DYNAMICS OF A FERROMAGNETIC GRANULAR SYSTEM

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ABSTRACT OF THE THESIS

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We study theoretically and numerically the dynamics of a one-dimensional ferromagnetic granular system. Corresponding to different types of potential in the chain, linear, weakly nonlinear and strongly nonlinear partial differential equations are derived respectively. The continuum limit is derived following the method used by Ishimori [14]. Specifically, we show that by giving initial dynamic force, a system endure anharmonic nearest neighbor interaction(NNI) and inverse power-law long range interaction(LRI) will generate nonlinear solitary waves. Both weakly and strongly nonlinear equations occupied unique properties. Furthermore, we find that the equations of motion varies with different values of the exponent parameter p in each case. Next, we focus on the discussion of the dipole-dipole interaction which corresponds to the ferromagnetic system. We show that though the main contribution to the solitary wave is the short range part, the long-range interaction effect the shape of the solitary wave as well as its propagation velocity.

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1 Introduction

In the last fifty years, started by Zabusky and Kruskal [37], the numerical [9, 30] and experimental [23, 4, 22] studies on the propagation of nonlinear solitary waves in one-dimensional chains of granular system, and in particular of spherical elastic particles, have thrived [33, 2, 26, 9, 11]. Solitary waves are lumps of energy, which can maintain their shapes while traveling at a constant speed. One of the most general governing equations of solitary waves is called the Korteweg-de Vries(K-dV) equation [16]. A wave equation have a soliton solution because of the balance of its nonlinearity and dispersion effect. In a one-dimensional granular system, the particle chain can generate nonlinear solitary waves by giving initial dynamic force. The nonlinearity arises due to the anharmonic short range contact between two adjacent particles. Many important equations regarding solitary waves are related to this system, including the Boussinesq equation, the nonlinear Schrödinger equation, the Benjamin-Ono equation and so on [7, 36].

However, the above equations can properly describe the system contains only local forces, which directly relate to the interaction between the adjacent spheres. For some materials and real systems, interactions between particles are more complicated that they may extend further than the nearest neighbors, which is so called the long-range interaction(LRI). This may include the long range Coulomb interaction, the dipole-dipole interaction and the quadrupole-quadrupole interaction [28]. As is well known, the dispersion relation of the lattice with the LRI is different from that of the lattice with only the NNI. Therefore, for these systems, they will have different solitary wave solutions. Particularly, If the particles are ferromagnetic material, then there exist a "inverse power"-type magnetic potential between each spheres, no matter whether they are contact directly or not. This is so called the nonlocal interaction. This paper is to investigate the effects of the magnetic force to the prototype of granular system and analyze the wave propagation in the chain of ferromagnetic spherical particles.

Much of the interest was given to the granular system is also because of the fact that the properties of granular particles are tunable by changing there mass, radius and material properties [6]. The tunable feature also gives such system many promising potential applications such as sound or shock absorbing [13, 5], sound focusing [31] and scramblers [29, 4]. Besides the ferromagnetic material, there are also many other

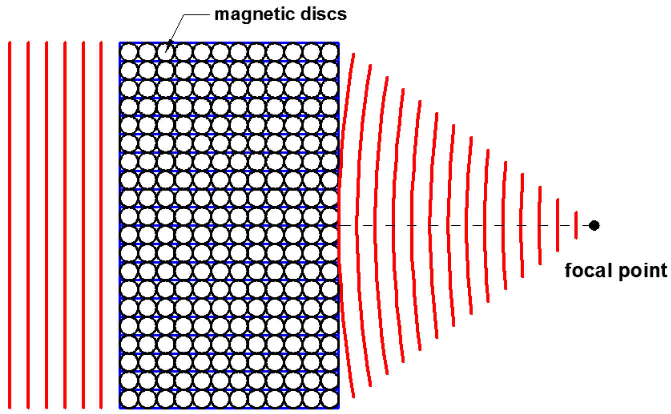

Figure 1: An illustration of application of ferromagnetic granular system as acoustic lens. This device can focus plane wave by adjusting the magnetic potential among the discs.

material and real systems, such as carbon nanotubes [8] and DNA [34] are affected by the nonlocal interaction greatly. The behavior of these systems cannot be accurately described by the classical local continuum theories. Thus, a well-defined model including nonlocal forces is very important to analysis the dynamic behavior of these material and systems. Works have been down related to this field can be traced back to nineteen-sixties. Kroner [17] proposed the elasticity theory of materials with long range cohesive force in 1967. Following him, many nonlocal constitutive models have been built by different authors, such as Kunin [18], Maugin [21], and Eringen [1]. However, many of these early works are done in the framework of lattice dynamics, not until recently, some models account for both lattice approach and continuum approach have been discussed by authors such as Rosenau [27], Pouget [25], Friesecke *et al* [12] and Lazar *et al.* [20]. To be specific, Ishimori [14]investigated a one-dimensional lattice with the Lennard-Jones long-range potential. He discussed the nonlinear waves in the continuum limit and found several different types of equations, depending on the power n. Despite those models, a further analysis of nonlocal interaction both in lattice dynamics and continuum field is required to show the effect of non-locality in particular material systems.

This paper focus on the ferromagnetic material system, we simplified the system as 1D granular chains in which each particle is affected by both local interaction force and nonlocal magnetic force. We first introduce the system as a chain of discrete contact spheres under lattice dynamics approach, then we introduce the continuum limit and, basing on the types of local interaction, we convert it into three different continuum models. These three models generate linear, weakly and strongly nonlinear wave equations respectively. Then we approximate the solitary wave solutions separately and analysis the dynamics properties of the systems.

The rest of the paper is as follow: In section 2, we introduce the model of 1D granular system, we derive the equation of motion for the discrete system and then introduce the continuum limits for three different cases, say harmonic case, weakly nonlinear case and strongly nonlinear case. Then we specify the model as the ferromagnetic system which endure dipole-dipole LRI. Basing on different types of local potential, three different wave equations in continuum limit and there analytical solutions are obtained in section 3, respectively. In section 4, we do the numerical simulation to verify our results and then comes the conclusion and discussion in section 5.

2 Equation of motion of 1D granular system

2.1 General equation of motion with long range interaction

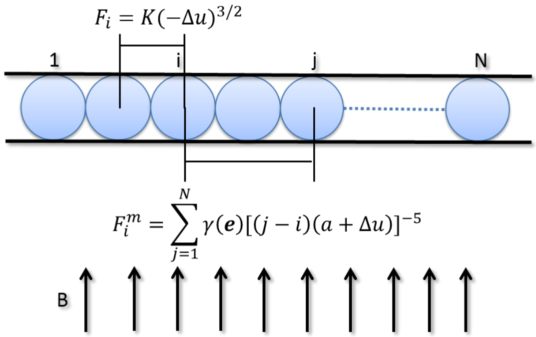

Figure 2: Ferromagnetic granular system under uniform magnetic field

Consider a one dimensional granular system with spacing a and lattice mass M, as shown in Fig. 2. Kinematically, we describe the system by the displacement $u_n(1 \leq n \leq N)$ of the nth lattice, where N is the total number of lattices in the system. The interactions in the lattice are specified by the total interaction energy:

$$V(u_0, \cdots, u_N) = V_1(u_0, \cdots, u_N) + V_2(u_0, \cdots, u_N) \tag{1}$$

where u_n is the displacement of the nth particle in x-direction, and

$$V_1 = \sum_{n=1}^{N} \psi(u_{n+1} - u_n) \tag{2}$$

is the total interaction energy between nearest neighbors,

$$V_2 = \frac{1}{2} \sum_{n,m=1}^{N} I_{nm}(u_n, u_m) \tag{3}$$

is the total LRI energy, where

$$I_{nm}(u_n, u_m) = \varphi\Big((m-n)a + u_m - u_n\Big), \tag{4}$$

is the LRI function which indicate that the nonlocal interaction potential between nth and mth $(1 \leq m \leq N)$ lattice points depends only on the inter-distance between two lattice point. Here, ψ and φ represent the NNI and LRI energy between each two lattices, respectively. For this system, the total kinetic energy should be

$$T = \sum_{n=2}^{N} \frac{1}{2} M \dot{u}_n^2, \tag{5}$$

where $\dot{u}_n$ denotes the velocity of the nth particle in the chain. Therefore the Lagrangian of the system can be expressed as

$$L = T - V = \sum_{n=2}^{N} \frac{1}{2} M \dot{u}_n^2 - \sum_{n=1}^{N} \psi(u_{n+1} - u_n) - \frac{1}{2} \sum_{n,m=1}^{N} I_{nm}(u_n, u_m). \tag{6}$$

The equation of motion for the nth particle of the discrete system obtained from the variation of (6) is

$$\begin{aligned} M\ddot{u}_n =& \psi'(u_{n+1} - u_n) - \psi'(u_n - u_{n-1}) \\ &+ \sum_{j=1}^{N} \{\varphi'[ja + u_{n+j} - u_n] - \varphi'[ja + u_n - u_{n-j}]\}, \end{aligned} \tag{7}$$

where $\ddot{u}_n$ denotes acceleration of the nth particle and $j = m - n$ is an integer. The above equation will be untractable if N is very large. We shall seek for the continuum description of the discrete system. Here we follow Nesterenko [24] and consider the space between each particle is small compared with the wavelengths. Under the long wavelength assumption, there exist a smooth function $\phi(\cdot, t) : (0, L) \to I\!R$ such that

$$\phi(x_n, t) = u_n(t), \qquad x_n = na, \qquad L = Na, \qquad \forall\, n = 0, \cdots, N.$$

Then we can derive the continuum representation of the total interaction energy of the system, which should be

$$V[\phi] = V_1[\phi] + V_2[\phi], \tag{8}$$

where

$$V_1[\phi] = \sum_{n=1}^{N} \psi[\phi(x_{n+1}) - \phi(x_n)], \qquad V_2[\phi] = \frac{1}{2} \sum_{n,m=1}^{N} \varphi[(m-n)a + \phi(x_m) - \phi(x_n)]. \tag{9}$$

The total kinetic energy of the system in continuum description is

$$T[\phi] = \sum_{n=2}^{N} \frac{1}{2} M \phi_t^2(x_n). \tag{10}$$

Finally, we can derive specified equations of motion for different types of systems, depending on different kinds of local and nonlocal potentials.

