $\square$

Proposition 5.2.28. *Let $m \in \mathbb{N}$. Then,*

$$\frac{2^{2m+2}\pi i(-1)^m}{(2m)!} \sum_{\substack{x(p)\\ x\not\equiv 0(p)}} \psi(x) \int_{I_\epsilon(\infty)} \log t \cdot \sum_{u(p)} \phi(t; 1-\langle (x-2u)/p\rangle)\phi^{(2m)}(2t; 1-\langle u/p\rangle)dt$$

$$= \frac{2^{2m+2}\pi^2(-1)^m}{p^{2m-1}} \sum_{k=0}^{2m} \frac{\{B_k - B_k(1/2)\}2^{2m-k}\psi(2) + B_k(1/2)}{(2m+1-k)k!\,(2m+1-k)!\,2^{2m-k}} \cdot B_{2m+1-k,\psi},$$

where ϵ is taken sufficiently small.

Proof. By Lemma 5.2.27 and Proposition 3.2.24, we get

$$\sum_{x=1}^{p-1} \psi(x) \int_{I_\epsilon(\infty)} \log t \cdot \sum_{u(p)} \phi(t; 1-\langle (x-2u)/p\rangle)\phi^{(2m)}(2t; 1-\langle u/p\rangle)dt$$

$$= -\pi i \sum_{j=1}^{2m+1} \frac{(-1)^j}{j}$$

$$\times\Bigg\{ \gamma^1_{j,m}(1) \sum_{\substack{x=1\\ x \text{ odd}}}^{p-1} B_j\left(\frac{x}{2p}+\frac{1}{2}\right)\psi(x) + \gamma^1_{j,m}(0) \sum_{\substack{x=1\\ x \text{ even}}}^{p-1} B_j\left(\frac{x}{2p}+\frac{1}{2}\right)\psi(x)$$

$$+ \gamma^2_{j,m}(1) \sum_{\substack{x=1\\ x \text{ odd}}}^{p-1} B_j\left(\frac{x}{2p}\right)\psi(x) + \gamma^2_{j,m}(0) \sum_{\substack{x=1\\ x \text{ even}}}^{p-1} B_j\left(\frac{x}{2p}\right)\psi(x)\Bigg\}$$

$$= -\pi i \sum_{j=1}^{2m+1} \frac{(-1)^j}{j} \Bigg\{ \gamma^1_{j,m}(1) \sum_{\substack{x=1\\ x \text{ even}}}^{2p-1} B_j\left(\frac{x}{2p}\right)\psi(x) + \gamma^1_{j,m}(0) \sum_{\substack{x=1\\ x \text{ odd}}}^{2p-1} B_j\left(\frac{x}{2p}\right)\psi(x)\Bigg\}$$

$$= -\pi i \sum_{j=1}^{2m+1} \frac{(-1)^j}{j} \Bigg\{ (\gamma^1_{j,m}(1) - \gamma^1_{j,m}(0)) \frac{\psi(2)B_{j,\psi}}{p^{j-1}} + \gamma^1_{j,m}(0) \frac{B_{j,\psi}}{(2p)^{j-1}} \Bigg\}$$

$$= \frac{-\pi i}{(2m+1)p^{2m-1}}$$

$$\times \sum_{j=1}^{2m+1} \frac{\binom{2m+1}{j}\{(B_{2m+1-j} - B_{2m+1-j}(1/2))\,2^{j-1}\psi(2) + B_{2m+1-j}(1/2)\}B_{j,\psi}}{2^{j-1}j}.$$

Thus the assertion of Proposition 5.2.28 immediately follows. $\square$

Proposition 5.2.29. *Let $m \in \mathbb{N}$. Then,*

$$
\begin{aligned}
&-4\pi i \sum_{j=0}^{m-1} \frac{(2m-2j-1)!\, 2^{2j}(-1)^j}{\{(m-j)!\,\}^2 (2j)!} \\
&\quad \times \sum_{\substack{x(p)\\ x\not\equiv 0(p)}} \psi(x) \int_{\Gamma_\epsilon} t^{2(j-m)} \sum_{u(p)} \phi(t; 1-\langle (x-2u)/p\rangle)\phi^{(2j)}(2t; 1-\langle u/p\rangle)dt \\
&= \frac{2^{2m+2}\pi^2}{p^{2m-1}} \sum_{j=0}^{m-1} \frac{(-1)^j}{\{(m-j)!\,\}^2} \sum_{k=0}^{2j+1} \frac{\{B_k - B_k(1/2)\}2^{2m-k}\psi(2) + B_k(1/2)}{(2m+1-k)k!\,(2j+1-k)!\,2^{2m-k}} \cdot B_{2m+1-k,\psi} \\
&\quad - \frac{2\pi^2}{m(m!\,)^2 p^{2m-1}} \cdot B_{2m,\psi},
\end{aligned}
$$

where ϵ is taken sufficiently small.

Proof. By Lemma 5.2.27 and Proposition 3.2.24, we get

$$
\begin{aligned}
&\sum_{x=1}^{p-1}\psi(x)\int_{\Gamma_\epsilon} t^{2(j-m)}\sum_{u(p)}\phi(t;1-\langle (x-2u)/p\rangle)\phi^{(2j)}(2t;1-\langle u/p\rangle)dt \\
&= 2\pi i\sum_{k=0}^{2j+1}\frac{(-1)^k\binom{k+2m-2j-1}{k}k!\,2^{2m-2j-1}}{(k+2m-2j)!} \\
&\times\Bigg\{\gamma_{k,j}^1(1)\sum_{\substack{x=1\\ x\text{ odd}}}^{p-1}B_{k+2m-2j}\left(\frac{x}{2p}+\frac{1}{2}\right)\psi(x)+\gamma_{k,j}^1(0)\sum_{\substack{x=1\\ x\text{ even}}}^{p-1}B_{k+2m-2j}\left(\frac{x}{2p}+\frac{1}{2}\right)\psi(x) \\
&\quad+\gamma_{k,j}^2(1)\sum_{\substack{x=1\\ x\text{ odd}}}^{p-1}B_{k+2m-2j}\left(\frac{x}{2p}\right)\psi(x)+\gamma_{k,j}^2(0)\sum_{\substack{x=1\\ x\text{ even}}}^{p-1}B_{k+2m-2j}\left(\frac{x}{2p}\right)\psi(x)\Bigg\} \\
&= 2\pi i\sum_{k=0}^{2j+1}\frac{(-1)^k\binom{k+2m-2j-1}{k}k!\,2^{2m-2j-1}}{(k+2m-2j)!} \\
&\qquad\times\left\{\gamma_{k,j}^1(1)\sum_{\substack{x=1\\ x\text{ even}}}^{2p-1}B_{k+2m-2j}\left(\frac{x}{2p}\right)\psi(x)+\gamma_{k,j}^1(0)\sum_{\substack{x=1\\ x\text{ odd}}}^{2p-1}B_{k+2m-2j}\left(\frac{x}{2p}\right)\psi(x)\right\} \\
&= 2\pi i\sum_{k=0}^{2j+1}\frac{(-1)^k\binom{k+2m-2j-1}{k}k!\,2^{2m-2j-1}}{(k+2m-2j)!} \\
&\qquad\times\left\{(\gamma_{k,j}^1(1)-\gamma_{k,j}^1(0))\frac{\psi(2)B_{k+2m-2j,\psi}}{p^{k+2m-2j-1}}+\gamma_{k,j}^1(0)\frac{B_{k+2m-2j,\psi}}{(2p)^{k+2m-2j-1}}\right\} \\
&= \frac{2\pi i}{(2j+1)p^{2m-1}}\sum_{k=0}^{2j+1}\frac{\binom{2j+1}{k}\{(B_k-B_k(1/2))\,2^{2m-k}\psi(2)+B_k(1/2)\}B_{2m+1-k,\psi}}{(2m+1-k)(2m-2j-1)!\,2^{2j+1-k}} \\
&\qquad+\frac{2\pi i B_1\delta_{j,0}}{(2m)!\,p^{2m-1}}B_{2m,\psi}.
\end{aligned}
$$

Thus the assertion of Proposition 5.2.29 immediately follows. □

Proposition 5.2.30. *Let* $m \in \mathbb{N}$. *Then,*

$$
\begin{aligned}
\sum_{\mu} \psi(\mu)\zeta_S(1-m; C_{123}, L(\mu)) &= iC(m)p^{2(m-1)} \\
&\times \left\{ \frac{2^{2m+2}\pi^2}{p^{2m-1}} \sum_{j=0}^{m} \frac{(-1)^j}{\{(m-j)!\,\}^2} \right. \\
&\qquad \times \sum_{k=0}^{2j+1-\delta_{j,m}} \frac{\{B_k - B_k(1/2)\}2^{2m-k}\psi(2) + B_k(1/2)}{(2m+1-k)k!\,(2j+1-k)!\,2^{2m-k}} \cdot B_{2m+1-k,\psi} \\
&\quad \left. + 2\pi^2 \sum_{x(p)} \psi(x) \sum_{n=1}^{2m} \mathscr{B}_{n,m} \cdot \phi_{2m+1-n,n}\left(2, p; \frac{x}{p}, 0\right) \right\} \\
&\quad - \frac{iC(m)}{2p} \left\{ \frac{4\pi^2}{m(m!\,)^2} - \frac{2\pi^2 \mathscr{M}_{(2m-1,0)}}{(2m)!} \right\} B_{2m,\psi} \\
&\quad - \frac{iC(m)}{2p} \left\{ \frac{-2\pi^2 \mathscr{M}_{(0,2m-1)} + 2\pi^2 \mathscr{N}_{(0,2m-1)}}{(2m)!} \right\} \psi(2) B_{2m,\psi},
\end{aligned}
$$

where μ *is over* 1 *and* κ, *and*

$$
\mathscr{B}_{n,m} = (-1)^n \cdot \frac{\mathscr{M}_{(2m-n,n-1)} - \mathscr{N}_{(2m-n,n-1)}}{(2m+1-n)!\,n!}. \tag{5.2.91}
$$

Proof. By assertion (i) of Proposition 5.2.26, Proposition 5.2.19, Proposition 5.2.21, Proposition 5.2.28, Proposition 5.2.29, and Lemma 5.2.22, it remains to show that

$$
\begin{aligned}
2\pi^2 \sum_{n=1}^{2m} (-1)^n \mathscr{B}_{n,m} \sum_{\substack{x(p) \\ x \not\equiv 0(p)}} \psi(x) \sum_{u(p)} B_{2m+1-n}(\langle (x-2u)/p \rangle) B_n(\langle u/p \rangle) \\
= 2\pi^2 \sum_{x(p)} \psi(x) \sum_{n=1}^{2m} \mathscr{B}_{n,m} \cdot \phi_{2m+1-n,n}\left(2, p; \frac{x}{p}, 0\right) \\
+ \frac{2\pi^2 B_1(1)\,\{-\mathscr{B}_{1,m} + \mathscr{B}_{2m,m}\psi(2)\}}{p^{2m-1}} \cdot B_{2m,\psi},
\end{aligned}
\tag{5.2.92}
$$

where we note that $\mathscr{N}_{(2m-1,0)} = 0\,(m \in \mathbb{N})$ by virtue of $\nu_1(u) = 0$. Taking Proposition

