INTERACTIONS BETWEEN TEACHERS’ USE OF COLLABORATIVE, DYNAMIC GEOMETRY ENVIRONMENT AND THEIR GEOMETRICAL KNOWLEDGE

by

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ABSTRACT OF THE DISSERTATION

Interactions between Teachers’ Use of Collaborative, Dynamic Geometry Environment and their Geometrical Knowledge

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Dynamic geometry environments allow learners to manipulate mathematical objects and to explore their properties and relations among them. In such environments, learners can observe mathematical objects in different forms and observe changes to each form while manipulating the objects. These affordances can significantly support learning of geometry, especially when coupled with collaborative problem solving. Using Vygotsky’s (1978) notion of mediated activity and Rabardel and Beguin’s (2005) theory of instrumental genesis, this dissertation explores how such an environment influences the learning of geometry. It investigates how middle and high school teachers appropriate a collaborative dynamic geometry environment called Virtual Math Teams with GeoGebra (VMTwG) and how their appropriation shapes their geometrical knowledge. Over the last three years, 23 middle and high school mathematics teachers collaborated
synchronously in VMTwG to solve open-ended geometrical tasks. In each year, different groups of mathematics teachers engaged in 15-week course and worked in small teams (2-4 teachers) to solve open-ended problems and construction tasks that a research team, including the researcher, from Rutgers University and Drexel University designed and revised.

The data used for this dissertation include discursive and inscriptive interactions of teams of mathematics teachers in VMTwG. The VMTwG environment records users chat messages, the dynamic geometry software (GeoGebra) actions, and system actions. To understand how mathematics teachers appropriate a collaborative, dynamic geometry environment and how their appropriation shapes their geometrical understanding, the researcher conducted three interrelated studies to investigate the following: (1) how teachers appropriate the dragging feature of dynamic geometry environments and how their appropriation influences the discursive development of their understanding of dependencies among geometric objects; (2) longitudinally, how teachers appropriate a collaborative, dynamic geometry environment and how this appropriation shapes their actions when solving geometrical tasks; and (3) what mediational roles of the VMTwG environment are evident as teachers solve geometrical tasks. Analyses show the teachers’ appropriation and application of the dragging feature of VMTwG allowed them to understand and identify dependencies among geometrical objects. Relying on their instrumentation of technological and mathematical affordances of the environment, teachers explored geometrical objects and relations, conjectured about them, and justified their conjectures. Finally, analysis also shows that in addition to Rabardel and Beguin’s
(2005) epistemic and pragmatic mediations, a third mediation, pedagogic mediation, was evident in teachers’ mathematical activities.

For the mathematics education community, this study provides understanding of how teachers use technological tools. It also informs the design and implementation of instructional programs that engage learners with such tools to extend their mathematical knowledge. This study suggests that further research is needed to understand how to orchestrate learners’ instrumentation of collaborative technological tools in which teachers do not have a conventional instructional role in the classroom.
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Chapter 1: Introduction

Geometry is an important area of mathematics. It supports mathematical understanding of many topics in other mathematical areas such as algebra, calculus, and analysis as well as forms of argumentation such as deductive reasoning and proof. It provides visual representations alongside the analytical representation of mathematical concepts, which promotes students learning by emphasizing and suppressing various aspects of the concepts (Davis, 1992; Goldenberg, 1988b; Noss, Healy, & Hoyles, 1997; Piez & Voxman, 1997). For example, graph of a function allows students to see the wholeness of the function and its behavior compared to other functions while a table of inputs and function’s outputs helps students understand the mapping feature of a function and shows its point-by-point growth (Goldenberg, 1988b). The importance of geometry also arises in many other fields such as architecture, engineering, computer science, biology, robotics, and art. Geometry’s importance creates the need to help students learn the subject meaningfully, which, in turns, calls for providing mathematics teachers with effective professional development programs that can help them learn about useful tools and how to use those tools to teach mathematical topics effectively.

Supporting mathematics teachers to use tools effectively in their classrooms is part of a general concern that researchers have attended to carefully from different perspectives. Researcher in supporting teachers includes investigating how teachers’ preparation and in-service professional development programs can help teachers improve their classroom practice. These programs cover different aspects of teachers’ knowledge that relate to the content they are going to teach, psychological and pedagogical issues of the level of their prospective students, and introductions to different educational
technologies that can support teachers’ instructional practice. Attending to different components of teachers’ knowledge is a practice that is encouraged by different educational institutions. The National Research Council (2010), in their report, Preparing Teachers, suggests that “current research and professional consensus correspond in suggesting that all mathematics teachers, even elementary teachers, rely on a combination of mathematics knowledge and pedagogical knowledge” (p. 114). Additionally, the National Mathematics Advisory Panel (2008) emphasizes that “teachers must know in detail the mathematical content they are responsible for teaching and its connections to other important mathematics” (p. xxi). Many researchers attempt to theorize teachers’ knowledge (Ball, Thames, & Phelps, 2008; Herbst & Kosko, 2014; Hill, Rowan, & Ball, 2005; Hill, Schilling, & Ball, 2004; Pino-Fan, Assis, & Castro, 2015; Shulman, 1986, 1987) to provide models for teachers’ preparation and professional development programs. Researchers agree that it is important to understand how pre-service and in-service teachers develop their knowledge for teaching in specific areas of mathematics such as geometry.

To teach geometry effectively in middle and high school, prospective or in-service mathematics teachers need to develop solid understanding of its content. Teachers’ understanding of geometry is part of their subject matter knowledge (Ball et al., 2008; Herbst & Kosko, 2014; Hill et al., 2005; Hill et al., 2004; Shulman, 1986, 1987), which, as many studies show, is significantly related to students’ achievement (Ball, Hill, & Bass, 2005; Hill et al., 2005; Rowan, Chiang, & Miller, 1997). Shulman’s (1986, 1987) contribution in teachers’ knowledge has profound impact on today’s research. Identifying teachers’ pedagogical content knowledge helps other researchers
extend his work to many other forms such as teachers’ mathematical knowledge for teaching (Ball et al., 2008; Hill et al., 2005; Hill et al., 2004), technological pedagogical content knowledge (Koehler & Mishra, 2008; Mishra & Koehler, 2006), and mathematical knowledge for teaching high school geometry (Herbst & Kosko, 2014). These models can guide professional development programs to help teachers improve their knowledge about mathematics, pedagogy, and technology.

Researchers have used models of teachers’ knowledge to investigate their geometrical knowledge. Some studies looked at teachers’ geometrical knowledge for teaching certain geometrical topic such as square and measurements (Steele, 2013; Zazkis & Leikin, 2008) and other studies tried to examine teachers’ geometrical knowledge for teaching in general (Chinnappan & Lawson, 2005; Sinclair & Yurita, 2008; Yanik, 2011). Other researchers investigated the use of technology and its influence on teachers’ geometrical knowledge. (e.g., Hohenwarter et al., 2009; Lavy & Shriki, 2010; Stols, 2012). However, literature of teachers’ geometrical knowledge lacks studies that continuously investigate teachers’ use of digital technologies and the change of their geometrical knowledge through a professional development program or intervention. Many studies only examined teachers’ pre-existing knowledge of different geometrical topics (Baturo & Nason, 1996; Bjuland, 2004; Chinnappan & Lawson, 2005; Stump, 2001; Swafford et al., 1997; Tchoshanov, 2011; Yanik, 2011; Zazkis & Leikin, 2008). Studies that looked at the development of teachers’ geometrical knowledge often assessed teachers’ knowledge either at the end of an intervention or at very few time points throughout the intervention that included use of some technologies in some studies (Cavey & Berenson, 2005; De Villiers, 2004; Hohenwarter et al., 2009; Lavy & Shriki,
2010; Sinclair & Yurita, 2008; Steele, 2013; Stols, 2012). The literature lacks studies that examine the integration of four components: collaboration, online learning environment, dynamic geometry, and teachers’ geometrical knowledge. Investigating teachers’ interaction in an environment that integrates these four components over a long period of time is the main drive for this study.

Looking at these components interacting together will provide mathematics teachers, mathematics educators, and other educators with insights into how integrating dynamic geometry, or other technologies, in online learning environments can help teachers develop content knowledge.

