Utilizing Video Analytics to Examine the Role of Representations in Problem Solving Across Grade Bands

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ABSTRACT OF THE DISSERTATION

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This study uses RUanalytics to examine the representations created by fourth and eighth grade students' work in solving open-ended activities of comparing and ordering fractions on a number line and finding surface area and volume of various models of stacked Cuisenaire rods. This research study has developed from data collected from a longitudinal and cross sectional study of two focus groups of students, conducted at Rutgers University. The research reported here builds on this earlier work by using video clips and full video to construct video narratives of student learning.

The methodology of this study is based on the viewing and analysis of archived video data, student transcripts and the creation of three RUanalytics, two showing problem solving of fourth graders and a third of eight graders. The video data were analyzed according to the model by Powell, Francisco and Maher (2003). The methodology was extended to include building video narratives with the RUanalytic tool.

The study was guided by a single research question: What representations do fourth and eighth grade students use to express their ideas in problem solving? As the fourth-grade students transitioned from operator to number understanding of fraction, they used a variety of representations in their problem solving: manipulatives, pictorial, written, symbolic, imagistic and spoken language. Eight-grade students also used manipulative, pictorial, written, symbolic notation, imagistic, experiential and spoken language to represent their data when working to find a formula for the surface area and volume of rods and stacked rods. By using a variety of representations and moving between and among them, students uncovered important mathematical ideas and relationships in their problem solving. This research provides evidence of the importance of open-ended problem solving tasks in which students have an opportunity to call upon a variety of personal representations to explore their ideas prior to formal instruction. By tracking student representations in problem solving, the RUanalytics show, students growing understanding of rational number ideas and early algebra concepts.

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Dedication

I dedicate this work to family for their support and encouragement throughout my studies. To accomplish a monumental task such as you always stand on the shoulders of giants and I have two such giants in my life my wife Abbey and daughter Jillian. Jillian, since your birth, Abbey and I have been constantly amazed at what you accomplish every day. Abbey your encouragement and support has made this moment possible. I love you very much.

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Chapter 1

1.1 Introduction

In *The Principles and Standards of School Mathematics*, “representation” is noted as central to the study of mathematics and mathematics education (NCTM 2000). The term “to represent” means correspond to, denote, depict, embody, label, mean, produce, refer to, or symbolize (NCTM 2000). Representation refers to the various ways that a student can externally depict their internal conceptualizations (Lesh, Post, & Behr, 1987). Abrahamson (2006) suggests that representations are “conceptual composites” (two or more connected ideas). These composite ideas may be not easily seen or understood just through use of the representation. Janvier (1987) comments, that representations are the “symbols (written), real objects, and mental images.” Goldin and Kaput 1992, distinguish between internal and external representations. External systems include such things as the real objects and written symbols referred to by Janvier, but also such things as the base-ten system of numbers, formal algebraic notation and computer based microworlds. Internal representations include personal constructs, the assignment of meaning to mathematical notations as well as the mental images referred to by Janvier. It is the interaction between the internal and external representations that is essential to effective teaching and learning (Goldin & Shteingold, 1998). These internal and external representational systems (Goldin, 2003) provide opportunities for students to better incorporate the math they are learning into their own schema.

As students study mathematics, it is essential that they gain understanding of essential concepts as they progress through school. A background in such concepts as rational numbers, problem solving and the ability to generalize solutions, are essential for students as they advance in their study of mathematics. Several recent studies at Rutgers University have investigated how young students build an understanding of mathematical ideas and provide justifications for their solutions to problems. For example, several studies traced student learning in the domain of fractions and their operations (Schmeelk, 2010; Yankelewitz, 2009; Reynolds, 2005; Steencken, 2001; Bulgar, 2000); others have studied different content domains such as algebra (Aboelnaga, 2011; Giordano, 2008; Mayansky, 2007; Spang, 2009) combinatorics (Tarlow, 2004), probability (Shay, 2009), geometry (Marchese, 2009), calculus (Pantozzi, 2009), and pre-calculus (Halien, 2011).

This prior research made use of raw video copied on videotapes, CD's, and DVD's. The video data enabled researchers to trace specific learning activities in detail. Researchers who observed the videotaping made field notes and collected student work to support their findings. The research reported here builds on this earlier work by using a new tool to access video clips and full-length video, now stored on the university's digital repository, to construct video narratives of student learning. The video data for these narratives come from the content domains of fractions, rational numbers, early algebra, and geometry. The lens for analysis focuses on students' use of multiple representations in building their solutions to problems and providing justifications for their solutions.

1.2 The Video Mosaic Collaborative (VMC)

Longitudinal and cross-sectional studies conducted by mathematics education researchers at Rutgers University over the last few decades have produced over 4500 hours of video data from multiple contexts that trace student development of mathematical ideas and ways of reasoning over time. The effort to preserve this collection of videos has resulted in an open source, digital repository housed at Rutgers University, called the Video Mosaic Collaborative (VMC). Currently more than 400 video clips from these studies are available (see http://www.videomosaic.org) as well as approximately 80 hours of full-length videos. The video data, task descriptions, transcripts, and student work are preserved from over 25-years of longitudinal and cross-sectional research studies, showing the mathematical learning of students from diverse populations and age levels engaged in doing mathematics. There are three studies that are the main sources of video data to the VMC. One is a longitudinal study that occurred in the Kenilworth NJ School district, which featured the same students from elementary through high school engaged in mathematical problems across a multitude of content strands. Another is a year-long study that occurred during the 1992-1993 school year in Colts Neck NJ, which studied students working with fractions and operations on fractions. Yet another is the three-year study of informal mathematics learning that occurred about a decade later in an urban middle school after-school program. The data for my studies came from the first two data sources.

The Rutgers longitudinal study was carried out with a cohort of students from first grade to twelfth and beyond in the district of Kenilworth, New Jersey, a working class community. The focus was studying the development of mathematical ideas and ways of reasoning in students.[1] For the first eight years, the setting was the K-8 public elementary school called Harding Elementary School. The research took place in a regular classroom setting. The classroom environment allowed for student exploration and thoughtful mathematical expression. The students worked collaboratively on open-ended mathematical tasks and were encouraged to pose

[1] Research support for the longitudinal study was provided by the U.S. National Science Foundation (awards MDR 9053597 and REC-9814846). The views expressed here are those of the authors.

conjectures, explore those ideas and develop theories that provide meaning to those tasks. The focus of the classroom setting was to provide an environment where students could build their own mathematical ideas (Maher, 2010).

In grades 9-12, the setting was informal, with problem-solving activities for students in grades 10-12 during after-school sessions in the high school. The goal was to provide an environment that invited students to build their own mathematical ideas as they worked collaboratively in small groups (Maher, 2010).

One of the contexts for cross-sectional research was the Colts Neck Study. This study occurred during the 1992-1993 school year and was a yearlong classroom-based teaching experiment designed to investigate how young students build their knowledge of fractions. This study was a partnership between Rutgers University and the Colts Neck Public School district, and a component of a larger three-year long National Science Foundation (NSF) study conducted by Davis and Maher in three New Jersey school districts. The Colts Neck Study took place during fifty-six, one to one and a half hour classroom sessions over the course of one year. The class was made up of 25 heterogeneously grouped students and their teacher, who observed all sessions. Fourth grade students were selected because the Conover Road School, where the study was conducted, did not include in their mathematics program operations with fractions.

1.3 The RUanalytic Tool

The VMC has a video editing and annotation tool called the RUanalytic tool, which allows the user to select a video resource and specify a start point and a stop point to create an event that can then be titled and annotated with text description (Agnew, Mills & Maher, 2010). The tool allows the user to string together a series of events, selecting from one or multiple resources in the larger video collection stored on the VMC, to tell a story or highlight specific concepts like student reasoning, teacher intervention of student learning, or building knowledge of specific mathematical concepts. Annotated events, when linked together, form a multimedia narrative that can be used independently for instruction, study, or research. Also, they can supplement a scholarly paper (e.g., Sigley & Wilkinson, 2015), enhance teaching (Van Ness, 2015), or be used as a tool in teacher professional development (Horwitz, 2011). Video narratives made using the RUanalytic tool can remain private in a user's workspace, be shared with others within the tool or via a temporary hyperlink, or undergo review for publication on the VMC site. When published, video analytics retain a cataloging connection to their source videos and receive a persistent URL, thus becoming new permanent resources preserved in the Video Mosaic repository.

While video analytics can be developed to help teachers, teacher educators and researchers, they can also be used as a tool to inform practice, illustrate theory, or show how student knowledge builds over time and through engagement in mathematical tasks in a collaborative setting.

Earlier research analyzing portions of the Rutgers Longitudinal Study data yielded an analytical framework involving several nonlinear phases (Powell, Francisco, and Maher (2003). Particular phases of work of analyzing the video data, specifically describing events of interest and transcribing them, facilitated preparation of a series of video clips for the repository. This research utilizes the earlier methodology and extends it through video analytic production, as discussed further in chapter four.

1.4 Three Studies on Representations

This research involves three studies, all of which analyze video data stored on the VMC repository. Each study, with accompanying RUanalytic, focuses on identifying the various types of representations that students used to build mathematical ideas and justifications for solutions to problems.

The single research question that guides all three studies is: *What representations do students use to express their ideas in problem solving?* Each study traces the representations used by students during their problem solving. In addition to the written analysis and presentation of findings, three RUanalytics published in the VMC support the analyses with text and visual evidence of how students were using representations in their problem solving.

The first two studies deal with fraction/rational number ideas explored by fourth grade students before their formal introduction of the concept of fractions in schools. Specifically, chapter five traces students' comparing and ordering of fractions as well as their placing of fractions on a number line segment. Associated with this first study is the RUanalytic, *Using Meredith's Models to Reason about Comparing and Ordering Unit Fractions* (retrieved from http://dx.doi.org/doi:10.7282/T33J3FQG) Chapter six then investigates an extension activity, tracing students' representations as the class works on placing fractions on an infinite number line. This study is supported by the RUanalytic, *Extending Fraction Placements from Segments to Number Line: Obstacles and Their Resolutions* (retrived from http://dx.doi.org/doi:10.7282/T39Z96SR).

Chapter seven examines activity from the longitudinal study that occurred in an eighth-grade classroom during an extended, single-class session approximately one hour in length.

