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Transport of Brownian particles in a narrow, slowly varying serpentine channel

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We study the transport of Brownian particles under a constant driving force and moving in channels that present a varying centerline but have constant aperture width (*serpentine* channels). We investigate two types of channels, *solid* channels, in which the particles are geometrically confined between solid walls and *soft* channels, in which the particles are confined by the potential energy landscape. We consider the limit of narrow, slowly varying channels, i.e., when the aperture and the variation in the position of the centerline are small compared to the length of a unit cell in the channel (wavelength). We use the method of asymptotic expansions to determine both the average velocity (or mobility) and the effective dispersion coefficient of the particles. We show that both solid and soft-channels have the same effects on the transport properties up to leading order correction. Including the next order correction, we obtain that the mobility in a solid-channel is smaller than that in a soft-channel. However, we discuss an alternative definition of the *effective* width of a soft channel that leads to equal mobilities up to second order terms. Interestingly, in both cases, the corrections to the mobility of the particles are independent of the Péclet number, and the Einstein-Smoluchowski relation is satisfied. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4917020]

I. INTRODUCTION

The transport of suspended particles confined to move in tubes and channels is important in a wide range of problems both in natural systems, e.g., particle transport in cells[1] or ion transport through a cell membrane,[2] as well as in engineered systems, including the development of separation and analytical microfluidic devices.[3–9] The confinement can be induced geometrically, leading to entropic barriers in *solid* channels, or by a potential energy landscape that energetically circumscribes the particles to a channel-like region, i.e., *soft* channels.

The unbiased Brownian motion of suspended particles confined to a symmetric channel (or pipeline) has been extensively studied in both solid and soft channels, using, for example, the Fick-Jacobs (F-J) approximation,[10] which reduces the dimensionality of the problem by averaging over the cross section. Zwanzig later modified the F-J approximation with a position-dependent effective diffusion coefficient that takes into account the curvature of the confining boundary.[2] Reguera and Rubi proposed a scaling law for the effective diffusion coefficient in order to improve the approximation in the case of boundaries with significant curvature (relatively large variations in the aperture of the channel).[11,12] Kalinay and Percus provided a different approach in which the projected one-dimensional problem can be systematically approximated by assuming that the diffusion time in the transverse direction is much smaller than that in the longitudinal direction.[13–17] Specifically, they derived an asymptotic expansion in the ratio between transverse and longitudinal diffusivities. In their work, the centerline of the channel is a straight line and the corrections result from spatial variations in the aperture of the channel. However, it is also important to consider the diffusion of suspended particles in a channel of constant width but with varying centerline, i.e., a serpentine channel,[18] particularly in microfluidic devices. Dagdug and Pineda[19] applied the projection method of Kalinay and Percus to channels with constant aperture and varying centerline, deriving a more general expression for the diffusion coefficient that also improves the accuracy of the approximation. Additionally, Kalinay and Percus showed that the projection method can also be applied in the case of soft channels.[17] Bradley performed an alternative asymptotic expansion based on the width of channel and derived an expression for the effective diffusivity of particles confined to a narrow channel, both in the case of a serpentine channel as well as for a channel with a varying aperture.[20] A generalization of Bradley's results to an arbitrary multidimensional system is given by Berezhkovskii and Szabo.[21] The results obtained from the projection method and the asymptotic analysis have been shown to agree up to second order[20] and a general discussion of their equivalence was recently provided by Kalinay.[22] Finally, García-Chung *et al.*[23] have recently proposed a novel method based on mapping a general two-dimensional channel into a straight channel by a coordinate transformation.

In recent years, the biased transport of suspended particles in the presence of an external force field has also received considerable attention due in part to the development of novel separation strategies in microfluidic devices.[8,24–26] In the simplest case, in which the external force is constant in the longitudinal direction, a straightforward extension of the F-J approximation has been used to evaluate the average velocity and the effective diffusivity of Brownian particles.[12,27,28] Kalinay showed that the projection method can also be applied in the case of biased diffusion.[29] Alternatively, the asymptotic analysis has been used to calculate higher order corrections to the F-J approximation in channels[30] and tubes[31] with slowly varying cross sections. Asymptotic analysis was also used to study effective (or *macro-*) transport properties of Brownian particles confined to narrow channels and driven by a longitudinal constant force,[32] a pressure-driven flow,[18] or an electro-osmotic flow.[33] Analogous behavior is observed in the biased transport of Brownian particles confined by a potential energy landscape, which is relevant to partition-induced separation in microfluidic devices.[7,34] In the case of soft-channels, we have previously shown that the leading order effect of the confining potential on the transport properties of the suspended particles is the same as that induced by solid walls, as long as the entropic barriers created by the varying aperture of the channel are the same.[35,36]

Here, we extend previous work to consider the case of biased transport of Brownian particles in a channel of constant width but varying centerline. In particular, we use asymptotic analysis to investigate the leading order correction to the effective dispersion coefficient. We also calculate higher order terms in the asymptotic expansion of the average velocity (or mobility). We consider both a solid-channel as well as a channel created by a confining potential, and compare the results.

II. TRANSPORT OF BROWNIAN PARTICLES IN A CURVED CHANNEL

Let us first describe the geometry of the channels considered in this work. There are three important characteristic length scales: L—the length of one period in the longitudinal direction, a—the average channel width, and δ_z—the amplitude of the variation in the position of the boundaries. Then, the problem can be categorized into three main cases: a *slowly varying* channel for $\delta_z/L \ll 1$, a *narrow* channel for $a/L \ll 1$, and a *weakly corrugated* channel for $\delta_z/a \ll 1$.

Here, we study the biased motion of a Brownian particle in a *narrow, slowly varying* channel ($a/L \ll 1$ and $\delta_z/L \ll 1$, $\delta_z/a \sim O(1)$) with a constant aperture but varying centerline. The bias is induced by a constant and uniform external force acting in the longitudinal direction and we consider both soft and solid-channels. In the case of a soft-channel, Brownian particles are confined by a potential that is periodic in the X-direction, $\bar{V}(X,Z) = \bar{V}(X+L,Z)$, and confines the particles in the Z-direction, $\bar{V}(X,Z) \to +\infty$ for $Z \to \pm\infty$. On the other hand, in the case of a solid-channel, Brownian particles are confined between solid walls, described by $Z = Z_+(X)$ and $Z = Z_-(X)$. (Note that the potential $\bar{V}(X,Z) = 0$ when considering the transport in a solid-channel.)

In the limit of negligible inertia effects, the motion of the particles is described by the Smoluchowski equation for the probability density $\bar{P}(X,Z,t)$,

$$\frac{\partial \bar{P}}{\partial t} + \nabla \cdot \bar{\mathbf{J}} = \delta(X,Z)\delta(t). \tag{1}$$

The probability flux, $\bar{\mathbf{J}}(X,Z,t)$, is given by

$$\bar{\mathbf{J}} = \frac{1}{\eta}\left[\left(F - \frac{\partial \bar{V}}{\partial X}\right)\bar{P} - k_B T\frac{\partial \bar{P}}{\partial X}\right]\vec{i} + \frac{1}{\eta}\left(-\frac{\partial \bar{V}}{\partial Z}\bar{P} - k_B T\frac{\partial \bar{P}}{\partial Z}\right)\vec{k}, \tag{2}$$

where $\vec{i}$ and $\vec{k}$ are the unit vectors along X and Z, respectively, η is the viscous friction coefficient, F is a uniform external force in the X-direction, k_B is the Boltzmann constant, T is the absolute temperature, and the Stokes-Einstein equation is used to write the diffusion coefficient in terms of η, $D = k_B T/\eta$. The inertia effects are negligible and, therefore, the velocity is simply the ratio of the force to the viscous friction coefficient η.

