DESIGN OF PIEZOELECTRIC GYRO-SENSOR USING
LANTHANUM GALLIUM SILLICATE (La$_3$Ga$_5$SiO$_{14}$) AND
TEMPERATURE BEHAVIOR OF LANGASITE

By
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A Dissertation submitted to the
Graduate School – New Brunswick
Rutgers, The State University of New Jersey
In partial fulfillment of the requirements
For the degree of
Doctor of Philosophy
Graduate Program in Civil and Environmental Engineering

Written under the direction of
Yook-Kong Yong

And approved by

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New Brunswick, New Jersey
January 2016
ABSTRACT OF THE DISSERTATION

Design of Piezoelectric Gyro-Sensor Using

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By GOBONG CHOI

Dissertation Director

Yook-Kong Yong

Lanthanum Gallium Silicate (La$_3$Ga$_5$SiO$_{14}$, Langasite, LGS) and its isomorphs are gaining in popularity due to their superior piezoelectric material properties. These materials have similar crystal properties as the quartz crystal, which is widely used in BAW applications such as resonators and piezoelectric vibratory gyroscopes. The langasite and langatate have better stability at high temperatures that have no phase transition up to the melting point at 1475°C and 1450°C, respectively. Additionally, quartz, langasite, and langatate crystals are trigonal class crystals that belonging to the same point group 32, for which well-known methods of trapping energy of piezoelectric resonator can be easily and accurately applied.

Quartz, langasite and langatate crystals are used for studying and designing piezoelectric vibratory gyroscopes. The three-dimensional finite element models are developed to simulate the characteristic of the vibratory gyroscopes. The piezoelectric double-ended tuning fork gyroscope is used to verify and validate the finite element analysis. The geometric and gyroscopic sensitivity of the finite element models of the quartz doubled-ended tuning fork gyroscope showed the excellent agreement with the
experimental data. The results show that langasite and langatate crystal are more effective than quartz crystal for high precision piezoelectric gyroscopes.

A length-extension vibratory gyroscope is our newly designed gyroscope, which utilizes a length extension mode as a driving mode and a flexure mode as a sensing mode to detect the Coriolis force generated by the rotation of the system. The sizes of the gyroscopes vary due to the material properties and length of the driving arms. The langasite and langatate gyroscopes are nearly the same size and are smaller than the quartz gyroscope. The results show that the newly designed length-extension gyroscope can be used as a gyro-sensor, and that langasite and langatate gyroscopes provide the stronger sensitivity to angular velocity than the quartz gyroscope. The length extension gyroscope is also able to detect the angular velocities about the other two axes(x- and y-axes).

The three-dimensional Langrangian formulations are used to calculate the frequency-temperature behaviors of the langasite. Normally, the piezoelectric effect is ignored in analysis of the quartz crystal due to their weak piezoelectric coupling. The piezoelectric coupling factor of the langasite crystal is much larger than that of the quartz crystal, which cannot be ignored. However, the analysis shows that the piezoelectric effect on the frequency-temperature behavior of langasite is negligible. The study on the temperature behaviors of the langatate cannot be completed, since the third-order non-linear elastic constants have not been published. The results show that no zero-temperature compensated cuts for A- and B- mode exist, while a more than sufficient number of the zero-temperature compensated orientations of langasite for C-mode have been identified. The temperature compensated cuts of langasite for any order higher than the first were not found.
Dedication

I dedicate this dissertation to my family and my wife for their constant support and unconditional love.

I love you all dearly.
Acknowledgements

I would like to express my sincere appreciation to my advisor, Dr. Yook-kong Yong for his constant guidance and encouragement, without which this work would not have been possible. For his unwavering support, I am truly grateful. To my committee, Dr. Perumalsamy N. Baragura, Dr. Mitsunori Denda, and Dr. Husam Najm, I am extremely grateful for the assistance and suggestions throughout my research. To all faculty members and staffs of civil engineering at Rutgers university, Thank you – especially to linda Szary and Gina Cullari for always listening and giving me words of encouragement.

I would also like to thank you all my friends and colleagues at the Rutgers University for their encouragement and moral support.

I especially thank my mom, dad and sister. My hard-working parents have sacrificed their lives for my sister and myself and provided unconditional love and care. I would not have made this far without them. My sister has been my best friend all my life and thank her for advice and support.

My deepest appreciation belong to my wife, Hanna Wee, for her support, patience and understanding.
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**Introduction**

In 1880, Jacques and Pierre Curie brothers discovered the piezoelectric effect. They found some materials, which generate electrical charges when a mechanical stress was applied. The word “piezo” is a Greek word means “to press or to squeeze” [1]. Therefore, piezoelectric means electricity generated from the pressure. Those piezoelectric materials have asymmetric crystal structure. The piezoelectric materials can be divide into three main categories; crystals, ceramics and thin films [2]. The most well-known and widely used piezoelectric material is quartz crystal (SiO$_2$).

Ever since micro-electromechanical systems are introduced, the piezoelectric gyro-sensors are dramatically gained the popularly due to its manufacture cost, power consumption and its size. Due to these reasons, the piezoelectric gyro-sensors are widely used in many industries such as automotive, aerospace, telecommunications, consumer electronics and entertainment industries. However, the piezoelectric gyro-sensors have to overcome many technical problems. The problems are related to the performance, complexity of the geometry and shock resistance due to the decrease of the size of the sensors. Quartz crystal gyro-sensors are more durable and have much simple geometry compared to other piezoelectric gyro-sensors based on ceramic piezoelectric materials.

The gyroscopic sensitivity of the gyroscope is one of the most important criteria in the design of gyro-sensor, which directly related to type of material and the geometry of the gyroscope. Quartz and ceramic type piezoelectric materials are widely used material for gyro-sensor. Lanthanum gallium silicate (La$_3$Ga$_5$SiO$_{14}$) and lanthanum gallium tantalite (La$_3$Ga$_{5.5}$Ta$_{0.5}$O$_{14}$), also known as langasite and langatate, respectively, are other piezoelectric materials that have same crystal properties as quartz crystal. They all belong
to point group 32 and trigonal class. Langasite and langatate crystals are fairly newer piezoelectric materials, which introduced in early 1980s by Russian scientists. Due to the number of reasons, the crystals are not utilized for any applications until recent years. Both crystals started to get attention after scientist found that they are suitable for high temperature condition and they perform better than quartz crystal due to their piezoelectric material properties. Both langasite and langatate have stronger electromechanical coupling factor, heavier mass density and higher temperature frequency coefficient constant than the quartz crystal. These material properties make langasite and langatate gyro-sensors perform superior to quartz gyroscope. The geometry of the gyroscope is also affecting the gyroscopic characteristics which can be analyzed by parametric modal analysis.

In this dissertation work, the langasite and langatate doubled-ended tuning fork gyro-sensor are analyzed and compared with the quartz gyroscope. Design of the gyroscopes and analyses were done using finite element method. The Furthermore, the length extension gyroscopes are introduced and analyzed with the same analysis method. The temperature behavior of the langasite is also analyzed by numerical method. Since, third order non-linear constants of langatate are not published; the temperature analysis on langatate has been omitted.

**Thesis Organization:**

The work carried out in this dissertation has been organized into six chapters.

- Chapter 1 provides a brief introductory of piezoelectric crystals
• Chapter 2 contains the introductory of piezoelectric vibratory gyroscope and its governing equations. The equations derived in this chapter are used to design and analyze the piezoelectric gyroscopes.

• Chapter 3 consists of evaluating the characteristics of the piezoelectric doubled-ended tuning fork. The finite element analysis is used to analyze the double-ended tuning forks gyro-sensors.

• Chapter 4 includes design of length extension gyroscopes. The gyroscopic and geometric sensitivity analyzes are also includes in this chapter.

• Chapter 5 contains the analysis of temperature nonlinearly behavior of langatate crystal

• Chapter 6 includes the conclusion and future improvement of this dissertation.
Chapter 1. Basics of Piezoelectric Crystals

1.1 Crystal systems and basic terminology

The crystal means a solid in which the atoms are arranged in a single pattern repeated through the body. The crystal atoms are occurring in small groups, all groups being exactly alike, similarly oriented, and regularly aligned in all three dimension. Each group are bounded by a parallelepiped and each parallelepiped can be considered as an ultimate building blocks of crystal. The crystal is formed by stacking the basic parallelepiped without any spaced between them. Such a building block is called a unit cell. Since the choice of a particular set of atoms to form a unit cell is arbitrary, it is evident that there is a wide range of choices in the shape and dimensions of the unit cell. In practice, the unit cell is selected by the symmetry of the crystal and by simply related to the actual crystal faces and X-ray reflections.

In crystallography, the properties of a crystal is represents in terms of natural coordinate system which the axes are indicated by the latter $a$, $b$ and $c$. For example, theses axes are equal length and are mutually perpendicular in a cubic crystal while the axes are unequal lengths and no two axes are mutually perpendicular in triclinic crystal. The faces of any crystal are all parallel to the planes whose intercepts on the $a$, $b$, and $c$ axis are small multiples of unit distances or infinity, in order that their reciprocals, when multiplied by a small common factor, are all small integers or zero, which represent the indices of the plane. In the orthorhombic, tetragonal, and cubic systems, these faces are normal to $a$, $b$ and $c$ axes. Even in the monoclinic and triclinic systems, these faces contain, respectively, the $b$ and $c$, $a$ and $c$, and $a$ and $b$ axes. As referred to the set of rectangular axes $X$, $Y$, $Z$, these indices are in general irrational except for cubic crystals.
The crystals are commonly classified into seven systems; triclinic, monoclinic, orthorhombic, tetragonal, trigonal, hexagonal, and cubic [3]. The seven systems are further divided into point groups, also known as classes, based on their symmetry with respect to a point. For example, the trigonal crystals all possess a three-fold axis, while those of the tetragonal and hexagonal crystal systems possess a four-fold and six-fold axis, respectively. The cubic crystal groups all have multiple threefold axes. The orthorhombic point groups have two-fold symmetry either 2 or \( m \) with respect to each of the X-, Y-, Z-directions of an orthogonal axis system, while the monoclinic point groups are limited to two-fold symmetry with respect to a single axis direction. Finally, the triclinic point groups can only have an axis of order 1. Overall, there are total 32-point groups. Table 1.1.1 and 1.1.2 gives the summary of crystal system and provides clarification by identifying the symmetry directions for each crystal system.
Table 1.1.1 Summary of crystal systems [IEEE Standards on Piezoelectricity]

<table>
<thead>
<tr>
<th>Crystal System</th>
<th>International Point Group</th>
<th>Axes Identification, Crystallographic</th>
<th>Axes Identification, Rectangular</th>
<th>e−−e− Axes (Note 3)</th>
<th>Schoenflies Symbol</th>
<th>Example</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triclinic</td>
<td>p₁ p₁ p₁</td>
<td>1 1</td>
<td>1010</td>
<td>Z X Z</td>
<td>C₁</td>
<td>Anisotropic</td>
<td>Cu₃NO₄</td>
</tr>
<tr>
<td>Monoclinic</td>
<td>m m m</td>
<td>2 2 2</td>
<td>1010</td>
<td>Z Y Z</td>
<td>C₂</td>
<td>Ethylene diamine tetracetic acid (EDT)</td>
<td>C₄H₈O₄N₂</td>
</tr>
<tr>
<td>Orthorhombic</td>
<td>n n n</td>
<td>2 2 2</td>
<td>1010</td>
<td>Z Y Z</td>
<td>C₃</td>
<td>Lithium dihydrogen sulfate</td>
<td>Li₂H₂SO₄</td>
</tr>
<tr>
<td>Tetragonal</td>
<td>4/2 4/2 4/2</td>
<td>2 2 2</td>
<td>1010</td>
<td>Z Y Z</td>
<td>C₄</td>
<td>Barium nitrate</td>
<td>Ba(NO₃)₂</td>
</tr>
<tr>
<td>Trigonal</td>
<td>3 3</td>
<td>2 2 2</td>
<td>1010</td>
<td>Z Y Z</td>
<td>C₃</td>
<td>Sodium carbonate</td>
<td>Na₂CO₃</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Crystal System</th>
<th>International Point Group</th>
<th>Axes Identification, Crystallographic</th>
<th>Axes Identification, Rectangular</th>
<th>e−−e− Axes (Note 3)</th>
<th>Schoenflies Symbol</th>
<th>Example</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hexagonal</td>
<td>6 6 6</td>
<td>2 2 2</td>
<td>1010</td>
<td>Z Y Z</td>
<td>C₆</td>
<td>Lithium carbonate</td>
<td>Li₂CO₃</td>
</tr>
<tr>
<td>Cubic</td>
<td>2 2 2</td>
<td>2 2 2</td>
<td>1010</td>
<td>Z Y Z</td>
<td>C₆</td>
<td>Barium nitrate</td>
<td>Ba(NO₃)₂</td>
</tr>
</tbody>
</table>

(See 2.2.3)
## Significant Symmetry Directions for the International (Hermann-Mauguin) Symbol

<table>
<thead>
<tr>
<th>Crystal System</th>
<th>First Symbol</th>
<th>Second Symbol</th>
<th>Third Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triclinic</td>
<td>None</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Monoclinic</td>
<td>b (1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Orthorhombic</td>
<td>a (1)</td>
<td>b (1)</td>
<td>c (1)</td>
</tr>
<tr>
<td>Tetragonal</td>
<td>c (1)</td>
<td>a (2)</td>
<td>[110] (2)</td>
</tr>
<tr>
<td>Trigonal</td>
<td>c (1)</td>
<td>a (3)</td>
<td></td>
</tr>
<tr>
<td>Hexagonal</td>
<td>c (1)</td>
<td>a (3)</td>
<td>Digonal (3)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>[1010 axis]</td>
</tr>
<tr>
<td>Cubic</td>
<td>a (3)</td>
<td>[111] (4)</td>
<td>[110] (6)</td>
</tr>
</tbody>
</table>

*The number after each direction indicates the multiplicity of that direction in the crystal.*

Table 1.1.2 Symmetry direction for each crystal system [IEEE Standards on Piezoelectricity]
1.2 Trigonal class piezoelectric material

The primary piezoelectric materials of interest in this dissertation are quartz, langasite and langatate, which all have a trigonal system [4] [5]. The trigonal system is one of the seven crystal systems in crystallography. Crystallography is the branch of the science of determining the arrangement of the atoms in the crystalline solid. Although quartz, langasite and langatate belong to the trigonal system, their unit cells are hexagonal. The Bravais-Miller system is often used to specify the hexagonal crystal system. This is the only crystal system that has a four-axis coordinate system as shown in figure 1.2.1. The system consist with three equivalent secondary axes, $a1$, $a2$ and $a3$, that are of equal length to each other of 120 degrees in a plane normal to $c$. These axes are either perpendicular to a plane of symmetry parallel to a two-fold axis or if there are neither two-fold axes perpendicular to $c$ nor plane of symmetry parallel to $c$, the $a$ axes become the smallest unit cell. The $x$-axis coincides in direction and sense with any one of $a$ axes. The $z$-axis is parallel to $c$ and the $y$-axis is perpendicular to $z$- and $x$-axis, which form a right-handed coordinate system. Positive-sense rules for $+Z$, $+X$, $+Y$ are listed in Table 1.2.1 for the trigonal and hexagonal system crystals. In addition, the quartz, langasite and langatate are belongs to the 32 symmetry classes which have a 3 fold rotational symmetry on one and a two-fold rotational symmetry on another axis.
Figure 1.2.1 Bravais-Miller system

Figure 1.2.2 Left and Right-handed quartz crystals, Trigonal class 32 [IEEE Standards on Piezoelectricity]
<table>
<thead>
<tr>
<th>Class</th>
<th>+Z</th>
<th>+X</th>
<th>+Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Positive $d_{33}$</td>
<td>Positive $d_{11}$</td>
<td>Form right-handed system</td>
</tr>
<tr>
<td>32</td>
<td>Arbitrary</td>
<td>Positive $d_{11}$</td>
<td>Form right-handed system</td>
</tr>
<tr>
<td>3$m$</td>
<td>Positive $d_{33}$</td>
<td>Form right-handed system</td>
<td>Positive $d_{22}$</td>
</tr>
<tr>
<td>6</td>
<td>Positive $d_{33}$</td>
<td>Arbitrary</td>
<td>Form right-handed system</td>
</tr>
<tr>
<td></td>
<td>Form right-handed system</td>
<td>Positive $d_{11}$</td>
<td>Positive $d_{22}$</td>
</tr>
<tr>
<td>622</td>
<td>Arbitrary</td>
<td>Arbitrary</td>
<td>Form right-handed system</td>
</tr>
<tr>
<td>6$m$m</td>
<td>Positive $d_{33}$</td>
<td>Arbitrary</td>
<td>Form right-handed system</td>
</tr>
<tr>
<td>6$m$2</td>
<td>Arbitrary</td>
<td>Positive $d_{11}$</td>
<td>Form right-handed system</td>
</tr>
</tbody>
</table>

Table 1.2.1 Positive sense rules for Z, X, and Y for trigonal and hexagonal crystals

[IEEE Standards on Piezoelectricity]
1.3 Governing Equations of Piezoelectricity

The governing equations of linear and non-linear piezoelectricity are listed in this section [6] [7] [8]. These equations are implemented in COMSOL Multiphysics finite element analysis software and Matlab to analyze the gyroscopes and study the frequency-temperature behavior of langasite crystal in succeeding chapters.

1.3.1 Linear Piezoelectricity

Strain-displacement relation

\[ \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \]  (1.3.1)

Equation of motion

\[ T_{i,j,j} = \rho \ddot{u}_{j} \]  (1.3.2)

\[ T_{ij} = T_{ij} \]  (1.3.3)

Electric field-potential relation

\[ E_{i} = -\varphi_{i} \]  (1.3.4)

Electrostatic

\[ D_{i,i} = 0 \]  (1.3.5)

Constitutive equations

\[ T_{ij} = e_{ijkl}^{E} \varepsilon_{kl} - e_{klj}^{E} E_{k} + \eta_{ijkl} \dot{\varepsilon}_{kl} \]  (1.3.6)

\[ D_{i} = e_{ikl} \varepsilon_{kl} + e_{ik}^{E} E_{k} \]  (1.3.7)

Where \( T_{ij} \) is the Cauchy stress tensor, \( u_{i} \) is the component of displacement, \( \varepsilon_{ij} \) is the lagrangian strain tensor, \( \rho \) is the mass density of the material and \( D_{i} \) is the electric displacement. \( E_{i} \) is the electric field, \( \varphi \) is the electric potential, \( e_{ijkl}^{E} \) is the elastic moduli of an anisotropic body, \( e_{ikl} \) is the piezoelectric constant describing the effect of an electric
field on the mechanical stress, $\epsilon_{ik}^s$ is the dielectric permittivity and $\eta_{ijkl}$ is the viscoelastic constant.

### 1.3.2 Non-linear Piezoelectricity

**Strain-displacement relation**

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i} + u_{i,k}u_{k,j})$$  \hspace{1cm} (1.3.8)

**Equation of motion**

$$(T_{ij} + T_{jk}u_{i,k})_j = \rho \ddot{u}_j$$  \hspace{1cm} (1.3.9)

$T_{ij} = T_{ij}$  \hspace{1cm} (1.3.10)

**Electric field-potential relation**

$$E_i = -\phi, i$$  \hspace{1cm} (1.3.11)

**Electrostatic**

$$D_{i,i} = 0$$  \hspace{1cm} (1.3.12)

**Constitutive equations**

$$T_{ij} = (C_{ijkl} + C_{ijklmn}\epsilon_{nm} - e_{ijklm}E_m)\varepsilon_{kl} - (e_{ijklmn}\varepsilon_{kl})E_m + \eta_{ijkl}\dot{\varepsilon}_{kl}$$  \hspace{1cm} (1.3.13)

$$D_i = (-e_{ikl} - e_{ijklm}\varepsilon_{mj})\varepsilon_{kl} + (\varepsilon_{ml}^e + \beta_{mkl}^e E_k)E_m$$  \hspace{1cm} (1.3.14)

where $e_{ijklm}$ is the nonlinear piezoelectric constants and $\beta_{mkl}^e$ is the electro-optical coefficients. The constitutive equation can be written in different forms, which listed in Appendix B. The stress-charge form of the piezoelectric constitutive equations, shown in equation 1.3.13 and 1.3.14 are implemented in finite element analysis.
Chapter 2. Microelectromechanical System (MEMS) Piezoelectric Vibratory Gyroscopes

2.1 Introduction

The vibratory gyroscope is a device, which utilizes the Coriolis force to measure the changes in angular momentum or angular velocity. The Coriolis force is generated by Coriolis affect, which only be seen in rotating non-inertial reference frame.

French physicist Leon Foucault introduced the gyroscope in 1852. He named a gyroscope combining two Greek words “gyro” meaning rotation and “skopeein” meaning to see. The earlier mechanical gyroscope consists of a spinning wheel that is suspended by three gimbals. However, the mechanical gyroscope has many disadvantages such as wear and bearing friction. Modern vibratory gyroscope eliminated these problems by removing the rotating parts.

There are many different types of vibratory gyroscopes in the current market. Wine-glass resonator/hemisphere resonator gyroscopes, cylindrical resonator gyroscope (CRG), tuning fork gyroscope and vibrating wheel gyroscope/disk resonator gyroscope (DPG) are the most widely used and well known vibratory gyroscopes. Alternately, optical gyroscopes such as fiber-optic gyroscope and ring laser gyroscope are available for high performance applications. Due to the high price of the optical gyroscopes, it can only be used for high performance applications. Inertial navigation and guidance system, radar sensors and structural health monitoring are the example of aerospace industry applications. A wide range of consumer electronics and entertainment industries
applications include stabilization in digital camera, smartphone and motion sensor for game console.

Microelectromechanical Systems (MEMS) is based on photo-lithography technology. In order to be considered as MEMS, it has to be in micrometer scale with electrical and mechanical functions. MEMS technologies became more desirable for many applications because of its lightweight, miniature size and manufacture cost.

