AN ABSTRACT APPROACH TO POINTWISE DECAY
ESTIMATES FOR DISPERSIVE EQUATIONS

by

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ABSTRACT OF THE DISSERTATION

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In the study of dispersive equations one is faced with the need to quantitatively estimate the decay rate of the solution in various norms. The pointwise decay estimates are particularly useful in applications, for instance, to prove asymptotic stability of solutions or to obtain the more general $L^p$ decay estimates.

In this dissertation we develop a new abstract theory to prove pointwise decay estimates in weighted spaces, starting only from a general commutator identity that should be satisfied by the Hamiltonian $H$ and a conjugate operator $A$. We show that for Schrödinger type equations generated by an abstract $H$, the identity $[H, iA] = \theta(H) + K$ combined with Kato-smoothness of $K$ and a regularity assumption of the type $H \in C^k(A)$, are sufficient to prove pointwise decay estimates of the Kato-Jensen type.

Our results apply at energy thresholds and do not explicitly use the kernel of the (unperturbed) Hamiltonian. Consequently, they are easier to implement on manifolds. We give several examples to show that such pointwise estimates follow effortlessly from the general theory.
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Dedication

To my parents, Bernardita and Manuel
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Chapter 1

Introduction

1.1 Dispersive estimates

The study of dispersive equations in the presence of potentials or obstacles, or on curved manifolds, is of great importance in PDE and has diverse applications in spectral theory and geometry. Informally speaking, the term “dispersive” refers to the fact that different frequencies in the equation will tend to propagate at different velocities, thus dispersing the solution over time [75].

A constant-coefficient linear dispersive PDE takes the form

\[ \partial_t \psi(x, t) = L\psi(x, t), \quad \psi(x, 0) = u(x), \]

with \( \psi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) and \( L \) a differential operator of order \( k \)

\[ L\psi(x) = \sum_{|\alpha| \leq k} c_\alpha \partial_x^\alpha \psi(x). \]

Important examples are the free Schrödinger equation

\[ i\hbar \partial_t \psi + \frac{\hbar^2}{2m} \Delta \psi = 0, \quad (\text{FSE}) \]

the wave equation

\[ \Box \psi = 0, \quad \psi(x, 0) = u(x), \quad \partial_t \psi(x, 0) = u_1(x), \quad (\text{WE}) \]

where \( \Box \) is the d’Alembertian operator defined as \( \Box = -\frac{1}{c^2} \partial_t^2 + \Delta \), and the Klein-Gordon equation

\[ \Box \psi = \frac{m^2 c^2}{\hbar^2} \psi, \quad \psi(x, 0) = u(x), \quad \partial_t \psi(x, 0) = u_1(x). \quad (\text{KGE}) \]

We will be mostly concerned with the Schrödinger equation, although some of the results will also apply to other important examples of dispersive equations.
The flow $\psi = e^{-itH}$ solves the Schrödinger equation

$$i\partial_t \psi = H\psi.$$  \hspace{1cm} (SE)

$H$ is assumed to be a self-adjoint operator, so $e^{itH}$ is a bounded operator defined in the sense of the functional calculus. Typically $H$ takes the form $H = -\Delta + V$ on $\mathbb{R}^n$, where $V$ is a real-valued potential which satisfies some decay conditions at infinity. These assumptions are mainly of two types:

- **Pointwise:** $|V(x)| \leq (1 + |x|)^{-\beta}$, for some $\beta > 0$
- **Integral:** $V \in L^p(\mathbb{R}^n)$ or a weighted variant thereof

Note that in general $\|e^{-itH}u\|_2 = \|u\|_2$ if $V$ is real-valued.

For the free case $V = 0$ one can use the Fourier transform to obtain the solution to (FSE) as

$$(e^{it\Delta}) (x) = C_n t^{-n/2} \int_{\mathbb{R}^n} e^{i|x-y|^2/4t} u(y) dy.$$  

Given the explicit nature of the integration kernel, one can derive the $L^1 \to L^\infty$ dispersive estimate $\|e^{it\Delta}u\|_\infty \leq C|t|^{-n/2}\|u\|_1$. In other words, the rate of decay $|t|^{-n/2}$ is immediate from the solution. By interpolation with the $L^2$ estimate above one gets the $L^p \to L^{p'}$ estimate

$$\sup_{t \neq 0} |t|^{n(1/p-1/2)} \|e^{it\Delta}u\|_{p'} \leq C\|u\|_p,$$  

with $1 \leq p \leq 2$.

It is well known that via a $T^*T$ argument this gives rise to the class of Strichartz estimates

$$\|e^{it\Delta}u\|_{L^q_t(L^r_x)} \leq C\|u\|_2,$$  

for all $2/q + n/p = n/2$, $2 < q \leq \infty$,

which reflects a smoothing effect in terms of gain of integrability rather than regularity. We mention that the important endpoint $q = 2$ cannot be obtained by this method; it was proven by Keel and Tao [45] using a double interpolation technique which can be applied to much more general situations.

For the case $V \neq 0$ no general solution to (SE) is available and one must proceed differently. Note that a nonzero potential might imply the existence of bound states which do not decay.
over time, and thus, the above estimates do not necessarily hold for all initial data. Instead we will study the evolution $e^{-itH}P_c$, where $P_c$ is the projection onto the continuous spectrum of $H$. The dispersive behavior of the evolution operator restricted to the continuous subspace of $H$ can be observed in the remarkable result due to Ruelle, Amrein, Georgescu and Enss, which shows that for any $R > 0$,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \| \chi_{\{|x| \leq R\}} e^{itH} P_c u \|_2^2 dt = 0,$$

(\text{RAGE})

that is, the Schrödinger flow leaves any bounded region in time average. See [36, 15] for a proof and several applications of the RAGE theorem to scattering theory.

It is then a natural task to determine under what general conditions the dispersive estimate $\|e^{-itH}P_c u\|_\infty \leq C |t|^{-n/2} \|u\|_1$ holds in this context.

### 1.2 Some results on decay estimates

We now review some important results in the context of pointwise decay estimates for time-independent Hamiltonians in different dimensions. For a complete survey of recent work we refer to the papers of Schlag [68] and Schlag and Soffer [70]. Throughout this dissertation we use the notation $\langle x \rangle = (1 + |x|^2)^{1/2}$.

For $H = -\Delta + V$ in dimension $n = 3$, Rauch [56] and Jensen and Kato [39] proved estimates in weighted $L^2(\mathbb{R}^3)$ spaces of the form

$$\|w(x)e^{itH}P_c w(x)u\|_2 \leq C |t|^{-3/2} \|u\|_2,$$

with $w(x) = e^{-c(x)}$ and $V$ exponentially decaying (Rauch) or $w(x) = \langle x \rangle^{-\sigma}$ and $V$ decaying at a power rate (Jensen and Kato). This estimate holds under the assumption that zero energy is regular, that is, zero is neither an eigenvalue nor a resonance of $H$. This refers to the validity of the bound

$$\sup_{\Im z > 0} \|w(x)(H + z)^{-1}w(x)\|_2 < \infty,$$

(0-cond)

in other words, that the resolvent of $H$ can be controlled in weighted spaces as $z$ approaches the origin of the complex plane. For a weight function of type $w(x) = \langle x \rangle^{-\sigma}$, a zero resonance
is a nonzero function $f$ such that

$$Hf = 0, \quad f \in \bigcap_{\epsilon > 0} L^{2, -1/2 - \epsilon}(\mathbb{R}^3),$$

with $L^{2, \sigma} = \langle x \rangle^\sigma L^2$. In the absence of zero eigenvalue and resonance, $(0\text{-cond})$ holds with $w(x) = \langle x \rangle^{-1/2 - \epsilon}$ for all $\epsilon > 0$.

Jensen and Kato also showed that the decay rate $|t|^{-3/2}$ fails if zero is a resonance and/or an eigenvalue. In fact, it was proven that if zero is a resonance but not an eigenvalue, then

$$C^{-1} < \sup_{\|u\|_2} \sup_{t \geq 1} |t|^{1/2} \|e^{itH} P_c u\|_2 < C < \infty,$$

in other words, one loses a power of $t$ in that case. Furthermore, this effect can also occur if zero is an eigenvalue even though $P_c$ projects away the corresponding eigenfunctions.

They obtained these results through a resolvent expansion in powers of $z$ around zero and then passing to the evolution operator using the Fourier transform. More precisely, the relevance of the zero energy resonance can be seen in the expansion of the resolvent of $H$ for odd dimensions as $z \to 0$ with $\Im z > 0$

$$R(z^2) := (-\Delta + V - z^2)^{-1} = z^{-2}B_{-2} + z^{-1}B_{-1} + B_0 + zB_1 + O(z^2),$$

where the residual term is understood in the operator norm on a weighted $L^2$ space. $B_{-2}$ is the orthogonal projection onto the zero eigenspace of $H$ and $B_{-1}$ is related to both the zero eigenspace and the resonant functions. In other words, if zero energy is a regular point then $B_{-2} = B_{-1} = 0$.

From the spectral representation

$$e^{itH} P_c = \frac{1}{2\pi i} \int_0^\infty e^{ith\lambda} (R(\lambda + i0) - R(\lambda - i0)) P_c d\lambda, \quad (1.2.1)$$

Jensen and Kato observe that $B_{-1} \neq 0$ leads to a local decay of order $|t|^{-1/2}$, otherwise, the power $|t|^{-3/2}$ is obtained. It can be seen from here that one loses a power of $t$ if zero is an eigenvalue even if we project away from the corresponding eigenspace. The expression in $(1.2.1)$ also evidences that main issue here is the contribution of $\lambda = 0$ coming from the power series expansion. In fact, for energies $\lambda > \lambda_0 > 0$ one can prove arbitrarily fast decay $|t|^{-k}$ using integration by parts.
General results on local decay for the Schrödinger flow were obtained by Murata [53]. He obtained expansions for the evolution $e^{itH}$ for all dimensions and for elliptic operators $H = -p(D) + V$, where $V$ is a compact operator in suitable weighted Sobolev spaces. The coefficients of the singular powers of the resolvent expansion are finite-rank operators which can be computed as solutions of some operator equations. For operators without resonance, Murata also obtained local decay estimates in dimensions one and two of orders $|t|^{-3/2}$ and $|t|^{-1} \log^{-2} t$, resp. These faster local decay rates play a crucial role in certain application to nonlinear stability results. The key aspect here is that, in contrast with the global decay rates, the improved local decay rates are integrable in time allowing to close certain bootstrap arguments involving the Duhamel formula.

Resolvent expansions at thresholds for dimension one were treated by Bollé, Danneels and Wilk in the case $\int V \, dx \neq 0$ in [9], and by Bollé, Danneels and Gesztesy in the case $\int V \, dx = 0$ [7] with an exponential decay condition on the potential. The two dimensional case was studied by the latter authors in [8], under the additional condition $\int V \, dx \neq 0$ and with exponential decay of the potential. The case $\int V \, dx = 0$ was not worked out with this method.

Jensen and Nenciu [40] developed a unified approach to resolvent expansions in all dimensions, which are particularly interesting for the cases $n = 1, 2$. It is assumed that the potential satisfies $V(x) = O(|x|^{-2-\epsilon})$ as $x \to \infty$ and the weight function is chosen as $w(x) = |V(x)|$. They used a repeated decomposition technique, localizing the singularity in subspaces of decreasing dimension. For the case $n = 1$ they obtain the expansion

$$R(z) = (H - z)^{-1} = z^{-1/2}C_{-1} + C_0 + z^{1/2}C_1 + O(z).$$

In the one-dimensional case zero cannot be an $L^2$-eigenvalue, but there may exist a nonzero solution to $H\psi = 0$ which satisfies $\psi \in L^\infty(\mathbb{R})$. In this scenario, $C_{-1} = ic_0\langle \psi | \cdot | \psi \rangle$, where $c_0$ is a constant. The case $n = 2$ is considerably more complicated. If zero is an eigenvalue but not a resonance they show that

$$R(z) = -z^{-1}P_0 + (\ln z)^{-1}C_{0,-1} + C_{0,0} + o(1),$$

where $P_0$ is the eigenprojection for eigenvalue zero of $H$, and $C_{0,-1}$ is an operator of rank at most three. For dimension $n \geq 3$ they obtain the same expansions of [39]. The methodology
of [40] is similar to the work of Murata in [53], however a particular choice of finite rank operators in terms of projections allows them to calculate explicitly the expansion coefficients, as opposed to the implicit nature of [53]. Also, compared to [8], a different factorization of $I + |V(x)|^{1/2}(-\Delta - z)^{-1}|V(x)|^{1/2}$ allows Jensen and Nenciu to work out the two-dimensional case regardless of the value of $\int V(x) \, dx$.

The method of [40] was then applied by Erdogan and Schlag for dimension three [19, 20] and by Schlag for dimension two [67]. In the latter case, he uses the expansion of the perturbed resolvent derived by Jensen and Nenciu in the presence of resonances and eigenvalues at zero. For potentials satisfying the decay $|V(x)| \leq C(1 + |x|)^{-\beta}$, $\beta > 3$, and an assumption at zero energy for $H$, Schlag obtains

$$\| e^{itH} P_{ac}(H) u \|_{\infty} \leq C|t|^{-1} \| u \|_1 \quad \text{for all } u \in L^1(\mathbb{R}^2).$$

The first authors that addressed global estimates for $H = -\Delta + V$ in $n \geq 3$ were Journé, Soffer and Sogge [42]. For $1 \leq p \leq 2$ they prove the estimate

$$\| e^{itH} P_c u \|_p \leq C t^{-n(1/p-1/2)} \| u \|_p.$$

The high-energy part of their proof involves a bootstrapping argument which uses Duhamel’s formula and two important estimates. The first one is related to the bound $\| e^{-it\Delta} V e^{it\Delta} \|_{p \rightarrow p} \leq \| \hat{V} \|_1$, which holds uniformly in $1 \leq p \leq \infty$. This explains the origin of the condition $\hat{V} \in L^1$ in their paper. The other ingredient to treat the high-energy part is a local smoothing property of the free Schrödinger equation, for which sufficient decay of $V$ is assumed. The assumption of no zero eigenvalue nor resonance is relevant in the low-energy analysis. In this region the resolvent expansions of Jensen and Kato play a crucial role.

An unpublished argument due to Ginibre allows one to pass from local decay to the global one by means of the Duhamel formula and under suitable conditions of decay on $V$. More precisely, assume that for some $\alpha > 1$ the estimate $\| \langle x \rangle^{-\alpha} e^{itH} P_c f \|_{L^2(\mathbb{R}^n)} \leq C \langle t \rangle^{-\alpha} \| \langle x \rangle^\alpha f \|_{L^2(\mathbb{R}^n)}$ holds, then the Duhamel formula iterated twice yields the same estimate without weights in the sense

$$\| e^{itH} P_c f \|_{L^\infty + L^2(\mathbb{R}^n)} \leq C \langle t \rangle^{-\alpha} \| f \|_{L^1 \cap L^2(\mathbb{R}^n)}. $$
The inclusion of the $L^2$ norm is undesirable for nonlinear problems and it is a subtle matter to remove it.

The work in [42] was later revisited by Rodnianski and Schlag [69], who proved $L^1(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ bounds using an integrability condition on $V$. Their pointwise estimates are based on a Born series expansion of the resolvent of $H = -\Delta + V$. Iterated integrals appear in the spectral resolution and these can be controlled under the assumption of small Kato-norm

$$\|V\|_K := \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x - y|} \, dy < 4\pi.$$ 

They also need a smallness condition on $V$ in terms of the Rollnick norm

$$\|V\|_R^2 := \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|V(x)||V(y)|}{|x - y|^2} \, dy < (4\pi)^2,$$

which ensures suitable spectral properties for $H$. In [28] Goldberg and Schlag remove the smallness assumption and extend the method for potentials decaying like $|V(x)| \leq C(x)^{-\beta}$ with $\beta > 3$. In this work they study the high-energy region expanding the resolvent as a finite Born series which can be controlled using stationary phase arguments and the limiting absorption principle. This refers to estimates of the form

$$\|(-\Delta - (\lambda^2 \pm i0))^{-1} f\|_{L^2, -\sigma} \leq C(\lambda)\|f\|_{L^2, \sigma}, \quad (1.2.2)$$

with $\lambda > 0$ and $\sigma > 1/2$. The low-energy regime is analyzed via the resolvent identity and the invertibility of $S_0 := I + (-\Delta - (\lambda^2 \pm i0))^{-1} V$, which is equivalent to zero being regular.

Yajima [79] and independently, Erdogan and Schlag [19] then adapted this methodology to the case of zero being an eigenvalue and/or a resonance. They obtained, under a decay assumption on $V$, the estimate

$$\|e^{itH}P_{ac} - t^{-1/2}F_t\|_{1 \to \infty} \leq C t^{-3/2},$$

with $F_t$ satisfying $\sup_t \|F_t\|_{1 \to \infty} < \infty$ and $\lim \sup_{t \to \infty} \|F_t\|_{1 \to \infty} > 0$.

Goldberg in [26] refines the method of [28] and shows that a decay estimate of order $|t|^{-3/2}$ follows for potentials $V \in L^{3/2+\epsilon}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. He requires an analog to (1.2.2) on $L^p(\mathbb{R}^3)$ spaces rather than weighted $L^2$ spaces, which is related to the Stein-Tomas theorem in Fourier analysis. This idea, combined with the result of Ionescu and Jerison [38] on absence of imbedded eigenvalues for $L^p$ potentials, was used by Goldberg and Schlag in [29]. Under the usual
restriction on zero energy, Goldberg proved in [27] that even $V \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ with $p < 3/2 < q$ suffices for a dispersive estimate. This class includes potentials of decay $|V(x)| \leq C(1 + |x|)^{-2-\epsilon}$ which are nearly critical with respect to the natural scaling of the Laplacian. No additional regularity of $V$ is assumed in this case. In contrast, Goldberg and Visan [30] show that for $n \geq 4$ the global $L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ estimate fails unless $V$ has some amount of regularity.

An alternative approach to decay estimates was proposed by Yajima [79], who proved $L^p$ boundedness of wave operators for $1 \leq p \leq \infty$, with the limit taken in the strong $L^2$-sense

$$W = \lim_{t \to \infty} e^{-itH} e^{-it\Delta}$$

for $n \geq 3$. The merit of the wave operators is that they intertwine the free evolution $H_0 = -\Delta$ and $H$ in the sense that

$$f(H)P_c = W f(H_0) W^*$$

for any Borel function on $\mathbb{R}$. In particular, $e^{itH}P_c = W e^{itH_0} W^*$ which yields the estimate

$$\|e^{itH}P_c u\|_\infty \leq C|t|^{-n/2} \|u\|_1$$

whenever $W : L^\infty \to L^\infty$ and $W^* : L^1 \to L^1$. Analogous results were also obtained for the wave and Klein-Gordon equations.

More recently, Beceanu and Goldberg [4] proved that for potentials in the closure of $C_c(\mathbb{R}^3)$ with respect to the Kato-norm, the following dispersive estimate holds

$$\|e^{-itH}(I - P_{pp}(H))u\| \leq C|t|^{-3/2}\|u\|_1,$$

under the assumption of no resonances nor eigenvalues of $H$ in $[0, \infty)$. The proof relies on a broad extension of Wiener’s $L^1$-inversion theorem to operator-valued functions, first observed in [3].

A common factor in most of the methodologies presented in this introduction, is the explicit use of the kernel of the unperturbed Hamiltonian. These techniques are less amenable to problems on manifolds, and therefore, most results in this case are of a different nature. See for example [64, 5, 6, 16, 76] for some results of the local decay type and Strichartz estimates. In the context of manifolds, the pointwise decay estimates are not known or not optimal. Hence, there seems to be a need to develop an abstract method to derive pointwise decay estimates without relying
on resolvent expansions. The methodology developed in Chapter 2 intends to be applicable to a large class of problems and it is based on the method of the conjugate operator. This framework has a long history and will be discussed in the next section.