2.2 Harmonic System

A 1D granular system includes nearest-neighbor interaction and long-range interactions can be defined as a harmonic system if both local and nonlocal potentials are harmonic. The system can be transformed to a spring-like system where each two spheres in the chain are connected by a linear spring. To derive the equation of motion of such system, first we can define the local potential density as

$$\psi[\phi] = \frac{1}{2} K_1 [\phi(x_{n+1}) - \phi(x_n)]^2, \tag{11}$$

where K_1 is the spring stiffness. For the nonlocal potential density $\varphi[\phi]$, we notice that the particle motion is a small variable compared with lattice spacing, so we can write it in a more general form:

$$\varphi[\phi] \approx \varphi[(m-n)a] + \frac{1}{2} \varphi^{''}[(m-n)a][\phi(x_m) - \phi(x_n)]^2 \tag{12}$$

where we have neglected the terms higher than $O(\phi(x_n)^3)$. The linear term vanished because the particles are in equilibrium when $\phi(x_n) = 0$ for all x_n. As a result, equation (7) can be written as

$$\begin{aligned} M\phi(x_n)_{tt} =& K_1[\phi(x_{n+1}) - 2\phi(x_n) + \phi(x_{n-1})] \\ &+ \sum_{j=1}^{N} \varphi^{''}(ja)[\phi(x_{n+j}) + \phi(x_{n-j}) - 2\phi(x_n)]. \end{aligned} \tag{13}$$

In the long wavelength approximation [14], we are looking for plane wave solutions of the form

$$\phi(x_n, t) \simeq e^{i(kna - \omega t)}. \tag{14}$$

Using this replacement, we can obtain the dispersion relation by substituting (14) into (13), which gives us

$$\begin{aligned} M{e^{i(kan-\omega t)}}_{tt} =& K_1[e^{i[ka(n+1)-\omega t]} - 2e^{i(kan-\omega t)} + e^{i[ka(n-1)-\omega t]}] \\ &+ \sum_{j=1}^{N} \varphi^{''}(ja)[e^{i[ka(n+j)-\omega t]} + e^{i[ka(n-j)-\omega t]} - 2e^{(ikan-\omega t)}]. \end{aligned} \tag{15}$$

Extracting $e^{i(kna-\omega t)}$ out on both sides, we obtain

$$\begin{aligned} M\omega^2(k)e^{i(kna-\omega t)} =& 2K_1[1 - \frac{e^{ika} + e^{-ika}}{2}]e^{i(kna-\omega t)} \\ &+ 2\sum_{j=1}^{N} \varphi^{''}(ja)[1 - \frac{e^{ikja} + e^{-ikja}}{2}]e^{i(kna-\omega t)}. \end{aligned} \tag{16}$$

The corresponding dispersion relation should be

$$M\omega^2(k) = 2K_1[1 - \cos(ka)] + 2L(ka) \tag{17}$$

where

$$L(ka) = \sum_{j=1}^{N} \varphi^{''}(ja)[1 - \cos(jka)], \tag{18}$$

is the LRI dispersion term. To derive the partial differential wave equation for the harmonic system, we have to specify the LRI term here. Generally, there are two different types of LRI potential in real systems or materials, the exponential law (also called Kac-Baker)type and the inverse power-law type. Both of them can be transformed into a harmonic one, but with different form of stiffness coefficient. Although sharing the same property in many aspects, they are essentially different. In this paper, we primarily consider the ferromagnetic granular system, which endures inverse-power law type LRI. Generally, this type of long-range potential density can be written as

$$\varphi(x) \simeq \frac{\gamma}{|x|^q}, \qquad (q \in \Re, q \geq 1). \tag{19}$$

where γ is the LRI potential coefficient. Thus

$$\varphi^{''}(x) = \frac{q(q+1)}{|x|^{q+2}}\gamma. \tag{20}$$

Therefore, according to equation (18), the long-range term of equation (17) can be written as

$$L(ka) = \sum_{j=1}^{N} \frac{[q(q+1)][1-\cos(jka)]}{(ja)^{q+2}} = JF_p(ka), \qquad p = q+2 \tag{21}$$

where

$$J = \frac{q(q+1)}{a^p}\gamma, \tag{22}$$

is the long range parameter which measures the strength of the LRI, and

$$F_p(ka) = \sum_{j=1}^{N} \frac{[1-\cos(jka)]}{j^p}, \tag{23}$$

is an even function of k. For given values of p, $F_p(ka)$ can be expanded into different forms(see appendix). Given the Taylor series of $\sin(ka)$ and $\cos(ka)$ that

$$\begin{aligned} \cos(ka) &\approx 1 - \frac{1}{2}(ka)^2 + \frac{1}{24}(ka)^4 + \frac{1}{720}(ka)^6 + \cdots, \\ \sin(ka) &\approx ka - \frac{1}{6}(ka)^3 + \frac{1}{120}(ka)^5 + \cdots. \end{aligned} \tag{24}$$

Keeping to the fourth order term, the dispersion relation can be written as

$$M\omega^2(k) \approx \quad 2K_1[\tfrac{1}{2}(ka)^2 - \tfrac{1}{24}(ka)^4] + 2J\textstyle\sum_{j=1}^{N}[\tfrac{1}{2}(jka)^2 - \tfrac{1}{24}(jka)^4]j^{-p}. \tag{25}$$

The equation of motion corresponds to this dispersion relation can be obtained from equation (25) that

$$\begin{aligned} M\partial_t^2 e^{i(kna-\omega t)} \approx & 2K_1[\frac{1}{2}(a\partial_x)^2 + \frac{1}{24}(a\partial_x)^4]e^{i(kna-\omega t)} \\ & + 2J\sum_{j=1}^{N}[\frac{1}{2}(ja\partial_x)^2 + \frac{1}{24}(ja\partial_x)^4]j^{-p}e^{i(kna-\omega t)}, \end{aligned} \tag{26}$$

which can be simplified as

$$M\phi_{tt} = a^2K_1[\phi_{xx} + \frac{1}{12}a^2\phi_{xxxx}] - 2JF_p(aD)\phi, \tag{27}$$

where

$$D = i\partial_x, \qquad F_p(aD) = \sum_{j=1}^{N}[-\frac{1}{2}(ja\partial_x)^2 - \frac{1}{24}(ja\partial_x)^4]j^{-p}. \tag{28}$$

The Hamiltonian of the system should be

$$H = \frac{1}{2a}\int_{-\infty}^{\infty}\{M\phi_t^2 + K_1[(a\phi_x)^2 - \frac{1}{12}a^4\phi_{xx}^2] - 2\phi J F_p(aD)\phi\}dx. \tag{29}$$

Equation (27) is actually a simple linear dispersive wave equation plus a nonlocal term. The effect of the LRI term on the equation of motion will varies with certain value of dispersive parameter p. This also indicates the properties of our harmonic system vary with different types of nonlocal potential. Later we will illustrate that the nonlocal potential will only influence the dispersion relation of the wave equations.

2.3 Weakly nonlinear system

Another essential type of granular system is called weakly nonlinear system. In this particular system, the local potential will include both ϕ^2 and ϕ^3 term. Thus equation (7) can be transformed to a nonlinear oscillators-like system [10]. First, we define the local potential density as

$$\psi[\phi] = \frac{1}{2}K_1[\phi(x_{n+1}) - \phi(x_n)]^2 - \frac{1}{6}K_2[\phi(x_{n+1}) - \phi(x_n)]^3, \tag{30}$$

Now, equation (7) can be written as follows:

$$\begin{aligned} M\phi(x_n)_{tt} =& K_1[\phi(x_{n+1}) - 2\phi(x_n) + \phi(x_{n-1})] \\ & - K_2[\phi(x_{n+1}) - 2\phi(x_n) + \phi(x_{n-1})][\phi(x_{n+1}) - \phi(x_{n-1})] \\ & + \sum_{j=1}^{N}\varphi''(ja)[\phi(x_{n+j}) + \phi(x_{n-j}) - 2\phi(x_n)]. \end{aligned} \tag{31}$$

Following the same steps as discussed in the harmonic section, we finally reach the dispersion relation for weakly nonlinear system:

$$M\omega^2(k) = 2K_1[1-\cos(ka)] - 2K_2\{[1-\cos(ka)][i\sin(ka)]\} + 2L(ka). \tag{32}$$

Similarly, the corresponding nonlinear partial differential wave equation should be

$$M\phi_{tt} = a^2K_1[\phi_{xx} + \frac{1}{12}a^2\phi_{xxxx}] - a^3K_2(\phi_x^2)_x - 2JF_p(aD)\phi. \tag{33}$$

We notice that the above equation is in the similar form of Boussinesq equation:

$$\phi_{tt} = \phi_{xx} + a\phi_{xxxx} - b\phi_x\phi_{xx}, \tag{34}$$

which has solitary wave solutions in two directions due to the combination of effects of its dispersive and nonlinear terms. The only difference between our equation and Boussinesq equation is the LRI term. Again, the LRI term will influence only the dispersion relation of the system so that a solitary wave solution can still be obtained from (33). Finally, the Hamiltonian of the system should be

$$\begin{aligned} H = \frac{1}{2a}\int_{-\infty}^{\infty} \{ & M\phi_t^2 + K_1[(a\phi_x)^2 - \frac{1}{12}a^4\phi_{xx}^2] \\ & - \frac{2}{3}K_2(a\phi_x)^3 - 2\phi JF_p(aD)\phi\}dx. \end{aligned} \tag{35}$$

2.4 Hertzian system

Hertzian system means the nearest-neighbor interaction between particles in the system is governed by Hertz potential [24, 32](Fig. 3). One of the most interesting characters of such system is that the local force between the spheres can not be linearized, means there is no linear term in the force at all. This feature makes it different from weakly nonlinear one and the system is so-called "strongly nonlinear system". Assuming a one-dimensional chain of spherical particles, which are barely contact at first. The interaction between two adjacent beads is governed by the Hertz's law:

$$F = K_h(-\Delta u)^{3/2}, \tag{36}$$

where $\Delta u(\Delta u < 0)$ is the change of distance between the centers of two spheres and K_h is the stiffness constant which can be expressed as

$$K_h = \frac{Ea^{1/2}}{3(1-\nu^2)}. \tag{37}$$

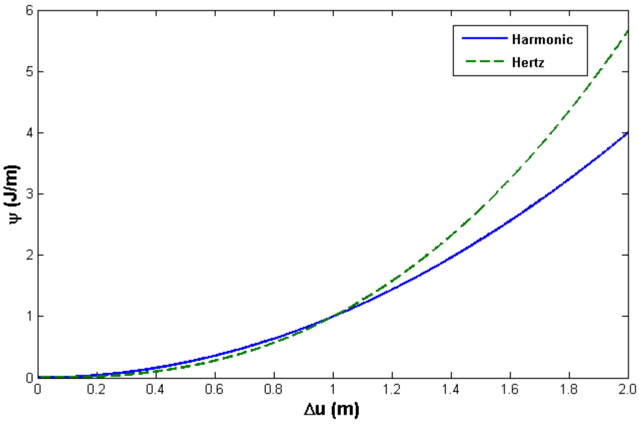

Figure 3: Comparing $\psi(\Delta u)$ for the Harmonic local potential and Hertz potential.