Instead of considering the problem in an unbounded domain in X, it is convenient to introduce the reduced probability density (and probability flux) which maps the infinite domain into a single period of the channel (see Refs. 37 and 38),

$$\tilde{P}(\bar{x},\bar{z},t) = \sum_{n_x=-\infty}^{+\infty} \bar{P}(\bar{x}+n_x L,\bar{z},t), \tag{3}$$

$$\tilde{\mathbf{J}}(\bar{x},\bar{z},t) = \sum_{n_x=-\infty}^{+\infty} \bar{\mathbf{J}}(\bar{x}+n_x L,\bar{z},t), \tag{4}$$

where the integer n_x indicates the number of periods along the channel, and $(\bar{x},\bar{z})$ is the same to (X,Z) except that $\bar{x}$ is defined in $[0,L]$. In other words, the reduced probability density $\tilde{P}(\bar{x},\bar{z},t)$ is only defined locally within a unit cell, $\Omega = \{(\bar{x},\bar{z}) \mid 0 \le \bar{x} \le L, -\infty < \bar{z} < \infty\}$ for a soft-channel and $\Omega = \{(\bar{x},\bar{z}) \mid 0 \le \bar{x} \le L, \bar{z}_- \le \bar{z} \le \bar{z}_+\}$ for a solid-channel. Therefore, the reduced probability density can be obtained by solving the Smoluchowski equation with periodic boundary conditions in $\bar{x}$. Note that, even though the global probability density function $\bar{P}(X,Z,t)$ does not have a meaningful asymptotic state at long times, the reduced probability density will approach a steady state solution.[37,38] In particular, the long-time asymptotic probability density (steady state), $\tilde{P}_\infty(\bar{x},\bar{z}) = \lim_{t\to\infty}\tilde{P}(\bar{x},\bar{z},t)$, is governed by the equation

$$\nabla \cdot \tilde{\mathbf{J}}_\infty = 0, \tag{5}$$

with the normalization condition for the reduced probability density,

$$\langle \tilde{P}_\infty \rangle \stackrel{\text{def}}{=} \int\int_\Omega \tilde{P}_\infty \mathrm{d}\bar{x}\mathrm{d}\bar{z} = 1. \tag{6}$$

The boundary conditions are periodic in $\bar{x}$,

$$\tilde{P}_\infty(0,\bar{z}) = \tilde{P}_\infty(L,\bar{z}), \tag{7}$$

and the zero-flux condition in $\bar{z}$, which depends on the type of the channel. For a soft-channel, it corresponds to a vanishingly small probability density and flux in the limit of large $\bar{z}$ values,

$$\tilde{J}^{\bar{z}}_\infty = \frac{1}{\eta}\left(-\frac{\partial \bar{V}}{\partial \bar{z}}\tilde{P}_\infty - k_B T\frac{\partial \tilde{P}_\infty}{\partial \bar{z}}\right) \xrightarrow[\bar{z}\to\pm\infty]{} 0. \tag{8}$$

In the case of a solid-channel, the zero-flux condition at the boundaries $\bar{z} = \bar{z}_\pm$ is given by

$$\tilde{J}_\infty \cdot \vec{N} = 0, \tag{9}$$

where $\vec{N}$ is the vector normal to the channel walls.

Let us now introduce the following dimensionless variables using the characteristic scales of the problem: $x = \bar{x}/L$, $z = \bar{z}/a$, $V = \bar{V}/(k_B T)$, as well as the re-scaled probability density $P_\infty = aL\tilde{P}_\infty$. The governing equation for the reduced probability then becomes

$$\epsilon^2 \frac{\partial}{\partial x}\left[\left(\mathrm{Pe} - \frac{\partial V}{\partial x}\right) P_\infty - \frac{\partial P_\infty}{\partial x}\right] + \frac{\partial}{\partial z}\left[-\frac{\partial V}{\partial z}P_\infty - \frac{\partial P_\infty}{\partial z}\right] = 0, \tag{10}$$

where $\epsilon = a/L$ is the aspect ratio of the channel, and Pe $= FL/k_B T$ is the Péclet number (a measure of the relative importance of convective and diffusive transport). The boundary conditions in dimensionless form are the periodic boundary condition in the x direction,

$$P_\infty(0,z) = P_\infty(1,z), \tag{11}$$

the normalization condition,

$$\langle P_\infty \rangle \overset{\text{def}}{=} \int\int_\Omega P_\infty \mathrm{d}x\mathrm{d}z = 1, \tag{12}$$

where $\Omega = \{(x,z) : 0 \le x \le 1, -\infty < z < \infty\}$ for a soft-channel and $\Omega = \{(x,z) : 0 \le x \le L, z_- \le z \le z_+\}$ for a solid-channel, and the zero-flux condition,

$$J^z_\infty(x, \pm\infty) = 0, \text{ for a soft-channel}, \tag{13a}$$

$$\epsilon^2 \frac{\mathrm{d}z_\pm}{\mathrm{d}x}\left(\mathrm{Pe} - \frac{\partial P_\infty}{\partial x}\right) + \frac{\partial P_\infty}{\partial z} = 0, \quad \text{at } z = z_\pm, \text{ for a solid-channel}. \tag{13b}$$

Once we obtain the asymptotic solution for the reduced probability distribution P_∞, we can calculate the average velocity along the channel by applying macrotransport theory,[39]

$$U^* = \int\int_\Omega J^x_\infty \mathrm{d}x\mathrm{d}z, \tag{14}$$

that is, the total flux in the x-direction averaged over a unit cell of the channel. Note that the average velocity is nondimensionalized by $\frac{k_B T}{\eta L}$ (or D/L) to obtain U^*.

The effective dispersion coefficient D^*, which characterizes the variance in the particle position relative to its mean position $U^* t$, can also be calculated from the asymptotic probability distribution P_∞, via the so-called B-field in *macrotransport theory*.[39] The B-field is defined by the following differential equation:[40–42]

$$\frac{\partial}{\partial z}\left(P_\infty \frac{\partial B}{\partial z}\right) - J^z_\infty \frac{\partial B}{\partial z} + \epsilon^2\left[\frac{\partial}{\partial x}\left(P_\infty \frac{\partial B}{\partial x}\right) - J^x_\infty \frac{\partial B}{\partial x}\right] = \epsilon^2 P_\infty U^*. \tag{15}$$

The boundary conditions for the B-field are

$$B(x=1,z) - B(x=0,z) = -1, \tag{16}$$

and

$$P_\infty \frac{\partial B}{\partial z} \xrightarrow[z\to\pm\infty]{} 0, \text{ for a soft-channel}, \tag{17a}$$

$$\epsilon^2 \frac{\mathrm{d}z_\pm}{\mathrm{d}x}\frac{\partial B}{\partial x} = \frac{\partial B}{\partial z} \text{ at } z = z_\pm(x), \text{ for a solid-channel}. \tag{17b}$$

