# ALGEBRAIC STUDIES OF SYMMETRIC OPERATORS

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A dissertation submitted to the Graduate School-Newark

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Graduate Program in Mathematical Sciences

written under the direction of

Professor William Keigher

and approved by

Newark, New Jersey

May, 2015

**Abstract**

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There was an old problem of G. C. Rota regarding the classification of all linear operators on associative algebras that satisfy algebraic identities. We only know very few of such operators at the beginning, for example, the derivative operator, average operator, difference operator and Rota-Baxter operator. Recently L. Guo, W. Sit and R. Zhang revisited Rota's problem in a paper by concentrating on two classes of operators: differential type operators and Rota-Baxter type operators. One of the Rota-Baxter type operators they found is the symmetric Rota-Baxter operator which symmetrizes the Rota-Baxter operator. In this dissertation, we initiate a systematic study of the symmetric Rota-Baxter operator, extending the previous works on the original Rota-Baxter operator. After giving basic properties and examples, we construct free symmetric Rota-Baxter algebras on an algebra and on a set by bracketed words and rooted trees separately. We then use the free symmetric Rota-Baxter algebra to obtain an extension of the well known dendriform algebra and its free objects. Finally, we extend our study to differential algebras. We construct the free symmetric differential Rota-Baxter algebra based on the previous free symmetric Rota-Baxter algebra on a set and the free symmetric differential algebra.

*To Professor Guo and Professor Keigher for all of their support,*

*and to my parents, who have waited a long time for this day.*


Acknowledgements:

There are so many people I wish to thank. Foremost, I thank Professor William Keigher for the unlimited time he spent with me, his encouragement whenever I needed it and his patience through the whole process. I really appreciate his helpful advice, discussions and his late nights working with me. I also thank Professor Li Guo who introduced me to this subject and suggested the outline of this dissertation. I appreciate his advice, direction and all of his understanding.

My sincere thanks go to those in the department of mathematics at Rutgers-Newark who recruited me, supported me and have been patient with me for five years. I thank all the friendly professors who were always willing to answer any questions that I have had and all the graduate students who could always talk with me and help me. In particular, I thank Professor Ulrich Oertel who freely helped me with my topology problems in a summer. I also thank Professor Robert Sczech, Professor Jane Gilman, Professor John Loftin and Professor Mao for all of their friendly and constructive talks and concerns. I thank Professor Mark Feighn for all of his observations and suggestions concerning my teaching assignments and Professor Lee Mosher for his assistance with my topology study. I highly appreciate Brad Issacson who was always a close friend and listened to all my complaints, Junqi

Wang who was always ready to help me with my mathematical questions and computer problems, as well as Hetal Kapadia who gave me productive suggestions, and the same thanks go to Shuai Jiang, Pei Wang, King Lee and Saikat Das. I also thank Susan Tel, Jane Manay and Katie Wcisel for their enormous time and energy for keeping our department running smoothly.

In addition, I thank Professor Zhijian Wu who served as my master advisor and gave me many suggestions not only in mathematics but also in being a real person. I am so grateful for all of his concerns. I also thank all the professors from my master program and my undergraduate university. It was they who educated me and shaped me for my life.

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# 1 Introduction

The subject of this thesis is motivated by Rota's long standing problem [56] [55] and its possible solution [37] as well as the research work of Rota-Baxter algebras. A Rota-Baxter algebra is an associative algebra equipped with a linear operator, called the Rota-Baxter operator, that generalizes the integral operator in analysis. The Rota-Baxter operator was introduced in 1960 [7] by G. Baxter to study the theory of fluctuations in probability. Later, other well-known mathematicians such as Atkinson, Cartier, and especially G. C. Rota have shown keen interest in Baxter algebras. Their fundamental papers brought the subject into the domains of algebra and combinatorics. The study of Baxter algebras continued through the 1960s and 1970s [13, 55, 54] and recently has led to remarkable results with applications to renormalization in quantum field theory [10, 12, 25, 26], multiple zeta values in number theory [42, 21], umbral calculus in combinatorics [33], and also in Loday's work on dendriform algebras [47] and Hopf algebras [1].

A long standing problem of Gian-Carlo Rota for associative algebras is the classification of all linear operators that can be defined on them [56] [55]. In the 1970s, there were only a few known operators, for example, the derivative operator, the difference operator, the average operator, and the Rota-Baxter operator. A few more appeared after Rota posed his problem. However, little progress was made to solve this problem in general. Guo, Sit and Zhang [37] recently formulated Rota's problem, in which they worked on Rota's problem in the framework of free operated algebras by viewing an associative algebra with a linear operator as one which satisfies a certain operated polynomial identity. They have also used rewriting systems, Gröbner-Shirshov bases and the help of computer algebra. In their research work, the authors have obtained a possibly complete list of 14 Rota-Baxter type operators and some other differential type operators as a partial solution to Rota's problem.

The symmetric Rota-Baxter operator is actually one of the 14 Rota-Baxter type operators. Some operators from this list have been studied, for example, average operators [16], RBNTD operators [6], Nijenhuis operators [39], and Rota-Baxter operators [32]. Others remain unknown, including symmetric Rota-Baxter operators which is the subject of the present research work.

Our approach to study this operator is based on algebraic constructions. We first give out some concrete examples from matrix algebras and semigroup algebras. The computing method was partially adopted from the work of L. Guo, M. Rosenkranz and S. Zheng [40]. Then we construct free symmetric Rota-Baxter algebras from the algebraic structures of bracketed words and rooted trees. This is an extension of previous works of P. Lei and L. Guo [39] and E. Fard and L. Guo [23, 24]. We continue our work on constructing free symmetric dendriform algebras in two cases. It is an extension of Loday's work on dendriform algebras. Finally, we extend our research onto the differential symmetric Rota-Baxter algebra, which is an extension of the work of L. Guo and W. Keigher [36].

## 2 Organization

The organization of this work is as follows. Standard materials or descriptions are drawn from [32, 36]. In Section 3, we review the necessary definitions and provide examples of symmetric Rota-Baxter operators using Maple and Mathematica. We begin Section 4 with the introduction of bracketed words and then proceed to an explicit construction of free symmetric Rota-Baxter algebras over an algebra. After that, we consider the case over a set by using rooted trees. The free symmetric Rota-Baxter algebra determines the symmetric dendriform algebra which is in Section 5, and we construct its free objects in two cases. In the last section, Section 6, we turn our attention to differential algebras. We compatibly define the symmetric differential algebra and construct the free symmetric differential Rota-Baxter algebra based on the results from Section 4.

# 3 Definitions and Examples

We will use $\mathbb{N}_{>0}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively to denote the set of positive integers, non-negative integers, integers, rational numbers, real numbers and complex numbers.

To fix the notations and to be self-contained, we briefly recall definitions. Refer to [32].

## 3.1 Definitions

In the following, by a ring we always mean a unitary ring, that is, a set $A$ with binary operations $+$ and $\cdot$ (which will often be suppressed) such that $(A, +)$ is an abelian group, $(A, \cdot)$ is a monoid and $\cdot$ is distributive over $+$. The unit of the monoid is called the identity element of $A$, denoted by $\mathbf{1}_A$ or simply 1. A ring homomorphism is assumed to preserve the unit. We use $\mathbf{k}$ to denote a commutative ring with identity element denoted by $\mathbf{1}$ or simply 1.

Let $A$ be a ring. A **(left)** $A$**-module** $M$ is an abelian group $M$ together with a **scalar multiplication** $A \times M \to M$ such that

$$a(x + y) = ax + ay,\ (a + b)x = ax + bx,\ (a\,b)x = a(bx),\ \forall a, b \in A,\ x, y \in M.$$

**Definition 3.1.** Let $\mathbf{k}$ be a commutative ring. A $\mathbf{k}$**-algebra** is a ring $A$ together with a unitary $\mathbf{k}$-module structure on the underlying abelian group of $A$ such that

$$k(ab) = (ka)b = a(kb),\ \forall\, k \in \mathbf{k},\ a, b \in A.$$

All **k**-algebras are taken to be unitary and noncommutative.

A Rota-Baxter algebra of weight zero (or simply, a Rota Baxter algebra) is thus an associative algebra equipped with a linear operator that generalizes the integral operator in analysis. Rota-Baxter algebras (initially known as Baxter algebras) originated in 1960 [7] from the probability study by G. Baxter to understand the Spitzer's identity in fluctuation theory. This concept drew the attention of many well-known mathematicians such as Atkinson, Cartier, and especially G. C. Rota, whose fundamental papers brought the subject into the areas of algebra and combinatorics around 1970. In 1980s, Lie algebras were studied independently by mathematical physicists C.N. Yang and R. Baxter under the name of the classical Yang Baxter Equation (CYBE). In 2000, Aguiar discovered that the Rota-Baxter algebra of weight zero and the associative analog of CYBE are related [2]. He also showed that the Rota-Baxter algebra of weight zero naturally carries the structure of a dendriform algebra which was introduced by Loday in his study of K-theory [47]. Also in 2000, Guo and Keigher showed that the free Rota-Baxter algebras can be constructed via generalization of the shuffle algebra [35], called the mixable shuffle algebra.

**Definition 3.2.** Let $\lambda$ be a given element of **k**. A **Rota-Baxter k-algebra of weight** $\lambda$, is a pair $(R, P)$ consisting of a **k**-algebra $R$ and a linear operator $P : R \to R$ that satisfies the **Rota-Baxter identity**

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \quad \forall x, y \in R. \tag{3.1}$$

Then $P$ is called a **Rota-Baxter operator of weight** $\lambda$.

In 1995 [56] [55], Rota posed a question about finding all linear operators that satisfy an algebraic identities on an associative algebra. More precisely, Rota's question involved an associative **k**-algebra $R$ with a **k**-linear unary operator $P$. The operations: addition, multiplication, scalar multiplication, and $P$, already are required to satisfy certain identities such as the commutative law of addition, the associative laws, the distributive law, and **k**-linearity for $P$. Rota wanted to find "all possible polynomial identities that could be satisfied by $P$ on an algebra" and to "classify all such identities". He also wanted to find "a complete list of such identities."

Taking this work forward, Guo, Sit and Zhang recently published a paper [37], in which they worked on Rota's problem and put together the framework of free operated algebras by viewing associative algebra with a linear operator, as one that is compatible with a certain operated polynomial identity. They have also used rewriting systems, Gröbner-Shirshov bases and the help of computer algebra. In that research work, the authors have obtained a possibly complete list of 14 Rota-Baxter type operators and some other differential type operators as a partial solution to Rota's problem.

The completeness of the list of Rota-Baxter type identities that Guo, Sit and Zhang found is still a conjecture and further work should be done. One of the identities in their framework is our identity: the symmetric Rota-Baxter operator. This new identity, which gives rise to a new class of associative **k**-algebras known as **symmetric Rota-Baxter algebras** is the **symmetric Rota-Baxter identity:**

$$P(x)P(y) = P(xP(y)) + P(yP(x)), \quad \forall x, y \in R. \tag{3.2}$$

where $R$ is a noncommutative $k$-algebra and this identity is automatically a case of weight 0.

It is easy to see that:

**Remark 3.3.** $P(R)$, the image of the symmetric Rota-Baxter operator, is commutative.

## 3.2 Examples

### 3.2.1 Classification of symmetric Rota-Baxter operators on the algebra of $2\times 2$ matrices over a field

Let $\mathbf{k}$ be a field of characteristic zero. In order to consider classifying symmetric Rota-Baxter operators over the algebra of the $2 \times 2$ matrices with entries in $\mathbf{k}$, we formulate the setup first and then use Maple to do computations. See the Maple codes in Appendix A.

**Setup:** Let $e_1 = (1,0,0,0), e_2 = (0,1,0,0), e_3 = (0,0,1,0)$ and $e_4 = (0,0,0,1)$. Since every $2\times 2$ matrix is a unique linear combination of the given basis, let

$$M_{2\times 2}(\mathbf{k}) := \sum_{m=1}^{4} \mathbf{k} e_m = \left\{ \sum_{m=1}^{4} a_m e_m \,\middle|\, a_m \in \mathbf{k} \right\} \tag{3.3}$$

denote the algebra of $2\times 2$ matrices.

Let $P : M_{2\times 2}(\mathbf{k}) \to M_{2\times 2}(\mathbf{k})$ be a symmetric Rota-Baxter operator. Since $P$ is

**k**-linear, we have

$$\begin{pmatrix} P(e_1) \\ P(e_2) \\ P(e_3) \\ P(e_4) \end{pmatrix} = \begin{pmatrix} a & u & i & m \\ b & f & j & n \\ c & g & k & w \\ d & h & l & p \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \quad (a, b, c, d, u, f, g, h, i, j, k, l, m, n, w, p \in \mathbf{k}). \quad (3.4)$$

The matrix $C := C_P$ above is called the **matrix of** $P$. Further, $P$ is a symmetric Rota-Baxter operator if and only if

$$P(e_i)P(e_j) = P(e_iP(e_j) + e_jP(e_i)) \quad (1 \leq i, j \leq 4). \quad (3.5)$$

By simplifying equations when $i, j = 1, 2, 3, 4$, we type them into Maple and compute out the following results. Note that the module of $2 \times 2$ matrices over $\mathbf{k}$ is isomorphic to the $k$-module $R = \mathbf{k} \times \mathbf{k} \times \mathbf{k} \times \mathbf{k}$.

**Proposition 3.4.** *Consider the* $\mathbf{k}$*-algebra* $R = \mathbf{k} \times \mathbf{k} \times \mathbf{k} \times \mathbf{k}$*, where the operators are defined componentwise. The matrices of symmetric Rota-Baxter operators of weight zero on* $R$ *with respect to the basis* $e_1 = (1, 0, 0, 0)$ *,* $e_2 = (0, 1, 0, 0)$*,* $e_3 = (0, 0, 1, 0)$ *and* $e_4 = (0, 0, 0, 1)$ *are given below, where all the parameters are in* $\mathbf{k}$*.*

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & g & 0 & w \\ 0 & 0 & 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} -f & \frac{-f^2}{b} & 0 & 0 \\ b & f & 0 & 0 \\ \frac{-f^2}{b} & \frac{-f^3}{b^2} & 0 & 0 \\ f & \frac{f^2}{b} & 0 & 0 \end{pmatrix} (b \neq 0),$$

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b & 0 & j & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, M_4 = \begin{pmatrix} e & e & -e & -e \\ -e & -e & e & e \\ e & e & -e & -e \\ -e & -e & e & e \end{pmatrix}.$$

We also computed the usual Rota-Baxter operators over this algebra of 2×2 matrices over $\mathbf{k}$

**Proposition 3.5.** *Consider the* $\mathbf{k}$*-algebra* $R = \mathbf{k} \times \mathbf{k} \times \mathbf{k} \times \mathbf{k}$*, where the operators are defined componentwise. The matrices of Rota-Baxter operators of weight zero on* $R$ *with respect to the basis* $e_1 = (1, 0, 0, 0)$ *,* $e_2 = (0, 1, 0, 0)$*,* $e_3 = (0, 0, 1, 0)$ *and* $e_4 = (0, 0, 0, 1)$ *are given below, where all the parameters are in* $\mathbf{k}$*.*

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & l & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b & \frac{-b^2}{j} & j & -b \\ 0 & 0 & 0 & 0 \\ \frac{-b^2}{j} & \frac{b^3}{j^2} & -b & \frac{b^2}{j} \end{pmatrix} (j \neq 0),$$

$$M_4 = \begin{pmatrix} \frac{b^2}{j} & \frac{-b^3}{j^2} & b & \frac{-b^2}{j} \\ b & \frac{-b^2}{j} & j & -b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (j \neq 0), M_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & l & 0 \end{pmatrix}, M_6 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_7 = \begin{pmatrix} 0 & 0 & i & k \\ -i & -k & 0 & 0 \\ 0 & 0 & k & \frac{k^2}{i} \\ -k & \frac{-k^2}{i} & 0 & 0 \end{pmatrix} (i \neq 0), M_8 = \begin{pmatrix} k & \frac{-k^2}{b} & 0 & 0 \\ b & -k & 0 & 0 \\ 0 & 0 & k & \frac{-k^2}{b} \\ 0 & 0 & b & -k \end{pmatrix} (b \neq 0), M_9 = \begin{pmatrix} 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -u & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ h & 0 & 0 & 0 \\ 0 & h & 0 & 0 \end{pmatrix}, M_{11} = \begin{pmatrix} 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, M_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & h & 0 & 0 \end{pmatrix},$$

$$M_{13} = \begin{pmatrix} 0 & w & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w \\ 0 & 0 & 0 & 0 \end{pmatrix}, M_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w \\ 0 & -w & 0 & 0 \end{pmatrix}.$$

### 3.2.2 Classification of symmetric Rota-Baxter operators on semigroup algebras of order two and three

We first formulate the setup for classifying symmetric Rota-Baxter operators on a general semigroup algebra of order $n$. Then we use the software Mathematica to do computations and give out two propositions. The computing method is partially adopted from [40] and the Mathematica codes are attached in Appendix B.

**Setup:** Let $S = \{e_1, \cdots, e_n\}$ be a finite semigroup with multiplication $\cdot$ that we often suppress. Let $\mathbf{k}$ be a commutative unitary ring and let

$$\mathbf{k}[S] := \sum_{m=1}^{n} \mathbf{k} e_m = \left\{ \sum_{m=1}^{n} a_m e_m \,\middle|\, a_m \in \mathbf{k}, 1 \le m \le n, m, n \in \mathbb{N}_{>0} \right\} \tag{3.6}$$

denote the semigroup algebra of $S$. The order $n$ of the semigroup $S$ is also said to be the **order** of the semigroup algebra $\mathbf{k}[S]$.

Let $P : \mathbf{k}[S] \to \mathbf{k}[S]$ be a symmetric Rota-Baxter operator. Since $P$ is $\mathbf{k}$-linear, we have

$$\begin{pmatrix} P(e_1) \\ P(e_2) \\ \cdots \\ P(e_n) \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ c_{21} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \cdots \\ e_n \end{pmatrix} \quad (c_{ij} \in \mathbf{k}, 1 \le i, j \le n). \tag{3.7}$$

The matrix $C := C_P := (c_{ij})_{1 \le i, j \le n}$ is called the **matrix of** $P$. Further, $P$ is a symmetric Rota-Baxter operator if and only if

$$P(e_i)P(e_j) = P(e_i P(e_j) + e_j P(e_i)) \quad (1 \le i, j \le n). \tag{3.8}$$

Let the Cayley (multiplication) table of the semigroup $S$ be given by

$$e_k \cdot e_\ell = \sum_{m=1}^{n} r_{k\ell}^m e_m \quad (1 \le k, \ell \le n), \tag{3.9}$$

where $r_{k\ell}^m \in \{0, 1\}$. Then we have

$$P(e_i)P(e_j) = \sum_{k=1}^{n} \sum_{\ell=1}^{n} c_{ik} c_{j\ell} e_k e_\ell = \sum_{m=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} r_{k\ell}^m c_{ik} c_{j\ell} e_m$$

and

$$
\begin{aligned}
P(e_jP(e_i)+e_iP(e_j)) &= \sum_{k=1}^{n} c_{ik}P(e_je_k)+\sum_{\ell=1}^{n} c_{j\ell}P(e_ie_\ell)\\
&= \sum_{k=1}^{n}\sum_{m=1}^{n} r_{jk}^{m}c_{ik}P(e_m)+\sum_{\ell=1}^{n}\sum_{m=1}^{n} r_{i\ell}^{m}c_{j\ell}P(e_m)\\
&= \sum_{k=1}^{n}\sum_{m=1}^{n} r_{jk}^{m}c_{ik}\Big(\sum_{\ell=1}^{n} c_{m\ell}e_\ell\Big)+\sum_{k=1}^{n}\sum_{m=1}^{n} r_{ik}^{m}c_{jk}\Big(\sum_{\ell=1}^{n} c_{m\ell}e_\ell\Big)\\
&= \sum_{m=1}^{n}\sum_{\ell=1}^{n}\sum_{k=1}^{n}(r_{jk}^{\ell}c_{ik}+r_{ik}^{\ell}c_{jk})c_{\ell m}e_m.
\end{aligned}
$$

Thus we obtain

**Theorem 3.6.** *Let $S=\{e_1,\cdots,e_n\}$ be a semigroup with its Cayley table given by Eq. (3.9). Let $\mathbf{k}$ be a commutative unitary ring and let $P:\mathbf{k}[S]\to\mathbf{k}[S]$ be a linear operator with matrix $C:=C_P=(c_{ij})_{1\le i,j\le n}$. Then $P$ is a symmetric Rota-Baxter operator of weight zero on $\mathbf{k}[S]$ if and only if the following equations hold.*

$$
\sum_{\ell=1}^{n}\sum_{k=1}^{n} r_{k\ell}^{m}c_{ik}c_{j\ell}=\sum_{\ell=1}^{n}\sum_{k=1}^{n}(r_{jk}^{\ell}c_{ik}+r_{ik}^{\ell}c_{jk})c_{\ell m}\quad (1\le i,j,m\le n). \tag{3.10}
$$

We will determine the matrices $C_P$ for all symmetric Rota-Baxter operators $P$ on $\mathbf{k}[S]$ of order two and three.

**Order 2:** As is well known [51], there are exactly five distinct nonisomorphic semigroups of order 2. We use $N_2, L_2, R_2, Y_2$ and $Z_2$ respectively to denote the null semigroup of order 2, the left zero semigroup, right zero semigroup, the semilattice of order 2 and the cyclic group of order 2. Since $L_2$ and $R_2$ are anti-isomorphic, there are exactly four distinct semigroups of order 2, up to isomorphism and anti-isomorphism, namely $N_2$, $Y_2$, $Z_2$ and $L_2$. The only noncommutative semigroup is $L_2$, which we will work on. We still show all the semigroups in the table below. Let $\{e_1, e_2\}$ denote the underlying set of each semigroup. Then the Cayley tables for these semigroups are as follows:

Table 1: The Cayley table of semigroups of order 2

| | | | |
|---|---|---|---|
| $N_2 := \begin{array}{c\|cc} \cdot & e_1 & e_2 \\ \hline e_1 & e_1 & e_1 \\ e_2 & e_1 & e_1 \end{array}$ | $Y_2 := \begin{array}{c\|cc} \cdot & e_1 & e_2 \\ \hline e_1 & e_1 & e_1 \\ e_2 & e_1 & e_2 \end{array}$ | $Z_2 := \begin{array}{c\|cc} \cdot & e_1 & e_2 \\ \hline e_1 & e_1 & e_2 \\ e_2 & e_2 & e_1 \end{array}$ | $L_2 := \begin{array}{c\|cc} \cdot & e_1 & e_2 \\ \hline e_1 & e_1 & e_1 \\ e_2 & e_2 & e_2 \end{array}$ |

**Proposition 3.7.** *Let* $\mathbf{k}$ *be a field of characteristics zero. All symmetric Rota-Baxter operators on an noncommutative semigroup algebra* $\mathbf{k}[S]$ *of order* 2 *have their matrices* $C_P$ *given below,where all the parameters are in* $\mathbf{k}$.

$$N_1 = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}.$$

**Order 3:** Up to isomorphism and anti-isomorphism, there are 18 semigroups of order 3 [17, 29, 30]. The Cayley tables of the 18 semigroups of order 3 can be found in [30]. See also [17, 50, 53]. We only consider the noncommutative cases and denote by $NCS$ the class of 6 noncommutative semigroups. The Cayley table is given below.

For symmetric Rota-Baxter operators on the corresponding semigroup algebras, we have the following classification proposition.

Table 2: The Cayley table of noncommutative semigroups of order 3

| | | | |
|---|---|---|---|
| | $NCS(1) := \begin{array}{c\|ccc} \cdot & e_1 & e_2 & e_3 \\ \hline e_1 & e_1 & e_1 & e_1 \\ e_2 & e_1 & e_2 & e_1 \\ e_3 & e_1 & e_3 & e_1 \end{array}$ | $NCS(2) := \begin{array}{c\|ccc} \cdot & e_1 & e_2 & e_3 \\ \hline e_1 & e_1 & e_1 & e_1 \\ e_2 & e_1 & e_2 & e_1 \\ e_3 & e_3 & e_3 & e_3 \end{array}$ | $NCS(3) := \begin{array}{c\|ccc} \cdot & e_1 & e_2 & e_3 \\ \hline e_1 & e_1 & e_1 & e_1 \\ e_2 & e_1 & e_2 & e_2 \\ e_3 & e_1 & e_3 & e_3 \end{array}$ |
| | $NCS(4) := \begin{array}{c\|ccc} \cdot & e_1 & e_2 & e_3 \\ \hline e_1 & e_1 & e_1 & e_1 \\ e_2 & e_2 & e_2 & e_2 \\ e_3 & e_1 & e_1 & e_1 \end{array}$ | $NCS(5) := \begin{array}{c\|ccc} \cdot & e_1 & e_2 & e_3 \\ \hline e_1 & e_1 & e_1 & e_1 \\ e_2 & e_2 & e_2 & e_2 \\ e_3 & e_3 & e_3 & e_3 \end{array}$ | $NCS(6) := \begin{array}{c\|ccc} \cdot & e_1 & e_2 & e_3 \\ \hline e_1 & e_1 & e_1 & e_1 \\ e_2 & e_1 & e_2 & e_3 \\ e_3 & e_3 & e_3 & e_3 \end{array}$ |

**Proposition 3.8.** *Let* $\mathbf{k}$ *be a field of characteristic zero. The matrices of the symmetric Rota-Baxter operators on noncommutative semigroup algebras of order three are given in Table 3, where all the parameters take values in* $\mathbf{k}$*.*

Table 3: Symmetric Rota-Baxter operators on noncommutative semigroup algebras of order 3

| **Semigroups** | **Matrices of symmetric Rota-Baxter operators on semigroup algebras** |
|---|---|
| $NCS(1)$ | $N_{1,1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a & 0 \end{pmatrix}.$ |
| $NCS(2)$ | $N_{2,1} = \begin{pmatrix} -a & -b & -c \\ 0 & 0 & 0 \\ a & b & c \end{pmatrix}, N_{2,2} = \begin{pmatrix} -a & -a & -b \\ 0 & 0 & 0 \\ a & a & b \end{pmatrix}, N_{2,3} = \begin{pmatrix} 0 & -a & -b \\ 0 & 0 & 0 \\ 0 & a & b \end{pmatrix}.$ |
| $NCS(3)$ | $N_{3,1} = \begin{pmatrix} 0 & 0 & 0 \\ a & b & c \\ -a & -b & -c \end{pmatrix}.$ |
| Continued on next page | |

Table 3: Symmetric Rota-Baxter operators on noncomumutative semigroup algebras of order 3

| **Semigroups** | **Matrices of symmetric Rota-Baxter operators on semigroup algebras** |
|---|---|
| $NCS(4)$ | $N_{4,1} = \begin{pmatrix} c & -a-b & d \\ -c-d & a & e \\ d & b & -d-e \end{pmatrix}, N_{4,2} = \begin{pmatrix} c & -a-b & d \\ -c & a & e \\ 0 & b & -d-e \end{pmatrix},$ $N_{4,3} = \begin{pmatrix} c & -a-b & d \\ -c & a & -2c \\ 0 & b & 2c-d \end{pmatrix}, N_{4,4} = \begin{pmatrix} 0 & -a & b \\ 0 & 0 & c \\ 0 & a & -b-c \end{pmatrix},$ $N_{4,5} = \begin{pmatrix} a & 0 & b \\ 0 & 0 & c \\ -a & 0 & -b-c \end{pmatrix}.$ |
| $NCS(5)$ | $N_{5,1} = \begin{pmatrix} a & c & e \\ b & d & f \\ -a-b & -c-d & -e-f \end{pmatrix}.$ |
| $NCS(6)$ | $N_{6,1} = \begin{pmatrix} b & -a & b \\ 0 & 0 & 0 \\ -b & a & -b \end{pmatrix}.$ |

**Remark 3.9.** We verified manually that all the above results are indeed symmetric Rota-Baxter operators.

# 4 Symmetric Rota-Baxter Algebras

## 4.1 Free symmetric Rota-Baxter algebras over an algebra

We start with the definition of free symmetric Rota-Baxter algebras.

**Definition 4.1.** Let $A$ be a noncommutative $\mathbf{k}$-algebra where $\mathbf{k}$ is a field. A free symmetric Rota-Baxter algebra over $A$ is a symmetric Rota-Baxter algebra $F_{\mathrm{S}}(A)$ with a symmetric Rota-Baxter operator $S_A$ and an algebra homomorphism $j_A : A \to F_{\mathrm{S}}(A)$ such that, for any symmetric Rota-Baxter algebra $S$ and any algebra homomorphism $f : A \to S$, there is a unique symmetric Rota-Baxter algebra homomorphism $\bar{f} : F_{\mathrm{S}}(A) \to S$ such that $\bar{f} \circ j_A = f$:

$$\begin{tikzcd} A \arrow[r, "j_A"] \arrow[dr, "f"'] & F_{\mathrm{S}}(A) \arrow[d, "\bar{f}"] \\ & S \end{tikzcd}$$

For the construction of free symmetric Rota-Baxter algebras, we follow the construction of free Rota-Baxter algebras [23, 32] using bracketed words. Alternatively, one can follow the approach of Gröbner-Shirshov bases [8]. Because of the lack of a uniform approach (see [37, 38] for some recent attempts in this direction) and to be notationally self contained, we give some details. We first display a $\mathbf{k}$-basis of the free symmetric Rota-Baxter algebra in terms of bracketed words in § 4.1.1. Then the product on the free symmetric Rota-Baxter algebra is given in § 4.1.2 and the universal property of the free symmetric Rota-Baxter algebra is proved in § 4.1.3.

### 4.1.1 A basis of the free symmetric Rota-Baxter algebra

Let $A$ be a noncommutative $\mathbf{k}$-algebra with a $\mathbf{k}$-basis $X$. We first display a $\mathbf{k}$-basis $\mathfrak{X}_\infty$ of $F_S(A)$ in terms of bracketed words from the alphabet set $X$.