Fourteen eighth graders, nine girls and five boys, participated in that session; however, my focus is on the activity of one group consisting of 2 girls and 2 boys. This study analyzes the representations generated by that group of four students as they worked together in solving open-ended tasks for finding surface area and volume, having available Cuisenaire Rods to build physical models. Associated with this study is the third RUanalytic, *Eight-Grade Students explore Surface Area and Volume Problems: The Role of Representations* (retrieved from http://dx.doi.org/doi:10.7282/T3V40X46). It traces student building of models, creating pictures and drawings, using words and symbols as well as other experienced-based representations.

Chapter 2 Theoretical Framework

The theoretical framework that guides this work parallels the conceptual framework built by Merlin Behr, Richard Lesh and Tom Post (1987). The framework described below provides the lens through which I study the developing ideas of students that are captured on video.

2.1 Types of Representations and Representational Systems

The use of representations is a vital component of the process of learning mathematics (Smith, 2003; Goldin, 2003; NCTM, 2000; Kaput 1987). Lesh, Post, & Behr (1987) identify five types of representations in mathematical problem solving: experiential, manipulatives, pictures or diagrams, spoken language, and written symbols. In experiential representations, knowledge is organized around real-world issues within context (community problems, for instance). Manipulative representations (Cuisenaire rods and fraction circles, for example) have very limited meanings without the relationships and operations that fit many situations. That is, the manipulatives by themselves will not produce better understanding; students must be guided in how they can make use of manipulative objects to enhance understanding of particular mathematical ideas based on the design of the objects and tasks to utilize them. Pictures or diagram representations are static models that can be internalized as images (photos and charts, for instance). Spoken language representations include many languages within languages or specialized domains (logic, for example). Written symbol representations involve written language (X + 3 = 7 and English sentence structure, for instance).

Each of these representations is extremely important. However, individually or in combinations, a variety of representations can become integrated parts of complex representational systems. Learning those systems and how to operate within them dominates school mathematics (Kaput, 1999). Students need to move fluidly from one representation to another in order to achieve deeper and more flexible understanding (Lesh, Post, & Behr, 1987; Goldin, 2001; NCTM, 2000). Some research suggests that visual/pictorial representations, like the drawing of diagrams, do not support students' understanding in mathematics (Cai & Lester, 2005), However, Stylianou (2002) suggests that visual or pictorial representation do indeed play a role in mathematical understanding. She studied the processes of professional mathematicians and found that they used visual representations to better inform their understanding of the problem at hand and to trigger next steps in the problem-solving process.

By making use of a variety of representations and moving successfully between and among them, students can develop a deeper understanding of a wide range of mathematical concepts. Such understanding is achieved when (a) s/he can recognize the idea embedded in a variety of qualitatively different representational systems, (b) s/he can flexibly manipulate the idea within given representational systems, and (c) s/he can accurately translate the idea from one system to another (Kilpatrick, Swafford & Findell, 2001, p. 119). According to these researchers, the teacher has an important role in supporting students' movement from one representational system to another.

The Lesh, Post, & Behr (1987) categories cover many forms of **external** representations. However, students also create internal representations to enhance their understanding (Goldin & Shteingold, 2001). Internal systems of representation are created within a student's mind and are used to assign mathematical meaning. Goldin and Kaput (1996) define **internal** representations as the **possible** mental configurations of learners, problem solvers, or students. Goldin (2003) further defines five categories of internal representations: 1) verbal/syntactic systems of natural language; 2) imagistic representations, which refer to systems in which the fundamental characters, signs, and configurations are neither verbal nor formal in nature, but bear some interpreted sensory resemblance to what is represented (Goldin & Kaput 1996); 3) formal notational systems of mathematics; 4) a system of planning and monitoring the problem solving process; and 5) an affective system that not only includes the general feeling of the problem solver, but also the states of feeling during the problem solving process. Because internal representations are the mental configurations of students, they are not readily apparent to teachers, who may have to infer an internal representation from students' external behavior (Goldin & Kaput, 1996).

The goal is for students to make meaning of the external representations and then use them to solve problems (Goldin, 2002). To make meaning of external representations, students must first process them internally, and then establish a connection between the internal and the external. Representations are inherently personal in their nature (Smith, 2003). Different students can and will interpret the same representation in a variety of ways. By inventing, adapting, or using their own representations and then fitting them into their own schema, students have an opportunity to construct their own internal representations.

Internal systems of representations, created within a learner's mind, can be used to assign mathematical meaning. These mental configurations are not observable to others and can only be inferred from external behavior (Goldin & Kaput, 1996). In order to successfully apply external representations for problem solving, learners must use them in a meaningful way, first by processing them internally, and then establishing a connection between the internal and the external. Goldin (2002) recommends that teachers focus on "the interaction between the internal and external" representations as a way to cultivate understanding. Cycling through a process that supports interactions between internal and external representations offers opportunities for individual growth in knowledge.

A student’s internal representations may possibly reflect the personal symbolization, construction, and assignment of meaning beyond external mathematical notations. However, the most difficult aspect of internal representations is that they cannot be observed. External representations, in contrast, can be observed. Teachers and researchers, then, infer if students are internalizing representations, based on their external representations and discourse used in regard to the subject. Attending to particular ways in which students appear to move back and forth between their internal and external representations thus provides teachers with an approach for supporting the development of mathematical understanding (Goldin, 2001).

2.2 Representations of Rational Numbers

According to Smith (2002), there is no area of mathematics as rich, complicated or difficult to teach as fractions, ratios and proportions. Mathematical concepts exist as an intricate web of interrelationships and this is certainly the case with building conceptual understanding of fraction, ratio, and proportion ideas. Fractions and ratios are “relational” numbers, expressing relationships between two discrete quantities. Proportion examines the relationship between two or more discrete quantities. These topics begin in elementary school with the operator understanding of fractions for young children. Their understanding of parts of units is expressed within a variety of representations: drawings, models, and images. The extension of operator understanding of fraction to number understanding appears subsequently in the school curriculum and resurfaces in later mathematics study in more abstract forms.

While the Common Core State Standards for Mathematics recommends that formal introduction to fraction operations occur in the fifth grade (CCSS, 2010), children informally begin to construct knowledge of fraction ideas at an early age. This concept might begin with the idea of sharing, or equally dividing, quantities between or among others, introducing the idea of fraction as an operator. Playing games or watching sports brings increased knowledge of fractions or ratios. Concepts such as baseball batting averages, completion percentages, and earned run averages are all fractions, ratios and proportions. These kinds of prior knowledge, developed through informal learning, can be an important foundation for subsequent, more formal learning. When teachers are aware of this, they can provide an opportunity for students to incorporate experiential, prior knowledge from real life applications of fractions and ratios to motivate understanding of the mathematical ideas behind the numbers.

For example, Steencken and Maher (2003) report on an instance of fairly sharing a candy bar to provide an imagistic representation of fraction as operator. When sharing candy bars fairly among the same number of students, for the same fractions to have the same value, the size of the original bars should be the same.

The fraction or rational number interpreted as a number takes on a different meaning. This occurs when comparing and ordering fractions or placing them on a number line. In so doing, the representation of the value of the fraction in terms of its magnitude, in contrast to its value as related to a specific unit, needs to be understood.

One obstacle for students to overcome is the difference between fractions and ratios. While fractions and ratios may appear similar, there are important differences that are essential for understanding. A fraction is defined as a “divided quantity” or a “partitioned quantity” (Smith, 2002). For example, if you give one pretzel to each of four friends, each friend has 1/4 of the total number of pretzels. However, when a divided quantity refers to a multiplicative relationship between two discrete quantities, this is called a ratio (Smith, 2002). For instance, in baseball your batting average is calculated by making a relationship between the total numbers of hits to the total number of at bats you have in a game. The word multiplicative is sometimes used because if you, in a five game series, have 10 hits out of 20 times at bat, then your average or ratio of hits to at bats is 0.500 or 1/2. The term proportionality is the comparison or reasoning with two ratios.

Each of these topics is more commonly encompassed in the topic of rational numbers. Behr, Lesh, Post, and Silver (1983, p. 91) note:

> Rational number concepts are among the most complex and important mathematical ideas children encounter during their postsecondary school years. The importance of these concepts can be examined from a variety of perspectives: *(a)* from a practical perspective, the ability to deal with these concepts usually transfers to situations and problems in the real-world; *(b)* from a psychological perspective, rational numbers provide an arena by which students can expand their mental structures necessary for continued intellectual development; and *(c)* from a mathematical perspective, rational-number understandings provide the foundation upon which elementary algebraic operations can later be based.

Rational numbers can be interpreted in many different ways: part whole, area or region, comparison, decimal, ratio, operator, measurement of continuous or discrete quantities, and indicated division. Because there are multiple ways for rational numbers to be interpreted, students typically encounter obstacles in their sense making of fractions. Of particular interest is how students navigate the transition from the more readily understood notion of fraction as an operator to the more abstract interpretation of fraction as a number.

Chapter 3 Review of the Literature

The earlier examinations of the Colts Neck research showed that fourth-grade students were able to build the idea of fraction as number (Schmeelk, 2010) and extend their knowledge to equivalent fractions (Steencken, 2001), comparing fractions (Reynolds, 2005) and division of fractions (Bulgar, 2002).

The Colts Neck Study was designed to encourage the students to build personal understandings of fraction ideas prior to being taught fractions through formal instruction. The researchers encouraged verbal representations, written representations, and using manipulatives to gain a better understanding of how students learn and understand fraction concepts prior to formal classroom instruction on that topic.

Steencken (2001) traced the growth of understanding fraction ideas. She found that the students showed an understanding of several mathematical concepts, such as fraction as operator, fraction as number, identifying the unit for comparing fractions, equivalent fractions and fraction comparisons. Steencken (2001) examined how the students built their fraction ideas through the representations they used. She also pointed out how students built on each other's ideas and challenged each other's ideas and required justification. Notably, she found that the students expressed fractional ideas more precisely throughout the sessions within their use of natural language, physical models and notation. Steencken (2001) also found that using the representation of the candy bar, which was an effective metaphor for introducing students to the idea of unit, served as a springboard to discussing the idea of comparing and ordering fractions and, later, to placing fractions on the number line.

Bulgar (2002) examined the nature of the researcher interventions, the representations that were used, and the student reasoning used to obtain solutions to problems. Bulgar (2002) identified a number of ways researchers intervened including: giving information to students, paraphrasing student ideas, asking another student to restate an idea posed by a student, asking questions, asking for justification and directing students to explore their own or another's idea. Bulgar (2002) found that the researcher was an important classroom figure but not responsible for students' acquisition of knowledge.

