ON BLOWUP TECHNIQUES AND THE PLURICOMPLEX GREEN'S FUNCTION

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Abstract

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The pluricomplex Green's functions on a compact Kähler manifold (M, ω) have been extensively studied over the past decades. Following and generalizing the blow up techniques in [PS12], we use pluripotential theory to show the existence and uniqueness of pluricomplex Green's functions with two types of prescribed singularities at a finite number of interior points.

To my parents,

and to Pei and Emma,

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1 Introduction, Examples and Organization

1.1 Model Examples

Let (M, ω) be a compact Kähler manifold of dimension n with smooth boundary ∂M and p be an interior point of M. A lot of research work has been done over the past decades regarding the existence and uniqueness of the solution $\phi \in PSH(M, \omega)$ such that

$$\begin{cases} \phi = 0 & \text{on } \partial M, \\ (\omega + i\partial\bar{\partial}\phi)^n = 0 & \text{on } M \backslash \{p\}, \\ (\omega + i\partial\bar{\partial}\phi)^n = V\delta_p & \text{at the point } p. \end{cases}$$

Here $V = \int_M \omega^n = \int_M (\omega + i\partial\bar{\partial}\phi)^n$ is the volume of M with respect to ω and δ_p is the Dirac measure centered at p, i.e. $\frac{1}{V}\int_M f\,(\omega + i\partial\bar{\partial}\phi)^n = f(p)$ for any $f \in C^\infty(M)$. For the case when the dimension $n = 1$, the answer is yes since we can always find the Green's function ϕ, $\omega + i\partial\bar{\partial}\phi = V\,\delta_p$, to the Laplace equation for a compact Riemann surface M.

For general dimension $n \geqslant 2$, it is still an open question.

Fix a bounded domain $D \subset \mathbb{C}^n$ and a fixed interior point $p \in D$, and consider the existence of a solution ϕ to the system of equations

$$\begin{cases} (\omega + i\partial\bar{\partial}\phi)^n = V\delta_p \\ \phi|_{\partial D} = 0 \end{cases}$$

In general we have the existence of ϕ but can not guarantee the uniqueness. See the following example.

Example 1.1.

On the unit disk D in $\mathbb{C}^n$, take $\phi = \log(|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)$ and p to be the origin. Clearly $\phi|_{\partial D} = 0$ and direct computation gives $(i\partial\bar{\partial}\log(|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2))^n = \delta_0$ near p. Away from p, some $z_i \neq 0$, without loss of generality we may assume $z_n \neq 0$ and thus

$$\begin{aligned}
0 &= (\,(\partial\bar{\partial}\log(|z_n|^2) + \partial\bar{\partial}\log(1 + |\frac{z_1}{z_n}|^2 + \cdots + |\frac{z_{n-1}}{z_n}|^2)\,)^n \\
&= (\,\partial\bar{\partial}\log(1 + |\frac{z_1}{z_n}|^2 + \cdots + |\frac{z_{n-1}}{z_n}|^2)\,)^n
\end{aligned}$$

Indeed, $\partial\bar{\partial}\log|z_n|^2 = 0$ when $z_n \neq 0$. Let $f_i(z) := \frac{z_i}{z_n}$ for $1 \leq i \leq n-1$. This gives a biholomorphic map

$$\pi := (f_1, \cdots, f_{n-1}) : \quad \mathbb{C}^n \backslash \{z_n = 0\} \longrightarrow \mathbb{C}^{n-1}$$

Let $\eta := i\partial\bar{\partial}\log(1 + |w_1|^2 + \cdots + |w_{n-1}|^2)$ be a smooth positive (1,1) form on $\mathbb{C}^{n-1}$.

We have that $\eta^n = 0$ on $\mathbb{C}^{n-1}$ and thus

$$(i\partial\bar{\partial}\log(1 + |\frac{z_1}{z_n}|^2 + \cdots + |\frac{z_{n-1}}{z_n}|^2))^n = (\pi^*\eta)^n = 0$$

One can further get uniqueness of such a solution ϕ by adding the assumption that ϕ has some prescribed logarithmic singularity near a single point p.

See the following example.

Example 1.2.

Suppose p is the only common zero of holomorphic functions $\{f_i\}_{i\leq n}$ defined in a bounded domain $D \subset \mathbb{C}^n$, then there exists a unique $\phi \in PSH(D)$ such that

$$\begin{cases} \phi|_{\partial D} = 0 \\ (\omega + i\partial\bar{\partial}\phi)^n = V\delta_p \\ \phi = \log(|f_1|^2 + \cdots + |f_n|^2) + g \quad \text{for some } g \in O(1) \text{ near p,} \end{cases}$$

More generally, for a compact Kähler manifold (M, ω) of dimension $n \geqslant 2$ with smooth boundary ∂M of real dimension $2n-1$, Phong and Sturm showed in [PS12] that for sufficiently small ϵ, there exists a unique $\phi \in PSH(M, \omega)$ such that

$$\begin{cases} \phi|_{\partial M} = 0 & \\ (\omega + i\partial\bar{\partial}\phi)^n = 0 & \text{on } M\backslash\{p\} \\ \phi = \epsilon\log(|f_1|^2 + \cdots + |f_n|^2) + g & \text{for some } g \in O(1) \text{ near } p \end{cases}$$

1.2 Questions

We look at the existence and uniqueness of the solutions to the following question.

Question 1.

For a compact Kähler manifold (M, ω) of dimension $n \geqslant 2$ with smooth boundary ∂M of real dimension $2n-1$, and sufficiently small δ, does there exist a unique $\phi \in PSH(M, \omega)$, such that

$$\begin{cases} \phi|_{\partial M} = 0 & \\ (\omega + i\partial\bar{\partial}\phi)^n = 0 & \text{on } M \backslash \{p\} \\ \phi = \delta \log(|f_1|^{2\beta_1} + \cdots + |f_n|^{2\beta_n}) + g & \text{for some } g \in O(1) \text{ near } p, \end{cases}$$

for any given constants $0 < \beta_i \leqslant 1$, $1 \leqslant i \leqslant n$.

Remark 1.3.

For the case where all β's are positive rational numbers, we will give an affirmative answer and prove the existence and uniqueness of such a ϕ in Corollary 4.10. Moreover, we show that $g \in L^\infty(M)$ is unique and Hölder continuous away from a neighborhood of p.

However, the question still remains open for general real positive constants $\beta_1, \cdots, \beta_n$.

We now check a few examples for Question 1 with respect to two basic types of manifolds: bounded domains in $\mathbb{C}^n$ and compact submanifolds of $\mathbb{CP}^n$.

Example 1.4 (Bounded domain in $\mathbb{C}^n$)**.**

Fix any positive real constants $\beta_1, \cdots, \beta_n$ as above and pick p to be the origin. We let $\phi(z) := \log(|z_1|^{2\beta_1} + \cdots + |z_n|^{2\beta_n})$ and look at $(i\partial\bar{\partial}\phi)^n$ on D, where

$$D = \{z \in \mathbb{C}^n : |z_1|^{2\beta_1} + \cdots + |z_n|^{2\beta_n} < 1\}$$

To see that $\phi \in PSH(D)$, it suffices to apply the following theorem.

Theorem 1.5 ([Dem, Theorem 5.6])**.**
Let $u_1, \cdots, u_p \in PSH(\Omega)$ and $\chi(t_1, \cdots, t_p) : \mathbb{R}^p \to \mathbb{R}$ be a convex function such that χ is non decreasing in each t_j. Then $\chi(u_1, \cdots, u_p)$ is plurisubharmonic on Ω.

Indeed, we let $\chi(t_1, \cdots, t_n) := \log(e^{t_1} + \cdots + e^{t_n})$ and clearly $\chi : \mathbb{R}^n \to \mathbb{R}$ is a convex function and non-decreasing in each t_i. For each $1 \leqslant i \leqslant n$, let $u_i := \beta_i \log |z_i|^2$ and thus u_i is plurisubharmonic. It follows from the above theorem that $\phi = \chi(u_1, \cdots, u_n)$ is plurisubharmonic.

We wish to show $(i\partial\bar{\partial}\phi)^n = 0$ on D away from the origin. For the part on D where all $z_i \neq 0$, differentiating twice gives

$$\begin{aligned}\partial_i\partial_{\bar{j}}\phi =& \partial_i\Big(\frac{\beta_j |z_j|^{2\beta_j - 2} z_j}{|z_1|^{2\beta_1} + \cdots + |z_n|^{2\beta_n}}\Big) \\ =& \frac{\delta_{ij}\beta_i\beta_j |z_j|^{2\beta_j-2}}{|z_1|^{2\beta_1} + \cdots + |z_n|^{2\beta_n}} - \frac{\beta_i\beta_j |z_i|^{2\beta_i-2}|z_j|^{2\beta_j-2}\overline{z_i}z_j}{(|z_1|^{2\beta_1} + \cdots + |z_n|^{2\beta_n})^2}\end{aligned}$$

In order to show $(\partial\bar{\partial}\phi)^n = 0$ there, it suffices to prove that $\det(\partial_i\partial_{\bar{j}}\phi) = 0$. Observe that as a matrix $(\,\partial_i\partial_{\bar{j}}\phi\,) = \operatorname{diag}(\beta_1, \cdots, \beta_n) \cdot B \cdot \operatorname{diag}(\beta_1, \cdots, \beta_n)$, where B is the matrix

$$B := \begin{pmatrix} \frac{|z_1|^{2\beta_1-2}}{|z_1|^{2\beta_1}+\cdots+|z_n|^{2\beta_n}} & & \\ & \ddots & \\ & & \frac{|z_n|^{2\beta_n-2}}{|z_1|^{2\beta_1}+\cdots+|z_n|^{2\beta_n}} \end{pmatrix}$$
$$- \begin{pmatrix} \frac{|z_1|^{2\beta_1-2}z_1}{|z_1|^{2\beta_1}+\cdots+|z_n|^{2\beta_n}} \\ \vdots \\ \frac{|z_n|^{2\beta_n-2}z_n}{|z_1|^{2\beta_1}+\cdots+|z_n|^{2\beta_n}} \end{pmatrix} \cdot \begin{pmatrix} \frac{|z_1|^{2\beta_1-2}\overline{z_1}}{|z_1|^{2\beta_1}+\cdots+|z_n|^{2\beta_n}} & \cdots & \frac{|z_n|^{2\beta_n-2}\overline{z_n}}{|z_1|^{2\beta_1}+\cdots+|z_n|^{2\beta_n}} \end{pmatrix}$$

Multiplying B by the non-zero vector $(z_1, \cdots, z_n)^{\mathrm{T}}$ gives

$$B \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} \frac{|z_1|^{2\beta_1-2}}{|z_1|^{2\beta_1}+\cdots+|z_n|^{2\beta_n}} & & \\ & \ddots & \\ & & \frac{|z_n|^{2\beta_n-2}}{|z_1|^{2\beta_1}+\cdots+|z_n|^{2\beta_n}} \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$
$$- \begin{pmatrix} \frac{|z_1|^{2\beta_1-2}z_1}{|z_1|^{2\beta_1}+\cdots+|z_n|^{2\beta_n}} \\ \vdots \\ \frac{|z_n|^{2\beta_n-2}z_n}{|z_1|^{2\beta_1}+\cdots+|z_n|^{2\beta_n}} \end{pmatrix} \cdot \begin{pmatrix} \frac{|z_1|^{2\beta_1-2}\overline{z_1}}{|z_1|^{2\beta_1}+\cdots+|z_n|^{2\beta_n}} & \cdots & \frac{|z_n|^{2\beta_n-2}\overline{z_n}}{|z_1|^{2\beta_1}+\cdots+|z_n|^{2\beta_n}} \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$
$$= 0$$

We see that $\det B = 0$ and thus $(\partial\bar{\partial}\phi)^n = 0$.

If some but not all $z_j = 0$, we see that $\phi(z) = \log(|z_1|^{2\beta_1} + \cdots + |z_n|^{2\beta_n})$ is locally bounded and thus $(i\partial\bar{\partial}\phi)^n$ is a positive closed current. Moreover, as a measure $(i\partial\bar{\partial}\phi)^n$ takes no mass at pluripolar sets. See [Bl1, Prop 2.1.3].

To show that ϕ lies in the domain of definition of Monge-Ampère operator

$\mathrm{MA}(\phi) := (i\partial\bar{\partial}\phi)^n$ for the entire domain D, it only remains to show that $(\partial\bar{\partial}\phi)^n$ is well defined at the origin. We apply the following theorem.

Theorem 1.6 ([Dem, Corollary 4.11]).

Let $u_1, \cdots, u_q$ be plurisubharmonic functions on X such that the unbounded locus $L(u_i)$ is contained in an analytic set $A_i \subset X$ for every i. Then $\partial\bar{\partial}u_1 \wedge \cdots \wedge \partial\bar{\partial}u_q$ is well defined as long as $A_{j_1} \cap \cdots \cap A_{j_m}$ has codimension at least m for all choices of indices $j_1 < \cdots < j_m$ in $\{1, \cdots, q\}$.

Notice that $L(\phi)$, the unbounded locus of $\phi = \log(|z_1|^{2\beta_1} + \cdots + |z_n|^{2\beta_n})$, is just the origin and thus has codimension n. So by the Theorem 1.7 above, we see that $(i\partial\bar{\partial}\phi)^p$ is a well defined positive closed (p, p) current for every $1 \leqslant p \leqslant n$ near the origin.

We conclude that $(\partial\bar{\partial}\phi)^n$ is well-defined on $D \subset \mathbb{C}^n$ and that on $D\backslash\{p\}$, $(i\partial\bar{\partial}\phi)^n = 0$.

Example 1.7 (Submanifolds of $\mathbb{CP}^n$).

Suppose we have $q+1$ holomorphic sections $s_0, \cdots, s_q \in H^0(\mathbb{CP}^n, O(k))$ and let

$$\phi = \frac{1}{k}\log(|s_0|^2_{h^k_{FS}} + \cdots + |s_q|^2_{h^k_{FS}})$$

it is easy to check that $\phi \in PSH(\mathbb{CP}^n, \omega_{FS})$.

In fact, we have a more general example below. Let $s_0, \cdots, s_q \in H^0(\mathbb{CP}^n, O(k))$ be $q+1$ holomorphic sections and let

$$\phi = \frac{1}{k}\log(|s_0|^{2\beta_0}_{h^k_{FS}} + \cdots + |s_q|^{2\beta_q}_{h^k_{FS}})$$

For any given constants $0 < \beta_i \leqslant 1$, we claim that $\phi \in PSH(\mathbb{CP}^n, \omega_{FS})$.

Proof. We only need to prove the claim for $k = 1$, for otherwise it is easy to prove through tensoring the power k.

i) First we observe that for any holomorphic function f on $\mathbb{P}^n$ and $\beta > 0$, $\beta \log |f|^2$ is plurisubharmonic.

   To show $\omega_{FS} + i\partial\bar{\partial}\phi \geqslant 0$ at any point $p \in \mathbb{P}^n$, we choose t, a trivializing section for $O(1)$ in a neighborhood near p. So locally $\omega_{FS} = i\partial\bar{\partial}\psi_{FS}$, where ψ_{FS} is defined by $|t|^2 = e^{-\psi_{FS}}$. Observe that if we let $f_i = \frac{s_i}{t}$, then $\{f_i\}$ are holomorphic functions near p.

ii) We have

$$\omega_{FS} + i\partial\bar{\partial}\phi \tag{1.1}$$

$$= i\partial\bar{\partial}(\psi_{FS} + \phi) \tag{1.2}$$

$$= i\partial\bar{\partial}\{\psi_{FS} + \log(|\frac{s_0}{t}|^{2\beta_0}|t|^{2\beta_0} + \cdots + |\frac{s_q}{t}|^{2\beta_q}|t|^{2\beta_q})\} \tag{1.3}$$

$$= i\partial\bar{\partial}\{\log\frac{1}{|t|^2} + \log(|\frac{s_0}{t}|^{2\beta_0}|t|^{2\beta_0} + \cdots + |\frac{s_q}{t}|^{2\beta_q}|t|^{2\beta_q})\} \tag{1.4}$$

$$= i\partial\bar{\partial}\log(|\frac{s_0}{t}|^{2\beta_0}|t|^{2\beta_0-2} + \cdots + |\frac{s_q}{t}|^{2\beta_q}|t|^{2\beta_q-2}) \tag{1.5}$$

$$= i\partial\bar{\partial}\log(|f_0|^{2\beta_0}e^{\psi_{FS}(1-\beta_0)} + \cdots + |f_q|^{2\beta_q}e^{\psi_{FS}(1-\beta_q)}) \tag{1.6}$$

iii) Let $u_i := \beta_i \log |f_i|^2 + (1-\beta_i)\psi_{FS}$ for each $0 \leq i \leq q$, then u_i is plurisubharmonic as a convex combination of two plurisubharmonic functions and equation (1.6) is equal to $i\partial\bar{\partial}\log(e^{u_0} + \cdots + e^{u_q})$.

By [Dem, Theorem 5.6], $\chi(u_0, \cdots, u_q) = \log(e^{u_0} + \cdots + e^{u_q})$ is plurisubharmonic and thus

$$\omega_{FS} + i\partial\bar{\partial}\phi = i\partial\bar{\partial}\chi(u_0, \cdots, u_q) \geqslant 0$$

Thus we conclude that ϕ is plurisubharmonic.

But does there exist such a unique pluricomplex Green's function ϕ on projective manifolds with smooth boundary such that ϕ is locally given by ϕ as above?

In general for any compact Kähler manifold (M, ω) with smooth boundary ∂M and all $\beta_i = 1$, the answer to both existence and uniqueness has been given by Phong and Sturm in [PS12].

In this thesis, we give an affirmative answer to Question 1 for the case where all the $\beta_1, \cdots, \beta_n$ are positive rational numbers. We also give the proof of existence and uniqueness of solutions to the Question 2 and Question 3 below.

Question 2. *Fix any constants $0 < \beta_i \leqslant 1$ and any sufficiently small $\delta > 0$. Does there exist a unique pluricomplex Green's function $G \in PSH(M, \omega)$, which is zero on ∂M and locally near p,*

$$G = \delta \log\{\sum_{j=1}^{n} |f_j|^{2\beta_j} (\sum_{i=1}^{n} |f_i|^2)^{1-\beta_j}\} + \phi \tag{1.7}$$

where ϕ is a bounded function defined on M?

The answer is yes, provided that the locally defined holomorphic functions $f_1, \cdots, f_n$ have p as their only common zero locus. Moreover we prove that ϕ

is C^α continuous away from any neighborhood of p, for any constant $0 < \alpha < \min\{\beta_j\}$.

Question 3. *Fix an integer $N \geqslant n \geqslant 2$, and n locally defined holomorphic functions $\{f_j\}_{1 \leqslant j \leqslant n}$ with p as their only common zero locus and $0 \leqslant \beta_{ij} \leqslant 2$. For any sufficiently small $\delta > 0$, does there exist a unique pluricomplex Green's function $G \in PSH(M, \omega)$ which vanishes on ∂M and locally near p,*

$$G = \delta \log\{\sum_{i=1}^{N} \prod_{j=1}^{n} |f_j|^{\beta_{ij}}\} + \phi \tag{1.8}$$

where ϕ is a bounded function defined on M?

The answer is yes, provided that for each fixed $1 \leqslant i \leqslant N$,

$$\beta_{i1} + \cdots + \beta_{in} = 2 \tag{1.9}$$

and that the singularity term

$$\log \sum_{i=1}^{N} \prod_{j=1}^{n} |f_j|^{\beta_{ij}} \text{ has } p \text{ as its only } -\infty \text{ pole.} \tag{1.10}$$

Under the above two conditions, we also prove that ϕ is C^α continuous away from any neighborhood of p, for any constant $0 < \alpha < \min\limits_{\beta_{ij} \neq 0}\{\beta_{ij}\}$.

Remark 1.8.

The homogeneity condition (1.9) is needed later when we apply blow-ups. The single point singularity condition in (1.10) guarantees that G lies in the domain of definition of the Monge-Ampère operator. See [Dem, Corollary 4.11].

1.3 Organization

This thesis is organized as follows:

In Section 2, we cover some necessary background. In Section 2.1, we review pluripotential theory on domains in $\mathbb{C}^n$ and on compact complex manifolds, including maximal function and Perron's envelop method. In Section 2.2 and 2.3, we review general blow up procedures of a manifold with respect to a compact submanifold and construct a metric on the blow-up space.

In Section 3, we prove some necessary lemmas, and construct Green's functions with the singularity prescribed by (1.7) to answer Question 2.

We give a similar proof in Section 4 for Question 3.

In section 4.6, we give some applications and answer Question 1 for positive rational indices $\{\beta_i\}_{1\leqslant i\leqslant n}$.

2 Definitions and Preliminaries

2.1 Pluripotential theory

We list here some useful and well known results from pluripotential theory for domains in $\mathbb{C}^n$ and complex manifolds M of dimension n, assuming $n \geqslant 2$. The results come from [Bl1][BT1][BT2][Dem][Dw][PSS12], where the reader may find detailed proofs.

Definition 2.1 (Plurisubharmonic functions on domains in $\mathbb{C}^n$)**.**
Let $\Omega \subset \mathbb{C}^n$ be a domain. An upper semi continuous function u defined on Ω is called plurisubharmonic or denoted as $u \in PSH(\Omega)$ if, for any complex line L such that $L \cap \Omega$ is nonempty, the restriction of u onto any connected component of $L \cap \Omega$ is either subharmonic or constantly $-\infty$.

Note that if u is smooth, then the definition is equivalent to that the complex Hessian $\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}(z)$ is nonnegative definite for any $z \in \Omega$.

Theorem 2.2 (Operations on plurisubharmonic functions)**.**
Convex combinations, finite maximums, decreasing limits and upper semicontinuous regularizations of supremums of plurisubharmonic functions are still plurisubharmonic.

Theorem 2.3 (Standard regularization and holomorphic mapping)**.**
Let $\eta_\epsilon = \eta(\frac{|z|}{\epsilon}) \in C^\infty(B(0,\epsilon))$ be the standard smoothing kernels in $\mathbb{C}^n$ and $u \in PSH(\Omega)$, then $u_\epsilon = u * \eta_\epsilon \in PSH(\Omega_1) \cap C^\infty(\Omega_1)$ for any relatively compact

subdomain $\Omega_1 \subset \Omega$ *in which* u_ϵ *is well-defined and* u_ϵ *decreases pointwise to* u *as* $\epsilon \longrightarrow 0$.

Let $F : \Omega_1 \longrightarrow \Omega$ *be a holomorphic mapping between two domains and* $u \in PSH(\Omega)$. *Then* $u \circ F$ *is plurisubharmonic on* Ω_1.

Definition 2.4 (Pluripolar sets)**.**

A subset $Z \subset \Omega$ is called pluripolar if it is the $-\infty$ pole locus of some $u \in PSH(\Omega)$, i.e. $Z = \{u = -\infty\}$. Note that an analytic subset $A \subset \Omega$ is locally pluripolar since it is locally given by the zero locus of some holomorphic function f.

Note that any countable union of pluripolar sets is pluripolar and by a Theorem of Josefson [Jos], any locally pluripolar set is globally pluripolar.

Theorem 2.5 (Extension over analytic subsets)**.**

Let $u \in PSH(\Omega \backslash A)$ *for some analytic subset* A *of codimension at least* 2, *then* $u \in PSH(\Omega)$. *If* A *is of codimension* 1, *then we need to assume that* u *is locally bounded.*

A proof can be done through first showing that u is locally bounded and then applying the upper semicontinuous regularization of supremums of a family of plurisubharmonic functions to extend the definition over A.

Theorem 2.6 ($\log r$-convexity)**.**

Let $u \in PSH(\Omega)$ *and fix a point* $z \in \Omega$. *Define* $V(r)$ *the average of* u *over a sphere of radius* r *centered at* z, *then* $V(r)$ *is a convex increasing function of* $\log r$.

Note that by Fubini's Theorem it is easy to see that plurisubharmonic functions are locally integrable, and the next theorem tells that many plurisubharmonic functions are exponentially integrable.

Theorem 2.7 (Exponentially integrable psh functions)**.**
Let $u \in PSH^-(\,\overline{B(0,1)}\,)$ be a negative plurisubharmonic function in a neighborhood of the unit ball such that $u(0) > -1$, then there exists a constant C such that

$$\int_{B(0,\frac{1}{2})} e^{-u} < C$$

Definition 2.8 (Differential forms with continuous coefficients)**.**
Denote $\mathcal{D}^k_{(p,q)}(\Omega)$ as the set of differential forms β of bidegree (p,q) with C^k continuous coefficients in Ω, i.e. locally in the form

$$\beta = \sum_{\substack{i_1<\cdots<i_p \\ j_1<\cdots<j_q}} \beta_{I,\bar{J}}\, dz_I \wedge d\bar{z}_J$$

where $I = \{i_1, \cdots, i_p\}$ and $J = \{j_1, \cdots, j_q\}$ and $\beta_{I,\bar{J}}$ are C^k continuous in Ω.

Definition 2.9 (Currents)**.**
The currents of bidegree (p,q), denoted as $\mathcal{T}_{(p,q)}(\Omega)$ are the set of differential forms α of bidegree (p,q) with distribution coefficients, i.e.

$$\alpha = \sum_{\substack{i_1<\cdots<i_p \\ j_1<\cdots<j_q}} \alpha_{I,\bar{J}}\, dz_I \wedge d\bar{z}_J$$

where $I = \{i_1, \cdots, i_p\}$ and $J = \{j_1, \cdots, j_q\}$ and the coefficients $\alpha_{I,\bar{J}}$ are locally defined distributions. Alternatively one can define currents by their action on smooth $(n-p, n-q)$ forms with test functions as coefficients. For example, $\alpha_{I,\bar{J}}$ can be complex measures on Ω. One can also define the current of integration over a complex submanifold V, i.e. a bidegree $(n-p, n-p)$ current $[V]$ that acts on any smooth differential (p,p) form ϕ in the following way

$$[V](\phi) := \int_V \phi$$

We are mostly interested in positive, closed (p,p) currents, which are defined below.

Definition 2.10 (Positive closed currents)**.**

A (p,p) current T is called Hermitian(or real), if $\bar{T} = T$. We call T to be of order k if its local distributional coefficients are of order k, i.e. its action can be extended to all test forms with coefficients in C^k smooth but not all that in C^{k-1}. T is called a positive current if $T(\beta) \geqslant 0$ for any simple positive test form β, i.e.

$$\beta = i\beta_1 \wedge \bar{\beta}_1 \wedge \cdots \wedge i\beta_{n-p} \wedge \bar{\beta}_{n-p}$$

for any test $(1,0)$ forms $\beta_1, \cdots, \beta_{n-p}$.

Note that any positive current is of order 0, i.e. its coefficients are complex measures. From now on, we suppose T is a positive closed (p,p) current.

Theorem 2.11 ($i\partial\bar{\partial}u$ as current)**.**

For any $u \in PSH(\Omega)$, $i\partial\bar{\partial}u$ is a closed positive $(1,1)$ current. If we further assume $u = \log|f|$ for some holomorphic function on Ω, then

$$i\partial\bar{\partial}u = 2\pi \cdot [\{f = 0\}]$$

which is the current of integration over the zero locus of f.

