Reliability Analysis and Maintenance Modeling of Multi-Component Systems Subject to Multiple Dependent Competing Failure Processes

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ABSTRACT OF THE DISSERTATION

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System reliability analyses, involving multiple failure processes, are important and challenging research topics, particularly when failure processes, such as degradation processes and random shocks, are competing and dependent. When component degradation models are extended to complex systems with multiple components, different perspectives of dependency should be considered in system reliability modeling. In this research, potential dependence patterns are investigated among multiple failure processes within and among components in systems and probabilistic models are developed to assess system reliability performance. For the reliability modeling of complex systems, if one component in the system degrades or fails prematurely, it is possible that other components will also degrade or fail prematurely given the shared working environment, which means component failure times are dependent. Furthermore, if a shock arrives to the system, and its impact on one component is large, it is likely that the shock impact on other components is also large. Therefore, the models are extended to perform quantitative analyses for system reliability considering that the damages to the two failure processes caused by shocks are dependent. From a multi-component system level perspective, the dependent

characteristics from shocks are complex and detailed, so the research is organized into several scenarios, i.e., dependent failure processes are considered in different ways. Stochastically dependent component degradation processes are also studied, and extended gamma process models are used to model the dependent degradation process. Based on these new reliability models, different maintenance policies are derived to provide cost effective maintenance plans. Different maintenance policies are investigated, including: (1) age replacement policies; (2) periodic inspection maintenance policies; (3) Condition-based maintenance policies; and (4) Individual component maintenance policy. This research has many meaningful research contributions. The dependent scenarios described above for shocks are studied for the first time in system reliability modeling, and it is also the initial system model based on stochastically dependent component degradation using the gamma process. Additionally, the on-condition maintenance plans and the individual component maintenance based on steady state system behavior are investigated for the first time. They represent more practical and cost effective policies for many engineering applications.

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I am very honored that I spent my time on doctoral study in United State, the largest developed and a beautiful country, which is the leader of science and technology all over the world. I am very honored that I spent my time on doctoral study in Rutgers University, an institute which has a long history, splendid culture, fine traditions, full of love and competition, and also has strong contemporary atmosphere of advanced education. From now on, no matter where I go, I will be proud to tell people that I am the alumni of Rutgers University, which has given me fabulous memories. I am very honored that during my study at Rutgers University, I met a kindly, respected, and knowledgeable elder, my mentor David Coit. Because of his zealous advice and guidance, I can make impossible things happen.

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List of Notation

The notation used in formulating the reliability and maintenance models is listed below.

$N(t)$ number of shock loads that have arrived by time t;

$N_i(t)$ number of shocks for i^{th} specific type having arrived by time t considering component shock set;

N number of components in a series or parallel system;

Λ arrival rate of random shocks;

λ_i arrival rate of ith type shocks to the system considering component shock set;

K number of required components in a k-out-of-n system;

ϕ_l Shock set for component l;

D_i threshold for catastrophic/hard failure of ith component;

D threshold for catastrophic/hard failure of component in the k-out-of-n system;

$W_{l,j,k}$ hard failure damage of the kth shock belonging to j^{th} type hitting component l considering component shock set;

W_{ij} size/magnitude of the jth shock load on the ith component;

W_j size/magnitude of the jth shock load on the component in the k-out-of-n system;

$\tilde{W}_{ij}$ purely random shock effect for the jth system shock to the ith component hard failure process (not proportional to system shock size)

$F_{l,j}(w)$ cumulative distribution function (cdf) of $W_{l,j,k}$ considering component shock set;

$F_{Wi}(w)$ cumulative distribution function (cdf) of W_{ij} ;

$F_W(w)$ cumulative distribution function (cdf) of W_j in the k-out-of-n system;

H_i critical wear degradation failure threshold of the ith component;

H critical wear degradation failure threshold of the component in the k-out-of-n system;

$X_i(t)$ wear volume of the ith component due to continuous degradation at t;

$X(t)$ wear volume of the component due to continuous degradation at t in the k-out-of-n system;

$X_{Si}(t)$ total wear volume of the ith component at t due to both continual wear and instantaneous damage;

$X_S(t)$ total wear volume of the component at t due to both continual wear and instantaneous damage in the k-out-of-n system;

$Y_{l,j,k}$ damage size to soft failure of component l caused by the kth shock load from the jth type in the shock set;

Y_{ij} damage size contributing to soft failure of the ith component caused by the jth shock load;

Y_j damage size contributing to soft failure of the component caused by the jth shock load in the k-out-of-n system;

$\tilde{Y}_{ij}$ purely random shock damage effect for the jth system shock to the ith component soft

	failure process (not proportional to system shock size)
$S_i(t)$	cumulative shock damage to soft failure process of the ith component at t;
$S(t)$	cumulative shock damage size of the component at t in the k-out-of-n system;
$G_i(x_i,t)$	cumulative distribution function (cdf) of $X_i(t)$;
$G(x,t)$	cumulative distribution function (cdf) of $X(t)$ in the k-out-of-n system;
$G_a(x_i/\bullet)$	cumulative distribution function (cdf) of $X_i(t)$, $X_i(t)$ is a Gamma process;
Z_j	j^{th} system shock size or magnitude
α_i	transmission parameter from system shock size to the ith component transmitted shock size (hard failure process)
γ_i	transmission parameter from system shock size to the ith component shock damage (soft failure process)
θ_i	scale parameter for Gamma process;
$v_i(t)$	intensity measure, is a non-decreasing, right-continuous function for $t > 0$;
$F_{X_i}(x_i,t)$	cdf of $X_{S_i}(t)$;
$F_X(x,t)$	cdf of $X_S(t)$;
$f_{Z_j}(z_j)$	pdf of the system shock size Z_j
$f_{Z_j}^{<m>}(z_j)$	pdf of the sum of m independent and identically $i.i.d.$ Z_j variables
$f_{Y_i}^{<k>}(y)$	pdf of the sum of k independent and identically distributed ($i.i.d.$) Y_{ij} variables
$f_Y(y)$	probability density function (pdf) of Y in the k-out-of-n system;
$f_Y^{<k>}(y)$	pdf of the sum of k independent and identically distributed (i.i.d.) Y variables in the k-out-of-n system;
$f_{Y_l}^{<k>}(y)$	pdf of the sum of $Y_{l,j,k}$ variables considering all number of shocks for all types in component l shock set;
$f_T(t)$	cdf of the failure time, T;
$F_T(t)$	
U_i	Initial degradation level for component i considering individual component maintenance policy;
$P(\tau)$	System survival probability in one inspection interval τ considering individual component maintenance policy;
V	periodic replacement interval;
$C(t)$	cumulative maintenance cost by time t;
$CR(V)$	average long-run maintenance cost rate of the age replacement policy;
$E[U]$	expected value of the renewal cycle length, U of the age replacement policy;
$E[G]$	expected value of the number of failures, G in a renewal cycle;

$E[TC]$	expected value of the total maintenance cost of the renewal cycle, TC;
C_R	replacement cost per unit;
C_F	cost of replacement caused by failure;
τ	periodic inspection interval;
$CR(\tau)$	average long-run maintenance cost rate of the second policy;
$E[K]$	expected value of the renewal cycle length, K of the second policy;
$E[N_I]$	expected value of the number of inspections N_I;
$E[\rho]$	expected value of system downtime or the time from a system failure to the next inspection when the failure is detected ρ;
C_R	replacement cost per unit;
C_I	cost associated with each inspection;
C_ρ	penalty cost rate during downtime;

1. Introduction

This research focuses on reliability of multi-component systems subject to multiple dependent competing failure processes, and associated maintenance policies and optimization. Previous research studied one single unit or a simple system, or relies on unrealistic assumptions of independent failure processes or shock damage within a component or among components. In this research, reliability models are derived for multi-component systems considering different dependent patterns within and among component failure processes, and different maintenance policies are defined and optimized to minimize maintenance cost per unit time.

System designers pursue new technologies because they potentially can provide desirable functions and features with superior reliability at a competitive cost. Therefore, reliability and cost are two critical factors that need to be analyzed and optimized.

Recent design and development innovations for many evolving technologies have been particularly impressive offering great potential. Commercialization of these new and exciting technologies has required advancements in material science, electrical, mechanical and biomedical engineering. The next stage of technological advancement requires that the manufacturing process is high-yield, low cost and highly-reliable. Traditional approaches of reliability analysis are sometimes inappropriate or inefficient for some new devices because either they are too reliable to observe failure time data in a reasonable time period, or the time period between design and product release is too short [1]. If new technologies are to be transitioned from low volume production or relatively simple design applications, new and innovative research focusing on system reliability issues must be considered [2]. Without new practical and effective models, the continued

advancement of these new devices may not reach its potential to satisfy consumers' demanding requirements and expectations.

For complex systems, reliability modeling can be a very complicated research topic involving a number of intricacies and difficulties. There are different factors that can influence reliability of engineering devices and systems. Environmental factors are some examples, e.g., temperature, humidity, wind speed, mechanical shocks etc. Also, aging, wearing, corrosion, mechanical fatigue and other physical changes can occur due to the regular operational and environmental exposure [3]. For many engineering applications, it is difficult to assess system reliability because traditional estimation methods, such as those based on observing failure times (even with accelerated life testing) are not appropriate or efficient, or has other limitations. However, system reliability is a very important issue [4], and to fully investigate system reliability, possible failure mechanisms of each component should be identified to further study their effects on the components and system functions. System reliability should be analyzed combining different failure mechanisms, since common issues can have effects on multiple failure mechanisms [5-7]. For example, material fatigue and aging under long-term repeated cyclical loading may lead to potential device deterioration, which in turn impacts the device reliability.

For many devices, reliability cannot be adequately measured and controlled. Reliability is time-dependent and can only be predicted or estimated but not measured exactly [8-10]. Considering liquefied natural gas pipeline for example, there are various factors that can degrade the reliability of natural gas pipelines. One major factor is corrosion, which is the gradual destruction of the surfaces of the pipelines as a result of chemical reactions with the environment [11]. Figure 1.1 shows the destruction that causes

the surfaces of the pipeline to degrade over time, and may ultimately lead to a failure. Some of the published research addressing pipeline reliability uses a simple linear defect growth model, i.e., a constant growth rate to estimate corrosion rate. Additionally, shocks from the environment and/or intervention from a third party can cause catastrophic failure suddenly, or cause incremental damage to the degradation process.

Figure 1.1: LNG pipeline failure due to corrosion

System reliability is a critical design characteristic that designers and manufacturers must aggressively address before introducing a new product to the market or during operation and service. For example, on April 20, 2010, the deep water horizon semi-submersible mobile offshore drilling rig explosion results in a massive offshore oil spill in the Gulf of Mexico. One main cause is the poor equipment reliability, which is a result of

drilling priorities taking precedence over maintenance [12]. Another example relates to commercial airline safety and reliability. A fleet was grounded in 1979, after a McDonnell Douglas DC-10 crashed shortly after take-off at O'Hare Airport in Chicago, killing 273 people [13]. In the last five years, the death risk for passengers in the United States has been one in 45 million flights. In other words, flying has become so reliable that a traveler could fly every day for an average of 123,000 years before being in a fatal crash [13]. These two cases demonstrate the importance of reliability on the world society and economy.

In traditional reliability analyses, failure-time-based reliability models are commonly used for many engineering applications. They are straightforward, but have limitations. In this research, failure mechanisms are investigated and combined with probabilistic modeling to develop new system reliability models. For reliability analysis, one thoroughly studied area is the reliability of systems subject to random shocks at random times [14]. Also, degradation is one of the common failure mechanisms that have been widely investigated [15]. Without loss of generality, reliability models are developed for systems subject to both degradation processes and random shock processes, considering that they are competing and dependent, and implement appropriate maintenance policies.

This chapter starts with the problem statement that describes the complicated system reliability problems. Then, motivation of this research is stated with inspiration from many practical engineering applications. Finally, objectives and contributions of this research are discussed.

1.1 Problem Statement

Modern products are developed to be more reliable with longer lifetime and higher performance, so it is very challenging to obtain accurate and sufficient time-to-failure data

in a cost effective way before releasing the product. When traditional failure-based reliability methods are not applicable due to the lack of failure data or other limitations, a methodology based on only limited or no actual failure data can be used, such as random shock and degradation based reliability analysis. Degradation is one of the common failure mechanisms that have been widely investigated [15-18]. Degradation-based reliability analysis is gaining extensive attention recently because degradation data can provide more information than failure time data, and can be used to predict the reliability performance even beyond the experimental time. Another extensively explored area is the reliability of systems subject to shocks occurring at random times with random magnitude [19-22].

Degradation modeling based on probabilistic modeling of a failure mechanism degradation path and comparison of a projected distribution to a pre-defined failure threshold has already been successfully applied for many applications [23]. Reliability analysis based on degradation modeling is a convenient and effective method for some highly reliable components or systems when observations of failures are rare. There are many examples of failure mechanisms where reliability prediction based on degradation modeling is an effective approach. Failure mechanisms are understood from a physical perspective and typical degradation measures include wear, drug stability, deterioration, degraded light intensity, crack propagation, resistance drift and loss of structural strength.

Random shock modeling is another important failure mode representing the sudden impacts from the external environment on the system. Shock models in system reliability are normally defined by the time between two consecutive shocks, the damage caused by a shock, and the dependence among the above elements [14]. Reliability models for systems have been extensively studied when they are exposed to external shock

environments. Various models developed are physically motivated. For instance, the extreme and cumulative shock models are appropriate for the fracture of brittle materials, and for the damage due to the earthquakes or volcanic activity, respectively.

There have been some notable reliability models developed based on the combination of random shock and degradation modeling [16]. Though studies have developed reliability models for combined degradation and random shocks, relatively scant research has been devoted to reliability analysis of systems with multiple components and multiple dependent competing failure processes. Multiple failure processes that a system experiences are often assumed to be independent in previous research, which restricts applications to systems when no interactions or correlations exist among those multiple failure processes. However, for complex systems where one failure process can affect another failure process or when multiple failure processes are simultaneously affected by some shared external stresses, the assumption of independence among multiple failure processes may not be valid, and the traditional reliability models may not accurately predict the system reliability. Therefore, new reliability models are needed for systems subject to multiple dependent failure processes.

In this study, the reliability models for systems subject to multiple dependent competing failure processes among multiple components is investigated. This is a very challenging problem if dependent shock damage effects or stochastically dependent component degradation processes are considered. The research investigates new reliability models for systems with dependent competing failure processes, dependent component failure times, and dependent damage to both   a hard failure process and a soft failure process caused by the shocks, and stochastically dependent component degradation paths.

Based on these new reliability models, appropriate maintenance policies are chosen and optimization models are developed to minimize a maintenance cost rate by determining replacement/inspection time intervals.

1.2 Motivation of Research

Earlier research on system reliability pertains to either random shocks or degradation modeling. Later, multiple failure processes have been studied as an effective method for system reliability analysis, which are applicable for independent failure processes. Although dependent and competing failure processes are recently considered, the dependent characteristics of those models is limited to single-unit systems or one component, and are based on the shared exposure to the shock processes. Scant research has been devoted to reliability analysis of systems with multiple dependent competing failure processes among components considering dependent component failure times, dependent shock damage to component failure processes, and stochastically dependent component degradation paths.

To sufficiently study system reliability based on component degradation, it is necessary and practical to consider many different dependent scenarios and patterns of multiple failure processes to better estimate system reliability. For many engineering applications, even very small errors can cause disasters or terrible consequences. For a multi-component system, if one component fails, it is probabilistically possible that the whole system has experienced a certain level of degradation and numbers of shocks, and other components may also fail soon as well. That is, these component failure times are probabilistically dependent often beyond the extent typically associated with competing failure processes due to shared with shock processes. Also, there are many engineering

applications, in which shocks with specific sizes or function can affect one or more components in the system but not necessarily all components. Therefore, classifying shocks according to their sizes, functions, acting points and categorizing them into different shock set is reasonable to achieve better reliability models. For example, a car can receive a random shock, which can be a thermal shock, mechanical shock, voltage shock, etc. Different shocks can affect different car components. Overall, it is realistic and practical to classify shocks into different sets based on shock size, function, and other factors for many engineering application devices.

Furthermore, dependent scenarios from different perspectives can be quantitatively considered, i.e., dependent shock damage to multiple failure processes within and among components, and stochastically dependent component degradation processes. For dependent shock damage, from a multi-component system level perspective, the dependent characteristic is complicated. Types of dependency are: (1) Shock damage to hard failure processes among components are dependent; (2) Shock damage to soft failure processes among components are dependent; (3) Shock damage to hard failure processes among components are dependent, and damage to soft failure processes are dependent, but they are mutually independent; and (4) Shock damage to hard failure processes among components are dependent, and damage to soft failure processes are dependent, they are also mutually dependent. For the degradation paths, dependent characteristics arise due to many reasons, including (a) components are physically touching each other, and one can directly influence the degradation of other components; (b) components are physically close to each other; (c) components exist in the shared environment, and the factors like

temperature, wind speed, voltage can affect all the component degradation paths at the same time; (d) components are in a load sharing design situation.

In previous research, different scenarios have been considered in modeling the reliability for systems subject to dependent competing failure processes, including: dependent component failure times, dependent shock damage to specific failure process and other dependent patterns. However, the scenarios mentioned above have not been considered in reliability modeling of complex systems. To achieve more accurate system reliability prediction, new system reliability models need to be developed to consider these different dependent characteristics beyond dependent failure processes due to the exposure to the same shocks. Based on the new reliability models, appropriate maintenance policies are optimized to assure system availability and performance and minimize system maintenance cost.

1.3 Research Contributions

In this research, new models are developed to analyze the reliability of complicated multi-component system subject to multiple dependent and competing failure processes. The main contribution is that we extend the single component and simple system to complex system and develop the new reliability models.

1. Reliability models are developed for multi-component complex system with each component subject to multiple failure processes and component failure times are shown to be dependent. Previous research pertained to either simple systems or required independent components.

2. In this research, the new reliability models are developed considering the dependent shock damage to two failure processes among components, which can be divided

into four different scenarios. In previous research, failure processes are only dependent due to shared shock process.

3. New system reliability models are developed considering stochastically dependent component degradation processes, and the gamma process is chosen to model the degradation process. Previous research has the limitation that system reliability modeling is based on the assumption that component degradation paths are independent.

4. Age replacement policy, periodic inspection maintenance policy and condition-based maintenance policy are common policies that have been applied to many systems. However, they are never considered based on the new reliability models for complex systems with components subject to multiple dependent competing failure processes, and in this research, these maintenance policies are considered and optimization problems are solved based on the new reliability models.

5. An individual component maintenance policy is studied based on steady state system behavior, and the system inspection interval is optimized, while previous research is based on the assumption that when one component fails, the whole system needs to be repaired or replaced.

2. Background and Literature Review

In this section, the literature is reviewed on reliability modeling for systems experiencing two or more failure processes (i.e., degradation processes and random shock processes) and maintenance policies associated with such systems.

2.1 Previous research on system reliability with multiple failure processes

2.1.1 Methods to model system degradation

Degradation is defined as the reduction in performance, reliability and life span of assets. Many failure mechanisms can be traced to an underlying degradation process. Degradation as a stochastic process can be modelled using several approaches [24].

Degradation can be categorized into two types: natural and forced degradation [25, 26]. Natural degradation is a time-dependent internal process in systems where gradual degradation brings the systems closer to failure. Forced degradation is external to systems, where its loading gradually increases so that a point is reached beyond which the systems fail.

Degradation models represent the underlying model of performance deterioration. A popular failure time model is the Weibull distribution; since it can represent different types of behavior including infant mortality and wear-out in the bathtub-tube curve [27]. However, the Weibull distribution is generally used for failure time, while a more advanced perspective is to have the distribution for a random degradation measure. Model-based approaches for degradation use mathematical dynamic models for a monitored asset. These approaches can be applicable in physics-based models and statistical models [28-31].

Knowledge-based approaches appear to be promising, because they require no defined models [29]. These approaches are employed where accurate mathematical models are difficult to define or select in the real world, or limitations of using model-based approaches become significant. Data-driven approaches are based upon statistical and learning techniques from pattern recognition [32]. Degradation models in reliability analysis can potentially be classified into two categories as shown in Figure 2.1, and are described in the following paragraphs [24].
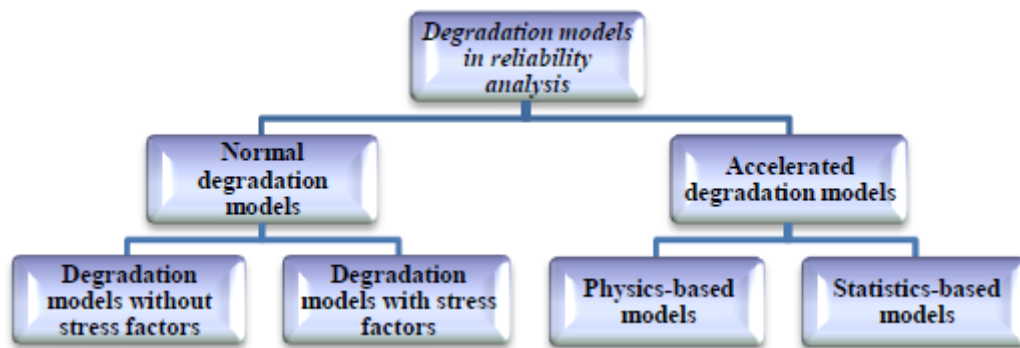
Figure 2.1: Classification of degradation models in reliability analysis

1. **Normal degradation models**: These models are used to estimate reliability with degradation data from normal operating conditions. Normal degradation models can be classified into two groups: degradation models with and without stress factors [24]. The difference is whether degradation is a function of defined stress and related reliability can be estimated at fixed levels of stress. These degradation models can be further classified as follows.

(1) General degradation path model: It fits the degradation observations using a regression model with random coefficients. Bagdonavičius et al. considered a degradation process using general nonparametric, nonlinear path models [33]. Lu et al. studied a model with

random regression coefficients and standard-deviation function for analyzing linear degradation data from semiconductors [34].

(2) Random process model: It fits degradation measures at each observation time by a specific distribution with time dependent parameters. Lu and Meeker fitted a random-effects model to fatigue degradation data and then used simulation-based methods to make inferences about the corresponding failure-time distribution [35]. In this method, multiple degradation data at a certain time have to be collected and treated as scattered points without orientation.

(3) Time series model: Lu et al. [36] developed a technique to predict individual system performance reliability in real-time considering multiple failure modes. It yields statistical results that reflect reliability characteristics of the population, includes on-line multivariate monitoring and forecasting of selected performance measures and conditional performance reliability estimates. The performance measures across time are treated as multi variate time series.

(4) Stress-strength interference model: In this model, there is random dispersion in the stress, which results from applied loads. Asset reliability corresponds to the event that strength exceeds stress can be developed. An et al. develops a model in which stress and strength are treated as discrete random variables, and a discrete stress-strength interference model is presented by using the universal generating function method [37].

(5) Stochastic degradation models: Brownian motion and gamma process models are continuous-time models that are appropriate for modeling continuous degradation process. These two models as well as Markov models are widely applied for degradation modelling. A Brownian motion model has an additive effect on the degradation. The conventional

Markov process model has been developed to the semi-Markov process model and the hidden Markov model to address more general reliability analysis problems [38]. For some applications with affecting the system and a self-recovering mechanism due to material resilience, Brownian motion is widely used to model the non-monotone deterioration with increasing tendency. The gamma process model has been increasingly used as a degradation process in maintenance optimization models, because strictly monotonically increasing property of the gamma process can be suitable for many engineering applications [39].

2. **Accelerated degradation models**: These models make inferences about reliability at normal conditions by using data obtained at accelerated time or stress conditions. To obtain data efficiently from a degradation test, it is often practical and efficient to employ an accelerated life test. Accelerated degradation models consist of physics-based models and the statistics-based models. Nelson [40] extensively describes both the physics-based models and statistics-based models. Furthermore, the statistical models with the accelerated failure time model are also reviewed in greater details [41].

2.1.2 Methods to model random shocks

Shock models in system reliability are normally defined by the time between two consecutive shocks, the damages caused by shocks, the system failure criteria and the dependence relationship among the above elements. Four categories of random shock models are classified: (*i*) an extreme shock model, where failure occurs when the magnitude of any shock exceeds a specified threshold; (*ii*) a cumulative shock model, where failure occurs when the cumulative damage from shocks exceeds a critical value; (*iii*) a run shock model, where failure occurs when there is a run of shocks exceeding a

critical magnitude; and (*iv*) a δ–shock model, where failure occurs when the time lag between two successive shocks is shorter than a threshold. This topic has been extensively studied and provides realistic formulations for modelling reliability systems exposed to a random environment [14].

1. Independence between the shock effect and the shock arrival time: The probability that the system still operates without failure after the kth shock can be defined. According to the time between consecutive shocks, these models are divided into four types: homogeneous Poisson process, non-homogeneous Poisson process, non-stationary pure birth process and renewal process. For the homogeneous Poisson process, conditions on these sequence of shocks are obtained to guarantee distribution properties of the survival function [42]. Some of these results are extended to non-homogeneous Poisson process [43] and non-stationary pure birth process [44]. Skoulakis [45] describes a general shock model for a reliability system.

2. Dependence between the effect of the shock and its arrival time: The damage caused by a shock is modelled by a random variable representing the shock's magnitude. Three major models are considered: extreme shock model, where the system fails as soon as the magnitude of a shock exceeds some given level; cumulative shock model, where the system fails when the cumulative shock magnitude exceeds some given level and run shock model, where the system works until k consecutive shocks with critical magnitude occur. An extension of the cumulative shock model is when the system failure depending on the cumulative damage of shocks whose magnitude exceeds a pre-specified threshold [46].Anderson developed a general model in which limit theorems apply for the first time the magnitude of a shock exceeds a threshold and the historical maximum magnitude are

given [47]. Anderson considered a shock model in which the time intervals between shocks are in the domain of attraction of a stable law-of-order less than a certain level or relatively stable [48]. Gut studied a theory for stopped two-dimensional random walk, which is well suited for cumulative shock models and failure is caused by a shock which is larger than a certain critical level [19, 20]. Shanthikumar and Sumita developed a general shock model associated with a correlated pair of renewal sequences, where system fails when shock damage exceeds threshold level [49]. They also studied some distribution properties of the system failure time in general shock models with a correlated renewal sequence [50]. Sumita investigated a class of cumulative shock models with a bivariate sequence of correlated random variables [51].

2.1.3 Model multiple failure processes

Common failure mechanisms and causes include wear, corrosion, fracture, shock loads, fatigue, etc. For many cases, a system suffers multiple failure mechanisms. Competing failure processes may occur and any of them can cause the system to fail. Competing means no matter which failure process occurs first, the system fails. These multiple competing failure processes may be independent or dependent.

Zuo et al. [52] develops a mixture model considering hard failures and degradation failures by assigning two failure mechanisms with different weights. A cumulative damage model is based on the cumulative damage theory for a degradation process exposed to discrete stresses and also the state of the process is assumed to be discrete. The common assumption of this model is that the shocks occur according to a Poisson process, and the amount of damage per shock is independently and identically distributed based on some arbitrarily selected common distribution [42, 53, 54]. Li and Pham studied a multi-state

degraded system reliability model subject to multiple competing failure processes, including two degradation processes and random shocks [55]. Hao and Su developed a new multiple competing failure models, in which multiple degradation processes and random shocks are considered [56]. Peng et al. developed a reliability model based on degradation and random shock modeling, which is then extended to a linear degradation path and normally distributed shock damage [57].

Wang et al. [58] investigated the system reliability with shock effect on the degradation process was considered and degradation analysis is conducted under fuzzy degradation data. Jiang et al. [59] developed reliability models for system subject to multiple dependent competing failure process correlated in two respects. The arrival of each shock impacts both failure processes, and also the shock process affects the hard failure threshold level. Two cases of dependency between the shock process and the hard failure threshold level were investigated. Lei [60] also developed reliability model for systems subject to multiple dependent competing failure processes with a changing, dependent failure threshold, which means when withstanding shocks, the system is deteriorating, and its resistance to failure is weakening. Rafiee [61] studied reliability models for devices subject to dependent competing failure processes of degradation and random shocks with a changing degradation rate according to particular random shock patterns.

2.2 Previous research on maintenance policy

Maintenance optimization models focus on finding either the optimal balance between costs and benefits of maintenance or the most appropriate time to execute maintenance [62]. It is conducted to achieve multiple maintenance targets, such as safety

control, reliability, system availability, and costs. Generally maintenance optimization models are classified according to the way they describe and represent natural variability and uncertainty in parameters, models and scenarios. Sherif and Smith [63] categorized the deterioration models into deterministic models and stochastic models. Stochastic models can be divided into risk and uncertainty. In the case of risk, it is assumed that a probability distribution of the time-to-failure is available, which is not true in the case of uncertainty. The optimization methods employed include linear and nonlinear programming, dynamic programming, Markov decision methods, decision analysis techniques, search techniques and heuristic approaches [64].

Maintenance policies can be classified as corrective maintenance, condition-based maintenance and preventive maintenance. Parameters considered in maintenance optimization are the cost of failure, the cost per time unit of downtime, the cost rate of corrective and preventive maintenance and the cost of repairable system replacement.

2.2.1 Models on optimization of preventive maintenance policies

Among the different types of maintenance policy, preventive maintenance is often chosen for large systems such as production systems, transport systems, etc. Preventive maintenance contains a set of management, administrative and technical actions to reduce the component age in order to improve the reliability of a system by replacing an older one with a newer one. These actions can be characterized according to their effects on the component age; the component becomes "as-good-as-new", the component age is reduced, or the state of the component is lightly affected only to ensure its necessary operating conditions, e.g., the component appears to be "as-bad-as-old". Preventive maintenance is a main maintenance policy used to reduce failure costs. Four categories can describe

preventive maintenance model classification: inspection models, minimal repair models, shock models, and miscellaneous replacement models.

Inspection models: System status is unknown unless an inspection is performed. Upon every inspection point, two decisions that have to be made: (1) whether to take maintenance action, and whether the system should be replaced or repaired to a certain state or whether the system should be left as is; (2) when the next inspection is to occur. Thus, the decision space of a maintenance inspection problem is two dimensional [65]. Barlow et al. [66] developed a basic pure inspection model for age replacement; i.e., no preventive maintenance is assumed, and the system is replaced only at failure. Luss [67] presented an approach to study systems where the degree of degradation can be measured through inspections, which reveals that the system is in one of several intermediate states of deterioration. Christer and Walter [68] developed models for optimal inspection and replacement for both perfect and imperfect inspection, in which they use delay time analysis. Anderson and Friedman [69] presented the optimal inspection times to minimize the total cost for a Brownian motion, in which inspections are costly as compared to other operating costs. At times inspection procedures themselves may pose a hazard to the system being checked. Chou and Butler [70] found optimal policies that maximize the expected lifetime of the system under inspection assuming that each inspection either causes immediate failure or else increases the failure rate.

Minimal repair: A repair or replacement of the failed component restores function to the system but the susceptibility of system failure remains as it was just before failure. In recent years, researchers have given more attention to the problem of optimal age replacement of complex systems subject to minimal repair at failure. Using the basic

minimal repair model, Tilquin and Cleroux [71] investigate an optimal replacement policy for a system when an adjustment cost is also considered. Boland and Proschan [72] studied a model where the minimal repair cost depends on the number of minimal repairs the system has suffered since the last replacement. Muth [73] developed a model in which minimal repair is executed if a failure occurred before a fixed time t^* and have system replacement at the first failure after t^*. Nakagawa [74] developed a minimal repair model combining the fixed-time and a counting replacement policy.

Replacement of system subject to shocks: Taylor [75] studies a maintenance policy of systems assuming that shocks occur according to a Poisson process and the damages caused by shocks are *i.i.d* exponential random variables. Feldman [76] assumes that the cumulative damage is a non-decreasing semi-Markov process and derives the expression for the long-run expected cost rate allowing the times between shocks to be arbitrarily distributed and dependent on the cumulated damage. Zuckerman [77] generalizes Taylor's model by not restricting the amount of damage caused by each shock to be exponential random variables. He allows the replacement cost before failure to be a non-decreasing function of the accumulated damage with a bound, i.e., the replacement cost of a failed system.

Miscellaneous replacement models: There are conditions in which systems can be observed continuously and failures detected immediately. However, at failure the system cannot be repaired and returned to an operating state with an unchanged failure rate. Instead, it has to be replaced at a cost. If the system is replaced before failure, then a lower replacement cost is incurred. Ansell et al. [78] developed an age replacement policy to model a system with increasing failure rate that is used over a finite time horizon. Derman

et al. [79] study the case in which the number of spares available is limited to *n*. If the system fails, it can neither be repaired nor replaced, but the system stops functioning. However, the system can be replaced before failure and continue functioning. Monahan [80] investigates a system whose degradation is partially observed, but inspection can be performed to observe the real state of the system. Nakagawa [81] describes ten replacement models by using combinations of age replacement, block replacement, and periodic replacement with minimal repair at failure.

2.2.2 Models on optimization of corrective maintenance policies

The actions that occur after the system fails are defined as corrective maintenance, which is thus a reactive strategy. The task of the maintenance in this scenario is usually to make repairs as soon as possible. Costs associated with corrective maintenance include repair costs, lost production and lost sales. To minimize the effects of lost production and accelerate repairs, actions such as increasing the size of maintenance teams, the use of back-up systems and implementation of emergency procedures can be considered. Unfortunately, such measures are relatively costly and/or only effective in the short-term. Although corrective maintenance has a direct influence on the components of a system, this was not sufficiently studied.

Sheut and Krajewski [82] studied a model that evaluates alternative corrective maintenance policies. Data analysis for the estimation of parameters of the failure process and effective corrective maintenance without preventive maintenance effect for repairable units is developed [83]. Lucia et al. [84] conducted an empirical assessment and improvement of the effort estimation model for corrective maintenance. Ding et al. [85] considers an aging multi-state system, whose failure rate varies with time. After any failure,

maintenance is performed with repair rate and cost of each repair determined by a corresponding corrective maintenance contract. Viles et al. [86] developed a functional network for the communication systems of trains and conducted a functional study and developed a tool for the improvement of the corrective maintenance. Goel [87] considers one unit system operating and the other as a cold standby. At random intervals, a check is done to determine the need for corrective maintenance.

2.2.3 Models on optimization of condition-based maintenance policies

Condition-based maintenance is a maintenance policy that assists maintenance decisions based on the information collected through condition monitoring. It consists of three main steps: data acquisition, data processing and maintenance decision-making. Condition-based maintenance attempts to avoid unnecessary maintenance tasks by taking maintenance actions only when there is evidence of abnormal behaviors of a physical asset. Condition-based maintenance can significantly reduce maintenance cost by reducing the number of unnecessary scheduled preventive maintenance operations, if it is properly established and effectively implemented.

Jardine [88] reviewed commonly used condition-based maintenance decision strategies such as trend analysis that is rooted in statistical process control, expert systems, and neural networks. Wang and Sharp [89] reviewed the recent development in modelling condition-based maintenance decision support. Wang [90] developed a condition-based maintenance model based on a random coefficient growth model where the coefficients of the regression growth model are assumed to follow known distribution functions. Grall et al. [91] assumed a multi-level control-limit rule replacement policy and obtained the optimal thresholds and inspection schedule by minimizing the expected maintenance cost

rate. Dieulle et al. [92] assumed a one-level replacement policy and a sequentially chosen inspection interval using a maintenance schedule function, and obtained the optimal threshold and inspection schedule by minimizing the global cost rate.

Amari and McLaughlin [93] used Markov chains to describe the condition-based maintenance model for a degradation system subject to periodic inspection. The optimal inspection frequency and maintenance threshold were found to maximize the system availability. Berenguer et al. [94] presented a condition-based maintenance policy for continuously deteriorating multi-component systems, which allows cost savings by performing simultaneous maintenance actions. Marseguerra et al. [95] utilized genetic algorithms to find the optimal thresholds by simultaneously optimizing profit and availability. Hosseini et al. [96] employed generalized stochastic Petri nets to represent a condition-based maintenance model for a system subject to deterioration failures and Poisson failures. Deterioration failures are assumed to be restored by a major repair and Poisson failures are restored by minimal repair. The optimal maintenance policy and inspection interval were then found to maximize system throughput.

For partially observable systems, Ohnishi et al. [97] applied a Markov decision process model for a discrete-time deterioration system to find the optimal replacement policy in which minimal repair is used to restore a failure if the decision is not to replace. Hontelez et al. [98] modeled a discrete Markov decision problem based on a continuous deterioration process to find the optimum maintenance policy with respect to cost. Aven [99] presented a counting process approach to determining the replacement policy minimizing the long-run expected cost. Barbera et al. [100] developed a condition-based maintenance model assuming exponential failure times with failure rate depending on

condition variables, and fixed inspection intervals. Christer et al. [101] developed a replacement cost model to obtain the optimal replacement policy given all available information.

Kumar and Westberg [102] developed a reliability-based approach for predicting the optimal maintenance time interval or the optimal threshold to minimize the total cost per unit time. Makis and Jardine [103] established a condition-based maintenance model using a Markov process to describe the evolution process of condition variables and a proportional hazard model to describe the failure mechanism that depends both on age and condition variable. The optimal replacement policy of the hazard control-limit type was determined by minimizing the long-run expected total cost rate. Barros et al. [104] considered an optimal condition-based maintenance policy for a two-unit parallel system of which unit-level monitoring information is imperfect and/or partial.

3. System Reliability Models with Degrading Components

In this section, system reliability models considering different dependent patterns are developed. First, reliability models of multi-component system subject to multiple dependent competing failure processes are developed, and component failure times are proven to be dependent. Based on this model, the system reliability model is extended to categorizing shocks into distinct shock sets according to different shock size, function and acting points. These studies focus on quantitative analyses, i.e., failure processes are defined to be 'dependent' due to the shared shock process. Later in this section, we extend to the case that system shock effect to the component failure processes are dependent, which can be divided into four different dependent scenarios, and system reliability models are developed accordingly. The four dependent patterns are: (1) Transmitted shock size affecting hard failure process among all components are dependent from each shock. (2) Damage to a soft failure process among all components are dependent from each shock. (3) Transmitted shock size affecting hard failure process among all components are dependent from each shock, and damage to soft failure process among all components are dependent from each shock. However, damage to hard failure and soft failure are independent. (4). Shock effects to the two failure processes among all components are dependent from each shock, and they are mutually dependent. Finally, the system reliability considering stochastically dependent component degradation processes is investigated, and the gamma process with a random scale parameter is chosen to model the degradation process.

For all of these reliability models, components in the systems are subject to two failure processes, i.e., the soft failure process caused by continuous smooth degradation and additional abrupt degradation damages due to a shock process, and hard failures caused

immediately from the system shock process. The failure processes are competing, since the system fails when any failure process occurs first, and they are dependent due to the shared shock process.

1. Hard failures due to shocks: Figure 3.1(b) shows that component i fails due to fracture when the jth load damage of the ith component exceeds hard failure threshold D_i. The probability that the ith component survives the jth shock is [57]:

$$P(W_{ij} < D_i) = F_{W_i}(D_i) \quad \text{for } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots \tag{1}$$

If the W_{ij} are assumed to be *i.i.d.* random variables distributed according to the normal distribution, $W_{ij} \sim N(\mu_{W_i}, \sigma^2_{w_i})$, then the probability that each component survival of a shock becomes[57]:

$$P_{Li} = F_{W_i}(D_i) = \Phi(\frac{D_i - \mu_{Wi}}{\sigma_{Wi}}) \quad \text{for} i = 1, 2, \ldots, n, \tag{2}$$

where $\Phi(\cdot)$ is the cumulative distribution function (cdf) of a standard normally distributed random variable. Eq. (1) is a generalized equation for probability of no hard failure. The normal distribution is not a required assumption for W_{ij}. Other distributions such as exponential or Weibull distributions can be considered according to practical engineering applications.

2. Soft failures due to degradation and shocks: Soft failures of the ith component can occur when the overall degradation of the component is beyond a threshold level H_i. As shown in Figure 3.1(a), the total degradation $X_{Si}(t)$, is the sum of the degradation due to continual wear and the instantaneous damages due to shocks. A linear degradation path is shown in Figure 3.1(a), $X_i(t) = \varphi_i + \beta_i t$, where the initial value φ_i and the degradation rate β_i

can be constant or random variables. This model is based on random effects models [35]

and suitable for other applications, whose degradation is linear or can be transformed into

a linear degradation path.
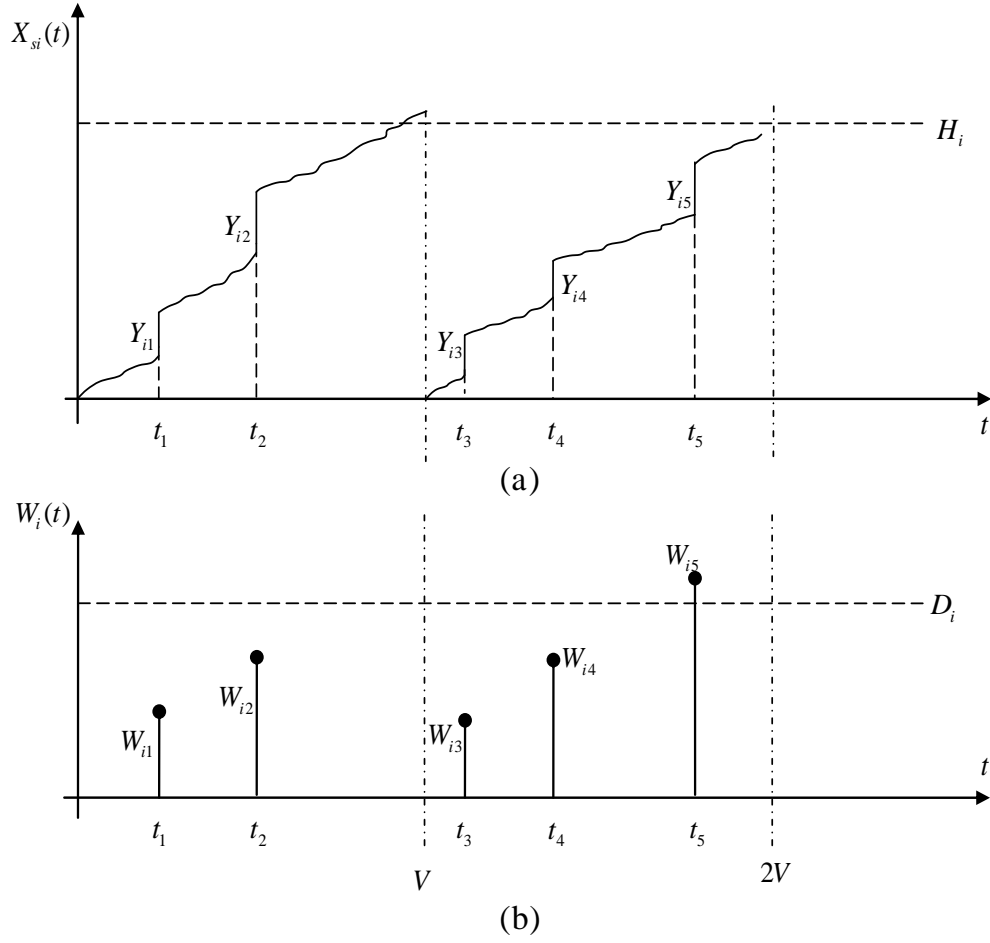
Figure 3.1: Two dependent competing failure processes for component i

(a) soft failure process, and (b) hard failure process[57]

Degradation shifts can accumulate instantaneously when a shock arrives. Each shock

impacts both failure processes of components. The cumulative damage due to random

shocks until time t, $S_i(t)$, is given as [57]

$$S_i(t) = \begin{cases} \sum_{j=1}^{N(t)} Y_{ij}, & \text{if } N(t) > 0, \\ 0, & \text{if } N(t) = 0, \end{cases} \tag{3}$$

where $N(t)$ is the total number of shocks that have arrived by time t, and $N(t)$ is a random variable. The overall degradation of the ith component is expressed as $X_{Si}(t) = X_i(t) + S_i(t)$.