Let $\lfloor$ and $\rfloor$ be symbols, called brackets, and let $X' = X \cup \{\lfloor, \rfloor\}$. Let $M(X')$ denote the free semigroup generated by $X'$.

**Definition 4.2.** Let $Y, Z$ be two subsets of $M(X')$. As in [23, 32], we define the **alternating product** of $Y$ and $Z$ to be

$$\Lambda(Y,Z) = \Big(\bigcup_{r\geq 1}(Y\lfloor Z\rfloor)^r\Big)\bigcup\Big(\bigcup_{r\geq 0}(Y\lfloor Z\rfloor)^r Y\Big)\bigcup\Big(\bigcup_{r\geq 1}(\lfloor Z\rfloor Y)^r\Big)\bigcup\Big(\bigcup_{r\geq 0}(\lfloor Z\rfloor Y)^r\lfloor Z\rfloor\Big) \tag{4.1}$$

Note that $\Lambda(Y,Z) \in M(X')$.

We construct a sequence $\mathfrak{X}_n (n \geq 0)$ of subsets of $M(X')$ by the following recursion. Let $\mathfrak{X}_0 = X$ and, for $n \geq 0$, define

$$\mathfrak{X}_{n+1} = \Lambda(X, \mathfrak{X}_n).$$

Further, define

$$\mathfrak{X}_\infty = \bigcup_{n\geq 0}\mathfrak{X}_n = \varinjlim \mathfrak{X}_n. \tag{4.2}$$

Here the second equation in Eq. (4.2) follows since $\mathfrak{X}_1 \supseteq \mathfrak{X}_0$ and, assuming $\mathfrak{X}_n \supseteq \mathfrak{X}_{n-1}$, we have

$$\mathfrak{X}_{n+1} = \Lambda(X, \mathfrak{X}_n) \supseteq \Lambda(X, \mathfrak{X}_{n-1}) \supseteq \mathfrak{X}_n.$$

By [23, 32] we have the disjoint union

$$\begin{aligned}\mathfrak{X}_\infty &= \Big(\bigsqcup_{r\geq 1}(X\lfloor\mathfrak{X}_\infty\rfloor)^r\Big)\bigsqcup\Big(\bigsqcup_{r\geq 0}(X\lfloor\mathfrak{X}_\infty\rfloor)^r X\Big)\\ &\bigsqcup\Big(\bigsqcup_{r\geq 1}(\lfloor\mathfrak{X}_\infty\rfloor X)^r\Big)\bigsqcup\Big(\bigsqcup_{r\geq 0}(\lfloor\mathfrak{X}_\infty\rfloor X)^r\lfloor\mathfrak{X}_\infty\rfloor\Big).\end{aligned}\tag{4.3}$$

Further, every $\mathbf{x}\in\mathfrak{X}_\infty$ has a unique decomposition

$$\mathbf{x}=\mathbf{x}_1\cdots\mathbf{x}_b,\tag{4.4}$$

where $\mathbf{x}_i$, $1\leq i\leq b$, is alternatively in $X$ or in $\lfloor\mathfrak{X}_\infty\rfloor$. This decomposition will be called the **standard decomposition** of $\mathbf{x}$.

For $\mathbf{x}$ in $\mathfrak{X}_\infty$ with standard decomposition $\mathbf{x}_1\cdots\mathbf{x}_b$, we define $b$ to be the **breadth** $b(\mathbf{x})$ of $\mathbf{x}$, we define the **head** $h(\mathbf{x})$ of $\mathbf{x}$ to be 0 (resp. 1) if $\mathbf{x}_1$ is in $X$ (resp. in $\lfloor\mathfrak{X}_\infty\rfloor$). Similarly define the **tail** $t(\mathbf{x})$ of $\mathbf{x}$ to be 0 (resp. 1) if $\mathbf{x}_b$ is in $X$ (resp. in $\lfloor\mathfrak{X}_\infty\rfloor$). And we also define **depth** $d(\mathbf{x})$:=min$\{n$, where $\mathbf{x}\in\mathfrak{X}_n\}$.

### 4.1.2 The product in a free symmetric Rota-Baxter algebra

Let

$$F_{\rm S}(A)=\bigoplus_{\mathbf{x}\in\mathfrak{X}_\infty}\mathbf{k}\mathbf{x}.$$

We now define a product $\diamond$ on $F_{\rm S}(A)$ by defining $\mathbf{x}\diamond\mathbf{x}'\in F_{\rm S}(A)$ for $\mathbf{x},\mathbf{x}'\in\mathfrak{X}_\infty$ and then extending bilinearly. Roughly speaking, the product of $\mathbf{x}$ and $\mathbf{x}'$ is defined to be the concatenation whenever $t(\mathbf{x})\neq h(\mathbf{x}')$. When $t(\mathbf{x})=h(\mathbf{x}')$, the product is defined by the product in $A$ or by the symmetric relation in Eq. (3.1).

To be precise, we use induction on the sum $n := d(\mathbf{x})+d(\mathbf{x}')$ of the depths of $\mathbf{x}$

and $\mathbf{x}'$. Then $n \geq 0$. If $n = 0$, then $\mathbf{x}, \mathbf{x}'$ are in $X$ and so are in $A$ and we define $\mathbf{x} \diamond \mathbf{x}' = \mathbf{x} \cdot \mathbf{x}' \in A \subseteq F_S(A)$. Here $\cdot$ is the product in $A$.

Suppose $\mathbf{x} \diamond \mathbf{x}'$ have been defined for all $\mathbf{x}, \mathbf{x}' \in \mathfrak{X}_\infty$ with $n \geq k \geq 0$ and let $\mathbf{x}, \mathbf{x}' \in \mathfrak{X}_\infty$ with $n = k + 1$.

First assume the breadth $b(\mathbf{x}) = b(\mathbf{x}') = 1$. Then $\mathbf{x}$ and $\mathbf{x}'$ are in $X$ or $\lfloor \mathfrak{X}_\infty \rfloor$. Since $n = k + 1$ is at least one, $\mathbf{x}$ and $\mathbf{x}'$ cannot be both in $X$. We accordingly define

$$\mathbf{x} \diamond \mathbf{x}' = \begin{cases} \mathbf{x}\mathbf{x}', & \text{if } \mathbf{x} \in X, \mathbf{x}' \in \lfloor \mathfrak{X}_\infty \rfloor, \\ \mathbf{x}\mathbf{x}', & \text{if } \mathbf{x} \in \lfloor \mathfrak{X}_\infty \rfloor, \mathbf{x}' \in X, \\ \lfloor \overline{\mathbf{x}} \diamond \mathbf{x}' \rfloor + \lfloor \overline{\mathbf{x}}' \diamond \mathbf{x} \rfloor, & \text{if } \mathbf{x} = \lfloor \overline{\mathbf{x}} \rfloor, \mathbf{x}' = \lfloor \overline{\mathbf{x}}' \rfloor \in \lfloor \mathfrak{X}_\infty \rfloor. \end{cases} \tag{4.5}$$

Here the product in the first and second case are by concatenation and in the third case is by the induction hypothesis since for the two products on the right hand side we have

$$\begin{aligned} d(\overline{\mathbf{x}}) + d(\lfloor \overline{\mathbf{x}}' \rfloor) &= d(\lfloor \overline{\mathbf{x}} \rfloor) - 1 + d(\lfloor \overline{\mathbf{x}}' \rfloor) = d(\mathbf{x}) + d(\mathbf{x}') - 1, \\ d(\overline{\mathbf{x}}') + d(\lfloor \overline{\mathbf{x}} \rfloor) &= d(\lfloor \overline{\mathbf{x}}' \rfloor) - 1 + d(\lfloor \overline{\mathbf{x}} \rfloor) = d(\mathbf{x}) + d(\mathbf{x}') - 1 \end{aligned}$$

which are all less than or equal to $k$.

Now assume $b(\mathbf{x}) > 1$ or $b(\mathbf{x}') > 1$. Let $\mathbf{x} = \mathbf{x}_1 \cdots \mathbf{x}_b$ and $\mathbf{x}' = \mathbf{x}'_1 \cdots \mathbf{x}'_{b'}$ be the standard decompositions from Eq. (4.4). We then define

$$\mathbf{x} \diamond \mathbf{x}' = \mathbf{x}_1 \cdots \mathbf{x}_{b-1} (\mathbf{x}_b \diamond \mathbf{x}'_1) \mathbf{x}'_2 \cdots \mathbf{x}'_{b'} \tag{4.6}$$

where $\mathbf{x}_b \diamond \mathbf{x}'_1$ is defined by Eq. (4.5) and the rest is given by concatenation. The concatenation is well-defined since by Eq. (4.5), and we have $h(\mathbf{x}_b) = h(\mathbf{x}_b \diamond \mathbf{x}'_1)$ and $t(\mathbf{x}'_1) = t(\mathbf{x}_b \diamond \mathbf{x}'_1)$. Therefore, $t(\mathbf{x}_{b-1}) \neq h(\mathbf{x}_b \diamond \mathbf{x}'_1)$ and $h(\mathbf{x}'_2) \neq t(\mathbf{x}_b \diamond \mathbf{x}'_1)$.

We have the following simple properties of $\diamond$ based on its definition above.

**Lemma 4.3.** *Let $\mathbf{x}, \mathbf{x}' \in \mathfrak{X}_\infty$. We have the following statements.*

(a) $h(\mathbf{x}) = h(\mathbf{x} \diamond \mathbf{x}')$ *and* $t(\mathbf{x}') = t(\mathbf{x} \diamond \mathbf{x}')$.

(b) *If* $t(\mathbf{x}) \neq h(\mathbf{x}')$, *then* $\mathbf{x} \diamond \mathbf{x}' = \mathbf{x}\mathbf{x}'$ *(concatenation).*

(c) *If* $t(\mathbf{x}) \neq h(\mathbf{x}')$, *then for any* $\mathbf{x}'' \in \mathfrak{X}_\infty$,

$$(\mathbf{x}\mathbf{x}') \diamond \mathbf{x}'' = \mathbf{x}(\mathbf{x}' \diamond \mathbf{x}''), \quad \mathbf{x}'' \diamond (\mathbf{x}\mathbf{x}') = (\mathbf{x}'' \diamond \mathbf{x})\mathbf{x}'.$$

Extending $\diamond$ bilinearly, we obtain a binary operation

$$\diamond : F_{\mathrm{S}}(A) \otimes F_{\mathrm{S}}(A) \to F_{\mathrm{S}}(A).$$

For $\mathbf{x} \in \mathfrak{X}_\infty$, define

$$S_A(\mathbf{x}) = \lfloor \mathbf{x} \rfloor. \tag{4.7}$$

Obviously $\lfloor \mathbf{x} \rfloor$ is again in $\mathfrak{X}_\infty$. Thus $S_A$ extends to a linear operator $S_A$ on $F_{\mathrm{S}}(A)$. Let

$$j_X : X \to \mathfrak{X}_\infty \to F_{\mathrm{S}}(A)$$

be the natural injection which extends to an algebra injection

$$j_A : A \to F_{\mathrm{S}}(A). \tag{4.8}$$

The following is our first main result which will be proved in the next subsection.

**Theorem 4.4.** *Let $A$ be a $\mathbf{k}$-algebra with a $\mathbf{k}$-basis $X$.*

(a) *The pair $(F_S(A), \diamond)$ is an algebra.*

(b) *The triple $(F_S(A), \diamond, S_A)$ is a symmetric Rota-Baxter algebra.*

(c) *The quadruple $(F_S(A), \diamond, S_A, j_A)$ is the free symmetric Rota-Baxter algebra on the algebra $A$.*

The following corollary of this theorem will be used later.

**Corollary 4.5.** *Let $M$ be a $\mathbf{k}$-module and let $T(M) = \bigoplus_{n\geq 1} M^{\otimes n}$ be the reduced tensor algebra over $M$. Then $F_S(T(M))$, together with the natural injection $i_M : M \to T(M) \xrightarrow{j_{T(M)}} F_S(T(M))$, is a free symmetric Rota-Baxtor algebra over $M$, in the sense that, for any symmetric Rota-Baxter algebra $A$ and $\mathbf{k}$-module map $f : M \to A$ there is a unique symmetric Rota-Baxter algebra homomorphism $\hat{f} : F_S(T(M)) \to A$ such that $\hat{f} \circ i_M = f$.*

*Proof.* This follows immediately from Theorem 4.4 and the fact that the construction of the free algebra on a module (resp. free symmetric Rota-Baxter algebra on an algebra, resp. free symmetric Rota-Baxter on a module) is the left adjoint functor of the forgetful functor from algebras to modules (resp. from symmetric Rota-Baxter algebras to algebras, resp. from symmetric Rota-Baxter algebras to modules), and the fact that the composition of two left adjoint functors is the left adjoint functor of the composition. □

### 4.1.3 The proof of Theorem 4.4

*Proof.* a. We just need to verify the associativity. For this we only need to verify

$$(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}''' = \mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''') \tag{4.9}$$

for $\mathbf{x}', \mathbf{x}'', \mathbf{x}''' \in \mathfrak{X}_\infty$. We will do this by induction on the sum of the depths $n := d(\mathbf{x}') + d(\mathbf{x}'') + d(\mathbf{x}''')$. If $n = 0$, then all of $\mathbf{x}', \mathbf{x}'', \mathbf{x}'''$ have depth zero and so are in $X$. In this case the product $\diamond$ is given by the product $\cdot$ in $A$ and so is associative.

Assume the associativity holds for $n \leq k$ and assume that $\mathbf{x}', \mathbf{x}'', \mathbf{x}''' \in \mathfrak{X}_\infty$ have $n = d(\mathbf{x}') + d(\mathbf{x}'') + d(\mathbf{x}''') = k + 1$.

If $t(\mathbf{x}') \neq h(\mathbf{x}'')$, then by Lemma 4.3,

$$(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}''' = (\mathbf{x}'\mathbf{x}'') \diamond \mathbf{x}''' = \mathbf{x}'(\mathbf{x}'' \diamond \mathbf{x}''') = \mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''').$$

A similar argument holds when $t(\mathbf{x}'') \neq h(\mathbf{x}''')$.

Thus we only need to verify the associativity when $t(\mathbf{x}') = h(\mathbf{x}'')$ and $t(\mathbf{x}'') = h(\mathbf{x}''')$.

We next reduce the breadths of the words.

**Lemma 4.6.** *If the associativity*

$$(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}''' = \mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''')$$

*holds for all $\mathbf{x}', \mathbf{x}''$ and $\mathbf{x}'''$ in $\mathfrak{X}_\infty$ of breadth one, then it holds for all $\mathbf{x}', \mathbf{x}''$ and $\mathbf{x}'''$ in $\mathfrak{X}_\infty$.*

*Proof.* We use induction on the sum of breadths $m := b(\mathbf{x}') + b(\mathbf{x}'') + b(\mathbf{x}''')$. Then

$m \geq 3$. The case when $m = 3$ is the assumption of the lemma. Assume the associativity holds for $3 \leq m \leq j$ and take $\mathbf{x}', \mathbf{x}'', \mathbf{x}''' \in \mathfrak{X}_\infty$ with $m = j + 1$. Then $j + 1 \geq 4$. So at least one of $\mathbf{x}', \mathbf{x}'', \mathbf{x}'''$ have breadth greater than or equal to 2.

First assume $b(\mathbf{x}') \geq 2$. Then $\mathbf{x}' = \mathbf{x}'_1\mathbf{x}'_2$ with $\mathbf{x}'_1$, $\mathbf{x}'_2 \in \mathfrak{X}_\infty$ and $t(\mathbf{x}'_1) \neq h(\mathbf{x}'_2)$. Thus by Lemma 4.3, we obtain

$$(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}''' = ((\mathbf{x}'_1\mathbf{x}'_2) \diamond \mathbf{x}'') \diamond \mathbf{x}''' = (\mathbf{x}'_1(\mathbf{x}'_2 \diamond \mathbf{x}'')) \diamond \mathbf{x}''' = \mathbf{x}'_1((\mathbf{x}'_2 \diamond \mathbf{x}'') \diamond \mathbf{x}''').$$

Similarly,

$$\mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''') = (\mathbf{x}'_1\mathbf{x}'_2) \diamond (\mathbf{x}'' \diamond \mathbf{x}''') = \mathbf{x}'_1(\mathbf{x}'_2 \diamond (\mathbf{x}'' \diamond \mathbf{x}''')).$$

Thus $(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}''' = \mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''')$ whenever $(\mathbf{x}'_2 \diamond \mathbf{x}'') \diamond \mathbf{x}''' = \mathbf{x}'_2 \diamond (\mathbf{x}'' \diamond \mathbf{x}''')$. The latter follows from the induction hypothesis. A similar proof works if $b(\mathbf{x}''') \geq 2$.

Finally if $b(\mathbf{x}'') \geq 2$, then $\mathbf{x}'' = \mathbf{x}''_1\mathbf{x}''_2$ with $\mathbf{x}''_1$, $\mathbf{x}''_2 \in \mathfrak{X}_\infty$ and $t(\mathbf{x}''_1) \neq h(\mathbf{x}''_2)$. By applying Lemma 4.3 repeatedly, we obtain

$$(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}''' = (\mathbf{x}' \diamond (\mathbf{x}''_1\mathbf{x}''_2)) \diamond \mathbf{x}''' = ((\mathbf{x}' \diamond \mathbf{x}''_1)\mathbf{x}''_2) \diamond \mathbf{x}''' = (\mathbf{x}' \diamond \mathbf{x}''_1)(\mathbf{x}''_2 \diamond \mathbf{x}''').$$

In the same way, we have

$$(\mathbf{x}' \diamond \mathbf{x}''_1)(\mathbf{x}''_2 \diamond \mathbf{x}''') = \mathbf{x}' \diamond (\mathbf{x}''_1(\mathbf{x}''_2 \diamond \mathbf{x}''')) = \mathbf{x}' \diamond ((\mathbf{x}''_1\mathbf{x}''_2) \diamond \mathbf{x}''') = \mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''').$$

This again proves the associativity. □

To summarize, our proof of the associativity has been reduced to the special case when $\mathbf{x}', \mathbf{x}'', \mathbf{x}''' \in \mathfrak{X}_\infty$ are chosen so that

(a) $n := d(\mathbf{x}') + d(\mathbf{x}'') + d(\mathbf{x}''') = k + 1 \geq 1$ with the assumption that the associativity holds when $n \leq k$.

(b) the elements have breadth one and

(c) $t(\mathbf{x}') = h(\mathbf{x}'')$ and $t(\mathbf{x}'') = h(\mathbf{x}''')$.

By item (b), the head and tail of each of the elements are the same. Therefore by item (c), either all the three elements are in $X$ or they are all in $\lfloor \mathfrak{X}_\infty \rfloor$. If all of $\mathbf{x}', \mathbf{x}'', \mathbf{x}'''$ are in $X$, then as already shown, the associativity follows from the associativity in $A$.

So it remains to consider the case when $\mathbf{x}', \mathbf{x}'', \mathbf{x}'''$ are all in $\lfloor \mathfrak{X}_\infty \rfloor$. Then $\mathbf{x}' = \lfloor \overline{\mathbf{x}}' \rfloor, \mathbf{x}'' = \lfloor \overline{\mathbf{x}}'' \rfloor, \mathbf{x}''' = \lfloor \overline{\mathbf{x}}''' \rfloor$ with $\overline{\mathbf{x}}', \overline{\mathbf{x}}'', \overline{\mathbf{x}}''' \in \mathfrak{X}_\infty$. Using Eq. (4.5) and bilinearity of the product $\diamond$, we have

$$\begin{aligned}
(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}''' &= \lfloor \overline{\mathbf{x}}' \diamond \mathbf{x}'' + \overline{\mathbf{x}}'' \diamond \mathbf{x}' \rfloor \diamond \mathbf{x}''' \\
&= \lfloor \overline{\mathbf{x}}' \diamond \mathbf{x}'' \rfloor \diamond \mathbf{x}''' + \lfloor \overline{\mathbf{x}}'' \diamond \mathbf{x}' \rfloor \diamond \mathbf{x}''' \\
&= \lfloor \overline{\mathbf{x}}' \diamond \mathbf{x}'' \diamond \mathbf{x}''' \rfloor + \lfloor \overline{\mathbf{x}}''' \diamond \lfloor \overline{\mathbf{x}}' \diamond \mathbf{x}'' \rfloor \rfloor + \lfloor \overline{\mathbf{x}}'' \diamond \mathbf{x}' \diamond \mathbf{x}''' \rfloor + \lfloor \overline{\mathbf{x}}''' \diamond \lfloor \overline{\mathbf{x}}'' \diamond \mathbf{x}' \rfloor \rfloor
\end{aligned}$$

Applying the induction hypothesis on $n$ to the first term and the third term, and then use Eq. (4.5) again, we obtain

$$\begin{aligned}
(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}''' &= \lfloor \overline{\mathbf{x}}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''') \rfloor + \lfloor \overline{\mathbf{x}}''' \diamond \lfloor \overline{\mathbf{x}}' \diamond \mathbf{x}'' \rfloor \rfloor + \lfloor \overline{\mathbf{x}}'' \diamond (\mathbf{x}' \diamond \mathbf{x}''') \rfloor + \lfloor \overline{\mathbf{x}}''' \diamond \lfloor \overline{\mathbf{x}}'' \diamond \mathbf{x}' \rfloor \rfloor \\
&= \lfloor \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \diamond \mathbf{x}''' \rfloor \rfloor + \lfloor \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}''' \diamond \mathbf{x}'' \rfloor \rfloor + \lfloor \overline{\mathbf{x}}''' \diamond \lfloor \overline{\mathbf{x}}' \diamond \mathbf{x}'' \rfloor \rfloor
\end{aligned}$$

$$+\lfloor \overline{\mathbf{x}}'' \diamond \lfloor \overline{\mathbf{x}}' \diamond \mathbf{x}''' \rfloor\rfloor + \lfloor \overline{\mathbf{x}}'' \diamond \lfloor \overline{\mathbf{x}}''' \diamond \mathbf{x}' \rfloor\rfloor + \lfloor \overline{\mathbf{x}}''' \diamond \lfloor \overline{\mathbf{x}}'' \diamond \mathbf{x}' \rfloor\rfloor.$$

By a similar computation, we obtain

$$\begin{aligned}\mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''') &= \lfloor \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \diamond \mathbf{x}''' \rfloor\rfloor + \lfloor \overline{\mathbf{x}}'' \diamond \lfloor \overline{\mathbf{x}}''' \diamond \mathbf{x}' \rfloor\rfloor + \lfloor \overline{\mathbf{x}}'' \diamond \lfloor \overline{\mathbf{x}}' \diamond \mathbf{x}''' \rfloor\rfloor \\ &\quad + \lfloor \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}''' \diamond \mathbf{x}'' \rfloor\rfloor + \lfloor \overline{\mathbf{x}}''' \diamond \lfloor \overline{\mathbf{x}}'' \diamond \mathbf{x}' \rfloor\rfloor + \lfloor \overline{\mathbf{x}}''' \diamond \lfloor \overline{\mathbf{x}}' \diamond \mathbf{x}'' \rfloor\rfloor.\end{aligned}$$

Now by induction, the $i$-th term in the expansion of $(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}'''$ matches with the $\sigma(i)$-th term in the expansion of $\mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''')$. Here the permutation $\sigma \in \Sigma_6$ is given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 6 & 3 & 2 & 5 \end{pmatrix}. \tag{4.10}$$

This completes the proof of Theorem 4.4.a.

b. The proof follows from the definition $S_A(\mathbf{x}) = \lfloor \mathbf{x} \rfloor$ and Eq. (4.5).

c. Let $(S, *, P)$ be a symmetric Rota-Baxter algebra with multiplication $*$. Let $f : A \to S$ be a $\mathbf{k}$-algebra homomorphism. We will construct a $\mathbf{k}$-linear map $\overline{f} : F_{\mathrm{S}}(A) \to S$ by defining $\overline{f}(\mathbf{x})$ for $\mathbf{x} \in \mathfrak{X}_\infty$. We achieve this by defining $\overline{f}(\mathbf{x})$ for $\mathbf{x} \in \mathfrak{X}_n$, $n \geq 0$, inductively on $n$. For $\mathbf{x} \in \mathfrak{X}_0 := X$, define $\overline{f}(\mathbf{x}) = f(\mathbf{x})$. Suppose $\overline{f}(\mathbf{x})$ has been defined for $\mathbf{x} \in \mathfrak{X}_n$ and consider $\mathbf{x}$ in $\mathfrak{X}_{n+1}$ which is, by definition and Eq. (4.3),

$$\begin{aligned}\Lambda(X, \mathfrak{X}_n) &= \Big(\bigcup_{r\geq 1} (X\lfloor \mathfrak{X}_n \rfloor)^r\Big) \bigcup \Big(\bigcup_{r\geq 0} (X\lfloor \mathfrak{X}_n \rfloor)^r X\Big) \\ &\quad \bigcup \Big(\bigcup_{r\geq 1} (\lfloor \mathfrak{X}_n \rfloor X)^r\Big) \bigcup \Big(\bigcup_{r\geq 0} (\lfloor \mathfrak{X}_n \rfloor X)^r \lfloor \mathfrak{X}_n \rfloor\Big).\end{aligned}$$

Let $\mathbf{x}$ be in the first union component $\bigcup_{r\geq 1}(X\lfloor \mathfrak{X}_n \rfloor)^r$ above. Then

$$\mathbf{x} = \prod_{i=1}^{r}(\mathbf{x}_{2i-1}\lfloor \mathbf{x}_{2i}\rfloor)$$

for $\mathbf{x}_{2i-1} \in X$ and $\mathbf{x}_{2i} \in \mathfrak{X}_n$, $1 \leq i \leq r$. By the construction of the multiplication $\diamond$ and the symmetric Rota-Baxter operator $S_A$, we have

$$\mathbf{x} = \diamond_{i=1}^{r}(\mathbf{x}_{2i-1} \diamond \lfloor \mathbf{x}_{2i}\rfloor) = \diamond_{i=1}^{r}(\mathbf{x}_{2i-1} \diamond S_A(\mathbf{x}_{2i})).$$

Define

$$\bar{f}(\mathbf{x}) = *_{i=1}^{r}(\bar{f}(\mathbf{x}_{2i-1}) * P(\bar{f}(\mathbf{x}_{2i}))). \tag{4.11}$$

where the right hand side is well-defined by the induction hypothesis. Similarly define $\bar{f}(\mathbf{x})$ if $\mathbf{x}$ is in the other union components. For any $\mathbf{x} \in \mathfrak{X}_\infty$, we have $S_A(\mathbf{x}) = \lfloor \mathbf{x} \rfloor \in \mathfrak{X}_\infty$, and by the definition of $\bar{f}$ in (Eq. (4.11)), we have

$$\bar{f}(\lfloor \mathbf{x} \rfloor) = \bar{f}(S_A(x)) = P(\bar{f}(\mathbf{x})). \tag{4.12}$$

So $\bar{f}$ commutes with the symmetric Rota-Baxter operator. Combining this equation with Eq. (4.11) we see that if $\mathbf{x} = \mathbf{x}_1 \cdots \mathbf{x}_b$ is the standard decomposition of $\mathbf{x}$, then

$$\bar{f}(\mathbf{x}) = \bar{f}(\mathbf{x}_1) * \cdots * \bar{f}(\mathbf{x}_b). \tag{4.13}$$

Note that this is the only possible way to define $\bar{f}(\mathbf{x})$ in order for $\bar{f}$ to be a symmetric Rota-Baxter algebra homomorphism extending $f$.