The purpose of this study is to understand how teachers appropriate technological tools while working on geometrical tasks and how they discursively develop their geometrical knowledge collaborating on their responses to these tasks in an online environment. The discursive development of teachers’ geometrical knowledge is generally defined as how teachers come to understand mathematical relations through solving geometrical problems discursively and collaboratively in online dynamic geometry environment. This study will try to answer a central question: How do teachers appropriate and interact with an online collaborative and dynamic geometry environment and evolve their geometrical knowledge?

This qualitative investigation contributes to understanding about how mathematics teachers appropriate and interact with an online collaborative and dynamic geometry environment and how their appropriation shapes their geometrical understanding. Three case studies were conducted to contribute to answering the research question. The first study investigated the dragging feature of dynamic geometry
environment (DGE) since the defining feature of DGEs is the ability to drag geometric objects (Arzarello, Olivero, Paola, & Robutti, 2002; Hölzl, 1996; Ruthven, Hennessy, & Deane, 2008; Scher, 2000; Sinclair & Yurita, 2008). It focused on examining how teachers appropriate dragging and how their appropriation influences the discursive development of their understanding of dependency among geometric objects.

Dependency is the salient mathematical idea behind dragging. The second investigated longitudinally how teachers appropriate collaborative, dynamic geometry environment and how this appropriation shapes their actions while solving geometrical tasks. The last study used Rabardel and Beguin’s (2005) categories of instrument genesis in an instrument-mediated activity to examine the mediational roles of the environment throughout 15 weeks of teachers working on geometrical tasks.

This dissertation study is a part of a larger research project that is funded by National Science Foundation (NSF)¹. The larger project aims to investigate the development of teachers and students’ productive mathematical discourse in a collaborative dynamic geometry environment. Collaborators from Drexel University and Rutgers University developed an online collaborative environment called Virtual Math Teams with GeoGebra (VMTwG) and engaged middle and high school mathematics teachers and their students in solving open-ended geometrical tasks. VMTwG includes a multi-user version of GeoGebra that allows teams of learners to share one GeoGebra space to construct and manipulate geometric objects. Learners can communicate with

¹ The support from the National Science Foundation, DRK-12 program, is under awards DRL-118773 and DRL-1118888. The findings and opinions reported are those of the authors and do not necessarily reflect the views of the funding agency.
each other through GeoGebra space as well as through a chat panel in VMTwG. The environment records learners’ chat postings and their actions in GeoGebra. These data can be viewed using a replayer that shows learners’ actions in GeoGebra and their chat postings at the same time. Data can also be viewed in a textual form that includes learners’ chat postings and detailed descriptions of their actions in GeoGebra.

The research team designed open-ended tasks that encourage learners to explore and discuss geometric objects and relations among them such as dependencies. Tasks also encouraged learners to share their thinking and support each other while solving geometrical problems (Powell & Alqahtani, 2015a). The research design involved an iterative process that allowed the research team to revise tasks in VMTwG based on how teachers and students responded to them. For five years, groups of middle and high school mathematics teachers engaged in professional development courses to learn geometry in VMTwG in the first half of the academic year then engage their students in VMTwG in the second half of the year. In each year, the research team analyzed the work of teachers and their students and revised the professional development course and how the tasks are designed and implemented.

This dissertation study uses the work of teachers in the professional development course to investigate a topic that is not a direct research goal of the larger project. This dissertation study focuses on data from three years of professional development during the time that the researcher was closely involved with running and designing these courses. The data for this dissertation study come from three professional development courses that occurred in the fall semesters of 2013, 2014, and 2015. The data that were generated from these courses were about 10,000 pages in addition to the inceptive data in
GeoGebra space. In these courses, 23 middle and high school mathematics teachers engaged collaboratively in VMTwG to work on geometrical tasks. The researcher collaborated with other researchers to analyze teachers’ work in VMTwG and answer the main question of this dissertation study. The studies that the researcher conducted are presented in this dissertation as three different articles, chapters 3, 4, and 5.

This dissertation study informs the larger project in understanding how teachers’ instrumentation of VMTwG occurs as they solve geometrical tasks. Understanding aspects of teachers’ instrumental appropriation of VMTwG contribute to improving the design of the larger study and the design of tasks. It also provides insight into how students may interact with each other in VMTwG and develop their mathematical knowledge.
Chapter 2: Literature Review

The theory of mathematical knowledge for teaching is overarching, including all realms of subject matter and pedagogy of mathematics such as arithmetic, algebra, calculus, differential equations, topology, and so on. This literature review examines teachers’ geometrical knowledge. Specifically, teachers’ common content knowledge and specialized content knowledge of geometry and dynamic geometry. I first describe my search process then present what I found in the literature about teachers’ common and specialized knowledge of geometry.

2.1 Search Process

The search process I followed started by identifying the journals that I will use to identify studies for this literature review. First, I used SCImago Journal & Country Rank (SCImago, 2007) website to identify mathematics education journals with high SJR indicators in the subject area of social sciences and subject category education. The SJR index indicates the visibility of a journal, which can be seen as an indicator of its importance. Here are the top nine journals in mathematics education that I included in the search:

1. Journal for Research in Mathematics Education
2. Educational Studies in Mathematics
3. Mathematical Thinking and Learning
4. Journal of Mathematical Behavior
5. International Journal of Science and Mathematics Education
6. Journal of Mathematics Teacher Education
7. Eurasia Journal of Mathematics, Science and Technology Education
8.  ZDM - International Journal on Mathematics Education

9.  For the Learning of Mathematics

In addition to mathematics education journals, I used the same website to find journals that are concerned with technology in education. I included the journals with high SJR indicator number. The journals I found were as the following:

1.  Computers and Education
2.  Journal of Computer Assisted Learning
3.  International Journal of Computer-Supported Collaborative Learning
4.  British Journal of Educational Technology
5.  Learning, Media and Technology
6.  Educational Technology and Society
7.  Educational Technology Research and Development
8.  Australasian Journal of Educational Technology
9.  Technology, Pedagogy and Education
10. Turkish Online Journal of Educational Technology

Table 2.1 shows the numbers of studies found in each of the 19 journals and their SJR indicator.

**Table 2.1** Selected journals with high SJR indicator and number for studies found.

<table>
<thead>
<tr>
<th>N</th>
<th>Journal</th>
<th>Number of Studies</th>
<th>SJR indicator</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Journal for Research in Mathematics Education</td>
<td>1</td>
<td>1.976</td>
</tr>
<tr>
<td>2</td>
<td>Educational Studies in Mathematics</td>
<td>6</td>
<td>1.042</td>
</tr>
<tr>
<td>3</td>
<td>Mathematical Thinking and Learning</td>
<td>0</td>
<td>1.040</td>
</tr>
<tr>
<td>4</td>
<td>Journal of Mathematics Teacher Education</td>
<td>2</td>
<td>0.874</td>
</tr>
<tr>
<td>5</td>
<td>Journal of Mathematical Behavior</td>
<td>3</td>
<td>0.853</td>
</tr>
</tbody>
</table>
I decided to look at other mathematics education and technology related journals that do not have SJR indication number but published studies related either to teachers’ geometrical knowledge or dynamic geometry or both. This resulted in an additional list of journals as the following:

1. Research in Mathematics Education
2. Journal of Computers in Mathematics and Science Teaching
4. Mathematics Teacher Education and Development
5. International Journal of Math Education in Science and Technology
6. Canadian Journal of Science, Mathematics and Technology Education
7. Contemporary Issues in Technology and Teacher Education
8. Research and Practice in Technology Enhanced Learning

Table 2.2 shows the number of studies found in each of these eight journals.

<table>
<thead>
<tr>
<th>N</th>
<th>Journal</th>
<th>Number of Studies</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Research in Mathematics Education</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>Journal of Computers in Mathematics and Science Teaching</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>International Journal of Computers for Mathematical Learning</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>Mathematics Teacher Education and Development</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>International Journal of Mathematical Education in Science and Technology</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>Canadian Journal of Science, Mathematics and Technology Education</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>Contemporary Issues in Technology and Teacher Education</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>Research and Practice in Technology Enhanced Learning</td>
<td>0</td>
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</tbody>
</table>

To search within each journal for studies that look at middle and high school teachers’ geometrical knowledge, I used both Education Resources Information Center (ERIC) and Google Scholar. To identify the studies, I used the following sets of keywords:

- Knowledge, Teachers
- Geometry, Teachers
- Geometric, Teachers
- Geometrical, Teachers
- Dynamic, Teachers
- Math, Teachers (for the non-math journals)
As Tables 2.1 and 2.2 show, there are few studies investigated middle or high school teachers’ geometrical knowledge. These studies varied in the type of geometrical knowledge they investigated. They also varied in how they assessed teachers’ geometrical knowledge. In the next section, I discuss how these studies investigated teachers’ knowledge in geometry and dynamic geometry.