Then, the effective dispersion coefficient is given by

$$D^* = \int\int_\Omega P_\infty\left[\left(\frac{\partial B}{\partial x}\right)^2 + \frac{1}{\epsilon^2}\left(\frac{\partial B}{\partial z}\right)^2\right]\mathrm{d}x\mathrm{d}z. \tag{18}$$

III. A NARROW, SLOWLY VARYING SOFT-CHANNEL CONFINED BY A PARABOLIC POTENTIAL

In this section, we consider a soft-channel in which particles are confined by a parabolic potential

$$\bar{V}(\bar{x},\bar{z}) = k_B T \pi\left(\frac{\bar{z} - \delta_z g(\bar{x}/L)}{a}\right)^2, \tag{19}$$

where $g(\bar{x})$ is a periodic function. We have shown in previous work that the configuration integral

$$\bar{I}(\bar{x}) = \int_{-\infty}^{\infty} e^{-\beta \bar{V}(\bar{x},\bar{z})}\mathrm{d}\bar{z}, \tag{20}$$

with $\beta = 1/k_B T$ plays a role analogous to the width of a solid-channel.[35] Therefore, we shall call it the effective width of the soft-channel, which in this case is a constant for the potential in Eq. (19), $\bar{I}(\bar{x}) = a$.

The potential in dimensionless form is given by

$$V(x,z) = \pi(z - \lambda g(x))^2, \tag{21}$$

where $\lambda = \delta_z/a$ is the ratio between the amplitude of the variations in the position of the centerline and the effective width of the soft-channel. The non-dimensional effective width of this soft-channel is $I(x) = 1$. In equilibrium, the distribution of particles is given by the Boltzmann distribution, $\exp(-V)$, showed in Fig. 1(a). Fig. 1(b) shows a schematic diagram of a soft-channel whose boundaries are two equipotential lines. Note that particles are not strictly confined by these two boundaries. However, there is large probability that a particle is in the region inside two equipotential lines between which the

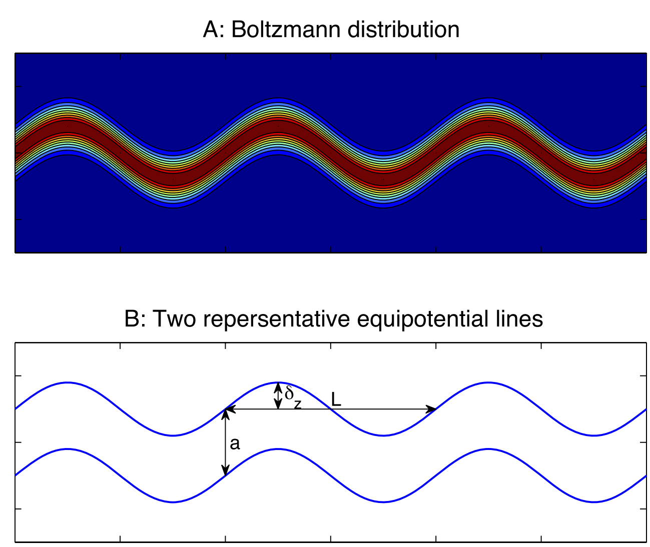

FIG. 1. (a) The Boltzmann distribution $\exp(-V)$, where $V(x,z) = \pi(z-\lambda g(x))^2$. (b) Schematic diagram of the curved channel confined by two equipotential lines $z = \lambda g(x) \pm 1/2$. The aspect ratio is $\epsilon = a/L$ and the ratio of the boundary amplitude to the width is $\lambda = \delta_z/a$.

distance is large. For example, if two soft-channel boundaries are equal potential lines $z = \lambda g(x) \pm 1/2$, the probability that a particle is moving inside this soft-channel is about 79% in equilibrium.

As discussed before, the aspect ratio is very small $\epsilon \ll 1$, and the amplitude of the variation in the position of the centerline is of the same order as the width of the channel $\lambda \sim O(1)$. Therefore, we propose a solution for the stationary probability distribution in the form of a regular perturbation expansion in the small aspect ratio ϵ,

$$P_\infty(x,z) \sim p_0 + \epsilon^2 p_1 + \epsilon^4 p_2 + \cdots. \tag{22}$$

The corresponding expansion for the probability flux is

$$\mathbf{J}_\infty(x,z) \sim \mathbf{J}_0 + \epsilon^2 \mathbf{J}_1 + \epsilon^4 \mathbf{J}_2 + \cdots. \tag{23}$$

At each order of the approximation, we first solve for the probability density $p_i(x,z)$, and we then calculate two important macroscopic transport properties: the average velocity given by Eq. (14) and the effective dispersion coefficient given by Eq. (18).

A. Average velocity in a narrow, slowly varying soft-channel

Substituting the expansion of P_∞ introduced above in Eq. (22) into Eq. (14) and applying the normalization and periodicity conditions given by Eqs. (12) and (11), respectively, we obtain the following expansion of the average velocity:

$$U^*_{soft} = \mathrm{Pe} - \int_0^1 \mathrm{d}x \int_{-\infty}^{\infty} \frac{\partial V}{\partial x} P_\infty \mathrm{d}z \sim u_0 + \epsilon^2 u_1 + \epsilon^4 u_2 + \cdots, \tag{24}$$

where

$$u_0 = \mathrm{Pe} - \int_0^1 \mathrm{d}x \int_{-\infty}^{\infty} \frac{\partial V}{\partial x} p_0 \mathrm{d}z, \tag{25a}$$

$$u_i = -\int_0^1 \mathrm{d}x \int_{-\infty}^{\infty} \frac{\partial V}{\partial x} p_i \mathrm{d}z, \quad \text{for} \quad i = 1,2,3\ldots. \tag{25b}$$

On the other hand, integrating both sides of Eq. (10) over the cross section and applying the far-field conditions, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_{-\infty}^{\infty}\left[\left(\mathrm{Pe} - \frac{\partial V}{\partial x}\right) P_\infty - \frac{\partial P_\infty}{\partial x}\right]\mathrm{d}z\right\} = 0. \tag{26}$$

This shows that the total flux in the x-direction $\bar{J}^x$ (the quantity inside the curly brackets) is, in steady state, constant along the channel. Furthermore, given the definition of the average velocity, $U^*_{soft} = \int_0^1 \bar{J}^x \mathrm{d}x$, we have that $\bar{J}^x = U^*_{soft}$. Therefore,

$$u_i = \int_{-\infty}^{\infty}\left[\left(\mathrm{Pe} - \frac{\partial V}{\partial x}\right) p_i - \frac{\partial p_i}{\partial x}\right]\mathrm{d}z, \tag{27}$$

which represents the constant flux at $O(\epsilon^{2i})$. This is a solvability condition for $p_i(x,z)$, which can also be derived from next order governing equation. In what follows, Eqs. (25a) and (25b) are used to calculate the average velocity u_i. We shall show that it is possible to obtain u_i by first finding $p_i(x,z)$ up to an unknown function of x. Then, this unknown part of $p_i(x,z)$, $C_i(x)$, is determined by means of Eq. (27).