Then, the potential energy between neighbors can be defined as:

$$V(\Delta u) = \frac{2}{5} K_h (-\Delta u)^{5/2}. \tag{38}$$

Thus, the total local interaction energy of the system should be

$$V_1[\phi] = \sum_{i=1}^{N} V(\Delta u_i), \tag{39}$$

where $\Delta u_i = u_{i+1} - u_i$. Then, according to equation (2), for this specific problem, we have

$$\psi(x) = \frac{2}{5} K_h (-x)^{5/2}, \qquad \forall\, x \in (0, L). \tag{40}$$

Following, we first derive the governing wave equation for the general strongly non-linear system, then we will specified it into Hertzian one. To make the long wavelength assumption valid, we again have to assume that the particle motions are very small when compared with the lattice spacing a. Therefore, according to the small parameter $\epsilon = a/L$, by Taylor expansion and truncation, we have

$$\begin{aligned} \phi(x_{n+1}) &= \phi(x_n) + a\phi_x(x_n) + \frac{a^2}{2}\phi_{xx}(x_n) + \frac{a^3}{6}\phi_{xxx}(x_n) + o(a^3) \\ &= \phi(x_n) + a\phi_x(x_n) + \eta, \end{aligned} \tag{41}$$

where

$$\eta = \frac{a^2}{2}\phi_{xx}(x_n) + \frac{a^3}{6}\phi_{xxx}(x_n) + o(a^3) \tag{42}$$

is introduced for brevity. Therefore, equation (9) can be rewritten as

$$V_1[\phi] = \sum_{n=1}^{N} \psi[\phi(x_{n+1}) - \phi(x_n)] \approx \frac{1}{a}\int_0^L \{\psi[a\phi_x(x) + \eta]\}dx. \tag{43}$$

Also, we have

$$\psi[a\phi_x(x) + \eta] = \psi[a\phi_x(x)] + \psi'[a\phi_x(x)]\eta + \frac{1}{2}\psi''[a\phi_x(x)]\eta^2 + o(\varepsilon^2). \tag{44}$$

Thus, we can write the stored energy functional in continuum representation as

$$\begin{aligned} V_1[\phi] &\approx \frac{1}{a}\int_0^L \{\psi[a\phi_x(x)] + \psi'[a\phi_x(x)]\eta + \frac{1}{2}\psi''[a\phi_x(x)]\eta^2\}dx \\ &\approx \frac{1}{a}\int_0^L \big[\psi(a\phi_x) + \psi'(a\phi_x)(\frac{a^2}{2}\phi_{xx} + \frac{a^3}{6}\phi_{xxx}) + \frac{1}{2}\psi''(a\phi_x)(\frac{a^2}{2}\phi_{xx} + \frac{a^3}{6}\phi_{xxx})^2\big]dx \\ &\approx \frac{1}{a}\int_0^L \big[\psi(a\phi_x) + \psi'(a\phi_x)(\frac{a^2}{2}\phi_{xx} + \frac{a^3}{6}\phi_{xxx}) + \frac{a^4}{8}\psi''(a\phi_x)\phi_{xx}^2\big]dx. \end{aligned} \tag{45}$$

Moreover, integrating by parts we have

$$\int_0^L \big[\psi'(a\phi_x)\phi_{xxx}\big]dx = \psi'(a\phi_x)\phi_{xx}\Big|_0^L - a\int_0^L \big[\psi''(a\phi_x)\phi_{xx}^2\big]dx. \tag{46}$$

Neglecting the boundary contribution for periodic solutions or solutions that vanishes at the boundary we have

$$V_1[\phi] = \int_0^L W_e(\phi_x, \phi_{xx})dx, \qquad W_e(\phi_x, \phi_{xx}) = \frac{1}{a}\psi(a\phi_x) - \frac{a^3}{24}\psi''(a\phi_x)|\phi_{xx}|^2. \tag{47}$$

For the nonlocal interaction potential term, according to (9), we have

$$
\begin{aligned}
V_2[\phi] &= \frac{1}{2}\sum_{n,m=1}^{N} \varphi[|(m-n)a+\phi(x_m)-\phi(x_n)|] \\
&\approx \frac{1}{2a^2}\int_{\mathbb{U}} \varphi\{y+\phi(y)-[x+\phi(x)]\}dxdy \\
&\approx \frac{1}{2a^2}\int_{\mathbb{U}} \big[\varphi(y-x)+\varphi'(y-x)[\phi(y)-\phi(x)] \\
&\qquad\qquad + \frac{1}{2}\varphi''(y-x)[\phi(y)-\phi(x)]^2+o(\varepsilon^2)\big]dxdy,
\end{aligned}
\tag{48}
$$

where we assumed $x_m = y = ma, x_n = x = na$, and

$$
\mathbb{U} = \{(x,y)\in\mathbb{R}^2 : |y-x|>a\}. \tag{49}
$$

Note that by symmetry, the second term of the expansion equals to zero. Therefore, neglecting higher order terms and within a trivial constant, the total nonlocal potential energy of the system can be written as

$$
V_2[\phi] = \frac{1}{4a^2}\int_{\mathbb{U}} \{2\varphi(y-x)+\varphi''(y-x)[\phi(y)-\phi(x)]^2\}dxdy. \tag{50}
$$

On the other hand, the total kinetic energy of the system is

$$
T[\phi] = \sum_{n=2}^{N}\frac{1}{2}M\phi(x_n)_t^2 = \int_0^L \frac{1}{2}\rho\phi_t^2 dx, \tag{51}
$$

where $\rho = M/a$ is the chain density. Therefore, the action function of the system is given by

$$
\begin{aligned}
S[\phi] = \int_{t_0}^{t_1} \Big\{ &\int_0^L \{\frac{1}{2}\rho\phi_t^2 - W_e(\phi_x,\phi_{xx})\}dx \\
&- \frac{1}{4a^2}\int_{\mathbb{U}} \{2\varphi(y-x)+\varphi''(y-x)[\phi(y)-\phi(x)]^2\}dxdy\Big\}dt.
\end{aligned}
\tag{52}
$$

Given the total interaction energy, we now can derive the continuum equation of motion of the lattice system. By assuming a small perturbation of the system, we have

$$
\begin{aligned}
\frac{d}{d\delta}S[\phi+\delta\phi_1]\Big|_{\delta=0} =& \\
&\frac{d}{d\delta}\int_{t_0}^{t_1}\Big\{\int_0^L\{\frac{1}{2}\rho(\phi+\delta\phi_1)_t^2-\frac{1}{a}\psi[a(\phi+\delta\phi_1)_x] \\
&\qquad +\frac{a^3}{24}\psi''[a(\phi+\delta\phi_1)_x]|(\phi+\delta\phi_1)_{xx}|^2\}dx \\
&\qquad -\frac{1}{4a^2}\int_{\mathbb{U}}\{\varphi''(y-x)[\phi(y)+\delta\phi_1(y)-\phi(x)-\delta\phi_1(x)]^2\}dxdy\Big\}dt \\
=&\frac{d}{d\delta}\int_{t_0}^{t_1}\Big\{\int_0^L\Big\{\rho\delta\phi_t\phi_{1t}-\frac{d}{dx}\delta\phi_1\{\psi'(a\phi_x) \\
&\qquad +\frac{a^3}{24}[a\psi'''(a\phi_x)\phi_{xx}^2+2\psi''(a\phi_x)\phi_{xxx}]\}\Big\}dx \\
&\qquad -\frac{1}{2a^2}\int_{\mathbb{U}}\{\varphi''(y-x)[\phi(y)-\phi(x)][\delta\phi_1(y)-\delta\phi_1(x)]\}dxdy\Big\}dt \\
=&\int_{t_0}^{t_1}\int_0^L\phi_1\Big\{\rho\phi_{tt}-\frac{d}{dx}\{\psi'(a\phi_x)+\frac{a^3}{24}[a\psi'''(a\phi_x)\phi_{xx}^2+2\psi''(a\phi_x)\phi_{xxx}]\} \\
&\qquad -\frac{1}{a^2}\int_{\{y:|y-x|\geq a\}}\{\varphi''(y-x)[\phi(y)-\phi(x)]\}dy\Big\}dxdt.
\end{aligned}
\tag{53}
$$

Finally, by the Hamilton's principle, immediately we know the equation of motion of the nth particle in continuum approximation. According to (53), we obtain

$$
\begin{aligned}
\rho\phi_{tt} =&\frac{d}{dx}\{\psi'(a\phi_x)+\frac{a^3}{24}[a\psi'''(a\phi_x)\phi_{xx}^2+2\psi''(a\phi_x)\phi_{xxx}]\} \\
&+\frac{1}{a^2}\int_{\{y:|y-x|\geq a\}}\{\varphi''(y-x)[\phi(y)-\phi(x)]\}dy,
\end{aligned}
\tag{54}
$$

$$\forall\ (x,t)\in(0,L)\times(0,+\infty).$$

Comparing with equation (12), we found that the long-range interaction parts of these two equations are the same under the continuum approximation. So we can rewrite equation (54) as

$$\rho\phi_{tt}=\frac{d}{dx}\{\psi'(a\phi_x)+\frac{a^3}{24}[a\psi'''(a\phi_x)\phi_{xx}^2+2\psi''(a\phi_x)\phi_{xxx}]\}-2J_sF_p(aD)\phi, \tag{55}$$

where

$$J_s=\frac{J}{a} \tag{56}$$

is the long range parameter for the strongly nonlinear system. The corresponding Hamiltonian of the system takes the form

$$H = \frac{1}{2}\int_{-\infty}^{\infty}\{\rho\phi_t^2 + 2W_e(\phi_x, \phi_{xx}) - \frac{2}{a}\phi J_s F_p(aD)\phi\}dx. \tag{57}$$

Now, we will specified the local potential energy to Hertzian type. From equation (40), we have

$$\begin{aligned} &\psi^{'}(a\phi_x) = -K_h(a)^{3/2}(-\phi_x)^{3/2}, \qquad \psi^{''}(a\phi_x) = \frac{3}{2}K_h(a)^{1/2}(-\phi_x)^{1/2}, \\ &\psi^{'''}(a\phi_x) = -\frac{3}{4}K_h(a)^{-1/2}(-\phi_x)^{-1/2}, \qquad \forall\, x \in (0, L). \end{aligned} \tag{58}$$

Substituting equation (58) into equation (55), we obtain the equation of motion of the nth particle in continuum approximation for this specific system as

$$\begin{aligned} &\rho\phi_{tt} + K_h a^{3/2}\frac{d}{dx}\Big\{(-\phi_x)^{3/2} + \frac{a^2}{32}(-\phi_x)^{-1/2}\phi_{xx}^2 - \frac{a^2}{8}(-\phi_x)^{1/2}\phi_{xxx}\Big\} + 2J_sF_p(aD)\phi \\ =&\frac{1}{c_n^2}\phi_{tt} - \{\frac{3}{2}(-\phi_x)^{1/2}\phi_{xx} + \frac{a^2}{8}(-\phi_x)^{1/2}\phi_{xxxx} - \frac{a^2}{8}\frac{\phi_{xx}\phi_{xxx}}{(-\phi_x)^{1/2}} - \frac{a^2}{64}\frac{\phi_{xx}^3}{(-\phi_x)^{3/2}}\} \\ &+ \frac{2J_sF_p(aD)}{c_n^2\rho}\phi \\ =&0. \end{aligned} \tag{59}$$

where

$$c_n^2 = \frac{K_h a^{3/2}}{\rho}. \tag{60}$$

Remark

The existence of strongly nonlinear system is due to the nonlinearity of local potential density. A system possess Hertzian interaction between adjacent lattices is described in this category because the Hertz contact force is nonlinear and cannot be linearized for the reason of lacking a small parameter. Instead of assuming the spheres in the chain are barely contact at the beginning of this section, we assume the pre-compression $\delta_0(\delta_0 < 0)$, however, is very large compared to the motion of spheres, we can then transformed the Hertzian system into a weakly nonlinear one.