The probability that the total degradation at time t is less than x_i, $F_X(x_i, t)$, can be derived as [57]

$$F_X(x_i, t) = P(X_{S_i}(t) < x_i) = \sum_{j=0}^{\infty} P(X_i(t) + S_i(t) < x_i \mid N(t) = j) P(N(t) = j) \tag{4}$$

Furthermore, if $G_i(x_i, t)$ is considered to be the cdf of $X_i(t)$ at t, $f_{Y_j}(y)$ to be the probability density function (pdf) of Y_{ij}, and $f_{Y_i}^{<k>}(y)$ to be the pdf of the sum of k i.i.d. Y_{ij} variables, then the cdf of $X_{Si}(t)$ in Eq. (4) can be derived using a convolution integral [57]:

$$F_X(x_i, t) = \sum_{j=0}^{\infty} \left(\int_0^{x_i} G_i(x_i - u, t) f_{Y_i}^{<j>}(u) \mathrm{d}u \right) P(N(t) = j). \tag{5}$$

If the shock damage on soft failure process of the ith component are i.i.d. normal random variables, $Y_{ij} \sim N(\mu_{Y_i}, \sigma_{Y_i}^2)$, the degradation path is linear with a constant initial value φ_i and a normal distributed degradation rate β_i, with $\beta_i \sim N(\mu_{\beta_i}, \sigma_{\beta_i}^2)$, and shocks arrives following a Poisson process with constant rate λ, then a more specific model can be determined based on Eq. (5) [57]:

$$F_X(x_i, t) = \sum_{j=0}^{\infty} \Phi\left(\frac{x_i - (\mu_{\beta_i} t + \varphi_i + j\mu_{Y_i})}{\sqrt{\sigma_{\beta_i}^2 t^2 + j\sigma_{Y_i}^2}} \right) \frac{\exp(-\lambda t)(\lambda t)^j}{j!} \tag{6}$$

Eq. (5) is a generalized equation for probability of no soft failure. The normal distribution is not a required assumption for soft failure process either, and other distributions like exponential distribution, Weibull distribution, etc, can also be considered according to engineering applications. The idea of using the Poisson process to model the shock process is well known in reliability literature. It is a very natural way of approaching modeling of the shock process. First, the shock process is part of environment, and it is not anticipated to have fundamental shifts. Second, the Poisson process is a logical choice, but the model can be extended to other pressures or serve as an approximation for other stationary processes.

Combining hard failure process and soft failure process, component reliability can be obtained as [57]:

$$R(t) = \sum_{j=0}^{\infty} \left(F_{W_i}(D_i) \right)^j \left(\int_0^{x_i} G_i(H_i - u, t) f_{Y_i}^{<j>}(u) \mathrm{d}u \right) P(N(t) = j) \tag{7}$$

3.1 System reliability model with dependent component degradation due to shared shock exposure

In this section, reliability of systems subject to dependent and competing failure processes is studied considering independent damage to soft/hard failure processes for all components. That is, when a shock arrives to the system, transmitted shock size to hard failure processes for all components are independent (independent W_{ij}), and shock damages to soft failure processes for all components are independent (independent Y_{ij}). However, due to the shared exposure to the same shock process, both failure processes among all components are proven to be dependent [105]. In this section, reliability models for multi-component systems subject to multiple dependent competing failure processes are

developed, and component failure times are proven to be dependent. Later in this section, the model is extended by categorizing the shocks into distinct shock sets according to different shock size, function and acting points.

3.1.1 System reliability model with dependent component failure time

A reliability model is developed for systems with dependent competing failure processes for each component and also dependent component failure times considering different structures/configurations. Two failure processes for each component are dependent due to the exposure to shared shocks, but damage to two failure processes by individual shocks are independent. Component failure times are also dependent, which means if one component fails prematurely, it is possible or likely that the whole system experienced certain levels of degradation and/or shocks, and it is probabilistically possible that other components will also fail soon.

Specific assumptions used for the reliability and maintenance modeling in this section are as follows:

1. When the threshold value H_i is exceeded by the total degradation of a component, soft failure occurs.

2. When the transmitted shock size on hard failure process exceeds the maximum strength D_i, hard failure occurs.

3. Random shocks arrive according to a Poisson process.

4. For series system, the system fails when the first component fails. For k-out-of-n system, the system works satisfactorily while at least k components survive both soft failure and hard failure processes. Parallel systems fail when all components experience either soft failure or hard failure. The reliability of a series-parallel system at time t is the

probability that at least one component within each sub-system survives both failure processes.

3.1.1.1 Series system

Figure 3.2 shows a series system with n components. The reliability of this system at time t is the probability that the each component in the system survives each of the $N(t)$ shock loads ($W_{ij}<D_i$ for $j=1, 2, \ldots$) and the total degradation is less than the threshold level ($X_{Si}(t)<H_i$):
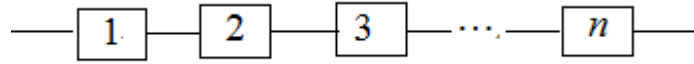
Figure 3.2: Series system example

The reliability can be expressed as the intersection of the events that each component has not failed up to time t. For each component to survive, it must withstand all $N(t)$ shocks and the total degradation much be less than the soft failure threshold.

$$R(t) = P\left\{\left[W_{11} < D_1, W_{12} < D_1,...,W_{1N(t)} < D_1, X_{S_1}(t) < H_1\right] \cap \right.$$
$$\left[W_{21} < D_2, W_{22} < D_2,...,W_{2N(t)} < D_2, X_{S_2}(t) < H_2\right] \cap ... \qquad (8)$$
$$\left.\left[W_{n1} < D_n, W_{n2} < D_n,...,W_{nN(t)} < D_n, X_{S_n}(t) < H_n\right]\right\}$$

3.1.1.1 Random effect model

The random effect model is based on linear degradation models, in which the degradation rate is distributed as normal distribution [35, 57]. The number of shocks $N(t)$ has an effect on each component. When $N(t)$ is large enough, the sum of damage to soft failure process of each component caused by shocks is large, and a failure is more frequent for all components. Alternatively, when there are relatively few shocks, times-to-failure are relatively longer for all components. Thus, this result causes component failure times to be probabilistically dependent. Conditioning on the number of shocks by time t [105,

106]:

$$R(t) = \sum_{m=0}^{\infty} P \left\{ \left[W_{11} < D_1, W_{12} < D_1, W_{1N(t)} < D_1, X_1(t) + \sum_{j=1}^{N(t)} Y_{1j} < H_1 \right] \cap \right.$$

$$\left[W_{21} < D_2, W_{22} < D_2, W_{2N(t)} < D_2, X_2(t) + \sum_{j=1}^{N(t)} Y_{2j} < H_2 \right] \cap \ldots$$

$$\left. \left[W_{n1} < D_n, W_{n2} < D_n, W_{nN(t)} < D_n, X_n(t) + \sum_{j=1}^{N(t)} Y_{2j} < H_n \right] \middle| N(t) = m \right\} P(N(t) = m)$$

(9)

Shocks arriving at random time intervals are modeled as a renewal process with Poisson process. When the system is shocked (at rate λ), all components experience a shock. Then, the system reliability function can be derived for the general case as [105, 106]:

$$R(t) = \sum_{m=0}^{\infty} \prod_{i=1}^{n} \left\{ P(W_{ij} < D_i)^m P \left[\left(X_i(t) + \sum_{j=1}^{m} Y_{ij} < H_i \right) \middle| N(t) = m \right] \right\} \frac{\exp(-\lambda t)(\lambda t)^m}{m!} \quad (10)$$

Based on Equations (1-5), the reliability function can be expressed as [105, 106]:

$$R(t) = \sum_{m=0}^{\infty} \prod_{i=1}^{n} \left\{ F_{W_i}(D_i)^m \int_0^{H_i} G_i(H_i - u) f_{Y_i}^{\langle m \rangle}(u) \mathrm{d}u \right\} \frac{\exp(-\lambda t)(\lambda t)^m}{m!}. \quad (11)$$

The reliability function for the more specific case with normally distributed W_{ij}, Y_{ij}, and β_i can be expressed as [105, 106]:

$$R(t) = \sum_{m=0}^{\infty} \prod_{i=1}^{n} \left\{ P_{Li}^{~m} \Phi \left(\frac{H_i - (\mu_{\beta i} t + \varphi_i + m\mu_{Y_i})}{\sqrt{\sigma_{\beta i}^2 t^2 + m\sigma_{Y_i}^2}} \right) \right\} \frac{\exp(-\lambda t)(\lambda t)^m}{m!} \quad (12)$$

where P_{Li} is given by Eq. (2) and $\Phi(\cdot)$ is the pdf of a standard normally distributed variable.

As another example simply to demonstrate that Eq. (11) is a general model, the reliability function with exponentially distributed W_{ij} (with parameter γ), exponentially distributed Y_{ij} (with parameter θ), and normally distributed β_i, can be expressed as [105,

106]:

$$R(t) = \sum_{m=0}^{\infty} \prod_{i=1}^{n} \left[\left[1 - e^{-\gamma D_i} \right]^m \int_0^{H_i} \Phi \left(\frac{H_i - (\mu_{\beta_i} t + \varphi_i + u)}{\sigma_{\beta_i} t} \right) \frac{u^{m-1} e^{-\frac{u}{\theta}}}{\theta^m \Gamma(m)} du \right] \frac{e^{-\lambda t} (\lambda t)^m}{m!}. \qquad (13)$$

Although the example is hypothetical, the parameters in Table 3.1 are estimated based on actual test data for MEMS [107]. According to Eq. (12), system reliability $R(t)$ is plotted in Figure 3.3. Also, in Figure 3.3, system reliability is presented for an analogous system with identical component failure processes but independent component failure processes. In this way, it is possible to observe one of the meaningful contributions from this new model, which is to explicitly consider components with dependent failure processes. As indicated in the figure, an incorrect component independence assumption would provide invalid results.

Figures 3.4 through 3.6 presents sensitivity analyses of μ_{Yi} or λ or both on $R(t)$. From Figures 3.4 through 3.6, decreased μ_{Yi} or decreased λ or both increases the reliability of both the dependent component system and the independent component system as expected. For the comparison, the decreased shock arrival rate is 10 times less frequent, and the mean shock damage is 30% of the previous amount for all components. When the damage to soft failure by shocks μ_{Yi} and shock arrival rate λ are decreased, the reliability for the dependent component system get closer to the reliability for the similar independent component system. This is also as expected because the covariance is smaller.

Table 3.1: Parameter values for series system reliability analysis

Parameters	component 1 & 2	component 3 & 4	Sources
H_i	$0.00125 \ \mu m^3$	$0.00127 \ \mu m^3$	Tanner and Dugger[107]
D_i	1.5Gpa(for polysilicon material)	1.4 Gpa(for polysilicon material)	Tanner and Dugger[107]
φ_i	0	0	Tanner and Dugger[107]
β_i	$\beta_i \sim N(\mu_{\beta i}, \sigma_{\beta i}^2) \mu_{\beta i}=8.4823 \times 10^{-9} \mu m^3$ $\sigma_{\beta i}=6.0016 \times 10^{-10} \mu m^3$	$\beta_i \sim N(\mu_{\beta i}, \sigma_{\beta i}^2) \mu_{\beta i}=8.4936 \times 10^{-9} \mu m^3$ $\sigma_{\beta i}=5.9011 \times 10^{-10} \mu m^3$	Tanner and Dugger[107]
λ	2.5×10^{-5}	2.5×10^{-5}	Assumption
Y_{ij}	$Y_{ij} \sim N(\mu_{Yi}, \sigma_{Yi}^2)$ $\mu_{Yi}=1 \times 10^{-4} \ \mu m^3$ $\sigma_{Yi}=2 \times 10^{-5} \ \mu m^3$	$Y_{ij} \sim N(\mu_{Yi}, \sigma_{Yi}^2)$ $\mu_{Yi}=0.9 \times 10^{-4} \ \mu m^3$ $\sigma_{Yi}=2.1 \times 10^{-5} \ \mu m^3$	Assumption
W_{ij}	$W_{ij} \sim N(\mu_{Wi}, \sigma_{Wi}^2)$ $\mu_{Wi}=1.2Gpa, \sigma_{Wi}=0.2GPa$	$W_{ij} \sim N(\mu_{Wi}, \sigma_{Wi}^2)$ $\mu_{Wi}=1.22 \ Gpa, \sigma_{Wi}=0.18 \ GPa$	Assumption
c_i	2×10^{-6}	2×10^{-6}	Assumption
b_i	2.2	2.2	Assumption
θ_i	1.5	1.5	Assumption

Sensitivity analysis was also performed on the ratio of system reliability with dependent and independent component failure processes, and the results are presented in Figure 3.7. System reliability was computed at $t = 2 \times 10^4$, and compared with the analogous system with independent components. The interesting parameters are the arrival rate of shocks λ and mean damage size for the ith component soft failure process from shocks, μ_{Yi}. The ratio R(dependent)/R(independent) increases approximately linearly as λ increase. That means, if the shock arrival rate is higher, and its effect on system reliability is more pronounced, the difference between R(dependent) and R(independent) is larger, and the dependency is more notable.
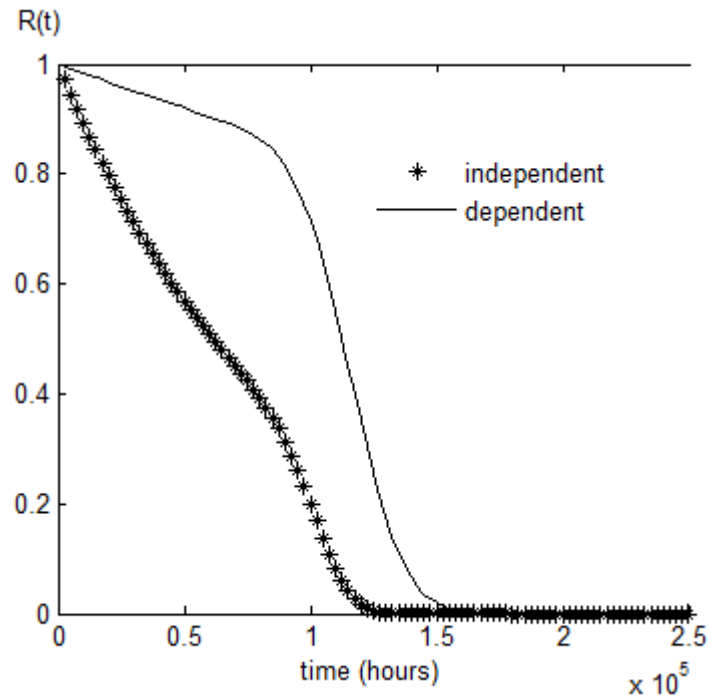
Figure 3.3: Plot of reliability function $R(t)$ of a series system with four components
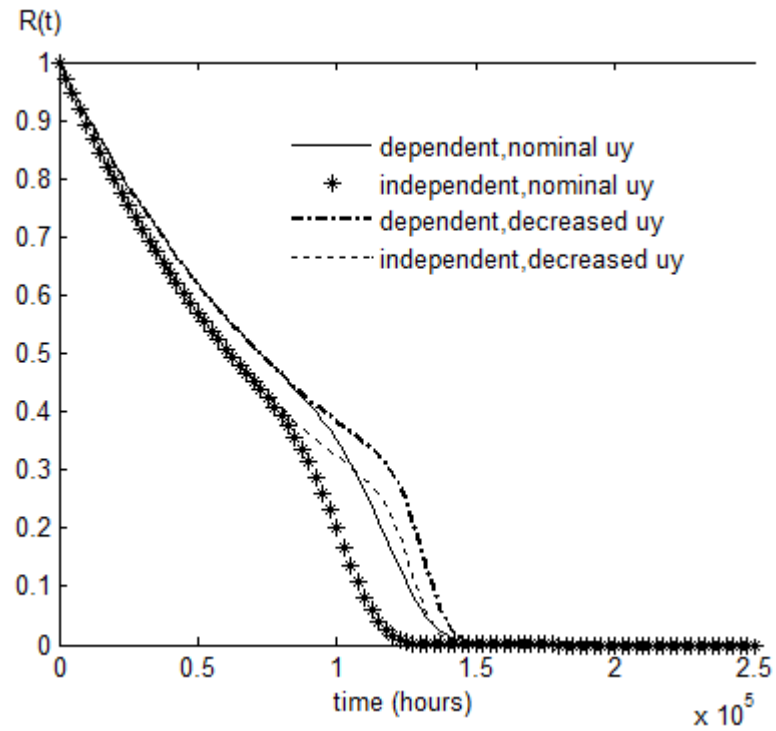
Figure 3.4: Sensitivity analysis of μY_i on $R(t)$
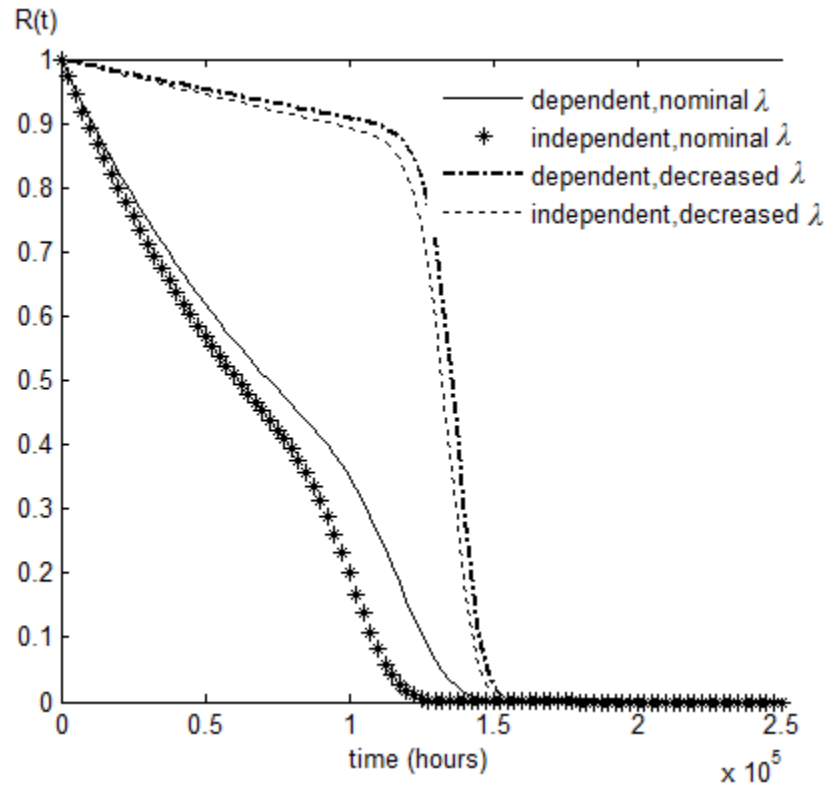
Figure 3.5: Sensitivity analysis of λ on $R(t)$
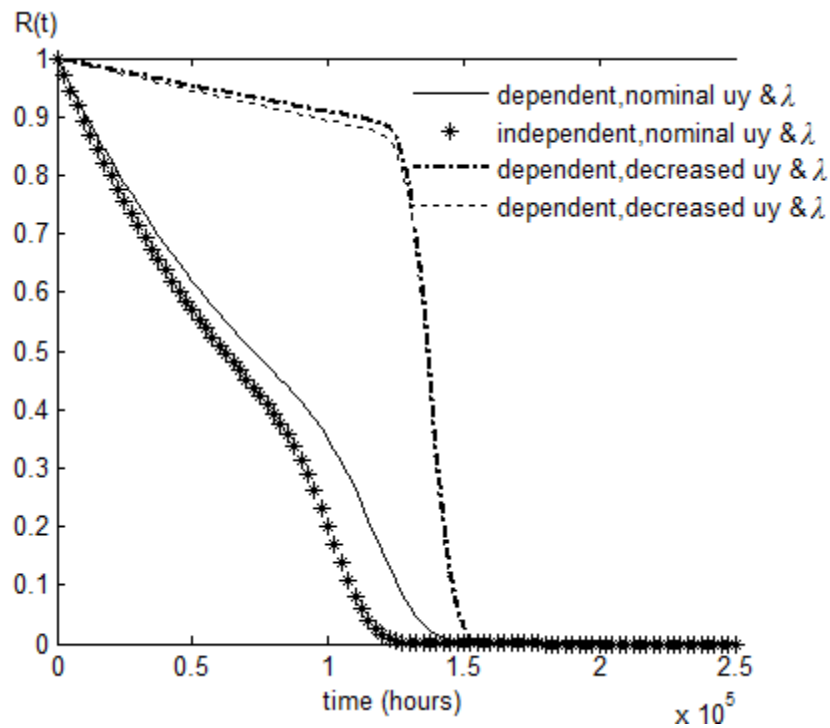
Figure 3.6: Sensitivity analysis of λ and μ_{Yi} on $R(t)$

Also, R(dependent)/R(independent) increases as μ_{Yi} increases, but not linearly. When

the shock damage size contribution to soft failure processes increases, system reliability is affected more intensely by each shock.
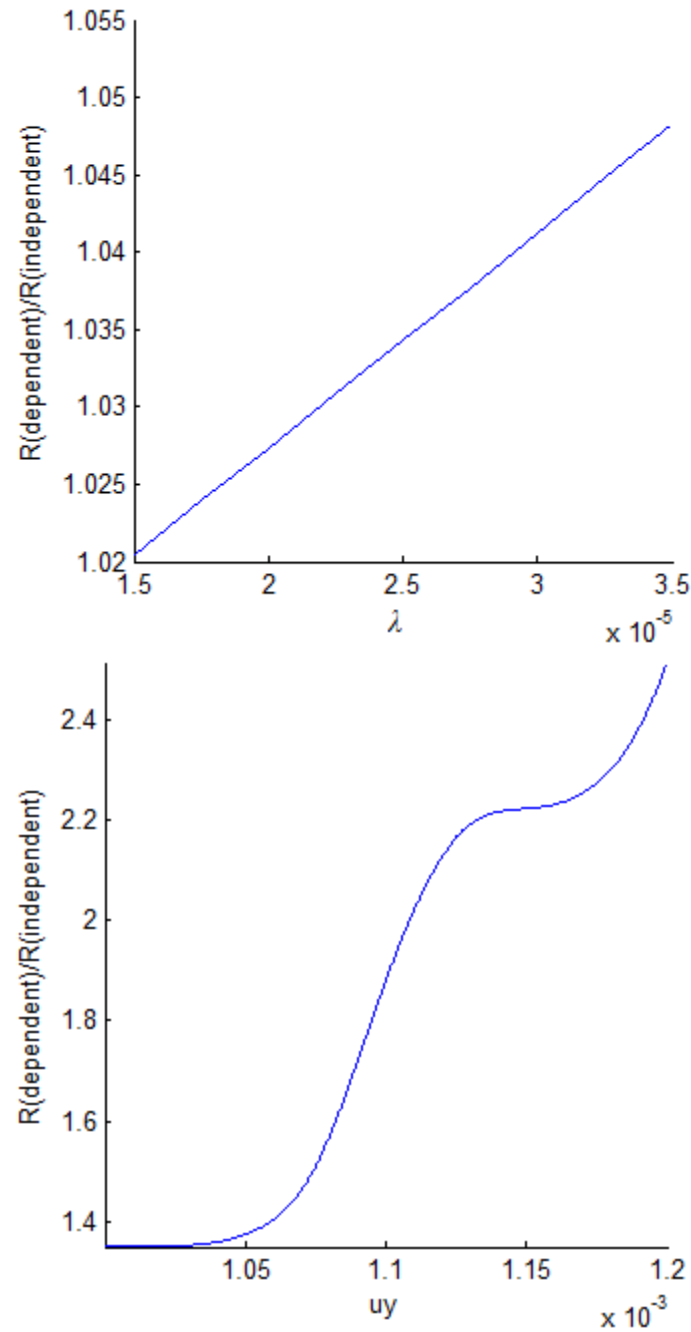
Figure 3.7: Ratio of $R(t)$ with dependent / independent failure processes versus λ and μ_{Y_i}

For a multi-component system, the number of shocks $N(t)$ has an effect on each component. When $N(t)$ is sufficiently large, the sum of the shock damage size contributing

to soft failure for each component is large, and there are also greater opportunities for hard failure; thus, a failure is more likely for all components. Alternatively, when there are relatively few shocks, times to failure are relatively longer for all components. Thus, the component failure processes are probabilistically dependent.

From the proof in Appendix, it can be concluded that component failure processes are dependent. If the covariance of two events is greater than zero, then the occurrences of these two events are correlated, and they are probabilistically dependent (but not necessarily physically dependent). With this idea, it is shown that soft failure processes for all components are dependent in the Appendix. Similar covariance derivations were performed for hard failure events. Because both the soft failure process and hard failure process for all components are probabilistically dependent, it can be logically concluded that component survival events are also dependent.

3.1.1.2 Gamma process for degradation

A gamma process is a random process with independent gamma distributed increments:

$X_{si}(t_2) - X_{si}(t_1) \sim \Gamma(v_i(t_2) - v_i(t_1), \theta_i)$. The difference between the degradation of component at two different times follows a gamma distribution with shape parameter $v_i(t_2) - v_i(t_1)$ and scale parameter θ_i.

Based on Eq. (8), using the gamma process to model component degradation path**:**

$$R(t) = \sum_{m=0}^{\infty} P \left\{ \left[W_{11} < D_1, W_{12} < D_1, ..., W_{1N(t)} < D_1, X_1(t, v_1(t), \theta_1) + \sum_{j=1}^{N(t)} Y_{1j} < H_1 \right] \cap ... \right. $$

$$\left[W_{21} < D_2, W_{22} < D_2, ..., W_{2N(t)} < D_2, X_2(t, v_2(t), \theta_2) + \sum_{j=1}^{N(t)} Y_{2j} < H_2 \right] \cap ...$$

$$\left. \left[W_{n1} < D_n, W_{n2} < D_n, ..., W_{nN(t)} < D_n, X_n(t, v_n(t), \theta_n) + \sum_{j=1}^{N(t)} Y_{nj} < H_n \right] \mid N(t) = m \right\}$$

$$\times P(N(t) = m)$$

(14)

$v_i(t)$ is a non-decreasing, right-continuous function for $t > 0$. Separating the hard failure process and soft failure processes, Eq. (14) can be re-written:

$$R(t) = \sum_{m=0}^{\infty} P\left(\left[W_{11} < D_1, ..., W_{1m} < D_1 \right] \cap \left[W_{21} < D_2, ..., W_{2m} < D_2 \right] \cap ... \left[W_{n1} < D_n, ..., W_{nm} < D_n \right] \right)$$

$$P\left(X_1(t, v_1(t), \theta_1) + \sum_{j=1}^{m} Y_{1j} < H_1, ..., X_n(t, v_n(t), \theta_n) + \sum_{j=1}^{m} Y_{nj} < H_n \right) P(N(t) = m)$$

(15)

An an example, by assuming W_{ij} to be *i.i.d.* random variables, and considering that soft failure processes are independent after conditioning:

$$R(t) = \sum_{m=0}^{\infty} \prod_{i=1}^{n} P\left(W_{ij} < D_i \right)^m P\left(X_i(t, v_i(t), \theta_i) + \sum_{j=1}^{m} Y_{ij} < H_i \right) \frac{e^{-\lambda t} (\lambda t)^m}{m!} \qquad (16)$$

Assuming shock damage to soft failure process Y_{ij} to be *i.i.d.* random variables with a normal distribution, $Y_{ij} \sim N(\mu_{Y_i}, \sigma_{Y_i}^2)$, and using convolution integral for the the sum of Y_{ij}:

$$R(t) = \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n} P\left(W_{ij} < D_i \right)^m \int_0^{H_i} P\left(X_i(t, v_i(t), \theta_i) + \sum_{j=1}^{m} Y_{ij} < H_i \mid \sum_{j=1}^{m} Y_{ij} = q \right) f_{Y_i}^{<m>}(q) dq \right] \frac{e^{-\lambda t} (\lambda t)^m}{m!}$$

(17)

$$R(t) = \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n} P\left(W_{ij} < D_i\right)^m \int_0^{H_i} P\left(X_i\left(t, v_i\left(t\right), \theta_i\right) < H_i - q\right) f_{Y_i}^{\langle m \rangle}(q) dq \right] \frac{e^{-\lambda t} \left(\lambda t\right)^m}{m!} \qquad (18)$$

The gamma process is used for degradation path $X_i(t)$:

$$R(t) = \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n} P\left(W_{ij} < D_i\right)^m \int_0^{H_i} Ga\left(X_i \mid v_i\left(t\right), \theta_i\right) f_{Y_i}^{\langle m \rangle}(q) dq \right] \frac{e^{-\lambda t} \left(\lambda t\right)^m}{m!} \qquad (19)$$

$$R(t) = \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n} P\left(W_{ij} < D_i\right)^m \int_0^{H_i} \left(1 - \frac{\Gamma\left(v_i(t), (H_i - q)\theta_i\right)}{\Gamma\left(v_i(t)\right)} \right) f_{Y_i}^{\langle m \rangle}(q) dq \right] \frac{e^{-\lambda t} \left(\lambda t\right)^m}{m!} \qquad (20)$$

Where $\Gamma\left(a, x\right) = \int_{z=x}^{\infty} z^{a-1} e^{-z} dz$ and $\Gamma\left(a\right) = \int_{z=0}^{\infty} z^{a-1} e^{-z} dz$, and $v_i(t)$ is a non-decreasing,

right-continuous function for $t > 0$. Eq. (19) is the general reliability model for system with

component stochastic degradation processes. Empirical studies show the deterioration at

time t is often proportional to a power law, which means $v_i(t) = c_i t^b$. (1) degradation of

concrete due to corrosion $b_i = 1$, and it is linear; (2) sulphate attack (parabolic, $b_i = 2$); (3)

diffusion-controlled aging (square root, $b_i = 0.5$), and other cases [108].

Considering $v_i(t) = c_i t^{bi}$, the equation becomes:

$$R(t) = \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n} P\left(W_{ij} < D_i\right)^m \int_0^{H_i} \left(1 - \frac{\Gamma\left(c_i t^b, (H_i - q)\theta_i\right)}{\Gamma\left(c_i t^b\right)} \right) f_{Y_i}^{\langle m \rangle}(q) dq \right] \frac{e^{-\lambda t} \left(\lambda t\right)^m}{m!} \qquad (21)$$

$$R(t) = \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n} P\left(W_{ij} < D_i\right)^m \int_0^{H_i} \left(1 - \frac{\int_{(H_i - q)\theta_i}^{\infty} z^{c_i t^b - 1} e^{-z} dz}{\int_0^{\infty} z^{c_i t^b - 1} e^{-z} dz} \right) f_{Y_i}^{\langle m \rangle}(q) dq \right] \frac{e^{-\lambda t} \left(\lambda t\right)^m}{m!} \qquad (22)$$

$$R(t) = \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n} P\left(W_{ij} < D_i\right)^m \int_0^{H_i} \left(\frac{\int_0^{(H_i - q)\theta_i} z^{c_i t^b - 1} e^{-z} dz}{\int_0^{\infty} z^{c_i t^b - 1} e^{-z} dz} \right) f_{Y_i}^{\langle m \rangle}(q) dq \right] \frac{e^{-\lambda t} \left(\lambda t\right)^m}{m!} \qquad (23)$$

A special case when $N(t)=0$ by time t, which means pure degradation, i.e., no shock arrives to the system, and with $b_i = 0.5$:

$$R(t) = \prod_{i=1}^{n} \left(\frac{\int_0^{H_i \vartheta_i} z^{c_i t^b -1} e^{-z} dz}{\int_0^\infty z^{c_i t^b -1} e^{-z} dz} \right) \tag{24}$$

Using the parameters in Table 3.1 for a gamma process, according to Eq. (22), system reliability $R(t)$ is plotted in Figure 3.8.
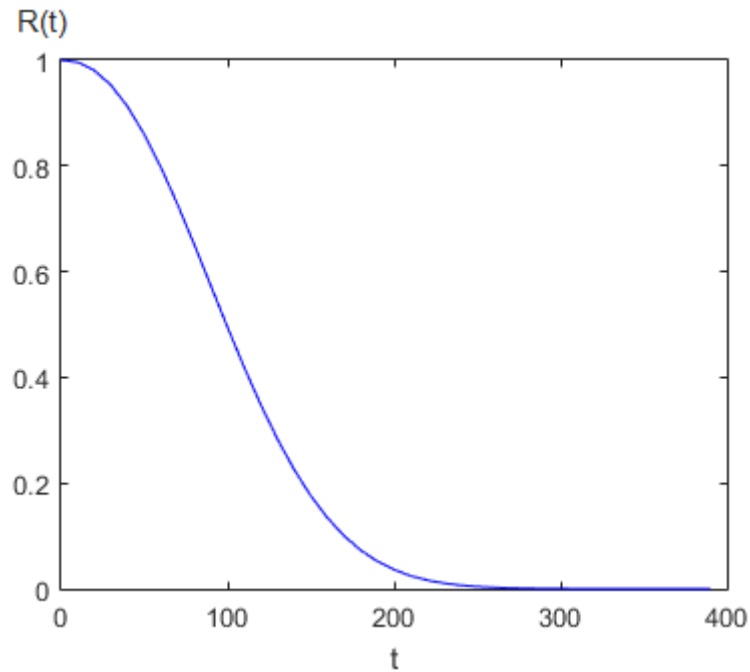
Figure 3.8: Plot of reliability function $R(t)$ of series system using Gamma process to model component degradation

3.1.1.2 *k*-out-of-*n* system

The reliability of *k*-out-of-*n* systems has been studied for many years. Adding redundancy is a traditional way to achieve reliability improvement. The components can achieve reliability of a working system by forming a partnership. In this part, a reliability model is developed for a *k*-out-of-*n* system, in which all components have the same

properties and experience two dependent/correlated failure processes and failure times of components are dependent and statistically correlated [109]. An age replacement policy is considered with a fixed replacement interval. We demonstrate the developed reliability model and maintenance policy for a *k*-out-of-*n* system subject to MDCFP using a representative example [109].

There has been significant and meaningful prior research done on *k*-out-of-*n* system reliability. Huang et al [110] presented a general multi-state *k*-out-of-*n* system model in his research.

For the *k*-out-of-*n* model, each component may fail due to two competing dependent failure modes, and the component fails when either of the two competing failure modes occurs. For a *k*-out-of-*n* system with all components having the same characteristics, the system fails when more than (*n*-*k*) components fail. Here, *S* is defined as the set of system components and $\phi(t)$ as a set of working components at time *t*, $\phi(t) \subset S$. In this model, all components are the same type, and the *i* subscript can be omitted from W_{ij}, Y_{ij}, etc.

$$\phi(t) = \left\{ \text{index of components satisfying} : W_1 < D, W_2 < D, ..., W_{N(t)} < D, X(t) + \sum_{j=1}^{N(t)} Y_j < H \right\}$$

$$= \left\{ \text{index of components satisfying} : \bigcap_{j=1}^{N(t)} (W_j < D), X(t) + \sum_{j=1}^{N(t)} Y_j < H \right\}$$

(25)

Figure 3.9 shows a *k*-out-of-*n* system. The system functions when at least *k* of those *n* components are available, i.e., have not failed. The reliability of this system at time *t* is the probability that at least *k* components survive each of the *N*(*t*) shock loads ($W_j < D$ for *j*=1, 2, …) and the total degradation of component *i* is less than the threshold level ($X_{Si}(t) < H_i$) .
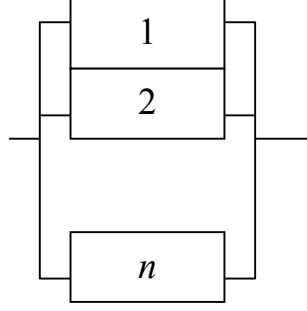
Figure 3.9: *k*-out-of-*n* system arranged in parallel.

System reliability can be derived based on the probability of the intersection of the events that the component degradation paths are within their safe regions, i.e., below the failure thresholds. Each component survives when the total wear volume is less than the degradation threshold and it survives from each shock [109].

$$
R(t) = P\big(|\phi(t)| \geq k\big)
$$

$$
R(t) = \sum_{m=0}^{\infty} P\big(|\phi(t)| \geq k \,|\, N(t) = m\big) P\big(N(t) = m\big)
$$

(26)

In Eq. (26), $|\phi(t)|$ represents the size or cardinality of set $\phi(t)$. In this model, shocks arriving at random time intervals are modeled as a Poisson process. When a shock impacts on the system at rate λ, all components are affected. Based on the random effect model, if the component survival probabilities are conditioned on the number of shocks, reliability is the sum of binomial probability mass functions [109]:

$$
R(t) = \sum_{m=0}^{\infty} \sum_{i=k}^{n} \binom{n}{i} P\left(\bigcap_{j=1}^{N(t)} (W_j < D), X(t) + \sum_{j=1}^{N(t)} Y_j < H \,|\, N(t) = m \right)^{i}
$$

$$
\times \left(1 - P\left(\bigcap_{j=1}^{N(t)} (W_j < D), X(t) + \sum_{j=1}^{N(t)} Y_j < H \,|\, N(t) = m \right) \right)^{n-i} P\big(N(t) = m\big)
$$

(27)

Since the hard failure damage size for each component by each shock is *i.i.d* random variable, the system reliability function can be derived as follows [109]:

$$R(t) = \sum_{m=0}^{\infty} \sum_{i=k}^{n} \binom{n}{i} \left[P\left(W_j < D\right)^m P\left(X(t) + \sum_{j=1}^{m} Y_j < H \,\middle|\, N(t) = m \right) \right]^i$$

$$\times \left[1 - P\left(W_j < D\right)^m P\left(X(t) + \sum_{j=1}^{m} Y_j < H \,\middle|\, N(t) = m \right) \right]^{(n-i)} \frac{e^{-\lambda t}(\lambda t)^m}{m!} \tag{28}$$

The reliability function can be generally expressed as[109]:

$$R(t) = \sum_{i=k}^{n} \binom{n}{i} \left[P\left(X(t) < H\right) \right]^i \times \left[1 - P\left(X(t) < H\right) \right]^{(n-i)} e^{-\lambda t}$$

$$+ \sum_{m=1}^{\infty} \sum_{i=k}^{n} \binom{n}{i} \left[F_W(D)^m \int_0^H G(H-u) f_Y^{\langle m \rangle}(u)\,du \right]^i \times \left[1 - F_W(D)^m \int_0^H G(H-u) f_Y^{\langle m \rangle}(u)\,du \right]^{(n-i)}$$

$$\times \frac{e^{-\lambda t}(\lambda t)^m}{m!}$$

$$\tag{29}$$

The reliability function for the more specific case with random effect degradation

path and normally distributed W_j, Y_j, and β, can be expressed as [109]:

$$R(t) = \sum_{i=k}^{n} \binom{n}{i} \left[\Phi\left(\frac{H - \mu_\beta t - \varphi}{\sigma_\beta t} \right) \right]^i \times \left[1 - \Phi\left(\frac{H - \mu_\beta t - \varphi}{\sigma_\beta t} \right) \right]^{(n-i)} e^{-\lambda t} +$$

$$\sum_{m=1}^{\infty} \sum_{i=k}^{n} \binom{n}{i} \left[F_W(D)^m \Phi\left(\frac{H - (\mu_\beta t + \varphi + m\mu_Y)}{\sqrt{\sigma_\beta^2 t^2 + m\sigma_Y^2}} \right) \right]^i \times \left[1 - F_W(D)^m \Phi\left(\frac{H - (\mu_\beta t + \varphi + m\mu_Y)}{\sqrt{\sigma_\beta^2 t^2 + m\sigma_Y^2}} \right) \right]^{(n-i)}$$

$$\times \frac{e^{-\lambda t}(\lambda t)^m}{m!}$$

$$\tag{30}$$

Using a gamma process to model the degradation, the reliability for *k*-out-of-*n*

system can be generally expressed:

$$R(t) = \sum_{i=k}^{n} \binom{n}{i} \left(\frac{\int_0^{H\theta} z^{ct^b-1}e^{-z}dz}{\int_0^{\infty} z^{ct^b-1}e^{-z}dz} \right)^i \times \left(1 - \frac{\int_0^{H\theta} z^{ct^b-1}e^{-z}dz}{\int_0^{\infty} z^{ct^b-1}e^{-z}dz} \right)^{(n-i)} e^{-\lambda t}$$

$$+ \sum_{m=1}^{\infty} \sum_{i=k}^{n} \binom{n}{i} \left[F_W(D)^m \int_0^H \left(\frac{\int_0^{(H-u)\theta} z^{ct^b-1}e^{-z}dz}{\int_0^{\infty} z^{ct^b-1}e^{-z}dz} \right) f_Y^{\langle m \rangle}(u)du \right]^i \left[1 - F_W(D)^m \int_0^H \left(\frac{\int_0^{(H-u)\theta} z^{ct^b-1}e^{-z}dz}{\int_0^{\infty} z^{ct^b-1}e^{-z}dz} \right) f_Y^{\langle m \rangle}(u)du \right]^{(n-i)}$$

$$\times \frac{e^{-\lambda t}(\lambda t)^m}{m!}$$

$$(31)$$

Table 3.2: Parameter values for *k*-out-of-*n* system reliability analysis

Parameters	All four components	Sources
H	0.00125 μm³	Tanner and Dugger [107]
D	1.5 Gpa (for polysilicon material)	Tanner and Dugger [107]
Φ	0	Tanner and Dugger [107]
B	$\beta \sim N(\mu_\beta, \sigma_\beta^2)$, μ_β=8.4823×10⁻⁹ μm³ σ_β=6.0016×10⁻¹⁰ μm³	Tanner and Dugger [107] Peng et al [57]
Λ	2.5×10⁻⁵	Assumption
Y_j	$Y_j \sim N(\mu_Y, \sigma_Y^2)$, μ_Y=1×10⁻⁴ μm³ σ_Y=2×10⁻⁵ μm³	Assumption
W_j	$W_j \sim N(\mu_W, \sigma_W^2)$, μ_W=1.2 GPa σ_W=0.2 GPa	Assumption

A system reliability model for *k*-out-of-*n* system is developed based on a random effect degradation model. Here is a numerical example for a 2-out-of-4 system with four of the same components. The parameters for reliability analysis are provided in Table 3.2. Based on Eq. (30), the reliability function $R(t)$ and the pdf of failure time $f_T(t)$ are plotted in Figure 3.10, where $f_T(t) = \dfrac{d}{dt} F_T(t)$.