3.2.1 into account, we get

$$\begin{aligned}
&2\pi^2 \sum_{n=1}^{2m} (-1)^n \mathscr{B}_{n,m} \sum_{\substack{x(p)\\ x\not\equiv 0(p)}} \psi(x) \sum_{u(p)} B_{2m+1-n}(\langle (x-2u)/p\rangle) B_n(\langle u/p\rangle) \\
&\quad = 2\pi^2 \sum_{x(p)} \psi(x) \sum_{n=1}^{2m} (-1)^n \mathscr{B}_{n,m} \sum_{u(p)} P_{2m+1-n}((x-2u)/p) P_n(u/p) \\
&\qquad + 2\pi^2 \sum_{x(p)} \psi(x) B_1(1) \left\{ (-1)\mathscr{B}_{1,m} P_{2m}\left(\frac{x}{p}\right) + \mathscr{B}_{2m,m} P_{2m}\left(\frac{2^{-1}x}{p}\right) \right\} \\
&\quad = 2\pi^2 \sum_{x(p)} \psi(x) \sum_{n=1}^{2m} \mathscr{B}_{n,m} \cdot \phi_{2m+1-n,n}\left(2, p; \frac{x}{p}, 0\right) \\
&\qquad + \frac{2\pi^2 B_1(1)\left\{-\mathscr{B}_{1,m} + \mathscr{B}_{2m,m}\psi(2)\right\}}{p^{2m-1}} \cdot B_{2m,\psi}.
\end{aligned}$$

Thus (5.2.92) is established, and consequently, completes the proof. □

Proposition 5.2.31. *Let $m \in \mathbb{N}$. Then,*

$$(i) \quad \sum_{\mu} \psi(\mu)\zeta_S(1-m; C_{13}, L(\mu)) = \frac{iC(m)}{p} \left\{ \frac{4\pi^2}{m(m!\,)^2} - \frac{2\pi^2 \mathscr{M}_{(2m-1,0)}}{(2m!\,)} \right\} B_{2m,\psi},$$

$$\begin{aligned}
(ii) \quad \sum_{\mu} \psi(\mu)\zeta_S(1-m; C_{23}, L(\mu)) = \frac{iC(m)}{p} &\left\{ \frac{2^{2m+1}\pi^2}{m} \sum_{j=0}^{m} \frac{(-1)^j}{\{(m-j)!\,\}^2 (2j)!} \right. \\
&\left. + \frac{-2\pi^2 \mathscr{M}_{(0,2m-1)} + 2\pi^2 \mathscr{N}_{(0,2m-1)}}{(2m)!} \right\} \psi(2) B_{2m,\psi},
\end{aligned}$$

where μ is over 1 and κ.

Proof. By Proposition 5.2.26, Proposition 5.2.20, and Proposition 5.2.21, both assertions follow from Proposition 3.2.1. □

Proposition 5.2.32. *Let $m \in \mathbb{N}$. Then,*

$$\begin{aligned} &L_{2,S}(1-m,\psi_{H,p}) = \\ &\quad \frac{(2m-1)!}{2^{2m}p}\left\{\frac{1}{2m+1}\sum_{j=0}^{m}\frac{(-1)^j}{\{(m-j)!\}^2(2j+1)!}\right\}B_{2m+1,\psi} \\ &\quad + \frac{(2m-1)!}{p}\sum_{k=1}^{m}\frac{1}{(2m+1-2k)(2k)!}\sum_{j=k}^{m}\frac{(-1)^j}{\{(m-j)!\}^2(2j+1-2k)!} \\ &\quad \times \frac{\{B_{2k}-B_{2k}(1/2)\}2^{2m-2k}\psi(2)+B_{2k}(1/2)}{2^{2m-2k}}\cdot B_{2m+1-2k,\psi} \\ &\quad + \frac{(2m-1)!\,p^{2(m-1)}}{2^{2m+1}}\sum_{x(p)}\psi(x)\sum_{n=1}^{2m}\mathscr{B}_{n,m}\cdot\phi_{2m+1-n,n}\left(2,p;\frac{x}{p},0\right), \end{aligned}$$

where $\mathscr{B}_{n,m}$ is given by (5.2.91) and $\phi_{r,s}(h,k;x,y)$ is given by (3.1.14).

Proof. By (5.2.80), Proposition 5.2.30, and Proposition 5.2.31, we have

$$\begin{aligned} &L_{2,S}(1-m,\psi_{H,p}) = \\ &\quad \frac{(2m-1)!}{p}\sum_{j=0}^{m}\frac{(-1)^j}{\{(m-j)!\}^2} \\ &\quad \times \sum_{k=0}^{2j+1-\delta_{j,m}}\frac{\{B_k-B_k(1/2)\}2^{2m-k}\psi(2)+B_k(1/2)}{(2m+1-k)k!\,(2j+1-k)!\,2^{2m-k}}\cdot B_{2m+1-k,\psi} \\ &\quad + \frac{(2m-1)!\,p^{2(m-1)}}{2^{2m+1}}\sum_{x(p)}\psi(x)\sum_{n=1}^{2m}\mathscr{B}_{n,m}\cdot\phi_{2m+1-n,n}\left(2,p;\frac{x}{p},0\right) \\ &\quad + \frac{(2m-1)!}{4p}\left\{\frac{1}{m}\sum_{j=0}^{m}\frac{(-1)^j}{\{(m-j)!\}^2(2j)!}\right\}\psi(2)B_{2m,\psi}. \end{aligned}$$

As $B_{2j+1} = B_{2j+1}(1/2) = 0\,(j \in \mathbb{N})$, the assertion of the proposition follows. □

In fact, much more can be said. We first prove a proposition, then a lemma.

Proposition 5.2.33. *Let $m \in \mathbb{N}$. Let $\phi_{r,s}(h,k;x,y)$, $\psi_{r,s}(h,k;x,y)$ denote the Carlitz Phi and Psi functions given by (3.1.14), respectively. Let $\mathscr{B}_{n,m}$ be the numbers* *given in (5.2.91). Set*

$$\mathscr{C}_{n,m} = \sum_{j=1}^{n-1} \binom{2m+1-j}{n-j} 2^{n-j} \cdot \mathscr{B}_{j,m} \qquad (n = 1, 2, \cdots, 2m). \tag{5.2.93}$$

Then, we have

$$\begin{aligned}
&\sum_{x(p)} \psi(x) \sum_{n=1}^{2m} \mathscr{B}_{n,m} \cdot \phi_{2m+1-n,n}\left(2, p; \frac{x}{p}, 0\right) \\
&\quad = \frac{1}{p^{2m-1}} \left\{ \sum_{n=1}^{2m} \frac{(-1)^{n+1}(2m+1-n)\mathscr{C}_{n,m}}{(n+1)2^n} \right\} B_{2m+1,\psi} \\
&\quad + \frac{2^{2m}}{p^{2m-1}} \sum_{k=1}^{m} \left\{ \sum_{n=2k-1}^{2m} \frac{(-1)^{n+1}\binom{n+1}{2k}(2m+1-n)\mathscr{C}_{n,m}}{(n+1)2^n} \right\} \\
&\qquad\qquad \times \frac{\{B_{2k} - B_{2k}(1/2)\}2^{2m-2k}\psi(2) + B_{2k}(1/2)}{2^{2m-2k}} \cdot B_{2m+1-2k,\psi} \\
&\quad + \frac{1}{p^{2m-1}} \sum_{n=2}^{2m} \frac{B_n\{n\mathscr{C}_{n,m} - 2(2m+2-n)\mathscr{C}_{n-1,m}\}}{n} \cdot B_{2m+1-n,\psi}.
\end{aligned}$$

Proof. By Proposition 3.2.33, we have

$$\sum_{n=1}^{2m} \mathscr{B}_{n,m} \cdot \phi_{2m+1-n,n}\left(2, p; \frac{x}{p}, 0\right) = \sum_{n=1}^{2m} \mathscr{C}_{n,m} \cdot \psi_{2m+1-n,n}\left(2, p; \frac{x}{p}, 0\right),$$

where the numbers $\mathscr{C}_{n,m}$ are given by (5.2.93). Applying Proposition 3.2.31, Proposition 3.2.32, and Propositon 3.2.1, we get

$$\begin{aligned}
&\sum_{x(p)} \psi(x) \sum_{n=1}^{2m} \mathscr{B}_{n,m} \cdot \phi_{2m+1-n,n}\left(2, p; \frac{x}{p}, 0\right) \\
&\quad = \frac{1}{p^{2m-1}} \Bigg\{ \sum_{n=1}^{2m} \frac{(2m+1-n)2^{2m-n}\mathscr{C}_{n,m}}{n+1} \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} \\
&\qquad\qquad \times \frac{\{B_k - B_k(1/2)\}2^{2m-k}\psi(2) + B_k(1/2)}{2^{2m-k}} \cdot B_{2m+1-k,\psi} \\
&\qquad + \left[\sum_{n=1}^{2m} (-1)^n (2m+1-n) 2^{2m-n-1} \mathscr{C}_{n,m}\right] \psi(2) B_{2m,\psi} \\
&\qquad + \sum_{n=2}^{2m} \frac{B_n\{n\mathscr{C}_{n,m} - 2(2m+2-n)\mathscr{C}_{n-1,m}\}}{n} \cdot B_{2m+1-n,\psi} \Bigg\}.
\end{aligned}$$

As $B_{2j+1} = B_{2j+1}(1/2) = 0\,(j \in \mathbb{N})$, the assertion of the proposition follows. □

Lemma 5.2.34. Let $\mathscr{C}_{n,m}\,(1 \leq n \leq 2m)$ be given by (5.2.93). Then,

$$\mathscr{C}_{2n,m} = \frac{2m+2-2n}{n} \cdot \mathscr{C}_{2n-1,m} \quad (1 \leq n \leq m).$$

Proof. By the definition of $\mathscr{C}_{n,m}$, we must show

$$\mathscr{B}_{2n,m} = -\frac{1}{2n} \sum_{j=1}^{2n-1} j \binom{2m+1-j}{2n-j} 2^{2n-j} \cdot \mathscr{B}_{j,m} \qquad (1 \leq n \leq m). \tag{5.2.94}$$

We first show

$$\begin{cases}
\mu_{2n}(u) = -\dfrac{1}{2n} \displaystyle\sum_{j=1}^{2n-1} (-1)^j j \binom{2n}{j} (1+u)^{2n-j} \mu_j(u), \\
\nu_{2n}(u) = -\dfrac{1}{2n} \displaystyle\sum_{j=1}^{2n-1} (-1)^j j \binom{2n}{j} (1+u)^{2n-j} \nu_j(u) \qquad (1 \leq n \leq m).
\end{cases} \tag{5.2.95}$$

By (5.2.48) and Proposition 3.2.27, we get

$$
\begin{aligned}
&-\frac{1}{2n}\sum_{j=1}^{2n-1}(-1)^j j\binom{2n}{j}(1+u)^{2n-j}\mu_j(u)\\
&=\sum_{j=1}^{2n-1}\sum_{k=0}^{\kappa_j}(-1)^{j-1}\binom{2n-1}{j-1}\binom{j-1}{2k}(1+u)^{2n-1-2k}(-u)^k\\
&=\sum_{k=0}^{\kappa_{2n}}(1+u)^{2n-1-2k}(-u)^k\sum_{j=2k+1}^{2n-1}(-1)^{j-1}\binom{2n-1}{j-1}\binom{j-1}{2k}\\
&=\sum_{k=0}^{\kappa_{2n}}\binom{2n-1}{2k}(1+u)^{2n-1-2k}(-u)^k\\
&=\mu_{2n}(u).
\end{aligned}
$$