2.2 Teachers’ Geometrical Knowledge

The studies included in this review looked at teachers’ subject matter knowledge in geometry and dynamic geometry. According to the model Ball et al. (2008) provide, mathematical knowledge for teaching (MKT), these studies mainly looked at the teachers’ common and specialized content knowledge. I was not able to find any study that investigated teachers’ horizon content knowledge. Most of the studies investigated teachers’ common content knowledge in geometry or dynamic geometry (Baturo & Nason, 1996; De Villiers, 2004; Hohenwarter et al., 2009; Koyuncu, Akyuz, & Cakiroglu, 2014; Lavy & Shriki, 2010; Sinclair & Yurita, 2008; Stols, 2012; Tchoshanov, 2011; Yanik, 2011; Zazkis & Leikin, 2008) and few studies investigated teachers’ specialized content knowledge (Bjuland, 2004; Chinnappan & Lawson, 2005). Some studies investigated both common and specialized content knowledge (Cavey & Berenson, 2005; Moore-Russo & Viglietti, 2011; Steele, 2013; Swafford et al., 1997).

When examining teachers’ specialized content knowledge, methodological questions arise about how to study it. Studies that focused on teachers’ specialized content knowledge (SCK) examined teachers’ geometrical knowledge in settings disconnected from classroom settings. As Powell and Hanna (2006, 2010) argue, teachers’ mathematical knowledge can effectively be seen through their practice, while
teachers are teaching or discussing their practice. Sinclair and Yurita’s study (2008) was the only study that looked at the teachers’ practice. They investigated changes in teacher’s mathematical discourse in classroom after introducing the teacher to a dynamic geometry software. Cavey and Berenson’s study (2005) looked at teachers’ geometrical knowledge, both CCK and SCK, through lesson planning. Lesson planning can be seen as part of practice and can provide an insight into teachers’ geometrical knowledge; however, the question about how much of this knowledge will be available while teaching remains unanswered.

In addition to the variety of types of geometrical knowledge that has been studied, the review revealed different themes in the way that teachers’ geometrical knowledge was investigated. Two main categories can be discussed about studying teachers’ geometrical knowledge: teachers’ pre-existing knowledge versus developing knowledge and using tests and surveys versus engaging the teaching in mathematical tasks to assess that knowledge. Figure 2.1 shows how studies fit with the emergent themes for this review.
Figure 2.1 Categories of studies that looked at teachers’ geometrical knowledge.

The intersections of those categories define the themes I chose for this review.

Pre-existing versus developing knowledge categories concern with the type of knowledge that was studied. These two categories of knowledge are mutually exclusive. Studying both categories of knowledge is possible, but none of this review’s studies looked at them both. Similarly, the horizontal dimension of categories—tests versus engaging in tasks—will be treated as mutually exclusive categories. There are studies that used two ways to assess teachers’ existing geometrical knowledge by administrating a test or distributing a survey then following that with some interviews (Hohenwarter et al., 2009; Leikin & Grossman, 2013; Steele, 2013; Swafford et al., 1997; Zengin & Tatar, 2015). The reason for deciding to treat the two categories in the second domain as mutually exclusive is that, even with those studies that use the two ways of assessments, researchers tend to treat one way as the main method for investigating teachers’ geometrical knowledge and use the other to support it. In this case, I use the main method of assessment to categorize
the study. In the following sections, I will present studies in the four emergent themes and then discuss lacunae in the literature about teachers’ knowledge in geometry and dynamic geometry.

2.2.1 Using tests or surveys to assess teachers’ pre-existing knowledge.

Studies that used tests or surveys to assess teachers’ pre-existing geometrical knowledge mostly look at teachers’ geometrical knowledge at one point of time and typically have big sample size. For example, Tchoshanov (2011) designed Teacher Content Knowledge Survey to test 102 middle school mathematics teachers’ knowledge. The survey consisted of 33 multiple-choice items that identify cognitive type of teachers’ content knowledge. He identified three cognitive types as the following:

1. Type 1: Knowledge of Facts and Procedures: memorization of facts, definitions, formulas, properties, and rules, and performing procedures.


3. Type 3: Knowledge of Models and Generalizations: generalization of mathematical statements, designing mathematical models, making and testing conjectures, and proving theorems (p. 148).

In addition, studies that used tests or surveys to assess teachers’ geometrical knowledge have a predefined understanding of this knowledge. Researchers design tools to test whether the teachers’ knowledge matches what they defined (Stump, 2001; Tsamir, Tirosh, Levenson, Barkai, & Tabach, 2015). A clear example of this research practice is a study conducted by Stump (2001). She examined teachers’ content...
knowledge of slope, using a survey that was based on her division of content knowledge into two parts:

- **Concept definition**: concept image and concept definition of slope as indicators of teachers’ knowledge of slope. For example, defining slope as geometric ratio (vertical change over horizontal change), algebraic ratio (the change in $y$ over the change in $x$), or using a physical property such as “slant”, “steepness”, “incline”, “pitch”, and “angle” (p. 129).

- **Mathematical understanding**: forming connections between formal or procedural knowledge and conceptual knowledge as well as understanding the relationships with other concepts. For example, seeing slope as rate of growth, equation with numerical parameters, or equation with literal parameters (p. 131).

She found that most of the teachers viewed slope as geometric ratio. Though this research practice reveals aspects of teachers’ geometrical knowledge, it may result in missing other components of teachers’ geometrical knowledge that the research did not account for while designing the instrument for assessing teachers’ knowledge.

Swafford et al. (1997) used van Heile’s theory to test the teachers’ geometrical knowledge before and after taking a course they designed that introduced the teachers to the theory of van Heile. Even though their study was an attempt to examine the changes in teachers’ knowledge, what they actually examined was the effectiveness of the course they designed.
2.2.2 Engaging teachers in tasks to assess their pre-existing knowledge.

The next group of studies did not use tests or surveys to assess teachers’ geometrical knowledge. Rather than requiring responses to certain questions, the studies in this group engaged teachers with geometrical tasks. The tasks included solving geometrical problems, discussing classmates’ solutions, creating lesson plans, or creating geometrical examples and problems. Many studies used interviews to assess teachers’ geometrical knowledge and had teachers solving problems (Baturo & Nason, 1996; Bjuland, 2004; Gómez-Chacón & Kuzniak, 2015; Yanik, 2011). Zazkis and Leikin (2008) asked the teachers to generate definitions of square and used those responses as insight into teachers’ geometrical knowledge.

Chinnappan and Lawson (2005) interviewed two experienced mathematics teachers (more than 15 years of experience) about their geometrical knowledge that is relevant to teaching. With each teacher, they conducted three one-hour interviews. In the first interview, the teachers were asked about geometric concepts (squares, rectangles, lines, similar triangles, congruent triangles, parallel lines, area, coordinates, triangles, right-angled triangles, regular hexagons, regular octagons, circles) and their understanding of teaching and learning of these concepts. In the second interview, the teachers were asked to solve four problems that involve the concepts from the first interview and think aloud while solving the problems. In the last interview, they asked series of probing questions to help teachers access some knowledge that they had not discussed in their previous two interviews. For example, one teacher did not mention symmetry while talking about the geometric figures and was reminded to discuss that.
This study is a good example of studies that did not use tests or surveys to assess teachers’ geometrical knowledge. However, it only used interviews to collect the data.

Engaging teachers in tasks to investigate their geometrical knowledge helps researchers understand many aspects of teachers’ knowledge. What is lacking in the literature of studies that used a method to assess teachers’ knowledge are studies that investigate teachers’ problem-solving process and studies that aim to understand how teachers arrive to their geometrical knowledge.