1.3 Commutators and regularity of operators

In spectral analysis, one of the most powerful tools is the method of the conjugate operator. In general terms, it states that in order to study the spectral properties of an operator $H$, it is advantageous to find an auxiliary operator $A$ such that the commutator $[H, iA]$ satisfies suitable properties. This idea can be traced back to the work of Putnam [55] and Kato [43], however at this stage the applications were rather restricted by the boundedness condition on $A$ and the global positivity requirement on the commutator. Later in the eighties, Mourre developed an abstract theory based on the conjugate operator method which he applied to 2- and 3-body Schrödinger operators [52]. In this work, he extended the framework to unbounded self-adjoint operators $A$ and localized the positivity condition to an interval $J$ in the spectrum of $H$. More precisely, he established the so-called Mourre estimate

$$E(J)[H, iA]E(J) \geq aE(J) + E(J)KE(J),$$

where $a > 0$, $E(J)$ is the spectral projection of $H$ and $K$ is a compact operator. The main result of [52] is the finiteness of the point spectrum of $H$ in the interval $J$ and the good control of the behavior of the resolvent $R(\lambda \pm i\mu) = (H - \lambda \mp i\mu)^{-1}$, where $\lambda \in J$ and $\mu \to 0$. To prove this result, he combines the Mourre estimate with a version of the virial theorem, that is, the fact that under suitable conditions on $H$ and $A$, the commutator $[H, iA]$ vanishes on eigenvectors of $H$ in the sense $\text{1}_\{\lambda\}(H)[H, iA]\text{1}_\{\lambda\}(H) = 0$. Since $H$ and $A$ are unbounded operators acting on a Hilbert space $\mathcal{H}$, the precise meaning of this identity is crucial, in fact, the commutator is defined a priori only as a quadratic form on $D(H) \cap D(A)$. We now mention three different types of assumptions that have been used in the literature to prove the virial theorem in an abstract setting. First, the assumptions of Mourre:

\(M\)

(i) $D(H) \cap D(A)$ is dense in $D(H)$

(ii) $e^{isA}$ preserves $D(H)$ and for each $u \in D(H)$ \(\sup_{|s| \leq 1} \|He^{isA}u\| < \infty\)
(iii) the quadratic form $[H, iA]$ on $D(H) \cap D(A)$ is bounded below, closeable and it extends to a bounded operator from $D(H)$ to $\mathcal{H}$.

The simplified assumptions of Mourre:

$$(M') \quad (i) \quad e^{isA} \text{ preserves } D(H)$$

$$(ii) \quad |(Hu, Au) - (Au, Hu)| \leq C(\|Hu\|^2 + \|u\|^2), \quad u \in D(H) \cap D(A).$$

And the $C^1(A)$ condition:

$$(ABG) \quad \text{there exists } z \text{ in the resolvent of } H \text{ such that the map } s \mapsto e^{isA}(z - H)^{-1}e^{-isA} \text{ is } C^1$$

for the strong topology of $B(\mathcal{H})$.

We refer to the book of Amrein, Boutet de Monvel and Georgescu [1] for a detailed exposition of Mourre’s method and to the paper of Georgescu and Gérard [22] for further discussion on these conditions.

The method of the conjugate operator has numerous applications and extensions; we mention here some of them. Perry, Sigal and Simon extended Mourre’s method to study the spectral properties of $N$-body Schrödinger operators [54]. They analyzed energy thresholds and showed the absence of singular continuous spectrum of $H$. Sigal and Soffer in [72] derived the so-called propagation estimates, which are based on the fact that $[H, iA]$ is the time derivative of $t \mapsto e^{itH}Ae^{-itH}$ at $t = 0$. These estimates allow to develop in a very natural way the scattering theory of perturbed Hamiltonians. Finally, we cite the work of Boutet de Monvel, Kazantseva and Mântoiu who initiated in [13] the so-called weakly conjugate operator method where the Mourre’s estimate is replaced by the condition $[H, iA] > 0$.

The abstract method developed in Chapter 2 and 3 of this dissertation relies on a commutator identity which is assumed to hold for the Hamiltonian $H$. We show that the $C^1(A)$ regularity is an appropriate framework to perform algebraic manipulations of commutators and then derive different pointwise decay estimates for the Schrödinger evolution.
Chapter 2

The abstract method\(^1\)

2.1 Introduction

The work of Mourre discussed in Section 1.3 and the later work in [73, 37, 25] imply decay estimates starting only from the Mourre estimate. However, this method does not apply at energy thresholds. One way around this problem is the Morawetz type estimates. They apply at thresholds, but limited to nontrapping potentials. The extension to repulsive potentials and low dimension was established as well in some cases. Mourre’s method was extended in many works to include thresholds [12, 48, 51, 49, 50, 11, 32, 33, 59, 60, 61, 62, 78, 66, 10, 74]. However these methods so far could not be versatile enough to include many common systems, mainly due to complicated assumptions or the use of abstract weighted spaces; they only imply local decay estimates.

In this chapter we develop a new abstract theory to prove pointwise decay estimates in weighted spaces, starting only from a general commutator identity that should be satisfied by the Hamiltonian. We show that for Schrödinger type equations generated by an abstract Hamiltonian \(H\), as well as Klein Gordon and wave equations, the identity \(i[H, A] = \theta(H)\) combined with the assumption of \(C^1(A)\) regularity of \(H\), are sufficient to prove pointwise decay estimates of the Kato-Jensen type. We then apply this theoretical framework to several examples. In Chapter 3 we extend the abstract method to deal with more general Hamiltonians.

\(^1\)V. Georgescu, M. L. and A. Soffer: Abstract theory of pointwise decay with applications to wave and Schrödinger equations. Submitted.
2.2 Evanescent states

Let $H$ be a self-adjoint operator on the Hilbert space $\mathcal{H}$ with spectral measure $E$. If $u \in \mathcal{H}$ let $E_u$ be the measure $E_u(J) = \|E(J)u\|^2$ and $\psi_u : \mathbb{R} \to \mathbb{C}$ the function $\psi_u(t) = \langle u | e^{itH}u \rangle = \int_{\mathbb{R}} e^{it\lambda} E_u(d\lambda)$. We are interested in vectors $u$ such that $\psi_u(t) \to 0$ as $t \to \infty$ and in the rapidity of this decay.

Note that $|\psi_u(t)|^2$ is a physically meaningful quantity if we think of $H$ as the Hamiltonian of a system whose state space is $\mathcal{H}$. Indeed, if $u, v$ are vectors of norm one then $|\langle v | e^{itH}u \rangle|^2$ is the probability of finding the system in the state $v$ at moment $t$ if the initial state is $u$, hence $|\psi_u(t)|^2$ is the probability that at moment $t$ the system be in the same state $u$ as at moment $t = 0$.

**Remark 2.2.1.** In this paper we are interested in the decay properties of the functions $\psi_u$ for $u$ in the absolute continuity subspace $\mathcal{H}_{ac}^H$ of $\mathcal{H}$ relatively to $H$. We shall see that $\psi_u \in L^2(\mathbb{R})$ for $u$ in a dense subspace of $\mathcal{H}_{ac}^H$ but in rather simple cases it may happen that $\psi_u \in L^1(\mathbb{R})$ only for $u = 0$. Formally speaking, the physically interesting quantity $|\psi_u(t)|^2$ generically decays more rapidly than $\langle t \rangle^{-1}$ but not as rapidly as $\langle t \rangle^{-2}$. Our results concern mainly the regularity of this decay, for example we give conditions such that $|\psi_u(t)|^2$ is really dominated by $\langle t \rangle^{-1}$.

Since $\psi_u$ is (modulo a constant factor) the Fourier transform of $E_u$, there is a strong relation between the decay of $\psi_u$ and the smoothness of $E_u$. If $u$ is absolutely continuous with respect to $H$ then $\psi_u \in C_0(\mathbb{R})$ (space of continuous functions which tend to zero at infinity). However, the decay may be quite slow if $E_u$ is not regular enough.

**Example 2.2.2.** Let $\Lambda$ be a real compact set with empty interior and strictly positive Lebesgue measure and let $H$ be the operator of multiplication by $x$ in $\mathcal{H} = L^2(\Lambda, dx)$. Then the spectrum of $H$ is purely absolutely continuous but $\psi_u \notin L^1(\mathbb{R})$ for all $u \in \mathcal{H} \setminus \{0\}$. Indeed, if $u \in \mathcal{H}$ and we extend it by zero outside $\Lambda$ then $\psi_u(t) = \int e^{itx} |u(x)|^2 dx$ hence if $\psi_u$ is integrable then $|u|^2$ is the Fourier transform of an integrable function, so it is continuous, so the set where $|u(x)|^2 \neq 0$ is open and contained in $\Lambda$, hence it is empty.

On the other hand, if $H$ has an absolutely continuous component then there are plenty of $u$
such that \( \psi_u \in L^2(\mathbb{R}) \): indeed, observe that \( \psi_u \in L^2(\mathbb{R}) \) if and only if \( E_u \) is an absolutely continuous measure with derivative \( E'_u \in L^2(\mathbb{R}) \) and then \( \| \psi_u \|_{L^2} = \sqrt{2\pi } \| E'_u \|_{L^2} \).

More generally, if we denote \( E_{v,u} \) the complex measure \( E_{v,u}(J) = \langle v | E(J) u \rangle \) then by the spectral theorem \( \langle v | e^{itH} u \rangle = \int e^{it\lambda} E_{v,u}(d\lambda) \), hence the left hand side belongs to \( L^2(\mathbb{R}) \) if and only if the measure \( E_{v,u} \) is absolutely continuous and has square integrable derivative \( E'_{v,u} \) and then we have

\[
\int_{\mathbb{R}} |\langle v | e^{itH} u \rangle|^2 dt = 2\pi \int |E'_{v,u}(\lambda)|^2 d\lambda.
\]

It is easy to prove the inequality \( |E'_{v,u}(\lambda)|^2 \leq E'_v(\lambda) E'_u(\lambda) \) or \( [u]_H \) is a complete norm on it. We mention that the relation \( |E'_{v,u}(\lambda)|^2 \leq E'_v(\lambda) E'_u(\lambda) \) also implies

\[
\left( \int_{\mathbb{R}} |\langle v | e^{itH} u \rangle|^2 dt \right)^{1/2} \leq [v]_H [u]_H.
\]

Lemma 2.2.3. If \( J \in B(\mathcal{H}) \) commutes with \( H \) then \( J \mathcal{E} \subset \mathcal{E} \) and \( [Ju]_H \leq \|J\| [u]_H \). If \( J_n = \theta_n(H) \) with \( \{\theta_n\} \) a uniformly bounded sequence of Borel functions such that \( \lim_n \theta_n(\lambda) = 1 \) for all \( \lambda \in \mathbb{R} \), then for any \( u \in \mathcal{E} \) we have \( \lim_n [Ju]_H = [u]_H \).

Proof. For the first part we use \( E'_Ju(\lambda) \leq \|J\|^2 E'_u(\lambda) \) (which is obvious) while for the second part \( E'_{\theta_n(H)u}(\lambda) = \theta^2_n(\lambda) E'(\lambda) \) and the dominated convergence theorem.

The quantity \( \int_{\mathbb{R}} |\psi_u(t)|^2 dt \) has a simple physical interpretation in the quantum setting: if \( u, v \) are two state vectors then \( \int_{\mathbb{R}} |\langle v | e^{itH} u \rangle|^2 dt \) is the total time spent by the system in the state \( v \) if the initial state is \( u \). Hence we may say that \( \int_{\mathbb{R}} |\langle u | e^{itH} u \rangle|^2 dt \) is the lifetime of the state \( u \). The elements of \( \mathcal{E}(H) \) are those of finite lifetime, or states in which the system spends a finite total time. We might call them self evanescent states, and they are absolutely continuous with respect to \( H \). Note that there is a Schrödinger Hamiltonian \( H \) and there is a state \( u \) which is singularly continuous with respect to \( H \) such that \( \psi_u(t) = O(|t|^{-1/2+\varepsilon}) \) for any \( \varepsilon > 0 \) [71].
Another interesting class $\mathcal{E}_\infty \equiv \mathcal{E}_\infty(H)$ is that of evanescent states defined by the condition

$$\int_\mathbb{R} |\langle v | e^{itH} u \rangle|^2 dt < \infty$$

for any $v$: such a state $u$ spends a finite time in any state $v$. The evanescent states disappear (or go to infinity) in a natural quantum mechanical sense, which explains the fundamental role they play in the Rosenblum Lemma [65] and later on in the Birman-Kato trace class scattering theory. A simple argument shows that $\mathcal{E}_\infty$ is the linear subspace of $\mathcal{E}$ consisting of vectors $u$ such that $E^*_u$ is a bounded function. In particular, $\mathcal{E}_\infty$ is dense in the absolutely continuity subspace associated to $H$.

**Example 2.2.4.** If $H = q = \text{operator of multiplication by } x \text{ in } L^2(\mathbb{R}, dx)$ then $\mathcal{E}(q) = L^1(\mathbb{R})$ and $\mathcal{E}_\infty(q) = L^\infty(\mathbb{R})$. Indeed, $\langle u | e^{itq} u \rangle = \int_\mathbb{R} e^{itx} |u(x)|^2 dx$ is an $L^2$ function of $t$ if and only if $|u|^2 \in L^2$ and then $|u|_q = (2\pi)^{1/4} \|u\|_{L^4}$. On the other hand, $\langle v | e^{itq} u \rangle = \int_\mathbb{R} e^{itx} \overline{v(x)} u(x) dx$ is an $L^2$ function of $t$ for any $v \in L^2$ if and only if $\bar{v} u \in L^2$ for any $v \in L^2$ hence if and only if $u \in L^\infty$.

### 2.3 Notes on commutators

Let $A$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$. If $S$ is a bounded operator on $\mathcal{H}$ then we denote $[A, S]_0$ the sesquilinear form on $D(A)$ defined by $[A, S]_0(u, v) = \langle Au | Sv \rangle - \langle u | SA v \rangle$.

As usual, we set $[S, A]_0 = -[A, S]_0$, $[S, iA]_0 = i[S, A]_0$, etc. We say that $S$ is of class $C^1(A)$, and we write $S \in C^1(A)$, if $[A, S]_0$ is continuous for the topology induced by $\mathcal{H}$ on $D(A)$ and then we denote $[A, S]$ the unique bounded operator on $\mathcal{H}$ such that $\langle u | [A, S] v \rangle = \langle Au | Sv \rangle - \langle u | SA v \rangle$ for all $u, v \in D(A)$. It is easy to show that $S \in C^1(A)$ if and only if $SD(A) \subset D(A)$ and the operator $SA - AS$ with domain $D(A)$ extends to a bounded operator $[A, S] \in B(\mathcal{H})$. Moreover, $S$ is of class $C^1(A)$ if and only if the following equivalent conditions are satisfied

1. the function $t \mapsto e^{-itA} S e^{itA}$ is Lipschitz in the norm operator topology
2. the function $t \mapsto e^{-itA} S e^{itA}$ is of class $C^1$ in the strong operator topology

and then we have $[S, iA] = \frac{d}{dt} e^{-itA} S e^{itA}|_{t=0}$.

Clearly $C^1(A)$ is a $*$-subalgebra of $B(\mathcal{H})$ and the usual commutator rules hold true: for any $S, T \in C^1(A)$ we have $[A, S]^* = -[A, S^*]$ and $[A, ST] = [A, S]T + S[A, T]$, and if $S$ is
bijective then $S^{-1} \in C^1(A)$ and $[A, S^{-1}] = -S^{-1}[A, S]S^{-1}$.

We often abbreviate $S' = [S, iA]$ if the operator $A$ is obvious from the context. Then we may write $(S')^* = (S^*)', (ST)' = S'T + ST'$, and $(S^{-1})' = -S^{-1}S'S^{-1}$.

We consider now the rather subtle case of unbounded operators. Note that we always equip the domain of an operator with its graph topology. If $H$ is a self-adjoint operator on $\mathcal{H}$ then $[A, H]_o$ is the sesquilinear form on $D(A) \cap D(H)$ defined by $[A, H]_o(u, v) = \langle Au|Hv \rangle - \langle Hu|Av \rangle$.

By analogy with the bounded operator case, one would expect that requiring denseness of $D(A) \cap D(H)$ in $D(H)$ and continuity of $[A, H]_o$ for the graph topology of $D(H)$ would give a good $C^1(A)$ notion. For example, this should imply the validity of the virial theorem, nice functions of $H$ (at least the resolvent) should also be of class $C^1$, etc. However this is not true, as the following example from [22] shows.

**Example 2.3.1.** In $\mathcal{H} = L^2(\mathbb{R}, dx)$ let $q = \text{operator of multiplication by } x$ and $p = -i \frac{d}{dx}$. Let $A = e^{i\omega p} - p$ and $H = e^{i\omega q}$ with $\omega = \sqrt{2\pi}$. This value of $\omega$ is chosen because $e^{i\omega p}e^{i\omega q} = e^{i\omega q}e^{i\omega p}$ on a very large set although the operators $e^{i\omega p}$ and $e^{i\omega q}$ do not commute. Then $D(A) \cap D(H)$ is dense in both $D(A)$ and $D(H)$ (moreover, $D(H) \cap H(A)$ is dense in $D(H)$), one has $[H, iA]_o = \omega H$ on $D(A) \cap D(H)$, but $(H + i)^{-1} \notin C^1(A)$.

A convenient definition of the $C^1(A)$ class for any self-adjoint operator is as follows. Let $R(z) = (H - z)^{-1}$ for $z$ in the resolvent set $\rho(H)$ of $H$. We say that $H$ is of class $C^1(A)$ if $R(z) \in C^1(A)$ for some (hence for all) $z \in \rho(H)$. In this case the space $R(z)D(A)$ is independent of $z \in \rho(H)$, it is a core of $H$, and is a dense subspace of $D(A) \cap D(H)$ for the intersection topology, i.e. for the norm $\|u\| + \|Au\| + \|Hu\|$. Moreover:

**Proposition 2.3.2.** Let $A, H$ be self-adjoint operators on a Hilbert space $\mathcal{H}$.

1. $H$ is of class $C^1(A)$ if and only if the next two conditions are satisfied:
   
   (a) $[A, H]_o$ is continuous for the topology induced by $D(H)$ on $D(A) \cap D(H)$,

   (b) there is $z \in \rho(H)$ such that $\{u \in D(A) \mid R(z)u \in D(A)\}$ is a core for $A$.

2. If $H \in C^1(A)$ then $D(A) \cap D(H)$ is dense in $D(H)$ hence $[A, H]_o$ extends to a uniquely determined continuous sesquilinear form $[A, H]$ on $D(H)$. We have:

$$[A, R(z)] = -R(z)[A, H]R(z) \quad \forall z \in \rho(H). \quad (2.3.2)$$
This is Theorem 6.2.10 in [1]. The condition (a) above is quite easy to check in general but not condition (b) because it involves a certain knowledge of the resolvent of $H$, which is a complicated object. We now describe criteria which allow one to avoid this problem.

We denote $H^1 = D(H)$ (equipped with the graph topology) and $H^{-1} = D(H)^*$ its adjoint space. The identification of the adjoint space $H^*$ of $H$ with itself via the Riesz Lemma gives us a scale $H^1 \subset H \subset H^{-1}$ with continuous and dense embeddings. If we define $H^s := [H^1, H^{-1}]_{(1-s)/2}$ for $-1 \leq s \leq 1$ by complex interpolation then $(H^s)^* = H^{-s}$ for any $s$ and $H^{1/2}$ is just $D(|H|^{1/2})$. Finally, we have continuous and dense embeddings

$$H^1 \subset H^{1/2} \subset H \subset H^{-1/2} \subset H^{-1}.$$  

If $H \in C^1(A)$ the continuous sesquilinear form $[A, H]$ on $D(H)$ is then identified with a linear continuous operator $H^1 \rightarrow H^{-1}$ and this is useful for example because it gives a simple interpretation to supplementary conditions like $[A, H]H^1 \subset H$. Observe that

$$H' := [H, iA] : H^1 \rightarrow H^{-1}$$

is a continuous symmetric operator. Now the following assertions are consequences of [1, Theorem 6.3.4, Lemma 7.5.3] and [22, Lemma 2].

1. If $e^{itA}H^1 \subset H^1 (\forall t)$ then $H \in C^1(A)$ if and only if condition (a) in part (1) of Proposition 2.3.2 is satisfied.

2. If $H \in C^1(A)$ and $H'\mathcal{H}^1 \subset \mathcal{H}$ then $e^{itA}H^1 \subset H^1 (\forall t)$ and the restrictions $e^{itA}|H^1$ give a strongly continuous group of operators on the Hilbert space $H^1$.

3. If $e^{itA}H^1 \subset H^1 (\forall t)$ then $D(A, H^1) = \{ u \in H^1 \cap D(A) \mid Au \in H^1 \}$ is a dense subspace of $H^1$ and $H$ is of class $C^1(A)$ with $H'\mathcal{H}^1 \subset \mathcal{H}$ if and only if $|\langle Au|Hv \rangle – \langle H u| Av \rangle| \leq C\| u \|_{H^1} \| v \|_{H^1}$ for all $u, v \in D(A, H^1)$.