Theorem 2.12 ($i\partial\bar{\partial} \wedge T$)**.**

For any $u \in PSH(\Omega) \cap L^{\infty}_{loc}(\Omega)$, $i\partial\bar{\partial}u \wedge T := i\partial\bar{\partial}(uT)$ is a positive closed current of bidegree $(p+1, p+1)$. As a result, if $u_1, \cdots, u_k$ are are bounded plurisubharmonic functions, the $i\partial\bar{\partial}u_1 \wedge \cdots \wedge i\partial\bar{\partial}u_k$ is a closed positive current.

We list a few properties of the Monge-Ampère operator $MA(u) := (i\partial\bar{\partial}u)^n$ for locally bounded, plurisubharmonic function u.

Definition 2.13 ($||T||_K$)**.**

Fix any compact subset K in the domain of definition of T. With the positive form $\beta := idz_1 \wedge d\bar{z}_1 + \cdots + idz_n \wedge d\bar{z}_n$, $||T||_K$ is defined by

$$||T||_K = \int_K T \wedge \beta^{n-p}$$

Theorem 2.14 (Chern-Levine-Nirenberg)**.**

Let $u_1, \cdots, u_k \in PSH(\Omega) \cap L^{\infty}$ and K be a compact subset of an open set U relatively compact in Ω, then there exist a constant which depends only on K, U, Ω such that

$$||i\partial\bar{\partial}u_1 \wedge \cdots \wedge i\partial\bar{\partial}u_k||_K \leqslant C(K, U, \Omega)||u_1||_{L^{\infty}(U)} \cdots ||u_k||_{L^{\infty}(U)}$$

Note that for $k = n$, one can define the relative Monge-Ampère capacity with respect to any Borel subset $E \subset \Omega$ as

$$c(E, \Omega) := \sup_{\substack{u \in PSH(\Omega) \\ -1 \leqslant u \leqslant 0}} \{\int_E (i\partial\bar{\partial}u)^n\}$$

With this, one can show that locally bounded plurisubharmonic functions do not carry Monge-Ampère mass on pluripolar sets. See [Bl1, Prop 2.2.3].

Theorem 2.15 (Continuity property, [BT2])**.**

Suppose $k+p \leqslant n$, $u_1, \cdots, u_k \in PSH(\Omega) \cap L^\infty_{loc}$. *Let* $\{u_j^{(i)}\}_{i=1}^\infty$ *be sequences of psh functions decreasing to* u_j, *for each* $1 \leqslant j \leqslant k$. *Then* $i\partial\bar{\partial}u_1^{(i)} \wedge \cdots \wedge i\partial\bar{\partial}u_k^{(i)} \wedge T$ *converges weakly to* $i\partial\bar{\partial}u_1 \wedge \cdots \wedge i\partial\bar{\partial}u_k \wedge T$.

Definition 2.16 (Perron Envelope)**.**

Let Ω be a bounded domain in $\mathbb{C}^n$ and $f \in L^\infty(\partial\Omega)$, the function

$$u_{f,\Omega} := \sup\{v \in PSH(\Omega) : {v^*}_{|\partial\Omega} \leqslant f\}$$

is called a Perron-Bremermann envelope of f in Ω.

It is important to notice that such $u_{f,\Omega}$ need not be upper semicontinuous in general. The following theorem applies to envelopes with continuous boundary function f.

Theorem 2.17 (Walsh [Wal])**.**

Let $f \in C(\partial\Omega)$ *and* $u = u_{f,\Omega}$ *be defined as above. Assume* ${u^*}_{|\partial\Omega} = {u_*}_{|\partial\Omega} = f$ *on* $\partial\Omega$, *then* u *is continuous on the entire* Ω.

Definition 2.18 (Maximal functions)**.**

A function $u \in PSH(\Omega)$ is called maximal if, for every $v \in PSH(\Omega)$ such that $v \leqslant u$ outside a compact subset $K \subset \Omega$, we have $v \leqslant u$ on the entire Ω.

Note for example, $\log ||z||$ is maximal in $\mathbb{C}^n \backslash \{0\}$ but not maximal in $\mathbb{C}^n$. Let

f be any holomorphic function on Ω, then $\log|f|$ and $|f|^{\gamma}$ with $\gamma \geqslant 0$ are maximal.

The following theorem states that maximal functions have zero Monge-Ampère mass in Ω. See [Bl1, Theorem 2.3.1].

Theorem 2.19.

Let u be a locally bounded plurisubharmonic function in Ω, then u is maximal in Ω if and only if $(i\partial\bar{\partial}u)^n = 0$ in Ω. In particular, being a locally bounded maximal plurisubharmonic function is a local property.

Definition 2.20 (ω-psh functions on manifolds)**.**

Let (M, ω) be a compact Kähler manifold and ω a Kähler form on M. Then a function ψ is called quasi-plurisubharmonic, or ω-plurisubharmonic, denoted as $\psi \in PSH(M, \omega)$, if locally $\omega + i\partial\bar{\partial}\psi = i\partial\bar{\partial}(\phi + \psi) \geqslant 0$, for a smooth local potential function ϕ of ω.

In general, if α is a closed (semi-)positive $(1, 1)$ current, we can still define $PSH(M, \alpha)$ to be the set of functions ψ such that $\alpha + i\partial\bar{\partial}\psi > 0(\geqslant 0)$ in the sense of currents.

Theorem 2.21 (The local $\partial\bar{\partial}$-lemma)**.**

Let β be a smooth, closed, real $(1, 1)$ form on a compact Kähler manifold (M, ω), then locally $\beta = i\partial\bar{\partial}\eta$ for some smooth real function η.

In particular, locally one can find a Kähler potential function f, such that $\omega = i\partial\bar{\partial}f$.

Theorem 2.22 (The global $\partial\bar{\partial}$-lemma)**.**

Let β be a smooth, d-exact, real $(1,1)$ form on a compact Kähler manifold (M,ω), then there exists a smooth real function η such that $\beta = i\partial\bar{\partial}\eta$.

Note that if we assume $\partial M = \emptyset$, by Stokes' theorem it is impossible to find a global Kähler potential function f for ω.

Definition 2.23 (Singular metric on line bundles)**.**

A singular Hermitian metric on a line bundle L is a metric which is given in any trivialization $\theta : L_{|U} \longrightarrow U \times \mathbb{C}$ by

$$||\xi|| = |\theta(\xi)|e^{-\phi(x)}, \quad x \in U, \quad \xi \in L_x$$

where $\phi \in L^1_{loc}(U)$ is an arbitrary function. Then there is a well-defined curvature current $c(L) := i\partial\bar{\partial}\phi$. See [Dem2].

Definition 2.24 (Lelong number)**.**

The Lelong number with respect to a plurisubharmonic function ϕ and a point $x \in M$ is defined as

$$\nu := \liminf_{z \to x} \frac{\phi(z)}{\log|z-x|}$$

Theorem 2.25 ([Dem2, lemma 2])**.**

If ϕ is plurisubharmonic on M, then $e^{-2\phi}$ is integrable in a neighborhood of x if $\nu(\phi,x) < 1$ and non integrable if $\nu(\phi,x) \geqslant n$.

2.2 Blow up along a submanifold

Let M be complex manifold of dimension n and Z a closed submanifold of dimension $m < n$.

Definition 2.26 (The (projectivised) normal bundle)**.**
The normal bundle of Z in M is the vector bundle over Z defined as the quotient $NZ := (TM)_{|_Z}/TZ$. The fibers of NZ are given at each point $p \in Z$ as $NZ_{|_p} = T_pM/T_pZ$. The projectivised normal bundle is defined as $\mathbb{P}(NZ) \to Z$ whose fibers are the projective spaces $\mathbb{P}(NZ_{|_p})$ associated to the fibers $NZ_{|_p}$.

Now we construct $\tilde{M}$, the blow up of M with center Z.

The idea is to replace each point $p \in Z$ by the projective space of all vectors at p that are normal to Z. Denote $E = \mathbb{P}(NZ)$ and let

$$\tilde{M} = (M \backslash Z) \cup E, \quad \text{be the disjoint union of two sets.} \tag{2.1}$$

Define a map $\pi : \tilde{M} \to M$ that extends the projection map $\mathbb{P}(NZ) \to Z$.

$$\pi = \begin{cases} \mathrm{Id}_{|M\backslash Z} : M\backslash Z \to M\backslash Z & \text{on } M\backslash Z \\ \text{the canonical projection} \ : \quad E_{|_p} \to p & \text{on } E \end{cases}$$

Now we define the manifold structure on $\tilde{M}$ by giving explicitly an atlas. A coordinate chart near a point $p \in M\backslash Z$ is naturally taken to be the original chart of M near p. Let the coordinates be $(x_1, \cdots, x_m, y_{m+1}, \cdots, y_n)$. We then fix a locally finite collection of small coordinates balls $\{U_\alpha\}_{U_\alpha \subset M}$ that covers

Z such that

$$U_\alpha \cap Z = \{(x_i^\alpha, y_j^\alpha) \in U_\alpha : y_j^\alpha = 0 \text{ for all } m+1 \leqslant j \leqslant n\} \tag{2.2}$$

It follows that $(x_1^\alpha, \cdots, x_m^\alpha)$ are the local coordinates on $Z \cap U_\alpha$. The span of normal vector fields $\{\partial_{y_j^\alpha}\}_{m+1 \leqslant j \leqslant n}$ yields a holomorphic frame of $NZ_{|U_\alpha}$ near p. Let us denote by $(t_{m+1}, \cdots, t_n)$ the coordinates along the fibers of NZ. Then

$$[(x_1^\alpha, \cdots, x_m^\alpha, 0, \cdots, 0), [t_{m+1}^\alpha : \cdots : t_n^\alpha])\} \tag{2.3}$$

are the coordinates of $\mathbb{P}(NZ)_{|Z\cap U_\alpha}$. To patch this with the coordinates of $U_\alpha \backslash Z$, which are $\{(x_i^\alpha, y_j^\alpha) \in U_\alpha : y^\alpha \neq 0\}$ and get a well defined local chart for an neighborhood $\tilde{U}_\alpha = \pi^{-1}(U_\alpha) \subset \tilde{M}$, we first set

$$\tilde{U}_{\alpha,j} = \{(x,y) \in U_\alpha \backslash Z : y_j \neq 0\} \cup \{((x,y),[t]) \in \mathbb{P}(NZ)_{|Z\cap U_\alpha} : t_j \neq 0\} \tag{2.4}$$

and clearly $\{\tilde{U}_{\alpha,j}\}_{m+1\leqslant j \leqslant n}$ covers of $\tilde{U}_\alpha$. Then for each α, define a complex manifold U'_α,

$$\begin{aligned} U'_\alpha := \{((z_1^\alpha, \cdots, z_n^\alpha), [t_{m+1}^\alpha : \cdots : t_n^\alpha]) \in U_\alpha \times \mathbb{P}^{n-m-1} : \\ z_k t_j = z_j t_k, \qquad \text{for all } j,k \geqslant m+1 \quad \} \end{aligned}$$

Clearly U'_α is covered by its $n-m$ open subsets $U'_{\alpha,j}$, where for each $m+1 \leqslant j \leqslant N$,

$$U'_{\alpha,j} := \{((z^\alpha),[t^\alpha]) \in U'_\alpha : z_j^\alpha \neq 0 \text{ or } t_j^\alpha \neq 0\} \tag{2.5}$$

For each pair $\tilde{U}_\alpha \to U'_\alpha$, we define a bijective holomorphic map $f_\alpha : \tilde{U}_\alpha \to U'_\alpha$

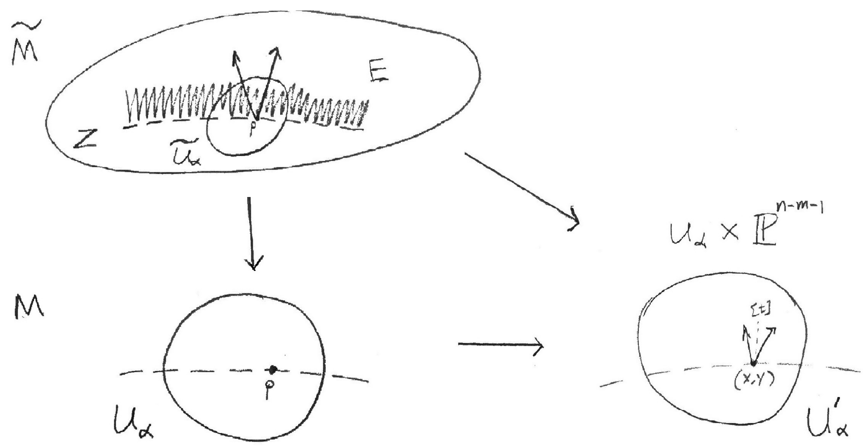

Figure 1: Coordinate neighborhood $\tilde{U}_\alpha \subset \tilde{M}$

as the following,

$$\text{if } y \neq 0, \qquad (x, y) \longmapsto (z, [t]) := ((x, y), [y])$$
$$\text{if } y = 0, \quad (x, 0, [t]) \longmapsto (z, [t]) := ((x, 0), [t])$$

We see that f_α factors through $f_{\alpha,j} : \tilde{U}_{\alpha,j} \to U'_{\alpha,j}$, which are defined as

$$\begin{aligned} \text{if } y \neq 0, \qquad (x, y) \longmapsto (z, [t]) &= ((x, y), [y]) \\ &= ((x, y_{m+1}, \cdots, y_j, \cdots, y_n), [\frac{y_{m+1}}{y_j}, \cdots, 1, \cdots, \frac{y_n}{y_j}]) \\ \text{if } y = 0, \quad (x, 0, [t]) \longmapsto (z, [t]) &= ((x, 0), [t]) \\ &= ((x, 0, \cdots, 0, \cdots, 0), [\frac{t_{m+1}}{t_j}, \cdots, 1, \cdots, \frac{t_n}{t_j}]) \end{aligned}$$

Combine $f_{\alpha,j}$ with the natural coordinates of $\tau'_{\alpha,j} : U'_{\alpha,j} \to \mathbb{C}^n$, which are given as

$$((x, y_{m+1}, \cdots, y_j, \cdots, y_n), [\frac{y_{m+1}}{y_j}, \cdots, 1, \cdots, \frac{y_n}{y_j}]) \longmapsto (x, \frac{y_{m+1}}{y_j}, \cdots, y_j, \cdots, \frac{y_n}{y_j})$$
$$((x, 0, \cdots, 0, \cdots, 0), [\frac{t_{m+1}}{t_j}, \cdots, 1, \cdots, \frac{t_n}{t_j}]) \longmapsto (x, \frac{t_{m+1}}{t_j}, \cdots, 0, \cdots, \frac{t_n}{t_j})$$

We eventually get a holomorphic coordinate chart of $\tilde{M}$ near E from $\tilde{\tau}_{\alpha,j} : \tilde{U}_{\alpha,j} \to \mathbb{C}^n$, where $\tilde{\tau}_{\alpha,j} := \tau'_{\alpha,j} \circ f_{\alpha,j}$ are defined by the following,

$$\begin{aligned} \text{for } z \in M \backslash Z, \qquad \tilde{\tau}_{\alpha,j}(z) &= (w_1, \cdots, w_n) \\ &:= (x_1, \cdots, x_m, \frac{y_{m+1}}{y_j}, \cdots, y_j, \cdots, \frac{y_n}{y_j}) \\ \text{for } (z, [t]) \in E_{|Z \cap U_\alpha}, \quad \tilde{\tau}_{\alpha,j}(z, [t]) &= (w_1, \cdots, w_n) \\ &:= (x_1, \cdots, x_m, \frac{t_{m+1}}{t_j}, \cdots, 0, \cdots, \frac{t_n}{t_j}) \end{aligned}$$

The inverse map $\tilde{\tau}_{\alpha,j}^{-1} : \mathbb{C}^n \cap \{w_j \neq 0\} \to \tilde{U}_{\alpha,j} \backslash E$ is given as $w_j \neq 0$ and

$$(w_1, \cdots, w_n) \longmapsto (w_1, \cdots, w_m, w_{m+1}w_j, \cdots, w_j, \cdots, w_n w_j) \tag{2.6}$$

We conclude that $\tilde{\tau}_{\alpha,j}$ are coordinate charts on $\tilde{M}$ and that locally, $E \cap \tilde{U}_{\alpha,j}$ is defined by the equation $w_j = 0$(See also [Dem]), therefore E is locally a smooth hypersurface in $\tilde{M}$.

We wish to show that $\tilde{M}$ is a complex manifold by showing that $\tilde{\tau}_{\alpha,i}$ and $\tilde{\tau}_{\beta,l}$ glues well on their overlap, i.e. the coordinate change $\tilde{U}_\alpha \to \tilde{U}_\beta : (z_i^\alpha)_{i \leqslant n} \mapsto (z_j^\beta)_{j \leqslant n}$ gives rise to a holomorphic coordinate change $(w_i^\alpha) \to (w_j^\beta)$.
Away from Z, say near point $p \in (\tilde{U}_\alpha \cap \tilde{U}_\beta) \backslash Z$, it is easy to see that $(w_i^\alpha) \to$

(w_j^β) is biholomorphic. Indeed, the map $\mathbb{C}^n \cap \{w_{i_0}^\alpha \neq 0\} \to \mathbb{C}^n \cap \{w_{l_0}^\beta \neq 0\}$ is the composition

$$(w_i^\alpha) \to (z_j^\alpha) \to (z_k^\beta) \to (w_l^\beta)$$

where every intermediate map is holomorphic.

For any point $p \in (\tilde{U}_\alpha \cap \tilde{U}_\beta) \cap Z$, we can find some $m+1 \leqslant i_0, l_0 \leqslant n$ such that $p \in (\tilde{U}_{\alpha,i_0} \cap \tilde{U}_{\beta,l_0}) \cap Z$. This means $t_{i_0}^\alpha \neq 0$ and $t_{l_0}^\beta \neq 0$. Then the map $(w_i^\alpha)_{|w_{i_0}=0} \to (w_j^\beta)_{|w_{l_0}=0}$ is the following composition of the maps 1, 2 and 3

$$\begin{aligned}
&(w_i^\alpha) = (z_1^\alpha, \cdots, z_m^\alpha, \frac{t_{m+1}^\alpha}{t_{i_0}^\alpha}, \cdots, \frac{t_{i_0-1}^\alpha}{t_{i_0}^\alpha}, 0, \frac{t_{i_0+1}^\alpha}{t_{i_0}^\alpha}, \cdots, \frac{t_n^\alpha}{t_{i_0}^\alpha})\\
&\qquad\qquad \downarrow 1\\
&((z_1^\alpha, \cdots, z_m^\alpha, y_{m+1}^\alpha, \cdots, y_n^\alpha), [\frac{t_{m+1}^\alpha}{t_{i_0}^\alpha}, \cdots, \frac{t_{i_0-1}^\alpha}{t_{i_0}^\alpha}, 1, \frac{t_{i_0+1}^\alpha}{t_{i_0}^\alpha}, \cdots, \frac{t_n^\alpha}{t_{i_0}^\alpha}])\\
=&((z_1^\alpha, \cdots, z_m^\alpha, y_{i_0}^\alpha \frac{t_{m+1}^\alpha}{t_{i_0}^\alpha}, \cdots, y_{i_0}^\alpha \frac{t_n^\alpha}{t_{i_0}^\alpha}), [\frac{t_{m+1}^\alpha}{t_{i_0}^\alpha}, \cdots, \frac{t_{i_0-1}^\alpha}{t_{i_0}^\alpha}, 1, \frac{t_{i_0+1}^\alpha}{t_{i_0}^\alpha}, \cdots, \frac{t_n^\alpha}{t_{i_0}^\alpha}])\\
&\qquad\qquad \downarrow 2\\
&((z_1^\beta, \cdots, z_m^\beta, y_{m+1}^\beta, \cdots, y_n^\beta), [\frac{t_{m+1}^\beta}{t_{l_0}^\beta}, \cdots, \frac{t_{l_0-1}^\beta}{t_{l_0}^\beta}, 1, \frac{t_{l_0+1}^\beta}{t_{l_0}^\beta}, \cdots, \frac{t_n^\beta}{t_{l_0}^\beta}])\\
=&((z_1^\beta, \cdots, z_m^\beta, y_{l_0}^\beta \frac{t_{m+1}^\beta}{t_{l_0}^\beta}, \cdots, y_{l_0}^\beta \frac{t_n^\beta}{t_{l_0}^\beta}), [\frac{t_{m+1}^\beta}{t_{l_0}^\beta}, \cdots, \frac{t_{l_0-1}^\beta}{t_{l_0}^\beta}, 1, \frac{t_{l_0+1}^\beta}{t_{l_0}^\beta}, \cdots, \frac{t_n^\beta}{t_{l_0}^\beta}])\\
&\qquad\qquad \downarrow 3\\
&(w_i^\beta) = (z_1^\beta, \cdots, z_m^\beta, \frac{t_{m+1}^\beta}{t_{l_0}^\beta}, \cdots, \frac{t_{l_0-1}^\beta}{t_{l_0}^\beta}, 0, \frac{t_{l_0+1}^\beta}{t_{l_0}^\beta}, \cdots, \frac{t_n^\beta}{t_{l_0}^\beta})
\end{aligned}$$

Since 1 and 3 are holomorphic, we only need to show that 2 is holomorphic. This follows from the fact that $(t_i^\alpha) \mapsto (t_l^\beta)$ is holomorphic near the point p. Indeed, they are the coordinates on the fibers of the normal bundle $NZ_{|\tilde{U}_\alpha \cap \tilde{U}_\beta \cap Z}$ and the rank of NZ is constant so the transition map $T_{\alpha,\beta} : (t_i^\alpha) \longmapsto (t_l^\beta)$, $T \in \Gamma(Z, GL(n-m, \mathbb{C}))$ has to be holomorphic. The proof is complete.

$\square$

We give two examples to illustrate the coordinates taken in equation (2.6).

Example 2.27 (Nodal in $\mathbb{C}^2$)**.**

Let $N = \{\, y^2 = x^3 + x^2 \,\} \subset \mathbb{C}^2$. Blow up $\mathbb{C}^2$ with respect to the origin by taking $x = \tilde{x}$ and $y = \tilde{x} \cdot \tilde{y}$, thus the blow up space $\tilde{N} = \{\, \tilde{y}^2 = \tilde{x} + 1 \,\} \subset \mathbb{C}^2 \times \mathbb{P}^1$.

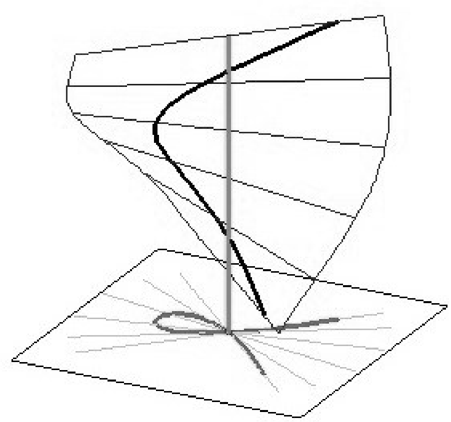

Figure 2: Blow up of $y^2 = x^3 + x^2$

The exceptional divisor E is given by $\tilde{x} = 0$. $E \subset \mathbb{P}^1$ consists of two points that represent the two tangent lines of N at the origin.

Example 2.28 (Cusp in $\mathbb{C}^2$)**.**

Let $N := \{\, y^2 = x^3 \,\} \subset \mathbb{C}^2$. Same as above, we take $x = \tilde{x}$ and $y = \tilde{x} \cdot \tilde{y}$, thus the blow up space

$$\tilde{N} = \{\tilde{y}^2 = \tilde{x}\} \subset \mathbb{C}^2 \times \mathbb{P}^1$$

The exceptional divisor $E \subset \mathbb{P}^1$ is just a single point, given by $\{\tilde{x} = 0\}$, which stands for the tangent line of N at the origin.

Definition 2.29 (The blow up of M with center Z)**.**

The map $\pi : \tilde{M} \to M$ is called the blow-up of M with center Z and E is called the exceptional divisor of $\tilde{M}$. From the above constructions, we see E is locally a smooth hypersurface of $\tilde{M}$ and that $\pi : \tilde{M} \backslash E \to M \backslash Z$ is biholomorphic.

2.3 Metric on general blow up spaces

We have constructed the blow up of $Z \subset M$ and now wish to give a metric on $Bl_Z M$. The following lemma is a general standard fact that the blow up space of M along a compact submanifold Z, admits a Kähler metric.

Lemma 2.30. *(Kähler metric on blow up space)*
For any compact Kähler manifold (M, ω) and a connected compact submanifold $Z \subset M$, we have a Kähler metric on $Bl_Z M$, the blow up of M with respect to center Z.

Proof. A detailed proof can be found in [Dem][PS12]. For completeness, we include a proof here to serve for our results.

i) The idea to cover Z with a family of small neighborhoods and locally construct a Kähler metric on $Bl_Z M$ as

$$\pi^*\omega - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_E \in [\pi^*\omega] + \epsilon\, c_1(O(-E))$$

for some Hermitian metric h_E on the line bundle $O(-E)$,where E is the exceptional divisor and $\pi : Bl_Z M \to M$ is the canonical surjection. For a small neighborhood near Z, we can find a small $\epsilon > 0$ such that

$$\pi^*\omega - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_E > 0 \tag{2.7}$$

Then use the compactness of Z to claim that there exists a finite collection of such neighborhoods that covers Z and choose an uniform ϵ

smaller than the minimum of the finitely many ϵ's to prove that (2.7) is true for the entire Z.

ii) Let $U_\alpha \subset M$ be a locally finite collection of coordinate neighborhoods which covers Z, and let $z^\alpha = (x^\alpha, y^\alpha)$ be coordinates on U_α such that

$$Z \cap U_\alpha = \{(x^\alpha, y^\alpha) \in U_\alpha : y^\alpha_{m+1} = \cdots = y^\alpha_n = 0\}$$

Choose a family of cut-off functions $\{\psi_\alpha\}$ with respect to Z and $\cup_\alpha U_\alpha$ such that $\psi_\alpha \in C^\infty(U_\alpha)$, $0 \leqslant \psi_\alpha \leqslant 1$ and $\sum\limits_\alpha \psi_\alpha = 1$ near Z.
Recall in equation (2.4) that the neighborhood $\tilde{U}_\alpha = \pi^{-1}(U_\alpha) \subset Bl_Z M$ is covered by n open sets $\tilde{U}_{\alpha,j}$ as,

$$\tilde{U}_{\alpha,j} = \{(x,y) \in U_\alpha \backslash Z : y_j \neq 0\} \cup \{((x,y),[t]) \in \mathbb{P}(NZ)_{|Z\cap U_\alpha} : t_j \neq 0\}$$

and that on each $\tilde{U}_{\alpha,j}$, we have a local coordinate chart $\tilde{\tau}_{\alpha,j} : \tilde{U}_{\alpha,j} \longrightarrow \mathbb{C}^n$ as the following,

for $z \in \tilde{U}_{\alpha,j} \backslash E, \quad y_j \neq 0,$

$$\begin{aligned}\tilde{\tau}_{\alpha,j}(z) &= (w_1, \cdots, w_n) \\ &:= (x_1, \cdots, x_m, \frac{y_{m+1}}{y_j}, \cdots, y_j, \cdots, \frac{y_n}{y_j})\end{aligned}$$

for $(z,[t]) \in \tilde{U}_{\alpha,j} \cap E, \quad t_j \neq 0,$

$$\begin{aligned}\tilde{\tau}_{\alpha,j}(z,[t]) &= (w_1, \cdots, w_n) \\ &:= (x_1, \cdots, x_m, \frac{t_{m+1}}{t_j}, \cdots, 0, \cdots, \frac{t_n}{t_j})\end{aligned}$$

iii) Now we give the metric h_E on $O(-E)$ over $\tilde{M}$. For any section $f \in O(-E)$, define $|f|^2_{h_E}$ over $\tilde{U}_{\alpha,j}$ as,

$$|f|^2_{h_E} = \frac{|f|^2}{\sum_\alpha \psi_\alpha\{|w_j|^2 \cdot (1 + \sum_{i\neq j, i=m+1}^{n} |w_i|^2)\} + 1 - \psi} \qquad \text{on } \tilde{U}_{\alpha,j} \tag{2.8}$$

$$= \frac{|f|^2}{\sum_\alpha \psi_\alpha\{|y_j|^2 \cdot (1 + \sum_{i\neq j, i=m+1}^{n} \frac{|t_i|^2}{|t_j|^2})\} + 1 - \psi} \qquad \text{on } \tilde{U}_{\alpha,j} \cap E \tag{2.9}$$

$$= \frac{|f|^2}{\sum_\alpha \psi_\alpha\{|y_j|^2 \cdot (1 + \sum_{i\neq j, i=m+1}^{n} \frac{|y_i|^2}{|y_j|^2})\} + 1 - \psi} \qquad \text{on } \tilde{U}_{\alpha,j} \backslash E \tag{2.10}$$

$$= \frac{|f|^2}{\sum_\alpha \psi_\alpha\{|y_{m+1}|^2 + \cdots + |y_n|^2\} + 1 - \psi} \qquad \text{on } \tilde{U}_{\alpha,j} \tag{2.11}$$

where we have replaced $\psi_\alpha \circ \pi$ with ψ_α.