Under this assumption, first we can expand equation (9) as

$$\begin{aligned}\psi[\delta_0+\phi(x_{n+1})-\phi(x_n)] =&\psi(\delta_0)+\frac{1}{2}\psi^{''}(\delta_0)[\phi(x_{n+1})-\phi(x_n)]^2\\&+\frac{1}{6}\psi^{'''}(\delta_0)[\phi(x_{n+1})-\phi(x_n)]^3,\end{aligned} \tag{61}$$

which is in the similar form with equation (30). Then, we follow the same steps discussed in weakly nonlinear case and finally we get the dispersion relation for this problem:

$$M\omega^2(k)=2\psi^{''}(\delta_0)[1-\cos(ka)]+2\psi^{'''}(\delta_0)\{[1-\cos(ka)][i\sin(ka)]\}+2L(ka). \tag{62}$$

The corresponding wave equation is

$$M\phi_{tt}=a^2\psi^{''}(\delta_0)[\phi_{xx}+\frac{1}{12}a^2\phi_{xxxx}]+a^3\psi^{'''}(\delta_0)(\phi_x^2)_x-2JF_p(aD)\phi. \tag{63}$$

From equation (40), we have that

$$\psi^{''}(\delta_0)=\frac{3}{2}K_h(-\delta_0)^{1/2},\qquad \psi^{'''}(\delta_0)=-\frac{3}{4}K_h(-\delta_0)^{-1/2},\qquad \forall\, x\in(0,L). \tag{64}$$

Substituting (64) into (63), we finally obtain the equation of motion of a Hertzian system with large pre-compression as

$$M\phi_{tt}=\frac{3}{4}K_h\{2(-\delta_0)^{1/2}a^2[\phi_{xx}+\frac{1}{12}a^2\phi_{xxxx}]-(-\delta_0)^{-1/2}a^3(\phi_x^2)_x\}-2JF_p(aD)\phi. \tag{65}$$

3 Ferromagnetic chains with nonlocal interactions

3.1 Nonlocal potential in ferromagnetic system

In this section, we specify our model as a one- dimensional chain of ferromagnetic balls with local elastic interactions and nonlocal magnetic interactions. To calculate the nonlocal interaction energy we recall that the magnetic flux induced by a magnetic dipole $\mathbf{m}$ at the origin is given by [15]

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}[\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{m}) - \mathbf{m}}{r^3}], \tag{66}$$

where μ_0 is the magnetic constant, r being the radius of spheres, and $\mathbf{n} = \mathbf{r}/|\mathbf{r}|$ is a unit vector. Assume that all balls are of the same diameter a and the same magnetization of magnitude M and direction $\mathbf{e}$, i.e.,

$$\mathbf{m} = M\frac{\pi a^3}{6}\mathbf{e}. \tag{67}$$

Then the total magnetic energy of the chain along $\mathbf{e}_x$ can be written as

$$\begin{aligned} U[u_i] &= -\frac{\mu_0\pi M^2 a^6}{144}\frac{1}{2}\sum_{m,n}[\frac{3(\mathbf{e}_x\cdot\mathbf{e})^2 - 1}{[|(n-m)(a+\delta_0) + (u_n - u_m)|]^3}] \\ &= \frac{1}{2}\sum_{m,n}\gamma(\mathbf{e})\kappa_{mn}[|(m-n)(a+\delta_0) + (u_m - u_n)|], \qquad m \neq n, \end{aligned}, \tag{68}$$

where

$$\begin{aligned} \gamma(\mathbf{e}) &= -\frac{\mu_0\pi M^2 a^6[3(\mathbf{e}_x\cdot\mathbf{e})^2 - 1]}{144}, \\ \kappa_{mn}(u_m - u_n) &= \frac{1}{[|(m-n)(a+\delta_0) + (u_m - u_n)|]^3}. \end{aligned} \tag{69}$$

Comparing with the general form of LRI term written as (19) and (22), we found the long-range parameter for the ferromagnetic system should be

$$J = \frac{q(q+1)}{a^p}\gamma(\mathbf{e}) = -\frac{\mu_0\pi M^2 a[3(\mathbf{e}_x\cdot\mathbf{e})^2 - 1]}{12}. \tag{70}$$

On the other hand, as mentioned in the previous chapter, according to different values of dispersive parameter p, the inverse-power law type long range interaction can be expanded into different form. In this specific problem, we have $p = 5$ and

$$\begin{aligned} F_5(ka) &= \sum_{j=1}^{\infty} \frac{1-\cos(jka)}{j^5} = \sum_{j=1}^{\infty} \{\frac{1}{2}(jka)^2 - \frac{1}{24}(jka)^4 + O[(jka)^4]\} j^{-5} \\ &= \frac{1}{2}\zeta(3)(ka)^2 + O[(ka)^4], \end{aligned} \tag{71}$$

where $\zeta(m) = \sum_{n=1}^{\infty} n^{-m}$ is the Riemann zeta function, and

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202.$$

Now, we start looking for the wave equations described ferromagnetic system with different types of local interaction.

3.2 Harmonic local interaction

From equation (25) and (71), we obtain the dispersion relation of the ferromagnetic system with harmonic local interaction:

$$M\omega^2(k) = [K_1 + \zeta(3)J](ka)^2 - \frac{1}{12}K_1(ka)^4, \tag{72}$$

where the corresponding wave equation is

$$\phi_{tt} - c^2\phi_{xx} - c_0^2\frac{a^2}{12}\phi_{xxxx} = 0, \tag{73}$$

where

$$c^2 = c_0^2(1+\zeta(3)\hat{J}), \qquad \hat{J} = \frac{J}{K_1}, \qquad c_0^2 = \frac{K_1 a^2}{M}. \tag{74}$$

The wave equation for harmonic system contains only a linear term and a higher order dispersion term. We assume equation (73) has plane wave solution:

$$\phi(x,t) = exp[i(\omega t - kx)], \tag{75}$$

then from (73) the dispersion relation is given by

$$\omega = \pm ck[1 - \frac{a^2}{12}c_0^2k^2]^{1/2}, \tag{76}$$

and the wave speed is given by

$$v = \frac{\omega}{k} = c[1 - \frac{a^2}{12}c_0^2 k^2]^{1/2}. \tag{77}$$

Since the velocity of each plane waves depends on k, an initial wave contains several sinusoidal waves cannot maintain its original shape as it travels through the medium. This shows the linear dispersive wave spreads out while it travels.

3.3 Weakly nonlinear local interaction

From equation (32) and (71), we obtain the dispersion relation of the ferromagnetic system with weakly nonlinear local interaction:

$$M\omega^2(k) = \quad [K_1 + \zeta(3)J](ka)^2 - [K_1/12](ka)^4 - K_2[i(ka)^3] \tag{78}$$

The corresponding nonlinear partial differential equation writes

$$\phi_{tt} - c^2\phi_{xx} - c_0^2\frac{a^2}{12}\phi_{xxxx} + \frac{K_2 a^3}{2M}(\phi_x^2)_x = 0, \tag{79}$$

where

$$c^2 = c_0^2[1 + \zeta(3)\hat{J}], \qquad \hat{J} = \frac{J}{K_1}, \qquad c_0^2 = \frac{K_1 a^2}{M}. \tag{80}$$

This is in the same form of Boussinesq type equation. Specifically, for ferromagnetic system with hertzian local contact and with large pre-compression, we can derive the dispersion relation from equation (62) and (71):

$$\begin{aligned} M\omega^2(ka) = [&\frac{3}{2}K_h(-\delta_0)^{1/2} + \zeta(3)J](ka)^2 - \frac{3}{4}iK_h(-\delta_0)^{-1/2}(ka)^3 \\ &- \frac{1}{8}K_h(-\delta_0)^{1/2}(ka)^4 + O[(ka)^4]. \end{aligned} \tag{81}$$

The corresponding wave equation is

$$\phi_{tt} - c^2\phi_{xx} - \frac{c_0^2}{8}[a^2\phi_{xxxx} + \frac{3a}{\delta_0}(\phi_x^2)_x] = 0, \tag{82}$$

where

$$c^2 = c_0^2[\frac{3}{2} + \zeta(3)\hat{J}], \qquad \hat{J} = \frac{J}{K_h(-\delta_0)^{1/2}}, \qquad c_0^2 = \frac{a^2}{K_h(-\delta_0)^{1/2}M}. \tag{83}$$

We notice that in the weakly nonlinear system, the particles in the chain endure both weakly nonlinear interaction and long range harmonic interaction. This kind of one-dimensional system has attracted many investigations in the last few years [7, 35]. Under our continuum approximation, the LRI term effect only the dispersion relation of the system. This can be observed from the equation (78) and (79) that no LRI corrections appear in the wave equation. Since introduced by Zabusky and Kruskal [37], the Korteweg-de Vires(KdV) equation can be derived from equation (79) under the same approximation. It describes solitary wave propagation in one direction:

$$z_t + cz_x + \frac{c_0^2 a^2}{24c} z_{xxx} + \frac{K_2 a^3}{2Mc} z_x z_{xx} = 0, \qquad z = -\phi_x. \tag{84}$$

It is nonlinear because of the product shown in the second summand. Since solitary wave propagate without any distortion of its shape, we can change to the moving frame by introducing the new variables

$$z(x,t) = f(x - vt) = f(y) \tag{85}$$

where v_w is the soliton velocity. Substituting (85) into (84), we have

$$(v - c)f_y - \frac{c_0^2 a^2}{24c} f_{yyy} - \frac{K_2 a^3}{2Mc} f f_y = 0. \tag{86}$$

The above equation is integrable, which leads us to

$$(v - c)f - \frac{c_0^2 a^2}{24c} f_{yy} - \frac{K_2 a^3}{4Mc} f^2 = c_1. \tag{87}$$

Integrating again on both sides, we obtain

$$\frac{1}{2}(v - c)f^2 - \frac{c_0^2 a^2}{48c} f_y^2 - \frac{K_2 a^3}{12Mc} f^3 = c_1 f + c_2. \tag{88}$$

where c_1, c_2 are constants. To investigate the behavior of the above equation, we applying boundary condition that

$$y \to \pm\infty, \qquad f \to 0, f_y \to 0, f_{yy} \to 0, \tag{89}$$

then $c_1 = c_2 = 0$. The above equation can be written as

$$f_y^2 = f^2\left(\frac{24c(v-c)}{c_0^2 a^2} - \frac{4K_2 a}{Mc_0^2} f\right). \tag{90}$$

The solution of the above ODE is well known as

$$f = \frac{6Mc(v-c)}{K_2 a^3} \mathrm{sech}^2\left\{\frac{1}{2a}\left[\frac{24c(v-c)}{c_0^2}\right]^{1/2} y\right\}. \tag{91}$$

Therefore, an exact solitary wave solution of (84) is

$$z = \frac{6Mc(v-c)}{K_2 a^3} \mathrm{sech}^2\left\{\frac{1}{2a}\left[\frac{24c(v-c)}{c_0^2}\right]^{1/2} (x - vt)\right\}, \tag{92}$$

or written in terms of the strain amplitude z_m as

$$z = z_m \mathrm{sech}^2\left[\left(\frac{K_2 a z_m}{c_0^2 M}\right)^{1/2} (x - vt)\right], \tag{93}$$

where

$$v = c + \frac{K_2 a^3}{6Mc} z_m \tag{94}$$

is the solitary wave velocity, and

$$W = \left(\frac{c_0^2 M}{K_2 a z_m}\right)^{1/2} \tag{95}$$

is the width of the solitary wave. The kink amplitude is

$$A_k = \int_{-\infty}^{\infty} z(x,t)dx = 2\sqrt{\frac{c_0^2 M z_m}{K_2 a}}. \tag{96}$$

The process of deriving the equation of motion for ferromagnetic system with weakly nonlinear local interaction is very robust and the differences between this case and

Hertzian contact with large pre-compression are only coefficients.