A sensitivity analysis was performed to assess the effects of *k* on $R(t)$ and $f_T(t)$. The results are shown in Figure 3.11. Given a fixed *n*, by changing *k*, both the reliability and the time to failure distribution are sensitive to the number of working components, *k*. When *k*

increases from 1 to 4, $R(t)$ shifts left. When $k=1$, it is a parallel system, and when $k=4$, it is a series system.
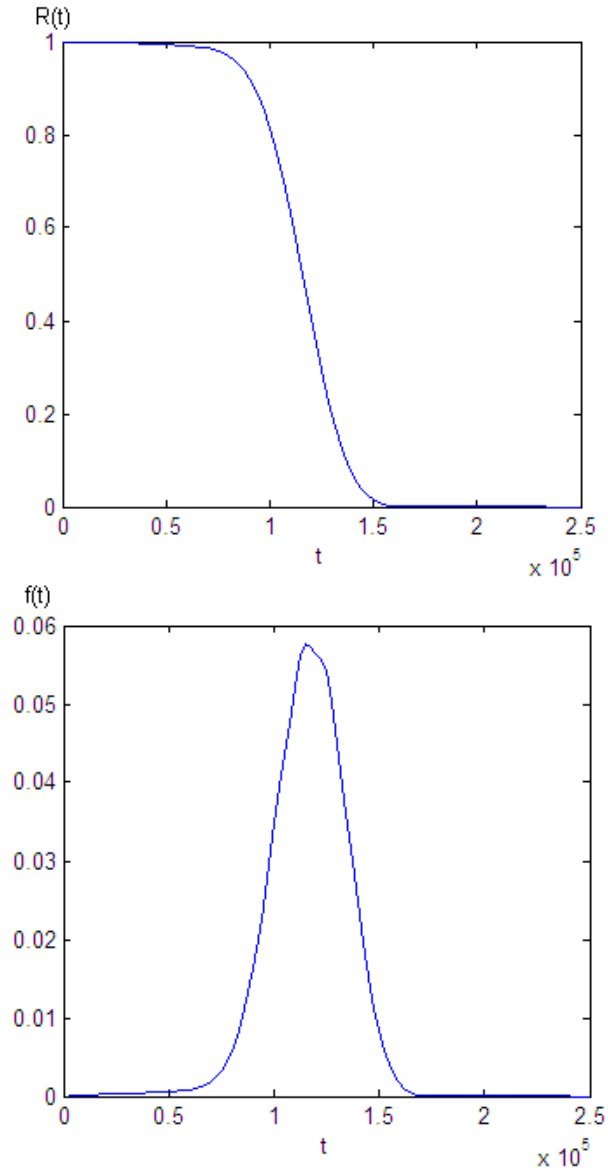
Figure 3.10: Plots of reliability function $R(t)$ and failure time distribution $f_T(t)$

As expected, Figure 3.10 indicates that when k is decreasing, the working requirements of the system are less demanding. When $k=1$, it means at least one component must be functioning, and the system is operational. The reliability of this system is higher than the reliability of systems when other k values are chosen. In Figure 3.11, before time 1.2×10^5, $f_T(t)$ increases for all cases and $f_T(t)$ for $k=3$ is higher than $f_T(t)$ for other k values except

k=4. Also, after time t=1.2×10^5, $f_T(t)$ decreases for all cases and $f_T(t)$ for k=1 is higher than

$f_T(t)$ for other k values. System reliability changes quickly around t=1.2×10^5.
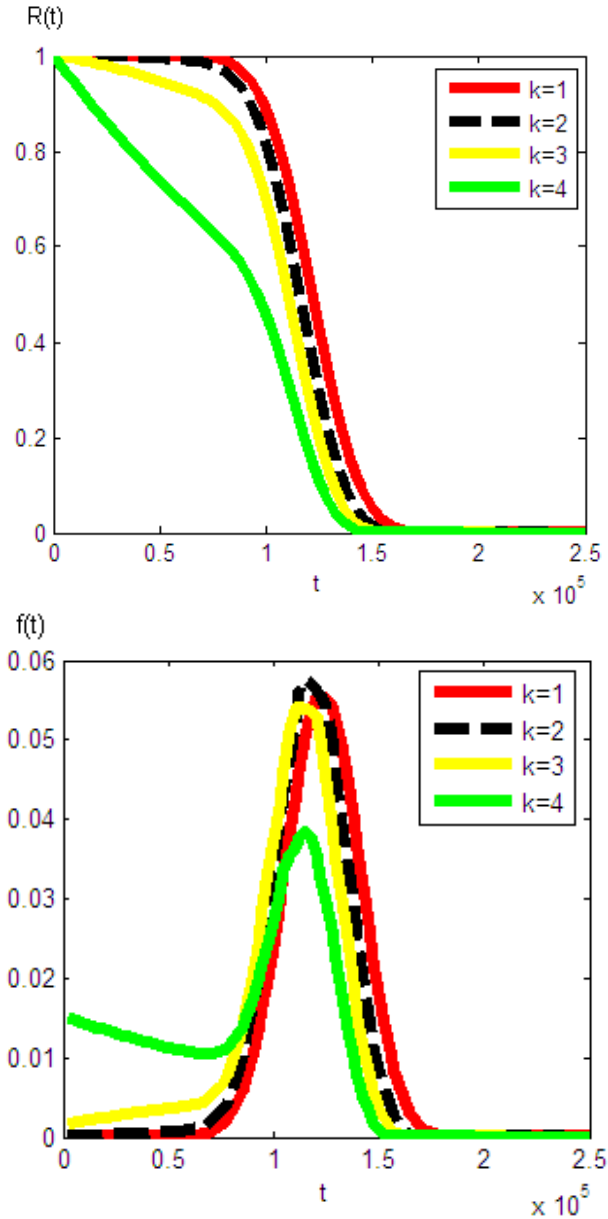
Figure 3.11: Sensitivity analysis of $R(t)$ and $f_T(t)$ on k

3.1.1.3 Parallel systems

Figure 3.12 presents a parallel system composed of n components. The reliability of

the parallel system at time t is the probability that at least one component of this system

survives each of the $N(t)$ shock loads ($W_{ij}<D_i$ for j=1, 2, …), and the total degradation of

that same component is less than the threshold level ($X_{S_i}(t){<}H_i$). The system fails when all components experience either soft failure or hard failure.
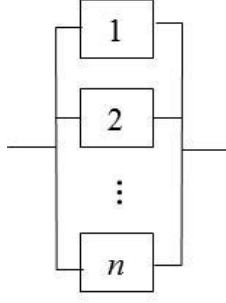
Figure 3.12: Parallel system example

System reliability for a parallel system based on random effect degradation model is given by the following equations for the two specific cases introduced in the previous section [105].

$$R(t) = 1 - \sum_{m=0}^{\infty} \prod_{i=1}^{n} \left[1 - F_{W_i}(D_i)^m \, \Phi \left(\frac{H_i - (\mu_{\beta_i} t + \varphi_i + m\mu_{Y_i})}{\sqrt{\sigma_{\beta_i}^2 t^2 + m\sigma_{Y_i}^2}} \right) \right] \frac{e^{-\lambda t} (\lambda t)^m}{m!}. \qquad (32)$$

Given exponentially distributed W_{ij} (with parameter γ), exponentially distributed Y_{ij} (with parameter θ), and normally distributed β_i, parallel system reliability be expressed as [105]

$$R(t) = 1 - \sum_{m=0}^{\infty} \prod_{i=1}^{n} \left[1 - \left[1 - e^{-\gamma D_i} \right]^m \int_0^{H_i} \Phi \left(\frac{H_i - (\mu_{\beta_i} t + \varphi_i + u)}{\sigma_{\beta_i} t} \right) \frac{u^{m-1} e^{-\frac{u}{\theta}}}{\theta^m \Gamma(m)} du \right] \frac{e^{-\lambda t} (\lambda t)^m}{m!}.$$

$$(33)$$

Alternatively, system reliability for a parallel system for gamma process degradation model is given by the following equations.

$$R(t) = 1 - \sum_{m=0}^{\infty} \prod_{i=1}^{n} \left[1 - P\left(W_{ij} < D_i\right)^m \int_0^{H_i} Ga\left(X_i \mid v_i(t), \theta_i \right) f_{Y_i}^{\langle m \rangle}(q) dq \right] \frac{e^{-\lambda t} (\lambda t)^m}{m!}. \qquad (34)$$

$$R(t) = 1 - \sum_{m=0}^{\infty} \prod_{i=1}^{n} \left[1 - P\left(W_{ij} < D_i\right)^m \int_0^{H_i} \left(1 - \frac{\int_0^{(H_i-q)\theta_i} z^{c_i t^b - 1} e^{-z} dz}{\int_0^{\infty} z^{c_i t^b - 1} e^{-z} dz} \right) f_{Y_i}^{\langle m \rangle}(q) dq \right] \frac{e^{-\lambda t}(\lambda t)^m}{m!}. \qquad (35)$$

3.1.1.4 Series-parallel systems

Figure 3.13 depicts a series-parallel system made up of s subsystems. S_l is the set of components in subsystem l with no component being used in more than one system ($S_l \cap S_k = \phi$ for all l, k), and each subsystem has n_l components with $n_l = |S_l|$. For the example depicted in the figure, $S_1 = \{1, 4, 7\}$, and $n_1 = 3$; $S_l = \{5, 6, 8, 12\}$, and $n_l = 4$; $S_k = \{2, 10, 11\}$, and $n_k = 3$; and $S_s = \{3, 9\}$, and $n_s = 2$.
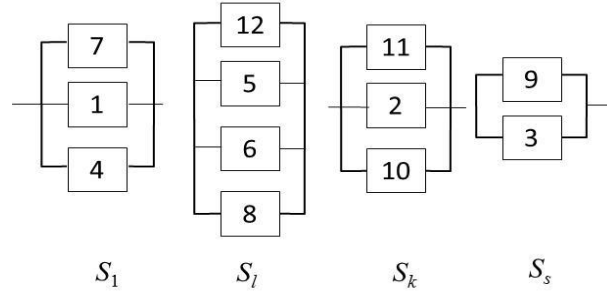
Figure 3.13: Series-parallel system example

The reliability of a series-parallel system at time t is the probability that at least one component within each subsystem survives each of the $N(t)$ shock loads ($W_{ij} < D_i$ for $j=1$, 2, ...), and the total degradation is less than the threshold level ($X_{S_i}(t) < H_i$) for that same component. The system fails when all components for at least one parallel subsystem experience either soft failure or hard failure.

System reliability for a series-parallel system is given by the following equations for general and exponential cases previously introduced [105]:

$$R(t) = \sum_{m=0}^{\infty} \prod_{l=1}^{s} \left(1 - \prod_{i \in S_l} \left[1 - F_{W_i}(D_i)^m \int_0^{H_i} G_i(H_i - u) f_{Y_i}^{\langle m \rangle}(u) du \right] \right) \frac{e^{-\lambda t}(\lambda t)^m}{m!} \qquad (36)$$

$$R(t) = \sum_{m=0}^{\infty} \prod_{l=1}^{s} \left(1 - \prod_{i \in S_l} \left[1 - \left[1 - e^{-\gamma D_i} \right]^m \int_0^{H_i} \Phi\left(\frac{H_i - (\mu_{\beta_i} t + \varphi_i + u)}{\sigma_{\beta_i} t} \right) \frac{u^{m-1} e^{-\frac{u}{\theta}}}{\theta^m \Gamma(m)} du \right] \right) \frac{e^{-\lambda t} (\lambda t)^m}{m!}$$

(37)

System reliability for a series-parallel system using the gamma process to model degradation is given by the following equations for the two specific cases previously introduced.

$$R(t) = \sum_{m=0}^{\infty} \prod_{l=1}^{s} \left(1 - \prod_{i \in S_l} \left[1 - F_{W_i}(D_i)^m \int_0^{H_i} \left(\frac{\int_0^{(H_i-u)\theta_i} z^{c_i t^b - 1} e^{-z} dz}{\int_0^{\infty} z^{c_i t^b - 1} e^{-z} dz} \right) f_{Y_i}^{\langle m \rangle}(u) du \right] \right) \frac{e^{-\lambda t} (\lambda t)^m}{m!}$$

(38)

$$R(t) = \sum_{m=0}^{\infty} \prod_{l=1}^{s} \left(1 - \prod_{i \in S_l} \left[1 - \left[1 - e^{-\gamma D_i} \right]^m \int_0^{H_i} \left(\frac{\int_0^{(H_i-u)\theta_i} z^{c_i t^b - 1} e^{-z} dz}{\int_0^{\infty} z^{c_i t^b - 1} e^{-z} dz} \right) \frac{u^{m-1} e^{-\frac{u}{\theta}}}{\theta^m \Gamma(m)} du \right] \right) \frac{e^{-\lambda t} (\lambda t)^m}{m!}$$ (39)

Overall, in this section multi-component system reliability with different system structure is studied, and the failure processes are considered dependent among components. This represents a fundamental extension and advancement because previous research and models were either limited to an individual component or it was assumed that the failure processes were independent. In Appendix A, the proof is provided to show that failure processes among components are dependent.

3.1.2 System reliability with distinct component shock sets

In this section, reliability is studied for multi-component systems subject to dependent competing risks of degradation wear and random shocks, with distinct shock sets [111, 112]. In practice, many systems are exposed to distinct and different types of shocks that

can be categorized according to their sizes, frequency, function, affected components or acting positions. In this new model, random shocks are classified into different sets. Shocks with specific sizes, frequency or function can selectively affect one or more components in the system but not necessarily all components. Additionally the shocks from the different shock sets can arrive at different rates and have different relative magnitudes. A MEMS (Micro-electromechanical systems) oscillator is a typical system subject to dependent and competing failure processes, and it is used in a numerical example to illustrate new reliability models.

Many systems or components are subject to competing risks of degradation processes and random shocks. Some researchers assume that when a shock comes, it affects all components in the system. However, shocks with specific sizes or function may selectively affect one or more components in the system, but not necessarily all components. Therefore, it is practical and realistic for some applications to classify random shocks into different sets based on their sizes and function, and the affected components.

Consider a battery used in a laptop computer that supplies electric power by a chemical reaction. It gradually weakens through usage, and becomes ineffective once the chemicals in the battery are exhausted, which can be considered as a soft failure process. When shocks arrive to the system, they also cause incremental damage to this degradation. In addition, environmental shocks, overheating or over-voltage can cause abrupt battery failure, which can be considered as a hard failure process. These two failure processes are competing, i.e., no matter which failure process happens first, the battery fails. Alternatively, striking the keyboard is a different type of shock which probably has no effect on battery life, but may impact other components of the laptop. There is significant interest in categorizing each

component's own shock set for the reliability modeling of a system subject to dependent and competing failure processes.

Scant research has been done on system reliability with specific and distinct component shock sets. The idea of component shock sets originates from many engineering applications. Considering a complex system of an automobile, shocks affecting a car can be categorized according to their attributes: mechanical shocks, thermal shocks, voltage shocks and other types of shocks. Within the category of mechanical shocks, different types of failures can be caused due to the sizes, function, affected components, e.g., fracture of steering and brake system, disconnection of fuel system, rupture of engine cooling fan blade or tire puncture.

Mathematical models are derived for system reliability for the distinct shock set problem. A non-linear maintenance optimization model is formulated and solved based on an iterative numerical search, i.e., golden section search method. Each component has its distinct shock set, meaning a set of shocks that impact the particular component. If two or more components share a common shock type in their shock sets, the times-to-failure of these components are dependent. This potentially causes components sharing the same type of shocks to fail more often as well.

We define for component l, the soft failure threshold is H_l, and the hard failure threshold is D_l $W_{i,j,k}$ is shock effect on hard failure processes caused by the kth shock of jth type belonging to component l shock set, and $Y_{i,j,k}$ is shock effect on soft failure processes caused by the kth shock of jth type belonging to component l shock set.

Figure 3.14 shows two failure processes for component l. This depicts just one of n components within the system. These two failure processes are dependent and competing.

Two life cycles are shown in the figure. In the first life cycle, component l fails due to soft failure, because component degradation exceeds the soft failure threshold level H_l. A random shock belonging to the component's shock set can cause hard failures when the hard failure threshold level D_l is exceeded. Therefore, in the second life cycle, component l fails due to hard failure, because damage to hard failure process $W_{l,2,3}$ (the third shock of second type belonging to component l shock set) exceeds the hard failure threshold D_l. Soft failure threshold H_l, and hard failure threshold D_l for component l are different fixed values. The component fails when either of the two failure processes occurs, i.e., they are competing.

Each component has its own shock set, defined as the set of those shock types that affect that component. Taking component l for example, Figure 3.14 shows that there are two different types of shocks that affect it: two shocks from Type 1 and three shocks from Type 2, and the third Type 2 shock causes hard failure. Although there are other types of shocks impacting the system during this time period, they have no effect on component l, due to the size, function or other aspects. The shock set for component l is denoted as $\phi_l = \{1,2\}$. In general, hard failure occurs when any shock k of the jth type associated with component l exceeds the threshold D_l.

All types of shocks having an effect on component l compose its shock set. When one type of shock affects multiple components, this shock type is represented in multiple shock sets according to the components affected. Failure times of components are dependent due to the shared common shock types in their shock sets [62].

The two failure processes have different failure thresholds (H_l is the failure threshold for the soft failure process and D_l is the failure threshold for the hard failure process). If

there are more than one shock type in shock set ϕ_l ($|\phi_l| > 1$), there could be separate hard failure thresholds for each relevant shock type (D_{lj} instead of D_l), but each relevant shock type for a particular component is assumed to be associated with the same hard failure threshold ($D_{lj} = D_l$ for all j).
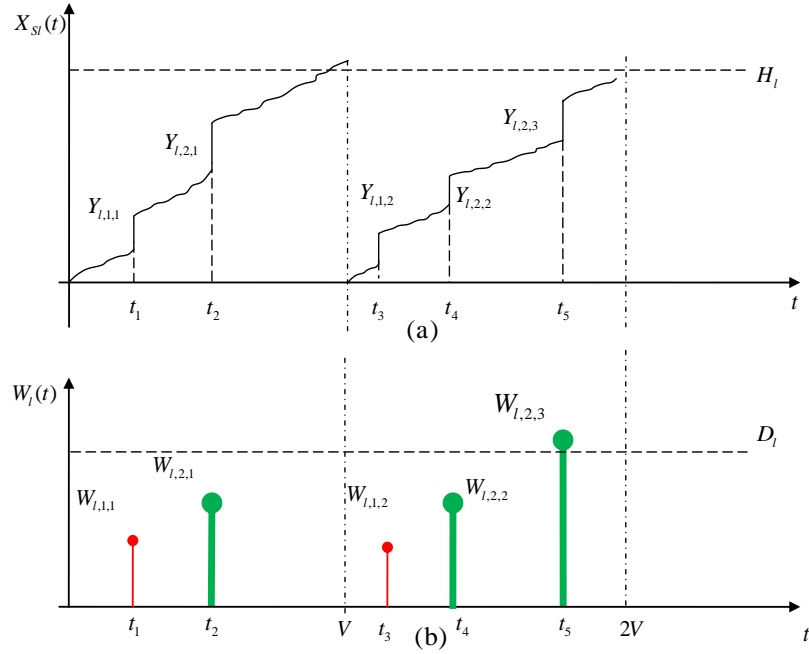
Figure 3.14: Dependent competing failure processes for component l with two types of shocks: (a) soft failure process and (b) hard failure process

The probabilistic model for the hard failure process is from Peng et al. [57]. The probability that component l in a n component system survives the kth shock of the jth type associated with this component is [111, 112]:

$$P(W_{l,j,k} < D_l) = F_{l,j,W}(D_l), \text{ for } l = 1, 2, \ldots, n, \ j \in \phi_l, \text{ and } k = 1, 2, \ldots N_j(t). \tag{40}$$

As an example, consider that $W_{l,j,k}$ are random variables following a normal distribution,

$W_{l,j,k} \sim N(\mu_{W_{l,j}}, \sigma^2_{W_{l,j}})$, then the probability of component l survival of a shock of the jth type becomes:

$$P(W_{l,j,k} < D_l) = F_{l,j,W}(D_l) = \Phi\left(\frac{D_l - \mu_{W_{l,j}}}{\sigma_{W_{l,j}}}\right) \text{ for } l = 1, 2, \ldots, n, \quad j \in \phi_l \tag{41}$$

where $\Phi(\bullet)$ is the cdf of a standard normally distributed random variable. Notice that $W_{l,j,k}$ can be negative given the normal distribution, although in practice it must be non-negative. Therefore, it is important to choose appropriate values for the mean and variance such that the probability that $W_{l,j,k}$ can be a negative value is almost 0.

Soft failure is the combined effect of pure degradation and degradation damage increment caused by shocks. This is the second of the competing failure processes. Soft failures of component l occurs when the total degradation is greater than H_l. Total degradation $X_{S_l}(t)$ is the sum of the degradation due to continual wear and the instantaneous damage shifts caused by all types of shocks that can have an effect on this component, i.e., all the components included in component l shock set. Cumulative damage due to random shocks until time t, $S_l(t)$, is given as [111, 112]:

$$S_l(t) = \begin{cases} \sum_{j \in \phi_l} \sum_{k=1}^{N_j(t)} Y_{l,j,k} & \text{otherwise} \\ 0 & \text{if } N_j(t) = 0, \ \forall j \end{cases} \tag{42}$$

$N_j(t)$ is the number of shocks of the jth type that have arrived by time t. $N_j(t)$ is a random variable so there is a random number of terms in the sum. The overall degradation of the lth component is $X_{S_l}(t) = X_l(t) + S_l(t)$. This is an extension of Peng et al. [57]. Then, the probability that the total degradation at time t is less than x_l can be derived as

$$P(X_{S_l}(t) < x_l) = P(X_l(t) + \sum_{j \in \phi_l} \sum_{k=1}^{N_j(t)} Y_{l,j,k} < x_l). \tag{43}$$

Different types of shock processes arriving to the system are assumed to be independent of each other. After conditioning on the numbers of shocks for all types,

different types of shocks arriving to the system are summed over all p. If some types of shocks do not belong to shock set of component l, they are not included in the shock set, $j \notin \phi_l$, and the effect is 0. Summing over the types of shocks not belonging to component l's shock set does not change the probability [111, 112]:

$$P(X_{S_l}(t) < x_l) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_p=0}^{\infty} P(X_l(t) + \sum_{j \in \phi_l} \sum_{k=1}^{m_j} Y_{l,j,k} < x_l \mid N_1(t) = m_1, \cdots N_p(t) = m_p)$$
$$\times P(N_1(t) = m_1) P(N_2(t) = m_2) \cdots P(N_p(t) = m_p). \tag{44}$$

Consider $G(x_l, t)$ to be the cdf of $X_l(t)$ at t, $f_{Y_l}^{<c>}(u)$ to be the pdf of the sum of $cY_{l,j,k}$ variables, $f_{Y_l}^{<\mathbf{m}_{\phi l}>}(u)$ to be the pdf of the sum of $Y_{l,j,k}$ variables as defined by vector $\mathbf{m}_{\phi l}$, a vecter of all m_j for $j \in \phi_l$. The cdf of $X_{Sl}(t)$ in Eq. (44) can be derived using a convolution integral [111, 112]:

$$P(X_{S_l}(t) < x_l) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_p=0}^{\infty} \left(\int_0^{x_l} G_l(x_l - u, t) f_{Y_l}^{<\mathbf{m}_{\phi l}>}(u) du \right) P(N_1(t) = m_1) \cdots P(N_p(t) = m_p).$$

$$\tag{45}$$

A Poisson process is an appropriate and common model for the shock arrival rates given a stationary usage environment. If the frequency rate of shocks is increasing or decreasing, then this model is not appropriate. The ith type of shocks follows a Poisson process with rate λ_i. Then Eq. (45) can be rewritten as:

$$P(X_{S_l}(t) < x_l) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_p=0}^{\infty} \left(\int_0^{x_l} G_l(x_l - u, t) f_{Y_l}^{<\mathbf{m}_{\phi l}>}(u) du \right) \prod_{i=1}^{p} \frac{e^{-\lambda_i t}(\lambda_i t)^{m_i}}{m_i !}. \tag{46}$$

If damages to the degradation process from shocks of all types associated with component l are normally distributed random variables, $Y_{l,j,k} \sim N(\mu_{Y_{l,j}}, \sigma_{Y_{l,j}}^2)$, and the degradation path is linear with a constant initial value φ_l and a normally distributed

degradation rate β_l with $\beta_l \sim N(\mu_{\beta_l}, \sigma_{\beta_l}^2)$, then the probability that component l does not experience soft failure before time t is:

$$P(X_{S_l}(t) < H_l) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_p=0}^{\infty} \Phi\left(\frac{H_l - (\mu_{\beta_l}t + \varphi_l + \sum_{j \in \phi_l} m_j \mu_{Y_{l,j}})}{\sqrt{\sigma_{\beta_l}^2 t^2 + \sum_{p=1}^{k} m_j \sigma_{Y_{l,j}}^2}}\right) \prod_{i=1}^{p} \frac{e^{-\lambda_i t}(\lambda_i t)^{m_i}}{m_i!} \quad (47)$$

The example system configuration described in this section is a series system. However, the concepts described in this paper can be extended to other system configurations. For a series system with n components, the system reliability is the probability of an intersection of events in which the degradation path of each component is within its safe region, i.e., below the soft failure threshold, and each component survives all types of shocks in its shock set ($W_{l,j,k} < D_l$, $j \in \phi_l$) [111, 112]:

$$R(t) = P\left\{\left[\bigcap_{j \in \phi_1}(W_{1,j,1} < D_1, W_{1,j,2} < D_1, ..., W_{1,j,N_j(t)} < D_1) \cap (X_1(t) + \sum_{j \in \phi_1} \sum_{k=1}^{N_j(t)} Y_{1,j,k} < H_1)\right] \cap \right.$$

$$\left[\bigcap_{j \in \phi_2}(W_{2,j,1} < D_2, W_{2,j,2} < D_2, ..., W_{2,j,N_j(t)} < D_2) \cap (X_2(t) + \sum_{j \in \phi_2} \sum_{k=1}^{N_j(t)} Y_{2,j,k} < H_2)\right] \cap ...$$

$$\left.\left[\bigcap_{j \in \phi_n}(W_{n,j,1} < D_n, W_{n,j,2} < D_n, ..., W_{n,j,N_j(t)} < D_n) \cap (X_n(t) + \sum_{j \in \phi_n} \sum_{k=1}^{N_j(t)} Y_{n,j,k} < H_n)\right]\right\}$$

$$(48)$$

When a shock belonging to component l shock set arrives to the system, it has effect on both of two failure processes, and $Y_{l,j,k}$ and $W_{l,j,k}$ are assumed to be independent. When more than two components share a common type of shocks in their shock sets, reliabilities of these components are dependent. This phenomenon makes the problem more complicated, since times-to-failure are dependent for the components due to sharing the same type of shocks. The reliability for the system can be obtained by conditioning on the

numbers of shocks of all types [111, 112]:

$$R(t) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_p=0}^{\infty} R(t|N_1(t) = m_1, N_2(t) = m_2, \cdots N_p(t) = m_p)$$
$$\times P\left(N_1(t) = m_1\right) P\left(N_2(t) = m_2\right) \cdots P\left(N_p(t) = m_p\right) \tag{49}$$

There are p types of shocks that can affect the whole system, and $|\phi_l| \leq p$, which means

items in component l's shock set should be less than or equal to the total types of shocks

that impact the whole system. The ith type of shocks arrives according to a Poisson process

with rate λ_i, for $i = 1, 2, \ldots, p$. The reliability function for a series system can be written as

follows [111, 112]:

$$R(t) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_p=0}^{\infty} \left\{ \prod_{l=1}^{n} R_l(t|N_1(t) = m_1, \cdots N_p(t) = m_p) \right\} \prod_{i=1}^{p} \frac{e^{-\lambda_i t} (\lambda_i t)^{m_i}}{m_i!} \tag{50}$$

The reliability of each component at time t is the probability that each component

survives all shocks in its shock set, and the total degradation is less than the threshold level

[111, 112]:

$$R(t) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_p=0}^{\infty} \left\{ \prod_{l=1}^{n} P\left(\bigcap_{j \in \phi_l} (W_{l,j,1} < D_l, W_{l,j,2} < D_l, \ldots, W_{l,j,m_j} < D_l) \right.\right.$$
$$\left.\left. \bigcap (X_l(t) + \sum_{j \in \phi_l} \sum_{k=1}^{m_j} Y_{l,j,k} < H_l) \right) \right\} \prod_{i=1}^{p} \frac{e^{-\lambda_i t} (\lambda_i t)^{m_i}}{m_i!} \tag{51}$$

Component failure times are dependent due to shared exposure of shocks, and it is the

random number of shocks that is responsible for the dependency. After conditioning on the

number of shocks for each shock set, the hard and soft failure processes for a component

become independent. Based on Equations (2) and (5), the reliability function can be

expressed as [111, 112]

$$R(t) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_p=0}^{\infty} \prod_{l=1}^{n} \left(\prod_{j \in \phi_l} P\left(W_{l,j,k} < D_l\right)^{m_j} \right) P\left(X_l(t) + \sum_{j \in \phi_l} \sum_{k=1}^{m_j} Y_{l,j,k} < H_l \right) \prod_{i=1}^{p} \frac{e^{-\lambda_i t}(\lambda_i t)^{m_i}}{m_i!}$$

(52)

Using convolutional integral for soft failure process based on random effect degradation modeling [111, 112]:

$$R(t) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_p=0}^{\infty} \left\{ \prod_{l=1}^{n} \left(\prod_{j \in \phi_l} F_{l,j,W}\left(D_l\right)^{m_j} \right) \int_0^{H_l} G_l(H_l - u, t) f_{Y_l}^{<\mathbf{m}_{\phi_l}>}(u) du \right\} \prod_{i=1}^{p} \frac{e^{-\lambda_i t}(\lambda_i t)^{m_i}}{m_i!}$$

(53)

In practice, $f_{Y_l}^{<k>}(y)$ can be difficult to evaluate for general shock distributions, but there are simple and closed-form expressions for particular examples. Even in other cases, Monte Carlo simulation can be used to estimate $f_{Y_l}^{<k>}(y)$. For example, if the degradation path is linear and $W_{l,j,k}$, $Y_{l,j,k}$, and β_l are normally distributed and independent of each other, i.e., $W_{l,j,k} \sim N(\mu_{Wl,j}, \sigma_{Wl,j})$., $Y_{l,j,k} \sim N(\mu_{Yl,j}, \sigma_{Yl,j})$., $\beta_l \sim N(\mu_{\beta l}, \sigma_{\beta l})$, the reliability function can be expressed as [116, 117]:

$$R(t) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_p=0}^{\infty} \left\{ \prod_{l=1}^{n} \prod_{j \in \phi_l} \Phi\left(\frac{D_l - \mu_{W_{l,j}}}{\sigma_{W_{l,j}}} \right)^{m_j} \Phi\left(\frac{H_l - \left(\mu_{\beta_l} t + \varphi_l + \sum_{j \in \phi_l} m_j \mu_{Y_{l,j}} \right)}{\sqrt{\sigma_{\beta_l}^2 t^2 + \sum_{j \in \phi_l} m_j \sigma_{Y_{l,j}}^2}} \right) \right\} \prod_{i=1}^{p} \frac{e^{-\lambda_i t}(\lambda_i t)^{m_i}}{m_i!}$$

(54)

Alternatively, the gamma process could be applied to model component degradation. Based on Eq. (52), the reliability function can be expressed as:

$$R(t) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_p=0}^{\infty} \left\{ \prod_{l=1}^{n} \left(\prod_{j \in \phi_l} F_{l,j,W}\left(D_l\right)^{m_j} \right) \int_0^{H_l} Ga(H_l - u) f_{Y_l}^{<\mathbf{m}_{\phi_l}>}(u) du \right\} \prod_{i=1}^{p} \frac{e^{-\lambda_i t}(\lambda_i t)^{m_i}}{m_i!}$$ (55)

$$R(t) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_p=0}^{\infty} \left\{ \prod_{l=1}^{n} \left(\prod_{j \in \phi_l} F_{l,j,W}(D_l)^{m_j} \right) \int_0^{H_l} \left(\frac{\int_0^{(H_l-u)\theta_l} z^{c_l t^b - 1} e^{-z} dz}{\int_0^{\infty} z^{c_l t^b - 1} e^{-z} dz} \right) f_{Y_l}^{<\mathbf{m}_{\phi_l}>}(u) du \right\} \prod_{i=1}^{p} \frac{e^{-\lambda_i t}(\lambda_i t)^{m_i}}{m_i!}$$

(56)

Here is an application of a system subject to dependent and competing failure processes with components distinct shock sets. MEMS oscillators shown in Figure 3.15 are timing devices that generate highly stable reference frequencies to sequence electronic systems, manage data transfer, define radio frequencies and measure elapsed time. Electronic systems are typical systems that suffer both mechanical shocks and voltage shocks. MEMS oscillators are replacing quartz crystal oscillators due to several advantages [113]. MEMS oscillators are attached to electronic circuits, often called sustaining amplifiers, to drive them in continuous motion. In most cases these circuits are located near the oscillators and in the same physical package.
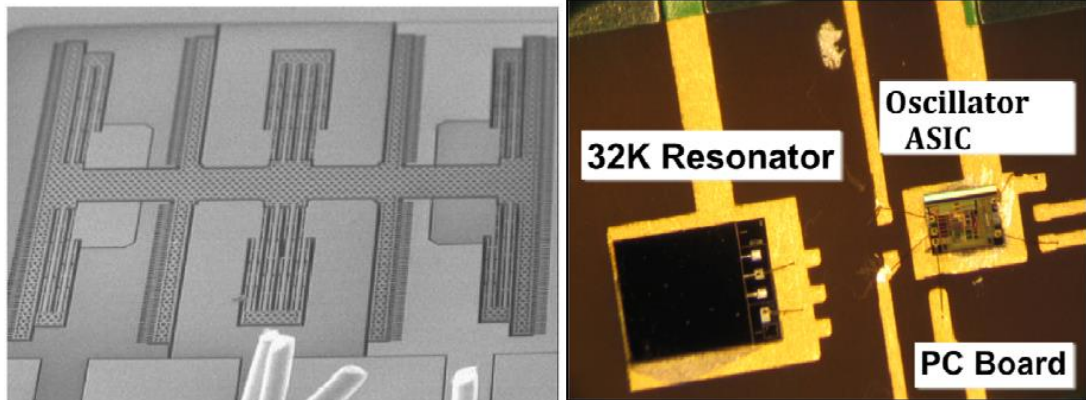

Figure 3.15: (a) 32.768 kHz comb-drive oscillator;
(b) wafer level packaged resonator with oscillator circuits on PCB [114]

To ensure the performance is satisfactory for important electronics applications, MEMS oscillators have to pass the standard Joint Electron Device Engineering Council

(JEDEC) reliability tests such as aging, solder reflow, thermal shock and autoclave.

MEMS oscillators vibrate at their natural resonant frequency. Due to the working loss of operation, MEMS oscillator mass decreases after a period of time. The decrease of mass can cause the frequency of vibration to increase, which is an obvious common phenomenon that exists in MEMS oscillators. The quality factor Q of an individual reactive component depends on the frequency ω at which it is evaluated, which is the oscillators' frequency of the circuit that it is used in. Q is a dimensionless parameter which increases when ω increases, and this conclusion can be easily obtained from Eq. (57), where ω is frequency in radians per second, L is the inductance, X_L is the inductive reactance, and R_L is the series resistance of the inductor. On the other hand, a system with a low quality factor is said to be over-damped, and such a system does not oscillate strongly [114].

$$Q_L = \frac{X_L}{R_L} = \frac{\omega L}{R_L} \tag{57}$$

A MEMS oscillator is a typical multi-component system subject to multiple competing and dependent failure processes. The change of frequency $\Delta\omega$ caused by each shock can be considered as a hard failure process, while the shifting quality factor Q can be considered as a soft failure process. As time passes, mass decreases, and frequency ω increases due to the decreasing mass. Then as a result, the quality factor Q increases due to the increasing ω. When there is high frequency and a very high value of Q, degradation wear is severe and considered as a soft failure. On the other hand, when there is a thermal shock, jitter or other vibration from the environment, it can cause a sudden change of frequency. If the shock or vibration is large enough to cause large $\Delta\omega$ instantaneously, hard failure occurs. As these shocks are quite different, it is reasonable to categorize them into different component shock sets.

MEMS oscillators are a particularly interesting example because the hard failure process and soft failure process are dependent, since Q and ω have a close relationship with each other. Here, it is important to state that the quality factor is a positive number at the beginning of operation; that is, the linear degradation has positive initial intercept, and a linear degradation model is chosen.
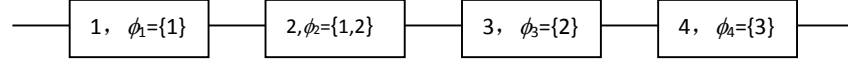
Figure 3.16: A series system with different shock set for each component

Consider a series system with four components in Figure 3.16, i.e., four comb-drive resonators in a system. Table 3.3 indicates the shock set information for these four components. In this example, the MEMS oscillator operates in a stable environment with shocks arriving as Poisson processes. There are four different types of shocks arriving to the system, and they are assumed to have constant arrival rates. Binary variables 0 and 1 in Table 3.3 represent the elements of the shock sets for each component. For example, the value of 1 in the table for component 2 and type 2 shock means that type 2 shock exists in the component 2 shock set, i.e., $2 \in \phi_l$. The parameters in Eq. (56) for reliability analysis are provided in Table 3.4. Without lack of generality, parameters of component 1 and 2 are assumed to be the same, and parameters for component 3 and 4 are the same. Most of the example parameters are from [114]. Based on Eq. (56), the reliability function $R(t)$ and the pdf of time-to-failure $f_T(t)$ are plotted in Figure 3.17. System reliability $R(t)$ changes dramatically when time is between 300 months and 600 months.