It remains to prove that the map $\bar{f}$ defined in Eq. (4.11) is indeed an algebra homomorphism. For this we only need to check the multiplicity

$$\overline{f}(\mathbf{x} \diamond \mathbf{x}') = \overline{f}(\mathbf{x}) * \overline{f}(\mathbf{x}') \tag{4.14}$$

for all $\mathbf{x}, \mathbf{x}' \in \mathfrak{X}_\infty$. For this we use induction on the sum of depths $n := d(\mathbf{x}) + d(\mathbf{x}')$. Then $n \geq 0$. When $n = 0$, we have $\mathbf{x}, \mathbf{x}' \in X$. Then Eq. (4.14) follows from the multiplicity of $f$. Assume the multiplicity holds for $\mathbf{x}, \mathbf{x}' \in \mathfrak{X}_\infty$ with $n \leq k$ and take $\mathbf{x}, \mathbf{x}' \in \mathfrak{X}_\infty$ with $n = k + 1$. Let $\mathbf{x} = \mathbf{x}_1 \cdots \mathbf{x}_b$ and $\mathbf{x}' = \mathbf{x}'_1 \cdots \mathbf{x}'_{b'}$ be the standard decompositions. Since $n = k + 1 \geq 1$, at least one of $\mathbf{x}_b$ and $\mathbf{x}'_{b'}$ is in $\lfloor \mathfrak{X}_\infty \rfloor$. Then by Eq. (4.5) we have,

$$\overline{f}(\mathbf{x}_b \diamond \mathbf{x}'_1) = \begin{cases} \overline{f}(\mathbf{x}_b \mathbf{x}'_1), & \text{if } \mathbf{x}_b \in X, \mathbf{x}'_1 \in \lfloor \mathfrak{X}_\infty \rfloor, \\ \overline{f}(\mathbf{x}_b \mathbf{x}'_1), & \text{if } \mathbf{x}_b \in \lfloor \mathfrak{X}_\infty \rfloor, \mathbf{x}'_1 \in X, \\ \overline{f}(\lfloor \overline{\mathbf{x}}_b \diamond \mathbf{x}'_1 \rfloor + \lfloor \overline{\mathbf{x}}'_1 \diamond \mathbf{x}_b \rfloor), & \text{if } \mathbf{x}_b = \lfloor \overline{\mathbf{x}}_b \rfloor, \mathbf{x}'_1 = \lfloor \overline{\mathbf{x}}'_1 \rfloor \in \lfloor \mathfrak{X}_\infty \rfloor. \end{cases}$$

In the first two cases, the right hand side is $\overline{f}(\mathbf{x}_b) * \overline{f}(\mathbf{x}'_1)$ by the definition of $\overline{f}$. In the third case, we have, by Eq. (4.12), the induction hypothesis and the symmetric Rota-Baxter relation of the operator $P$ on $S$,

$$\begin{aligned} \overline{f}(\lfloor \overline{\mathbf{x}}_b \diamond \mathbf{x}'_1 \rfloor + \lfloor \overline{\mathbf{x}}'_1 \diamond \mathbf{x}_b \rfloor) =& \overline{f}(\lfloor \overline{\mathbf{x}}_b \diamond \mathbf{x}'_1 \rfloor) + \overline{f}(\lfloor \overline{\mathbf{x}}'_1 \diamond \mathbf{x}_b \rfloor) \\ =& P(\overline{f}(\overline{\mathbf{x}}_b \diamond \mathbf{x}'_1)) + P(\overline{f}(\overline{\mathbf{x}}'_1 \diamond \mathbf{x}_b)) \\ =& P(\overline{f}(\overline{\mathbf{x}}_b) * \overline{f}(\mathbf{x}'_1)) + P(\overline{f}(\overline{\mathbf{x}}'_1) * \overline{f}(\mathbf{x}_b)) \\ =& P(\overline{f}(\overline{\mathbf{x}}_b) * \overline{f}(\lfloor \overline{\mathbf{x}}'_1 \rfloor)) + P(\overline{f}(\overline{\mathbf{x}}'_1) * \overline{f}(\lfloor \overline{\mathbf{x}}_b \rfloor)) \\ =& P(\overline{f}(\overline{\mathbf{x}}_b) * P(\overline{f}(\overline{\mathbf{x}}'_1))) + P(\overline{f}(\overline{\mathbf{x}}'_1) * P(\overline{f}(\overline{\mathbf{x}}_b))) \\ =& P(\overline{f}(\overline{\mathbf{x}}_b)) * P(\overline{f}(\overline{\mathbf{x}}'_1)) \end{aligned}$$

$$= \overline{f}(\lfloor \overline{\mathbf{x}}_b \rfloor) * \overline{f}(\lfloor \overline{\mathbf{x}}'_1 \rfloor)$$
$$= \overline{f}(\mathbf{x}_b) * \overline{f}(\mathbf{x}'_1).$$

Therefore $\overline{f}(\mathbf{x}_b \diamond \mathbf{x}'_1) = \overline{f}(\mathbf{x}_b) * \overline{f}(\mathbf{x}'_1)$. Then

$$\begin{aligned}
\overline{f}(\mathbf{x} \diamond \mathbf{x}') &= \overline{f}(\mathbf{x}_1 \cdots \mathbf{x}_{b-1}(\mathbf{x}_b \diamond \mathbf{x}'_1)\mathbf{x}'_2 \cdots \mathbf{x}'_{b'}) \\
&= \overline{f}(\mathbf{x}_1) * \cdots * \overline{f}(\mathbf{x}_{b-1}) * \overline{f}(\mathbf{x}_b \diamond \mathbf{x}'_1) * \overline{f}(\mathbf{x}'_2) * \cdots * \overline{f}(\mathbf{x}'_{b'}) \\
&= \overline{f}(\mathbf{x}_1) * \cdots * \overline{f}(\mathbf{x}_{b-1}) * \overline{f}(\mathbf{x}_b) * \overline{f}(\mathbf{x}'_1) * \overline{f}(\mathbf{x}'_2) * \cdots * \overline{f}(\mathbf{x}'_{b'}) \\
&= \overline{f}(\mathbf{x}) * \overline{f}(\mathbf{x}').
\end{aligned}$$

This is what we need. □

## 4.2 Free symmetric Rota-Baxter algebras over a module or a set

We first obtain a symmetric Rota–Baxter algebra structure on planar rooted forests and their various subsets. This allows us to give a uniform construction of free symmetric Rota–Baxter algebras in different settings (modules, sets, etc) in §4.2.3. For other variations of this construction, see [5, 22, 34]. The following standard descriptions for rooted forests are partially drawn from [36].

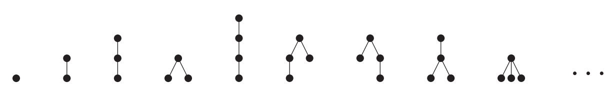
### 4.2.1 Planar rooted forests

For the convenience of the reader and for fixing notations, we recall basic concepts and facts of planar rooted trees. For references, see [18, 60].

A (free) tree is an undirected graph that is connected and contains no cycles. A

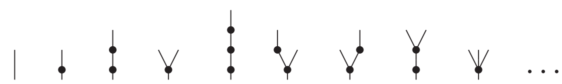
**rooted tree** is a free tree in which a particular vertex has been distinguished as the **root**. Such a distinguished vertex endows the tree with a directed graph structure when the edges of the tree are given the orientation of pointing away from the root. If two vertices of a rooted tree are connected by such an oriented edge, then the vertex on the side of the root is called the **parent** and the vertex on the opposite side of the root is called a **child**. A vertex with no children is called a **leaf**. By our convention, in a tree with only one vertex, this vertex is a leaf, as well as the root. The number of edges in a path connecting two vertices in a rooted tree is called the **length** of the path. The **depth** $\mathrm{d}(T)$ (or **height**) of a rooted tree $T$ is the length of the longest path from its root to its leafs. A **planar rooted tree** is a rooted tree with a fixed embedding into the plane.

There are two ways to draw planar rooted trees. In the first way, all vertices are represented by a dot and the root is usually at the top of the tree. The following list shows the first few of them.



Note that we distinguish the sides of the trees (i.e. the sixth tree above is different from the seventh), so the trees are planar. The tree $\bullet$ with only the root is called the **empty tree**. This method is used, for example, in the above references [18, 60] and in the Hopf algebra of non-planar rooted trees of Connes and Kreimer [9, 10].

In the second way the leaf vertices are removed with only the edges leading to them left, and the root, placed at the bottom in opposite to the first drawing, gets an extra edge pointing down. The following list shows the first few of them.



This is used, for example in the Hopf algebra of planar rooted trees of Loday and Ronco [46, 48] and noncommutative variation of the Connes-Kreimer Hopf algebra [41, 31]. In the following we will mostly use the first method.

Let $\mathcal{T}$ be the set of planar rooted trees and let $\mathcal{F}$ be the free semigroup generated by $\mathcal{T}$ in which the product is denoted by $\sqcup$, called the concatenation. Thus each element in $\mathcal{F}$ is a noncommutative product $T_1 \sqcup \cdots \sqcup T_n$ consisting of trees $T_1, \cdots, T_n \in \mathcal{T}$, called a **planar rooted forest**. We also use the abbreviation

$$T^{\sqcup n} = \underbrace{T \sqcup \cdots \sqcup T}_{n \text{ terms}}. \tag{4.15}$$

**Remark 4.1.** For the rest of this chapter and §6, a tree or forest means a planar rooted one unless otherwise specified.

We use the (grafting) **brackets** $\lfloor T_1 \sqcup \cdots \sqcup T_n \rfloor$ to denote the tree obtained by **grafting**, that is, by adding a new root together with an edge from the new root to the root of each of the trees $T_1, \cdots, T_n$. This is the $B^+$ operator in the work of Connes and Kreimer [10]. The operation is also denoted by $T_1 \vee \cdots \vee T_n$ in some other literatures, such as in Loday and Ronco [46, 48]. Note that our operation $\sqcup$ is different from $\vee$. Their relation is

$$\lfloor T_1 \sqcup \cdots \sqcup T_n \rfloor = T_1 \vee \cdots \vee T_n.$$

As an opposite operation, we use the (degrafting) **bar** $\overline{F}$ to denote the forest obtained by **degrafting**, that is, by deleting the root with its edges.

See [34] for a general framework of such algebraic structures with operators.

The **depth** of a forest $F$ is the maximal depth $\mathrm{d} = \mathrm{d}(F)$ of trees in $F$. Clearly, $\mathrm{d}(\lfloor F \rfloor) = \mathrm{d}(F) + 1$. The trees in a forest $F$ are called root branches of $\lfloor F \rfloor$. Furthermore, for a forest $F = T_1 \sqcup \cdots \sqcup T_b$ with trees $T_1, \cdots, T_b$, we define $b = b(F)$ to be the **breadth** of $F$. Let $\ell(F)$ be the number of leafs of $F$. Then

$$\ell(F) = \sum_{i=1}^{b} \ell(T_i). \tag{4.16}$$

We will often use the following recursive structure on forests. For any subset $X$ of $\mathcal{F}$, let $\langle X \rangle$ be the sub-semigroup of $\mathcal{F}$ generated by $X$. Let $\mathcal{F}_0 = \langle \bullet \rangle$, consisting of forests $\bullet^{\sqcup n}, n \geq 0$. These are also the forests of depth zero. Then recursively define

$$\mathcal{F}_n = \langle \{\bullet\} \cup \lfloor \mathcal{F}_{n-1} \rfloor \rangle. \tag{4.17}$$

It is clear that $\mathcal{F}_n$ is the set of forests with depth less or equal to $n$. From this observation, we see that $\mathcal{F}_n$ form a linear ordered direct system: $\mathcal{F}_n \supseteq \mathcal{F}_{n-1}$, and

$$\mathcal{F} = \cup_{n \geq 0} \mathcal{F}_n = \varinjlim \mathcal{F}_n. \tag{4.18}$$

### 4.2.2 Symmetric Rota-Baxter operators on rooted forests

In this section, we are going to define the product $\diamond$ on $\mathbf{k}\,\mathcal{F}$, making $\{\mathbf{k}\mathcal{F}, \diamond\}$ into a symmetric Rota–Baxter algebra. We define $\diamond$ by giving a set map

$$\diamond : \mathcal{F} \times \mathcal{F} \to \mathbf{k}\,\mathcal{F}$$

and then extend it bilinearly. For this, we use the depth filtration $\mathcal{F} = \cup_{n\geq 0}\mathcal{F}_n$ in Eq. (4.18) and apply induction on $i + j$ to define

$$\diamond : \mathcal{F}_i \times \mathcal{F}_j \to \mathbf{k}\,\mathcal{F}.$$

When $i + j = 0$, we have $\mathcal{F}_i = \mathcal{F}_j = \langle \bullet \rangle$. With the notation in Eq. (4.15), we define

$$\diamond : \mathcal{F}_0 \times \mathcal{F}_0 \to \mathbf{k}\,\mathcal{F},\ \bullet^{\sqcup m} \diamond \bullet^{\sqcup n} := \bullet^{\sqcup (m+n-1)}. \tag{4.19}$$

For a given $k \geq 0$, suppose that $\diamond : \mathcal{F}_i \times \mathcal{F}_j \to \mathbf{k}\,\mathcal{F}$ is defined for $i + j \leq k$. Consider forests $F, F'$ with $\mathrm{d}(F) + \mathrm{d}(F') = k + 1$.

First assume that $F$ and $F'$ are trees. Note that a tree is either $\bullet$ or is of the form $\lfloor \overline{F} \rfloor$ for a forest $\overline{F}$ of smaller depth. Thus we can define

$$F \diamond F' = \begin{cases} F, & \text{if } F' = \bullet, \\ F', & \text{if } F = \bullet, \\ \lfloor \overline{F} \diamond F' \rfloor + \lfloor \overline{F}' \diamond F \rfloor, & \text{if } F = \lfloor \overline{T} \rfloor, F' = \lfloor \overline{T}' \rfloor, \end{cases} \tag{4.20}$$

since for the two products on the right hand of the third equation, the sums

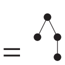
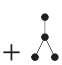
$$\mathrm{d}(\overline{F}) + \mathrm{d}(F'), \quad \mathrm{d}(\overline{F}') + \mathrm{d}(F) \tag{4.21}$$

are both equal to $k$. Note that in every case above, $F \diamond F'$ is a tree or a sum of trees. Now consider arbitrary forests $F = T_1 \sqcup \cdots \sqcup T_b$ and $F' = T_1' \sqcup \cdots \sqcup T_{b'}'$ with $\mathrm{d}(F) + \mathrm{d}(F') = k + 1$. We then define

$$F \diamond F' = T_1 \sqcup \cdots \sqcup T_{b-1} \sqcup (T_b \diamond T_1') \sqcup T_2' \cdots \sqcup T_{b'} \tag{4.22}$$

where $T_b \diamond T_1'$ is defined by Eq. (4.20). By the remark after Eq. (4.21), $F \diamond F'$ is in $\mathbf{k}\mathcal{F}$. This completes the definition of the set map $\diamond$ on $\mathcal{F} \times \mathcal{F}$.

As an example, we have

 $\diamond$  $= \lfloor \bullet \sqcup \bullet \rfloor \diamond \lfloor \bullet \rfloor = \lfloor (\bullet \sqcup \bullet) \diamond \lfloor \bullet \rfloor \rfloor + \lfloor \bullet \diamond \lfloor \bullet \sqcup \bullet \rfloor \rfloor =$  $+$  (4.23)

We record the following simple properties of $\diamond$ for later applications.

**Lemma 4.7.** *Let $F, F', F''$ be forests.*

(a) $(F \sqcup F') \diamond F'' = F \sqcup (F' \diamond F''), \quad F'' \diamond (F \sqcup F') = (F'' \diamond F) \sqcup F'.$

(b) $\ell(F \diamond F') = \ell(F) + \ell(F') - 1.$

So $\mathbf{k}\mathcal{F}$ with the operations $\sqcup$ and $\diamond$ forms a 2-associative algebra in the sense of [49, 52].

*Proof.* (a). Let $F = T_1 \sqcup \cdots \sqcup T_b$, $F' = T_1' \sqcup \cdots \sqcup T_{b'}'$ and $F'' = T_1'' \sqcup \cdots \sqcup T_{b''}''$ be the decomposition of the forests into trees. Since $\sqcup$ is an associative product, by

Eq. (4.22) we have,

$$\begin{aligned}
(F \sqcup F') \diamond F'' &= (T_1 \sqcup \cdots \sqcup T_b \sqcup T'_1 \sqcup \cdots \sqcup T'_{b'}) \diamond (T''_1 \sqcup T''_2 \sqcup \cdots \sqcup T''_{b''}) \\
&= T_1 \sqcup \cdots \sqcup T_b \sqcup T'_1 \sqcup \cdots \sqcup T'_{b'-1} \sqcup (T'_{b'} \diamond T''_1) \sqcup T''_2 \sqcup \cdots \sqcup T''_{b''} \\
&= (T_1 \sqcup \cdots \sqcup T_b) \sqcup (T'_1 \sqcup \cdots \sqcup T'_{b'-1} \sqcup (T'_{b'} \diamond T''_1) \sqcup T''_2 \sqcup \cdots \sqcup T''_{b''}) \\
&= F \sqcup (F' \diamond F'').
\end{aligned}$$

The proof of the second equation is the same.

(b). We prove by induction on the sum $m := \mathrm{d}(F) + \mathrm{d}(F')$. When $m = 0$, it follows from Eq. (4.19). Assume that the equation holds for all $F$ and $F'$ with $m \leq k$ and consider $F$ and $F'$ with $\mathrm{d}(F) + \mathrm{d}(F') = k + 1$.

If $F$ and $F'$ are trees, then the equation holds by Eq. (4.20), the induction hypothesis and the fact that $\ell(\lfloor \overline{F} \rfloor) = \ell(\overline{F})$ for a forest $\overline{F}$. Then for forests $F$ and $F'$, the equation follows from Eq. (4.22) and Eq. (4.16) $\square$

Note that $\ell(F \diamond F')$ is defined as the number of leafs of either tree of the right hand side if there are more than one tree, and each tree on the right hand side has the same number of leafs.

Extending $\diamond$ bilinearly, we obtain a binary operation

$$\diamond : \mathbf{k}\mathcal{F} \otimes \mathbf{k}\mathcal{F} \to \mathbf{k}\mathcal{F}.$$

For $F \in \mathcal{F}$, we use the grafting operation to define

$$P_{\mathcal{F}}(F) = \lfloor F \rfloor. \tag{4.24}$$

Then $P_{\mathcal{F}}$ extends to a linear operator on $\mathbf{k}\,\mathcal{F}$.

The following is our first main result in this chapter and will be proved in the next subsection.

**Theorem 4.8.** (a) *The pair* $(\mathbf{k}\,\mathcal{F}, \diamond)$ *is an associative algebra.*

(b) *The triple* $(\mathbf{k}\,\mathcal{F}, \diamond, P_{\mathcal{F}})$ *is a symmetric Rota–Baxter algebra.*

### 4.2.3 The proof of Theorem 4.8

*Proof.* (a). By Definition (4.20), $\bullet$ is the identity under the product $\diamond$. So we just need to verify the associativity. For this we only need to verify

$$(F \diamond F') \diamond F'' = F \diamond (F' \diamond F'') \tag{4.25}$$

for forests $F, F', F'' \in \mathcal{F}$. We will accomplish this by induction on the sum of the depths $n := \mathrm{d}(F) + \mathrm{d}(F') + \mathrm{d}(F'')$. If $n = 0$, then all of $F, F', F''$ have depth zero and so are in $\mathcal{F}_0 = \langle \bullet \rangle$, the sub-semigroup of $\mathcal{F}$ generated by $\bullet$. Then we have $F = \bullet^{\sqcup i}$, $F' = \bullet^{\sqcup i'}$ and $F'' = \bullet^{\sqcup i''}$, for $i, i', i'' \geq 1$. Then the associativity follows from Eq. (4.19) since both sides of Eq. (4.25) is $\bullet^{\sqcup(i+i'+i''-2)}$.

Let $k \geq 1$. Assume Eq. (4.25) holds for $n \leq k$ and assume that $F, F', F'' \in \mathcal{F}$ satisfy $n = \mathrm{d}(F) + \mathrm{d}(F') + \mathrm{d}(F'') = k + 1$. We next reduce the breadths of the forests.

**Lemma 4.9.** *If the associativity*

$$(F \diamond F') \diamond F'' = F \diamond (F' \diamond F'')$$

*holds when $F, F'$ and $F''$ are trees, then it holds when they are forests.*

*Proof.* We use induction on the sum of breadths $m := b(F) + b(F') + b(F'')$. Then $m \geq 3$. The case when $m = 3$ is the assumption of the lemma. Assume the associativity holds for $3 \leq m \leq j$ and take $F, F', F'' \in \mathcal{F}$ with $m = j + 1$. Then $j + 1 \geq 4$. So at least one of $F, F', F''$ has breadth greater than or equal to 2.

First assume $b(F) \geq 2$. Then $F = F_1 \sqcup F_2$ with $F_1,\ F_2 \in \mathcal{F}$. Thus by Lemma 4.7,

$$(F \diamond F') \diamond F'' = ((F_1 \sqcup F_2) \diamond F') \diamond F'' = (F_1 \sqcup (F_2 \diamond F')) \diamond F'' = F_1 \sqcup ((F_2 \diamond F') \diamond F'').$$

Similarly,

$$F \diamond (F' \diamond F'') = (F_1 \sqcup F_2) \diamond (F' \diamond F'') = F_1 \sqcup (F_2 \diamond (F' \diamond F'')).$$

Thus

$$(F \diamond F') \diamond F'' = F \diamond (F' \diamond F'')$$

whenever

$$(F_2 \diamond F') \diamond F'' = F_2 \diamond (F' \diamond F'')$$

which follows from the induction hypothesis. A similar proof works if $b(F'') \geq 2$. Finally if $b(F') \geq 2$, then $F' = F'_1 \sqcup F'_2$ with $F'_1,\ F'_2 \in \mathcal{F}$. Using Lemma 4.7 repeatedly, we have

$$(F \diamond F') \diamond F'' = (F \diamond (F'_1 \sqcup F'_2)) \diamond F'' = ((F \diamond F'_1) \sqcup F'_2) \diamond F'' = (F \diamond F'_1) \sqcup (F'_2 \diamond F'').$$

In the same way, we have $F \diamond (F' \diamond F'') = (F \diamond F'_1) \sqcup (F'_2 \diamond F'')$. This again proves the associativity. $\square$

To summarize, our proof of the associativity (4.25) has been reduced to the special case when the forests $F, F', F'' \in \mathcal{F}$ are chosen such that

(a) $n := \mathrm{d}(F)+\mathrm{d}(F')+\mathrm{d}(F'') = k+1 \geq 1$ with the assumption that the associativity holds when $n \leq k$, and

(b) the forests are of breadth one, that is, they are trees.

If either one of the trees is $\bullet$, the identity under the product $\diamond$, then the associativity is clear. So it remains to consider the case when $F, F', F''$ are all in $\lfloor \mathcal{F} \rfloor$. Then $F = \lfloor \overline{F} \rfloor, F' = \lfloor \overline{F}' \rfloor, F'' = \lfloor \overline{F}'' \rfloor$ with $\overline{F}, \overline{F}', \overline{F}'' \in \mathcal{F}$. To deal with this case, we prove the following general fact on symmetric Rota–Baxter operators on not necessarily associative algebras.

**Lemma 4.10.** *Let $R$ be a* **k***-module with a multiplication $\cdot$ that is not necessarily associative. Let $\lfloor\ \rfloor_R : R \to R$ be a* **k***-linear map such that the symmetric Rota–Baxter identity holds:*

$$\lfloor x \rfloor_R \cdot \lfloor x' \rfloor_R = \lfloor x \cdot \lfloor x' \rfloor_R \rfloor_R + \lfloor x' \cdot \lfloor x \rfloor_R \rfloor_R \forall\, x,\ x' \in R. \tag{4.26}$$

*Let $x, x'$ and $x''$ be in $R$. If*

$$(x \cdot x') \cdot x'' = x \cdot (x' \cdot x''),$$

*then we say that $(x, x', x'')$ is an* **associative triple** *for the product $\cdot$. For any $y, y', y'' \in R$, if all the triples*

$$(y, \lfloor y' \rfloor_R, \lfloor y'' \rfloor_R),\ (y', \lfloor y \rfloor_R, y''),\ (y', \lfloor y'' \rfloor_R, \lfloor y \rfloor_R),\ (y'', \lfloor y' \rfloor_R, \lfloor y \rfloor_R) \tag{4.27}$$

*are associative triples for* $\cdot$, *then* $(\lfloor y\rfloor_R, \lfloor y'\rfloor_R, \lfloor y''\rfloor_R)$ *is an associative triple for* $\cdot$.

*Proof.* Using Eq. (4.26) and bilinearity of the product $\cdot$, we have

$$
\begin{aligned}
(\lfloor y\rfloor_R \cdot \lfloor y'\rfloor_R) \cdot \lfloor y''\rfloor_R &= (\lfloor y \cdot \lfloor y'\rfloor_R + \lfloor y' \cdot \lfloor y\rfloor_R\rfloor_R) \cdot \lfloor y''\rfloor_R \\
&= \lfloor y \cdot \lfloor y'\rfloor_R\rfloor_R \cdot \lfloor y''\rfloor_R + \lfloor y' \cdot \lfloor y\rfloor_R\rfloor_R \cdot \lfloor y''\rfloor_R \\
&= \lfloor (y \cdot \lfloor y'\rfloor_R) \cdot \lfloor y''\rfloor_R\rfloor_R + \lfloor y'' \cdot \lfloor y \cdot \lfloor y'\rfloor_R\rfloor_R\rfloor_R \\
&\quad + \lfloor (y' \cdot \lfloor y\rfloor_R) \cdot \lfloor y''\rfloor_R\rfloor_R + \lfloor y'' \cdot \lfloor y' \cdot \lfloor y\rfloor_R\rfloor_R\rfloor_R.
\end{aligned}
$$

Applying the associativity of the triples in Eq. (4.27) to $(y \cdot \lfloor y'\rfloor_R) \cdot \lfloor y''\rfloor_R$ and $(y' \cdot \lfloor y\rfloor_R) \cdot \lfloor y''\rfloor_R$ above and then using Eq. (4.26) again, we have

$$
\begin{aligned}
&(\lfloor y\rfloor_R \cdot \lfloor y'\rfloor_R) \cdot \lfloor y''\rfloor_R \\
= \; &\lfloor y\lfloor y'\lfloor y''\rfloor_R\rfloor_R\rfloor_R + \lfloor y\lfloor y''\lfloor y'\rfloor_R\rfloor_R\rfloor_R + \lfloor y''\lfloor y\lfloor y'\rfloor_R\rfloor_R\rfloor_R \\
&+ \lfloor y'\lfloor y\lfloor y''\rfloor_R\rfloor_R\rfloor_R + \lfloor y'\lfloor y''\lfloor y\rfloor_R\rfloor_R\rfloor_R + \lfloor y''\lfloor y'\lfloor y\rfloor_R\rfloor_R\rfloor_R.
\end{aligned}
$$

By a similar calculation, we have

$$
\begin{aligned}
&\lfloor y\rfloor_R \cdot (\lfloor y'\rfloor_R \cdot \lfloor y''\rfloor_R) \\
= \; &\lfloor y\lfloor y'\lfloor y''\rfloor_R\rfloor_R\rfloor_R + \lfloor y'\lfloor y''\lfloor y\rfloor_R\rfloor_R\rfloor_R + \lfloor y'\lfloor y\lfloor y''\rfloor_R\rfloor_R\rfloor_R \\
&+ \lfloor y\lfloor y''\lfloor y'\rfloor_R\rfloor_R\rfloor_R + \lfloor y''\lfloor y'\lfloor y\rfloor_R\rfloor_R\rfloor_R + \lfloor y''\lfloor y\lfloor y'\rfloor_R\rfloor_R\rfloor_R.
\end{aligned}
$$

Now by the associativity of the triples in Eq. (4.27), the $i$-th term in the expansion of $(\lfloor y\rfloor_R \cdot \lfloor y'\rfloor_R)\cdot \lfloor y''\rfloor_R$ matches with the $\sigma(i)$-th term in the expansion of $\lfloor y\rfloor_R \cdot (\lfloor y'\rfloor_R \cdot \lfloor y''\rfloor_R)$. Here the permutation $\sigma \in \Sigma_6$ is

$$\begin{pmatrix} i \\ \sigma(i) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 6 & 3 & 2 & 5 \end{pmatrix}. \tag{4.28}$$

This proves the lemma. □

To continue the proof of Theorem 4.8, we apply Lemma 4.10 to the situation where $R$ is $\mathbf{k}\mathcal{F}$ with the multiplication $\cdot = \diamond$, the symmetric Rota–Baxter operator $\lfloor\ \rfloor_R = \lfloor\ \rfloor$ and the triple $(y, y', y'') = (\overline{F}, \overline{F}', \overline{F}'')$. By the induction hypothesis on $n$, all the triples in Eq. (4.27) and are associative for $\diamond$. So by Lemma 4.10, the triple $(F, F', F'')$ is associative for $\diamond$. This completes the induction and therefore the proof of the first part of Theorem 4.8.

(b). We just need to prove that $P_{\mathcal{F}}(F) = \lfloor F\rfloor$ is a symmetric Rota–Baxter operator This is immediate from Eq. (4.20). □

We will construct the free symmetric Rota–Baxter algebra on a **k**-module or on a set by expressing elements in the symmetric Rota–Baxter algebra in terms of forests from § 4.2, in addition with angles decorated by elements from the **k**-module or set. These decorated forests will be introduced in § 4.2.4. The free symmetric Rota–Baxter algebra will be constructed in § 4.2.5. When the **k**-module is taken to be free on a set, we obtain the free symmetric Rota–Baxter algebra on the set. This will be discussed in § 4.2.6.

#### 4.2.4 Rooted forests with angular decorations by a module

Let $M$ be a non-zero $\mathbf{k}$-module. Let $F$ be in $\mathcal{F}$ with $\ell$ leafs. We let $M^{\otimes F}$ denote the tensor power $M^{\otimes(\ell-1)}$ labeled by $F$. In other words,

$$M^{\otimes F} = \{(F; \mathfrak{m}) \mid \mathfrak{m} \in M^{\otimes(\ell-1)}\} \tag{4.29}$$

with the $\mathbf{k}$-module structure coming from the second component and with the convention that $M^{\otimes 0} = \mathbf{k}$. We can think of $M^{\otimes F}$ as the tensor power of $M$ with exponent $F$ with the usual tensor power $M^{\otimes n}, n \geq 0$, corresponding to $M^{\otimes F}$ when $F$ is the forest $\bullet^{\sqcup(n+1)}$.

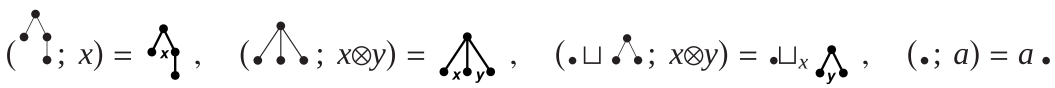
**Definition 4.11.** We call $M^{\otimes F}$ the **module of the forest $F$ with angular decoration by $M$**, and call $(F; \mathfrak{m})$, for $\mathfrak{m} \in M^{\otimes(\ell(F)-1)}$, an **angularly decorated forest $F$** with the decoration tensor $\mathfrak{m}$.