Similar reasoning can be applied to $\nu_{2n}(u)$, thus establishing (5.2.95). From (5.2.95), it follows that

$$
\begin{cases}
\dfrac{(1+u)^{2m-2n}\mu_{2n}(u)}{(2m+1-2n)!\,(2n)!}=-\dfrac{1}{2n}\displaystyle\sum_{j=1}^{2n-1}(-1)^j j\binom{2m+1-j}{2n-j}\dfrac{(1+u)^{2m-j}\mu_j(u)}{(2m+1-j)!\,j!},\\[2ex]
\dfrac{(1+u)^{2m-2n}\nu_{2n}(u)}{(2m+1-2n)!\,(2n)!}=-\dfrac{1}{2n}\displaystyle\sum_{j=1}^{2n-1}(-1)^j j\binom{2m+1-j}{2n-j}\dfrac{(1+u)^{2m-j}\nu_j(u)}{(2m+1-j)!\,j!}.
\end{cases}
$$

Therefore, from (5.2.74), (5.2.77), we see that

$$
\begin{cases}
\dfrac{\mathscr{M}_{(2m-2n,2n-1)}}{(2m+1-2n)!\,(2n)!}=-\dfrac{1}{2n}\displaystyle\sum_{j=1}^{2n-1} j\binom{2m+1-j}{2n-j}2^{2n-j}\dfrac{(-1)^j\mathscr{M}_{(2m-j,j-1)}}{(2m+1-j)!\,j!},\\[2ex]
\dfrac{\mathscr{N}_{(2m-2n,2n-1)}}{(2m+1-2n)!\,(2n)!}=-\dfrac{1}{2n}\displaystyle\sum_{j=1}^{2n-1} j\binom{2m+1-j}{2n-j}2^{2n-j}\dfrac{(-1)^j\mathscr{N}_{(2m-j,j-1)}}{(2m+1-j)!\,j!}.
\end{cases}
$$

This implies (5.2.94), and consequently, completes the proof. □

Proposition 5.2.35. *Let $m \in \mathbb{N}$. Then,*

(*i*) $L_{2,S}(1-m, \psi_{H,p}) =$

$$\frac{(2m-1)!}{2^{2m}p}\left\{\frac{1}{2m+1}\sum_{j=0}^{m}\frac{(-1)^j}{\{(m-j)!\}^2(2j+1)!} + \sum_{n=1}^{2m}\frac{(-1)^{n+1}(2m+1-n)\mathscr{C}_{n,m}}{(n+1)2^{n+1}}\right\}B_{2m+1,\psi}$$
$$+\frac{(2m-1)!}{p}\sum_{k=1}^{m}\left\{\frac{1}{(2m+1-2k)(2k)!}\sum_{j=k}^{m}\frac{(-1)^j}{\{(m-j)!\}^2(2j+1-2k)!} + \sum_{n=2k-1}^{2m}\frac{(-1)^{n+1}\binom{n+1}{2k}(2m+1-n)\mathscr{C}_{n,m}}{(n+1)2^{n+1}}\right\}$$
$$\times\frac{\{B_{2k}-B_{2k}(1/2)\}2^{2m-2k}\psi(2)+B_{2k}(1/2)}{2^{2m-2k}}\cdot B_{2m+1-2k,\psi},$$

where $\mathscr{C}_{n,m}$ is given by (5.2.93).

(*ii*) *Accordingly,* $L_{2,S}(1-m, \psi_{H,p}) \in \mathbb{Q}$.

(*iii*) *If* $p \equiv 1(4)$, *then* $L_{2,S}(1-m, \psi_{H,p}) = 0$,

If $p \equiv 3(4)$, *then, in particular,*

$$L_{2,S}(0, \psi_{H,p}) = \frac{11}{72p}B_{3,\psi} + \frac{1-3\psi(2)}{24p}B_{1,\psi}.$$

Proof. We first prove the assertion (i). From Proposition 5.2.32 and Proposition 5.2.33, it remains to show

$$\frac{(2m-1)!}{2^{2m+1}p}\sum_{n=2}^{2m}\frac{B_n\{n\mathscr{C}_{n,m}-2(2m+2-n)\mathscr{C}_{n-1,m}\}}{n}\cdot B_{2m+1-n,\psi} = 0.$$

This follows immediately from Lemma 5.2.34 and the fact that $B_{2n+1} = 0\,(n \in \mathbb{N})$. Thus we obtain the expression (i) for $L_{2,S}(1-m, \psi_{H,p})$. By (5.2.93), (5.2.91), Lemma 5.2.18, and (5.2.78), we see that $\mathscr{C}_{n,m} \in \mathbb{Q}\,(1 \leq n \leq 2m)$. Therefore, the assertion (ii) follows from (i). If $p \equiv 1(4)$, then $B_{k,\psi} = 0$ for any odd k. Thus the first assertion of

(iii) follows from (i). By (5.2.74), (5.2.48), and (5.2.75), we have

$$\mathscr{M}_{(1,0)} = \frac{1}{2}\mathscr{M}_{(0,1)} = \frac{1}{\pi i}\int_{I_\epsilon(1)} \log u \cdot u^{-2}(1-u^2)du = -4,$$

and by (5.2.77), (5.2.48) (where we see that $\nu_1(u) = \nu_2(u) = 0$), we have $\mathscr{N}_{(1,0)} = \mathscr{N}_{(0,1)} = 0$. Hence, by (5.2.91), $\mathscr{B}_{1,1} = 2$, $\mathscr{B}_{2,1} = -4$, and by (5.2.93), $\mathscr{C}_{1,1} = 2$, $\mathscr{C}_{2,1} = 4$. Thus the second assertion of (iii) follows from (i). □

We are now in position to prove Theorem 5.2 introduced at the beginning of Section 5.2.

Proof of Theorem 5.2. This follows immediately from Proposition 5.2.24, Proposition 5.2.35, and (5.2.81). □

We now give a formula for all of the special values of $L_2(s, \psi_{H,p})$ at non-positive integer values of s.

Theorem 5.2.36. *Let $m \in \mathbb{N}$. With the numbers $c(k_1, k_2, k_3)$, $S_p(k_1, k_2, k_3)$, $\mathscr{C}_{n,m}$*

given by (5.2.84), (5.2.93), we have

$$
\begin{aligned}
L_2(1-m,\psi_{H,p}) = &-\frac{(2m-1)!\,p^{2(m-1)}}{2^{2m}} \sum_{k_1,k_2,k_3}'' c(k_1,k_2,k_3)S_p(k_1,k_2,k_3) \\
&+ \frac{(2m-1)!}{2^{2m}p}\Bigg\{\frac{1}{2m+1}\sum_{j=0}^{m}\frac{(-1)^j}{\{(m-j)!\}^2(2j+1)!} \\
&\quad + \sum_{n=1}^{2m}\frac{(-1)^{n+1}(2m+1-n)\mathscr{C}_{n,m}}{(n+1)2^{n+1}}\Bigg\}B_{2m+1,\psi} \\
&+ \frac{(2m-1)!}{3\cdot 2^{2m+2}p}\Bigg\{\Bigg[\frac{2}{2m-1}\sum_{j=1}^{m}\frac{(-1)^j}{\{(m-j)!\}^2(2j-1)!} \\
&\quad + \sum_{n=1}^{2m}\frac{(-1)^{n+1}n(2m+1-n)\mathscr{C}_{n,m}}{2^n}\Bigg]\times\Big(3\cdot 2^{2m-2}\psi(2)-1\Big) \\
&\quad + p\Big(3c(1,1,2m-1)-c(1,2m-1,1)\delta_{p,3}\Big)\Bigg\}B_{2m-1,\psi}, \\
&+ \frac{(2m-1)!}{p}\sum_{k=2}^{m}\Bigg\{\frac{1}{(2m+1-2k)(2k)!}\sum_{j=k}^{m}\frac{(-1)^j}{\{(m-j)!\}^2(2j+1-2k)!} \\
&\quad + \sum_{n=2k-1}^{2m}\frac{(-1)^{n+1}\binom{n+1}{2k}(2m+1-n)\mathscr{C}_{n,m}}{(n+1)2^{n+1}}\Bigg\} \\
&\quad \times\frac{\{B_{2k}-B_{2k}(1/2)\}2^{2m-2k}\psi(2)+B_{2k}(1/2)}{2^{2m-2k}}\cdot B_{2m+1-2k,\psi},
\end{aligned}
$$

where (k_1,k_2,k_3) runs over all triples of integers with $k_2 \geq 0$, $k_1,k_3 \geq 1$, and $k_1+k_2+k_3 = 2m+1$.

Proof. This follows immediately from Proposition 5.2.24, Proposition 5.2.35, and (5.2.81). □

While Arakawa[1] gave an explicit formula for the special value of $L_2^*(s,\psi_{H,p})$ at $s=0$, he only proved the rationality of the other special values of $L_2^*(s,\psi_{H,p})$ at negative integer values of s. Therefore, we will give a formula for all of the special values of $L_2^*(s,\psi_{H,p})$ at non-positive integer values of s. The process of finding the special values of $L_2^*(s,\psi_{H,p})$ is similar to that of $L_2(s,\psi_{H,p})$, so we omit the proof of the following theorem.

We put

$$S_p^*(k_1,k_2,k_3)=\sum_{\alpha,\gamma(p)} P_{k_1}\left(\frac{\alpha^2-2\alpha\gamma}{p}\right)P_{k_2}\left(\frac{2\alpha\gamma}{p}\right)P_{k_3}\left(\frac{\gamma^2-\alpha^2}{p}\right). \tag{5.2.96}$$

Theorem 5.2.37. *Let $m\in\mathbb{N}$. With the numbers $c(k_1,k_2,k_3)$, $S_p^*(k_1,k_2,k_3)$, $\mathscr{C}_{n,m}$ given by (5.2.84), (5.2.96), (5.2.93), we have*

$$\begin{aligned}
L_2^*(1-m,\psi_{H,p}) &= -\frac{(2m-1)!\,p^{2(m-1)}}{2^{2m}}\sideset{}{''}\sum_{k_1,k_2,k_3}\frac{c(k_1,k_2,k_3)}{2^{k_2-1}}S_p^*(k_1,k_2,k_3)\\
&\quad+\frac{(2m-1)!}{2^{2m-1}p}\left\{\frac{1}{2m+1}\sum_{j=0}^{m}\frac{(-1)^j}{\{(m-j)!\}^2(2j+1)!}\right.\\
&\quad\left.+\sum_{n=1}^{2m}\frac{(-1)^{n+1}(2m+1-n)\mathscr{C}_{n,m}}{(n+1)2^{n+1}}\right\}B_{2m+1,\psi}\\
&\quad+\frac{(2m-1)!}{3\cdot 2^{2m+2}p}\left\{\frac{2}{2m-1}\sum_{j=1}^{m}\frac{(-1)^j}{\{(m-j)!\}^2(2j-1)!}\right.\\
&\quad+\sum_{n=1}^{2m}\frac{(-1)^{n+1}n(2m+1-n)\mathscr{C}_{n,m}}{2^n}\\
&\quad\left.+p\left(3c(1,1,2m-1)+\frac{c(1,2m-1,1)}{2^{2m-2}}\delta_{p,3}\right)\right\}B_{2m-1,\psi},\\
&\quad+\frac{(2m-1)!}{2^{2m-1}p}\sum_{k=2}^{m}\left\{\frac{1}{(2m+1-2k)(2k)!}\sum_{j=k}^{m}\frac{(-1)^j}{\{(m-j)!\}^2(2j+1-2k)!}\right.\\
&\quad\left.+\sum_{n=2k-1}^{2m}\frac{(-1)^{n+1}\binom{n+1}{2k}(2m+1-n)\mathscr{C}_{n,m}}{(n+1)2^{n+1}}\right\}B_{2k}\cdot B_{2m+1-2k,\psi},
\end{aligned}$$

where (k_1,k_2,k_3) runs over all triples of integers with $k_2\geq 0$, $k_1,k_3\geq 1$, and $k_1+k_2+k_3=2m+1$.