4. Assume $e^{itA}H^{1/2} \subset H^{1/2} (\forall t)$. Then $D(A, H^{1/2}) := \{ u \in H^{1/2} \cap D(A) \mid Au \in H^{1/2} \}$ is dense in $H^{1/2}$ and if the quadratic form $\langle H u|Au \rangle – \langle Au|Hu \rangle$ on $D(A, H^{1/2})$ is continuous for the topology induced by $H^{1/2}$ then $H \in C^1(A)$. 

We mention that Hypotheses 1, 2’ and 3 on page 62 of [15] imply that \( H \) is of class \( C^1(A) \), cf. relation (4.10) there.

We now give some “pathological” examples which clarify the notion of \( C^1 \) regularity.

**Example 2.3.3.** Let \( \mathcal{H} = L^2(\mathbb{R}) \) and \( A = p \). It is clear that the operator of multiplication by a rational real function is of class \( C^1(p) \), in fact of class \( C^\infty(p) \) in a natural sense. For example, if \( H = q^{-m} \) then \((H + i)^{-1} = q^m (1 + iq^m)^{-1} \) is clearly a bounded operator of class \( C^1(p) \) if \( m \in \mathbb{N} \) and \([q^{-m}, ip] = mq^{-m-1} \) as continuous forms on \( D(q^{-m}) \). The worst case is attained when \( m = 1 \): then \( H' = H^2 \) hence \( H' + i : \mathcal{H}^1 \to \mathcal{H}^{-1} \) is an isomorphism, in particular \( H^* \mathcal{H}^1 \) is not included in any of the smaller spaces \( \mathcal{H}^s \) with \( s > -1 \). If \( m \geq 1 \) is an odd integer then \( H \) is of class \( C^1(A) \) and \( H' = mH^{1+1/m} \) where \( x^{1/m} := -|x|^{1/m} \) if \( x < 0 \); now we have \( H^* \mathcal{H}^1 \subset \mathcal{H}^{-1/m} \) and this is optimal.

**Remark 2.3.4.** Example 2.3.3 shows that if \( H \in C^1(A) \) then neither \( e^{itA} \) nor \( (A+i\lambda)^{-1} \) leave invariant \( D(H) \) in general.

If \( H \in C^1(A) \) then \( D(A) \cap D(H) \) is dense in \( D(H) \) but is not dense in \( D(A) \) in general.

**Example 2.3.5.** Let \( H = q^{-m} \) with \( m \geq 1 \) and \( A = p \) as in Example 2.3.3. Then \( D(A) \) is the Sobolev space consisting of functions \( u \in L^2(\mathbb{R}) \) with derivative \( u' \in L^2(\mathbb{R}) \), so we have \( D(A) \subset C_0(\mathbb{R}) \) continuously. Thus if \( u \in D(A) \cap D(H) \) then \( u \) is a continuous function such that \( \int |u(x)|^2 x^{-2m} dx < \infty \) which implies \( u(0) = 0 \). But \( \{ u \in D(A) \mid u(0) = 0 \} \) is a closed hyperplane of codimension one in the Hilbert space \( D(A) \).

By taking \( m \) large in the preceding example we see that for any \( \varepsilon > 0 \) there is a self-adjoint operator \( H \) of class \( C^1(A) \) with \( H^* \mathcal{H}^1 \subset \mathcal{H}^{-\varepsilon} \) such that \( D(A) \cap D(H) \) is not dense in \( D(A) \). Thus the next result is optimal.

**Proposition 2.3.6.** If \( H \in C^1(A) \) and \( H^* \mathcal{H}^1 \subset \mathcal{H} \) then \( D(A) \cap D(H) \) is dense in \( D(A) \). More precisely, if we set \( R_\varepsilon = (1 + i\varepsilon H)^{-1} \) for \( \varepsilon > 0 \) then \( R_\varepsilon D(A) \subset D(A) \cap D(H) \) and \( \lim_{\varepsilon \to 0} R_\varepsilon = 1 \) in the Hilbert space \( D(A) \).

**Proof.** We have \( R_\varepsilon D(A) \subset D(A) \cap D(H) \) and \([A, R_\varepsilon] = \varepsilon R_\varepsilon H^* R_\varepsilon \) by Proposition 2.3.2. Then \( \varepsilon \|H^* R_\varepsilon\| \leq \|H^* R_1\| \| (\varepsilon + iH) R_\varepsilon \| \leq C \| H^* R_\varepsilon = \varepsilon H^* R_\varepsilon \| \) if \( u \in D(H) \). Thus \( \lim_{\varepsilon \to 0} \lim_{\varepsilon \to 0} [A, R_\varepsilon] = 0 \) hence \( AR_\varepsilon u \to Au \) for any \( u \in D(A) \).
This $C^1(A)$ property transfers from $H$ to some functions of $H$: for example, it is easy to prove that $\phi(H) \in C^1(A)$ if $\phi \in C^2(\mathbb{R})$ and $|\phi(\lambda)| + |\phi'(\lambda)| + |\phi''(\lambda)| \leq C(\lambda)^{-2}$. But obviously $e^{iH} \notin C^1(A)$ in general.

**Theorem 2.3.7.** Let $H$ be a self-adjoint operator of class $C^1(A)$ and $t \in \mathbb{R}$. Then the restriction of $[A, e^{itH}]_0$ to $D(A) \cap D(H)$ extends to a continuous form $[A, e^{itH}]$ on $D(H)$ and, in the strong topology of the space of sesquilinear forms on $D(H)$, we have:

$$
[A, e^{itH}, A] = \int_0^t e^{i(t-s)H} H' e^{isH} \, ds. \tag{2.3.3}
$$

**Proof.** Clearly it suffices to assume $t = 1$. For $n \geq 1$ integer let $R_n = (1 - iH/n)^{-1}$. Then $R_n$ has norm $\leq 1$ and $e^{iH} = s\lim_{n \to \infty} R_n^0$ in both spaces $\mathcal{H}$ and $D(H)$. Since $H$ is of class $C^1(A)$ we have $R_n \in C^1(A)$ and $[A, R_n] = \frac{i}{n} R_n[A, H] R_n$, so $R_n^0 \in C^1(A)$ and

$$
[A, R_n^0] = \sum_{k=0}^{n-1} R_n^k [A, R_n] R_n^{n-1-k} = \frac{i}{n} \sum_{k=1}^n R_n^k[A, H] R_n^{n+1-k}.
$$

It is clear that $\langle u|[A, R_n^0]v \rangle \to \langle u|[A, e^{iH}]_0 v \rangle$ as $n \to \infty$ for all $u, v \in D(A)$. Thus it remains to be shown that for all $u, v \in D(H)$:

$$
\frac{1}{n} \sum_{k=1}^n \langle R_n^k u|[A, H] R_n^{n+1-k} v \rangle \to \int_0^1 (e^{-isH} u|[A, H] e^{i(1-s)H} v) \, ds. \tag{2.3.4}
$$

We have

$$
\left\| \sum_{k=1}^n R_n^k[A, H] R_n^{n+1-k} \right\|_{\mathcal{H}^1 \to \mathcal{H}^1} \leq n \left\| [A, H] \right\|_{\mathcal{H}^1 \to \mathcal{H}^1},
$$

hence it suffices to prove that (2.3.4) holds for $u, v$ in a dense subspace of $D(H)$. So we may assume that $u, v$ have compact support with respect to $H$.

Let $a$ be a number such that $|\log(1 + z) - z| \leq a|z|^2$ if $z \in \mathbb{C}$ and $|z| < 1/2$. If $\phi_n(x) = (1 - i x/n)^{-1}$ then for $x$ in a real compact set, $1 \leq k \leq n$, and $n$ large, we have

$$
|\phi_n(x)^k - e^{i \frac{ka}{n}}| = |e^{k \log(1 - i \frac{x}{n}) + i \frac{ka}{n}} - 1| \leq Ck|\log(1 - i x/n) + i x/n| \leq Cka|x/n|^2
$$

where $C$ is a number depending only on the set where varies $x$. Thus the last term above is an $O(x^2/n)$ and so we get

$$
\left\| R_n^k u - e^{-i \frac{k}{n} H} u \right\|_{D(H)} = O\left(\frac{1}{n}\right).
$$
A similar argument gives
\[ \| R_n^{n+1-k} v - e^{it \frac{n+1-k}{n} H} V \|_{D(H)} = O \left( \frac{1}{n} \right). \]
Hence
\[ \frac{1}{n} \sum_{k=1}^{n} \langle R_n^k u | [A, H] R_n^{n+1-k} v \rangle = \frac{1}{n} \sum_{k=1}^{n} \langle e^{-i \frac{k}{n} H} u | [A, H] e^{-i \frac{k}{n} H} e^{it \frac{n+1}{n} H} v \rangle + O \left( \frac{1}{n} \right). \]
Finally, we have \( e^{i \frac{n+1}{n} H} v \rightarrow e^{i H} v \) in \( D(H) \) and the \( D(H) \)-valued functions \( s \mapsto e^{-i s H} u \) and \( s \mapsto e^{-i s H} v \) are continuous. This proves (2.3.3).

The relation (2.3.3) also holds in \( B(D(H), D(H)^*) \) in the strong topology and then one may easily prove relations like the next one hold in \( B(\mathcal{H}) \):
\[ [A, e^{it H} R(z)^2] = R(z) [A, e^{it H}] R(z) + [A, R(z)] e^{it H} R(z) + e^{it H} R(z) [A, R(z)]. \] (2.3.5)
If \( H' D(H) \subset \mathcal{H} \) then the right hand side of (2.3.3) will clearly belong to \( B(D(H), \mathcal{H}) \) hence we shall also have \( [A, e^{it H}] \in B(D(H), \mathcal{H}) \) and (2.3.3) will hold strongly in \( B(D(H), \mathcal{H}) \).

We say that \( H' \), or \( [A, H] \), commutes with \( H \) if for any \( t \in \mathbb{R} \) the relation \( H' e^{it H} = e^{it H} H' \) holds in \( B(D(H), D(H)^*) \). This is clearly equivalent to \( H' \varphi(H) = \varphi(H) H' \) for any bounded Borel function \( \varphi : \mathbb{R} \rightarrow \mathbb{C} \). Note also that \( H' \) commutes with \( H \) if and only if there is \( z \in \rho(H) \) such that \( [A, R(z)] \) commutes with \( R(z) \) (this condition is independent of \( z \)). If we set \( R = R(z) \), we then have \( R' = -RH'R = -H'R^2 \).

If \( H' \) commutes with \( H \) then Proposition 2.3.7 can be significantly improved. If \( k \in \mathbb{N} \) let \( C^k_{\text{bi}}(\mathbb{R}) \) be the space of functions in \( C^k(\mathbb{R}) \) whose derivatives of orders \( \leq k \) are bounded.

**Proposition 2.3.8.** Let \( H \) be self-adjoint of class \( C^1(A) \) such that \( H' \) commutes with \( H \) and let \( \varphi \in C^1_{\text{bi}}(\mathbb{R}) \). Then the restriction of \( [A, \varphi(H)] \) to \( D(A) \cap D(H) \) extends to a continuous form \( [A, \varphi(H)] \) on \( D(H) \) and \( [A, \varphi(H)] = [A, H] \varphi'(H) = \varphi'(H) [A, H] \). In other terms:
\[ \varphi(H)' = \varphi'(H) H' = H' \varphi'(H), \quad \text{in particular} \quad (e^{it H})' = it H' e^{it H}. \] (2.3.6)

**Proof.** Due to Theorem 2.3.7 we have \( [A, e^{it H}] = it [A, H] e^{it H} \) for any real \( t \), hence the proposition is true if \( \varphi(\lambda) = e^{it \lambda} \). This clearly implies the proposition if \( \varphi \) is the Fourier transform of a bounded measure \( \hat{\varphi} \) such that \( \int |x \hat{\varphi}(x)| dx < \infty \). The general case follows by a standard limiting procedure. \( \square \)
Example 2.3.9. Consider once again the situation from Example 2.3.3. Then \([p,e^{iH}]_0\) is a restriction of \(-mq^{-m-1}e^{iH}\) hence is not a bounded operator but it extends to a continuous form on \(D(H)\). In the worst case \(m = 1\) we get \([e^{iH},A] = H^2e^{iH}\), hence the result of Theorem 2.3.7 is optimal. If \(\varphi\) is a \(C^1\) function then \(\varphi(H) = H^2\varphi'(H)\) hence \(\varphi(H)'\) cannot be bounded unless \(|\varphi'(\lambda)| \leq C(\lambda)^{-2}\).

2.4 Commutators and decay

From Proposition 2.3.8 we get the following decay result.

Proposition 2.4.1. Let \(H \in C^1(A)\) such that \(H' := [H,iA]\) commutes with \(H\) and let \(u \in D(H) \cap D(A)\). Then \(|(u[H' e^{iH}\mu]| \leq 2|t|^{-1}\|Au\||u||. In particular, if \(H' = B^*B\) for some continuous \(B : D(H) \to \mathcal{H}\) commuting with \(H\), then \(|\psi_{Bu}(t)| \leq C_u(t)^{-1}\) for \(u \in D(H) \cap D(A)\). If \(B\) is bounded on \(\mathcal{H}\) then this holds for all \(u \in D(A)\).

Proof. The first part is obvious. The fact that \(B\) commutes with \(H\) means \(e^{itH}B = Be^{itH}\)
for any \(t\) and this clearly implies that \([A,H]\) commutes with \(H\). Then \(\langle Bu|e^{itH}Bu\rangle = \langle u|[H,iA]|e^{itH}\mu\rangle\) hence the second and the third part are consequences of the first one. \(\square\)

Remark 2.4.2. Some of the next results are abstract versions of the following estimate: if \(H = h(q)\) and \(A = -p\) in \(L^2(\mathbb{R})\) then \(H' = h'(q)\) hence if \(|h'| \geq c > 0\) and \(h''/h'^2\) is bounded then an integration by parts gives \(|\int e^{ith(x)}|u(x)|^2dx| \leq C_u\|t\|^{-1}\) if \(u \in D(p)\).

We shall say that a densely defined operator \(S\) on \(\mathcal{H}\) is *boundedly invertible* if \(S\) is injective, its range is dense, and its inverse extends to a continuous operator on \(\mathcal{H}\). If \(S\) is symmetric this means that \(S\) is essentially self-adjoint and 0 is in the resolvent set of its closure.

Proposition 2.4.3. Let \(H \in C^1(A)\) such that \(H'\) commutes with \(H\) and \(H'D(H) \subset \mathcal{H}\). Assume that \(H'\), when considered as operator on \(\mathcal{H}\), is boundedly invertible and \(H'^{-1}\) extends to a bounded operator of class \(C^1(A)\). Then \(|\psi_u(t)| \leq C_u\|t\|^{-1}\) if \(u \in D(A)\).

Proof. From Proposition 2.3.8 we get \([e^{itH},A] = tH'e^{itH}\) as operators \(D(H) \to D(H)^*\) hence \([e^{itH},A]\) is a bounded operator \(D(H) \to \mathcal{H}\) and we have \([e^{itH},A]H'^{-1} = te^{itH}\) on the
range of $H'$. We denote $K$ the continuous extension to $\mathcal{H}$ of $H'^{-1}$ and note that $K$ commutes with $H$ because $H'e^{itH} = e^{itH'}H'$ hence $H'^{-1}e^{itH} = e^{itH'}H'^{-1}$ for all $t$. If $u \in D(A)$ and $Ku \in D(H)$ then $Ku \in D(A)$ because $K \in C^1(A)$ hence

$$t\psi_u(t) = \langle u|e^{itH}, A]Ku \rangle = \langle e^{-itH}u|AKu \rangle - \langle Au|e^{itH}Ku \rangle.$$

This implies

$$|t\psi_u(t)| \leq \|u\|\|AKu\| + \|Au\|\|Ku\| \leq \|[A,K]\|\|u\|^2 + 2\|K\|\|u\|\|Au\|.$$

Now let $u$ be an arbitrary element of $D(A)$. Let $R_\varepsilon = (1 + i\varepsilon H)^{-1}$ and $u_\varepsilon = R_\varepsilon u$. Then $u_\varepsilon \in D(A)$ because $R_\varepsilon D(A) \subset D(A)$ and $Ku_\varepsilon = R_\varepsilon Ku \in D(H)$ hence we have

$$|t\psi_{u_\varepsilon}(t)| \leq \|[A,K]\|\|u_\varepsilon\|^2 + 2\|K\|\|u_\varepsilon\|\|Au_\varepsilon\|.$$

Since $[A,R_\varepsilon] = \varepsilon H'R_\varepsilon^2$ we get $|t\psi_u(t)| \leq \|[A,K]\|\|u\|^2 + 2\|K\|\|u\|\|Au\|$ by making $\varepsilon \to 0$ in the preceding inequality. □

**Remark 2.4.4.** We may restate the assumptions of Proposition 2.4.3 as follows: $H$ is of class $C^2(A)$, $H'D(H) \subset \mathcal{H}$, and $H'$ when seen as operator on $\mathcal{H}$ is essentially self-adjoint and 0 is not in its spectrum.

**Remark 2.4.5.** The good decay $\psi_u(t) = O(t^{-1})$ obtained in Proposition 2.4.3 depends on a quite strong condition on $H'$ which in particular forces $H'$ to be an essentially self-adjoint operator on $\mathcal{H}$ whose spectrum does not contain zero. In the “classical” case mentioned in the Remark 2.4.2 this means $|h'(x)| \geq c > 0$ which is rather natural when one has to estimate an integral like $\psi(t) = \int e^{ith(x)}f(x)dx$ for large positive $t$: **points of stationary phase should be avoided, otherwise we cannot expect more than $\psi(t) = O(t^{-1/2})$.**

We now consider operators satisfying some special commutation relations but allow $H'$ to have zeros, e.g. we treat the simplest case $H' = cH$. Note that Example 2.3.1 shows that requiring only an algebraic relation like $[H, iA] = cH$ is highly ambiguous; the property $H \in C^1(A)$ is then necessary and is not automatically satisfied.

**Remark 2.4.6.** In many of the applications of the conjugate operator method, see for example Section 2.7, the operator $A$ is unbounded in energy space. However, it is possible to introduce
an energy cut-off for $A$ that does not alter the $C^1(A)$ condition for $H$ and preserves the behavior of the commutation relation at thresholds. For instance, consider $H \in C^1(A)$ bounded from below and such that $H' = cH$. Define the operators $g(H) = \langle H \rangle^{-1/4}$ and $\tilde{A} = g(H)Ag(H)$.

Then, it is easy to see that $H \in C^1(\tilde{A})$ and $[H, i\tilde{A}] = H'g(H)^2$. This regularization technique will be explored in Chapter 3.

**Remark 2.4.7.** The subsequent results will hold for self evanescent states $u \in \mathcal{E}$, but they also rely on the condition $Au \in \mathcal{E}$. The latter assumption is not satisfied in general, in fact, it is implied by a stronger localization condition for $u$. To elude this, it will be assumed that there is a projection $P$ which commutes with $H$, and such that $u \in \text{Ran}P$. Then the condition $Au \in \mathcal{E}$ can be replaced by $PAPu \in \mathcal{E}$, which is easier to satisfy. For instance, one can choose $P$ as the projection on the continuous spectrum of $H$ and the proofs presented below can be slightly modified to obtain the same decay estimates. This idea will be explicitly used in Chapter 3.

**Proposition 2.4.8.** Let $H \in C^1(A)$ such that $H' = cH$ with $c \neq 0$ and let $u \in D(A)$ such that $u, Au \in \mathcal{E}$. Then $|\psi_u(t)| \leq C_u(t)^{-1/2}$.