In order to calculate the average velocity u_i, we first need to calculate the probability density p_i. Substituting the expansion of P_∞ into Eq. (10), we determine the leading order governing equation

$$\frac{\partial}{\partial z}\left(-\frac{\partial V}{\partial z} p_0 - \frac{\partial p_0}{\partial z}\right) = \frac{\partial J_0^z}{\partial z} = 0. \tag{28}$$

The corresponding leading order boundary and normalization conditions, derived from Eqs. (6)-(8), are

$$J_0^z(x,\pm\infty) = 0, \tag{29a}$$

$$p_0(0,z) = p_0(1,z), \tag{29b}$$

$$\langle p_0 \rangle = 1. \tag{29c}$$

Equation (28) shows that the flux J_0^z is independent of z, which in combination with the zero flux condition for $z \to \pm\infty$, implies that $J_0^z = 0$. Then, the leading order solution of the probability density is

$$p_0(x,z) = f_0(x) e^{-V(x,z)}, \tag{30}$$

where $f_0(x)$ is an unknown function which satisfies the periodicity and normalization conditions. As we mentioned before, without knowing the explicit solution of $f_0(x)$, we can still calculate the leading order average velocity according to Eq. (25a) by taking advantage of the relation $\frac{\partial V}{\partial x} = -\lambda \frac{\mathrm{d}g}{\mathrm{d}x}\frac{\partial V}{\partial z}$,

$$u_0 = \mathrm{Pe}. \tag{31}$$

The explicit solution of $f_0(x)$ can be determined from Eq. (27),

$$\mathrm{Pe} = \int_{-\infty}^{\infty}\left[\left(\mathrm{Pe} - \frac{\partial V}{\partial x}\right) p_0 - \frac{\partial p_0}{\partial x}\right]\mathrm{d}z. \tag{32}$$

By substituting the solution of p_0 and evaluating the integral, we obtain

$$\mathrm{Pe} = \left(\mathrm{Pe} f_0 - \frac{\mathrm{d}f_0}{\mathrm{d}x}\right) I(x), \tag{33}$$

where $I(x) = \int_{-\infty}^{\infty} e^{-V}\mathrm{d}z = 1$. This leads to $f_0(x) = 1$. Thus, the leading order term of the probability density is

$$p_0(x,z) = e^{-V}, \tag{34}$$

i.e., the Boltzmann distribution. This means that a small variation in the centerline of the channel does not affect the leading order solution. Therefore, we seek higher order corrections, p_n for $n = 1,2,3,\ldots$, which are obtained by solving Eq. (10) for increasing powers of ϵ. Specifically, grouping the terms of order ϵ^{2n}, after replacing P_∞ in Eq. (10) by its expansion, we obtain

$$\frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z} p_n + \frac{\partial p_n}{\partial z}\right) = \frac{\partial}{\partial x}\left[\left(\mathrm{Pe} - \frac{\partial V}{\partial x}\right) p_{n-1} - \frac{\partial p_{n-1}}{\partial x}\right]. \tag{35}$$

The corresponding boundary and normalization conditions are

$$J_n^z(x,\pm\infty) = 0, \tag{36a}$$

$$p_n(0,z) = p_n(1,z), \tag{36b}$$

$$\langle p_n \rangle = 0. \tag{36c}$$

Substituting $p_0 = e^{-V}$ into Eq. (35) for $n = 1$, we obtain

$$\frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z} p_1 + \frac{\partial p_1}{\partial z}\right) = \mathrm{Pe}\,\frac{\partial p_0}{\partial x}. \tag{37}$$

Integrating both sides of the equation twice with respect to z, we find the solution of the probability density term at $O(\epsilon^2)$,

$$p_1(x,z) = f_1(x,z)e^{-V(x,z)}, \tag{38}$$

where $f_1(x,z) = -\lambda \mathrm{Pe}\,\frac{\mathrm{d}g}{\mathrm{d}x}\,z + C_1(x)$. The function $C_1(x)$ can be determined from Eq. (27),

$$\frac{\mathrm{d}C_1}{\mathrm{d}x} - \mathrm{Pe}C_1 = -\mathrm{Pe}\,\lambda^2 g(x)\left(\mathrm{Pe}\,\frac{\mathrm{d}g}{\mathrm{d}x} - \frac{\mathrm{d}^2 g}{\mathrm{d}x^2}\right) - u_1, \tag{39}$$

where $C_1(x)$ satisfies $\int_0^1 C_1(x)\mathrm{d}x = 0$ derived from the normalization condition and the periodic condition $C_1(0) = C_1(1)$.

Analogous to the leading order term, we can evaluate u_1 without knowing $C_1(x)$,

$$\begin{aligned} u_1 &= -\int_0^1 \mathrm{d}x \int_{-\infty}^{\infty} \frac{\partial V}{\partial x} p_1 \mathrm{d}z \\ &= -\int_0^1 \mathrm{d}x \int_{-\infty}^{\infty} \left(-\lambda \frac{\mathrm{d}g}{\mathrm{d}x}\frac{\partial V}{\partial z}\right) f_1(x,z)e^{-V}\mathrm{d}z \\ &= \int_0^1 \mathrm{d}x \int_{-\infty}^{\infty} \left(\lambda \frac{\mathrm{d}g}{\mathrm{d}x}\right)\frac{\partial f_1}{\partial z}e^{-V}\mathrm{d}z \\ &= -\lambda^2 \mathrm{Pe}\int_0^1 \left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2 \mathrm{d}x. \end{aligned} \tag{40}$$

Continuing with the same approach, we can solve the problem at $O(\epsilon^4)$. The probability density is

$$p_2(x,z) = f_2(x,z)e^{-V}, \tag{41}$$

where

$$\begin{aligned} f_2(x,z) = &\frac{\mathrm{Pe}\,\lambda^2}{2}\frac{\mathrm{d}g}{\mathrm{d}x}\left(\mathrm{Pe}\frac{\mathrm{d}g}{\mathrm{d}x} - \frac{\mathrm{d}^2 g}{\mathrm{d}x^2}\right)(z-\lambda g(x))^2 \\ &+ \frac{\mathrm{Pe}\,\lambda}{2\pi}\left(\mathrm{Pe}\,\frac{\mathrm{d}^2 g}{\mathrm{d}x^2} - \frac{\mathrm{d}^3 g}{\mathrm{d}x^3}\right)z - u_1\lambda\frac{\mathrm{d}g}{\mathrm{d}x}z + C_2(x). \end{aligned} \tag{42}$$

The corresponding average velocity is

$$u_2 = \frac{\mathrm{Pe}\,\lambda^2}{2\pi}\int_0^1 \left(\frac{\mathrm{d}^2 g}{\mathrm{d}x^2}\right)^2 dx + \mathrm{Pe}\lambda^4\left(\int_0^1 \left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2 dx\right)^2. \tag{43}$$

In summary, the average velocity up to $O(\epsilon^4)$ is

$$\begin{aligned} U^*_{soft} \sim &\ \mathrm{Pe} - \epsilon^2\lambda^2\mathrm{Pe}\int_0^1 \left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2 \mathrm{d}x \\ &+ \epsilon^4\mathrm{Pe}\left[\frac{\lambda^2}{2\pi}\int_0^1 \left(\frac{\mathrm{d}^2 g}{\mathrm{d}x^2}\right)^2 \mathrm{d}x + \lambda^4\left(\int_0^1 \left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2 \mathrm{d}x\right)^2\right] \\ &+ O(\epsilon^6). \end{aligned} \tag{44}$$