![Figure 2.1.1 Schematic diagrams of the cylindrical gyroscopes [9].](image1)

![Figure 2.1.2 Schematic diagram of double tuning fork gyroscope [10].](image2)
2.2 Dynamics of Vibratory Gyroscopes

The fundamental equations of the vibratory gyroscope are developed in this chapter. The simple theoretical gyroscope is comprised of a driving-mode oscillator, sensing-mode detector and the proof mass. The driving mode oscillator comprises of proof mass which oscillating in the driving direction at driving resonant frequency and it is suspended by suspension system.

The sensing mode accelerometer is comprises of proof mass suspended by suspension system. It allows the proof mass to oscillate in the sensing direction, which is perpendicular to the driving direction. The Coriolis force is generated in the sensing direction when the gyroscope is subjected to an angular rotation. The sense electrodes detect the change in angular momentum and output the signal.
A simple theoretical z-axis gyroscope, shown in figure 2.2.1, is used to illustrate the dynamics and operational principles of the vibratory gyroscope. The gyroscope subjected to rotation about z-axis is considered. The equation of the motion of the vibratory gyroscope can be derived by taking second time derivative of the position vector [1].

The equations of the motion of the moving body in a rotating reference frame consist of following definitions:

A : Inertial frame

B : Rotating reference frame

\( \mathbf{r}_A \): Position vector relative to inertial frame A

\( \mathbf{r}_B \): Position vector relative to inertial frame B

\( \mathbf{v}_A \): Velocity vector relative to inertial frame A, \( \mathbf{v}_A = \dot{\mathbf{r}}_A \)

\( \mathbf{v}_B \): Velocity vector relative to inertial frame B, \( \mathbf{v}_B = \dot{\mathbf{r}}_B \)
Figure 2.2.2 Illustration of the position vector relative to the inertial reference frame and the rotating reference frame.

\( a_A \): Acceleration vector relative to inertial frame A, \( a_A = \dot{v}_A = \ddot{r}_A \)

\( a_B \): Acceleration vector relative to inertial frame B, \( a_B = \dot{v}_B = \ddot{r}_B \)

\( \Theta \): Orientation vector of rotating reference frame relative to inertial frame

\( \Omega \): Angular velocity vector of rotating reference frame, \( \Omega = \dot{\theta} \)

\( R \): Position vector of rotating frame

By taking the second time derivative of the position vector, the equations of the motions of moving body in a rotating frame is obtained and they are given as

\[
\begin{align*}
r_A(t) &= R(t) + r_B(t) \\
\dot{r}_A(t) &= \dot{R}(t) + \dot{r}_B(t) + \dot{\theta} \times r_B(t)
\end{align*}
\]

(2.2.1) \hspace{1cm} (2.2.2)
\[ \ddot{\mathbf{r}}_A(t) = \ddot{\mathbf{r}}_A(t) + \ddot{\mathbf{r}}_B(t) + \hat{\Omega} \times \dot{\mathbf{r}}_B(t) + \hat{\Omega} \times (\dot{\hat{\Omega}} \times \mathbf{r}_B(t)) + \hat{\Theta} \times \mathbf{r}_B(t) + \hat{\Theta} \times \dot{\mathbf{r}}_B(t) \]

(2.2.3)

The figure 2.2.2 represent the position vector relative to the inertial frame A and the rotating frame. Equation 2.2.3 can be reduced by writing the equation in terms of velocity vector, \( \mathbf{v} \), acceleration vector, \( \mathbf{a} \), and linear acceleration of the reference frame B, A and is it given as

\[ \mathbf{a}_A = \mathbf{a} + \mathbf{a}_B + \hat{\Omega} \times \mathbf{r}_B + \hat{\Omega} \times (\hat{\Omega} \times \mathbf{r}_B) + 2 \hat{\Omega} \times \mathbf{v}_B \]

(2.2.4)

Above equation 2.2.4 can be divided into three parts where first three terms are the local acceleration, \( \mathbf{a} + \mathbf{a}_B + \hat{\Omega} \times \mathbf{r}_B \), following two terms are centripetal acceleration, \( \hat{\Omega} \times (\hat{\Omega} \times \mathbf{r}_B) \), and last two terms are the Coriolis acceleration, \( 2 \hat{\Omega} \times \mathbf{v}_B \). The Coriolis acceleration only appears in the rotating reference frame. We can further simplify the equation by multiplying the mass with the equation 2.2.4 to find the equation of the motion of the proof mass moving with the rotating frame reference frame and it is shown as

\[ \mathbf{F}_{\text{ext}} = m [\mathbf{a} + \mathbf{a}_B + \hat{\Omega} \times \mathbf{r}_B + \hat{\Omega} \times (\hat{\Omega} \times \mathbf{r}_B) + 2 \hat{\Omega} \times \mathbf{v}_B] \]

(2.2.5)

where \( \mathbf{F}_{\text{ext}} \) is the total external force applied on the proof mass, \( m \) is the proof mass, \( \mathbf{a} \) is the linear acceleration and \( \mathbf{v}_B \) and \( \mathbf{a}_B \) are the velocity and acceleration vectors of the proof mass with respect to the reference frame.

In a simple theoretical z-axis gyroscope, the proof mass is allow to move in two principle oscillation directions which are the driving direction along the x-axis and the sense direction along the y-axis as shown in figure 2.2.1. The schematic diagram of the 2-DOF vibratory gyroscope in rotating reference frame rotation with respect to the inertial frame is shown in figure 2.2.3. The equations of the motion along the drive and sense
directions, which shown in equation 2.2.6, can be written by decomposing the motion of the proof mass into the two principle oscillation directions and neglecting the linear acceleration.

\[
m\ddot{x} + c_x\dot{x} + (k_x - m(\Omega_y^2 + \Omega_z^2))x + m(\Omega_x\Omega_y - \dot{\Omega}_z)y = \tau_x + 2m\Omega_z\dot{y}
\]

\[
m\ddot{y} + c_y\dot{y} + (k_y - m(\Omega_x^2 + \Omega_z^2))y + m(\Omega_y\Omega_y + \dot{\Omega}_z)x = \tau_y - 2m\Omega_z\dot{x}
\]

(2.2.6)

where \(c_x\) and \(c_y\) are the damping coefficient in the drive direction and sense direction, \(k_x\) and \(k_y\) are the spring constant in the drive direction and sense direction and \(\dot{\Omega}_z\) is the angular acceleration rotation about z-axis. \(\tau_x\) is the external harmonic excitation force that drives the proof mass, \(m\), at the driving resonant frequency in the drive direction and \(\tau_y\) is the total external force which comprise of parasitic and external inertial forces in the sense direction. Finally, \(2m\Omega_z\dot{y}\) and \(2m\Omega_z\dot{x}\) are the Coriolis terms applied to the proof mass induced by rotation of rotating reference frame with respect to inertial frame.

Equation 2.2.6 can be further simplified by assuming a constant angular rate, \(\dot{\Omega}_z = 0\) and angular rate is much lower that driving resonant frequency which \(\Omega_x^2\), \(\Omega_y^2\) and \(\Omega_y\Omega_y\) become negligible. In addition, the term \(2m\Omega_z\dot{y}\) become negligible since sense-mode Coriolis force response is much smaller than drive-mode motion. Therefore, the simplified equations motion of 2 degrees of freedom vibratory gyroscope are

\[
m\ddot{x} + c_x\dot{x} + k_xx = \tau_x
\]

\[
m\ddot{y} + c_y\dot{y} + k_yy = \tau_y - 2m\Omega_z\dot{x}
\]

(2.2.7)

The first equation of equation 2.2.7 represents the equation of motion in drive direction while the second equation of equation 2.2.7 represents the equation of motion in sense
direction. The second term of right hand side of the second equation of equation 2.2.7 is the practical Coriolis applied on the gyroscope that is proportional to the angular rate.
Figure 2.2.3 The schematic diagram of the 2-DOF vibratory gyroscope in rotating reference frame rotation with respect to the inertial frame.
2.3 Drive-Mode

The vibratory gyroscope requires the momentum to detect the Coriolis force. Moment is usually generated by 1-D.O.F resonator or oscillator. The equation of motion of 1-DOF resonator along the x-axis given as

\[ m \ddot{x} + c \dot{x} + k x = F(t) \]  \hspace{1cm} (2.3.1)

If resonator generates the harmonic excitation force, \( F = F_0 \sin \omega t \), the equation 2.8 can be rewritten as

\[ m_d \ddot{x} + c_d \dot{x} + k_d x = F_d \sin \omega t \]  \hspace{1cm} (2.3.2)

where \( m_d \) is drive mass proof, \( c_d \) is the viscous and thermoelastic damping and \( k_d \) is the drive-mode suspension stiffness and \( \omega \) is the angular frequency. The steady state the response of above equation is

\[ x = x_0 \sin(\omega t + \phi_d) \]  \hspace{1cm} (2.3.3)

where \( x_0 \) is the amplitude and \( \phi_d \) is the phase of the drive-mode steady state response.

\[ x_0 = \frac{F_d}{k_d \sqrt{1 - \left(\frac{\omega}{\omega_d}\right)^2 + \left[ \frac{1}{Q_d \omega_d} \right]^2}} \]  \hspace{1cm} (2.3.4)

\[ \phi_d = -\tan^{-1} \left( \frac{1}{Q_d \omega_d} \frac{\omega}{\omega_d} \right) \]  \hspace{1cm} (2.3.5)

\[ \omega_d = \sqrt{\frac{k_d}{m_d}} \]  \hspace{1cm} (2.3.6)

\[ Q_d = \frac{m_d \omega_d}{c_d} \]  \hspace{1cm} (2.3.7)

where the \( \omega_d \) is the drive-mode resonant frequency. \( Q_d \) is the drive-mode Quality factor, which defined as maximum ratio of the amplitude to the static deflection. The gyroscopic sensitivity of the gyroscope is directly proportional to the drive-mode oscillation amplitude, which directly scale the quality factor \( Q \). Therefore, the gyroscopes are usually
operate at the drive resonant frequency. The phase at the resonant become -90 and the amplitude at the resonant is

\[ x_{0_{res}} = Q_d \frac{r_d}{m_d \omega_d^2} \]  

(2.3.8)

### 2.4 Coriolis Response

In a vibratory gyroscope, the Coriolis force, \( F_c \), is generated by the resonator which driving at the drive-mode resonant frequency. The equations of the motion of Coriolis response in the sense-mode direction can be derived by considering the system which drive-mode operates at drive resonant frequency, \( \omega_d \), with the drive motion regulated to be in sinusoidal form as shown in equation 2.4.1.

\[
x = x_0 \sin(\omega_d t + \phi_d) \\
x = x_0 \omega_d \sin(\omega_d t + \phi_d) \\
F_c = -2m_c \Omega_z \dot{x} = -2m_c \Omega_z x_0 \omega_d \cos(\omega_d t + \phi_d) 
\]  

(2.4.1)  
(2.4.2)  
(2.4.3)

where \( m_c \) is the part of the proof mass that contributed to Coriolis force. Considering sense-mode oscillator is also 1-DOF, the equation of motion in y-axis is

\[
m_s \ddot{y} + c_s \dot{y} + k_s y = F_c 
\]  

(2.4.4)

By substituting equation 2.4.3 into equation 2.4.4, the equation of motion of the 1-DOF sense-mode oscillator in y-axis become

\[
m_s \ddot{y} + c_s \dot{y} + k_s y = -2m_c \Omega_z x_0 \omega_d \cos(\omega_d t + \phi_d) 
\]  

(2.4.5)

where \( m_s \) is the part of proof mass that responds to the Coriolis force. In single-mass vibratory gyroscope system, \( m_c, m_s \) and \( m_d \) are equal. Similar to the drive-mode, amplitude and phase of the sense-mode response can be found as
\[ y_0 = \frac{\Omega_z m_c \omega_d}{m_s \omega_s^2} \frac{2x_0}{k_d \sqrt{\left[ 1 - \left( \frac{\omega_d}{\omega_s} \right)^2 \right]^2 + \left[ \frac{1}{Q_s \omega_s \omega_d} \right]^2}} \quad (2.4.6) \]

\[ \phi_s = -\tan^{-1} \left( \frac{\omega_d \omega_s}{1 - \left( \omega_d \omega_s \right)^2} \right) + \phi_d \quad (2.4.7) \]

where

\[ \omega_s = \sqrt{\frac{k_s}{m_s}} \quad (2.4.8) \]

\[ Q_s = \frac{s \omega_s}{c_s} \quad (2.4.9) \]

The \( \omega_s \) is sense-mode resonant frequency and \( Q_s \) is the sense-mode Quality factor. The phase at the sense-mode resonant frequency is \(-90^\circ\) and the amplitude at the sense-mode resonant is

\[ y_{0_{res}} = \Omega_z \frac{2Q_s x_0 m_c}{m_s \omega_s} \quad (2.4.10) \]

Achieving the higher gyroscopic sensitivity is the most important criteria in designing a gyroscope. In order to get the maximum gyroscopic sensitivity, most of the gyroscope operates at peak or near the peak of the sense-mode resonance frequency, as illustrated in figure 2.4.1. This technical is called mode-matching. When the drive-mode resonant frequency gets closer to the sense-mode resonant frequency, the sense-mode amplitude increases dramatically. Even though matching the drive-mode and sense-mode enhance the gyroscopic sensitivity, but it makes the system unstable and shifts the resonant frequencies. Gyroscopic sensitivity can be also improve by increasing the drive-mode oscillation amplitude and increasing the quality factor \( Q \) by decreasing damping. Figure 2.4.2 and 2.4.3 illustrate how quality factor affects the sensitivity of the gyroscope. We can consider the system operating at resonant frequency of \( \omega = 1000 \) kHz and assume that
the drive-mode and sense-modes are mode-matched. If sense-mode is shift by 5 kHz from the resonant frequency, the gain drops by 29% when $Q = 1000$. For the same system with the higher quality factor, $Q = 10,000$, the gain drops by 90%. It is clear shows that the higher gyroscopic sensitivity can be achieve by increase of quality factor; however, the system become very sensitive. In addition, maximizing the mass $m_c$ and minimizing $m_s$ enhance the performance of the gyroscope [1].

Figure 2.4.1 Drive and sense mode response of the gyroscope [1].
Figure 2.4.2 Sense mode response of the gyroscope at a resonant frequency of $\omega_s=10$ kHz and a $Q = 1000$ [1].

Figure 2.4.3 Sense mode response of the gyroscope at a resonant frequency of $\omega_s=10$ kHz and a $Q = 10,000$ [1].
Chapter 3. Piezoelectric doubled-ended tuning fork gyroscope

3.1 Introduction

The finite element analysis has been used to design the gyroscopes for many years and they are very accurate [12]. In this chapter, the piezoelectric double-ended tuning fork gyroscope, which designed by Sato, Ono, and Tomikawa [13] [14] [15] and [16], will be analyze by finite element analysis. A piezoelectric double-ended tuning fork gyroscope has many advantages such that it can be easily manufacture due to its small size and simple geometry, it is very durable and reliable under external shock since it is made of a single quartz crystal and it has stable vibration characteristic [13]. Moreover, the gyro-sensor has flatly supports at each ends that make the tight support. Even though the gyroscope can detect the dual axis, y- and z-axes rotation, we only looked at z-axis rotation for most of this chapter. The first part of this chapter is to study the geometric the gyroscopic characteristics of gyroscope. In the second part of this chapter, the mounting and support structure will be investigate. Furthermore, we analyze the langasite and langatate double-ended tuning fork gyroscopes in addition to the quartz double-ended tuning fork gyroscope. The langasite and langatate gyroscopes will have higher gyroscopic sensitivity than quartz gyroscope due to their higher electromechanical coupling coefficient and mass density [17] [18]. Since langasite and langatate have same crystal class as quartz, the well-known methods of trapping energy of resonator can be easily and accurately applied [19] [20]. The material properties of the langasite, langatate and quartz are listed in Appendix C.
3.2 Structure of the double-ended tuning fork gyroscope

A single rotated z-cut quartz, langasite and langatate crystals are used for a double-ended tuning fork gyroscopes. The schematic diagrams of the piezoelectric double-ended tuning fork gyroscope and electrodes configuration diagram are shown in figure 3.2.1. The gyroscope is axisymmetric about both x- and y-axis. It is comprised of three main components and the mounting support. The first main component is driving arms located at the center of the gyroscope where it includes the driving electrodes. The second main component is the z-axis detecting parts that are located between the base and the driving arms. The last component is the supporting bases that are located at each end of the sensor and they are fixed with silicone rubber adhesive. Dimensions of the main components of gyroscope are listed in table 3.2.1.

Figure 3.2.1 Structure of a double-ended tuning fork gyroscope.
### Table 3.2.1 Dimensions of the double-ended tuning fork gyroscope

<table>
<thead>
<tr>
<th>Component</th>
<th>Dimension</th>
<th>Component</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>16.0</td>
<td>W</td>
<td>2.00</td>
</tr>
<tr>
<td>L1</td>
<td>7.46</td>
<td>W1</td>
<td>0.31</td>
</tr>
<tr>
<td>L2</td>
<td>1.38</td>
<td>W2</td>
<td>1.60</td>
</tr>
<tr>
<td>L3</td>
<td>0.89</td>
<td>W3</td>
<td>0.451</td>
</tr>
<tr>
<td>LB</td>
<td>2.00</td>
<td>Wi</td>
<td>0.75</td>
</tr>
<tr>
<td>t Silicon rubber</td>
<td>0.03</td>
<td>t</td>
<td>0.30</td>
</tr>
</tbody>
</table>

Unit: mm

### 3.3 Principle of Operation of Piezoelectric Double-Ended Tuning Fork Gyroscope

The principle of operation of piezoelectric double-ended tuning fork gyroscope is shown in figure 3.3.1. It shows the two modes of vibrations that are used by the gyroscope to detect the angular velocity about z-axis. The driving mode is a flexure mode in the x-y plane, as shown in Figure 3.3.1 (a). It is a symmetrical mode driven by drive electrodes on drive arms. The z-axis detecting mode is an asymmetric flexure mode in the x-y plane, as shown in Figure 3.3.1 (b).

The driving electrodes on the drive arms are configured to vibrate the drive arms symmetrical in the x-y plane. When the gyroscope is subjected to a rotation about the z-axis, a pair of Coriolis forces $F_{cz}$ is generated proportional to the angular velocity $\Omega_z$ and vibration velocity $V_x$. The moment $M_{cz}$ are produced at each end of the driving arms and displace them in opposite direction which will cause an asymmetric mode in the horizontal x-y plane. The z-axis detection electrodes on the arm $L3$ measures the change of charge or voltage to output change in angular velocity.
Figure 3.3.1 Operation principle of double-ended tuning fork gyroscope
3.4 Finite Element Analysis

The geometric sensitivity and gyroscopic sensitivity characteristics of the double-ended tuning fork gyroscope are simulated using Comsol Multiphysics 4.3a. As shown in figure 3.4.1, a total 16,467 tetrahedral mesh elements and 303,770 degree of freedom are used in the FEM analysis. The gyroscopic sensitivity is dependent on the matching of the resonant frequency of drive mode and detection mode. In order to have maximum gyroscopic sensitivity of the gyro sensor, the natural frequency of the $z$-axis detection mode must match closely to the natural frequency of the driving mode. The drive electrodes are located at the drive arms and the sensing electrodes are located at the arm $L_3$, which are designed to sense the changes in electric charge or voltage. For modal analyses, all the electrodes are grounded, while for the frequency response analysis, the driving electrodes are excited at $\pm 1$ V. The thickness of 0.2μm of gold films are applied to all the electrodes.

![Figure 3.4.1 FEM mesh of the double-ended tuning fork gyroscope.](image)

While the thickness of electrode films could be changed for fine-tuning of the driving or detecting frequencies, we need to determine as the first order an optimal geometry of the resonator where the driving and detecting frequencies are well matched. The frequency of the $z$-axis detection mode is highly dependent on the width $W_3$ when the detection arm
length $L_3$ is fixed, that is, the frequency of the detection mode increases with $W_3$ for a fixed $L_3$.

Figure 3.4.2, 3.4.3 and 3.4.4 show the frequency spectrum of the driving mode and detection mode versus the width $W_3$ for quartz, langasite and langatate, respectively. From the three figures, the optimal $W_3$ is determined for each of the material quartz, langasite and langatate. We observe that the driving mode is highly coupled to the detection mode when their modal branches intersected in the frequency spectrum. Hence, in order to reduce the coupling between the two modes but at the same time maintain adequate performance of the detection mode, we select $W_3$ width that is slightly away from the intersection of the two modal branches. The $W_3$ width chosen is shown in table 3.4.1 for each material quartz, langasite and langatate, respectively, along with their driving frequency, detection frequency, and frequency difference $\Delta f$. The change in width $W_3$ has little effect on the driving frequency. The thickness, $t$, of the gyroscope is kept constant at 0.30 mm.
Figure 3.4.2 frequency spectrum of quartz of the driving and detecting modes as a function of W3
Figure 3.4.3 Frequency spectrum of langasite of the driving and detecting modes as a function of $W_3$. 

![Frequency spectrum graph](image)
Figure 3.4.4 Frequency spectrum of langatate of the driving and detecting modes as a function of $W_3$. 
Table 3.4.1 The driving frequency and detecting frequency of the doubled ended tuning fork gyroscope of each material

<table>
<thead>
<tr>
<th>Material</th>
<th>$f_{driving}$</th>
<th>$f_{detecting}$</th>
<th>$\Delta f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quartz, $W3=0.445\text{mm}$</td>
<td>30733</td>
<td>25270</td>
<td>23224</td>
</tr>
<tr>
<td>Langasite, $W3=0.452\text{mm}$</td>
<td>30740</td>
<td>25264</td>
<td>23230</td>
</tr>
<tr>
<td>Langatate, $W3=0.450\text{mm}$</td>
<td></td>
<td></td>
<td>6</td>
</tr>
</tbody>
</table>

Unit: Hz

The gyro-characteristics of the gyroscope also depends on the frequency separation, $\Delta f$. When driving frequency is too close to the detecting resonant frequency, the system becomes unstable which can cause a shift in detecting resonance frequency. Moreover, the coupling of driving mode and detecting mode will change the mode shapes of both modes. Operating the system too far away from the detection mode frequency will reduce the sensitivity of the gyroscope. It is very important in this simulation for comparing the relative merits of each material that we keep the same frequency separation for each material. The equation of the frequency separation is

$$\Delta f = f_{detecting} - f_{driving} \quad (3.4.1)$$

where $f_{detecting}$ and $f_{driving}$ are the detecting frequency and driving frequency in Hz.