**Proof.** We have $\psi_u \in L^2(\mathbb{R})$ because $u \in \mathcal{E}$ hence, according to Corollary A.2, it suffices to show that the function $(\delta\psi)(t) = t\psi_u(t)$ also belongs to $L^2(\mathbb{R})$. If $u \in D(|H|^{1/2})$ then $t\psi_u(t) = \langle u|itHe^{itH}u \rangle$ so that by using Proposition 2.3.8 we get:

$$-ict\psi_u(t) = \langle u|tcHe^{itH}u \rangle = \langle u|[A, itH]e^{itH}u \rangle = \langle u|[A, e^{itH}]u \rangle.$$  

Then, if $u \in D(A)$ we get $ict\psi_u(t) = \langle Au|e^{itH}u \rangle - \langle u|e^{itH}Au \rangle$ hence (2.2.1) implies:

$$c^2\|\delta\psi_u\|^2 \leq 2\|\psi_u\|_{L^2}\|\psi_{Au}\|_{L^2} = 2[u]_{H}^2[Au]_{H}^2.$$  

(2.4.7)

So the proposition is proved under the supplementary condition $u \in D(H)$ and the estimate (2.4.7) depends only on $c$. Now consider an arbitrary $u \in D(A)$ such that $u, Au \in \mathcal{E}$ and for $\varepsilon > 0$ let $R_\varepsilon = (1 + i\varepsilon H)^{-1}$. Then from Proposition 2.3.2 we get $R_\varepsilon u \in D(A)$ and $[A, R_\varepsilon] = R_\varepsilon[i\varepsilon H, A]R_\varepsilon = c\varepsilon HR_\varepsilon^2$. If we set $u_\varepsilon = R_\varepsilon u$ then the estimate (2.4.7) gives $c^2||\delta\psi_{u_\varepsilon}\|^2 \leq 2[u_\varepsilon]_{H}^2[Au_\varepsilon]_{H}^2$. Finally, let $\varepsilon \to 0$ and use Fatou’s lemma in the left hand side and Lemma 2.2.3 on the right hand side to get (2.4.7) without the condition $u \in D(H)$. □
Proposition 2.4.9. Let $H \in C^1(A)$ such that $H \geq 0$ and $H' = H(1 + H)^{-1}$. If $u \in D(A)$ and $u, Au \in \mathcal{E}$ then $|\psi_u(t)| \leq C_u\langle t \rangle^{-1/2}$.

Proof. Choose $\varphi \in C^\infty_c(\mathbb{R})$ with $0 \leq \varphi \leq 1$ and equal to one on a neighborhood of zero and set $\phi = \varphi(H), \phi^\perp = 1 - \phi^2$. Then $\psi_u(t) = \psi_{\varphi u}(t) + \langle u|\phi^\perp e^{itH}u \rangle$ and we have $|\langle u|\phi^\perp e^{itH}u \rangle| \leq C\langle t \rangle^{-1}$ by an argument almost identical to that of the proof of Proposition 2.4.8 (note that $\xi(H)D(A) \subset D(A)$ if $\xi$ is a smooth function constant near infinity). Hence it suffices to prove the proposition under the supplementary assumption that the $H$-support of $u$ (i.e. $\text{supp } E_u$) is compact. As before, we have $t\psi_u'(t) = \langle u|itHe^{itH}u \rangle$ so that by using Proposition 2.3.8 and with the notation $\Lambda = \sqrt{H + 1}$ we get:

$$-it\psi_u'(t) = \langle u|tHe^{itH}u \rangle = \langle \Lambda u|tH(H + 1)^{-1}e^{itH}\Lambda u \rangle = \langle \Lambda u|[e^{itH}, A]\Lambda u \rangle.$$

We have $\Lambda u \in D(A) \cap \mathcal{E}$ because $\Lambda u = \theta(H)u$ for some smooth function with compact support $\theta$ and $u \in D(A) \cap \mathcal{E}$. Thus

$$|t\psi_u'(t)| = |\langle u|tHe^{itH}u \rangle| \leq |\langle \Lambda u|e^{itH}A\Lambda u \rangle| + |\langle A\Lambda u|e^{itH}\Lambda u \rangle|.$$

Finally, the relation

$$A\Lambda u = A\theta(H)u = [A, \theta(H)]u + \theta(H)Au = i\theta'(H)H(H + 1)^{-1}u + \theta(H)Au$$

gives $A\Lambda u \in \mathcal{E}$. Thus $\delta\psi_u \in L^2(\mathbb{R})$ hence Corollary A.2 implies $|\psi_u(t)| \leq C_u\langle t \rangle^{-1/2}$. □

The next result is of the same nature but more general; the proof is essentially the same.

Theorem 2.4.10. Let $H \in C^1(A)$ such that $H' = \theta(H)$ with $\theta$ real of class $C^1$ with bounded derivative and such that: (1) if $|\lambda| \geq \varepsilon > 0$ then $|\theta(\lambda)| \geq c_\varepsilon > 0$, (2) $\lambda/\theta(\lambda)$ extends to a $C^1$ function on $\mathbb{R}$. If $u \in D(A)$ and $u, Au \in \mathcal{E}$ then $|\psi_u(t)| \leq C_u\langle t \rangle^{-1/2}$.

Proof. Let $\varphi \in C^\infty_c(\mathbb{R})$ real and equal to one on a neighborhood of zero and let us set $\phi = \varphi(H), \phi^\perp = 1 - \phi^2$, so that $\psi_u(t) = \psi_{\varphi u}(t) + \langle u|\phi^\perp e^{itH}u \rangle$. We first show that the second term is $O(t^{-1})$. We have $\phi^\perp H'^{-1} = \xi(H)$ with $\xi(\lambda) = (1 - \varphi^2(\lambda))/\theta(\lambda)$ hence

$$t\langle u|\phi^\perp e^{itH}u \rangle = \langle \phi^\perp H'^{-1}u|tHe^{itH}u \rangle = \langle \xi(H)u|tH'e^{itH}u \rangle = \langle \xi(H)u|[e^{itH}, A]u \rangle.$$
Until here $u$ was an arbitrary element of $\mathcal{H}$. If $u \in D(H) \cap D(A)$ then we can expand the commutator and get

$$|t\langle u|\phi^-e^{itH}u\rangle| = |\langle e^{-itH}\xi(H)u|Au\rangle - \langle A\xi(H)u|e^{itH}u\rangle|$$

$$\leq \|Au\| \|\xi(H)u\| + \|A\xi(H)u\| \|u\|.$$ 

Since $\xi$ is a bounded function of class $C^1$ with bounded derivative we can use Proposition 2.3.6 and get $\xi(H)' = \xi'(H)H' = (\xi\theta)(H)$. We have $\xi\theta = -\theta/\theta$ outside a compact neighborhood of zero, hence $\xi(H)'$ is a bounded operator, so $\xi(H)$ is of class $C^1(A)$, hence $\xi(H)D(A) \subset D(A)$. Then, since $H'D(H) \subset \mathcal{H}$, this estimate remains true for any $u \in D(A)$ by Proposition 2.3.6. Thus $\|\langle u|\phi^-e^{itH}u\rangle\| \leq C_u\langle t\rangle^{-1}$ for any $u \in D(A)$.

From now on we change notations: $\phi u$ will be denoted $u$. So we may assume $\text{supp} E_u \subset [-1,1]$ and $u, Au \in \mathcal{E}$, cf. Lemma 2.2.3, and we want to prove that the function $t\psi'_u(t)$ belongs to $L^2(\mathbb{R})$. Let $\eta$ be the $C^1$ function on $\mathbb{R}$ which extends $\lambda/\theta(\lambda)$, let $\zeta \in C^\infty_c(\mathbb{R})$ such that $\zeta(H)u = u$, and let us set $\tilde{\eta} = \eta\zeta$. Then $\tilde{\eta}(H)u \in D(A) \cap \mathcal{E}$ and $A\tilde{\eta}(H)u = [A, \tilde{\eta}(H)]u + \tilde{\eta}(H)Au \in \mathcal{E}$. Finally

$$-it\psi'_u(t) = \langle u|tHe^{itH}u\rangle = \langle u|\eta(H)te^{itH}u\rangle = \langle \tilde{\eta}(H)u|e^{itH}, A\zeta(H)u\rangle$$

$$= \langle e^{-itH}\tilde{\eta}(H)u|A\zeta(H)u\rangle - \langle A\tilde{\eta}(H)u|e^{itH}\zeta(H)u\rangle$$

and from (2.2.1) we get the square integrability of $t\psi'_u(t)$. \hfill \Box

### 2.5 Higher order commutators

The decay estimates obtained so far on $\psi_u(t)$ are at most of order $O(t^{-1})$ and it is clear that to obtain $O(t^{-k})$ for some integer $k > 1$ we need conditions of the form $u \in D(A^k)$ and assumptions on the higher order commutators of $A$ with $H$. We recall here the necessary formalism.

Let $A$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and $k \in \mathbb{N}$. We say that $S$ is of class $C^k(A)$, and we write $S \in C^k(A)$, if the map $\mathbb{R} \ni t \mapsto e^{-itA}Se^{itA} \in B(\mathcal{H})$ is of class $C^k$ in the strong operator topology. It is clear that $S \in C^{k+1}(A)$ if and only if $S \in C^1(A)$ and...
If \( S \in C^2(A) \) we set \((S')' = S'' = S^{(2)}\), etc. Clearly \( C^k(A) \) is a \(*\)-subalgebra of \( B(\mathcal{H}) \) and if \( S \in B(\mathcal{H}) \) is bijective and \( S \in C^k(A) \) then \( S^{-1} \in C^k(A) \).

For any \( S \in B(\mathcal{H}) \) let \( \mathcal{A}(S) = [S, iA] \) considered as a sesquilinear form on \( D(A) \). We may iterate this and define a sesquilinear form on \( D(A^k) \) by:

\[
S^{(k)} \equiv A^k(S) = i^k \sum_{i+j=k} \frac{k!}{i!j!} (-A)^i S A^j.
\]

Then \( S \in C^k(A) \) if and only if this form is continuous for the topology induced by \( \mathcal{H} \) on \( D(A^k) \). We keep the notation \( \mathcal{A}^k(S) \) or \( S^{(k)} \) for the bounded operator associated to its continuous extension to \( \mathcal{H} \).

Strictly speaking, the operator \( \mathcal{A} \) acting in \( B(\mathcal{H}) \) must be defined as the infinitesimal generator of the group of automorphisms \( \mathcal{U} = \{U_t\}_{t \in \mathbb{R}} \) of \( B(\mathcal{H}) \) given by \( U_t(S) \equiv e^{tA}(S) = e^{-itA} S e^{itA} \).

This group is not of class \( C_0 \) and so \( \mathcal{A} \) is not densely defined. Then \( C^k(A) \) is just the domain of \( \mathcal{A}^k \). One may also define \( C^\alpha(A) \) if \( \alpha \) is not an integer as the Besov space of order \( (\alpha, \infty) \) associated to \( \mathcal{U} \).

We denote \( B_1(\mathcal{H}) \) the Banach algebra of trace class operators on \( \mathcal{H} \). Its dual is identified with the space \( B(\mathcal{H}) \) of all bounded operators on \( \mathcal{H} \) with the help of the bilinear form \( \text{Tr}(S \rho) \). It is clear that the restrictions of the \( \mathcal{U}_t \) to \( B_1(\mathcal{H}) \subset B(\mathcal{H}) \) give a group of automorphisms of \( B_1(\mathcal{H}) \) and that this group is of class \( C_0 \). We do not distinguish in notation between \( \mathcal{U} \) and \( \mathcal{A} \) and their restrictions to \( B_1(\mathcal{H}) \) but note that, for example, the domain of \( \mathcal{A} \) in \( B_1(\mathcal{H}) \) is the set of \( S \in C^1(A) \cap B_1(\mathcal{H}) \) such that \( \mathcal{A}(S) \in B_1(\mathcal{H}) \). Moreover, if \( S = |u \rangle \langle v| \) and \( u, v \in D(A^k) \) then \( S \) belongs to the domain of \( \mathcal{A}^k \) in \( B_1(\mathcal{H}) \).

Now let \( H \) be a self-adjoint operator on \( \mathcal{H} \) and \( R(z) = (H - z)^{-1} \) for \( z \) in the resolvent set \( \rho(H) \) of \( H \). We say that \( H \) is of class \( C^k(A) \) if \( R(z_0) \in C^k(A) \) for some \( z_0 \in \rho(H) \); then we shall have \( R(z) \in C^k(A) \) for all \( z \in \rho(H) \) and more generally \( \varphi(H) \in C^k(A) \) for a large class of functions \( \varphi \) (e.g. rational and bounded on the spectrum of \( H \)).

For each real \( m \) let \( S^m(\mathbb{R}) \) be the set of symbols of class \( m \) on \( \mathbb{R} \), i.e. the set of functions \( \varphi : \mathbb{R} \to \mathbb{C} \) of class \( C^\infty \) such that \( \left| \varphi^{(k)}(\lambda) \right| \leq C_k |\lambda|^{-m-k} \) for all \( k \in \mathbb{N} \). Note that \( S^m \cdot S^n \subset S^{m+n} \) and \( \varphi^{(j)} \in S^{m-j} \) if \( \varphi \in S^m \) and \( j \in \mathbb{N} \).
Proposition 2.5.1. Let \( H \) be a self-adjoint operator of class \( C^1(A) \) with \( H' = \theta(H) \) for some \( \theta \in S^2(\mathbb{R}) \). Then \( H \) is of class \( C^\infty(A) \). Let \( \delta_\theta \) be the first order differential operator given by \( \delta_\theta = \theta(\lambda) \frac{d}{d\lambda} \). If \( \theta \in S^1(\mathbb{R}) \) and \( \varphi \in S^0(\mathbb{R}) \) then \( \varphi(H) \) is of class \( C^\infty(A) \) and

\[
A^k(\varphi(H)) = \left( \delta_\theta^k \varphi \right)(H) \quad \forall k \in \mathbb{N}.
\]

(2.5.8)

Proof. We begin with a general remark. By using Proposition 2.3.8 we see that if \( H \) is of class \( C^1(A) \) and \( H' = \theta(H) \) for some real Borel function \( \theta \), and if \( \varphi \in C^1_b(\mathbb{R}) \), then \( \varphi'(H) = A(\varphi(H)) = \theta(H) \varphi'(H) = (\delta_\theta \varphi)(H) \). In particular, if \( \delta_\theta \varphi \) is a bounded function then \( \varphi(H) \) is of class \( C^1(A) \).

If we take \( \theta \in S^2 \) and \( \varphi(\lambda) = (\lambda + i)^{-1} \) then \( \varphi \in S^{-1} \) hence \( \theta \varphi' \in S^0 \). Thus the operator \( R = (H + i)^{-1} = \varphi(H) \) satisfies \( R' = \psi(H) \) with \( \psi \in S^0 \). Now we may apply the preceding argument with \( \varphi \) replaced by \( \psi \) and get \( \psi \in C^1(A) \), so \( R' \in C^1(A) \), etc. This proves that \( H \) is of class \( C^\infty(A) \).

In the preceding argument we clearly may take any \( \varphi \in S^{-1} \). If \( \theta \in S^1 \) then the same argument works for any \( \varphi \in S^0 \) and gives the last assertion of the proposition.

\( \square \)

Remark 2.5.2. If \( \theta \in S^m \) and \( \varphi \in S^{-(m-1)} \) with \( 1 \leq m \leq 2 \) the last assertion of the proposition remains true (with the same proof).

We finish this section with some comments in connection with relation (2.5.8). At a formal level (2.5.8) means

\[
\begin{aligned}
e^{-itA} \varphi(H) e^{itA} &\equiv e^{itA} (\varphi(H)) = (e^{it\theta} \varphi)(H). \\
\end{aligned}
\]

(2.5.9)

We shall explain without going into details how one may rigorously interpret this relation and how one may use it to get decay estimates.

Let \( \xi_t \) be the flow of diffeomorphisms of the real line defined by the vector field \( \delta_\theta = \theta(\lambda) \frac{d}{d\lambda} \).

This means that \( \frac{d}{dt} \xi_t(\lambda) = \theta(\xi_t(\lambda)) \) and \( \xi_0(\lambda) = \lambda \) for all \( \lambda \in \mathbb{R} \) (we assume that such a global flow exists). Then if \( \varphi : \mathbb{R} \to \mathbb{C} \) is a smooth function we have \( \frac{d}{dt} \varphi \circ \xi_t = (\delta_\theta \varphi) \circ \xi_t \) or \( \varphi \circ \xi_t = e^{it\theta} \varphi \). Hence (2.5.9) may be written \( e^{-itA} \varphi(H) e^{itA} = (\varphi \circ \xi_t)(H) \). This can be easily checked independently of what we have done before.
Let $M(\mathbb{R})$ be the space of all bounded Borel measures on $\mathbb{R}$. We associate to $H$ a continuous linear map $\Phi : B_1(\mathcal{H}) \to M(\mathbb{R})$ defined as follows: if $\rho \in B_1(\mathcal{H})$ then $\int \varphi \Phi(\rho) = \text{Tr}(\varphi(H)\rho)$ for any bounded Borel function $\varphi$. Then

$$\text{Tr}(\varphi(H)U_{-t}(\rho)) = \text{Tr}(e^{-itA}\varphi(H)e^{itA}\rho) = \text{Tr}((\varphi \circ \xi_t)(H)\rho)$$

which means that the measure $\Phi(U_{-t}(\rho))$ is equal to the image of the measure $\Phi(\rho)$ through the map $\xi_t$. Or, if we denote $V_t$ the map $M(\mathbb{R}) \to M(\mathbb{R})$ which sends a measure $\mu$ into its image $\xi_t^* (\mu)$ through $\xi_t$, we have $\Phi \circ U_{-t} = V_t \circ \Phi$.

Thus, if $\rho$ belongs to the Besov space $B_1(\mathcal{H})_{s,p}$ associated to the group of automorphisms $U_t$ of $B_1(\mathcal{H})$ then $\Phi(\rho)$ belongs to the Besov space $M(\mathbb{R})_{s,p}$ associated to the group of automorphisms $V_t$ of $M(\mathbb{R})$ (notations as in [1]). This gives smoothness properties of the measure $\Phi(\rho)$ with respect to the differential operator $\delta_\theta$ in terms of smoothness properties of $\rho$ with respect to the operator $A$. In particular, since $\text{Tr}(e^{itH}\rho) = \int e^{it\lambda} \Phi(\rho)(d\lambda)$ is just the Fourier transform of the measure $\Phi(\rho) \equiv \Phi(\rho)$, this allows us to control the decay as $t \to \infty$ of $t \mapsto \text{Tr}(e^{itH}\rho)$ in terms of the local behavior of the measure $\Phi(\rho)$.

The operators $V_t$ can be explicitly computed in many situations and the preceding strategy gives optimal results. For example, in the simplest case $[H, iA] = 1$ we get for any $s > 0$

$$\|\langle A \rangle^{-s} e^{itH} \langle A \rangle^{-s} \| \leq C_s \langle t \rangle^{-s} \quad (2.5.10)$$

If $[H, iA] = H$ then such a good decay is impossible because zero is a threshold (see remark 2.6.2) but if $\eta$ is a smooth function equal to zero near zero and to one near infinity then

$$\|\langle A \rangle^{-s} e^{itH} \eta(H) \langle A \rangle^{-s} \| \leq C_s \langle t \rangle^{-s}. \quad (2.5.11)$$

This may be extended to a large class of functions $\theta$.

### 2.6 Higher order decay

The expressions $\psi_u(t) = \langle u|e^{itH}u \rangle$ that we considered until now are quadratic in $u$ and this complicates the higher order computations. To elude this we note that $\psi_u(t) = \text{Tr}(e^{itH}\rho)$ with $\rho = |u\rangle\langle u|$, expression which makes sense for any $\rho \in B_1(\mathcal{H})$ and is linear in $\rho$.

We begin with an extension to higher orders of Proposition 2.4.3.
Theorem 2.6.1. Let $k \in \mathbb{N}$ and $s \in [0, k]$ real. Assume that $H$ is of class $C^{k+1}(A)$ and $H'$ commutes with $H$, satisfies $H'D(H) \subset \mathcal{H}$, and is boundedly invertible. Then for each vector $u \in D(|A|^s)$ we have $\psi_u(t) = O(t^{-s})$.

Proof. By an interpolation argument it suffices to prove $|\psi_u(t)| \leq C_k(\|u\| + \|A^k u\|)^2(t)^{-k}$ for $u$ in a dense subspace of $D(A^k)$. Formally this is quite straightforward starting with the formula $(itH')^{-1}A(e^{itH}) = e^{itH}$ and then iterating it $k$ times; we next sketch the rigorous proof. We change slightly the notations from the proof of Proposition 2.4.3 and denote $K$ the continuous extension to $\mathcal{H}$ of $-iH^{-1}$. Then $K$ commutes with $H$, is of class $C^k(A)$, and we have $KA(e^{itH}) = A(e^{itH})K = te^{itH}$. Let $u \in D(A^k)$ and $\rho = |u\rangle\langle u|$ or a more general trace class operator. Let $L_K$ and $R_K$ be the operators of right and left multiplication by $K$, which act both in $B(\mathcal{H})$ and in $B_1(\mathcal{H})$. Then $R_KA(e^{itH}) = te^{itH}$ hence

$$t\psi_u(t) = \text{Tr}\left((R_KA)(e^{itH})\rho\right) = \text{Tr}\left(A(e^{itH})(K\rho)\right) = -\text{Tr}\left(e^{itH}A(K\rho)\right) = -\text{Tr}\left(e^{itH}(AL_K)\rho\right).$$

This is easy to justify since $Ku \in D(A^k)$ because $K$ is of class $C^k(A)$. In exactly the same way, starting with $(R_KA)^k(e^{itH}) = t^k e^{itH}$ we get

$$t^k \psi_u(t) = \text{Tr}\left((R_KA)^k(e^{itH})\rho\right) = (-1)^k\text{Tr}\left(e^{itH}(AL_K)^k\rho\right).$$

Finally, it remains to note that $\|(AL_K)^k\rho\|_{B_1(\mathcal{H})} \leq C_k(\|u\| + \|A^k u\|)^2$.