Note that, to this order of the approximation, the average velocity depends linearly on the Péclet number. Equivalently, the normalized mobility is constant and independent of the Péclet number,

$$\begin{aligned} \mu_{soft} = \frac{U^*}{\mathrm{Pe}} \sim &\ 1 - \epsilon^2\lambda^2\int_0^1 \left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2 \mathrm{d}x \\ &+ \epsilon^4\left[\frac{\lambda^2}{2\pi}\int_0^1 \left(\frac{\mathrm{d}^2 g}{\mathrm{d}x^2}\right)^2 \mathrm{d}x + \lambda^4\left(\int_0^1 \left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2 \mathrm{d}x\right)^2\right] \\ &+ O(\epsilon^6). \end{aligned} \tag{45}$$

B. Effective dispersion coefficient in a narrow, slowly varying soft-channel

In order to calculate the effective dispersion coefficient D^*_{soft}, we need to solve the B-field in Eq. (15). Asymptotic expansions are proposed in the following form:

$$B(x,z) \sim B_0 + \epsilon^2 B_1 + \epsilon^4 B_2 + \cdots, \tag{46}$$

$$D^*_{soft} \sim D_0 + \epsilon^2 D_1 + \epsilon^4 D_2 + \cdots, \tag{47}$$

where

$$D_0 = \int_0^1 \mathrm{d}x \int_{-\infty}^{\infty} p_0\left(\frac{\partial B_0}{\partial x}\right)^2 \mathrm{d}z, \tag{48}$$

$$\begin{aligned} D_1 = \int_0^1 \mathrm{d}x \int_{-\infty}^{\infty} &\left\{p_0\left[2\frac{\partial B_0}{\partial x}\frac{\partial B_1}{\partial x} + \left(\frac{\partial B_1}{\partial z}\right)^2\right]\right. \\ &\left. + p_1\left(\frac{\partial B_0}{\partial x}\right)^2\right\}\mathrm{d}z. \end{aligned} \tag{49}$$

The leading order governing equation derived from Eq. (15) is, after simplifications,

$$\frac{\partial}{\partial z}\left(p_0\frac{\partial B_0}{\partial z}\right) = 0, \tag{50}$$

and the boundary conditions at $O(1)$ are

$$p_0\frac{\partial B_0}{\partial z} \xrightarrow[z\to\pm\infty]{} 0, \tag{51}$$

$$B_0(1,z) - B_0(0,z) = -1. \tag{52}$$

Therefore, B_0 is a function of x only. The exact solution, up to an arbitrary additive constant, can be derived by integrating the governing equation over the cross section at the next order in the expansion,[35] that is

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty}\left(P_\infty\frac{\partial B}{\partial x} - BJ^x_\infty\right)\mathrm{d}z = U^*_{soft}\int_{-\infty}^{\infty} P_\infty \mathrm{d}z. \tag{53}$$

The leading order of this equation can be simplified to obtain

$$\frac{\mathrm{d}^2 B_0}{\mathrm{d}x^2} - \mathrm{Pe}\,\frac{\mathrm{d}B_0}{\mathrm{d}x} = \mathrm{Pe}, \tag{54}$$

which gives

$$\frac{dB_0}{dx} = -1. \tag{55}$$

Then, the leading order of the effective dispersion coefficient is

$$D_0 = \int_0^1 \mathrm{d}x \int_{-\infty}^{\infty} p_0\left(\frac{\mathrm{d}B_0}{\mathrm{d}x}\right)^2 \mathrm{d}z = 1. \tag{56}$$

This result is consistent with the leading order term for the average velocity, which was also not affected by the variation in the position of the channel centerline.

The governing equation at $O(\epsilon^2)$ is also derived from Eq. (15). After some simplifications, we obtain

$$\begin{aligned} &\frac{\partial}{\partial z}\left(p_0\frac{\partial B_1}{\partial z}\right) + \frac{\partial}{\partial x}\left(p_0\frac{\partial B_0}{\partial x}\right) \\ &\quad - \left[\left(\mathrm{Pe} - \frac{\partial V}{\partial x}\right)p_0 - \frac{\partial p_0}{\partial x}\right]\frac{\partial B_0}{\partial x} = p_0 u_0, \end{aligned} \tag{57}$$

with the boundary conditions

$$p_0\frac{\partial B_1}{\partial z}\xrightarrow[z\to\pm\infty]{} 0, \tag{58}$$

$$B_1(1,z)-B_1(0,z)=0. \tag{59}$$

Substituting the functions of B_0, p_0, and u_0 into Eq. (57) and integrating twice with respect to z, we obtain

$$B_1(x,z)=-\lambda\frac{\mathrm{d}g}{\mathrm{d}x}z+K_1(x), \tag{60}$$

with the condition $K_1(0)=K_1(1)$ derived from the condition $B_1(0,z)=B_1(1,z)$. After some simplifications, the effective dispersion coefficient at $O(\epsilon^2)$ is given by

$$D_1=\int_0^1 dx\int_{-\infty}^{\infty}\left\{p_0\left[2\frac{\partial B_0}{\partial x}\frac{\partial B_1}{\partial x}+\left(\frac{\partial B_1}{\partial z}\right)^2\right]+p_1\left(\frac{\partial B_0}{\partial x}\right)^2\right\}\mathrm{d}z=-\lambda^2\int_0^1\left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2 dx. \tag{61}$$

Therefore, the effective dispersion coefficient up to $O(\epsilon^2)$ is given by

$$D^*_{soft}\sim 1-\epsilon^2\lambda^2\int_0^1\left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2\mathrm{d}x. \tag{62}$$

This recovers the Einstein-Smoluchowski relation in the dimensionless form, which is $D^*_{soft}=\mu^*_{soft}$.

IV. TRANSPORT IN A NARROW, SLOWLY VARYING SOLID-CHANNEL

In this section, we consider a solid-channel with upper and lower walls described by $z_\pm=\lambda g(x)\pm 1/2$. The function $\lambda g(x)$ corresponds to the centerline of the channel, and it is periodic $g(0)=g(1)$. The channel width, $w(x)=z_+-z_-=1$, is equal to the effective width $I(x)=1$ of the soft-channel considered in Sec. III. The dimensionless governing equation, with $V(x,z)=0$, becomes

$$\epsilon^2\frac{\partial}{\partial x}\left(\mathrm{Pe}P_\infty-\frac{\partial P_\infty}{\partial x}\right)-\frac{\partial^2 P_\infty}{\partial z^2}=0. \tag{63}$$

The periodic boundary condition in x, the zero-flux condition at the boundaries, and the normalization condition are given by

$$P_\infty(0,z)=P_\infty(1,z), \tag{64a}$$

$$-\epsilon^2\lambda g'(x)\left(\mathrm{Pe}P_\infty-\frac{\partial P_\infty}{\partial x}\right)=\frac{\partial P_\infty}{\partial z},\ \text{at } z=\lambda g(x)\pm\frac{1}{2}, \tag{64b}$$

$$\int_0^1 \mathrm{d}x\int_{\lambda g(x)-1/2}^{\lambda g(x)+1/2}P_\infty \mathrm{d}z=1. \tag{64c}$$