3.4 Hertzian local interaction

Following , we refer to a dynamic system lacking of a generic linearization with definite wave speed as a strongly nonlinear system. In contrast to a weakly nonlinear system that can be seen as the classic Boussinesq equation plus a long-range interaction part, the strongly nonlinear system, e.g., the ferromagnetic system with Hertzian local interaction, admits no nontrivial linearization. In a strongly nonlinear system, the linear part in the equation disappear, which means the linear wave cannot propagate in the chain anymore. This situation was described as "sonic vacuum" by Nesterenko [24] in 1992. The reason we get two different wave equations for Hertzian local contact system is the lack of one small parameter in the strongly nonlinear case, only long wave approximation still valid. Therefore, the wave equation introduced here has unique properties. For the strongly nonlinear case, the above wave equation cannot describe the system properly. Instead of using (32), we need to use (59) to derive the equation of motion, which should be

$$\begin{aligned}&\frac{1}{c_n^2}\phi_{tt} - \{\frac{3}{2}(-\phi_x)^{1/2}\phi_{xx} + \frac{a^2}{8}(-\phi_x)^{1/2}\phi_{xxxx} - \frac{a^2}{8}\frac{\phi_{xx}\phi_{xxx}}{(-\phi_x)^{1/2}} - \frac{a^2}{64}\frac{\phi_{xx}^3}{(-\phi_x)^{3/2}}\} \\ &- \zeta(3)\hat{J}_s\phi_{xx} = 0.\end{aligned} \tag{97}$$

where

$$\hat{J}_s = \frac{J_s}{K_h a^{-1/2}}. \tag{98}$$

Similarly, we are looking for the stationary solutions of (97), so we need to assume that $\phi(x,t) = f(x - v_s t) = f(y)$, where v_s is the phase velocity. However, instead of expressing the phase velocity $v_s(\eta_m)$ as a function of its strain amplitude, we should use $v_s(\eta_m, \hat{J}_s)$, where we plug in the effect of the LRI interaction. and equation (97) can be written as

$$\begin{aligned}(\frac{v_s(\eta_m,\hat{J}_s)^2 - c_n^2\zeta(3)\hat{J}_s}{c_n^2})f_{0xx} =& \frac{3}{2}(-f_{0x})^{1/2}f_{0xx} + \frac{a^2}{8}(-f_{0x})^{1/2}f_{0xxxx} \\ &- \frac{a^2}{8}\frac{f_{0xx}f_{0xxx}}{(-f_{0x})^{1/2}} - \frac{a^2}{64}\frac{f_{0xx}^3}{(-f_{0x})^{3/2}},\end{aligned} \tag{99}$$

The solution of (99) can be obtained by following the procedure discussed by Nesterenko [24]. Using the replacement $\eta = -f_{0x}$, substituting it in (99), we have

$$\begin{aligned}(\frac{v_s(\eta_m, \hat{J}_s)^2 - c_n^2\zeta(3)\hat{J}_s}{c_n^2})\eta_x =& \frac{3}{2}\eta^{1/2}\eta_x + \frac{a^2}{8}\eta^{1/2}\eta_{xxx} + \frac{a^2}{8}\frac{\eta_x\eta_{xx}}{\eta^{1/2}} - \frac{a^2}{64}\frac{\eta_x^3}{\eta^{3/2}} \\ =& \frac{3}{2}\eta^{1/2}\eta_x + \frac{a^2}{8}\frac{(\eta\eta_{xx})_x}{\eta^{1/2}} - \frac{a^2}{64}\frac{\eta_x^3}{\eta^{3/2}} \\ =& \frac{3}{2}\eta^{1/2}\eta_x + [\frac{a^2}{8}\eta^{1/2}\eta_{xxx} + \frac{a^2}{16}\eta_x\eta_{xx}\eta^{-1/2}] \\ &+ [\frac{a^2}{16}\eta_x\eta_{xx}\eta^{-1/2} - \frac{a^2}{64}\eta_x^3\eta^{-3/2}].\end{aligned} \tag{100}$$

The above equation is integrable. With the condition that $\eta(x = +\infty) = \eta_0, \eta_x(x = +\infty) = 0, \eta_{xx}(x = +\infty) = 0$, we have

$$(\frac{v_s(\eta_m, \hat{J}_s)^2 - c_n^2\zeta(3)\hat{J}_s}{c_n^2})\eta = \eta^{3/2} + \frac{a^2}{8}\eta^{1/2}\eta_{xx} + \frac{a^2}{32}\eta^{-1/2}\eta_x^2 + C_1. \tag{101}$$

If we do the replacement of variable $\eta = z^{4/5}$, equation (101) can be changed into

$$(\frac{v_s(\eta_m, \hat{J}_s)^2 - c_n^2\zeta(3)\hat{J}_s}{c_n^2})z^{4/5} = z^{6/5} + \frac{a^2}{10}z^{1/5}z_{xx} + C_1, \tag{102}$$

where C_1 is a constant. The above equation can be rewritten as

$$w^{4/5} = w^{6/5} + w^{1/5}w_{\chi\chi} + C_2, \tag{103}$$

with the replacement

$$z = (\frac{v_s(\eta_m, \hat{J}_s)^2 - c_n^2\zeta(3)\hat{J}_s}{c_n^2})^{5/2}w, \qquad \chi = \frac{\sqrt{10}}{a}x. \tag{104}$$

A convenient form can be obtained from (103)

$$\begin{aligned} w_{\chi\chi} &= -\frac{\partial}{\partial w}W(w), \\ W(w) &= -\frac{5}{8}w^{8/5} + \frac{1}{2}w^2 + C_3 w^{4/5}.\end{aligned} \tag{105}$$

The general solution of (105) for periodical motion is well known as(Landau and Lifshitz [19])

$$\chi = \chi_0 + \int_{w_0}^{w} \frac{dw}{\sqrt{-2[W(w) - W(w_{max})]}}, \tag{106}$$

where w_{max} corresponds to the maximum strain in the periodic wave. What we discussed here is the case when $C_3 = 0$, which indicate the pre-compression of the system equals to zero. In a strongly nonlinear granular system contains only nearest-neighbor interaction, sound is not available to propagate due to the absence of quadratic term in the equation of motion. However, quadratic term comes from the LRI part is included in our system, which makes it possible for sound to travel through the system with the speed of

$$c^2 = c_n^2 \zeta(3) \hat{J}_s. \tag{107}$$

The solution for this particular system is written as

$$w = (\frac{5}{4})^{5/2} \cos^5(\frac{1}{5}\chi), \tag{108}$$

Therefore, the solution of (101) can be written as

$$\eta = \{\frac{5[v_s(\eta_m, \hat{J}_s)^2 - c_n^2 \zeta(3) \hat{J}_s]}{4c_n^2}\}^2 \cos^4(\frac{\sqrt{10}}{5a} y). \tag{109}$$

For periodic waves, this solution being a sequence of positive humps connected at the points with zero strains. The solitary solution however, can be taken as one hump of the periodic solution. The spatial size of the soliton is therefore

$$L_s = (\frac{5a}{\sqrt{10}})\pi \approx 5a. \tag{110}$$

which indicates the width of solitary waves is limit as five particles spacing. We can also derive the kink amplitude expression by integrating (109) on one solitary wave interval

$([5a\pi/2\sqrt{10}, 15a\pi/2\sqrt{10}])$, which gives us

$$\begin{aligned}\int_{5a\pi/2\sqrt{10}}^{15a\pi/2\sqrt{10}} \eta(y)dy =& \eta_m\{120\pi a/\sqrt{10} + 8\sqrt{10}a[sin(3\pi) - sin(\pi)] \\ &+ \sqrt{10}a[sin(6\pi) - sin(2\pi)]\}/64 \\ =&[\frac{15\pi a}{8\sqrt{10}}]\eta_m,\end{aligned} \tag{111}$$

where

$$\eta_m = \{\frac{5[v_s(\eta_m, \hat{J}_s)^2 - c_n^2\zeta(3)\hat{J}_s]}{4c_n^2}\}^2 = \{\frac{5[v_s(\eta_m, \hat{J}_s)^2 - c^2]}{4c_n^2}\}^2, \tag{112}$$

is the strain amplitude. Then we have the solitary wave speed

$$v_s(\eta_m, \hat{J}_s) = [\frac{4c_n^2\eta_m^{1/2} + 5c_n^2\zeta(3)\hat{J}_s}{5}]^{1/2} = [\frac{4c_n^2\eta_m^{1/2}}{5} + c^2]^{1/2}. \tag{113}$$

Since we also have the relationships between the wave front velocity v_m, solitary wave velocity v_s and maximum strain η_m:

$$v_m = v_s(\eta_m, \hat{J}_s)\eta_m, \tag{114}$$

we can express the solitary wave velocity as a function of v_m and $\hat{J}$:

$$v_s(v_m, \hat{J}_s) = [\frac{4}{5}c_n^2[\frac{v_m}{v_s(v_m, \hat{J}_s)}]^{1/2} + c^2]^{1/2}. \tag{115}$$

The above equation shows the nonlinear dependency of solitary wave velocity on wave front velocity and range parameter.

4 Numerical simulation

We use the fourth-order Runge-Kutta(RK4) finite difference method to simulate the system [3]. This method is fourth-order accurate in time. The RK4 method is implemented in Matlab. After initialising the system, the code integrates forward in time with step size $h = 10^{-7}s$. The geomoetrical parameters for the simulation are from material properties. We discussed this method in details in Appendix B.

Our system is simplified as a 1D chain of stainless steel spheres being arranged horizontally with zero initial compression, which means they are in equilibrium positions at the beginning. The diameter of the balls are $d = 0.005m$ with the density $\rho = 7780kg/m^3$, Young's modulus $E = 193Gpa$ and Poisson ratio $\nu = 0.3$. The chain contains $N + 1$ particles. Here we pick $N = 70$. The 19th and 20th particle are the strikers and the last sphere of each end of the chain has infinite radius which act as a wall and will remain stationary during the simulation. The initial velocities of all spheres are set to zero at the beginning. Then the system is perturbed by given the same amount of speed$(0.5m/s, 1m/s, 5m/s, 10m/s)$ in opposite directions to the two strikers so that we can investigate the propagation of waves in the chain. We assumed that the friction and energy dissipation are negligible during the simulation.