Table 3.3: Shock set information for system reliability analysis

	Type 1 shock	Type 2 shock	Type 3 shock
Component 1	1	0	0
Component 2	1	1	0
Component 3	0	1	0

Component 4	0	0	1
shock arrival rate	1.30×10^{-2}	1.32×10^{-2}	1.31×10^{-2}

Table 3.4: Parameter values for system reliability with different component shock set

Parameters	component 1 & 2	component 3 & 4	Sources
H_l	8100	8200	Assumption
D_l	92 MHz	93 MHz	Assumption
φ_l	4000	3700	Hsu [114]
β_l	$\beta_l \sim N(\mu_\beta,\sigma_\beta^2)$ μ_β=0.90 σ_β=0.080	$\beta_l \sim N(\mu_\beta,\sigma_\beta^2)$ μ_β=0.93 σ_β=0.078	Hsu [114]
$Y_{l,j,k}$	$Y_{l,j,k}\sim N(\mu_Y,\sigma_Y^2)$ $\mu_{Y1,1}$ =400, $\sigma_{Y1,1}$ =15 $\mu_{Y2,1}$ =410, $\sigma_{Y2,1}$ =18 $\mu_{Y2,2}$ =390, $\sigma_{Y2,2}$ =17	$Y_{l,j} \sim N(\mu_Y,\sigma_Y^2)$ $\mu_{Y3,2}$ =408, $\sigma_{Y3,2}$ =16 $\mu_{Y4,3}$ =402, $\sigma_{Y4,3}$ =15	Hsu [114]
$W_{l,j,k}$	$W_{l,j,k}\sim N(\mu_W,\sigma_W^2)$ $\mu_{W1,1}$ =72.6 MHz, $\sigma_{W1,1}$ =6.3 MHz $\mu_{W2,1}$ =71.8 MHz, $\sigma_{W2,1}$ =6.1 MHz $\mu_{W2,2}$ =70.2 MHz, $\sigma_{W2,2}$ =5.8 MHz	$W_{l,j} \sim N(\mu_W,\sigma_W^2)$ $\mu_{W3,2}$ =71.1 MHz, $\sigma_{W3,2}$ =6.0 MHz $\mu_{W4,3}$ =70.3 MHz, $\sigma_{W4,3}$ =5.9 MHz	Hsu [114]
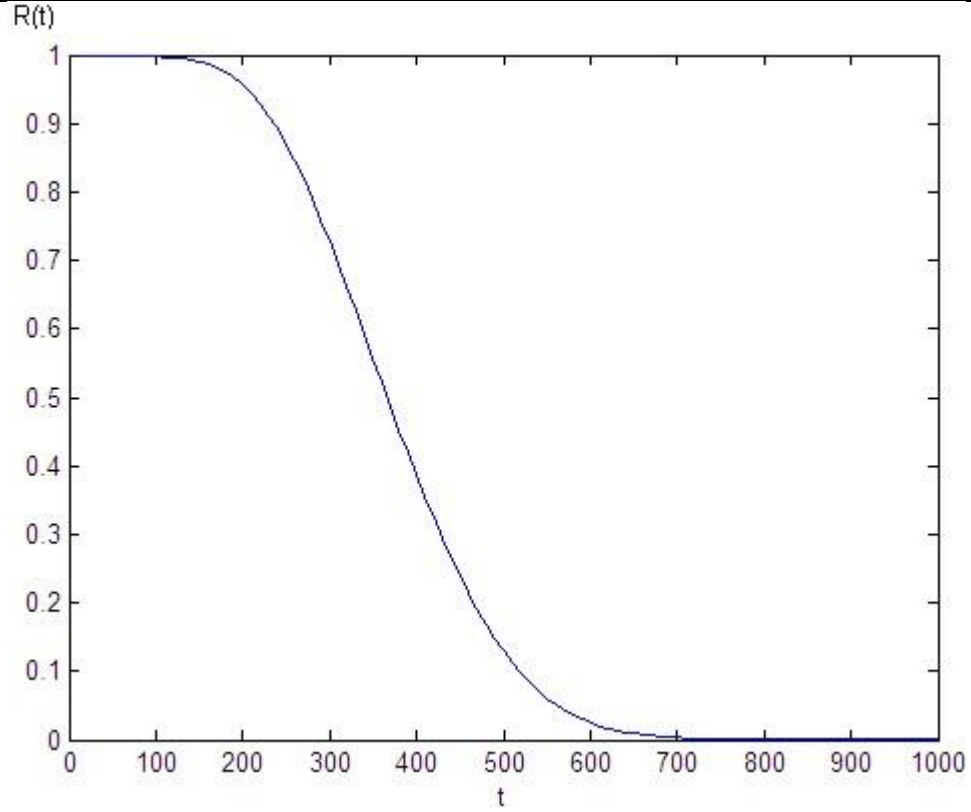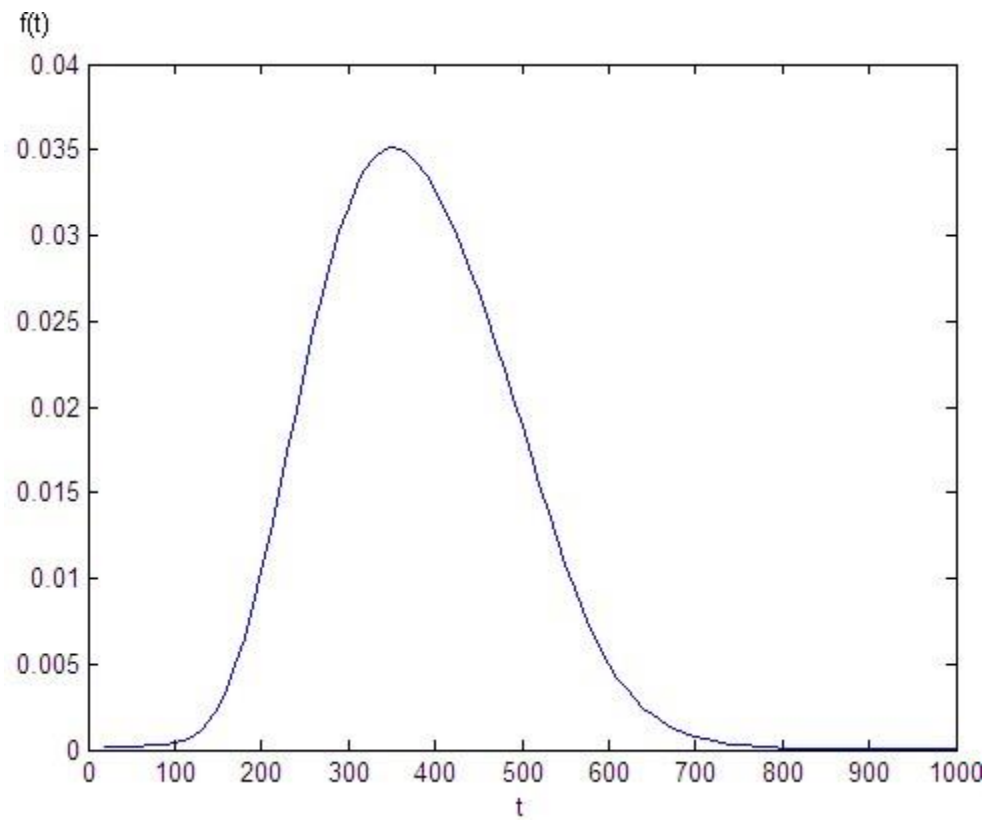
Figure 3.17: Reliability function $R(t)$ and time-to-failure distribution $f_T(t)$
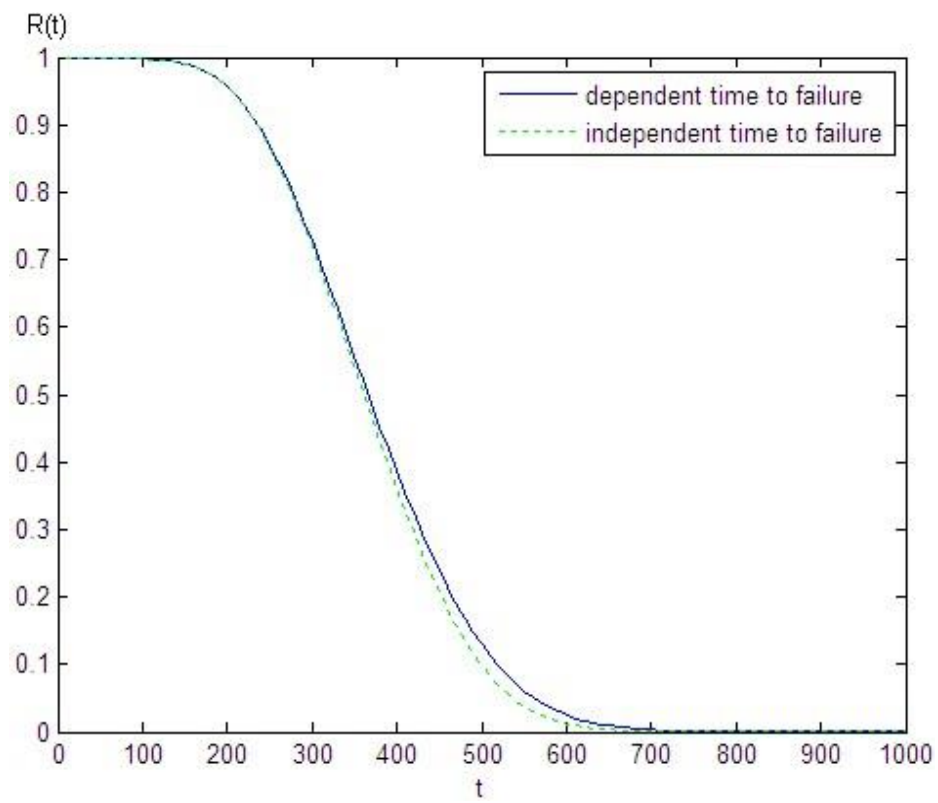
Figure 3.18: Reliability for system with dependent and independent component failure times
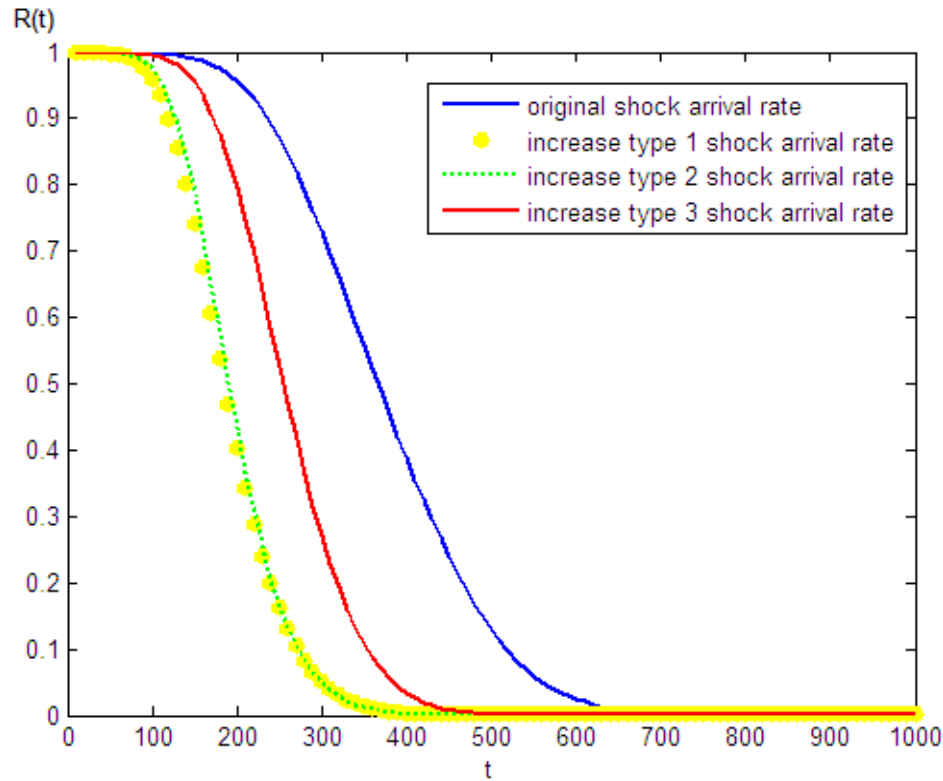
Figure 3.19: Sensitivity analysis of $R(t)$ and $f_T(t)$ on λ

If more than two components share the same shock type in their shock sets, time-to-failure of these components are dependent. This is evaluated by analyzing another system with the same configuration and parameters as the series system, except that component 2 does not suffer type 1 shock, but instead suffers from a type 4 shock that has exactly the same parameters as a type 1 shock. Similarly, component 3 does not suffer type 2, but type 5 instead with the same parameters as shock type 2. Since no common types of shocks are shared in shock sets for the new system, times-to-failure of all components are independent.

Figure 3.18 indicates that the reliability for the system with dependent component failure time is similar from the reliability for a system with independent component failure time, but not the same. The difference is relatively small due to the parameter values in the example. However, if other values are chosen for parameters or use another application,

the difference would be larger.

A sensitivity analysis was performed to assess the effects of different shock arrival rates on $R(t)$. In Figure 3.19, it can be observed that increasing the arrival rate of any type of shocks can decrease system reliability. When the increase in arrival rates of these three types of shocks is the same, type 1 and type 2 shocks affect reliability more dramatically than type 3 shocks. That is because both type 1 and type 2 shocks affect two components, but type 3 shocks only affects one component.

To summarize, in Section 3.1, reliability is studied for a system subject to dependent and competing failure processes focusing on qualitative analysis. That is, failure processes are dependent due to the exposure of shared shocks. System reliability with dependent failure processes among components is developed considering different system configuration. Then, a new reliability model is developed considering that different components possess distinct shock sets, since shocks with specific sizes or function may selectively affect one or more components in the system, but not necessarily all components.

3.2 System reliability with dependent *W* and/or dependent *Y*

System reliability was previously investigated with two failure processes for each component. The failure processes and component failure times are dependent due to the shared shock process. However, the correlation of transmitted shock magnitude or damage to any specific failure process among all components from the same shock was not considered. This can potentially cause poor reliability prediction. If that assumption is not appropriate. The dependence of shock sizes and damages to multiple failure processes from a quantitative analysis perspective is important in reliability analysis of system. In this

section, four different dependent patterns/scenarios of dependent shock damages on multiple failure processes for all components are considered [115]. For any specific shock arriving to the multi-component system, it is probabilistically likely that damages on one specific failure process (hard or soft) among all components by this specific shock are dependent. Also, it is probabilistically likely that for each shock, damages to both components failure processes, i.e., soft failure processes and hard failure processes are separately or mutually dependent at the same time. Without considering dependent shock damages to failure processes and quantifying the correlation, reliability prediction cannot be satisfied for many engineering applications.

In mechanics, an 'impact' is a high force or shock applied on system, which usually has a greater effect than a lower force effect. Furthermore, given the same impact on the whole system, all components with different materials, different connection modes or different damping ratios can behave in quite different ways. Ductile materials like steel tend to be more brittle at high loading rates, and spalling may occur on the reverse side to the impact location if penetration does not occur. Considering a multi-component system as shown in Figure 3.20, the jth shock arrives to the system with shock size Z_j. Springs or effective springs connect each component within the whole system. Given different damping ratio for each connection, transmitted shock size to each component W_{ij} (transmitted shock size to i^{th} component by j^{th} shock) can be obtained, and accordingly obtain shock damage Y_{ij} (i^{th} component degradation increment caused by j^{th} shock).
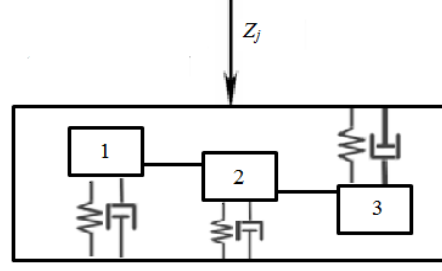
Figure 3.20: Different component effect caused by the same shock arriving to the system

3.2.1 Dependent W and independent Y: Scenario 1
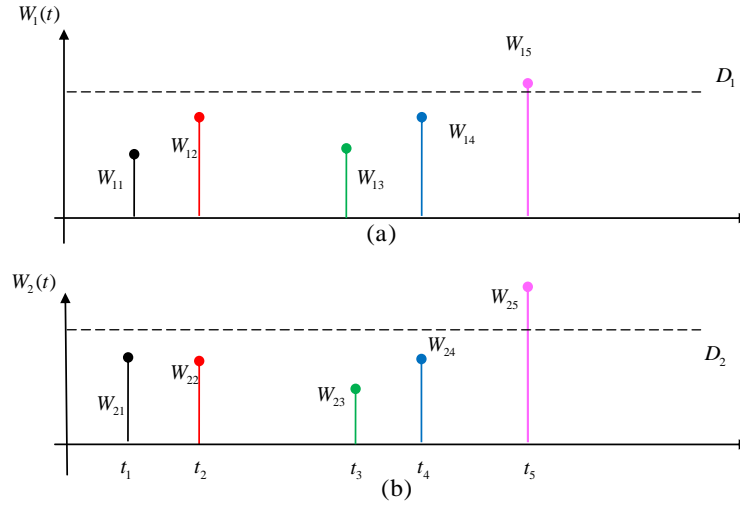
Figure 3.21: Dependent shock damage to hard failure processes in scenario 1

In this scenario, system reliability is studied considering dependent transmitted shock sizes to hard failure processes among all components. Considering a series system with n components, Fig. 3.21 shows hard failure process for component 1 and 2 (other n - 2 components have similar hard failure processes). Given a shock arriving to the system, the transmitted shock size to each component according to transmission parameter α_i can be obtained. Components are exposed to different level effects at the same time t_1 when the first shock arrives to the system, and W_{11}, W_{21},..., W_{n1} for all n components are considered to be dependent due to the shared exposure to the first shock to the whole system. It is similar for other shocks, i.e., W_{1m}, W_{2m},..., W_{nm} for all n components are dependent

due to the shared exposure to the mth shock arriving to system.

There are different ways to consider the dependent characteristic, like additive dependent, proportional correlated and other more complicated models. Here it is formulated as: $W_{ij} = \tilde{W}_{ij} + \alpha_i Z_j$, in which Z_j is the magnitude of jth shock, and α_i is the transmission parameter from system shock size to transmitted shock size of component i affecting the hard failure process. $\tilde{W}_{ij}$ is the shock contribution to i^{th} component hard failure process by j^{th} shock not proportional to system shock size (which can be defined to be 0 for all j if shock transmission is directed proportional). As for shock damages to soft failure process caused by shocks, i.e., $Y_{1m}, Y_{2m},\ldots, Y_{nm}$, they are assumed to be independent in this scenario, and their dependent case is studied in a later scenario. For a series system, reliability is the probability that by time t, degradation for each component is less than a soft failure threshold and transmitted shock sizes to hard failure process is less than hard failure threshold level.

According to Eq.(8), with $\tilde{W}_{ij} = \tilde{W}_{ij} + \alpha_i Z_j$, $X_{si}(t) = X_i(t) + \sum_{j=1}^{N(t)} Y_{ij}$, and conditioning on the number of shocks by time t:

$$R(t) = \sum_{m=0}^{\infty} P\left\{\left[\tilde{W}_{11} + \alpha_1 Z_1 < D_1, \tilde{W}_{12} + \alpha_1 Z_2 < D_1, \tilde{W}_{1m} + \alpha_1 Z_m < D_1, X_1(t) + \sum_{j=1}^{m} Y_{1j} < H_1\right] \cap \right.$$
$$\left[\tilde{W}_{21}\alpha_2 + Z_1 < D_2, \tilde{W}_{22} + \alpha_2 Z_2 < D_2, \tilde{W}_{2m} + \alpha_2 Z_m < D_2, X_2(t) + \sum_{j=1}^{m} Y_{2j} < H_2\right] \cap \ldots$$
$$\left.\left[\tilde{W}_{n1} + \alpha_n Z_1 < D_n, \tilde{W}_{n2} + \alpha_n Z_2 < D_n, \tilde{W}_{nm} + \alpha_n Z_m < D_n, X_n(t) + \sum_{j=1}^{m} Y_{nj} < H_n\right]\right\} P(N(t) = m)$$

$$(58)$$

Rearrange the grouping terms from components to shocks, and system reliability can be expressed as:

$$R(t) = \sum_{m=0}^{\infty} P\left\{ \left[\tilde{W}_{11} + \alpha_1 Z_1 < D_1, \tilde{W}_{21} + \alpha_2 Z_1 < D_2, ..., \tilde{W}_{n1} + \alpha_n Z_1 < D_n, X_1(t) + \sum_{j=1}^{m} Y_{1j} < H_1 \right] \cap \right.$$

$$\left[\tilde{W}_{12} + \alpha_1 Z_2 < D_1, \tilde{W}_{22} + \alpha_2 Z_2 < D_2, ..., \tilde{W}_{n2} + \alpha_n Z_2 < D_n, X_2(t) + \sum_{j=1}^{m} Y_{2j} < H_2 \right] \cap ...$$

$$\left. \left[\tilde{W}_{1m} + \alpha_1 Z_m < D_1, \tilde{W}_{2m} + \alpha_2 Z_m < D_2, ..., \tilde{W}_{nm} + \alpha_n Z_m < D_n, X_n(t) + \sum_{j=1}^{m} Y_{nj} < H_n \right] \right\} P(N(t) = m)$$

(59)

The two failure processes are then independent after conditioning of the number of shocks

$$R(t) = \sum_{m=0}^{\infty} \prod_{j=1}^{m} P\left(\tilde{W}_{1j} + \alpha_1 Z_j < D_1, ..., \tilde{W}_{nj} + \alpha_n Z_j < D_n \right) \prod_{i=1}^{n} P\left(X_i(t) + \sum_{j=1}^{m} Y_{ij} < H_i \right) P(N(t) = m)$$

(60)

Conditioning on the size of jth shock and integrating over all the z_j value and assuming shocks following a Poisson process:

$$R(t) = \sum_{m=0}^{\infty} \prod_{j=1}^{m} \int_{z_j} P\left(\tilde{W}_{1j} + \alpha_1 Z_j < D_1, ..., \tilde{W}_{nj} + \alpha_n Z_j < D_n | Z_j = z_i \right) f_{Z_j}(z_j) dz_j$$

$$\times \prod_{i=1}^{n} P\left(X_i(t) + \sum_{j=1}^{m} Y_{ij} < H_i \right) \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

(61)

$$R(t) = \sum_{m=0}^{\infty} \prod_{j=1}^{m} \int_{z_j} P\left(\tilde{W}_{1j} < D_1 - \alpha_1 z_j, ..., \tilde{W}_{nj} < D_n - \alpha_n z_j \right) f_{Z_j}(z_j) dz_j$$

$$\times \prod_{i=1}^{n} P\left(X_i(t) + \sum_{j=1}^{m} Y_{ij} < H_i \right) \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

(62)

Since both $\tilde{W}_{ij}$ and Z_j are *iid* random variables, Eq. (52) can be simplified as follow:

$$R(t) = \sum_{m=0}^{\infty} \left(\int_z \prod_{i=1}^{n} P\left(\tilde{W}_{ij} < D_i - \alpha_i z \right) f_{Z_j}(z) dz \right)^m \prod_{i=1}^{n} P\left(X_i(t) + \sum_{j=1}^{m} Y_{ij} < H_i \right) \frac{\exp(-\lambda t)(\lambda t)^m}{m!} \quad (63)$$

Based on a random effect model for degradation, using convolutional integral, Eq. (63) can be expressed as:

$$R(t) = \sum_{m=0}^{\infty} \left(\int_z \prod_{i=1}^{n} F_{\tilde{W}_i}\left(D_i - \alpha_i z\right) f_{Z_j}(z) dz \right)^m \prod_{i=1}^{n} \int_0^{H_i} G(H_i - u) f_{Y_i}^{\langle m \rangle}(u) du \frac{\exp(-\lambda t)(\lambda t)^m}{m!} \quad (64)$$

As an example, if it is assumed that W_{ij}, Y_{ij}, and β_i follow normal distributions, the reliability function for the more specific case can be expressed as:

$$R(t) = \sum_{m=0}^{\infty} \left(\int_z \prod_{i=1}^{n} \Phi\left(\frac{D_i - \alpha_i z - \mu_{\tilde{W}_i}}{\sigma_{\tilde{W}_i}} \right) f_{Z_j}(z) dz \right)^m \prod_{i=1}^{n} \Phi\left(\frac{H_i - (\mu_{\beta_i} t + \varphi_i + m\mu_{Y_i})}{\sqrt{\sigma_{\beta_i}^2 t^2 + m\sigma_{Y_i}^2}} \right) \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

$$(65)$$

Considering the gamma process to model degradation, with $v_i(t) = c_i t^{b_i}$ as the component gamma process shape parameter or degradation path with $b_i = 0.5$

$$R(t) = \sum_{m=0}^{\infty} \left(\int_z \prod_{i=1}^{n} F_{\tilde{W}_i}\left(D_i - \alpha_i z\right) f_{Z_j}(z) dz \right)^m \prod_{i=1}^{n} \int_0^{H_i} \left(\frac{\int_0^{(H_i - u)\theta_i} z^{c_i \sqrt{t} - 1} e^{-z} dz}{\int_0^{\infty} z^{c_i \sqrt{t} - 1} e^{-z} dz} \right) f_{Y_i}^{\langle m \rangle}(u) du \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

$$(66)$$

3.2.1.1 Numerical example

A three component series system numerical example is used to illustrate the reliability models of different dependent shock damage scenarios. The example is a hypothetical system for demonstration purposes. Values of parameters are shown in Table 3.5 for the random effect model.

Table 3.5: Parameter values for multi-component system reliability analysis for four different dependent shock damage scenarios

Parameters	component 1	component 2	component 3
H_l	0.001265	0.00127	0.00126
D_l	1.5	1.4	1.45
φ_l	0	0	0
β_l	$\beta_l \sim N(\mu_\beta,\sigma_\beta^2)$ μ_β=8.4886×10⁻⁸ σ_β=1.9216×10⁻⁸	$\beta_l \sim N(\mu_\beta,\sigma_\beta^2)$ μ_β=8.4936×10⁻⁸ σ_β=1.9011×10⁻⁸	$\beta_l \sim N(\mu_\beta,\sigma_\beta^2)$ μ_β=8.4876×10⁻⁸ σ_β=1.9520×10⁻⁸
$Y_{l,j}$	$Y_{l,j} \sim N(\mu_Y,\sigma_Y^2)$ μ_{Y1}=0.93×10⁻⁴ σ_{Y1}=2.07×10⁻⁵	$Y_{l,j} \sim N(\mu_Y,\sigma_Y^2)$ μ_{Y2}=0.9×10⁻⁴ σ_{Y2}=2.1×10⁻⁵	$Y_{l,j} \sim N(\mu_Y,\sigma_Y^2)$ μ_{Y3}=0.95×10⁻⁴ σ_{Y3}=2.05×10⁻⁵
$W_{l,j}$	$W_{l,j} \sim N(\mu_W,\sigma_W^2)$ μ_{W1}=0.9 σ_{W1}=0.2	$W_{l,j} \sim N(\mu_W,\sigma_W^2)$ μ_{W2}=0.92 σ_{W2}=0.18	$W_{l,j} \sim N(\mu_W,\sigma_W^2)$ μ_{W3}=0.91 σ_{W3}=0.19
γ_i	5.1×10⁻⁵	5.2×10⁻⁵	5.0×10⁻⁵
α_i	0.04	0.042	0.041
λ	3×10⁻⁵		
μ_z	5		
σ_z	0.5		

Based on Eq. (65) in Section 3.2.1, the reliability function $R(t)$ and the pdf of time-to-failure $f_T(t)$ are plotted in Figures 3.22 and 3.23 for dependent transmitted shock size on hard failure process case. System reliability decreases smoothly before $t=1\times10^4$, and changes dramatically between time 1×10^4 and 1.5×10^4.
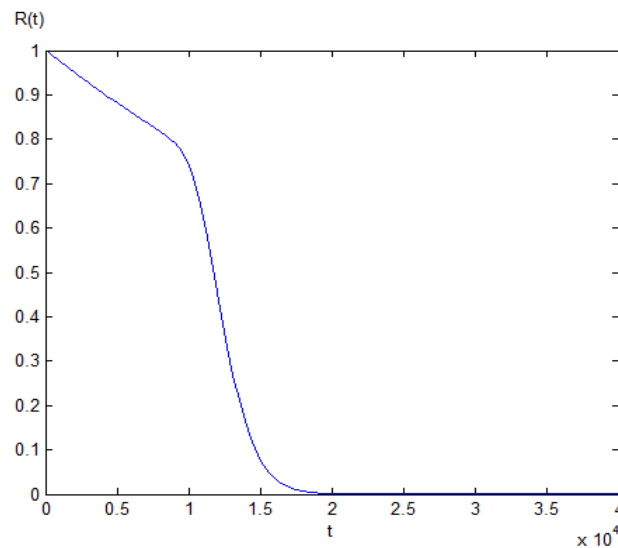
Figure 3.22: System reliability in scenario 1
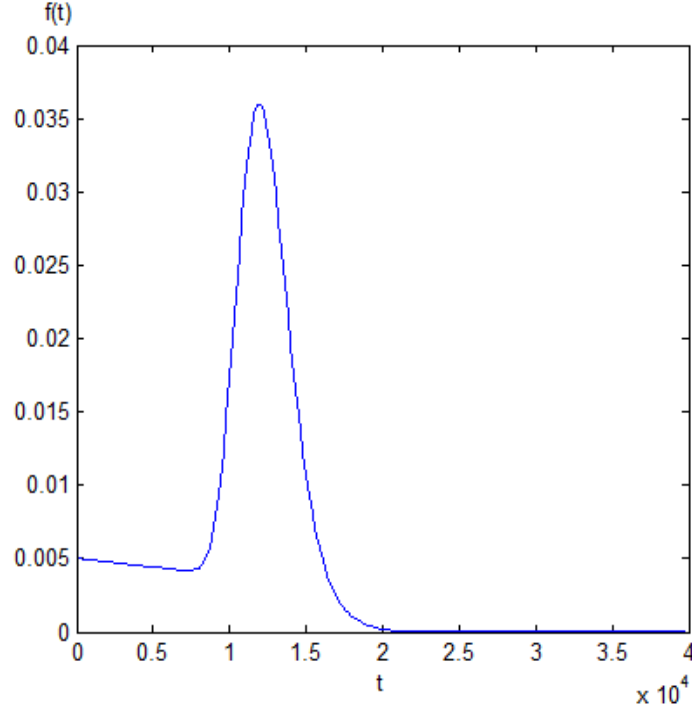
Figure 3.23: Time to failure in scenario 1

3.2.2 Independent *W* and dependent *Y*

For scenario 2, the system reliability with dependent shock damage to soft failure process among components is studied. For a series system with *n* components, Figure 3.24 shows that when a specific shock arriving to the system at time t_1, damage increments to degradation of component 1 and 2 caused by the first shock, Y_{11} and Y_{21} are dependent. It is also similar for other (*n* - 2) components in the system, which means the shock damages to soft failure process of all components are linear dependent, i.e., Y_{11}, Y_{21},…, Y_{n1} for all *n* components by the first shock are linear dependent. Similarly, Y_{1m}, Y_{2m},…, Y_{nm} for all *n* components by m^{th} shock arriving to system are linear dependent. It is formulated as: $Y_{ij} = \tilde{Y}_{ij} + \gamma_i Z_j$, in which Z_j is j^{th} shock size, and γ_i is the transmission parameter from system shock size to component *i* soft failure damage. $\tilde{Y}_{ij}$ is the damage to i^{th} component soft failure process by j^{th} shock not proportional to shock size. The transmitted shock sizes caused by

these shocks, i.e., W_{1m}, W_{2m},…, W_{nm}, are assumed to be independent in this case.
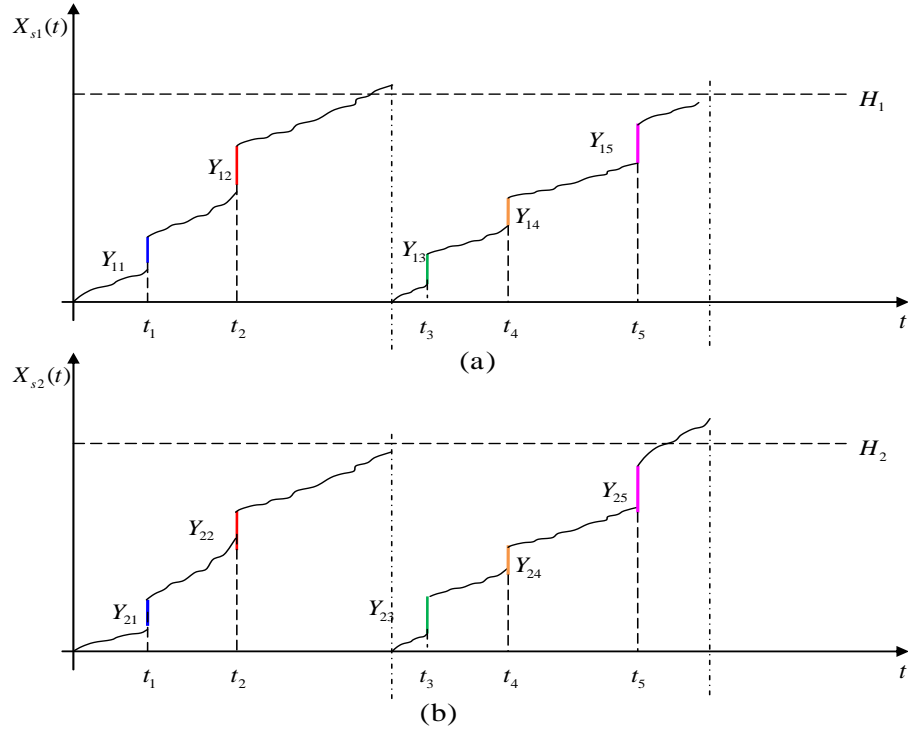
Figure 3.24: Dependent shock damage to soft failure processes in scenario 2

For a series system, the initial reliability equation is shown in Eq. (8), with

$Y_{ij} = \tilde{Y}_{ij} + \gamma_i Z_j$ and conditioning on the number of shocks:

$$R(t) = \sum_{m=0}^{\infty} P\left\{\left[W_{11} < D_1, W_{12} < D_1, W_{1N(t)} < D_1, X_1(t) + \sum_{j=1}^{N(t)}\left(\tilde{Y}_{1j} + \beta_1 Z_j\right) < H_1\right] \cap \right.$$

$$\left[W_{21} < D_2, W_{22} < D_2, W_{2N(t)} < D_2, X_2(t) + \sum_{j=1}^{N(t)}\left(\tilde{Y}_{2j} + \beta_2 Z_j\right) < H_2\right] \cap ...$$

$$\left.\left[W_{n1} < D_n, W_{n2} < D_n, W_{nN(t)} < D_n, X_n(t) + \sum_{j=1}^{N(t)}\left(\tilde{Y}_{nj} + \beta_n Z_j\right) < H_n\right] \mid N(t) = m\right\} P(N(t) = m)$$

$$(67)$$

Hard failure processes for all components are independent after conditioning on the

number of shocks by time t, and soft failure processes are dependent due to the shared item

Z_1, Z_2 ,…, $Z_{N(t)}$:

$$R(t) = \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n} \left(P(W_{i1} < D_i), ..., P(W_{im} < D_i) \right) \right]$$
$$P\left(X_1(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{1j} + \beta_1 Z_j \right) < H_1, ..., X_n(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{nj} + \beta_n Z_j \right) < H_n \right) \frac{\exp(-\lambda t)(\lambda t)^m}{m!} \tag{68}$$

Since W_{ij} are *i.i.d* random variables:

$$R(t) = \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n} P(W_{ij} < D_i) \right]^m P\left(X_1(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{1j} + \beta_1 Z_j \right) < H_1, ..., X_n(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{nj} + \beta_n Z_j \right) < H_n \right) \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$
$$\tag{69}$$

Conditioning on the sum of shock damage for all shocks arriving to system by time t, and integrating over all the value of this sum:

$$R(t) = \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n} P(W_{ij} < D_i) \right]^m \int_0^{\infty} P\left(X_1(t) + \sum_{j=1}^{m} \tilde{Y}_{1j} + \beta_1 \sum_{j=1}^{m} Z_j < H_1, ..., X_n(t) + \sum_{j=1}^{m} \tilde{Y}_{nj} + \beta_n \sum_{j=1}^{m} Z_j < H_n \mid \sum_{j=1}^{m} Z_j = u \right)$$
$$\times f_Z^{\langle m \rangle}(u) du \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$
$$\tag{70}$$

$$R(t) = \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n} P(W_{ij} < D_i) \right]^m \int_0^{\infty} P\left(X_1(t) + \sum_{j=1}^{m} \tilde{Y}_{1j} < H_1 - \beta_1 u, ..., X_n(t) + \sum_{j=1}^{m} \tilde{Y}_{nj} < H_n - \beta_n u \right) f_Z^{\langle m \rangle}(u) du$$
$$\times \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$
$$\tag{71}$$

$$R(t) = \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n} F_{W_i}(D_i) \right]^m \int_0^{\infty} \prod_{i=1}^{n} P\left(X_i(t) + \sum_{j=1}^{m} \tilde{Y}_{ij} < H_i - \beta_i u \right) f_Z^{\langle m \rangle}(u) du \frac{\exp(-\lambda t)(\lambda t)^m}{m!} \tag{72}$$

Also, conditioning on the damage to i^{th} component soft failure process by all shock not depending on shock size, $\sum_{j=1}^{m} \tilde{Y}_{ij}$:

$$R(t) = \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n} F_{W_i}(D_i) \right]^m \int_{u=0}^{\infty} \left(\prod_{i=1}^{n} \int_{y=0}^{\infty} P\left(X_i(t) + \sum_{j=1}^{m} \tilde{Y}_{ij} < H_i - \beta_i u | \sum_{j=1}^{m} \tilde{Y}_{ij} = y \right) f_{\tilde{Y}_i}^{\langle m \rangle}(y) dy \right) f_Z^{\langle m \rangle}(u) du$$

$$\times \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

(73)

$$R(t) = \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n} F_{W_i}(D_i) \right]^m \int_{u=0}^{\infty} \left(\prod_{i=1}^{n} \int_{y=0}^{\infty} P\left(X_i(t) < H_i - y - \beta_i u \right) f_{\tilde{Y}_i}^{\langle m \rangle}(y) dy \right) f_Z^{\langle m \rangle}(u) du \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

(74)

If random effect model is considered for degradation, define $G(x_i, t)$ to be the cdf

of $X_i(t)$ at t:

$$R(t) = \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n} F_{W_i}(D_i) \right]^m \int_{u=0}^{\infty} \left(\prod_{i=1}^{n} \int_{y=0}^{\infty} G(H_i - y - \beta_i u) f_{\tilde{Y}_i}^{\langle m \rangle}(y) dy \right) f_Z^{\langle m \rangle}(u) du \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

(75)

For examples where W_{ij}, Y_{ij}, and Z_j follow normal distributions, the reliability function

for the more specific case can be expressed as:

$$R(t) = \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n} \Phi\left(\frac{D_i - \mu_{W_i}}{\sigma_{W_i}} \right) \right]^m \int_{u=0}^{\infty} \left(\prod_{i=1}^{n} \int_{y=0}^{\infty} \Phi\left(\frac{H_i - \left(\mu_{\gamma_i} t + \varphi_i + m\beta_i \mu_Z + m\mu_{\tilde{Y}_i} \right)}{\sqrt{\sigma_{\gamma_i}^2 t^2 + m\beta_i \sigma_Z^2 + m\sigma_{\tilde{Y}_i}^2}} \right) f_{\tilde{Y}}^{\langle m \rangle}(y) dy \right) f_Z^{\langle m \rangle}(u) du$$

$$\times \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

(76)

Using the gamma process model for the degradation:

$$R(t) = \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n} F_{W_i}(D_i) \right]^m \int_{u=0}^{\infty} \left(\prod_{i=1}^{n} \int_{y=0}^{\infty} \left(\frac{\int_0^{(H_i-y-\beta_i u)\theta_i} z^{c_i \sqrt{t}-1} e^{-z} dz}{\int_0^{\infty} z^{c_i \sqrt{t}-1} e^{-z} dz} \right) f_{\tilde{Y}_i}^{\langle m \rangle}(y) dy \right) f_Z^{\langle m \rangle}(u) du \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

(77)

3.2.2.1 Numerical example

A three component series system is used to illustrate the reliability models of different dependent shock damage scenarios. Values of the parameters are shown in Table 3.5. Based on Eq. (76) in Section 3.2.2, the reliability function $R(t)$ and the pdf of time-to-failure $f_T(t)$ are shown in Figure 3.25 and 3.26 for dependent damage on soft failure process case. System reliability does not change notably before time 1×10^4.
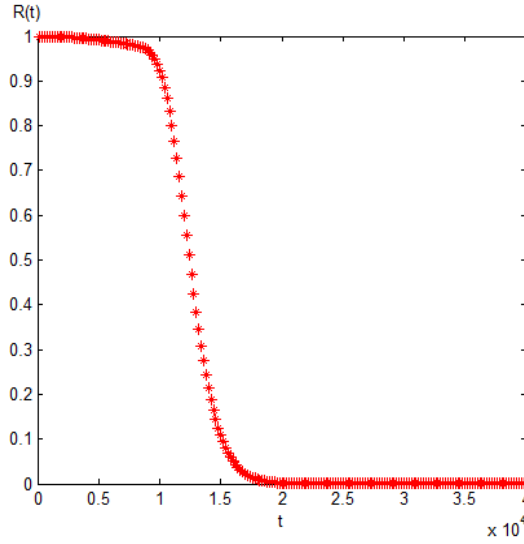

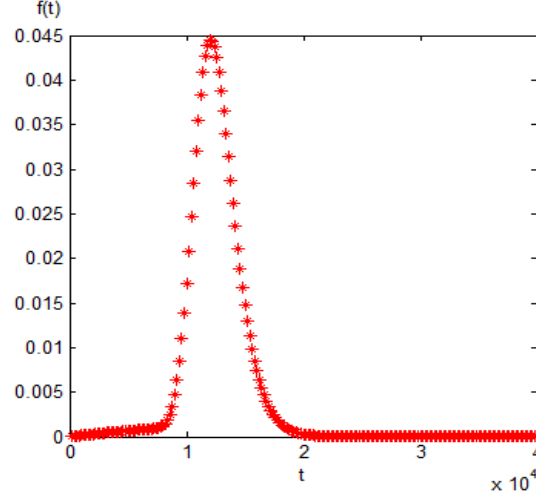Figure 3.25: System reliability in scenario 2

Figure 3.26: Time to failure in scenario 2

3.2.3 Dependent *W* and dependent *Y*

In last two sections, when a specific system shock arrives to the system, all components receive shock effects, and the shock effects to one failure process for all components are dependent while the effect to the other failure process for all components are independent. In this scenario, the shock effects to the two failure process are separately dependent. From Figure 3.27, j^{th} shock arriving to the system, it causes damage to all n components soft failure processes in the series system $Y_{1j}, Y_{2j}, \ldots, Y_{nj}$ and also transmitted shock sizes $W_{1j}, W_{2j}, \ldots, W_{nj}$. This study case assumes $Y_{1j}, Y_{2j}, \ldots, Y_{nj}$ are dependent, $W_{1j}, W_{2j}, \ldots, W_{nj}$ dependent. However, damages to these two failure processes are mutually independent, i.e., W_{ij} and Y_{ij} are independent. It also works for other n-2 components in the system. It is formulated as $W_{ij} = \tilde{W}_{ij} + \alpha_i Z_{1j}$, $Y_{ij} = \tilde{Y}_{ij} + \gamma_i Z_{2j}$, in which Z_{1j} is j^{th} system shock size related to hard failure process and Z_{2j} is an associated j^{th} shock size related to soft failure process, and Z_{1j} and Z_{2j} are independent. α_i is the transmission parameter from shock size to hard failure process damage of ith component, and γ_i is the transmission parameter from shock size to soft failure process damage to i^{th} component. $\tilde{W}_{ij}$ is the damage to i^{th}

component hard failure process by j^{th} shock not proportional to shock size, and $\tilde{Y}_{ij}$ is the damage to i^{th} component soft failure process by j^{th} shock not proportional to shock size. The shared item Z_{1j} makes W_{ij} dependent, and the shared item of Z_{2j} makes Y_{ij} dependent. Since W_{ij} and Y_{ij} share no common item, damages to these two processes are mutually independent.