2.2.3 Using tests or surveys to assess teachers’ developing knowledge.

The category of investigating teachers’ developing knowledge includes studies that engaged teachers in some type of course, intervention, or professional development program. Some researchers engaged teachers in one type of program then used tests or surveys to assess their geometrical knowledge. One example of this type of study is a study done by Hohenwarter et al. (2009). Using a dynamic geometry software, GeoGebra, researchers engaged 44 middle and high school mathematics teachers in four workshops about different geometric and algebraic topics. At the end of the last workshop, they used a survey to assess teachers’ mathematical knowledge. The survey consisted of 24 questions aligned with the Florida Sunshine State Standards.

Similar to the critique mentioned above about using tests and survey to assess teachers’ pre-existing geometrical knowledge, this kind of studies measure a pre-defined form of knowledge. A questionnaire or a test captures limited aspects of teachers’ common and specialized mathematical knowledge.
2.2.4 Engaging teachers in tasks to assess their developing knowledge.

The last theme includes studies that looked at teachers’ developing geometrical knowledge through engaging them with geometrical tasks. As I mentioned before, these studies engage teachers in some sort of a course or intervention to see the growth of their knowledge. Some studies in this theme used different ways to assess teachers’ geometrical knowledge. Cavey and Berenson (2005) engaged prospective teachers in a five-week lesson plan study about right triangle trigonometry and investigated their knowledge growth. Using a sequence of interviews and lesson planning, researchers gained insight into teachers’ specialized content knowledge in geometry. Moore-Russo and Viglietti (2011) used teachers’ proofs and reflections about a set of geometry instructional animations. Teachers’ reflections about the animations gave the researchers’ insight into teaching certain geometrical topics as well as deepened teachers’ understating of those topics.

Other studies assessed teachers’ geometrical knowledge in multiple ways (Lavy & Shriki, 2010; Steele, 2013). The main purpose of these studies was to provide a framework for assessing teachers’ mathematical knowledge. Lavy and Shriki (2010) engaged 25 preservice teachers in problem-posing activities using “What If Not?” (WIN) strategy in a dynamic geometry environment to examine the development of participants’ mathematical and meta-mathematical knowledge. They defined meta-mathematical knowledge as the “ability to examine the correctness of a mathematical product, its relevance or elegance, and to contemplate the relations between concepts” (p. 12). The participants worked in pairs to solve geometrical problems using the WIN strategy then created portfolios that were the main source of data. To gain an insight into the
prospective teachers’ mathematical and meta-mathematical knowledge, the researchers analyzed steps 2-4 of WIN strategy by looking at the list of attributes and negated attributes and alternatives that teachers created. Their analysis showed that the teachers’ mathematical knowledge development was related to “the formal definitions of the concepts and the objects involved” (p. 22) and the development of meta-mathematical knowledge involved structuring coherent mathematical situations, validity of the mathematical situation, and the importance of providing formal mathematical proofs.

Similarly, Steele (2013) designed tasks to examine the mathematical knowledge for teaching geometry and measurement (teacher’s knowledge of length, perimeter, and area) by looking at teachers’ common content knowledge (CCK) and specialized content knowledge (SCK). Working with 25 teachers in a 6-week intervention, the researcher identified key mathematical ideas for teaching length, perimeter, and area. He created lists of ideas about CCK and SCK for teaching length, perimeter, and area. For each of these ideas, he created a task to assess mathematical knowledge for teaching. The main goal of this study was to provide principles for developing measures of teachers’ mathematical knowledge for teaching and providing examples of measuring mathematical knowledge for teaching length, perimeter, and area. However, while principles might be transferable to other mathematical topics, identifying the key concepts and creating tasks with coding rubrics might not be easily transferable to new topics.

Only one study looked at changes in teachers’ mathematical discourse as an indicator of change in geometrical knowledge. Sinclair and Yurita (2008) investigated the impact of implementing dynamic geometry in teaching on teachers’ geometrical
knowledge. The researchers captured the classroom discourse of one teacher while teaching geometry with paper-and-pencil and then with using dynamic geometry software. They found significant change in teacher’s discourse, from static to dynamic and having the students to make the shift from static to dynamic discourse.

Investigating changes in teachers’ geometrical discourse, whether in practice settings or in professional-development settings, can provide insightful understanding of teachers’ geometrical knowledge. As this review showed, only one study used discursive data to gain insight into teachers’ geometrical knowledge.

2.3 Conclusion

The review about teachers’ knowledge of geometry and dynamic geometry shows that few studies looked at teachers’ geometrical knowledge, especially in dynamic geometry environment (DGE). These studies either investigated teachers’ pre-existing knowledge of geometry and dynamic geometry or the development of teachers’ knowledge. However, most of the studies that investigated the development of teachers’ knowledge engaged teachers in an intervention and reported on changes in teachers’ knowledge either at the end of the intervention or at few time points throughout the intervention. The literature lacks studies that continuously investigate the change of teachers’ geometrical knowledge through professional development programs or interventions, over time. This dissertation study investigates longitudinally how teachers interact in collaborative dynamic geometry environments and how they extend their geometrical knowledge. The review of the literature on teachers’ knowledge of geometry and dynamic geometry also shows that studies used one of two methods to assess teachers’ knowledge. First group of studies used tests or surveys to assess teachers’
knowledge. This method of assessment neglects many aspects of teachers’ knowledge. The second group engaged teachers in tasks to assess their geometrical knowledge such as solving problems, lesson planning, providing mathematical definitions and examples, and teaching in a classroom. Additionally, the review shows that there are few studies examined teachers’ knowledge in dynamic geometry environments and in collaborative settings. It is important to understand how dynamic geometry tools influence teachers’ understanding of different geometric concepts.

This dissertation study looks at teachers’ collaborative interactions in a DGE and the formation of their geometrical knowledge. After reviewing the literature on teachers’ knowledge of geometry and dynamic geometry, I found little research that has been done on dynamic geometry or investigated teachers’ discourse interactions to understanding their geometrical knowledge. Looking at teachers’ interactions in a DGE is one of the major contributions of this study. In addition, many studies only looked at pre-existing geometrical knowledge of teachers and most of the studies that looked at teachers’ developing geometrical knowledge did not investigate the construction of that knowledge. Understanding the developments of teachers’ geometrical knowledge requires investigating teachers’ engagements with geometrical tasks over a long period of time. The following three chapters are studies that investigate teachers’ appropriation of collaborative DGE and extensions in their geometrical understanding.

Each of the following three chapters consists of a study that draws on Vygostky’s (1978) notion of mediated activity and Rabardel and Beguin’s (2005) theory of instrumental genesis to investigate teachers’ mathematical activity in VMTwG. The studies explore how teachers appropriate longitudinally the VMTwG environment and
how their appropriation influences their knowledge of geometry while working on open-ended geometrical tasks. The design and data production for these studies rely on the importance of social interactions and collaboration while learning mathematics. They also rely on the dialogic view of mathematics and the different components of mathematics curriculum. We view mathematics as a dialogic subject that results from discussing mathematical objects, relations among objects, and dynamics of these relations (Gattegno, 1987). Mathematics involves two types of knowledge that Hewitt (1999) calls necessary and arbitrary knowledge. Following his notion of these types of mathematical knowledge, we focus on mathematical ideas that learners can derive by attending to and noticing mathematical properties and relations (necessary knowledge) rather than focusing on the semiotic conventions such as names, labels, and notations (arbitrary knowledge).

These theoretical considerations informed data production and analysis in this dissertation study. Teachers were encouraged to engage collaboratively to explore and manipulate mathematical objects and discuss their properties and relations among them. Data analysis focused on identifying these moments when teachers are engaged with mathematical discourse about mathematical objects and relations and relate them to teachers’ actions in the dynamic geometry environment. The three studies investigated teachers’ interactions in VMTwG differently to understanding their instrumentation process and the development of their mathematical knowledge. The next chapter presents the first study that investigates how teachers appropriate dragging in dynamic geometry and how their appropriation influences their identification and construction of geometric dependencies.
Chapter 3: Instrumental Appropriation of a Collaborative, Dynamic-Geometry Environment and Geometrical Understanding

Abstract: To understand learners’ appropriation of technological tools and geometrical understanding, we draw on the theory of instrumental genesis (Lonchamp, 2012; Rabardel & Beguin, 2005), which seeks to explain how learners accomplish tasks interacting with tools. To appropriate a tool, learners develop their own knowledge of how to use it, which turns the tool into an instrument that mediates an activity between learners and a task. The tool used in this study is the Virtual Math Teams with GeoGebra (VMTwG) environment. It contains a chat panel and multiuser version of GeoGebra. The learners are seven middle and high school mathematics teachers who participated in a professional development course in which they collaborated synchronously in VMTwG to solve geometrical tasks. We use conventional content analysis to analyze the work of a team consisting of two high school teachers. Analysis shows that the teachers’ appropriation and application of the dragging feature of VMTwG shaped their understanding of geometrical relations, particularly dependencies. This informs the broader question of how and what mathematical knowledge learners’ construct using certain technologies.