Notice that in (2.9), $w_j = y_j$ on $\tilde{U}_{\alpha,j} \cap E$ by the choice of coordinate charts $\tilde{\tau}_{\alpha,j}$ and there the denominator seems to be zero on

$$E \cap \tilde{U}_{\alpha,j} = \{w_j = y_j = 0\} \cap \tilde{U}_{\alpha,j}$$

But since $f \in O(-E)$ is locally given as a holomorphic function that vanishes on $E \cap \tilde{U}_{\alpha,j}$, by letting $\tilde{U}_\alpha$ to be sufficiently small, we may assume $f = w_j \cdot g$ in $\tilde{U}_{\alpha,j}$ for some holomorphic function g, and we have

$$|f|^2_{h_E} = \frac{|f|^2}{\sum\limits_\alpha \psi_\alpha\{|y_{m+1}|^2 + \cdots + |y_n|^2\} + 1 - \psi} \quad \text{on } \tilde{U}_{\alpha,j} \tag{2.12}$$

$$= \frac{|g|^2}{\sum\limits_\alpha \psi_\alpha\{\frac{|y_{m+1}|^2}{|y_j|^2} + \cdots + \frac{|y_n|^2}{|y_j|^2}\} + 1 - \psi} \quad \text{on } \tilde{U}_{\alpha,j} \tag{2.13}$$

$$= \frac{|g|^2}{\sum\limits_\alpha \psi_\alpha\{1 + \sum\limits_{i\neq j, i=m+1}^{n} |w_i|^2\} + 1 - \psi} \quad \text{on } \tilde{U}_{\alpha,j} \tag{2.14}$$

Also we see that (2.12) is independent of α and j, so $|f|^2_{h_E}$ defined above glues well on the overlap of any pair of charts $\tilde{\tau}_{\alpha_1,j} : \tilde{U}_{\alpha_1,j} \to \mathbb{C}^n$ and $\tilde{\tau}_{\alpha_2,k} : \tilde{U}_{\alpha_2,k} \to \mathbb{C}^n$ for different α_1, α_2, j, k.

iv) We see that h_E is a well defined smooth metric on $O(-E)$ and that near $E \subset \tilde{M}$, $\psi = 1$ and thus

$$-i\partial\bar{\partial}\log h_E = i\partial\bar{\partial}\log(|y_{m+1}|^2 + \cdots + |y_n|^2) \tag{2.15}$$

$$= \omega_{FS|_{\mathbb{P}^{n-m-1}}} \tag{2.16}$$

Therefore we conclude that locally on U_α, equation (2.7) defines a Kähler metric as

$$\pi^*\omega - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_E > 0 \text{ , for any } 0 < \epsilon < \epsilon_\alpha$$

where ϵ_α is a small constant depending on U_α. To see it is positive for the entire E, notice that $Z \subset M$ is compact and so $\{U_\alpha\}$ is a finite cover. It suffices to choose $\epsilon < \min\{\epsilon_\alpha\}$.

□

3 Case I

The singuarity in Question 2 gives rise to Theorem 3.1, which also applies to finitely many interior points $p_1, \cdots, p_N$ in M. It is a generalization of [PS12, Theorem 1] where all $\beta_i = 1$. The proof that deals with a single point p is given in Theorems 3.8.

3.1 Theorem I

Theorem 3.1. *Let ω be a Kähler form on compact complex manifold M of dimension $n \geqslant 2$ and assume $\partial M \neq 0$ is smooth. Fix N interior points $\{p_1, \cdots, p_N\}$, $N{\cdot}n$ constants $0 < \beta_{mj} \leqslant 1$ and $N{\cdot}n$ local holomorphic functions $\{f_{mj}\}$ such that for each $1 \leqslant m \leqslant N$, $\{f_{mj}\}_{1 \leqslant j \leqslant n}$ are defined in a neighborhood of p_m, with p_m as their only common zero in this neighborhood.*

Then there exists a constant ϵ_0 such that for all $0 < \epsilon_m < \epsilon_0$, $1 \leqslant m \leqslant N$, there exists a unique function $G(z; p_1, \cdots, p_N) \in PSH(M, \omega)$ satisfying that $(\omega + \frac{i}{2}\partial\bar{\partial}G)^n = 0$ on $M \backslash \{p_1, \cdots, p_n\}$ and that

$$\begin{cases} G = 0 & \text{on} \quad \partial M \quad (3.1) \\ G = \epsilon_m \log\{\sum_{j=1}^{n} |f_{mj}|^{2\beta_{mj}} (\sum_{i=1}^{n} |f_{mi}|^2)^{1-\beta_{mj}}\} + O(1) & \text{near} \quad p_m \quad (3.2) \end{cases}$$

Moreover, $G \in C^\alpha(K)$ for any compact $K \subset M \backslash \{p_1, \cdots, p_N\}$ and any positive constant $\alpha < \min_{m \leqslant N, j \leqslant n} \beta_{mj}$.

With an iteration of N times, each of which deals with a singularity at an isolated point p_m and yields an exceptional divisor E_m, we may reduce to the case of $N = 1$ which will be shown in Theorem 3.8 in Section 3.5.

Indeed, we can apply N steps of iterated blow up procedures. On the m-th step, the blow up map is a biholomorphism away from the particular point p_m. So the boundary points get mapped to the new boundary(which is the biholomorphic image via the m-th blow up map) and the images of the isolated points $p_{m+1}, \cdots, p_N$ in the interior of the first $m-1$ blow up spaces, stays isolated in the interior of the m-th blow up space. Furthermore, they stay away from the exceptional divisors E_m and $E_1, \cdots, E_{m-1}$. Then one can continue with the process for $p_{m+1}, \cdots, p_N$ and eventually apply Theorem 3.8 to the single point p_N.

3.2 Singular (1,1) form on $X(M, p, f_1, \cdots, f_n)$

Now we consider M near p with the given data (M, p, f_i) and apply the general blow up arguments to the special case of blowing up M near p, which is the only common zero of the local holomorphic functions $f_1, \cdots, f_n$. We first construct a singular (1,1) form on the blow up space $X = X(M, p, f_i)$ in some ambient space W, where the blow up spaces X and W were previously given by [PS12].

Lemma 3.2 (Singular (1,1) form $\omega_\delta(M, p, f_1, \cdots, f_n, \beta_1, \cdots, \beta_n)$)**.**
Given (M, p, f_i) the data of the Kähler manifold (M, ω), an interior point p, and local holomorphic functions f_i, $1 \leqslant i \leqslant n$ with p as their only common zero. Then there exists a complex analytic space $X = X(M, p, f_1, \cdots, f_n)$ and a biholomorphism $\pi_1 : X \backslash X_0 \to M \backslash \{p\}$, mapping $X_0 \mapsto p$ for some X_0 biholomorphically equivalent to $\mathbb{P}^{n-1}$ with following property:

The form ω_δ defined on $M\backslash\{p\}$ as

$$\omega_\delta = \omega + \delta\frac{i}{2}\partial\bar{\partial}\{\psi \log \sum_{j=1}^{n} |f_j(z)|^{2\beta_j} (\sum_{i=1}^{n} |f_j(z)|^2)^{1-\beta_j} + 1 - \psi\} \tag{3.3}$$

has the pull back $\pi_1^*\omega_\delta$ on $X\backslash X_0$ that extends to a closed strictly positive singular (1,1) form on X. Here ψ is some cut-off function supported in some small neighborhood of p.

Proof. i) Fix $\epsilon_0 > 0$ and since p is the only common zero of the local holomorphic functions $f_1, \cdots, f_n$. We can find a small open neighborhood $U \subset M$ such that $\sum_{j=1}^{n} |f_i|^2 \neq 0$ on $U\backslash\{p\}$. Choose another open neighborhood $U_0 \subset\subset U$ and define

$$U_{\epsilon_0} := U_0 \cap \{z : \sum_{j=1}^{n} |f_i(z)|^2 < \frac{{\epsilon_0}^2}{4}\}$$

We blow up U_{ϵ_0} with center p and denote the blow up space V_{ϵ_0}, as

$$\begin{aligned} V_{\epsilon_0} &:= \mathrm{BL}_{<f_1,\cdots,f_n>} U_{\epsilon_0} \\ &= \{((z_1, \cdots, z_n), [t_1, \cdots, t_n]) \in U_{\epsilon_0} \times \mathbb{P}^{n-1} : t_i f_j(z) = t_j f_i(z)\} \end{aligned}$$

Set theoretically gluing V_{ϵ_0} with $M\backslash\{p\}$ on $U_{\epsilon_0}\backslash\{p\}$ defines space X by

$$X := (V_{\epsilon_0} \cup M\backslash\{p\}) / \sim_{: q_1 \in M\backslash\{p\} \sim (q_2,[t]) \in V_{\epsilon_0} \Longleftrightarrow q_1 = q_2}$$

Clearly $U_{\epsilon_0}\backslash\{p\}$ is covered by n open subsets $\{z : f_j(z) \neq 0\} \cap U_{\epsilon_0}$ and

V_{ϵ_0} is covered by n open subsets

$$\{(z,[t]) : t_j \neq 0\} \cap V_{\epsilon_0}$$

The surjection $\mathrm{proj}_1 : V_{\epsilon_0} \subset U_{\epsilon_0} \times \mathbb{P}^{n-1} \to U_{\epsilon_0}$ defines X_0 as the inverse image of p, which is biholomorphic to $\mathbb{P}^{n-1}$.

ii) Now construct a complex manifold W such that X is locally an analytic subset of W. Perturb the holomorphic functions $f_1, \cdots, f_n$ by $\xi \in B_{\frac{\epsilon_0}{2}}(0) \subset \mathbb{C}^n$. Let the graph of the map $(f_1, \cdots, f_n)$ over U_{ϵ_0} be

$$Z = \overline{\{(z,\xi) \in U_{\epsilon_0} \times B_{\frac{\epsilon_0}{2}} : \xi_i = f_i(z) \text{ for all } 1 \leqslant i \leqslant n\}}$$

Note that $Z \subset \overline{M} \times \overline{B_{\epsilon_0}}$ is a smooth compact submanifold of dimension n and the image of Z under the projection map $\mathrm{proj}_1 : Z \to U_0$ is a subset of U_0 and is thus compactly supported in U. Define local holomorphic functions $g_1(z,\xi), \cdots, g_n(z,\xi)$ over $U_0 \times B_{\epsilon_0}$ as

$$g_i(z) := f_i(z) - \xi_i \text{ for } 1 \leqslant i \leqslant n$$

By the triangle inequality, $\sum\limits_{j=1}^{n} |g_j(z)|^2 < {\epsilon_0}^2$ on $U_{\epsilon_0} \times B_{\frac{\epsilon_0}{2}}$. Clearly in $U_0 \times B_{\epsilon_0}$, we have

$$Z = \{g_1 = \cdots = g_n = 0\} \cap \overline{U_{\epsilon_0} \times B_{\frac{\epsilon_0}{2}}}$$

and since Z is compact, Z has an neighborhood T, defined as

$$T := \{(z,\xi) \in U_0 \times B_{\epsilon_0} : \min_{(z',\xi')\in Z} \{\sum_{j=1}^{n} |f_j(z) - f_j(z')|^2 + |\xi_j - {\xi'}_j|^2 < \frac{{\epsilon_0}^2}{5}\}\}$$

By letting ϵ_0 be sufficiently small, we have that $T \subset\subset U_0 \times B_{\epsilon_0} \subset M \times B_{\epsilon_0}$

iii) Let $W_{\frac{\epsilon_0}{2}}$ be the blow up of $U_{\epsilon_0} \times B_{\frac{\epsilon_0}{2}}$ with center Z, denoted as

$$\begin{aligned} W_{\frac{\epsilon_0}{2}} &:= \mathrm{BL}_{<g_1,\cdots,g_n>}(U_{\epsilon_0} \times B_{\frac{\epsilon_0}{2}}) \\ &= \{ (z,\xi,[t]) \in U_{\epsilon_0} \times B_{\frac{\epsilon_0}{2}} \times \mathbb{P}^{n-1} : t_i\, g_j(z,\xi) = t_j\, g_i(z,\xi) \} \end{aligned}$$

Let W be the set theoretically union of $W_{\frac{\epsilon_0}{2}}$ and $(M \times B_{\epsilon_0})\backslash Z$ glued via the canonical surjection

$$\begin{aligned} \mathrm{proj}_1 : \quad W_{\frac{\epsilon_0}{2}} &\longrightarrow U_{\epsilon_0} \times B_{\frac{\epsilon_0}{2}} \\ (z,\xi,[t]) &\longmapsto (z,\xi) \end{aligned}$$

As $Z \subset M \times B_{\epsilon_0}$ is a smooth submanifold, we see that W is a smooth manifold. The exceptional divisor E_g is locally the inverse image of Z under the blow up map, which is denoted as

$$\begin{aligned} \pi_1 : W &\longrightarrow M \times B_{\epsilon_0} \\ E_g &\longmapsto Z \end{aligned}$$

On W, let $W_{\epsilon_0} := {\pi_1}^{-1}(U_0 \times B_{\epsilon_0})$ and we see that locally near E_g,

$E_g \subset {\pi_1}^{-1}(T) \subset\subset W_{\epsilon_0}$ and clearly W_{ϵ_0} is covered by its n open subsets

$$W_{\epsilon_0,j} = \{(z,\xi,[t]) \in U_0 \times B_{\epsilon_0} \times \mathbb{P}^{n-1} : t_j \neq 0 \text{ or } g_j(z,\xi) \neq 0\}$$

With the same construction as (2.4) in section 2.2, we see that $W_{\epsilon_0,j}$ admits a local holomorphic chart $\tau_j : W_{\epsilon_0,j} \to \mathbb{C}^n$ that glues biholomorphically on the overlap with one another. Away from E_g, $W \backslash E_g$ is biholomorphic to $(M \times B_{\epsilon_0}) \backslash Z$.

iv) Since Z is a compact submanifold of $M \times B_{\epsilon_0}$, blowing up Z in $M \times B_{\epsilon_0}$ and applying constructions similar to lemma 2.30, we get a singular Hermitian metric h_{E_g} on $O(-E_g)$ over W as the following.

Suppose W_{ϵ_0} is covered by a finite collection of neighborhoods $\{W_\alpha\}_\alpha$ in W. Choose a family of cut-off functions χ_{W_α} with respect to $\overline{\pi_1^{-1}(T)}$ and W_{ϵ_0} such that $\chi_{W_\alpha} \in C^\infty(W_{\epsilon_0})$, $0 \leqslant \chi_{W_\alpha} \leqslant 1$ and

$$\chi_W := \sum_\alpha \chi_{W_\alpha} = 1 \text{ on } \overline{\pi_1^{-1}(T)}$$

Let $T_\alpha := \pi(W_\alpha)$ and thus $T \subset\subset \cup_\alpha T_\alpha$. Moreover, each W_α is covered by n open subsets

$$W_{\alpha,j} = \{(z,\xi,[t]) \in T_\alpha \times \mathbb{P}^{n-1} : t_j \neq 0 \text{ or } g_j(z,\xi) \neq 0\} \cap W_\alpha$$

For a section f in $O(-E_g)$, since $f_{|E_g} = 0$, we may assume that locally $f = g_j \cdot \tilde{f}$ in the neighborhood $W_{\alpha,j}$ for some holomorphic function $\tilde{f}$.

Define $|f|^2_{h_{E_g}}$ on $\tilde{W}_{\alpha,j}$ as

$$=\frac{|f|^2}{\sum\limits_{\alpha}\chi_{W_\alpha}\{\sum\limits_{k=1}^{n}|g_k|^{2\beta_k}(\sum\limits_{i=1}^{n}|g_i|^2)^{1-\beta_k}\}+1-\chi_W} \tag{3.4}$$

$$=\frac{|g_j\cdot\tilde{f}|^2}{\sum\limits_{\alpha}\chi_{W_\alpha}\{|g_j|^2\sum\limits_{k=1}^{n}|\frac{g_k}{g_j}|^{2\beta_k}(\sum\limits_{i=1}^{n}|\frac{g_i}{g_j}|^2)^{1-\beta_k}\}+1-\chi_W} \quad \text{on } W_{\alpha,j}\backslash E_g \tag{3.5}$$

$$=\frac{|\tilde{f}|^2}{\sum\limits_{\alpha}\chi_{W_\alpha}\{\sum\limits_{k=1}^{n}|\frac{g_k}{g_j}|^{2\beta_k}(\sum\limits_{i=1}^{n}|\frac{g_i}{g_j}|^2)^{1-\beta_k}\}+1-\chi_W} \quad \text{on } W_{\alpha,j}\backslash E_g \tag{3.6}$$

$$=\frac{|\tilde{f}|^2}{\sum\limits_{\alpha}\chi_{W_\alpha}\{\sum\limits_{k=1}^{n}|\frac{t_k}{t_j}|^{2\beta_k}(\sum\limits_{i=1}^{n}|\frac{t_i}{t_j}|^2)^{1-\beta_k}\}+1-\chi_W} \quad \text{on } W_{\alpha,j} \tag{3.7}$$

We see from (3.4) that h_{E_g} defined in this way glues well for all α and j. It is a smooth(resp. singular) Hermitian metric on $O(-E_g)$ if and only if all $\beta_i = 1$(resp. some $\beta_i < 1$). With h_{E_g}, we can define a Kähler form(resp. positive closed singular (1,1) form) on W later in step vi).

v) Define X to be a subset of W by setting $\xi_1 = \cdots = \xi_n = 0$, and locally X is defined by

$$X_{\epsilon_0} := \{(z, 0, [t]) \in U_{\epsilon_0} \times B_{\frac{\epsilon_0}{2}} \times \mathbb{P}^{n-1} : t_i\, f_j(z) = t_j\, f_i(z)\}$$

which is an analytic subspace of the smooth manifold W. Note that X might not be smooth in general, for which we apply Hironaka's theorem with iterated blow ups later in next lemma. Now we see that this defines a map $\pi_1 : X \to M$ which is a biholomorphism between $X\backslash X_0$ and $M\backslash\{p\}$, where $X_0 = {\pi_1}^{-1}(p)$ is biholomorphically equivalent to $\mathbb{P}^{n-1}$.

Moreover, near X_0, X is covered by its n open subsets

$$X_{\epsilon_0,j} = \{(z,0,[t]) \in X_{\epsilon_0} : t_j \neq 0 \text{ or } f_j(z) \neq 0\}$$

By letting $U_1 := \mathrm{proj}_1(T)$, we have a small open neighborhood U_1 near p on M satisfying $U_1 \subset\subset U_0$. Choose any cut-off function $\psi \in C^\infty(U)$ such that

$$\psi \circ \ \mathrm{proj}_1 \circ \pi_1 = \chi_W : W_{\epsilon_0} \longrightarrow \mathbb{R}$$

then clearly

$$\psi_{|_{\overline{U_1}}} \equiv 1 \text{ and } \psi_{|_{U \setminus U_0}} \equiv 0$$

Indeed, consider the projection map $\mathrm{proj}_1 : U_0 \times B_{\epsilon_0} \to U_0$ and the blow up map $\pi_1 : W_{\epsilon_0} \to U_0 \times B_{\epsilon_0}$, the image of $\mathrm{proj}_1 \circ \pi_1$ lies in U_0. The composition of proj_1 and ψ gives

$$\chi_T := \psi \circ \mathrm{proj}_1 : \pi_1(W_{\epsilon_0}) \longrightarrow U_0 \longrightarrow \mathbb{R}$$

This defines a cut-off function χ_T in $U_0 \times B_{\epsilon_0}$ near T, such that $\chi_T \equiv 1$ on $\overline{T}$.

vi) We wish to define a positive closed singular (1,1) form on X in the sense of currents. We do so by first constructing a (1,1) form on W, the ambient manifold where X lies in.

Given the Kähler metric ω on M, we extend it naturally to a Kähler metric on $M \times B_{\epsilon_0}$, denoted as $\omega' := \omega + \omega_0$, where ω_0 is the standard

Euclidean metric on B_{ϵ_0}. Choose the cut-off functions $\chi_{_W}$, $\chi_{_T}$ defined in step iv) and v), we define a closed singular (1,1) form ω''_δ on $W\backslash E_g$,

$$\omega''_\delta := {\pi_1}^*\omega' + \delta\frac{i}{2}\partial\bar{\partial}\{\chi_{_W}\log\sum_{j=1}^{n}|f_j(z)-\xi_j|^{2\beta_j}(\sum_{i=1}^{n}|f_i(z)-\xi_i|^2)^{1-\beta_j}+1-\chi_{_W}\} \tag{3.8}$$

Thus on $(M\times B_{\epsilon_0})\backslash Z$, which is biholomorphically equivalent to $W\backslash E_g$,

$$\omega''_\delta = \omega' + \delta\frac{i}{2}\partial\bar{\partial}\{\chi_{_T}\log\sum_{j=1}^{n}|f_j(z)-\xi_j|^{2\beta_j}(\sum_{i=1}^{n}|f_i(z)-\xi_i|^2)^{1-\beta_j}+1-\chi_{_T}\} \tag{3.9}$$

On E_g, we define the form ω''_δ as

$$\omega''_\delta := {\pi_1}^*\omega' + \delta\frac{i}{2}\partial\bar{\partial}\{\log\sum_{j=1}^{n}|t_j|^{2\beta_j}(\sum_{i=1}^{n}|t_i|^2)^{1-\beta_j}\} \tag{3.10}$$

In step vii) we will show that the ω''_δ defined in (3.8)(3.10), is strictly positive closed (1,1) form with bounded potential function in the sense of currents.

But first we need to show that the two definitions are coherent on the overlap of a neighborhood away from E_g and some open set contained in $\chi_{_W}=1$ near E_g.

To see this, first notice that on $W\backslash E_g$, $g_j(z,\xi)=f_j(z)-\xi_j\neq 0$ for some $1\leqslant j\leqslant n$. Without loss of generality, we may assume that $g_1(z,\xi)=f_1(z)-\xi_1\neq 0$. Then we must have $t_1\neq 0$ and $\frac{g_i}{g_1}=\frac{t_i}{t_1}$, for every $1\leqslant i\leqslant n$.

Hence on the overlap we have,

$$
\begin{aligned}
\omega''_\delta &= \pi_1^*\omega' + \delta\frac{i}{2}\partial\bar\partial\{\chi_W \log\sum_{j=1}^n |f_j(z)-\xi_j|^{2\beta_j}(\sum_{i=1}^n |f_i(z)-\xi_i|^2)^{1-\beta_j} + 1 - \chi_W\}\\
&= \pi_1^*\omega' + \delta\frac{i}{2}\partial\bar\partial\{\log\sum_{j=1}^n |f_j(z)-\xi_j|^{2\beta_j}(\sum_{i=1}^n |f_i(z)-\xi_i|^2)^{1-\beta_j}\}\\
&= \pi_1^*\omega' + \delta\frac{i}{2}\partial\bar\partial\{\log\sum_{j=1}^n |g_j|^{2\beta_j}(\sum_{i=1}^n |g_i|^2)^{1-\beta_j}\}\\
&= \pi_1^*\omega' + \delta\frac{i}{2}\partial\bar\partial\{\log\sum_{j=1}^n \frac{|g_j|^{2\beta_j}}{|g_1|^{2\beta_j}}(\sum_{i=1}^n \frac{|g_i|^2}{|g_1|^2})^{1-\beta_j}\} + \delta\frac{i}{2}\partial\bar\partial\log|g_1|^2\\
&= \pi_1^*\omega' + \delta\frac{i}{2}\partial\bar\partial\{\log\sum_{j=1}^n \frac{|g_j|^{2\beta_j}}{|g_1|^{2\beta_j}}(\sum_{i=1}^n \frac{|g_i|^2}{|g_1|^2})^{1-\beta_j}\}\\
&= \pi_1^*\omega' + \delta\frac{i}{2}\partial\bar\partial\{\log\sum_{j=1}^n \frac{|t_j|^{2\beta_j}}{|t_1|^{2\beta_j}}(\sum_{i=1}^n \frac{|t_i|^2}{|t_1|^2})^{1-\beta_j}\}\\
&= \pi_1^*\omega' + \delta\frac{i}{2}\partial\bar\partial\{\log\sum_{j=1}^n \frac{|t_j|^{2\beta_j}}{|t_1|^{2\beta_j}}(\sum_{i=1}^n \frac{|t_i|^2}{|t_1|^2})^{1-\beta_j}\} + \delta\frac{i}{2}\partial\bar\partial\log|t_1|^2\\
&= \pi_1^*\omega' + \delta\frac{i}{2}\partial\bar\partial\{\log\sum_{j=1}^n |t_j|^{2\beta_j}(\sum_{i=1}^n |t_i|^2)^{1-\beta_j}\}
\end{aligned}
$$

vii) To show $\omega''_\delta > 0$ on W, first notice that $\pi_1^*\omega' > 0$ on $W\backslash E_g \simeq (B\times M)\backslash Z$ as it is the pull back of ω' by a biholomorphic map. So by compactness of Z we can let $\delta > 0$ to be sufficiently small to make $\omega''_\delta > 0$.