3.5 The Gyroscopic Sensitivity

The gyroscopic sensitivities of the quartz, langasite and langatate double-ended tuning fork were obtained by frequency response analysis. For the models that we were analyzing are only in angular velocity about the z-axis, we applied the following gyroscopic equations in our finite element models [21].
\[ f_x = F_x + \rho \omega^2 u_x + j2\rho \Omega_z \omega u_y \]  \hspace{1cm} (3.5.1)

\[ f_y = F_y + \rho \omega^2 u_y - j2\rho \Omega_z \omega u_x \] \hspace{1cm} (3.5.2)

where \( f \) is the total force acting on the element, \( F \) is the external force, \( \rho \) is the mass of the element, \( u_y \) is the displacement in \( y \)-direction and \( u_x \) is the displacement of \( x \)-direction. The third terms on the right hand side are the Coriolis forces generated by the rotation about \( z \)-axis with angular velocity \( \Omega_z \). Frequency response analyses were performed without Coriolis-body force and angular velocity, \( \Omega_z \). Then, we applied the body forces \( F_b \) as Coriolis-body forces on the entire structures and these are given as

\[ F_{bx} = 2\rho \omega \Omega_x u_y \] \hspace{1cm} (3.5.3)

\[ F_{by} = -2\rho \omega \Omega_x u_x \] \hspace{1cm} (3.5.4)

\[ F_{bz} = 0 \] \hspace{1cm} (3.5.5)

where \( F_{bx}, F_{by}, \) and \( F_{bz} \) are the body forces applied to the structure in the \( x \)-, \( y \)-, and \( z \)-directions, respectively. The difference in voltage on the detecting electrodes were used to calculate the gyro-sensitivity. The output voltage as a function of the angular velocities, \( \Omega_z \), which represent the gyroscopic sensitivity, of each material are shown in figure 3.5.1.

The sensitivity of quartz, langasite, and langatate to angular velocity \( \Omega_z \) were 0.191mV/(deg/s), 0.469mV/(deg/s) and 0.784mV/(deg/s), respectively. As the values indicate, sensitivity of langatate and langasite were about 4 times and 2 times higher than quartz, respectively. Also, it can be seen that all three materials exhibit a linear relationship between output voltage and angular velocity.
3.5 The gyroscopic sensitivity of quartz, langasite and langatate double-ended tuning fork gyroscope rotation about the z-axis

3.6 Effect of the mounting support of quartz double-ended tuning fork gyroscope

The vibration modes of piezoelectric crystal involve not only the geometry of the resonator or gyroscope, but also the supporting and mounting structure [22]. The quality factor of the gyroscope can decrease significantly with the small changes of mounting support. In this section, the geometry characteristics of the quartz double-ended tuning fork gyroscope with dual axis is analyzed as well as the gyroscopic sensitivity of the gyroscope rotation about z-axis as a function of thickness of mounting support. Furthermore, we look into the equivalent circuit parameter of the Butterworth Van Dyke resonator. The same double-ended tuning fork gyroscopes are used in this section except we exclude the langasite and langatate double-ended tuning fork gyroscope.
The structure of the quartz double-ended tuning fork gyroscope used in this section has the same dimension as in previous section. However, the arrangement of the electrodes has been changed so that it can detect the y- and z-axis as shown in figure 3.6.1. The electrodes are thin films made of gold. The thickness of the driving electrodes could be adjusted for fine-tuning the driving mode frequency by using the mass loading effect of the electrodes. The gyroscope was designed so that the moment and the displacement of the mounting support are minimized while the moment of the detecting arms was maximized. The driving, y-axis sense, and z-axis sense electrodes are denoted as \( D_r \pm, D_y \pm, \) and \( D_z \pm, \) respectively.

The operation principle of the quartz double-ended tuning fork gyroscopes is the same as in previous section except it can detect the y-axis, in addition to the z-axis. For the angular velocity rotation about z-axis, the principle of the operation is the same as in section 3.3. Figure 3.6.2 shows the three modes of vibrations that are employed by the gyroscope to detect the angular velocities about the y- and z-axes of the device: (a) At the top is the driving mode which is a flexure mode in the x-y plane, (b) at the middle is the y-axis (\( \Omega_y \)) detection mode which is a flexure mode in the y-z plane, and (c) at the bottom is the z-axis (\( \Omega_z \)) detection mode which is a asymmetric flexure mode in the x-y plane.

The driving electrodes are designed to vibrate the center arms in the horizontal x-y plane as shown in figure 3.6.2 (a). The top arms vibrate in the opposite x-direction from the bottom arm. The Coriolis force, \( F_{cy} \), is generated in z-direction when an angular velocity \( \Omega_y \) is applied to the gyroscope. Due to the Coriolis Effect, the center arms then vibrate in y-z plane as shown in figure 3.6.2 (b). The y-axis detecting electrodes on the center arms of the gyroscope detect the Coriolis force \( F_{cy} \).
Figure 3.6.1 The electrode configuration of the double-ended tuning fork
Figure 3.6.2 The three modes of vibration of the gyroscope: (a) Driving mode, (b) y-axis detection mode and (c) z-axis detection mode.
The 3-D Structural Mechanics, Piezoelectric Devices (pzd) module was employed to model the structure of the quartz gyroscope. The FEM model of the gyroscope is shown in figure 3.6.3. Tetrahedral elements were used in the mesh. There were 50,594 elements and 321,757 degrees of freedom. Both eigenfrequency and frequency response analyses were performed. The eigenfrequency analysis was employed to (1) determine the optimum gyroscope geometry for the driving mode and two detection modes, and (2) to calculate the electrical parameters of the gyroscope. The frequency response analysis was needed to calculate the Coriolis forces and the detection sensitivity of the gyroscope.

![FEM mesh of the double-ended tuning fork gyroscope.](image)

3.6.1 Eigenfrequency analysis

Since the sensitivity of the gyroscope is dependent on the matching of the natural frequencies of the detecting modes to the natural frequency of the driving mode, parametric eigenfrequency analyses were performed on the geometry of the tuning fork. The parameters were the height of the mounting supports and the thickness of the quartz blank. These parameters affect the frequencies of the driving mode and the detecting modes. For example, figure 3.6.4 shows the changes in the y-detection and z-detection
mode frequencies and the driving mode frequency as a function of the thickness of the resonator-gyroscope when $W_3 = 0.491$ mm. We observed that it is difficult to get all three modal frequencies to match, especially the $y$-detection mode frequency. At the resonator thickness $t = 0.36031$ mm the three modes come closest together but there is coupling of the $y$-detection mode with the driving and $z$-detection modes. This coupling creates cross-talking of the $y$-detection mode with the $z$-detection mode. Hence, in this paper we focused on a one axis detection gyroscope, namely, the $z$-axis angular velocity detection while putting aside for now the $y$-axis angular velocity detection.

The analysis also shows that it is possible to match very well the frequency of the $z$-detection mode to the frequency of the driving mode. Figure 3.6.5 shows that a good frequency match could be obtained at a resonator thickness of 0.3 mm. The normalized resonant frequency of the $z$-axis detection mode depends on the width $W_3$. The width $W_3$ is adequate at $W_3=0.491$ mm, as shown in figure 3.6.6. The change in the width $W_3$ has very little effect on the driving and $y$-axis detection modes frequencies. We also observed that the resonant frequency of the $z$-axis detection mode can be tuned by the width $W_3$. We can further optimize the frequency matching of the $z$-detection mode. Figure 3.6.7 shows the change in frequency of the $z$-detection mode as a function of the spacer height. The blue line is for $W_3=0.491$ mm while the red line is for $W_3=0.49$ mm.

Based upon our parametric studies, we chose the resonator dimensions to be those provided in table 3.6.1. (Please see figure 3.2.1 and 3.6.1 for the definition of the dimension)
Table 3.6.1 Dimensions of the double-ended tuning fork gyroscope.

<table>
<thead>
<tr>
<th>Item</th>
<th>Dimension</th>
<th>Item</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>16.0</td>
<td>W</td>
<td>2.00</td>
</tr>
<tr>
<td>L1</td>
<td>7.46</td>
<td>W1</td>
<td>0.31</td>
</tr>
<tr>
<td>L3</td>
<td>0.89</td>
<td>W3</td>
<td>0.43</td>
</tr>
<tr>
<td>LB</td>
<td>2.00</td>
<td>Wi</td>
<td>0.75</td>
</tr>
<tr>
<td>t (blank thickness)</td>
<td>0.30</td>
<td>Spacer</td>
<td>0.28</td>
</tr>
<tr>
<td>Silicon rubber</td>
<td>0.03</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 3.6.4 Resonant frequencies of the driving, y- and z-axis detection modes as a function of the resonator thickness when W3=0.491mm
Figure 3.6.5 Resonant frequencies of the z-axis detection mode and the driving mode with the resonator thickness when W3=0.43mm
Figure 3.6.6 Resonant frequencies of the driving, z- and y-axis detection modes as function of the W3
Figure 3.6.7 Change in resonant frequencies of the z-axis detection mode for height of the spacer.
3.6.2 Butterworth Van Dyke Electrical Parameters

The piezoelectric gyroscope could be represented in an equivalent circuit parameters of the Butterworth van Dyke resonator as shown in figure 3.6.8. [23]

![Equivalent electrical circuit parameters of the piezoelectric resonator-gyroscope represented as Butterworth van Dyke resonator](image)

The static capacitance $C_0$ is obtained from a COMSOL electrostatic analysis by putting 1V on the drive electrodes. $C_0$ is then equal to the total surface charge on the drive electrodes as shown in equation 3.6.1. The electrical parameters $C_1$, $L_1$ and $R_1$ can be obtained from our eigenfrequency analysis of the resonator-gyroscope. The following four quantities are first obtained from the eigenfrequency analysis:

1. $\omega_r$, the short circuit resonance frequency,
2. $Q$, the quality factor of the mode $\omega_r$,
3. $q$, the charge on the drive electrode, and
4. $E$, an energy norm.

$Q$ and $\omega_r$ are obtained directly from the COMSOL eigenfrequency model $pzd.Q_{eig}$ and $pzd.omega$, respectively, while $q$ and $E$ are calculated using equation 3.6.1 and 3.6.2, respectively.
\[ q = | \int_{\text{drive electrode}} (pzd. nD) dA | \] 

\[ E = | \int_{\text{All domains}} (pzd. rho * pdz. disp^2) d\Omega | \]

The four quantities \( \omega_r, Q, q, \) and \( E \) are used to calculate the electrical parameters \( C_1, L_1, \) and \( R_1 \) using equation 3.6.3, 3.6.4 and 3.6.5 below

\[ c_1 = \frac{q^2}{E \omega_r^2} \] 

\[ (3.6.3) \]

\[ L_1 = \frac{1}{c_1 \omega_r^2} \] 

\[ (3.6.4) \]

\[ R_1 = \frac{1}{Q c_1 \omega_r} \] 

\[ (3.6.5) \]

The calculated Butterworth van Dyke electrical parameters for our gyroscope are provided in Table 3.6.2.

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>30781.69</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1 ) (F)</td>
<td>1.0347 E -15</td>
</tr>
<tr>
<td>( R_1 ) (( \Omega ))</td>
<td>3332.4965</td>
</tr>
<tr>
<td>( C_0 ) (F)</td>
<td>1.8394 E -10</td>
</tr>
<tr>
<td>( L_1 ) (H)</td>
<td>25837.8089</td>
</tr>
</tbody>
</table>

It is important that the dissipation or damping of the gyroscope be modeled because of its effect on the electrical parameters and sensitivity of the gyroscope. The quality factor \( Q \) obtained from the COMSOL model is dependent on the damping coefficients of the quartz elastic constants, and the dissipation in the silicon rubber adhesive used to bond the
gyroscope to a device substrate. Equation 3.6.5 shows that the motional resistance $R_1$ is inversely proportional to the $Q$ if the motional capacitance $C_1$ is constant. Since the model damping will affect the coupling between resonant modes, and the resonant frequencies of the modes themselves, the dissipation at the silicon rubber adhesive bonds was studied and found to be an important factor in the sensitivity of the gyroscope when subjected to an angular velocity about the z-axis.

3.6.3 Frequency response analysis of the double-ended gyroscope

The sensitivity of the gyroscope to angular velocity about the z-axis was simulated with the frequency domain or harmonic response analysis. The frequency response of the driving mode with 1V at the drive electrodes is shown in figure 3.6.9. The simulation method is the same as in previous sections 3.5 and 3.6. We observe a change in the charge at the z-detection electrodes when the gyroscope is rotated about the z-axis. For example in a frequency response analysis shown in figure 3.6.10 we see the change in charge at the z-detection electrodes when $\Omega_z = 120 \text{ deg. /s}$. Therefore, the gyroscope could sense the angular velocity $\Omega_z$ about the z-axis as a change in charge at the z-axis detection electrodes. The magnitude of the change is greatest at the resonant frequency of the driving mode.

The sensitivity of the gyroscope is dependent on the matching of the resonant frequency of the z-axis detection mode to the resonant frequency of the driving mode. When the two resonant frequencies are not well matched the sensitivity of the gyroscope decreases. We studied the effect of small changes in the thickness of the silicon rubber adhesive at the bottom of the spacers of the gyroscope (please see figure 3.6.1). The adhesive thickness has significant effects on the z-axis detection mode frequency but has
little or no effect on the driving mode frequency. The change in the z-axis detection mode frequency resonator will affect the sensitivity of gyroscope when subjected to an angular velocity about its z-axis. This is shown in figure 3.6.11 where we observe that a small change of 4μm in adhesive thickness could change the sensitivity by as much as 44%. Therefore the adhesive thickness will be a significant factor in improving the production yields of this gyroscope.

Figure 3.6.9 Frequency response of the driving mode with 1V at the drive electrodes
Figure 3.6.10 Change in charge at the z-axis detection electrodes when the gyroscope experiences an angular velocity $\Omega_z = 120$ deg./s about z-axis
Figure 3.6.11 The z-axis detection sensitivity of the gyroscope as a function of adhesive thickness
3.1 Summary

Using FEM simulation, we have investigated the geometry of double-ended tuning fork gyroscope and gyro-characteristics of a double-ended tuning fork gyroscope that utilized either quartz, or langasite, or langatate crystal respectively. Our study confirms that the langasite and langatate gyroscopes provide higher gyro-sensitivity than the quartz gyroscope. The gyro-sensitivity is also a function of geometry and frequency separation between the driving mode frequency and the detection mode frequency. In order to achieve higher sensitivity, adequate frequency separation is needed so that both the driving mode and detection mode remain pure while at the same time the detection mode could sense the Coriolis force at the driving mode frequency. We conclude that the langasite and langatate crystals are better gyroscopic materials than quartz.

Furthermore, a COMSOL model of a piezoelectric quartz double ended tuning fork gyroscope was successfully implemented. The gyroscope has two detection modes; the first mode detects the angular velocity about a z-axis perpendicular to the tuning fork plane (X-Y plane), while the second mode detects the angular velocity about a y-axis that is the longitudinal axis along the length of the tuning fork. The quartz double-ended tuning fork gyro sensor characteristics were studied using COMOL software. Both eigenfrequency and frequency response analyses were performed. The eigenfrequency analysis was employed to (1) determine the optimum gyroscope geometry for the driving mode and two detection modes, and (2) to calculate the electrical parameters of the gyroscope. The frequency response analysis was needed to calculate the Coriolis forces and the detection sensitivity of the gyroscope. An optimal geometry of the gyroscope was found by eigenfrequency analyses with geometric parameters such as the dimensions of
the tuning fork tines, spacers, and resonator thickness. The dimensions of the gyroscope provided in Table 3.6.1 were obtained using eigenfrequency analyses. The frequency response analysis was used to simulate the response of the gyroscope to angular velocity about its z-axis. The Coriolis forces were treated as body forces in the COMSOL model, which cause change in charge at the detection electrodes. The sign and magnitude of the change in charge at the detection electrodes are proportional to the sign and magnitude of the angular velocity of the gyroscope. We observed that good frequency matching of the driving mode with the detection modes was essential for the gyroscope to function well as a sensor for angular velocity. A sensitivity analysis was performed for detection of z-axis angular velocity as a function of the silicon rubber adhesive thickness at the bottom of the spacers. The z-axis detection mode was found to be sensitive to the adhesive height. Conversely the driving mode was found to be insensitive to the adhesive height. Therefore, the adhesive height is an important factor in the matching of the z-detection mode frequency to the driving mode frequency. The adhesive height would affect the production yields of this gyroscope.
Chapter 4. Piezoelectric length-extension gyroscope

4.1 Introduction

A piezoelectric length extension gyroscope is a newly designed gyroscope, which utilizes the length extension mode and flexure mode to detect the Coriolis force [24]. The z-cut, third-overtone, extensional quartz resonator was extensively studied by Yong, Lee and Chung [25] [26]. Moreover, the length-extension mode are studied by many other researchers [27] [28]. It can be made of a single piezoelectric crystal with MEMs fabrication technology. It has small size structure that can be easily mount and manufactured. The geometry of a length extension gyroscope is very simple compared to other vibratory gyroscopes. The length extension gyroscope utilizes the length extension mode to reduce the energy dissipation that uses the symmetric mode shape with the zero displacement nodes at the center of the driving arms. In this chapter 4, the piezoelectric length extension gyroscope is analyze by finite element analysis. Furthermore, gyroscopic sensitivity of quartz, langasite and langatate length extension gyroscopes are compared.

4.2 Structure of the Piezoelectric Length Extension Vibratory Gyroscope

The piezoelectric length extension vibratory is comprises of three main components, driving parts, sensing parts and supporting arms and base, as shown in figure 4.2.1 and figure 4.2.2.

4.2.1 Driving Parts

The driving arms are located at the middles of the sensing arms. The crystallographic axis and electric polarization direction of each crystal are considered for designing the length extension gyroscopes. The driving part electrodes are placed on the side wall (y-z
plane) of the driving arms so the drive arms can extend and contract in drive direction (y-axis). In order to get maximum gyroscopic sensitivity, gyroscopes are designed so that the drive mode shape is identical to the resonant frequency mode shape, the first length extension mode. The each driving arms are driven by ± 1 V which applied on one side of the electrodes whereas opposite sides of the electrodes are ground. The advantage of using length extension mode is energy dissipation of the gyroscope. The drive-mode vibration mode has very small influence by the support conditions, which high Q-factor of the gyroscope can be easily achieved.

4.2.2 Sensing Parts

The sensing part is the part that connects the driving arms and the support part while it measures the Coriolis force signals of the gyroscope. The sensing bridge is located at the end of the supporting arms, as shown in figure 4.2.2. Similar to the driving arms, the electrodes on the sensing bridge are located on the sidewalls (x-y plane) of the sensing bridge. The electrodes are placed so that the resonant frequency mode shape, the flexure mode, is similar to the sensing mode shape. In consideration of the crystallographic axis and electric polarization direction of each crystal, the sensing electrodes, which measure and output the Coriolis force signal of a gyroscope, are placed on one side of the sidewalls and the electrodes on the other side is grounded. The result is that higher sensitivity of the gyroscope can be achieved.

4.2.3 Support Part and Base

The supporting part is located between the base and sensing part, as shown in figure 4.2.2. The length of the support part, $L_2$, is determined by the length of driving arm, $L_D$. The small gap between the end of the driving arm and the supporting part is adequate,
since the displacement of the driving arms is insignificant. The width of the length, Ws, has to be long enough so that displacement of the support part along the x-axis is minimized. The end of the base is fixed to the fixed end conditions.

Figure 4.2.1 Structure of the piezoelectric length extension vibratory.

Figure 4.2.2 Dimensions of the piezoelectric length extension gyroscope
4.3 Principle of Operation of the Piezoelectric Length Extension Gyroscope

The principle of operation of a length extension gyroscope is shown in figure 4.3.1. The each mode of two vibration modes is used as driving mode and sensing mode. Both modes are used to detect the angular velocity rotation about z-axis. The driving mode is a first extension mode of driving arm in the x-y plane. The driving arm is expands and contracts along y-axis by the external harmonic excitation force, as shown in figure 4.3.2 (a). The midpoint of the driving arm is the zero displacement node. The second vibration mode is the z-axis detection mode that is an anti-symmetric flexure mode in x-y plane, as shown in figure 4.3.2 (b). When the gyroscope is subjected to a rotation about z-axis, a pair of Coriolis forces, $F_c$, on the driving arms is generated proportional to the angular rate of gyroscope, $\Omega_z$, mass density, $m$, and vibration velocity, $V_y$. A pair of equal and opposite Coriolis force on each driving arms, which generated by the rotation about z-axis, create the moment, $M_c$, on the center of the driving arms. The moment, $M_c$, induce the flexure mode of the sensing arms. The z-axis detection electrodes on the sensing arm measure the Coriolis force signals. The driving frequency, $f_d$, and sensing frequency, $f_s$, have to tuned closed to each other in order to achieve higher gyroscopic sensitivity.
Figure 4.3.1 The principle of operation of a length extension gyroscope

(a) Driving mode

(b) Sensing mode

Figure 4.3.2 Drive-mode and sense-mode vibration modes of the piezoelectric length extension gyroscope
4.4 Finite Element Simulation

In this section, the length extension gyroscope is simulated and analyzed by finite element analysis method. Quartz, langasite and langatate crystals are used as materials for the length extension gyroscopes. The three materials can be easily compared, since they have same crystal properties. The material properties of the quartz, langasite and langatate are listed in Appendix C. The eigenfrequency analysis and frequency domain analysis are performed to study the geometric characteristics and gyro-characteristics of each gyroscope. The following equations 4.4.1, 4.4.2, 4.4.3, and 4.4.4 are implemented in COMSOL finite element method software. The equation 4.4.1 is the strain-displacement relation and the equation 4.4.2 is the electric field-potential relation.

\[ S_{ij} = \frac{1}{2} (u_{j,i} + u_{i,j}) \]  \hspace{1cm} (4.4.1)
\[ E_i = -\varphi_i \]  \hspace{1cm} (4.4.2)

where \( S_{ij} \) is the mechanical strains, \( u_i \) is the mechanical displacements, and \( E_i \) is the electrical field and \( \varphi_i \) is the electric potential.

\[ T_k = -e_{Kk}E_k + c^{E}_{kj}S_j \]  \hspace{1cm} (4.4.3)
\[ D_i = \varepsilon_{ik}E_k + e_{ij}S_j \]  \hspace{1cm} (4.4.4)

The equation 4.4.3 and 4.4.4 are the piezoelectric constitutive equations in stress-charge form. \( T_k, e_{Kk}, c^{E}_{kj}, D_i, \varepsilon_{ik} \) are the mechanical stress, piezoelectric constants, linear elastic constants, electric displacements, and dielectric permittivity, respectively.
Figure 4.4.1 FEM mesh of the length extension gyroscope

Figure 4.4.2 The locations of the drive and sensing electrodes of the gyroscope
The figure 4.4.1 shows the FEM model of the proposed gyroscope. The 10 layers of the triangle shaped elements were used in mesh for all three materials. The total number of elements and degrees of freedom which used in FEM is listed in table 4.4.1. The drive and sense electrodes are also included in the FEM simulations, which shown in figure 4.4.2. All the electrodes are coated with 0.2μm thickness gold film.