We focus our analysis on the effects of long-range potential and particle speed to the wave propagation velocity in different systems. Numerically, we calculate the wave propagation velocity by measuring the time a solitary wave need to travel between two particles in the chain. We try to use the particles in the middle of the chain (30th, 40th and 50th, 60th) to calculate results and take the average to minimize errors. Fig. 4 shows the relationship between the wave propagation velocity c and the long-range parameter $\hat{J}$ in harmonic system. c_0 is the wave propagation velocity in the case $\hat{J} = 0$, which corresponds to the system without long-range potential. We plot the ratio between c and c_0 to shows the effect of long-range potential more clearly. The analytical relationship between c and $\hat{J}$ is illustrated by equation (74). We can see from the plot that the numerical results match the theoretical results perfectly. Remarkably, an interesting behavior in this case is the wave propagation velocity decrease sharply when $\hat{J}$ goes near the limit $-0.832(-1/\zeta(3))$. This also indicate a system can have sharp resolution by tuning the LRI effect in certain range. Our simulation also demonstrate that not the

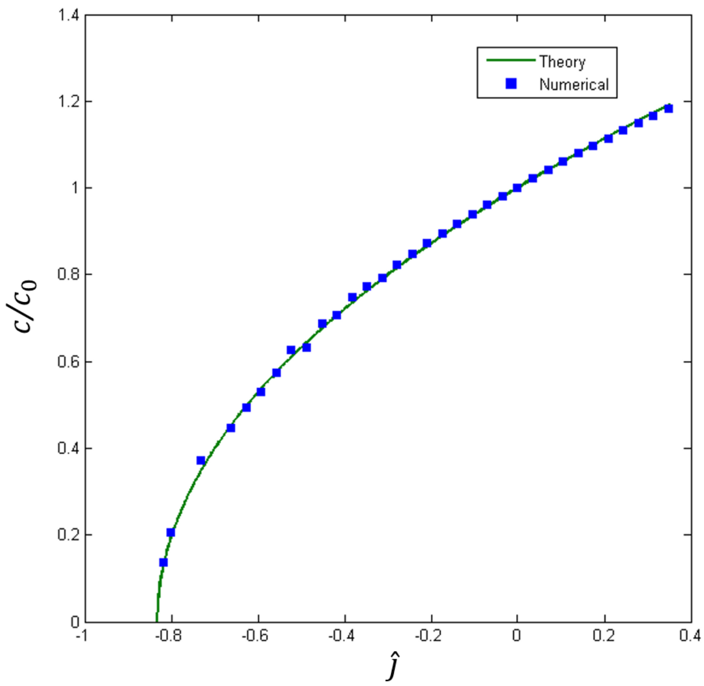

Figure 4: The ratio of wave propagation velocity c/c_0 vs different values of the long-range parameter $\hat{J}$ in harmonic system, where the initial velocity of the strikers are $v_i = \pm 5m/s$.

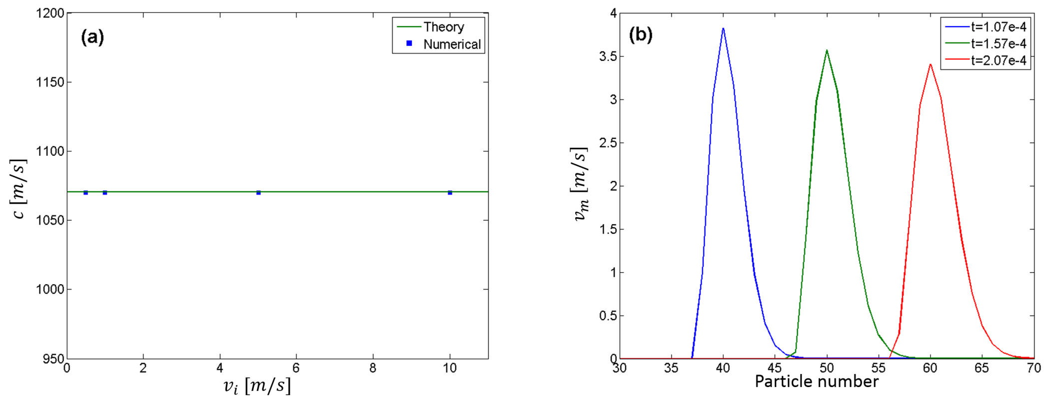

Figure 5: For the harmonic system, plot (a) shows the relationship between the wave propagation velocity c and the initial velocity of the strikers v_i in the circumstance $\hat{J} = 0.14$. Plot (b) shows the wave front velocity v_m in the chain at different time in the circumstance $\hat{J} = 0.07$, $v_i = 5m/s$.

same with nonlinear waves, the wave front speed does not effect the wave propagation velocity in harmonic system, which can be seen in the plot (a) of Fig. 5. The wave propagation velocities are nearly the same when $v_i = 0.5m/s$ and $v_i = 10m/s$. The plot (b) of Fig. 5 shows the wave front velocity v_m decreases rapidly from $3.81m/s$ to $3.42m/s$ while wave propagating through the chain between $1.07 \times 10^{-4}s$ and $2.07 \times 10^{-4}s$ and the wave length is about 10 particle diameters in this certain case.

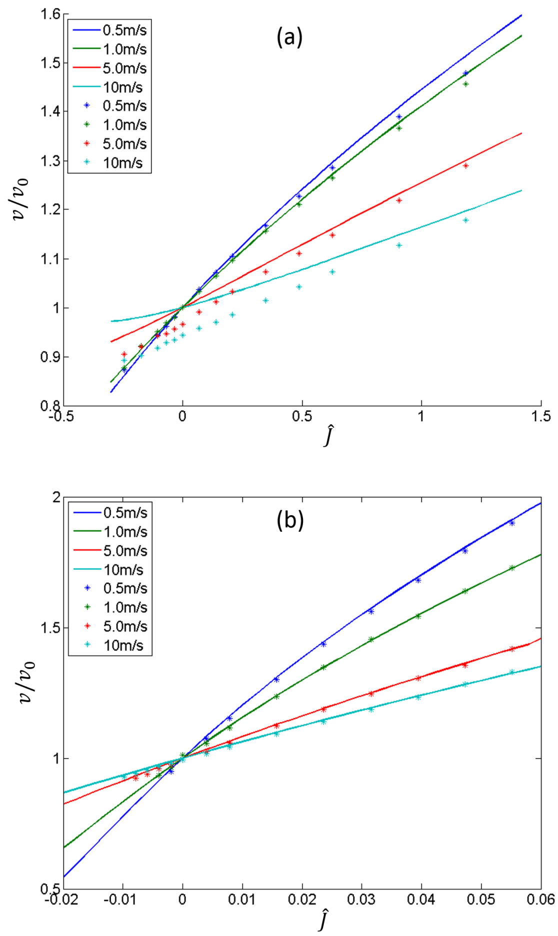

Figure 6: The ratio of wave propagation velocity v/v_0 vs different values of the long-range parameter $\hat{J}$ in two different systems. Solid lines and discrete asterisks represent theoretical and numerical results respectively. Plot (a) shows the weakly nonlinear system and plot (b) shows the strongly nonlinear system. Different curves correspond to waves propagating at different wave front velocities v_m in both cases.

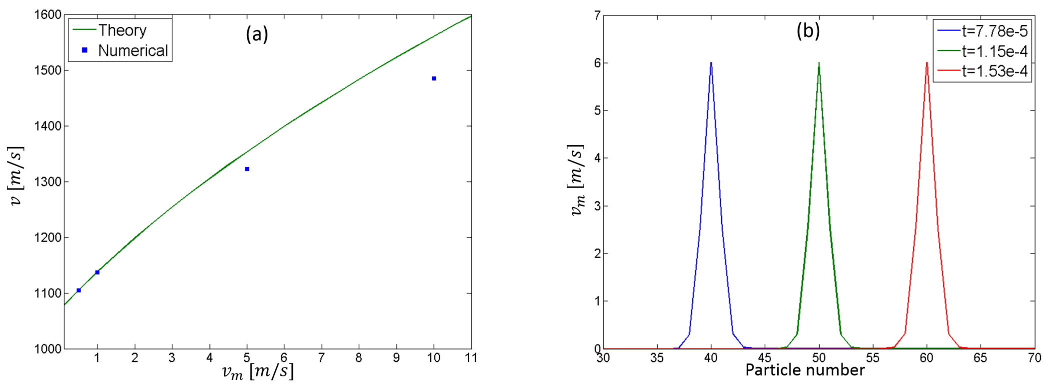

Figure 7: For the weakly nonlinear system, plot (a) shows the propagation velocity v of solitary waves at different front velocities v_m. Plot (b) shows v_m in the chain at different time. Both in the circumstance $\hat{J} = 0.07$.

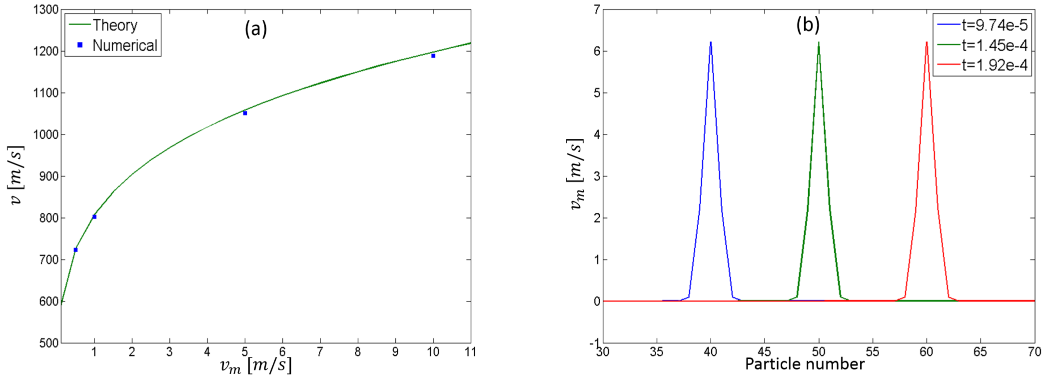

Figure 8: For the hertz system, plot (a) shows the propagation velocity v of solitary waves at different front velocities v_m. Plot (b) shows v_m in the chain at different time. Both in the circumstance $\hat{J} = 0.004$.

Figure 6 shows the relationship between the wave propagation velocities v and long-range parameter $\hat{J}$ under different wave front velocities v_m in weakly nonlinear and hertz system, respectively. v_0 represent wave propagation velocities in the circumstance $\hat{J} = 0$ in both cases. we can see v increase as long as $\hat{J}$ going larger in both cases, which implies that the long-range interaction force among the lattices will increase the speed of wave propagating through them. Plots (a) of Fig. 7 and of Fig. 8 illustrate that the v grow with v_m under the same value of $\hat{J}$ both in weakly nonlinear system and hertz system. This result also demonstrate that while the wave propagating through the particle chain, the velocities of the lattices will influence the wave propagation velocity. Plots (b) of Fig. 7 and of Fig. 8 show that v_m almost maintain the same values while wave propagating through the chain in both weakly nonlinear system and hertz system, which lead to the preservation of the solitary waves profile. The wave lengths of both cases are limited to approximately 5 particle diameters, which is consistent with the theoretical prediction (110).