Also define the depth and breadth of $(F; \mathfrak{m})$ by

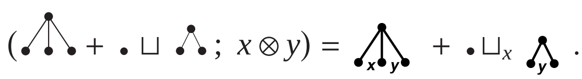
$$\mathrm{d}(F; \mathfrak{m}) = \mathrm{d}(F), \quad b(F; \mathfrak{m}) = b(F).$$

Definition 4.11 is justified by the following tree interpretation of $M^{\otimes F}$. Let $(F; \mathfrak{m})$ be an angularly decorated forest with a pure tensor $\mathfrak{m} = a_1 \otimes \cdots \otimes a_{\ell-1} \in M^{\otimes(\ell-1)}$, $\ell \geq 2$. We picture $(F; \mathfrak{m})$ as the forest $F$ with its angles between adjacent leafs (either from the same tree or from adjacent trees) decorated by $a_1, \cdots, a_{\ell-1}$ from the left most angle to the right most angle. If $\ell(F) = 1$, so $F$ is a ladder tree with only one leaf, then $(F; a)$, $a \in \mathbf{k}$, is interpreted as the multiple $aF$ of the ladder tree

$F$. For example, we have



When $\mathfrak{m} = \sum_i \mathfrak{m}_i$ is not a pure tensor, but a sum of pure tensors $\mathfrak{m}_i$ in $M^{\otimes(\ell-1)}$, we can picture $(F; \mathfrak{m})$ as a sum $\sum_i (F; \mathfrak{m}_i)$ of the forest $F$ with decorations from the pure tensors. Likewise, if $F$ is a linear combination $\sum_i c_i F_i$ of forests $F_i$ with the same number of leaves $\ell$ and if $\mathfrak{m} = a_1 \otimes \cdots \otimes a_{\ell-1} \in M^{\otimes(\ell-1)}$, we also use $(F; \mathfrak{m})$ to denote the linear combination $\sum_i c_i (F_i; \mathfrak{m})$. For example,



Let $(F; \mathfrak{m})$ be an angular decoration of the forest $F$ by a pure tensor $\mathfrak{m}$. Let $F = T_1 \sqcup \cdots \sqcup T_b$ be the decomposition of $F$ into trees. We consider the corresponding decomposition of decorated forests. If $b = 1$, then $F$ is a tree and $(F; \mathfrak{m})$ has no further decompositions. If $b > 1$, then there is the relation

$$\ell(F) = \ell(T_1) + \cdots + \ell(T_b).$$

Denote $\ell_i = \ell(T_i), 1 \leq i \leq b$. Then

$$(T_1; a_1 \otimes \cdots \otimes a_{\ell_1 - 1}),\ (T_2; a_{\ell_1+1} \otimes \cdots \otimes a_{\ell_1+\ell_2-1}), \cdots, (T_b; a_{\ell_1+\cdots+\ell_{b-1}+1} \otimes \cdots \otimes a_{\ell_1+\cdots+\ell_b})$$

are well-defined angularly decorated trees for the trees $T_i$ with $\ell(T_i) > 1$. If $\ell(T_i) = 1$, then $a_{\ell_{i-1}+\ell_i-1} = a_{\ell_{i-1}}$ and we use the convention $(T_i; a_{\ell_{i-1}+\ell_i-1}) = (T_i; \mathbf{1})$. With this convention, we have,

$$\begin{aligned}(F; a_1 \otimes \cdots \otimes a_{\ell-1}) \;&=\; (T_1; a_1 \otimes \cdots \otimes a_{\ell_1 - 1}) \sqcup_{a_{\ell_1}} (T_2; a_{\ell_1+1} \otimes \cdots \otimes a_{\ell_1+\ell_2-1}) \sqcup_{a_{\ell_1+\ell_2}} \\ &\quad \cdots \sqcup_{a_{\ell_1+\cdots+\ell_{b-1}}} (T_b; a_{\ell_1+\cdots+\ell_{b-1}+1} \otimes \cdots \otimes a_{\ell_1+\cdots+\ell_b}).\end{aligned}$$

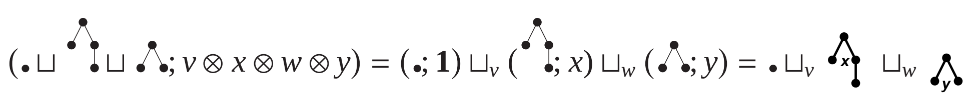
We call this the **standard decomposition** of $(F; \mathfrak{m})$ and abbreviate it as

$$(F; \mathfrak{m}) = (T_1; \mathfrak{m}_1) \sqcup_{u_1} (T_2; \mathfrak{m}_2) \sqcup_{u_2} \cdots \sqcup_{u_{b-1}} (T_b; \mathfrak{m}_b). \tag{4.30}$$

In other words,

$$(T_i; \mathfrak{m}_i) = \begin{cases} (T_i; a_{\ell_1+\cdots+\ell_{i-1}+1} \otimes \cdots \otimes a_{\ell_1+\cdots+\ell_i - 1}), & \ell_i > 1, i < b, \\ (T_i; a_{\ell_1+\cdots+\ell_{i-1}+1} \otimes \cdots \otimes a_{\ell_1+\cdots+\ell_i}), & \ell_i > 1, i = b, \\ (T_i; \mathbf{1}), & \ell_i = 1 \end{cases} \tag{4.31}$$

and $u_i = a_{\ell_1+\cdots+\ell_i}$. For example,



We display the following simple property for later applications.

**Lemma 4.12.** *Let $F \neq \bullet$. In the standard decomposition (4.30) of $(F; \mathfrak{m})$, if $T_i = \bullet$ for some $1 \leq i \leq b$, then $b > 1$ and the corresponding factor $(T_i; \mathfrak{m}_i)$ is $(T_i; \mathbf{1})$.*

*Proof.* Let $F \neq \bullet$ and let $F = T_1 \sqcup \cdots \sqcup T_b$ be its standard decomposition. Suppose $T_i = \bullet$ for some $1 \leq i \leq b$ and $b = 1$. Then $F = T_i = \bullet$, a contradiction. So $b > 1$, and by our convention, $(T_i; \mathfrak{m}_i) = (T_i; \bullet)$. $\square$

### 4.2.5 Free symmetric Rota–Baxter algebras on a module as decorated forests

We define the $\mathbf{k}$-module

$$Ш^{NC}(M) = \bigoplus_{F \in \mathcal{F}} M^{\otimes F}.$$

and define a product $\overline{\diamond}$ on $Ш^{NC}(M)$ by using the product $\diamond$ on $\mathcal{F}$ in Section 4.2.2. Let $T(M) = \oplus_{n \geq 0} M^{\otimes n}$ be the tensor algebra and let $\overline{\otimes}$ be its product, so for $\mathfrak{m} \in M^{\otimes n}$ and $\mathfrak{m}' \in M^{\otimes n'}$, we have

$$\mathfrak{m} \overline{\otimes} \mathfrak{m}' = \begin{cases} \mathfrak{m} \otimes \mathfrak{m}' \in M^{\otimes n+n'}, & \text{if } n > 0, n' > 0, \\ \mathfrak{m}\mathfrak{m}' \in M^{\otimes n'}, & \text{if } n = 0, n' > 0, \\ \mathfrak{m}'\mathfrak{m} \in M^{\otimes n}, & \text{if } n > 0, n' = 0, \\ \mathfrak{m}'\mathfrak{m} \in \mathbf{k}, & \text{if } n = n' = 0. \end{cases} \tag{4.32}$$

Here the products in the second and third case are scalar product and in the fourth case is the product in $\mathbf{k}$. In other words, $\overline{\otimes}$ identifies $\mathbf{k} \otimes M$ and $M \otimes \mathbf{k}$ with $M$ by the structure maps $\mathbf{k} \otimes M \to M$ and $M \otimes \mathbf{k} \to M$ of the $\mathbf{k}$-module $M$.

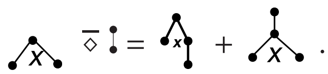
**Definition 4.13.** For tensors $D = (F; \mathfrak{m}) \in M^{\otimes F}$ and $D' = (F'; \mathfrak{m}') \in M^{\otimes F'}$, define

$$D \overline{\diamond} D' = (F \diamond F'; \mathfrak{m} \overline{\otimes} \mathfrak{m}'). \tag{4.33}$$

The right hand side is well-defined since $\mathfrak{m} \overline{\otimes} \mathfrak{m}'$ has tensor degree

$$\deg(\mathfrak{m} \overline{\otimes} \mathfrak{m}') = \deg(\mathfrak{m}) + \deg(\mathfrak{m}') = \ell(F) - 1 + \ell(F') - 1$$

which equals $\ell(F \diamond F') - 1$ by Lemma 4.7.(b). For example, from Eq. (4.23) we have



(4.34)

By Eq. (4.19) – (4.22), we have a more explicit expression.

$$D\overline{\diamond}D' = \begin{cases} (\bullet; cc'), & \text{if } D = (\bullet; c), D' = (\bullet; c'), \\ (F; c'\mathfrak{m}), & \text{if } D' = (\bullet, c'), F \neq \bullet, \\ (F'; c\mathfrak{m}'), & \text{if } D = (\bullet, c), F' \neq \bullet, \\ (F \diamond F'; \mathfrak{m} \otimes \mathfrak{m}'), & \text{if } F \neq \bullet, F' \neq \bullet. \end{cases} \tag{4.35}$$

We can describe $\overline{\diamond}$ even more explicitly in terms of the standard decompositions in Eq. (4.30) of $D = (F; \mathfrak{m})$ and $D' = (F'; \mathfrak{m}')$ for pure tensors $\mathfrak{m}$ and $\mathfrak{m}'$:

$$D = (F; \mathfrak{m}) = (T_1; \mathfrak{m}_1) \sqcup_{u_1} (T_2; \mathfrak{m}_2) \sqcup_{u_2} \cdots \sqcup_{u_{b-1}} (T_b; \mathfrak{m}_b),$$

$$D' = (F'; \mathfrak{m}') = (T'_1; \mathfrak{m}'_1) \sqcup_{u'_1} (T'_2; \mathfrak{m}'_2) \sqcup_{u'_2} \cdots \sqcup_{u'_{b'-1}} (T'_{b'}; \mathfrak{m}'_{b'}).$$

Then by Eq. (4.19) – (4.22) and Eq. (4.33) – (4.35), it is easy to see that the product $\overline{\diamond}$ can be defined by induction on the sum of the depths $\mathrm{d} = \mathrm{d}(F)$ and $\mathrm{d}' = \mathrm{d}(F')$ as follows: If $\mathrm{d} + \mathrm{d}' = 0$, then $F = \bullet^{\sqcup i}$ and $F' = \bullet^{\sqcup j}$ for $i, j \geq 1$. If $i = 1$, then $D = (F; \mathfrak{m}) = (\bullet; c) = c(\bullet; \mathbf{1})$ and we define $D\overline{\diamond}D' = cD' = (F'; c\mathfrak{m}')$. Similarly define $D\overline{\diamond}D'$ if $j = 1$. If $i > 1$ and $j > 1$, then $(F; \mathfrak{m}) = (\bullet; \mathbf{1}) \sqcup_{u_1} \cdots \sqcup_{u_{b-1}} (\bullet; \mathbf{1})$ with $u_1, \cdots, u_{b-1} \in M$. Similarly, $(F'; \mathfrak{m}') = (\bullet; \mathbf{1}) \sqcup_{u'_1} \cdots \sqcup_{u'_{b'-1}} (\bullet; \mathbf{1})$. Then define

$$(F; \mathfrak{m}) \overline{\diamond} (F'; \mathfrak{m}') = (\bullet; \mathbf{1}) \sqcup_{u_1} \cdots \sqcup_{u_{b-1}} (\bullet; \mathbf{1}) \sqcup_{u'_1} \cdots \sqcup_{u'_{b'-1}} (\bullet; \mathbf{1}).$$

Suppose $D\overline{\diamond}D'$ has been defined for all $D=(F;\mathfrak{m})$ and $D'=(F';\mathfrak{m}')$ with $\mathrm{d}(F)+\mathrm{d}(F')\leq k$ and consider $D$ and $D'$ with $\mathrm{d}(F)+\mathrm{d}(F')=k+1$. Then we define

$$D\overline{\diamond}D' = (T_1;\mathfrak{m}_1)\sqcup_{u_1}\cdots\sqcup_{u_{b-1}}((T_b;\mathfrak{m}_b)\overline{\diamond}(T_1';\mathfrak{m}_1'))\sqcup_{u_1'}\cdots\sqcup_{u'_{b'-1}}(T'_{b'};\mathfrak{m}'_{b'}) \tag{4.36}$$

where

$$\begin{aligned}&(T_b;\mathfrak{m}_b)\overline{\diamond}(T_1';\mathfrak{m}_1') \\ =&\begin{cases}(\bullet;\mathbf{1}), & \text{if } T_b=T_1'=\bullet \text{ (so } \mathfrak{m}_b=\mathfrak{m}_1'=\mathbf{1}),\\ (T_b,\mathfrak{m}_b), & \text{if } T_1'=\bullet, T_b\neq\bullet,\\ (T_1',\mathfrak{m}_1'), & \text{if } T_1'\neq\bullet, T_b=\bullet,\\ \lfloor(\overline{F}_b;\mathfrak{m}_b)\overline{\diamond}(T_1';\mathfrak{m}_1')\rfloor+\lfloor(\overline{F}_1';\mathfrak{m}_1')\overline{\diamond}(T_b;\mathfrak{m}_b)\rfloor, & \text{if } T_1'=\lfloor\overline{F}_1'\rfloor\neq\bullet, T_b=\lfloor\overline{F}_b\rfloor\neq\bullet.\end{cases}\end{aligned} \tag{4.37}$$

In the last case, we have applied the induction hypothesis on $\mathrm{d}(F)+\mathrm{d}(F')$ to define the terms in the brackets on the right hand side. Further, for $(F;\mathfrak{m})\in M^{\otimes F}$, define $\lfloor(F;\mathfrak{m})\rfloor=(\lfloor F\rfloor;\mathfrak{m})$. This is well-defined since $\ell(F)=\ell(\lfloor F\rfloor)$.

The product $\overline{\diamond}$ is clearly bilinear. So extending it biadditively, we obtain a binary operation

$$\overline{\diamond}: Ш^{\mathrm{NC}}(M)\otimes Ш^{\mathrm{NC}}(M)\to Ш^{\mathrm{NC}}(M).$$

For $(F;\mathfrak{m})\in M^{\otimes F}$, define

$$P_M(F;\mathfrak{m})=\lfloor(F;\mathfrak{m})\rfloor=(\lfloor F\rfloor;\mathfrak{m})\in M^{\otimes\lfloor F\rfloor}. \tag{4.38}$$

As commented above, this is well-defined. Thus $P_M$ defines a linear operator on $Ш^{\mathrm{NC}}(M)$. Note that the right hand side is also $(P_{\mathcal{F}}(F);\mathfrak{m})$ with $P_{\mathcal{F}}$ defined in

Eq. (4.24). Let

$$j_M : M \to Ш^{\mathrm{NC}}(M) \tag{4.39}$$

be the $\mathbf{k}$-module map sending $a \in M$ to $(\bullet \sqcup \bullet; a)$.

**Theorem 4.14.** *Let $M$ be a $\mathbf{k}$-module.*

(a) *The pair $(Ш^{\mathrm{NC}}(M), \overline{\diamond})$ is an associative algebra.*

(b) *The triple $(Ш^{\mathrm{NC}}(M), \overline{\diamond}, P_M)$ is a symmetric Rota–Baxter algebra.*

(c) *The quadruple $(Ш^{\mathrm{NC}}(M), \overline{\diamond}, P_M, j_M)$ is the free symmetric Rota–Baxter algebra on the module $M$. More precisely, for any symmetric Rota–Baxter algebra $(R, P)$ and module morphism $f : M \to R$, there is a unique symmetric Rota–Baxter algebra morphism $\overline{f} : Ш^{\mathrm{NC}}(M) \to R$ such that $f = \overline{f} \circ j_M$.*

*Proof.* (a) By definition, $(\bullet, \mathbf{1})$ is the unit of the multiplication $\overline{\diamond}$. For the associativity of $\overline{\diamond}$ on $Ш^{\mathrm{NC}}(M)$ we only need to prove

$$(D \overline{\diamond} D') \overline{\diamond} D'' = D \overline{\diamond} (D' \overline{\diamond} D'')$$

for any angularly decorated forests $D = (F; \mathfrak{m}) \in M^{\otimes F}, D' = (F'; \mathfrak{m}') \in M^{\otimes F'}$ and $D'' = (F''; \mathfrak{m}'') \in M^{\otimes F''}$. Then by Eq. (4.33), we have

$$(D \overline{\diamond} D') \overline{\diamond} D'' = ((F \diamond F') \diamond F''; (\mathfrak{m} \overline{\otimes} \mathfrak{m}') \overline{\otimes} \mathfrak{m}''),$$

$$D \overline{\diamond} (D' \overline{\diamond} D'') = (F \diamond (F' \diamond F''); \mathfrak{m} \overline{\otimes} (\mathfrak{m}' \overline{\otimes} \mathfrak{m}'')).$$

The first components of the two right hand sides agree since the product $\diamond$ is associative by Theorem 4.8. The second component of the two right hand sides agree because the product $\overline{\otimes}$ in Eq. (4.32) for the tensor algebra $T(M) := \bigoplus_{n\geq 0} M^{\otimes n}$ is also associative. This proves the associativity of $\overline{\diamond}$.

(b). The symmetric Rota–Baxter relation of $\lfloor\ \rfloor$ on $Ш^{\mathrm{NC}}(M)$ follows from the symmetric Rota–Baxter relation of $\lfloor\ \rfloor$ on $\mathbf{k}\mathcal{F}$ in Theorem 4.8. More specifically, it is the last equation in Eq (4.37).

(c). Let $(R, P)$ be a symmetric Rota–Baxter algebra. Let $*$ be the multiplication in $R$ and let $\mathbf{1}_R$ be its unit. Let $f : M \to R$ be a $\mathbf{k}$-module map. We will construct a $\mathbf{k}$-linear map $\overline{f} : Ш^{\mathrm{NC}}(M) \to R$ by defining $\overline{f}(D)$ for $D = (F; \mathfrak{m}) \in M^{\otimes F}$. We will achieve this by induction on the depth $\mathrm{d}(F)$ of $F$.

If $\mathrm{d}(F) = 0$, then $F = \bullet^{\sqcup i}$ for some $i \geq 1$. If $i = 1$, then $D = (\bullet; c)$, $c \in \mathbf{k}$. Define $\overline{f}(D) = c\mathbf{1}_R$. In particular, define $\overline{f}(\bullet; \mathbf{1}) = \mathbf{1}_R$. Then $\overline{f}$ sends the unit to the unit. If $i \geq 2$, then $D = (F; \mathfrak{m})$ with $\mathfrak{m} = a_1 \otimes \cdots \otimes a_n \in M^{\otimes n}$ where $n + 1$ is the number of leafs $\ell(F)$. Then we define $\overline{f}(D) = f(a_1) * \cdots * f(a_n)$. In particular, $\overline{f} \circ j_M = f$.

Assume that $\overline{f}(D)$ has been defined for all $D = (F; \mathfrak{m})$ with $\mathrm{d}(F) \leq k$ and let $D = (F; \mathfrak{m})$ with $\mathrm{d}(F) = k + 1$. So $F \neq \bullet$. Let $D = (T_1; \mathfrak{m}_1) \sqcup_{u_1} \cdots \sqcup_{u_{b-1}} (T_b; \mathfrak{m}_b)$ be the standard decomposition of $D$ given in Eq. (4.30). For each $1 \leq i \leq b$, $T_i$ is a tree, so it is either $\bullet$ or is of the form $\lfloor \overline{F}_i \rfloor$ for another forest $\overline{F}_i$. By Lemma 4.12, if $T_i = \bullet$, then $b > 1$ and $\mathfrak{m}_i = \mathbf{1}$. We accordingly define

$$\overline{f}(T_i; \mathfrak{m}_i) = \begin{cases} \mathbf{1}_R, & \text{if } T_i = \bullet, \\ P(\overline{f}(\overline{F}_i; \mathfrak{m}_i)), & \text{if } T_i = \lfloor \overline{F}_i \rfloor. \end{cases} \tag{4.40}$$

In the later case, $(\overline{F}_i; \mathfrak{m}_i)$ is a well-defined angularly decorated forest since $\overline{F}_i$ has the same number of leafs as the number of leafs of $T_i$, and then $\overline{f}(\overline{F}_i; \mathfrak{m}_i)$ is defined by the induction hypothesis since $\mathrm{d}(\overline{F_i}) = \mathrm{d}(T_i) - 1 \leq k$. Therefore we can define

$$\overline{f}(D) = \overline{f}(T_1; \mathfrak{m}_1) * f(u_1) * \cdots * f(u_{b-1}) * \overline{f}(T_b; \mathfrak{m}_b). \tag{4.41}$$

For any $D = (F; \mathfrak{m}) \in M^{\otimes F}$, we have $P_M(D) = (\lfloor F \rfloor; \mathfrak{m}) \in Ш^{NC}(M)$, and by the definition of $\overline{f}$ in Eq. (4.40) and (4.41), we have

$$\overline{f}(\lfloor D \rfloor) = P(\overline{f}(D)). \tag{4.42}$$

So $\overline{f}$ commutes with the symmetric Rota–Baxter operators.

Further, Eq. (4.40) and (4.41) are clearly the only way to define $\overline{f}$ in order for $\overline{f}$ to be a symmetric Rota–Baxter algebra homomorphism that extends $f$.

It remains to prove that the map $\overline{f}$ defined in Eq. (4.41) is indeed an algebra homomorphism. For this we only need to check the multiplicativity

$$\overline{f}(D \overline{\diamond} D') = \overline{f}(D) * \overline{f}(D') \tag{4.43}$$

for all angularly decorated forests $D = (F; \mathfrak{m}), D' = (F'; \mathfrak{m}')$ with pure tensors $\mathfrak{m}$ and $\mathfrak{m}'$. Let

$$(F; \mathfrak{m}) = (T_1; \mathfrak{m}_1) \sqcup_{u_1} (T_2; \mathfrak{m}_2) \sqcup_{u_2} \cdots \sqcup_{u_{b-1}} (T_b; \mathfrak{m}_b)$$

and

$$(F'; \mathfrak{m}') = (T'_1; \mathfrak{m}'_1) \sqcup_{u'_1} (T'_2; \mathfrak{m}'_2) \sqcup_{u'_2} \cdots \sqcup_{u'_{b'-1}} (T'_{b'}; \mathfrak{m}'_{b'})$$

be their standard decompositions.

We first note that, since $\overline{f}$ sends the identity $(\bullet; \mathbf{1})$ of $\text{Ш}^{\mathrm{NC}}(M)$ to the identity $\mathbf{1}_R$ of $R$, the multiplicativity is clear if either one of $D$ or $D'$ is in $(\bullet; \mathbf{k})$, that is, if either one of $F$ or $F'$ is $\bullet$. So we only need to verify the multiplicativity when $F \neq \bullet$ and $F' \neq \bullet$.

We further make the following reduction. By Eq. (4.41) and Eq. (4.36), we have

$$\begin{aligned}\overline{f}(D \overline{\diamond} D') &= \overline{f}(T_1; \mathfrak{m}_1) * f(u_1) * \cdots * f(u_{b-1}) \\ &\quad * \overline{f}((T_b; \mathfrak{m}_b) \overline{\diamond} (T_1'; \mathfrak{m}_1')) * f(u_1') * \cdots * f(u'_{b'-1}) * \overline{f}(T'_{b'}; \mathfrak{m}'_{b'})\end{aligned}$$

and

$$\begin{aligned}\overline{f}(D) * \overline{f}(D') &= \overline{f}(T_1; \mathfrak{m}_1) * f(u_1) * \cdots * f(u_{b-1}) \\ &\quad * \overline{f}(T_b; \mathfrak{m}_b) * \overline{f}(T_1'; \mathfrak{m}_1') * f(u_1') * \cdots * f(u'_{b'-1}) * \overline{f}(T'_{b'}; \mathfrak{m}'_{b'}).\end{aligned}$$

We thus have

$$\overline{f}((F; \mathfrak{m}) \overline{\diamond} (F'; \mathfrak{m}')) = \overline{f}(F; \mathfrak{m}) * \overline{f}(F'; \mathfrak{m}') \tag{4.44}$$

if and only if

$$\overline{f}((T_b, ; \mathfrak{m}_b) \overline{\diamond} (T_1'; \mathfrak{m}_1')) = \overline{f}(T_b; \mathfrak{m}_b) * \overline{f}(T_1'; \mathfrak{m}_1'). \tag{4.45}$$

So we only need to prove Eq. (4.45). For this we use induction on the sum of depths $n := \mathrm{d}(T_b) + \mathrm{d}(T_1')$ of $T_b$ and $T_1'$. Then $n \geq 0$. When $n = 0$, we have $T_b = T_1' = \bullet$. So by Lemma 4.12, we have $b > 1, b' > 1$, and

$$(T_b; \mathfrak{m}_b) = (T_1'; \mathfrak{m}_1') = (T_b; \mathfrak{m}_b) \overline{\diamond} (T_1'; \mathfrak{m}_1') = (\bullet; \mathbf{1}).$$

Then

$$\overline{f}(T_b;\mathfrak{m}_b)=\overline{f}(T_1';\mathfrak{m}_1')=\overline{f}((T_b;\mathfrak{m}_b)\overline{\diamond}(T_1';\mathfrak{m}_1'))=\mathbf{1}_R.$$

Thus Eq. (4.45) and hence Eq. (4.44) holds.

Assume that the multiplicativity holds for $D$ and $D'$ in $M^{\otimes\mathcal{F}}$ with $n=\mathrm{d}(T_b)+\mathrm{d}(T_1')\leq k$ and take $D,D'\in M^{\otimes\mathcal{F}}$ with $n=k+1$. So $n\geq 1$. Then at least one of $\mathrm{d}(T_b)$ and $\mathrm{d}(T_1')$ is not zero. If exactly one of them is zero, so exactly one of $T_b$ and $T_1'$ is $\bullet$, then by Eq. (4.37),

$$(T_b;\mathfrak{m}_b)\overline{\diamond}(T_1';\mathfrak{m}_1')=\begin{cases}(T_b;\mathfrak{m}_b), & \text{if } T_1'=\bullet, T_b\neq\bullet,\\ (T_1';\mathfrak{m}_1'), & \text{if } T_1'\neq\bullet, T_b=\bullet.\end{cases}$$

Then

$$\overline{f}((T_b;\mathfrak{m}_b)\overline{\diamond}(T_1';\mathfrak{m}_1'))=\begin{cases}\overline{f}(T_b;\mathfrak{m}_b), & \text{if } T_1'=\bullet, T_b\neq\bullet,\\ \overline{f}(T_1';\mathfrak{m}_1'), & \text{if } T_1'\neq\bullet, T_b=\bullet.\end{cases}$$

Then Eq. (4.45) and hence (4.44) holds since one factor in $\overline{f}(T_b;\mathfrak{m}_b)*\overline{f}(T_1';\mathfrak{m}_1')$ is $\mathbf{1}_R$.

If neither $\mathrm{d}(T_b)$ nor $\mathrm{d}(T_1')$ is zero, then $T_b=\lfloor\overline{F}_b\rfloor$ and $T_1'=\lfloor\overline{F}_1'\rfloor$ for some forests $\overline{F}_b$ and $\overline{F}_1'$ in $\mathcal{F}$. Then $(T_b;\mathfrak{m}_b)=\lfloor(\overline{F}_b;\mathfrak{m}_b)\rfloor$ and $(T_1';\mathfrak{m}_1')=\lfloor(\overline{F}_1';\mathfrak{m}_1')\rfloor$. We will take care of this case by the following lemma.

**Lemma 4.15.** *Let $(R_1,P_1)$ and $(R_2,P_2)$ be not necessarily associative* **k***-algebras $R_1$ and $R_2$ together with* **k***-linear endomorphisms $P_1$ and $P_2$ that each satisfies the symmetric Rota–Baxter identity in Eq. (3.1). Let $g:R_1\to R_2$ be a* **k***-linear map such that*

$$g\circ P_1=P_2\circ g. \tag{4.46}$$

*Let $x, y \in R_1$ be such that*

$$g(xP_1(y)) = g(x) \cdot g(P_1(y)),\ g(yP_1(x)) = g(y) \cdot g(P_1(x)). \tag{4.47}$$

*Here we have suppressed the product in $R_1$ and denote the product in $R_2$ by $\cdot$. Then $g(P_1(x)P_1(y)) = g(P_1(x)) \cdot g(P_1(y))$.*

*Proof.* By the symmetric Rota–Baxter relations of $P_1$ and $P_2$, Eq. (4.46) and Eq. (4.47), we have

$$\begin{aligned}
g(P_1(x)P_1(y)) &= g(P_1(xP_1(y)) + P_1(yP_1(x))) \\
&= g(P_1(xP_1(y))) + g(P_1(yP_1(x))) \\
&= P_2(g(xP_1(y))) + P_2(g(yP_1(x))) \\
&= P_2(g(x) \cdot g(P_1(y))) + P_2(g(y) \cdot g(P_1(x))) \\
&= P_2(g(x) \cdot P_2(g(y))) + P_2(g(y) \cdot P_2(g(x))) \\
&= P_2(g(x)) \cdot P_2(g(y)) \\
&= g(P_1(x)) \cdot g(P_1(y)).
\end{aligned}$$

□

Now we apply Lemma 4.15 to our proof with $(R_1, P_1) = (Ш^{NC}(M), \lfloor\ \rfloor)$, $(R_2, P_2) = (R, P)$ and $g = \overline{f}$. By the induction hypothesis, Eq. (4.47) holds for $x = (\overline{F}_b; \mathfrak{m}_b)$ and $y = (\overline{F}'_1; \mathfrak{m}'_1)$. Therefore by Lemma 4.15,

$$\overline{f}((T_b; m_b)\overline{\diamond}(T'_1; m'_1)) = \overline{f}(\lfloor(\overline{F}_b; \mathfrak{m}_b)\rfloor\overline{\diamond}\lfloor(\overline{F}'_1; \mathfrak{m}'_1)\rfloor) = \overline{f}(\lfloor(\overline{F}_b; \mathfrak{m}_b)\rfloor) * \overline{f}(\lfloor(\overline{F}'_1; \mathfrak{m}'_1)\rfloor) = \overline{f}(T_b) * \overline{f}(T'_1).$$

Thus Eq. (4.44) holds for $n = k + 1$. This completes the induction and the proof of Theorem 4.14. $\square$

### 4.2.6 Free symmetric Rota–Baxter algebras on a set

Here we use the tree construction of free symmetric Rota–Baxter algebra on a module above to obtain a similar construction of a free symmetric Rota–Baxter algebra on a set and display a canonical basis of the free symmetric Rota–Baxter algebra in terms of forests decorated by the set.