Remark 2.6.2. The following example shows that such a good decay as in Theorem 2.6.1 cannot be expected if $H'$ is not boundedly invertible. In the Hilbert space $\mathcal{H} = L^2(0, \infty)$ let $H$ be the operator of multiplication by the independent variable $x$ and let $A$ be the self-adjoint realization of $\frac{1}{2}(x\frac{d}{dx} + \frac{d}{dx})$. Then $H$ is of class $C^\infty(A)$ and $H' = [H, iA] = H$. Let $u$ be a $C^\infty$ function on $(0, \infty)$ which is zero for $x > 2$ and equal to $x^{-\theta}$ for $x < 1$ with $0 < \theta < 1/2$. Then $u \in D(|A|^s)$ for all $s > 0$ but $\psi_u(t) \sim \int_0^1 e^{itx}x^{-2\theta}dx \sim t^{2\theta-1}$ for $t \to \infty$, hence the decay can be made as bad as possible. On the other hand, Example 2.2.4 explains why the space $\mathcal{E}$ helps to improve the behavior.

We now give a higher order version of Theorem 2.4.10. Recall that $\theta \in S^m(\mathbb{R})$ is an elliptic symbol if there is $c > 0$ such that $|\theta(\lambda)| \geq c|\lambda|^m$ near infinity. Then $\eta/\theta \in S^{-m}(\mathbb{R})$ for any $C^\infty$ function $\eta$ with support in the region where $\theta \neq 0$ and equal to one near infinity.
Theorem 2.6.3. Let $H \in C^1(A)$ such that $H' = \theta(H)$ for some elliptic symbol $\theta \in S^m(\mathbb{R})$ with $0 \leq m \leq 1$. Assume: (1) $\theta(\lambda) \neq 0$ if $\lambda \neq 0$ and (2) $\lambda/\theta(\lambda)$ extends to a $C^\infty$ function on $\mathbb{R}$. Let $k$ be an odd integer and let $u \in \mathcal{H}$ be of the form $|H|^{(k-1)/4}v$ for some $v \in D(A^k)$ such that $A^j v \in \mathcal{E}$ for $0 \leq j \leq k$. Then $|\psi_u(t)| \leq C_u(t)^{-k/2}$.

Proof. Denote $S^0_{(0)}(\mathbb{R})$ the set of $a \in S^0(\mathbb{R})$ such that $a(\lambda) = 0$ near zero. We first prove the following: if $n \in \mathbb{N}$ and $a \in S^0_{(0)}(\mathbb{R})$ then there are $a_0, a_1, \ldots, a_n \in S^0(\mathbb{R})$ such that:

$$t^n a(H)e^{itH} = \sum_{j=0}^{n} A^j (a_j(H)e^{itH}).$$  \hspace{1cm} (2.6.12)

Of course, the $a_j$ also depend on $n$. If $n = 1$ we write (see also the proof of Theorem 2.4.10):

$$ta(H)e^{itH} = -i \frac{a(H)}{\theta(H)} A(e^{itH}) = A(a_1(H)e^{itH}) + a_0(H)e^{itH}$$  \hspace{1cm} (2.6.13)

where $a_1 = \frac{a'}{\theta}$ and $a_0 = -\theta a_1'$ (use Proposition 2.5.1). We mention that we use without comment the relation $A(ST) = A(S)T + S A(T)$ with the further simplification that in our context $S$ and $T$ are functions of $H$ hence commute. Now assume (2.6.12) is true and let us prove it with $n$ replaced by $n + 1$. Let $b \in C^\infty$ equal to zero near zero and to 1 near infinity and such that $a_j = a_j b$ for all $j$. Then

$$t^{n+1} a(H)e^{itH} = \sum_{j=0}^{n} A^j (a_j(H)tb(H)e^{itH})$$

Now we use (2.6.13) and replace $tb(H)e^{itH} = A(b_1(H)e^{itH}) + b_0(H)e^{itH}$. Thus

$$a_j(H)tb(H)e^{itH} = a_j(H)A(b_1(H)e^{itH}) + a_j(H)b_0(H)e^{itH}$$

$$= A(a_j(H)b_1(H)e^{itH}) + (a_j(H)b_0(H) - A(a_j(H))b_1(H)) e^{itH}$$

which clearly gives the required result.

Now we begin the proof of the theorem. As in the proof of Theorem 2.4.10 we consider separately the case when $u$ is zero near energy zero and that when $u = \varphi(H)u$ for some $\varphi \in C_c^\infty$. The first case is an immediate consequence of (2.6.12) because there is $a \in S^0_{(0)}(\mathbb{R})$ such that $a(H)u = u$ hence (recall the notation $\rho = |u\rangle \langle u|$)

$$t^k \langle u|e^{itH}u \rangle = \text{Tr} \left( t^ka(H)e^{itH} \rho \right) = \sum_{j=0}^{k} (-1)^j \text{Tr} \left( a_j(H)e^{itH} A^j(\rho) \right)$$  \hspace{1cm} (2.6.14)
which implies $\psi_u(t) = O(t^{-k})$ because obviously $\rho \in D(A^k)$ if $u \in D(A^k)$.

Note that the facts established above hold for an arbitrary $u \in \mathcal{H}$. The condition involving $v$ is needed to have some control on the behavior of $u$ at zero energy, which cannot be arbitrary as explained in Remark 2.6.2. When we localize near zero energy we replace $u$ by $\varphi(H)u$ with $\varphi \in C_0^\infty$ equal to one on a neighborhood of zero. If $u = |H|^m v$ with $m = (k - 1)/4$ and $v \in D(A^k)$ such that $A^j v \in \mathcal{E}$ for $0 \leq j \leq k$ then $\varphi(H)u = |H|^m \varphi(H)v$. By Proposition 2.5.1 $H$ is of class $C^\infty(A)$ so $\varphi(H)D(A^j) \subset D(A^j)$ for any $j$ and $A^j \varphi(H)v \in \mathcal{E}$ by Lemma 2.2.3.

Thus for the rest of the proof we may assume that the support of $u$ in a spectral representation of $H$ is included in $[-1, 1]$ and $u = |H|^m v$ with $v \in D(A^k)$ such that $A^j v \in \mathcal{E}$ for $0 \leq j \leq k$.

It is clear that $v$ has the same $H$-support as $u$. Our purpose is to check the assumptions of the Corollary A.3 for $\psi = \psi_u$. There are two conditions to be verified: the functions $t^{k+1} \psi_u(t)$ and $t^{k+1} \psi_u'(t)$ should be in $L^2(\mathbb{R})$. We treat only the second one, the first is treated similarly. If $\ell = 2m + 1 = (k + 1)/2$ then

$$t^\ell \psi_u'(t) = \langle u|t^\ell H e^{itH}u \rangle = \langle |H|^m v|t^\ell H e^{itH} |H|^m v \rangle = \langle v|t^\ell H^\ell \text{sgn}^{\ell+1}(H)e^{itH}v \rangle.$$  

Let $\eta$ be a $C^\infty$ function with compact support such that $\eta(\lambda) = \lambda/\theta(\lambda)$ on $[-1, 1]$. Then $\lambda = \eta(\lambda)/\theta(\lambda)$ on $[-1, 1] \setminus \{0\}$ hence on $[-1, 1]$ so we have

$$i^{\ell-1}t^\ell \psi_u'(t) = \langle \eta(H)^\ell v|t^\ell \theta(H)^\ell e^{itH}v \rangle = \langle \eta(H)^\ell v|(itH)^\ell e^{itH}v \rangle.$$  

Recall that we have $A(e^{itH}) = itH' e^{itH}$ in a sense described in Proposition 2.3.8. But under the present conditions we have much more because $H'D(H) \subset \mathcal{H}$ hence $e^{itA}$ leaves invariant the domain of $H$ and induces there a $C_0$-group (see the assertion (2) page 16). In particular, the set of $u \in D(H) \cap D(A^j)$ such that $A^j u \in D(H)$ for any $j \in \mathbb{N}$ is dense in $D(H)$ (and is a core for $A$). Moreover, the $A^j(H)$ are bounded operators if $j \geq 2$. This allows us to compute $A^j(e^{itH})$ inductively as usual. Our next computations look slightly formal but it is straightforward, although a little tedious, to rigorously justify each step.

Above we fixed $\ell$ to the value $(k + 1)/2$ but now we allow it to take any value smaller than this.
one. For the case $\ell = 1$ see the proof of Theorem 2.4.10. For $\ell = 2$ we write

$$A^2(e^{itH}) = A(itHe^{itH}) = itH'' e^{itH} + (itH')^2 e^{itH} = \frac{H''}{H} itHe^{itH} + (itH')^2 e^{itH} = \frac{H''}{H} A(e^{itH}) + (itH')^2 e^{itH}$$

By “localizing” Proposition 2.5.1 we get

$$H'' = A(H') = A(\theta(H)) = \theta(H) \theta'(H) = H' \theta'(H)$$

hence $H'' H' = \theta'(H)$. Thus

$$(itH')^2 e^{itH} = A^2(e^{itH}) - \theta'(H) A(e^{itH}).$$

Then if we set $\rho = |\eta^2(H)v\rangle \langle v|$ we get

$$it^2 \psi'_u(t) = \text{Tr} ((itH')^2 e^{itH} \rho) = \text{Tr} (A^2(e^{itH}) \rho) - \text{Tr} (\theta'(H) A(e^{itH}) \rho)$$

$$= \text{Tr} (e^{itH} A^2(\rho)) - \text{Tr} (e^{itH} A(\rho \theta'(H))).$$

The right hand side belongs to $L^2(\mathbb{R})$ by the argument from Theorem 2.4.10, which finishes the proof in the case $\ell = 2$. The general case does not involve any new idea: by writing conveniently $\frac{H''(t)}{H'}$ one may express $(itH')^\ell e^{itH}$ as a linear combination of functions of $H$ times commutators $A^j(e^{itH})$ and one may proceed as above.

\[\square\]

2.7 Applications

We will use the previous results to obtain decay estimates for $\psi_u(t) = \langle u | e^{itH} u \rangle$ in several situations. Note that Example 2.3.1 and Proposition 2.3.2 show that the commutation relation is not enough to prove the $C^1(A)$ condition for $H$. For instance, in addition to the continuity of $[A, H]_0$ on $D(A) \cap D(H)$, it suffices to verify the invariance of domain $R(z) D(A) \subset D(A)$. In other cases it is convenient to verify the simplified assumptions of Mourre [52], which are stronger than the $C^1(A)$ property [1]:

(a) $e^{i\theta A} D(H) \subset D(H)$

(b) There is a subspace $\mathcal{S} \subset D(A) \cap D(H)$ which is a core for $H$ such that $e^{i\theta A} \mathcal{S} \subset \mathcal{S}$ and the form $[H, iA]$ on $\mathcal{S}$ extends to a continuous operator $D(H) \to \mathcal{H}$.

Recall the subspace $\mathcal{E} = \{ u \in \mathcal{H} | [u]_H < \infty \}$, where $[u]_H = (\int_{\mathbb{R}} |\psi_u(t)|^2 dt)^{1/4}$. 

2.7.1 Example 1: Laplacian in $\mathbb{R}^n$.

Let $H = -\Delta$ in $L^2(\mathbb{R}^n)$ with domain the Sobolev space $\mathcal{H}^2(\mathbb{R}^n)$ and $A = -\frac{1}{2}(x \cdot \nabla + \nabla \cdot x)$ the generator of dilations which is essentially self-adjoint on the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$. Condition (a) is a consequence of the formula $e^{\theta A}(H + i)^{-1} = (e^{-2\theta H} + i)^{-1}e^{-i\theta A}$, and (b) is satisfied since $\mathcal{S}$ is a core for $H$ which is trivially invariant under the dilation group. Integration by parts on $\mathcal{S}$ shows that $[H, iA] = 2H$. We conclude from Proposition 2.4.8 that for $u \in D(A)$ such that $u, Au \in \mathcal{E}$, $\psi_u$ satisfies the decay estimate $|\psi_u(t)| \leq C_u \langle t \rangle^{-1/2}$. Higher-order decay estimates follow from Theorem 2.6.3.

2.7.2 Example 2: $H = -\partial_{xx} + \partial_{yy}$ in $\mathbb{R}^2$.

Let $H = -\partial_{xx} + \partial_{yy}$ and $A = -\frac{1}{2}(x \cdot \nabla + \nabla \cdot x)$ in $L^2(\mathbb{R}^2)$. With the help of a Fourier transformation we see that $H$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^2)$. Clearly $[H, iA] = 2H$, hence the estimate of Example 1 holds. One may treat similarly the case when the operator $H$ in $L^2(\mathbb{R}^n)$ is an arbitrary homogeneous polynomial of order $m$ in the derivatives $i\partial_1, \ldots, i\partial_n$ with constant coefficients: then $[H, iA] = mH$.

2.7.3 Example 3: Electric field in $\mathbb{R}^n$.

Here we study the case $H = -\Delta + \vec{h} \cdot x$ and $A = i\vec{h} \cdot \nabla$ in $\mathbb{R}^n$, where $\vec{h}$ is a fixed unitary vector. We take again $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ as a core for $H$ and then it is easy to check the commutation relation $[H, iA] = 1$. Therefore Proposition 2.4.1 provides the estimate $|\psi_u(t)| \leq C_u \langle t \rangle^{-1}$ for $u \in D(A)$, where $C_u = 2\|u\|\|Au\|$. Further estimates follow from Theorem 2.6.1.

2.7.4 Example 4: $H = -x^{2-\theta} \Delta - \Delta x^{2-\theta}$ in $\mathbb{R}_+^n$.

For $0 < \theta < 2$ consider $H = -x^{2-\theta} \Delta - \Delta x^{2-\theta}$ and $A = -\frac{1}{2}(x \cdot \nabla + \nabla \cdot x)$ in $\mathbb{R}_+^n$. Then $\mathcal{S} = C_\infty_c(\mathbb{R}_+^n)$ is a core for $H$ and the domain conditions follow from the formula $e^{-i\alpha A}He^{i\alpha A} = e^{i\alpha}H$. The commutation relation is $[H, iA] = \theta H$, which yields the estimate of Example 1.
2.7.5 Example 5: Fractional Laplacian in $\mathbb{R}^n$.

For $0 < s < 2$, let $H = (−Δ)^{s/2}$ with domain the Sobolev space $\mathcal{H}^s(\mathbb{R}^n)$ and consider $A = −\frac{i}{2}(x \cdot \nabla + \nabla \cdot x)$. Then $\mathcal{S} = C_c^\infty(\mathbb{R}_+)$ is a core for $H$ and homogeneity of $H$ with respect to $A$ implies that $[H, iA] = sH$. The estimate of Example 1 follows.

2.7.6 Example 6: Multiplication by $\lambda$ in $L^2(\mathbb{R}_+, d\mu)$.

Let $H = \lambda$ and $A = −\frac{i}{2}(\lambda \partial_\lambda + g(\lambda))$ on $L^2(\mathbb{R}_+, d\mu)$, where $g$ is to be determined. Assume that $d\mu = h(\lambda)d\lambda$, for some non-vanishing function $h$ of class $C^1(\mathbb{R}_+)$. It can be shown that if $g$ satisfies the relation $g(\lambda) = \lambda \frac{h'}{h} + 1$, then $A$ is self-adjoint in $L^2(\mathbb{R}_+, d\mu)$. For instance, if $h(\lambda) = \lambda^N$ then choose $g(\lambda) = N + 1$. If $g$ is a bounded function, $\mathcal{S} = C_c^\infty(\mathbb{R}_+)$ is a core for $A$ and the commutation relation is $[H, iA] = 2H$. For $z \in \rho(H)$ the function $(\lambda - z)^{-1}$ is smooth and has bounded derivative on $\mathbb{R}_+$, hence the domain invariance $R(z)D(A) \subset D(A)$ can be easily checked. Therefore $H$ is of class $C^1(A)$, which gives the estimate of Example 1.

2.7.7 Example 7: Dirac operator in $L^2(\mathbb{R}^3; \mathbb{C}^4)$.

We consider the Dirac operator for a spin-1/2 particle of mass $m > 0$ given by $H = \alpha \cdot P + \beta m$ on $\mathcal{H} = L^2(\mathbb{R}^3; \mathbb{C}^4)$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta$ denote the $4 \times 4$ Dirac matrices. The domain of $H$ is the Sobolev space $\mathcal{H}^1(\mathbb{R}^3; \mathbb{C}^4)$ and it is known that $\sigma(H) = \sigma_{ac}(H) = (−\infty, −m] \cup [m, \infty)$. See the book of Thaller [77].

The Foldy-Wouthuysen transformation $U_{FW}$ maps the free Dirac operator into a $2 \times 2$ block form. Consider the Newton-Wigner position operator $Q_{NW}$ defined as the inverse FW transformation of multiplication by $x$, that is, $Q_{NW} = U_{FW}^{-1,1}Q_{FW}$. Using $A = Q_{NW}$, then $H$ is of class $C^1(A)$ and direct calculation shows that $[H, iA] = √H^2 − m^2H^{-1}$ [63]. The following decay estimate follows from this commutation relation.

**Proposition 2.7.1.** Let $H$ and $A$ as above. Then for $u \in D(A) \cap \mathcal{E}$ such that $Au \in \mathcal{E}$, one has the estimate $|\psi_u(t)| \leq C_u(t)^{-1/2}$.

**Proof.** Let $\varphi \in C_c^\infty([m, \infty))$ real and equal to one on a small interval $[m, m + \epsilon]$ and set
\[ \phi = \varphi(H), \phi^\perp = 1 - \phi^2. \] For simplicity we assume \( u \) in the subspace of positive energies, then \( \psi_u = \psi_{\phi u} + \langle \phi^\perp u | e^{iH} u \rangle \). For the high-energy region

\[
t\langle u | \phi^\perp e^{iH} u \rangle = \langle \phi^\perp u | te^{iH} u \rangle = \langle \phi^\perp u | H(H^2 - m^2)^{-1/2} [e^{iH}, A] u \rangle = \langle H(H^2 - m^2)^{-1/2} \phi^\perp e^{-iH} u | Au \rangle - \langle AH(H^2 - m^2)^{-1/2} \phi^\perp u | e^{iH} u \rangle,
\]

and it follows that \( |\langle u | \phi^\perp e^{iH} u \rangle| \leq C \langle t \rangle^{-1/2} \).

For energy close to \( m \), assume that the support of \( u \) in a spectral representation of \( H \) is contained in a compact interval.

Note that \( [e^{i(H-m)}, A] = t \sqrt{H^2 - m^2} H^{-1} e^{i(H-m)} \) as continuous forms on \( D(H) \).

Define the auxiliary function \( \psi(t) = \langle u | e^{it(H-m)} u \rangle \). Then

\[
-ipt' = \langle u | (H - m)te^{i(H-m)} u \rangle = \langle (H - m)^{1/2} u | H(H + m)^{-1/2} [e^{i(H-m)}, A] u \rangle = \langle (H - m)^{1/2} H(H + m)^{1/2} e^{-i(H-m)} u | Au \rangle - \langle A(H - m)^{1/2} H(H + m)^{-1/2} u | e^{i(H-m)} u \rangle.
\]

The right hand side is in \( L^2_t \) because \( H \in C^1(A) \) and \( u \) is compactly supported so Lemma 2.2.3 applies. We conclude that \( |\psi(t)| \leq C \langle t \rangle^{-1/2} \) for all \( t \) and since \( |\psi_u| = |\psi| \) the result is proven. \( \square \)

### 2.7.8 Example 8: Wave equation in \( \mathbb{R}^n \).

For \( H > 0 \) consider the equation

\[
(WE) \quad \begin{cases} 
\partial_{tt} u + H^2 u = 0 \\
u(0) = f \\
\partial_t u(0) = g.
\end{cases}
\]

Assume \( H = L^2(\mathbb{R}^n) \). Define \( u_1(t) := \cos(tH) \), \( u_2(t) := \frac{\sin(tH)}{H} \). Then \( u(t) = u_1(t) f + u_2(t) g \) is a solution to \( (WE) \).
For \( f, g \in \mathcal{H} \) define the function \( \psi_{f,g}(t) := \langle f | u_1(t) f \rangle + \langle f | u_2(t) g \rangle \) and the subspace \( \mathcal{E} = \{ u \in \mathcal{H} \mid [u]_H < \infty \} \), where \([h]_H = \| \langle h | u_1(t) h \rangle \|_{L^2_t}^{1/2} + \| \langle h | u_2(t) h \rangle \|_{L^2_t}^{1/2}\).