5 Conclusion

We investigated the dynamics of the one-dimensional ferromagnetic granular system with both local and nonlocal interactions. Systems with harmonic, cubic and Hertz local potentials were discussed separately and several different wave equations in continuum limit have been found. We generalized the granular systems with different types of potentials into three classical kinds, namely the harmonic system, the weakly nonlinear system and the strongly nonlinear system. We showed that the exponent parameter p has significant effect to the dispersion relation of the wave equations in each case. Namely, for $p = 5$, which corresponds to the ferromagnetic granular, a $sech^2$ shape KdV solitary wave is found for the weakly nonlinear system and a cos^4 shape solitary wave is found for the strongly nonlinear system. Simulation results show that the shape of the waves and their velocities change slightly during propagation as prediction. Also, our numerical simulation results verified the relationships between wave propagation velocity, wave front velocity and the LRI parameter $\hat{J}$ in both cases. Most importantly, we demonstrated that the effects of the ferromagnetic long-range potential to the shape of the solitary wave as well as to its propagation velocity. This tunable feature makes the system could have potential applications in the design of acoustic lenses which can be used in sound focusing devices.

Appendix A: Details of derivation of

Here, we list out the expansions of $F_p(ka)$ for different values of p, which defined as the LRI term in the text.

$$F_p(ka) = \sum_{j=1}^{\infty} \frac{1-\cosh(jka)}{j^p}. \tag{116}$$

For $p = 2$

$$F_2(ka) = \frac{\pi}{2}|ka| - \frac{1}{4}(ka)^2. \tag{117}$$

For $p = 3$

$$F_3(ka) \approx -\frac{\pi}{2}|ka|^2 \log|ka| + \frac{3}{4}(ka)^2 + \frac{1}{288}(ka)^4. \tag{118}$$

For $p = 4$

$$F_4(ka) = \frac{1}{2}\zeta(2)(ka)^2 \log|ka| - \frac{\pi}{12}|ka|^3 + \frac{1}{12}(ka)^4. \tag{119}$$

For $p = 5$

$$F_5(ka) \approx \frac{1}{2}\zeta(3)(ka)^2 + \frac{1}{24}(ka)^4 \log|ka|. \tag{120}$$

For $p > 5$

$$F_p(ka) \approx \frac{1}{2}\zeta(p-2)(ka)^2 - \frac{1}{24}\zeta(p-4)(ka)^4. \tag{121}$$

Appendix B: Numerical method

In numerical analysis, the Runge-Kutta methods play an important role in all iterative methods, which were first developed by C. Runge and M. W. Kutta around 1900. Among these methods, the fourth-order Runge-Kutta method(Also known as RK4), which is used here, is reasonably simple and robust and is a good general candidate for numerical solution of differential equations when combined with an intelligent adaptive step-size routine. This specific method is well known, but will be described briefly here for completeness. Consider a initial value problem that

$$\dot{y} = f(t, y), \qquad\qquad y(t_0) = y_0.$$

By given a time-step size h, we have

$$\begin{aligned} y^{\tau+1} &= y^{\tau} + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \\ t^{\tau+1} &= t^{\tau} + h, \end{aligned}$$

where τ is the time steps and

$$\begin{aligned} k_1 &= f(t^{\tau}, y^{\tau}), \\ k_2 &= f(t^{\tau} + \frac{h}{2}, y^{\tau} + \frac{h}{2}k_1), \\ k_3 &= f(t^{\tau} + \frac{h}{2}, y^{\tau} + \frac{h}{2}k_2), \\ k_4 &= f(t^{\tau} + h, y^{\tau} + hk_3). \end{aligned}$$

Here $y^{\tau+1}$ is the approximation of the next value. This method iteratively calculate four increments and take the weighted average of them so that the total accumulated error is order $O(h^4)$. For our specific problem, we have the general equation of motion in the form

$$\begin{aligned} M\ddot{u}_n =& \psi'(u_{n+1} - u_n) - \psi'(u_n - u_{n-1}) \\ &+ \sum_{j=1}^{N} \{\varphi'[ja + u_{n+j} - u_n] - \varphi'[ja + u_n - u_{n-j}]\}. \end{aligned}$$

So we have

$$\ddot{u}_n = f(t, \dot{u}_n), \qquad \dot{u}(t_0) = \dot{u}_0.$$

$$\begin{aligned} \dot{u}_n^{\tau+1} &= \dot{u}_n^{\tau} + \frac{h}{6}(k_{1n} + 2k_{2n} + 2k_{3n} + k_{4n}), \\ t^{\tau+1} &= t^{\tau} + h, \end{aligned}$$

where

$$
\begin{aligned}
k_{1n} =& f(t^{\tau}, \dot{u}_n^{\tau}) \\
=& \frac{1}{M}\Big\{\psi'(u_{n+1}^{\tau} - u_n^{\tau}) - \psi'(u_n^{\tau} - u_{n-1}^{\tau}) \\
&+ \sum_{j=1}^{N}\{\varphi'[ja + u_{n+j}^{\tau} - u_n^{\tau}] - \varphi'[ja + u_n^{\tau} - u_{n-j}^{\tau}]\}\Big\}, \\
k_{2n} =& f(t^{\tau+\frac{h}{2}}, \dot{u}_n^{\tau+\frac{h}{2}} + \frac{h}{2}k_{1n}) \\
=& \frac{1}{M}\Big\{\psi'(u_{n+1}^{\tau+\frac{h}{2}} - u_n^{\tau+\frac{h}{2}} + \frac{h}{2}k_{1n}) - \psi'(u_n^{\tau+\frac{h}{2}} - u_{n-1}^{\tau+\frac{h}{2}} + \frac{h}{2}k_{1n}) \\
&+ \sum_{j=1}^{N}\{\varphi'[ja + u_{n+j}^{\tau+\frac{h}{2}} - u_n^{\tau+\frac{h}{2}} + \frac{h}{2}k_{1n}] - \varphi'[ja + u_n^{\tau+\frac{h}{2}} - u_{n-j}^{\tau+\frac{h}{2}} + \frac{h}{2}k_{1n}]\}\Big\}, \\
k_{3n} =& f(t^{\tau+\frac{h}{2}}, \dot{u}_n^{\tau+\frac{h}{2}} + \frac{h}{2}k_{2n}) \\
=& \frac{1}{M}\Big\{\psi'(u_{n+1}^{\tau+\frac{h}{2}} - u_n^{\tau+\frac{h}{2}} + \frac{h}{2}k_{2n}) - \psi'(u_n^{\tau+\frac{h}{2}} - u_{n-1}^{\tau+\frac{h}{2}} + \frac{h}{2}k_{2n}) \\
&+ \sum_{j=1}^{N}\{\varphi'[ja + u_{n+j}^{\tau+\frac{h}{2}} - u_n^{\tau+\frac{h}{2}} + \frac{h}{2}k_{2n}] - \varphi'[ja + u_n^{\tau+\frac{h}{2}} - u_{n-j}^{\tau+\frac{h}{2}} + \frac{h}{2}k_{2n}]\}\Big\}, \\
k_{4n} =& f(t^{\tau+h}, \dot{u}_n^{\tau+h} + hk_{3n}) \\
=& \frac{1}{M}\Big\{\psi'(u_{n+1}^{\tau+h} - u_n^{\tau+h} + hk_{3n}) - \psi'(u_n^{\tau+h} - u_{n-1}^{\tau+h} + hk_{3n}) \\
&+ \sum_{j=1}^{N}\{\varphi'[ja + u_{n+j}^{\tau+h} - u_n^{\tau+h} + hk_{3n}] - \varphi'[ja + u_n^{\tau+h} - u_{n-j}^{\tau+h} + hk_{3n}]\}\Big\}.
\end{aligned}
$$

For harmonic local potential

$$\varphi[\Delta x] = \frac{1}{2}K_1[\Delta x]^2.$$

For cubic local potential

$$\varphi[\Delta x] = \frac{1}{2}K_1[\Delta x]^2 - \frac{1}{6}K_2[\Delta x]^3.$$

For Hertzian local potential

$$\varphi[\Delta x] = \frac{2}{5}K_h[\Delta x]^{5/2}.$$

For dipole-dipole LRI

$$\psi[\Delta x] \simeq \frac{\gamma}{|\Delta x|^5}$$

By given the time steps τ, step size h and initial condition $\dot{u}_0$, we can calculate the positions and velocities of any particles in the chain at any time steps.

Appendix C: Core simulation code in MATLAB

We only include the Hertz system code. Other two cases are very similar to this one.

Main file

```
clear all
% BASIC PARAMETERS
time_steps = 3000;  %time steps
d_t = 1e-7;  %step size
N = 80;    %Number of spheres in the chain
a = 0.005; %Particle diameter
rad = a/2; %Particle radius
ca2 = 4.999e9*a^(5/2)/(5.092e-4); %calculate c
combodata = zeros(time_steps,110*(N+1)); % create a matrix to store all
velocities data

    dvm = 0;  % create a matrix to store all velocities data
    dB = 0;
    % create a vector to store initial velocity data
    init_vel = zeros(1, 120);
    B = zeros(1, 120);   % create a vector to store J data
    J_1 = zeros(1,120);  % create a vector to store Jhat data
    va = zeros(1, 120);  % create a vector to store analysis velocity data
    % create a vector to store numerical velocity data
    V_numerical = zeros(1,230);
    % create a vector to store analysis velocity ratio data
    va_ratio = zeros(1, 120);
    % create a vector to store numerical velocity ratio data
    V_numerical_ratio = zeros(1, 120);
     % create a vector to store maximum velocity of the 50th particle data
    v50m = zeros(1 ,120);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%INITIALIZATION
%Start a Loop to assign different initial velocity
%and LRI parameter to the system
  for dvm = 0
      for dB = 2
          %init_vel(1,12*dvm+dB) = dvm+1;
          init_vel(1,12*dvm+dB) = 10;
          B(1,12*dvm+dB) = 1000*(dB-1)*((1.257e-6)*pi*9.2e5^2*a^6)/144;
          %calculate real J
          J_1(1,12*dvm+dB)=12*B(1,12*dvm+dB)/(a^(11/2)*4.999e9);
          %calculate Jhat
%Start a function to assign initial values of parameters to the system
    [mass, E, R, poisson, position, velocity, acceleration, ...
        overlaps, A, ...
        ] = ...
        initialise(time_steps, N, init_vel(1,12*dvm+dB));
%Start a function to iterate in time and store the velocity,
%position and ovelaps data after calculated by funtion RK
        for i = 2:time_steps

            [velocity(i,:), position(i,:), overlaps(i,:)] = ...
                RK(position(i-1,:), ...
                velocity(i-1,:), j, N, ...
                R, a, B(1,12*dvm+dB),...
                mass, E, poisson, i, d_t, ...
                position(1,N), A);

        end
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Calculate the numerical wave propagation velocity by seraching two
%particles with the same velocity and divided distace by time.
           for i = 1:2
           for j = 1:3000
               if velocity(j,10*i+20)>(dvm+1)*0.2
                   break;
               end
           end
               t1 = j*d_t;
           for j = 1:3000
               if velocity(j,10*(i+1)+20)>(dvm+1)*0.2
                   break;
               end
           end
               t2 = j*d_t;
       V_numerical(1,2*(12*dvm+dB-1)+i) = 10*a/(t2-t1);
       V_numerical_ratio(1,2*(12*dvm+dB-1)+i) = ...
           V_numerical(1,2*(12*dvm+dB-1)+i)/va(1,1);

           end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Look for the maximu velocity of the 50th particle
       for j = 1:3000
           if velocity(j,50)>v50m(1,12*dvm+dB)
               v50m(1,12*dvm+dB) = velocity(j,50);
           end
       end
%Calculate the analysis wave propagation velocity and its ratio
       va(1,12*dvm+dB) = ((4*ca2*(v50m(1,12*dvm+dB)/...
           V_numerical(1,2*(12*dvm+dB-1)+2))^0.5...
                          +5*ca2*1.202*J_1(1,12*dvm+dB))/5)^0.5;

       va_ratio(1,12*dvm+dB) = va(1,12*dvm+dB)/va(1,1);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Save all velocity simulation with one set initial velocity
%and LRI parameterresults to combodata
       combodata(1:time_steps,(dvm*11+dB-1)*...
           (N+1)+1:(dvm*11+dB-1)*(N+1)+N) = velocity(1:time_steps,:);
       combodata(1:time_steps,(dvm*11+dB)*(N+1)) = 0;
     end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Create a blank space to seperate each set of data
       init_vel(1,(12*dvm+dB)+1) = 0;
       B(1,(12*dvm+dB)+1) = 0;
       J_1(1,(12*dvm+dB)+1) = 0;
       va(1,(12*dvm+dB)+1) = 0;
       V_numerical(1,2*(12*dvm+dB)+1) = 0;
 end
```