On E_g, we only have $\pi_1^*\omega' \geqslant 0$ and in order to show $\omega''_\delta > 0$ we need to prove that the (1,1) form in (3.10) denoted as

$$\Omega(\beta, t) = \frac{i}{2}\partial\bar\partial\{\log\sum_{j=1}^n |t_j|^{2\beta_j}(\sum_{i=1}^n |t_i|^2)^{1-\beta_j}\}$$

is strictly positive.

If all $\beta_i = 1$, $\Omega(\beta, t) = \omega_{FS}$, the Fubini-Study metric over the projective space $\mathbb{P}^{n-1}$ and hence is strictly positive. Since Z is compact, we can get strict positivity of ω''_δ for any arbitrary small constant δ. In fact, ω''_δ is a smooth Kähler metric on W, see [PS12].

If all $\beta_i < 1$, choose a constant $C_1 = \max_{1 \leqslant i \leqslant n}\{\beta_i\} < 1$,

$$\begin{aligned}
\Omega(\beta, t) &= \frac{i}{2}\partial\bar{\partial}\{\log \sum_{j=1}^{n} |t_j|^{2\beta_j} (\sum_{i=1}^{n} |t_i|^2)^{1-\beta_j}\} \\
&= \frac{i}{2}\partial\bar{\partial}\{\log \sum_{j=1}^{n} |t_j|^{2\beta_j} (\sum_{i=1}^{n} |t_i|^2)^{1-C_1+C_1-\beta_j}\} \\
&= \frac{i}{2}\partial\bar{\partial}\{\log \sum_{j=1}^{n} |t_j|^{2\beta_j} (\sum_{i=1}^{n} |t_i|^2)^{C_1-\beta_j}\} + \frac{i}{2}\partial\bar{\partial}\log (\sum_{i=1}^{n} |t_i|^2)^{1-C_1} \\
&= \frac{i}{2}\partial\bar{\partial}\{\log \sum_{j=1}^{n} |t_j|^{2\beta_j} (\sum_{i=1}^{n} |t_i|^2)^{C_1-\beta_j}\} + (1 - C_1)\,\omega_{FS}
\end{aligned}$$

We show that the first term in the above line is semi-positive. Let

$$\begin{aligned}
u_j(t, \beta_j) &= \log\{|t_j|^{2\beta_j} (\sum_{i=1}^{n} |t_i|^2)^{C_1-\beta_j}\} \\
&= \beta_j \log |t_j|^2 + (C_1 - \beta_j) \log \sum_{i=1}^{n} |t_i|^2
\end{aligned}$$

Then $\partial\bar{\partial}u_j(t, \beta_j) \geqslant 0$ and u_j is plurisubharmonic. Now apply [Dem, Theorem 5.6] to see that

$$\log \sum_{j=1}^{n} |t_j|^{2\beta_j} (\sum_{i=1}^{n} |t_i|^2)^{C_1-\beta_j} = \log(e^{u_1} + \cdots + e^{u_n}) \tag{3.11}$$

is plurisubharmonic. Therefore $\Omega(\beta, t) \geqslant (1 - C_1)\,\omega_{FS} > 0$.

For general β values, where some but not all $\beta_i = 1$, we can still go through the same steps as above by letting

$$u_j(t, \beta_j) = \log\{|t_j|^{2\beta_j}(\sum_{i=1}^{n} |t_i|^2)^{1-\beta_j}\} \tag{3.12}$$

$$= \beta_j \log |t_j|^2 + (1 - \beta_j) \log \sum_{i=1}^{n} |t_i|^2 \tag{3.13}$$

Clearly $i\partial\bar{\partial}u_j \geqslant 0$ for each $1 \leqslant j \leqslant n$ and thus

$$\Omega(\beta, t) = \frac{i}{2}\partial\bar{\partial}\log(e^{u_1} + \cdots + e^{u_n}) \geqslant 0$$

We have semi-positivity, and it still remains to show strict-positivity.

Define Θ as

$$\begin{aligned}\Theta(u_1, \cdots, u_n) &= \log(e^{u_1} + \cdots + e^{u_n}) \\ &= \log \sum_{j=1}^{n} |t_j|^{2\beta_j}(\sum_{i=1}^{n} |t_i|^2)^{1-\beta_j}\end{aligned}$$

Clearly we have that $D_j\Theta > 0$ for every $1 \leqslant j \leqslant n$. As there exists some $\beta_{i_0} < 1$, we have

$$\begin{aligned}i\partial\bar{\partial}u_{i_0} &\geqslant (1 - \beta_{i_0})i\partial\bar{\partial}\log \sum_{i=1}^{n} |t_i|^2 \\ &= (1 - \beta_{i_0})\omega_{FS|_{\mathbb{P}^{n-1}}}\end{aligned}$$

we have that u_{i_0} is strictly plurisubharmonic.

To show Θ is strictly plurisubharmonic, it suffices to show that locally $\xi^p\overline{\xi^q}\partial_p\partial_{\overline{q}}\Theta \geqslant \lambda \cdot \xi\overline{\xi}$ for some small positive constant λ and any vector $\xi \in \mathbb{C}^n$. This can be seen from

$$
\begin{aligned}
&\xi^p\overline{\xi^q}\partial_p\partial_{\overline{q}}\Theta(u_1, \cdots, u_n) \\
=&\xi^p\overline{\xi^q}\partial_p(\sum_{j=1}^{n} D_j\Theta\partial_{\overline{q}}u_j) \\
=&\xi^p\overline{\xi^q}(\sum_{j=1}^{n} D_j\Theta\partial_p\partial_{\overline{q}}u_j \quad + \quad \sum_{i=1}^{n}\sum_{j=1}^{n}(D_iD_j\Theta)\partial_p u_i\partial_{\overline{q}}u_j\,) \\
=&\sum_{j=1}^{n} D_j\Theta(\,\xi^p\overline{\xi^q}\partial_p\partial_{\overline{q}}u_j\,) \quad + \quad \sum_{i=1}^{n}\sum_{j=1}^{n}(D_iD_j\Theta)(\,\xi^p\partial_p u_i\,)(\,\overline{\xi^q}\partial_{\overline{q}}u_j\,) \\
\geqslant&\sum_{j=1}^{n} D_j\Theta(\,\xi^p\overline{\xi^q}\partial_p\partial_{\overline{q}}u_j\,) \\
\geqslant&\lambda_0 \cdot \xi\overline{\xi}
\end{aligned}
$$

where λ_0 is a local constant depending on the smallest eigenvalue of $(\,\partial_k\bar{\partial}_l u_{i_0}\,)$. And $\lambda_0 > 0$, as u_{i_0} is strictly plurisubharmonic.

viii) Now we have a singular (1,1) form on $M\backslash\{p\}$ as

$$\omega_\delta := \omega + \delta\frac{i}{2}\partial\bar{\partial}\{\psi\log\sum_{j=1}^{n}|f_j(z)|^{2\beta_j}(\sum_{i=1}^{n}|f_i(z)|^2)^{1-\beta_j} + 1 - \psi\} \quad (3.14)$$

On W, we have constructed a strictly positive singular (1,1) form

$$\omega''_\delta = \pi_1^*\omega' + \delta\frac{i}{2}\partial\bar{\partial}\{\chi_W\log\sum_{j=1}^{n}|f_j(z)-\xi_j|^{2\beta_j}(\sum_{i=1}^{n}|f_i(z)-\xi_i|^2)^{1-\beta_j} + 1 - \chi_W\}$$

We get a positive singular (1,1) form ω'_δ by restricting ω''_δ from W to X

and letting $\xi_1 = \cdots = \xi_n = 0$.

$$\omega'_\delta := \omega''_{\delta\,|_{\xi=0}} \tag{3.15}$$

$$= \pi_1^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{j=1}^{n} |f_j(z)|^{2\beta_j} (\sum_{i=1}^{n} |f_i(z)|^2)^{1-\beta_j} + 1 - \chi\} \tag{3.16}$$

where $\chi := \chi_{W\,|_{\xi=0}}$ and notice that as a result of the restriction,

$$\pi_1{}^*\omega' = \pi_1{}^*(\,\omega + \omega_0\,)$$

changed into $\pi_1{}^*\omega$. Moreover, we have that ω'_δ is strictly positive on the entire X. On $X\backslash X_0$, ω'_δ is the same as $\pi_1{}^*\omega_\delta$, the pull back of ω_δ from $M\backslash\{p\}$.

To summarize, $\pi_1^*\omega_\delta$, the pull back of ω_δ from $M\backslash\{p\}$ to $X\backslash X_0$, extends to X. This can be seen from that fact that $\psi \circ \pi_1 = \chi_W = 1$ near X_0 in a neighborhood of the exceptional divisor E_g in W and that

$$\begin{aligned} \omega'_{\delta|_{X_0}} =& \pi_1{}^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi_W \log \sum_{j=1}^{n} |t_j|^{2\beta_j} (\sum_{i=1}^{n} |t_i|^2)^{1-\beta_j} + 1 - \chi_W\} \\ =& \pi_1{}^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi_W \log \sum_{j=1}^{n} |f_j(z)|^{2\beta_j} (\sum_{i=1}^{n} |f_i(z)|^2)^{1-\beta_j} + 1 - \chi_W\} \end{aligned}$$

The proof of lemma 3.2 is complete.

$\square$

3.3 Singular (1,1) form on iterated blow up space X'

The X constructed in a local way in lemma 3.2 is an analytic subspace of W and X is in general not smooth. To get a smooth manifold, we apply the iterated blow up technics[PS12] given by a deep theorem of Hironaka(See[H]) to resolve the singularities $X_{sing} \subset X$ in the ambient space W. Their process is to blow up W finite times and resolve the singularities of X. From there we have an iterated blow up space W', a smooth manifold X' which lies inside the ambient space W' and a positive closed (1,1) current Ω' on X'.

Lemma 3.3 (Hironaka's Theorem on iterated blow-ups, see [H][PS12])**.**
Let W be a complex manifold and $X \subset W$ be a complex analytic space. Then there exists an iterated blow up space $\pi_2 : W' \to W$ with the exceptional divisor $E \subset W'$ that resolves the singularities of X in the following way:
Let

$$X' = \overline{\pi_2^{-1}(X)\backslash E} \subset W' \tag{3.17}$$

Then X' is a smooth manifold and $\pi_2 : X' \to X$, the restriction of the map π_2 on X', is surjective. Moreover, we have a divisor E' with normal crossings in X' as

$$E' = E \cap X' = \pi^{-1}(X_{sing}) \tag{3.18}$$

and an isomorphism on $X'\backslash E'$

$$\pi_2 : X'\backslash E' \to X_{reg} \tag{3.19}$$

From lemma 3.3, the singularities in $X_{sing} \subset X_0$ are resolved with the blow up map $\pi_2 : W' \longrightarrow W$ and the construction of $E' \subset X' \subset W'$.

Now we wish to give a closed positive singular (1,1) form Ω' on X'.

Lemma 3.4 (Existence of $h_{E'}$ and Ω' on X')**.**
Same as in lemma 3.2, fix a singular (1,1) form ω'_δ on X,

$$\omega'_\delta = \pi_1^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{j=1}^{n} |f_j(z)|^{2\beta_j}(\sum_{i=1}^{n} |f_i(z)|^2)^{1-\beta_j} + 1 - \chi\}$$

Then there exists a closed strictly positive singular (1,1) form Ω' on X',

$$\Omega' := \pi_2^*(\omega'_\delta) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} > 0 \tag{3.20}$$

for some small constant ϵ and some smooth metric $h_{E'}$ on $O(-E')$, where E' is some effective divisor on X'.

Proof.

i) From lemma 3.2, on W we constructed a positive closed singular (1,1) form ω''_δ, together with $\pi_1^*(\omega_\delta)$ as its restriction to X. Apply lemma 3.3 to get $E' \subset X' \subset W'$ and $\pi_2 : X' \to X$ that resolves the singularities X_{sing} which lies in $X_0 \subset E_g \subset W$. Here E_g is the compact smooth submanifold of W defined in lemma 3.2.

ii) Choose a finite collection of small neighborhoods $\{U_\alpha\}_\alpha$ that covers E_g. Since ω''_δ is strictly positive and locally in U_α we have

$$\omega''_\delta = i\partial\bar{\partial}\phi_\alpha$$

where ϕ_α is a strictly plurisubharmonic function in U_α. This means that locally in U_α, there exists a small positive constant λ_α such that for all $\xi \in \mathbb{C}^n$,

$$i\partial_k\partial_{\bar{j}}\phi_\alpha\, \xi^k\overline{\xi^j} \geqslant \lambda_\alpha\, |\xi|^2$$

Fix a smooth Kähler form $\omega_{\delta,W}$ on W as in (3.23). Then locally in U_α, $\omega_{\delta,W} = g_{k\bar{j}}(x)\, dz^k \wedge \overline{dz^j}$. Let $\Lambda(x)$ be the largest eigen value of $(g_{i\bar{j}}(x))$ and let

$$\Lambda := \max_\alpha\{\sup_{x\in U_\alpha} \Lambda(x)\}$$

and fix any large positive constant A, such that $\frac{1}{A} \leqslant \frac{\min_\alpha \lambda_\alpha}{\Lambda}$. We see that $\omega''_\delta \geqslant \frac{\omega_{\delta,W}}{A}$ in the neighborhood $\cup_\alpha U_\alpha$ of E_g.

iii) Now apply [PS12, lemma 7] to the Kähler form $\frac{\omega_{\delta,W}}{A}$. It follows that there exists an effective divisor E_1 on W' supported on E(i.e. locally $E_1 = \sum\limits_\nu m_\nu E_\nu$ for some integers m_ν and some divisors E_ν in E), together with a smooth metric h_{E_1} on $O(-E_1)$ such that

$$\pi_2^*(\frac{\omega_{\delta,W}}{A}) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E_1} > 0$$

is a Kähler form on W' for ϵ sufficient small.

Define

$$\Omega_{W'} := \pi_2^*(\omega''_\delta) - \epsilon \frac{i}{2}\partial\bar{\partial}\log h_{E_1}$$

Then we claim that $\Omega_{W'}$ is a strictly positive singular Hermitian (1,1) form on W'. Indeed, the strict positivity can be seen from that $\pi_2^*(\omega''_\delta)$ is semi-positive on W' and strictly positive away from the exceptional divisor E. And near E, in the neighborhood ${\pi_2}^{-1}(\cup_\alpha U_\alpha)$, we have that $\pi_2^*(\omega''_\delta) \geqslant \frac{\pi_2^*(\omega_{\delta,W})}{A}$ and thus

$$\pi_2^*(\omega''_\delta) - \epsilon \frac{i}{2}\partial\bar{\partial}\log h_{E_1} \geqslant \frac{\pi_2^*(\omega_{\delta,W})}{A} - \epsilon \frac{i}{2}\partial\bar{\partial}\log h_{E_1} > 0$$

iv) Since we have that $X' \subset W'$ is a resolution of singularity of $X \subset W$ with the exceptional divisor $E' \subset X'$. Restrict the divisor E_1 to X' and get an effective divisor E_2 supported in $E' \subset X'$. Restrict the smooth metric h_{E_1} from the line bundle $O(-E_1)$ to $O(-E_2)$ and get a smooth metric h_{E_2} on $O(-E_2)$ over X'. Then we have a strictly positive singular (1,1) form Ω' on X' as

$$\Omega' := \pi_2^*(\omega'_\delta) - \epsilon \frac{i}{2}\partial\bar{\partial}\log h_{E_2} > 0 \tag{3.21}$$

Here the positivity can be seen from the fact that Ω' is the restriction of $\Omega_{W'}$ from W' to its submanifold X' and that $\Omega_{W'}$ defined in step iii) is strictly positive.

□

Remark 3.5.

Note that for the case where $\beta_1 = \cdots \beta_n = 1$, Phong-Sturm showed in [PS12] that the following Kähler forms

$$\omega_{\delta,X} :=\pi_1^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{i=1}^{n} |f_i(z)|^2 + 1 - \chi\} \tag{3.22}$$

$$\omega_{\delta,W} :=\pi_1^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi_W \log \sum_{i=1}^{n} |f_i(z) - \xi_i|^2 + 1 - \chi_W\} \tag{3.23}$$

get pulled back to $\pi_2^*(\omega_{\delta,X})$ and $\pi_2^*(\omega_{\delta,W})$, which are smooth semi-positive (1,1) forms on X' and W'. Thus by defining

$$\Omega_0 := \pi_2^*(\pi_1^*(\omega) + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{i=1}^{n} |f_i|^2 + 1 - \chi\}) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E_2} > 0$$

we have a smooth Kähler form Ω_0 on X'. So we can see that X' is a smooth Kähler manifold with Ω_0, which happens to be same as Ω' for this special case.

As a generalization (for $\beta_1 \leqslant 1, \cdots, \beta_n \leqslant 1$) to their result, Ω' defined in (3.21) is a family of strictly positive closed (1,1) currents $\{\Omega'_\beta\}_{\beta_i \leqslant 1}$ on X'.

To summarize this section, we have the following lemma.

Lemma 3.6. *Given (M, p, f_i) the data of a compact Kähler manifold (M, ω) with smooth boundary ∂M, an interior point p, and local holomorphic functions f_i, $1 \leqslant i \leqslant n$ with p as their only common zero.*

Then there exists a compact complex manifold $X' = X'(M, p, f_i)$ with Kähler form Ω_0 and a holomorphic map $\pi : X' \longrightarrow M$, sending $\partial X' \longrightarrow \partial M$ with the following properties:

a) There is a closed, strictly positive singular (1,1) form Ω' on X', an effective divisor E' and an $\epsilon > 0$ sufficiently small such that

$$\Omega' := \pi^*(\omega_\delta) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} > 0 \text{ for some smooth metric } h_{E'} \text{ on } O(-E')$$

b) The restriction $\pi|_{X'\backslash E'}$ defines a surjective holomorphic map $\pi : X'\backslash E' \to M\backslash\{p\}$ and

$$\begin{aligned}\pi_*\Omega' =&\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\psi\log\sum_{j=1}^{n}|f_j(z)|^{2\beta_j}(\sum_{i=1}^{n}|f_i(z)|^2)^{1-\beta_j}\\ &+1-\psi\} - \epsilon\frac{i}{2}\partial\bar{\partial}\log(h_{E'}\circ\pi^{-1})\end{aligned}$$

where $\psi(z)$ is some cut-off function which is 1 in a neighborhood of p and compactly supported in a slightly larger neighborhood.

Proof. i) Fix a neighborhood U, and some $U_0 \subset\subset U$, take the cut off function $\psi(z)$ supported in U_0 same as in lemma 3.2. We have thus a closed positive singular (1,1) form ω_δ on $M\backslash\{p\}$ defined as

$$\omega_\delta = \omega + \delta\frac{i}{2}\partial\bar{\partial}\{\psi\log\sum_{j=1}^{n}|f_j(z)|^{2\beta_j}(\sum_{i=1}^{n}|f_i(z)|^2)^{1-\beta_j} + 1 - \psi\}$$

Apply lemma 3.2 and we have the analytic subspace

$$X = X(M, p, f_1, \cdots, f_n)$$

in the ambient space W and a biholomorphic map $\pi_1 : X\backslash X_0 \to M\backslash\{p\}$

such that $\pi_1^*(\omega_\delta)$ extends to a singular (1,1) form ω'_δ on X.

ii) Since $X \subset W$ is only an analytic subspace with $X_0 \subset E_g \subset W$, where E_g is the exceptional divisor in $W = \mathrm{BL}_Z(M \times B_{\epsilon_0})$ over $Z = \{g_1(z,\xi) = \cdots = g_n(z,\xi) = 0\} \subset M \times B_{\epsilon_0}$ and

$$X_0 = E_g \cap \{\xi_1 = \cdots \xi_n = 0\}$$

We see that $X_{sing} \subset X_0$ and $X \backslash X_0 \subset X_{reg}$.

iii) Then we apply lemma 3.3 (Hironaka's theorem) to get an iterated blow up space $\pi_2 : W' \to W$, which is a smooth manifold with the exceptional divisor $E \subset W'$ and a smooth submanifold $X' \subset W'$ such that the restricted map

$$\pi_{2|_{X'}} : X' \to X$$

is surjective and that

$$\pi_{2|_{X' \backslash E'}} : X' \backslash E' \to X_{reg}$$

is biholomorphic. Here $E' = E \cap X' = {\pi_2}^{-1}(X_{sing}) \cap X'$.

iv) Apply lemma 3.4 to pull back ${\pi_1}^*(\omega_\delta)$ via the surjective blow up map

$$\pi_{2|_{X'}} : X' \to X$$

which resolves the singularity X_{sing} with the exceptional divisor E'. We get a smooth metric $h_{E'}$ on the line bundle $O(-E')$ over X', such that for sufficient small $\epsilon > 0$,

$$\pi_2{}^*(\pi_1^*(\omega_\delta)) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} > 0$$

Set $\Omega' = \pi_2{}^*(\pi_1^*(\omega_\delta)) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'}$ and this proves a).

v) Now we prove b). First notice that since $X_{reg} = X\backslash X_{sing} \supset X\backslash X_0$, so the map $\pi_1 : X_{reg} \to M\backslash\{p\}$ is surjective. Taking its composition with the biholomorphism

$$\pi_2 : X'\backslash E' \to X_{reg}$$

defines a surjective holomorphic map

$$\pi = \pi_1 \circ \pi_2 : X'\backslash E' \to M\backslash\{p\}$$

Clearly π sends $\partial X'$ to ∂M and π is a biholomorphism between $M\backslash\{p\}$ and its inverse image, which is

$$\pi^{-1}(M\backslash\{p\}) = \pi_2^{-1}(X\backslash X_0) \subset X'\backslash E'$$

So we can push forward Ω' to $M\backslash\{p\}$ and get

$$\begin{aligned}\pi_*\Omega' =& \omega + \delta\frac{i}{2}\partial\bar{\partial}\{\psi\log\sum_{j=1}^{n}|f_j(z)|^{2\beta_j}(\sum_{i=1}^{n}|f_i(z)|^2)^{1-\beta_j} \\ &+ 1 - \psi\} - \epsilon\frac{i}{2}\partial\bar{\partial}\log(h_{E'}\circ\pi^{-1})\end{aligned}$$

□

3.4 A solution $\phi \in PSH(X', \Omega')$

Through the iterated blow up procedures, we have $E \subset W'$ and $E' \subset X'$ with Ω', a strictly positive singular Hermitian form on X'. And as shown in remark 3.5 (See also [PS12]), there is a Kähler form on X', defined as

$$\Omega_0 := \pi_2^*(\pi_1^*(\omega) + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{i=1}^{n} |f_i|^2 + 1 - \chi\}) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E_2} > 0 \quad (3.24)$$

Note that on $X' \backslash E'$,

$$\begin{aligned}\Omega' = \pi_2^*(\pi_1^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{j=1}^{n} |f_j(z)|^{2\beta_j} (\sum_{i=1}^{n} |f_i(z)|^2)^{1-\beta_j} \\ + 1 - \chi\}) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} \\ = \pi^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi_W \log \sum_{j=1}^{n} |f_j \circ \pi_2|^{2\beta_j} (\sum_{i=1}^{n} |f_i \circ \pi_2|^2)^{1-\beta_j} \\ + 1 - \chi_W\}) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'}\end{aligned}$$

and that on E',

$$\Omega' = \pi^*\omega + \pi_2{}^*(\delta\frac{i}{2}\partial\bar{\partial}\{\chi_W \log \sum_{j=1}^{n} |t_j|^{2\beta_j} (\sum_{i=1}^{n} |t_i|^2)^{1-\beta_j} + 1 - \chi_W\}) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'}$$

where $\chi := \chi_W \circ \pi_2$ is a cut-off function in a neighborhood of $E' \subset X'$. Then locally, $\Omega' = i\partial\bar{\partial}\theta$ where θ is a bounded continuous strictly plurisubharmonic potential function.

We wish to construct a solution to the following degenerate Monge-Ampère equation with respect to Ω' on X'.

Lemma 3.7. *Let (X', Ω_0) be the compact Kähler manifold with smooth boundary and let the strictly positive singular Hermitian (1,1) form Ω' on X' be defined in lemma 3.6.*
Then there exists a unique $\phi \in PSH(X', \Omega') \cap L^\infty(X')$ such that

$$(\Omega' + \frac{i}{2}\partial\bar{\partial}\phi)^n = 0 \ \textit{on } X' \tag{3.25}$$

$$\phi = 0 \ \textit{on } \partial X' \tag{3.26}$$

Moreover, $\phi \in C^\alpha(K')$ for any compact subset K' of $X' \backslash E'$ and any constant $0 < \alpha < \min_i \{\beta_i\}$.

Proof. (*Existence, a first proof using [PS09][PSS12].*)

i) Let

$$\Omega_1 := \pi^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{i=1}^n |f_i \circ \pi_2|^2 + 1 - \chi\} \geqslant 0 \tag{3.27}$$

be a smooth semi-positive $(1,1)$ form on X'. From the above, we see that Ω_1 satisfies the following condition

$$\Omega_1 - \frac{i}{2}\partial\bar{\partial}\log h_{E'} = \Omega_0 > 0$$

By [PS09, Theorem 2](or [PSS12, Theorem 14]), there exists a unique $\phi_1 \in PSH(X', \Omega_1) \cap L^\infty(X')$ to the following

$$(\Omega_1 + \frac{i}{2}\partial\bar{\partial}\phi_1)^n = 0 \tag{3.28}$$

$$\Omega_1 + \frac{i}{2}\partial\bar{\partial}\phi_1 \geqslant 0 \tag{3.29}$$

$$\phi_{1|_{\partial X'}} = 0 \tag{3.30}$$

Moreover, $\phi_1 \in C^{\alpha_1}(X' \backslash E')$ for any $0 < \alpha_1 < 1$.

ii) On X', we see that

$$\begin{aligned}
\Omega' = \pi^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{j=1}^{n} |f_j \circ \pi_2|^{2\beta_j} (\sum_{i=1}^{n} |f_i \circ \pi_2|^2)^{1-\beta_j} \\
+ 1 - \chi\}\,) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} \\
= \pi^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{i=1}^{n} |f_i \circ \pi_2|^2 + 1 - \chi\} \\
+ \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{j=1}^{n} \frac{|f_j \circ \pi_2|^{2\beta_j}}{(\sum_{i=1}^{n} |f_i \circ \pi_2|^2)^{\beta_j}}\} - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} \\
= \Omega_1 + \frac{i}{2}\partial\bar{\partial}F
\end{aligned}$$

where we have let

$$\Omega_1 := \pi^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{i=1}^{n} |f_i \circ \pi_2|^2 + 1 - \chi\} \geqslant 0$$

be a smooth semi-positive (1,1) form on X' and

$$F := \delta\chi \log \sum_{j=1}^{n} \frac{|f_j \circ \pi_2|^{2\beta_j}}{(\sum_{i=1}^{n} |f_i \circ \pi_2|^2)^{\beta_j}} - \epsilon \log h_{E'}$$

Clearly we see that $F = 0$ on $\partial X'$, as χ and $\log h_{E'}$ both vanish away from a neighborhood of E'. Let $\phi := \phi_1 - F$, then ϕ is a solution to the degenerate Monge-Ampère equation (3.25). Indeed,

$$\begin{aligned} 0 =& (\Omega_1 + \frac{i}{2}\partial\bar{\partial}\phi_1)^n \\ =& (\Omega_1 + \frac{i}{2}\partial\bar{\partial}F + \frac{i}{2}\partial\bar{\partial}\phi)^n \\ =& (\Omega' + \frac{i}{2}\partial\bar{\partial}\phi)^n \end{aligned}$$

Since ϕ_1 and F both vanish at $\partial X'$, we have $\phi_{|_{\partial X'}} = 0$ and this proves (3.26).

iii) To show that ϕ is bounded, it suffices to show that F is bounded on X'. Since $h_{E'} > 0$ is smooth on X', which is compact, it follows that $\log h_{E'}$ is uniformly bounded. To see F is bounded, we notice that

$$F = \delta F_1 - \epsilon \log h_{E'}$$

where we let

$$F_1 := \chi \log \sum_{j=1}^{n} \frac{|f_j \circ \pi_2|^{2\beta_j}}{(\sum_{i=1}^{n} |f_i \circ \pi_2|^2)^{\beta_j}} \tag{3.31}$$

F_1 is uniformly bounded on X', as can be seen from that

$$\begin{aligned}
\sum_{i=1}^{n} |f_i \circ \pi_2|^2 \leqslant n \cdot \max_i\{|f_i \circ \pi_2|^2\} \quad \text{and that} \\
-C_1 \cdot \log n \leqslant \log \max_{j=1}^{n} \frac{|f_j \circ \pi_2|^{2\beta_j}}{(\sum_{i=1}^{n} |f_i \circ \pi_2|^2)^{\beta_j}} \\
\leqslant \log \sum_{j=1}^{n} \frac{|f_j \circ \pi_2|^{2\beta_j}}{(\sum_{i=1}^{n} |f_i \circ \pi_2|^2)^{\beta_j}} \\
\leqslant \log n
\end{aligned}$$

where the uniform constant $C_1 = \max_{1 \leqslant i \leqslant n}\{\beta_i\} > 0$. Thus we get that $F \in L^\infty(X')$ and that $\phi = \phi_1 - F \in L^\infty(X')$.

iv) Fix any compact subset K' in $X' \backslash E'$ and any constant $0 < \alpha < \min_{1 \leqslant j \leqslant n}\{\beta_j\}$. We have that $F_1 \in C^\alpha(K')$ and thus $F \in C^\alpha(K')$. Since $\phi_1 \in C^{\alpha_1}(X' \backslash E')$ for any $0 < \alpha_1 < 1$, it follows that $\phi = \phi_1 - F \in C^\alpha(K)$.