5.3 Ibukiyama's Evaluation of $L_2(s, \psi_{H,p})$

First, we review the definition of $L_2(s, \psi_{H,p})$. Let L_2 (resp L_2^*) denote the lattice formed by 2×2 integral symmetric (resp. half-integral symmetric) matrices, and let $L_{2,+}$, $L_{2,+}^*$ be the subsets consisting of all positive definite matrices of L_2, L_2^*, respectively. We fix an odd prime integer p. We denote by ψ the Legendre symbol mod p: $\psi(a) = \left(\frac{a}{p}\right)$. For each $T \in L_2^*$, we define $\psi_{H,p}(T)$ as follows: We put $\psi_{H,p}(T) = 0$, if $\det(T) \not\equiv 0 \bmod p$. When $\det(T) \equiv 0(p)$, we have ${}^t gTg \equiv \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right) \bmod p$ for some $g \in SL_2(\mathbb{F}_p)$ and $a \in \mathbb{F}_p$, and we put $\psi_{H,p}(T) = \psi(a)$. Then $L_2(s, \psi_{H,p})$ is defined by

$$L_2(s, \psi_{H,p}) = \sum_{T \in L_{2,+}/SL_2(\mathbb{Z})} \frac{\psi_{H,p}(T)}{\epsilon(T)(\det(T))^s},$$

where $L_{2,+}/SL_2(\mathbb{Z})$ denotes the representatives of $SL_2(\mathbb{Z})$-equivalence classes in $L_{2,+}$ and $\epsilon(T) = \#\{g \in SL_2(\mathbb{Z}) \mid {}^t gTg = T\}$.

The following result was proved by Ibukiyama.

Theorem 5.3 [9]. *With notation and assumptions being the same as above, we get*

$$L_2(s, \psi_{H,p}) = -\frac{B_{1,\psi}}{p^s}\zeta(2s-1).$$

Before proving this, we first prove a lemma. If $T \in L_{2,+}^*$, then $4\det(T) = -d_K f^2$ for some positive integer f and the fundamental discriminant d_K of some imaginary quadratic field K. For any such $d = d_K f^2$, we denote by $\mathscr{P}(d)$ the set of primitive matrices of $L_{2,+}^*$ with $4\det(T) = -d$:

$$\mathscr{P}(d) = \left\{ \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in L_{2,+}^* :\ 4ac - b^2 = -d,\ (a,b,c) = 1 \right\}.$$

Denote by $\mathscr{S}(d) = \mathscr{P}(d)/\sim$ the set of $SL_2(\mathbb{Z})$-equivalence classes in $\mathscr{P}(d)$.

Remark. If $T = \left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right) \in \mathscr{P}(d)$, then $\psi_{H,p}(T) = \psi(a)$ if $p \nmid a$ and $\psi_{H,p}(T) =$

$\psi(c)$ if $p \mid a$. Indeed, if $p \mid a$, then since $\mathrm{rank}(T \bmod p) = 1$, we have $p \mid b$ and $c \in \mathbb{F}_p^\times$, and if $p \nmid a$, then we have

$$\begin{pmatrix} 1 & 0 \\ -\frac{b}{2a} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} 1 & -\frac{b}{2a} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & \frac{4ac-b^2}{4a} \end{pmatrix} \equiv \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \bmod p.$$

This shows that for each equivalence class in $\mathscr{S}(d)$, we can choose a representative $T = \left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right) \in \mathscr{P}(d)$ such that $\psi_{H,p}(T) = \psi(a)$. This will prove useful in the lemma below.

We state without proof the following Proposition.

Proposition 5.3.1 [2]. *Let K be a quadratic number field, and let d_K denote its fundamental discriminant. Let χ be a primitive Dirichlet character with conductor $|d_K|$, and let χ_K be the primitive Dirichlet character with conductor $|d_K|$ associated to the quadratic number field K: $\chi_K(n) = \left(\frac{d_K}{n}\right)$. Let $O_{K,f} = \mathbb{Z} + fO_K$ denote the order of K with conductor f. Then,*

$$\chi = \chi_K \Leftrightarrow \chi(N(I)) = 1 \textit{ for any ideal } I \textit{ of } O_{K,f} \textit{ with } (N(I), d_K) = 1.$$

Lemma 5.3.2 [9]. *With notation and assumptions being the same as above, we get*

(i) *If $\mathbb{Q}(\sqrt{d}) \neq \mathbb{Q}(\sqrt{\psi(-1)p})$, then* $\displaystyle\sum_{T \in \mathscr{S}(d)} \psi_{H,p}(T) = 0,$

(ii) *If $\mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{\psi(-1)p})$ (which can only occur if $p \equiv 3(4)$), then $\psi_{H,p}(T) = 1$ for any $T \in \mathscr{P}(d)$.*

Proof. This can be proved by using the well-known relation between $\mathscr{S}(d)$ and proper ideal classes of the order $O_{K,f}$ of K with conductor f. We review this shortly. For any $T = \left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right) \in \mathscr{P}(d)$, put $I(T) = \mathbb{Z}a + \mathbb{Z}(b + \sqrt{d})/2$. Then, $I(T)$ is the proper primitive $O_{K,f}$ ideal and $\mathrm{Norm}(I(T)) = \#(O_{K,f}/I(T)) = a$. Through this mapping, we get a bijection $\mathscr{S}(d) \cong \mathrm{Cl}(O_{K,f})$, where $\mathrm{Cl}(O_{K,f})$ is the class group of the proper $O_{K,f}$ ideals prime to f. If $p \nmid d$, then by definition $\psi_{H,p} = 0$ for any $T \in \mathscr{S}(d)$, and the above assertion is trivial. Next we assume that $p \mid d$. From the remark made before Proposition 5.3.1, we see that for each equivalence class in $\mathscr{S}(d)$, we can choose a representative $T \in \mathscr{P}(d)$ such that $\psi_{H,p}(T) = \psi(\mathrm{Norm}(I(T)))$ with $(\mathrm{Norm}(I(T)), p) = 1$. If $\mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{\psi(-1)p})$, then $\psi = \chi_{Q(\sqrt{\psi(-1)p}}$, and it follows by Proposition 5.3.1 that $\psi_{H,p}(T) = 1$ for any $T \in \mathscr{S}(d)$. If on the other hand, $\mathbb{Q}(\sqrt{d}) \neq \mathbb{Q}(\sqrt{\psi(-1)p})$, then $\psi \neq \chi_{Q(\sqrt{\psi(-1)p}}$, and it follows again by Proposition 5.3.1 that ψ induces a nontrivial character on $\mathrm{Cl}(O_{K,f})$. Since $\mathrm{Cl}(O_{K,f})$ is a group, we get the assertion (i). □

Proof of Theorem 5.3. First, it is obvious that $\psi_{H,p}(eT) = \psi(e)\psi_{H,p}(T)$ for any integer e prime to p and any $T \in L^*_{2,+}$. Moreover, we see that $\epsilon(eT) = \epsilon(T)$ for any $T \in L^*_{2,+}$. Noting that $L_{2,+} = \{T \in L^*_{2,+} \mid 4\det(T) \equiv 0(2)\}$, we can reduce $L_2(s, \psi_{H,p})$ to a sum over primitive matrices of $L^*_{2,+}$:

$$\begin{aligned}
L_2(s, \psi_{H,p}) &= \sum_{\substack{T \in L^*_{2,+}/SL_2(\mathbb{Z}) \\ 4\det(T) \equiv 0(2)}} \frac{\psi_{H,p}(T)}{\epsilon(T)(\det(T))^s} \\
&= \sum_{e=1}^{\infty} \frac{\psi(e)}{e^{2s}} \sum_{\substack{d=1 \\ d \text{ even}}}^{\infty} \sum_{T \in \mathscr{S}(-d)} \frac{\psi_{H,p}(T)}{\epsilon(T)(d/4)^s} \\
&= 2^{2s} L(2s, \psi) \sum_{\substack{d=1 \\ d \text{ even}}}^{\infty} \sum_{T \in \mathscr{S}(-d)} \frac{\psi_{H,p}(T)}{\epsilon(T) d^s}.
\end{aligned}$$

Since $\epsilon(T)$ has a common value for all elements in $\mathscr{P}(-d)$, we denote this by $\epsilon(-d)$. Applying Lemma 5.3.2, we see that only the part for T with $4\det(T) = d = pf^2$

$(f \in \mathbb{N})$ remains alive. Hence, we get

$$L_2(s, \psi_{H,p}) = \frac{2^{2s}}{p^s} L(2s, \psi) \sum_{\substack{f=1 \\ f \text{ even}}}^{\infty} \frac{|\mathscr{S}(-pf^2)|}{\epsilon(-pf^2) f^{2s}}.$$

By virtue of the relation between $\mathscr{S}(d_K f^2)$ and proper ideal classes of the order $O_{K,f}$ of K with conductor f, $|\mathscr{S}(-pf^2)|$ is the class number $h(-pf^2) = \#\mathrm{Cl}(O_{K,f})$ of the order $O_{K,f}$ of the imaginary quadratic number field $K = \mathbb{Q}(\sqrt{-p})$. If $p \equiv 3(4)$ and $K = \mathbb{Q}(\sqrt{-p})$, it is well known that the class number $h(-pf^2)$ of $O_{K,f}$ is given by

$$h(-pf^2) = \frac{h(-p)f}{[O_K^\times : O_{K,f}^\times]} \prod_{\substack{q|f \\ q \text{ prime}}} \left(1 - \frac{\psi(q)}{q}\right),$$

where O_K denotes the maximal order of K. Since $\epsilon(T) = \#(O_{K,f}^\times)$ for $T \in \mathscr{P}(-pf^2)$, and $B_{1,\psi} = -2h(-p)/\#(O_K^\times)$, we have

$$\begin{aligned}
L_2(s, \psi_{H,p}) &= -\frac{2^{2s-1} B_{1,\psi}}{p^s} L(2s, \psi) \sum_{\substack{f=1 \\ f \text{ even}}}^{\infty} \frac{1}{f^{2s-1}} \prod_{\substack{q|f \\ q \text{ prime}}} \left(1 - \frac{\psi(q)}{q}\right) \\
&= -\frac{2^{2s-1} B_{1,\psi}}{p^s} L(2s, \psi) \sum_{\substack{f=1 \\ f \text{ even}}}^{\infty} \frac{1}{f^{2s-1}} \prod_{\substack{q|f \\ q \text{ prime}}} \left(1 + \frac{\psi(q)\mu(q)}{q}\right) \\
&= -\frac{2^{2s-1} B_{1,\psi}}{p^s} L(2s, \psi) \sum_{\substack{f=1 \\ f \text{ even}}}^{\infty} \frac{1}{f^{2s-1}} \sum_{m|f} \frac{1}{m} \psi(m)\mu(m).
\end{aligned}$$

Letting $f = 2mn$ in the sum above, we get

$$\begin{aligned}
L_2(s, \psi_{H,p}) &= -\frac{B_{1,\psi}}{p^s} L(2s, \psi) \sum_{n,m=1}^{\infty} \frac{1}{n^{2s-1} m^{2s}} \psi(m)\mu(m) \\
&= -\frac{B_{1,\psi}}{p^s} L(2s, \psi) \zeta(2s-1) \prod_{\substack{q \\ q \text{ prime}}} \left(1 - \frac{\psi(q)}{q^{2s}}\right) \\
&= -\frac{B_{1,\psi}}{p^s} \zeta(2s-1).
\end{aligned}$$

□

We give two corollaries to Theorem 5.3.