For a series system, a general system reliability equation is shown in Eq. (8). With $W_{ij} = \tilde{W}_{ij} + \alpha_i Z_{1j}$, $Y_{ij} = \tilde{Y}_{ij} + \gamma_i Z_{2j}$ and conditioning on the number of shocks, Eq. (78) can be obtained.
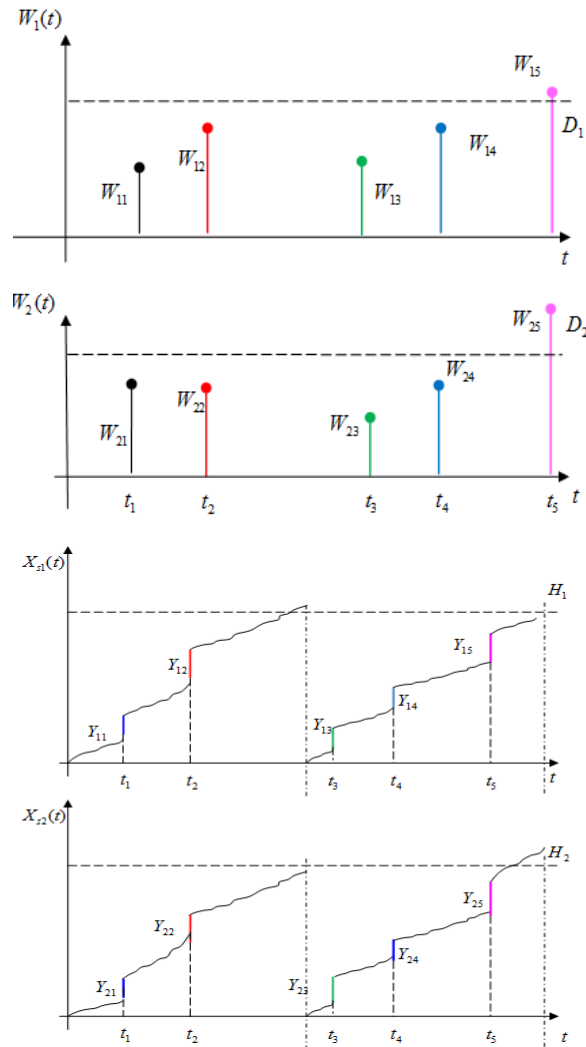

Figure 3.27: Separately dependent shock damage to two failure processes in scenario 3

$$R(t) = \sum_{m=0}^{\infty} P \left\{ \left[\tilde{W}_{11} + \alpha_1 Z_{11} < D_1, \tilde{W}_{21} + \alpha_2 Z_{11} < D_2, ..., \tilde{W}_{n1} + \alpha_n Z_{11} < D_n, X_1(t) + \sum_{j=1}^{N(t)} \left(\tilde{Y}_{1j} + \beta_1 Z_{2j} \right) < H_1 \right] \cap \right.$$

$$\left[\tilde{W}_{12} + \alpha_1 Z_{12} < D_1, \tilde{W}_{22} + \alpha_2 Z_{12} < D_2, ..., \tilde{W}_{n2} + \alpha_n Z_{12} < D_n, X_2(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{2j} + \beta_2 Z_{2j} \right) < H_2 \right] \cap ...$$

$$\left. \left[\tilde{W}_{1m} + \alpha_1 Z_{1m} < D_1, \tilde{W}_{2m} + \alpha_2 Z_{1m} < D_2, ..., \tilde{W}_{nm} + \alpha_n Z_{1m} < D_n, X_n(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{nj} + \beta_n Z_{2j} \right) < H_n \right] \right\} P(N(t) = m)$$

$$(78)$$

$$R(t) = \sum_{m=0}^{\infty} P \left(\left[\tilde{W}_{11} + \alpha_1 Z_{11} < D_1, ..., \tilde{W}_{n1} + \alpha_n Z_{11} < D_n \right] \cap ... \cap \left[\tilde{W}_{1m} + \alpha_1 Z_{1m} < D_1, ..., \tilde{W}_{nm} + \alpha_n Z_{1m} < D_n \right] \right)$$

$$\times P \left(X_1(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{1j} + \beta_1 Z_{2j} \right) < H_1, ..., X_n(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{nj} + \beta_n Z_{2j} \right) < H_n \right) P(N(t) = m)$$

$$(79)$$

$$R(t) = \sum_{m=0}^{\infty} \left[P \left(\tilde{W}_{1j} + \alpha_1 Z_{1j} < D_1, \tilde{W}_{2j} + \alpha_2 Z_{1j} < D_2, ..., \tilde{W}_{nj} + \alpha_n Z_{1j} < D_n \right)^m \right]$$

$$\times P \left(X_1(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{1j} + \beta_1 Z_{2j} \right) < H_1, ..., X_n(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{nj} + \beta_n Z_{2j} \right) < H_n \right) P(N(t) = m)$$

$$(80)$$

$$R(t) = \sum_{m=0}^{\infty} \left[\int_0^{\infty} P \left(\tilde{W}_{1j} + \alpha_1 Z_{1j} < D_1, \tilde{W}_{2j} + \alpha_2 Z_{1j} < D_2, ..., \tilde{W}_{nj} + \alpha_n Z_{1j} < D_n \mid Z_{1j} = u_1 \right) f_{Z_{1j}}(u_1) du_1 \right]^m$$

$$P \left(X_1(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{1j} + \beta_1 Z_{2j} \right) < H_1, ..., X_n(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{nj} + \beta_n Z_{2j} \right) < H_n \right) P(N(t) = m)$$

$$(81)$$

Replace Z_{1j} with u_1, and subtract from both sides in the equation, and then integrating results in

$$R(t) = \sum_{m=0}^{\infty} \left[\int_0^{\infty} P\left(\tilde{W}_{1j} < D_1 - \alpha_1 u_1, \tilde{W}_{2j} < D_2 - \alpha_2 u_1, ..., \tilde{W}_{nj} < D_n + \alpha_n u_1 \right) f_{Z_{1j}}(u_1) du_1 \right]^m$$

$$\times P\left(X_1(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{1j} + \beta_1 Z_{2j} \right) < H_1, ..., X_n(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{nj} + \beta_n Z_{2j} \right) < H_n \right) P(N(t) = m)$$

$$(82)$$

$$R(t) = \sum_{m=0}^{\infty} \left[\int_0^{\infty} \prod_{i=1}^{n} P\left(\tilde{W}_{ij} < D_i - \alpha_i u_1 \right) f_{Z_{1j}}(u_1) du_1 \right]^m$$

$$\times P\left(X_1(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{1j} + \beta_1 Z_{2j} \right) < H_1, ..., X_n(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{nj} + \beta_n Z_{2j} \right) < H_n \right) P(N(t) = m)$$

$$(83)$$

With $F_{\tilde{W}_i}(w)$ as the cdf of $\tilde{W}_{ij}$, and conditioning on the specific value u_2 for $\sum_{j=1}^{m} Z_{2j}$,

and integrating over all the value of u_2:

$$R(t) = \sum_{m=0}^{\infty} \left[\int_0^{\infty} \prod_{i=1}^{n} F_{\tilde{W}_i}\left(D_i - \alpha_i u_1 \right) f_{Z_{1j}}(u_1) du_1 \right]^m$$

$$\times \int_0^{\infty} P\left(X_1(t) + \sum_{j=1}^{m} \tilde{Y}_{1j} + \beta_1 \sum_{j=1}^{m} Z_{2j} < H_1, ..., X_n(t) + \sum_{j=1}^{m} \tilde{Y}_{nj} + \beta_n \sum_{j=1}^{m} Z_{2j} < H_n \mid \sum_{j=1}^{m} Z_{2j} = u_2 \right) f_{Z_2}^{\langle m \rangle}(u_2) du_2$$

$$\times \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

$$(84)$$

Replace $\sum_{j=1}^{m} Z_{2j}$ with u_2, and subtract it from both sides in the equation, and then

integrating

$$R(t) = \sum_{m=0}^{\infty} \left[\int_0^{\infty} \prod_{i=1}^{n} F_{\tilde{W}_i}\left(D_i - \alpha_i u_1 \right) f_{Z_{1j}}(u_1) du_1 \right]^m$$

$$\times \int_0^{\infty} P\left(X_1(t) + \sum_{j=1}^{m} \tilde{Y}_{1j} < H_1 - \beta_1 u_2, ..., X_n(t) + \sum_{j=1}^{m} \tilde{Y}_{nj} < H_n - \beta_n u_2 \right) f_{Z_2}^{\langle m \rangle}(u_2) du_2 \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

$$(85)$$

$$R(t) = \sum_{m=0}^{\infty} \left[\int_0^{\infty} \prod_{i=1}^{n} F_{\tilde{W}_i} \left(D_i - \alpha_i u_1 \right) f_{Z_{1j}}(u_1) du_1 \right]^m \int_0^{\infty} \prod_{i=1}^{n} P\left(X_i(t) + \sum_{j=1}^{m} \tilde{Y}_{ij} < H_i - \beta_i u_2 \right) f_{Z_2}^{\langle m \rangle}(u_2) du_2$$

$$\times \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

(86)

Similarly, conditioning on the specific value y for $\sum_{j=1}^{m} \tilde{Y}_{ij}$, and integrating over all

the value of y:

$$R(t) = \sum_{m=0}^{\infty} \left[\int_0^{\infty} \prod_{i=1}^{n} F_{\tilde{W}_i} \left(D_i - \alpha_i u_1 \right) f_{Z_{1j}}(u_1) du_1 \right]^m \int_0^{\infty} \left(\prod_{i=1}^{n} \int_0^{\infty} P\left(X_i(t) + \sum_{j=1}^{m} \tilde{Y}_{ij} < H_i - \beta_i u_2 \mid \sum_{j=1}^{m} \tilde{Y}_{ij} = y \right) \right)$$

$$\times f_{\tilde{Y}_i}^{\langle m \rangle}(y) dy \right) f_{Z_2}^{\langle m \rangle}(u_2) du_2 \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

(87)

Replace $\sum_{j=1}^{m} \tilde{Y}_{ij}$ with y, and subtract it from both sides in the equation, Eq. (87) can

be expressed as:

$$R(t) = \sum_{m=0}^{\infty} \left[\int_0^{\infty} \prod_{i=1}^{n} F_{\tilde{W}_i} \left(D_i - \alpha_i u_1 \right) f_{Z_{1j}}(u_1) du_1 \right]^m \int_0^{\infty} \left(\prod_{i=1}^{n} \int_0^{\infty} P\left(X_i(t) < H_i - \beta_i u_2 - y \right) \right.$$

$$\left. f_{\tilde{Y}_i}^{\langle m \rangle}(y) dy \right) f_{Z_2}^{\langle m \rangle}(u_2) du_2 \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

(88)

Considering the random effect model for degradation, and define $G(x_i, t)$ to be the

cdf of $X_i(t)$ at t, Eq. (88) can be expressed:

$$R(t) = \sum_{m=0}^{\infty} \left[\int_0^\infty \prod_{i=1}^n F_{\tilde{W}_i} (D_i - \alpha_i u_1) f_{Z_{1j}} (u_1) du_1 \right]^m \int_0^\infty \prod_{i=1}^n \int_0^\infty G(H_i - y - \beta_i u_2) f_{\tilde{Y}_i}^{\langle m \rangle}(y) dy f_{Z_2}^{\langle m \rangle}(u_2) du_2$$

$$\times \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

$$(89)$$

For example where W_{ij}, Y_{ij}, and Z_{2j} follow normal distribution, the reliability function

for the more specific case can be expressed as:

$$R(t) = \sum_{m=0}^{\infty} \left[\int_0^\infty \prod_{i=1}^n \Phi \left(\frac{D_i - \alpha_i u_1 - \mu_{\tilde{W}_i}}{\sigma_{\tilde{W}_i}} \right) f_{Z_{1j}}(u_1) du_1 \right]^m \int_0^\infty \left(\prod_{i=1}^n \int_0^\infty \Phi \left(\frac{H_i - \left(\mu_{\gamma_i} t + \varphi_i + m\mu_{\tilde{Y}_i} + \beta_i \mu_{Z_2} \right)}{\sqrt{\sigma_{\gamma_i}^2 t^2 + m\sigma_{\tilde{Y}_i}^2 + \beta_i \sigma_{Z_2}^2}} \right) \right.$$

$$\left. \times f_{\tilde{Y}_i}^{\langle m \rangle}(y) dy \right) f_{Z_2}^{\langle m \rangle}(u_2) du_2 \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

$$(90)$$

Alternatively, using Gamma process to model degradation:

$$R(t) = \sum_{m=0}^{\infty} \left[\int_0^\infty \prod_{i=1}^n F_{\tilde{W}_i} (D_i - \alpha_i u_1) f_{Z_{1j}} (u_1) du_1 \right]^m \int_0^\infty \prod_{i=1}^n \int_0^\infty \left(\frac{\int_0^{(H_i - y - \beta_i u_2)\theta_i} z^{c_i \sqrt{t} - 1} e^{-z} dz}{\int_0^\infty z^{c_i \sqrt{t} - 1} e^{-z} dz} \right) f_{\tilde{Y}_i}^{\langle m \rangle}(y) dy f_{Z_2}^{\langle m \rangle}(u_2) du_2$$

$$\times \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

$$(91)$$

3.2.3.1 Numerical example

A three component series system is again used to illustrate the reliability models of

different dependent shock damage scenarios. Values of the parameters are shown in Table.

3.5.

Based on Eq. (90) in Section 3.2.3, the reliability function $R(t)$ and the pdf of time-to-

failure $f_T(t)$ are shown in Figure 3.28 and 3.29 for dependent transmitted shock size and

dependent damage on soft failure process case, in which they are separately dependent.
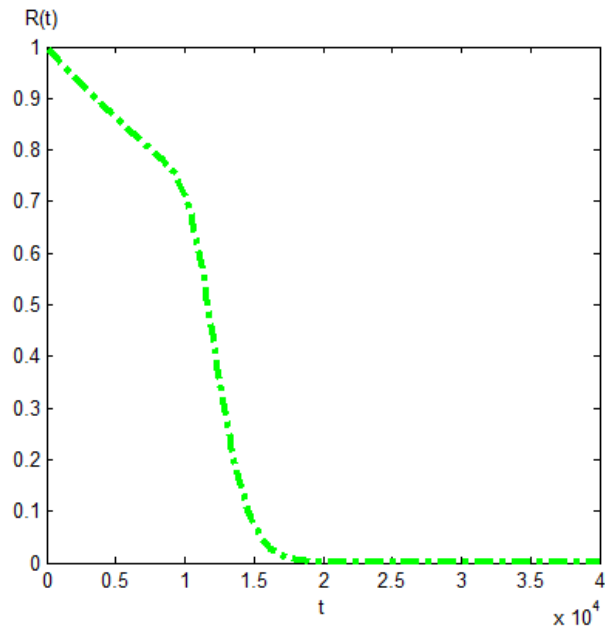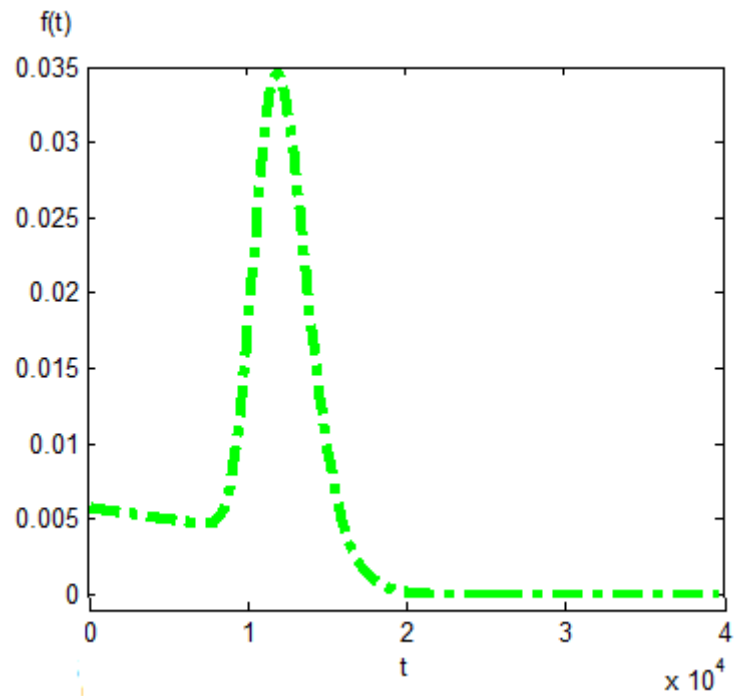

Figure 3.28: System reliability in scenario 3

Figure 3.29: Time to failure in scenario 3

3.2.4 System reliability with dependent *W, Y*

In this scenario, shock damage to hard failure process among components is considered to be dependent, shock damage to soft failure process among components is dependent, and shock damage to two failure processes is also mutually dependent, i.e., W_{ij} and Y_{ij} are dependent. For the jth shock arriving to the system, it causes degradation damage increments to all components in the system $Y_{1j}, Y_{2j},\ldots, Y_{nj}$ and transmitted shock sizes to hard failure processes $W_{1j}, W_{2j},\ldots, W_{nj}$. This scenario is formulated as $W_{ij} = \tilde{W}_{ij} + \alpha_i Z_j$, $Y_{ij} = \tilde{Y}_{ij} + \gamma_i Z_j$, in which Z_j is j^{th} system shock size, and α_i is the transmission parameter from system shock size to hard failure process transmitted shock size, and γ_i is the transmission parameter from shock size to soft failure process damage. $\tilde{W}_{ij}$ and $\tilde{Y}_{ij}$ are damage to two failure processes not proportional to shock size. The common item Z_j makes Y_{ij} dependent, W_{ij} dependent and each pair of W_{ij} and Y_{ij} dependent, which means damages of these two failure processes are mutually dependent. Figure 3.30 shows the scenario with component 1 and 2. It also works for other n - 2 components in the system.
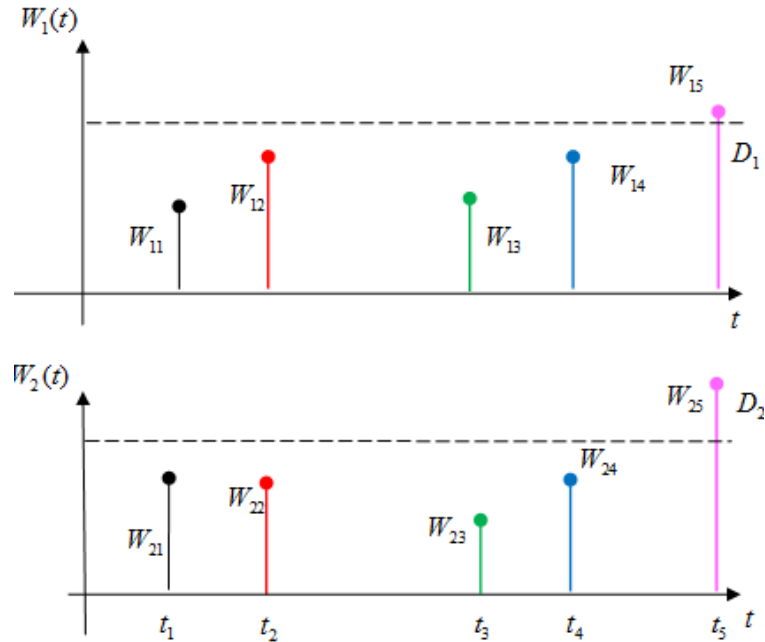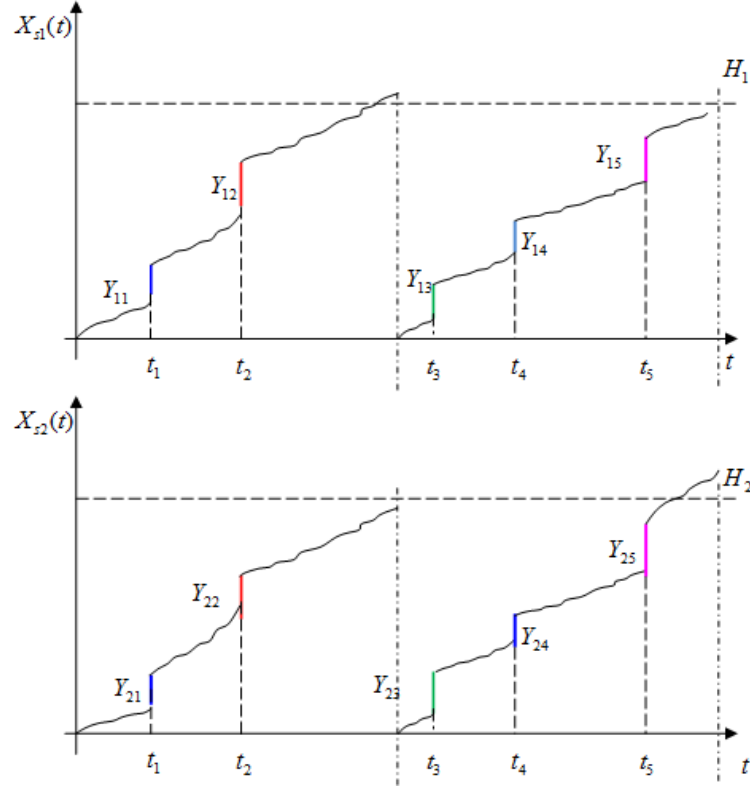
Figure 3.30: Mutually dependent shock damage to two failure processes in scenario 4

For a series system, With $W_{ij} = \tilde{W}_{ij} + \alpha_i Z_j$, $Y_{ij} = \tilde{Y}_{ij} + \gamma_i Z_j$ and conditioning on the number of shocks

$$R(t) = \sum_{m=0}^{\infty} P\left\{ \left[\tilde{W}_{11} + \alpha_1 Z_1 < D_1, \tilde{W}_{21} + \alpha_2 Z_1 < D_2, ..., \tilde{W}_{n1} + \alpha_n Z_1 < D_n, X_1(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{1j} + \beta_1 Z_j \right) < H_1 \right] \cap \right.$$

$$\left[\tilde{W}_{12} + \alpha_1 Z_2 < D_1, \tilde{W}_{22} + \alpha_2 Z_2 < D_2, ..., \tilde{W}_{n2} + \alpha_n Z_2 < D_n, X_2(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{2j} + \beta_2 Z_j \right) < H_2 \right] \cap ...$$

$$\left. \left[\tilde{W}_{1m} + \alpha_1 Z_m < D_1, \tilde{W}_{2m} + \alpha_2 Z_m < D_2, ..., \tilde{W}_{nm} + \alpha_n Z_m < D_n, X_n(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{nj} + \beta_n Z_j \right) < H_n \right] \right\}$$

$$\times P(N(t) = m)$$

(92)

Recursively conditioning on the specific values z_j for sizes of shocks Z_j arriving to the system, and integrating over all the value of z_j:

$$R(t) = \sum_{m=0}^{\infty} \int_{z_m} ... \int_{z_1} P \left\{ \left[\tilde{W}_{11} + \alpha_1 Z_1 < D_1, \tilde{W}_{21} + \alpha_2 Z_1 < D_2, ..., \tilde{W}_{n1} + \alpha_n Z_1 < D_n, X_1(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{1j} + \beta_1 Z_j \right) < H_1 \right] \cap \right.$$

$$\left[\tilde{W}_{12} + \alpha_1 Z_2 < D_1, \tilde{W}_{22} + \alpha_2 Z_2 < D_2, ..., \tilde{W}_{n2} + \alpha_n Z_2 < D_n, X_2(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{2j} + \beta_2 Z_j \right) < H_2 \right] \cap ...$$

$$\left. \left[\tilde{W}_{1m} + \alpha_1 Z_m < D_1, \tilde{W}_{2m} + \alpha_2 Z_m < D_2, ..., \tilde{W}_{nm} + \alpha_n Z_m < D_n, X_n(t) + \sum_{j=1}^{m} \left(\tilde{Y}_{nj} + \beta_n Z_j \right) < H_n \right] | Z_1 = z_1, ..., Z_m = z_m \right\}$$

$$\times f_{Z_1}(z_1) dz_1 ... f_{Z_m}(z_m) dz_m P(N(t) = m)$$

(93)

After conditioning on Z_j, two failure processes for all components are independent.

Eq. (93) can be expressed as:

$$R(t) = \sum_{m=0}^{\infty} \int_{z_m} ... \int_{z_1} \left[\prod_{j=1}^{m} P \left(\tilde{W}_{1j} < D_1 - \alpha_1 z_j, ..., \tilde{W}_{nj} < D_n - \alpha_n z_j \right) \right] \prod_{i=1}^{n} P \left(X_i(t) + \sum_{j=1}^{m} \tilde{Y}_{ij} < H_i - \beta_i \sum_{j=1}^{m} z_j \right)$$

$$\times f_{Z_1}(z_1) dz_1 ... f_{Z_m}(z_m) dz_m \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

(94)

$$R(t) = \sum_{m=0}^{\infty} \int_{z_m} ... \int_{z_1} \left[\prod_{i=1}^{n} F_{\tilde{W}_i} \left(D_i - \alpha_i z_j \right) \right]^m \prod_{i=1}^{n} P \left(X_i(t) + \sum_{j=1}^{m} \tilde{Y}_{ij} < H_i - \beta_i \sum_{j=1}^{m} z_j \right)$$

$$\times f_{Z_1}(z_1) dz_1 ... f_{Z_m}(z_m) dz_m \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

(95)

Conditioning on the specific value y for $\sum_{j=1}^{m} \tilde{Y}_{ij}$, and integrating over all the value of

y

$$R(t) = \sum_{m=0}^{\infty} \int_{z_m} ... \int_{z_1} \left[\prod_{i=1}^{n} F_{\tilde{W}_i} \left(D_i - \alpha_i z_j \right) \right]^m \prod_{i=1}^{n} \int_{y} P \left(X_i(t) + \sum_{j=1}^{m} \tilde{Y}_{ij} < H_i - \beta_i \sum_{j=1}^{m} z_j \mid \sum_{j=1}^{m} \tilde{Y}_{ij} = y \right)$$

$$\times f_{Y_i}^{\langle m \rangle}(y) dy f_{Z_1}(z_1) dz_1 ... f_{Z_m}(z_m) dz_m \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

(96)

Replace $\sum_{j=1}^{m} \tilde{Y}_{ij}$ with y, and subtract it from both sides in the equation, Eq. (86) can

now be expressed as

$$R(t) = \sum_{m=0}^{\infty} \int_{z_m} \cdots \int_{z_1} \left[\prod_{i=1}^{n} F_{\tilde{W}_i}\left(D_i - \alpha_i z_j \right) \right]^m \prod_{i=1}^{n} \int_y P\left(X_i(t) < H_i - \beta_i \sum_{j=1}^{m} z_j - y \right) f_{Y_i}^{\langle m \rangle}(y) dy$$

$$\times f_{Z_1}(z_1) dz_1 \ldots f_{Z_m}(z_m) dz_m \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

(97)

If random effect model is considered for degradation, and define $G(x_i, t)$ to be the

cdf of $X_i(t)$ at t:

$$R(t) = \sum_{m=0}^{\infty} \int_{z_m} \cdots \int_{z_1} \left[\prod_{i=1}^{n} F_{\tilde{W}_i}\left(D_i - \alpha_i z_j \right) \right]^m \prod_{i=1}^{n} \int_y G\left(H_i - \beta_i \sum_{j=1}^{m} z_j - y \right)$$

$$\times f_{Y_i}^{\langle m \rangle}(y) dy f_{Z_1}(z_1) dz_1 \ldots f_{Z_m}(z_m) dz_m \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

(98)

For example where W_{ij}, Y_{ij}, Z_j follow normal distribution, the reliability function for the

more specific case can be expressed as:

$$R(t) = \sum_{m=0}^{\infty} \int_{z_m} \cdots \int_{z_1} \left[\prod_{i=1}^{n} \Phi\left(\frac{D_i - \alpha_i z_j - \mu_{\tilde{W}_i}}{\sigma_{\tilde{W}_i}} \right) \right]^m \prod_{i=1}^{n} \int_y \Phi\left(\frac{H_i - \left(\mu_{\gamma_i} t + \varphi_i + m\left(\mu_{\tilde{Y}_i} + \beta_i \mu_{Z_j} \right) \right)}{\sqrt{\sigma_{\gamma_i}^2 t^2 + m\left(\beta_i \sigma_{Z_j}^2 + \sigma_{\tilde{Y}_i}^2 \right)}} \right)$$

$$\times f_{Y_i}^{\langle m \rangle}(y) dy f_{Z_1}(z_1) dz_1 \ldots f_{Z_m}(z_m) dz_m \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

(99)

Eq. (99) involves infinite number of integrals when the number of shocks by time t

goes to infinity. This problem cannot be solved numerically, but can be solved efficiently

using simulation.

Using a gamma process to model degradation:

$$R(t) = \sum_{m=0}^{\infty} \int_{z_m} \cdots \int_{z_1} \left[\prod_{i=1}^{n} F_{\tilde{W}_i} \left(D_i - \alpha_i z_j \right) \right]^m \prod_{i=1}^{n} \int_y G\left(H_i - \beta_i \sum_{j=1}^{m} z_j - y \right)$$

$$\times f_{Y_i}^{\langle m \rangle}(y) dy f_{Z_1}(z_1) dz_1 \ldots f_{Z_m}(z_m) dz_m \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

(100)

3.2.3.1 Numerical example

A three component series system is used to illustrate the reliability models of different dependent shock damage scenarios. Values of parameters are shown in Table 3.5. For mutually dependent scenario, according to Eq. (99) in Section 3.3, infinite integrals are involved in system reliability equation. For this case, Monte Carlo simulation is used to estimate system reliability. The reliability function $R(t)$ and the pdf of time-to-failure $f_T(t)$ are shown in Figures 3.31 and 3.32.
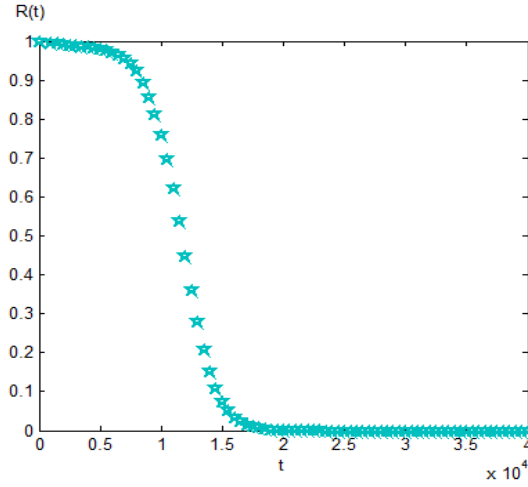

Figure 3.31: System reliability in scenario 4
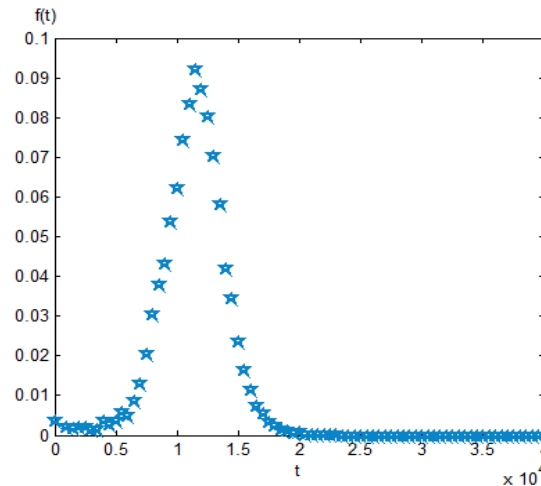
Figure 3.32: Time to failure in scenario 4

3.3 System reliability with stochastically dependent component degradation

In this section, the system reliability is studied for complex multi-component systems with each component subject to multiple failure processes, and component degradation paths are stochastically dependent. This is an entirely new formulation that has not been considered before. The gamma process is used to model the stochastic process of component degradation. In this new model, degradation paths among components are considered to be dependent, which is very challenging but also a very practical situation that could not be neglected.

The dependent characteristic of stochastic degradation paths among components is a challenging issue because it causes the complexity of system reliability modeling and calculation difficulties. However, for some systems, it is practical and realistic concern that should not be neglected. The dependent degradation paths characteristics arise due to different reasons: (1) Components co-exist in the same shared environment, and the factors, such as temperature, humidity and voltage can affect all the component degradation paths

at the same time. For example, in an offshore wind farm, many wind turbines experience similar wind speed and directions, and potentially tidal wave. These are the main factors having effect on wind turbine fatigue and corrosion, which are the main failure mechanisms for wind turbine blade, gear box, etc; (2) Degradation status of some components can directly influence the degradation of other components, and in return, the degradation of other components may also affect the original instigating components or other components, which means they are dependent or correlated. For another example, in an electronic system, a transformer is an electrical device which transfers energy through electromagnetic induction, and its degradation can affect the performance of the central processing unit (CPU). Also, the degradation situation of the CPU can influence the degradation of transformer in return, or have effect on the degradation level of other components, such as a resistance-capacitance filter (RC filter).

There are different models to model stochastic deterioration. Either the simple failure rate function or more complicated stochastic processes such as a random deterioration rate, Markov process, Brownian motion with drift (also called the Wiener process), the compound Poisson process, and the gamma process can be considered.

The Brownian motion with drift is a stochastic process with independent, real-valued increments and decrements having a normal distribution, and the limitation of this model is that it is not monotonically increasing or decreasing. Thus, it is not appropriate for many hardware design applications whose degradation is a monotone function. The compound Poisson process is a stochastic process with *i.i.d* jumps which occur according to a Poisson process. It has a finite number of jumps in finite time intervals. Therefore, it is suitable for modelling usage such as damage due to sporadic shocks. A gamma process is a stochastic

process with independent, non-negative increments having a gamma distribution with an identical scale parameter. It has an infinite number of jumps in finite time intervals, and it is suitable to model gradual damage monotonically accumulating over time in a sequence of tiny increments, such as wear, fatigue, corrosion, crack growth, erosion, consumption, etc. An advantage of modeling deterioration processes through gamma processes is that the required mathematical calculations are relatively straightforward. In this section, gamma process is used to model components degradation path.

3.3.1 Reliability modeling for series system

An example system configuration is a series system with n components, in which a component fails when either of the two dependent and competing failure modes occurs, and all components in the system behave similarly. The reliability of this series system at time t is the probability that each component survives each of the $N(t)$ shock loads ($W_{ij} < D_i$ for $j=1, 2, ...$) and the total degradation of each component is less than the soft failure threshold level ($X_{Si}(t) < H_i$) . A gamma process is a random process with independent gamma distributed increments: $X_{si}(t_2) - X_{si}(t_1) \sim \Gamma(v_i(t_2) - v_i(t_1), \theta_i)$. The difference between the degradation of component at two different times follows a gamma process with shape parameter ($v_i(t_2) - v_i(t_1)$) and scale parameter θ_i.

Based on Eq. (8), using gamma process to model component degradation path:

$$R(t) = P\left\{\left[W_{11} < D_1, W_{12} < D_1, ..., W_{1N(t)} < D_1, X_{s1}(t, v_1(t), \theta_1) < H_1\right] \cap \left[W_{21} < D_2, W_{22} < D_2, ..., W_{2N(t)} < D_2, \right.\right.$$
$$\left.\left. X_{s2}(t, v_2(t), \theta_2) < H_2\right] \cap \left[W_{n1} < D_n, W_{n2} < D_n, ..., W_{nN(t)} < D_n, X_{sn}(t, v_n(t), \theta_n) < H_n\right]\right\}$$

(101)

Using the gamma process to model component degradation path

$$R(t) = \sum_{m=0}^{\infty} P \left\{ \left[W_{11} < D_1, W_{12} < D_1, ..., W_{1N(t)} < D_1, X_1(t, v_1(t), \theta_1) + \sum_{j=1}^{N(t)} Y_{1j} < H_1 \right] \cap ... \right.$$

$$\left[W_{21} < D_2, W_{22} < D_2, ..., W_{2N(t)} < D_2, X_2(t, v_2(t), \theta_2) + \sum_{j=1}^{N(t)} Y_{2j} < H_2 \right] \cap ...$$

$$\left. \left[W_{n1} < D_n, W_{n2} < D_n, ..., W_{nN(t)} < D_n, X_n(t, v_n(t), \theta_n) + \sum_{j=1}^{N(t)} Y_{nj} < H_n \right] \middle| N(t) = m \right\}$$

$$\times P(N(t) = m)$$

(102)

$v_i(t)$ is a non-decreasing, right-continuous function for $t > 0$. Separating the hard failure process and soft failure process, Eq. (102) can be re-written:

$$R(t) = \sum_{m=0}^{\infty} P\left\{ [W_{11} < D_1, ..., W_{1m} < D_1] \cap [W_{21} < D_2, ..., W_{2m} < D_2] \cap ... [W_{n1} < D_n, ..., W_{nm} < D_n] \right\}$$

$$P\left(X_1(t, v_1(t), \theta_1) + \sum_{j=1}^{m} Y_{1j} < H_1, ..., X_n(t, v_n(t), \theta_n) + \sum_{j=1}^{m} Y_{nj} < H_n \right) P(N(t) = m)$$

(103)

Assuming W_{ij} to be *i.i.d.* random variables:

$$R(t) = \sum_{m=0}^{\infty} P\left(W_{i1} < D_i, W_{i2} < D_i, ..., W_{ij} < D_i \right)^m P\left(X_1(t, v_1(t), \theta_1) + \sum_{j=1}^{m} Y_{1j} < H_1, \right.$$

$$\left. X_2(t, v_2(t), \theta_2) + \sum_{j=1}^{m} Y_{2j} < H_2, ..., X_n(t, v_n(t), \theta_n) + \sum_{j=1}^{m} Y_{nj} < H_n \right) P(N(t) = m)$$

(104)

We use the gamma process to model component stochastic degradation path. As mentioned above, degradation paths among components are dependent. Define θ as random variable. By assuming $\theta_i = \alpha_i \theta$, the common θ can achieve the dependent characteristic. Conditioning on θ and integrating yields

$$R(t) = \sum_{m=0}^{\infty} \prod_{i=1}^{n} F_{W_i} (D_i)^m \int_{\theta} P\left(X_1(t, v_1(t), \alpha_1 \theta) + \sum_{j=1}^{m} Y_{1j} < H_1, \right.$$

$$X_2(t, v_2(t), \alpha_2 \theta) + \sum_{j=1}^{m} Y_{2j} < H_2, ..., X_n(t, v_n(t), \alpha_n \theta) + \sum_{j=1}^{m} Y_{nj} < H_n \mid \theta = \vartheta \left. \right) f_\theta(\vartheta) d\vartheta \ P(N(t) = m)$$

$$(105)$$

$$R(t) = \sum_{m=0}^{\infty} \prod_{i=1}^{n} F_{W_i} (D_i)^m \int_{\theta} \prod_{i=1}^{n} P\left(X_i(t, v_i(t), \alpha_i \vartheta) + \sum_{j=1}^{m} Y_{ij} < H_i \right) f_\theta(\vartheta) d\vartheta \ \frac{e^{-\lambda t} (\lambda t)^m}{m!}$$

$$(106)$$

Conditioning on the sum of Y_{ij} and integrating yield:

$$R(t) = \sum_{m=0}^{\infty} \prod_{i=1}^{n} F_{W_i} (D_i)^m \int_{\theta} \prod_{i=1}^{n} \int_0^{H_i} P\left(X_i(t, v_i(t), \alpha_i \vartheta) + \sum_{j=1}^{m} Y_{ij} < H_i \mid \sum_{j=1}^{m} Y_{ij} = q \right)$$

$$\times f_{Y_i}^{<m>}(q) dq \ f_\theta(\vartheta) d\vartheta \ \frac{e^{-\lambda t} (\lambda t)^m}{m!} \tag{107}$$

$$R(t) = \sum_{m=0}^{\infty} \prod_{i=1}^{n} F_{W_i} (D_i)^m \int_{\theta} \prod_{i=1}^{n} \int_0^{H_i} P\left(X_i(t, v_i(t), \alpha_i \vartheta) < H_i - q \right)$$

$$\times f_{Y_i}^{\langle m \rangle}(q) dq f_\theta(\vartheta) d\vartheta \ \frac{e^{-\lambda t} (\lambda t)^m}{m!} \tag{108}$$

With the gamma process used for degradation path $X_i(t)$:

$$R(t) = \sum_{m=0}^{\infty} \prod_{i=1}^{n} F_{W_i} (D_i)^m \int_{\theta} \prod_{i=1}^{n} \int_0^{H_i} Ga\left(X_i \mid v_i(t), \alpha_i \vartheta \right) f_{Y_i}^{\langle m \rangle}(q) dq \ f_\theta(\vartheta) d\vartheta \ \frac{e^{-\lambda t} (\lambda t)^m}{m!}$$

$$(109)$$

$$R(t) = \sum_{m=0}^{\infty} \prod_{i=1}^{n} F_{W_i} (D_i)^m \int_{\theta} \prod_{i=1}^{n} \int_0^{H_i} \left(1 - \frac{\Gamma\left(v_i(t), (H_i - q) \alpha_i \vartheta \right)}{\Gamma\left(v_i(t) \right)} \right) f_{Y_i}^{\langle m \rangle}(q) dq f_\theta(\vartheta) d\vartheta \ \frac{e^{-\lambda t} (\lambda t)^m}{m!}$$

$$(110)$$

Where $\Gamma(a,x) = \int\limits_{z=x}^{\infty} z^{a-1}e^{-z}dz$ and $\Gamma(a) = \int\limits_{z=0}^{\infty} z^{a-1}e^{-z}dz$, and $v_i(t)$ is a non-decreasing,

right-continuous function for $t > 0$. Eq. (98) is the general reliability model for system with dependent component stochastic degradation processes. Empirical studies show the deterioration at time t is often proportional to a power law, which means $v_i(t)=c_i t^{b_i}$. (1) degradation of concrete due to corrosion $b_i=1$, and it is linear; (2) sulphate attack (parabolic $b_i=2$); (3) diffusion-controlled ageing (square root, $b_i=0.5$), and other cases. Substitute $v(t)$ with $c_i t^{b_i}$

$$R(t) = \sum_{m=0}^{\infty} \prod_{i=1}^{n} F_{W_i}(D_i)^m \int_\theta \prod_{i=1}^{n} \int_0^{H_i} \left(1 - \frac{\Gamma\left(c_i t^b, (H_i - q)\alpha_i \vartheta\right)}{\Gamma\left(c_i t^b\right)} \right) f_{Y_i}^{\langle m \rangle}(q)dq\, f_\theta(\vartheta)d\vartheta\; \frac{e^{-\lambda t}(\lambda t)^m}{m!}$$

(111)

$$R(t) = \sum_{m=0}^{\infty} \prod_{i=1}^{n} F_{W_i}(D_i)^m \int_\theta \prod_{i=1}^{n} \int_0^{H_i} \left(1 - \frac{\int\limits_{(H_i-q)\alpha_i\vartheta}^{\infty} z^{c_i t^b - 1}e^{-z}dz}{\int\limits_0^{\infty} z^{c_i t^b - 1}e^{-z}dz} \right) f_{Y_i}^{\langle m \rangle}(q)dq\, f_\theta(\vartheta)d\vartheta\; \frac{e^{-\lambda t}(\lambda t)^m}{m!}$$

(112)

$$R(t) = \sum_{m=0}^{\infty} \prod_{i=1}^{n} F_{W_i}(D_i)^m \int_\theta \prod_{i=1}^{n} \int_0^{H_i} \left(\frac{\int\limits_0^{(H_i-q)\alpha_i\vartheta} z^{c_i t^b - 1}e^{-z}dz}{\int\limits_0^{\infty} z^{c_i t^b - 1}e^{-z}dz} \right) f_{Y_i}^{\langle m \rangle}(q)dq\, f_\theta(\vartheta)d\vartheta\; \frac{e^{-\lambda t}(\lambda t)^m}{m!}$$

(113)

A special case is when $m=0$ by time t, which means pure degradation, i.e., no shock come to the system. With $b_i= 0.5$:

$$R(t) = \int_\theta \prod_{i=1}^{n} \left(\frac{\int\limits_0^{H_i\alpha_i\vartheta} z^{c_i \sqrt{t} - 1}e^{-z}dz}{\int\limits_0^{\infty} z^{c_i \sqrt{t} - 1}e^{-z}dz} \right) f_\theta(\vartheta)d\vartheta$$

(114)

3.3.2 Reliability modeling for parallel system

The reliability of a parallel system at time t is the probability that at least one component of this system survives each of the $N(t)$ shock loads ($W_{ij}<D_i$ for $j=1, 2, \ldots$),and the total degradation of that same component is less than the threshold level ($X_{S_i}(t) < H_i$). The system fails when all components experience either soft failure or hard failure. System reliability for a parallel system with dependent degradation paths is given by the following equations.

$$R(t) = 1 - \sum_{m=0}^{\infty} \int_{\theta} \prod_{i=1}^{n} \int_{0}^{H_i} \left[1 - F_{W_i}(D_i)^m Ga\left(X_i \mid v_i(t), \alpha_i \vartheta\right) \right] f_{Y_i}^{\langle m \rangle}(q) dq\, f_{\theta}(\vartheta) d\vartheta \frac{e^{-\lambda t}(\lambda t)^m}{m!}$$

(115)

$$R(t) = 1 - \sum_{m=0}^{\infty} \int_{\theta} \prod_{i=1}^{n} \int_{0}^{H_i} \left[1 - F_{W_i}(D_i)^m \left(1 - \frac{\Gamma\left(v_i(t),(H_i-q)\alpha_i\vartheta\right)}{\Gamma\left(v_i(t)\right)} \right) \right] f_{Y_i}^{\langle m \rangle}(q) dq\, f_{\theta}(\vartheta) d\vartheta \frac{e^{-\lambda t}(\lambda t)^m}{m!}$$

(116)

3.3.3 Reliability modeling for series-parallel system

The reliability of a series-parallel system at time t is the probability that at least one component within each subsystem survives each of the $N(t)$ shock loads ($W_{ij} < D_i$ for $j= 1$, $2, \ldots$), and the total degradation is less than the threshold level ($X_{S_i}(t) < H_i$) for that same component. The system fails when all components for at least one parallel subsystem experience either soft failure or catastrophic failure.