**Proposition 2.7.2.** Let \( H \) and \( A \) be self-adjoint operators, assume \( H \in C^1(A) \) and the commutation relation \([H, iA] = cH\), with \( c \neq 0\). Then for \( f, g \in D(A) \cap \mathcal{E} \) such that \( Af, Ag \in \mathcal{E} \), one has the estimate \( |\psi_{f,g}(t)| \leq C_{f,g}(t)^{-1/2} \).

**Proof.** Similarly to Proposition 2.3.8, the following two sesquilinear forms restricted to \( D(A) \cap D(H) \) extend to continuous forms on \( D(H) \) satisfying the identities

\[
\begin{bmatrix}
\cos(tH), iA \\
\sin(tH)/H, iA
\end{bmatrix} = \begin{bmatrix} -ctH \sin(tH) \\ ct \cos(tH) - c \frac{\sin(tH)}{H} \end{bmatrix}.
\]

We will use Corollary A.2 for \( f, g \in D(|H|^{1/2}) \). Clearly \( \psi_{f,g} \in L^2(\mathbb{R}) \). Now we calculate

\[
ct \psi'_{f,g}(t) = -\langle f | ctH \sin(tH) f \rangle + \langle f | ct \cos(tH) g \rangle
\]
\[
= \langle f | [u_1, iA] f \rangle + \langle f | [u_2, iA] g \rangle + c \langle f, u_2 g \rangle
\]
\[
= \langle u_1 f | iAf \rangle + \langle iAf | u_1 f \rangle + \langle u_2 f | iAg \rangle + \langle iAg | u_2 g \rangle + c \langle f, u_2 g \rangle.
\]

Thus \( c \| \psi_{f,g} \|_{L^2} \leq C[f]_H ([g]_H + [Ag]_H + [Af]_H) \). For \( f, g \) not necessarily in \( D(H) \) we can proceed analogously to Proposition 2.4.8 using \( u_\epsilon = R_\epsilon u \) and letting \( \epsilon \to 0 \).

\[\square\]

**2.7.9 Example 9: Klein-Gordon equation in \( \mathbb{R}^n \).**

Now we draw our attention to (WE) in the case \( H = \sqrt{-\Delta + m^2} \), for \( m > 0 \). The vector space is again defined as \( \mathcal{E} = \{ u \in \mathcal{H} \mid [u]_H < \infty \} \), where \([h]_H = \| \langle h | e^{itH} h \rangle \|_{L^2_t}^{1/2}\). Let \( A \) be the generator of dilations, then \( H \) is of class \( C^1(A) \) and it can be formally shown that \([H, iA] = H - m^2 H^{-1}\).

Let \( u_1, u_2 \) be as in (WE), define \( \psi_{f,g}^1(t) = \langle f | u_1(t) f \rangle \) and \( \psi_{f,g}^2(t) = \langle f | u_2(t) g \rangle \). We are interested in the decay rate of \( \psi_{f,g} := \psi_{f,g}^1(t) + \psi_{f,g}^2(t) \).

**Proposition 2.7.3.** For \( H \) and \( A \) defined as above and \( f, g \in D(A) \cap \mathcal{E} \) such that \( Af, Ag \in \mathcal{E} \), then \( |\psi_{f,g}(t)| \leq C_{f,g}(t)^{-1/2} \).
For the second term, we write

\[ \psi \]

which again yields the estimate

\[ \psi \]

Now we prove the desired estimate. Observe that

\[ \psi \]

We conclude that

\[ \psi \]

By Lemma 2.2.3 we conclude that \( \| \delta \psi \|_{L^2} \leq C([f]_H^2 + [f]_H[A]_H + [Af]_H^2) \). For general \( g \in D(A) \), replace it by \( g_\epsilon = R_\epsilon g \) and let \( \epsilon \to 0 \).

We conclude that \( |\psi(t)| \leq C_{f,g}(t)^{-1/2} \). Notice that \( |\langle f|e^{itH}g \rangle| = |\psi(t)| \) satisfies the same inequality.

Now we prove the desired estimate. Observe that \( \psi_{f,g}^{1}(t) = \frac{1}{2} (\langle f|e^{itH}f \rangle + \langle f|e^{-itH}f \rangle) \), therefore \( |\psi_{f,g}^{1}(t)| \leq C_f(t)^{-1/2} \).

For the second term, we write \( \psi_{f,g}^{2}(t) = \frac{1}{2} (\langle f|H^{-1}e^{itH}g \rangle + \langle f|H^{-1}e^{-itH}g \rangle) \) and redefine the auxiliary function \( \psi(t) := \langle f|H^{-1}e^{it(H-m)}g \rangle \), which is in \( L^2(\mathbb{R}) \) by the spectral theorem. Now

\[
-\frac{d}{dt}\psi'(t) = \langle f|H^{-1}(H - m)e^{it(H-m)}g \rangle \\
= \langle f|e^{it(H-m)}A(H + m)^{-1}g \rangle \\
= \langle e^{-it(H-m)}f|A(H + m)^{-1}g \rangle + \langle Af|e^{it(H-m)}(H + m)^{-1}g \rangle,
\]

which again yields the estimate \( |\psi(t)| \leq C_{f,g}(t)^{-1/2} \), concluding the proof. \( \square \)
Chapter 3

The abstract method: perturbed Hamiltonians

3.1 Introduction

The abstract method presented in Chapter 2 can only deal with commutation relations \([H, iA]\) that are equal to functions of \(H\) (or commuting with \(H\)). This is certainly restrictive since for general Hamiltonians (for example, with potential \(V\)) there is no known procedure to find or construct a suitable conjugate operator \(A\). In this chapter we show that our methodology based on commutators can be extended in order to capture more general situations. More precisely, we prove that a commutation relation of type \([H, iA] = Q(H) + K\), in addition to regularity of \(H\) and Kato-smoothness of \(K\), still guarantees time decay rates of diverse order. The methodology is based on the construction of a modified conjugate operator \(\tilde{A}\) that reduces the problem to the estimates of Chapter 2. Consequently, these results also apply at energy thresholds and do not rely on resolvent estimates. We will conclude this chapter discussing applications to the (SE) with potential of critical decay and for the free (SE) on an asymptotically flat manifold.

3.2 Preliminaries

Let \(A\) and \(H\) be two self-adjoint operators on a Hilbert space \(\mathcal{H}\). We are interested in extending the results of Chapter 2 to the case where \([H, iA]\) is not necessarily equal to a function of \(H\). In this work we will assume a more general commutation relation which will yield similar decay estimates by means of a suitable adaptation of the conjugate operator \(A\).

We now review some standard definitions in functional analysis. As usual, we write \(\langle x \rangle = (1 + x^2)^{1/2}\). Denote \(\mathcal{H}^1 = D(H)\) the domain of \(H\) and consider its adjoint space \(\mathcal{H}^{-1} = D(H)^*\).

Finally, we discuss the notion of Kato-smoothness. We shall say that a closed operator \( C \) determined continuous sesquilinear form \( 6.2.10 \) in \([1]\) shows that \( D(H) \) of class \( C_a \) with relative norm \( z \) if the above identity holds in \( H \) function \( H \) always equip the domain of an operator with its graph topology. If \( Q \) is unbounded and \( D(Q) \supset H^1 \) then we say that \( Q \) commutes with \( H \) if the above identity holds in \( B(H^1, H) \) for \( Q \).

Consider \( Q \) a densely defined operator on \( H \) with \( D(Q) \supset H^1 \). We say that \( Q \) is \( H \)-bounded with relative norm \( a \) if for some \( a, b \in \mathbb{R} \) one has \( \|Q \psi\| \leq a\|H \psi\| + b\|\psi\| \), for all \( \psi \in H^1 \).

If \( S \) is a bounded operator on \( H \) then we denote \([A, S]_o\) the sesquilinear form on \( D(A) \) defined by \([A, S]_o(u, v) = \langle Au|Sv\rangle - \langle u|SAv\rangle\). We say that \( S \) is of class \( C^1(A) \), and we write \( S \in C^1(A) \), if \([A, S]_o\) is continuous for the topology induced by \( H \) on \( D(A) \) and then we denote \([A, S] \) the unique bounded operator on \( H \) such that \( \langle u|[A, S]|v\rangle = \langle Au|Sv\rangle - \langle u|SAv\rangle \) for all \( u, v \in D(A) \). We consider now the rather subtle case of unbounded operators. Note that we always equip the domain of an operator with its graph topology. If \( H \) is a self-adjoint operator on \( H \) then \([A, H]_o\) is the sesquilinear form on \( D(A) \cap D(H) \) defined by \([A, H]_o(u, v) = \langle Au|Hv\rangle - \langle Hu|Av\rangle\). A convenient definition of the \( C^1(A) \) class for any self-adjoint operator is as follows. Let \( R(z) = (H - zI)^{-1} \) for \( z \) in the resolvent set \( \rho(H) \) of \( H \). We say that \( H \) is of class \( C^1(A) \) if \( R(z) \in C^1(A) \) for some (hence for all) \( z \in \rho(H) \). In this case, Proposition 6.2.10 in [1] shows that \( D(A) \cap H^1 \) is dense in \( H^1 \) and hence \([A, H]_o\) extends to a uniquely determined continuous sesquilinear form \([A, H]\) on \( H^1 \). Further properties and examples of the \( C^1 \) regularity can be found in Chapter 2 and the paper [22].

Finally, we discuss the notion of Kato-smoothness. We shall say that a closed operator \( E \) is \( H \)-smooth on the range of a bounded operator \( P \) if and only if for each \( \psi \in H \) and each \( \epsilon \neq 0 \), \( R(\lambda + i\epsilon)P \psi \in D(E) \) for almost all \( \lambda \in \mathbb{R} \) and moreover

\[
\|E\|^2_H = \sup_{\|\psi\|=1} \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \left( \|ER(\lambda + i\epsilon)P \psi\|^2 + \|ER(\lambda - i\epsilon)P \psi\|^2 \right) d\lambda < \infty.
\]

In particular \( E \) is \( H \)-smooth on the range of \( P \) if and only if for all \( \psi \in H \), \( e^{itH}P \psi \in D(E) \) for almost every \( t \in \mathbb{R} \) and

\[
\int_{-\infty}^{\infty} \|Ee^{-itH}P \psi\|^2 dt \leq 2\pi\|E\|^2_H \|P \psi\|^2.
\]
In several applications the smoothness can be controlled more precisely using Sobolev norms as follows. Let $M$ be an $n$-dimensional Riemannian manifold with a smooth Riemannian metric $g_{ij}$. Consider the Hilbert space $\mathcal{H} := L^2(M)$ with the inner product defined as $\langle \phi | \psi \rangle = \int_M \phi(x) \overline{\psi(x)} dg$, where $dg := \sqrt{\det g_{ij}}$. Denote by $\| \cdot \|$ the norm induced by this inner product, that is, $\| \phi \| = \int_M |\phi(x)|^2 dg$. The self-adjoint operator $H$ defined on $\mathcal{H}$ enjoys the standard functional calculus and one can define the homogeneous Sobolev norms $\| \psi \|_{\dot{H}^s(M)} := \| H^{s/2} \psi \|$, for $0 \leq s \leq 1$.

We then generalize the notion of $H$-smoothness taking the supremum on a subspace of $\mathcal{H}$ instead of the whole space. We will say that a closed operator $E$ is $|H|^s$-smooth on the range of a bounded operator $P$ if and only if for all $\psi \in \dot{H}^s(M)$, $e^{itH} P \psi \in D(E)$ for almost every $t \in \mathbb{R}$ and

$$
\int_{-\infty}^{\infty} \| E e^{-itH} P \phi \|^2 dt \leq C_E \| P \psi \|_{\dot{H}^s(M)}^2.
$$

### 3.3 Assumptions

Let $H$ and $A$ be two self-adjoint operators on the Hilbert space $\mathcal{H} = L^2(M)$, where $M$ is a smooth $n$-dimensional Riemannian manifold equipped with the standard $L^2$-inner product $\langle \cdot | \cdot \rangle$ and norm $\| \cdot \|$. Let $P$ be an orthogonal projection and $Q$ be an $H$-bounded operator with relative norm $a \geq 0$. $P$ and $Q$ commute with $H$. Let $E, F$ be linear operators such that $D(E) \cap D(F) \supset P \mathcal{H}^1$. The main assumptions are established as follows.

1. **(H)**
   - $H$ is of class $C^1(A)$
   - $H$ and $A$ satisfy the commutation relation $P[H, iA]P = P(Q + K)P$ for $K \equiv F^* E$, in the sense that for all $\phi, \psi \in \mathcal{H}^1$
     
     $$
     \langle \phi, P[H, iA]P \psi \rangle = \langle P\phi, QP\psi \rangle + \langle FP\phi, EP\psi \rangle.
     $$
   - $K$ is symmetric on $\mathcal{H}^1$, that is, $\langle \phi | K \psi \rangle = \langle K \phi | \psi \rangle$
   - $K \mathcal{H}^1 \subset \mathcal{H}$ and $|H|^{-s/2} K |H|^{-s/2}$ is a bounded operator on $\mathcal{H}$, for some $s > 0$
   - $E$ and $F$ are $|H|^s$-smooth on the range of $P$
**Remark 3.3.1.** Since $H$ is of class $C^1(A)$, the sesquilinear form $[H, iA]_o$ on $\mathcal{H}^1 \cap D(A)$ extends to a continuous operator in $B(\mathcal{H}^1, \mathcal{H}^{-1})$. Then the commutation relation of (H) implies that the sesquilinear form $(P\phi, QP\psi) + (FP\phi, EP\psi)$ restricted to $\mathcal{H}^1 \cap D(A)$ also extends to a bounded operator in the same space. Therefore, the commutation relation can be written at the level of operators in $B(\mathcal{H}^1, \mathcal{H}^{-1})$ as $P[H, iA]P = P(Q + K)P$, for a densely defined symmetric operator $K$.

For some of the decay estimates we will require an additional smoothness condition.

*(Ha)* The sesquilinear form $P[A, K]_oP$ defined on $D(A) \cap D(K)$ satisfies the identity

$$P[A, K]_oP(\phi, \psi) = (F'P\phi, E'P\psi),$$

for all $\phi, \psi \in D(A) \cap D(K)$. Here $E', F'$ hold the same properties of $E, F$ in assumptions (H). Moreover, $P[A, K]_oP$ extends to a densely defined operator $K'$ such that $\langle H \rangle^{-s/2}K'(\langle H \rangle)^{-s/2}$ is bounded on $\mathcal{H}$.

The commutation relation of (H) is the crucial assumption. It replaces the identity of type $[H, iA] = \varphi(H)$ in Chapter 2, with a much more general expression. Indeed, it now admits an operator $Q$ (controlled by $H$) in addition to a Kato-smooth perturbation. Observe that the $C^1(A)$ regularity provides a suitable framework in this case as well.

The strategy to derive decay estimates under these generalized conditions will be based on the construction of a new conjugate operator $\tilde{A}$ which will simplify the commutation identity, thus reducing the problem to our preceding estimates. The remaining assumptions of (H) will justify this construction and other algebraic manipulations.

### 3.4 The conjugate operator

This section aims to construct the conjugate operator $\tilde{A}$. Using the functional calculus define for $s > 0$ the cut-off $h_s(H) := \langle H \rangle^{-s/2}$. For any operator $X$ denote $X_h := h_s(H)Xh_s(H)$ (omitting the parameter $s$ in $X_h$ for brevity).
We first consider the operator $A_h$. Note that since $H \in C^1(A)$, the operator $h_s(H)$ is of class $C^1(A)$ for $s$ large enough (see the comment before Theorem 2.3.7 in Chapter 2). Moreover, by Lemma 6.2.9 in [1], the $C^1(A)$ condition for $h_s(H)$ holds for any $s > 0$. This follows from the continuity of the form $[A, h_s(H)]$ for the topology induced by $\mathcal{H}$, which is a consequence of assumptions (H). Note that $D(A_h) \supset D(A)$ and therefore $A_h$ is a densely defined symmetric operator. With some abuse of notation we denote its closure by $\tilde{A}_h$. Lemma 7.2.15 in [1] proves that $A_h$ is self-adjoint and $D(A)$ is a core.

The next step is to add a linear perturbation to $A_h$. Define for each $t \in \mathbb{R}$ the operator $h_s(H)U_t h_s(H) := \int_0^t e^{-isH} PK_h P e^{isH} \, ds$, which is bounded under assumptions (H). We now show that the limiting operator $B_h := s\lim_{t \to \infty} h_s(H)U_t h_s(H)$ is well-defined and bounded as well. Let $\phi, \psi$ in $\mathcal{H}$.

\[
|\langle \phi | h_s(H)(U_t - U_r) h_s(H) \psi \rangle | = \left| \int_r^t \langle h_s(H) P \phi | e^{-isH} K e^{isH} h_s(H) P \psi \rangle \, ds \right| \\
\leq \int_r^t |\langle F e^{i s H} h_s(H) P \phi | e^{i s H} h_s(H) P \psi \rangle| \, ds \\
\leq \left( \int_0^\infty \| F e^{i s H} h_s(H) P \phi \|^2 \, ds \right)^{1/2} \left( \int_r^t \| F e^{i s H} h_s(H) P \psi \|^2 \, ds \right)^{1/2} \\
\leq C_F \| h_s(H) P \phi \|_{H^s(M)} \left( \int_r^t \| F e^{i s H} h_s(H) P \psi \|^2 \, ds \right)^{1/2} \\
\leq C_F \| \phi \| \left( \int_r^t \| F e^{i s H} h_s(H) P \psi \|^2 \, ds \right)^{1/2}.
\]  

(3.4.1)

Since the integrand of the last step is in $L^1(\mathbb{R})$ and $\phi$ is arbitrary, we conclude that the sequence converges strongly to the bounded operator $B_h$.

Finally, define $\tilde{A} := A_h + B_h$. Note that $D(\tilde{A}) = D(A_h) \supset D(A)$. The following result justifies this construction.

**Proposition 3.4.1.** Assume (H). Then $\tilde{A}$ defined as above is self-adjoint. Moreover, $H$ is of class $C^1(\tilde{A})$ and the continuous sesquilinear form $[H, i \tilde{A} \rangle_0$ on $\mathcal{H}^1 \cap D(\tilde{A})$ is identified with an operator in $B(\mathcal{H}^1, \mathcal{H})$ satisfying the commutation relation $P[H, i \tilde{A}] P = Q_h P$.

**Proof.** Note that $B_h$ is bounded and symmetric by construction, thus the self-adjointness of $A_h$ is guaranteed by the Kato-Relich theorem. The $C^1(\tilde{A})$ property follows from the identity
\[ [H, A_h] = [H, A]_h \] as operators in \( B(\mathcal{H}^1, \mathcal{H}^{-1}) \) which is a consequence of Proposition 7.2.16, in addition to Proposition 6.2.10 in [1].