Bulgar (2002) also explored how students expressed their ideas. In particular, she examined the representations used by students and identified a variety of types including verbal, physical models, written symbols, drawings and gestures. An interesting finding is that student reasoning and justification included the use of experiential representations, as well as metaphors. Ultimately, Bulgar (2002) showed evidence that students had understanding of division of fractions.

Reynolds (2005) studied a series of sessions of the same class of students, occurring between those analyzed by Steencken (2001) and Bulgar (2002), which focused on tasks to compare fractions. Reynolds (2005) explored student conjectures and traced strategies used to support or debunk ideas. She also documented the development of students' language through the sessions. She found that certain conjectures by students brought about new conjectures and that student's built on each other's ideas. Ultimately, Reynolds (2005) showed that students were able to appropriately compare fractions, building on the idea of equivalence.

Yankelewitz (2009) applied a different lens to analyze the series of 17 fraction sessions initially studied in earlier dissertations (Bulgar, 2002; Reynolds, 2005; Steencken, 2001). She identified different forms of mathematical reasoning that emerged from students' work on the tasks, as well as factors of both the task and the classroom environment that encouraged the development of their reasoning. She also identified critical events as those moments when students provided a solution and justified their solution using an argument (Yankelewitz, Muller, Maher 2010). Yankelewitz (2009) found that students use many forms of reasoning including the following: generic reasoning, reasoning by cases, recursive reasoning, reasoning using upper and lower bounds. An interesting finding is that the students used direct reasoning most of the time and less frequently used indirect arguments. Moreover, Yankelewitz and colleagues (2010) observed that student arguments did not follow conventional mathematical proof methods. Rather the students used their own conventional speech. It was further explained that many of the counterarguments were indirect and emerged by students challenging the claims of others, whereas direct arguments were predominately used when students needed to provide justification of their own claims.

Tracing the development of different yet related ideas, Schmeelk (2010) explored student's understanding of the transition of fraction as an operator to fraction as a number. For this she examined the representations and reasoning used by students as they extended their fraction ideas to incorporate rational numbers. Her data were four sessions of the Colts Neck Study specifically focusing on comparing and ordering fractions, placing fractions on a number line segment, and placing fractions on an infinite number line. To accomplish these activities students used not only verbal representations, but they also used manipulatives, pictorial representations and diagrams to gain understanding of fraction as a number. Schmeelk (2010) concluded that students' movement from the rod (manipulative representation) to the drawing (pictorial) representation shows a natural transition to comparing and ordering fractions on a number line. This is evidence that students can build an understanding of rational number concepts prior to formal instruction. This finding is particularly significant because it documented that students could learn naturally without having a teacher impose representations on them.

Shifting settings, Marchese (2009) examined data obtained from the Kenilworth Longitudinal Study that centered on two sessions in 1996 with a class consisting of fourteen students. The objective was to examine how students solved an open-ended activity consisting of a series of tasks designed for finding surface area and volume of Cuisenaire rods in a variety of configurations. It also was of interest to see whether students would be able to create a generalization based on their solutions. Marchese's (2009) analysis focused on the representations students used to solve the tasks before them and yielded insight into how students use representations to generalize solutions to surface area and volume problems.

Chapter 4 Methodology

As a guide for analyzing the video data, seven nonlinear phases as described by Powell, Francisco, Maher (2003) are suggested. These are: viewing attentively the video data; describing the video data; identifying critical events; transcribing; coding; constructing a storyline; and composing a narrative (Powell et al, 2003, p. 413). The components that are applicable for constructing RUanalytics for this study are elaborated below.

At the time this study began, some but not all the video data that were needed to construct video narratives for this research were available on the VMC. Thus, work was needed to prepare the full-length videos for ingestion to the VMC repository. This required several steps, since VMC video resources are cataloged with extensive metadata that include descriptions of video episodes and transcripts.

First, the full-length videos were watched several times. While studying them, I took notes on events of interest that were occurring in each video. As part of my analysis and to prepare for the cataloging process, an excel spreadsheet was used to mark the time codes for events that were considered interesting. These events, with indicated time codes, were then described. In addition to these descriptions, problems of sound or clarity were noted. These notes helped with describing the video and identifying critical events in the video. Powell et al (2003, p. 416) define a critical event as “when it demonstrates a significant or contrasting change from previous understanding, a conceptual leap from earlier understanding.” Identifying and describing those critical events assisted me with constructing my narrative.

The next step involved making transcripts of the video and then submitting them for verification by another researcher, who verified and tagged errors. Once the transcripts were verified, they were prepared for ingestion along with the video resources to the VMC repository.

My next step involved coding the data, focusing on various forms of representations. Utilizing the categories of external representations identified by Lesh, Post and Behr (1987), I looked for and coded each representation used and, because I worked from the full video, I was able to identify the results of the representation used.

By connecting sequences of critical events, researchers can both construct a story line and compose a narrative for further analysis, which corresponds to steps 6 and 7 of the analytical model (Powell, Francisco & Maher, 2003). However, since these phases were articulated, significant advancements have been made in Internet speed, and video streaming technology. Drawing upon the technological advancements of the VMC repository and the RUanalytic tool, the method I used to connect those critical events into a story line with analytical narrative is through the construction of RUanalytics. I first describe how the tool works, and then describe my process in using it.

The RUanalytic tool allows the user to create a video case from video events that are annotated with text. The RUanalytic (Agnew et al, 2010) allows the user to string together any number of video clips from the larger video collection stored on the VMC to tell a story or highlight specific concepts like student reasoning, teacher intervention of student learning, or a specific content lesson. These events, when linked together, form a multimedia narrative that can be used independently for instruction, study, or research. Also, they can supplement a scholarly paper (e.g., Sigley & Wilkinson, 2015), enhance teaching (e.g., Van Ness, 2015), or be used to enhance teacher professional development (e.g., Horwitz 2011) with video. In utilizing the full video in creation of the RUanalytic, rather than making use of edited clips, one preserves a reliable chronology of the events during the activity and can be sure that all critical events in the activity are captured in the analysis.

The creation of a RUanalytic is not a single step process. There are many decisions that need to be made even before you begin to cut the larger video clips into smaller pieces.

One needs to identify a theoretical or conceptual framework, what content strand to use, and the goal(s) of the analytic. These three components help identify what makes an event critical. Successful identification of critical events from larger video segments and describing their significance for a clearly articulated purpose is what allows the constructed RUanalytic to achieve its goals.

As explained in Chapter 1, the VMC houses a substantial collection of videos from several research studies. Its repository also provides a cutting-edge tool, the RUanalytic tool, enabling a user to extract one or more specific portions of video data for further analyses. Creating an analytic can be a laborious process unless you have an effective plan. With thousands of hours of video to choose from, having a system by which one can narrow down what is useful for one's efforts is essential. Below is the method I used for constructing all three analytics that appear in this study. This methodology was first developed in 2011 during the process of creating my first analytic that became published (Horwitz, 2011). In order effectively to use the tool, the following steps were used in the process:

1. Become familiar with videos from the collection.

This was a process that began during my doctoral study. Studying videos for my classwork and using them as a resource for small research projects made me quite familiar with a great deal of video that is currently on the VMC. Indeed, the scholarly interests of graduate students and professional researchers have catalyzed continuing expansion of VMC resources from the larger video collection. In addition to video resources, the VMC stores metadata that includes transcripts and student work. Becoming familiar with the various resources available on the VMC allows the user to make use video and metadata for their studies. Trolling for video and related resources requires an extensive amount of time and study. Acquiring knowledge of what is available and identifying the videos is a first step.

2. Identify the purpose of the RUanalytic and the population you are addressing

Identifying your purpose will inform the design of your RUanalytic. Analytics may be framed differently for different audiences, such as for use in professional development for in-service teachers (see, for example, Horwitz, 2011). One might then frame the activity and point out the unique aspects of the situations that teachers may not typically have the opportunity to observe in a traditional classroom. When designed for this purpose, the RUanalytic could be used to educate teachers who may not be familiar with a particular task or the variety of approaches that students may use to build a solution for it.

RUanalytics can also be an important supplement to a research paper. As it is in this case, RUanalytics that focus on research questions will provide data for specific claims that are made. As an example beyond this dissertation research, Sigley and Wilkinson (2015) authored a video narrative and journal article featuring a longitudinal analysis of one student's problem solving over time and his growth in acquiring precise use of mathematical language.

The next few steps need to be enumerated, however, not necessarily in a linear fashion. For example, certain mathematical content strands, such as combinatorics, have been studied at multiple age levels and grade bands. Also, my framework is more conceptual than theoretical; however, it was thoroughly researched and written in draft form prior to the creation of any of the three analytics.

3. Identify a framework to guide the analysis

4. Indicate the context, including mathematical content and grade level(s)

5. Identify student(s) or groups of students whose learning is to be traced

Being that my goal was to examine representations across grade bands it was essential to choose content important to specific grade bands. In the case of this project, I focus on fractions in the elementary grade band and early algebra concepts in the middle school grade band.

The content strand also takes into account in some cases age of the students along with content. For example, if you are going to use fractions tasks you will be using tasks mostly given to fourth grade students. With other content strands, like combinatorics, there are video of students at all grade levels.

6. Watch the video multiple times

If you are using a full-length video of mathematical activity, as I did with the early algebra study presented in chapter 6, you need to watch the entire video several times, taking notes on sections that you plan to revisit. Many of the full-length videos on the VMC are more than an hour long, so extensive notes on specific sections, including time-code markers you plan to revisit, are essential for a more efficient examination and analysis of the video.

If you are using a video that has already been partitioned into clips, then watching all the pertinent clips is essential. This is where settling in on your lens is important. You need to examine the video through your chosen lens, identifying the specific portions that are important to achieve your purpose. This needs to be done a number of times, taking notes and transcribing carefully those events you plan to use.

7. Plan out the events of the RUanalytic using text

Using time-code markers to set the counter for start and end times for each of the events was how I began to plan out my analytics. As I sectioned off the video I planned to use, I began to construct event descriptions (the text designed to accompany the video). The event descriptions should reflect your lens and center on what is critical in the event. After clipping the series of events in the tool and drafting descriptions for each analytic in a separate document, I would enlist a colleague with a critical eye to watch the analytic and read the descriptions to ensure that the analytic and text achieved the stated goals.