A. Average velocity in a narrow, slowly varying solid-channel

Analogous to the analysis presented for soft-channels, we focus on the limiting case of $\epsilon\ll 1$ and $\lambda\sim O(1)$. First, we propose a solution in the form of an asymptotic expansion of P_∞,

$$P_\infty\sim\rho_0(x,z)+\epsilon^2\rho_1(x,z)+\epsilon^4\rho_2(x,z)+\cdots. \tag{65}$$

After the probability density P_∞ is determined, the average velocity can be evaluated by

$$U^*_{solid}=\int_0^1\mathrm{d}x\int_{\lambda g(x)-1/2}^{\lambda g(x)+1/2}\left(\mathrm{Pe}P_\infty-\frac{\partial P_\infty}{\partial x}\right)\mathrm{d}z \sim v_0+\epsilon^2 v_1+\epsilon^4 v_2+\cdots, \tag{66}$$

where

$$v_0=\mathrm{Pe}-\int_0^1\mathrm{d}x\int_{\lambda g(x)-1/2}^{\lambda g(x)+1/2}\frac{\partial\rho_0}{\partial x}\mathrm{d}z, \tag{67a}$$

$$v_i=-\int_0^1\mathrm{d}x\int_{\lambda g(x)-1/2}^{\lambda g(x)+1/2}\frac{\partial\rho_i}{\partial x}\mathrm{d}z,\ \text{for } i=1,2,3,\ldots. \tag{67b}$$

On the other hand, integrating both sides of Eq. (63) and applying the zero-flux boundary conditions, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{\lambda g(x)-1/2}^{\lambda g(x)+1/2}\left(\mathrm{Pe}P_\infty-\frac{\partial P_\infty}{\partial x}\right)\mathrm{d}z\right]=0. \tag{68}$$

Therefore, as expected, the quantity inside square brackets, which is the total flux in x-direction, is constant along the channel. Since the integral of the total flux in x-direction is the average velocity, we obtain

$$U^*_{solid}=\int_{\lambda g(x)-1/2}^{\lambda g(x)+1/2}\left(\mathrm{Pe}P_\infty-\frac{\partial P_\infty}{\partial x}\right)\mathrm{d}z \tag{69}$$

or

$$v_i=\int_{\lambda g(x)-1/2}^{\lambda g(x)+1/2}\left(\mathrm{Pe}\,\rho_i-\frac{\partial\rho_i}{\partial x}\right)\mathrm{d}z =\mathrm{Pe}\,\bar\rho_i-\left(\frac{\mathrm{d}\bar\rho_i}{\mathrm{d}x}-\lambda\frac{\mathrm{d}g}{\mathrm{d}x}\rho_i|_{z=\lambda g(x)+1/2}+\lambda\frac{\mathrm{d}g}{\mathrm{d}x}\rho_i|_{z=\lambda g(x)-1/2}\right), \tag{70}$$

where $\bar\rho_i$ is the marginal probability density, $\bar\rho_i=\int_{\lambda g(x)-1/2}^{\lambda g(x)+1/2}\rho_i\mathrm{d}z$. Then, integrating both sides of the equation above with respect to x from 0 to 1, the first term on the right hand side of the equation cancels for $i\geq 1$, due to the normalization condition; the second term of the right hand side is also identically zero, due to the periodicity in x. Therefore, we obtain an alternative expression for the average velocity,

$$v_i=\int_0^1\lambda\frac{\mathrm{d}g}{\mathrm{d}x}\left(\rho_i|_{z=\lambda g(x)+1/2}-\rho_i|_{z=\lambda g(x)-1/2}\right)\mathrm{d}x,\quad \text{for } i=1,2,3,\ldots. \tag{71}$$

This equation shows that the average velocity is completely determined by the probability density on the upper and lower boundaries. We shall use this expression to calculate the average velocity.

It is straightforward to show that the leading order term for the probability density is uniform, $\rho_0=1$ inside the channel. The corresponding leading order contribution to the average velocity is $v_0=\mathrm{Pe}$. The higher order terms of the probability density are governed by

$$\frac{\partial^2\rho_i}{\partial z^2}=\frac{\partial}{\partial x}\left(\mathrm{Pe}\,\rho_{i-1}-\frac{\partial\rho_{i-1}}{\partial x}\right),\ \text{for } i=1,2,3,\ldots, \tag{72}$$

and satisfy both the zero-flux boundary condition

$$\frac{\partial \rho_i}{\partial z} = -\lambda \frac{\mathrm{d}g}{\mathrm{d}x}\left(\mathrm{Pe}\,\rho_{i-1} - \frac{\partial \rho_{i-1}}{\partial x}\right), \text{ for } i = 1,2,3,\ldots, \quad \text{at } z_\pm = \lambda g(x) \pm 1/2, \tag{73}$$

as well as the normalization condition,

$$\int_0^1 \mathrm{d}x \int_{\lambda g(x)-1/2}^{\lambda g(x)+1/2} \rho_i \mathrm{d}z = 0, \text{ for } i = 1,2,3,\ldots. \tag{74}$$

Substituting ρ_0 into Eq. (72), we obtain $\partial^2\rho_1/\partial z^2 = 0$, which corresponds to a solution of the form

$$\rho_1(x,z) = a_1^1(x)z + a_0^1(x), \tag{75}$$

where $a_1^1(x)$ is determined by the zero-flux condition

$$a_1^1(x) = -\lambda \mathrm{Pe}\frac{\mathrm{d}g}{\mathrm{d}x}, \tag{76}$$

and the normalization condition implies $\int_0^1 a_0^1(x)\mathrm{d}x = 0$. The governing equation for $a_0^1(x)$ can be derived by integrating the $O(\epsilon^4)$ terms of the governing equation with respect to z over the cross section

$$\mathrm{Pe}a_0^1 - \frac{\mathrm{d}a_0^1}{\mathrm{d}x} = v_1 - \left(\mathrm{Pe}a_1^1 - \frac{\mathrm{d}a_1^1}{\mathrm{d}x}\right)\lambda g(x). \tag{77}$$

However, it is not necessary to determine $a_0^1(x)$ for calculating the average velocity. In fact, from Eq. (71), we obtain

$$v_1 = -\lambda^2 \mathrm{Pe}\int_0^1 \left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2 \mathrm{d}x. \tag{78}$$

Before calculating higher order contributions, we present the general procedure to calculate ρ_i for $i = 1,2,3,\ldots$. Since ρ_1 is a first order polynomial in terms of z, by induction, we propose ρ_i to be a polynomial of degree $(2i-1)$ in terms of z,

$$\rho_i(x,z) = a_{2i-1}^i(x)\frac{z^{2i-1}}{(2i-1)!} + a_{2i-2}^i(x)\frac{z^{2i-2}}{(2i-2)!} + \cdots + a_1^i(x)\frac{z}{1!} + a_0^i(x). \tag{79}$$

The coefficient $a_0^i(x)$ can be determined from the normalization condition for the next order term in the asymptotic solution. The coefficient $a_1^i(x)$ can be determined by the zero-flux condition in Eq. (73), and all other coefficients $a_j^i(x)$ can be determined from the coefficients of the lower order term in the asymptotic solution

$$a_j^i(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{Pe}a_{j-2}^{i-1}(x) - \frac{\mathrm{d}a_{j-2}^{i-1}(x)}{\mathrm{d}x}\right], \text{ for } j = 2,3,\ldots,2i-1.$$