System reliability for a series-parallel system is given by the following equations for the two specific cases previously introduced.

$$R(t) = \sum_{m=0}^{\infty} \int_{\theta} \prod_{l=1}^{s} \left\{ 1 - \prod_{i \in S_l} \int_{0}^{H_i} \left[1 - F_{W_i}(D_i)^m Ga\left(X_i \mid v_i(t), \alpha_i \vartheta\right) \right] f_{Y_i}^{\langle m \rangle}(q) dq \right\} f_{\theta}(\vartheta) d\vartheta \frac{e^{-\lambda t}(\lambda t)^m}{m!}$$

$$(117)$$

$$R(t) = \sum_{m=0}^{\infty} \int_{\theta} \prod_{l=1}^{s} \left\{ 1 - \prod_{i \in S_l} \int_0^{H_i} \left[1 - F_{W_i}(D_i)^m \left(1 - \frac{\Gamma\left(v_i(t), (H_i - q)\alpha_i \vartheta\right)}{\Gamma\left(v_i(t)\right)} \right) \right] f_{Y_i}^{\langle m \rangle}(q) dq \right\} f_{\theta}(\vartheta) d\vartheta \frac{e^{-\lambda t}(\lambda t)^m}{m!}$$

$$(118)$$

3.3.4 Numerical example

Considering a series system with four components, the parameters for reliability analysis are provided in Table 3.6. For this example, W_{ij} and Y_{ij} follow normal distributions, and $X_i(t)$ is modeled as a gamma process with $v_i(t)$ and $\theta_i = \alpha_i \theta$. Without loss of generality, parameters of component 1 and 2 are assumed to be the same, and parameters of component 3 and 4 are the assumed to be same. Figure 3.33 illustrates the reliability of the system. Figure 3.34 shows time-to-failure pdf.

Table 3.6. Parameter values for multi-component system reliability analysis

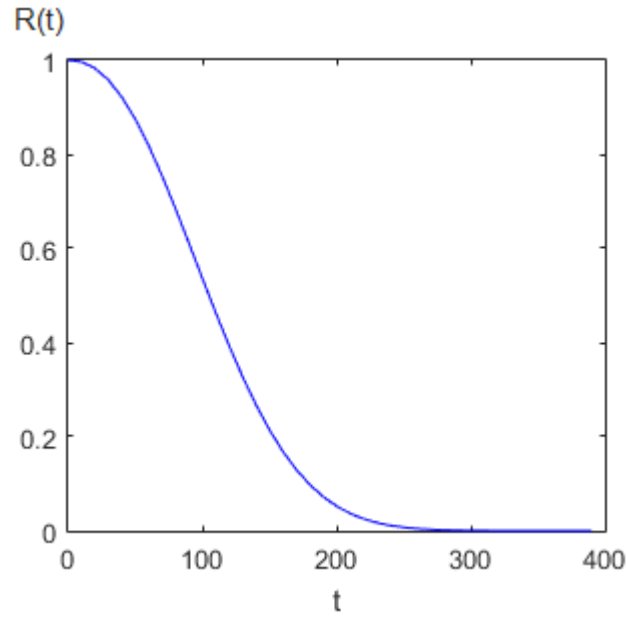| Parameter	component 1 & 2	component 3 & 4
H_i	0.00125	0.00127
D_i	1.5	1.6
φ_i	0	0
c_i	2.5×10^{-6}	2.5×10^{-6}
λ	2.5×10^{-5}	2.5×10^{-5}
Y_{ij}	$Y_{ij} \sim N(\mu_{Yi}, \sigma_{Yi}^2)$ $\mu_{Yi} = 7 \times 10^{-5}$, $\sigma_{Yi} = 1.6 \times 10^{-5}$	$Y_{ij} \sim N(\mu_{Yi}, \sigma_{Yi}^2)$ $\mu_{Yi} = 6 \times 10^{-5}$, $\sigma_{Yi} = 1.5 \times 10^{-5}$
W_{ij}	$W_{ij} \sim N(\mu_{Wi}, \sigma_{Wi}^2)$ $\mu_{Wi} = 1.2$, $\sigma_{Wi} = 0.16$	$W_{ij} \sim N(\mu_{Wi}, \sigma_{Wi}^2)$ $\mu_{Wi} = 1.22$, $\sigma_{Wi} = 0.18$
α_i	2	2
b	2.2	2.2

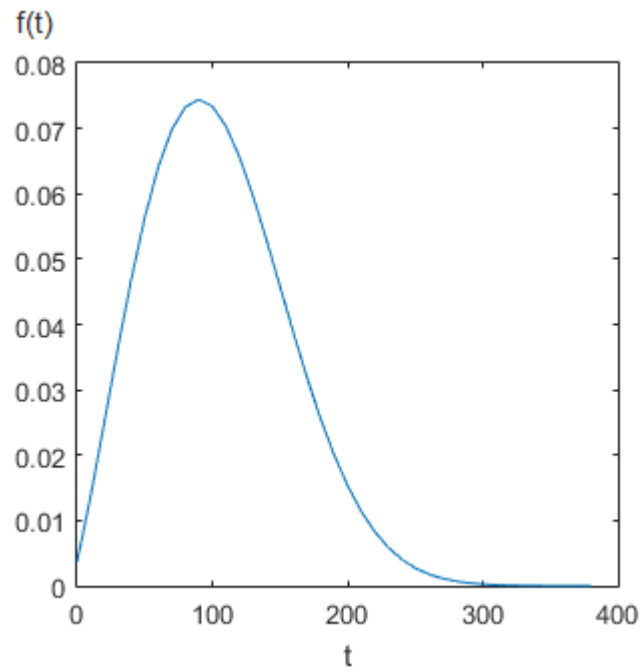Figure 3.33: Four components series system reliability *R(t)*

Figure 3.34: Time-to-failure distribution for the system

Figures 3.33 and 3.34 show that system reliability decreases dramatically before time reaches $t=200$. After that, reliability is below 0.1 and changes slowly. This new model is important for system maintenance policy and maintenance frequency decisions. To better

understand parameter effect on system reliability, sensitivity analysis is conducted The

interesting parameters are α_i, and two parameters in $v_i(t)$: c_i and b_i.

The gamma process is chosen to model component stochastic degradation path.

Degradation paths among components are dependent through assuming parameter: $\theta_i = \alpha_i \theta$

, i.e., the common item θ can achieve the dependent characteristic. For Eq. (103), if α_i

increases, the upper bound of the integral for z increases, and the whole value of the

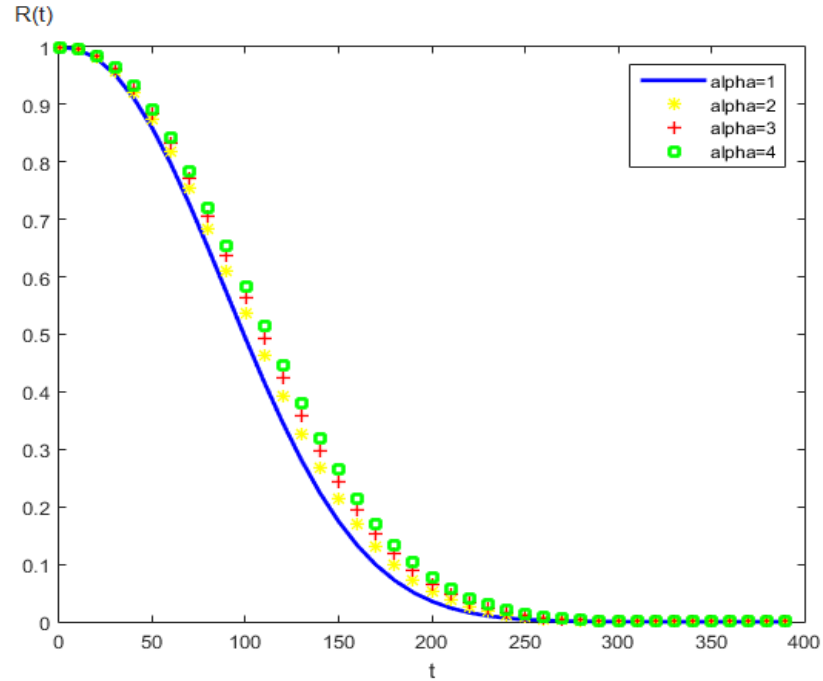numerator increase. Hence, system reliability increases. It is reflected in Figure 3.35.

Figure 3.35: System reliability sensitivity analysis for parameter α_i
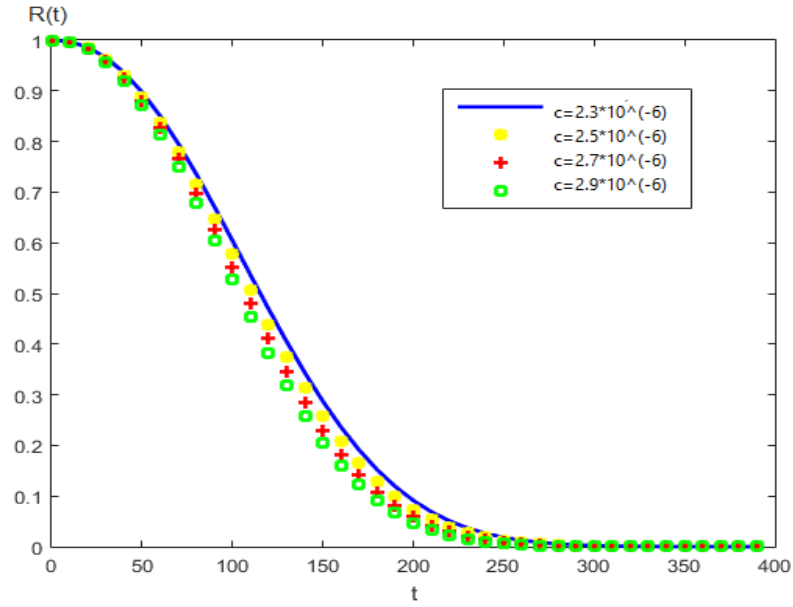
Figure 3.36: System reliability sensitivity analysis for parameter c_i

Empirical studies show the deterioration at time t is often proportional to a power law, which means $v_i(t)=c_i t^{bi}$. Given the same b_i value, when c_i increases, components in the system degrade faster. Therefore the system has the higher probability to fail given the same time. Figure 3.36 shows when c_i value increases, system reliability decreases.
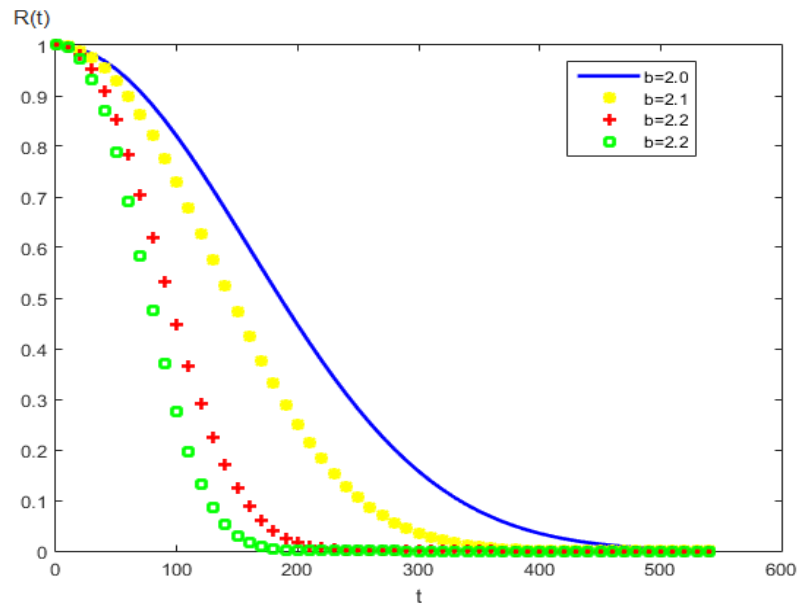

Figure 3.37: System reliability sensitivity analysis for parameter α_i

The simple stochastic process is defined as a time-dependent function for which the average rate of degradation per unit time is a random quantity. For example, $X_i(t)=At$, where the average degradation rate A has a probability distribution. Empirical studies show the degradation at time t is often proportional to a power law, which means $c_i t^{bi}$. Different b_i value can affect reliability. From **Figure** 3.37, when b_i value increases, i.e., degradation rate increases, each component in the system degrades faster. Therefore, the system reliability value for higher b_i is smaller for any specific time t.

For a parallel system with two components, the parameters for reliability analysis are provided in Table 3.6. Figure 3.38 illustrates the reliability of the system. Figure 3.39 shows the time-to-failure pdf distribution.
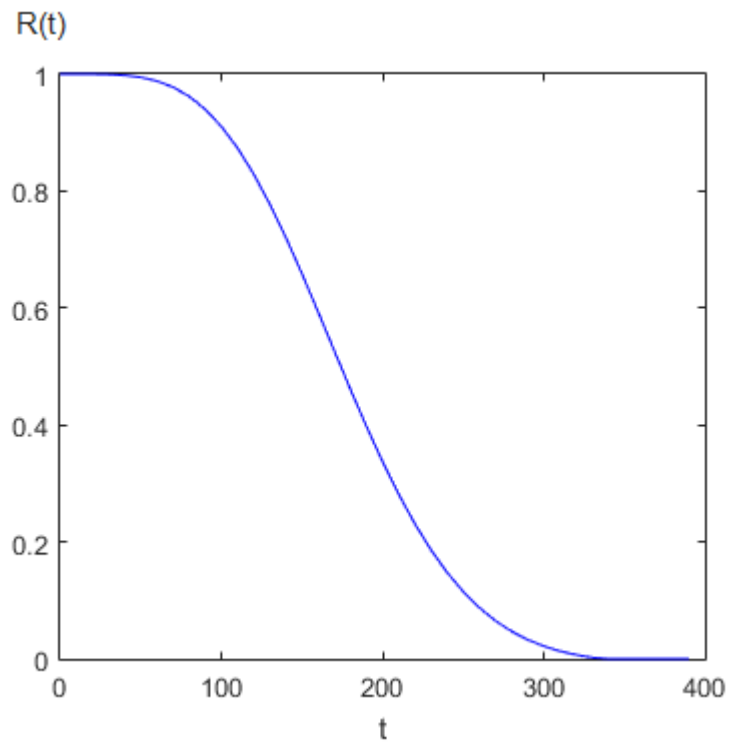

Figure 3.38: Two component parallel system reliability $R(t)$
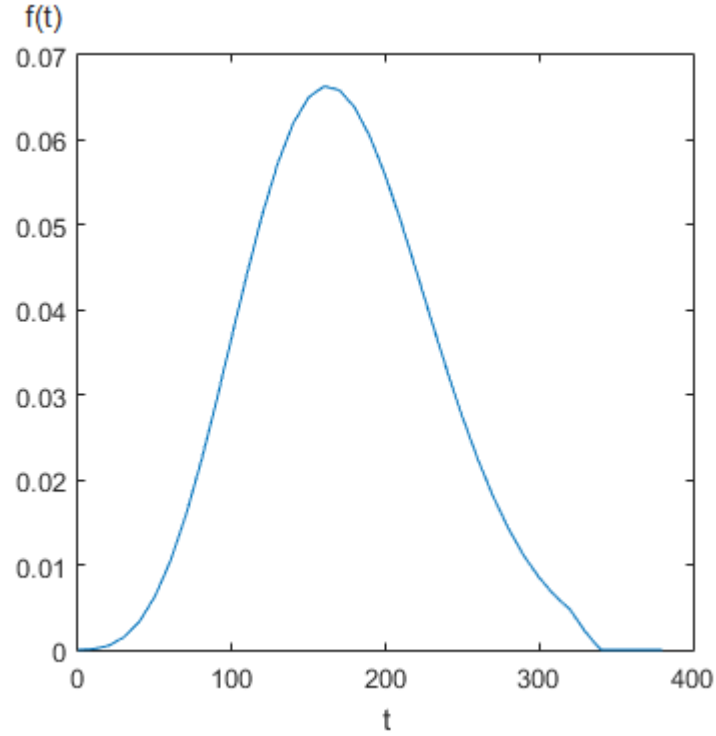
Figure 3.39: Time-to-failure distribution for the parallel system

We developed a new reliability model for systems with dependent component degradation paths, and the gamma process is chosen for the stochastic degradation modeling. For systems with components that share the similar environment or degradation status of some components that can directly or indirectly affect the degradation of other components, it is more practical and realistic to model the reliability in this way, which can provide better and accurate estimation. A numerical example of a four component series system is illustrated to demonstrate the reliability modeling. Sensitivity analysis is conducted for parameter α_i in the gamma process, and parameter c_i and b_i in degradation model. Higher α_i, and lower c_i and b_i value increase the system reliability.

4. Maintenance Policies and Optimization

Systems used in the production of goods and delivery of services constitute the vast majority of most industry's capital. These systems are subject to deterioration with usage and age. Most of them are maintained or repairable systems. For some systems, such as aircraft, submarines, military systems, and nuclear systems, it is important to avoid failure during actual operation because it can be dangerous or disastrous. Therefore, both corrective and preventive maintenance to these systems is necessary since it can improve availability and minimize life cycle costs. The growing importance of maintenance has generated an increasing interest in the development and implementation of optimal maintenance strategies for improving system performance, preventing the occurrence of system failures, and reducing maintenance costs of deteriorating systems. In this section, different maintenance policies are considered and the maintenance optimization problems are solved based on the new reliability models developed in Section 3.

New reliability models have been developed for complex system with components subject to multiple failure processes. Since they are new reliability models considering more complicated dependent scenarios, no maintenance policies have been developed for the system under these conditions. In this section, first, the traditional age replacement policy and periodic inspection policy are determined for the systems based on the new reliability modeling. Furthermore, condition-based maintenance policy combining advantages of other maintenance policies is considered for multi-component system based on the new reliability modeling, which has not been studied before. All these policies are based on the condition that if one component fails, the whole system is replaced. Though it is challenging, an individual maintenance policy is more practical, realistic and cost

effective for many engineering system applications. In this section, the individual component maintenance policy is studied and the inspection interval is optimized based on steady state system behavior using two methods, including: (1) uniform distribution approximation of initial degradation; and (2) geometric distribution of component survival intervals. It is the first time individual component maintenance policy is considered for a complex system with components subject to multiple failure processes.

Specific assumptions used for the maintenance modeling in this section are as follows:

1. In Section 4.1, models are developed for systems that are assumed to be packaged and sealed together, making it impossible or impractical to repair or replace individual components within the system, e.g., MEMS, encapsulated printed circuit cards. In Section 4.3, the individual component maintenance instead of group maintenance is studied, which is more cost effective.

2. For age replacement maintenance policy, the system is preventively replaced at a fixed age. However, if the system fails before the specified age, it is replaced correctively immediately. Replacements are assumed to be instantaneous and perfect, when it is applicable.

3. For a periodic inspection maintenance policy, the system is inspected at periodic intervals. If the system fails before the specified inspection interval, it is not replaced until the next inspection. There is penalty cost associated with failures of the system during downtime, e.g., cost associated with loss of production or opportunity costs.

4. A condition-based maintenance policy is studied with the following policy: (1) The system is defined to be within a safe region upon an inspection when degradation

is below a defined on-condition threshold, and nothing is done; (2) It is working, but has a high probability to fail soon when degradation is between the on-condition threshold and a failure threshold upon the inspection. In this case, the system is replaced preventively; (3) At any time, the system can fail when degradation is above the failure threshold. Failure is not detected until the next inspection schedule, and the system is correctively replaced with a new one upon the inspection.

5. For a series systems, the system fails when the first component fails. For *k*-out-of-*n* system, the system works satisfactorily when at least *k* components survive both soft failure and hard failure processes. Parallel systems fail when all components experience either soft failure or hard failure. The reliability of a series-parallel system at time *t* is the probability that at least one component within each subsystem survives both failure processes.

4.1 Age replacement and periodic inspection maintenance policy

Maintenance policies are determined for systems to achieve high system availability and low cost. The objective function studied in this research is the maintenance cost rate, which is the ratio of total maintenance cost and time duration associated with the cost. We are interested in the long time maintenance cost rate, but one life cycle can be considered as equivalent because successive life cycles behave similarly. Considering multi-component systems subject to multiple dependent competing failure processes, there can be different applicable maintenance strategies for systems subject to multiple failure processes based on the new reliability models. Maintenance policies can be developed for any or all of the reliability models developed in Section 3, but only some examples are

demonstrated in this section. For series systems, two different maintenance policies are considered. First, an age replacement policy with a fixed replacement interval is considered for a system with multiple components each exposed to two competing dependent failure processes. Then, a periodic inspection maintenance policy is considered for the same system.

4.1.1 Series system

For the age replacement policy, to evaluate the performance of the maintenance policy, an average long-run maintenance cost rate model is used, in which the periodic replacement interval V is the decision variable. In the model at time V, the system is replaced with a new one, with all new components. However, if the system fails before time V, it is replaced immediately. The average long-run total maintenance cost per unit time can be evaluated by:

$$\lim_{t \to \infty}(C(t)/t) = \frac{\text{Expected maintenance cost between two replacements}}{\text{Expected time between two replacements}} = \frac{E(TC)}{E(U)}$$

(119)

where TC is the total maintenance cost of a renewal cycle, and U is the length of a cycle that takes a value of V or time-to-failure T if $T < V$ [57]. Then the expected total maintenance cost is given as

$$E(TC) = C_F F_T(V) + C_R$$

(120)

where C_F is the cost (both direct and indirect) that can be attributed to a unanticipated failure, C_R is the replacement cost, and V is the time of the periodic replacement. Expected time between two replacements or expected cycle length is

$$E(U) = VR(V) + \int_0^V tf_T(t)\mathrm{d}t = \int_0^V R(t)dt$$

(121)

Based on Eqs. (12, 119-121), the average long-run maintenance cost rate as a function of *V*, *CR(V)*, is given as

$$
CR(V) = \frac{C_F\left(1 - \sum_{m=0}^{\infty}\prod_{i=1}^{n}\left\{P_{Li}{}^m\Phi(\frac{H_i - (\mu_{\beta i}V + \varphi_i + m\mu_{Yi})}{\sqrt{\sigma_{\beta i}^2 V^2 + m\sigma_{Yi}^2}})\right\}\frac{\exp(-\lambda V)(\lambda V)^m}{m!}\right) + C_R}{\int_0^V\left(\sum_{m=0}^{\infty}\prod_{i=1}^{n}\left\{P_{Li}{}^m\Phi(\frac{H_i - (\mu_{\beta i}t + \varphi_i + m\mu_{Yi})}{\sqrt{\sigma_{\beta i}^2 t^2 + m\sigma_{Yi}^2}})\right\}\frac{\exp(-\lambda t)(\lambda t)^m}{m!}\right)dt}
$$

(122)

To obtain an analytical result of the optimal solution, the first derivative of the objective function is calculated in Eq. (122), as given below:

$$
CR'(V) = \frac{C_F f_T(V)\left[V - \int_0^V F_T(t)dt\right] - \left[C_F F_T(V) + C_R\right]\left[1 - F_T(V)\right]}{\left[\int_0^V R(t)dt\right]^2} = 0
$$

(123)

Different optimization techniques that are found in the literature can be broadly classified into three categories: calculus-based techniques, enumerative techniques, and stochastic search algorithms. Numerical methods, also called calculus-based methods, use a set of necessary and sufficient conditions that must be satisfied by the solution of the optimization problem. Enumerative techniques involve evaluating each and every point of the finite, or discretized infinite, search space in order to arrive at the optimal solution. Guided random or stochastic search techniques are based on enumerative methods, but they use additional information about the search space to guide the search to potentially more promising regions of the search space. For the age replacement policy optimization problem, the objective function is cost rate or maintenance cost per unit time, and the only decision variable is system replacement interval. In this case, an enumerative technique is

used.

The second model is a maintenance policy with an inspection interval. At intervals of time τ, the system is inspected. If the system fails before time τ, it is not replaced until the next inspection. If the system is still operating satisfactorily with no failed components, nothing is done. For this model, average long-run maintenance cost rate model is also used, in which the periodic inspection interval τ is the decision variable:

$$\lim_{t \to \infty}(C(t)/t) = \frac{\text{Expected maintenance cost between two replacements}}{\text{Expected time between two replacements}} = \frac{E(TC)}{E(K)}$$

(124)

where TC is the total maintenance cost of a renewal cycle, and K is the length of a cycle that takes a value of a multiple of τ. Then, the expected total maintenance cost is given as

$$E(TC) = C_I E(N_I) + C_\rho E(\rho) + C_R$$

(125)

The expected value of the number of inspections N_I is

$$E(N_I) = \sum_{i=1}^{\infty} i(F_T(i\tau) - F_T((i-1)\tau))$$

(126)

In which, $F_T(t) = 1 - R(t)$, which is the probability of failure. The expected value of system downtime or the expected time from a system failure to the next inspection when the failure is detected is

$$E(\rho) = \sum_{i=1}^{\infty} E[\rho \mid N_I = i]P(N_I = i) = \sum_{i=1}^{\infty}\left(F_T(i\tau) - F_T((i-1)\tau)\right)\int_{(i-1)\tau}^{i\tau}(i\tau - t)dF_T(t)$$

(127)

The expected time between two replacements or expected cycle length is

$$E(K) = \sum_{i=1}^{\infty} E[K \mid N_I = i]P(N_I = i) = \sum_{i=1}^{\infty} i\tau(F_T(i\tau) - F_T((i-1)\tau))$$

(128)

Based on Eq. (125) through (128), the average long-run maintenance cost rate as a function of t is given as

$$CR(\tau) = \frac{C_I E(N_I) + C_\rho E(\rho) + C_R}{E(K)}$$

$$= \frac{C_I \sum_{i=1}^{\infty} i\left(F_T(i\tau) - F_T((i-1)\tau)\right) + C_\rho \sum_{i=1}^{\infty} \left(\left(F_T(i\tau) - F_T((i-1)\tau)\right)\int_{(i-1)\tau}^{i\tau}(i\tau - t)dF_T(t)\right) + C_R}{\sum_{i=1}^{\infty} i\tau(F_T(i\tau) - F_T((i-1)\tau))} \quad (129)$$

Here a numerical example is presented to illustrate the maintenance models. A series system with four components is a typical example for demonstration purposes. Enumerative techniques are used for evaluating each and every point of the finite, or discretized infinite, search space in order to arrive at the optimal solution, and this method is used to solve maintenance optimization problem.

The first maintenance policy is the age replacement policy, i.e., replace the system at a fixed interval V. Choosing $C_F=\$150$ and $C_R=\$30$, according to Eq. (123), the minimum average long-run maintenance cost rate of $\$1.958\times10^{-3}$/cycle, which is obtained at $V^*=0.9\times10^5$, the optimal replacement interval for dependent component system. Figure 4.1 illustrates $CR(V)$ as a function of V.
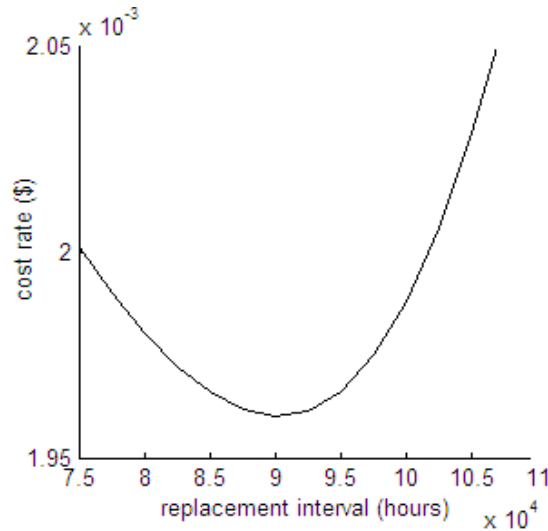
Figure 4.1: Cost rate versus replacement interval for dependent component system

The second maintenance policy is to inspect a system at intervals of τ, and to replace the system when it is observed to have failed. Choosing C_I=\$100, C_ρ=\$200, and C_R=\$200, according to Eq. (129), the minimum average long-run maintenance cost rate of \$0.242/cycle, which is obtained at $\tau^*=1.731\times10^3$ for the dependent component system, as shown in Figure 4.2.
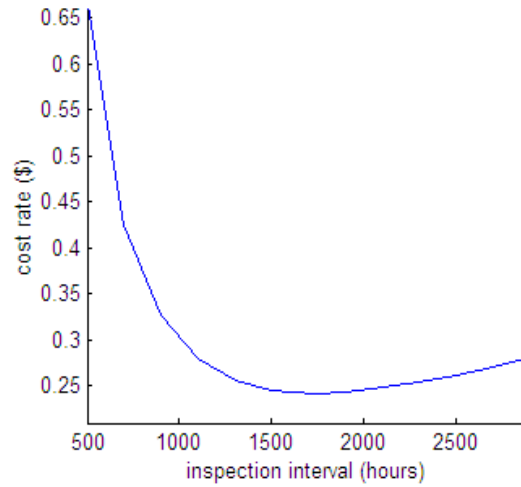

Figure 4.2: Cost rate versus inspection interval for dependent component system

4.1.2 *k*-out-of-*n* System

The reliability of *k*-out-of-*n* systems is studied, and a corresponding reliability model is presented in Section 3.1.1.2. An age replacement policy is considered for *k*-out-of-*n* systems. Choosing C_F=\$150and C_R=\$30, the minimum average long-run maintenance cost rate is \$1.5343\times10^{-3}/cycle, which is obtained at $V^*=1.354\times10^5$, the optimal number of revolutions for periodic replacement. Figure 4.3 illustrates $CR(V)$ as a function of V.

A sensitivity analysis was performed to analyze the effects of the model parameters on the optimal solutions. The parameters of interest from the model are C_R and C_F. The results are shown in Figures 4.4 and 4.5, respectively.
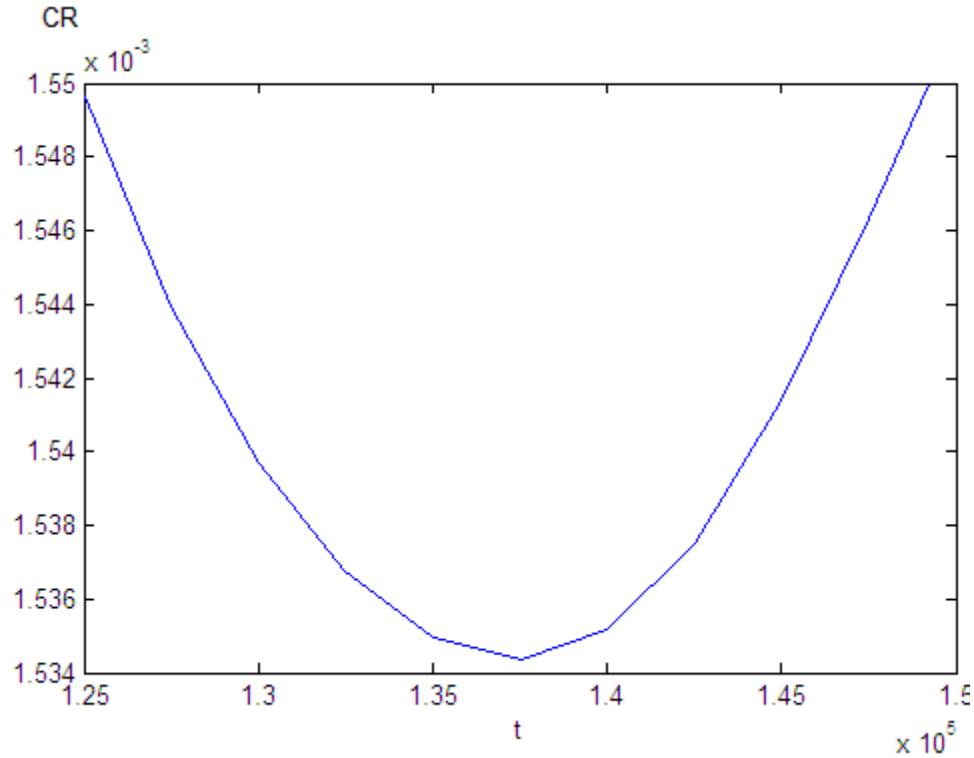
Figure 4.3: Average long-run maintenance cost rate versus replacement interval

When C_F increases from \$120 to \$720 as shown in Figure 4.4, the minimum average long-run maintenance cost rate, $CR(V^*)$, increases from \$0.00128 to \$0.00636, and the optimal replacement interval does not change significantly. This implies that a higher failure cost leads to a higher potential of cost, and this range of failure costs does not have a great effect on the replacement interval.

As shown in Figure 4.5, when C_R increases from \$30 to \$930, the minimum average long-run maintenance cost rate increases from \$0.00153 to \$0.00939, and the optimal replacement interval increases from 1.400×10^5 to 1.475×10^5 revolutions. This indicates that a larger replacement cost results in a longer replacement interval and higher cost rate. As a result, the system should be replaced less frequently when the cost of replacement is higher.
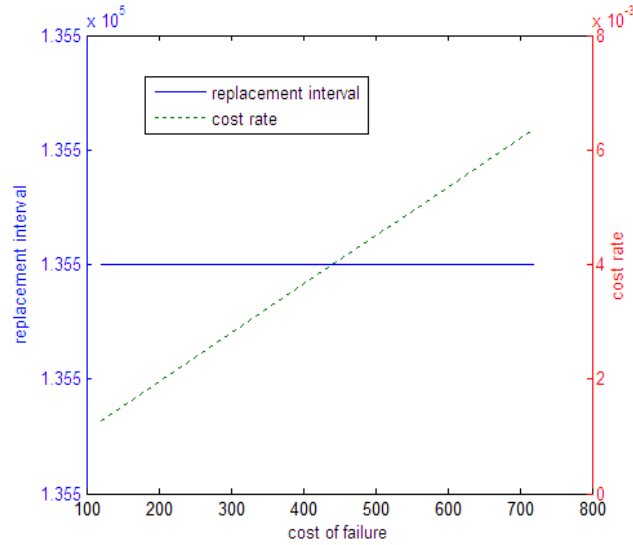
Figure 4.4: Sensitivity analysis of $CR(V^*)$ and V^* on C_F
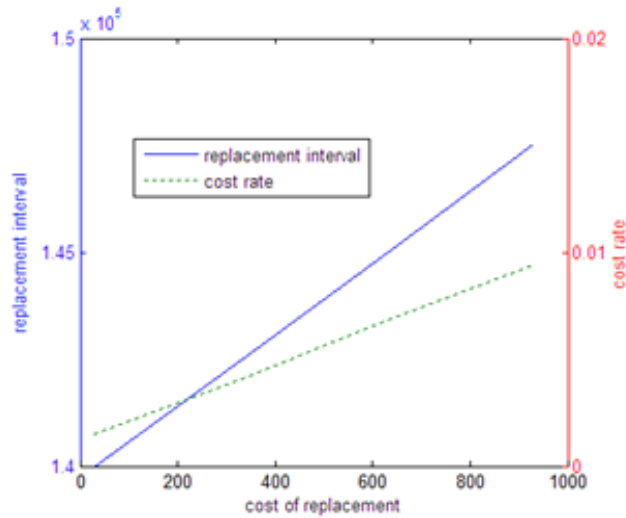
Figure 4.5: Sensitivity analysis of $CR(V^*)$ and V^* on C_R

4.1.3 Shock sets

For a multi-component system with each component has its own shock set, i.e., a shock can affect one or more components but not necessarily all components in the system, preventive maintenance optimization models are developed for the system. Decision variables for two different maintenance scheduling problems, the preventive maintenance replacement time interval, and the preventive maintenance inspection time interval, are determined by minimizing a defined system cost rate. Sensitivity analysis is performed to

provide insight into the behavior of the maintenance policies. The MEMS (Micro-electromechanical systems) oscillator example studied in Section 3.1.2 is used again to illustrate maintenance policies.

Two different preventive maintenance policies are considered for the system. The first maintenance policy is to replace the system at a fixed interval V. Choosing C_F=$150and C_R=$30, the minimum average long-run maintenance cost rate is $0.1518/cycle, which is obtained at V*=91 months, the optimal periodic replacement interval. Figure 4.6 illustrates $CR(V)$ as a function of V.
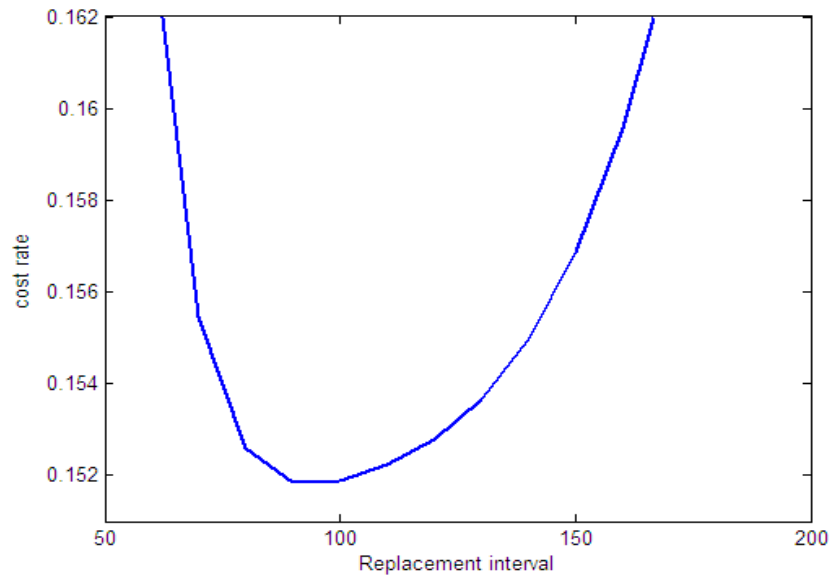
Figure 4.6: Average long-run maintenance cost rate versus replacement interval

A sensitivity analysis was performed to analyze the effects of the model parameters on the optimal solutions. The model parameters of interest are C_R and C_F. The results are shown in Figures 4.7 and 4.8, respectively. There are results for five different points, and then straight lines connect those points on the graph. This is to provide a better indication of the trend, and it is not intended to precisely indicate the function between the five points.

When C_F increases from $50 to $150 as shown in Figure 4.7, the minimum average

long-run maintenance cost rate, $CR(V^*)$, increases from \$0.1506 to \$0.1518, and the optimal replacement interval decreases from 100 to 92 months. This implies that a higher failure cost leads to a higher potential cost rate and lower replacement interval. As a result, the system should be replaced more frequently when the cost of failure is higher.