8. Input a draft of the related text into the analytic tool

Once my RUanalytics had been drafted and reviewed multiple times, only then did it make sense to input the text and narrative into the RUanalytic tool. Overall the RUanalytic tool is easy to use and is continually improving. However, at the time my research was conducted, editing within the analytic tool was a difficult process due to small print and no ability to save multiple copies of an analytic. For new users and on-demand technical support, there is a tutorial on the web:

(https://www.youtube.com/watch?v=XePFAXnY9JQ&index=1&list=PLk-JdlB998fMTIBUy9q8Te5tBjZVSgwjL).

Chapter 5 Comparing and Ordering Unit Fractions

5.1 Introduction:

This chapter focuses on a set of activities that took place on November 10, 1993 on the twelfth of forty-four fraction sessions during a yearlong study of how fourth-grade students build fraction ideas. Accompanying the activities is a RUanalytic focusing on the placement of fractions on a line segment, *Using Meredith's models to reason about comparing and ordering unit fractions,* (Horwitz, 2015a, available at http://dx.doi.org/doi:10.7282/T33J3FQG). The first activity begins with students comparing and ordering unit fractions and is followed by students placing unit fractions on a line segment between zero and one, which subsequently gets extended to the interval between zero and two.

5.2 Introductory tasks

In the activities preceding this session, students explored various problems concerning the relationship between whole numbers and fractions. Students built models using Cuisenaire rods to identify number names for the various fractions in relation to the particular rod that was given the number name, "one". Students also explored the concept of "infinitely many" (Schmeelk & Horwitz, 2015).

The concept of comparing and ordering fractions began with a problem that involved the sharing of candy bars of different sizes. This provided a metaphor for the importance of retaining the same unit if the candy were to be equitably shared. The idea of "fair sharing" triggered a discussion about which fraction of the candy bar was bigger or smaller when candy bars of different sizes were compared (Van Ness & Alston, 2015). The idea of comparing and ordering fractions is continued in this session as students are challenged to place fractions on a line segment.

During the session preceding the analytic below, Researcher Maher asked the class to create a number line segment, with endpoints labeled 0 and 1, and to place the fractions 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8 1/9 and 1/10 on that segment. Students discussed the task and began construction of the line segments as a class activity. They continued working on their line segment constructions at home and returned with their completed work this is where the events captured in this chapter begin.

5.3 RUanalytic: ***Using Meredith's models to reason about comparing and ordering unit fractions***

Event 1: Placing unit fractions on a line segment

At the beginning of this session, Researcher Maher shared with the class the number lines that Meredith created as illustrated in Figure 1 (pictorial representation). Notice that on the first of her five line segments, Meredith labeled points to indicate the position of each of the fractions that Researcher Maher asked students to place on the interval. Below that line segment were four more line segments to show each the fraction placements of $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ and $\frac{1}{5}$, indicating the partitioned segments as halves, thirds, fourths, and fifths. Researcher Maher commented that she had some questions for Meredith about her work. She then asked Meredith to come up to the overhead at the front of the room and explain what she had drawn (see Figure 5.1).

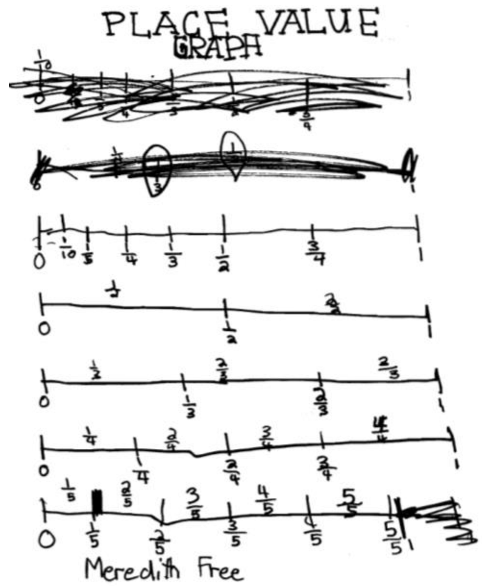

Figure 5.1: Meredith's segment representations of unit fractions between 0 and 1

Event 2: Meredith’s set of partitioned line segments

After reviewing other students’ number lines, Researcher Maher puts Meredith’s unique pictorial representation on the overhead projector for class discussion. After her initial line segment on which she had placed points for each of the assigned unit fractions and ¾, Meredith made four separate line segments to show each fraction placement. These segments were partitioned into halves, thirds, fourths, and fifths. Meredith’s decision to use separate segments to represent these fractions was different from what was done by the other students who had only made unit fraction placements on a single line segment. In this event the class attempts to make sense of Meredith’s representations of fractions on those line segments.

Event 3: Erik finds Meredith's representations confusing

Once students had a chance to view Meredith's number lines, they expressed their difficulties – thereby offering new representations as spoken language – in trying to understand her pictorial representations. Erik first voiced confusion.

> Erik: Well, see, if someone was to look at this for the first time, on the third number line, for the first time, I know when I did I got confused because I thought that in the middle of the section where she put the one third, two thirds and three thirds. I thought that, that is where they would be. So that's why. I think that is what Michael did, too. So, that is why he asked the question.
>
> Researcher Maher: How many of you thought that?

Jessica later clarified Erik's comment:

> Jessica: I think what Erik means is that he thought Meredith was making a whole new number line; like, she thought two thirds was the half.[because on the third number line two-thirds is written in the middle of the line]

Notice that, in Figure 5.1, each line segment produced by Meredith showed not only unit fractions but also partitioned the segment into equal components.

Event 4: Erik and Alan discuss Meredith's representation

In Figure 5.2 (below), we see that Erik came to the overhead and questioned where 2/3 was positioned in the second third of the segment displaying thirds. Researcher Maher asked him to show what part of the segment represented 2/3. Erik pointed to where he would start and end his 2/3 of the segment. He commented:

> ...because the way you said that one segment. Well, if you use both those segments like the segment here [indicating zero] to here [indicating two thirds], that would be two thirds, but you said the segment here [the segment from one-third to two thirds]. It would be the two thirds the segment, but it would be only one segment.

Alan then explained that Meredith was labeling the region in between the marks 1/3 and 2/3. Alan then supported his argument by saying that 2/3 should encompass both the first third and second third segments.

Researcher Maher then asked Alan where he would place 3/3, indicating, “I see over here the numbers zero, one third, two thirds, and one... Where would you put three thirds?” Alan responded, “If you put three thirds, you would put it just in that big area,” indicating that you would put 3/3 in the region between zero to one. Erik said to put 3/3 at the end of the line, and Alan suggested that 3/3 represented the length of the whole line segment between zero and one.

> Alan: You are doing it the way Erik’s talking about it, you would put one third over that area [indicating from zero to one-third] and you would put two thirds in that area [indicating from zero to two-thirds] and three thirds in that area [indicating the entire line segment], because this [pointing with his pencil from zero to one-third] would be representing one third, both of those [moving his pencil between zero to two thirds] would be two thirds and three of those [moving his pencil across the entire line segment] would be three thirds.
>
> Researcher Maher: Ok, so if you were to put the number three thirds on the line? I see over here [indicating the numbers one-third, two–thirds, and three-thirds that appear below the line] the numbers zero, one third, two thirds, and one... Where would you put three thirds?
>
> Alan: If you put three thirds, you would put it just in that big area [moving his pencil across the entire line segment] because it would be...
>
> Erik: [*Erik points to 1 on overhead and murmurs*] No, you wouldn’t. It would be right, there [indicating the number 1].
>
> Alan: Right that [indicating where one is already marked] would be the mark of the three third, but all three of those [indicating the whole line segment] are the three thirds.

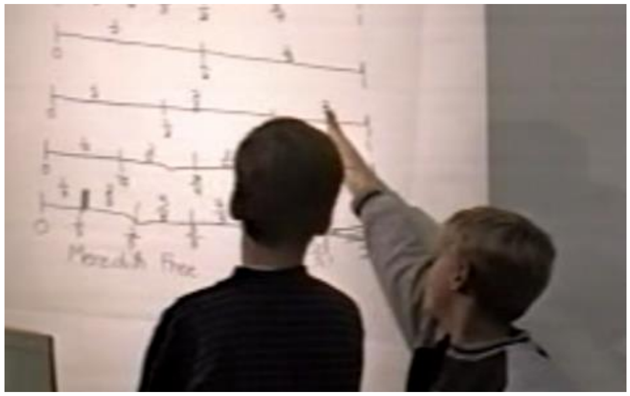

Figure 5.2: Alan and Erik gesture to the region between 0 and 2/3

Event 5: Alan's partitioned line segments

The students were now faced with a dilemma: Where should 3/3's be placed on the line segment? As the conversation continued, Alan drew new pictorial representation in an effort to answer the question. He drew a segment on the overhead and partitioned it into thirds (see Figure 5.3). As event 5 progressed, Alan then drew three more line segments: the first as long as the first third which he labels as one-third; the second, as long as two-thirds, which he labels 2/3; and the third as long as the original segment, which he labels as 3/3.

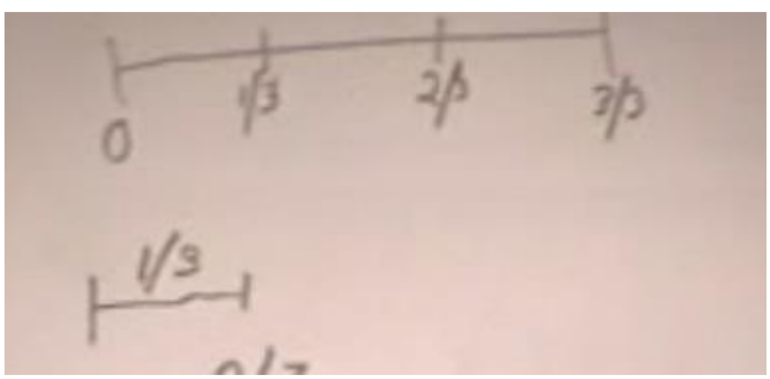

Figure 5.3: Alan labels the 0 to 3/3 interval

Event 6: Students collaborate to refine their rule

Researcher Maher raises the question about where students would place the number one on Alan's number segment shown in Figure 5.3. She comments that there are differences in views: some believe the line segment should be longer and some believe that the number line should not be any longer. Discussion follows.