Corollary 5.3.3. *Let $m \in \mathbb{N}$. Then,*

$$L_2(1-m, \psi_{H,p}) = \frac{p^{m-1} B_{2m} B_{1,\psi}}{2m}.$$

In particular, we get $L_2(0, \psi_{H,p}) = \frac{1}{12} B_{1,\psi}$.

Proof. This follows immediately from the fact that $\zeta(1-2m) = -\frac{B_{2m}}{2m}$ $(m \in \mathbb{N})$.

□

Corollary 5.3.4. *$L_2(s, \psi_{H,p})$ is absolutely convergent for Re(s)$> \frac{3}{2}$ and can be continued analytically to a meromorphic function in the whole complex plane which is holomorphic everywhere except at $s = 1$, where $L_2(s, \psi_{H,p})$ has a simple pole with residue $-\frac{B_{1,\psi}}{2p}$.*

Proof. This follows immediately from the properties of the Riemann zeta function and the fact that $\lim_{s \to 1}(s-1)\zeta(s) = 1$. □

Since the evaluation of $L_2^*(s, \psi_{H,p})$ is similar to that of $L_2(s, \psi_{H,p})$, we omit the proof of the following theorem.

Theorem 5.3.5 [9]. *With notation and assumptions being the same as above, we get*

$$L_2^*(s, \psi_{H,p}) = -\frac{2^{2s-1} B_{1,\psi}}{p^s} \zeta(2s-1).$$

We give two corollaries to Theorem 5.3.5.

Corollary 5.3.6 [9]. *Let $m \in \mathbb{N}$. Then,*

$$L_2^*(1-m, \psi_{H,p}) = \frac{p^{m-1} B_{2m} B_{1,\psi}}{2^{2m} \cdot m}.$$

Proof. This follows immediately from the fact that $\zeta(1-2m) = -\frac{B_{2m}}{2m}$ $(m \in \mathbb{N})$.

□

Corollary 5.3.7 [1]. *$L_2^*(s, \psi_{H,p})$ is absolutely convergent for Re(s)$> \frac{3}{2}$ and can be continued analytically to a meromorphic function in the whole complex plane which is holomorphic everywhere except at $s = 1$, where $L_2^*(s, \psi_{H,p})$ has a simple pole with residue $-\frac{B_{1,\psi}}{p}$.*

Proof. This follows immediately from the properties of the Riemann zeta function and the fact that $\lim_{s\to 1}(s-1)\zeta(s) = 1$. □

5.4 Proof of Theorem 5 and its Corollaries

We shall now prove Theorem 5 introduced at the beginning of Chapter 5. We fix an odd prime p. We denote by ψ the Legendre symbol mod p: $\psi(a) = \left(\frac{a}{p}\right)$.

Proof of Theorem 5 (The Dual Lee-Weintraub Identity). By Theorem 5.1, Theorem 5.2, and Theorem 5.3, we get

$$\begin{aligned} I_p &= \sqrt{\psi(-1)p}\left(-6p\left(Q_p + \frac{1}{6}h_p\right) - \frac{3}{4}B_{3,\psi} - \frac{3}{4}B_{2,\psi} - \frac{9-3\psi(2)-2p\delta_{p,3}}{4}B_{1,\psi}\right) \\ &= \sqrt{\psi(-1)p}\left(-6p\,L_2(0,\psi_{H,p}) + \frac{1}{6}B_{3,\psi} - \frac{3}{4}B_{2,\psi} + \frac{3p-4}{2}B_{1,\psi}\right) \\ &= \sqrt{\psi(-1)p}\left(\frac{1}{6}B_{3,\psi} - \frac{3}{4}B_{2,\psi} + (p-2)B_{1,\psi}\right). \end{aligned}$$

□

As an immediate corollary to Theorem 5, we prove the following conjecture of Ibukiyama (see Example 8.19 and Remark 8.20 in [2]).

Corollary 5.4.1 *We put* $T = \{(a,b,c) \in \mathbb{Z}^3 \mid 1 \le a,b,c \le p-1,\ ab+bc+ca \equiv 0(p)\}$ *and*

$$A_p = \sum_{(a,b,c)\in T} abc\,\psi(abc).$$

Then, we get

$$A_p = -\frac{p^2}{6}B_{3,\psi} - \frac{3p^3}{4}B_{2,\psi} + \frac{p^2(p+1)}{2}B_{1,\psi}.$$

Proof. Observe that this is the same A_p given by (5.1.3). By (5.1.4) and Theorem 5, we get

$$\begin{aligned} A_p &= \frac{p^2}{\psi(-1)}\left(\frac{1}{\sqrt{\psi(-1)p}}I_p - \frac{3(p-1)}{4}(B_{2,\psi} + 2B_{1,\psi})\right) \\ &= -\frac{p^2}{6}B_{3,\psi} - \frac{3p^3}{4}B_{2,\psi} + \frac{p^2(p+1)}{2}B_{1,\psi}. \end{aligned}$$

□

We put

$$S = \{(a,b,c) \in \mathbb{Z}^3 \mid 1 \le a,b,c \le p-1,\ 4ab - (c-a-b)^2 \equiv 0(p)\},$$
$$T = \{(a,b,c) \in \mathbb{Z}^3 \mid 1 \le a,b,c \le p-1,\ ab + bc + ca \equiv 0(p)\},$$

and consider the following four character sums

$$S_p = \sum_{(a,b,c)\in S} P_1\left(\frac{a}{p}\right) P_1\left(\frac{b}{p}\right) P_1\left(\frac{c}{p}\right) \psi(abc),$$
$$T_p = \sum_{(a,b,c)\in T} P_1\left(\frac{a}{p}\right) P_1\left(\frac{b}{p}\right) P_1\left(\frac{c}{p}\right) \psi(abc)$$
$$\mathscr{S}_p = \sum_{(a,b,c)\in S} c_1\left(\frac{a}{p}\right) c_1\left(\frac{b}{p}\right) c_1\left(\frac{c}{p}\right) \psi(abc),$$
$$\mathscr{T}_p = \sum_{(a,b,c)\in T} c_1\left(\frac{a}{p}\right) c_1\left(\frac{b}{p}\right) c_1\left(\frac{c}{p}\right) \psi(abc).$$

Proposition 5.4.2. *With notation and assumptions being the same as above, we have*

$$(i) \quad S_p = \sum_{k,t(p)} P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{t^2}{p}\right) P_1\left(\frac{(k+t)^2}{p}\right),$$
$$(ii) \quad T_p = \sum_{k,t(p)} P_1\left(\frac{kt}{p}\right) P_1\left(\frac{k^2+kt}{p}\right) P_1\left(\frac{t^2+kt}{p}\right)$$
$$(iii) \quad \mathscr{S}_p = \sum_{k,t(p)} c_1\left(\frac{k^2}{p}\right) c_1\left(\frac{t^2}{p}\right) c_1\left(\frac{(k+t)^2}{p}\right),$$
$$(iv) \quad \mathscr{T}_p = \sum_{k,t(p)} c_1\left(\frac{kt}{p}\right) c_1\left(\frac{k^2+kt}{p}\right) c_1\left(\frac{t^2+kt}{p}\right).$$

Proof. All of the assertions are clear in the case of $p \equiv 1(4)$ since everything vanishes. (The left sides vanish by the usual even/odd argument and the right sides vanish by replacing (k,t) by $(k\sqrt{-1}, t\sqrt{-1})$). Thus we assume that $p \equiv 3(4)$. We observe the 1-to-1 correspondence between the sets

$$S = \{(a,b,c) \in \mathbb{Z}^3 \mid 1 \leq a,b,c \leq p-1,\ 4ab-(c-a-b)^2 \equiv 0(p)\}$$

and

$$\{(k, kt^2, k(t+1)^2) \in (\mathbb{F}_p^\times)^3 \mid 1 \leq k,t \leq p-1,\ t \neq p-1\}.$$

Substituting $(k, kt^2, k(t+1)^2)$ for (a,b,c) in S_p, we get

$$S_p = \sum_{k,t(p)} P_1\left(\frac{k}{p}\right) P_1\left(\frac{kt^2}{p}\right) P_1\left(\frac{k(t+1)^2}{p}\right) \psi(k).$$

Observe that, without the character, the sum on the right would vanish due to the usual even/odd argument. Hence, adding this vanishing sum to both sides yields

$$S_p = \sum_{k,t(p)} P_1\left(\frac{k^2}{p}\right) P_1\left(\frac{k^2t^2}{p}\right) P_1\left(\frac{k^2(t+1)^2}{p}\right).$$

Thus the assertion (i) follows from replacing t by $k^{-1}t$ where $k \not\equiv 0$. The assertion (iii) is similarly verified, so the proof is omitted. Next, we observe the 1-to-1 correspondence between the sets

$$T = \{(k,l,m) \in \mathbb{Z}^3 \mid 1 \leq k,l,m \leq p-1,\ kl+lm+mk \equiv 0(p)\}$$

and

$$\{(-kt, k(t+1), kt(t+1)) \in (\mathbb{F}_p^\times)^3 \mid 1 \leq k,t \leq p-1,\ t \neq p-1\}.$$

Substituting $(-kt, k(t+1), kt(t+1))$ for (a,b,c) in T_p, we get

$$T_p = \sum_{k,t(p)} P_1\left(\frac{kt}{p}\right) P_1\left(\frac{k(t+1)}{p}\right) P_1\left(\frac{kt(t+1)}{p}\right) \psi(k).$$

As before, observe that without the character, the sum on the right would vanish due to the usual even/odd argument. Hence, adding this vanishing sum to both sides yields

$$T_p = \sum_{k,t(p)} P_1\left(\frac{k^2t}{p}\right) P_1\left(\frac{k^2(t+1)}{p}\right) P_1\left(\frac{k^2t(t+1)}{p}\right).$$