The targeted the driving frequency of the gyroscopes to be around 1 MHz. The gyroscope uses a length extension mode as a driving mode. The driving arm is symmetric along the x-axis. Hence, we considered a longitudinal vibration of elastic beam of fixed-free case to simplify the problem. The equation of the motion for free vibration of elastic beam is

\[ m \frac{\partial^2 u}{\partial t^2} - EA \frac{\partial^2 u}{\partial x^2} = 0 \] (4.4.5)

Applying the fixed-free boundary conditions to the equation 4.4.5, we can obtain the equation for the frequency of a longitudinal vibration of beam when the structure is fixed at its one end.

\[ f \equiv \frac{1}{2\pi} \sqrt{\frac{EA}{mL_D^3}} = \frac{1}{2\pi L} \sqrt{\frac{E}{\rho}} \] (4.4.6)

The driving frequencies of each material are depends on the length of the driving arms, L_D, elastic modulus, E, and mass density. Since elastic modulus and mass density of each materials are constant, the driving frequencies are depends on the length of the driving arms of the gyroscope. Therefore, the lengths of the drive arms of each gyroscope are different; such that the length of the driving arms of langasite and langatate gyroscopes are shorter than quartz gyroscope. Hence, overall size of the langasite and langatate gyroscopes can be minimized.
4.4.1 Eigenfrequency analysis

The eigenfrequency analyses were used to determine the length of the drive arms as well as overall size of each gyroscope. The specific sizes of each gyroscope are listed in Table 4.4.2.

Table 4.4.1 Conditions of FEM simulation

<table>
<thead>
<tr>
<th>Material</th>
<th>Quartz</th>
<th>Langasite</th>
<th>Langatate</th>
</tr>
</thead>
<tbody>
<tr>
<td># of elements</td>
<td>26,950</td>
<td>10,110</td>
<td>5,170</td>
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<tr>
<td># of degree of freedom</td>
<td>490,310</td>
<td>188,918</td>
<td>100,214</td>
</tr>
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</table>

Table 4.4.2 Dimension of the length extension gyroscope of quartz, langasite and langatate.

<table>
<thead>
<tr>
<th></th>
<th>Quartz</th>
<th>Langasite</th>
<th>Langatate</th>
</tr>
</thead>
<tbody>
<tr>
<td>W</td>
<td>2.500</td>
<td>1.074</td>
<td>1.074</td>
</tr>
<tr>
<td>Wₚ</td>
<td>1.258</td>
<td>0.574</td>
<td>0.574</td>
</tr>
<tr>
<td>W₀</td>
<td>0.100</td>
<td>0.100</td>
<td>0.100</td>
</tr>
<tr>
<td>L</td>
<td>3.000</td>
<td>2.000</td>
<td>2.000</td>
</tr>
<tr>
<td>L₀</td>
<td>2.510</td>
<td>2.016</td>
<td>1.998</td>
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<tr>
<td>L₂</td>
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<td>0.300</td>
<td>0.300</td>
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<tr>
<td>Lₘ</td>
<td>0.100</td>
<td>0.100</td>
<td>0.100</td>
</tr>
<tr>
<td>t</td>
<td>0.060</td>
<td>0.060</td>
<td>0.060</td>
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</table>

Units: mm

In the simulation, the cut angles of each crystal are different. It is realized that the z-cut crystals are not the optimum cut angle for all materials. Hence, we analyzed the each material by rotating the geometry along the x-axis to find out the geometry where the displacement along the z-axis of the driving arm is minimum. It is not only reduces the vibration of the system but also reduce the energy dissipation which increases the
sensitivities of the gyroscopes. The optimum cut angles of the quartz, langasite and langatate are $\phi = 0^\circ$, $\theta = -9^\circ$, $\phi = 0^\circ$, $\theta = 24^\circ$ and $\phi = 0^\circ$, $\theta = 24^\circ$, respectively.

The sensitivity of the gyroscopes is depending on the matching of the driving frequency and the sensing frequency. Therefore, parametric eigenfrequency analyses were performed to obtain the optimum length of the drive arm of each material. Thus, the parameters which used in this simulation were the length of the driving arms. The changing the length of the drive arms not only affect the driving frequencies but also sensing frequencies as shown in figure 4.4.3, 4.4.4 and 4.4.5. The rates of change of the sensing frequencies are higher than driving frequencies. The adequate length of the drive arms of the quartz gyroscope, langasite gyroscope and langatate gyroscope are 2.5 mm, 2.016 mm and 1.998 mm, respectively. The diving and sensing frequencies can be adjust by changing the length of the sensing arm, $W_s$ or width of the sensing arm, $L_s$. However, changing those two parameter to match the driving and sensing frequencies cannot be accomplished due to the limitation of its geometry. Hence, it can only be used for fine tuning. The mechanical coupling between the driving mode and sensing mode of the quartz gyroscope is very small compared to the other two materials, this makes it easier to match the driving and sensing frequency very closed to each other when the length of the driving arm is around 2.515mm as shown in figure 4.4. The mechanical coupling of langasite gyroscope is larger than quartz gyroscope but it’s smaller than langatate gyroscope. Two modes can be matched when the length of the driving arm is near 2.018mm as shown in figure 4.4.4. As shown in figure 4.4.5, the piezoelectric coupling factor of langatate is larger enough so that two modes are never cross each other.
Previously, we discussed that matching the driving and sensing frequencies increase the gyro-sensitivity of the gyroscope. However, the system becomes unstable when the driving frequency gets too close to the sensing frequency which requires the frequency separation of driving and sensing frequency. The frequency separations of quartz, langasite and langatate gyroscopes are 29.4 Hz, 504 Hz and 519.5 Hz, respectively. The separation frequency equation is simple and it is

$$\Delta f = f_{\text{detecting}} - f_{\text{driving}}$$  \hspace{1cm} (4.4.7)$$

where $f_{\text{detecting}}$ is the detecting frequency and $f_{\text{driving}}$ is the frequency.
Figure 4.4.3 Frequency spectrum of the quartz gyroscope as a function of the length of the drive arm, $L_D$
Figure 4.4.4 Frequency spectrum of the langasite gyroscope as a function of the length of the drive arm, $L_D$
Figure 4.4.5 Frequency spectrum of the langatate gyroscope as a function of the length of the drive arm, $L_D$. 
4.4.2 Frequency response analysis

We further analyzed the length extension gyroscope with frequency response analyses using the optimized geometries of the gyroscopes of each material. The equations of the motion of the element of the rotating system rotating about z-axis are implemented in FEM simulation and they are

\[ f_x = F_x + \rho \omega^2 u_x + j2\rho \Omega_x \omega u_y + \rho \Omega_z^2 x \]  
(4.4.8)

\[ f_y = F_y + \rho \omega^2 u_y - j2\rho \Omega_x \omega u_x + \rho \Omega_z^2 y \]  
(4.4.9)

\[ f_z = 0 \]  
(4.4.10)

where \( f, F, u, \Omega \) and \( \omega \) are respectively the total force acting on the element, external forces, element displacement, and angular velocity and angular frequency. The \( x \) and \( y \) are the coordinates of the element. The first term and second terms of the right hand side of equation 4.4.8 and 4.4.9 are the external force and inertia forces induced by vibration. The third terms on the right hand side of equation 4.4.8 and 4.4.9 are the Coriolis force terms that we are interested in this study. Finally, the last terms of the equations are the centrifugal forces. The Coriolis forces are applied as body forces, \( F_b \), on the entire gyroscopes in frequency response analyses to simulate the gyroscope rotation about z-axis and they are

\[ F_{bx} = 2\rho \omega \Omega_x u_y \]  
(4.4.11)

\[ F_{by} = -2\rho \omega \Omega_x u_x \]  
(4.4.12)

\[ F_{bz} = 0 \]  
(4.4.13)

where \( F_{bx}, F_{by} \) and \( F_{bz} \) are respectively the body forces applied to the entire structure in x-, y-, and z-directions. Since the vibration direction of the drive arms of the gyroscope is parallel to the angular velocity, z-axis, the Coriolis force in the z-direction is zero.
Similarly, we can also apply the Coriolis forces for rotation about x- and y-axis. The absolute values of the charges on the driving and sensing electrodes of each material as a function of the driving frequency, \( f_{dr} \), are shown in figure 4.4.6, 4.4.7 and 4.4.8. Also, the gyroscope response of each gyroscope to angular velocity of 60 degrees/s about x-, y- and z-axis are shown in figure 4.4.9, 4.4.10 and 4.4.11. These results show that the charge are the maximum at the resonance frequency, \( f_{dr} \), of the driving mode. It also shows that the sensitivity at the resonance will get amplified the most. The sensitivities of each gyroscope are also calculated by frequency response analyses. The electric potential was used as output signal. The gyro-sensitivities of the gyroscopes of each material are shown in figure 4.4.12, 4.4.13 and 4.4.14. The sensitivities of the quartz, langasite and langatate gyroscopes are \( S_z = 2.78 \times 10^{-4} \text{V/(deg/s)} \), \( S_z = 1.60 \times 10^{-2} \text{V/(deg/s)} \), \( S_z = -5.76 \times 10^{-3} \text{V/(deg/s)} \), respectively. The sensitivity of the quartz gyroscope was calculated to be very small as expected. The sensitivities of the langasite and langatate are about 57 times and 20 times more sensitive than the quartz gyroscope. We also simulate the gyro-sensitivities of the gyroscopes rotation about the x-and y-axis. Both gyro-sensitivities were very small as expected. The quality factor, \( Q \), is also calculated. The summary of the gyro-sensitivity of the gyroscopes are listed in Table 4.4.3. The results clearly show that the length extension gyroscope performed as a gyro-sensor.
Figure 4.4.6 Frequency responses in charge at the driving and sensing electrodes for the quartz gyroscope.
Figure 4.4.7 Frequency responses in charge at the driving and sensing electrodes for the langasite gyroscope.
Figure 4.4.8 Frequency responses in charge at the driving and sensing electrodes for the langatate gyroscope.
Figure 4.4.9  Gyroscope response of the quartz gyroscope to angular velocity of 60 degree/s about x-, y- and z-axis.
Figure 4.4.10 Gyroscope response of the langasite gyroscope to angular velocity of 60 degree/s about x-, y- and z-axis.
Figure 4.4.11 Gyroscope response of the langatate gyroscope to angular velocity of 60 degree/s about x-, y- and z-axis.
Figure 4.4.12 Gyroscopic sensitivity of the quartz gyroscope rotation about x-, y- and z-axis.
Figure 4.4.13 Gyroscopic sensitivity of the langsite gyroscope rotation about x-, y- and z-axis.
Figure 4.4.14 Gyroscopic sensitivity of the langtate gyroscope rotation about x-, y- and z-axis
Table 4.4.3 The driving and sensing frequencies of the gyroscopes and sensitivities of the gyroscopes of each material.

<table>
<thead>
<tr>
<th>Units</th>
<th>Quartz</th>
<th>Langasite</th>
<th>Langatate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{dr.}$</td>
<td>Hz</td>
<td>1034888</td>
<td>999496</td>
</tr>
<tr>
<td>$f_{sens.}$</td>
<td>Hz</td>
<td>1034854</td>
<td>999999.7</td>
</tr>
<tr>
<td>$S_x$</td>
<td>V/(deg/s)</td>
<td>8.46e-6</td>
<td>1.52e-4</td>
</tr>
<tr>
<td>$S_y$</td>
<td>V/(deg/s)</td>
<td>1.70e-6</td>
<td>8.90e-5</td>
</tr>
<tr>
<td>$S_z$</td>
<td>V/(deg/s)</td>
<td>2.78e-4</td>
<td>-1.6e-2</td>
</tr>
<tr>
<td>$Q$</td>
<td></td>
<td>230048</td>
<td>970375</td>
</tr>
</tbody>
</table>
4.5 Frequency-Temperature Analysis of the Langasite

The frequency-temperature behavior of the gyroscope is an important criterion of a design of the langasite, since small shift in drive frequency can significantly reduce the gyroscopic sensitivity of the gyroscope. However, many researchers or engineers do not aware of this effect when they design the gyroscope. In this chapter, the frequency-temperature behavior of the langasite length-extension gyroscope is studied. We only considered the langasite gyroscope, since we do not have all the material properties of the langatate such as third-order non-linear elastic constants or higher order temperature constants.

We implemented the non-linear questions from chapter 5 into Comsol Multiphysics software to compute the frequency-temperature curves of the gyroscopes. The simulation involves two steps, stationary and eigenfrequency. In the stationary analysis, only the solid mechanics module is involved, which determines the material behavior affects by the temperature changes. In the eigenfrequency analysis, the changes of drive frequency of the gyroscope as a function of temperature is calculated. Please see chapter 5 for more details.

The figure 4.5.1 and 4.5.2 show the temperature–frequency curve of the quartz and langasite length-extension gyroscope. Both gyroscopes are not temperature compensated, as the curves nearly linear. The cubic and parabolic curves can be expected for quartz and langasite gyroscopes, respectively, when the gyroscopes are temperature compensated. The langasite curve can only be parabola in this case, since we do not have third-order temperature coefficients of langasite. For more detail of the non-linear parameters and variable of Comsol software please see Appendix D.
Figure 4.5.1 The temperature–frequency curve of the quartz gyroscope

Figure 4.5.2 The temperature–frequency curve of the langasite gyroscope
4.6 SUMMARY

A novel design of the length extension gyroscope is proposed and it is analyzed with the finite element analysis software. The calculated optimum cut angle for a singly rotated quartz crystal have $\phi=0^\circ$, $\theta=-9^\circ$, a singly rotated langasite crystal have $\phi=0^\circ$, $\theta=24^\circ$ and a singly rotated langatate have $\phi=0^\circ$, $\theta=24^\circ$, respectively. The sizes of the gyroscopes vary due to the material properties and length of the driving arms. Overall sizes of the langasite and langatate gyroscopes are smaller than the quartz gyroscope and these two are nearly the same size. The gyro-sensitivity rotations about z-axis of quartz, langasite and langatate gyroscopes were $2.78 \times 10^{-4} V/(\text{deg/s})$, $-1.60 \times 10^{-2} V/(\text{deg/s})$ and $-1.6 \times 10^{-2} V/(\text{deg/s})$. The results show that the finite element method to design the gyroscope are reliable and the newly designed length extension gyroscope can be used as a gyro-sensor. Moreover, the langasite and langatate gyroscope provided stronger sensitivity to angular velocity than the quartz gyroscope as we expected. The gyro-sensitivities of the gyroscopes are also a function of the geometry of the gyroscope. The length extension gyroscope is also able to detect the angular velocities about the other two axes (x- and y-axes).
Chapter 5. The Frequency-Temperature Behavior of Langasite

5.1 Introduction

Frequency control and stability of the resonator or piezoelectric sensor at high frequency or higher sensitivity is one of the most important criteria in the design of piezoelectric sensor. The quartz crystal has been used for many applications due to its temperature stability. For example, the AT-cut is the most well-known and most used type in resonator. It has a frequency temperature coefficient described by a cubic function of temperature whereas most other crystal cuts give a parabolic temperature characteristic. The frequency-temperature curve of the fundamental thickness-shear vibration of AT-cut is shown in Figure 5.1.1.

The langasite and langatate are highly attractive piezoelectric materials for the high temperature environment due to their temperature frequency coefficient (TFC) and their temperature stability. Especially, the langasite and its isomorphs have the quite unique temperature compensate property that is similar to the quartz crystal [29] [30] [31]. Therefore, these crystals are of current interest for BAW applications [32]. The complete sets of the materials properties of quartz, langasite and langastate are listed in Appendix C. The analysis on the temperature-frequency behavior of langatate has not performed in this dissertation, since third-order non-linear elastic constants of langatate have not published.

The non-linear field equations of thermo-elasticity in Langrangian formulation are used to develop the three dimensional linear field equations for small vibration superposed on thermally induced deformations by steady and uniform temperature change. This Lagrangian formulation consist with three states, the natural state, initial state and final state [33] [34] [35] [36].
Figure 5.1.1 Frequency-temperature curves of the fundamental thickness-shear vibration of AT-cut plate. [34]
Figure 5.1.2 positions of a material point at the natural, initial and final state. [34]
5.2 Equation of Motion for Small Vibrations Superposed on Thermally Induced Deformations

The non-linear equations of thermo-elasticity in Lagrangian formation consist of three steps, natural state, initial state and final state. The incremental equations are conveniently derived with the following definitions:

- $T_0$: Reference temperature
- $T$: Final temperature from the reference temperature $T-T_0$
- $\theta$: Temperature increment respect to reference temperature
- $U_i$: Displacement at initial state
- $\bar{U}_i$: Total displacement from the natural state to final state
- $u_i$: Incremental displacement
- $E_{ij}$: Lagrangian strain tensor at initial state
- $\bar{E}_{ij}$: Total strain tensor from the natural state to final state
- $e_{ij}$: Incremental strain
- $T_{ij}$: Kirchoff - Piola stress tensor at initial state
- $\bar{T}_{ij}$: Total stress tensor from the natural state to final state
- $t_{ij}$: Incremental stress
- $P_i$: Surface traction at initial state
- $\bar{P}_i$: Total traction from the natural state to final state
- $p_i$: Incremental traction
- $n_i$: Unit outward normal vector to $S$
- $C_{ijkl}$: Second-order elastic stiffnesses of the crystal
\( C_{ijklmn} \): Third-order elastic stiffnesses of the crystal

\( C_{ijklmpq}^{(4)} \): Four-order elastic stiffnesses of the crystal

\( C_{ijkl}^{(1)} \): First temperature derivative of the second-order elastic stiffnesses of the crystal

\( \tilde{C}_{ijkl}^{(2)} \): Effective Second temperature derivative of the second-order elastic stiffnesses of the crystal

\( \lambda_i^\theta \): Stress coefficient of temperature

\( \alpha_i^\theta \): Thermal expansion coefficient

The first state is the natural state where crystal is at rest, free of stress and strain with a uniform temperature \( T_0 \) as shown in figure 5.1.2. The rectangular Cartesian coordinate system is used to illustrate the position of a material point as shown in figure 5.1.2. \( x_i \) (\( i=1,2,3 \)) is the position of a material point and \( T_0 \) is the reference temperature.