As shown in Figure 4.8, when C_R increases from \$10 to \$90, the minimum average long-run maintenance cost rate increases from \$0.07 to \$0.43, and the optimal replacement interval increases from 92 to 104 months. This indicates that a larger replacement cost results in a longer replacement interval and higher cost rate. As a result, the system should be replaced less frequently when the cost of replacement is higher.
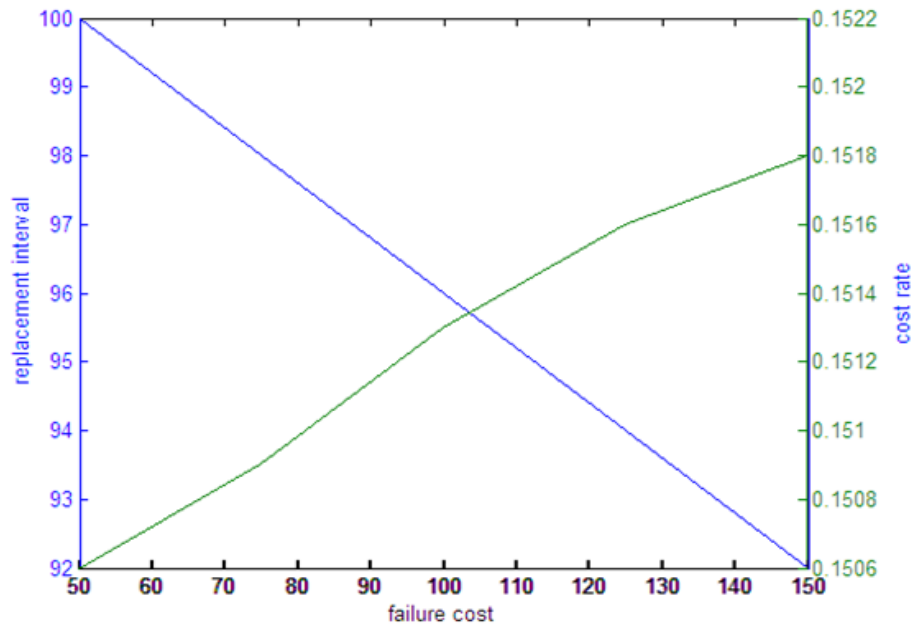
Figure 4.7: Sensitivity analysis of $CR(V^*)$ and V^* on C_F
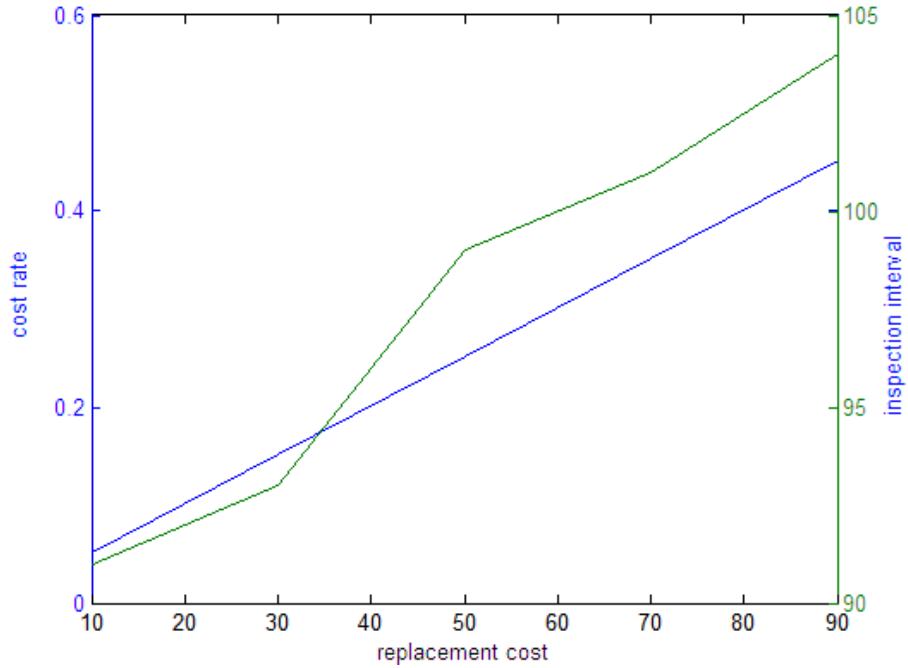
Figure 4.8: Sensitivity analysis of $CR(V^*)$ and V^* on C_R

The second maintenance policy is to inspect the system at intervals of τ and to replace

the system when it is observed to have failed. Choosing $C_I=\$200$, $C_\rho=\$4$ and $C_R=\$300$, the

minimum average long-run maintenance cost rate is \$0.9978/cycle, which is obtained at

$\tau^*=94$ months, the optimal number of months for periodic replacement. Figure 4.9

illustrates $C_R(\tau)$ as a function of τ.

Figure 4.9: Average long-run maintenance cost rate versus replacement interval

A sensitivity analysis was performed to analyze the effects of the model parameters on the optimal solutions. The model parameters of interest include C_I, C_ρ and C_R. The results are shown in Figures 4.10 to 4.12. Similar to the discussion about Figures 4.8 and 4.9, there are only results for five points, and straight lines are used to connect those points to provide an indication of the overall trend.

When C_I increases from \$200 to \$400 as shown in Figure 4.10, the minimum average maintenance cost rate, $CR(\tau^*)$, increases from \$0.9978 to \$3.0542, and the optimal inspection interval increases from 94 to 101 months. This implies that a higher inspection cost leads to a higher potential cost rate and higher inspection interval. As a result, the system should be inspected less frequently when the cost of inspection is high.

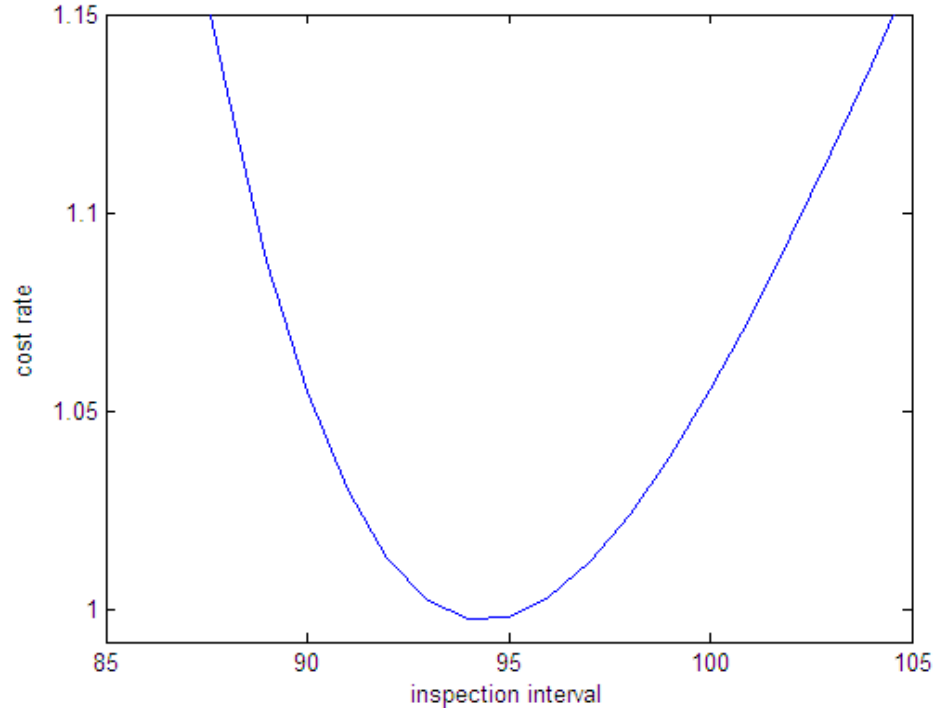As shown in Figure 4.11, when C_ρ increases from \$2.5 to \$4.0, the minimum average long-run maintenance cost rate decreases from \$2.4986 to \$0.9978, and the optimal inspection interval decreases from 120 to 94 months. This indicates that a higher cost of

downtime results in a shorter inspection interval, and lower cost rate. As a result, the system should be inspected more frequently when the cost of inspection is high.

When C_R increases from \$300 to \$500 as shown in Figure 4.12, the minimum average long-run maintenance cost rate, $CR(\tau^*)$, increases from \$0.9978 to \$3.0318, and the optimal inspection interval increases from 94 to 99 month. This implies that a higher replacement cost leads to a higher potential of cost rate and higher inspection interval. As a result, the system should be inspected less frequently when the cost of replacement is high.
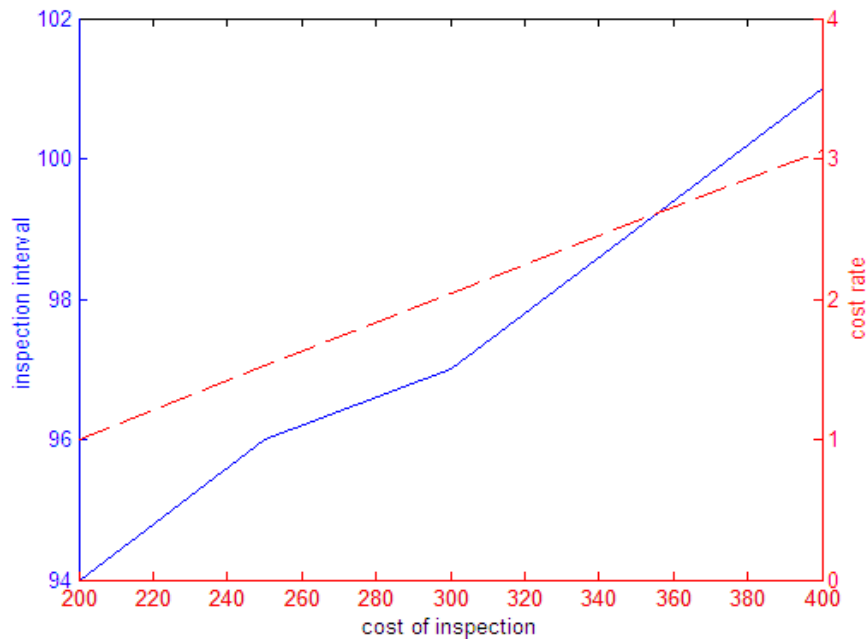
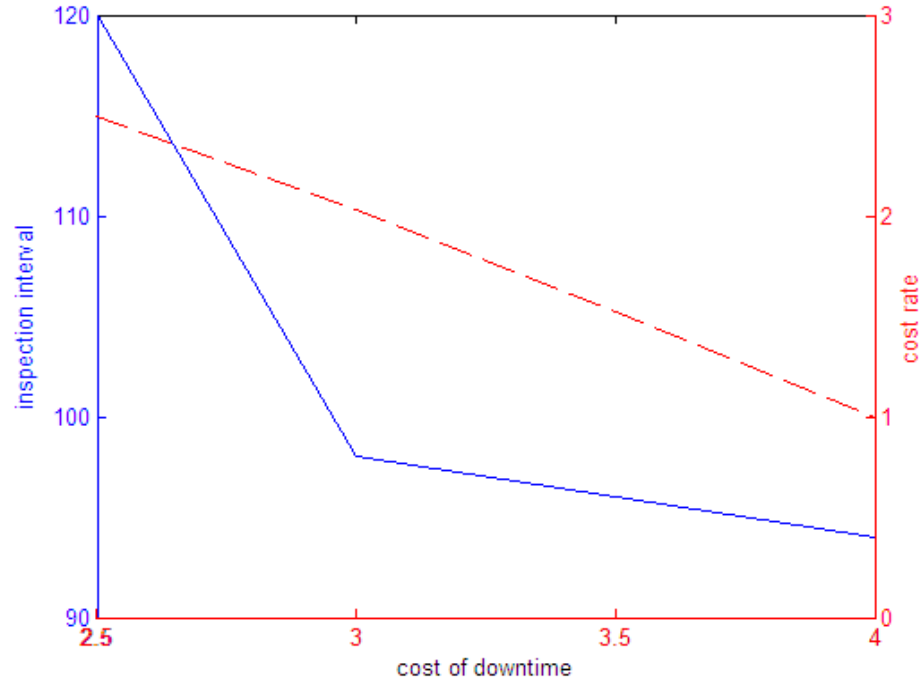Figure 4.10: Sensitivity analysis of $CR(\tau^*)$ and τ^* on C_I

Figure 4.11: Sensitivity analysis of $CR(\tau^*)$ and τ^* on C_ρ
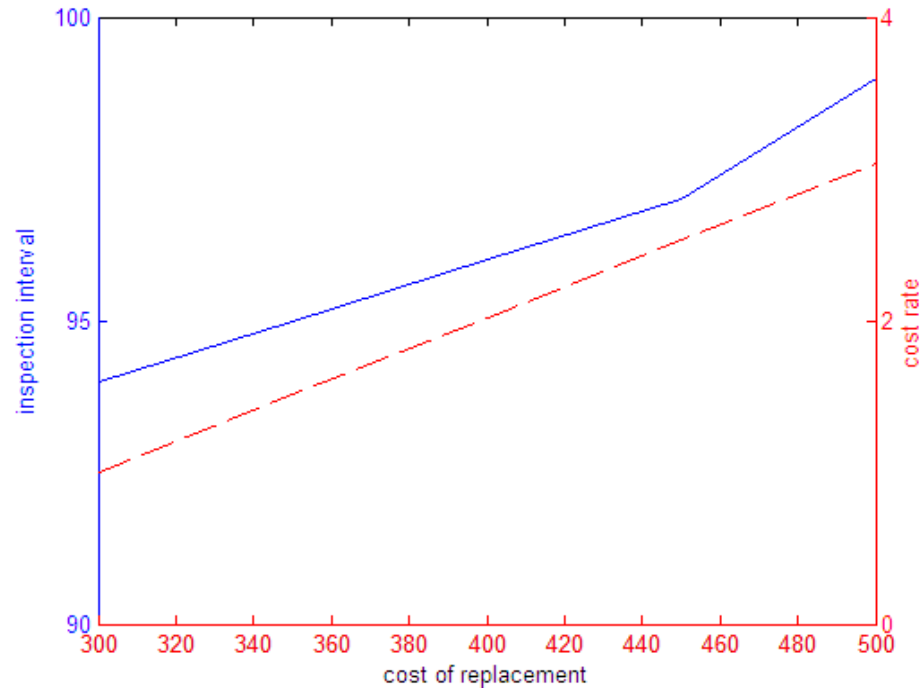

Figure 4.12: Sensitivity analysis of $CR(\tau^*)$ and τ^* on C_R

4.2 Condition-Based Maintenance

In this section, a maintenance optimization model is presented to determine on-condition failure thresholds and inspection intervals for complex multi-component systems

with each component experiencing multiple failure processes due to simultaneous exposure to degradation and shock loads. On-condition maintenance optimization is considered for systems of degrading components, which offers cost benefits over time-based preventive maintenance or replace-on-failure policies. For systems of degrading components, this can be a particularly difficult problem because of the dependent degradation and dependent failure times.

Traditional system models are based on failure time distribution, and it cannot be used to determine on-condition thresholds, however, condition-based maintenance policy can be applied to systems based on the degradation modeling. In previous research, preventive maintenance and periodic inspection models have been considered, but for systems whose costs due to failure are high, it is prudent to avoid the event of failure, i.e., the components or system should be repaired or replaced before the failure occurs. The determination of optimal on-condition thresholds for all components is effective to avoid the event of failure and to minimize cost. Low on-condition thresholds are often inefficient because they waste component life, and high on-condition thresholds are risky because the system is prone to costly failure.

For this new model, a new optimization model is formulated and solved to determine optimal on-condition thresholds and inspection intervals. Corresponding system models with time-to-failure distribution cannot be used to do this. It is important to note that when a system is inspected, we are inspecting all components. An inspection interval may be optimal for one component, but might not be for another component, so the optimization requires a compromise. The on-condition maintenance optimization model is demonstrated on a series system example with dependent degrading components because of shared shock

exposure (independent W_{ij} and Y_{ij}). Specific maintenance optimization policies could be developed based on any of the reliability models from Section 3, but it is only demonstrated on one type of model. The new model offers cost benefits and performance improvement over time-based preventive maintenance or replace-on-failure policies. Specific assumptions in this section are as follows:

1. The model is for systems that are sealed or packaged together, making it impossible or impractical to repair or replace individual components within the system, e.g., MEMS.

2. An on-condition threshold is defined for the soft failure process as H_i^2, which is less than or equal to the failure threshold of H_i^1 (note that 1 and 2 are superscripts and not exponents).

3. For the maintenance, the system is inspected at periodic intervals and no continuous monitoring is performed. Replacements are assumed to be instantaneous and perfect.

4. Upon an inspection, if the overall degradation of all n components is lower than their individual on-condition threshold H_i^2, then the system is within the high safety level area, and nothing is done.

5. If the degradation of any component is between H_i^1 and H_i^2, the system does not fail, but is prone to high failure risk, then the system is replaced with a new one preventively.

6. If the system fails, that is, the total degradation of any component is higher than H_i^1 before the next specified inspection interval, it is not immediately detected and not replaced until the next inspection. There is a penalty cost associated with

the failure of system during downtime, e.g., cost associated with loss of production, opportunity costs, etc.

For some systems, when the cost/consequence of failure are excessive compared to a comparable preventive repair cost, replacement cost or other kinds of cost, it is prudent to prevent the failure from happening or replace the equipment at the earliest convenience after it has sufficiently aged, rather than allowing the failure to occur and possibly cause severe consequences. The concept of condition-monitoring and on-condition thresholds for the components is adopted as criteria to evaluate and measure system status, and therefore, increase the opportunity to detect the components' critical and degraded situation and to avoid the failure events.
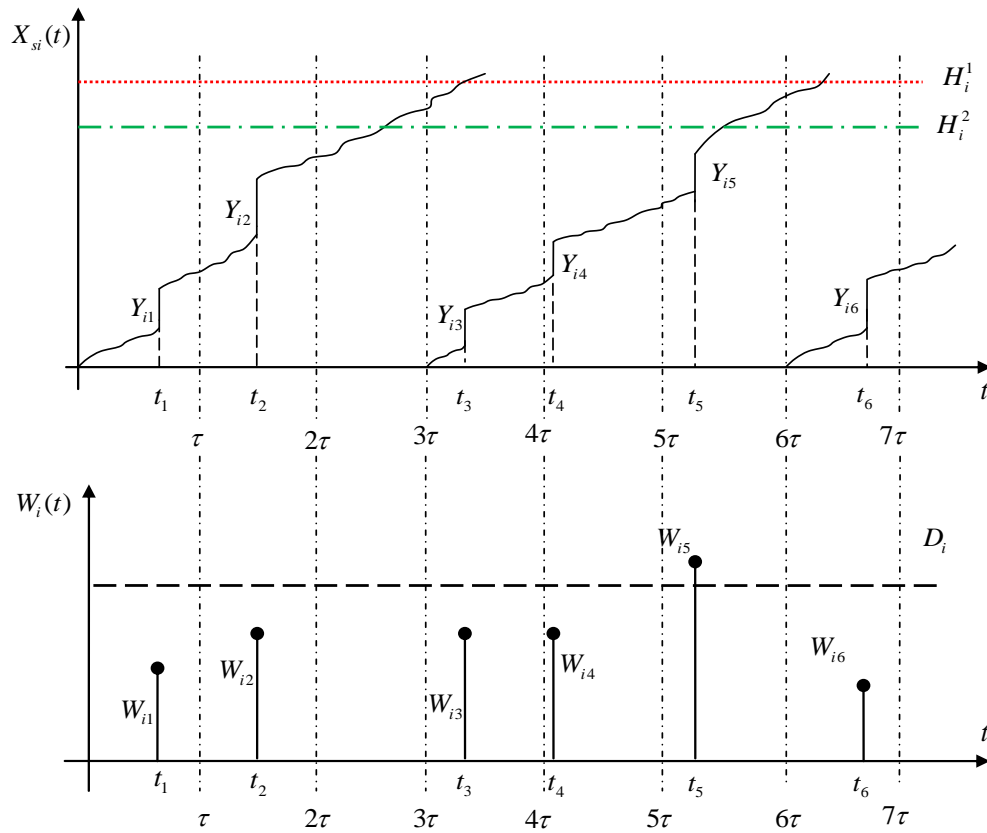

Figure 4.13: Two thresholds divide system status into three regions

In Figure 4.13, H_i^1 is the soft failure threshold for component i and H_i^2 is the on-

condition threshold for component i, and $H_i^2 \le H_i^1$ (considering H_i^1 and H_i^2, the superscript is not an exponent). At each inspection point, the condition for each component is determined and compared to a threshold. The action taken depends on a selection of condition-based operational status data and the defined maintenance condition rules. In Figure 4.13, given a fixed on-condition threshold H_i^2 for component i (lower bar and dash line in soft failure process), rulers related to this on-condition degradation threshold are adopted to take action based on the component degradation state.

At each inspection interval, if no hard failure occurs, and at the same time, total degradation of the ith component is less than H_i^2, the on-condition threshold for i^{th} component, the component is in the safe region. The safe region is defined as the combination of soft failure process and hard failure process both below their respective thresholds and this status is defined as event A shown in Table 3.3. If no hard failure occurs, and total degradation is between H_i^2 and H_i^1 for the ith component, this component still has not failed. However, probabilistically it may fail within a short period, and this status can be indicated as the combination of soft failure process area between H_i^2 and H_i^1, and hard failure process area below the hard failure threshold, which is defined as event B shown in Table 4.1. If there has been a hard failure or the total degradation of any component i is greater than H_i^1 (higher dash line in soft failure process), this situation is defined as a failure. The status can be defined as the union of the soft failure process area above H_i^1, and hard failure process area above D_i, and this status is defined as event C.

Define the probability that component total degradation less than x by time t as $\Psi(x)$:

$$\Psi(x) = \sum_{m=0}^{\infty} \left(\int_0^x G_i(x-u,t) f_{Y_i}^{<m>}(u) du \right) \frac{\exp(-\lambda t)(\lambda t)^m}{m!} \qquad (130)$$

Considering the safe region case for example, conditioning on m shocks arriving to the system by time t with probability $\dfrac{\exp(-\lambda t)(\lambda t)^m}{m!}$, the probability of no hard failure is $P_{Li}{}^m$, and the probability that total degradation is less than H_i^2 is $\Psi(H_i^2) = \int_0^{H_i^2} G_i(H_i^2 - u, t) f_{Y_i}^{<m>}(u)du$. Combining both the soft failure process and hard failure process, the probability for event A, the component is in safe region is

$$P(A) = \sum_{m=0}^{\infty} P_{Li}^m \left(\int_0^{H_i^2} G_i(H_i^2 - u, t) f_{Y_i}^{<m>}(u)du \right) \frac{\exp(-\lambda t)(\lambda t)^m}{m!}.$$

Similarly, for event B, the component is still working, but it is likely to fail within the next inspection interval. The probability of no hard failure is $P_{Li}{}^m$, and the probability that total degradation is between H_i^1 and H_i^2 is $\Psi(H_i^1) - \Psi(H_i^2)$, which can also be expressed as: $\int_0^{H_i^1} G_i(H_i^1 - u, t) f_{Y_i}^{<m>}(u)du - \int_0^{H_i^2} G_i(H_i^2 - u, t) f_{Y_i}^{<m>}(u)du$. Combining both the soft failure process and the hard failure process, the probability for event B can be obtained. For event C, either soft failure happens or hard failure happens, and the probability equals to one minus the probability that neither of these two failure happens. Finally, the maintenance policy is summarized in Table 4.1.

Table. 4.1 component status defined with two soft failure thresholds and one hard failure threshold

Event	Description	Probability
A	component is in safe region: do nothing	$P(A) = \sum_{m=0}^{\infty} P_{Li}^m \left(\int_0^{H_i^2} G_i(H_i^2 - u, t) f_{Y_i}^{<m>}(u)du \right) \dfrac{\exp(-\lambda t)(\lambda t)^m}{m!}$

B	component is working, but probabilistically fails soon: replace preventively	$P(B) = \sum_{m=0}^{\infty} P_{Li}^{m} \left(\Psi(H_i^1) - \Psi(H_i^2) \right) \dfrac{\exp(-\lambda t)(\lambda t)^m}{m!}$
C	component fails: replace correctively	$P(C) = \sum_{m=0}^{\infty} \left(1 - P_{Li}^{m} \int_{0}^{H_i^1} G(H_i^1 - u, t) f_Y^{<m>}(u) du \right) \dfrac{\exp(-\lambda t)(\lambda t)^m}{m!}$

To evaluate the performance of the condition-based maintenance policy, the average long-run maintenance cost rate model is used, in which the periodic inspection interval τ for the whole system and on-condition thresholds H_i^2 for all components are the decision variables. Upon an inspection, the system is changed with a new one when a hard failure has occurred or total degradation is greater than the on-condition threshold. The expected value of the number of inspections N_I, in which $\mathbf{H}^2 = (H_1^2, H_2^2, ..., H_n^2)$ is given by

$$E(N_I) = \sum_{m=1}^{\infty} m(F_T^{\mathbf{H}^2}(m\tau) - F_T^{\mathbf{H}^2}((m-1)\tau)) \qquad (131)$$

In which $F_T^{\mathbf{H}^2}(t)$ is the cdf for the time when system is replaced for any reason, either preventive or corrective maintenance. It can be expressed entirely as a function of $\mathbf{H}^2$ (and not $\mathbf{H}^1$) because any degradation beyond results in a replacement. It is also the probability that no hard failure occurs but component degradation above on-condition threshold H_i^2.

From Figure 4.14, system downtime is the time duration between a failure occurrence and the next time an inspection is performed and a failure detected. Conditioning on the event that a failure occures at time t between the $(m-1)^{\text{th}}$ and m^{th} inspection $[(m-1)\tau, m\tau]$ with probability $\left(F_T^{\mathbf{H}^2}(m\tau) - F_T^{\mathbf{H}^2}((m-1)\tau) \right)$, and defining the failure time as t', the system downtime is $m\tau - t'$. The expected value of system downtime or the expected time from a

system failure to the next inspection when the failure is detected, can then be determined

as $\int\limits_{(m-1)\tau}^{m\tau} (m\tau - t)dF_T^{\mathbf{H}^1}(t) \cdot$ In which, $F_T^{\mathbf{H}^1}(t)$ is the cdf of system failure time without preventive

replacement, i.e., failure occurs when degradation reaches H_i^1. It is also the probability that

either there is a hard failure or component degradation is above the soft failure threshold

$\mathbf{H}_i^1$ for some components.

Summing over the probability that failure can occur in any inspection interval, the

expected system downtime can be obtained as follows:

$$E(\rho) = \sum_{m=1}^{\infty} E[\rho \mid N_I = m]P(N_I = m) = \sum_{m=1}^{\infty} \left(\left(F_T^{\mathbf{H}^2}(m\tau) - F_T^{\mathbf{H}^2}((m-1)\tau) \right) \int\limits_{(m-1)\tau}^{m\tau} (m\tau - t)dF_T^{\mathbf{H}^1}(t) \right)$$

(132)

The expected time between two replacements or expected cycle length is

$$E(K) = \sum_{m=1}^{\infty} E[K \mid N_I = m]P(N_I = m) = \sum_{i=1}^{\infty} m\tau(F_T^{\mathbf{H}^2}(m\tau) - F_T^{\mathbf{H}^2}((m-1)\tau)) \qquad (133)$$
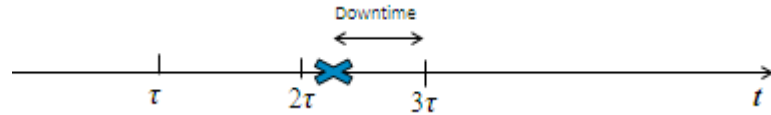
Figure 4.14: System downtime under periodic inspection maintenance policy

Condition-based maintenance offers the promise of enhancing the effectiveness of

maintenance programs in an effective way. The practical case is considered that when the

penalty cost due to downtime is relatively higher than the corresponding preventive

maintenance cost, it is better to replace the whole system before the wear volumes of

components reach their critical degradation thresholds. For some systems, it is best to just

let them fail, but those cases are not considered in this paper.

Optimization of the on-condition degradation threshold can achieve the idea of

replacing the system before failure by providing the criteria to detect the degradation of component beyond the on-condition threshold. If the on-condition threshold is too low and far away from the failure threshold level, then the whole system has to be replaced more frequently, and there is extra cost due to the waste of system life. Alternatively, if the threshold is too high, then the system may fail before the next inspection leading to potentially expensive downtime cost. Therefore, the on-condition degradation thresholds for all components and an inspection interval for the whole system are chosen to be decision variables in this maintenance optimization problem.

To evaluate the performance of the condition-based maintenance policy, an average long-run maintenance cost rate model is again used, in which the periodic inspection interval τ for the whole system and on-condition thresholds H_i^2 for all components are the decision variables. At time τ, and subsequent inspection intervals of time τ, the entire assembled system is inspected. If the system is still operating satisfactorily with no component wear volume above the on-condition threshold, nothing is done. If degradation thresholds for all components are below the fixed critical degradation thresholds H_i^1 but some are above the on-condition threshold H_i^2, the whole system is replaced preventively. If hard failure occurs or at least one component's wear volume is above the critical degradation threshold H_i^1 prior to inspection, then the system is not replaced with a new one correctively until the next inspection. Still, the average long-run maintenance cost per unit time is evaluated.

The defined cost rate is $CR(\tau, \mathbf{H}^2)$ and it is defined as the ratio of expected total cost to the expected cycle duration.

$$CR(\tau, \mathbf{H}^2) = E(TC) / E[K] \tag{134}$$

The expected total maintenance cost is then given as:

$$E(TC) = C_I E(N_I) + C_\rho E(\rho) + C_R \tag{135}$$

where C_I is the cost of each inspection. C_R is the replacement cost, C_ρ is the penalty cost incurred during down time, and τ is the time interval for periodic inspection. Based on Eq. (131) to (135), the average long-run maintenance cost rate is given as

$$CR(\tau, \mathbf{H}^2)$$

$$= \frac{C_I \sum_{m=1}^{\infty} m\left(F_T^{\mathbf{H}^2}(m\tau) - F_T^{\mathbf{H}^2}((m-1)\tau)\right) + C_\rho \sum_{m=1}^{\infty}\left(\left(F_T^{\mathbf{H}^2}(m\tau) - F_T^{\mathbf{H}^2}((m-1)\tau)\right) \int_{(m-1)\tau}^{m\tau} (m\tau - t)dF_T^{\mathbf{H}^1}(t)\right) + C_R}{\sum_{m=1}^{\infty} m\tau(F_T^{\mathbf{H}^2}(m\tau) - F_T^{\mathbf{H}^2}((m-1)\tau))}$$

$$\tag{136}$$

For this maintenance optimization problem, there are n components in the system, and $(n+1)$ decision variables; namely n on-condition thresholds for all components and periodic inspection interval for the whole system. The objective is to minimize the maintenance cost rate, and constraints are that on-condition thresholds for all components should be less than or equal to their critical failure thresholds, and inspection interval should be a positive value. Therefore, the maintenance optimization problem can be formulated as follows:

$$
\begin{aligned}
\min \quad & CR(\tau, \mathbf{H}^2) \\
\text{s.t.} \quad & 0 \le H_1^2 \le H_1^1, \\
& 0 \le H_2^2 \le H_2^1, \\
& \quad ... \\
& 0 \le H_n^2 \le H_n^1, \\
& \tau \ge 0,
\end{aligned}
\tag{137}
$$

It is a difficult non-linear optimization problem with continuous decision variables and a convex feasible region. For constrained nonlinear optimization problems, there are many available algorithms to obtain optimal solutions.

To solve the approximate problem, an interior point method is used (as implemented as the *fmincon* algorithm in the MATLAB optimization toolbox). The method consists of a self-concordant barrier function used to encode the convex set. Contrary to the simplex method, it reaches an optimal solution by traversing the interior of the feasible region. The algorithm uses one of two main types of steps iteration [116]. By default, the algorithm first attempts to take a direct step within the feasible region. A direct step attempts to solve the Karush Kuhn Tucker (KKT) equations for the approximate problem via a linear approximation, which is also called a Newton step. Given the approximate problem, its Lagrangian and Hessian matrix can be obtained. By solving the KKT equations, we can get the direct step and the solution for the next iteration. If it cannot take a direct step, it attempts a conjugate gradient step, and minimizes a quadratic approximation to the approximate problem in a trust region, subject to linearized constraints. One case where it does not take a direct step is when the approximate problem is not locally convex near the current iteration. At each iteration, the algorithm decreases a merit function. A new solution point is reached after taking the step and a new iteration is started. By countinuing with successive iterations, optimal solution can be obtained when a pre-defined stopping criterion is met.

A series system with four components which has dependent degradation paths due to shared shock exposure (independent W_{ij} and Y_{ij}) is considered as an example. The parameters for reliability analysis are provided in Table 4.2. For this example, W_{ij} and Y_{ij} follow normal distributions. Without loss of generality, parameters of component 1 and 2 are assumed to be the same, and parameters of component 3 and 4 are assumed to be the same. This is a conceptual example to demonstrate the reliability function and maintenance

models. Although the example is conceptual, H_i^1, D_i and $X_i(t)=\varphi_i+\beta_i t$ are estimated based on documented degradation trends. In this part, maintenance optimizations are performed for both the series system and all the individual components making up the system separately, and the results are discussed.

Table 4.2: Parameter values in condition-based maintenance model

Parameter	component 1 & 2	component 3 & 4	Sources
H_i^1	0.00125 μm^3	0.00127 μm^3	Tanner and Dugger [107]
D_i	1.5 Gpa	1.4 Gpa	Tanner and Dugger [107]
φ_i	0	0	Tanner and Dugger [107]
β_i	$\beta_i \sim N(\mu_{\beta i}, \sigma_{\beta i}^2)$ $\mu_{\beta i}$=8.4823×10^{-9} μm^3 $\sigma_{\beta i}$=6.0016×10^{-10} μm^3	$\beta_i \sim N(\mu_{\beta i}, \sigma_{\beta i}^2)$ $\mu_{\beta i}$=8.4936×10^{-9} μm^3 $\sigma_{\beta i}$=5.9011×10^{-10} μm^3	Tanner and Dugger [107]
λ	2.5×10^{-5}	2.5×10^{-5}	Assumption
Y_{ij}	$Y_{ij} \sim N(\mu_{Yi}, \sigma_{Yi}^2)$ μ_{Yi} =1×10^{-4} μm^3 σ_{Yi} =2×10^{-5} μm^3	$Y_{ij} \sim N(\mu_{Yi}, \sigma_{Yi}^2)$ μ_{Yi} =0.9×10^{-4} μm^3 σ_{Yi}=2.1×10^{-5} μm^3	Assumption
W_{ij}	$W_{ij} \sim N(\mu_{Wi}, \sigma_{Wi}^2)$ μ_{Wi} =1.2 GPa, σ_{Wi} =0.2 GPa	$W_{ij} \sim N(\mu_{Wi}, \sigma_{Wi}^2)$ μ_{Wi} =1.22 GPa, σ_{Wi} =0.18 GPa	Assumption

First, the maintenance policy is considered for the series system with four components, i.e., the whole system is inspected at one interval of τ and the whole system is replaced when the wear volume is above H_i^2 for any component. Choosing C_I=\$58, C_ρ=\$300 and C_R=\$200, after 35 steps of iteration, the minimum average long-run maintenance cost rate for system is \$0.007481954, which is obtained at periodic inspection interval τ^*=1219 hours, and on-condition degradation threshold are $H_1^{2*}= H_2^{2*}$=0.0010027, and $H_3^{2*}= H_4^{2*}$=0.0009214. Figure 4.15 illustrates the iteration process of decision variables, i.e., inspection interval, on-condition degradation threshold for component 1 and 2, and on-condition degradation threshold for component 3 and 4. Figure 4.16 shows the progress of the optimization iterations for the objective function,
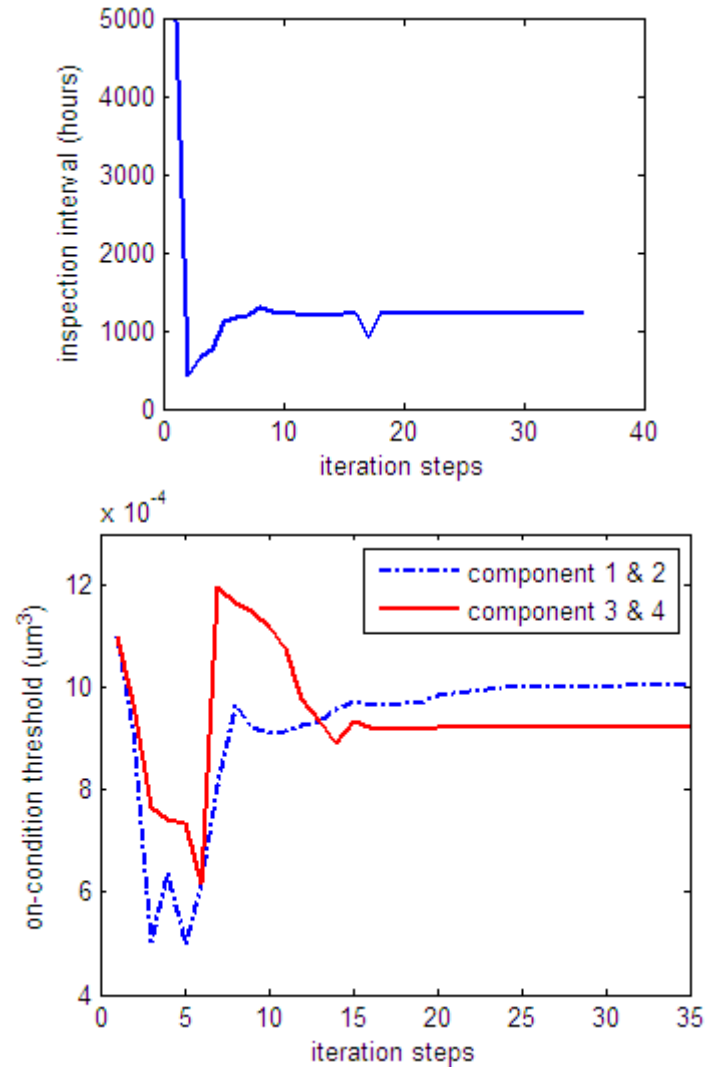
i.e., the system maintenance cost rate.

Figure 4.15: Iteration process for five decision variables: inspection interval τ^*, and on-condition threshold for all components
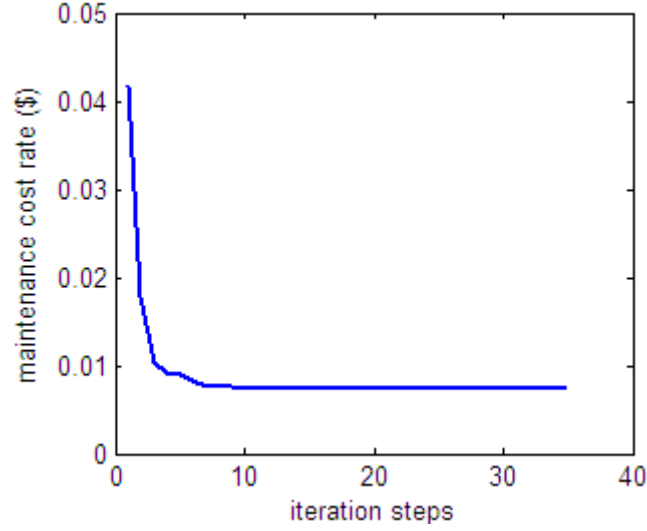
Figure 4.16: Iteration process of maintenance cost rate for system with four components

To evaluate the results, the maintenance policy for individual and independent components is considered. That is, four components are treated as individual systems, and the individual four components are inspected at their own inspection intervals and components are replaced when failure occurs. Since components 1 and 2 share the same parameter, the maintenance optimization for them are the same. Choosing C_I=$58, C_ρ=$300 and C_R=$200, with 16 steps of iteration, the minimum average long-run maintenance cost rate for component 1 and 2 is $0.005041, which is obtained at periodic inspection interval $\tau_{1,2}{}^*$=1862 hours, and on-condition degradation threshold for components $H_1{}^{2*}$= $H_2{}^{2*}$=0.000623. Figure 4.17 illustrates the iteration process of two decision variables: the inspection interval and the on-condition degradation threshold for component 1 and 2. Figure 4.18 shows the iteration for objective functions, i.e., the maintenance cost rate.
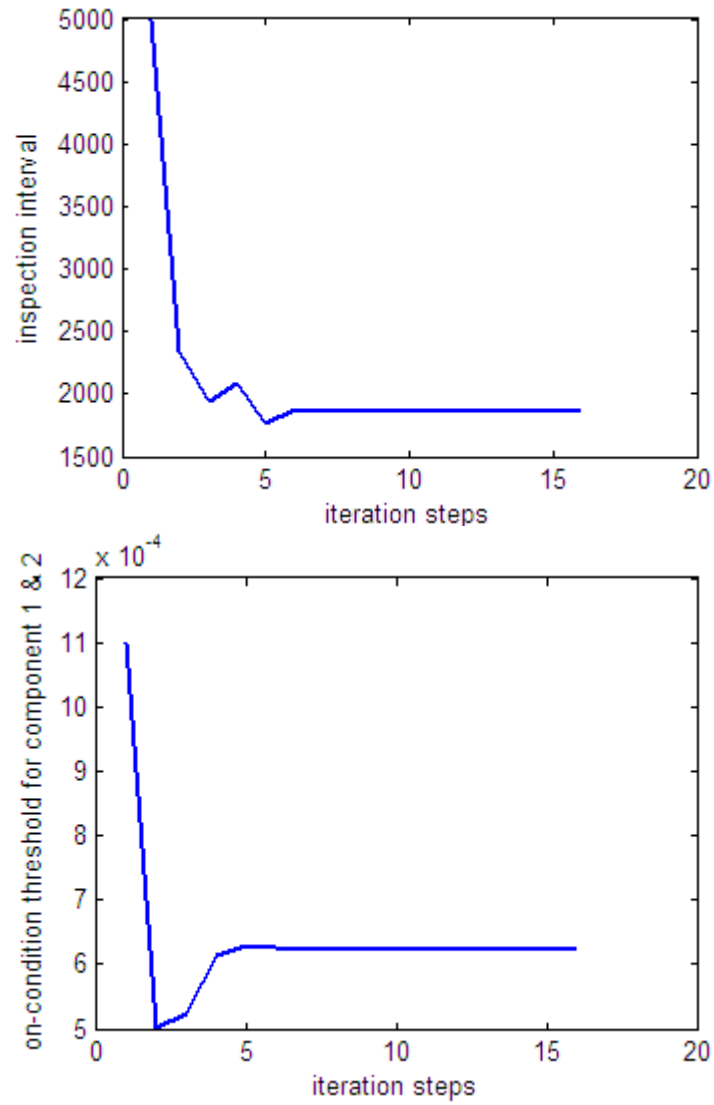
Figure 4.17: Iteration process two decision variables: inspection interval τ^*, and on-condition threshold for components 1 and 2

Figure 4.18: Iteration process of maintenance cost rate for component 1 and 2

Similarly, individual components 3 and 4 are inspected at their own inspection intervals and the components are replaced when failure occurs. Given the same cost C_I=\$58, C_p=\$300 and C_R=\$200, with 19 steps of iteration, the minimum average long-run maintenance cost rate for components 3 and 4 is \$0.00599, which is obtained at periodic inspection interval $\tau_{3,4}$*=1536 hours, and on-condition degradation threshold for components H_3^2*= H_4^2*=0.000757. Figure 4.19 illustrates the iteration process of two decision variables: inspection interval and on-condition degradation threshold for component 3 and 4. Figure 4.20 shows the iterations for objective function, i.e., the maintenance cost rate.