2 This paper is accepted to be published in the International Journal of Education in Mathematics, Science and Technology in April 2016.
3.1 Introduction

Understanding geometry is important in itself and for understanding other areas of mathematics. It contributes to logical and deductive reasoning about spatial objects and relationships. Geometry provides visual representations alongside the analytical representation of a mathematical concept (Goldenberg, 1988b; Piez & Voxman, 1997). Pairing learning geometry with technological tools of Web 2.0 can allow learners to investigate collaboratively geometric objects, properties, and relations and develop flexible understanding of geometry. The Common Core State Standards for Mathematics underscores that mathematics educators should seek to develop students’ mathematical practices so that they “use appropriate tools strategically,” including dynamic geometry environments (Common Core State Standards Initiative, 2010, p. 7). Although teaching with technology is recommended, meta-analytic studies show that teaching mathematics with technology cannot guarantee positive influence on learning (Kaput & Thompson, 1994; Wenglinsky, 1998). Consequently, careful investigations are required to understand the appropriation of technology and how it shapes mathematics learning. To contribute to this understanding, we describe the influence of learners’ appropriation of online dynamic geometry tools on their geometrical understanding. This paper responds to the question: How does learners’ appropriation of an online, collaborative dynamic geometry environment shape their geometrical understanding?

3.2 Literature Review

Researchers investigated the use of technology in learning mathematics for different purposes. The first group of researchers focused on investigating the effect of introducing certain technologies in learning mathematics. For example, Hohenwarter et
al. (2009) engaged 44 middle and high school mathematics teachers in four workshops to learn about different geometric and algebraic topics in a dynamic geometry environment (DGE), GeoGebra, to investigate changes in teachers’ mathematical knowledge in general. Similarly, Sinclair and Yurita (2008) investigated teacher’s changes in mathematical discourse after introducing the use of dynamic geometry software in classroom. These studies did not focus primarily on how the teachers interacted with DGE; rather, their main goals were investigating the impact of introducing such technologies on learning and teaching mathematics.

Other studies focused on certain aspects of interacting with DGEs. Arzarello et al. (2002) studied the dragging action in DGE and the cognitive processes behind each different type of dragging. They identify two levels of cognitive processes linked to dragging: ascending (moving from drawings to theory) and descending processes (moving from theory to drawings). Ascending processes allow users to investigate the drawings freely to look for patterns and invariants. Descending processes are used with a theory in mind to validate or test properties. Dragging within those two cognitive levels can vary between wondering dragging, bound dragging, guided dragging, dummy locus dragging, line dragging, linked dragging, and dragging test (Arzarello et al., 2002). Baccaglini-Frank and Mariotti (2010) used the work of Arzarello et al. to develop a model that tries to explain cognitive processes behind different types of dragging. They used four different types of dragging: wondering dragging, maintaining dragging, dragging with trace activated, and dragging test. Wondering dragging is dragging that aims to look for regularities while maintaining dragging is dragging base points so that the dynamic figure maintain certain properties. Dragging with trace activated is dragging
base points with trace activated on them. Drag test is dragging base points to test whether certain properties will meet certain conditions (Baccaglini-Frank & Mariotti, 2010).

The last group of studies looked at the instrumental transformation of technological tools to instruments that mediate users’ activity. An example of these studies is Guin and Trouche’s (1998) study. They investigated the instrumentation process that a group of students used to transform graphing calculators to mathematical instrument. They conclude that instrumentation is a complex and slow process and that interacting with technological tools in learning mathematics might not result in transforming the tool into an instrument. Similarly, Guin and Trouche (2002) and Trouche (2003) show how learners transform technological tools into mathematical instruments with more focus on the teacher’s and the environment’s roles that support this transformation.

Based on this review, there is need for studies that investigate how users, through the instrumentation process, interact with each other in DGEs. The users’ interactions with DGEs have an influence on their thinking and learning of geometry (Hegedus & Moreno-Armella, 2010; Rabardel & Beguin, 2005), which makes investigating how learners appropriate an online, collaborative dynamic geometry environment important. It can help mathematics educators understand how DGEs shape learners’ geometrical understanding.

3.3 Theoretical Perspective

To understand learners’ appropriation of technological tools, we draw on a Vygotskian perspective about goal-directed, instrument-mediated action and activity.
Instrumental genesis (Lonchamp, 2012; Rabardel & Beguin, 2005) theorizes how learners interact with tools that mediate their activity on a task. To appropriate a tool, users (teachers, students, or learners in general) develop their own knowledge of how to use it, which turns the tool into an instrument that mediates activity between users and a task. The basic concept of the theory is that users engage in an activity in which actions are performed upon an object (matter, reality, object of work…) in order to achieve a goal using an artifact (technical or material component). Rabardel and Beguin (2005) emphasize that the instrument is not just the tool or the artifact, the material device or semiotic construct, it is “a mixed entity, born of both the user and the object: the instrument is a composite entity made up of an artifact component and a scheme component.” (p. 442). An instrument is a two-fold entity, part artifactual and part psychological as utilization schemes. The user acquires a utilization scheme and applies it to the artifact.

Artifacts are subject to two kinds of utilization schemes. The first kind of utilization schemes is usage schemas, which are directly related to the artifact. It constitutes the basic knowledge of how to operate or use the artifact. For example, the knowledge about driving a car for an experienced driver such as changing gears or turning the steering wheel, or the knowledge of the components of a digital camera and knowing how to use them. The second kind of utilization schemes is instrument-mediated action schemes, which are more related to the transformations that can be done to the object. These schemes are concerned with the activity that will lead the users, using the artifact, to reach a desired goal. For the example of driving a car, the instrument-mediated action scheme will be more focused on the other variables on the road that a
driver needs to be aware of and react to their existence to be able to reach the final
destination (Lonchamp, 2012; Rabardel & Beguin, 2005). The activity can be individual
or collective depending on the number of users engaged in the activity.

Just interacting with an artifact is not an instrument-mediated activity. In
instrument-mediated activity, instruments mediate users’ activity or action to achieve a
certain goal. While engaging in an activity, users monitor consciously the continuous
transformation of an object towards their goal. This mediator role that instruments play
governs the user-object relations, which might take epistemic or pragmatic forms. The
epistemic mediation form focuses on the object and its properties. In this form, the
instrument helps the user understands the object and its structure. On the other hand, in
the pragmatic mediation form, the user is mainly concerned with the required actions
while using the instrument to transform the object into the desired final result
(Lonchamp, 2012; Rabardel & Beguin, 2005). The final result is the final transformation
of the object, which might not match exactly the initial goal. The user may find certain
form of the transformed object satisfying enough and, therefore, end the activity.

During an activity that is mediated by an instrument, it is understandable that the
artifact affects the activity; however, users play a major role in shaping the activity. The
users’ interactions with an artifact shape the activity. Two different users can approach an
artifact differently, develop different utilization schemes, and create two different
activities and instruments.

The transformation of an artifact or tool into an instrument, or instrumental
genesis, occurs through two important dialectical processes that account for potential
changes in the instrument and in the learners, instrumentalization and instrumentation. The instrumentalization process is defined as “the process in which the learner enriches the artifact properties” (Rabardel & Beguin, 2005, p. 444). In this process, the user selects and modifies the properties of the artifact, for example, using a wrench as a hammer. The second process of instrumental genesis is instrumentation. This process is about the development of the learner side of the instrument. The development of the learner is basically the assimilation of an artifact to a scheme and the adaptation of utilization schemes. With the example of using a wrench as a hammer, the learner already had acquired the utilization scheme of a hammer and when a hammer was not available at the time of the action, the learner chose the wrench and associated it to the hammer utilization scheme. This is an example of “direct assimilation of artifact into a utilization scheme” (Rabardel & Beguin, 2005, p. 446), which changes the meaning of the artifact. During the act of assimilation, the learner employs previous utilization schemes to new artifact. In our example, acquiring the hammer utilization scheme led the learner to choose the wrench and not another tool because he is aware of the functions of the hammer and its structure that makes the learner look for a similar tool that can take the same scheme. In the situation where a new artifact cannot be assimilated to previously acquired utilization scheme, the learner adapts utilization schemes and makes the necessary modifications to it.