**Remark 4.2.** Either by the general principle of forgetful functors or by an easy direct check, the free symmetric Rota–Baxter algebra on a set $X$ is the free symmetric Rota–Baxter algebra on the free $\mathbf{k}$-module $M = \mathbf{k}\,X$. Thus we can easily obtain a construction of the free symmetric Rota–Baxter algebra on $X$ by decorated forests from the construction of $Ш^{\mathrm{NC}}(M)$ in § 4.2.5.

For any $n \geq 1$, the tensor power $M^{\otimes n}$ has a natural basis $X^n = \{(x_1, \cdots, x_n) \mid x_i \in X,\ 1 \leq i \leq n\}$. Accordingly, for any rooted forest $F \in \mathcal{F}$, with $\ell = \ell(F) \geq 2$, the set

$$X^F := \{(F; (x_1, \cdots, x_{\ell-1})) := (F; x_1 \otimes \cdots \otimes x_{\ell-1}) \mid x_i \in X,\ 1 \leq i \leq \ell - 1\}$$

form a basis of $M^{\otimes F}$ defined in Eq. (4.29). Note that when $\ell(F) = 1$, $M^{\otimes F} = \mathbf{k}\,F$ has a basis $X^F := \{(F; \mathbf{1})\}$. In summary, every $M^{\otimes F}, F \in \mathcal{F}$, has a basis

$$X^F := \{(F; \vec{x}) \mid \vec{x} \in X^{\ell(F)-1}\}, \tag{4.48}$$

with the convention that $X^0 = \{\mathbf{1}\}$. Thus the disjoint union

$$X^{\mathcal{F}} := \coprod_{F\in\mathcal{F}} X^F. \tag{4.49}$$

forms a basis of

$$Ш^{NC}(X) := Ш^{NC}(M).$$

We call $X^{\mathcal{F}}$ the set of **angularly decorated rooted forests with decoration set** $X$. As in Section 4.2.4, they can be pictured as rooted forests with adjacent leafs decorated by elements from $X$.

Likewise, for $(F;\vec{x}) \in X^{\mathcal{F}}$, the decomposition (4.30) gives the **standard decomposition**

$$(F;\vec{x}) = (T_1;\vec{x}_1) \sqcup_{u_1} (T_2;\vec{x}_2) \sqcup_{u_2} \cdots \sqcup_{u_{b-1}} (T_b;\vec{x}_b) \tag{4.50}$$

where $F = T_1 \sqcup \cdots \sqcup T_b$ is the decomposition of $F$ into trees and $\vec{x}$ is the vector concatenation of the elements of $\vec{x}_1, u_1, \vec{x}_2, \cdots, u_{b-1}, \vec{x}_b$ which are not the unit $\mathbf{1}$. As a corollary of Theorem 4.14, we have

**Theorem 4.16.** *For $D = (F;(x_1,\cdots,x_b))$, $D' = (F';(x'_1,\cdots,x'_{b'}))$ in $X^{\mathcal{F}}$, define*

$$D\overline{\diamond}D' = \begin{cases} (\bullet;\mathbf{1}), & \text{if } F = F' = \bullet, \\ D, & \text{if } F' = \bullet, F \neq \bullet, \\ D', & \text{if } F = \bullet, F' \neq \bullet, \\ (F\diamond F';(x_1,\cdots,x_b,x'_1,\cdots,x'_{b'})), & \text{if } F \neq \bullet, F' \neq \bullet, \end{cases} \tag{4.51}$$

*where $\diamond$ is defined in Eq. (4.20) and (4.22). Define*

$$P_X : Ш^{\mathrm{N}C}(X) \to Ш^{\mathrm{N}C}(X), \quad P_X(F; (x_1, \cdots, x_b)) = (\lfloor F \rfloor; (x_1, \cdots, x_b)),$$

*and*

$$j_X : X \to Ш^{\mathrm{N}C}(X), \quad j_X(x) = (\bullet \sqcup \bullet; (x)), \quad x \in X.$$

*Then the quadruple* $(Ш^{\mathrm{N}C}(X), \overline{\diamond}, P_X, j_X)$ *is the free symmetric Rota–Baxter algebra on* $X$.

*Proof.* The product $\overline{\diamond}$ in Eq. (4.51) is defined to be the restriction of the product $\overline{\diamond}$ in Eq. (4.35) to $X^{\mathcal{F}}$. Since $X^{\mathcal{F}}$ is a basis of $Ш^{\mathrm{N}C}(X)$, the two products coincide. So $Ш^{\mathrm{N}C}(X)$ and $Ш^{\mathrm{N}C}(M)$ are isomorphic as symmetric Rota-Baxter algebras. Then as commented in Remark 4.2, $Ш^{\mathrm{N}C}(X)$ is the free symmetric Rota–Baxter algebra on $X$. $\square$

# 5 Symmetric Dendriform Algebras

## 5.1 Rota-Baxter algebras and dendriform algebras

### 5.1.1 Dendriform algebras and tridendriform algebras

The concept of a dendriform algebra was introduced by Loday [46] in 1995 with motivation from algebraic $K$-theory.

**Definition 5.1.** ([46]) A **dendriform k-algebra** (previously also called a dendriform dialgebra) is a **k**-module $D$ with two binary operations $\prec$ and $\succ$ that satisfy the following relations.

$$\begin{aligned}(x \prec y) \prec z &= x \prec (y \prec z + y \succ z),\\ (x \succ y) \prec z &= x \succ (y \prec z), \quad x, y, z \in D,\\ (x \prec y + x \succ y) \succ z &= x \succ (y \succ z).\end{aligned} \tag{5.1}$$

Dendriform algebras have been further studied with connections to several areas in mathematics and physics, including operads, homological algebra, Hopf algebra, Lie and Leibniz algebra, combinatorics, arithmetic and quantum field theory.

A few years later, Loday and Ronco defined the tridendriform algebra in their study [48] of polytopes and Koszul duality.

**Definition 5.2.** ([48]) A **tridendriform k-algebra** (previously also called a dendriform trialgebra) is a **k**-module $T$ equipped with three binary operations $\prec, \succ$ and $\cdot$ that satisfy the following relations.

$$(x \prec y) \prec z = x \prec (y \star z),$$

$$
\begin{aligned}
(x \succ y) \prec z &= x \succ (y \prec z), \\
(x \star y) \succ z &= x \succ (y \succ z), \\
(x \succ y) \cdot z &= x \succ (y \cdot z), \quad x, y, z \in T, \\
(x \prec y) \cdot z &= x \cdot (y \succ z), \\
(x \cdot y) \prec z &= x \cdot (y \prec z), \\
(x \cdot y) \cdot z &= x \cdot (y \cdot z).
\end{aligned}
\tag{5.2}
$$

Here we have used the notation

$$
\star = \prec + \succ + \cdot\,. \tag{5.3}
$$

**Proposition 5.3.** [32]

(a) *Let $(D, \prec, \succ)$ be a dendriform algebra. The operation $\star := \star_D$ on $D$ defined by*

$$
x \star y := x \prec y + x \succ y, \quad x, y \in D, \tag{5.4}
$$

*is associative.*

(b) *Let $(T, \prec, \succ, \cdot)$ be a tridendriform algebra. The operation $\star := \star_T$ on $T$ defined by*

$$
x \star y := x \prec y + x \succ y + x \cdot y, \quad x, y \in T, \tag{5.5}
$$

*is associative.*

Thus, a dendriform algebra and tridendriform algebra share the property that the sum of the binary operations $\star := \prec + \succ$ for a dendriform algebra or $\star := \prec + \succ + \cdot$ for a tridendriform algebra is associative. Such a property is called a "splitting the associativity".

*Proof.* We just prove Item a. The proof of Item b is similar.

Adding the left hand sides of Eqs. (5.1) , we obtain

$$
\begin{aligned}
&(x \prec y) \prec z + (x \succ y) \prec z + (x \prec y + x \succ y) \succ z \\
=&(x \prec y + x \succ y) \prec z + (x \prec y + x \succ y) \succ z \\
=&(x \star y) \prec z + (x \star y) \succ z \\
=&(x \star y) \star z.
\end{aligned}
$$

Similarly, adding the right hand sides of these equations, we obtain $x \star (y \star z)$. Thus we have proved the associativity of $\star$. ◻

### 5.1.2 From Rota-Baxter algebras to dendriform algebras

**Theorem 5.4.** (a) [3] *A Rota-Baxter algebra* $(R, P)$ *of weight zero defines a dendriform algebra* $(R, \prec_P, \succ_P)$*, where*

$$x \prec_P y = xP(y),\ \ x \succ_P y = P(x)y,\ \ \forall x, y \in R. \tag{5.6}$$

(b) [19] *A Rota-Baxter algebra* $(R, P)$ *of weight* $\lambda$ *defines a tridendriform algebra* $(R, \prec_P, \succ_P, \cdot_P)$*, where*

$$x \prec_P y = xP(y),\ \ x \succ_P y = P(x)y,\ \ x \cdot_P y = \lambda xy,\ \ \forall x, y \in R. \tag{5.7}$$

*Proof.* It is straightforward to verify 5.1 and 5.2 under the definitions of 5.6 and 5.7. ◻

### 5.1.3 Introduction to non-symmetric operad theory

Operad theory was originating from work in algebraic topology by Boardman and Vogt, and J. Peter May (to whom their name is due). It has more recently found many applications, drawing for example on work by Maxim Kontsevich on graph homology.

For details on binary quadratic non-symmetric operads, see [32, 45]. The following materials are partially from the class notes of the advanced topic class of algebra at Rutgers-Newark in 2014 Spring.

The concept of operads is similar to the concept of algebras. Let $k$ be a field and $V$ be a $k$-vector space. Then $Hom(V, V)$ is a $k$-vector space with a composition operator under associativity rule, which is actually a $k$-algebra. Generalizing this, we call any $k$-vector space $A$ a $k$-algebra if $A$ has a binary operation satisfied associativity. Meanwhile, we call $V$ a representation of $A$ if there is a $k$-algebra homomorphism $f : A \to Hom(V, V)$. And $V$ is actually also called an $A$-module. This is the equivalent definition for a module. We will check this later for the operad case.

Then what will happen if we have multi-linear maps in $Hom(V^{\otimes n}, V)$? And what is the composition rule between different dimensions of multi-linear maps?

We have the following analogue.

**Definition 5.5.** [32]**Partial composition** Let $\mathcal{E}_n := \mathcal{E}_{V,n} := Hom(V^{\otimes n}, V), n \geq 1$, for

$\mu \in \mathcal{E}_m, \nu \in \mathcal{E}_n$ and $1 \leq i \leq m$, define the composition

$$(\mu \circ_i \nu) := (\mu \circ_{m,n,i} \nu) : V^{\otimes(m+n-1)} \to V$$

by

$$(\mu \circ_i \nu)(x_1, x_2, \cdots, x_{m+n-1}) = \mu(x_1, \cdots x_{i-1}, \nu(x_i, \cdots x_{i+m-1}), x_{i+m}, \cdots, x_{m+n-1}).$$

So we have the composition between different dimensions of multi-linear maps

$$\circ_i : \mathcal{E}_m \otimes \mathcal{E}_n \to \mathcal{E}_{m+n-1}, 1 \leq i \leq m.$$

Further, for $\lambda \in \mathcal{E}_l, \mu \in \mathcal{E}_m$ and $\nu \in \mathcal{E}_n$, we have

**Proposition 5.6.** [32]**Associativities for higher dimentions**

*(i)* $(\lambda \circ_i \mu) \circ_{i-1+j} \nu = \lambda \circ_i (\mu \circ_j \nu), \quad 1 \leq i \leq \ell, 1 \leq j \leq m.$ **(Sequential composition)**

*(ii)* $(\lambda \circ_i \mu) \circ_{k-1+m} \nu = (\lambda \circ_k \nu) \circ_i \mu, \quad 1 \leq i < k \leq \ell.$ **(Parallel composition)**

*(iii)* *There is an element* $\mathrm{id} \in \mathcal{E}_1$ *such that* $\mathrm{id} \circ \mu = \mu$ *and* $\mu \circ \mathrm{id} = \mu$ *for* $\mu \in \mathcal{E}_n, n \geq 0.$ **(Identity)**

**Definition 5.7.** [32]**Non-symmetric operad**

Let **k** be a field,

(a) A **graded vector space** is a sequence $\mathcal{P} := \{\mathcal{P}_n\}_{n\geq 0}$ of **k**-vector spaces $\mathcal{P}_n, n \geq 0$;

(b) A **non-symmetric (ns) operad** is a graded vector space $\mathcal{P} = \{\mathcal{P}_n\}_{n\geq 0}$ equipped with **partial compositions**:

$$\circ_i := \circ_{m,n,i} : \mathcal{P}_m \otimes \mathcal{P}_n \longrightarrow \mathcal{P}_{m+n-1}, \quad 1 \leq i \leq m, \tag{5.8}$$

such that, for $\lambda \in \mathcal{P}_\ell, \mu \in \mathcal{P}_m$ and $\nu \in \mathcal{P}_n$, Proposition 5.6 hold.

**Definition 5.8. Suboperad**

Let $\mathcal{P} = \bigoplus_{k\geq 1} \mathcal{P}_k$ be a non-symmetric operad,

(a) A graded vector space $\mathcal{Q} = \bigoplus_{k\geq 1} \mathcal{Q}_k$ is called a graded subspace of $\mathcal{P}$ if $\mathcal{Q}_k \subseteq \mathcal{P}_k$;

(b) A graded subspace $\mathcal{Q} = \bigoplus_{k\geq 1} \mathcal{Q}_k$ is called a sub-operad of $\mathcal{P} = \bigoplus_{k\geq 1} \mathcal{P}_k$ if $\mathcal{Q}$ is closed under partial composition $\circ_i|_{\mathcal{Q}_k}$. Or equivalently, Proposition 5.6 holds for $\circ_i|_{\mathcal{Q}_k}$.

**Definition 5.9. Generated suboperad** Let $B \subseteq \mathcal{P}(= \bigoplus_{k\geq 1} \mathcal{P}_k)$, the suboperad of $\mathcal{P}$ generated by $B$ denoted as $Op(B) = Op_{\mathcal{P}}(B)$ is the smallest suboperad of $\mathcal{P}$ containing $B$. And $Op(B) = \bigcap_{B\subseteq \mathcal{B} \leqslant \mathcal{P}} \mathcal{B}$.

Note: $B \subseteq \mathcal{P}$, so $\bigcap_{B\subseteq \mathcal{B}\leqslant\mathcal{P}} \mathcal{B} \neq \emptyset$.

**Definition 5.10. Operad ideal** A suboperad $\mathcal{Q}$ of $\mathcal{P}$ is called an operad ideal of $\mathcal{P}$ if

$$\mathcal{Q}_m \circ_i \mathcal{P}_n \subseteq \mathcal{Q}_{m+n-1}, \mathcal{P}_m \circ_i \mathcal{Q}_n \subseteq \mathcal{Q}_{m+n-1}, \forall m, n, \mathcal{P}_n \in \mathcal{P}, \mathcal{Q}_m \in \mathcal{Q}.$$

**Definition 5.11. Generated operad ideal** An operad idea $\mathcal{B}$ of $\mathcal{P}$ is called to be generated by $B \subseteq \mathcal{P}$ if $\mathcal{B}$ is the smallest ideal of $\mathcal{P}$ containing $B$ denoted by $OpId(B)$, where $OpId(B) = \bigcap_{B\subseteq\mathcal{B}\leqslant\mathcal{P}} \mathcal{B}$.

**Definition 5.12. Binary operad** A non-symmetric operad $\mathcal{P} = \{\mathcal{P}_n\}_{n\geq 0}$ is called **binary** if $\mathcal{P}_1 = \mathbf{k}.\mathrm{id}$ and $\mathcal{P}_n, n \geq 3$ are induced from $\mathcal{P}_2$ by partial compositions and is denoted as $\mathcal{P} = Op_{\mathcal{P}}(\mathcal{P}_2)$.

**Definition 5.13. Free binary operad** Let $V(= \mathcal{P}_2)$ be a vector space. A binary operad $\mathcal{P}(V) = \bigoplus_{k\geq 1} \mathcal{P}_k(V)$ is called **free binary operad** on $V$ with $i : V \to \mathcal{P}_2$ if the diagram

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "f"'] & \mathcal{P}(V) \arrow[d, "\bar{f}"] \\ & \mathcal{Q} \end{tikzcd}$$

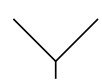
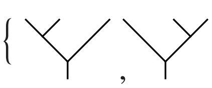
commutes.

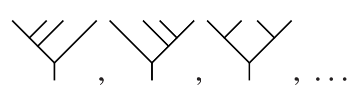
In another word, $\forall$ binary operad $\mathcal{Q} = \bigoplus_{k\geq 1} \mathcal{Q}_k$ and linear map $f : V \to \mathcal{Q}_2$, $\exists!$ operad homomorphism $\bar{f} : \mathcal{P}(V) \to \mathcal{Q}$, s.t. $f = \bar{f} \circ i$.

**Note**: Free binary operad is also called Magma Operad.

We will construct the free binary operad by binary planar trees, and denote it as $\mathfrak{M}(V)$.

Let $V$ be a $k$-vector space of binary operations. We consider binary planar trees with vertices decorated by elements of $V$. Here are the first few of them without decoration.

$Y_1(V) = \{\,|\,\}$, $Y_2 = \{$  $\}$, $Y_3 = \{$  $\}$

$Y_4 = \{$  $\}$.

Then,

$$\mathfrak{M}_1(V) = k.id = kY_1 = k\,|;$$

$$\mathfrak{M}_2(V) = k\{Y_2|\forall v \in V\};$$

$$\mathfrak{M}_3(V) = k\{Y_3|\forall v_1, v_2 \in V\};$$

$$\mathfrak{M}_4(V) = k\{Y_4|\forall v_1, v_2, v_3 \in V\};$$

$$\ldots$$

$$\mathfrak{M}_n(V) = k\{Y_n \mid v_1, \cdots, v_n \in V\}.$$

Then, define $\mathfrak{M}(V) = \bigoplus_{k\geq 1} \mathfrak{M}_k(V)$. Natually, we have $\mathfrak{M}_m \circ_i \mathfrak{M}_n \to \mathfrak{M}_{m+n-1}, \forall m, n \geq 1, 1 \leq i \leq m$ which satisfies Proposition 5.6.

**Theorem 5.14. Free binary trees** $\mathfrak{M}(V)$ *with* $i : V \to \{Y_2 \mid \forall v \in V\} = \mathfrak{M}_2$ *is the free binary operad generated by* $V$ *in terms of planar trees.*

**Proposition 5.15.** *For any non-symmetric binary operad* $\mathcal{P} = \bigoplus_{k\geq 1} \mathcal{P}_k$*, there is a vector space* $V(= \mathcal{P}_2)$ *and an operad ideal* $(R)$ *of the free binary operad* $\mathcal{P}(V)(= \mathfrak{M}(V))$ *such that* $\mathcal{P} = \mathcal{P}(V)/(R)$*.*

Here, $V$ is called the space of generators and $R$ is called the space of relations. And $\mathcal{P} = \mathcal{P}(V)/(R)$ is thus determined by $(V, R)$.

**Example 5.16. Associative operad** Let $V = k.\star$, and $R = Y_21 - Y_22$, then $\mathcal{F}_{Asso}(V)/(R)$

is the Associative Operad.

**Definition 5.17. $\mathcal{P}$-algebra** Let $\mathcal{P} = \bigoplus_{n\geq 1} \mathcal{P}$ be an operad, and let $U$ be a vector space with $k$-nary operator $\alpha_k (\in Hom(U^{\otimes k}, U))$. Then, $U = \{U, \alpha_k\}$ is called a $\mathcal{P}$-algebra if $\exists$ an operad homomorphism

$$f : \mathcal{P} \to \bigoplus_{n\geq 1} Hom(U^{\otimes n}, U),$$

where $\bigoplus_{n\geq 1} Hom(U^{\otimes n}, U)$ is naturally an endomorphism operad from the beginning of this section.

**Note**: $U$ is also called a representation of the operad $\mathcal{P}$. This is similar to $R$-module as a representation of $R$-algebra.

**Example 5.18. Associative algebra** Let $U = \{U, \alpha\}$ be a vector space with binary operator $\alpha$, and $\varphi$ be an operad homomorphism from $\mathcal{F}_{Asso}(V)/(R)$ (defined in Example 5.16 to $\bigoplus_{n\geq 1} Hom(U^{\otimes n}, U)$. Then, $U = \{U, \alpha\}$ is an associative algebra.

**Definition 5.19. Binary quadratic operad** A binary operad $\mathcal{P} = \mathcal{F}(V)/(R)$ is called quadratic if $R \subseteq \mathcal{F}_3(V)$ where $\mathcal{F}_3 = \mathcal{F}_2 \circ_1 \mathcal{F}_2 \oplus \mathcal{F}_2 \circ_2 \mathcal{F}_2 = V \circ_1 V \oplus V \circ_2 V = V^{\otimes 2} \oplus V^{\otimes 2}$.

**Note**: A typical element of $V^{\otimes 2}$ is of the form $\sum_{i=1}^{k} \odot_i^{(1)} \otimes \odot_i^{(2)}$ with $\odot_i^{(1)}, \odot_i^{(2)} \in V, 1 \leq i \leq k$. Thus a typical element of $V^{\otimes 2} \oplus V^{\otimes 2}$ is of the form

$$\left( \sum_{i=1}^{k} \odot_i^{(1)} \otimes \odot_i^{(2)}, \sum_{j=1}^{m} \odot_j^{(3)} \otimes \odot_j^{(4)} \right), \quad \odot_i^{(1)}, \odot_i^{(2)}, \odot_j^{(3)}, \odot_j^{(4)} \in V, 1 \leq i \leq k, 1 \leq j \leq m, k, m \geq 1.$$

Here, the ”,” in the parenthesis means minus$''-''$, and the first part means to do the first operation firstly and the second part means to do the second operation firstly.

**Corollary 5.20. Binary Quadratic Algebras**

*$U = \{U, \alpha\}$ is called a binary quadratic algebra if $\exists$ a linear map $f : V \to Hom(U^{\otimes 2}, U)$ s.t. $\overline{f}(R) = 0$. Here $\overline{f}$ is induced by $f$ : $\overline{f}(\mu \circ_i \nu) := f(\mu) \circ_i f(\nu)$.*

### 5.1.4 From Rota-Baxter algebras to dendriform algebras revisited

In this section, we consider an inverse of Theorem 5.4 in the following sense. Suppose $(R, P)$ is a Rota-Baxter algebra and defines binary operations

$$x \prec_P y := xP(y), \;\; x \succ_P y := P(x)y.$$

By Theorem 5.4, the two operations satisfy the dendriform algebra relations in Definition 5.6. Our inverse question is, what other relations could $(R, \prec_P, \succ_P)$ satisfy? By the non-symmetric operad theory, we can make the question precise. We then determine all relations that are consistent with the Rota-Baxter operator.

**Theorem 5.21.** *Let $V = \mathbf{k}\{\prec, \succ\}$ be the vector space with basis $\{\prec, \succ\}$ and let $\mathcal{P} = \mathcal{P}(V)/(R)$ be a binary quadratic non-symmetric operad. The following statements are equivalent.*

(a) *For every Rota-Baxter algebra $(T, P)$ with weight 0, the triple $(T, \prec_P, \succ_P)$ is a $\mathcal{P}$-algebra.*

(b) *The relation space $R$ of $\mathcal{P}$ is contained in the subspace of $V^{\otimes 2} \oplus V^{\otimes 2}$ spanned by*

$$(\prec \otimes \prec, \prec \otimes \star),$$

$$(\star \otimes \succ, \succ \otimes \succ),$$

$$(\succ \otimes \prec, \succ \otimes \prec), \tag{5.9}$$

*where $\star = \prec + \succ$. More precisely, any $\mathcal{P}$-algebra $A$ satisfies the relations*

$$(x \prec y) \prec z = x \prec (y \star z), \quad (x \star y) \succ z = x \succ (y \succ z),$$
$$(x \succ y) \prec z = x \succ (y \prec z). \tag{5.10}$$

**Note**: We call the operad $\mathcal{P}$ defined by the relations in Eq. 5.9 **Rota-Baxter dendriform operad**, and call a triple $\{T, \prec, \succ\}$ satisfying Eq. 5.10 a **Rota-Baxter dendriform algebra**, which actually corresponds to the general dendriform algebra.

*Proof.* With $V = \mathbf{k}\{\prec, \succ\}$, we have

$$V^{\otimes 2} \oplus V^{\otimes 2} = \bigoplus_{\odot_1, \odot_2, \odot_3, \odot_4 \in \{\prec, \succ\}} \mathbf{k}(\odot_1 \otimes \odot_2, \odot_3 \otimes \odot_4).$$

Thus any element $r$ of $V^{\otimes 2} \oplus V^{\otimes 2}$ is of the form

$$\begin{aligned} r \quad := \quad & a_1(\prec \otimes \prec, 0) + a_2(\prec \otimes \succ, 0) + a_3(\succ \otimes \prec, 0) + a_4(\succ \otimes \succ, 0) \\ & + b_1(0, \prec \otimes \prec) + b_2(0, \succ \otimes \prec) + b_3(0, \prec \otimes \succ) + b_4(0, \succ \otimes \succ), \end{aligned}$$

where the coefficients are in $\mathbf{k}$.

(a $\Rightarrow$ b) Let $\mathcal{P} = \mathcal{P}(V)/(R)$ be an operad satisfying the condition in Item a. Let $r$ be in $R$ expressed in the above form. Then for any Rota-Baxter algebra $(T, P)$, the triple $(T, \prec_P, \succ_P)$ is a $\mathcal{P}$-algebra. Thus

$$a_1(x \prec_P y) \prec_P z + a_2(x \prec_P y) \succ_P z + a_3(x \succ_P y) \prec_P z + a_4(x \succ_P y) \succ_P z$$
$$+b_1 x \prec_P (y \prec_P z) + b_2 x \succ_P (y \prec_P z) + b_3 x \prec_P (y \succ_P z) + b_4 x \succ_P (y \succ_P z) = 0, \forall x, y, z \in T.$$

By the definitions of $\prec_P, \succ_P$ in Eq.( 5.6), we have

$$a_1 xP(y)P(z) + a_2 P(xP(y))z + a_3 P(x)yP(z) + a_4 P(P(x)y)z$$
$$+b_1 xP(yP(z)) + b_2 P(x)(yP(z)) + b_3 xP(P(y)z) + b_4 P(x)(P(y)z) = 0.$$

Since $P$ is a Rota-Baxter operator, we further have:

$$a_1 xP(P(y)z) + a_1 xP(yP(z)) + a_2 P(xP(y))z + a_3 P(x)yP(z) + a_4 P(P(x)y)z$$
$$+b_1 xP(yP(z)) + b_2 P(x)(yP(z)) + b_3 xP(P(y)z) + b_4 P(P(x)y)z + b_4 P(xP(y))z = 0.$$

Collecting similar terms, we obtain

$$(a_1 + b_1)xP(yP(z)) + (a_1 + b_3)xP(P(y)z) + (a_2 + b_4)P(xP(y))z$$
$$+(a_3 + b_2)P(x)yP(z) + (a_4 + b_4)P(xP(y))z = 0$$

Now we take the special case when $(T, P)$ is the free Rota-Baxter algebra $(F_T T(M)), P_{T(M)})$ defined in Corollary 4.5 for our choice of $M = \mathbf{k}\{x, y, z\}$ and $P_{T(M)}(u) = \lfloor u \rfloor$. Then the above equation is just

$$(a_1 + b_1)x\lfloor y \lfloor z \rfloor \rfloor + (a_1 + b_3)x\lfloor \lfloor y \rfloor z \rfloor + (a_2 + b_4)\lfloor x \lfloor y \rfloor \rfloor z$$

$$+(a_3+b_2)\lfloor x\rfloor y\lfloor z\rfloor+(a_4+b_4)\lfloor x\lfloor y\rfloor\rfloor z=0$$

Note that the set of elements

$$x\lfloor y\lfloor z\rfloor\rfloor,\ x\lfloor\lfloor y\rfloor z\rfloor,\ \lfloor x\lfloor y\rfloor\rfloor z,\ \lfloor x\rfloor y\lfloor z\rfloor,\ \lfloor x\lfloor y\rfloor\rfloor z$$

is a subset of the basis $\mathfrak{X}_\infty$ of the free Rota-Baxter algebra $F_T(T(M))$ and hence is linearly independent. Thus the coefficients must be zero, that is,

$$a_1=-b_1=-b_3,\ a_2=a_4=-b_4,\ a_3=-b_2$$

Substituting these equations into the general relation $r$, we find that the any relation $r$ that can be satisfied by $\prec_P,\succ_P$ for all Rota-Baxter algebras $(T,P)$ is of the form

$$\begin{aligned} r &= a_1\big((x\prec y)\prec z-x\prec(y\succ z)-x\prec(y\prec z)\big)\\ &\quad +a_3\big((x\succ y)\prec z-x\succ(y\prec z)\big)\\ &\quad -b_4\big(x\succ(y\succ z)-(x\prec y)\succ z-(x\succ y)\succ z\big), \end{aligned}$$

where $a_1,a_3,b_4\in\mathbf{k}$ can be arbitrary. Thus $r$ is in the subspace prescribed in Item b, as needed.

(b $\Rightarrow$ a) We check directly that all the relations in Eq 5.10 are satisfied by $(T,\prec_P,\succ_P)$ for every Rota-Baxter algebra $(T,P)$.