Initialization file

```
%This is the initialization fucntion to set basic parameters of the system
function [mass, E, R, poisson, position, velocity, ...
    acceleration, ...
    overlaps, A, force_30, force_60, distance_30_60] = ...
    initialise(time_steps, N, init_vel)

%Material properties
rho_steel = 7780;
rho_PTFE = 2178;
rho_brass = 8500;
E_steel = 1.93*10^11;
E_rubber = 30000000;
E_brass = 1.03 * 10^11;
E_PTFE = 1.46*10^9;
a = 0.005;
rad = a/2;
poisson_steel = 0.3;
poisson_rubber = 0.49;
poisson_brass = 0.34;
poisson_PTFE = 0.46;

mass = zeros(1,N);
E = zeros(1,N);
R = zeros(1,N);
poisson = zeros(1,N);
%Set Young's modulus, raidus, poisson ratio and mass values to the balls
for i = 2:N
    E(i) = E_steel;
    R(i) = rad;
    poisson(i) = poisson_steel;
    mass(i) = (4/3)*pi*rho_steel*(R(i))^3;
end

E(1) = E_steel;
R(1) = rad;
poisson(1) = poisson_steel;
mass(1) = (4/3)*pi*rho_steel*(R(1))^3;

%E(N+1) = E_steel;
E(N+1) = E_brass;
%R(N+1) = 1000;
%poisson(N+1) = poisson_steel;
poisson(N+1) = poisson_brass;
%mass(N) = (4/3)pi*rho_steel*(R(N))^3;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%create a group of matrices to store the data of position, velocity,
%accelaration and overlaps
position = zeros(N,time_steps)';
velocity = zeros(N,time_steps)';
acceleration = zeros(N, time_steps)';

overlaps = zeros(N+1,time_steps)';
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%ovelap between the first ball in the chain and the Wall is always zero
overlaps(:,1) = 0;
%All particles are in equilibrium at the beginning
```

```
position(1,1) = 0;
%Set initial velocity to the 19th and 20th particles in the chain
velocity(1,20) = init_vel;

velocity(1,19) = -init_vel;

%Calculate Hertz interaction parameter
A = zeros(1,N);

A(1:(N-1)) = 4*E(1:(N-1)).*E(2:N).*sqrt(R(1:(N-1))...
    .*R(2:N)./...
    (R(1:(N-1))+R(2:N)))./(3*E(2:N).*...
    (1-poisson(1:(N-1)).^2)+...
    3*E(1:(N-1)).*(1-poisson(2:N).^2));
A(N) = (4*E(N)*E(N+1)*sqrt(R(N)))/(3*E(N+1)*...
    (1-poisson(N)^2)+...
    3*E(N)*(1-poisson(N+1)^2));
end
```

RK4 file

```
%This is the RK function to integrate the equation of motion in time and
%calculate velocity and postion of particles in each step.
function [v_new, x_new, overlaps_new] = RK(x_old, v_old, j, ...
    N, R, a, B,...
    mass, E, poisson, i, d_t, Nth_ball_init_pos, A)
rad = a/2;
%create a group of vectors to save position and velocity data in each step
k1_x = zeros(1,N); k1_v = zeros(1,N);
k2_x = zeros(1,N); k2_v = zeros(1,N);
k3_x = zeros(1,N); k3_v = zeros(1,N);
k4_x = zeros(1,N); k4_v = zeros(1,N);

k_overlaps = zeros(4,N+1);
%calculate k1x
k1_x = d_t * v_old;
%calculate ovelaps between each pair of balls
[k_overlaps(1, :)]= ...
    contacts (N, Nth_ball_init_pos, x_old, R);

overlaps_new = contacts(N, Nth_ball_init_pos, x_old, R);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%caculate LRI force
T = 0;
for t = 2:N

      ne = (x_old(t)-x_old(1)+(t-1)*a).^(-4)-((t-1)*a).^(-4);
      T = T+ne;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%calculate k1v
k1_v(1)    = d_t/mass(1)*(A(N)*k_overlaps(1,1)^(3/2)-3*B*T...
    -A(1)*k_overlaps(1,2)^(3/2));

for j = 2:N
                 plus = 0;
                 for p = 1:j-1
                     ad = (x_old(j)-x_old(p)+(j-p)*a).^(-4)...
                         -((j-p)*a).^(-4);
                     plus = plus+ad;

                 end
                 Q = 0;
                 for q = j+1:N
                     mi = (x_old(q)-x_old(j)+(q-j)*a).^(-4)...
                         -((q-j)*a).^(-4);
                     Q = Q+mi;
                 end

               k1_v(j) = d_t/mass(j)*(3*B*plus+A(j-1).*k_overlaps(1,j)...
                         .^(3/2)-3*B*Q-A(j).*k_overlaps(1,j+1).^(3/2));

end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%save position and velocity data to temporay varaiables
xx = x_old+0.5*k1_x;
```

```
vv = v_old+0.5*k1_v;
%calculate k2x
k2_x = d_t*vv;
%calculate ovelaps between each pair of balls
[k_overlaps(2, :)]= ...
    contacts(N, Nth_ball_init_pos, xx, R);
%caculate LRI force
T = 0;
for t = 2:N
      ne = (xx(t)-xx(1)+(t-1)*a).^(-4)-((t-1)*a).^(-4);
      T = T+ne;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%calculate k2v
k2_v(1)    = d_t/mass(1)*(A(N)*k_overlaps(2,1)^(3/2)-3*B*T...
    -A(1)*k_overlaps(2,2)^(3/2));


for j = 2:N
                 plus = 0;
                 for p = 1:j-1
                     ad = (xx(j)-xx(p)+(j-p)*a).^(-4)-((j-p)*a).^(-4);
                     plus = plus+ad;

                 end

                 Q = 0;
                 for q = j+1:N
                     mi = (xx(q)-xx(j)+(q-j)*a).^(-4)-((q-j)*a).^(-4);
                     Q = Q+mi;
                 end

               k2_v(j)  = d_t/mass(j)*(3*B*plus+A(j-1).*k_overlaps(2,j)...
                          .^(3/2)-3*B*Q-A(j).*k_overlaps(2,j+1).^(3/2));


end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%save position and velocity data to temporay varaiables
xx = x_old+0.5*k2_x;
vv = v_old+0.5*k2_v;
%calculate k3x
k3_x = d_t*vv;
%calculate ovelaps between each pair of balls
[k_overlaps(3,:)] = ...
    contacts(N, Nth_ball_init_pos, xx, R);
%caculate LRI force
T = 0;
for t = 2:N
      ne = (xx(t)-xx(1)+(t-1)*a).^(-4)-((t-1)*a).^(-4);
      T = T+ne;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%calculate k3v
```

```
k3_v(1)    = d_t/mass(1)*(A(N)*k_overlaps(3,1)^(3/2)-3*B*T-
A(1)*k_overlaps(3,2)^(3/2));

for j = 2:N
                 plus = 0;
                 for p = 1:j-1
                     ad = (xx(j)-xx(p)+(j-p)*a).^(-4)-((j-p)*a).^(-4);
                     plus = plus+ad;

                 end
                 Q = 0;
                 for q = j+1:N
                     mi = (xx(q)-xx(j)+(q-j)*a).^(-4)-((q-j)*a).^(-4);
                     Q = Q+mi;
                 end

               k3_v(j) = d_t/mass(j)*(3*B*plus+A(j-1).*k_overlaps(3,j)...
                          .^(3/2)-3*B*Q-A(j).*k_overlaps(3,j+1).^(3/2));

end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%save position and velocity data to temporay varaiables
xx = x_old + k3_x;
vv = v_old + k3_v;
%calculate k4x
k4_x = d_t*vv;
%calculate ovelaps between each pair of balls
[k_overlaps(4, :)] = ...
    contacts(N, Nth_ball_init_pos, xx, R);
%caculate LRI force
T = 0;
for t = 2:N
      ne = (xx(t)-xx(1)+(t-1)*a).^(-4)-((t-1)*a).^(-4);
      T = T+ne;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%calculate k4v
k4_v(1)    = d_t/mass(1)*(A(N)*k_overlaps(4,1)^(3/2)-3*B*T...
    -A(1)*k_overlaps(4,2)^(3/2));

for j = 2:N
                 plus = 0;
                 for p = 1:j-1
                     ad = (xx(j)-xx(p)+(j-p)*a).^(-4)-((j-p)*a).^(-4);
                     plus = plus+ad;

                 end
                 Q = 0;
                 for q = j+1:N
                     mi = (xx(q)-xx(j)+(q-j)*a).^(-4)-((q-j)*a).^(-4);
                     Q = Q+mi;
                 end

               k4_v(j) = d_t/mass(j)*(3*B*plus+A(j-1).*k_overlaps(4,j)...
                          .^(3/2)- 3*B*Q-A(j).*k_overlaps(4,j+1).^(3/2));
```

```
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Save velocity and position data
v_new = v_old +(1/6)*(k1_v+2*k2_v+2*k3_v+k4_v);
x_new = x_old +(1/6)*(k1_x+2*k2_x+2*k3_x+k4_x);
```

Overlaps file

```
%This is the function to calculate overlaps between balls
function [new_overlaps] = contacts(N, Nth_ball_init_pos, xx, R)

new_overlaps = zeros(1,N+1);
%overlap between the first ball and the wall
new_overlaps(1) = max(0, 0-xx(1));
%overlaps between each pair of balls in the chain
new_overlaps(2:N) = max( 0, xx(1:(N-1))-xx(2:N));
%overlap between the first ball and the wall
new_overlaps(N+1) = max(0, xx(N)-0);
```

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