Through the two processes of instrumental genesis, instrumentation and instrumentalization, dialectically the tools influence the thinking of the learner and the learner influences the design of the tools. On the one hand, the structure and functionality of tools shape how the learner uses the tool, which result in shaping the learner’s
thinking. On the other hand, the learner’s interactions with the tool also shapes the tool and how it is used. In the case of dynamic geometry environments (DGEs), instrumentation occurs when users develop utilization schemes of how to use the environment. Utilization schemes include learning how to use a DGE’s tools to construct and manipulate geometric objects. It also includes understanding the functions of those tools and their links to the theory of geometry and using the tools to explore, conjecture, and justify relations among geometric objects. The instrumentalization process concerns the design and the use of DGEs. Concerning the design, developers instrumentalize by deciding on what functions to include, how to organize the environment, how it reacts to users’ actions, how to support users’ activity, and so forth. Concerning the use of DGEs, instrumentalization occurs with users’ decisions about how to use the environment. Users may use the environment and its functions as the developers intended. However, to achieve certain goals, users may use DGEs’ tools differently from how developers intended. For example, constructing two parallel lines can be accomplished using “Parallel Line” tool, but users may use “Regular Polygon” tool to construct a square and then construct two lines on two opposite sides of the square. Of these two dialectically related processes, for this study, we are concerned with instrumentation in DGEs since we are interested in understanding how users develop their utilization schemes and learn to use the tools as intended. This understanding can provide insights into learners’ geometrical knowledge and its development.

Further insights can be derived from how learners respond to the environment’s actions. The feedback that the software gives to the user after manipulating dynamic objects affects the user’s interaction with the software. The environment reacts to the
users’ actions through engineered infrastructure that responds according to the theory of geometry. These reactions can inform the users’ actions and shape their thinking, which provide insights into how DGEs are used to mediate users’ activity. During users’ activity, dragging the base points or “hot-spots” of a dynamic figure can change the geometrical properties of the figure and can provide insights into its construction process. Hot-spots are “points that can be used to construct mathematical figures, e.g. join two points with a segment, or construct a piecewise graph, and then used to dynamically change the construction.” (Hegedus & Moreno-Armella, 2010, p. 26).

The relationship between the user and the DGE is a result of co-action between the two (Hegedus & Moreno-Armella, 2010). The notion of co-action has two sides: (a) the user’s action can guide DGE and (b) DGE’s reaction can guide the user. A dynamic geometry software allows users to act on it and, in turn, reacts to their actions. As users drag (click, hold, and slide) a hotspot of a geometric figure, the DGE redraws and updates information on the screen, preserving all constructed mathematical relations among objects of the figure. In redrawing, the DGE creates a family of not only visually but also mathematically similar figures. Users may then attend to the reaction of the DGE and experience and understand underlying mathematical relations such as dependencies. DGEs “remember” underlying mathematical relations among various objects of a construction. For instance, if a point P is the midpoint of a segment AB, then as the length or position of segment AB changes, P’s relationship to AB remains invariant, namely that P is equidistant from the line segment’s endpoints A and B.

Dynamic geometry environments are tools that learners can appropriate through an instrumentation process. Learners will need to acquire utilization schemes – usage
schemes and instrument-mediated activity schemes – to appropriate the tool. In DGE, the usage schemes includes knowledge about the software use and its functionalities. The second level of utilization schemes for a DGE includes knowledge of geometry and dependencies. When learners appropriate a DGE as an instrument, they will be able to use it to demonstrate geometric concepts and solve geometrical problems. This appropriation may result in knowledge of how to use dynamic geometry software as well as knowledge of geometry. The geometrical knowledge can be a special type of knowledge shaped by DGE. Within DGEs, Straesser (2002) sees that geometry is “lived in differently, broader scope, has a new, more flexible structure, [and] offers easy access to certain heuristic strategies.” (p. 331). Balacheff and Kaput (1996) claim that characteristics of DGEs result in creating new mathematics, a geometry that is different from Euclidian geometry in the plane.

3.4 Methods

Data come from a project that integrates a cyberlearning environment with digital tools for collaborative geometrical explorations grounded in a pedagogical approach that engages learners in developing significant mathematical discourse. The project investigates learners’ actions as they occur through an iterative coevolution of the technology and curricular resources in the context of engaging, reflective collaborative learning experiences of significant mathematical discourse by in-service teachers and their students. The data for this paper come for an online professional development course for middle and high school teachers that occurred over 15 weeks in the fall 2014. In small teams, seven New Jersey middle and high school mathematics teachers engaged in interactive, discursive learning of dynamic geometry through collaborating to solve
tasks in a computer-supported, collaborative-learning environment: Virtual Math Teams with GeoGebra (VMTwG). They also were given articles to read and discuss about collaboration (Mercer & Sams, 2006; Rowe & Bicknell, 2004), mathematical practices (Common Core State Standards Initiative, 2010), accountable talk (Resnick, Michaels, & O’Connor, 2010), technological pedagogical content knowledge (Mishra & Koehler, 2006), implementing technology in mathematics classroom (McGraw & Grant, 2005), and validating dynamic geometry constructions (Stylianides & Stylianides, 2005) and analyzed logs of their VMTwG interactions to examine, reflect, and modify their collaborative and mathematical practices.

VMTwG, a product of a collaborative research project among investigators at Rutgers University and Drexel University, is an interactional, synchronous space. It contains support for chat rooms with collaborative tools for mathematical explorations, including a multi-user version of GeoGebra, where team members can define dynamic objects and drag the hotspots around on their screens. VMTwG records users’ chat postings and GeoGebra actions. The research team designed dynamic-geometry tasks that encourage participants to discuss and collaboratively manipulate and construct dynamic-geometry objects, notice dependencies and other relations among the objects, make conjectures, and build justifications (Powell & Alqahtani, 2015a).

For this research, we analyze the work of Team 3, which consists of two high school mathematics teachers. Before this course, both teachers did not have previous experience with dynamic geometry. The teachers met in VMTwG synchronously for two hours twice a week. We selected data of this team’s interactions since they demonstrated conspicuously how team members built an understanding of the dragging affordances of
the environment. To understand how teachers interact with VMTwG environment and how the environment shapes their geometrical knowledge, we used the discursive data generated from their work on three tasks, Task 8 (see Appendix A), which they worked on in the fourth week, Task 16, which they worked on in the fifth week, and Task 21, which they worked on in the seventh week. Task 8 (see Figure 3.1) asks the teachers to discuss the construction of equilateral triangle and then to construct it. Task 16 (see Figure 3.2) asks the teachers to drag dynamically different triangles and discuss the dependencies involved in their construction. Task 21 (see Figure 3.4) asks teachers to construct a perpendicular line that passes through an arbitrary point. The discursive data that were collected include the logs of teachers’ chat communications and their GeoGebra interactions. Using conventional content analysis (Hsieh & Shannon, 2005), we analyzed their discursive data to understand the developmental process of instrument appropriation and the implications of that appropriation. In addition, we used the construct of co-action to understand when, why, and how do teachers interact with base points; what feedback do they perceive and what do they do with this feedback; and how does the feedback shape their subsequent actions.

3.5 Results

The analysis focuses on understanding how the teachers’ appropriation of VMTwG shapes their geometrical understanding. Specifically, our results show how through co-action teachers’ interaction with VMTwG leads to shaping their understanding of affordances of dragging in GeoGebra while working on constructing an equilateral triangle (Task 8). The result from Task 16 shows how teachers used aspects of the environment to identify dependencies among geometric objects. Our results in a later
VMTwG session also show how the VMTwG environment shapes the knowledge that the teachers develop while working to construct a perpendicular line that goes through an arbitrary point as well as their heuristic to solve the problem (Task 21).