Next state is the initial state which crystal is subject to a steady and uniform temperature increase from \( T_0 \) to \( T \). In this state, the material is allows expand freely and the position of a material is moved from \( x_i \) to \( y_i \) due to the increases of the temperature. The non-linear equation of the thermo-elasticity in Lagrangian formulation can be written as:

(a) Displacement

\[
U_i = y_i - x_i
\]  
(5.2.1)

(b) Strain

\[
E_{ij} = \frac{1}{2} (U_{j,i} + U_{i,j} + U_{k,i} U_{k,j})
\]  
(5.2.2)
c) Stress-strain-temperature relations

\[ T_{ij} = \]
\[ C_{ijkl}^\theta E_{kl} + \frac{1}{2} C_{ijklmn}^\theta E_{kl} E_{mn} + \frac{1}{6} C_{ijklmnop}^\theta E_{kl} E_{mn} E_{pq} + \]
\[ \frac{1}{24} C_{ijklmnopqrs}^\theta E_{kl} E_{mn} E_{pq} E_{rs} - \lambda_{ij}^\theta \] (5.2.3)

(d) Stress equation of motion

\[ (T_{ij} + T_{jk} U_{ik})_{ij} = \rho \ddot{U}_i \] (5.2.4)

(e) Traction-stress tensor relation

\[ P_i = n_j (T_{ij} + T_{jk} U_{ik}) \text{ on } S \] (5.2.5)

In the stress-strain-temperature relations equation, equation 5.2.3, material properties can be assumed to be temperature dependent and have polynomial form.

\[ C_{ijkl}^\theta = C_{ijkl} + C_{ijkl}^{(1)} + \frac{1}{2} C_{ijkl}^{(2)} \theta^2 + \frac{1}{6} C_{ijkl}^{(3)} \theta^3 \] (5.2.6)

\[ C_{ijklmn}^\theta = C_{ijklmn} + C_{ijklmn}^{(1)} \theta + \frac{1}{2} C_{ijklmn}^{(2)} \theta \] (5.2.7)

\[ C_{ijklmnop}^\theta = C_{ijklmnop} + C_{ijklmnop}^{(1)} \theta \] (5.2.8)

\[ C_{ijklmnopqrs}^\theta = C_{ijklmnopqrs} \] (5.2.9)

\[ \lambda_{ij}^\theta = \lambda_{ij}^1 \theta + \lambda_{ij}^2 \theta^2 + \lambda_{ij}^3 \theta^3 + \lambda_{ij}^4 \theta^4 \] (5.2.10)

where

\[ C_{ijkl}^{(n)} = \left. \frac{\partial^n C_{ijkl}}{\partial T^n} \right|_{T_0}, \quad n = 1,2,3 \] (5.2.11)

\[ C_{ijklmn}^{(n)} = \left. \frac{\partial^n C_{ijklmn}}{\partial T^n} \right|_{T_0}, \quad n = 1,2 \] (5.2.12)

\[ C_{ijklmnop}^{(n)} = \left. \frac{\partial^n C_{ijklmnop}}{\partial T^n} \right|_{T_0}, \quad n = 1 \] (5.2.13)
The equation 5.2.11, 5.2.12 and 5.2.13 are temperature derivatives of the elastic stiffnesses evaluated at the reference temperature in the natural state. The linear thermal expansion coefficients are symmetric and have polynomial form as shown in equation 5.2.14.

\[ \alpha_{ij}^\theta = \alpha_{ij}^1 \theta + \alpha_{ij}^2 \theta^2 + \alpha_{ij}^3 \theta^3 \]  
(5.2.14)

\[ \frac{1}{2} (U_{j,i} + U_{i,j}) = \alpha_{ij}^\theta = \alpha_{ji}^\theta \]  
(5.2.15)

If the equation 5.2.14 is the linear thermal expansion which measured instead of stress coefficient, \( \lambda_{ij}^\theta \), the equation 5.2.2 can be written in terms of thermal coefficient expansion.

\[ E_{ij} = \alpha_{ij}^\theta + \frac{1}{2} \alpha_{kl}^\theta \alpha_{kj}^\theta \]  
(5.2.16)

In the initial fields the plate is at rest and allowed free expansion. Also, any higher term are neglected since higher order terms are much smaller than first term in equation 5.2.16.

\[ U_{j,i} = U_{i,j} = E_{ij} = \alpha_{ij}^\theta \]  
(5.2.17)

\[ T_{ij} = 0 \]  
(5.2.18)

\[ \ddot{U}_i = 0 \]  
(5.2.19)

In the final state, small vibration is superposed with the thermally induced deformation. The governing equations for final state is similir to initial state. In this state, the material point is moved from \( y_i \) to \( z_i \). Therefore, incremental displacement can be calculated by simply taking the different between the final state displacement and the initial state displacement, \( u_i = z_i - y_i \) as shown in equation 5.2.20. The variable at the final state is denoted by “barred” quantity.

\[ \overline{U}_i = U_i + u_i = z_i - x_i \]  
(5.2.20)
Similarity, strain, stress and traction at the final state can be defined as:

\[ E_{ij} = E_{ij} + e_{ij} \]  
\[ T_{ij} = T_{ij} + t_{ij} \]  
\[ P_l = P_l + p_l \]

The governing equations for final state are similar to initial state and they are:

\[ E_{ij} = \frac{1}{2} (U_{j,i} + U_{i,j} + U_{k,i} U_{k,j}) \]  
\[ T_{ij} = C_{ijkl}^{\theta} E_{kl} + \frac{1}{2} C_{ijklmn}^{\theta} E_{kl} E_{mn} + \frac{1}{6} C_{ijklmnpq}^{\theta} E_{kl} E_{mn} E_{pq} + \frac{1}{24} C_{ijklmnpqrs}^{\theta} E_{kl} E_{mn} E_{pq} E_{rs} - \lambda_{ij}^{\theta} \]  
\[ (T_{ij} + T_{jk} U_{l,k})_{,j} = \rho \ddot{U}_i \]  
\[ P_l = n_j (T_{ij} + U_{l,k}) \quad \text{on } S \]

As the temperature at the final state does not change, the governing equation for incremental fields can be calculated by subtracting initial state from final state and they are defined as:

\[ e_{ij} = \frac{1}{2} (u_{j,i} + u_{i,j} + U_{k,j} u_{k,i} + U_{k,i} u_{k,j}) \]  
\[ t_{ij} = \left( C_{ijkl}^{\theta} + C_{ijklmn}^{\theta} E_{mn} + \frac{1}{2} C_{ijklmnpq}^{\theta} E_{mn} E_{pq} + \frac{1}{6} C_{ijklmnpqrs}^{\theta} E_{mn} E_{pq} E_{rs} \right) e_{kl} \]  
\[ (t_{ij} + t_{jk} U_{l,k} + T_{jk} U_{l,k})_{,j} = \rho_0 \ddot{U}_i \]
\[ p_i = n_j(t_{ij} + t_{jk}U_{i,k} + T_{jk}u_{i,k}) \text{ on } S \]  
(5.2.31)

In the governing equation of incremental fields, the incremental deforemation are assumed to be small and product terms are neglected.

By substituting equation 5.2.17 into 5.2.28, the incremental strain-displacement relation can be defined as:

\[ e_{ij} = \frac{1}{2} (u_{j,i} + u_{i,j} + a_{kj,i}u_{k,i} + a_{ki,j}u_{k,j}) \]  
(5.2.32)

By substituting equation 5.2.6,5.2.7 and 5.2.8 into equation 5.2.29, the stress-strain-temperature relations can be defined as:

\[ t_{ij} = \left( C_{ijkl} + D_{ijkl}^{(1)}\theta + D_{ijkl}^{(2)}\theta^2 + D_{ijkl}^{(3)}\theta^3 \right) e_{kl} \]  
(5.2.33)

where

\[ D_{ijkl}^{(1)} = C_{ijkl}^{(1)} + C_{ijklmn}\alpha_{mn}^{(1)} \]

\[ D_{ijkl}^{(2)} = \frac{1}{2} \sigma_{ijkl}^{(2)} + C_{ijklmn}\alpha_{mn}^{(2)} \]

\[ D_{ijkl}^{(3)} = \frac{1}{6} \sigma_{ijkl}^{(3)} + C_{ijklmn}\alpha_{mn}^{(3)} \]

\[ e_{ijkl}^{(2)} = C_{ijkl}^{(2)} + 2C_{ijklmn}\alpha_{mn}^{(1)} + C_{ijklmnq}\alpha_{mn}^{(1)}\alpha_{pq}^{(1)} \]

\[ e_{ijkl}^{(3)} = C_{ijkl}^{(3)} + 3C_{ijklmn}\alpha_{mn}^{(1)} + 6C_{ijklmn}\alpha_{mn}^{(2)} \]

\[ + 6C_{ijklmnq}\alpha_{mn}^{(1)}\alpha_{pq}^{(1)} + 3C_{ijklmnq}\alpha_{mn}^{(1)}\alpha_{pq}^{(1)} + C_{ijklmnqrs}\alpha_{mn}^{(1)}\alpha_{pq}^{(1)}\alpha_{rs}^{(1)} \]

Similarly, the stress equations of motion can be calculated and they are defined as:
\[(t_{ij} + \alpha_{ik}^{\theta} t_{jk})_j = \rho_0 \ddot{u}_i \]  \hspace{1cm} (5.2.34)

\[p_i = n_j (t_{ij} + \alpha_{ik}^{\theta} t_{jk}) \text{ on } S \]  \hspace{1cm} (5.2.35)

Finally, the incremental displacement equation of the motion and the traction boundary condition can be defined by substituting equation 5.2.16 and 5.2.32 into equation 5.2.33 and then, equation 5.2.34 into 5.2.35.

\[G_{ijkl}u_{kjl} = \rho_0 \ddot{u}_i \]  \hspace{1cm} (5.2.36)

\[p_i = n_j G_{ijkl}u_{k,l} \text{ on } S \]  \hspace{1cm} (5.2.37)

where

\[G_{ijkl} = C_{ijkl} + G_{ijkl}^{(1)} \theta + \frac{1}{2} G_{ijkl}^{(2)} \theta^2 \]  \hspace{1cm} (5.2.38)

\[G_{ijkl}^{(1)} = G_{ijkl}^{(1)} \alpha_{is}^{(1)} + G_{ijkl}^{(1)} \alpha_{is}^{(1)} \right) \hspace{1cm} (5.2.39)

\[G_{ijkl}^{(2)} = C_{ijkl} \alpha_{is}^{(1)} + C_{ijkl} \alpha_{is}^{(1)} + C_{ijkl} \alpha_{is}^{(1)} + C_{ijkl} \alpha_{is}^{(1)} + C_{ijkl} \alpha_{is}^{(1)} + C_{ijkl} \alpha_{is}^{(1)} + C_{ijkl} \alpha_{is}^{(1)} + C_{ijkl} \alpha_{is}^{(1)} \] (5.2.40)

\[G_{ijkl}^{(3)} = \frac{1}{6} \tilde{G}_{ijkl}^{(3)} \right) \hspace{1cm} (5.2.41)

Since the \(D_{ijkl}^{(n)}\) and \(A_{ijkl}^{(2)}\) have same symmetry as \(C_{ijkl}\) from equation 5.2.33, 5.2.36, 5.2.37 and 5.2.38, we can say

\[G_{ijkl}^{(n)} = G_{klij}^{(n)} \]  \hspace{1cm} (5.2.42)

but

\[G_{ijkl}^{(n)} \neq G_{jikl}^{(n)} \neq G_{ijkl}^{(n)} \]
The non-linear equations become to linear equations when $\theta=0$. For the small vibration superposed on homogeneous thermal strain only required $C_{ijkl}^{(1)}$ and $\tilde{C}_{ijkl}^{(2)}$ to describe the temperature dependence of elastic stiffnesses up to the second power of $\theta$ based on above equations. As previously mentioned, we cannot calculate higher than second power of $\theta$ since there is no measured values of effective third-order temperature derivative.

5.2.1 Thickness Vibration

We let $x_i (i = 1, 2, 3)$ be the crystallographical axes of the crystals. By IEEE, a doubly rotated crystal plate is defined by $(y, x, \omega, \ell) \phi, \theta$. The crystal plate has thickness of $2b$ as shown in figure 5.3 which $n_i$ denote the unit outward normal to the faces of the plate in the natural state. Refer to the crystallographic axes and the figure 5.3, the unit normal components are:

$$n_1 = - \cos \theta \sin \phi$$
$$n_2 = \cos \theta \sin \phi$$
$$n_3 = \sin \theta$$

(5.2.43)

For traction free face conditions, equation 5.2.37 becomes

$$p_i = n_j G_{ijkl} u_{k,l} = 0 \quad \text{at} \quad n_j X_j = \pm b$$

(5.2.44)

For solution of equation 5.2.36, for harmonic and antisymmetric thickness vibrations

$$u_k = A_k \sin \xi n_p X_p e^{i\omega t}$$

(5.2.45)

By substituting equation 5.2.45 into the 5.2.36 and 5.2.44, we can get

$$(Q_{ik} - \lambda \delta_{ik}) A_k = 0$$

(5.2.46)

where

$$\xi = \frac{n \pi}{2b}, \quad n = 1, 3, 5, ...$$
\[ Q_{ik} = Q_{ki} = G_{ijkl} n_j n_l \]

\[ \lambda = \rho_0 \frac{\omega^2}{\xi^2} = \rho_0 \left( \frac{2b \omega}{n\pi} \right)^2 \]

The equation 5.2.46 is the eigenvalue problem with the measured quantity, \( Q_{ik} \). The amplitude \( A_k \) and eignenvalue \( \lambda \) are temperature dependent with the equation 5.2.46.

\[ A_k = A_k^{(0)} + A_k^{(1)} \theta + A_k^{(2)} \theta^2 + A_k^{(3)} \theta^3 \]

\[ \lambda = \lambda^{(0)} + \lambda^{(1)} \theta + \lambda^{(2)} \theta^2 + \lambda^{(3)} \theta^3 \]  

(5.2.47)

By inserting equation 5.2.47 into equation 5.2.46, the eigenvalue problems of powers of \( \theta \) can be obtained.

\[ \theta^0 : \left[ Q_k^{(0)} - \lambda^{(0)} \delta_{ik} \right] A_k^{(0)} = 0 \]

\[ \theta^1 : \left[ Q_k^{(0)} - \lambda^{(0)} \delta_{ik} \right] A_k^{(1)} + \left[ Q_k^{(1)} - \lambda^{(1)} \delta_{ik} \right] A_k^{(0)} = 0 \]

\[ \theta^2 : \left[ Q_k^{(0)} - \lambda^{(0)} \delta_{ik} \right] A_k^{(2)} + \left[ Q_k^{(1)} - \lambda^{(1)} \delta_{ik} \right] A_k^{(1)} + \left[ Q_k^{(2)} - \lambda^{(2)} \delta_{ik} \right] A_k^{(0)} = 0 \]  

(5.2.48)

where

\[ Q_{ik}^{(0)} = G_{ijkl}^{(0)} n_j n_l \]

\[ Q_{ik}^{(1)} = G_{ijkl}^{(1)} n_j n_l \]

\[ Q_{ik}^{(2)} = G_{ijkl}^{(2)} n_j n_l \]  

(5.2.49)

The \( \lambda^{(0)} \) can be easily calculate by solving the eigenvalue problem which is the first equation of 5.2.48 and \( A_k^{(0)} \) can be solve uniquely by requiring normalization

\[ A_k^{(0)} A_k^{(0)} = 1 \]  

(5.2.50)

By utilizing equation 5.2.50 and second equation of 5.2.48, we can get following relation:

\[ \lambda^{(1)} = A_l^{(0)} Q_{ik}^{(1)} A_k^{(0)} \]  

(5.2.51)
$A_k^{(1)}$ can be solved uniquely by requiring $A_k^{(1)}$ orthogonal to $A_k^{(0)}$, since

$$[Q_k^{(1)} - \lambda^{(1)} \delta_{lk}]A_k^{(0)}$$

term in equation 5.2.48 is orthogonal to the eigenvector of $[Q_k^{(0)} - \lambda^{(0)} \delta_{lk}]$.

$$A_k^{(0)} A_k^{(1)} = 0 \quad (5.2.52)$$

We can solve $\lambda^{(2)}$ by multiplying $A_i^{(0)}$ with the third equation of 5.2.48.

$$\lambda^{(2)} = A_i^{(0)} Q_{ik}^{(2)} A_k^{(0)} + A_i^{(0)} Q_{ik}^{(1)} A_k^{(1)} \quad (5.2.53)$$

Similarity, multiplying third equation of 5.2.47 by $A_i^{(0)}$ and utilizing

$$A_k^{(0)} = \frac{1}{2} A_k^{(1)} A_k^{(1)} \quad (5.2.54)$$

We get

$$\lambda^{(3)} = A_i^{(0)} Q_{ik}^{(3)} A_k^{(0)} + A_i^{(0)} (Q_{ik}^{(1)} - \lambda^{(1)} \delta_{lk}) A_k^{(2)} + A_i^{(0)} Q_{ik}^{(2)} A_k^{(1)} \quad (5.2.55)$$

Further substituting the $\omega = 2\pi f$ into the fourth equation of 5.2.46, we can get following relation:

$$f^2 = \frac{n^2}{4\rho_0 b^2} \left[ \lambda^0 + \lambda^{(1)} \theta + \lambda^{(2)} \theta^2 + \lambda^{(3)} \theta^3 \right] \quad (5.2.56)$$

where

$$f_0^2 = \frac{n^2 \lambda^0}{4\rho_0 b^2}$$

By taking the first, second and third derivatives of equation 5.2.56 with respect to temperature $T$ and substituting Bechmann’s temperature coefficient of frequency, we can obtain $\lambda^{(1)}, \lambda^{(2)}$ and $\lambda^{(3)}$ in terms of $T f^{(n)}$.

$$\lambda^{(1)} = 2 \lambda^{(0)} T f^{(1)} \quad (5.2.57)$$

$$\lambda^{(2)} = 2 \lambda^{(0)} T f^{(2)} + \lambda^{(0)} [T f^{(1)}]^2 \quad (5.2.58)$$
\[ \lambda^{(2)} = 2 \lambda^{(0)} T f^{(3)} + 2 \lambda^{(0)} T f^{(1)} T f^{(2)} \]  

(5.2.59)

where

\[ T f^{(n)} = \frac{1}{n! f_0 ^n} \frac{\partial^n f}{\partial T^n} \bigg|_{T=0} \]

Further substituting equation 5.2.51, 5.2.53 and 5.2.55 into equation 5.2.57, 5.2.58 and 5.2.59 and rearranging the in terms of known measured values, we have

\[ 2 \lambda^{(0)} T f^{(1)} = \left[ C_{ijkl} n_j n_i A_i^{(0)} A_k^{(0)} + \left[ C_{ijkl} a_{is}^{(1)} + C_{ijkl} a_{ks}^{(1)} + \alpha_{ijklm} a_{mn}^{(1)} \right] n_j n_i A_i^{(0)} A_k^{(0)} \right] \]

(5.2.60)

\[ 4 \lambda^{(0)} T f^{(2)} = \tilde{C}_{ijkl} n_j n_i A_i^{(0)} A_k^{(0)} + 2 \lambda^{(0)} [T f^{(1)}]^2 \]

\[ + 2 \left[ C_{ijkl} a_{is}^{(1)} + C_{ijkl} a_{ks}^{(1)} + C_{ijkl} a_{is}^{(1)} a_{ks}^{(1)} + C_{ijkl} a_{is}^{(1)} + C_{ijkl} a_{ks}^{(1)} + C_{ijkl} a_{is}^{(1)} a_{ks}^{(1)} + C_{ijkl} a_{is}^{(1)} a_{ks}^{(1)} + C_{ijkl} a_{is}^{(1)} a_{ks}^{(1)} \right] n_j n_i A_i^{(0)} A_k^{(0)} \]

(5.2.61)

\[ 12 \lambda^{(0)} T f^{(3)} = \tilde{C}_{ijkl} n_j n_i A_i^{(0)} A_k^{(0)} - 12 \lambda^{(0)} T f^{(1)} T f^{(2)} \]

\[ + 6 \left( A_i^{(1)} Q_{ik}^{(1)} A_k^{(2)} + A_i^{(0)} Q_{ik}^{(2)} A_k^{(1)} + \lambda^{(0)} T f^{(1)} A_k^{(1)} A_k^{(1)} \right) \]

\[ + 6 \left[ a_{mn}^{(3)} + 2 C_{ijkl} a_{ks}^{(3)} + 2 \left( C_{ijkl} a_{is}^{(1)} a_{ks}^{(1)} \right) a_{ks}^{(2)} \right] \]

\[ + 2 \left( \frac{1}{2} C_{ijkl} a_{is}^{(2)} a_{ks}^{(2)} a_{ks}^{(2)} + 2 C_{ijkl} a_{is}^{(1)} a_{ks}^{(1)} a_{ks}^{(2)} \right) \]

\[ + \left( C_{ijkl} a_{is}^{(1)} A_i^{(1)} A_k^{(1)} + C_{ijkl} a_{is}^{(1)} A_i^{(0)} A_k^{(0)} \right) \]

(5.2.62)
Figure 5.2.1 A doubly rotated crystal plate and normal with respect to crystallographic axes. [34]
5.3 Equation of Motion for Small Vibrations Superposed on Thermally Induced Deformations with the Piezoelectric Effect.

In recent study, researcher found that the langasite is better material for BAW application than quartz crystal due to its piezoelectric properties and temperature stability. In temperature frequency analysis of quartz crystal, the piezoelectric effects are neglected since effects are marginal. On the other hand, the piezoelectric coefficient of langasite is much larger than quartz crystal. In this chapter, we will derive the three-dimensional equation of motion for small vibration superposed on thermally induced deformations with the piezoelectric effect.

The equations of motion for small vibrations superposed on thermally induced deformations with the piezoelectric effect are conveniently derived with the following definitions:

\[ \theta : \text{Temperature increment respect to reference temperature } T-T_0 \]
\[ s_{ij} : \text{Incremental strain} \]
\[ t_{ij} : \text{Incremental stress} \]
\[ p_i : \text{Incremental traction} \]
\[ E_k : \text{Electric field} \]
\[ \varphi, k : \text{Electrical potential} \]
\[ D_i : \text{Electric charge density displacement} \]
\[ e_{kij} : \text{Piezoelectric coupling coefficients} \]
\[ \varepsilon_{ik} : \text{Dielectric coefficients} \]
\[ n_i : \text{Unit outward normal vector to } S \]
\[ C_{ijkl} : \text{Second-order elastic stiffnesses of the crystal} \]
\( C_{ijklmn} \): Third-order elastic stiffnesses of the crystal

\( C_{ijklmnpq} \): Four-order elastic stiffnesses of the crystal

\( C_{ijkl}^{(1)} \): First temperature derivative of the second-order elastic stiffnesses of the crystal

\( \tilde{C}_{ijkl}^{(2)} \): Effective Second temperature derivative of the second-order elastic stiffnesses of the crystal

\( \tilde{C}_{ijkl}^{(3)} \): Effective third temperature derivative of the second-order elastic stiffnesses of the crystal

\( \lambda_{ij}^{\theta} \): Stress coefficient of temperature

\( \alpha_{ij}^{\theta} \): Thermal expansion coefficient

The Lagrangian formulation of incremental equations, which derived from previous section, will be used in this section. The following incremental equations and constitutive equations include piezoelectric effect and they are similar to the ones in previous section.