We now prove the commutation relation. Using the functional calculus define \( R_\epsilon = (1 + i\epsilon H)^{-1} \) and the bounded operator \( H_\epsilon := HH_\epsilon = (i\epsilon)^{-1}(1 - R_\epsilon) \). Note that \( P[H_\epsilon, K_h]_\circ P \) is a continuous sesquilinear form on \( \mathcal{H} \) satisfying

\[
P[H_\epsilon, K_h]_\circ P = i\epsilon^{-1}P[R_\epsilon, K_h]_\circ P = R_\epsilon P[H, K_h]_\circ PR_\epsilon
\]

in form sense. Now we calculate

\[
[H_\epsilon, i\hbar_s(H)U_t\hbar_s(H)]_\circ = \frac{1}{\epsilon} \int_0^t e^{-i\epsilon H} [R_\epsilon, iPK_h P]_\circ e^{i\epsilon H} ds
\]

\[
= R_\epsilon \left( \int_0^t e^{-i\epsilon H} P[H, iK_h]_\circ Pe^{i\epsilon H} ds \right) R_\epsilon
\]

\[
= R_\epsilon e^{-i\epsilon H} PK_h Pe^{i\epsilon H} R_\epsilon - R_\epsilon PK_h PR_\epsilon. \tag{3.4.2}
\]

Note that for any \( \phi, \psi \in \mathcal{H}^1 \) one has \( (\phi, e^{-i\epsilon H} PK_h Pe^{i\epsilon H} \psi) \to 0 \) for a subsequence \( t_k \to \infty \). This follows from the estimate

\[
\int_0^\infty |(\phi, e^{-i\epsilon H} PK_h Pe^{i\epsilon H} \psi)| dt = \int_0^\infty |(Fe^{i\epsilon H} h_s(H)P\phi, Ee^{i\epsilon H} h_s(H)P\psi)| dt
\]

\[
\leq C \int_0^\infty \left( \|Fe^{i\epsilon H} h_sP\phi\|^2 + \|Fe^{i\epsilon H} h_sP\psi\|^2 \right) dt
\]

\[
\leq C_F \|\phi\|^2 + C_E \|\psi\|^2.
\]

Then let \( \epsilon \to 0 \) on the RHS of (3.4.2) which converges to \(-PK_hP\) in the weak form sense on \( \mathcal{H} \). Finally, by making \( t_k \to \infty \) in \([H_\epsilon, i\hbar_s(H)U_t\hbar_s(H)]_\circ\) and then \( \epsilon \to 0 \) in weak form sense in \( \mathcal{H}^1 \), we conclude that the form \([H, iB_h]_\circ\) with domain \( \mathcal{H}^1 \) extends to a bounded sesquilinear form on \( \mathcal{H} \) satisfying \([H, iB_h]_\circ = -PK_hP\).

Now, as sesquilinear forms in \( D(H) \cap D(\tilde{A}) \) we have \([H, i\tilde{A}]_\circ = [H, iA_h]_\circ + [H, iB_h]_\circ\). By the previous discussion we conclude that \([H, i\tilde{A}]_\circ\) is continuous for the topology induced by \( H \) on \( \mathcal{H}^1 \cap D(\tilde{A}) \) and moreover

\[
P[H, i\tilde{A}]P = P[H, iA]_hP - PK_hP
\]

\[
= Ph_s(H)(Q + K)h_s(H)P - PK_hP
\]

\[
= PQ_hP,
\]
which concludes the proof.

Once we have established a suitable commutation relation for $H$ and $\tilde{A}$ in Proposition 3.4.1, we recall two important commutator identities obtained in Chapter 2. The proofs are analogous and use $\tilde{A}$ as the conjugate operator.

**Proposition 3.4.2.** Let $H$ be a self-adjoint operator of class $C^1(\tilde{A})$. Then the restriction of $[\tilde{A}, e^{itH}]_0$ to $D(\tilde{A}) \cap \mathcal{H}^1$ extends to a continuous form $[\tilde{A}, e^{itH}]_0$ and, in the strong topology of space of sesquilinear forms on $\mathcal{H}^1$, we have

$$
[e^{itH}, \tilde{A}] = \int_0^te^{i(t-s)H}[H, i\tilde{A}]e^{isH}ds.
$$

Moreover, under the conditions of (H) we have $P[e^{itH}, \tilde{A}]P = tPQ_hPe^{itH}$ as operators in $B(\mathcal{H}^1, \mathcal{H})$.

**Proof.** Same as Theorem 2.3.7.

**Proposition 3.4.3.** Let $H$ be a self-adjoint operator of class $C^1(\tilde{A})$. Then $D(\tilde{A}) \cap \mathcal{H}^1$ is dense in $\mathcal{H}^1$ and

$$
[\tilde{A}, R(z)] = -R(z)[\tilde{A}, H]R(z) \quad \text{for all } z \in \rho(H).
$$

**Proof.** Proposition 6.2.10 in [1].

In the next section, it will be useful to “commute $A$ through $B$” in the important case $Q = cH$, for some $c \neq 0$. To give precise meaning to this, we consider the sesquilinear form $C_0$ on $D(\tilde{A})$ defined as $C_0(\phi, \psi) := P[\tilde{A}, B_h]P(\phi, \psi) = (\tilde{A}P\phi, B_hP\psi) - (B_hP\phi, A \tilde{P}\psi)$. The next result justifies our assertion.

**Proposition 3.4.4.** Under conditions (H) and (Ha) in the case $Q = cH$, the sesquilinear form $C_0$ defined on $D(\tilde{A})$ as above can be extended to a bounded operator in $\mathcal{H}$, which we denote by $C$.

**Proof.** Step 1: Commutator identity
As before, define $R_\epsilon = (1 + i\epsilon H)^{-1}$ and the bounded operator $H_\epsilon := HH_\epsilon$. For $\phi, \psi$ in $D(\tilde{A})$ consider the form

$$D_\epsilon(\phi, \psi) := \frac{i((\tilde{A}H_\epsilon P\phi, B_\epsilon R_\epsilon^* P\psi) - (B_\epsilon H_\epsilon P\phi, \tilde{A}PR_\epsilon^* \psi))}{D_1^1} - \frac{i((\tilde{A}PR_\epsilon \phi, B_\epsilon H_\epsilon^* P\psi) - (B_\epsilon R_\epsilon P\phi, \tilde{A}H_\epsilon^* P\psi))}{D_2^2}. \quad (3.4.3)$$

We now use Prop. 3.8 in [24] to calculate the commutator identities $P[H_\epsilon, i\tilde{A}]P = cH_\epsilon R_\epsilon^2 P$ and $[H_\epsilon, iB_\epsilon] = -PR_\epsilon K_h R_\epsilon P$, which will be used to expand the expression above.

$$D_1^1 = (P[i\tilde{A}, H_\epsilon]P\phi, B_\epsilon \psi) + i(H_\epsilon \tilde{A}P\phi, B_\epsilon \psi) - i(\phi, H_\epsilon^* B_\epsilon \tilde{A}P\psi)$$

$$= -(cH_\epsilon R_\epsilon^2 P\phi, B_\epsilon \psi) - (\tilde{A}P\phi, H_\epsilon^* iB_\epsilon \psi) + (\phi, H_\epsilon^* iB_\epsilon \tilde{A}P\psi)$$

$$= -(cH_\epsilon R_\epsilon^2 P\phi, B_\epsilon \psi) - (\tilde{A}P\phi, [H_\epsilon^*, iB_\epsilon]\psi)$$

$$= -(\phi, PR_\epsilon^* K_h R_\epsilon^* \tilde{A}P\psi) + (\phi, iB_\epsilon H_\epsilon^* \tilde{A}P\psi)$$

Thus, $D_\epsilon(\phi, \psi) = [B_\epsilon, cH_\epsilon R_\epsilon^2 P]_\circ(\phi, \psi) + [\tilde{A}P, PR_\epsilon K_h R_\epsilon P]_\circ(\phi, \psi)$. Note that each operator on the RHS converges strongly in the operator sense as $\epsilon \to 0$.

The first term can be extended to the bounded operator $-icPK_h^2 P$ (see proof of Prop. 3.4.1). On the other hand, by conditions (H) and (Ha) the second term can be expanded into Kato-smooth operators

$$[\tilde{A}P, PK_h P]_\circ(\phi, \psi) = [\tilde{A}h P, PK_h P]_\circ(\phi, \psi) + [B_h P, PK_h P]_\circ(\phi, \psi)$$

$$= (F'Ph\phi, E'Ph\psi) + (FPB_h P\phi, EPh\psi) - (EPh\phi, FPhB_h P\psi),$$

and thus the sesquilinear form can be extended to a bounded operator in $\mathcal{H}$ satisfying

$$\|[\tilde{A}P, PK_h P]\| \leq C(E, F, E', F').$$

We conclude that the sesquilinear form $D_\epsilon$ can be extended to a bounded operator in $\mathcal{H}$ denoted by $C_0$. 
Step 2: Convergence as $t \to \infty$

Fix $\epsilon > 0$ and for $\phi, \psi \in D(A)$ denote $\phi_\epsilon = R_\epsilon \phi$, $\psi_\epsilon = R_\epsilon \psi$. Define the sesquilinear form $C_t(\phi, \psi) := (A e^{itH} P \phi, B e^{itH} P \psi) - (B e^{itH} P \phi, A e^{itH} P \psi)$. Note that

$$\frac{d}{dt} C_t(\phi_\epsilon, \psi_\epsilon) = D_\epsilon(e^{itH} \phi, e^{itH} \psi),$$

so we use an estimate similar to (3.4.1) to prove convergence in $t$.

$$|(C_t - C_r)(\phi_\epsilon, \psi_\epsilon)|$$

$$= \left| \int_r^t D_\epsilon(e^{i\tau H} \phi, e^{i\tau H} \psi) d\tau \right|$$

$$\leq \int_r^t |\langle P \phi_\epsilon | e^{i\tau H} cK_h e^{i\tau H} P \psi_\epsilon \rangle| d\tau + \int_r^t |\langle \phi | [AP, PR_\epsilon K_h R_\epsilon P] \psi_\epsilon \rangle| d\tau$$

$$\leq C(F) \|\phi\| \left( \int_r^t \|e^{i\tau H} h_s P \psi_\epsilon\|^2 ds \right)^{1/2} + C(F') \|\phi\| \left( \int_r^t \|E e^{i\tau H} h_s P \psi_\epsilon\|^2 ds \right)^{1/2}.$$

By Kato-smoothness of the RHS we conclude that the sequence is Cauchy in $t$ (for fixed $\epsilon$) and moreover, $C_t(\phi_\epsilon, \psi_\epsilon) \to 0$ as $t \to \infty$.

We proceed analogously to prove the result of the theorem.

$$|C_0(\phi_\epsilon, \psi_\epsilon)| \leq \limsup_t \left\{ |C_{t, \epsilon}(\phi, \psi)| + \int_0^t D_\epsilon(e^{i\tau H} \phi_\epsilon, e^{i\tau H} \psi_\epsilon) d\tau \right\}$$

$$\leq C(E, F, E', F') \|\phi_\epsilon\| \|\psi_\epsilon\|.$$

Since $R_\epsilon D(A)$ is dense in $\mathcal{H}$, we conclude that the sesquilinear form

$$C_0(\phi, \psi) = (AP \phi, BP \psi) - (BP \phi, AP \psi)$$

restricted to $D(A)$ extends to a bounded operator on $\mathcal{H}$. \qed

**Corollary 3.4.5.** $B_h$ leaves invariant the domain of $\tilde{A}$, that is, $B_h D(\tilde{A}) \subset D(\tilde{A})$.

**Proof.** Let $\epsilon > 0$ and $\psi \in D(\tilde{A})$, $\phi \in \mathcal{H}$. Denote $R_\epsilon(\tilde{A}) = (1 + i\tilde{A})^{-1}$ and $\tilde{A}_\epsilon = \tilde{A} R_\epsilon(\tilde{A})$. 


Recall also that \( PB_h P = B_h \) and \([P, \hat{A}]\) extends to a bounded operator since \( H \in C^1(\hat{A}) \).

\[
|\langle \phi | \hat{A} B_h \psi \rangle| = |\langle P \hat{A} \Re(e^\hat{A}^* \phi) B_h \psi \rangle| \\
= |\langle [P, \hat{A}](1 - i\epsilon \hat{A})^{-1} \phi | B_h \psi \rangle + \langle \hat{A} P(1 - i\epsilon \hat{A})^{-1} \phi | B_h \psi \rangle| \\
\leq C \| \phi \| \| \psi \| + |\langle (1 - i\epsilon \hat{A})^{-1} \phi | P[\hat{A}, B_h] \psi \rangle| + |\langle B_h P(1 - i\epsilon \hat{A})^{-1} \phi | P\hat{A} \psi \rangle| \\
\leq C \| \phi \| (\| \psi \| + \| \hat{A} \psi \|),
\]

where the constant \( C \) is independent of \( \epsilon \). Thus, \( \| \hat{A} \epsilon B_h \psi \| \leq C(\| \psi \| + \| \hat{A} \psi \|) \) and we conclude by Fatou’s lemma.

\[
\square
\]

### 3.5 Decay estimates

**Definition 3.5.1.** For \( u \in \mathcal{H} \), define the function \( \psi_u(t) := \langle u, e^{itH} u \rangle \), \( t \in \mathbb{R} \) and the set

\[
\mathcal{E} = \{ u \in \mathcal{H} : \psi_u \in L^2(\mathbb{R}) \}.
\]

For \( u \in \mathcal{E} \) denote \( [u]_H = \| \psi_u \|_{L^2}^{1/2} \).

In Chapter 2 it was shown that \( \mathcal{E} \) is a dense linear subspace of the absolutely continuity subspace of \( H \) and \([\cdot]_H\) is a complete norm on it. We will now consider the following additional assumption on the space \( \mathcal{E} \), which will be relevant for Propositions 3.5.3 and 3.5.6 where no assumption of positivity of \( Q \) is made.

\( (H_b) \) For any \( u \in \mathcal{H} \) one has \( (A_h + i)^{-1} u \in \mathcal{E} \).

We now proceed to prove the main results of this chapter.

**Theorem 3.5.2.** Assume \( (H) \) with \( Q \geq 0 \). Then for \( u \in D(|H|^{s/2}) \) such that \( P u \in D(A) \cap D(Q^{1/2}) \) we have the estimate \( \| \psi_{Q^{1/2} P u}(t) \| \leq C_u(t)^{-1} \).
Proof. Assume first that \( u \in \mathcal{H}^1 \) and define \( v = (H)^{s/2}u \).

\[
\begin{align*}
t \psi_{Q^{1/2}P_u}(t) &= \langle Q^{1/2}P_u | te^{itH} Q^{1/2}P_u \rangle \\
&= \langle v | tPQhe^{itH}Pv \rangle \\
&= \langle v | P[e^{itH}, \tilde{A}]Pv \rangle \\
&= \langle e^{-itH}Pv | \tilde{A}Pv \rangle - \langle \tilde{A}Pv | e^{itH}Pv \rangle. \tag{3.5.4}
\end{align*}
\]

We now expand the first term of the last expression, the second one is analogous.

\[
|\langle e^{-itH}Pv | \tilde{A}Pv \rangle| = |\langle e^{-itH}Pv | Apv \rangle + \langle e^{-itH}Pv | B_{\epsilon}Pv \rangle| \leq \|u\| \|PAPv\| + C \|Q^{s/2}Pu\|.
\]

Hence \( |t \psi_{Q^{1/2}P_u}(t)| \leq 2(\|u\| \|PAPu\| + C \|Q^{s/2}Pu\|). \)

For general \( u \) we define \( u_\epsilon := R_\epsilon u \in \mathcal{H}^1 \). Note that \( Pu_\epsilon \in D(Q^{1/2}) \) since \( Q \) commutes with \( H \) and thus from (3.5.4) we obtain the estimate

\[
|t \psi_{Q^{1/2}P_{u_\epsilon}}(t)| \leq 2 \|v_\epsilon\| \|P\tilde{A}Pv_\epsilon\|. \tag{3.5.5}
\]

Since \( H \in C^1(\tilde{A}) \), from Proposition 12 in [24] we obtain that \( PR_\epsilon v \in D(\tilde{A}) \). Moreover, Proposition 3.4.3 implies that \([\tilde{A}, R_\epsilon] = \epsilon R_\epsilon [H,i\tilde{A}]R_\epsilon \), hence \( P[\tilde{A}, R_\epsilon]P = \epsilon PR_\epsilon QR_\epsilon P \) as operators in \( \mathcal{H} \). Now we use that \( Q \) is \( H \)-bounded with relative norm \( a \), which yields

\[
\|P\tilde{A}Pv_\epsilon\| = \|P[\tilde{A}, R_\epsilon]Pv\| + \|PR_\epsilon \tilde{A}Pv\| \leq \epsilon (a \|HR_\epsilon Pv\| + b \|R_\epsilon Pv\|) + \|\tilde{A}Pv\| \leq (a + \epsilon b) \|Pv\| + \|PAPu\| + C \|Pv\| \leq C \|Q^{s/2}Pu\| + \|PAPu\|.
\]

Finally, let \( \epsilon \to 0 \) and use Fatou’s lemma on the LHS side of (3.5.5) to conclude

\[
|t \psi_{Q^{1/2}P_u}(t)| \leq 2 \|Q^{s/2}u\| \left( C \|Q^{s/2}Pu\| + \|PAPu\| \right).
\]

\[
\square
\]

**Theorem 3.5.3.** Assume \( (H), (Ha) \) and \( (Hb) \) in the special case \( Q = cH \) and \( s = 1/2 \). Then for \( u \in D(A) \) such that \( Pu \) and \( PAPu \) are in \( \mathcal{E} \), one has \( |\psi_{P_u}(t)| \leq C_u \langle t \rangle^{-1/2} \).
Let \( u = u_1 + u_2 \), where \( u_1 = \chi(H)u \) and \( u_2 = (1 - \chi(H))u \). Note that \( \psi_{u_1} = \psi_{pu_1} + \psi_{pu_2} \).

We will show that \( \psi_{u_1} \equiv O(t^{-1/2}) \) and \( \psi_{u_2} \equiv O(t^{-1}) \).

By Corollary A.2 it suffices to prove that \( \delta \psi_{pu_1} \equiv \psi_{pu_1}^{(t)}(t) \) is in \( L^2(\mathbb{R}) \). Define \( v = \langle H \rangle^{s/2} u_1 \).

We now expand the first term of the above expression, the second one is analogous.

\[
\langle e^{-itH} | P \tilde{A} P v \rangle = \langle e^{-itH} | P A_h P v \rangle + \langle e^{-itH} | P B_h P v \rangle
\]

\[
= \langle e^{-itH} | PA_h u_1 \rangle + \langle e^{-itH} | P(A_h + i)B_h P v \rangle.
\]

Note that \( (A_h + i)B_h P v = \[A_h,B_h\]Pv + B_h(A_h + i)Pv \), which is in \( H \) by Proposition 3.4.4 and the fact that \( v \in D(A_h) \).

Thus \( c\|\delta \psi_{pu_1}\| \leq 2\|[Pu_1]_H\left|(PA_h u_1) + \left[(A + i)^{-1}(C\langle H \rangle^{s/2}u_1 + B_h\langle H \rangle^{s/2}AP u_1)\right]\right|_H\), and we conclude \( |\psi_{u_1}(t)| \leq C_{u_1}(t)^{-1/2} \).

To estimate the decay of \( u_2 \) we now consider \( v \in H \) such that \( u_2 = |H|^{1/2}\langle H \rangle^{-1/2}v \). Then

\[
ct \psi_{pu_2}(t) = \langle Pu_2 | cte^{itH} Pu_2 \rangle
\]

\[
= \langle Pu_2 | cte^{itH} H_h \text{sgn}(H) P v \rangle
\]

\[
= \langle v | P[e^{itH}, \tilde{A}] \text{sgn}(H) P v \rangle. \tag{3.5.6}
\]

Note that \( g(H) := |H|^{-1/2}\langle H \rangle^{1/2}(1 - \chi(H)) \) is a bounded smooth function, so \( \|v\| \leq \|u\| \) and \( PA_h P = g(H) H_h P + P g(H) \tilde{A} P \). Hence the first term of the commutator in (3.5.6) is bounded \( \|\langle e^{-itH} | P \tilde{A} P v \rangle\| \leq \|u\| \left(\|u\| + \|\tilde{A} P u\|\right) \) and we conclude that \( |\psi_{pu_2}(t)| \leq C_u(t)^{-1} \) as desired. \(\Box\)
**Remark 3.5.4.** In case assumptions (H), (Ha) and (Hb) are met with \(s = 0\), the construction of the conjugate operator is simpler because there is no need to introduce the cut-off \(h_s\) (and then \(\tilde{A} = A + B\)). Hence Theorems 3.5.2 and 3.5.3 remain valid. We state them here for completeness, the proofs are analogous.

**Proposition 3.5.5.** Assume (H) with \(Q \geq 0\) and \(s = 0\). Then for \(u \in \mathcal{H}\) such that \(Pu \in D(A) \cap D(Q^{1/2})\) we have the estimate \(|\psi_{Q^{1/2}Pu}(t)| \leq C_u\langle t\rangle^{-1}\).

**Proposition 3.5.6.** Assume (H), (Ha) and (Hb) in the special case \(Q = cH\) and \(s = 0\). Then for \(u \in D(A)\) such that \(Pu\) and \(PAPu\) are in \(E\), one has \(|\psi_Pu(t)| \leq C_u\langle t\rangle^{-1/2}\).

We now study the particular case \(H = H_0 + V(x)\) on \(\mathcal{H} = L^2(\mathbb{R}^n)\), where \(H_0\) is a self-adjoint operator and \(V(x)\) is smooth real-valued function. Let \(P\) be a projection that commutes with \(H\). Let us consider the following assumptions.