3.5.1 Appropriating dragging.

Task 8 asks teachers to drag objects and then to discuss (in the chat window) what they notice about the given figure and then construct a similar one in GeoGebra (see Figure 3.1). Among other things, previous tasks engaged the teachers in noticing as they dragged hotspots variances and invariances of objects and relations of figures.

Figure 3.1 Task 8: Constructing Equilateral Triangle.

This task was intended to extend the teachers’ experience with dragging and geometrical dependencies. Before this session, the teachers had already worked on some basic geometric objects (such as lines, lines segments, circles, and circles whose radius is
dynamically dependent on a line segment) and were asked to drag and notice relationships among the objects. Those tasks mainly aimed to familiarize the teachers with the functionality of the VMTwG environment and to the cognitive habit of noticing and wondering about the behavior of object and relations among objects. That is, the tasks engaged the teachers in becoming aware of co-active relations between their actions and reactions of the VMTwG environment. Below, their chat posting shows that they focused on relationships that were visually apparent. It also shows that the teachers, Gouri and Sophiak, revisited their understanding of dragging after being instructed to create an equilateral triangle.

<table>
<thead>
<tr>
<th>#</th>
<th>User</th>
<th>Chat Post</th>
</tr>
</thead>
<tbody>
<tr>
<td>26</td>
<td>Sophiak</td>
<td>It seems that point C is fixed but pts A&amp;B are not. I am thinking somehow A&amp;B were used to create the circles which is why the make the circles bigger or smaller.</td>
</tr>
<tr>
<td>27</td>
<td>Sophiak</td>
<td>How about you try to explore now?</td>
</tr>
<tr>
<td>28</td>
<td>Gouri</td>
<td>ok I'll continue on with #2 [the second instruction in Task 8] as well</td>
</tr>
<tr>
<td>29</td>
<td>Sophiak</td>
<td>No, I would like to create the objects as well. I think it is valuable if we both explore</td>
</tr>
<tr>
<td>30</td>
<td>Gouri</td>
<td>C does seem fixed/constrained</td>
</tr>
<tr>
<td>31</td>
<td>Gouri</td>
<td>sure - how about i do it and then you do it as well after?</td>
</tr>
<tr>
<td>32</td>
<td>Sophiak</td>
<td>Sounds good. Please type what you do.</td>
</tr>
<tr>
<td>33</td>
<td>Gouri</td>
<td>So far I created 2 circles</td>
</tr>
<tr>
<td>34</td>
<td>Gouri</td>
<td>and overlapped the D point as the raius point for E</td>
</tr>
<tr>
<td>35</td>
<td>Gouri</td>
<td>one more try</td>
</tr>
<tr>
<td>36</td>
<td>Gouri</td>
<td>ok - i deleted the other circle because i dont need it</td>
</tr>
<tr>
<td>37</td>
<td>Gouri</td>
<td>I somehow thought i could create all 3 points, abc through two circles</td>
</tr>
<tr>
<td>38</td>
<td>Sophiak</td>
<td>How did you create F?</td>
</tr>
<tr>
<td>39</td>
<td>Gouri</td>
<td>I added a point</td>
</tr>
</tbody>
</table>
Gouri then the polygon tool for the triangle
Sophiak Did you want to explore your picture to see if it behaves the same way as the original?
Gouri ok
Gouri [after dragging for few minutes] I noticed that it's the points that make the circle dynamic
Gouri and not the circle (in black) itself

Analysis of this excerpt reveals two aspects of this team’s instrumentation process: collaboration and tool use, which parallels their mathematical understanding.

From a collaboration point of view, the team was trying to establish collaborative norms by starting tasks by exploring the pre-constructed figures and then reproducing those figures. In lines 27 and 29, Sophiak suggests explicitly that Gouri explores before she constructs. This team’s evolving collaborative norm seems to start with each member exploring and sharing noticings and then each member constructing the figures.

With regards to teachers’ actions towards solving the task, the teachers started by stating their noticings of the construction. In line 26, Sophiak mentions that point C is fixed and points A and B are not and states that points A and B are used to construct the two circles. She states that since dragging points A and B effects the circles then they are used in constructing the circles. It indicates how Sophiak views the relationship between dependency and construction and how she is starting to identify the hotspots of the figure. Her comment at line 26 seems to indicate that she is connecting prior experience (A and B’s independent role) with other tasks to what she notices about the size of the circles. The co-action—dragging points A and B with the change in the circumference of the circles—provides epistemic mediation since Sophiak acquires knowledge about the
relationship between the base points, A and B, and the circumference of the circles. The second team member, Gouri, takes control, agrees that point C is “fixed/constrained”, and tries to construct a similar figure. She successfully creates a similar figure to the task’s figure. In lines 33 to 40, Gouri describes to Sophiak the process of her construction, and following Sophiak’s suggestion in line 41, drags and tests Gouri’s construction and the pre-constructed figure. She states after dragging in lines 43 and 45 that “the points that make the circle dynamic and not the circle (in black) itself”. These comments suggest that Gouri was concerned with what is being dragged in a dynamic geometry environment and what makes it dynamic.

This event shows that the two teachers are distinguishing between dragging that affects other geometrical properties in addition to the location of an object–dragging the points that relates to the construction–and dragging that only affects the location of an object–dragging the circumference of a circle. The second teacher here is also showing her understanding of the hotspots of the figure. The DGE’s reaction informed the teachers’ dragging. The co-action between the teachers and the environment helped the teachers develop an understanding of the dragging functionality in DGE. This shows how teachers appropriate the environment through developing their understanding of dragging and dependencies. In this session, the teachers started to pay more attention to how constructions in DGE take place.

3.5.2 Identifying dependencies.

In the fifth week of the course, Team 3 worked on Task 16. The teachers were asked to drag dynamically different triangles and discuss the dependencies involved in the construction (see Figure 3.2).
The teachers show fluent use of dragging dependency while discussing the triangles. They stated that the first triangle does not have any dependencies involved in its construction. While discussing the second triangle (poly2), they dragged the vertices D, E, and F vigorously then checked the Algebra View in GeoGebra to look for more relationships among triangle DEF objects. Algebra View is an analytical view that shows some properties of the objects in the graphic view. It also shows the hidden objects and their properties (see Figure 3.3). Team 3 was able to use Algebra View to explore the objects and their relationships.

**Figure 3.2** Task 16: Triangles with Dependencies.
The following excerpt from Team 3’s work on Task 16 shows how the team effectively used dragging and Algebra View to identify dependencies involved in the construction of triangle DEF.

<table>
<thead>
<tr>
<th>#</th>
<th>User</th>
<th>Chat Post</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>Gouri</td>
<td>I don't really see many dependencies between the triangles in this case</td>
</tr>
<tr>
<td>12</td>
<td>Sophiak</td>
<td>I thought the same thing, poly 1 seems independent</td>
</tr>
<tr>
<td>13</td>
<td>Gouri</td>
<td>[dragged point E] Poly2 is completely different</td>
</tr>
<tr>
<td>14</td>
<td>Sophiak</td>
<td>What do you mean by different?</td>
</tr>
<tr>
<td>15</td>
<td>Gouri</td>
<td>[dragged point E] When I drag point E, point F is dependent in that it follows E</td>
</tr>
<tr>
<td>16</td>
<td>Sophiak</td>
<td>Does it move from either point F or D?</td>
</tr>
<tr>
<td>17</td>
<td>Gouri</td>
<td>[dragged points D, E, and F and made the circle that was used in the construction visible] Points E and F are on a circle</td>
</tr>
<tr>
<td>18</td>
<td>Gouri</td>
<td>they are radii and move accordingly</td>
</tr>
<tr>
<td>19</td>
<td>Sophiak</td>
<td>How did you notice that?</td>
</tr>
<tr>
<td>20</td>
<td>Gouri</td>
<td>[dragged points E and D] D is the center of the circle</td>
</tr>
<tr>
<td>21</td>
<td>Gouri</td>
<td>I went to algebra view and unhid the conic e</td>
</tr>
</tbody>
</table>
As mentioned above, the team states that the first triangle does not have dependencies in lines 11 and 12. In line 13 and 15, Gouri drags point E, an independent point, and notices that point F is changing, which makes point F dependent on point E. Then Sophiak asks about the other points in line 16. That question trigged the action by Gouri to drag all the vertices of the triangle. In lines 17 and 18, Gouri realizes that there is a hidden circle used in the construction and states that DE and DF “are radii and move accordingly” when changing the circle. In line 20, she states that point D is the center of the circle. The second teacher, Sophiak asks “How did you notice that?” In line 21, Gouri states that she used Algebra View to unhide the circle used in this construction. This allowed Sophiak to know that the triangle is an isosceles triangle and justifies that in line 26 saying “it will always be isosceles since DE and DF are radii of the same circle.” The team’s work on this task shows that this team is using VMTwG with its different component efficiently to explore mathematical objects and their relations and to justify those relations.