Figure 4.19: Iteration process two decision variables: inspection interval τ^*, and on-condition threshold for components 3 and 4
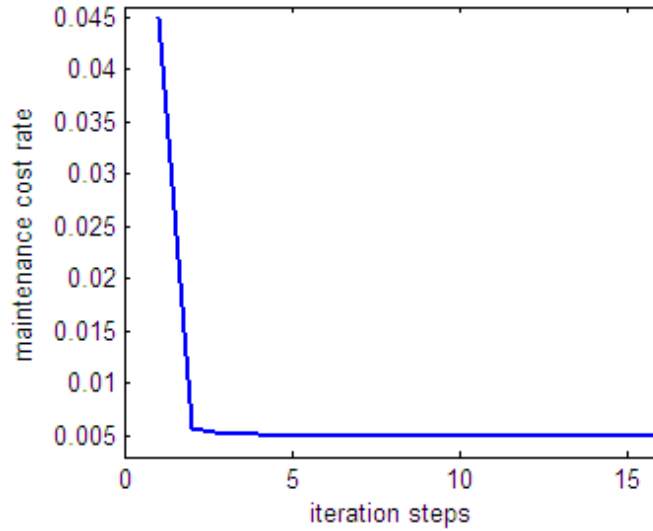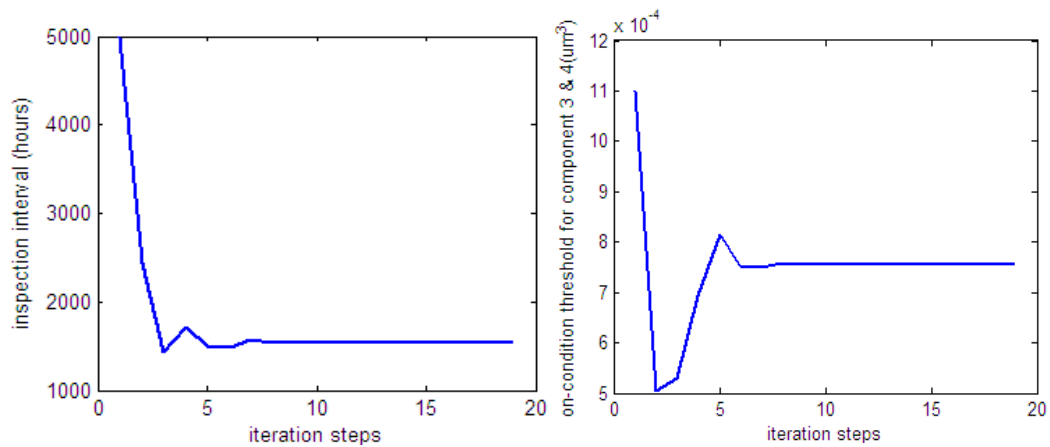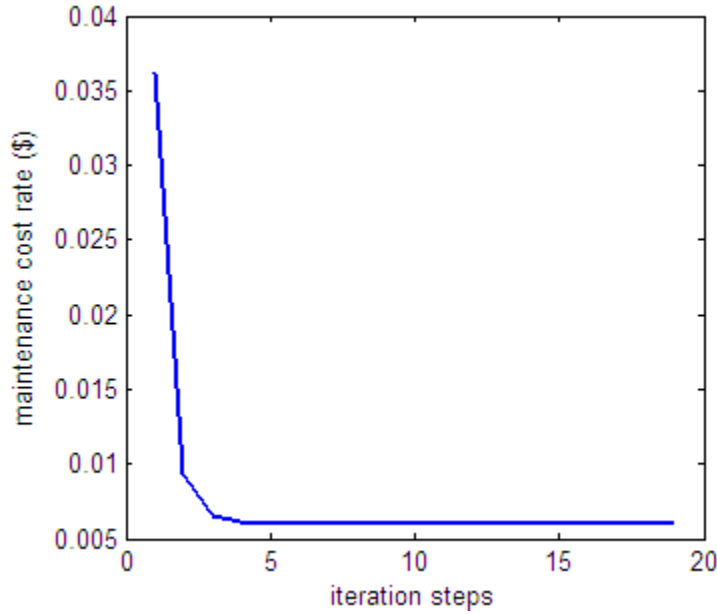

Figure 4.20: Iteration process of maintenance cost rate for components 3 and 4

Inspection intervals for either component 1 and 2 or component 3 and 4 are greater than the inspection interval for the series system, which means we have to compromise to inspect the system more frequently if there are more components in the system. Since time-to-failure for all components are different, and the series system reliability is less than individual component reliability given any fixed time, the system should be inspected more often to increase the probability of avoiding failure and relatively high downtime cost. Also, inspection intervals for components 1 and 2 are greater than the interval for components 3 and 4. This is mainly because the degradation rate and the shock damage effect for components 1 and 2 are lower compared to components 3 and 4, which means the reliability for components 1 and 2 is higher than components 3 and 4. Therefore, components 3 and 4 need to be inspected more often, and accordingly, the inspection interval for components 3 and 4 is smaller.

Optimal on-condition threshold values for either the component 1 and 2 or component

3 and 4 maintenance case is less than their optimal on-condition threshold value for system maintenance case. Inspection interval and on-condition threshold are two variables of the trade-off. If the component/system is inspected quite often with a high on-condition threshold, there may still have a high probability to detect system status and replace it preventively. Alternatively, if they are inspected less often, we can compensate by using a lower on-condition threshold value to achieve an effective preventive maintenance. It has been explained why the inspection interval for an individual component is greater than inspection interval for the series system above. According to the trade-off, optimal on-condition thresholds for individual component are lower than their value in the series system.

From the result, the maintenance cost rate for either component 1 and 2 or component 3 and 4 is less than the maintenance cost rate for the whole system. This is mainly due to the higher replacement cost for series system. Every time replacement is performed for the series system, and all the components need to be replaced. However, the summation of cost rate for all four individual components is higher than cost rate for the series system with the same four components. All four components in the series system are inspected, and this cost is the same as the cost we spend inspecting each individual component. Therefore, the inspection cost rate for all four individual component maintenance is higher than inspection cost rate for the series system with exactly the same four components.

4.3 Individual Component Maintenance

The earlier models were applicable for cases when one component fails in the system, upon the scheduled inspection or replacement time point, the whole system is replaced. For many systems, this is not cost efficient or practical to maintain the system in

this way. In this section, when one component fails in the system, individual component maintenance is performed rather than group maintenance. Individual component maintenance modeling is a challenging problem, since all components do not share the same renewal life.

In this study, the steady state system behavior is studied considering long-run working status and successive failures and replacements. Every time interval τ, the system is inspected. If all components are working well, nothing is done. It is assumed that at most one component can fail within any inspection interval, which is reasonable for most reliable systems. Upon the inspection, if one component has failed, the failed component is replaced with new one and a new life starts for this component. The replacement can be achieved immediately.

Individual component maintenance is based on the individual component reliability, i.e., component survival probability in any inspection interval based on system steady state behavior. Given this maintenance policy, the system survival probability during each inspection interval can be obtained under steady state status. For any inspection interval during steady state, U_i is defined as the initial component degradation level for component i at the beginning of the inspection interval and $f_{Ui}(u)$ is defined as the pdf of U_i. Individual component maintenance is investigated based on two methods, including: (1) component inspection interval survival probability can be obtained based on the approximation that the degradation level of component i at the beginning of any inspection interval follows uniform distribution, $U_i \sim \text{uniform}(0, H_i)$; (2) Based on a geometric distribution of survival inspection intervals, the pdf of initial component degradation level at the beginning of inspection interval can be approximated.

For a series system, the probability that each component survives all the shocks arriving to the system within the inspection interval, and system total degradation is below the failure threshold can be determined based on the distribution of the degradation at the beginning of the interval. Component total degradation contains three parts: initial degradation at the beginning of the inspection interval, pure degradation during the interval and the degradation caused by all the shocks arriving to system within the inspection interval. The number of shock prior to the inspection interval is unknown but the corresponding cumulative shock damages are part of U_i, and it is known that they are less than H_i (or the system would have been replaced at a previous inspection).

$P(\tau)$ is defined as the probability that all components survives both failure processes during the inspection interval under steady state, given initial degradation amounts U_i. Conditioning on the number of shocks arriving to the system during the interval τ, the system survival probability for a single inspection interval τ can be determined. The survival probability for a series system is the intersection of the events that no component fails.

$$
\begin{aligned}
P(\tau) = \sum_{m=0}^{\infty} P &\left\{ \left[W_{11} < D_1, W_{12} < D_1, W_{1N(\tau)} < D_1, U_1 + \beta_1 \tau + \sum_{j=1}^{N(\tau)} Y_{1j} < H_1 \right] \cap \right. \\
&\left[W_{21} < D_2, W_{22} < D_2, W_{2N(\tau)} < D_2, U_2 + \beta_2 \tau + \sum_{j=1}^{N(\tau)} Y_{2j} < H_2 \right] \cap ... \\
&\left. \left[W_{n1} < D_n, W_{n2} < D_n, W_{nN(\tau)} < D_n, U_n + \beta_n \tau + \sum_{j=1}^{N(\tau)} Y_{2j} < H_n \right] \middle| N(\tau) = m \right\} P(N(\tau) = m)
\end{aligned}
$$

(138)

Component failure processes are independent after conditioning on the number of shocks

$$P(\tau) = \sum_{m=0}^{\infty} \prod_{i=1}^{n} P\left\{\left[W_{i1} < D_i, W_{i2} < D_i, W_{im} < D_i, U_i + \beta_i\tau + \sum_{j=1}^{m} Y_{ij} < H_i\right]\right\} P(N(t) = m) \qquad (139)$$

Shock arrivals occur as a Poisson process, and shock damage to hard failure process are *i.i.d.*

$$P(\tau) = \sum_{m=0}^{\infty} \prod_{i=1}^{n} P\left\{\left[W_{i1} < D_i, W_{i2} < D_i, W_{im} < D_i, U_i + \beta_i\tau + \sum_{j=1}^{m} Y_{ij} < H_i\right]\right\} \frac{e^{-\lambda\tau}(\lambda\tau)^m}{m!} \qquad (140)$$

$$P(\tau) = \sum_{m=0}^{\infty} \prod_{i=1}^{n} P(W_{ij} < D_i)^m P\left(U_i + \beta_i\tau + \sum_{j=1}^{m} Y_{ij} < H_i\right) \frac{e^{-\lambda\tau}(\lambda\tau)^m}{m!} \qquad (141)$$

Conditioning on the initial status for degradation of each component, the system survival probability is:

$$\begin{aligned}
P(\tau) &= \sum_{m=0}^{\infty} \prod_{i=1}^{n} P(W_{ij} < D_i)^m \int_0^{H_i} P\left(U_i + \beta_i\tau + \sum_{j=1}^{m} Y_{ij} < H_i \mid U = u\right) f_{U_i}(u)\,du \frac{e^{-\lambda\tau}(\lambda\tau)^m}{m!} \\
&= \sum_{m=0}^{\infty} \prod_{i=1}^{n} P(W_{ij} < D_i)^m \int_0^{H_i} P\left(\beta_i\tau + \sum_{j=1}^{m} Y_{ij} < H_i - u\right) f_{U_i}(u)\,du \frac{e^{-\lambda\tau}(\lambda\tau)^m}{m!}
\end{aligned} \qquad (142)$$

There are two models developed in this section that can provide the pdf of initial component degradation $f_{U_i}(u)$ under system steady state, which can assist in the calculation of system survival probability $P(\tau)$ and the individual component maintenance cost.

4.3.1 Uniform initial component degradation

To obtain the system survival probability, first, we need to know $f_{U_i}(u)$. Consider the system is working under steady state. If the system is inspected at any specific time, the degradation of any component upon the inspection interval can be any value between 0 and component soft failure threshold level H_i. Based on this, we consider the

approximation that the degradation level of component i at the beginning of the any

inspection interval follows uniform distribution, $U_i \sim$ uniform$(0,H_i)$, i.e., $f_{U_i}(u) = \dfrac{1}{H_i}$ for

$0 \leq U_i \leq H_i$.

$$P(\tau) = \sum_{m=0}^{\infty} \prod_{i=1}^{n} P(W_{ij} < D_i)^m \frac{1}{H_i} \int_0^{H_i} P\left(\beta_i \tau + \sum_{j=1}^{m} Y_{ij} < H_i - u \right) du \frac{e^{-\lambda \tau} (\lambda \tau)^m}{m!} \tag{143}$$

Further conditioning on the shock damage to soft failure process:

$$P(\tau) = \sum_{m=0}^{\infty} \prod_{i=1}^{n} P(W_{ij} < D_i)^m \frac{1}{H_i} \int_0^{H_i} \int_0^{H_i - u} P\left(\beta_i \tau + \sum_{j=1}^{m} Y_{ij} < H_i - u \mid \sum_{j=1}^{m} Y_{ij} = q \right) f_{Y_i}^{\langle m \rangle}(q) dq du \frac{e^{-\lambda \tau} (\lambda \tau)^m}{m!}$$

$$\tag{144}$$

A more specific equation can be developed when the normal distribution for Y and

β, is appropriate.

$$P(\tau) = \sum_{m=0}^{\infty} \prod_{i=1}^{n} P(W_{ij} < D_i)^m \frac{1}{H_i} \int_0^{H_i} \Phi\left(\frac{H_i - u - m\mu_{Y_i} - \mu_{\beta i} \tau}{\sqrt{\sigma_{\beta_i}^2 \tau^2 + m\sigma_{Y_i}^2}} \right) du \frac{e^{-\lambda \tau} (\lambda \tau)^m}{m!} \tag{145}$$

The total cost include inspection cost, downtime cost and replacement cost

$$E(TC) = C_I + \sum_{i=1}^{n} C_{R_i} \left(1 - P_i(\tau) \right) + C_\rho E(\rho) \tag{146}$$

Where C_{R_i} is the replacement cost for component i and P_i is the probability that

component i survives inspection interval τ during steady state. $E(\rho)$ can be determined as

$$E(\rho) = \int_0^\tau (\tau - t) \frac{d(1 - P(t))}{dt} = \int_0^\tau (\tau - t) f_T(t) dt \tag{147}$$

In which $P(t) = \sum_{m=0}^{\infty} P(W_{ij} < D_i)^m \frac{1}{H_i} \int_0^{H_i} \Phi\left(\frac{H_i - u - m\mu_{Y_i} - \mu_{\beta i} t}{\sqrt{\sigma_{\beta_i}^2 t^2 + m\sigma_{Y_i}^2}} \right) du \frac{e^{-\lambda \tau} (\lambda t)^m}{m!}$

To minimize the long-run maintenance cost rate:

$$CR(\tau) = \frac{C_I + \sum_{i=1}^{n} C_{Ri}\left(1 - P_i(\tau)\right) + C_\rho E(\rho)}{\tau} \tag{148}$$

4.3.2 Initial degradation based on geometric distribution of survival intervals

To better understand the initial component degradation at the beginning of an inspection interval, it is important to know its life history, i.e., upon the beginning of the inspection interval, how long the component has survived. The number of previous survival intervals is a non-negative integer, that is random, but can be considered to be distributed as geometric distribution with probability mass function $p(1-p)^v$, with v = number of survival intervals. $E[T_i]$ is the expected component failure time, and $p_i \approx \tau/E[T_i]$ is the approximate probability that component i survives one randomly selected inspection interval under steady state. Therefore, we can obtain $f_{U_i}(u_i)$ as:

$$f_{U_i}(u_i) = \sum_{v=0}^{v_{max}} f_{U_i}(u_i \mid \text{survival intervals} = v)p_i(1-p_i)^v \tag{149}$$

We can obtain $f_{U_i}(u_i \mid \text{survival intervals} = v)$ based on the previous developed reliability models. The probability distribution function of the initial degradation upon the inspection point can be estimated based on an aggregation of degradation for known survival intervals. For a typical example, Figure 4.21 (a) to Figure 4.25 (a) show the individual pdf of initial component degradation considering a specific number of component survival intervals before the inspection interval we are interested in. For example, the blue line shows the pdf of component initial degradation based on the condition that this component has already survived one interval before the interval we are

interested in, and red lines mean component has survived two inspection intervals, and so on. Figure 4.21 (b) to Figure 4.25 (b) show the combined pdf of initial component degradation given all cases of survival intervals. The geometric distribution is an approximation, but logical given there is no prior knowledge of u_i. Comparing all the figure in each column, and when p_i value becomes small, the combined pdf is more stable, and it begins to resemble a uniform distribution. The right figures can be directly used in the simulation of $f_{U_i}(u_i)$, which is the sufficient condition for system survival probability $P(\tau)$ and individual component maintenance cost calculation. For the cases that the combined pdf cannot be fitted with a function, Monte Carlo simulation can be used to test different value for number of survival interval before the inspection and calculate the initial degradation.

Figure 4.21: individual (a) and combined (b) pdf of component initial degradation for

p=0.1

Figure 4.22: individual (a) and combined (b) pdf of component initial degradation for p=0.05

Figure 4.23: individual (a) and combined (b) pdf of component initial degradation for
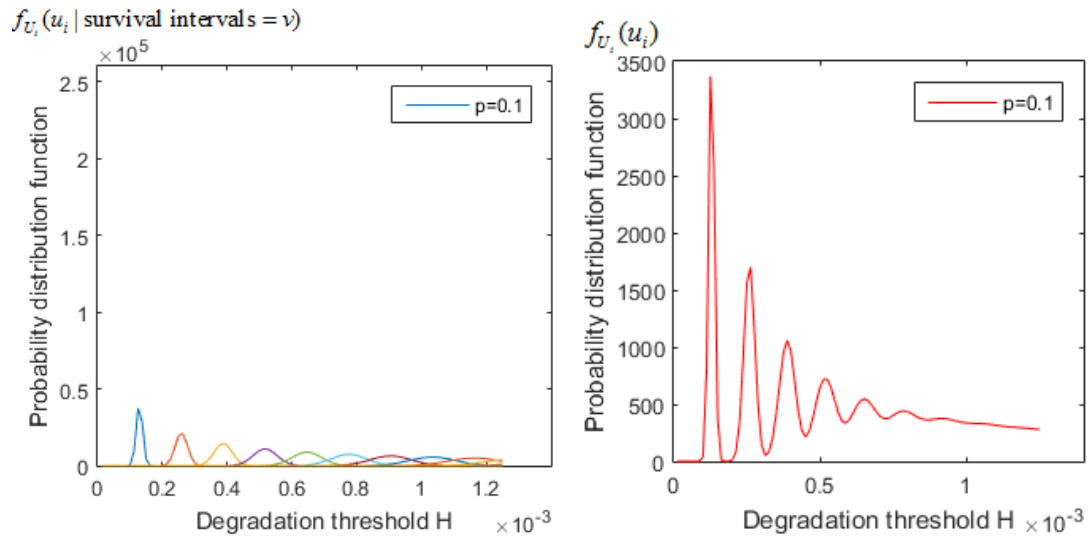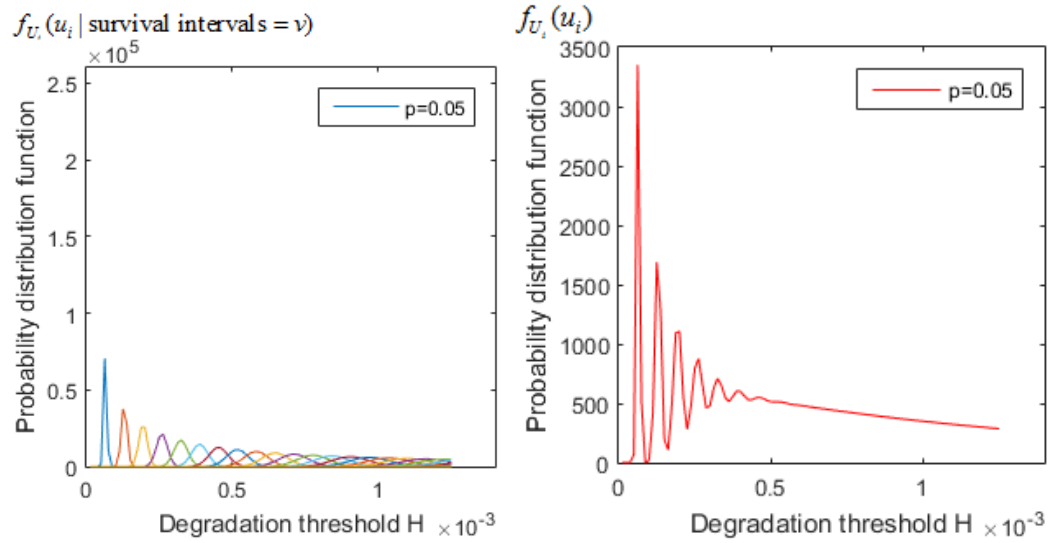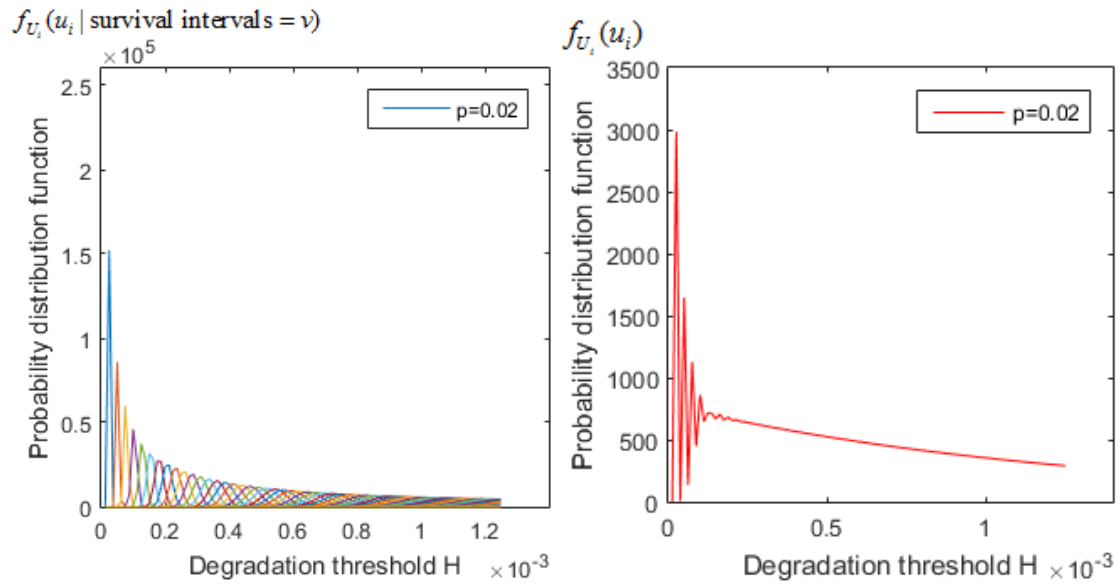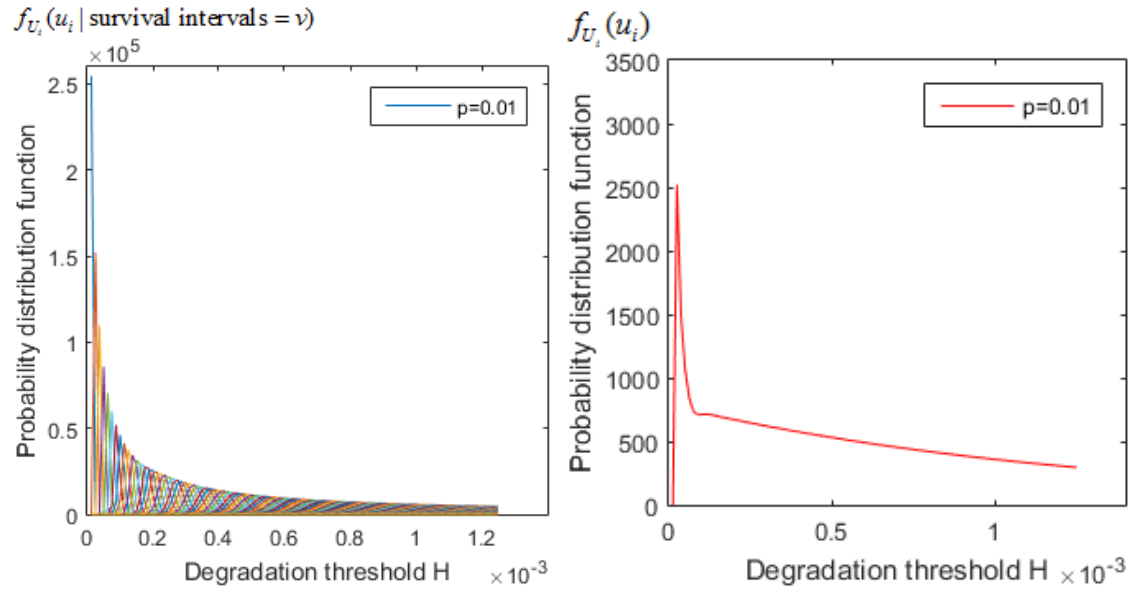
p=0.02

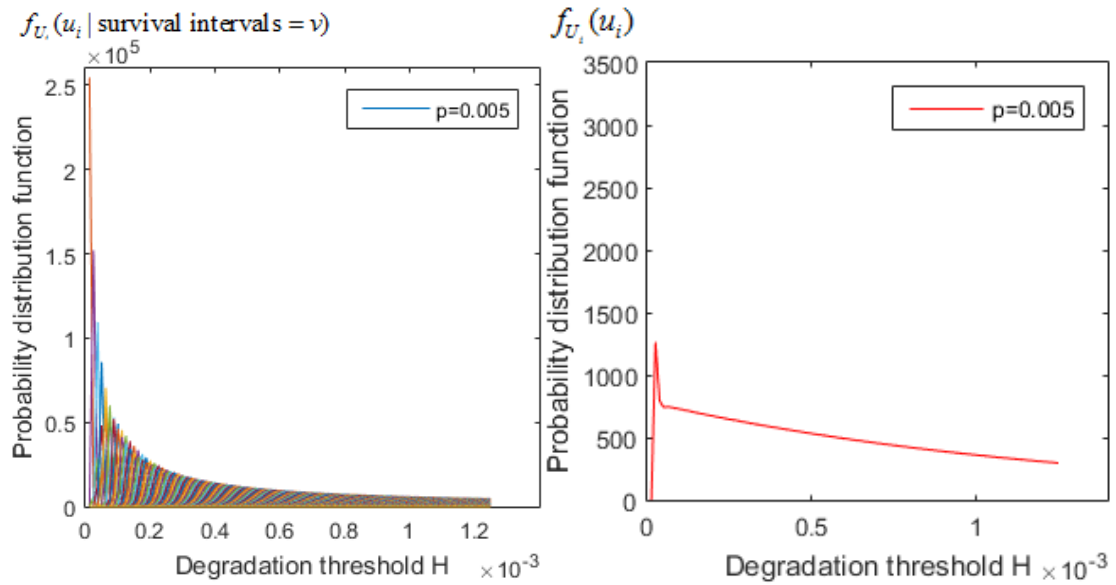Figure 4.24: individual (a) and combined (b) pdf of component initial degradation for
$p=0.01$

Figure 4.25: individual (a) and combined (b) pdf of component initial degradation for
$p=0.005$

4.3.3 Numerical examples

A numerical example is used to illustrate the individual component maintenance model based on both approximation methods for $f_{U_i}(u_i)$ the uniform initial degradation approximation, use the parameters in Table 3.1, and considering inspection cost C_I=3000, downtime cost rate C_ρ=50, and replacement cost for components are the same C_r=2000; Figure 4.26 shows that if the system is inspected every 8500 time units, the minimal maintenance cost rate can be obtained.
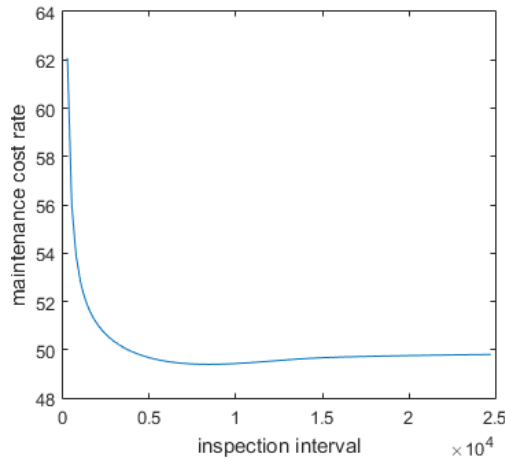
Figure 4.26: Individual component maintenance cost rate optimization result

To better understand parameter effects on the individual component maintenance cost rate, sensitivity analysis is conducted. The interesting parameters are inspection cost C_I, downtime cost rate C_ρ, and component replacement cost C_{Ri}.

From Figure 4.27, when inspection cost increases, the optimal inspection interval shifts to the right, which means when the system is inspected less frequently, and the cost savings can be achieved. Also, when inspection costs increase, the objective function, i.e., optimal maintenance cost rate, increases.
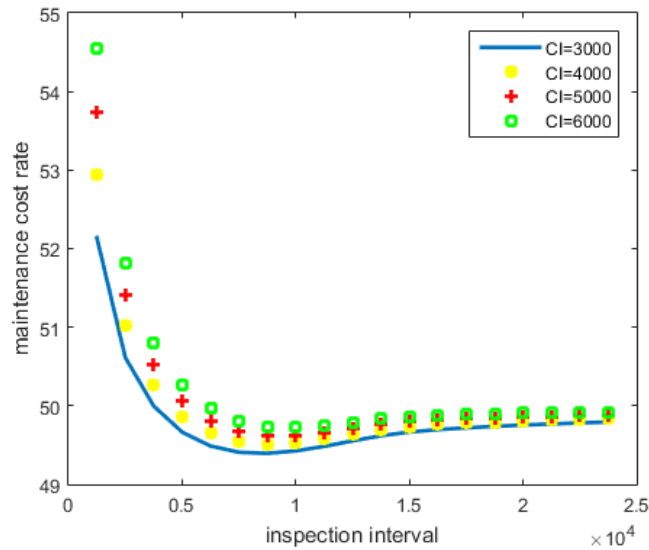
Figure 4.27: System reliability sensitivity analysis for inspection cost

From Figure 4.28, when downtime cost rate increases, the optimal inspection interval shifts to the left, which means when the system is inspected more frequently, and cost savings can be achieved. Also, when the downtime cost rate increases, the objective function, i.e., optimal maintenance cost rate increases.

Figure 4.28: System reliability sensitivity analysis for downtime cost rate

From Figure 4.29, when the component replacement cost rate increases, the optimal inspection interval does not significantly change. Also, when the component replacement

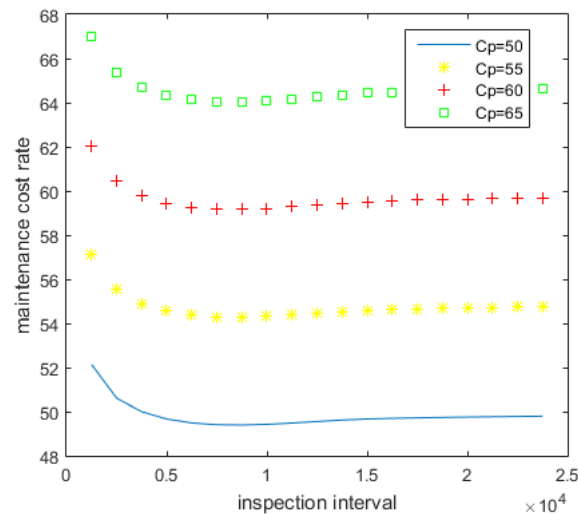cost increases, the objective function, i.e., optimal maintenance cost rate, increases.
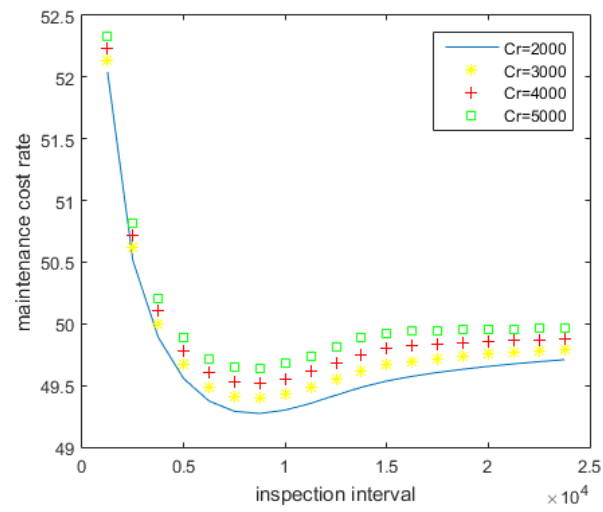

Figure 4.29: System reliability sensitivity analysis for component replacement cost

5. Conclusions

This research develops reliability models for multi-component systems subject to multiple dependent competing failure processes and solves the maintenance optimization problem for different maintenance policies. Multiple failure processes for each component are dependent, and also time-to-failure for all components are also dependent.

Reliability models are developed for systems with component degradation models for different structures/configurations, including series systems, parallel systems, k-out-of-n systems, and series-parallel systems. Two special cases are considered: (1) When a shock arrives to the system, it affects both hard failure process and soft failure process, and the effects on the two failure processes are considered to be dependent. (2) Each component within the system has its own shock set, i.e., shocks are divided into different sets according to different sizes, functions, acting points and other characteristics.

Next, the developed reliability models were extended to system reliability models considering dependent shock damage sizes to the two failure processes. For the system reliability model, failure processes can be dependent/correlated due to the shared exposure to the same shocks arriving to the system. However, if one shock arriving to the system with multiple components, it is probabilistically likely that its damage impact to all component hard failure processes by this specific shock are dependent, and it is also probabilistically possible that its damage impact to all component soft failure processes by this specific shock are dependent. Therefore, a more detailed dependency pattern is considered by assuming transmitted shock sizes for some or all components caused by the same shock are dependent, and shock damage for some or all components caused by the

same shock are dependent. Further, a more detailed reliability model is studied considering W_{ij} dependent, Y_{ij} dependent, W_{ij} and Y_{ij} mutually dependent.

Component degradation paths are considered to be dependent since components share the same working environment and also degradation of one component can have an effect on the degradation of other components. Gamma process models have been used to model the degradation paths. System reliability models have been developed considering stochastically dependent component degradation paths.

Finally, an age replacement policy and a periodic inspection policy are then chosen as prospective maintenance strategies. Maintenance policies and optimization have been conducted for series system, k-out-of-n system, and shock set cases. By optimizing the age replacement interval and periodic inspection interval, long-run system maintenance cost rate is minimized. Furthermore, a condition-based maintenance model is developed by defining a relatively lower degradation threshold compared to the critical soft failure threshold. It is an effective maintenance policy for systems because the failure cost is much higher than other kinds of cost. Then, an individual component maintenance policy rather than group maintenance is considered. The system survival probability is modeled under the long-run steady state assumption and the system periodic inspection interval is optimized.

Dependent shock effects and dependent component degradation paths have not been considered before in reliability modeling for system with components subject to multiple failure processes. Also no maintenance policies have been applied to such systems before. Especially, in this research, the individual component maintenance policy is

considered for such systems, which is more realistic and cost efficient for many engineering applications.

There can be many extensions and potential future research based on this work, including: (1) combined dependent component degradation with concurrent dependent shock damage; (2) stochastic degradation modeling, i.e., geometric Brownian motion; (3) other individual maintenance policies, such as age replacement policy, condition-based maintenance policy and related multi-objective optimization modeling, etc.

Appendix A: proof of component dependent failure times

In Section 3.1.1, previous research is extended by studying a multi-component system and failure processes among components that are also dependent. Here, this extension is demonstrated by deriving the covariance of the degradation for any two components, and proving that it is greater than zero given the presence of shocks. Furthermore, events for component survival from hard failure are defined, and it is demonstrated that the covariance of these events for any two components is positive.

For a multi-component system, the number of shocks $N(t)$ has an effect on each component. When $N(t)$ is sufficiently large, the sum of the shock damage size contributing to soft failure for each component is large, and there are also greater opportunities for hard failure; thus, a failure is more likely for all components. Alternatively, when there are relatively few shocks, times to failure are relatively longer for all components. Thus, the component failure processes are probabilistically dependent. The dependency of soft failure processes for all components is proved.

If the covariance of two events is greater than zero, then the occurrences of these two events are correlated. Probabilistically dependent (but not necessarily physically dependent). With this idea, soft failure processes for all components are dependent. The covariance of total degradation of component i and component j is

$$Cov(X_{S_i}(t), X_{S_j}(t)) = E\left[X_{S_i}(t)X_{S_j}(t)\right] - E\left[X_{S_i}(t)\right]E\left[X_{S_j}(t)\right]. \qquad (A1)$$

For component i, $X_{S_i}(t) = X_i(t) + S_i(t)$, and this equation applies for all components.

$$Cov(X_{S_i}(t), X_{S_j}(t)) = \left(E[X_i(t)X_j(t)] - E[X_i(t)]E[X_j(t)] \right)$$

$$+ \left(E[X_i(t)\sum_{k=1}^{N(t)} Y_{jk}] - E[X_i(t)]E[\sum_{k=1}^{N(t)} Y_{jk}] \right) + \left(E[X_j(t)\sum_{k=1}^{N(t)} Y_{ik}] - E[X_j(t)]E[\sum_{k=1}^{N(t)} Y_{ik}] \right)$$

$$+ \left(E[\sum_{k=1}^{N(t)} Y_{ik} \sum_{k=1}^{N(t)} Y_{jk}] - E[\sum_{k=1}^{N(t)} Y_{ik}]E[\sum_{k=1}^{N(t)} Y_{jk}] \right).$$

(A2)

Because the value of the first three items are zero

$$Cov(X_{S_i}(t), X_{S_j}(t)) = \left(E\left[\sum_{k=1}^{N(t)} Y_{ik} \sum_{k=1}^{N(t)} Y_{jk}\right] - E\left[\sum_{k=1}^{N(t)} Y_{ik}\right] E\left[\sum_{k=1}^{N(t)} Y_{jk}\right] \right)$$

$$= E\left[(Y_{i1} + ... + Y_{iN(t)})(Y_{j1} + ... + Y_{jN(t)}) \right] - E\left[(Y_{i1} + ... + Y_{iN(t)}) \right] E\left[(Y_{j1} + ... + Y_{jN(t)}) \right].$$

(A3)

As an example, consider that Y_{ij} are *i.i.d* random variables that follow a parametric

distribution, and shocks arrive as a Poisson process:

$$Cov(X_{S_i}(t), X_{S_j}(t)) = E\left[N(t)^2 \right] \mu_{Y_i} \mu_{Y_j} - E\left[N(t) \right]^2 \mu_{Y_i} \mu_{Y_j}$$

$$= \mu_{Y_i} \mu_{Y_j} Var(N(t)) = \mu_{Y_i} \mu_{Y_j} \lambda t > 0$$

(A4)

Similar covariance derivations were performed for hard failure events. If H_i = event no

hard failure of component i, and $H_i \in \{0,1\}$, $H_j \in \{0,1\}$, the covariance of events H_i and H_j

is greater than zero for two components i and j.

Applying the law of total probability, and define $a = \Pr(W_{il} < D_i)$, $b = \Pr(W_{jl} < D_j)$

, and $N_l = \Pr(N(t) = l)$

$$P(H_i) = P\left(W_{i1} < D_i, W_{i2} < D_i, W_{iN(t)} < D_i \right) = \sum_{l=1}^{\infty} P\left(W_i < D_i \right)^l P\left(N(t) = l \right) = \sum_{l=0}^{\infty} a^l N_l \quad (A5)$$

$$P(H_j) = P\left(W_{j1} < D_j, W_{j2} < D_j, W_{jN(t)} < D_j \right) = \sum_{l=1}^{\infty} P\left(W_j < D_j \right)^l P\left(N(t) = l \right) = \sum_{l=0}^{\infty} b^l N_l$$

$$(A6)$$

$$\Pr\left(H_i \cap H_j\right) = \sum_{l=0}^{\infty} P\left\{\left(\bigcap_{k=1}^{N(t)} W_{ik} < D_i\right) \cap \left(\bigcap_{k=1}^{N(t)} W_{jk} < D_j\right) \mid N(t) = l\right\} P\left\{N(t) = l\right\}$$

$$(A7)$$

$$= \sum_{l=0}^{\infty} P\left(W_{ik} < D_i\right)^l P\left(W_{jk} < D_j\right)^l P\left(N(t) = l\right) = \sum_{l=0}^{\infty} a^l b^l N_l$$

We want to prove $(\sum_{l=0}^{\infty} a^l N_l)(\sum_{l=0}^{\infty} b^l N_l) \neq \sum_{i=0}^{\infty} a^l b^l N_l$, that is $P(H_i \cap H_j) \neq P(H_i)P(H_j)$, or that

$Cov(H_i, H_j) > 0$

Consider shocks follow a Poisson process with rate λ.

$$(\sum_{l=0}^{\infty} a^l N_l)(\sum_{l=0}^{\infty} b^l N_l) = \left(\sum_{l=0}^{\infty} a^l \frac{e^{-\lambda t}(\lambda t)^l}{l!}\right)\left(\sum_{l=0}^{\infty} b^l \frac{e^{-\lambda t}(\lambda t)^l}{l!}\right) = e^{-2\lambda t}\left(\sum_{l=0}^{\infty} \frac{(\lambda a t)^l}{l!}\right)\left(\sum_{l=0}^{\infty} \frac{(\lambda b t)^l}{l!}\right)$$

$$= e^{-2\lambda t} e^{\lambda a t} e^{\lambda b t} = e^{\lambda t(a+b-2)}$$

$$(A8)$$

$$\sum_{l=0}^{\infty} a^l b^l N_l = \left(\sum_{l=0}^{\infty} a^l b^l \frac{e^{-\lambda t}(\lambda t)^l}{l!}\right) = \left(e^{-\lambda t}\sum_{l=0}^{\infty} \frac{(\lambda a b t)^l}{l!}\right) = \left(e^{-\lambda t} e^{\lambda a b t}\right) = e^{\lambda t(ab-1)} \qquad (A9)$$

It can be proved easily that $e^{\lambda t(a+b-2)} < e^{\lambda t(ab-1)}$ for $0 < a < 1$ and $0 < b < 1$, and

$$P\left(H_i \cap H_j\right) > P\left(H_i\right)P\left(H_j\right). \qquad (A10)$$

$Cov(H_i, H_j)$ can similarly be demonstrated to be positive, considering that $E[H_j] = P(H_i)$, $E[H_j] = P(H_j)$, and $E[H_i \cap H_j] = P(H_i \cap H_j)$.

Because both the soft failure process and hard failure process for all components are probabilistically dependent, it can be concluded that component survival events are also dependent.

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