\(\text{\textbf{(H1)}}\)

(i) There is a self-adjoint first-order operator \(A\) so that \([H_0, iA] = cH_0\) for some \(c \neq 0\)

(ii) The functions \(V(x)\) and \(w(x) := [V(x), iA]\) are bounded

Conditions (H1) ensure that \(H\) is of class \(C^1(A)\) and it follows that the sesquilinear form \([H, iA]_0 = cH - cV(x) + w(x)\) on \(D(H) \cap D(A)\) can be identified with an operator \([H, iA]\) in \(B(\mathcal{H}^1, \mathcal{H})\). In order to satisfy assumptions (H) with \(Q = cH\) and \(K = -cV(x) + w(x)\) it remains to show the Kato-smoothness condition. We can derive this from decay estimates as follows.

Let \(\sigma > 0\) and \(P_{ac}(H)\) be the projection onto the space of absolute continuity of \(H\). Consider the local decay estimate

\[
\int_{\mathbb{R}} \|\langle x\rangle^{-\sigma/2}e^{-itH}P_{ac}u\|^2 dt \leq C\|u\|, \tag{3.5.7}
\]

for all \(u \in \mathcal{H}\) and some \(C > 0\). Then from Proposition 3.5.3 we obtain the following result.

**Proposition 3.5.7.** Let \(H = H_0 + V(x)\) and \(A\) as in (H1), with \(V\) such that

\[
\sup_{x \in \mathbb{R}^n} \left(\langle x\rangle^\sigma|V(x)| + \langle x\rangle^{\sigma+1}|
abla V(x)|\right) < \infty.
\]

Assume also the local decay estimate (3.5.7). Then for \(u \in D(A)\) such that \(P_{ac}AP_{ac}u\) is in \(E\), then \(|\psi_{P_{ac}u}(t)| \leq C_u(t)^{-1/2}\).
3.6 Higher-order decay estimates

We now improve our results by iteration of the previous method. In order to obtain higher-order decay estimates, the main difficulty lies in extending Proposition 3.4.4 for higher powers of $A_h$ and $B_h$. This task is extremely laborious with the current methods so it will not be pursued here. Proposition 3.4.4 allows to construct the operator $\tilde{A}^2 = (A_h + B_h)^2$ on $D(\tilde{A}^2) = D(A_h^2) \supset D(A^2)$. This will be enough to increase the time decay rate by a power of one as shown in the propositions below.

In this section, we will restrict our framework to the case $s = 0$ and $P = \text{Id}$ in (H), (Ha) and (Hb). So here $\tilde{A} = A + B$ defined on $D(\tilde{A}) = D(A)$, $B$ is bounded on $\mathcal{H}$ and the commutation relation reads $[H, i\tilde{A}] = Q$.

The necessary formalism to deal with higher-order estimates is the regularity of type $C^k(\tilde{A})$ which was described in Section 2.5 of Chapter 2. Recall the definition $\tilde{A}(S) = [S, i\tilde{A}]$ as a sesquilinear form on $D(\tilde{A})$.

Recall that a densely defined operator $S$ on $\mathcal{H}$ is said to be *boundedly invertible* if $S$ is injective, its range is dense, and its inverse extends to a continuous operator on $\mathcal{H}$. If $S$ is symmetric this means that $S$ is essentially self-adjoint and 0 is in the resolvent set of its closure.

**Proposition 3.6.1.** Let $H$ be of class $C^2(A)$. Assume (H) and (Ha) with $s = 0$ and $Q = Q(H)$ such that $Q'$ and $Q''Q$ are bounded functions. Then for $u \in \mathcal{H}$ such that $u \in D(A^2) \cap D(Q)$ we have the estimate $|\psi_{Qu}(t)| \leq C_u(t)^{-2}$.

**Proof.** Note that as sesquilinear forms in $D(Q^2) \cap D(\tilde{A})$ one has the identity

$$t^2 Q^2 e^{itH} = \tilde{A}^2(e^{itH}) - Q'(H)\tilde{A}(e^{itH}).$$

Note also that $\tilde{A}^2(H) = Q'(H)Q(H)$, hence $H \in C^2(\tilde{A})$ by Propositions 7.2.16 and 6.2.10 in [1].
Assume first that \( u \in D(Q^2) \), then

\[
|t^2\psi_{Qu}(t)| = |\langle u | t^2 Q^2 e^{itH} u \rangle | \\
\leq |\langle u | \tilde{A}^2 (e^{itH}) u \rangle | + |\langle u | Q' (H) \tilde{A} (e^{itH}) u \rangle | \\
\leq C \|u\| \|\tilde{A}^2 u\| + \|\tilde{A} u\|^2 + \|\tilde{A} Q' (H) u\| \\
\leq C \|u\| (\|\tilde{A} u\| + \|\tilde{A}^2 u\|) + \|\tilde{A} u\|^2.
\]

By Proposition 3.4.4, \((A + B)^2 u\) is well defined and \( \|\tilde{A}^2 u\| \leq \|A^2 u\| + C \|Au\| + \|u\| \), thus \( |\psi_{Qu}(t)| \leq C_u (t)^{-2} \), with \( C_u = C \|u\| (\|\tilde{A} u\| + \|\tilde{A}^2 u\|) + \|Au\|^2 \).

For general \( u \in D(A^2) \cap D(Q) \) we use \( u_\epsilon = R_\epsilon u \in \mathcal{H}^1 \). Note that \( u_\epsilon \in D(\tilde{A}^2) \) because \( R_\epsilon \in C^2(\tilde{A}) \). Proceeding like in the proof of Theorem 3.5.2 and letting \( \epsilon \to 0 \) we obtain the desired result.

\[ \square \]

Remark 3.6.2. Note that if the operator \( Q(H) \) is assumed boundedly invertible, the estimate of Proposition 3.6.1 holds for \( \psi_u(t) \), with \( u \in D(A^2) \).

Proposition 3.6.3. Let \( H \) be of class \( C^1(A) \). Assume \((H), (Ha)\) and \((Hb)\) in the special case \( Q = cH \) and \( s = 0 \). Let \( u \in \mathcal{H} \) be of the form \( u = |H|^{1/2} v \), for some \( v \in D(A^2) \) such that \( A^j v \in E, j = 0, 1, 2 \) and \( AB v \in E \). Then \( |\psi_u(t)| \leq C_u (t)^{-3/2} \).

Proof. Define \( \varphi \in C_c^\infty (\mathbb{R}) \) equal to one around zero and \( 0 \leq \varphi \leq 1 \). We decompose 
\( u := u_1 + u_2 \), where \( u_1 = \varphi(H) u \) and \( u_2 = \sqrt{1 - \varphi^2(H)} u \). Note that \( \psi_u = \psi_{u_1} + \psi_{u_2} \). We will show that \( \psi_{u_1} = O(t^{-3/2}) \) and \( \psi_{u_2} = O(t^{-2}) \). To study \( \psi_{u_1} \) we rely on Corollary A.3, that is, we need to prove that both \( t\psi_{u_1}(t) \) and \( t^2 \psi'_{u_1}(t) \) are functions in \( L^2(\mathbb{R}) \). We only show the latter, the former is an analogous calculation. Set \( v_1 = \varphi(H) v \).

\[
|c^2 t^2 \psi'_{u_1}(t)| = |\langle u_1 | c^2 t^2 H e^{itH} u_1 \rangle | \\
= |\langle v_1 | c^2 t^2 H^2 e^{itH} \text{sign}(H) v_1 \rangle | \\
\leq |\langle v_1 | \tilde{A}^2 (e^{itH}) v_1 \rangle | + |c \langle v_1 | \tilde{A} (e^{itH}) v_1 \rangle |.
\]

We now expand the first term of the above expression, the second one is analogous. Note \( H \in C^1(\tilde{A}) \) and \( Q(H) = cH \), hence \( H \in C^\infty(\tilde{A}) \) by Proposition 2.5.1. Since \( \varphi \in C_c^\infty (\mathbb{R}) \),
we can commute $\varphi(H)$ through $\tilde{A}$ and so we can replace $v_1$ with $v$ (up to a bounded function of $H$).

$$\langle v | \tilde{A}^2(e^{itH})v \rangle = \langle e^{-itH}v | \tilde{A}^2v \rangle - 2\langle \tilde{A}v | e^{itH}\tilde{A}v \rangle + \langle \tilde{A}^2v | e^{itH}v \rangle$$

Note that $\tilde{A}v = Av + (A + i)^{-1}(A + i)Bv \in \mathcal{E}$ by assumption (Hb). Similarly, $\tilde{A}^2v \in \mathcal{E}$ because

$$\tilde{A}^2v = A^2v + ABv + BAv + B^2v = A^2v + ABv + (A + i)^{-1}(A + i)BAv + (A + i)^{-1}(A + i)B^2v.$$ 

This yields that $t^2\psi'(u_1(t)) \in L^2(\mathbb{R})$ as desired. Now for $u_2$, we consider $v_2 \in \mathcal{H}$ such that $u_2 = cH\langle H \rangle^{-1}v_2$. Then

$$|t^2\psi(u_2(t))| = |\langle u_2 | t^2e^{itH}u_2 \rangle| = |\langle v_2 | c^2t^2H^2e^{itH}\langle H \rangle^{-2}v_2 \rangle| \leq C(||v_2||||\tilde{A}^2v_2|| + ||\tilde{A}v_2||^2) \leq C||u_2||(||u_2|| + ||Au_2|| + ||A^2u_2||) + ||Au_2||^2,$$

which concludes the proof.

\[\square\]

3.7 Applications

3.7.1 Potential of critical decay

Here we consider the equation $i\partial_t u + \Delta u - V(x)u = 0$ in $\mathbb{R}^n$ ($n \geq 3$), with initial condition $u(0, x) = f(x)$. In [14], Burq, Planchon, Stalker and Tahvildar-Zadeh use a resolvent estimate to obtain weighted $L^2$ estimates for time-independent potentials $V(x) \in C^1(\mathbb{R}^n \setminus \{0\})$ satisfying the following assumptions.

(A1) $\sup_{x \in \mathbb{R}^n} |x|^2|V| < \infty$

(A2) The operator $\Delta + |x|^2V + \lambda^2$ is positive on every sphere, i.e., there is a $\delta > 0$ such that for every $r > 0$,

$$\int_{|x|=r} |\nabla u(x)|^2 + (\lambda^2 + |x|^2V(x))|u(x)|^2d\sigma(x) \geq \delta^2 \int_{|x|=r} |u(x)|^2d\sigma(x)$$
(A3) The operator $\Delta + |x|^2 \tilde{V} + \lambda^2$ is positive on every sphere, i.e., (A2) holds with $\tilde{V}$ in place of $V$.

Here $\Delta$ represents the spherical Laplacian, $\tilde{V} := \partial_r (r V(x))$ and $\lambda = (n - 2)/2$.

In Section 3 of that paper the following Morawetz estimate is obtained

$$\| |x|^{-1} e^{-itH} f \|_{L^2_t L^2_x} \leq C \| f \|_{L^2}.$$  (3.7.8)

We will use this result to obtain pointwise estimates using the generator of dilations $A = -\frac{i}{2} (x \cdot \nabla + \nabla \cdot x)$ as the conjugate operator. In order to verify the $C^1(A)$ regularity for $H$ and the other conditions of (H) we will consider the following additional assumptions.

(A4) $V$ is nonnegative and locally integrable in $\mathbb{R}^n$

(A5) $\sup_{x \in \mathbb{R}^n} |x|^2 |\nabla V| < \infty$ and $\| (x \cdot \nabla V) \langle V \rangle^{-1} \|_{L^\infty} < \infty$

Condition (A4) simplifies our analysis here, but the positivity might not be necessary. The general case will be pursued elsewhere. Theorem 4.6a in the book of Kato [44] proves that the domain of the Friedrich extension of $H = -\Delta + V(x)$ is characterized as the set of all

i) $u \in \mathcal{H}$ such that $\nabla u$ belongs to $(L^2(\mathbb{R}^n))^n$, ii) $\int V(x) |u|^2 dx < \infty$, and iii) $\Delta u$ exists and $-\Delta u + V(x)u$ belongs to $\mathcal{H}$.

It is easy to check that the dilation group leaves $D(H)$ invariant, in fact ii) and iii) are preserved under condition (A5). In this scenario, the $C^1(A)$ regularity follows from the commutation relation $[H, iA] = -2\Delta - x \cdot \nabla V = 2H - (2V + x \cdot \nabla V)$.

Note that here $Q = 2H$ and $K = -(2V + x \cdot \nabla V)$.

Now write $K = \text{sign}(K) |K|^{1/2} |K|^{1/2}$. Assumptions (A1) and (A4) yield the bound

$$\| |K|^{1/2} e^{itH} f \|_{L^2} \leq C \| |x|^{-1} e^{itH} f \|_{L^2},$$

and then estimate (3.7.8) implies that $K$ is the product of two Kato-smooth operators. Therefore, the assumptions of (H) hold with $s = 0$ and the estimate of Proposition 3.5.6 reads as follows.

**Proposition 3.7.1.** Let $H$ and $A$ be as above and $P$ a projection commuting with $H$. Assume conditions (A1)-(A5) and (Hb). Then for $u \in D(A)$ such that $Pu$ and $PAPu$ are in $\mathcal{E}$, one has the estimate $|\psi_{Pu}(t)| \leq C_u(t)^{-1/2}$. 
Remark 3.7.2. We can use the results of [14] to characterize the space $\mathcal{E}$. For instance, the endpoint Strichartz estimate $\|e^{-itH}u\|_{L^2_t(L^6_x)} \leq C\|u\|_{L^2}$ in dimension $n = 3$ yields that for $u \in L^{6/5}(\mathbb{R}^3)$, we have $\|\psi u\|_{L^2_x} \leq \|u\|_{L^{6/5}}\|e^{-itH}u\|_{L^2_t(L^6_x)} \leq C\|u\|_{L^{6/5}}\|u\|_{L^2}$. Another choice is to use the estimate (3.7.8) in dimension $n$, to show that if $|x|u \in L^2(\mathbb{R}^n)$, one has $\|\psi u\|_{L^2_x} \leq \|u\|_{L^2}\|u\|_{L^2}$.

3.7.2 Laplacian on manifold

Let $(M, g) = (\mathbb{R}^3, g)$ be a compact perturbation of $\mathbb{R}^3$, i.e. $M$ is $\mathbb{R}^3$ endowed with a smooth metric $g$ which equals the Euclidean metric outside of a ball $B(0, R_0) = \{x \in \mathbb{R}^3 : |x| \leq R_0\}$ for some fixed $R_0$. In [64] global-in-time decay estimates were obtained by Rodnianski and Tao for solutions to the Schrödinger equation $iu_t = Hu$, where $H = -\frac{i}{2}\Delta_M$ is the Laplace-Beltrami operator on $M$. It has been shown that $H$ defined on $\mathcal{S} := C^\infty(M)$ (smooth functions of compact support) is essentially self-adjoint and its domain is the Sobolev space $H^2(M)$. The main result of their paper is the following.

Theorem. Let $M$ be a smooth compact perturbation of $\mathbb{R}^3$ which is nontrapping and smoothly diffeomorphic to $\mathbb{R}^3$. Then for any Schwartz solution $u(x,t)$ and any $\sigma > 0$ we have

$$
\int_\mathbb{R} \|\langle x \rangle^{-1/2 - \sigma}\nabla e^{-itH}u_0\|_{L^2(M)}^2 + \|\langle x \rangle^{-3/2 - \sigma}e^{-itH}u_0\|_{L^2(M)}^2 dt \leq C_{\sigma,M}\|u_0\|^2_{H^{1/2}(M)}.
$$

(3.7.9)

We choose the conjugate operator $A = -i/2(x \cdot \nabla + \nabla \cdot x)$ defined on $\mathcal{S}$. Note that the dilation group leaves $H^2(M)$ invariant. Assuming that metric is smooth and bounded, the $C^1(A)$ condition follows from the commutation relation $[H, iA] = 2H + K$, where $K$ is a second order operator supported in $B(0, R_0)$. More explicitly, a straightforward calculation on $\mathcal{S}$ shows that $K$ is an an operator of the form $K = k_1(x)H + ik_2(x)\nabla + ik_3(x)$, where the $k_i$ are smooth and bounded real functions supported in $B(0, R_0)$. Observe also that $\langle H \rangle^{-1/2}K\langle H \rangle^{-1/2}$ is bounded. Taking $E = F = \sqrt{|K|}$, the Kato-smoothness is a direct consequence of the estimate (3.7.9) and therefore the assumptions of (H) are met with $s = 1/2$. Condition (Ha) holds since $[A, K]$ is a second order differential operator as well. Theorem 3.5.3 yields the following result.
Proposition 3.7.3. Let $H$ and $A$ be above and $P$ a projection commuting with $H$. Assume that condition (Hb) holds for the space $E$. Then for $u \in D(A)$ such that $Pu$ and $PAPu$ are in $E$, one has $|\psi_{Pu}(t)| \leq C_u(t)^{-1/2}$. 
Appendix A

We prove here an auxiliary estimate. We consider functions $g$ defined on $\mathbb{R}_+ = (0, \infty)$ and denote $\|g\|_p$ their $L^p$ norms. Let $\delta$ the operator $(\delta g) = xg'(x)$ acting in the sense of distributions and set $\tilde{g}(t) = \int_0^\infty e^{itx} g(x) dx$ for $t > 0$ (improper integral).

**Lemma A.1.** \(|\tilde{g}(t)| \leq |t|^{-1/2}2^{3/2}(p-1)^{-1/2p}\|g\|_p^{1/2}\|\delta g\|_q^{1/2} \text{ if } 1 < p < \infty \text{ and } \frac{1}{p} + \frac{1}{q} = 1.\)

**Proof.** We may assume that $g \in L^p$ and $\delta g \in L^q$. For any $s > 0$ we have

\[
\left| \int_0^s e^{itx} g(x) dx \right| \leq s^{1/q}\|g\|_p. \tag{A.1}
\]

Since $g \in L^p$ with $p < \infty$, there is a sequence $a_n \to \infty$ such that $g(a_n) \to 0$ (otherwise $|g(x)| \geq c > 0$ on a neighborhood of infinity, so $|g|^p$ cannot be integrable). Since $p > 1$, after integrating over $(s, a_n)$ and then making $n \to \infty$, we also obtain

\[
|g(s)| \leq \int_s^\infty |g'(x)| dx \leq (p-1)^{-1/p} s^{1/p-1}\|\delta g\|_q \tag{A.2}
\]

by H"older inequality. Then

\[
\int_s^\infty e^{itx} g(x) dx = \lim_{a \to \infty} \int_s^a \left( \frac{d}{dx} \frac{1}{it} e^{itx} \right) g(x) dx
= \lim_{a \to \infty} \left[ \frac{e^{ita}g(a) - e^{its}g(s)}{it} - \frac{1}{it} \int_s^a e^{itx} g'(x) dx \right].
\]

We take here $a = a_n$ and make $n \to \infty$ to get

\[-it \int_s^\infty e^{itx} g(x) dx = e^{its}g(s) + \int_s^\infty e^{itx} g'(x) dx \]

and then by using (A.2) two times we obtain

\[
\left| \int_s^\infty e^{itx} g(x) dx \right| \leq 2(p-1)^{-1/p} s^{-1/q} t^{-1}\|\delta g\|_q.
\]

Let $\varepsilon > 0$ and $s = \varepsilon^q/t$. Then (A.1) and the last inequality give

\[
|\tilde{g}(t)| \leq \varepsilon t^{-1/q}\|g\|_p + 2(p-1)^{-1/p} \varepsilon^{-1} t^{-1/p}\|\delta g\|_q.
\]
The infimum over $\varepsilon > 0$ of an expression $\varepsilon a + \varepsilon^{-1} b$ is $2\sqrt{ab}$. This finishes the proof. \hfill $\square$

**Corollary A.2.** If $\psi \in L^2(\mathbb{R})$ and $t\psi'(t) \in L^2(\mathbb{R})$ then $|\psi(t)| \leq C_\psi |t|^{-1/2}$ for $t \in \mathbb{R} \setminus \{0\}$.

**Proof.** We use Lemma A.1 with $p = 2$ and $g$ equal to the Fourier transform of $\psi$. \hfill $\square$

**Corollary A.3.** If a function $\psi$ is such that $t^{k-1/2} \psi(t)$ and $t^{k+1/2} \psi'(t)$ belong to $L^2(\mathbb{R})$ for some $k \geq 1$ then $|\psi(t)| \leq C_\psi |t|^{-k/2}$ for all $t \in \mathbb{R} \setminus \{0\}$. 

References