Researcher Maher then asked Meredith to begin the discussion with the placement of the number one. Meredith came to the overhead and using the line segments that Alan drew earlier and said, "—three thirds is the same as saying one. Four fourths is the same as saying one. A hundred hundredths is the same thing as saying one."

The idea of equivalent names for naming one is significant because, as Meredith continues to extend these examples, Michael builds on the idea and offers a generalization of Meredith's statements, claiming," I think that if you have a number with the same number on top as in the bottom, then it is always going to be equal to the number named one."

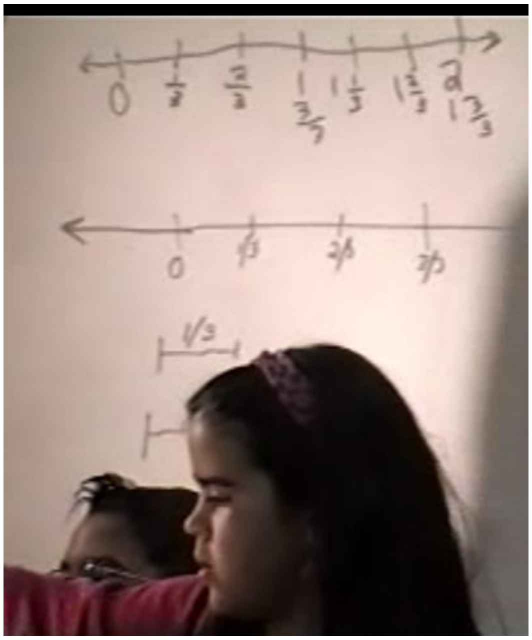

Figure 5.4: Meredith introduces mixed numbers

Event 7: Meredith introduces mixed numbers

Meredith has now drawn an expanded number line segment (pictorial representation) that begins at zero and ends at two. While Brian is talking, notice that Meredith extended her labels from 1/3, 2/3, 3/3 with 1 1/3, 1 2/3, and placed 1 3/3 under the number 2, as seen in Figure 5.4. After she finished, Alan continued his discussion with Brian about the placement of fractions and their value on the number line segment.

> Brian: There is no fraction value between zero and one third that are thirds, because…
>
> Alan: If you have no value between zero and one third, then look, you eliminate that how many spaces would you have? Two.
>
> Brian: What I would think would be that if you start on the lower number [one-third] and then keep going up and until you hit that [his finger on two-thirds]. Anything between there [between one-third and two-thirds] would be the one-third and the two-thirds right there [pointing to the space between two thirds and three thirds] And, if you keep going up to there you will hit that [his finger on three-thirds] and, that [the space between two-thirds and three-thirds] will be two thirds and if you put a bar right there [after three thirds] it would count as one, two thirds between 1 would be three thirds. Because, like I was saying, if you started there at zero, then zero does not have value …

Notice how Brian did not claim to see any other numbers (that are thirds) between zero and one-third, one-third and two-thirds and so on. It is as if Brian expects that as he progresses along the line segment and iterates thirds, he is summing thirds. It is interesting that Brian wants to add one more mark on the number line segment after three-thirds to represent the number one.

In an effort to explore Brian's assertion, Alan then takes the three red rods and lines them up next to one dark green rod (manipulatives) and creates a new representation of the number line segment (the dark green rod) broken into thirds (the red rods). In this representation he is able to pull the first red rod and explains to Brian that if his assertion is true than that red rod has no value.

Event 8: Sarah, Beth and Audra show where they would place 3/3

As the session was drawing to a close, the class had not reached consensus about the placement of 3/3 on the number line segment. While there were strong arguments put forth by both Alan and Meredith, other students, such as Brian, had different ideas for representing the placement of 3/3. Researcher Maher invited another group of students (Sarah, Beth and Audra) to share their ideas. The created their own picture using the yellow rod to represent one-third. It is interesting that they utilized the yellow Cuisenaire rod to show that the lengths between 1/3, 2/3 and 3/3 are the same. They first positioned the number 3/3 above the segment. When Researcher Maher asked them to place the number 3/3 below, they wrote 3/3 to the left of 1. Figure 5.5 shows the group's initial placement of the number 3/3 on the overhead projector.

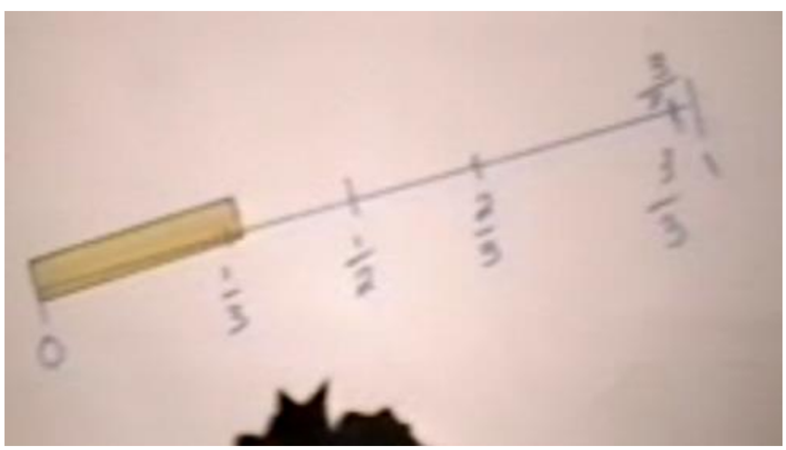

Figure 5.5: Sara, Beth and Audra place 3/3

Audra: We thought that we did not have to put anything else on the number line, because if we put this [moving her pencil from zero to one] from zero to one and you would mark one third here [her pencil on the one-third mark] because if you used a ruler here to measure it or something one third would go here [her pencil on one-third], two thirds would go here [her pencil on the two-thirds mark], three thirds would go here [her pencil on the three-thirds mark above the line just before the mark for the number one], because the length will be the same as Cuisenaire rods or something.

Researcher Maher: What I am not clear about when I am looking up there, I am not sure, you wrote a number on top, and I thought we all agreed that we would write our number line with the numbers in the bottom. Where would you place the three-thirds?

Audra: We would place it here [Group puts three thirds slightly to the left of one.]

Researcher Maher: Could you write it underneath the number line and tell us why you would place it there. Because, I think some of us are interested in where you would place it. And, you are telling me you would place it with 1 to the right of three-thirds? So this clearly defines where we have differences of opinions, right? Is that true? So, how many of you are agreeing that we should place 1 to the right of three thirds? How many of you believe that?

Students: [Student response not on camera view.]

Researcher Maher: Oh. How many of you believe that it should go to the left of three-thirds?

Students: [One student on camera, Bryan, raises hand.]

Researcher Maher: How many of you believe it should go right on top of the one?

Students: [Nine students raise their hand.]

Audra: That’s what we meant. We just could not get it right on top to fit.

Researcher Maher: You meant to put it on top?

Students: You just put it under it.

Researcher Maher: Oh, I see, you just could not fit it in. So, Jessica, how could they do it to put it by the one? [Jessica walks up to overhead.] You want to go show them how to do it? They meant the same thing, Okay. So, I am hearing that we have some agreement here then; you all agree that three thirds would go under one. It would go in the same spot. How many agree with that? [Six students on camera raise their hands] And, I would like to hear again why that would work, could you tell me Erin? Why you would put it there and not to the right or to the left? Any idea? Want to think about it?

So here we see that students are beginning to come to consensus that three thirds should be in the same spot as one on the number line segment. While their original diagram did not show this, they clarified their intent by saying they did know it should be in the same spot, they just did not know how to express it. With each number representing a specific spot on the line the group, Sara Beth and Audra, show evidence of the transition to fractions being a numbers on the line.

5.4 Findings

What representations do students use to express their ideas?

Table 1, below, documents the representations used in comparing and ordering fractions and placing them on a number line segment. The students used manipulative, pictorial, written, symbolic notation, and spoken language to represent their information and solve the tasks. The table organizes the representations by the event in which they appear in the RUanalytic.

Table 1. Representations used in comparing and ordering fractions and placing them on a number line segment

Event number	Representation	Task
1	Pictorial	Creating a number line segment
2	Pictorial and spoken language	Student analysis of Meredith’s number lines
3	Pictorial and spoken language	Analysis of Meredith’s number lines
4	Pictorial and spoken language	Where does 2/3 go on the number line? 3/3?

5	Pictorial	Where does 3/3 go on the number line segment
6	Symbolic	Equivalent number names for the number one
7	Pictorial, symbolic and manipulative	Mixed numbers from zero to two,
8	Pictorial and manipulative	Creation of a number line segment from zero to one. Where does the number three thirds go on that segment.

5.5 Summary

Meredith exhibited multiple notions of fraction ideas. When Meredith presented her number line to the class, she explained to the class that she had used multiple line segments to represent fractions (halves, thirds, fourths and fifths). What was so special about her series of number lines was that on the top of her number lines she labeled the area (fraction as operator) and on the bottom of the line she labeled the fraction number (fraction as number). Many students in the class were confused by this notation. When discussing the infinite number line, Meredith correctly placed negative fourths between zero and negative one on the number line, despite having no formal training in negative numbers. Later in that same activity, Meredith explains equivalent fraction ideas by explaining why one half can also be named two fourths. Alan also showed he understood fraction as operator, as he described how one third could go in any of three places. H e also showed he understood fraction as number when he placed “the number one-third” on the number line.

This study documents the representations students used during the activity of placing fractions on a number line segment. Through collaboration, students generated their own external and internal representational systems as they worked through this activity whose goal was the transition of fraction as an operator to fraction as a number. The activity began with Researcher Maher showing Meredith's number line segments to the class and the student's attempts at analyzing that pictorial representation. Throughout the activity students used several types of external representations - pictorial, symbolic, manipulative, and spoken language - in an effort to create their own internal representational systems. The RUanalytic, Using Meredith's models to reason about comparing and ordering unit fractions, presents dynamic evidence where we can see the mathematical sense making process unfold as student representations are generated and used in a natural way to express and support their ideas, challenge explanations given by other students and justify their reasoning.

Chapter 6 Extending Fraction Placements from Segments to Number Line

This chapter focuses on a set of activities that took place on November 10, 1993 on the twelfth of forty-four fraction sessions during a yearlong study designed to investigate how 4th grade students build fraction ideas. The facilitator on this day was Researcher Carolyn Maher. Showing the activities detailed in this chapter, there is a RUanalytic focusing on placement of fractions on a number line, titled, *Extending Fraction Placements from Segments to Number Line: Obstacles and their Resolutions*, (Horwitz, 2015b, available at http://dx.doi.org/doi:10.7282/T39Z96SR).