This completes the first proof of the existence part.

(*Existence, a second proof using Perron's envelope method.*)

We give a slightly more general proof, where essentially the boundary function is given by $\phi_1 \in PSH(X', \Omega_0)$, which is continuous near the boundary. We are allowed to apply the Perron envelope method with respect to Ω' and boundary condition ϕ_1. We see that $\phi = \phi_1$ away from a neighborhood of E' in X'. Locally in this neighborhood, fix a bounded plurisubharmonic function θ such that $\frac{i}{2}\partial\bar{\partial}\theta = \Omega'$. Then by [Wal], $\phi + \theta$ is upper semi-continuous and indeed ϕ lies in $PSH(X', \Omega')$.

I) Define $\omega_1 = \Omega'$ and

$$E(\omega_1, f) = \{\psi : \omega_1 + i\partial\bar{\partial}\psi \geqslant 0, {\psi^*}_{|\partial X'} \leqslant f_{|\partial X'}\}$$

to be the Perron's envelope of subsolutions with respect to ω_1 and any continuous function f defined near $\partial X'$. It is easy to see that $E(\omega_1, f)$ is not empty. Indeed, it contains constant functions $\psi = -C$ for any sufficiently large $C \geqslant 0$. Then consider the envelope with the boundary condition

$$E(\omega_1, \phi_1) = \{\psi : \omega_1 + i\partial\bar{\partial}\psi \geqslant 0, {\psi^*}_{|\partial X'} \leqslant \phi_1\}$$

II) Take the point-wise supreme for all $\psi \in E(\omega_1, \phi_1)$ and define

$$\phi = \sup_{\psi}\{\psi \in E(\omega_1, \phi_1)\} = \sup\{\psi : \omega_1 + i\partial\bar{\partial}\psi \geqslant 0, {\psi^*}_{|\partial X'} \leqslant \phi_1\} \quad (3.32)$$

Now $\phi \in PSH(X', \omega_1)$ is a globally defined function on X' and $\phi_{|\partial X'} = {\phi_1}_{|\partial X'} = 0$.

III) To show that the ϕ defined above is a solution to the degenerate Monge-Ampère equation

$$(\omega_1 + i\partial\bar{\partial}\phi)^n = 0 \quad \text{ on } X'$$

It suffices to show ϕ is maximal with respect to all subsolutions on any small neighborhood U.

IV) Pick any point p_1 and any small neighborhood U that contains p_1, then locally in U, $\omega_1 = i\partial\bar{\partial}\theta$ where θ is a bounded strictly plurisubharmonic function. We wish to show that $(\omega_1 + i\partial\bar{\partial}\phi)^n = i\partial\bar{\partial}(\theta+\phi)^n = 0$ on U, so it suffices to show that $\phi + \theta \in PSH(U)$ is maximal. Recall that u is maximal on U iff for any $v \in PSH(U)$ satisfying $v \leqslant u$ outside a compact subset K of U, we have $v \leqslant u$ in U. See in Blocki's book [Bl1].

V) Fix any function $v \in PSH(U)$ and compact subset K of U such that $v \leqslant \theta + \phi$ on $U \backslash K$. Define

$$\tilde{v} = \max(v, \theta + \phi)$$

be a plurisubharmonic function on U. Clearly we have $\tilde{v} = \theta + \phi$ on $U \backslash K$. Let

$$\tilde{\phi} = \tilde{v} - \theta \text{ on } U \tag{3.33}$$

and extend $\tilde{\phi}$ to the manifold X' by letting $\tilde{\phi} = \phi$ on $X' \backslash U$. Observe that outside U,

$$\omega_1 + i\partial\bar{\partial}\tilde{\phi} = \omega_1 + i\partial\bar{\partial}\phi \geqslant 0$$

and in U,

$$\omega_1 + i\partial\bar{\partial}\tilde{\phi} = i\partial\bar{\partial}\theta + i\partial\bar{\partial}(\tilde{v} - \theta) = i\partial\bar{\partial}\tilde{v} \geqslant 0$$

We obtain that $\tilde{\phi} \in PSH(X', \omega_1)$.

VI) From the above, we see that $\tilde{\phi}$ is in the envelope $E(\omega_1, \phi_1)$. And since ϕ

is defined in step II) to be the supremum of $E(\omega_1, \phi_1)$,

$$\begin{aligned} & \tilde{\phi} \leqslant \phi && \text{everywhere in } X' \\ \Longrightarrow\quad & \tilde{v} - \theta \leqslant \phi && \text{in } U \\ \Longrightarrow\quad & v \leqslant \tilde{v} \leqslant \theta + \phi && \text{in } U \end{aligned}$$

This proves that $\theta + \phi$ is maximal and therefore $(\omega_1 + i\partial\bar{\partial}\phi)^n = i\partial\bar{\partial}(\theta + \phi)^n = 0$ in U. Since U is any arbitrary small neighborhood on X', we conclude that $(\omega_1 + i\partial\bar{\partial}\phi)^n = 0$ on X'.

This completes the second proof of existence.

(*Uniqueness.*)

Fix any solution $\phi \in PSH(X', \Omega') \cap L^\infty(X')$ that satisfies (3.25) and (3.26) and let Ω_1 and F be as in (3.27) and (3.31). Notice that on X',

$$\Omega_1 + \frac{i}{2}\partial\bar{\partial}F + \frac{i}{2}\partial\bar{\partial}\phi = \Omega' + \frac{i}{2}\partial\bar{\partial}\phi \geqslant 0$$
$$(\Omega_1 + \frac{i}{2}\partial\bar{\partial}F + \frac{i}{2}\partial\bar{\partial}\phi)^n = (\Omega' + \frac{i}{2}\partial\bar{\partial}\phi)^n = 0$$

By letting $\phi_1 := \phi + F$, we have $\phi_1 \in PSH(X', \Omega_1)$. Besides, since $F \in L^\infty(X')$ and F vanishes on the boundary $\partial X'$, we have that $\phi_1 = \phi + F \in L^\infty(X')$ and that

$$\phi_{1_{|\partial X'}} = \phi_{|\partial X'} = 0$$

So ϕ_1 is a bounded solution to the following Dirichlet problem for the totally degenerate Monge-Ampère equation on X'

$$(\Omega_1 + \frac{i}{2}\partial\bar{\partial}\phi_1)^n = 0 \tag{3.34}$$

$$\Omega_1 + \frac{i}{2}\partial\bar{\partial}\phi_1 \geqslant 0 \tag{3.35}$$

$$\phi_{1|_{\partial X'}} = 0 \tag{3.36}$$

Moreover, since Ω_1 satisfies the following condition

$$\Omega_1 - \epsilon \log h_{E'} = \Omega_0 > 0 \text{ on } X'$$

By uniqueness part of [PS09, Theorem 2] (or [PSS12, Theorem 14]), $\phi_1 \in PSH(X', \Omega') \cap L^\infty(X')$ is unique. We get the uniqueness of ϕ, up to the choice of δ and $\epsilon \log h_{E'}$.

□

3.5 Proof of Theorem I

We rephrase Theorem 3.1 and reduce it to the singularities near a single interior point $p \in M$. We fix any positive real numbers $\beta_1, \cdots, \beta_n \leqslant 1$.

Theorem 3.8. *Let ω be a Kähler metric on compact complex manifold M of dimension $n \geqslant 2$ and assume $\partial M \neq \emptyset$ is smooth. Fix n holomorphic functions $\{f_j\}$ such that $\{f_j\}_{1 \leqslant j \leqslant n}$ are locally defined in a neighborhood of p, with p as their only common zero in this neighborhood.*

Then there exists a small constant $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, there *exists a unique solution* $G(z; p, f_1, \cdots, f_n) \in PSH(M, \omega)$ *to the following*

$$\begin{cases} G = 0 & \text{on } \partial M \quad (3.37) \\ (\omega + \frac{i}{2}\partial\bar{\partial}G)^n = 0 & \text{on } M\backslash\{p\} \quad (3.38) \\ G = \delta \log\{\sum_{j=1}^{n} |f_j(z)|^{2\beta_j} (\sum_{i=1}^{n} |f_i|^2)^{1-\beta_j}\} + \phi & \text{near } p \quad (3.39) \end{cases}$$

for some unique $\phi \in L^\infty(M)$ *that vanishes on the boundary. Moreover,* G *and* ϕ *lies in* $C^\alpha(K)$ *for any compact subset* $K \subset M\backslash\{p\}$ *and any constant* $0 < \alpha < \min_j\{\beta_j\}$*. The uniqueness is with respect to a given constant* δ *and a choice of cut-off function in a small neighborhood near* p*.*

Proof. (Existence)

Fix any constant $0 < \alpha < \min_j\{\beta_j\}$.

i) In lemma 3.3 we applied the iterated blow up map

$$\pi : W' \to M\backslash\{p\}$$

with the exceptional divisor E and an n dimensional smooth submanifold $X' \subset W'$ and an effective divisor E' supported in E and away from $\pi^{-1}(\partial M) = \partial X'$.

ii) From lemma 3.6, we have a strictly positive closed singular (1,1) form

Ω' defined on X' as

$$\begin{aligned}\Omega' &= \pi^*(\omega_\delta) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} > 0 \\ &= \pi_2^*(\pi_1^*(\omega_\delta)) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} > 0\end{aligned}$$

for some smooth metric $h_{E'}$ on $O(-E')$. Here

$$\omega_\delta = \omega + \delta\frac{i}{2}\partial\bar{\partial}\{\psi\log\sum_{j=1}^{n}|f_j(z)|^{2\beta_j}(\sum_{i=1}^{n}|f_i(z)|^2)^{1-\beta_j} + 1 - \psi\}$$

is the same as that in lemma 3.2.

iii) Now apply lemma 3.7, which showed that there exists a unique solution $\Phi \in PSH(X', \Omega') \cap L^\infty(X')$ to the degenerate Monge-Ampère equation

$$(\Omega' + \frac{i}{2}\partial\bar{\partial}\Phi)^n = 0 \text{ on } X' \tag{3.40}$$

$$\Phi_{|\partial X'} = 0 \tag{3.41}$$

and $\Phi \in C^\alpha(K')$ for any compact subset K' in $X'\backslash E'$. Then we see that on X',

$$(\,\pi^*(\omega_\delta) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} + \frac{i}{2}\partial\bar{\partial}\Phi\,)^n = 0 \tag{3.42}$$

$$\pi^*(\omega_\delta) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} + \frac{i}{2}\partial\bar{\partial}\Phi \geqslant 0 \tag{3.43}$$

iv) Take the composition of $\Phi - \epsilon\log h_{E'}$ with π^{-1}, which maps biholomorphically from $M\backslash\{p\}$ to $(\pi^{-1})(M\backslash\{p\}) \subset X'\backslash E'$ and define

$$\phi := (\Phi - \epsilon \log h_{E'}) \circ \pi^{-1} : M\backslash\{p\} \longrightarrow X'\backslash E' \longrightarrow \mathbb{R} \tag{3.44}$$

We get that $\phi \in PSH(M, \omega_\delta) \cap L^\infty(M)$ and $\phi \in C^\alpha(K)$ for any compact subset of $K \subset M\backslash\{p\}$. Push (3.42) forward from $X'\backslash E'$ to $M\backslash\{p\}$ via π, which is a surjective holomorphic map. We have on $M\backslash\{p\}$,

$$\{\, \pi_*(\pi^*(\omega_\delta)) + \frac{i}{2}\partial\bar{\partial}(\, (\Phi - \epsilon \log h_{E'}) \circ \pi^{-1})\, \}^n = 0 \tag{3.45}$$

$$(\omega_\delta + \frac{i}{2}\partial\bar{\partial}\phi\,)^n = 0 \tag{3.46}$$

and thus

$$\begin{aligned} 0 =& \{\, \omega + \delta \frac{i}{2}\partial\bar{\partial}\{\psi \log \sum_{j=1}^{n} |f_j(z)|^{2\beta_j} (\sum_{i=1}^{n} |f_i(z)|^2)^{1-\beta_j} \\ & \qquad + 1 - \psi\} + \frac{i}{2}\partial\bar{\partial}\phi\, \}^n \end{aligned}$$

Note that it might seem in the definition of (3.44) that ϕ depends on the constant ϵ, which is given by the $-\epsilon \log h_{E'}$ term in Ω'. However, we can see from the proof of lemma 3.7 that Φ contains a copy of $\epsilon \log h_{E'}$, we conclude that $\phi = (\Phi - \epsilon \log h_{E'}) \circ \pi^{-1}$ is independent of the choice of $\epsilon \log h_{E'}$.

v) From (3.46), we extract and define G and normalize by adding a constant

$-\delta$ to ensure the boundary conditions,

$$G := \delta\{\psi \log \sum_{j=1}^{n} |f_j(z)|^{2\beta_j} (\sum_{i=1}^{n} |f_i(z)|^2)^{1-\beta_j} + 1 - \psi\} + \phi - \delta \quad (3.47)$$

$$= \delta\{\psi \log \sum_{j=1}^{n} |f_j(z)|^{2\beta_j} (\sum_{i=1}^{n} |f_i(z)|^2)^{1-\beta_j} - \psi\} + \phi \quad (3.48)$$

Clearly $G \in PSH(M \backslash \{p\}, \omega) \cap C^{\alpha}(K)$ and on $M \backslash \{p\}$ we have

$$(\omega + \frac{i}{2}\partial\bar{\partial}G)^n = 0$$

vi) On ∂M, as the cut-off function $\psi_{|\partial M} = 0$ and

$$\phi_{|\partial M} = (\Phi - \epsilon\partial\bar{\partial}\log h_{E'}) \circ \pi^{-1}{}_{|\partial M}$$
$$= (\Phi - \epsilon\partial\bar{\partial}\log h_{E'})_{|\partial X'} = 0$$

we see that $G_{|\partial M} = 0$. Moreover, we have that $\psi = 1$ in a neighborhood of p and there

$$G = \delta \log\{\sum_{j=1}^{n} |f_j(z)|^{2\beta_j} (\sum_{i=1}^{n} |f_i|^2)^{1-\beta_j}\} + \phi - \delta$$

On M, $\phi - \delta$ is bounded and lies in $C^{\alpha}(K)$ and this proves the log singularity at p formulated by (3.39).

vii) In order to show that $G \in PSH(M, \omega)$, it suffices to show that G extends over p as a plurisubharmonic function. We can see this by letting

$$\tilde{G}_\epsilon = G(z) + \epsilon \log|z-p| \qquad \text{on } M\backslash\{p\} \tag{3.49}$$

$$\tilde{G}_\epsilon = -\infty \qquad \text{on } p \tag{3.50}$$

For fixed $\epsilon > 0$, $\tilde{G}_\epsilon$ is ω-plurisubharmonic over M as near p,

$$\limsup_{z\to p} \tilde{G}_\epsilon(z) = \limsup_{z\to p}(G(z) + \epsilon \log|z-p|) = -\infty \leqslant \tilde{G}_\epsilon(p)$$

Then denote $u(z) = (\sup_{\epsilon>0} \tilde{G}_\epsilon)^*$ and we have $u(z) \in PSH(M, \omega)$ due to the general fact that upper semicontinuous regularizations of supremums of plurisubharmonic functions are still plurisubharmonic. See [Dw, Corollary 5.3]. Moreover, we see that

$$u(z) = G(z) \qquad \text{on } M\backslash\{p\} \tag{3.51}$$

$$u(z) = \limsup_{z\to p} G(z) \qquad \text{at } p \tag{3.52}$$

By redefining G as u, we have completed the proof of existence part of the theorem.

(Uniqueness, we prove it by contradiction.)

viii) Fix G, a solution defined in the existence part. Suppose that there exists another $G_1(z; p, f_1, \cdots, f_n) \in PSH(M, \omega)$ that vanishes on ∂M

and satisfies the following

$$0 = (\omega + \frac{i}{2}\partial\bar{\partial}G_1)^n \qquad \text{on } M\backslash\{p\} \quad (3.53)$$

$$G_1 = \delta\{\psi_1 \log \sum_{j=1}^{n} |f_j|^{2\beta_j} (\sum_{i=1}^{n} |f_i|^2)^{1-\beta_j} - \psi_1\} + \phi_1 \quad \text{near } p \quad (3.54)$$

where $\phi_1 \in L^\infty(M)$ and vanishes on ∂M. Here the δ in G_1 is the same as that in G and ψ_1 is some other cut-off function supported in another neighborhood U_1 of p. Without loss of generality we can replace both U_1 and U by a smaller neighborhood and assume the cut-off function ψ_1 is the same as ψ. Then we show $G_1 = G$ by showing that $\phi_1 = \phi$ on the entire M.

On $M\backslash p$, we have that

$$\phi_1 = G_1 - \delta\{\psi \log \sum_{j=1}^{n} |f_j|^{2\beta_j} (\sum_{i=1}^{n} |f_i|^2)^{1-\beta_j} - \psi\} \quad (3.55)$$

$$= G_1 - G + \phi(z) \quad (3.56)$$

Let the smooth metric $h_{E'}$ over $O(-E')$, and $\Phi := \phi \circ \pi + \epsilon \log h_{E'} \in PSH(X', \Omega')$, together with the $\epsilon \log h_{E'}$ and ϕ taken as the same as those defined in the existence part.

Composing $\phi_1 : M \to \mathbb{R}$ with the the iterated blow up map $\pi : X' \to M$ which has been constructed together with $E' \subset \ X' \subset W'$, we define a

$\Phi_1 \in PSH(X'\backslash E', \Omega')$ as

$$\Phi_1 := \phi_1 \circ \pi + \epsilon \log h_{E'}$$

Clearly $\Phi_1 \in L^\infty(X')$, for that by assumption we have $\phi_1 \in L^\infty(M)$.

ix) We wish to show $\phi_1 = \phi$ on M by showing $\Phi_1 = \Phi$ on X'. Since Φ_1 is bounded and E' is a subset of a pluripolar set in X', we see that Φ_1 extends over E' by applying an extension theorem of Demailly. See ([Dem, Theorem 5.24]. Therefore, we have

$$\Phi_1 \in PSH(X', \Omega')$$

And the boundary condition ${\Phi_1}_{|_{\partial X'}} = 0$ can be seen from that ${\phi_1}_{|_{\partial M}} = 0$ and that $\log h_{E'|_{\partial X'}} = 0$, as $\log h_{E'}$ is supported in a neighborhood of E'. Now we claim that

$$(\Omega' + \frac{i}{2}\partial\bar{\partial}\Phi_1)^n = 0 \text{ on } X' \tag{3.57}$$

This is true on $X'\backslash E'$, as can be seen from the fact the the restricted map

$$\pi_{|_{X'\backslash E'}} : \quad X'\backslash E' \longrightarrow M\backslash\{p\}$$

is a holomorphic surjective map. Indeed, since by assumption that on

$M \backslash \{p\}$,

$$0 = (\omega + \frac{i}{2}\partial\bar{\partial}G_1)^n \tag{3.58}$$

$$= (\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\psi \log \sum_{j=1}^{n} |f_j|^{2\beta_j} (\sum_{i=1}^{n} |f_i|^2)^{1-\beta_j} + 1 - \psi\} + \frac{i}{2}\partial\bar{\partial}\phi_1)^n \tag{3.59}$$

By pulling back with the map π, we have on $X' \backslash E'$,

$$\begin{aligned} 0 =& (\pi^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{j=1}^{n} |f_j \circ \pi|^{2\beta_j} (\sum_{i=1}^{n} |f_i \circ \pi|^2)^{1-\beta_j} \\ &\quad + 1 - \chi\} + \frac{i}{2}\partial\bar{\partial}\phi_1 \circ \pi)^n \\ =& (\pi^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{j=1}^{n} |f_j \circ \pi|^{2\beta_j} (\sum_{i=1}^{n} |f_i \circ \pi|^2)^{1-\beta_j} \\ &\quad + 1 - \chi\} - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} + \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} + \frac{i}{2}\partial\bar{\partial}\phi_1 \circ \pi)^n \\ =& (\Omega' + \frac{i}{2}\partial\bar{\partial}\Phi_1)^n \end{aligned}$$

x) We show that $(\Omega' + \frac{i}{2}\partial\bar{\partial}\Phi_1)^n = 0$ on E' as well. Locally we can define a potential function θ_1 such that, $\Omega' + \frac{i}{2}\partial\bar{\partial}\Phi_1 = \Omega_0 + \frac{i}{2}\partial\bar{\partial}F_1 + \frac{i}{2}\partial\bar{\partial}\Phi_1 = \frac{i}{2}\partial\bar{\partial}\theta_1$, where Ω_0 and F_1 are defined as in (3.24)(3.31). Since Ω_0 is smooth Kähler form on X' and the functions F_1 and Φ_1 are bounded on X', we see that θ_1 is a locally bounded plurisubharmonic function on X'. Now consider the general fact that for any locally bounded plurisubharmonic function the Monge-Ampère measure takes no mass at pluripolar sets and their subsets. See [Bl1, Prop 2.2.3, Theorem 3.1].

So we get that $(\partial\bar{\partial}\theta_1)^n = 0$ on E' and therefore $(\Omega' + \frac{i}{2}\partial\bar{\partial}\Phi_1)^n = 0$ on entire X'. We now apply the uniqueness part of lemma 3.7 and consider Φ and Φ_1 are two functions both satisfying (3.25)(3.26), so we must have $\Phi = \Phi_1$. This shows that $\phi = (\Phi - \epsilon \log h_{E'}) \circ \pi^{-1} = (\Phi_1 - \epsilon \log h_{E'}) \circ \pi^{-1} = \phi_1$ on M. Thus $G = G_1$ and the uniqueness part is proved.

$\square$

4 Case II

In this section we give an answer to Question 3 in Theorem 4.8, and as a corollary we give an immediate answer to Question 1. The same proof as that of case I applies to this setting, with slight modifications that deals with the singularities in the (1,1) forms. For completeness, we give a proof that starts from scratch and include all intermediate steps. Most lemmas from section 3 still hold true here.

4.1 Theorem II

Now fix any positive integer Γ and we have our theorem 4.1, which prescribes a second type of singularity at isolated interior points. Note that the following theorem is a generalization of [PS12, Theorem 1], where $N_\gamma = n$ and $\beta_{\gamma,11} = \beta_{\gamma,22} = \cdots = \beta_{\gamma,nn} = 2$ for every $\gamma \leqslant \Gamma$.

Theorem 4.1. *Let ω be a Kähler form on compact complex manifold M of dimension $n \geqslant 2$ and assume $\partial M \neq \emptyset$ is smooth. Fix Γ interior points $\{p_1, \cdots, p_\gamma, \cdots, p_\Gamma\}$ in M. Given any $\Gamma \cdot n$ holomorphic functions $\{f_{\gamma,j}\}_{\gamma \leqslant \Gamma, j \leqslant n}$, any Γ positive integers $\{N_\gamma \geqslant n : 1 \leqslant \gamma \leqslant \Gamma\}$ and any finite set of constants $\{0 \leqslant \beta_{\gamma,ij} \leqslant 2 : 1 \leqslant \gamma \leqslant \Gamma,\ 1 \leqslant i \leqslant N_\gamma$ and $1 \leqslant j \leqslant n\}$ that satisfies*

- *a) For each γ fixed, $f_{\gamma 1}, \cdots, f_{\gamma,n}$ are locally defined in a neighborhood of p_γ, such that p_γ is their only common zero and that the function $\sum_{i=1}^{N_\gamma} \prod_{j=1}^{n} |f_{\gamma,j}|^{\beta_{\gamma,ij}}$ has p_γ as its only zero locus in this neighborhood.*

- *b) For each γ fixed and every $1 \leqslant i \leqslant N_\gamma$, $\beta_{\gamma,i1} + \cdots + \beta_{\gamma,in} = 2$*

Then there exists a constant δ_0 such that for any Γ positive numbers smaller than δ_0, i.e. any Γ-tuple $(\delta_1, \cdots, \delta_\Gamma)$ in

$$\{(\delta_1, \cdots, \delta_\Gamma) : 0 < \delta_\gamma < \delta_0 \text{ for every } 1 \leqslant \gamma \leqslant \Gamma\}$$

there exists a unique function

$$G(z; p_1, \cdots, p_\Gamma) \in PSH(M, \omega) \cap C^\alpha(K)$$

for any compact subset K of $M\backslash\{p_1, \cdots, p_N\}$ and any constant $0 < \alpha < \min\{\beta_{\gamma,ij} : \beta_{\gamma,ij} > 0\}$ with the following properties

$$\begin{cases} G = 0 & \text{on } \partial M & (4.1) \\ (\omega + \frac{i}{2}\partial\bar{\partial}G)^n = 0 & \text{on } M\backslash\{p_1, \cdots, p_\Gamma\} & (4.2) \\ G = \delta_\gamma \log \sum_{i=1}^{N_\gamma} \prod_{j=1}^{n} |f_{\gamma,j}|^{\beta_{\gamma,ij}} + O(1) & \text{near } p_\gamma & (4.3) \end{cases}$$

With an iteration of Γ times, each of which deals with the singularity at an isolated point, say p_γ, and yields an exceptional divisor E_γ, we may reduce to the case of $\Gamma = 1$ which will be shown in theorem 4.8 in Section 4.5. Indeed, we can start the iteration by defining $M_0 = M$ and $\omega_0 = \omega$. Suppose that right before the γ-th step, we have $E_1, \cdots, E_{\gamma-1} \subset M_{\gamma-1}$ and $\omega_{\gamma-1}$ on $M_{\gamma-1}$. On the γ-th step, we can construct the γ-th blow up space M_γ and a singular (1,1) form ω_γ on M_γ with respect to $\omega_{\gamma-1}$ on $M_{\gamma-1}$, $p_\gamma \in M_{\gamma-1}$ and $E_\gamma \subset M_\gamma$. And since the γ-th blow up map $\pi_\gamma : (M_\gamma, E_\gamma, \omega_\gamma) \longrightarrow (M_{\gamma-1}, p_\gamma, \omega_{\gamma-1})$ is a biholomorphism away from the particular point p_γ, the boundary gets mapped to the new boundary (which is the biholomorphic image under the γ-th blow up map) in the γ-th blow up space M_γ and the isolated points $p_{\gamma+1}, \cdots, p_\Gamma$, which were in the interior of the first $\gamma - 1$ blow up spaces, stay isolated in the interior of the γ-th blow up space. Moreover, they stay away from the exceptional divisors E_γ and $E_1, \cdots, E_{\gamma-1}$.