Thus the assertion (ii) follows from replacing t by $k^{-1}t$ where $k \not\equiv 0$. The assertion (iv) is similarly verified, so the proof is omittted. □

Recall that the sets S and T are dual to each other since the quadratic forms in their respective congruence conditions are dual to each other. This implies the following proposition,

Proposition 5.4.3. *Let f_1, f_2, f_3 be periodic functions with period p, and let $\hat{f}_1, \hat{f}_2, \hat{f}_3$ be their respective finite Fourier transforms. Then, we have*

$$\begin{aligned}
&\sum_{(a,b,c)\in T} f_1\left(\frac{a}{p}\right) f_2\left(\frac{a}{p}\right) f_3\left(\frac{a}{p}\right) \psi(abc) \\
&= \frac{\psi(-1)p\,\tau(\psi)}{p^3}\Bigg\{ \sum_{(a,b,c)\in S} \hat{f}_1\left(\frac{a}{p}\right) \hat{f}_2\left(\frac{a}{p}\right) \hat{f}_3\left(\frac{a}{p}\right) \psi(abc) \\
&+ \hat{f}_1(0)\sum_{a(p)} \hat{f}_2\left(\frac{a}{p}\right) \hat{f}_3\left(\frac{a}{p}\right) \psi(a) + \hat{f}_2(0)\sum_{a(p)} \hat{f}_1\left(\frac{a}{p}\right) \hat{f}_3\left(\frac{a}{p}\right) \psi(a) \\
&+ \hat{f}_3(0)\sum_{a(p)} \hat{f}_1\left(\frac{a}{p}\right) \hat{f}_2\left(\frac{a}{p}\right) \psi(a) - \frac{1}{p}\sum_{a,b,c(p)} \hat{f}_1\left(\frac{a}{p}\right) \hat{f}_2\left(\frac{b}{p}\right) \hat{f}_3\left(\frac{c}{p}\right) \psi(c) \\
&- \frac{p^3 f_3(0)}{p\,\tau(\psi)}\left(f_1(0)\sum_{a(p)} f_2\left(\frac{a}{p}\right) \psi(a) + f_2(0)\sum_{a(p)} f_1\left(\frac{a}{p}\right) \psi(a) \right)\Bigg\}.
\end{aligned}$$

Proof. Recall the 1-to-1 correspondence between the sets T and $\{(-kt, k(t+1), kt(t+1)) \in (\mathbb{F}_p^\times)^3 \mid 1 \le k, t \le p-1,\ t \ne p-1\}$. Substituting $(-kt, k(t+1), kt(t+1))$

for (a,b,c), we get

$$\sum_{(a,b,c)\in T} f_1\left(\frac{a}{p}\right) f_2\left(\frac{a}{p}\right) f_3\left(\frac{a}{p}\right) \psi(abc)$$

$$= \psi(-1)\sum_{k,t(p)} f_1\left(\frac{-kt}{p}\right) f_2\left(\frac{k(t+1)}{p}\right) f_3\left(\frac{kt(t+1)}{p}\right) \psi(kt^2(t+1)^2)$$

(5.4.1)

$$= \psi(-1)\Bigg\{\sum_{k,t(p)} f_1\left(\frac{-kt}{p}\right) f_2\left(\frac{k(t+1)}{p}\right) f_3\left(\frac{kt(t+1)}{p}\right) \psi(k)$$

$$- f_3(0)\left(f_1(0)\sum_{k(p)} f_2\left(\frac{k}{p}\right)\psi(k) + f_2(0)\sum_{k(p)} f_1\left(\frac{k}{p}\right)\psi(k)\right)\Bigg\}.$$

Let

$$M_p = \sum_{k,t(p)} f_1\left(\frac{-kt}{p}\right) f_2\left(\frac{k(t+1)}{p}\right) f_3\left(\frac{kt(t+1)}{p}\right) \psi(k).$$

Applying the finite Fourier transform to f_1, f_2, f_3 (Proposition 3.1.12), we get

$$M_p = \frac{1}{p^3}\sum_{k,l,m(p)} \hat{f}_1\left(\frac{k}{p}\right) \hat{f}_2\left(\frac{l}{p}\right) \hat{f}_3\left(\frac{m}{p}\right) \sum_{k,t(p)} \zeta^{-dkt+ek(t+1)+fkt(t+1)}\psi(k)$$

$$= \frac{1}{p^3}\sum_{k,l,m(p)} \hat{f}_1\left(\frac{k}{p}\right) \hat{f}_2\left(\frac{l}{p}\right) \hat{f}_3\left(\frac{m}{p}\right) \sum_{k,t(p)} \zeta^{k\{-dt+e(t+1)+ft(t+1)\}}\psi(k)$$

$$= \frac{\tau(\psi)}{p^3}\sum_{k,l,m(p)} \hat{f}_1\left(\frac{k}{p}\right) \hat{f}_2\left(\frac{l}{p}\right) \hat{f}_3\left(\frac{m}{p}\right) \sum_{t(p)} \psi(mt^2+(l+m-k)t+l).$$

Let $\Delta = (k-l-m)^2 - 4ml$. By Proposition 3.2.16, we see that

$$\sum_{t(p)} \psi(mt^2+(l+m-k)t+l) = \begin{cases} p\psi(k), & \text{if } m\equiv 0(p),\ l\equiv k(p), \\ 0, & \text{if } m\equiv 0(p),\ l\not\equiv k(p), \\ -\psi(m), & \text{if } m\not\equiv 0(p),\ \Delta\not\equiv 0(p) \\ (p-1)\psi(m), & \text{if } m\not\equiv 0(p),\ \Delta\equiv 0(p). \end{cases}$$

Hence, we get

$$M_p = \frac{p\,\tau(\psi)}{p^3}\Bigg\{\hat{f}_3(0)\sum_{k(p)} \hat{f}_1\left(\frac{k}{p}\right)\hat{f}_2\left(\frac{k}{p}\right)\psi(k) - \frac{1}{p}\sum_{k,l,m(p)} \hat{f}_1\left(\frac{k}{p}\right)\hat{f}_2\left(\frac{l}{p}\right)\hat{f}_3\left(\frac{m}{p}\right)\psi(m) + \sum_{\substack{k,l,m(p)\\ \Delta\equiv 0(p)}} \hat{f}_1\left(\frac{k}{p}\right)\hat{f}_2\left(\frac{l}{p}\right)\hat{f}_3\left(\frac{m}{p}\right)\psi(m)\Bigg\}. \tag{5.4.2}$$

Observe that

$$\begin{aligned}\sum_{\substack{k,l,m(p)\\ \Delta\equiv 0(p)}} &\hat{f}_1\left(\frac{k}{p}\right)\hat{f}_2\left(\frac{l}{p}\right)\hat{f}_3\left(\frac{m}{p}\right)\psi(m)\\ &= \sum_{(k,l,m)\in S} \hat{f}_1\left(\frac{k}{p}\right)\hat{f}_2\left(\frac{l}{p}\right)\hat{f}_3\left(\frac{m}{p}\right)\psi(m)\\ &+ \hat{f}_1(0)\sum_{m(p)} \hat{f}_2\left(\frac{m}{p}\right)\hat{f}_3\left(\frac{m}{p}\right)\psi(m) + \hat{f}_2(0)\sum_{m(p)} \hat{f}_1\left(\frac{m}{p}\right)\hat{f}_3\left(\frac{m}{p}\right)\psi(m).\end{aligned}$$

Since $\psi(m) = \psi(klm)$ for any $(k,l,m) \in S$, the assertion of the proposition follows from plugging back into (5.4.2) and (5.4.1). □

Thus, the sum of products of f's taken over T is "roughly" equal to the sum of products of $\hat{f}$'s taken over S. When f_1, f_2, f_3 are all odd p-periodic functions (and therefore so are $\hat{f}_1, \hat{f}_2, \hat{f}_3$ by Proposition 3.1.14), then $f_1(0) = f_2(0) = f_3(0) = \hat{f}_1(0) = \hat{f}_2(0) = \hat{f}_3(0) = 0$, and Proposition 5.4.3 reveals that

$$\begin{aligned}\sum_{(a,b,c)\in T} &f_1\left(\frac{a}{p}\right) f_2\left(\frac{a}{p}\right) f_3\left(\frac{a}{p}\right)\psi(abc)\\ &= \frac{\psi(-1)p\,\tau(\psi)}{p^3}\sum_{(a,b,c)\in S} \hat{f}_1\left(\frac{a}{p}\right)\hat{f}_2\left(\frac{a}{p}\right)\hat{f}_3\left(\frac{a}{p}\right)\psi(abc).\end{aligned} \tag{5.4.3}$$

From the work of Ibukiyama[9], we get

Proposition 5.4.4. *With notation and assumptions being the same as above, we* *have*

$$(i) \quad S_p = -\frac{1}{6p}B_{3,\psi} - \frac{p-2}{4p}B_{1,\psi},$$
$$(ii) \quad \mathscr{T}_p = 8\pi^3 p^{3/2}\left(\frac{1}{6p}B_{3,\psi} + \frac{p-2}{4p}B_{1,\psi}\right).$$

Proof. The assertion is clear for $p \equiv 1(4)$ since everything vanishes by the usual even/odd argument. Thus we assume that $p \equiv 3(4)$. We put

$$D_p = \sum_{(a,b,c)\in S} abc\,\psi(abc).$$

From Corollary 1.3 of Ibukiyama[9], we have, for $p \equiv 3(4)$,

$$D_p = -\frac{p^2}{6}B_{3,\psi} + \frac{p^2(3p-1)(p-2)}{4}B_{1,\psi}. \tag{5.4.4}$$

Expressing everything in terms of periodic Bernoulli polynomials, we get

$$\begin{aligned}
\frac{1}{p^3}D_p &= \sum_{(a,b,c)\in S} \frac{a}{p}\frac{b}{p}\frac{c}{p}\,\psi(abc) \\
&= \sum_{(a,b,c)\in S}\left(P_1\left(\frac{a}{p}\right)+\frac{1}{2}\right)\left(P_1\left(\frac{b}{p}\right)+\frac{1}{2}\right)\left(P_1\left(\frac{c}{p}\right)+\frac{1}{2}\right)\psi(abc) \\
&= S_p \\
&+ \frac{1}{2}\sum_{(a,b,c)\in S}\Bigg(P_1\left(\frac{a}{p}\right)P_1\left(\frac{b}{p}\right) + P_1\left(\frac{b}{p}\right)P_1\left(\frac{c}{p}\right)\psi(abc) \\
&\quad + P_1\left(\frac{c}{p}\right)P_1\left(\frac{a}{p}\right)\Bigg)\psi(abc) \\
&+ \frac{1}{4}\sum_{(a,b,c)\in S}\left(P_1\left(\frac{a}{p}\right)+P_1\left(\frac{b}{p}\right)+P_1\left(\frac{c}{p}\right)\right)\psi(abc) \\
&+ \frac{1}{8}\sum_{(a,b,c)\in S}\psi(abc).
\end{aligned}$$

By the usual even/odd argument, the second and fourth sums vanish, leaving us with

$$\frac{1}{p^3}D_p = S_p + \frac{1}{4}\sum_{(a,b,c)\in S}\left(P_1\left(\frac{a}{p}\right) + P_1\left(\frac{b}{p}\right) + P_1\left(\frac{c}{p}\right)\right)\psi(abc).$$