6.1 Introduction

In the activity preceding this session students explored placing unit fractions on a number line segment beginning at zero and ending with one, as was presented in detail in Chapter 5. Students used Cuisenaire rods to partition the line segment equally and placed numbers in the exact location where they believed they would go on the line segment.

The concept of comparing and ordering fractions arose in this yearlong intervention with an earlier problem that involved the fair sharing of candy bars of different sizes (Steencken & Maher 2003; Van Ness & Alston 2015). This activity provided students with a physical representation of two different sized units and reaffirmed why it is important to retain the same unit if the candy were to be equitably shared. This activity also triggered a discussion of which fractions are bigger or smaller when candy bars of different sizes are compared. The idea of comparing and ordering fractions was continued when students compared fractions using the same unit for reference, and later when students began placing unit fractions on a line segment.

This idea is explored further here, as students are once again challenged to place fractions on an infinite number line.

6.2 Introductory Tasks

Researcher Maher has attached a number line drawn on a long sheet of paper to the board. The line is labeled with integers from -3 to +3. Arrows are drawn on either end of the segment to indicate that the number line continues in both directions. Prior to the start of the activity, Researcher Maher prompts students to have a particular fraction in mind, and then indicates that students will be asked to place the fractions below the number line in the appropriate spot. When there was more than one equivalent fraction that should be in the same spot on the number line, students were instructed to place the fractions underneath the previous fraction(s) already labeled. She invites students to volunteer to place their fractions on the line.

As mentioned, this activity begins as students had just finished placing unit fractions on a number line segment. Now they were being asked to place fractions on an infinite number line. This change in the activity was significant because students had not yet formally worked with negative numbers in the math classroom. Also, this would be the first time students would be asked to place fractions on a number line with multiple integers already labeled on it.

6.3 RUanalytic: Extending Fraction Placements from Segments to Number Line: Obstacles and their Resolutions

This RUanalytic offers eight events, summarized below, that identify the variety of representations used by the students as they worked together on the problems. They illustrate some of the student understandings and obstacles encountered in their transition from fraction as an operator to fraction as a number. The narrative respects the chronology of the events presented, showing multiple external representations that are made visible in students' collaborative problem solving.

Event 1: Students begin to place their fractions on the infinite number line

Audra volunteers to place the first fraction, having chosen the fraction one-half, on the number line described above (pictorial representation). She places one-half directly under the number zero and looks back at the class for confirmation on her placement (see Figure 6.1).

Event 2: Does 1/2 really equal 0?

Discussion (spoken language) on her placement begins. Michael comments, "No. That's not right. That is half between negative and positive, but that is not a half on the number line." Brian offers, "It should be one half between zero and three because on that side is the negative side." David and Erik are quick to agree with Audra. David comments, "I agree with Audra because since it is integers it would go both ways, zero is one half of the whole thing that keeps on going, cause that where you start you can keep on going either way, but that is the middle."

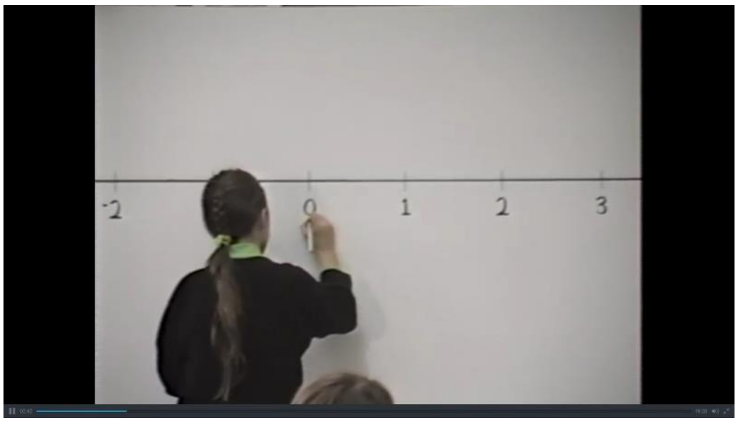

Figure 6.1 Audra places one-half under the zero

Erik adds to the conversation, "I agree with Audra and David, because there is no way. I heard Michael say that the half would have to be on the positive side; it's integers; they keep going; if it is on the positive side, it's not going to be equal halves, the negatives would be larger than the positives and if you even make the number line bigger, zero is right in the middle, so it is going to have to be half."

Event 3: Researcher Maher places 4 on the number line

Researcher Maher changes the pictorial representation that the students are discussing as she placed the number +4 on the number line. In the background you hear a student say, "You can't do a half anymore..."

Researcher Maher:	[*Walks to overhead and extends the positive numbers to include four but does not extend the negative numbers.*]
Researcher Maher:	I want to make that my number line.
Students:	That is not half anymore.
Researcher Maher:	Alan?
Alan:	Since you added on the four, then, that mean you now have four numbers positive but three numbers negative
Researcher Maher:	By the way, do I really have four numbers positive?
Students:	You have four numbers negative. You have five numbers positive … zero, one, two, three, and four.
Researcher Maher:	Is that what we have on the number line, numbers up to four?
Students:	[*Murmur.*]

With what appears to be an asymmetric number line, students struggle to keep the placement of the number one half at the point that would divide the line in half, saying that because the extra integer was added to the line that one half now has to be placed elsewhere. Even Audra, who originally placed the ½ on the line agrees, “This can’t be one half.”

Event 4: Where do you place -1/2 and 3/4?

Researcher Maher asked the class where they would place $-\frac{1}{2}$. Alan (see Figure 6.2) then came forward with a suggestion. He placed -1/2 midway between -1 and -2, apparently because the negative side of the line is labeled from 0 to -3 and this particular point would divide the segment from 0 to -3 into two equal parts. Then the question is raised about where to place three fourths. Alan can be seen partitioning the positive part of the number line into quarters. Meredith comments that she disagrees with Alan, and Researcher Maher asks her to come to the front to offer an alternate idea. Similar to her earlier work in extending the number line segment from zero to two (as seen in chapter 5), Meredith marks quarters from zero to one and writes in ¾. Notice that she labels the marks she made as $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}$ and $\frac{4}{4}$, and then continues with $1\frac{1}{4}, 1\frac{2}{4}, 1\frac{3}{4}$ and $1\frac{4}{4}$.

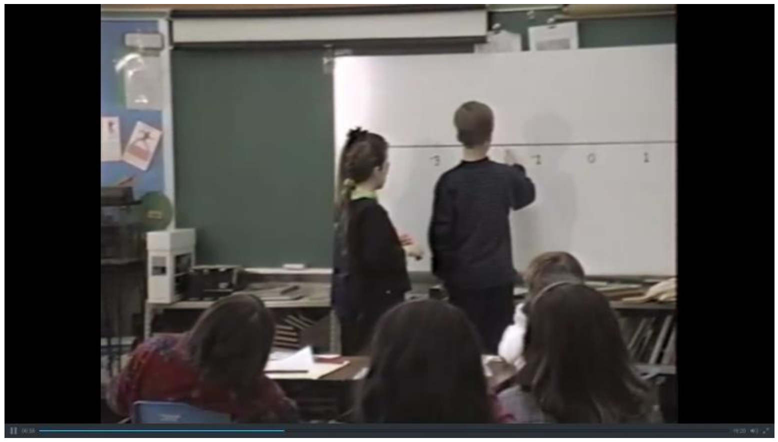

Figure 6.2 Alan discusses where to place negative one-half

Event 5: Where should the number one half be placed?

While students have placed several fractions on the number line, the location for the number one half remained under the zero. Researcher Maher labeled the number +4 on the number line and asked, “How many of you want to put it (the number ½) somewhere else?” Amy responds, “You could keep it (the number ½) there, but you have to add negative four.”

Event 6: David’s explanation of Meredith’s placements

Here we see David commenting on Meredith’s placement of fractions on the number line. He says, “I think Audra is using the whole thing while Meredith is using from zero to two.” When David refers to the “whole thing” he appears to be referring to Audra’s using the entire number line as her unit or “whole,” with the implication that one half should be placed in the center of that segment. Meredith is then asked, by Researcher Maher, to put -1/4, -2/4, -3/4, and -4/4 on the line. Meredith then placed these points on the number line as requested, mirroring the positive points she has already placed on the line.

Event 7: James uses different names for the same number

Here James suggests that one half be placed between the numbers zero and one. In so doing, the number one half would be equivalent to the number two fourths, which Meredith has already placed on the infinite number line.

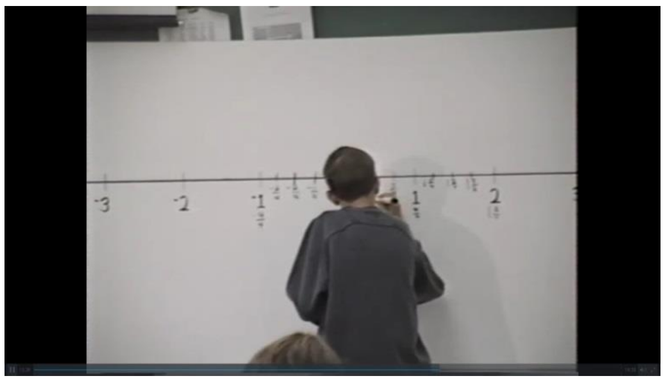

Figure 6.3 James places one-half under two-fourths

Event 8: Students place mixed numbers on the number line

At the beginning of the event, students are discussing their ideas and confusion about placing the number ½ on the number line. Andrew comments about a source of confusion:

> What Audra is confused about is the length of the number line. Right now you have five positive numbers and four negative numbers counting the zero. So they are not exactly the same, so you would have to put the one half a little more over to the positive side like about right where the two fourths is, now that we have three negative numbers and four positive numbers.

James offers a different idea; he comments that 1/2 should go, in his words, "in the middle of two things…" He continues to comment, "So, it is confusing to see in which place between three to four, two to three, one to two, zero to one, negative one to zero, negative two to negative one. I think that is the confusion."

When Researcher Maher points to the midpoint between the numbers 2 and 3 and asks if you could place the number ½ there, Andrew then responds that it should be 2 and ½.

Meredith then supports James's idea by making an analogy to a ruler (manipulative) by pointing out each number 1 ½, 2 ½, etc. She takes one out (See Figure 6.4) and comments, "It is like a ruler, here it has the inches one half, one and one half, two and one half, three and one half, four and one half, five and one half, and so on."