Then continue with the process for $p_{\gamma+1}, \cdots, p_\Gamma$. Eventually, we solve for a bounded and continuous solution to the Monge-Ampère equation in the Γ-th blow up space M_Γ with respect to E_Γ and the singular (1,1) form ω_Γ. Then pull the solution back by ${\pi_1}^{-1} \circ \cdots \circ {\pi_\Gamma}^{-1}$ from M_Γ to $M_{\Gamma-1} \backslash \{p_\Gamma\}$, to $M_{\Gamma-2} \backslash \{p_{\Gamma-1}, p_\Gamma\}$, $\cdots$, and eventually to $M \backslash \{p_1, \cdots, p_{\Gamma-1}, p_\Gamma\}$.

From now on, we reduce to the case of $\Gamma = 1$ and consider M near p with the given data of (M, p, f_i), the constants $\{0 \leqslant \beta_{ij} \leqslant 2 : 1 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant n\}$ satisfying that $\beta_{i1} + \cdots + \beta_{in} = 2$ for each fixed $1 \leqslant i \leqslant N$ and that $\sum\limits_{i=1}^{N} \prod\limits_{j=1}^{n} |f_j|^{\beta_{ij}}$ has p as its only zero point.

4.2 Singular (1,1) form on $X(M, p, f_1, \cdots, f_n)$

Now we consider M near p with the given data (M, p, f_i). Same as in lemma 3.2, we first construct a singular (1,1) form on the blow up space $X = X(M, p, f_i)$, which might not be smooth, in some ambient space W.

Lemma 4.2 (Singular Hermitian (1,1) form on $X(M, p, f_1, \cdots, f_n)$)**.** *Given (M, p, f_i) the data of a compact Kähler manifold (M, ω), an interior point p, n local holomorphic functions f_i, $1 \leqslant i \leqslant n$ with p as their only* *common zero. Fix any constants* $\{0 \leqslant \beta_{ij} \leqslant 2 : 1 \leqslant i \leqslant N$ *and* $1 \leqslant j \leqslant n\}$ *satisfying that* $\beta_{i1}+\cdots+\beta_{in} = 2$ *for each fixed* $1 \leqslant i \leqslant N$ *and that* $\sum_{i=1}^{N}\prod_{j=1}^{n}|f_j|^{\beta_{ij}}$ *has* p *as its only zero point.*

Then there exists a complex analytic space $X = X(M, p, f_i)$ *and a biholomorphism* $\pi_1 : X\backslash X_0 \to M\backslash\{p\}$, *sending* $X_0 \to p$ *for some* X_0 *biholomorphically equivalent to* $\mathbb{P}^{n-1}$ *with following property:*

The form ω_δ *defined on* $M\backslash\{p\}$ *as*

$$\omega_\delta = \omega + \delta\frac{i}{2}\partial\bar{\partial}\{\psi \log \sum_{i=1}^{N}\prod_{j=1}^{n}|f_j|^{\beta_{ij}} + 1 - \psi\} \tag{4.4}$$

has the pull back $\pi_1^*\omega_\delta$ *on* $X\backslash X_0$ *that extends to a closed strictly positive singular (1,1) form on* X.

Proof. i) Fix $\epsilon_0 > 0$ and since p is the only common zero of the local holomorphic functions $f_1, \cdots, f_n$ for some small open neighborhood $U \subset M$, i.e. $\sum_{j=1}^{n}|f_i|^2 \neq 0$ on $U\backslash\{p\}$. Fix U and choose another open neighborhood $U_0 \subset\subset U$ and define

$$U_{\epsilon_0} := U_0 \cap \{z : \sum_{j=1}^{n}|f_i(z)|^2 < \frac{{\epsilon_0}^2}{4}\}$$

We blow up U_{ϵ_0} with center p and denote as V_{ϵ_0},

$$\begin{aligned} V_{\epsilon_0} &:= \mathrm{BL}_{<f_1,\cdots,f_n>}U_{\epsilon_0} \\ &= \{((z_1, \cdots, z_n), [t_1, \cdots, t_n]) \in U_{\epsilon_0} \times \mathbb{P}^{n-1} : t_i f_j(z) = t_j f_i(z)\} \end{aligned}$$

Set theoretically gluing V_{ϵ_0} with $M\backslash\{p\}$ on $V_{\epsilon_0}\backslash\{p\}$ defines space X by

$$X := (V_{\epsilon_0} \cup M\backslash\{p\})/\sim_{:\, q_1 \in M\backslash\{p\} \sim (q_2,[t]) \in V_{\epsilon_0} \Longleftrightarrow q_1 = q_2}$$

$U_{\epsilon_0}\backslash\{p\}$ is covered by n open subsets $\{z : f_j(z) \neq 0\} \cap U_{\epsilon_0}$ and V_{ϵ_0} is covered by n open subsets

$$\{(z,[t]) : t_j \neq 0\} \cap V_{\epsilon_0}$$

The surjection $\mathrm{proj}_1 : V_{\epsilon_0} \subset U_{\epsilon_0} \times \mathbb{P}^{n-1} \to U_{\epsilon_0}$ defines X_0 as the inverse image of p, thus X_0 is biholomorphic to $\mathbb{P}^{n-1}$.

ii) Now construct a complex manifold W such that X is locally an analytic subset of W. Perturb the holomorphic functions $f_1, \cdots, f_n$ by $\xi \in B_{\frac{\epsilon_0}{2}}(0) \subset \mathbb{C}^n$ in the following way. Let the graph of the map $(f_1, \cdots, f_n)$ over U_{ϵ_0} be

$$Z = \overline{\{(z,\xi) \in U_{\epsilon_0} \times B_{\frac{\epsilon_0}{2}} : \xi_i = f_i(z) \text{ for all } 1 \leqslant i \leqslant n\}}$$

Note that $Z \subset \overline{M} \times \overline{B_{\epsilon_0}}$ is a smooth compact submanifold of dimension n and the image of Z under the projection map $\mathrm{proj}_1 : Z \to U_0$ is a subset of U_0 and is thus compactly supported in U. Define local holomorphic functions $g_1(z,\xi), \cdots, g_n(z,\xi)$ over $U_0 \times B_{\epsilon_0}$ as

$$g_i(z) := f_i(z) - \xi_i \text{ for } 1 \leqslant i \leqslant n$$

By the triangle inequality, $\sum_{j=1}^{n} |g_j(z)|^2 < {\epsilon_0}^2$ on $U_{\epsilon_0} \times B_{\frac{\epsilon_0}{2}}$. Clearly in $U_0 \times B_{\epsilon_0}$, we have

$$Z = \{g_1 = \cdots = g_n = 0\} \cap \overline{U_{\epsilon_0} \times B_{\frac{\epsilon_0}{2}}}$$

and there, Z has an neighborhood T , defined as

$$T := \{(z, \xi) \in U_0 \times B_{\epsilon_0} : \min_{(z',\xi') \in Z} \{ \sum_{j=1}^{n} |f_j(z) - f_j(z')|^2 + |\xi_j - {\xi'}_j|^2 < \frac{{\epsilon_0}^2}{5} \} \}$$

we have that $T \subset\subset U_0 \times B_{\epsilon_0} \subset M \times B_{\epsilon_0}$

iii) Let $W_{\frac{\epsilon_0}{2}}$ be the blow up of $U_{\epsilon_0} \times B_{\frac{\epsilon_0}{2}}$ with center Z, denoted as

$$\begin{aligned} W_{\frac{\epsilon_0}{2}} &: = \mathrm{BL}_{<g_1, \cdots, g_n>} (U_{\epsilon_0} \times B_{\frac{\epsilon_0}{2}}) \\ &= \{ (z, \xi, [t]) \in U_{\epsilon_0} \times B_{\frac{\epsilon_0}{2}} \times \mathbb{P}^{n-1} : t_i \, g_j(z, \xi) = t_j \, g_i(z, \xi) \} \end{aligned}$$

Let W be the set theoretically union of $W_{\frac{\epsilon_0}{2}}$ and $(M \times B_{\epsilon_0}) \backslash Z$, glued via the canonical surjection

$$\begin{aligned} \mathrm{proj}_1 : \quad W_{\frac{\epsilon_0}{2}} &\longrightarrow U_{\epsilon_0} \times B_{\frac{\epsilon_0}{2}} \\ (z, \xi, [t]) &\longmapsto (z, \xi) \end{aligned}$$

As $Z \subset M \times B_{\epsilon_0}$ is a smooth submanifold, we see that W is a smooth manifold by lemma 2.30. The exceptional divisor E_g is locally the inverse image of Z under the blow up map, which is denoted as

$$\pi_1 : W = \mathrm{BL}_{<g_1,\cdots,g_n>}(\, M \times B_{\epsilon_0}\,) \longrightarrow M \times B_{\epsilon_0}$$
$$E_g \longmapsto Z$$

On W, let $W_{\epsilon_0} := {\pi_1}^{-1}(U_0 \times B_{\epsilon_0})$ and we see that locally near E_g, $E_g \subset {\pi_1}^{-1}(T) \subset\subset W_{\epsilon_0}$. Clearly W_{ϵ_0} is covered by n open subsets

$$W_{\epsilon_0,j} = \{(z, \xi, [t]) \in W_{\epsilon_0} : t_j \neq 0 \text{ or } g_j(z, \xi) \neq 0\} \cap W_{\epsilon_0}$$

With the same construction as in (2.4) in section 2.2, we see that $W_{\epsilon_0,j}$ admits a local holomorphic chart $\tau_j : W_{\epsilon_0,j} \to \mathbb{C}^n$ that glues biholomorphically on the overlap with one another. Away from E_g, $W \backslash E_g$ is biholomorphic to $(M \times B_{\epsilon_0})\backslash Z$.

iv) Since Z is a compact submanifold of $M \times B_{\epsilon_0}$, blowing up Z in $M \times B_{\epsilon_0}$ and applying constructions similar to lemma 2.30, we get a singular metric h_{E_g} on $O(-E_g)$ over W as the following.

Suppose W_{ϵ_0} is covered by a finite collection of neighborhoods $\{W_\alpha\}_\alpha$ in W. Choose a family of cut-off functions χ_{W_α} with respect to $\overline{\pi_1^{-1}(T)}$ and W_{ϵ_0} such that $\chi_{W_\alpha} \in C^\infty(W_{\epsilon_0})$, $0 \leqslant \chi_{W_\alpha} \leqslant 1$ and

$$\chi_W := \sum_\alpha \chi_{W_\alpha} = 1 \text{ on } \overline{\pi_1^{-1}(T)}$$

Let $T_\alpha := \pi_1(W_\alpha)$ and thus $T \subset\subset \cup_\alpha T_\alpha$. Moreover, each W_α is covered by n open subsets

$$W_{\alpha,j} = \{(z,\xi,[t]) \in W_\alpha : t_j \neq 0 \text{ or } g_j(z,\xi) \neq 0\}$$

For a section f in $O(-E_g)$, since $f_{|E_g} = 0$, we may assume that locally $f = g_j \cdot \tilde{f}$ in the neighborhood $W_{\alpha,j}$ for some holomorphic function $\tilde{f}$.

Define $|f|^2_{h_{E_g}}$ on $W_{\alpha,j}$ as

$$= \frac{|f|^2}{\sum_\alpha \chi_{W_\alpha}\{\sum_{i=1}^N \prod_{k=1}^n |g_k|^{\beta_{ik}}\} + 1 - \chi_W} \tag{4.5}$$

$$= \frac{|g_j \cdot \tilde{f}|^2}{\sum_\alpha \chi_{W_\alpha}\{|g_j|^2 \sum_{i=1}^N \prod_{k=1}^n |\frac{g_k}{g_j}|^{\beta_{ik}}\} + 1 - \chi_W} \quad \text{on } W_{\alpha,j} \backslash E_g \tag{4.6}$$

$$= \frac{|\tilde{f}|^2}{\sum_\alpha \chi_{W_\alpha}\{\sum_{i=1}^N \prod_{k=1}^n |\frac{g_k}{g_j}|^{\beta_{ik}}\} + 1 - \chi_W} \quad \text{on } W_{\alpha,j} \backslash E_g \tag{4.7}$$

$$= \frac{|\tilde{f}|^2}{\sum_\alpha \chi_{W_\alpha}\{\sum_{i=1}^N \prod_{k=1}^n |\frac{t_k}{t_j}|^{\beta_{ik}}\} + 1 - \chi_W} \quad \text{on } W_{\alpha,j} \tag{4.8}$$

We see from (4.5) that h_{E_g} defined in this way glues well for all α and j. It is a globally well defined smooth(resp. singular) Hermitian metric on $O(-E_g)$ if $N = n$, and all $\beta_{ii} = 2$ or 0 (resp. $N \geqslant n$ and some $0 < \beta_{ij} < 2$). See [Dem2]. With h_{E_g}, we can define a Kähler form(resp. singular Hermitian form) on W later in step vii).

v) Define X to be the subset of W where $\xi_1 = \cdots = \xi_n = 0$, and locally near X_0, X is given by

$$X_{\epsilon_0} := \{(z, 0, [t]) \in U_{\epsilon_0} \times B_{\frac{\epsilon_0}{2}} \times \mathbb{P}^{n-1} : t_i\, f_j(z) = t_j\, f_i(z)\}$$

which is an analytic subspace of the smooth manifold W. Note that X might not be smooth in general, for which case we apply Hironaka's theorem with iterated blow ups later in next lemma. Now we see that this defines a map $\pi : X \to M$ which is a biholomorphism between $X \backslash X_0$ and $M \backslash \{p\}$, where $X_0 = \pi^{-1}(p)$ is biholomorphically equivalent to $\mathbb{P}^{n-1}$. And near X_0, X is covered by its n open subsets

$$X_{\epsilon_0, j} = \{(z, 0, [t]) \in X_{\epsilon_0} : t_j \neq 0 \text{ or } f_j(z) \neq 0\}$$

vi) By letting $U_1 := \mathrm{proj}_1(T)$, we have a small open neighborhoods U_1 near p on M satisfying $U_1 \subset\subset U_0$. Choose any cut-off function $\psi \in C^\infty(U)$ such that

$$\psi \circ \ \mathrm{proj}_1 \circ \pi_1 = \chi_W : W_{\epsilon_0} \longrightarrow \mathbb{R}$$

then clearly

$$\psi_{|_{\overline{U_1}}} \equiv 1 \text{ and } \psi_{|_{U \backslash U_0}} \equiv 0$$

Indeed, consider the projection map $\mathrm{proj}_1 : U_0 \times B_{\epsilon_0} \to U_0$ and the blow up map $\pi_1 : W_{\epsilon_0} \to U_0 \times B_{\epsilon_0}$, the image of $\mathrm{proj}_1 \circ \pi_1$ lies in U_0. The composition of proj_1 and ψ gives

$$\chi_T := \psi \circ \mathrm{proj}_1 : \pi_1(W_{\epsilon_0}) \longrightarrow U_0 \longrightarrow \mathbb{R}$$

which is a cut-off function $\chi_{_T}$ in $U_0 \times B_{\epsilon_0}$ near T, such that $\chi_{_T} \equiv 1$ on $\overline{T}$.

vii) We wish to define a positive closed singular (1,1) form on X in the sense of currents. We do so by first constructing a singular (1,1) form on W, the ambient manifold where X lies in. Given ω the Kähler metric on M, we extend it naturally to a Kähler metric on $M \times B_{\epsilon_0}$, denoted as $\omega' := \omega + \omega_0$, where ω_0 is the standard Euclidean metric on B_{ϵ_0}. Choose the cut-off functions $\chi_{_W}$, $\chi_{_T}$ given in step iv) and vi), we define a singular (1,1) form ω''_δ on $W \backslash E_g$,

$$\omega''_\delta := \pi_1^* \omega' + \delta \frac{i}{2} \partial\bar{\partial} \{ \chi_{_W} \log \sum_{i=1}^{N} \prod_{k=1}^{n} |f_k - \xi_k|^{\beta_{ik}} + 1 - \chi_{_W} \} \tag{4.9}$$

Thus on $(M \times B_{\epsilon_0}) \backslash Z$, which is biholomorphically equivalent to $W \backslash E_g$

$$\omega''_\delta := \omega' + \delta \frac{i}{2} \partial\bar{\partial} \{ \chi_{_T} \log \sum_{i=1}^{N} \prod_{k=1}^{n} |f_k - \xi_k|^{\beta_{ik}} + 1 - \chi_{_T} \} \tag{4.10}$$

In a small neighborhood near E_g in W, we have $\chi_{_W} = 1$ and define the singular (1,1) form ω''_δ as

$$\omega''_\delta := \pi_1^* \omega' + \delta \frac{i}{2} \partial\bar{\partial} \{ \log \sum_{i=1}^{N} \prod_{k=1}^{n} |t_k|^{\beta_{ik}} \} \tag{4.11}$$

In step viii) we will show that ω''_δ defined in (4.9)(4.11), is strictly positive closed (1,1) current with locally bounded potential function on W, in the sense of currents. But first we need to show that the two definitions are coherent on the overlap of a neighborhood away from E_g and the open set $\{\chi_W = 1\}^\circ$, which is near E_g. To see this, first notice that away from E_g, $g_j(z,\xi) = f_j(z) - \xi_j \neq 0$ for some $1 \leqslant j \leqslant n$. Without loss of generality, we may assume that $g_1(z,\xi) = f_1(z) - \xi_1 \neq 0$. Then we must have $t_1 \neq 0$ and $\frac{g_i}{g_1} = \frac{t_i}{t_1}$, for all $1 \leqslant i \leqslant n$ and hence

$$\begin{aligned}
\omega''_\delta &= \pi_1^*\omega' + \delta\frac{i}{2}\partial\bar\partial\{\chi_W \log\sum_{i=1}^N\prod_{k=1}^n |f_k - \xi_k|^{\beta_{ik}} + 1 - \chi_W\} \\
&= \pi_1^*\omega' + \delta\frac{i}{2}\partial\bar\partial\{\log\sum_{i=1}^N\prod_{k=1}^n |f_k - \xi_k|^{\beta_{ik}}\} \\
&= \pi_1^*\omega' + \delta\frac{i}{2}\partial\bar\partial\{\log\sum_{i=1}^N\prod_{k=1}^n |g_k|^{\beta_{ik}}\} \\
&= \pi_1^*\omega' + \delta\frac{i}{2}\partial\bar\partial\{ \log\sum_{i=1}^N\prod_{k=1}^n |\frac{g_k}{g_1}|^{\beta_{ik}}\} + \delta\frac{i}{2}\partial\bar\partial\log|g_1|^2 \\
&= \pi_1^*\omega' + \delta\frac{i}{2}\partial\bar\partial\{ \log\sum_{i=1}^N\prod_{k=1}^n |\frac{g_k}{g_1}|^{\beta_{ik}}\} \\
&= \pi_1^*\omega' + \delta\frac{i}{2}\partial\bar\partial\{ \log\sum_{i=1}^N\prod_{k=1}^n |\frac{t_k}{t_1}|^{\beta_{ik}}\} \\
&= \pi_1^*\omega' + \delta\frac{i}{2}\partial\bar\partial\{ \log\sum_{i=1}^N\prod_{k=1}^n |\frac{t_k}{t_1}|^{\beta_{ik}}\} + \delta\frac{i}{2}\partial\bar\partial\log|t_1|^2 \\
&= \pi_1^*\omega' + \delta\frac{i}{2}\partial\bar\partial\{ \log\sum_{i=1}^N\prod_{k=1}^n |t_k|^{\beta_{ik}}\}
\end{aligned}$$

viii) To show $\omega''_\delta > 0$ on W, first notice that $\pi_1^*\omega'$ is strictly positive on $W\backslash E_g \simeq (B\times M)\backslash Z$ as it is the pull back of ω' by a biholomorphic map. On E_g, we have $\pi_1^*\omega' \geqslant 0$ and in order to show $\omega''_\delta > 0$ and it only remains to prove that the (1,1) form in (4.11) denoted as

$$\Omega(\beta, t) := \frac{i}{2}\partial\bar{\partial}\{\log \sum_{i=1}^{N} \prod_{k=1}^{n} |t_k|^{\beta_{ik}}\}$$

is strictly positive.

If $N = n$ and all $\beta_{11} = \cdots = \beta_{nn} = 2$, $\Omega(\beta, t) = \omega_{FS}$, is the Fubini-Study metric on the projective space $\mathbb{P}^{n-1}$ and hence is strictly positive. Since Z is compact, we can get strict positivity of ω''_δ for any sufficiently small constant δ. In fact, ω''_δ is a smooth Kähler form on W in this case. See [PS12].

For general β_{ij} and $N \geqslant n$, we claim that for the function $\sum\limits_{i=1}^{N} \prod\limits_{j=1}^{n} |f_j|^{\beta_{ij}}$ which has p as its unique zero point in the neighborhood U on M, there exists n indices $1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant N$ such that

$$\beta_{i_1 1} = \beta_{i_2 2} = \cdots = \beta_{i_n n} = 2$$

We prove the claim by contradiction. Suppose that the claim is false for some k, say $k = 1$, i.e. $\beta_{i1} < 2$ for all $1 \leqslant i \leqslant N$, which is equivalent to say $\max\limits_{i=1}^{N}\{\beta_{i1}\} < 2$. By assumption of the lemma, for each fixed i, $\sum\limits_{j=1}^{n} \beta_{ij} = 2$. Thus we must have $\sum\limits_{j=2}^{N} \beta_{ij} > 0$ for all $1 \leqslant i \leqslant N$. Pick a sufficiently small coordinate ball $U_1 \subset U$ on M that contains p. Clearly on $U\backslash\{f_1 = 0\}$,

$$\sum_{i=1}^{N}\prod_{j=1}^{n}|f_j|^{\beta_{ij}} \tag{4.12}$$

$$=|f_1|^{\beta_{11}}\cdots|f_n|^{\beta_{1n}}+\cdots+|f_1|^{\beta_{N1}}\cdots|f_n|^{\beta_{Nn}} \tag{4.13}$$

$$=|f_1|^2\cdot|f_1|^{-2}\cdot\sum_{i=1}^{N}\prod_{j=1}^{n}|f_j|^{\beta_{ij}} \tag{4.14}$$

$$=|f_1|^2\cdot\sum_{i=1}^{N}\prod_{j=1}^{n}|\frac{f_j}{f_1}|^{\beta_{ij}} \tag{4.15}$$

$$=|f_1|^2\cdot\{\,|\frac{f_2}{f_1}|^{\beta_{12}}\cdots|\frac{f_n}{f_1}|^{\beta_{1n}}+\cdots+|\frac{f_2}{f_1}|^{\beta_{N1}}\cdots|\frac{f_n}{f_1}|^{\beta_{Nn}}\} \tag{4.16}$$

The above term (4.16) vanishes on the uncountable subset $U_1\cap\{f_2 = \cdots = f_n = 0\}$ of U. This contradicts the fact that p is the only zero point of (4.12) in U.

Now that the claim is true, we may assume without loss of generality that

$$i_1 = 1, i_2 = 2, \cdots, i_n = n$$

and therefore $\beta_{11} = \cdots = \beta_{nn} = 2$. So we have on $W\backslash E_g$,

$$\frac{i}{2}\partial\bar{\partial}\log\sum_{i=1}^{N}\prod_{k=1}^{n}|g_k|^{\beta_{ik}}$$
$$=\frac{i}{2}\partial\bar{\partial}\log\{\,|g_1|^2+|g_2|^2+\cdots+|g_n|^2+\sum_{i=n+1}^{N}\prod_{k=1}^{n}|g_k|^{\beta_{ik}}\}$$

and near E_g,

$$\begin{aligned}\Omega(\beta,t) =& \frac{i}{2}\partial\bar{\partial}\log\sum_{i=1}^{N}\prod_{k=1}^{n}|t_k|^{\beta_{ik}} && (4.17)\\ =& \frac{i}{2}\partial\bar{\partial}\log\{\sum_{i=1}^{n}\prod_{k=1}^{n}|t_k|^{\beta_{ik}}+\sum_{i=n+1}^{N}\prod_{k=1}^{n}|t_k|^{\beta_{ik}}\} && (4.18)\\ =& \frac{i}{2}\partial\bar{\partial}\log\{|t_1|^2+|t_2|^2+\cdots+|t_n|^2+\sum_{i=n+1}^{N}\prod_{k=1}^{n}|t_k|^{\beta_{ik}}\} && (4.19)\end{aligned}$$

Define u_n, $\{u_i : i \geqslant n+1\}$ and Θ as

$$\begin{aligned} u_n &:= \log(|t_1|^2+|t_2|^2+\cdots+|t_n|^2),\\ u_i &:= \log\prod_{k=1}^{n}|t_k|^{\beta_{ik}}\\ &= \sum_{k=1}^{n}\frac{\beta_{ik}}{2}\log|t_k|^2\ ,\\ \Theta(u_n,u_{n+1},\cdots,u_N) &:= \log(e^{u_n}+e^{u_{n+1}}+\cdots+e^{u_N}\ )\\ &= \log\{\ |t_1|^2+|t_2|^2+\cdots+|t_n|^2+\sum_{i=n+1}^{N}\prod_{k=1}^{n}|t_k|^{\beta_{ik}}\}\ . \end{aligned}$$

Clearly we have $D_i\Theta > 0$, and that u_i is plurisubharmonic for all $i \geqslant n+1$ and since $i\partial\bar{\partial}u_n = i\partial\bar{\partial}\log(|t_1|^2+|t_2|^2+\cdots+|t_n|^2)$, u_n is strictly plurisubharmonic. Therefore $\Theta(\,u_n(t),\cdots,u_N(t)\,)$ is plurisubharmonic by [Dem, Theorem 5.6], and $\Omega(\beta,t) = i\partial\bar{\partial}\Theta \geqslant 0$.