In principle, higher order terms can be obtained using the proposed procedure repeatedly. Here, for simplicity, we only show the results up to $O(\epsilon^4)$,

$$\rho_2(x,z) = a_3^2(x)\frac{z^3}{3!} + a_2^2(x)\frac{z^2}{2!} + a_1^2(x)\frac{z}{1!} + a_0^2(x), \tag{80}$$

where $a_3^2(x)$ and $a_2^2(x)$ are derived from $a_1^1(x)$ and $a_0^1(x)$, respectively,

$$a_3^2(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{Pe}a_1^1(x) - \frac{\mathrm{d}a_1^1(x)}{\mathrm{d}x}\right] = -\lambda\mathrm{Pe}\left[\mathrm{Pe}\frac{\mathrm{d}^2g}{\mathrm{d}x^2} - \frac{\mathrm{d}^3g}{\mathrm{d}x^3}\right], \tag{81}$$

$$a_2^2(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{Pe}a_0^1(x) - \frac{\mathrm{d}a_0^1(x)}{\mathrm{d}x}\right] = \lambda^2\mathrm{Pe}\frac{\mathrm{d}}{\mathrm{d}x}\left[g(x)\left(\mathrm{Pe}\frac{\mathrm{d}g}{\mathrm{d}x} - \frac{\mathrm{d}^2g}{\mathrm{d}x^2}\right)\right]. \tag{82}$$

Then, $a_1^2(x)$ is determined by the zero-flux boundary condition

$$a_1^2(x) = -\lambda\frac{\mathrm{d}g}{\mathrm{d}x}\left(\mathrm{Pe}\,\rho_1 - \frac{\partial\rho_1}{\partial x}\right)\bigg|_{z=\lambda g(x)+1/2} - \left(a_3^2(x)\frac{z^2}{2!} + a_2^2(x)\frac{z}{1!}\right)\bigg|_{z=\lambda g(x)+1/2}. \tag{83}$$

Finally, the next order correction to the average velocity, v_2, is evaluated from Eq. (71),

$$v_2 = \frac{1}{12}\mathrm{Pe}\,\lambda^2\int_0^1\left(\frac{\mathrm{d}^2g}{\mathrm{d}x^2}\right)^2\mathrm{d}x + \mathrm{Pe}\,\lambda^4\left[\int_0^1\left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2\mathrm{d}x\right]^2. \tag{84}$$

Adding these contributions, the dimensionless mobility up to $O(\epsilon^4)$ is given by

$$\mu_{solid} = \frac{U^*_{solid}}{\mathrm{Pe}} \sim 1 - \epsilon^2\lambda^2\int_0^1\left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2\mathrm{d}x + \epsilon^4\left\{\frac{1}{12}\lambda^2\int_0^1\left(\frac{\mathrm{d}^2g}{\mathrm{d}x^2}\right)^2\mathrm{d}x + \lambda^4\left[\int_0^1\left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2\mathrm{d}x\right]^2\right\}. \tag{85}$$

First, we note that the mobility up to $O(\epsilon^4)$ is independent of the Péclet number, as in the case of soft-channels. Comparing this mobility, μ_{solid}, with μ_{soft} obtained for the soft-channel, it is clear that the resistance of the solid-channel is higher than that of the soft-channel. Specifically, the effect of the second derivative of the position of the centerline of the solid channel is nearly one half of that in a soft-channel. This conclusion is different from that obtained from the transport in a weakly corrugated symmetric channel with a varying width, in which the resistance of the solid-channel could be smaller than that of the soft-channel for small Péclet numbers.[36] However, we can make $(\epsilon^4\frac{1}{12}\lambda^2)_{solid} = (\epsilon^4\frac{1}{2\pi}\lambda^2)_{soft}$ by letting $\epsilon_{soft} = \sqrt{\pi/6}\epsilon_{solid}$ and $\lambda_{soft} = \sqrt{6/\pi}\lambda_{solid}$. For example, in a dimensional case, we can increase the solid-channel width from $w(x) = b$ to $b\sqrt{6/\pi}$ or reducing the soft-channel width from $l(x) = b$ to $b/\sqrt{6/\pi}$. Thus, if $l(x) = w(x)/\sqrt{6/\pi}$, the average velocity in both soft and solid-channels is the same up to $O(\epsilon^4)$. Fig. 2 shows two equivalent channels up to $O(\epsilon^4)$ where the solid-channel is bounded by two red lines $z_\pm = \lambda g(x) \pm \frac{\sqrt{6/\pi}}{2}$ and the soft-channel is represented by the concentration of particles, that is the Boltzmann distribution $e^{-\pi(z-\lambda g(x))^2}$.

B. Effective dispersion coefficient in a narrow, slowly varying solid-channel

In order to calculate the effective dispersion coefficient, D^*_{solid}, we need to solve the B-field equation

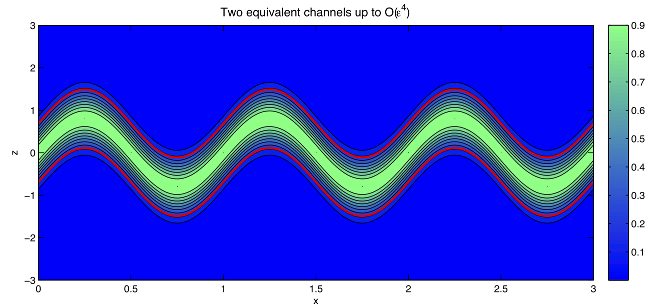

FIG. 2. Two equivalent channels up to $O(\epsilon^4)$: the solid-channel is bounded by two red lines $z_\pm = \lambda g(x) \pm \frac{\sqrt{6/\pi}}{2}$ and the concentration in the soft-channel is given by the Boltzmann distribution $e^{-\pi(z-\lambda g(x))^2}$, where $\epsilon = 0.1$ and $\lambda = 0.8$.

$$\frac{\partial}{\partial z}\left(P_\infty \frac{\partial B}{\partial z}\right) - \left(-\frac{\partial P_\infty}{\partial z}\right)\frac{\partial B}{\partial z} + \epsilon^2\left[\frac{\partial}{\partial x}\left(P_\infty \frac{\partial B}{\partial x}\right) - \left(\mathrm{Pe}P_\infty - \frac{\partial P_\infty}{\partial x}\right)\frac{\partial B}{\partial x}\right] = \epsilon^2 P_\infty U^*_{solid}, \tag{86}$$

with boundary conditions

$$\frac{\partial B}{\partial z} = \epsilon^2 \lambda \frac{dg}{dx}\frac{\partial B}{\partial x}, \text{ at } z = \lambda g(x) \pm \frac{1}{2}, \tag{87a}$$

$$B(1,z) - B(0,z) = -1. \tag{87b}$$

Proposing then an asymptotic expansion for the B-field,

$$B(x,z) \sim \mathcal{B}_0(x,z) + \epsilon^2\mathcal{B}_1(x,z) + \epsilon^4\mathcal{B}_2(x,z) + \cdots, \tag{88}$$

and solving for $\mathcal{B}_i(x,z)$, the effective dispersion coefficient is given by

$$D^*_{solid} = \int_0^1 \mathrm{d}x \int_{\lambda g(x)-1/2}^{\lambda g(x)+1/2}\left[\left(\frac{\partial B}{\partial x}\right)^2 + \frac{1}{\epsilon^2}\left(\frac{\partial B}{\partial z}\right)^2\right]\mathrm{d}z \sim \mathcal{D}_0 + \epsilon^2\mathcal{D}_1 + \epsilon^4\mathcal{D}_2 + \cdots, \tag{89}$$

where

$$\mathcal{D}_0 = \int_0^1 \mathrm{d}x \int_{\lambda g(x)-1/2}^{\lambda g(x)+1/2} \rho_0\left(\frac{\partial \mathcal{B}_0}{\partial x}\right)^2\mathrm{d}z, \tag{90}$$

$$\mathcal{D}_1 = \int_0^1 \mathrm{d}x \int_{\lambda g(x)-1/2}^{\lambda g(x)+1/2}\left\{\rho_0\left[2\frac{\partial \mathcal{B}_0}{\partial x}\frac{\partial \mathcal{B}_1}{\partial x} + \left(\frac{\partial \mathcal{B}_1}{\partial z}\right)^2\right] + \rho_1\left(\frac{\partial \mathcal{B}_0}{\partial x}\right)^2\right\}\mathrm{d}z. \tag{91}$$