Recall the 1-to-1 correspondence between the sets S and $\{(k, kt^2, k(t+1)^2) \in (\mathbb{F}_p^\times)^3 \mid 1 \le k, t \le p-1,\ t \ne p-1\}$. Substituting $(k, kt^2, k(t+1)^2)$ for (a,b,c), we get

$$\frac{1}{p^3}D_p = S_p + \frac{1}{4}\sum_{k,t(p)}\left(P_1\left(\frac{k}{p}\right) + P_1\left(\frac{kt^2}{p}\right) + P_1\left(\frac{k(t+1)^2}{p}\right)\right)\psi(kt^2(t+1)^2).$$

Then by Proposition 3.2.1, we obtain

$$S_p = \frac{1}{p^3}D_p - \frac{3(p-2)}{4}B_{1,\psi}.$$

Thus the assertion (i) of the proposition follows from (5.4.4). From (5.4.3), when $f_1 = f_2 = f_3 = c_1$, $\hat{f}_1 = \hat{f}_2 = \hat{f}_3 = (2\pi i)P_1$, we get

$$\mathscr{T}_p = -\frac{p\sqrt{-p}\,(2\pi i p)^3}{p^3}S_p.$$

Thus the assertion (ii) follows from (i). $\square$

As another corollary to Theorem 5, we have

Corollary 5.4.5. *With notation and assumptions being the same as above, we have*

$$(i)\quad T_p = -\frac{1}{6p}B_{3,\psi} - \frac{p-2}{4p}B_{1,\psi},$$

$$(ii)\quad \mathscr{S}_p = 8\pi^3 p^{3/2}\left(\frac{1}{6p}B_{3,\psi} + \frac{p-2}{4p}B_{1,\psi}\right).$$

Proof. The assertion is clear when $p \equiv 1(4)$ since both sides vanish by the usual even/odd argument. Thus we assume that $p \equiv 3(4)$. From Proposition 5.4.2, we have

$$T_p = \sum_{k,t(p)} P_1\left(\frac{kt}{p}\right) P_1\left(\frac{k^2+kt}{p}\right) P_1\left(\frac{t^2+kt}{p}\right).$$

Observe that the sum on the right-hand side is precisely the sum X_p given by (5.1.6). Therefore, by (5.1.7) and Corollary 5.4.1, we get

$$\begin{aligned} T_p &= \frac{1}{p^3}A_p + \frac{3}{4}B_{2,\psi} - \frac{3}{4}B_{1,\psi} \\ &= -\frac{1}{6p}B_{3,\psi} - \frac{p-2}{4p}B_{1,\psi}. \end{aligned}$$

Thus the assertion (i) is established. From (5.4.3), when $f_1 = f_2 = f_3 = P_1$, $\hat{f}_1 = \hat{f}_2 = \hat{f}_3 = -\frac{1}{2\pi i}c_1$, we get

$$T_p = \frac{p\sqrt{-p}}{(2\pi i p)^3}\mathscr{S}p.$$

Thus the assertion (ii) follows from (i). □

We now prove the following conjecture that I made several years ago. It was this conjecture, dubbed "The $S = T$ Conjecture", that led me down the path of Chapter 5.

The $S = T$ Theorem. *With notation and assumptions being the same as above, we have*

$$S_p = T_p \text{ and } \mathscr{S}_p = \mathscr{T}_p.$$

Proof. This follows immediately from Proposition 4.5.4 and Corollary 4.5.5. □

5.5 Arakawa Identities

From Theorem 5.2.36 and Corollary 5.3.3, and from Theorem 5.2.37 and Corollary 5.3.6, we see that the special values of $L_2(1-m,\psi_{H,p})\,(m\in\mathbb{N})$ and $L_2^*(1-m,\psi_{H,p})\,(m\in\mathbb{N})$ give rise to formulas expressing sums of products of periodic Bernoulli polynomials in terms of generalized Bernoulli numbers. These formulas are of great importance, interest, and significance, since I strongly believe that they cannot be easily obtained by using elementary techniques from algebra and number theory. For this reason, I call these formulas Arakawa Identities. This infinite sequence of Arakawa Identities is the main theorem of this dissertation.

Theorem 5.5 (Arakawa Identities). *We fix an odd prime p. We denote by ψ the Legendre symbol mod p: $\psi(a)=\left(\frac{a}{p}\right)$. Let $m\in\mathbb{N}$. We put*

$$A_p(k_1,k_2,k_3)=\sum_{\alpha,\gamma(p)} P_{k_1}\left(\frac{\alpha^2-2\alpha\gamma}{p}\right)P_{k_2}\left(\frac{\alpha\gamma}{p}\right)P_{k_3}\left(\frac{\gamma^2-\alpha^2}{p}\right),$$

$$A_p^*(k_1,k_2,k_3)=\sum_{\alpha,\gamma(p)} P_{k_1}\left(\frac{\alpha^2-2\alpha\gamma}{p}\right)P_{k_2}\left(\frac{2\alpha\gamma}{p}\right)P_{k_3}\left(\frac{\gamma^2-\alpha^2}{p}\right),$$

$$T(m)=\{(k_1,k_2,k_3)\in\mathbb{Z}^3\mid k_2\geq 0,\ k_1,k_3\geq 1,\ k_1+k_2+k_3=2m+1\}.$$

Then, we have

$$
\begin{aligned}
(i)\quad & \sum_{(k_1,k_2,k_3)\in T(m)} c(k_1,k_2,k_3)A_p(k_1,k_2,k_3) \\
&= \frac{1}{p^{2m-1}}\Bigg\{\frac{1}{2m+1}\sum_{j=0}^{m}\frac{(-1)^j}{\{(m-j)!\}^2(2j+1)!} \\
&\quad + \sum_{n=1}^{2m}\frac{(-1)^{n+1}(2m+1-n)\mathscr{C}_{n,m}}{(n+1)2^{n+1}}\Bigg\}B_{2m+1,\psi} \\
&+ \frac{1}{12p^{2m-1}}\Bigg\{\Bigg[\frac{2}{2m-1}\sum_{j=1}^{m}\frac{(-1)^j}{\{(m-j)!\}^2(2j-1)!} \\
&\quad + \sum_{n=1}^{2m}\frac{(-1)^{n+1}n(2m+1-n)\mathscr{C}_{n,m}}{2^n}\Bigg]\Big(3\cdot 2^{2m-2}\psi(2)-1\Big) \\
&\quad + p\Big(3c(1,1,2m-1)-c(1,2m-1,1)\delta_{p,3}\Big)\Bigg\}B_{2m-1,\psi} \\
&+ \frac{2^{2m}}{p^{2m-1}}\sum_{k=2}^{m-1}\Bigg\{\frac{1}{(2m+1-2k)(2k)!}\sum_{j=k}^{m}\frac{(-1)^j}{\{(m-j)!\}^2(2j+1-2k)!} \\
&\quad + \sum_{n=2k-1}^{2m}\frac{(-1)^{n+1}\binom{n+1}{2k}(2m+1-n)\mathscr{C}_{n,m}}{(n+1)2^{n+1}}\Bigg\} \\
&\quad \times \frac{\{B_{2k}-B_{2k}(1/2)\}2^{2m-2k}\psi(2)+B_{2k}(1/2)}{2^{2m-2k}}\cdot B_{2m+1-2k,\psi} \\
&+ \frac{2^{2m}}{(2m)!\,p^{2m-1}}\Bigg\{(-1)^m\Big(\{B_{2m}-B_{2m}(1/2)\}\psi(2)+B_{2m}(1/2)\Big)(1-\delta_{m,1}) \\
&\quad - p^m B_{2m}\Bigg\}B_{1,\psi},
\end{aligned}
$$

where $c(k_1,k_2,k_3)$, $\mathscr{C}_{n,m}$ *are given by (5.2.84), (5.2.93).*

(*ii*)
$$\sum_{(k_1,k_2,k_3)\in T(m)} \frac{c(k_1,k_2,k_3)}{2^{k_2-1}} A_p^*(k_1,k_2,k_3)$$
$$= \frac{2}{p^{2m-1}}\left\{\frac{1}{2m+1}\sum_{j=0}^{m}\frac{(-1)^j}{\{(m-j)!\}^2(2j+1)!} + \sum_{n=1}^{2m}\frac{(-1)^{n+1}(2m+1-n)\mathscr{C}_{n,m}}{(n+1)2^{n+1}}\right\}B_{2m+1,\psi}$$
$$+ \frac{1}{12p^{2m-1}}\left\{\frac{2}{2m-1}\sum_{j=1}^{m}\frac{(-1)^j}{\{(m-j)!\}^2(2j-1)!} + \sum_{n=1}^{2m}\frac{(-1)^{n+1}n(2m+1-n)\mathscr{C}_{n,m}}{2^n} + p\left(3c(1,1,2m-1) + \frac{c(1,2m-1,1)}{2^{2m-2}}\delta_{p,3}\right)\right\}B_{2m-1,\psi}$$
$$+ \frac{2}{p^{2m-1}}\sum_{k=2}^{m-1}\left\{\frac{1}{(2m+1-2k)(2k)!}\sum_{j=k}^{m}\frac{(-1)^j}{\{(m-j)!\}^2(2j+1-2k)!} + \sum_{n=2k-1}^{2m}\frac{(-1)^{n+1}\binom{n+1}{2k}(2m+1-n)\mathscr{C}_{n,m}}{(n+1)2^{n+1}}\right\}B_{2k}\cdot B_{2m+1-2k,\psi}$$
$$+ \frac{2}{(2m)!\,p^{2m-1}}\left\{(-1)^m(1-\delta_{m,1}) - p^m\right\}B_{2m}\cdot B_{1,\psi},$$

where $c(k_1,k_2,k_3)$, $\mathscr{C}_{n,m}$ *are given by (5.2.84), (5.2.93).*

Proof. Noting that $m\mathscr{C}_{2m,m} - 2\mathscr{C}_{2m-1,m} = 0$ by Lemma 5.2.34, the assertion (i) follows immediately from Theorem 5.2.36 and Corollary 5.3.3, and the assertion (ii) follows immediately from Theorem 5.2.37 and Corollary 5.3.6. □

The only obstruction to explicitly determining Arakawa Identities is determining the numbers $c(k_1,k_2,k_3)$, $\mathscr{C}_{n,m}$ given by (5.2.84), (5.2.93). While these numbers may seem complicated, all of our hard work in Section 5.2 has allowed us to find these numbers in a rather simple and systematic manner.

Recipe for explicitly determining Arakawa Identities. Let $m \in \mathbb{N}$. We first determine $c(k_1,k_2,k_3)$ for $(k_1,k_2,k_3) \in T(m)$. If $k_2 \geq 1$, then $c(k_1,k_2,k_3)$ is easily determined by (5.2.84), (5.2.24), (5.2.26), (5.2.27), and (5.2.28). If $k_2 = 0$, then $c(k_1, 0, k_3)$ is easily determined by (5.2.84), (5.2.34), (5.2.32), Lemma 5.2.11, (5.2.30), and (5.2.28). The numbers $\mathscr{C}_{n,m}$ $(1 \leq n \leq 2m)$ are also easily determined by (5.2.93), (5.2.91), (5.2.74), (5.2.77), (5.2.48), (5.2.41), (5.2.75), and (5.2.28). □

We give three corollaries corresponding to the Arakawa Identities for $m = 1, 2, 3$, respectively. We first prepare a lemma.