To show that $\Omega(\beta,t)$ is strictly positive, it suffices to show that $\xi^p\overline{\xi^q}\partial_p\partial_{\overline{q}}\Theta \geqslant \lambda\cdot\xi\overline{\xi}$ for some positive constant λ and any vector $\xi\in\mathbb{C}^n$. This can be seen from

$$\begin{aligned}
&\xi^p\overline{\xi^q}\partial_p\partial_{\overline{q}}\Theta(u_n,\cdots,u_N)\\
=&\xi^p\overline{\xi^q}\partial_p(\sum_{j=n}^{N}D_j\Theta\cdot\partial_{\overline{q}}u_j)\\
=&\xi^p\overline{\xi^q}(\sum_{j=n}^{N}D_j\Theta\cdot\partial_p\partial_{\overline{q}}u_j \quad + \quad \sum_{i=n}^{N}\sum_{j=n}^{N}(D_iD_j\Theta)\cdot\partial_p u_i\partial_{\overline{q}}u_j\,)\\
=&\sum_{j=n}^{N}D_j\Theta(\,\xi^p\overline{\xi^q}\partial_p\partial_{\overline{q}}u_j\,) \quad + \quad \sum_{i=n}^{N}\sum_{j=n}^{N}(D_iD_j\Theta)(\,\xi^p\partial_p u_i\,)(\,\overline{\xi^q}\partial_{\overline{q}}u_j\,)\\
\geqslant&\sum_{j=n}^{N}D_j\Theta(\,\xi^p\overline{\xi^q}\partial_p\partial_{\overline{q}}u_j\,)\\
\geqslant&\lambda_0\cdot\xi\overline{\xi}
\end{aligned}$$

where λ_0 is a local constant depending on the smallest eigenvalue of $(\,\partial_k\bar{\partial}_l u_n\,)$. And $\lambda_0>0$, as u_n is strictly plurisubharmonic.

ix) Now we have a singular (1,1) form on $M\backslash\{p\}$ as

$$\omega_\delta := \omega + \delta\frac{i}{2}\partial\bar{\partial}\{\psi\log\sum_{i=1}^{N}\prod_{j=1}^{n}|f_j(z)|^{\beta_{ij}} + 1 - \psi\} \tag{4.20}$$

where ψ is the cut-off function given in step vi). On W, we have constructed a strictly positive singular (1,1) form

$$\omega''_\delta = \pi_1^*\omega' + \delta\frac{i}{2}\partial\bar{\partial}\{\chi_W\log\sum_{i=1}^{N}\prod_{j=1}^{n}|f_j(z)-\xi_j|^{\beta_{ij}} + 1 - \chi_W\}$$

On X, we get a positive closed singular (1,1) form ω'_δ by restricting ω''_δ

from W to X and letting $\xi_1 = \cdots = \xi_n = 0$,

$$\omega'_\delta := \omega''_{\delta\,|_{\xi=0}} \tag{4.21}$$

$$= \pi_1^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{i=1}^{N}\prod_{j=1}^{n}|f_j(z)|^{\beta_{ij}} + 1 - \chi\} \tag{4.22}$$

$\chi := \chi_{W\,|_{\xi=0}}$. As a result of the restriction, ${\pi_1}^*\omega' = {\pi_1}^*(\,\omega + \omega_0\,)$ changed into ${\pi_1}^*\omega$ and χ_W into χ.

Moreover, we have that ω'_δ is strictly positive and that on $X\backslash X_0$, ω'_δ is the same as ${\pi_1}^*\omega_\delta$, the pull back of ω_δ from $M\backslash\{p\}$.

To summarize, $\pi_1^*\omega_\delta$, the pull back of ω_δ from $M\backslash\{p\}$ to $X\backslash X_0$, extends to entire X, as can bee seen from that fact that $\psi \circ \pi_1 = \chi_W = 1$ near X_0 in a neighborhood of the exceptional divisor E_g in W and that

$$\omega'_{\delta|_{X_0}} = \pi_1^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi_W \log \sum_{i=1}^{N}\prod_{j=1}^{n}|t_j|^{\beta_{ij}} + 1 - \chi_W\}$$
$$= \pi_1^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi_W \log \sum_{i=1}^{N}\prod_{j=1}^{n}|f_j(z)|^{\beta_{ij}} + 1 - \chi_W\}$$

$\square$

4.3 Singular (1,1) form on iterated blow up space X'

The X given in a local way in section 4.2 is an analytic subspace of W and X is in general not smooth. To get a smooth manifold X', we can apply the iterated blow up technics[PS12] by applying a deep theorem of Hironaka to revolve the singularities X_{sing} within the ambient space W. Their idea is to blow up W finite times and resolve the singularities of X. From there we get an iterated blow up space W', a smooth manifold X' which lies inside W' and a strictly-positive closed (1,1) current Ω' on X'

Lemma 4.3 (Hironaka's Theorem on iterated blow-ups, see [H][PS12])**.**
Let W be a complex manifold and $X \subset W$ be a complex analytic space. Then there exists an iterated blow up $\pi_2 : W' \to W$ with the exceptional divisor $E \subset W'$ that resolves the singularities of X in the following way: let

$$X' = \overline{\pi_2^{-1}(X) \backslash E} \subset W' \tag{4.23}$$

Then X' is a smooth manifold and $\pi_2 : X' \to X$, the restriction of the map π_2 to X', is surjective. Moreover, we have a divisor in X'

$$E' = E \cap X' = \pi^{-1}(X_{sing}) \tag{4.24}$$

which is a divisor with normal crossings and an isomorphism on $X' \backslash E'$

$$\pi : X' \backslash E' \to X_{reg} \tag{4.25}$$

From lemma 4.3, the singularities in $X_{sing} \subset X$ are resolved by the blow up map $\pi_2 : W' \longrightarrow W$ and the construction of $E' \subset X' \subset W'$. Now we wish to give a closed, strictly-positive (1,1) form Ω' on X'.

Lemma 4.4 (Existence of $h_{E'}$ and Ω' on X')**.**
Let $\omega'_\delta = \pi_1^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{i=1}^{N}\prod_{j=1}^{n}|f_j(z)|^{\beta_{ij}} + 1 - \chi\}$ *as in lemma 4.2,* $E \subset W'$ *and* $E' \subset X'$ *as above.*
There exists a closed, singular strictly positive (1,1) form Ω' *on* X'*, defined as*

$$\Omega' = \pi_2^*(\omega'_\delta) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} > 0 \tag{4.26}$$

for some smooth metric $h_{E'}$ *on* $O(-E')$*, where* E' *is some effective divisor on* X'*.*

Proof.

i) Recall that in lemma 4.2, we constructed a strictly positive singular Hermitian (1, 1) form ω''_δ on W, together with $\pi^*(\omega_\delta)$ as its restriction to X. Apply lemma 4.3 and get $E' \subset X' \subset W'$ and $\pi_2 : X' \to X$ that resolves the singularities X_{sing} which lies in $X_0 \subset E_g \subset W$. Here E_g is the compact smooth submanifold of W defined in lemma 4.2.

ii) Choose a finite collection of small neighborhoods $\{U_\alpha\}_\alpha$ that covers E_g. Since ω''_δ is strictly positive and locally in U_α we have

$$\omega''_\delta = i\partial\bar{\partial}\phi_\alpha$$

where ϕ_α is a strictly plurisubharmonic function in U_α. This means that locally in U_α, there exists a small positive constant λ_α such that for all $\xi \in \mathbb{C}^n$,

$$i\partial_i\partial_{\bar{j}}\phi_\alpha\, \xi^i\overline{\xi^j} \geqslant \lambda_\alpha\, |\xi|^2$$

Fix a smooth Kähler form $\omega_{_{\delta,W}}$ on W as in (4.29). Then locally in U_α, $\omega_{_{\delta,W}} = g_{i\bar{j}}(x)\, dz^i \wedge \overline{dz^j}$. Let $\lambda(x)$ be the largest eigen value of $(g_{i\bar{j}}(x))$ and

$$\Lambda := \max_\alpha\{\sup_{x\in U_\alpha} \lambda(x)\}$$

Choose any large positive constant A, such that $\frac{1}{A} \leqslant \frac{\min\limits_\alpha \lambda_\alpha}{\Lambda}$. We see that $\omega''_\delta \geqslant \frac{\omega_{\delta,W}}{A}$ in the neighborhood $\cup_\alpha U_\alpha$ of E_g.

iii) Now apply [PS12, lemma 7] to the Kähler form $\frac{\omega_{\delta,W}}{A}$. It follows that there exists an effective divisor E_1 on W' supported on E(i.e. locally $E_1 = \sum\limits_\nu m_\nu E_\nu$ for some integers m_ν and some divisors E_ν in E), together with a smooth metric h_{E_1} on $O(-E_1)$ such that

$$\pi_2^*(\frac{\omega_{\delta,W}}{A}) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E_1} > 0$$

is a Kähler form on W' for ϵ sufficient small.

Define

$$\Omega_{W'} := \pi_2^*(\omega''_\delta) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E_1}$$

Then we claim that $\Omega_{W'}$ is a strictly positive singular Hermitian (1,1) form on W'.

Indeed, the strict positivity can be seen from that $\pi_2^*(\omega''_\delta)$ is semi-positive on W' and strictly positive away from the exceptional divisor E. And near E, in the neighborhood $\pi_2{}^{-1}(\cup_\alpha U_\alpha)$, we have that $\pi_2^*(\omega''_\delta) \geqslant \frac{\pi_2^*(\omega_{\delta,W})}{A}$ and thus

$$\pi_2^*(\omega''_\delta) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E_1} \geqslant \frac{\pi_2^*(\omega_{\delta,W})}{A} - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E_1} > 0$$

iv) Since we have that $X' \subset W'$ is a resolution of singularity of $X \subset W$ with the exceptional divisor $E' \subset X'$. Then we restrict the divisor E_1 to X' to get an effective divisor E_2 supported in $E' \subset X'$ and restrict the metric h_{E_1} on the line bundle $O(-E_1)$ over W' to a smooth metric h_{E_2} on $O(-E_2)$ over X'. Then we define a strictly positive singular (1,1) form Ω' on X' as

$$\Omega' := \pi_2^*(\omega'_\delta) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E_2} > 0 \tag{4.27}$$

Here the positivity can be seen from the fact that Ω' is the restriction of $\Omega_{W'}$ from W' to its submanifold X' and that $\Omega_{W'}$ defined in step iii) is strictly positive.

□

Remark 4.5.

Notice that for the special case where $N = n$ and $\beta_{11} = \cdots = \beta_{nn} = 2$, Phong-Sturm showed in [PS12] that the smooth Kähler forms

$$\omega_{\delta,X} := \pi_1^*\omega + \delta\frac{i}{2}\partial\bar\partial\{\chi\log\sum_{i=1}^{n}|f_i(z)|^2 + 1 - \chi\} \tag{4.28}$$

$$\omega_{\delta,W} := \pi_1^*\omega + \delta\frac{i}{2}\partial\bar\partial\{\chi_W\log\sum_{i=1}^{n}|f_i(z) - \xi_i|^2 + 1 - \chi_W\} \tag{4.29}$$

pull back to $\pi_2^*(\omega_{\delta,X})$ and $\pi_2^*(\omega_{\delta,W})$, which are smooth semi-positive (1,1) forms on X' and W'. Thus we see that by defining

$$\Omega_0 := \pi_2^*(\pi_1^*(\omega) + \delta\frac{i}{2}\partial\bar\partial\{\chi\log\sum_{i=1}^{n}|f_i|^2 + 1 - \chi\}) - \epsilon\frac{i}{2}\partial\bar\partial\log h_{E_2} > 0$$

we have a smooth Kähler form Ω_0 on X'. So one can see that X' is a smooth Kähler manifold with Ω_0, which happens to be same as Ω' for the case where $N = n$ and all $\beta_{11} = \cdots = \beta_{nn} = 2$. As a generalization ($N \geqslant n$ and $0 \leqslant \beta_{ij} \leqslant 2$, for all $i \leqslant N, j \leqslant n$) to their result, the Ω' defined in (4.27) is a family of positive closed (1,1) currents $\{\Omega'_\beta\}$ on X'.

As a summary to this section, we have the following lemma

Lemma 4.6. *Given (M, p, f_i) the data of a compact Kähler manifold (M, ω) with smooth boundary ∂M, an interior point p, and local holomorphic functions f_i, $1 \leqslant i \leqslant n$ with p as their only common zero. Same as in lemma 4.2, fix the singular (1,1) form on M*

$$\omega_\delta = \omega + \delta\frac{i}{2}\partial\bar\partial\{\psi\log\sum_{i=1}^{N}\prod_{j=1}^{n}|f_j|^{\beta_{ij}} + 1 - \psi\}$$

Then there exists a compact complex manifold $X' = X'(M, p, f_i)$ with Kähler form Ω_0 and a holomorphic map $\pi : X' \longrightarrow M$, sending $\partial X' \longrightarrow \partial M$ with the following properties:

a) *There is a closed, strictly positive singular (1,1) form Ω' on X', given as*

$$\Omega' := \pi^*(\omega_\delta) - \epsilon \frac{i}{2}\partial\bar{\partial}\log h_{E'} > 0$$

for some effective divisor E' in X', some smooth metric $h_{E'}$ on $O(-E')$ and any $\epsilon > 0$ sufficiently small.

b) *The restriction $\pi|_{X'\backslash E'}$ defines a surjective holomorphic map*

$$\pi : X'\backslash E' \to M\backslash\{p\}$$

and

$$\pi_*\Omega' = \omega + \delta\frac{i}{2}\partial\bar{\partial}\{\psi \log \sum_{i=1}^{N}\prod_{j=1}^{n}|f_j(z)|^{\beta_{ij}} + 1 - \psi\} - \epsilon\frac{i}{2}\partial\bar{\partial}(\log h_{E'} \circ \pi^{-1})$$

Proof. i) We have a closed positive singular (1,1) form ω_δ on $M\backslash\{p\}$, where

$$\omega_\delta = \omega + \delta\frac{i}{2}\partial\bar{\partial}\{\psi \log \sum_{i=1}^{N}\prod_{j=1}^{n}|f_j(z)|^{\beta_{ij}} + 1 - \psi\}$$

Apply lemma 4.2 and we have the analytic subspace $X = X(M, p, f_1, \cdots, f_n)$ in the ambient space W together with a biholomorphic map $\pi_1 : X\backslash X_0 \to M\backslash\{p\}$ such that $\pi_1^*(\omega_\delta)$ extends to a singular (1,1) form ω'_δ on X.

ii) Since $X \subset W$ is only an analytic subspace with $X_0 \subset E_g \subset W$, where E_g is the exceptional divisor in $W = \mathrm{BL}_Z(M \times B_{\epsilon_0})$ over $Z = \{g_1(z,\xi) = \cdots = g_n(z,\xi) = 0\} \subset M \times B_{\epsilon_0}$ and

$$X_0 = E_g \cap \{\xi_1 = \cdots \xi_n = 0\}$$

We see that $X_{sing} \subset X_0$ and $X \backslash X_0 \subset X_{reg}$.

iii) Then apply lemma 4.3(Hironaka's theorem, [PS12]) to get an iterated blow up space $\pi_2 : W' \to W$, which is a smooth manifold with the exceptional divisor $E \subset W'$ and a smooth submanifold $X' \subset W'$ such that the restricted map

$$\pi_{2|_{X'}} : X' \to X$$

is surjective and that

$$\pi_{2|_{X' \backslash E'}} : X' \backslash E' \to X_{reg}$$

is biholomorphic. Here $E' = E \cap X' = {\pi_2}^{-1}(X_{sing}) \cap X'$.

iv) Given ${\pi_1}^*(\omega_\delta)$, which extends to the positive closed singular (1,1) form ω'_δ on X and the blow up map $\pi_{2|_{X'}} : X' \to X$ that resolves the singularity X_{sing} with exceptional divisor E', we apply lemma 4.5 to get a smooth metric $h_{E'}$ on the line bundle $O(-E')$ over X', such that ${\pi_2}^*(\pi_1^*(\omega_\delta)) - \epsilon \frac{i}{2}\partial\bar{\partial} \log h_{E'} > 0$ for sufficiently small $\epsilon > 0$. Set $\Omega' = {\pi_2}^*(\pi_1^*(\omega_\delta)) - \epsilon \frac{i}{2}\partial\bar{\partial} \log h_{E'}$ and this proves a).

v) Now we prove b). First notice that since $X_{reg} = X\backslash X_{sing} \supset X\backslash X_0$, so the map

$$\pi_1 : X_{reg} \to M\backslash\{p\}$$

is surjective. Taking its composition with the biholomorphism

$$\pi_2 : X'\backslash E' \to X_{reg}$$

defines a surjective holomorphic map

$$\pi = \pi_1 \circ \pi_2 : X'\backslash E' \to M\backslash\{p\}$$

Clearly π sends $\partial X'$ to ∂M and π is a biholomorphism between $M\backslash\{p\}$ and its inverse image, which is

$$\pi^{-1}(M\backslash\{p\}) = \pi_2^{-1}(X\backslash X_0) \subset X'\backslash E'$$

So we can push forward Ω' to $M\backslash\{p\}$ and get

$$\begin{aligned} &\pi_*\Omega'|_{M\backslash\{p\}} \\ =&\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\psi \log \sum_{i=1}^{N}\prod_{j=1}^{n}|f_j(z)|^{\beta_{ij}} + 1 - \psi\} - \epsilon\frac{i}{2}\partial\bar{\partial}(\log h_{E'} \circ \pi^{-1}) \end{aligned}$$

Here $\psi(z)$ is the cut off function supported in the neighborhood $U_0 \subset\subset U$, where ψ, U_0 and U defined in lemma 4.2.

□

4.4 A solution $\phi \in PSH(X', \Omega')$

Through the iterated blow up procedures, we constructed $E \subset W'$ with $\Omega_{W'}$ and $E' \subset X'$ with Ω', a strictly positive singular Hermitian form on X'. And as shown in remark 4.5 (See [PS12]), there is a Kähler metric on X', defined as

$$\Omega_0 := \pi_2^*(\pi_1^*(\omega) + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{i=1}^{n} |f_i|^2 + 1 - \chi\}) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E_2} > 0 \quad (4.30)$$

Note that on $X' \backslash E'$ we have

$$\begin{aligned}\Omega' &= \pi_2^*(\pi_1^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi_W \log \sum_{i=1}^{N}\prod_{j=1}^{n} |f_j(z)|^{\beta_{ij}} + 1 - \chi_W\}) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} \\ &= \pi^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{i=1}^{N}\prod_{j=1}^{n} |f_j \circ \pi_2|^{\beta_{ij}} + 1 - \chi\}) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'}\end{aligned}$$

and that on E',

$$\Omega' = \pi^*\omega + \pi_2{}^*(\delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{i=1}^{N}\prod_{j=1}^{n} |t_j|^{\beta_{ij}} + 1 - \chi\}) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'}$$

where $\chi = \chi_W \circ \pi_2$ is a cut-off function in a neighborhood of $E' \subset X'$. Then locally $\Omega' = i\partial\bar{\partial}\theta$, where θ is some continuous bounded strictly plurisubharmonic potential function.

We wish to construct a solution to the following degenerate Monge-Ampère equation with respect to Ω' on X'.

Lemma 4.7. *Let (X', Ω_0) be the compact Kähler manifold with smooth boundary and the strictly positive singular Hermitian (1,1) form Ω' on X' be defined as above.*
Then there exists a unique $\phi \in PSH(X', \Omega') \cap L^{\infty}(X')$ such that

$$(\Omega' + \frac{i}{2}\partial\bar{\partial}\phi)^n = 0 \textit{ on } X' \tag{4.31}$$

$$\phi = 0 \textit{ on } \partial X' \tag{4.32}$$

Moreover, $\phi \in C^{\alpha}(K')$ for any compact subset K' of $X' \backslash E'$ and any constant $0 < \alpha < \min\limits_{\beta_{ij} \neq 0}\{\beta_{ij}\}$.

Proof. (*Existence, a first proof using [PS09][PSS12].*)

i) Let

$$\Omega_1 := \pi^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{i=1}^{n} |f_i \circ \pi_2|^2 + 1 - \chi\} \geqslant 0 \tag{4.33}$$

be a smooth semi-positive $(1,1)$ form on X'. From the above, it satisfies the condition

$$\Omega_1 - \frac{i}{2}\partial\bar{\partial}\log h_{E'} = \Omega_0 > 0$$

By [PS09, Theorem 2](or [PSS12, Theorem 14]), there exists a unique

$$\phi_1 \in PSH(X', \Omega_1) \cap L^{\infty}(X')$$

such that

$$(\Omega_1 + \frac{i}{2}\partial\bar{\partial}\phi_1)^n = 0 \tag{4.34}$$

$$\Omega_1 + \frac{i}{2}\partial\bar{\partial}\phi_1 \geqslant 0 \tag{4.35}$$

$$\phi_{1|_{\partial X'}} = 0 \tag{4.36}$$

Moreover, $\phi_1 \in C^{\alpha_1}(X'\backslash E')$ for any $0 < \alpha_1 < 1$.

ii) On X',

$$\begin{aligned}
\Omega' &= \pi^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{i=1}^{N}\prod_{j=1}^{n}|f_j(z)|^{\beta_{ij}} + 1 - \chi\} - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} \\
&= \pi^*\omega + \delta\frac{i}{2}\partial\bar{\partial}(\chi \log\{\,|f_1 \circ \pi_2|^2 + \cdots + |f_n \circ \pi_2|^2 + \sum_{i=n+1}^{N}\prod_{k=1}^{n}|f_k \circ \pi_2|^{\beta_{ik}}\} \\
&\qquad\qquad + 1 - \chi\,) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} \\
&= \pi^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{i=1}^{n}|f_i \circ \pi_2|^2 + 1 - \chi\} \\
&\qquad + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log(1 + \sum_{i=n+1}^{N}\frac{\prod_{k=1}^{n}|f_k \circ \pi_2|^{\beta_{ik}}}{\sum_{k=1}^{n}|f_k \circ \pi_2|^2})\} - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} \\
&= \Omega_1 + \frac{i}{2}\partial\bar{\partial}F
\end{aligned}$$

Here we have let

$$\Omega_1 = \pi^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{i=1}^{n}|f_i \circ \pi_2|^2 + 1 - \chi\} \geqslant 0$$

be a smooth semi-positive (1,1) form on X' and

$$F := \delta\chi\log(1+\sum_{i=n+1}^{N}\frac{\prod_{k=1}^{n}|f_k\circ\pi_2|^{\beta_{ik}}}{\sum_{k=1}^{n}|f_k\circ\pi_2|^2}) - \epsilon\log h_{E'} \tag{4.37}$$

We see that $F = 0$ on $\partial X'$, as $\chi = 0$ and $\log h_{E'} = 0$ away from a neighborhood of E'. Let $\phi := \phi_1 - F$, then on X'

$$\begin{aligned}(\Omega' + \frac{i}{2}\partial\bar{\partial}\phi)^n &= (\Omega_1 + \frac{i}{2}\partial\bar{\partial}F + \frac{i}{2}\partial\bar{\partial}(\phi_1 - F))^n\\ &= (\Omega_1 + \frac{i}{2}\partial\bar{\partial}\phi_1)^n\\ &= 0\end{aligned}$$

thus ϕ is a solution to the degenerate Monge-Ampère equation (4.31). Since ϕ_1 and F both vanishes at $\partial X'$, we have $\phi_{|\partial X'} = 0$ and this proves (4.32).

iii) To show that ϕ is bounded, it suffices to show F is bounded on X'. Since $h_{E'} > 0$ is smooth on X', which is compact, we have that $\log h_{E'}$ is bounded on X'. It only remains to show that

$$F_1 := \chi\log(1+\sum_{i=n+1}^{N}\frac{\prod_{k=1}^{n}|f_k\circ\pi_2|^{\beta_{ik}}}{\sum_{k=1}^{n}|f_k\circ\pi_2|^2}) \quad \text{is uniformly bounded on } X'$$

This can be seen from that

$$0 \leqslant \prod_{k=1}^{n} |f_k \circ \pi_2|^{\beta_{ik}} \leqslant \max_k\{|f_k \circ \pi_2|^2\} \quad \text{and that}$$

$$\begin{aligned}
0 &\leqslant \log\Big(1 + \sum_{i=n+1}^{N} \frac{\prod_{k=1}^{n} |f_k \circ \pi_2|^{\beta_{ik}}}{\sum_{k=1}^{n} |f_k \circ \pi_2|^2} \Big) \\
&\leqslant \log\Big(1 + \sum_{i=n+1}^{N} \frac{\max_k\{|f_k \circ \pi_2|^2\}}{\sum_{k=1}^{n} |f_k \circ \pi_2|^2}\Big) \\
&\leqslant \log(1 + N - n)
\end{aligned}$$

Thus F is bounded on X' and $\phi = \phi_1 - F \in L^\infty(X')$.

iv) For any compact subset K' of $X' \backslash E'$ and any fixed constant $0 < \alpha < \min_{\beta_{ij} \neq 0}\{\beta_{ij}\}$. We have that $F_1 \in C^\alpha(K')$ and hence $F \in C^\alpha(K')$. Since $\phi_1 \in C^{\alpha_1}(X' \backslash E')$ for any $0 < \alpha_1 < 1$, it follows that $\phi = \phi_1 - F \in C^\alpha(K')$.

This completes the first proof of the existence part.

(*Existence, a second proof using Perron's envelope method*)

We give a slightly more general proof, where essentially the boundary function is given by $\phi_1 \in PSH(X', \Omega_0)$, which is continuous near the boundary. And we see that $\phi = \phi_1$ away from a neighborhood of E' in X'. Locally in this neighborhood, fix potential function θ such that $\frac{i}{2}\partial\bar{\partial}\theta = \Omega'$. Then by [Wal], $\phi + \theta$ is upper semi-continuous and ϕ lies in $PSH(X', \Omega')$.

I) Define $\omega_1 = \Omega'$ and

$$E(\omega_1, f) = \{\psi : \omega_1 + i\partial\bar{\partial}\psi \geqslant 0, {\psi^*}_{|\partial X'} \leqslant f_{|\partial X'}\}$$

to be the Perron's envelope of subsolutions with respect to ω_1 and any continuous function f defined near $\partial X'$. It is easy to see that $E(\omega_1, f)$ is not empty. Indeed, it contains constant functions $\psi = -C$ for any sufficiently large $C \geqslant 0$. Then consider the envelope with zero boundary condition

$$E(\omega_1, \phi_1) = \{\psi : \omega_1 + i\partial\bar{\partial}\psi \geqslant 0, {\psi^*}_{|\partial X'} \leqslant \phi_1\}$$

II) Take the point-wise supreme for all $\psi \in E(\omega_1, \phi_1)$ and define

$$\phi = \sup_{\psi}\{\psi \in E(\omega_1, \phi_1)\} = \sup\{\psi : \omega_1 + i\partial\bar{\partial}\psi \geqslant 0, {\psi^*}_{|\partial X'} \leqslant \phi_1\} \quad (4.38)$$

Now $\phi \in PSH(X', \omega_1)$ is a globally defined function on X' and $\phi_{|\partial X'} = \phi_1|_{\partial X'} = 0$.