(a) To check the first relation in Eq. (5.10), we have

$$\begin{aligned}(x \prec_P y) \prec_P z &= xP(y)P(z)\\ &= xP(yP(z)) + xP(P(y)z)\\ &= x \prec_P (y \prec_P z) + x \prec_P (y \succ_P z),\end{aligned}$$

(b) For the second relation in Eq. (5.10), we similarly have

$$\begin{aligned}x \succ_P (y \succ_P z) &= P(x)P(y)z\\ &= P(xP(y))z + P(P(x)y)z\\ &= (x \prec_P y) \succ_P z + (x \succ_P y) \succ_P z,\end{aligned}$$

(c) For the third relation in Eq. (5.10), we have

$$(x \succ_P y) \prec_P z \;=\; (P(x)y)P(z) = P(x)(yP(z)) = x \succ_P (y \prec_P z).$$

Thus if the relation space $R$ of an operad $\mathcal{P} = \mathcal{P}(V)/(R)$ is contained in the subspace spanned by the vectors in Eq. 5.9, then the corresponding relations are linear combinations of the equations in Eq. 5.10 and hence are satisfied by $(T, \prec_P, \succ_P)$ for each Rota-Baxter algebra $(T, P)$. Therefore $(T, \prec_P, \succ_P)$ is a $\mathcal{P}$-algebra. This completes the proof. $\square$

## 5.2 “Classical” definition of symmetric dendriform algebras

**Definition 5.22.** (**Classical definition**) A **symmetric dendriform k-algebra** is a **k**-module $D$ with two binary operations $\prec$ and $\prec'$ that satisfy the following relations:

$$\begin{aligned} (x \prec y) \prec z &= x \prec (y \prec z + y \prec' z), \\ (x \prec' y) \prec z &= x \prec' (y \prec z), \quad x, y, z \in D, \\ (x \prec y + x \prec' y) \prec' z &= x \prec' (y \prec' z), \end{aligned} \tag{5.11}$$

where $\prec' := \succ$ .

**Corollary 5.23.** *Let* $(D, \prec, \prec')$ *be a symmetric dendriform algebra. The operation* $\star := \star_D$ *on* $D$ *defined by*

$$x \star y := x \prec y + x \prec' y, \quad x, y \in D, \tag{5.12}$$

*is associative.*

*Proof.* Adding the left hand sides of Eqs. (5.11) , we obtain

$$\begin{aligned} &(x \prec y) \prec z + (x \prec' y) \prec z + (x \prec y + x \prec' y) \prec' z \\ =&(x \prec y + x \prec' y) \prec z + (x \prec y + x \prec' y) \prec' z \\ =&(x \star y) \prec z + (x \star y) \prec' z \\ =&(x \star y) \star z. \end{aligned}$$

Similarly, adding the right hand sides of these equations, we obtain $x \star (y \star z)$. Thus we have proved the associativity of $\star$. □

**Theorem 5.24. From symmetric Rota-Baxter algebra to symmetric dendriform algebras**

*A symmetric Rota-Baxter algebra* $(S, P)$ *of weight zero defines a symmetric dendriform algebra* $(S, \prec_P, \prec'_P)$*, where*

$$x \prec_P y = xP(y),\ \ x \prec'_P y = yP(x), \forall x, y \in S. \tag{5.13}$$

*Proof.* We will check Eq 5.11 under Eq 5.13 and use the symmetric Rota-Baxter identity.

$(x \prec y) \prec z = xP(y)P(z) = xP(yP(z) + zP(y)) = x \prec (y \prec z + y \prec' z)$;

$(x \prec' y) \prec z = yP(x)P(z) = yP(z)P(x) = (y \prec z)P(x) = x \prec' (y \prec z)$;

$(x \prec y + x \prec' y) \prec' z = (xP(y) + yP(x)) \prec' z = zP(xP(y) + yP(x)) = zP(x)P(y) = zP(y)P(x)$

$= z \prec y \prec x = (y \prec' z) \prec x = x \prec' (y \prec' z), \quad \forall x, y, z \in D.$ □

**Remark 5.25. It is important to note that**: $x \prec'_P y = y \prec_P x$.

Remark 5.25 tells that $\prec'$ is the permutation of $\prec$ under the context of symmetric Rota-Baxter algebras. In terms of symmetric group action, $\prec' = \prec^{(12)}$ .

**Example 5.26. (A concrete example)**

Let $\mathbf{k}M_{2x2}(E) = \mathbf{k}\{\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, where $a, b \in E$ and $E$ is a symmetric dendriform algebra$\}$, with the redefined matrix multiplication operation $\prec$, then $\mathbf{k}M_{2x2}(E)$ is a symmetric dendriform algebra.

## 5.3 Introduction to symmetric operad theory

Symmetric binary operads theory is to some extent parallel to the non-symmetric case and we will mainly use binary trees to describe its definition and free properties. For more details on this, see [32, 45]. This section is also partially from the class notes of the advanced topic class of algebra at Rutgers-Newark in 2014 Spring.

**Note**: Section 5.1.3 will be treated as the background of this section.

By replacing graded vector spaces $\mathcal{P}_n$ with graded $\mathcal{S}_n$-modules and adding the symmetry compatibilities, we have

**Definition 5.27.** [45]**Symmetric operad** A symmetric operad is an $\mathcal{S}-$module $\mathcal{P} := \{\mathcal{P}_n\}_{n\geq 0}$ equipped with partial compositions:

$$\circ_i := \circ_{m,n,i} : \mathcal{P}_m \otimes \mathcal{P}_n \longrightarrow \mathcal{P}_{m+n-1}, \quad 1 \leq i \leq m, \tag{5.14}$$

such that, for $\lambda \in \mathcal{P}_\ell, \mu \in \mathcal{P}_m$ and $\nu \in \mathcal{P}_n$, the following relations hold.

(i) $(\lambda \circ_i \mu) \circ_{i-1+j} \nu = \lambda \circ_i (\mu \circ_j \nu), \quad 1 \leq i \leq \ell, 1 \leq j \leq m$. (Sequential composition)

(ii) $(\lambda \circ_i \mu) \circ_{k-1+m} \nu = (\lambda \circ_k \nu) \circ_i \mu, \quad 1 \leq i < k \leq \ell$. (Parallel composition)

(iii) There is an element $\mathrm{id} \in \mathcal{P}_1$ such that $\mathrm{id}\circ\mu = \mu$ and $\mu\circ\mathrm{id} = \mu$ for $\mu \in \mathcal{P}_n, n \geq 0$. (Identity)

(iv) For $\sigma \in \mathcal{S}_n$, $\mu \circ_i \nu^\sigma = (\mu \circ_i \nu)^{\sigma'}, \sigma' \in \mathcal{S}_{m+n-1}$;
for $\sigma \in \mathcal{S}_m$, $\mu^\sigma \circ_i \nu = (\mu \circ_{\sigma(i)} \nu)^{\sigma''}, \sigma'' \in \mathcal{S}_{m+n-1}$. (Symmetry compatibility)

**Definition 5.28.** [45]**Free symmetric operads** Let $M$ be a $\mathcal{S}$-module, the free symmetric operad generated by $M$ is denoted as $\mathcal{F}(M)$ equipped with an $\mathcal{S}$-module morphism $\eta(M) : M \to \mathcal{F}(M)$ which satisfies the following universal condition: for any $\mathcal{S}$-module $f : M \to \mathcal{P}$, where $\mathcal{P}$ is any symmetric operad, extends uniquely into an operad morphism $\bar{f} : \mathcal{F}(M) \to \mathcal{P}$ :

$$\begin{tikzcd} M \arrow[r, "\eta(M)"] \arrow[dr, "f"'] & \mathcal{F}_(M) \arrow[d, "\bar{f}"] \\ & \mathcal{P} \end{tikzcd}$$

Now let's use binary trees to construct the free operads.

**Definition 5.29. Labeled Trees** Let $\mathcal{T}$ denote the set of planar binary trees $\{|, Y_2, Y_3, Y_4, \cdots\}$. If $t \in \mathcal{T}$ has $n$ leaves, we call $t$ an *n-tree* denoted as $t_n$. For each vertex $v$ of $t$, let $In(v)$ denote the set of inputs of $v$.

**Definition 5.30. Decorated Trees** Let $V$ be a set and let $t$ be an *n-tree*. $t(V)$="$t$ with vertices decorated by elements in V and with leaves decorated by inputs." We call $t(V)$ a labeled $n$-tree and $\mathcal{T}(V) = \bigsqcup_{t\in\mathcal{T}} t(V)$. And we let $Vin(t)$ denote the set of labels of the vertices of $t$ and $Lin(t)$ denote the set of labels of leaves of $t$.

**Corollary 5.31.** *Let $V$ be a set of binary operations: $V = V_2$. Then the* **free non-symmetric binary operad** *generated by $V$ is given by the vector space $\mathcal{T}_{ns}(V) = \bigoplus_{t\in\mathcal{T}} t[V]$, where $t[V]$ is the non-symmetric treewise tensor module associated to $t$ and is given by $t[V] := \bigotimes_{v\in Vin(t)} V_{|In(v)|}$.*

Here, $t_1[V] = \{|\}$ is trivial. $t_2[V] = \{Y_2|v \in V\}$. $t_3[V] = \{Y_3|v1, v2 \in V\} = (t_2[V] \otimes t_2[V]) \oplus (t_2[V] \otimes t_2[V]) = V^{\otimes 2} \oplus V^{\otimes 2}$. Thus, $\dim(t_3[V])$=8 when $V$ only has two binary operations.

For symmetric case, we replace planar binary trees by binary trees "in space".

**Definition 5.32.** (**In space**) Let $\mathbb{T}$ ="binary trees in 3-D space"= $\{|; Y_1; Y_2; \cdots\}$ where each vertex can rotate $360^o$.

So in this case, $t_2[V] = t_2[V]$ and so on. For more details, see Section 5.8.5 in [45]. Similary, we define

**Definition 5.33.** Let $V$ be a set of binary operations: $V = V_2$ and $k$ be a field. Then the **free symmetric binary operad** generated by $V$ is given by $\mathbb{T}(V) = \bigoplus_{\mathbf{t}\in\mathbb{T}}(\mathbf{t}(V)\otimes k)$ where $\mathbf{t}(V)$ is the symmetric treewise tensor module associated to $\mathbf{t}$.

**Note**: We not only consider the rotation of the vertices but also the rotation of the inputs. This is why it is hard to draw $\mathbf{t}(V)$.

Here, $\mathbf{t_1}(V)$ is trivial also. $\mathbf{t_2}(V) = V \otimes_{S_2} k = \{Y_1, xy|v \in V, \text{ and } x, y \in k\}$, where $S_2$ acting on inputs $x, y$ cancels the rotation of the only vertex.

The complicated case is about $\mathbf{t_3}(V)$. Suppose $V$ has only two binary operations $\prec, \prec'$.

**Claim** that $\mathbf{t_3}(V) = V \otimes_{S_2} ((V \otimes k) \oplus (k \otimes V)) \otimes_{S_2} k[S_3]$ and $\dim(\mathbf{t_3}(V)) = 12$.

*Proof.* For the non-symmetric case, we already know that $\dim t_2[V] = 8$. Now we have to consider the rotation of vertices and the rotation of inputs which makes things a little more complicated. For 3 inputs, we have 6 cases under permutations. Then, we have $2 \times 2 \times 6$ for one piece $Y \circ_1 Y$ and totally 48 cases. But the rotation of two vertices partially cancel the permutation of 3 inputs. So the final dimension should be $48/2/2 = 12$. □

Explicitly, we have three types of elements which generates $\mathbf{t_3}$:

For an operad where the space of generators $V$ is equal to $\mathbf{k}[S_2] = \mu.\mathbf{k} \oplus \mu'.\mathbf{k}$ with $\mu.(12) = \mu'$, we will adopt the convention in [45] [p. 199]and denote the 12 elements of $\mathcal{T}(V)(3)$ by $v_i, 1 \leq i \leq 12$, in the following table.

| | | | | | |
|---|---|---|---|---|---|
| $v_1$ | $\mu \circ_{\mathrm{I}} \mu \leftrightarrow (xy)z$ | $v_5$ | $\mu \circ_{\mathrm{III}} \mu \leftrightarrow (zx)y$ | $v_9$ | $\mu \circ_{\mathrm{II}} \mu \leftrightarrow (yz)x$ |
| $v_2$ | $\mu' \circ_{\mathrm{II}} \mu \leftrightarrow x(yz)$ | $v_6$ | $\mu' \circ_{\mathrm{I}} \mu \leftrightarrow z(xy)$ | $v_{10}$ | $\mu' \circ_{\mathrm{III}} \mu \leftrightarrow y(zx)$ |
| $v_3$ | $\mu' \circ_{\mathrm{II}} \mu' \leftrightarrow x(zy)$ | $v_7$ | $\mu' \circ_{\mathrm{I}} \mu' \leftrightarrow z(yx)$ | $v_{11}$ | $\mu' \circ_{\mathrm{III}} \mu' \leftrightarrow y(xz)$ |
| $v_4$ | $\mu \circ_{\mathrm{III}} \mu' \leftrightarrow (xz)y$ | $v_8$ | $\mu \circ_{\mathrm{II}} \mu' \leftrightarrow (zy)x$ | $v_{12}$ | $\mu \circ_{\mathrm{I}} \mu' \leftrightarrow (yx)z$ |

Now, we are good enough to proceed to next two sections.

## 5.4 Operadic definition of symmetric dendriform algebras

The dendriform algebra is introduced by Loday [45]. The concept of symmetric dendriform algebras can be modified in a similar way.

**Definition 5.34.** A **symmetric dendriform algebra** $E$ over $K$ is a $K$-vector space $E$ equipped with two binary operations

$$\prec: E \otimes E \to E,$$
$$\prec': E \otimes E \to E,$$

which satisfy the following axioms:

$$(x \prec y) \prec z = x \prec (y \star z), \tag{5.15}$$
$$(x \star y) \prec' z = x \prec' (y \prec' z), \tag{5.16}$$
$$(y \prec z) \prec x = y \prec (z \star x), \tag{5.17}$$

$$(y \star z) \prec' x = y \prec' (z \prec' x), \tag{5.18}$$

$$(z \prec x) \prec y = z \prec (x \star y), \tag{5.19}$$

$$(z \star x) \prec' y = z \prec' (x \prec' y), \tag{5.20}$$

where $\prec' := \prec^{(12)}$, $K$ is a commutative ring, and $\star = \prec + \prec'$, $\forall x, y, z \in E$.

**Remark 5.35.** Here, the order of arguments does matter. And we call the order of $(xyz)$ **Type I**, the order of $(yzx)$ **Type II** and the order of $(zxy)$ **Type III**.

**Proposition 5.36.** *This definition gives the following properties:*

$$(x \prec y) \prec z = (x \prec z) \prec y, \tag{5.21}$$

$$(y \prec z) \prec x = (y \prec x) \prec z, \tag{5.22}$$

$$(z \prec x) \prec y = (z \prec y) \prec x. \tag{5.23}$$

*Proof.* Eq. 5.15 and Eq. 5.18 gives Eq. 5.21. Similarly,Eq. 5.17 and Eq. 5.20 gives Eq. 5.22 and Eq. 5.16 and Eq. 5.19 gives Eq. 5.23. □

**Corollary 5.37.** *The symmetric dendriform algebra automatically has the following three normal relations derived from its definition:*

$$(x \prec' y) \prec z = x \prec' (y \prec z), \tag{5.24}$$

$$(y \prec' z) \prec x = y \prec' (z \prec x), \tag{5.25}$$

$$(z \prec' x) \prec y = z \prec' (x \prec y). \tag{5.26}$$

*Proof.* By observation and since $\prec' = \prec^{(12)}$, Eq. 5.20 and Eq. 5.17 gives Eq. 5.24:

$$(x \prec' y) \prec z = z \prec' (x \prec' y) = (z \star x) \prec' y = y \prec (z \star x) = (y \prec z) \prec x = x \prec' (y \prec z).$$

Similarly, Eq. 5.16 and Eq. 5.19 gives Eq. 5.25, and Eq. 5.18 and Eq. 5.15 gives Eq. 5.26:

$$(y \prec' z) \prec x = x \prec' (y \prec' z) = (x \star y) \prec' z = z \prec (x \star y) = (z \prec x) \prec y = y \prec' (z \prec x);$$

$$(z \prec' x) \prec y = y \prec' (z \prec' x) = (y \star z) \prec' x = x \prec (y \star z) = (x \prec y) \prec z = z \prec' (x \prec y).$$

□

**Proposition 5.38.** *Any symmetric dendriform algebra $E$ is an associative algebra under the operation $\star$ defined by $x \star y := x \prec y + x \prec' y$.*

*Proof.* By adding up the three equalities Eq. 5.15, Eq 5.16 and Eq. 5.24, we get $(x \star y) \star z$ on the left hand side and $x \star (y \star z)$ on the right hand side as Type I, and similarly for Type II and Type III, whence the statement. □

Symmetric dendriform algebras share similar properties as general dendriform algebras. Except the theorem above that they are both associative, the next theorem is an analogue of the results of Ebrahimi-Fard [23] that a Rota-Baxter algebra gives a dendriform algebra or a tridendriform algebra.

**Theorem 5.39.** *A symmetric Rota-Baxter algebra $(S, P)$ defines a symmetric dendriform algebra $(S, \prec_P, \prec'_P)$, where $x \prec_P y := xP(y)$ and $x \prec'_P y := yP(x)$.*

*Proof.* We only check Type I which are Eq. 5.15, Eq 5.16 and Eq. 5.24, and Type II and Type III are similar by changing the order of arguments.

$(x \prec_P y) \prec_P z = xP(y)P(z) = xP(yP(z) + zP(y)) = x \prec_P (y \prec_P z + y \prec'_P z) = x \prec_P (y \star_P z)$;

$(x \prec'_P y) \prec_P z = yP(x)P(z) = yP(z)P(x) = (y \prec_P z)P(x) = x \prec'_P (y \prec_P z)$;

$(x \star_P y) \prec'_P z = (x \prec_P y + x \prec'_P y) \prec'_P z = zP(xP(y) + yP(x)) = zP(x)P(y) = zP(y)P(x) = (y \prec'_P z)P(x) = x \prec'_P (y \prec'_P z)$. □

## 5.5 From symmetric Rota-Baxter algebras to symmetric dendriform algebras revisited

In this section, we consider an inverse of Theorem 5.39 in the following sense. Suppose $(S, P)$ is a symmetric Rota-Baxter algebra and defines binary operations

$$x \prec_P y := xP(y),\ x \prec'_P y := yP(x).$$

By Theorem 5.39, the two operations satisfy the symmetric dendriform algebra relations in Definition 5.34. Our inverse question is, what other relations could $(S, \prec_P, \prec'_P)$ satisfy? By the symmetric operad theory, we can make the question precise. We then determine all relations that are consistent with the symmetric Rota-Baxter operator.

**Theorem 5.40.** *Let $V = \mathbf{k}\{\prec, \prec'\}$ be the vector space with basis $\{\prec, \prec'\}$ and let $\mathcal{P} = \mathcal{P}(V)/(R)$ be a binary quadratic symmetric operad. The following statements are equivalent.*

(a) *For every symmetric Rota-Baxter algebra* $(S, P)$*, the triple* $(S, \prec_P, \prec'_P)$ *is a* $\mathcal{P}$*-algebra.*

(b) *The relation space* $R$ *of* $\mathcal{P}$ *is contained in the subspace of* $V^{\otimes 2} \oplus V^{\otimes 2}$ *spanned by*

$$
\begin{aligned}
&(\prec \otimes_i \prec, \prec \otimes_i \star),\\
&(\prec' \otimes_i \prec, \prec' \otimes_i \prec),\\
&(\star \otimes_i \prec', \prec' \otimes_i \prec'),
\end{aligned}
\tag{5.27}
$$

*where* $\star = \prec + \prec'$ *and* $i \in \{I, II, III\}$*. More precisely, any* $\mathcal{P}$*-algebra* $A$ *satisfies the relations*

$$
\begin{aligned}
&(x \prec y) \prec z = x \prec (y \star z), \quad (x \prec' y) \prec z = x \prec' (y \prec z), \quad (x \star y) \prec' z = x \prec' (y \prec' z),\\
&(y \prec z) \prec x = y \prec (z \star x), \quad (y \prec' z) \prec x = y \prec' (z \prec x), \quad (y \star z) \prec' x = y \prec' (z \prec' x),\\
&(z \prec x) \prec y = z \prec (x \star y), \quad (z \prec' x) \prec y = z \prec' (x \prec y), \quad (z \star x) \prec' y = z \prec' (x \prec' y).
\end{aligned}
\tag{5.28}
$$

**Note**: We call the operad $\mathcal{P}$ defined by the relations in Eq. 5.27 **symmetric Rota-Baxter dendriform operad**, and call a triple $\{S, \prec, \prec'\}$ satisfying Eq. 5.28 a **symmetric Rota-Baxter dendriform algebra**, which actually corresponds to the general symmetric dendriform algebra.

*Proof.* With $V = \mathbf{k}\{\prec, \prec'\}$, we have

$$V^{\otimes 2} \oplus V^{\otimes 2} = \bigoplus_{\odot_1,\odot_2,\odot_3,\odot_4 \in \{\prec,\prec'\}} \mathbf{k}(\odot_1 \otimes \odot_2, \odot_3 \otimes \odot_4).$$

Thus any element $r$ of $V^{\otimes 2} \oplus V^{\otimes 2}$ is of the form

$$\begin{aligned} r \;:=\; & a_1(\prec \otimes \prec, 0) + a_2(\prec \otimes \prec', 0) + a_3(\prec' \otimes \prec, 0) + a_4(\prec' \otimes \prec', 0) \\ & + b_1(0, \prec \otimes \prec) + b_2(0, \prec' \otimes \prec) + b_3(0, \prec \otimes \prec') + b_4(0, \prec' \otimes \prec') \end{aligned}$$

where the coefficients are in $\mathbf{k}$.

(a $\Rightarrow$ b) Let $\mathcal{P} = \mathcal{P}(V)/(R)$ be an operad satisfying the condition in Item a. Let $r$ be in $R$ expressed in the above form. Then for any symmetric Rota-Baxter algebra $(S, P)$, the triple $(S, \prec_P, \prec'_P)$ is a $\mathcal{P}$-algebra. Thus $\forall x, y, z \in S$ which will cover three types of the order of $x, y, z$,

$$\begin{aligned} & a_1(x \prec_P y) \prec_P z + a_2(x \prec_P y) \prec'_P z + a_3(x \prec'_P y) \prec_P z + a_4(x \prec'_P y) \prec'_P z \\ & + b_1 x \prec_P (y \prec_P z) + b_2 x \prec'_P (y \prec_P z) + b_3 x \prec_P (y \prec'_P z) + b_4 x \prec'_P (y \prec'_P z) = 0 \end{aligned}$$

By the definitions of $\prec_P, \prec'_P$ in Theorem 5.39, we have

$$\begin{aligned} & a_1 xP(y)P(z) + a_2 zP(xP(y)) + a_3 yP(x)P(z) + a_4 zP(yP(x)) \\ & + b_1 xP(yP(z)) + b_2 yP(z)P(x) + b_3 xP(zP(y)) + b_4 zP(y)P(x) = 0. \end{aligned}$$

Since $P$ is a symmetric Rota-Baxter operator, we further have

$$a_1xP(yP(z)) + a_1xP(zP(y)) + a_2zP(xP(y)) + a_3yP(xP(z)) + a_3yP(zP(x)) + a_4zP(yP(x))$$
$$+b_1xP(yP(z)) + b_2yP(zP(x)) + b_2yP(xP(z)) + b_3xP(zP(y)) + b_4zP(yP(x)) + b_4zP(xP(y)) = 0.$$

Collecting similar terms, we obtain

$$(a_1 + b_1)xP(yP(z)) + (a_1 + b_3)xP(zP(y)) + (a_2 + b_4)zP(xP(y))$$
$$+(a_3 + b_2)yP(xP(z)) + (a_3 + b_2)yP(zP(x)) + (a_4 + b_4)zP(yP(x)) = 0$$

Now we take the special case when $(S, P)$ is the free symmetric Rota-Baxter algebra $(F_S T(M), P_{T(M)})$ defined in Corollary 4.5 for our choice of $M = \mathbf{k}\{x, y, z\}$ and $P_{T(M)}(u) = \lfloor u \rfloor$. Then the above equation is just

$$(a_1 + b_1)x\lfloor y\lfloor z\rfloor\rfloor + (a_1 + b_3)x\lfloor z\lfloor y\rfloor\rfloor + (a_2 + b_4)z\lfloor x\lfloor y\rfloor\rfloor$$
$$+(a_3 + b_2)y\lfloor x\lfloor z\rfloor\rfloor + (a_3 + b_2)y\lfloor z\lfloor x\rfloor\rfloor + (a_4 + b_4)z\lfloor y\lfloor x\rfloor\rfloor = 0$$

Note that the set of elements

$$x\lfloor y\lfloor z\rfloor\rfloor, x\lfloor z\lfloor y\rfloor\rfloor, z\lfloor x\lfloor y\rfloor\rfloor, y\lfloor x\lfloor z\rfloor\rfloor, y\lfloor z\lfloor x\rfloor\rfloor, z\lfloor y\lfloor x\rfloor\rfloor$$

is a subset of the basis $\mathfrak{X}_\infty$ of the free symmetric Rota-Baxter algebra $F_S(T(M))$ and hence is linearly independent. Thus the coefficients must be zero, that is,

$$a_1 = -b_1 = -b_3, a_2 = a_4 = -b_4, a_3 = -b_2$$

Substituting these equations into the general relation $r$, we find that the any relation $r$ that can be satisfied by $\prec_P, \prec'_P$ for all symmetric Rota-Baxter algebras $(S, P)$ is of the form

$$\begin{aligned} r &= a_1\Big((x \prec y) \prec z - x \prec (y \prec' z) - x \prec (y \prec z)\Big) \\ &\quad + a_3\Big((x \prec' y) \prec z - x \prec' (y \prec z)\Big) \\ &\quad + b_4\Big(x \prec' (y \prec' z) - (x \prec y) \prec' z - (x \prec' y) \prec' z\Big), \end{aligned}$$

where $a_1, a_3, b_4 \in \mathbf{k}$ can be arbitrary. Thus $r$ is in the subspace prescribed in Item b, as needed.

(b $\Rightarrow$ a) We check directly that all the relations in Eq. (5.28) are satisfied by $(S, \prec_P, \prec'_P)$ for every symmetric Rota-Baxter algebra $(S, P)$.

(a) To check the first relation in Eq. (5.28), we have

$$\begin{aligned} (x \prec_P y) \prec_P z &= xP(y)P(z) \\ &= xP(yP(z)) + xP(P(y)z) \\ &= x \prec_P (y \prec_P z) + x \prec_P (y \succ_P z), \end{aligned}$$

(b) For the second relation in Eq. (5.28), we similarly have

$$\begin{aligned} x \succ_P (y \succ_P z) &= P(x)P(y)z \\ &= P(xP(y))z + P(P(x)y)z \\ &= (x \prec_P y) \succ_P z + (x \succ_P y) \succ_P z, \end{aligned}$$

(c) For the third relation in Eq. (5.28), we have

$$(x \succ_P y) \prec_P z \quad = \quad (P(x)y)P(z) = P(x)(yP(z)) = x \succ_P (y \prec_P z).$$

Thus if the relation space $R$ of an operad $\mathcal{P} = \mathcal{P}(V)/(R)$ is contained in the subspace spanned by the vectors in Eq. (5.27), then the corresponding relations are linear combinations of the equations in Eq. (5.28) and hence are satisfied by $(S, \prec_P, \prec'_P)$ for each symmetric Rota-Baxter algebra $(S, P)$. Therefore $(S, \prec_P, \prec'_P)$ is a $\mathcal{P}$-algebra. This completes the proof of Theorem 5.40. □

## 5.6 Free symmetric dendriform algebras

### 5.6.1 Case I: One generator

Let $V = k\{x\}$ be the generator space with only one generator $x$ and $k$ is a field. Define $x \prec x = x \prec' x = xP(x) = x\lfloor x \rfloor$ where $P$ is the symmetric Rota-Baxter operator. Since $\prec' = \prec^{(12)}$, we will stick to $\prec$ in this case.

**Definition 5.41. Symmetric dendriform words** are words generated by $k\{x, \lfloor, \rfloor\}$, and are defined recursively as $x_1 = x$, $x_{k+1} = x \prec x_k = x\lfloor x_k \rfloor$.

For any two symmetric dendriform words $x_m$ and $x_n$, define $x_m \prec x_n = C_{m,n}x_{m+n}$ where $C_{m,n}$ is a coefficient. We will find this is well-defined later as Corollary 5.54 in Case II.

**Lemma 5.42.** $C_{m,n} = C_{m-1,n} + C_{n,m-1}$.

*Proof.* $x_m \prec x_n = x_m\lfloor x_n\rfloor = x\lfloor x_{m-1}\rfloor\lfloor x_n\rfloor = xP(x_{m-1})P(x_n) = xP(x_{m-1}P(x_n) + x_nP(x_{m-1})) = x \prec (x_{m-1} \prec x_n + x_n \prec x_{m-1}) = x \prec (C_{m-1,n}x_{m+n-1} + C_{n,m-1}x_{m+n-1}) = (C_{m-1,n} + C_{n,m-1})x \prec x_{m+n-1} = (C_{m-1,n} + C_{n,m-1})x_{m+n}$.
Since $x_m \prec x_n = C_{m,n}x_{m+n}$, we are done by recursion. □

**Corollary 5.43.** *$C_{1,n} = 1$ and $C_{m,1} = m$.*

*Proof.* By definition above, $x_{n+1} = x_1 \prec x_n = C_{1,n}x_{1+n}$, so $C_{1,n} = 1$.
$C_{m,1} = C_{m,1} + C_{1,m} = C_{m,1} + 1 = \cdots = C_{1,1} + m = 1 + m = m + 1$. □

This property is similar to Pascal triangle. we have the following corollary and matrices.