Lemma 5.5.1. *With notation and assumptions being the same as above, we have*

$$\begin{cases} A_p(i,0,j) = (-1)^{i+j} A_p(j,0,i), \\ A_p^*(i,0,j) = (-1)^{i+j} A_p^*(j,0,i) \quad (i,j \in \mathbb{N}). \end{cases}$$

Proof. This follows by replacing (α, γ) by $(\alpha + \gamma, \gamma)$ in the definitions of A_p, A_p^*. □

Corollary 5.5.2. *With notation and assumptions being the same as above, we get*

$$(i) \quad 4A_p(1,1,1) - \frac{2}{3}A_p(1,0,2) = \frac{11}{18p}B_{3,\psi} + \frac{4p+1-3\psi(2)-2p\delta_{p,3}}{6p}B_{1,\psi}.$$

$$(ii) \quad 4A_p^*(1,1,1) - \frac{4}{3}A_p^*(1,0,2) = \frac{11}{9p}B_{3,\psi} + \frac{5p-1+2p\delta_{p,3}}{6p}B_{1,\psi}.$$

Proof. Computing the Arakawa Identities for $m = 1$, we obtain

$$\begin{aligned}
&c(1,0,2)A_p(1,0,2) + c(1,1,1)A_p(1,1,1) + c(2,0,1)A_p(2,0,1) \\
&\quad = \frac{1}{p}\left\{\frac{5}{18} + \frac{\mathscr{C}_{1,1}}{4} - \frac{\mathscr{C}_{2,1}}{24}\right\}B_{3,\psi} + \frac{1}{12p}\left\{\left(-2 + \mathscr{C}_{1,1} - \frac{\mathscr{C}_{2,1}}{2}\right)(3\psi(2) - 1)\right. \\
&\qquad \left. + pc(1,1,1)(3 - \delta_{p,3}) - 4p\right\}B_{1,\psi},
\end{aligned}$$

$$\begin{aligned}
&2c(1,0,2)A_p^*(1,0,2) + c(1,1,1)A_p^*(1,1,1) + 2c(2,0,1)A_p^*(2,0,1) \\
&\quad = \frac{2}{p}\left\{\frac{5}{18} + \frac{\mathscr{C}_{1,1}}{4} - \frac{\mathscr{C}_{2,1}}{24}\right\}B_{3,\psi} \\
&\qquad + \frac{1}{12p}\left\{-2 + \mathscr{C}_{1,1} - \frac{\mathscr{C}_{2,1}}{2} + pc(1,1,1)(3 + \delta_{p,3}) - 2p\right\}B_{1,\psi}.
\end{aligned}$$

We see from (5.2.84) and Proposition 5.2.12 that $c(1,0,2) = 2/3$, $c(1,1,1) = 4$, $c(2,0,1) = 4/3$, and from the proof of the assertion (iii) in Proposition 5.2.35, that $\mathscr{C}_{1,1} = 2$, $\mathscr{C}_{2,1} = 4$. Hence, we get

$$\begin{aligned}
&\frac{2}{3}A_p(1,0,2) + 4A_p(1,1,1) + \frac{4}{3}A_p(2,0,1) = \frac{11}{18p}B_{3,\psi} + \frac{2p + 1 - 3\psi(2) - p\delta_{p,3}}{6p}B_{1,\psi}, \\
&\frac{4}{3}A_p^*(1,0,2) + 4A_p^*(1,1,1) + \frac{8}{3}A_p^*(2,0,1) = \frac{11}{9p}B_{3,\psi} + \frac{5p - 1 + 2p\delta_{p,3}}{6p}B_{1,\psi}.
\end{aligned}$$

By Lemma 5.5.1, we have $A_p(2,0,1) = -A_p(1,0,2)$ and $A_p^*(2,0,1) = -A_p^*(1,0,2)$. Thus the assertions of the corollary immediately follow. □

Corollary 5.5.3. *With notation and assumptions being the same as above, we get*

$$\begin{aligned}
(i)\quad &\frac{4}{3}A_p(1,2,2) + \frac{8}{3}A_p(1,3,1) + \frac{2}{3}A_p(2,1,2) + \frac{8}{3}A_p(2,2,1) + \frac{8}{9}A_p(3,1,1) \\
&= \frac{1}{p^3}\left(\frac{203}{1800}B_{5,\psi} + \frac{1 - 12\psi(2) - 8p\delta_{p,3}}{36}B_{3,\psi} + \frac{8p^2 + 7 - 15\psi(2)}{360}B_{1,\psi}\right). \\
(ii)\quad &\frac{2}{3}A_p^*(1,2,2) + \frac{2}{3}A_p^*(1,3,1) + \frac{2}{3}A_p^*(2,1,2) + \frac{4}{3}A_p^*(2,2,1) + \frac{8}{9}A_p^*(3,1,1) \\
&= \frac{1}{p^3}\left(\frac{203}{900}B_{5,\psi} + \frac{-1 + 2p\delta_{p,3}}{36}B_{3,\psi} + \frac{p^2 - 1}{360}B_{1,\psi}\right).
\end{aligned}$$

Proof. We first prove the assertion (i). Following the recipe for explicitly determining Arakawa Identities (where the details are left to the interested reader), we find that $c(1,1,3)=0$, $c(1,2,2)=4/3$, $c(1,3,1)=8/3$, $c(2,1,2)=2/3$, $c(2,2,1)=8/3$, $c(3,1,1)=8/9$, $c(1,0,4)=-4/81$, $c(2,0,3)=-4/81$, $c(3,0,2)=4/81$, $c(4,0,1)=4/81$, and $\mathscr{C}_{1,2}=5/12$, $\mathscr{C}_{2,2}=5/3$, $\mathscr{C}_{3,2}=34/9$, $\mathscr{C}_{4,2}=34/9$. Hence, we obtain the following Arakawa Identity for $m=2$:

$$\begin{aligned}
&\frac{4}{3}A_p(1,2,2)+\frac{8}{3}A_p(1,3,1)+\frac{2}{3}A_p(2,1,2)+\frac{8}{3}A_p(2,2,1)+\frac{8}{9}A_p(3,1,1)\\
&\quad-\frac{4}{81}A_p(1,0,4)-\frac{4}{81}A_p(2,0,3)+\frac{4}{81}A_p(3,0,2)+\frac{4}{81}A_p(4,0,1)\\
&\quad=\frac{1}{p^3}\left(\frac{203}{1800}B_{5,\psi}+\frac{1-12\psi(2)-8p\delta_{p,3}}{36}B_{3,\psi}+\frac{8p^2+7-15\psi(2)}{360}B_{1,\psi}\right).
\end{aligned}$$

By Lemma 5.5.1, we get $A_p(4,0,1)=-A_p(1,0,4)$ and $A_p(3,0,2)=-A_p(2,0,3)$. Moreover, from Proposition 3.2.34, we see that $A_p(1,0,4)=-A_p(2,0,3)$. Thus the assertion (i) follows. The assertion (ii) is similarly verified, so the proof is omitted.

□

Corollary 5.5.4. *With notation and assumptions being the same as above, we get*

$$\begin{aligned}
(i)\quad &\frac{32}{135}A_p(1,3,3)+\frac{4}{5}A_p(1,4,2)+\frac{24}{25}A_p(1,5,1)+\frac{8}{45}A_p(2,2,3)+\frac{44}{45}A_p(2,3,2)\\
&+\frac{8}{5}A_p(2,4,1)+\frac{8}{135}A_p(3,1,3)+\frac{8}{15}A_p(3,2,2)+\frac{176}{135}A_p(3,3,1)\\
&+\frac{2}{15}A_p(4,1,2)+\frac{8}{15}A_p(4,2,1)+\frac{8}{75}A_p(5,1,1)-\frac{152}{18225}A_p(1,0,6)\\
&-\frac{76}{6075}A_p(2,0,5)-\frac{26}{3645}A_p(3,0,4)\\
&=\frac{1}{p^5}\left\{\frac{1759}{176400}B_{7,\psi}+\frac{1-48\psi(2)-32p\delta_{p,3}}{400}B_{5,\psi}+\frac{49-420\psi(2)}{32400}B_{3,\psi}\right.\\
&\left.+\frac{-32p^3+31-63\psi(2)}{15120}B_{1,\psi}\right\}.
\end{aligned}$$

$$(ii)\quad \frac{8}{135}A_p^*(1,3,3)+\frac{1}{10}A_p^*(1,4,2)+\frac{3}{50}A_p^*(1,5,1)+\frac{4}{45}A_p^*(2,2,3)+\frac{11}{45}A_p^*(2,3,2)$$
$$+\frac{1}{5}A_p^*(2,4,1)+\frac{8}{135}A_p^*(3,1,3)+\frac{4}{15}A_p^*(3,2,2)+\frac{44}{135}A_p^*(3,3,1)$$
$$+\frac{2}{15}A_p^*(4,1,2)+\frac{4}{15}A_p^*(4,2,1)+\frac{8}{75}A_p^*(5,1,1)-\frac{304}{18225}A_p^*(1,0,6)$$
$$-\frac{152}{6075}A_p^*(2,0,5)-\frac{52}{3645}A_p^*(3,0,4)$$
$$=\frac{1}{p^5}\left\{\frac{1759}{88200}B_{7,\psi}+\frac{-1+2p\delta_{p,3}}{400}B_{5,\psi}-\frac{7}{32400}B_{3,\psi}-\frac{p^3+1}{15120}B_{1,\psi}\right\}.$$

Proof. Taking into account $A_p(i,0,j)=-A_p(j,0,i)$, $A_p^*(i,0,j)=-A_p^*(j,0,i)$ $(i+j=2m+1)$ which follows from Lemma 5.5.1, these are precisely the Arakawa Identities for $m=3$. Following the recipe for explicitly determining Arakawa Identities (where the details are left to the enthusiastic reader), we find that $c(1,1,5)=0$, $c(1,2,4)=0$, $c(1,3,3)=32/135$, $c(1,4,2)=4/5$, $c(1,5,1)=24/25$, $c(2,1,4)=0$, $c(2,2,3)=8/45$, $c(2,3,2)=44/45$, $c(2,4,1)=8/5$, $c(3,1,3)=8/135$, $c(3,2,2)=8/15$, $c(3,3,1)=176/135$, $c(4,1,2)=2/15$, $c(4,2,1)=8/15$, $c(5,1,1)=8/75$, $c(1,0,6)=32/18225$, $c(2,0,5)=16/6075$, $c(3,0,4)=14/3645$, $c(4,0,3)=8/729$, $c(5,0,2)=92/6075$, $c(6,0,1)=184/18225$, and $\mathscr{C}_{1,3}=11/270$, $\mathscr{C}_{2,3}=11/45$, $\mathscr{C}_{3,3}=241/270$, $\mathscr{C}_{4,3}=241/135$, $\mathscr{C}_{5,3}=446/225$, $\mathscr{C}_{6,3}=892/675$. Whence the Arakawa Identities for $m=3$ yield the desired results. □

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