(a) Strain- displacement relation

\[
s_{kl} = \frac{1}{2} (u_{k,l} + u_{l,k} + \alpha_{r,k}^{\theta} u_{r,l} + \alpha_{r,l}^{\theta} u_{r,k})
\]  

(5.3.1)

(b) Electric field – potential relation

\[
E_k = -\varphi_{,k}
\]  

(5.3.2)

(c) Constitutive equations

\[
t_{ij} = C_{ijkl}^{\theta} s_{kl} - e_{kl} E_k
\]  

(5.3.3)

\[
D_l = e_{ikl} s_{kl} - \varepsilon_{lk} E_k
\]  

(5.3.4)
(d) Stress equation of motion

\[ (t_{ij} + \alpha_{ir}^\theta t_{rj})_{,j} = \rho \ddot{u}_i \]  
(5.3.5)

\[ p_i = n_j(t_{ij} + \alpha_{ik}^\theta t_{jk}) \quad \text{on } S \]  
(5.3.6)

(e) Electrostatics

\[ D_{t,i} = 0 \]  
(5.3.7)

The constitutive equations can be expressed in terms of displacement by substituting equation (5.3.1) into equation (5.3.3) and (5.3.4).

\[ t_{ij} = C_{ijkl}^\theta (u_{k,l} + \alpha_{rkl}^\theta u_{r,l}) - e_{kij} E_k \]  
(5.3.8)

\[ D_t = e_{t,kl} (u_{k,l} + \alpha_{rkl}^\theta u_{r,l}) + \epsilon_{ik} k \]  
(5.3.9)

where

\[ C_{ijkl}^\theta = C_{pkl} + D_{ijkl}^{(1)} \theta + D_{ijkl}^{(2)} \theta^2 + D_{ijkl}^{(3)} \theta^3 \]

\[ D_{ijkl}^{(1)} = C_{ijkl}^{(1)} + C_{ijklmn} (\alpha_{mn}^{(1)}) \]

\[ D_{ijkl}^{(2)} = \frac{1}{2} C_{ijkl}^{(2)} + C_{ijklmn} (\alpha_{mn}^{(2)}) \]

\[ D_{ijkl}^{(3)} = \frac{1}{6} C_{ijkl}^{(3)} + C_{ijklmn} (\alpha_{mn}^{(3)}) \]

\[ \alpha_{r\theta} = \alpha_{r\theta}^{(1)} \theta + \alpha_{r\theta}^{(2)} \theta^2 + \alpha_{r\theta}^{(3)} \theta^3 \]

Further substituting the equation (5.3.8) into equation (5.3.5), we obtain

\[ \left[ C_{ijkl}^\theta + \alpha_{rkl}^\theta C_{ijkl} + \alpha_{rkl}^\theta C_{ijkl} + \alpha_{rkl}^\theta \alpha_{sk}^\theta C_{rjkl} \right] u_{k,l,j} - [e_{kij} + \alpha_{r\theta}^\theta e_{krj}] E_k,j = \rho \ddot{u}_i \]

or

\[ G_{ijkl}^\theta u_{k,l,j} - H_{kl}^\theta E_{k,j} = \rho \ddot{u}_i \]  
(5.3.10)
where

\[
G_{ijkl}^\theta = C_{ijkl}^\theta + \alpha_{rk}^\theta C_{ijkl}^{rl} + \alpha_{ri}^\theta C_{ijkl}^{rl} + \alpha_{ri}^\theta \alpha_{sk}^\theta C_{ijkl}^{rs}
\]

\[
G_{ijkl}^{(1)} = C_{ijkl} + G_{ijkl}^{(1)} \theta + G_{ijkl}^{(2)} \theta^2 + G_{ijkl}^{(3)} \theta^3
\]

\[
G_{ijkl}^{(2)} = D_{ijkl}^{(1)} + \alpha_{rk}^{(1)} C_{ijkl}^{rl} + \alpha_{ri}^{(1)} C_{ijkl}^{rl}
\]

\[
G_{ijkl}^{(3)} = D_{ijkl}^{(2)} + \alpha_{rk}^{(2)} C_{ijkl}^{rl} + \alpha_{rl}^{(2)} C_{ijkl}^{rl} + D_{ijkl}^{(1)} \alpha_{rl}^{(1)} + D_{ijkl}^{(1)} \alpha_{ri}^{(1)}
\]

\[
G_{ijkl}^{(3)} = D_{ijkl}^{(3)} + \alpha_{rk}^{(3)} C_{ijkl}^{rl} + \alpha_{rl}^{(3)} C_{ijkl}^{rl} + D_{ijkl}^{(2)} \alpha_{rl}^{(1)} + D_{ijkl}^{(2)} \alpha_{ri}^{(1)} + D_{ijkl}^{(2)} \alpha_{ri}^{(1)}
\]

\[
H_{kij}^\theta = e_{kij} + \alpha_{rl}^\theta e_{krj}
\]

In the equation (5.3.10), the piezoelectric coupling coefficient, the material properties are assumed to be temperature dependent. By further substituting equation (5.3.9) into equation (5.3.4), we get

\[
(e_{kli} + \alpha_{rl}^\theta e_{krj} + e_{ik} e_{k,i} = 0) \quad (5.3.11)
\]

or

\[
H_{kij}^\theta u_{k,li} + e_{ik} E_{k,i} = 0 \quad (5.3.12)
\]

where

\[
H_{kli}^\theta = e_{kli} + \alpha_{rl}^\theta e_{krj}
\]

\[
= e_{kli} + e_{krl} \alpha_{rl}^{(1)} \theta + e_{ikl} \alpha_{rl}^{(2)} \theta^2 + e_{ikl} \alpha_{rl}^{(3)} \theta^3
\]

\[
= e_{ikl} + H_{ikl}^{(1)} \theta + H_{ikl}^{(2)} \theta^2 + H_{ikl}^{(3)} \theta^3
\]
5.3.1 Thickness Vibration

Similar to the section 5.2.1, we kept the same notation so that $x_i \ (i = 1, 2, 3)$ be the crystallographical axes of the crystals. By conventional IEEE notation, a doubly rotated crystal plate is defined by $(y, x, \omega, l) \phi, \theta$. The crystal plate has thickness of $2b$ as shown in figure 5.3 which $n_i$ denote the unit outward normal to the faces of the plate in the natural state. Refer to the crystallographic axes and the figure 5.3, the unit normal components are same as equation 5.4.1:

$$n_1 = -\cos \theta \sin \phi$$
$$n_2 = \cos \theta \cos \phi$$
$$n_3 = \sin \theta$$

(5.3.12)

For traction free face boundary conditions, the equation 5.3.2 and 5.3.6 becomes

$$p_i = n_j \left[ G_{ijkl} u_{k,l} + H_{kij} \phi_k \right] = 0 \quad \text{at} \quad n_j X_j = \pm b$$

(5.3.13)

$$\phi = V \quad \text{at} \quad n_j X_j = \pm b$$

(5.3.14)

The solution for harmonic and anti-symmetric thickness vibrations are

$$u_k = A_k \sin \xi n_p X_p e^{i\omega t}$$

(5.3.15)

$$\phi = V \sin \xi n_p X_p e^{i\omega t}$$

(5.3.16)

and

$$u_{k,l} = \xi n_l \cos \xi n_p X_p e^{i\omega t} \cdot A_k$$

(5.3.17)

$$u_{k,ij} = -\xi^2 n_l n_j \sin \xi n_p X_p e^{i\omega t} \cdot A_k$$

(5.3.18)

$$\ddot{u}_i = -\omega^2 \sin \xi n_p X_p e^{i\omega t} \cdot A_k$$

(5.3.19)

$$\phi_{,k} = V \xi n_k \cos \xi n_p X_p e^{i\omega t}$$

(5.3.20)

$$\phi_{,kj} = -V \xi^2 n_k n_j \sin \xi n_p X_p e^{i\omega t}$$

(5.3.21)
By further substituting equation 5.3.18 and 5.3.21 into equation 5.3.11, we get

\[ H_{kij}^\theta \left[ -A_i \xi^2 n_j n_k \sin \xi n_p X_p e^{i\omega t} \right] - \varepsilon_{jk} \left[ -V \xi^2 n_j n_k \sin \xi n_p X_p e^{i\omega t} \right] = 0 \]  

(5.3.22)

By simplifying the equation 5.3.22, we obtain electric potential in terms of piezoelectric coupling coefficient, dielectric coefficient and amplitude.

\[ V = \gamma_i^\theta A_i \]  

(5.3.23)

where

\[ V = \frac{n_j n_k H_{kij}^\theta}{n_r n_s \varepsilon_{rs}} A_i \]  

(5.3.24)

\[ \gamma_i^\theta = \frac{n_m n_n H_{nim}^\theta}{n_r n_s \varepsilon_{rs}} \]  

(5.3.25)

The equation 5.3.21 can be written in terms of \( \gamma_i^\theta \) by substituting 5.3.23 into equation 5.3.21.

\[ \phi_{kj} = -\gamma_q^\theta A_q \xi^2 n_k n_j \sin \xi n_p X_p e^{i\omega t} \]  

(5.3.26)

Now, we can substitute equation 5.3.18, 5.3.19 and 5.3.26 into equation 5.3.10 to setup the eigenvalue problem.

\[ \xi^2 Q_{ik} A_k = \rho \omega^2 A_i \]  

(5.3.27)

where

\[ Q_{ik} = G_{ijkl}^\theta n_j n_l + H_{qip}^\theta \gamma_k^\theta n_p n_q \]  

(5.3.28)

\[ H_{qip}^\theta \gamma_k^\theta n_p n_q = \frac{H_{pij}^\theta H_{qkl}^\theta}{\varepsilon_{rs} n_r n_s} n_p n_q n_j n_l \]  

(5.3.29)

Further substituting the equation 5.3.29 into 5.3.28, we obtain piezoelectrically stiffened Christoffel constant \( \tilde{\alpha}_{ijkl}^\theta \).
\[ Q_{lk} = G_{ijkl}^0 \, n_j n_l = Q_{ki} \]  
(5.3.30)

\[ \tilde{G}_{ijkl}^\theta = G_{ijkl}^\theta + \frac{H_{ijkl}^\theta \, n_p \, n_q}{\epsilon_{rs} \, n_r \, n_s} \]  
(5.3.31)

where

\[ \tilde{G}_{ijkl}^\theta = \tilde{G}_{ijkl} + \tilde{G}_{ijkl}^{(1)} \theta + \tilde{G}_{ijkl}^{(2)} \theta^2 + \tilde{G}_{ijkl}^{(3)} \theta^3 \]

\[ \tilde{G}_{ijkl} = C_{ijkl} + \frac{e_{pij} \, e_{qkl} \, n_p \, n_q}{\epsilon_{rs} \, n_r \, n_s} \]

\[ \tilde{G}_{ijkl}^{(1)} = C_{ijkl}^{(1)} + \frac{e_{pij} \, H_{qkl}^{(1)} \, n_p \, n_q}{\epsilon_{rs} \, n_r \, n_s} + \frac{H_{ijkl}^{(1)} \, e_{qkl} \, n_p \, n_q}{\epsilon_{rs} \, n_r \, n_s} \]

\[ \tilde{G}_{ijkl}^{(2)} = C_{ijkl}^{(2)} + \frac{e_{pij} \, H_{qkl}^{(2)} \, n_p \, n_q}{\epsilon_{rs} \, n_r \, n_s} + \frac{H_{ijkl}^{(2)} \, e_{qkl} \, n_p \, n_q}{\epsilon_{rs} \, n_r \, n_s} + \frac{H_{ijkl}^{(1)} \, H_{qkl}^{(1)} \, n_p \, n_q}{\epsilon_{rs} \, n_r \, n_s} \]

\[ \tilde{G}_{ijkl}^{(3)} = C_{ijkl}^{(3)} + \frac{e_{pij} \, H_{qkl}^{(3)} \, n_p \, n_q}{\epsilon_{rs} \, n_r \, n_s} + \frac{H_{ijkl}^{(3)} \, e_{qkl} \, n_p \, n_q}{\epsilon_{rs} \, n_r \, n_s} + \frac{H_{ijkl}^{(1)} \, H_{qkl}^{(1)} \, n_p \, n_q}{\epsilon_{rs} \, n_r \, n_s} + \frac{H_{ijkl}^{(2)} \, H_{qkl}^{(1)} \, n_p \, n_q}{\epsilon_{rs} \, n_r \, n_s} \]

Upon substituting the equations 5.3.31 and 5.3.30 into equation 5.3.27, we find

\[ [\xi^2 \tilde{G}_{ijkl}^\theta \, n_j \, n_l] A_k = \rho \omega^2 A_i \]

or

\[ [Q_{lk} - \lambda \delta_{lk}] A_k = 0 \]  
(5.3.32)

where

\[ \lambda = \rho \, \frac{\omega^2}{\xi}, \quad \xi = \frac{n \pi}{2 b}, \quad n = 1, 3, 5, \ldots \]  
(5.3.33)

The equation 5.3.32 is the eigenvalue problem including the temperature effect and piezoelectric effect on crystal. The traction free boundary condition can be further simplify by substituting equation 5.3.17 and 5.3.20 into equation 5.3.13.
\[ p_i = n_j \left[ A_k G_{ijkl}^\theta \xi n_l \cos \xi n_p X_p e^{i\omega t} + H_{kij}^\theta V \xi n_k \cos \xi n_p X_p e^{i\omega t} \right] \]
\[ = n_j \left[ A_k G_{ijkl}^\theta \xi n_l + H_{kij}^\theta V \xi n_k \right] \cos \xi n_p X_p e^{i\omega t} = 0 \text{ at } n_j X_n = \pm b \] (5.3.34)

We let
\[ t_i = \left[ G_{ijkl}^\theta n_j n_l A_k + H_{kij}^\theta n_k V \right] \] (5.3.35)
and then,
\[ p_i = \xi t_i \cos \xi n_p X_p e^{i\omega t} = 0 \text{ at } n_j X_j = \pm b \] (5.3.36)

Since \( e^{i\omega t} \), \( \xi \) and \( t_i \) cannot equal to zero, \( \cos \xi b \) has to be zero. Therefore,
\[ \xi = \frac{\pi}{2b}, \frac{3\pi}{2b}, \frac{5\pi}{2b}, ... \]
\[ \lambda = \rho \left( \frac{2b\omega t}{n\pi} \right)^2, \quad n = 1, 2, 3, ... \] (5.3.37)

Now, we can repeat the same procedure as in section 5.2.1 and solve the eigenvalue problem.

From the equation 5.3.32, the amplitude \( A_k \) and eigenvalue \( \lambda \) are assumed to be temperature dependent and utilizing Rayleigh-Schrodinger’s pertubation method, we get
\[ A_k = A_k^{(0)} + A_k^{(1)} \theta + A_k^{(2)} \theta^2 \]
\[ \lambda = \lambda^{(0)} + \lambda^{(1)} \theta + \lambda^{(2)} \theta^2 \] (5.3.38)

By substituting the equation 5.3.28 into equation 5.3.32, the eigenvalue problems of powers of \( \theta \) can be obtained and they are
\[ \theta^0 : \left[ Q_k^{(0)} - \lambda^{(0)} \delta_{lk} \right] A_k^{(0)} = 0 \]
\[ \theta^1 : \left[ Q_k^{(0)} - \lambda^{(0)} \delta_{lk} \right] A_k^{(1)} + \left[ Q_k^{(1)} - \lambda^{(1)} \delta_{lk} \right] A_k^{(0)} = 0 \]
\[ \theta^2 : \left[ Q_k^{(0)} - \lambda^{(0)} \delta_{ik} \right] A_k^{(2)} + \left[ Q_k^{(1)} - \lambda^{(1)} \delta_{ik} \right] A_k^{(2)} + \left[ Q_k^{(2)} - \lambda^{(2)} \delta_{ik} \right] A_k^{(0)} = 0 \]

\[ \theta^3 : \left[ Q_k^{(0)} - \lambda^{(0)} \delta_{ik} \right] A_k^{(3)} + \left[ Q_k^{(1)} - \lambda^{(1)} \delta_{ik} \right] A_k^{(2)} + \left[ Q_k^{(2)} - \lambda^{(2)} \delta_{ik} \right] A_k^{(1)} + \left[ Q_k^{(3)} - \lambda^{(3)} \delta_{ik} \right] A_k^{(0)} = 0 \]

(5.3.39)

where

\[ Q_{ik}^{(0)} = g_{ijkl} n_j n_l \]
\[ Q_{ik}^{(1)} = g_{ijkl} n_j n_l \]
\[ Q_{ik}^{(2)} = g_{ijkl} n_j n_l \]
\[ Q_{ik}^{(3)} = g_{ijkl} n_j n_l \]  

(5.3.40)

Similar to the previous section 5.2.1, the each eigenvalues \( \lambda^{(n)} \) are uniqlely calculated.

The \( \lambda^{(0)} \) and \( A_k^{(0)} \) can be easily calculate by solving the first eigenvalue problem of equation 5.3.39 and satifying the equation 5.3.41.

\[ A_k^{(0)} A_k^{(0)} = 1 \]  

(5.3.41)

By multiplying \( A_i^{(0)} \) with the second equation of 5.3.39 and utilizing equation 5.3.41 and first euquation of 5.3.39, we can get following relation:

\[ \lambda^{(1)} = A_i^{(0)} Q_{ik}^{(1)} A_k^{(0)} \]  

(5.3.42)

\( A_k^{(1)} \) can be solve uniquely by requiring \( A_k^{(1)} \) orthogonal to \( A_k^{(0)} \), since \( \left[ Q_k^{(1)} - \lambda^{(1)} \delta_{ik} \right] A_k^{(0)} \) term in equation 5.3.39 is orthogonal to the eigenvector of \( \left[ Q_k^{(0)} - \lambda^{(0)} \delta_{ik} \right] \).

\[ A_k^{(0)} A_k^{(1)} = 0 \]  

(5.3.43)
We further multiply the $A_i^{(0)}$ with the third equation of 5.3.39 and utilizing equation 5.3.41 and 5.3.43, we obtain
\[
\lambda^{(2)} = A_i^{(0)} Q_{lk}^{(2)} A_k^{(0)} + A_i^{(0)} Q_{lk}^{(1)} A_k^{(1)} \tag{5.3.44}
\]
Finally, $\lambda^{(3)}$ can be calculated by repeating the same procedure, which is multiplying $A_i^{(0)}$ with the forth equation of 5.3.39, and employing the constraint, equation 5.3.45.
\[
A_k^{(2)} A_k^{(1)} = -\frac{1}{2} A_k^{(1)} A_k^{(1)} \tag{5.3.45}
\]
\[
\lambda^{(3)} = A_i^{(0)} Q_{lk}^{(3)} A_k^{(0)} + A_i^{(0)} \left( Q_k^{(1)} - \lambda^{(1)} \delta_{lk} \right) A_k^{(2)} + A_i^{(0)} Q_{lk}^{(2)} A_k^{(1)} \tag{5.3.46}
\]
These above formulations are the normal procedures to compute the $\lambda^{(n)}$ and $A_k^{(n)}$ when $Q_k^{(n)}$ are known. In the next chapter, we are going to compute the temperature coefficient of frequency using the calculated quantity, such as $\lambda^{(n)}$ and $A_k^{(n)}$, from this section.

### 5.3.2 Temperature coefficient of frequency

In this section, we will utilize the same method, which previously used in section 5.2.1 by Yong, Lee and Tiersten, to find the relation between Bechmann’ temperature coefficient of frequency $Tf^n$ and the eigen value $\lambda^{(n)}$.
\[
\lambda^{(1)} = 2 \lambda^{(0)} Tf^{(1)} \tag{5.3.47}
\]
\[
\lambda^{(2)} = 2 \lambda^{(0)} Tf^{(2)} + \lambda^{(0)} [Tf^{(1)}]^2 \tag{5.3.48}
\]
\[
\lambda^{(3)} = 2 \lambda^{(0)} Tf^{(3)} + 2 \lambda^{(0)} Tf^{(1)} Tf^{(2)} \tag{5.3.49}
\]
where
\[
Tf^{(n)} = \frac{1}{n! f_0} \frac{\partial^n f}{\partial T^n} |_{T_0}
\]
The above equations can be rearrange so that \( T_f^{(n)} \) are a function of \( \lambda^{(n)} \) and they are

For the first mode:

\[
T_f^{(1)} = \frac{\lambda^{(1)}}{2\lambda^{(0)}}
\]

(5.3.50)

For the second mode:

\[
T_f^{(2)} = \frac{1}{2} \frac{\lambda^{(2)} - \lambda^{(0)}(T_f^{(1)})^2}{\lambda^{(0)}} = \frac{4\lambda^{(2)} - \lambda^{(1)}\lambda^{(1)}}{8\lambda^{(0)}\lambda^{(0)}}
\]

(5.3.51)

For the third mode:

\[
T_f^{(3)} = \frac{1}{2} \frac{\lambda^{(3)} - 2\lambda^{(0)}T_f^{(1)}T_f^{(2)}}{\lambda^{(0)}} = \frac{\lambda^{(3)}}{\lambda^{(0)}} + \frac{1}{4\lambda^{(0)}}\lambda^{(1)}\lambda^{(1)} + \frac{1}{16\lambda^{(0)}\lambda^{(0)}}\lambda^{(1)}\lambda^{(1)}
\]

(5.3.52)

### 5.4 Temperature Compensated Cuts of Langasite

In this chapter, the temperature coefficients of frequency \( T_f^{(n)} \) are computed at the entire region of the langasite crystal. From the computational results, the temperature compensated cuts of all three thickness vibration modes, the thickness-extensional or A-mode, fast thickness- shear modes or B-mode and slow thickness- shear or C-mode are identified by loci plots. The calculated lambda, \( \lambda \), values from equation 5.3.40 are directly related to each mode of vibration, such that lambda and resonance frequencies follow below relations,

\[
f_A > f_B > f_C
\]

(5.4.1)

\[
\lambda_1^0 > \lambda_2^0 > \lambda_3^0
\]

(5.4.2)

where

\[
\lambda_1^0 = f_A \quad & \quad \lambda_2^0 = f_B \quad & \quad \lambda_3^0 = f_C
\]

The temperature compensated cut of \( T_f^3 \) cannot be calculated, since the third-order temperature derivative of the langasite has not been reported. [37] Therefore, the analysis of the temperature compensated cut of \( T_f^1 \) and \( T_f^2 \) are only included in this works. We implemented the equations from the previous section to calculate the \( T_f^1 \) and \( T_f^2 \) of each
vibration mode. The complete sets of the material properties of the langasite obtain by Bourquin [38] and Molocha [39] are listed in Appendix C.