**Corollary 5.44.** *Let B denote the symmetric Pascal matrix consisting of binomial coefficients as follows. Then the matrix C formed by $C_{m,n}$ is a submatrix of B by deleting the first column of B. Thus, $C_{m,n} = B_{m,n+1}$.*

We take a $5 \times 5$ symmetric Pascal matrix as an example:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{pmatrix}.$$

**Lemma 5.45. Symmetric Pascal Matrix**[44] *Let $B_{m,n}$ denotes the $(m,n)$-entry of the symmetric Pascal matrix of binomial coefficients, $B_{m,n} = \begin{pmatrix} m+n-2 \\ m-1 \end{pmatrix}$.*

**Corollary 5.46.** $C_{m,n} = B_{m,n+1} = \begin{pmatrix} m+n-1 \\ m-1 \end{pmatrix}$.

**Theorem 5.47.** *$C_{m,n}$ holds for any symmetric dendriform algebra.*

*Proof.* Let $D$ be any symmetric dendriform algebra and $a \in D$. Define $a_1 = a$ and $a_{k+1} = a_1 \prec a_k = a_1 \lfloor a_k \rfloor$. Then for any $a_m$ and $a_n$ in $D$, define $a_m \prec a_n = C_{m,n} a_{m+n}$. Since

$a_m \prec a_n = (a_1 \prec a_{m-1}) \prec a_n = a_1 \prec (a_{m-1} \prec a_n + a_{m-1} \prec' a_n) = a_1 \prec (C_{m-1,n} a_{m+n-1} + C_{n,m-1} a_{m+n-1})$

$= (C_{m-1,n} + C_{n,m-1}) a_1 \prec a_{m+n} = (C_{m-1,n} + C_{n,m-1}) a_{m+n}$, and $a_m \prec a_n = C_{m,n} x_{m+n}$,

we have $C_{m+1,n} = C_{m,n} + C_{n,m}$ again. □

**Theorem 5.48.** *The space of symmetric dendriform words denoted as $sDW(V)$ is a symmetric dendriform algebra.*

*Proof.* We only need to verify the relation $(x_m \prec x_n) \prec x_l = x_m \prec (x_n \prec x_l + x_l \prec x_n)$ for any $x_m$, $x_n$, and $x_l \in sDW(V)$. By Cor 5.46 and Lemma 5.42,

LHS$=C_{m,n} x_{m+n} \prec x_l = C_{m,n} C_{m+n,l} x_{m+n+l} = \begin{pmatrix} m+n-1 \\ m-1 \end{pmatrix} \begin{pmatrix} m+n+l-1 \\ m+n-1 \end{pmatrix} x_{m+n+l} = \frac{(m+n-1)!}{(m-1)!n!} \frac{(m+n+l-1)!}{(m+n-1)!l!}$,

while

RHS$=x_m (C_{n,l} x_{n+l} + C_{l,n} x_{l+n}) = (C_{n,l} + C_{l,n}) x_m \prec x_{n+l} = (C_{n,l} + C_{l,n}) C_{m,n+l} x_{m+n+l} = C_{n+1,l} C_{m,n+l} x_{m+n+l} = \begin{pmatrix} n+l \\ n \end{pmatrix} \begin{pmatrix} m+n+l-1 \\ m-1 \end{pmatrix} x_{m+n+l} = \frac{(n+l)!}{n!l!} \frac{(m+n+l-1)!}{(m-1)!(n+l)!} x_{m+n+l}$.

By comparing the two sides coefficients, we find the two sides equal. □

**Theorem 5.49. Free symmetric dendriform algebras** *$sDW(V)$ is a free symmetric dendriform algebra over $V = k\{x\}$ with a set map $j : V \to sDW(V)$ such that, for any symmetric dendriform algebra $D$ and any set map $f : V \to D$, there is a unique symmetric dendriform algebra homomorphism $\bar{f} : sDW(V) \to D$ such that $\bar{f} \circ j = f$:*

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[dr, "f"'] & sDW(V) \arrow[d, "\bar{f}"] \\ & D \end{tikzcd} .$$

*Proof.* We have verified that $sDW(V)$ is a symmetric dendriform algebra by Theorem 5.48. We only need to define $\bar{f}$ and prove $\bar{f}$ is a symmetric dendriform homomorphism and it is unique.

Define $\bar{f}(x) = a \in D$ and let $a_1 = a$ and $a_{k+1} = a_1 \prec a_k$. Then define $\bar{f}(x_k) = a_k$. Note this is the only way to define $\bar{f}$ to be a symmetric dendriform homomorphism. On one hand, we have $\bar{f}(x_m \prec x_n) = \bar{f}(C_{m,n}x_{m+n}) = C_{m,n}\bar{f}(x_{m+n}) = C_{m,n}a_{m+n}$; on the other hand, we have $\bar{f}(x_m) \prec \bar{f}(x_n) = a_m \prec a_n = C_{m,n}a_{m+n}$ by Theorem 5.47. So $\bar{f}$ commutes with $\prec$ and thus is a symmetric dendriform algebra homomorphism. □

Since $C_{m,n}$ holds for any symmetric dendriform algebra, we can give an abstract definition for the free symmetric dendriform algebra which has no relation with symmetric Rota-Baxter identity. Define $F = \bigoplus_{k\geq 1} x_k$ over one generator $x_1$ with the recursion $x_{k+1} = x_1 \prec x_k$ and the defined operation $x_m \prec x_n = C_{m,n}x_{m+n}$.

**Theorem 5.50.** *$(F, \prec, \prec')$ is a free symmetric dendriform algebra over one generator $x_1$.*

*Proof.* The proof is similar to Theorem 5.49. □

**Example 5.51.** (**A natural example**)

(A special Polynomial algebra): Let $k[x]$ be a general polynomial algebra, and redefine its product as $x^m \cdot x^n = \binom{m+n-1}{m-1} x^{m+n}$. Then $\{k[x], \cdot\}$ is a symmetric dendriform algebra.

### 5.6.2 Case II: Multiple generators

Given a set $X$, let $V = kX$ be the generators space and $k$ is a field. Define $X_k, k \geq 1$ by recursion: $X_1 = X$, $X_{k+1} = X \prec X_k = XP(X_k) = X_1\lfloor X_k \rfloor$. Then, any element $\mathcal{X} \in X_k$ is in the form of $x_1\lfloor x_2 \lfloor x_3 \lfloor \ldots \lfloor x_k \rfloor\rfloor \ldots \rfloor$, where $x_i \in X$. And we let $m = l(\mathcal{X})$ be the length of the word $\mathcal{X}$.

**Definition 5.52.** **Multiple symmetric dendriform words(msDW)** are words generated by $k\{X, \lfloor, \rfloor\}$, and are defined recursively by $X_1 = X$ and $X_{k+1} = X \prec X_k = X\lfloor X_k \rfloor$.

**Definition 5.53.** **The products in the symmetric dendriform algebra**

$\forall m, n \geq 1$, let $\mathcal{X} \in X_m$ and $\mathcal{Y} \in X_n$, define

$$\prec: X_m \otimes X_n \to X_{m+n}$$

$$\prec': X_m \otimes X_n \to X_{m+n}$$

by induction on $m \geq 1$.

For $m = 1$, define $\mathcal{X} \prec \mathcal{Y} = x_1 \prec \mathcal{Y} = x_1\lfloor \mathcal{Y} \rfloor$

For $\mathcal{X} \prec' \mathcal{Y} = x_1 \prec' \mathcal{Y} = \mathcal{Y} \prec x_1$, we use induction on $n = l(\mathcal{Y}) \geq 1$.

For $n = 1$, $x_1 \prec' \mathcal{Y} = x_1 \prec' y_1 = y_1 \prec x_1 = y_1\lfloor x_1 \rfloor$.

Suppose for $n = l(\mathcal{Y}) \leq t, t \geq 1$ $x_1 \prec' \mathcal{Y} = \mathcal{Y}\lfloor x_1 \rfloor$ is defined, then when $n = l(\mathcal{Y}) = t + 1$,

$x_1 \prec' \mathcal{Y} = \mathcal{Y} \prec x_1 = y_1\lfloor \mathcal{Y}' \rfloor \prec x_1 = y_1\lfloor \mathcal{Y}' \rfloor\lfloor x_1 \rfloor = y_1\lfloor \mathcal{Y}'\lfloor x_1 \rfloor + x_1\lfloor \mathcal{Y}' \rfloor\rfloor = y_1(\lfloor x_1 \prec' \mathcal{Y}' \rfloor + \lfloor x_1 \prec \mathcal{Y}' \rfloor)$. For now, we are good at $m = 1$.

Next, suppose $\mathcal{X} \prec \mathcal{Y}$ and $\mathcal{X} \prec' \mathcal{Y}$ is defined for $m = l(\mathcal{X}) \leq s, s \geq 1$, then we consider the case when $m = s + 1$.

$\mathcal{X} \prec \mathcal{Y} = x_1\lfloor \mathcal{X}' \rfloor \prec \mathcal{Y} = x_1\lfloor \mathcal{X}' \rfloor\lfloor \mathcal{Y} \rfloor = x_1\lfloor \mathcal{X}'\lfloor \mathcal{Y} \rfloor + \mathcal{Y}\lfloor \mathcal{X}' \rfloor\rfloor = x_1\lfloor \mathcal{X}' \prec \mathcal{Y} + \mathcal{X}' \prec' \mathcal{Y}' \rfloor$.

For $\mathcal{X} \prec' \mathcal{Y} = \mathcal{Y} \prec \mathcal{X}$, we use induction on $n = l(\mathcal{Y}) \geq 1$ again.

For $n = 1$, $\mathcal{X} \prec' y_1 = y_1 \prec \mathcal{X} = y_1\lfloor \mathcal{X} \rfloor$.

Suppose for $n = l(\mathcal{Y}) \leq t, t \geq 1$, $\mathcal{X} \prec' \mathcal{Y}$ is defined. Then, when $n = l(\mathcal{Y}) = t + 1$,

$\mathcal{X} \prec' \mathcal{Y} = \mathcal{Y} \prec \mathcal{X} = y_1\lfloor \mathcal{Y}' \rfloor\lfloor \mathcal{X} \rfloor = y_1\lfloor \mathcal{Y}'\lfloor \mathcal{X} \rfloor + \mathcal{X}\lfloor \mathcal{Y}' \rfloor\rfloor = y_1\lfloor \mathcal{X} \prec' \mathcal{Y}' + \mathcal{X} \prec \mathcal{Y}' \rfloor$.

By now, we are good for $m = s + 1$.

Therefore, we are done for the whole induction process and we defined for $\forall m, n \geq 1$,

$$\mathcal{X} \prec \mathcal{Y} = x_1\lfloor \mathcal{X} \prec \mathcal{Y} + \mathcal{X}' \prec' \mathcal{Y} \rfloor$$

$$\mathcal{X} \prec' \mathcal{Y} = y_1\lfloor \mathcal{X} \prec \mathcal{Y}' + \mathcal{X} \prec' \mathcal{Y}' \rfloor.$$

**Corollary 5.54.** *sDW(V) in Case I is well defined where $X = \{x\}$.*

**Theorem 5.55.** $(msDW(V), \prec, \prec')$ *is a symmetric dendriform algebra.*

*Proof.* We will check the first relation $(x \prec y) \prec z = x \prec (y \prec z + z \prec y)$ by induction on the sum of the lengths of any three words. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be any three words with the sum of their lengths $n \geq 3$.

When $n = 3$, $(x \prec y) \prec z = xP(y)P(z) = xP(zP(y) + yP(z)) = x \prec (z \prec y + y \prec z)$.

Suppose the relation holds for $n = k$, then when $n = k + 1$, we have by definition

$(\mathcal{X} \prec \mathcal{Y}) \prec \mathcal{Z} = x_1(\mathcal{X}' \prec \mathcal{Y} + \mathcal{Y} \prec \mathcal{X}') \prec \mathcal{Z} = x_1((\mathcal{X}' \prec \mathcal{Y} + \mathcal{Y} \prec \mathcal{X}') \prec \mathcal{Z} + \mathcal{Z} \prec (\mathcal{X}' \prec \mathcal{Y} + \mathcal{Y} \prec \mathcal{X}')) =$

$x_1 \prec ((\mathcal{X}' \prec \mathcal{Y}) \prec \mathcal{Z}) + x_1 \prec ((\mathcal{Y} \prec \mathcal{X}') \prec \mathcal{Z}) + x_1 \prec (\mathcal{Z} \prec (\mathcal{X}' \prec \mathcal{Y})) + x_1 \prec (\mathcal{Z} \prec (\mathcal{Y} \prec \mathcal{X}'));$

$\mathcal{X} \prec (\mathcal{Y} \prec \mathcal{Z} + \mathcal{Z} \prec \mathcal{Y}) = x_1 \prec (\mathcal{X}' \prec (\mathcal{Y} \prec \mathcal{Z} + \mathcal{Z} \prec \mathcal{Y}) + (\mathcal{Y} \prec \mathcal{Z} + \mathcal{Z} \prec \mathcal{Y}) \prec \mathcal{X}') =$

$x_1 \prec (\mathcal{X}' \prec (\mathcal{Y} \prec \mathcal{Z} + \mathcal{Z} \prec \mathcal{Y}) + x_1 \prec ((\mathcal{Y} \prec \mathcal{Z}) \prec \mathcal{X}') + x_1 \prec ((\mathcal{Z} \prec \mathcal{Y}) \prec \mathcal{X}').$

Comparing the two equations,

$(\mathcal{X}' \prec \mathcal{Y}) \prec \mathcal{Z} = \mathcal{X}' \prec (\mathcal{Y} \prec \mathcal{Z} + \mathcal{Z} \prec \mathcal{Y})$ and $\mathcal{Z} \prec (\mathcal{Y} \prec \mathcal{X}') = (\mathcal{Z} \prec \mathcal{Y}) \prec \mathcal{X}'$ since the sum of the lengths of $\mathcal{X}'$, $\mathcal{Y}$ and $\mathcal{Z}$ is $n$ and therefore we can use induction on $n$; and $(\mathcal{Y} \prec \mathcal{X}') \prec \mathcal{Z} = (\mathcal{Y} \prec \mathcal{Z}) \prec \mathcal{X}'$ by Corollary 5.36.

This completes our proof. □

**Proposition 5.56.** $(msDW(V), \prec, \prec')$ *is a free symmetric dendriform algebra over* $V = kX$ *with a set map* $j : V \to msDW(V)$ *such that, for any symmetric dendriform algebra* $D$ *and any set map* $f : V \to D$*, there is a unique symmetric dendriform algebra homomorphism* $\bar{f} : msDW(V) \to D$ *such that* $\bar{f} \circ j = f$*:*

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[dr, "f"'] & sDW(V) \arrow[d, "\bar{f}"] . \\ & D \end{tikzcd}$$

*Proof.* Let $a_i = f(x_i)$ for $x_i \in X$. Define $\bar{f}(x_i) = a_i$, and $\bar{f}(\mathcal{X}) = \bar{f}(x_1 \prec \mathcal{X}') = a_1 \prec \bar{f}(\mathcal{X}')$ inductively. Then, we will check $\bar{f}(\mathcal{X} \prec \mathcal{Y}) = \bar{f}(\mathcal{X}) \prec \bar{f}(\mathcal{Y})$ by induction on the sum of lengths $n$ of $\mathcal{X}$ and $\mathcal{Y} \in msDW(V)$.

When $n = 2$, $\bar{f}(\mathcal{X} \prec \mathcal{Y}) = \bar{f}(x \prec y) = f(x) \prec f(y)$ for $x, y \in X$ as defined.

Suppose for $n = k \geq 2$, the homomorphism holds. Then, for $n = k + 1$,

$\overline{f}(\mathcal{X} \prec \mathcal{Y}) = \overline{f}(x_1 \prec (\mathcal{X}' \prec \mathcal{Y} + \mathcal{Y} \prec \mathcal{X}')) = \overline{f}(x_1 \prec (\mathcal{X}' \prec \mathcal{Y})) + \overline{f}(x_1 \prec (\mathcal{Y} \prec \mathcal{X}')) =$

$\overline{f}(x_1) \prec \overline{f}(\mathcal{X}' \prec \mathcal{Y}) + \overline{f}(x_1) \prec \overline{f}(\mathcal{Y} \prec \mathcal{X}') = \overline{f}(x_1) \prec (\overline{f}(\mathcal{X}') \prec \overline{f}(\mathcal{Y})) + \overline{f}(x_1) \prec (\overline{f}(\mathcal{Y}) \prec$

$\overline{f}(\mathcal{X}')) =$

$(\overline{f}(x_1) \prec \overline{f}(\mathcal{X}')) \prec \overline{f}(\mathcal{Y}) = \overline{f}(\mathcal{X}) \prec \overline{f}(\mathcal{Y}).$

This completes our induction and thus $\overline{f}$ is an symmetric dendriform algebra homomorphism. $\square$

# 6 Symmetric Differential Rota-Baxter Algebras

## 6.1 Introduction

### 6.1.1 Motivation

As an analogy to the study of differential Rota-Baxter algebras, we study the symmetric differential Rota-Baxter algebras by mainly completing the construction of the free symmetric differential algebra and the free symmetric differential Rota-Baxter algebras. Again, we only consider the non-commutative case.

We first briefly introduce the idea of differential Rota-Baxter algebras[36]. The differential Rota-Baxter algebra is the algebraic structure reflecting the relation between the differential operator and the integral operator as in the First Fundamental Theorem of Calculus.

As is well-known, the two principal components of calculus are the differential calculus which studies the differential operator $d(f)(x) = \frac{df}{dx}(x)$ and the integral calculus which studies the integral operator $P(f)(x) = \int_a^x f(t)dt$. The discrete versions of these two operators are the difference operator and summation operator.

The abstraction of the differential operator and difference operator led to the development of differential algebra and difference algebra [15, 43]. Likewise, the integral operator $P$ and summation operator have been abstracted to give the notion of Rota-Baxter operators and Rota-Baxter algebras [7, 54, 55].

In the last few years, major progress has been made in both differential algebra and Rota-Baxter algebra, with applications in broad areas in mathematics and physics [4, 5, 14, 10, 20, 24, 26, 27, 35, 58, 59]. For instance, both operators played important roles in the recent developments in renormalization of quantum field theory [10, 11, 28].

The differential operator and the integral operator are related by the First Fundamental Theorem of Calculus stating that (under suitable conditions)

$$\frac{d}{dx}\Big(\int_a^x f(t)dt\Big) = f(x). \tag{6.1}$$

Thus the integral operator is the right inverse of the differential operator, so that $(d \circ P)(f) = f$. A similar relation holds for the difference operator and summation operator (see Example 6.1.(e)). It is therefore natural to introduce the notion of differential integral algebra, or more generally the notion of differential Rota-Baxter algebra, that provides a framework to put differential/difference algebra and Rota-Baxter algebra together in the spirit of Eq. (6.1).

Quite often, problems on differential equations and differential algebra are studied by translating them into integral problems. This transition uses in disguise the underlying structure of differential Rota-Baxter algebra. In fact, Baxter [7] defined his algebra and gave an algebraic proof of the Spitzer identity in probability guided by such a point of view for first order linear ODEs. This view was further stressed by Rota [55] in connection with finding $q$-analogues of classical identities of special functions. The framework introduced in[36] should provide a natural setting to study such problems. The reader is also invited to consult the paper [57], where a similar structure was independently defined under a different context and was applied to study boundary problems for linear ODE in differential algebras. Similar situations might happen for symmetric cases.

### 6.1.2 Definitions and preliminary examples

We now introduce the concept of a differential symmetric Rota-Baxter algebra consisting of an algebra with both a symmetric differential operator and a symmetric Rota-Baxter operator with a compatibility condition between these two operators.

**Definition 6.1.** Let $\mathbf{k}$ be a unitary commutative ring.

(a) A **symmetric differential k-algebra** is an associative $\mathbf{k}$-algebra $R$ together with a linear operator $d : R \to R$ such that

$$d(xy) = d(x)y + d(y)x, \forall\, x, y \in R, \tag{6.2}$$

and

$$d(1) = 0. \tag{6.3}$$

Such an operator is called a **symmetric differential operator** or a **symmetric derivation**. Let $(R, d)$ and $(S, e)$ be any two symmetric differential algebras. Then $f : (R, d) \to (S, e)$ is a **symmetric differential algebra homomorphism** if $f : R \to S$ is a $k$-algebra homomorphism and $e(f(x)) = f(d(x))$ for all $x \in R$.

(b) A **symmetric Rota-Baxter k-algebra** is an associative $\mathbf{k}$-algebra $R$ together with a linear operator $P : R \to R$ such that

$$P(x)P(y) = P(xP(y)) + P(yP(x)), \forall\, x, y \in R. \tag{6.4}$$

Such an operator is called a **symmetric Rota-Baxter operator**.

(c) A **symmetric differential Rota-Baxter k-algebra** is an associative **k**-algebra $R$ together with a symmetric differential operator $d$ and a symmetric Rota-Baxter operator $P$ such that

$$d \circ P = \mathrm{id}_R. \tag{6.5}$$

**Note**: Any commutative differential algebras in the usual sense are symmetrical differential algebras. So through the whole chapter or even this paper, we will only consider non-communtative cases.

We next give some simple examples of differential, Rota-Baxter and differential Rota-Baxter algebras[36]. By Theorem 6.4, every symmetric differential algebra naturally gives rise to a symmetric differential Rota-Baxter algebra.

**Example 6.1.** (a) A 0-derivation and a 0-differential algebra is a derivation and differential algebra in the usual sense [43].

(b) Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Let $R = \mathrm{Cont}(\mathbb{R})$ denote the $\mathbb{R}$-algebra of continuous functions $f : \mathbb{R} \to \mathbb{R}$, and consider the usual "difference quotient" operator $d_\lambda$ on $R$ defined by

$$(d_\lambda(f))(x) = (f(x + \lambda) - f(x))/\lambda. \tag{6.6}$$

Then it is immediate that $d_\lambda$ is a $\lambda$-derivation on $R$. When $\lambda = 1$, we obtain the usual difference operator on functions. Further, the usual derivation is $d_0 := \lim_{\lambda \to 0} d_\lambda$.

(c) A difference algebra [15] is defined to be a commutative algebra $R$ together with an injective algebra endomorphism $\phi$ on $R$. It is simple to check that $\phi - \mathrm{id}$ is a differential operator of weight 1.

(d) By the First Fundamental Theorem of Calculus in Eq. (6.1), $(\mathrm{Cont}(\mathbb{R}), d/dx, \int_0^x)$ is a differential Rota-Baxter algebra of weight 0.

(e) Let $0 < \lambda \in \mathbb{R}$. Let $R$ be an $\mathbb{R}$-subalgebra of $\mathrm{Cont}(\mathbb{R})$ that is closed under the operators

$$P_0(f)(x) = -\int_x^\infty f(t)dt, \quad P_\lambda(f)(x) = -\lambda \sum_{n\geq 0} f(x+n\lambda).$$

For example, $R$ can be taken to be the $\mathbb{R}$-subalgebra generated by $e^{-x}$: $R = \sum_{k\geq 1} \mathbb{R}e^{-kx}$. Then $P_\lambda$ is a Rota-Baxter operator of weight $\lambda$ and, for the $d_\lambda$ in Eq. (6.6),

$$d_\lambda \circ P_\lambda = \mathrm{id}_R, \forall\, 0 \neq \lambda \in \mathbb{R},$$

reducing to the fundamental theorem $d_0 \circ P_0 = \mathrm{id}_R$ when $\lambda$ goes to 0. So $(R, d_\lambda, P_\lambda)$ is a differential Rota-Baxter algebra of weight $\lambda$.

## 6.2 Symmetric differential algebras

We first give some basic properties of symmetric differential algebras, followed by a study of free symmetric differential algebras.

### 6.2.1 Basic properties for symmetric differential operators

Some basic properties of differential operators can be easily generalized to symmetric differential operators. The following proposition generalizes the power rule in differential calculus and the well-known result of Leibniz [43, p.60].

**Proposition 6.2.** *Let* $(R, d)$ *be a symmetric differential* $\mathbf{k}$*-algebra of weight* $0$*.*

(a) *Let* $x \in R$ *and* $n \in \mathbb{N}_+$*. Then*

$$d(x^n) = nd(x)x^{n-1}.$$

(b) *Let* $x, y \in R$*, and let* $n \in \mathbb{N}^+$*. Then*

$$d^{n+1}(xy) = \sum_{i=0}^{n}\left(\binom{n}{i}d^{i+1}(x)d^{n-i}(y) + \binom{n}{i}d^{i+1}(y)d^{n-i}(x)\right). \tag{6.7}$$

*Proof.* (a) When $n = 1$, it is trivial. Suppose the equation holds for $n \geq 1$, then apply Eq. 6.2, we have $d(x^{n+1}) = d(x^n x) = d(x^n)x + d(x)(x^n) = nd(x)x^{n-1}x + d(x)(x^n) = (n + 1)d(x)x^n$.

(b) We proceed the proof by induction on $n$.

When $n = 1$, by using Eq. 6.2 twice, we have $d^2(xy) = d(d(xy)) = d(d(x)y+d(y)x) = d(d(x)y)+d(d(y)x) = d^2(x)y+d(y)d(x)+d^2(y)x+d(x)d(y) = d(x)d(y)+d^2(x)d^0(y)+d(y)d(x)+d^2(y)d^0(x)$ satisfied Eq. 6.7 for $n = 1$ and we use the convention $d^0(x) = x$. Suppose Eq. 6.7 holds for $n \geq 1$, then for the case $n + 1$, we have

$d^{n+2}(xy) = d(d^{n+1}(xy)) = d(\sum_{i=0}^{n}(\binom{n}{i}d^{i+1}(x)d^{n-i}(y) + \binom{n}{i}d^{i+1}(y)d^{n-i}(x)))$ by the assumption.

We also have $d^{n+2}(xy) = \sum_{i=0}^{n+1}(\binom{n+1}{i}d^{i+1}(x)d^{n+1-i}(y)+\binom{n+1}{i}d^{i+1}(y)d^{n+1-i}(x))$ by Eq. 6.2.

We will prove the first half part $d(\sum_{i=0}^{n}\binom{n}{i}d^{i+1}(x)d^{n-i}(y)) = \sum_{i=0}^{n+1}\binom{n+1}{i}d^{i+1}(x)d^{n+1-i}(y)$ by Eq. 6.2, changing variables and Pascal rule $\binom{n}{i}+\binom{n}{i-1} = \binom{n+1}{i}$, and the second half part is the same process.

$d(\sum_{i=0}^{n}\binom{n}{i}d^{i+1}(x)d^{n-i}(y)) = \sum_{i=0}^{n}\binom{n}{i}d^{i+2}(x)d^{n-i}(y)+\sum_{i=0}^{n}\binom{n}{i}d^{n+1-i}(y)d^{i+1}(x) =$(by Eq. 6.2)

$\sum_{t=1}^{n+1}\binom{n}{t-1}(d^{t+1}(x)d^{n+1-t}(y))+\sum_{j=n}^{0}\binom{n}{j}(d^{j+1}(x)d^{n+1-j}(y)) =$ (let $t := n-i$ and $j := i+1$)

$\sum_{i=1}^{n+1}\binom{n}{i-1}(d^{i+1}(x)d^{n+1-i}(y)) + \sum_{i=0}^{n}\binom{n}{i}(d^{i+1}(x)d^{n+1-i}(y)) =$ (let $i = j = t$ )

$\binom{n}{n}d^{n+2}(x)d^{0}(y)+\sum_{i=1}^{n}\binom{n}{i-1}(d^{i+1}(x)d^{n+1-i}(y))+\binom{n}{0}d^{1}(x)d^{n+1}(y)+\sum_{i=1}^{n}\binom{n}{i}(d^{i+1}(x)d^{n+1-i}(y)) =$

$\binom{n+1}{n+1}d^{n+2}(x)d^{0}(y) + \sum_{i=1}^{n}(\binom{n}{i-1} + \binom{n}{i})d^{i+1}(x)d^{n+1-i}(y) + \binom{n+1}{0}d^{1}(x)d^{n+1}(y) =$

$\sum_{i=0}^{n+1}\binom{n+1}{i}d^{i+1}(x)d^{n+1-i}(y)$.

This completes the induction. $\square$

### 6.2.2 Free symmetric differential algebras

Using the same construction as for free differential algebras of weight 0, we obtain free symmetric differential algebras.

**Theorem 6.3.** *Let $X$ be a set. Let*

$$\Delta(X) = X \times \mathbb{N} = \{x^{(n)} \mid x \in X, n \geq 0\}.$$

*Let $\mathbf{k}^{NC}\{X\}$ be the free noncommutative algebra $\mathbf{k}\langle \Delta X\rangle$ on the set $\Delta X$. Define $d_X^{NC} : \mathbf{k}^{NC}\{X\} \to \mathbf{k}^{NC}\{X\}$ as follows. Let $w = u_1 \cdots u_k, u_i \in \Delta X$, $1 \leq i \leq k$, be a* *noncommutative word from the alphabet set* $\Delta(X)$*. If* $k = 1$*, so that* $w = x^{(n)} \in \Delta(X)$*, define* $d_X(w) = x^{(n+1)}$*. If* $k > 1$*, recursively define*

$$d_X(w) = d_X(u_1)u_2 \cdots u_k + d_X(u_2 \cdots u_k)u_1. \tag{6.8}$$

*Further define* $d_X(1) = 0$ *and then extend* $d_X$ *to* $\mathbf{k}\{X\}$ *by linearity. Then* $(\mathbf{k}^{NC}\{X\}, d_X^{NC})$ *is the free symmetric differential algebra on the set* $X$*.*

*Proof.* Let $(R, d)$ be a noncommutative symmetric differential algebra and let $f : X \to R$ be a set map. We extend $f$ to a symmetric differential algebra homomorphism $\bar{f} : \mathbf{k}^{NC}\{X\} \to R$ as follows.

Let $w = u_1 \cdots u_k, u_i \in \Delta X, 1 \le i \le k$, be an noncommutative word from the alphabet set $\Delta X$. If $k = 1$, then $w = x^{(n)} \in \Delta X$. Define

$$\bar{f}(w) = d^n(f(x)). \tag{6.9}$$

Note that this is the only possible definition in order for $\bar{f}$ to be a differential algebra homomorphism. If $k > 1$, recursively define

$$\bar{f}(w) = \bar{f}(u_1)\bar{f}(u_2 \cdots u_k).$$

Further define $\bar{f}(1) = 1$ and then extend $\bar{f}$ to $\mathbf{k}\{X\}$ by linearity. This is the only possible definition in order for $\bar{f}$ to be an algebra homomorphism.