Figure 6.4 Meredith uses a ruler to show that there is one half between each whole number

6.3 Findings

Table 2 documents the representations used in placing fractions on an infinite number line. The students used imagistic, manipulative, pictorial, written, and spoken language to represent their information and solve the tasks. The table organizes the representations by the event in which they appear in the RUanalytic.

Table 2. Representations used in placing fractions on an infinite number line

Event number	Representation	Task
1	Pictorial and written	Placing ½ on the number line
2	Spoken language, pictorial, imagistic	Where does ½ go on the number line
3	Pictorial and imagistic	Researcher Maher places 4

		on the existing number line
4	Pictorial and written	Where do you place -1/2 and ¾
5	Imagistic and spoken language	Where should ½ be placed
6	Imagistic and spoken language	Placing mixed numbers on the number line
7	Spoken language and written	Placing ½ on the number line
8	Spoken language and Manipulative	There is a ½ between any two whole numbers

6.4 Summary

This study documents the representations students used during the activity of placing fractions on an infinite number line. Through collaboration, students generated their own external and internal representational systems as they worked through this activity whose goal was the transition of fraction as an operator to fraction as a number. The activity began with an infinite number line posted in the front of the room. This number line had multiple integers already written on it starting at -3 and going to +3. The number line also had arrows on either end indicating that it continued in both directions. Throughout the activity students used several types of external representations - pictorial, symbolic, manipulative, imagistic, and spoken language - in an effort to create their own internal representational systems. The RUanalytic, Extending Fraction Placements from Segments to Number Line: Obstacles and their Resolutions, presents dynamic evidence where we can see the mathematical sense making process unfold as student representations are generated and used in a natural way to express and support their ideas, challenge explanations given by other students and justify their reasoning.

Chapter 7 The Role of Representations in Early Algebra: Exploration of Surface Area and Volume

7.1 Introduction

The purpose of this study was to examine how eighth-grade students use different representations to illustrate their developing understanding of the concepts of surface area and volume.

Accompanying this study is a RUanalytic, events chosen for the RUanalytic focus on the following categories of external representations: manipulative/physical, written symbols, experiential, spoken language and pictures and diagrams (Lesh, Post and Behr 1987). The RUanalytic is entitled; *Eight-Grade Students explore Surface Area and Volume Problems: The Role of Representations*, (Horwitz, 2015c, available at http://dx.doi.org/doi:10.7282/T3V40X46).

Researcher Carolyn Maher presented the following tasks:

1. Find the surface area of one rod.

2. Find the volume of one rod.

3. Find the volume of any number of stacked rods of a particular length.

4. Find the surface area of any number of stacked rods of a particular length.

7.2 RUanalytic: Student Use of Representations in Solving Surface Area and Volume Problems.

This RUanalytic offers nine events, summarized below, that identify the variety of representations used by the students as they worked together on the problems. They illustrate some of the student understandings and obstacles encountered in finding a general rule for surface area and volume problems. The narrative respects the chronology of the events presented, showing multiple external representations that are made visible in students' collaborative problem solving.

Event 1: What is surface area?

Students began with Cuisenaire rods (manipulatives) on their table. Researcher Maher asks them to imagine the white rod as a [rubber] stamp. She then asked the students how many "stamps" it would take to cover each rod. This idea of stamping generates an experiential representation of surface area because it draws on the student's prior experience of knowing what a stamp is and how to stamp. Also, stamping produces a flat, two-dimensional representation of surface area. It is important that students are focused on the concept surface area as being two dimensions. Internally, the process of "stamping" also enables the generation of an imagistic representation (Goldin & Kaput, 1996), since students can "imagine" a [rubber] stamp and "stamping". The quality of the representation is based on imagining an image produced by a stamp.

The students, familiar with the idea of a stamp, can use this prior experience to represent a square unit. Researcher Maher asks the students to think about the notion of stamping, and how many stamps would it take to cover the outside of each rod. Students begin with a light green rod, which has a length of three centimeters. While the researcher is explaining the activity, notice that Brian takes a white rod and moves it, like a stamp, over another rod. The combination of these three representations provides students with a starting point for the upcoming activity on surface area.

Event 2: Students find the surface area of the light green rod by stamping

As the event begins we see Michael rotating the light green rod, as the others watched, saying 3, 6, 9, 12 and 14. Romina, Michele, Brian and Michael agree that the surface area of a light green rod is 14 square units by carrying out the stamping process.

Event 3: Michael suggests a formula

In the prior event, Researcher Maher pointed out that there are 10 rods in the set and now expands the question by asking, "Can you tell me a quick way of finding the surface area of every rod in that box?" Students move from using the physical rods in order to calculate a particular surface area by "counting the stamps," to the construction of a symbolic representation or formula.

Michael, Romina, Brian and Michelle come up with a formula quickly. Their formula, written out by Michael, is (length X 4)+2. Notice that they generalize the formula to find the surface area of any rod in the box.

Figure 7.1. Michael writes down his formula for Surface Area

Event 4: Introducing cubic units

Researcher Maher returns to the table and introduces the white rod as a representation of one cubic unit. She states, “I’m going to talk about this, this cube. Right? [Holds white cube] The size of this [white] cube, I’m going to call this a unit. Okay? This would be a volume would be one unit cubed. Could you figure out what the volume is of every other rod in the box?”

Romina comments, “So just the length?” Mike offers the correct formula, “Length times width times height.” (Symbolic representation) Brian picking up the light green rod says, “And this (light green rod) would be three units cubed.”

Event 5: Romina uses multiple representations to communicate with the group

Here we see Brian and Michael questioning Romina about her assertion that length equals volume. Brian asks, “Length equals volume? Where’d you get that?” Michael pulls out a yellow rod and says, “This (the yellow rod) is the length of five.” Romina then takes the comment and builds a model, “if you put that [a yellow rod] wouldn’t you think that [reaching for white rods]. Okay, what’s the length of this [lining up white rods to equal length of the yellow rod]? Wouldn’t you think that [still lining up the white rods], if I saw that, if I saw that [pointing to the white train next to the yellow rod], wouldn’t you go like this [using her finger to count]? Wouldn’t you think it’s 5 cubes?”

Because Romina’s model does take units into account Brian responds, “Okay. I know what you’re doing.”

Here we see Romina moving between two representations. She is verbally explaining what she means by L=V and also using the rods as manipulatives to illustrate her words. If one were to only examine Romina's words, her explanation is difficult to follow because she uses her conventional language. However, with the added illustration of the manipulative (see Figure 7.2), she is able to take units into account and convey her thinking to the others in the group, as evidenced by Brian's last statement that he understands what she means.

The confusion in representing square and cubic units symbolically provides an obstacle to students' communication with each other. In earlier events, we see students imagining a two-dimensional stamp and stamping their way to the surface area of a rod. We then notice students use the white cube to represent one cube unit and subsequently to line them up to get the volume of a rod. However, the students were not yet facile in their use of conventional representations of units. Using the mathematical terminology and symbolic notation became an obstacle in providing an accurate symbolic representation of a formula for either volume or surface area. So, not only are the representations important, but in order to communicate mathematically, students need to be able to express their representations using conventional language notations (Lesh, Post, & Behr, 1987; Goldin, Shteingold, 2001; NCTM, 2000).

Figure 7.2. Romina uses the Cuisenaire Rods to explain her reasoning

Event 6: Students collaborate to refine their rule

In this event, students discuss how to compute the volume of each of the rods and what to write as a rule. Romina claims, "Length equals Volume". Michelle offers, "Well, wouldn't this be like even though it is the length equals the volume, you have to state that it's length times width times height so you put length times 1 times 1 even though it is just length to show." Notice how students are beginning to attend to the significance of expressing the appropriate dimensions.

Event 7: Students attempt to explain their idea, L=V, to Researcher Maher

Romina explains her computation for the volume of a red rod, indicating, "Okay, if you use this [one white rod]. This is what we're using as a measuring tool [white rod]. And one, two of these [white rods] equals that one [red rod]. So then this one [red rod] is 2 units cubed."

Researcher Maher responds by attending to Romina's observation that it's the length that told her how many white rods to use in making the train. She challenges the students to write a mathematical rule to attend to the units, indicating, "But you're not really saying length equals volume because that doesn't make any sense because we know the length of this [red rod]. The length is two units. The volume is cubic units. See. The volume of this [white rod] is one cubic unit. So the volume of this [red rod] is not equal to length of two units. The volume is equal to two cubic units. Is that true? So watch those units a little bit and see if you can write it."

Researcher Maher's response stresses the importance of attending to units in measuring length, area and volume.

Event 8: Students check their formula with a second example

The regular classroom teacher, who has been observing the students working on the task, now joins the group and asks, “What did you say the volume of the rod is?” Romina responds, “three” and the teacher responds, “Because it is three long, right?” Romina agrees, and then the teacher stacked another light green rod on top of the first one resulting in a volume of six “cubes” (see Figure 7.3). Romina agrees that the volume would not be three because the teacher has changed the height. Michelle adds, “You have to state what the width and the height is to calculate volume even if it is just one.” The teacher points out to Romina that L=V is giving a formula for a single rod. She agrees. Length and Volume are not equal in all cases.

Figure 7.3. Classroom teacher provides a counterexample

Event 9: Students work to come up with a surface area and volume of the stack of the dark green rods

After completing the last task of creating a formula for volume, Researcher Maher then asked students to “stack” several rods together. She selects three light green rods and asks students to come up with a way of finding the surface area and volume for the three rods "evenly stacked together."

This event begins with Brian, Michael, Romina and Michelle working on the task of calculating surface area of the stacked rods. Romina first lines up three white rods alongside the green stack and count to find the length. Brian tries to write out a formula, while Michelle is using the rod model and counting the open faces showing in the model. Michelle says that, "Okay. The length is 3. So it's 3 times 8. So that's another 6 (attending to the two ends of the stack of rods). 30. Okay, is that right? 8 plus 8 because there's 8 sides showing. There's 1, 2, 3, 4, 5, 6 , 7, 8, so that's 24. 25, 26, 27, 28, 29, 30."

Figure 7.4. Romina and Michelle collaborate on the surface area of three stacked rods

7.3 Findings

Table 3 documents the representations used in solving the task of finding a formula to find the surface area of single rods. The students used manipulative, pictorial, written, symbolic notation, imagistic, experiential and spoken language to represent their information and solve the tasks. The table presents the representations according to the events in the RUanalytic in which they appear.