It is straightforward to calculate the effective dispersion coefficient up to $O(\epsilon^2)$,

$$D^*_{solid} \sim 1 - \epsilon^2\lambda^2\int_0^1\left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2\mathrm{d}x. \tag{92}$$

We have shown before that the corrections to the mobility are independent of the Péclet number, and the Einstein-Smoluchowski relation $D^*_{solid} = \mu^*_{solid}$ is also valid in the case of solid-channels.

V. DISCUSSION AND CONCLUSIONS

Physically, the dimensional average velocity discussed above can be expressed as the ratio between the distance traveled in the longitudinal direction and the nominal holdup time within the curved channel,[18]

$$\bar{U}^* = \frac{\text{longitudinal distance traveled}}{\text{nominal holdup time}} = \frac{L}{t^*}. \tag{93}$$

The nominal holdup time is the average transit time between the two ends of the channel, separated a distance L. The driving force F is constant along the $\bar{x}$-direction. Then, to calculate the velocity along the channel centerline, we first write its tangent, $(\mathrm{d}\bar{x}, \mathrm{d}\bar{z})$ in the dimensional form. Then, the differential arclength is $\mathrm{d}s = \sqrt{\mathrm{d}\bar{x}^2 + \mathrm{d}\bar{z}^2}$ and the velocity along the centerline is $\frac{\mathrm{d}\bar{x}}{\mathrm{d}s}\frac{F}{\eta}$. Therefore, the nominal holdup time is

$$t^* = \int_0^L \frac{ds}{\frac{\mathrm{d}\bar{x}}{\mathrm{d}s}\frac{F}{\eta}} = \frac{\eta}{F}\int_0^L\left[1 + \left(\frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{x}}\right)^2\right]\mathrm{d}\bar{x} = \frac{L\eta}{F}\int_0^1\left[1 + \epsilon^2\lambda^2\left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2\right]\mathrm{d}x, \tag{94}$$

and the dimensionless average velocity can be approximated by the average velocity for particles moving along the centerline of the channel,

$$U^*_c = \frac{\bar{U}^*}{k_BT/(\eta L)} = \mathrm{Pe}\,\frac{1}{1 + \epsilon^2\lambda^2\int_0^1\left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2\mathrm{d}x} = \mathrm{Pe}\left\{1 - \epsilon^2\lambda^2\int_0^1\left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2\mathrm{d}x + \left[\epsilon^2\lambda^2\int_0^1\left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2\mathrm{d}x\right]^2 + O(\epsilon^6)\right\}. \tag{95}$$

This simple physical argument recovers the exact results of the average velocity up to $O(\epsilon^2)$. Even for the result at $O(\epsilon^4)$, the effect due to the first derivative of the centerline function is the same.

In the absence of external forces, the diffusion of Brownian particles in the longitudinal direction is reduced depending on the tortuosity of the system, defined as the square of the ratio of the arc-length of the serpentine channel L_S to the distance between the two ends of the channel L.[18] Specifically, in the absence of external force, the effective dispersion of the particles reduces to the effective diffusivity along the channel and can be approximated by assuming the particles are restricted to move along the centerline of the channel,

$$D^*_c = \left(\frac{L}{L_S}\right)^2 = \frac{1}{\left(\int_0^1\sqrt{1 + \lambda^2\epsilon^2\left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2}\mathrm{d}x\right)^2}. \tag{96}$$

By expanding the above expression with $\epsilon \ll 1$, we obtain the series up to $O(\epsilon^2)$,

$$D^*_c = 1 - \lambda^2\epsilon^2\int_0^2\left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2\mathrm{d}x + O(\epsilon^4). \tag{97}$$

The fact that this calculation gives the same correction as that obtained in Eqs. (62) and (92) shows that the dominant effect on the dispersion of particles in a narrow, serpentine channel, in the presence of an external bias, is the reduction in the dispersion due to the tortuosity of the channel.

In summary, we have used the method of asymptotic expansions to calculate two important transport properties in the motion of suspended particles in a narrow, slowly varying serpentine channel: the average velocity (or mobility) and the effective dispersion coefficient. We compare the results for two types of channels, solid-channels that confine the particles with solid walls and soft-channels created by a confining potential.

In soft channels, the mobility in Eq. (45) is

$$\mu_{soft} \sim 1 - \epsilon^2\lambda^2 \int_0^1 \left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2 \mathrm{d}x + \epsilon^4 \left[\frac{\lambda^2}{2\pi} \int_0^1 \left(\frac{\mathrm{d}^2 g}{\mathrm{d}x^2}\right)^2 \mathrm{d}x + \lambda^4 \left(\int_0^1 \left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2 \mathrm{d}x\right)^2\right] + O(\epsilon^6), \quad (98)$$

and the effective dispersion coefficient in Eq. (62) is

$$D^*_{soft} \sim 1 - \epsilon^2\lambda^2 \int_0^1 \left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2 \mathrm{d}x + O(\epsilon^4). \quad (99)$$

In solid channels, the mobility in Eq. (85) is

$$\mu_{solid} \sim 1 - \epsilon^2\lambda^2 \int_0^1 \left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2 \mathrm{d}x + \epsilon^4 \left\{\frac{1}{12}\lambda^2 \int_0^1 \left(\frac{\mathrm{d}^2 g}{\mathrm{d}x^2}\right)^2 \mathrm{d}x + \lambda^4 \left[\int_0^1 \left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2 \mathrm{d}x\right]^2\right\} + O(\epsilon^6), \quad (100)$$

and the effective dispersion coefficient in Eq. (92) is

$$D^*_{solid} \sim 1 - \epsilon^2\lambda^2 \int_0^1 \left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2 \mathrm{d}x + O(\epsilon^4). \quad (101)$$

Our results show that the leading order correction for both the mobility and the dispersion coefficient in both solid and soft channels is given by the effective tortuosity of the serpentine channel, which is independent of the Peclet number. As a result, the Einstein-Smoluchowski relation between effective mobility and diffusion is satisfied for both types of channels. The higher order correction, at $O(\epsilon^4)$, shows that the resistance of the solid-channel to particle transport is larger, but the difference can be eliminated by changing the width of one of the channels. Interestingly, in both types of channels, the mobility up to $O(\epsilon^4)$ is independent of the Peclet number.

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