The temperature coefficients of frequency are calculate with the data obtained by Bourquin. The calculated values of each temperature coefficient of frequency are shown in figure 5.4.1, 5.4.2, 5.4.3, 5.4.4, 5.4.5 and 5.4.6. The figure 5.4.1 and 5.4.2 represent the first- and second-order temperature coefficient of frequency for A-mode. The figure 5.4.3 and 5.4.4 represent the first- and second-order temperature coefficient of frequency for B-mode. The figure 5.4.5 and 5.4.6 show the first- and second-order temperature coefficient of frequency for C-mode. There are no zero-valued temperature coefficients for A- and B-mode are found based on our calculation. Figure 5.4.7 shows the loci of C-mode temperature compensated orientations for langasite. A more than sufficient number of the zero-temperature compensated orientations of first-order temperature coefficient of frequency of langasite for C-mode have been identified. Figure 5.4.8 and 5.4.9 show a typical examples of the behavior of first- and second-order frequency-temperature coefficient of langasite for C-mode when $\theta = 10^\circ, 20^\circ$ and $30^\circ$. The blue, red and green curves are represent the temperature coefficient of frequencies when $\theta = 10^\circ, 20^\circ$ and $30^\circ$, respectively. No zero values of second-order temperature of frequency of langasite are found.
Figure 5.4.1 First-order temperature coefficient of frequency of langasite for A-mode
Figure 5.4.2 Second-order temperature coefficient of frequency of langasite for A-mode
Figure 5.4.3 First-order temperature coefficient of frequency of langasite for B-mode
Figure 5.4.4 Second-order temperature coefficient of frequency of langasite for B-mode
Figure 5.4.5 First-order temperature coefficient of frequency of langasite for C-mode
Figure 5.4.6 Second-order temperature coefficient of frequency of langasite for C-mode
Figure 5.4.7 Temperature compensated loci of langasite for C-mode.
Figure 5.4.8 First-order of temperature coefficient of frequency of the C-mode for langasite when $\theta = 10^\circ$, $20^\circ$ and $30^\circ$. 
Figure 5.4.9 Second-order of temperature coefficient of frequency of the C-mode for langasite when $\theta = 10^{\circ}, 20^{\circ}$ and $30^{\circ}$
In addition, the temperature coefficients of frequency are calculated with the data obtained by Molacha [39]. The calculated values of each temperature coefficient of frequency are shown in figure 5.4.10, 5.4.11, 5.4.12, 5.4.13, 5.4.14 and 5.4.15. The figure 5.4.10 and 5.4.11 represent the first- and the second-order temperature coefficients of frequency for A-mode. The figure 5.4.12 and 5.4.13 represent the first- and second-order temperature coefficient of frequency for B-mode. The figure 5.4.14 and 5.4.15 show the first- and second-order temperature coefficient of frequency for C-mode. There are no zero-valued temperature coefficients for A- and B-mode are found based on the calculation. Figure 5.4.16 shows the first-order temperature compensated loci of the langasite for C-mode. The red crosses and the green diamonds represent the calculated temperature coefficients of frequency with the data obtained from ref. [38] and [39], respectively. The temperature compensated cuts are closely matched except when \( \theta \approx \pm 70^\circ \) and \( \phi \approx \pm 40^\circ \) to 60\(^\circ\). Figure 5.4.17 shows the first- and the second-order temperature compensated loci of the langasite for C mode. The green circles and red crosses represent the first-order temperature coefficient of frequency, which computed with the data obtain by Bourquin [38] and Malocha [39], and the blue diamonds represent the second-order temperature coefficient of frequency with the data obtain from ref. [40]. A more than sufficient number of the temperature compensated orientations up to the second-order of langasite for C-mode have been identified. Our computation shows that no zero values of the second-order temperature coefficients of frequency are identify with the data obtain from ref. [38]. The points, where the red crosses and the blue diamonds intersect, are the crystal orientations, which the first- and the second-order temperature coefficient of frequencies equal to zero. These intersect points provide the ideal langasite cuts for a
resonator which operates at extremely high temperature. Figure 5.4.18 and 5.4.19 show the behavior of the first- and second-order temperature coefficient of langasite for C-mode when $\phi$ varies from $10^\circ$ to $30^\circ$ at increment of $10^\circ$ and $\theta$ varies from $-120^\circ$ to $120^\circ$. As calculated results illustrate in fig.5.4.18 and 5.4.19, the number of zero-valued temperature coefficient of frequency are identified.

Figure 5.4.20 shows the examples of the frequency-temperature behavior of the langasite. The $\theta$ is fixed at $-35^\circ$, while $\phi$ varies from $30.36^\circ$ to $30.40^\circ$ at increment of $0.02^\circ$. The results show that the langasite has excellent frequency-temperature stability.
Figure 5.4.10 The first-order temperature coefficient of frequency of langasite for A-mode
Figure 5.4.11 The second-order temperature coefficient of frequency of langasite for A-mode
Figure 5.4.12 The first-order temperature coefficient of frequency of langasite for B-mode
Figure 5.4.13 The second-order temperature coefficient of frequency of langasite for B-mode
Figure 5.4.14 The first-order temperature coefficient of frequency of langasite for C-mode
Figure 5.4.15 The first-order temperature coefficient of frequency of langasite for C-mode
Figure 5.4.16 Loci of Tf1=0 for the thickness mode C of langasite
Figure 5.4.17 Loci of $Tf_1=0$ and $Tf_2=0$ for the thickness mode C of langasite

Figure 5.4.18 Calculated first-order temperature coefficient of frequency of langasite when $\phi=10, 20, 30$
Figure 5.4.19 Calculated second-order temperature coefficient of frequency of langasite when $\phi=10, 20, 30$
Figure 5.4.20 Frequency – temperature curve when $\theta = 33.5$
5.5 The Material Properties at High Temperature

It is well known that the resonant frequency of the piezoelectric resonators or devices not only depends on its crystalline orientations, material anisotropy and mounting support, but also on temperature. The resonant frequency shifting due to the changes of the temperature can be compensated or corrected with a known temperature dependence of the resonant frequencies or by measuring the temperature of the piezoelectric crystal by some independent means. The temperature coefficients are usually measured at the room temperature, 25°C and those material properties values that associate with the reference temperature are only valid under a relatively small temperature range around the room temperature. For example, the oil and gas industry requires reliable and stable sensors to operate at high temperatures because the temperature is a function of depth. The langasite sensor has the potential to allow the oil/gas companies to drill deeper as they seek to find sources of production.

In this section, the Lagrangian equations, which previously derived in Chapter 5.2 and 5.3, are used to predict the material properties at the elevated temperature. Only the effective elastic constants are temperature dependent and requires the computation of the material constants. The six-independent second-order linear elastic constants and the thermal expansion coefficients are analytically calculated with the data obtain from [41] by shifting the reference temperature from 25°C to 600°C. The newly calculated materials constants are used to compute the temperature coefficient of frequency. The temperature compensated orientations of the langasite for C-mode at reference temperature are compared with the temperature compensated orientations that are calculated in a wide temperature range.
By assuming the relationship between the temperature and the elastic constant of langasite at any reference temperature is the same, each second-order linear elastic constant is computed with the first equation of Eq. 5.2.6. The Taylor series expansion is used to predict the second-order linear elastic constants at high temperatures. The changes of each elastic constant of the langasite as a function of temperature are shown in fig. 5.5.1 to 5.59. Each second-order elastic constant of langasite and thermal expansion coefficients are computed from -50°C to 500°C at increments of 25°C. Base on the calculation of the stiffness of langasite, $C_{11}$, $C_{12}$, $C_{13}$, $C_{14}$, $C_{33}$ and $C_{44}$ decrease when temperature increases. $C_{66}$ is the only second-order linear elastic constant that increases when temperature increases.

Similarly, the thermal expansion coefficients at a wide temperature range, from -50°C to 600°C, are computed with Eq. 5.1.14. Figure 5.5.8 and 5.5.9 show the changes of the thermal expansion coefficient as a function of temperature. Both thermal expansion coefficients increase when the temperature increases. In addition, the rate of change in the thermal expansion coefficient increases with the temperature.

By shifting the reference temperature using the regression analysis technique, we obtained new set of second-order elastic constants: first-and second-order temperature derivative of the elastic constants and thermal expansion coefficients at temperatures from 25°C to 600°C. The calculated second order elastic constants are compared with the data obtained by Weihnacht [42], as shown in Fig. 5.5.10. The comparison between the newly calculated sets of second-order elastic constants agrees well with experimental data, except elastic constant $C_{33}$. Figure 5.5.11 and 5.5.12 show the first order temperature derivative of second-order elastic constant and thermal expansion
coefficients of langasite as a function of reference temperature. The second temperature
derivative of second order elastic constants and the second order thermal expansion
coefficients are temperature independent due to the nature of the curve fitting process.
Once the third-order temperature coefficients are determined, the second-order
temperature derivative and second-order thermal expansion coefficients of langasite can
be calculated as a function of temperature.
The temperature coefficient of frequency of the langasite plates is calculated with the
present sets of elastic constants and the thermal expansion coefficients at high
temperatures. The temperature compensated orientations of langasite are investigated in
the entire region of the crystalline. The fast- and slow-thickness shear modes are not
included in this study, as zero-valued temperature coefficient of frequency of those two
modes are not found. Figure 5.5.13 shows a loci of C-mode temperature compensated
orientations for langasite at various temperatures from 25°C to 315°C. The results shows
that when the reference temperature increases, the zero-valued temperature coefficient of
frequency curves get smaller until it reaches the temperature of around 315°C. At any
temperature above 350 °C, the temperature compensated orientations do not exist. Figure
5.5.14, 5.5.15 and 5.5.16 show the typical examples of the behavior of the temperature
coefficient of frequency for the C-mode of langasite when \( \phi=30^\circ, 45^\circ \) and 60\(^\circ \),
respectively. The number of points crossing the zero-value of temperature coefficient of
frequency decreases as the reference temperature increases as shown in fig. 5.5.14, 5.5.15
and 5.5.16.
Figure 5.5.1 The second-order linear elastic constant $C_{11}$ as a function of temperature

Figure 5.5.2 The second-order linear elastic constant $C_{12}$ as a function of temperature
Figure 5.5.3 The second-order linear elastic constant $C_{13}$ as a function of temperature

Figure 5.5.4 The second-order linear elastic constant $C_{14}$ as a function of temperature
Figure 5.5.5 The second-order linear elastic constant $C_{66}$ as a function of temperature

Figure 5.5.6 The second-order linear elastic constant $C_{44}$ as a function of temperature
Figure 5.5.7 The second-order linear elastic constant $C_{33}$ as a function of temperature
Figure 5.5.8 Thermal expansion coefficient $\varepsilon_{11}$ as a function of temperature

Figure 5.5.9 Thermal expansion coefficient $\varepsilon_{33}$ as a function of temperature
Figure 5.5.10 Temperature dependence of the second-order elastic constants of langasite
Figure 5.5.11 Temperature dependence of first temperature derivative of the second-order linear elastic constant of langasite
Figure 5.5.12 Temperature dependence of thermal expansion coefficients

**Graph Details:**
- **Y-axis:** Thermal expansion coefficient (1e-6)
- **X-axis:** Reference temperature

**Graph Lines:**
- Blue line: $\alpha_{11}(1)$
- Red line: $\alpha_{33}(1)$
Figure 5.5.13 Loci of C-mode temperature compensated orientation for langasite
Figure 5.5.14 Temperature dependence of the temperature coefficient of frequency of the C-mode for langasite when $\phi=30^\circ$. 
Figure 5.5.15 Temperature dependence of the temperature coefficient of frequency of the C-mode for langasite when $\phi=45^\circ$
Figure 5.5.16 Temperature dependence of the temperature coefficient of frequency of the C-mode for langasite when $\phi=60^\circ$
5.6 Summary

The frequency-temperature behavior of the langasite crystal is studied with the three-dimensional equations of incremental motion superposed on homogeneous thermal strain including piezoelectric effect. Normally, the piezoelectric effect is neglected in analysis of quartz crystal due to their very small piezoelectric coupling. The piezoelectric coupling factor of the langasite crystal is much larger than its quartz crystal and cannot be ignored. However, the analysis shows that piezoelectric effect on the frequency-temperature behavior of langasite is negligible. The temperature coefficient of frequency of langasite of all three modes of vibration, A-, B- and C-mode, is calculated. The results show that no temperature compensated orientation of the A- and B-modes exist. For C-mode, the temperature compensated cuts are found at several locations.

Only the effective elastic constants are a function of the temperature in the equations for small vibrations superposed on thermally-induced deformations by steady and uniform temperature changes. The effective elastic constants consist of the second-order elastic constants, first-and effective second-order temperature derivative of second-order elastic constants, third-order elastic constant, and the first-and second-order thermal expansion coefficient, in addition to piezoelectric and dielectric coefficients. These values are measured at the room temperature, but they can change when measured at different temperatures. The second order elastic constants and the thermal expansion coefficients are calculated with regression analysis at high temperatures. The results show that this theoretical method can be used to predict the elastic constants and the thermal expansion coefficient at high temperatures as newly computed elastic constants agree very well with the experimental data, except $C_{33}$. This theoretical method to predict the
temperature dependency of the material properties of langasite can be more accurately implemented once higher-order temperature coefficients are known. The loci shows that the temperature compensated orientations change as temperature increases. No temperature compensated orientations can be found at temperatures above 350°C. The loci can be used for design of sensors or devices that operates at high temperatures.
**Chapter 6. Future works**

In this dissertation, we studied double-ended tuning fork gyroscope and introduced the newly designed gyroscope called, length-extension gyroscope. Using quartz, langasite and langatate crystals, we simulated each gyroscope and compared the geometric and gyroscopic characteristics. Furthermore, frequency-temperature behavior of the langasite has been investigated at the reference temperature, as well as elevated temperatures.

In the future study, the third-order temperature coefficient of frequency of langasite can be calculated once the full set of the third-order temperature coefficient are measured. This will provide more accurate results. The temperature coefficient of frequency of langatate can be investigated when the third-order nonlinear elastic constants are reported.
Bibliography


[17] J. Yang, "Some analytical results on piezoelectric gyroscopes," in Frequency


Appendix A.

A.1 Matrix Notation

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<th>Einstein tensor notation</th>
<th>Voigt notation</th>
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<tr>
<td>33 (zz)</td>
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<td>12 or 21 (xy or yx)</td>
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The Voigt notation is an alternative way of representing and simplifying higher-order tensor. The notation is often used in continuum mechanics.
Appendix B.

B.1 Alternate forms of Constitution equations

The standard constitutive equations can be written in four different forms.

Stress-Charge form

\[ T_{ij} = c^E_{ijkl} s_{kl} - e_{kij} E_k \]
\[ D_i = e_{ikl} S_{kl} + \varepsilon^S_{ik} E_k \]

Stress-Voltage form

\[ T_{ij} = c^D_{ijkl} s_{kl} - h_{kij} D_k \]
\[ E_i = -h_{ikl} s_{kl} + \beta^S_{ik} D_k \]

Strain-Charge form

\[ S_{ij} = s^E_{ijkl} T_{kl} + d_{kij} E_k \]
\[ D_i = d_{ikl} T_{kl} + \varepsilon^T_{ik} E_k \]

Strain–Voltage form

\[ S_{ij} = s^D_{ijkl} T_{kl} + g_{kij} D_k \]
\[ D_i = -g_{ikl} T_{kl} + \beta^T_{ik} D_k \]

The relations between coefficients appearing in the constitutive equations are

\[ c^E_{pr} s^E_{qr} = \delta_{pg} \quad c^D_{pr} s^D_{qr} = \delta_{pg} \]
\[ \beta^S_{ik} \varepsilon^S_{jk} = \delta_{ij} \quad \beta^T_{ik} \varepsilon^T_{jk} = \delta_{ij} \]
\[ c^E_{pq} = c^E_{pq} + e_{kp} h_{kq} \quad s^D_{pq} = s^E_{pq} - d_{kp} g_{kq} \]
\[ \varepsilon^T_{ij} = \varepsilon^S_{ij} + D_{lq} e_{jq} \quad \beta^T_{ij} = \beta^S_{ij} - g_{lq} h_{jq} \]
\begin{align*}
e_{ip} &= d_{iq}c_{qp}^E \\
g_{ip} &= \beta_{ik}^Td_{kp} \\
d_{ip} &= \epsilon_{ik}g_{kp} \\
h_{ip} &= g_{iq}c_{qp}^{D}
\end{align*}

where \(i,j,k = 1,2,3\) and \(p,q,r = 1,2,3,4,5,6\)
Appendix C.  Properties of Materials

C.1 Quartz

The material properties of quartz is determined by Yong [34] [33], Thurston [43], Bechmann [44] and Lamb and Richter [45].

The mass density of quartz is 2649 kg/m$^3$

Second-order elastic constant: (10$^{10}$ N/m$^2$)

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First-order temperature derivative of elastic constants: (10$^6$ N/m$^2$/°C)

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Effective second-order temperature derivative of elastic constants: (10$^3$ N/m$^2$/(°C)$^2$)
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Effective third-order temperature derivative of elastic constants: \((\text{N/m}^2/°\text{C})^3\)

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Third-order thermal expansion coefficients: \((10^{12} \text{ 1/C}^3)\)

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Dielectric constant: \((10^{12} \text{ 1/Coulooms/(Volts}^\text{g} \cdot \text{m})}\)

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C.2 Langasite

The material constants for langasite have been determined by Molocha [39], Sergeev [46], Bourquin [41] and Sorokin [47].

Mass density of langasite: \(\rho = 5743 \text{ kg/m}^3\)

Second-order linear elastic constant: \((10^{10} \text{ N/m}^2)[39]\)

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Second-order linear elastic constant: \((10^{10} \text{ N/m}^2)\) [41]

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Second-order thermal expansion coefficients: \((10^9 \text{ } 1/\text{C}^2)\)

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Dielectric constant: \((10^{12} \text{ Coulombs}/(\text{Volts}\cdot\text{m}))\)

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Piezoelectric constants: \((10^{-3} \text{ Coulombs}/\text{m}^2)\)

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C.3 Langata
e

The material constants for langata have been determined by Molocha [39], Sergeev [46], Bourquin [41] and Zhang [48].

Mass density of langata: \(\rho = 6150 \text{ kg/m}^3\)

Second-order elastic constant: \((10^{10} \text{ N/m}^2)\)

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Viscosity constant: \( (10^{-3} \text{ N-sec/m}^2) \)

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First-order thermal expansion coefficients: \( (10^{-6} \text{ 1/C}) \)

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Second-order thermal expansion coefficients: \( (10^{-9} \text{ 1/C}^2) \)

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Dielectric constant: \( (10^{-12} \text{ Coulombs/(Volts* m)}) \)

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Piezoelectric constants: \( (10^{-3} \text{ Coulombs/m}^2) \)

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C.4 Gold

The material constants for langatate have been determined by Radwan [49].