*Table 3.*Representations used in task: surface area of multiple single rods.

Event number	Representation	Task
1	Imagistic, experiential, manipulative	Surface area
2	Spoken language and manipulative	Surface area
3	Symbolic, spoken language and written	Surface area

Table 4 documents the representations used in solving the task of finding a volume of a single rod. The students used manipulative, pictorial, written, symbolic notation, imagistic and spoken language to represent their information and solve the tasks. The table presents the representations according to the events in the RUanalytic in which they appear.

Table 4. Representations used in task: Volume of single rod.

Event number	Representation used	Concept
4	Manipulative, symbolic, spoken language	Volume of a single rod, cube units
5	Symbolic, spoken language and manipulative	Volume of a single rod, cube units
6	Symbolic and spoken language	Volume of a single rod, cube units
7	Spoken language, symbolic, manipulative	Cube units
8	Spoken	Volume of

	language, symbolic, manipulative	two rods stacked

Table 5 documents the representations used in solving the task of finding a formula to find the surface area of evenly stacked rods. The students used manipulative, written, symbolic notation and spoken language representations to represent their information and solve the tasks in this section of the activity. The table presents the representations according to the events in the RUanalytic in which they appear.

Table 5. Representations for finding surface area of evenly stacked rods

Event number	Representation used	Concept
9	Manipulative, spoken language, symbolic, written	Volume of stacked rods

Table 6 shows generalizations made by students throughout the surface area and volume activity. The following section elaborates on how students made connections among different representations, which supported their production of generalizations.

Table 6. Generalizations made by the group throughout the activity.

Student	Event	Generalization
Brian	3	"Length times four plus two."
Mike	4	Length times width times height
Romina	5	Length equals volume
Romina	8	"Length in units times height

		in units times width in units equals volume.”

7.4 Connecting the physical manipulatives with the symbolic notation of the formula

Analysis of the video data shows that students were able to make some connections between the physical model of the rods and the formula, written using symbolic notation. In the beginning of the activity, the researcher used the rods to illustrate a real-life situation for the students: the imagistic representation of a stamp covering the light green rod. With the prior experience that the group has had using the rods, they were able to quickly calculate an answer for the surface area of the light green rod. The group used the rods to create models and test hypotheses about their formulas for finding the surface area and volume. In events 5 and 6, when Romina is attempting to explain her hypothesis that length equals volume, she used the Cuisenaire rods to communicate clearly with the rest of the group. In event 8, when she is explaining her hypothesis to her regular classroom teacher, she uses rods to provide a counterexample that causes her to confront the fallacy in her original hypothesis. After this, in event 9 the students use the rods to correctly model and calculate the surface area of an evenly stacked rod structure of four light green rods.

7.4.1 Connecting the spoken language with the symbolic notation

This group while discussing surface area and volume relied heavily on spoken language. In event 3, when asked if they could find a formula, Michael suggests length times width times height and is ignored by the group. In event 6, when the students are trying to refine their rule, which Romina had written as “L=V”, Michelle says: “Well, wouldn’t this be like - even though it is the length equals the volume - you have to state that its length times width times height so you put length times one times one even though it is just length to show.”

7.4.2 Students built on prior experiences to create new ones.

Brian built on the group's prior experience in building a formula for finding the surface area of a single rod, expressed as “length times four plus two”, (Event 3), when he developed a formula, expressed as “length times the number of rods times four plus two”, for finding the surface area of an evenly stacked rod structure (Event 9). While Brian’s formula for the evenly stacked rod structure was not accurate, he built this rule from the previous formula of surface area of a single rod, adding the new variable, the stack of rods, which had been introduced in this part of the task.

7.4.3 Researcher intervention in students creating a formula for finding volume of a single rod

Researcher Maher stressed the importance of units in the volume formula (event 7). The formula in question was Romina’s "length equals volume", which Romina had represented using symbolic notation of “L=V”. Researcher Maher questioned the claim that the length equals volume. As indicated in event 7, Researcher Maher chose to use a verbal representation supported by the manipulatives on the table. In this event she comments, “But you’re not really saying length equals volume because that doesn’t make any sense because we know the length of this [red rod]. The length is two units.

The volume is cubic units. See. The volume of this [white rod] is one cubic unit. So the volume of this [red rod] is not equal to length of two units. The volume is equal to two cubic units. Is that true? So watch those units a little bit and see if you can write it."

The students' classroom teacher intervened (event 8) when Romina was reluctant to be moved from her claim that length equals volume. He offered a counter example with the rods and then followed up with a verbal representation. He first placed an additional light green rod on top of the light green rod that the students were using to calculate volume. He then asked if the volume had changed. Romina agreed the volume changed even though the length was still the same and makes reference to the prior discussion with Researcher Maher in event 7.

7.5 Conclusions and Implications for Teaching

This episode provides an example of how eighth-grade student's work together to create representations while solving an open-ended algebraic task and attempt to generalize by positing formulae. Generating and utilizing multiple representations and moving fluidly between and among them are important in building a deeper understanding of algebra ideas such as surface area and volume. The episode provides a rich example of students' use of representations in the development of general solutions in early algebra development.

These activities illustrated in this session show the students' natural movement between and among many representations. It also provides evidence of students utilizing those representations to explain their algebraic ideas and their reasoning as they attempted to build generalizations about how to calculate surface area and volume of sets of rods.

7.6 Summary

This study documents students learning through an open-ended activity without having representations imposed on them by their teacher through formal instruction. This activity was divided into multiple tasks where students used all categories of external representations to progress through the tasks. Through collaboration, students generated their own external and internal representational systems as they worked through each task in this activity. The ultimate goal of the activity was to see if students could generalize their findings from the specific tasks into formulas that could be used in all situations. We see dynamic evidence of this presented in the RUanalytic, in which student representations are generated and used in natural ways to express and support their ideas, challenge explanations given by other students and justify their reasoning.

Chapter 8 Conclusions

8.1 Representations used to express ideas

The three studies presented in Chapters 5, 6 and 7 show that, with minimal instructional intervention, students generate their own external representations in an effort to express their ideas during exploratory mathematical activities. Opportunities to work collaboratively and communicate using natural language to discuss, and sometimes debate, the meaning of the representations they produce supports students' mathematical learning.

The two fractions studies show that through open-ended activities that invite investigation with appropriate tools available, young children can naturally build an understanding of comparing and ordering fractions on a number line and naturally utilize multiple representations before formal instruction. These studies trace and document student understanding of rational numbers as they naturally move from comparing and ordering fractions to placing fractions on a line segment to placing fractions on an infinite number line.

Similarly open-ended activities inviting exploration with appropriate tools promote the development of algebraic reasoning as shown by students' natural movement between and among many representations. The third study provides evidence of students utilizing those representations to explain their algebraic ideas and their reasoning as they attempted to build generalizations about how to calculate surface area and volume of sets of rods.

8.2 Summary

The students in all three studies were given the opportunity to work on open-ended tasks, in which they used a variety of representations to build models and generalize their solutions. Moreover, students made connections across multiple representations to make sense of the mathematics. Mathematical relationships are discoverable only as students are given the opportunity to explore mathematical ideas as emphasized by Davis (1984). This suggests the value of inviting students to explore and providing them with the freedom to uncover mathematical ideas and relationships *using various representations that they bring to the problem solving*. It is particularly significant that representations were not imposed on them during the course of their investigations.

The students also used a variety of natural written representations to express their fraction ideas. They drew representations including the number line, Cuisenaire rods, pie charts, rulers and personal notations.

Manipulative tools were another form of representations used by students to express their fraction ideas. Students used models, number lines, Cuisenaire Rods, rulers and candy bars.

The Colts Neck classroom studies give evidence that young children, thoughtfully engaged in discussions about fraction ideas, can use their knowledge to order fractions on a number line. What is notable is that student built ideas through argumentation and discussion and without traditional instruction. In a student-centered environment where students are given appropriate tasks, freedom to select personal representations, time to explore and play with ideas, and when communication is encouraged and respected, students thus engaged build powerful mathematical ideas. The studies provide detailed evidence about how connections were made between fraction ideas and traces how those ideas were extended to negative numbers.

8.3 Significance of the Research

This research in the first two studies is significant both for its evidence of how students develop their understanding of rational numbers without using traditional instructional methods and for examining representations that naturally resonate with students (rather than examining representations that are *imposed* on students) as they thoughtfully engage in rational number discussions. As stated in the opening of this chapter, everything mathematical can be thought of in terms of representations. As human understanding cannot be thought of as "snapshots" (Davis 1992), alluding to human mathematical understanding takes an indirect approach by examining thinking patterns and representations used by an individual. Thus, through studying representations, this dissertation gives evidence of students growing understanding of rational number ideas.

8.4 Limitations and Implications for Further Research

The findings reported in the three studies are based on video data collected two decades ago by making use of a video database in which all actions of students were not captured. Also, limitation in technology and the focus of the videographer sometimes resulted in missing student actions where the camera did not capture the work or sound, making some students difficult to hear speaking. Not all of the student constructed models were captured. When the camera was stationary, the data captured was either student's interaction or their work, and it was not always clear as to what students were producing when hearing their talk.

The focus of these studies was on the representations produced by the students in their problem solving. Future work might examine the nature of researcher interventions, forms of reasoning that evolved, group dynamics, development of argumentation, attention to teacher questioning, and student questioning of each other as well as of assessment mechanisms with respect to the student-centered approach to learning. For example, in earlier work, some researchers examined different video data for the forms of reasoning used by students to justify their ideas. Mueller (2007), Mueller and Maher (2009) and Yankelewitz (2009) report ways that students build their personal understanding of number ideas. Forms of reasoning that were identified in the earlier studies, including direct reasoning, reasoning by cases, reasoning using upper and lower bounds, and reasoning by contradiction. These argument forms could provide an analytical framework for future research that examines the development of students' forms of reasoning under conditions that promote collaboration and engagement as they work in their placement of rational numbers on the number line.

While the findings from the early algebra study are significant in demonstrating the representations that eighth-grade students bring forth in their open-ended mathematical investigations, it might be noted that the students, in earlier years, had been exposed to basic, early algebra ideas (Giordano, 2008; Spang, 2009). Familiarity with early algebra concepts suggests that the findings from this activity cannot be generalized to other students who have not had previous experience with early algebra ideas as well as with building rod models with Cuisenaire rods.

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