Since $\mathbf{k}\{X\}$ is the free noncommutative algebra on $\Delta X$, $\bar{f}$ is an algebra homomorphism. So it remains to verify that, for all noncommutative words $w = u_1 \cdots u_k$

from the alphabet set $\Delta X$,

$$\overline{f}(d_X^{NC}(w)) = d(\overline{f}(w)), \tag{6.10}$$

for which we use induction on $k$. The case when $k = 1$ follows immediately from Eq. (6.9). For the inductive step, by Eq. (6.8):

$$\begin{aligned}\overline{f}(d_X^{NC}(w)) &= \overline{f}(d_X^{NC}(u_1)u_2\cdots u_k) + \overline{f}(d_X^{NC}(u_2\cdots u_k)u_1)\\ &= \overline{f}(d_X^{NC}(u_1))\overline{f}(u_2\cdots u_k) + \overline{f}(d_X^{NC}(u_2\cdots u_k))\overline{f}(u_1).\end{aligned}$$

Then by Eq. (6.9), the induction hypothesis on $k$ and the symmetric differential algebra relation for $d$, the last sum above equals to $d(\overline{f}(w))$:

$\overline{f}(d_X^{NC}(u_1))\overline{f}(u_2\cdots u_k)+\overline{f}(d_X^{NC}(u_2\cdots u_k))\overline{f}(u_1) = d(\overline{f}(u_1))\overline{f}(u_2\cdots u_k)+d(\overline{f}(u_2\cdots u_k))\overline{f}(u_1)$
$= d(\overline{f}(u_1)\overline{f}(u_2\cdots u_k)) = d(\overline{f}(w))$. □

## 6.3 Free symmetric differential Rota-Baxter algebras

**Theorem 6.4.** *Let $(\mathbf{k}^{NC}\{X\}, d_X^{NC}) = (\mathbf{k}\langle\Delta X\rangle, d_X^{NC})$ be the free symmetric differential algebra on a set $X$, constructed in Theorem 6.3. Let $Ш^{\mathrm{NC}}(\Delta X)$ be the free symmetric Rota-Baxter algebra on $\Delta X$, constructed in Theorem 4.14.*

(a) *There is a unique extension $\overline{d}_X^{NC}$ of $d_X^{NC}$ to $Ш^{\mathrm{NC}}(\Delta X)$ so that $(Ш^{\mathrm{NC}}(\Delta X), \overline{d}_X^{NC}, P_{\Delta X})$ is a differential symmetric Rota-Baxter algebra.*

(b) *The symmetric differential Rota-Baxter algebra $Ш^{\mathrm{NC}}(\Delta X)$ thus obtained is the free differential Rota-Baxter algebra over $X$.*

*Proof.* (a). We define a symmetric derivation $\overline{d}_X^{NC}$ on $Ш^{\mathrm{NC}}(\Delta X)$ as follows. Let $F \in \mathcal{F}$ and let $D \in (\Delta X)^F$ be the forest $F$ with angular decoration by $\vec{y} \in (\Delta X)^{\ell(F)-1}$.

Let

$$D = (F;\vec{y}) = (T_1;\vec{y}_1)y_{i_1}(T_2;\vec{y}_2)y_{i_2}\cdots y_{i_{b-1}}(T_b;\vec{y}_b)$$

be the standard decomposition of $D$ in Eq. (4.30). We define $\overline{d}_X^{NC}$ by induction on the breadth $b = b(F)$ of $F$. If $b = 1$, then $F$ is a tree so either $F = \bullet$ or $F = \lfloor \overline{F} \rfloor$ for a forest $\overline{F}$. Accordingly we define

$$\overline{d}_X^{NC}(F;\vec{y}) = \begin{cases} 0, & \text{if } F = \bullet, \\ (\overline{F};\vec{y}), & \text{if } F = \lfloor \overline{F} \rfloor \end{cases} \tag{6.11}$$

We note that this is the only way to define $\overline{d}_X^{NC}$ in order to obtain a differential symmetric Rota-Baxter algebra since $\bullet$ is the identity and $(F;\vec{y}) = \lfloor(\overline{F};\vec{y})\rfloor$.

If $b > 1$, then $F = T_1 \sqcup F_t$ for another forest $F_t = T_2 \sqcup \cdots \sqcup F_b$ (t in $F_t$ stands for the tail). So

$$D = (F;\vec{y}) = (T_1;\vec{y}_1)y_{i_1}(F_t;\vec{y}_t) = D_1 y_{i_1} D_t$$

where $D_1 = (T_1;\vec{y}_1)$ and $D_t = (T_2;\vec{y}_2)y_{i_2}\cdots y_{i_{b-1}}(T_b;\vec{y}_b)$. We then define

$$\overline{d}_X^{NC}(D) = \overline{d}_X^{NC}(T_1;\vec{y}_1)y_{i_1}(F_t;\vec{y}_t) + d_X^{NC}(y_{i_1})(F_t;\vec{y}_t)(T_1;\vec{y}_1) + \overline{d}_X^{NC}(F_t;\vec{y}_t)y_{i_1}(T_1;\vec{y}_1) \tag{6.12}$$

where $\overline{d}_X^{NC}(T_1;\vec{y}_1)$ is defined in Eq. (6.11) and $\overline{d}_X^{NC}(F_t;\vec{y}_t)$ is defined by the induction hypothesis. Note that by Eq. (4.22),

$$(T_1;\vec{y}_1)y_{i_1}(F_t;\vec{y}_t) = (T_1;\vec{y}_1)\overline{\diamond}(\bullet y_{i_1}\bullet)\overline{\diamond}(F_t;\vec{y}_t).$$

So if $\overline{d}_X^{NC}$ were to satisfy the symmetric Leibniz rule Eq. (6.2) with respect to the product $\overline{\diamond}$, then we must have

$$\begin{aligned}\overline{d}_X^{NC}(D) &= \overline{d}_X^{NC}(T_1;\vec{y}_1)\overline{\diamond}(\bullet y_{i_1}\bullet)\overline{\diamond}(F_t;\vec{y}_t) + \overline{d}_X^{NC}(\bullet y_{i_1}\bullet)\overline{\diamond}(F_t;\vec{y}_t)\overline{\diamond}(T_1;\vec{y}_1) \\ &\quad + \overline{d}_X^{NC}(F_t;\vec{y}_t)\overline{\diamond}(\bullet y_{i_1}\bullet)\overline{\diamond}(T_1;\vec{y}_1).\end{aligned} \tag{6.13}$$

Since $\overline{d}_X^{NC}$ is to extend $d_X^{NC} : \mathbf{k}^{NC}\{X\} \to \mathbf{k}^{NC}\{X\}$, we have

$$\overline{d}_X^{NC}(\bullet\, y_{i_1}\bullet) = \overline{d}_X^{NC}(j_{\Delta X}(y_{i_1})) = j_{\Delta X}(d_X^{NC}(y_{i_1})) = \bullet\, d_X^{NC}(y_{i_1})\,\bullet\,.$$

So by Eq. (4.22), Eq. (6.14) agrees with Eq. (6.12). Thus $\overline{d}_X^{NC}(D)$ is the unique map that satisfies the symmetric Leibniz rule (6.2).

We also have the short hand notation,

$$\overline{d}_X^{NC}(D) = \overline{d}_X^{NC}(D_1)y_{i_1}D_t + \overline{d}_X^{NC}(y_{i_1}D_t)D_1 \tag{6.14}$$

where

$$\overline{d}_X^{NC}(y_{i_1}D_t) := d_X^{NC}(y_{i_1})D_t + \overline{d}_X^{NC}(D_t)y_{i_1}$$

Similarly, we can also write $D = D_h y_{i_{b-1}} D_b$ where $D_h$ ($h$ stands for the head) is an angularly decorated forest and $D_b$ is an angularly decorated tree. Then

$$\overline{d}_X^{NC}(D) = \overline{d}_X^{NC}(D_h y_{i_{b-1}})D_b + \overline{d}_X^{NC}(D_b)D_h y_{i_{b-1}} \tag{6.15}$$

In fact, write

$$D = v_1 v_2 \cdots v_{2b-1},$$

where

$$v_j = \begin{cases} D_{(j+1)/2}, & j \text{ odd}, \\ y_{i_{j/2}}, & j \text{ even}. \end{cases}$$

Then using Eq. (6.12) and an induction on $b$, we obtain the "general symmetric Leibniz formula" with respect to the concatenation product:

$$\overline{d}_X^{NC}(D) = \sum_{k \subseteq [2b-1]} v'_k v_{k+1} \cdots v_{2b-1} v_{k-1} v_{k-2} \cdots v_1, \tag{6.16}$$

where $[2b-1] = \{1, \cdots, 2b-1\}$ and

$$v'_k = \begin{cases} \overline{d}_X^{NC}(v_k), k \in I, k \text{ odd}, \\ d_X^{NC}(v_k), k \in I, k \text{ even}. \end{cases}$$

We now prove that $\overline{d}_X^{NC}$ is a symmetric derivation with respect to the product $\overline{\diamond}$. Let $D$ and $D'$ be angularly decorated forests and write

$$D = (F; \vec{y}) = (T_1; \vec{y}_1) y_{i_1} (T_2; \vec{y}_2) y_{i_2} \cdots y_{i_{b-1}} (T_b; \vec{y}_b) = D_h y_{i_{b-1}} D_b$$

and

$$D' = (F'; \vec{y}') = (T'_1; \vec{y}'_1) y'_{i_1} (T'_2; \vec{y}'_2) y'_{i_2} \cdots y'_{i_{b'-1}} (T'_{b'}; \vec{y}'_{b'}) = D'_1 y'_{i_1} D'_t$$

be as above with angularly decorated trees $D_b$, $D'_1$, angularly decorated forests $D_h$, $D'_t$ and $y_{i_{b-1}}, y'_{i_1} \in \Delta X$. Then by Eq. (4.22) (see [24] for further details), $D \overline{\diamond} D'$ has the standard decomposition

$$D \overline{\diamond} D' = (T_1; \vec{y}_1) y_{i_1} \cdots y_{i_{b-1}} ((T_b; \vec{y}_b) \overline{\diamond} (T'_1; \vec{y}'_1)) y'_{i_1} \cdots y'_{i_{b'-1}} (T'_{b'}; \vec{y}'_{b'})$$

$$= \quad D_h y_{i_{b-1}}(D_b \diamond D'_1) y'_{i_1} D'_t \tag{6.17}$$

where

$$D_b \overline{\diamond} D'_1 = (T_b; \vec{y}_b)\overline{\diamond}(T'_1; \vec{y}'_1) \tag{6.18}$$

$$= \begin{cases} (\bullet; \mathbf{1}), & \text{if } T_b = T'_1 = \bullet \text{ (so } \vec{y}_b = \vec{y}'_1 = \mathbf{1}\text{)}, \\ (T_b; \vec{y}_b), & \text{if } T'_1 = \bullet, T_b \neq \bullet, \\ (T'_1; \vec{y}'_1), & \text{if } T'_1 \neq \bullet, T_b = \bullet, \\ \lfloor(\overline{F}_b; \vec{y})\overline{\diamond}(T'_1; \vec{y}')\rfloor + \lfloor(\overline{F}'_1; \vec{y}')\overline{\diamond}(T_b; \vec{y})\rfloor, & \text{if } T'_1 = \lfloor\overline{F}'_1\rfloor \neq \bullet, T_b = \lfloor\overline{F}_b\rfloor \neq \bullet. \end{cases}$$

By Eq. (6.17) and Eq. (6.16), we have

$$\begin{aligned} \overline{d}_X^{NC}(D\overline{\diamond}D') &= \overline{d}_X^{NC}((D_h y_{i_{b-1}})(D_b\overline{\diamond}D'_1)(y'_{i_1}D'_t)) \\ &= \overline{d}_X^{NC}(D_h y_{i_{b-1}})(D_b\overline{\diamond}D'_1)(y'_{i_1}D'_t) + \overline{d}_X^{NC}(D_b\overline{\diamond}D'_1)(y'_{i_1}D'_t)(D_h y_{i_{b-1}}) \qquad (6.19) \\ &\quad + \overline{d}_X^{NC}(y'_{i_1}D'_t)(D_b\overline{\diamond}D'_1)(D_h y_{i_{b-1}}) \end{aligned}$$

Using Eq. (6.18), we have

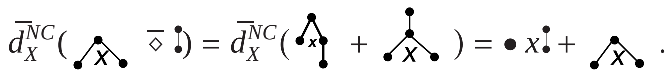
$$\overline{d}_X^{NC}(D_b\overline{\diamond}D'_1) = \overline{d}_X^{NC}(D_b)\overline{\diamond}D'_1 + \overline{d}_X^{NC}(D'_1)\overline{\diamond}D_b \tag{6.20}$$

Applying this to Eq. (6.19), we find that the resulting expansion for $\overline{d}_X^{NC}(D\overline{\diamond}D')$ agrees with the expansion of

$$\overline{d}_X^{NC}(D)\overline{\diamond}D' + \overline{d}_X^{NC}(D')\overline{\diamond}D$$

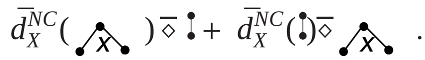
after applying Eq. (6.14) to $\overline{d}_X^{NC}(D)$ and applying Eq. (6.15) to $\overline{d}_X^{NC}(D')$.

As an example, from Eq. (4.34), we have



(6.21)

This agrees with



(b). Let $(R, *, d, P)$ be any differential symmetric Rota-Baxter algebra and let $\varphi : X \to R$ be any function. Since $Ш^{\rm NC}(\Delta X)$ is a free symmetric Rota-Baxter algebra, there exists a unique symmetric Rota-Baxter algebra homomorphism $\tilde{\varphi} : (Ш^{\rm NC}(\Delta X), P_{\Delta X}) \to (R, P)$ such that $\varphi = \tilde{\varphi} \circ j_X$. We next show $\tilde{\varphi}$ is a symmetric differential algebra map. That is to show $d(\tilde{\varphi}(D)) = \tilde{\varphi}(\bar{d}_X^{NC}(D))$ for any $D \in Ш^{\rm NC}(\Delta X)$.

**Case I:** If $D = (T; \vec{y})$, where $T$ is a tree, then $T = \bullet$ or $T = \lfloor \overline{T} \rfloor$ for a forest $\overline{T}$.

When $T = \bullet$, $d(\tilde{\varphi}(\bullet; \vec{y})) = d(1) = 0$ since $\tilde{\varphi}$ is a homomorphism. By Eq. 6.11, $\tilde{\varphi}(\bar{d}_X^{NC}(\bullet; \vec{y})) = \tilde{\varphi}(0) = 0$. So the trivial case holds.

When $T = \lfloor \overline{T} \rfloor$, by the decompostion in Eq. 4.41, the definition in Eq. 4.42 and Eq. 6.11,

$$d(\tilde{\varphi}(D)) = d(\tilde{\varphi}(T; \vec{y})) = d(P(\tilde{\varphi}(\lfloor \overline{T} \rfloor; \vec{y}))) = \tilde{\varphi}(\lfloor \overline{T} \rfloor; \vec{y}) = \tilde{\varphi}(\bar{d}_X^{NC}(T; \vec{y})). \qquad (6.22)$$

**Case II:** If $D = (F; \vec{y}) = (T_1; \vec{y_1}) y_{i_1} \cdots y_{i_{b-1}} (T_b; \vec{y_b})$, where $F$ is a forest with breadth greater than one, then by the general Leibniz equation 6.16 and Eq. 6.22, we have

$d(\tilde{\varphi}(D)) = d(\tilde{\varphi}((T_1; \vec{y_1}) y_{i_1} \cdots y_{i_{b-1}} (T_b; \vec{y_b}))) = d(\tilde{\varphi}(T_1; \vec{y_1}) * \varphi(y_{i_1}) * \cdots * \tilde{\varphi}(T_b; \vec{y_b}))$

$= d(V_1 * \cdots * V_{2b-1}) = \sum_{k=1}^{2b-1} d(V_k) * V_{k+1} * \cdots * V_{2b-1} * V_{k-1} * \cdots * V_1,$

and

$\tilde{\varphi}(\overline{d}_X^{NC}(D) = \tilde{\varphi}(\overline{d}_X^{NC}((T_1; \vec{y_1})y_{i_1} \cdots y_{i_{b-1}}(T_b; \vec{y_b})) = \tilde{\varphi}(\sum_{k=1}^{2b-1} \overline{d}_X^{NC}(V_k) \diamond V_{k+1} \cdots \diamond V_{2b-1} \diamond V_{k-1} \diamond \cdots \diamond V_1) = \sum_{k=1}^{2b-1} \tilde{\varphi}(\overline{d}_X^{NC}(V_k)) * \tilde{\varphi}(V_{k+1}) * \cdots * \tilde{\varphi}(V_{2b-1}) * \tilde{\varphi}(V_{k-1}) * \cdots * \tilde{\varphi}(V_1)$.

The two above equations agree. So we have proved that $d(\tilde{\varphi}(D)) = \tilde{\varphi}(\overline{d}_X^{NC}(D))$. This shows that $\tilde{\varphi}$ is a symmetric differential algebra homomorphism. $\square$

# Appendix

## A. The Maple codes for Section 3.2.1.

```
solve({i*m + j*w = (m + k)*i + (n + l)*m, i*n + j*p = (m + k)*j + (n + l)*n, k*m + l*w
    = (m + k)*k + (n + l)*w, k*n + l*p = (m + k)*l + (n + l)*p, a^2 + b*c = 2*a^2 + 2*b
    *u, a*b + b*d = 2*a*b + 2*b*f, a*c + d*c = 2*a*c + 2*b*g, c*b + d^2 = 2*a*d + 2
    *b*h, u^2 + f*g = 2*g*a + 2*h*u, u*f + f*h = 2*g*b + 2*h*f, g*u + h*g = 2*g*c
    + 2*h*g, g*f + h^2 = 2*g*d + 2*h^2, i^2 + j*k = 2*i^2 + 2*j*m, i*j + j*l = 2*i*j + 2
    *j*n, k*i + l*k = 2*i*k + 2*j*w, k*j + l^2 = 2*i*l + 2*j*p, m^2 + n*w = 2*w*i + 2
    *p*m, m*n + n*p = 2*w*j + 2*p*n, w*m + p*w = 2*w*k + 2*w*l, w*n + p^2 = 2*w
    *p + 2*p^2, a*u + b*g = (u + c)*a + (f + d)*u, a*f + b*h = (u + c)*b + (f + d)*f, c
    *u + d*g = (u + c)*c + (f + d)*g, c*f + d*h = (u + c)*d + (f + d)*h, b*k = u*j + k*i
    + e*m, a*j + b*l = b*i + f*j + j*k + u*l, d*k = g*j + k^2 + w*l, c*j + d*l = i*d + j*h
    + k*l + l*p, u*m + f*w = w*a + p*u + g*i + h*m, u*n + f*p = w*b + p*f + g*j + h
    *n, g*m + h*w = w*c + p*g + g*k + h*w, g*n + h*p = w*d + p*h + g*l + h*p, a*m
    + b*w = m*a + n*u + c*i + d*m, a*n + b*p = m*b + n*f + c*j + d*n, c*m + d*w = m
    *c + n*g + c*k + d*w, c*n + d*p = m*d + n*h + c*l + d*p, a*m + b*w = m*a + n*u
    + c*i + d*m, a*n + b*p = m*b + n*f + c*j + d*n, c*m + d*w = m*c + n*g + c*k + d
    *w, c*n + d*p = m*d + n*h + c*l + d*p, u*i + f*k = k*a + l*u + g*i + h*m, u*j + f*l
    = k*b + l*f + g*j + h*n, g*i + h*k = k*c + l*g + g*k + h*w, g*j + h*l = k*d + l*h + g
    *l + h*p
}, {a, b, c, d, u, f, g, h, i, j, k, l, m, n, w, p})
```

$$\{a=0, b=0, c=0, d=0, f=0, g=g, h=0, i=0, j=0, k=0, l=0, m=0, n=0, p=0, u=0, w=w\}, \left\{a=-f, b=b, c=-\frac{f^2}{b}, d=f, f=f, g=-\frac{f^3}{b^2}, h=\frac{f^2}{b}, i=0, j=0, k=0, l=0, m=0, n=0, p=0, u=-\frac{f^2}{b}, w=0\right\}, \{a=0, b=b, c=0, d=0, f=0, g=0, h=0, i=0, j=j, k=0, l=0, m=0, n=0, p=0, u=0, w=0\}, \{a=e, b=-e, c=e, d=-e, f=-e, g=e, h=-e, i=-e, j=e, k=-e, l=e, m=-e, n=e, p=e, u=e, w=-e\} \tag{1}$$

```
solve({a^2 + b*c = 2*a^2 + b*u + c*i, a*b + b*d = 2*a*b + b*f + c*j, c*a + d*c = 2*a*c
    + b*g + c*k, c*b + d^2 = 2*a*d + b*h + c*l, u^2 + f*g = g*(a + m) + (h + u)*u, u*f
    + f*h = g*(b + n) + (h + u)*f, g*u + h*g = g*(c + w) + (h + u)*g, g*f + h^2 = g*(d
    + p) + (h + u)*h,
i^2 + j*k = j*a + l*i + i^2 + j*m, i*j + j*l = j*b + (l + i)*j + j*n, k*i + l*k = j*c + (l + i)
    *k + j*w, k*j + l^2 = j*d + (l + i)*l + j*p,
 m^2 + n*w = n*u + w*i + 2*p*m, m*n + n*p = n*f + w*j + 2*p*n, w*m + p*w = n*g + w
    *k + 2*p*w, w*n + p^2 = n*h + w*l + 2*p^2
 , a*u + g*b = u*a + (a + f)*u, a*f + b*h = u*b + (a + f)*f, c*u + d*g = u*c + (a + f)
    *g, c*f + d*h = u*d + (a + f)*h,
a*i + b*k = (b + i)*a + j*u + d*i, a*j + b*l = (b + i)*b + j*f + d*j, c*i + d*k = (b + i)*c
    + j*g + d*k, c*j + d*l = (b + i)*d + j*h + d*l,
a*m + b*w = m*a + (b + n)*u + d*m, a*n + b*p = m*b + (b + n)*f + d*n, c*m + d*w = m
    *c + (b + n)*g + d*w, c*n + d*p = m*d + (b + n)*h + d*p,
u*a + f*c = (u + c)*a + d*u + g*i, u*b + f*d = (u + c)*b + d*f + g*j, g*a + h*c = (u + c)
    *c + d*g + g*k, g*b + h*d = (u + c)*d + d*h + g*l,
u*i + f*k = (f + k)*a + l*u + h*i, u*j + f*l = (f + k)*b + l*f + h*j, g*i + h*k = (f + k)*c
    + l*g + h*k, g*i + h*l = (f + k)*d + l*h + h*l,
u*m + f*w = w*a + (f + p)*u + h*m, u*n + f*p = w*b + (f + p)*f + h*n, g*m + h*w = w
    *c + (f + p)*g + h*w, g*n + h*p = w*d + (f + p)*h + h*p,
```

Figure 1: The procedure to compute symmetric Rota-Baxter operators on 2x2 matrices

$i*a+j*c=i*a+(k+a)*i+b*m, i*b+j*d=i*b+(k+a)*j+b*n, k*a+l*c=i*c+(k+a)*k+b*w, k*b+l*d=i*d+(k+a)*l+b*p,$
$i*u+j*g=i*u+u*i+(k+f)*m, i*f+j*h=i*f+u*j+(k+f)*n, k*u+l*g=i*g+u*k+(k+f)*w, k*f+l*h=i*h+u*l+(k+f)*p,$
$i*m+j*w=j*u+m*i+(l+n)*m, i*n+j*p=j*f+m*j+(l+n)*n, k*m+l*w=j*g+m*k+(l+n)*w, k*n+l*p=j*h+m*l+(l+n)*p,$
$m*a+n*c=m*a+(w+c)*i+d*m, m*b+n*d=m*b+(w+c)*j+d*n, w*a+p*c=m*c+(w+c)*k+d*w, w*b+p*d=m*d+(w+c)*l+d*p,$
$m*u+n*g=m*u+g*i+(w+h)*m, m*f+n*h=m*f+g*j+(w+h)*n, w*u+p*g=m*g+g*k+(w+h)*w, w*f+p*h=m*h+g*l+(w+h)*p,$
$m*i+n*k=n*a+(p+k)*i+l*m, m*j+n*l=n*b+(p+k)*j+l*n, w*i+p*k=n*c+(p+k)*k+l*w, w*j+p*l=n*d+(p+k)*l+l*p\}, \{a, b, c, d, u, f, g, h, i, j, k, l, m, n, w, p\})$

$$\{a=0, b=0, c=0, d=0, f=0, g=0, h=0, i=0, j=j, k=0, l=l, m=0, n=0, p=0, u=0, w=0\}, \{a=0, b=0, c=0, d=0, f=0, g=0, h=0, i=i, j=j, k=0, l=0, m=0, n=0, p=0, u=0, w=0\}, \left\{a=0, b=b, c=0, d=-\frac{b^2}{j}, f=-\frac{b^2}{j}, g=0, h=\frac{b^3}{j^2}, i=0, j=j, k=0, l=-b, m=0, n=-b, p=\frac{b^2}{j}, u=0, w=0\right\}, \left\{a=\frac{b^2}{j}, b=b, c=0, d=0, f=-\frac{b^2}{j}, g=0, h=0, i=b, j=j, k=0, l=0, m=-\frac{b^2}{j}, n=-b, p=0, u=-\frac{b^3}{j^2}, w=0\right\}, \{a=0, b=0, c=0, d=0, f=0, g=0, h=0, i=0, j=0, k=0, l=l, m=0, n=-l, p=0, u=0, w=0\}, \{a=0, b=0, c=0, d=0, f=0, g=0, h=0, i=i, j=0, k=0, l=0, m=0, n=i, p=0, u=0, w=0\}, \left\{a=0, b=-i, c=0, d=-k, f=-k, g=0, h=-\frac{k^2}{i}, i=i, j=0, k=k, l=0, m=k, n=0, p=0, u=0, w=\frac{k^2}{i}\right\}, \left\{a=k, b=b, c=0, d=0, f=-k, g=0, h=0, i=0, j=0, k=k, l=b, m=0, n=0, p=-k, u=-\frac{k^2}{b}, w=-\frac{k^2}{b}\right\}, \{a=0, b=0, c=-u, d=0, f=0, g=0, h=0, i=0, j=0, k=0, l=0, m=0, n=0, p=0, u=u, w=0\}, \{a=0, b=0, c=h, d=0, f=0, g=0, h=h, i=0, j=0, k=0, l=0, m=0, n=0, p=0, u=0, w=0\}, \{a=0, b=0, c=0, d=0, f=0, g=g, h=0, i=0, j=0, k=0, l=0, m=0, n=0, p=0, u=u, w=0\}, \{a=0, b=0, c=0, d=0, f=0, g=g, h=h, i=0, j=0, k=0, l=0, m=0, n=0, p=0, u=0, w=0\}, \{a=0, b=0, c=0, d=0, f=0, g=0, h=0, i=0, j=0, k=0, l=0, m=0, n=0, p=0, u=w, w=w\}, \{a=0, b=0, c=0, d=0, f=0, g=0, h=-w, i=0, j=0, k=0, l=0, m=0, n=0, p=0, u=0, w=w\} \tag{2}$$

Figure 2: The procedure to compute Rota-Baxter operators on 2x2 matrices

**B. The Mathematica codes[40] for Section 3.2.2.**

```
Scl[x_] := First[Flatten[x]]
Vec[x_] := Flatten[x]
```

$$c = \begin{pmatrix} \begin{pmatrix} \alpha 1 \\ \beta 1 \end{pmatrix} & \begin{pmatrix} \alpha 2 \\ \beta 2 \end{pmatrix} \\ \begin{pmatrix} \gamma 1 \\ \delta 1 \end{pmatrix} & \begin{pmatrix} \gamma 2 \\ \delta 2 \end{pmatrix} \end{pmatrix};$$

```
MakeExpression[RowBox[{x_, SubscriptBox["⊙", c_], y_}], StandardForm] :=
 MakeExpression[
  RowBox[{"CircleDot", "[", RowBox[{x, ",", c, ",", y}], "]"}], StandardForm]
MakeBoxes[CircleDot[x_, c_, y_], StandardForm] :=
 MakeBoxes[RowBox[{x, SubscriptBox["⊙", c], y}], StandardForm]
x_ ⊙c_ y_ := Vec[xᵀ.c.y]
e[i_, n_] := List /@ UnitVector[n, i]
With[{n = 2},
  Table[e[i, n] ⊙c e[j, n] - Sum[c[[i, k, j]] e[k, n], {k, 1, n}], {i, n}, {j, n}]] // Flatten
RBA[p_, c_, x_, y_] := (p.x) ⊙c (p.y) - p.(y ⊙c (p.x)) - p.(x ⊙c (p.y))
RBA[p_, c_] := With[{n = Length[c // First]},
  Flatten[Table[Expand[RBA[p, c, e[i, n], e[j, n]]],
    {i, n}, {j, n}], 2]]
RBA[c_] := With[{n = Length[c // First]}, With[{p = Array[Symbol["p"], {n, n}]},
   {RBA[p, c], p}]]
RBA[c]
FindRBA[c_] := With[{sys = RBA[c]},
  Solve[Map[# == 0 &, First[sys]], Flatten[Last[sys]]]]
FindRBA[c]
SGM[t_] := With[{n = Length[t // First]},
  Table[If[t[[i, j]] == k, 1, 0], {i, n}, {k, n}, {j, n}]]
```

Figure 3: The procedure to compute symmetric Rota-Baxter operators on semigroup algebras

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