**Mass density of gold**: $\rho = 19,300 \text{ kg/m}^3$

Elastic constants (Gpa)

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Appendix D. Comsol Parameters and Variable

D.1 Variable

\( \text{Theta} = T - T_{\text{ref}} \) Temperature in Celcius
\( \theta_2 = \theta^2 \)
\( \theta_3 = \theta^3 \)
\( \alpha_{11} = \alpha_{11} + \alpha_{f11} \theta + \alpha_{c11} \theta^2 \) 1/K Thermal expansion coefficient
\( \alpha_{12} = \alpha_{12} + \alpha_{f12} \theta + \alpha_{c12} \theta^2 \) 1/K Thermal expansion coefficient
\( \alpha_{13} = \alpha_{13} + \alpha_{f13} \theta + \alpha_{c13} \theta^2 \) 1/K Thermal expansion coefficient
\( \alpha_{22} = \alpha_{22} + \alpha_{f22} \theta + \alpha_{c22} \theta^2 \) 1/K Thermal expansion coefficient
\( \alpha_{23} = \alpha_{23} + \alpha_{f23} \theta + \alpha_{c23} \theta^2 \) 1/K Thermal expansion coefficient
\( \alpha_{33} = \alpha_{33} + \alpha_{f33} \theta + \alpha_{c33} \theta^2 \) 1/K Thermal expansion coefficient

\( C_{T11} = N_{111} \text{solid.eX} + N_{112} \text{solid.eY} + N_{113} \text{solid.eZ} + 2(N_{114} \text{solid.eYZ} + N_{115} \text{solid.eXZ} + N_{116} \text{solid.eXY}) \)
\( C_{T12} = N_{121} \text{solid.eX} + N_{122} \text{solid.eY} + N_{123} \text{solid.eZ} + 2(N_{124} \text{solid.eYZ} + N_{125} \text{solid.eXZ} + N_{126} \text{solid.eXY}) \)
\( C_{T13} = N_{131} \text{solid.eX} + N_{132} \text{solid.eY} + N_{133} \text{solid.eZ} + 2(N_{134} \text{solid.eYZ} + N_{135} \text{solid.eXZ} + N_{136} \text{solid.eXY}) \)
\( C_{T14} = N_{141} \text{solid.eX} + N_{142} \text{solid.eY} + N_{143} \text{solid.eZ} + 2(N_{144} \text{solid.eYZ} + N_{145} \text{solid.eXZ} + N_{146} \text{solid.eXY}) \)
\( C_{T15} = N_{151} \text{solid.eX} + N_{152} \text{solid.eY} + N_{153} \text{solid.eZ} + 2(N_{154} \text{solid.eYZ} + N_{155} \text{solid.eXZ} + N_{156} \text{solid.eXY}) \)
\( C_{T16} = N_{161} \text{solid.eX} + N_{162} \text{solid.eY} + N_{163} \text{solid.eZ} + 2(N_{164} \text{solid.eYZ} + N_{165} \text{solid.eXZ} + N_{166} \text{solid.eXY}) \)
\( C_{T22} = N_{221} \text{solid.eX} + N_{222} \text{solid.eY} + N_{223} \text{solid.eZ} + 2(N_{224} \text{solid.eYZ} + N_{225} \text{solid.eXZ} + N_{226} \text{solid.eXY}) \)
\( C_{T23} = N_{231} \text{solid.eX} + N_{232} \text{solid.eY} + N_{233} \text{solid.eZ} + 2(N_{234} \text{solid.eYZ} + N_{235} \text{solid.eXZ} + N_{236} \text{solid.eXY}) \)
CT24
= N241*solid.eX+N242*solid.eY+N243*solid.eZ+2*(N244*solid.eYZ+N245*solid.eXZ+N246*solid.eXY)

CT25
= N251*solid.eX+N252*solid.eY+N253*solid.eZ+2*(N254*solid.eYZ+N255*solid.eXZ+N256*solid.eXY)

CT26
= N261*solid.eX+N262*solid.eY+N263*solid.eZ+2*(N264*solid.eYZ+N265*solid.eXZ+N266*solid.eXY)

CT33
= N331*solid.eX+N332*solid.eY+N333*solid.eZ+2*(N334*solid.eYZ+N335*solid.eXZ+N336*solid.eXY)

CT34
= N341*solid.eX+N342*solid.eY+N343*solid.eZ+2*(N344*solid.eYZ+N345*solid.eXZ+N346*solid.eXY)

CT35
= N351*solid.eX+N352*solid.eY+N353*solid.eZ+2*(N354*solid.eYZ+N355*solid.eXZ+N356*solid.eXY)

CT36
= N361*solid.eX+N362*solid.eY+N363*solid.eZ+2*(N364*solid.eYZ+N365*solid.eXZ+N366*solid.eXY)

CT44
= N441*solid.eX+N442*solid.eY+N443*solid.eZ+2*(N444*solid.eYZ+N445*solid.eXZ+N446*solid.eXY)

CT45
= N451*solid.eX+N452*solid.eY+N453*solid.eZ+2*(N454*solid.eYZ+N455*solid.eXZ+N456*solid.eXY)

CT46
= N461*solid.eX+N462*solid.eY+N463*solid.eZ+2*(N464*solid.eYZ+N465*solid.eXZ+N466*solid.eXY)

CT55
= N551*solid.eX+N552*solid.eY+N553*solid.eZ+2*(N554*solid.eYZ+N555*solid.eXZ+N556*solid.eXY)

CT56
= N561*solid.eX+N562*solid.eY+N563*solid.eZ+2*(N564*solid.eYZ+N565*solid.eXZ+N566*solid.eXY)

CT66
= N661*solid.eX+N662*solid.eY+N663*solid.eZ+2*(N664*solid.eYZ+N665*solid.eXZ+N666*solid.eXY)

C11 = C011+Ca11*theta+0.5*Cb11*theta2+0.1666666666*Cc11*theta3+CT11 Pa Nonlinear elastic constants with damping
\begin{align*}
C_{12} &= C_{012} + C_{12}^a \theta + 0.5 C_{b12} \theta^2 + 0.16666666666666666 C_{c12} \theta^3 + C_{12}^T \\
C_{13} &= C_{013} + C_{13}^a \theta + 0.5 C_{b13} \theta^2 + 0.16666666666666666 C_{c13} \theta^3 + C_{13}^T \\
C_{14} &= C_{014} + C_{14}^a \theta + 0.5 C_{b14} \theta^2 + 0.16666666666666666 C_{c14} \theta^3 + C_{14}^T \\
C_{15} &= C_{015} + C_{15}^a \theta + 0.5 C_{b15} \theta^2 + 0.16666666666666666 C_{c15} \theta^3 + C_{15}^T \\
C_{16} &= C_{016} + C_{16}^a \theta + 0.5 C_{b16} \theta^2 + 0.16666666666666666 C_{c16} \theta^3 + C_{16}^T \\
C_{22} &= C_{022} + C_{22}^a \theta + 0.5 C_{b22} \theta^2 + 0.16666666666666666 C_{c22} \theta^3 + C_{22}^T \\
C_{23} &= C_{023} + C_{23}^a \theta + 0.5 C_{b23} \theta^2 + 0.16666666666666666 C_{c23} \theta^3 + C_{23}^T \\
C_{24} &= C_{024} + C_{24}^a \theta + 0.5 C_{b24} \theta^2 + 0.16666666666666666 C_{c24} \theta^3 + C_{24}^T \\
C_{25} &= C_{025} + C_{25}^a \theta + 0.5 C_{b25} \theta^2 + 0.16666666666666666 C_{c25} \theta^3 + C_{25}^T \\
C_{26} &= C_{026} + C_{26}^a \theta + 0.5 C_{b26} \theta^2 + 0.16666666666666666 C_{c26} \theta^3 + C_{26}^T \\
C_{33} &= C_{033} + C_{33}^a \theta + 0.5 C_{b33} \theta^2 + 0.16666666666666666 C_{c33} \theta^3 + C_{33}^T \\
C_{34} &= C_{034} + C_{34}^a \theta + 0.5 C_{b34} \theta^2 + 0.16666666666666666 C_{c34} \theta^3 + C_{34}^T \\
C_{35} &= C_{035} + C_{35}^a \theta + 0.5 C_{b35} \theta^2 + 0.16666666666666666 C_{c35} \theta^3 + C_{35}^T \\
C_{36} &= C_{036} + C_{36}^a \theta + 0.5 C_{b36} \theta^2 + 0.16666666666666666 C_{c36} \theta^3 + C_{36}^T \end{align*}
C44 = \( C_{044} + C_{a44}\theta + 0.5C_{b44}\theta^2 + 0.1666666666C_{c44}\theta^3 + C_{T44} \) Pa Nonlinear elastic constants with damping

C45 = \( C_{045} + C_{a45}\theta + 0.5C_{b45}\theta^2 + 0.1666666666C_{c45}\theta^3 + C_{T45} \) Pa Nonlinear elastic constants with damping

C46 = \( C_{046} + C_{a46}\theta + 0.5C_{b46}\theta^2 + 0.1666666666C_{c46}\theta^3 + C_{T46} \) Pa Nonlinear elastic constants with damping

C55 = \( C_{055} + C_{a55}\theta + 0.5C_{b55}\theta^2 + 0.1666666666C_{c55}\theta^3 + C_{T55} \) Pa Nonlinear elastic constants with damping

C56 = \( C_{056} + C_{a56}\theta + 0.5C_{b56}\theta^2 + 0.1666666666C_{c56}\theta^3 + C_{T56} \) Pa Nonlinear elastic constants with damping

C66 = \( C_{066} + C_{a66}\theta + 0.5C_{b66}\theta^2 + 0.1666666666C_{c66}\theta^3 + C_{T66} \) Pa Nonlinear elastic constants with damping

e11 = e_{011} + e_{a11}\theta + e_{b11}\theta^2 + e_{c11}\theta^3 \text{ C/m}^2 \text{ Piezoelectric stress constant}
e12 = e_{012} + e_{a12}\theta + e_{b12}\theta^2 + e_{c12}\theta^3 \text{ C/m}^2 \text{ Piezoelectric stress constant}
e13 = e_{013} + e_{a13}\theta + e_{b13}\theta^2 + e_{c13}\theta^3 \text{ C/m}^2 \text{ Piezoelectric stress constant}
e14 = e_{014} + e_{a14}\theta + e_{b14}\theta^2 + e_{c14}\theta^3 \text{ C/m}^2 \text{ Piezoelectric stress constant}
e15 = e_{015} + e_{a15}\theta + e_{b15}\theta^2 + e_{c15}\theta^3 \text{ C/m}^2 \text{ Piezoelectric stress constant}
e16 = e_{016} + e_{a16}\theta + e_{b16}\theta^2 + e_{c16}\theta^3 \text{ C/m}^2 \text{ Piezoelectric stress constant}
e21 = e_{021} + e_{a21}\theta + e_{b21}\theta^2 + e_{c21}\theta^3 \text{ C/m}^2 \text{ Piezoelectric stress constant}
e22 = e_{022} + e_{a22}\theta + e_{b22}\theta^2 + e_{c22}\theta^3 \text{ C/m}^2 \text{ Piezoelectric stress constant}
e23 = e_{023} + e_{a23}\theta + e_{b23}\theta^2 + e_{c23}\theta^3 \text{ C/m}^2 \text{ Piezoelectric stress constant}
e24 = e_{024} + e_{a24}\theta + e_{b24}\theta^2 + e_{c24}\theta^3 \text{ C/m}^2 \text{ Piezoelectric stress constant}
e25 = e_{025} + e_{a25}\theta + e_{b25}\theta^2 + e_{c25}\theta^3 \text{ C/m}^2 \text{ Piezoelectric stress constant}
e26 = e_{026} + e_{a26}\theta + e_{b26}\theta^2 + e_{c26}\theta^3 \text{ C/m}^2 \text{ Piezoelectric stress constant}
e31 = e_{031} + e_{a31}\theta + e_{b31}\theta^2 + e_{c31}\theta^3 \text{ C/m}^2 \text{ Piezoelectric stress constant}
e32 = e_{032} + e_{a32}\theta + e_{b32}\theta^2 + e_{c32}\theta^3 \text{ C/m}^2 \text{ Piezoelectric stress constant}
e33 = e_{033} + e_{a33}\theta + e_{b33}\theta^2 + e_{c33}\theta^3 \text{ C/m}^2 \text{ Piezoelectric stress constant}
e34 = e_{034} + e_{a34}\theta + e_{b34}\theta^2 + e_{c34}\theta^3 \text{ C/m}^2 \text{ Piezoelectric stress constant}
\[ e_{35} = e_{035} + ea_{35} \theta + eb_{35} \theta^2 + ec_{35} \theta^3 \text{ C/m}^2 \] Piezoelectric stress constant

\[ e_{36} = e_{036} + ea_{36} \theta + eb_{36} \theta^2 + ec_{36} \theta^3 \text{ C/m}^2 \] Piezoelectric stress constant

\[ \epsilon_{11} = \epsilon_{011} + \epsilon_{a11} \theta + \epsilon_{b11} \theta^2 + \epsilon_{c11} \theta^3 \text{ F/m} \] Dielectric permittivity

\[ \epsilon_{12} = \epsilon_{012} + \epsilon_{a12} \theta + \epsilon_{b12} \theta^2 + \epsilon_{c12} \theta^3 \text{ F/m} \] Dielectric permittivity

\[ \epsilon_{13} = \epsilon_{013} + \epsilon_{a13} \theta + \epsilon_{b13} \theta^2 + \epsilon_{c13} \theta^3 \text{ F/m} \] Dielectric permittivity

\[ \epsilon_{22} = \epsilon_{022} + \epsilon_{a22} \theta + \epsilon_{b22} \theta^2 + \epsilon_{c22} \theta^3 \text{ F/m} \] Dielectric permittivity

\[ \epsilon_{23} = \epsilon_{023} + \epsilon_{a23} \theta + \epsilon_{b23} \theta^2 + \epsilon_{c23} \theta^3 \text{ F/m} \] Dielectric permittivity

\[ \epsilon_{33} = \epsilon_{033} + \epsilon_{a33} \theta + \epsilon_{b33} \theta^2 + \epsilon_{c33} \theta^3 \text{ F/m} \] Dielectric permittivity

**D.2 Parameters**

\[ \text{C}_\text{freq} \] 1e6 Center frequency of device, Hz

\[ \text{C}_\omega = \text{C}_\text{freq} \times 2 \pi \text{ C}_\text{freq} \text{ in rad/s} \]

\[ T \] 25 Temperature in Celcius

\[ \text{Tref} \] 25 Reference temperature in Celcius

\[ \text{gold}_\text{rho} \] 19300

\[ \text{gold}_\text{thickness} \] 0.2e-6

\[ \text{C}_{011} = 1.884900 \times 10^11 + i \times \text{C}_\omega \times 4.43 \times 10^{-04} \text{ Pa} \] Elastic constants with damping

\[ \text{C}_{012} = 1.133960 \times 10^11 + i \times \text{C}_\omega \times 6.185 \times 10^{-05} \text{ Pa} \] Elastic constants with damping

\[ \text{C}_{013} = 8.755398 \times 10^10 + i \times \text{C}_\omega \times 1.185 \times 10^{-04} \text{ Pa} \] Elastic constants with damping

\[ \text{C}_{014} = 6.796592 \times 10^10 + i \times \text{C}_\omega \times 9.128 \times 10^{-05} \text{ Pa} \] Elastic constants with damping

\[ \text{C}_{015} = 0.000000 + i \times \text{C}_\omega \times 0.000000 \text{ Pa} \] Elastic constants with damping

\[ \text{C}_{016} = 0.000000 + i \times \text{C}_\omega \times 0.000000 \text{ Pa} \] Elastic constants with damping

\[ \text{C}_{022} = 1.773070 \times 10^11 + i \times \text{C}_\omega \times 4.859 \times 10^{-04} \text{ Pa} \] Elastic constants with damping

\[ \text{C}_{023} = 1.096556 \times 10^11 + i \times \text{C}_\omega \times 5.594 \times 10^{-06} \text{ Pa} \] Elastic constants with damping

\[ \text{C}_{024} = 4.435348 \times 10^10 + i \times \text{C}_\omega \times 4.699 \times 10^{-05} \text{ Pa} \] Elastic constants with damping

\[ \text{C}_{025} = 0.000000 + i \times \text{C}_\omega \times 0.000000 \text{ Pa} \] Elastic constants with damping

\[ \text{C}_{026} = 0.000000 + i \times \text{C}_\omega \times 0.000000 \text{ Pa} \] Elastic constants with damping

\[ \text{C}_{033} = 2.473117 \times 10^11 + i \times \text{C}_\omega \times 4.377 \times 10^{-04} \text{ Pa} \] Elastic constants with damping

\[ \text{C}_{034} = 1.329184 \times 10^10 + i \times \text{C}_\omega \times 3.017 \times 10^{-05} \text{ Pa} \] Elastic constants with damping
C035 0.000000e+00+i*C_omega*0.000000e+00 Pa Elastic constants with damping
C036 0.000000e+00+i*C_omega*0.000000e+00 Pa Elastic constants with damping
C044 6.648565e+10+i*C_omega*1.866576e-04 Pa Elastic constants with damping
C045 0.000000e+00+i*C_omega*0.000000e+00 Pa Elastic constants with damping
C046 0.000000e+00+i*C_omega*0.000000e+00 Pa Elastic constants with damping
C055 4.129200e+10+i*C_omega*2.191898e-04 Pa Elastic constants with damping
C056 1.374128e+10+i*C_omega*-5.208517e-05 Pa Elastic constants with damping
C066 5.462800e+10+i*C_omega*1.288102e-04 Pa Elastic constants with damping
Ca11 -4.424200e+06 Pa/K 1st temperature derivative of C0
Ca12 -1.325355e+07 Pa/K 1st temperature derivative of C0
Ca13 -1.393350e+06 Pa/K 1st temperature derivative of C0
Ca14 -3.389128e+05 Pa/K 1st temperature derivative of C0
Ca15 0.000000e+00 Pa/K 1st temperature derivative of C0
Ca16 0.000000e+00 Pa/K 1st temperature derivative of C0
Ca22 1.939984e+06 Pa/K 1st temperature derivative of C0
Ca23 -3.964216e+06 Pa/K 1st temperature derivative of C0
Ca24 2.433941e+06 Pa/K 1st temperature derivative of C0
Ca25 0.000000e+00 Pa/K 1st temperature derivative of C0
Ca26 0.000000e+00 Pa/K 1st temperature derivative of C0
Ca33 5.422548e+06 Pa/K 1st temperature derivative of C0
Ca34 6.424904e+06 Pa/K 1st temperature derivative of C0
Ca35 0.000000e+00 Pa/K 1st temperature derivative of C0
Ca36 0.000000e+00 Pa/K 1st temperature derivative of C0
Ca44 -4.170158e+05 Pa/K 1st temperature derivative of C0
Ca45 0.000000e+00 Pa/K 1st temperature derivative of C0
Ca46 0.000000e+00 Pa/K 1st temperature derivative of C0
Ca55 3.940561e+06 Pa/K 1st temperature derivative of C0
Ca56 -4.351932e+06 Pa/K 1st temperature derivative of C0
Ca66 -6.929611e+05 Pa/K 1st temperature derivative of C0
Cb11 -1.475700e+04 Pa/K^2 2nd temperature derivative of C0
Cb12 -3.691453e+04 Pa/K^2 2nd temperature derivative of C0
Cb13 -1.751347e+04 Pa/K^2 2nd temperature derivative of C0
Cb14 2.140374e+04 Pa/K^2 2nd temperature derivative of C0
Cb15 0.000000e+00 Pa/K^2 2nd temperature derivative of C0
Cb16 0.000000e+00 Pa/K^2 2nd temperature derivative of C0
Cb22 -2.125137e+04 Pa/K^2 2nd temperature derivative of C0
Cb23 -1.043062e+03 Pa/K^2 2nd temperature derivative of C0
Cb24 6.598400e+02 Pa/K^2 2nd temperature derivative of C0
Cb25 0.000000e+00 Pa/K^2 2nd temperature derivative of C0
Cb26 0.000000e+00 Pa/K^2 2nd temperature derivative of C0
Cb33 -3.025106e+02 Pa/K^2 2nd temperature derivative of C0
Cb34 3.429790e+02 Pa/K^2 2nd temperature derivative of C0
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Cb36 0.000000e+00 Pa/K^2 2nd temperature derivative of C0
Cb44 2.251938e+03 Pa/K^2 2nd temperature derivative of C0
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Cb56 -2.281400e+03 Pa/K^2 2nd temperature derivative of C0
Cb66 1.957817e+04 Pa/K^2 2nd temperature derivative of C0
Cc11 0.000000e+00 Pa/K^3 3rd temperature derivative of C0
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N665 0.000000e+00 Pa Nonlinear elastic constant
N666 0.000000e+00 Pa Nonlinear elastic constant

\begin{align*}
e011 &\quad -4.020000e-01 \text{ C/m}^2 \text{ Piezoelectric stress constant} \\
e012 &\quad 4.328472e-01 \text{ C/m}^2 \text{ Piezoelectric stress constant} \\
e013 &\quad -3.084722e-02 \text{ C/m}^2 \text{ Piezoelectric stress constant} \\
e014 &\quad -6.171600e-02 \text{ C/m}^2 \text{ Piezoelectric stress constant} \\
e015 &\quad 0.000000e+00 \text{ C/m}^2 \text{ Piezoelectric stress constant} \\
e016 &\quad 0.000000e+00 \text{ C/m}^2 \text{ Piezoelectric stress constant} \\
e021 &\quad 0.000000e+00 \text{ C/m}^2 \text{ Piezoelectric stress constant} \\
e022 &\quad 0.000000e+00 \text{ C/m}^2 \text{ Piezoelectric stress constant} \\
e023 &\quad 0.000000e+00 \text{ C/m}^2 \text{ Piezoelectric stress constant} \\
e024 &\quad 0.000000e+00 \text{ C/m}^2 \text{ Piezoelectric stress constant} \\
e025 &\quad -2.587002e-01 \text{ C/m}^2 \text{ Piezoelectric stress constant} \\
e026 &\quad 2.868193e-01 \text{ C/m}^2 \text{ Piezoelectric stress constant} \\
e031 &\quad 0.000000e+00 \text{ C/m}^2 \text{ Piezoelectric stress constant} \\
e032 &\quad 0.000000e+00 \text{ C/m}^2 \text{ Piezoelectric stress constant} \\
e033 &\quad 0.000000e+00 \text{ C/m}^2 \text{ Piezoelectric stress constant} \\
e034 &\quad 0.000000e+00 \text{ C/m}^2 \text{ Piezoelectric stress constant} \\
e035 &\quad 1.151807e-01 \text{ C/m}^2 \text{ Piezoelectric stress constant} \\
e036 &\quad -1.277002e-01 \text{ C/m}^2 \text{ Piezoelectric stress constant} \\
ea11 &\quad 3.290000e-04 \text{ C/(m}^2\text{.K)} \text{ 1st temperature coefficient of e0} \\
ea12 &\quad -5.287275e-04 \text{ C/(m}^2\text{.K)} \text{ 1st temperature coefficient of e0} \\
ea13 &\quad 1.997275e-04 \text{ C/(m}^2\text{.K)} \text{ 1st temperature coefficient of e0}
\end{align*}
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eps012 0.000000e+00 F/m Dielectric permittivity
eps013 0.000000e+00 F/m Dielectric permittivity
eps022 2.454830e+01 F/m Dielectric permittivity
eps023 1.106914e+01 F/m Dielectric permittivity
Dielectric permittivity

eps033 $4.448170 \times 10^1$ F/m

eps011 $3.229000 \times 10^{-4}$ F/(m.K)

eps012 $0.000000 \times 10^0$ F/(m.K)

eps013 $0.000000 \times 10^0$ F/(m.K)

eps022 $1.475558 \times 10^{-4}$ F/(m.K)

eps023 $-3.938296 \times 10^{-4}$ F/(m.K)

eps033 $-5.616558 \times 10^{-4}$ F/(m.K)

eps11 $-1.073000 \times 10^{-6}$ F/(m.K^2)

eps12 $0.000000 \times 10^0$ F/(m.K^2)

eps13 $0.000000 \times 10^0$ F/(m.K^2)

eps22 $-8.055086 \times 10^{-6}$ F/(m.K^2)

eps23 $6.007954 \times 10^{-7}$ F/(m.K^2)

eps33 $2.764086 \times 10^{-7}$ F/(m.K^2)

First thermal expansion coefficient

alf011 $5.630000 \times 10^{-6}$ 1/K

Second thermal expansion coefficient

alfb11 $5.979000 \times 10^{-9}$ 1/K^2

Third thermal expansion coefficient

epsc11 $0.000000 \times 10^0$ F/(m.K^3)

epsc12 $0.000000 \times 10^0$ F/(m.K^3)

epsc13 $0.000000 \times 10^0$ F/(m.K^3)

epsc22 $0.000000 \times 10^0$ F/(m.K^3)

epsc23 $0.000000 \times 10^0$ F/(m.K^3)

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