III) To show that the ϕ defined above is a solution to the degenerate Monge-Ampère equation

$$(\omega_1 + i\partial\bar{\partial}\phi)^n = 0 \text{ on } X'$$

It suffices to show that ϕ is maximal with respect to all subsolutions on any small neighborhood U.

Pick any point p_1 and any small neighborhood U that contains p_1, then locally in U we have $\omega_1 = i\partial\bar{\partial}\theta$, where θ is a bounded strictly plurisubharmonic function. We wish to show that $(\omega_1 + i\partial\bar{\partial}\phi)^n = i\partial\bar{\partial}(\theta+\phi)^n = 0$ on U, so it suffices to show that $\phi + \theta \in PSH(U)$ is maximal. Recall that u is maximal on U iff for any $v \in PSH(U)$ satisfying $v \leqslant u$ outside a compact subset K of U, we have $v \leqslant u$ in U. See in Blocki's book [Bl1].

IV) Fix any function $v \in PSH(U)$ and compact subset K of U such that $v \leqslant \theta + \phi$ on $U \backslash K$. Define

$$\tilde{v} = \max(v, \theta + \phi)$$

be a plurisubharmonic function on U. Clearly $\tilde{v} = \theta + \phi$ on $U \backslash K$. Let

$$\tilde{\phi} := \tilde{v} - \theta \text{ on } U \tag{4.39}$$

and extend $\tilde{\phi}$ to the manifold X' by letting $\tilde{\phi} = \phi$ on $X' \backslash U$. Observe that outside U,

$$\omega_1 + i\partial\bar{\partial}\tilde{\phi} = \omega_1 + i\partial\bar{\partial}\phi \geqslant 0$$

and in U,

$$\omega_1 + i\partial\bar{\partial}\tilde{\phi} = i\partial\bar{\partial}\theta + i\partial\bar{\partial}(\tilde{v} - \theta) = i\partial\bar{\partial}\tilde{v} \geqslant 0$$

We obtain that $\tilde{\phi} \in PSH(X', \omega_1)$.

V) From the above, $\tilde{\phi}$ is in the envelope $E(\omega_1, \phi_1)$. And since ϕ is defined in step II) to be the supremum of $E(\omega_1, \phi_1)$, we have

$$\begin{aligned} & \tilde{\phi} \leqslant \phi && \text{everywhere in } X' \\ \Longrightarrow\ & \tilde{v} - \theta \leqslant \phi && \text{in } U \\ \Longrightarrow\ & v \leqslant \tilde{v} \leqslant \theta + \phi && \text{in } U \end{aligned}$$

This proves that $\theta + \phi$ is maximal and therefore $(\omega_1 + i\partial\bar{\partial}\phi)^n = i\partial\bar{\partial}(\theta + \phi)^n = 0$ in U. Since U is any arbitrary small neighborhood on X', we conclude that $(\omega_1 + i\partial\bar{\partial}\phi)^n = 0$ on X'.

(*Uniqueness.*)

Fix any solution $\phi \in PSH(X', \Omega')$ that satisfies (4.31) and (4.32) and let Ω_1 and F be as in (4.33) and (4.37). Notice that we have on X',

$$\Omega_1 + \frac{i}{2}\partial\bar{\partial}F + \frac{i}{2}\partial\bar{\partial}\phi = \Omega' + \frac{i}{2}\partial\bar{\partial}\phi \geqslant 0$$
$$(\Omega_1 + \frac{i}{2}\partial\bar{\partial}F + \frac{i}{2}\partial\bar{\partial}\phi)^n = (\Omega' + \frac{i}{2}\partial\bar{\partial}\phi)^n = 0$$

By letting $\phi_1 := \phi + F$, we have $\phi_1 \in PSH(X', \Omega_1)$. Besides, since $F \in L^\infty(X')$ and vanishes on the boundary $\partial X'$, we have $\phi_1 = \phi + F \in L^\infty(X')$ and that

$$\phi_{1|_{\partial X'}} = \phi_{|_{\partial X'}} = 0$$

So ϕ_1 is a bounded solution to the following Dirichlet problem for the totally degenerate Monge-Ampère equation on X'

$$(\Omega_1 + \frac{i}{2}\partial\bar{\partial}\phi_1)^n = 0 \tag{4.40}$$

$$\Omega_1 + \frac{i}{2}\partial\bar{\partial}\phi_1 \geqslant 0 \tag{4.41}$$

$$\phi_{1_{|\partial X'}} = 0 \tag{4.42}$$

Moreover, since Ω_1 satisfies the following condition

$$\Omega_1 - \epsilon \log h_{E'} = \Omega_0 > 0 \text{ on } X'$$

By uniqueness part of [PS09, Theorem 2] (or [PSS12, Theorem 14]), we see that ϕ_1 is unique. The uniqueness of $\phi \in PSH(X', \Omega') \cap L^\infty(X')$ is up to the choice of δ and $\epsilon \log h_{E'}$.

$\square$

4.5 Proof of Theorem II

We rephrase Theorem 4.1 and reduce it to the case when Γ, the number of isolated points $\{p_\gamma\}$, equals 1.

Theorem 4.8. *Let ω be a Kähler metric on compact complex manifold M of dimension $n \geqslant 2$ and assume $\partial M \neq \emptyset$ is smooth. Fix n local holomorphic functions $\{f_j\}_{1 \leqslant j \leqslant n}$ that are defined in a neighborhood of p, with p as their only common zero in this neighborhood and constants $\{0 \leqslant \beta_{ij} \leqslant 2 : 1 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant n\}$ satisfying that $\beta_{i1} + \cdots + \beta_{in} = 2$ for each fixed $1 \leqslant i \leqslant N$ and that $\sum_{i=1}^{N} \prod_{j=1}^{n} |f_j|^{\beta_{ij}}$ has p as its only zero point.*
Then there exists a small constant $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, there exists a unique solution $G(z; p, f_1 \cdots, f_n) \in PSH(M, \omega)$ to the following

$$\begin{cases} G = 0 & \text{on } \partial M & (4.43) \\ (\omega + \frac{i}{2}\partial\bar{\partial}G)^n = 0 & \text{on } M \backslash \{p\} & (4.44) \\ G(z; p, f) = \delta \log\{\sum_{i=1}^{N} \prod_{j=1}^{n} |f_j(z)|^{\beta_{ij}}\} + \phi & \text{near } p & (4.45) \end{cases}$$

for some unique $\phi \in L^\infty(M)$ that vanishes on the boundary. Moreover, G and ϕ lies in $C^\alpha(K)$ for any compact subset $K \subset M \backslash \{p\}$ and any constant $0 < \alpha < \min_{\beta_{ij} \neq 0} \{\beta_{ij}\}$. The uniqueness is up to the constant δ and a choice of cut-off function in a small neighborhood near p.

Proof. (Existence)

Fix any constant $0 < \alpha < \min_{\beta_{ij} \neq 0} \{\beta_{ij}\}$.

i) In lemma 4.4, we constructed the iterated blow up map

$$\pi : W' \to M\backslash\{p\}$$

with the exceptional divisor E and an n dimensional smooth submanifold $X' \subset W'$ and an effective divisor E' supported in $E \cap X$ and away from $\pi^{-1}(\partial M) = \partial X'$.

ii) From lemma 4.6, we have a strictly positive closed singular (1,1) form Ω' defined on X' as

$$\begin{aligned}\Omega' &= \pi^*(\omega_\delta) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} > 0 \\ &= \pi_2^*(\pi_1^*(\omega_\delta)) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} > 0\end{aligned}$$

for some smooth metric $h_{E'}$ on $O(-E')$. Here

$$\omega_\delta := \omega + \delta\frac{i}{2}\partial\bar{\partial}\{\psi\log\sum_{i=1}^{N}\prod_{j=1}^{n}|f_j|^{\beta_{ij}} + 1 - \psi\}$$

is the same as that in lemma 4.2.

iii) Now apply lemma 4.7, which showed that there exists a unique solution $\Phi \in PSH(X', \Omega') \cap L^\infty(X')$ to the degenerate Monge-Ampère equation

$$(\Omega' + \frac{i}{2}\partial\bar{\partial}\Phi)^n = 0 \text{ on } X' \tag{4.46}$$

$$\Phi_{|_{\partial X'}} = 0 \tag{4.47}$$

Moreover, $\Phi \in C^{\alpha}(K')$ for any compact subset K' in $X'\backslash E'$. Then we see that on X',

$$(\,\pi^*(\omega_\delta) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} + \frac{i}{2}\partial\bar{\partial}\Phi\,)^n = 0 \tag{4.48}$$

$$\pi^*(\omega_\delta) - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} + \frac{i}{2}\partial\bar{\partial}\Phi \geqslant 0 \tag{4.49}$$

iv) Take the composition of $\Phi - \epsilon\log h_{E'}$ with π^{-1}, which maps biholomorphically from $M\backslash\{p\}$ to $(\pi^{-1})(M\backslash\{p\}) \subset X'\backslash E'$ and define

$$\phi := (\Phi - \epsilon\log h_{E'}) \circ \pi^{-1} : M\backslash\{p\} \longrightarrow X'\backslash E' \longrightarrow \mathbb{R} \tag{4.50}$$

We get that $\phi \in PSH(M, \omega_\delta) \cap L^{\infty}(M)$ and $\phi \in C^{\alpha}(K)$ for any compact subset K in $M\backslash\{p\}$. Push (4.48) forward with π from $X'\backslash E'$ to $M\backslash\{p\}$ and since π is a surjective holomorphic map, we have that on $M\backslash\{p\}$

$$\{\,\pi_*(\pi^*(\omega_\delta)) + \frac{i}{2}\partial\bar{\partial}(\,(\Phi - \epsilon\log h_{E'}) \circ \pi^{-1})\,\}^n = 0 \tag{4.51}$$

$$(\omega_\delta + \frac{i}{2}\partial\bar{\partial}\phi\,)^n = 0 \tag{4.52}$$

$$\{\,\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\psi\log\sum_{i=1}^{N}\prod_{j=1}^{n}|f_j(z)|^{\beta_{ij}} + 1 - \psi\} + \frac{i}{2}\partial\bar{\partial}\phi\,\}^n = 0 \tag{4.53}$$

Note that it might seem in the definition of ϕ in (4.50) that ϕ depends on the constant ϵ, which is determined by the $-\epsilon\log h_{E'}$ term in Ω'. But we can see from the proof of lemma 4.7 that Φ contains a copy of $\epsilon\log h_{E'}$, thus $\phi = (\Phi - \epsilon\log h_{E'}) \circ \pi^{-1}$ is independent of the choice of $\epsilon\log h_{E'}$.

v) From the (4.53), we extract and define G and normalize by adding a constant $-\delta$ to ensure the boundary conditions,

$$G(z,p,f_1,\cdots,f_n) = \delta\{\psi \log \sum_{i=1}^{N} \prod_{j=1}^{n} |f_j(z)|^{\beta_{ij}} + 1 - \psi\} + \phi - \delta \quad (4.54)$$

$$= \delta\{\psi \log \sum_{i=1}^{N} \prod_{j=1}^{n} |f_j(z)|^{\beta_{ij}} - \psi\} + \phi \quad (4.55)$$

Clearly $G \in PSH(M\backslash\{p\},\omega) \cap C^{\alpha}(K)$.

vi) From (4.53), we have that $(\omega + \frac{i}{2}\partial\bar{\partial}G)^n = 0$ on $M\backslash\{p\}$. To see that G vanishes on ∂M, it suffices to check that the cut-off function $\psi_{|\partial M} = 0$ and that

$$\phi_{|\partial M} = (\Phi - \epsilon \log h_{E'}) \circ \pi^{-1}{}_{|\partial M}$$
$$= (\Phi - \epsilon \log h_{E'})_{|\partial X'} = 0$$

we see that $G_{|\partial M} = 0$. Moreover, we have that $\psi = 1$ in a neighborhood of p,

$$G = \delta\{\log \sum_{i=1}^{N} \prod_{j=1}^{n} |f_j(z)|^{\beta_{ij}}\} + \phi - \delta$$

and $\phi - \delta \in L^{\infty}(M)$. This proves the log singularity at p formulated by (4.45).

vii) In order to show that $G \in PSH(M,\omega)$, it suffices to show that G extends over p as a plurisubharmonic function.

We can see this by letting

$$\tilde{G}_\epsilon = G(z) + \epsilon \log|z-p| \qquad \text{on } M\backslash\{p\} \tag{4.56}$$

$$\tilde{G}_\epsilon = -\infty \qquad \text{on } p \tag{4.57}$$

For fixed $\epsilon > 0$, $\tilde{G}_\epsilon$ is ω-plurisubharmonic over M as near p,

$$\limsup_{z\to p} \tilde{G}_\epsilon(z) = \limsup_{z\to p}(G(z) + \epsilon \log|z-p|) = -\infty \leqslant \tilde{G}_\epsilon(p)$$

Then denote $u(z) = (\sup_{\epsilon>0} \tilde{G}_\epsilon)^*$ and we have $u(z) \in PSH(M,\omega)$ because of the general fact that upper semicontinuous regularizations of supremums of plurisubharmonic functions are still plurisubharmonic. See [Dw, Corollary 5.3]. Moreover, we see that

$$u(z) = G(z) \qquad \text{on } M\backslash\{p\} \tag{4.58}$$

$$u(z) = \limsup_{z\to p} G(z) \qquad \text{at } p \tag{4.59}$$

By redefining G as u, we have completed the proof of existence part.

(Uniqueness) We prove it by contradiction.

viii) Let G be a solution defined in the existence part. Suppose that there exists another

$$G_1(z;p,f_1\cdots,f_n) \in PSH(M,\omega)$$

such that

$$\begin{cases} G_1 = 0 & \text{on } \partial M \quad (4.60) \\ (\omega + i\partial\bar{\partial}G_1)^n = 0 & \text{on } M\backslash\{p\} \quad (4.61) \\ G_1 = \delta\{\psi_1 \log \sum_{i=1}^{N} \prod_{j=1}^{n} |f_j(z)|^{\beta_{ij}} - \psi_1\} + \phi_1 & \text{near } p \quad (4.62) \end{cases}$$

where $\phi_1 \in L^\infty(M)$ and vanishes on ∂M. Here the δ in G_1 is the same as that in G and ψ_1 is some other cut-off function supported in another neighborhood U_1 of p. Without loss of generality we can replace both U_1 and U by a smaller neighborhood, and assume the cut-off function ψ_1 is the same as ψ. Then we show $G_1 = G$ by showing that $\phi_1 = \phi$.

On M, we have that

$$\phi_1 = G_1 - \delta\{\psi \log \sum_{i=1}^{N} \prod_{j=1}^{n} |f_j(z)|^{\beta_{ij}} - \psi\} \tag{4.63}$$

$$= G_1 - G + \phi(z) \tag{4.64}$$

Let the smooth metric $h_{E'}$ over $O(-E')$, and $\Phi \in PSH(X', \Omega')$ as before, i.e. $\Phi := \phi \circ \pi + \epsilon \log h_{E'}$ where the $\epsilon \log h_{E'}$ and ϕ are taken to be the same as that defined in the existence part.

Compose $\phi_1 : M \to \mathbb{R}$ with the iterated blow up map $\pi : X' \to M$ which has been constructed together with $E' \subset X' \subset W'$. Define

$$\Phi_1 := \phi_1 \circ \pi + \epsilon \log h_{E'}$$

Then $\Phi_1 \in PSH(X'\backslash E', \Omega')$. And since by assumption $\phi_1 \in L^\infty(M)$, we have

$$\Phi_1 \in L^\infty(X')$$

ix) We wish to show $\phi_1 = \phi$ on M by showing $\Phi_1 = \Phi$ on X'. Since Φ_1 is bounded and E' is a subset of a pluripolar set in X', we see that Φ_1 extends uniquely over E' by applying an extension theorem of Demailly(See [Dem, Theorem 5.24]). Therefore, $\Phi_1 \in PSH(X', \Omega')$ and the boundary condition ${\Phi_1}_{|_{\partial X'}} = 0$ can be seen from that ${\phi_1}_{|_{\partial M}} = 0$ and that $\log {h_{E'}}_{|_{\partial X'}} = 0$, as $\log h_{E'}$ is supported in a neighborhood of E'.

Now we claim that

$$(\Omega' + \frac{i}{2}\partial\bar{\partial}\Phi_1)^n = 0 \text{ on } X' \tag{4.65}$$

This is true on $X'\backslash E'$, as can be seen from the fact the the restricted map

$$\pi_{|_{X'\backslash E'}} : \quad X'\backslash E' \longrightarrow M\backslash\{p\}$$

is holomorphic and surjective. Since by assumption that on $M\backslash\{p\}$

$$0 = (\omega + \frac{i}{2}\partial\bar{\partial}G_1)^n \tag{4.66}$$

$$= (\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\psi \log \sum_{i=1}^{N}\prod_{j=1}^{n}|f_j(z)|^{\beta_{ij}} + 1 - \psi\} + \frac{i}{2}\partial\bar{\partial}\phi_1)^n \tag{4.67}$$

By pulling back with the map π, we have that on $X'\backslash E'$,

$$0 =(\pi^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{i=1}^{N}\prod_{j=1}^{n} |f_j \circ \pi|^{\beta_{ij}} + 1 - \chi\} + \frac{i}{2}\partial\bar{\partial}\phi_1 \circ \pi)^n \tag{4.68}$$

$$=(\pi^*\omega + \delta\frac{i}{2}\partial\bar{\partial}\{\chi \log \sum_{i=1}^{N}\prod_{j=1}^{n} |f_j \circ \pi|^{\beta_{ij}} + 1 - \chi\} - \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} \tag{4.69}$$

$$+ \epsilon\frac{i}{2}\partial\bar{\partial}\log h_{E'} + \frac{i}{2}\partial\bar{\partial}\phi_1 \circ \pi)^n \tag{4.70}$$

$$=(\Omega' + \frac{i}{2}\partial\bar{\partial}\Phi_1)^n \tag{4.71}$$

x) We show that $(\Omega' + \frac{i}{2}\partial\bar{\partial}\Phi_1)^n = 0$ on E' as well. Locally we can define a potential function θ_1 such that, $\Omega' + \frac{i}{2}\partial\bar{\partial}\Phi_1 = \Omega_0 + \frac{i}{2}\partial\bar{\partial}F_1 + \frac{i}{2}\partial\bar{\partial}\Phi_1 = \frac{i}{2}\partial\bar{\partial}\theta_1$, where Ω_0 and F_1 are defined as in (4.30)(4.37). Since Ω_0 is smooth Kähler form on X' and the functions F_1 and Φ_1 are bounded on X', we see that θ_1 is a locally bounded plurisubharmonic function on X'. Now apply the general fact that for any locally bounded plurisubharmonic function the Monge-Ampère measure takes no mass at pluripolar sets and their subsets. See [Bl1, Prop 2.2.3, Theorem 3.1]. So we see that $(\partial\bar{\partial}\theta_1)^n = 0$ on E' and therefore $(\Omega' + \frac{i}{2}\partial\bar{\partial}\Phi_1)^n = 0$ on entire X'.

We now apply the uniqueness part of lemma 4.7 and consider Φ and Φ_1 are two functions both satisfying (4.31)(4.32), so we must have $\Phi = \Phi_1$. This proves the uniqueness part of the theorem.

$\square$

4.6 Applications

As a direct application of Theorem 4.8, we give an answer to Question 1 in the case where $\beta_1, \cdots, \beta_n$ are all positive rational numbers. We first assume that all β's are equivalent, i.e. $\beta_1 = \cdots = \beta_n = \beta = \frac{r}{k} > 0$, where r and k are relatively prime positive integers.

Corollary 4.9. *Let ω be a Kähler metric on compact complex manifold M of dimension $n \geqslant 2$ and assume $\partial M \neq \emptyset$ is smooth. Fix any positive rational number $\beta = \frac{r}{k} > 0$ and any local holomorphic functions $\{f_j\}_{1 \leqslant j \leqslant n}$ that are defined in a neighborhood of p, with p as their only common zero in this neighborhood.*

Then there exists a small constant $\delta_1 > 0$ such that for all $0 < \delta < \delta_1$, there exists a unique solution $G(z; p, f_1 \cdots, f_n) \in PSH(M, \omega)$ to the following

$$\begin{cases} G = 0 & \text{on } \partial M & (4.72) \\ (\omega + i\partial\bar{\partial}G)^n = 0 & \text{on } M\backslash\{p\} & (4.73) \\ G(z; p, f) = \delta \cdot \psi \, \log(|f_1|^\beta + \cdots + |f_n|^\beta)\} + \phi & \text{near } p & (4.74) \end{cases}$$

where $G \in C^\alpha(K)$ and $\phi \in L^\infty(M) \cap C^\alpha(K)$ for any compact subset $K \subset M\backslash\{p\}$ and any constant $0 < \alpha < \frac{1}{k}$. The uniqueness is with respect to a given choice of δ and a choice of a cut-off function ψ supported in a small neighborhood near p.

Proof. i) Fix any positive integer k, a neighborhood U sufficiently small such that p is the only common zero of $f_1, f_2, \cdots, f_n$ and some local coordinate system $(z_1, z_2, \cdots, z_n)$ on U.

We look at the following function

$$\begin{aligned} g_0 &= 2k\log(|f_1|^{\frac{1}{k}} + \cdots + |f_n|^{\frac{1}{k}}) \\ &= \log(|f_1|^2 + |f_2|^2 + \cdots + |f_n|^2 + \sum_{\substack{i_1+i_2+\cdots+i_n=2k \\ i_1<2k,\cdots,i_n<2k}} |f_1|^{\frac{i_1}{k}}|f_2|^{\frac{i_2}{k}}\cdots|f_n|^{\frac{i_n}{k}}) \end{aligned}$$

Clearly g_0 satisfies the homogeneity condition and the single point vanishing condition that are required by theorem 4.8, and therefore there exists a small $\delta_0 > 0$ such that for all $\delta < \delta_0$, there exists a unique pluricomplex Green's function $G_0 \in PSH(M, \omega)$, such that

$$\begin{aligned} G_0 &= \psi \cdot \delta\, g_0 + \phi_0 && (4.75) \\ &= \psi \cdot \delta\, 2k\log(|f_1|^{\frac{1}{k}} + \cdots + |f_n|^{\frac{1}{k}}) + \phi_0 && (4.76) \end{aligned}$$

in a neighborhood U_0 of p. Here ψ is some cut-off function supported in the neighborhood U_0 and without loss of generality, we still denote it as U. Moreover, ϕ_0 is unique in $L^\infty(M)$ and G_0, ϕ_0 are in $C^\alpha(K)$ for compact subset K of $M\backslash\{p\}$ and any constant $0 < \alpha < \frac{1}{k}$.

ii) By taking $\delta_1 = 2k \cdot \delta_0$ and letting $g_1 = \log(|f_1|^{\frac{1}{k}} + \cdots + |f_n|^{\frac{1}{k}})$, we see that for all $\delta < \delta_1$, there exists a unique pluricomplex Green's function $G_1 \in PSH(M, \omega)$, such that

$$\begin{aligned} G_1 &= \psi \cdot \delta g_1 + \phi_1 && (4.77) \\ &= \psi \cdot \delta \log(|f_1|^{\frac{1}{k}} + \cdots + |f_n|^{\frac{1}{k}}) + \phi_1 && (4.78) \end{aligned}$$

in the neighborhood U of p and G_1 vanishes at the boundary ∂M. Here ϕ_1 has the same properties of ϕ_0.

iii) Observe that we have such existence and uniqueness of G_1 for any choices of $\{f_1, \cdots, f_n\}$, provided that p is the only common zero of $f_1, f_2, \cdots, f_n$. Now pick any other positive integer $r > 0$, clearly p is the only common zero of the holomorphic functions ${f_1}^r, {f_2}^r, \cdots, {f_n}^r$. Replace $\{f_1, f_2, \cdots f_n\}$ by $\{{f_1}^r, {f_2}^r, \cdots, {f_n}^r\}$ and we conclude that for all $\delta < \delta_1$, there exists a unique pluricomplex Greens function $G \in PSH(M, \omega)$, such that

$$G = \psi \cdot \delta \log(|f_1|^{\frac{r}{k}} + \cdots + |f_n|^{\frac{r}{k}}) + \phi \tag{4.79}$$

in the neighborhood U of the p and G vanishes at the boundary. Here ϕ has the same properties of ϕ_1. This proves the corollary.

□

The same argument in the corollary 4.9 works for all positive rational numbers $\beta_1, \cdots, \beta_n$, not necessarily equivalent. Now fix positive rational numbers $\beta_1 = \frac{r_1}{k_1}, \cdots, \beta_n = \frac{r_n}{k_n}$. We will have α-Hölder continuity, for any $0 < \alpha < \frac{1}{k_1 \cdot k_2 \cdots k_n}$.

Corollary 4.10.

Let $M, \omega, p, f_1, \cdots, f_n$ be the same as Corollary 4.9 and $\beta_1, \cdots, \beta_n$ be n positive rational numbers as above.

Then there exists a small constant $\delta_1 > 0$ such that for all $0 < \delta < \delta_1$, there *exists a unique solution* $G(z; p, f_1, \cdots, f_n) \in PSH(M, \omega)$ *to the following*

$$\begin{cases} G = 0 & \textit{on } \partial M \quad (4.80) \\ (\omega + i\partial\bar{\partial}G)^n = 0 & \textit{on } M\backslash\{p\} \quad (4.81) \\ G(z; p, f) = \delta \cdot \psi \, \log(|f_1|^{\beta_1} + \cdots + |f_n|^{\beta_n})\} + \phi & \textit{near } p \quad (4.82) \end{cases}$$

where $G \in C^{\alpha}(K)$ *and* $\phi \in L^{\infty}(M) \cap C^{\alpha}(K)$ *for any compact subset* $K \subset M\backslash\{p\}$ *and any constant* $0 < \alpha < \frac{1}{k_1 \cdot k_2 \cdots k_n}$. *The uniqueness is with respect to a given choice of* δ *and a choice of a cut-off function* ψ *supported in a small neighborhood near* p.

Proof. Fix $D := k_1 \cdot k_2 \cdots k_n$, any constant $0 < \alpha < \frac{1}{D}$ and any compact subset K in $M\backslash\{p\}$. For each $1 \leqslant i \leqslant n$, let $R_i := r_i \cdot \frac{D}{k_i}$. Define n holomorphic functions

$$F_1 := {f_1}^{R_1}, \cdots, F_n := {f_n}^{R_n}$$

which clearly have p as the only vanishing locus. Then apply Corollary 4.9 to $F_1, \cdots, F_n$ and we see that there exists a δ_1 such that for all $\delta < \delta_1$, there exists a unique pluricomplex Green's function $G \in PSH(M, \omega) \cap C^{\alpha}(K)$ such that

$$G = \psi\delta \log(|F_1|^{\frac{1}{D}} + \cdots + |F_n|^{\frac{1}{D}}) + \phi \tag{4.83}$$

$$= \psi\delta \log(|f_1|^{\frac{r_1}{k_1}} + \cdots + |f_n|^{\frac{r_n}{k_n}}) + \phi \tag{4.84}$$

in a neighborhood U of p. Here ψ is some cut-off function supported in the neighborhood U and ϕ is a unique function in $L^{\infty}(M) \cap C^{\alpha}(K)$.

$\square$

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