### 3.5.3 Constructing perpendicular lines.

Teachers’ understanding of dragging different types of objects, hotspots and other objects, in DGE helped them appropriate the environment, which influenced the type of knowledge that teachers developed in later sessions in the course. To further illustrate
this, we now discuss the teachers’ work on Task 21 (see Figure 3.4). Between Task 8 (constructing equilateral triangle) and Task 21, the teachers used the compass tool to copy line segments and to construct different types of triangles. Task 21 invited them to construct a perpendicular line that passes through an arbitrary point on a given line (see Figure 3.4).

In the preceding task, Task 20, the teachers constructed a line perpendicular to a give line (see Figure 3.5). In this task, the intent was to enable the teachers to develop insight and skill to solve Task 21. However, after working on Task 20, this team of teachers were unable to solve Task 21 in their first attempt. As Mason and Johnston-Wilder (2006) note that “What is intended, what is activated (implemented), and what is attained or constructed by the learners are often rather different” (p. 27). In written feedback, we suggested that in their next synchronous session they revisit Task 20 and try to use its technique to solve Task 21.
The teachers met again after three days to solve Task 21. To do so, they created a new GeoGebra workspace or tab within Task 21 and labeled it, “Task21-again” (see Figure 3.6) and successfully solved it. Interestingly, however, their solution did not rely on the solution of Task 20 in the way we anticipated. We intended that Task 20 would help the teachers see that one way to solve Task 21 is by constructing a circle that has the arbitrary point as center then mark the two intersection points of this circle with the given line to identify the radius for two other circles that will intersect in two points. Connecting those two intersecting points will create a perpendicular line that passes through the arbitrary point. Instead, the teachers develop another approach.

The teachers used some insights from Task 20 to construct perpendicular lines multiple times. They started by constructing a line AB and an arbitrary point C (see
Figure 3.6). Then using the technique from Task 20, they constructed a line EF perpendicular line to AB (constructed circles with common radius AB, marked their intersections points E and F, then hid the circles) and dragged points A and B to test the construction. On that line, they marked point G and, employing the Task-20 technique, used it and point E to construct line IJ perpendicular to EF, which make IJ parallel to AB. After that, they construct circle EC and marked the intersection point of this circle with line IJ, point K. They dragged point C to test the behavior of the construction. Finally, they construct line KC, which is perpendicular to AB and passes through the arbitrary point C.

![Image of GeoGebra interface showing the construction process](image)

**Figure 3.6** Team 3’s Solution to Task 21 In VMTwG.

Proving that KC is perpendicular to AB is beyond the scope of this study; however, it can be done easily using triangle congruency. The teachers struggled for about an hour to solve this task. They passed the control of GeoGebra to each other and
tried to make sense of each other’s actions. They referred to Task 20 a few times and discussed how they could use it in solving Task 21. They collectively constructed their final solution. After each step of their construction, they dragged points A, B, and C to make sure that at each stage their construction maintained properties they intended. Their appropriation of dragging—what to drag, how to drag, and what to expect—was dominant in their problem solving of Task 21.

3.6 Discussion

A team of two high school teachers was introduced to collaborative, online, dynamic geometry environment, VMTwG, in a 15-weeklong professional course. During this course, the members of this team interacted in VMTwG to notice variances and invariances of objects and relations in pre-constructed figures or figures that they constructed and to solve open-ended geometry problems. Our analysis of their interactions allowed us to understand how they appropriated the environment and how this appropriation influenced their geometrical knowledge. At the beginning of the course, the teachers started by focusing on appropriating the dragging affordance of DGEs. They paid special attention to the characteristics of the objects that being dragged. Their interactions indicate that they see the significance of dragging the hotspots of a construction (Hegedus & Moreno-Armella, 2010). The co-action of the VMTwG environment that occurs while they drag different objects in Task 8 helped the teachers identify the hotspots and use them to test their construction and become aware of dependencies. The need for more than wondering dragging (Arzarello et al., 2002; Baccaglini-Frank & Mariotti, 2010) in this task motivated the teachers to develop more purposeful dragging. They used maintain dragging to check if the triangle maintains its
properties and later on, drag to test the validity of their construction (the drag test).

Similarly, the teachers used combination of wondering dragging and maintain dragging to identify dependencies in Task 16. Their maintain dragging was more evident while trying to understand dependencies in the isosceles triangle.

Their process of appropriating VMTwG started with dragging and then looking at and understanding dependencies in dynamic constructions. Team 3’s interaction with VMTwG suggests that while appropriating DGE, constructing dynamic figures in DGEs comes last, after dragging and dependencies. The Team’s interaction also shows that the teachers attended to the analytical descriptions of objects in GeoGebra by using the Algebra View. They used this view in conjunction with explorations through dragging and visualization to satisfy their wondering and verify what they believed true.

While trying to solve Task 21, as mentioned above, the teachers referred to Task 20 multiple times and collectively constructed their solution for Task 21. They dragged the hotspots of their construction after each step. Their understanding of dragging hotspots helped them solve the task. While constructing a perpendicular line that passes through an arbitrary point, the teachers constructed a rectangle. Their procedure anticipates work they will be doing in a later task.

Understanding how instrumentation process occurs in DGEs informs implementing DGEs in learning and teaching mathematics. As Guin and Trouche (1998) found, instrumentation is a complex, slow process. The appropriation of dragging is an important aspect of transforming dynamic geometry tool into an instrument. The environment’s reaction to dragging helps users identifying the independent, partially
independent, and completely constrained points in any geometric figure, but users need to be attentive to dragging points and see what geometrical relations and properties remain invariant. It is also important to anticipate DGE to shape users’ knowledge by developing relational understanding of geometric notions.

Further research is needed to understand what other aspects of DGEs are important in the instrumentation process. Investigating how the appropriation of different aspects or tools of DGEs might influence learners’ knowledge is also needed. The instrumentalization process with DGEs is another important area of research since it can inform DGE design and implementation in teaching and learning mathematics. Understanding how learners use DGEs informs how researchers and educators design tasks for this environment and how to support learners’ appropriation of this technological tool. Additionally, research is needed to investigate how teachers implement DGE collaboratively in their teaching to support students’ strategic use of DGE tools and the discursive development of their geometrical ideas.

The next chapter builds on the results of this study. It includes an investigation of teachers’ mathematical activity in 15-week professional development course. These longitudinal data of teachers’ work in VMTwG were analyzed by identifying mathematical moments that teachers engaged in and link them to teachers’ actions in VMTwG. Similar to this study, the next study found that teachers relied on the dragging affordance of DGE to explore mathematical objects and relations. It also showed that teachers relied on other technological and cognitive tools (Algebra view and congruency and similarity) to explore and discuss the properties of mathematical objects and relations among them.
Chapter 4: Teachers’ Instrumental Genesis and Their Geometrical Understanding

Abstract: Investigating how learners appropriate technological tools while engaging in mathematical tasks provides an insight into how learners’ tool-mediated activities shape their mathematical knowledge. The theory of instrumental genesis (Lonchamp, 2012; Rabardel & Beguin, 2005) explains how learners appropriate technological tools and accomplish tasks while interacting with these tools. In this study, we investigate how teachers appropriate a collaborative, dynamic geometry environment, the Virtual Math Teams with GeoGebra (VMTwG), and how their appropriation shape their geometrical knowledge. The environment contains a chat panel and multiuser version of GeoGebra that teachers share. The participants are seven middle and high school mathematics teachers who participated in a 15-week professional development course in which they collaborated synchronously in VMTwG to solve geometrical tasks. We analyzed the work of one team of two high school teachers. Our analysis shows that the teachers used three technological and conceptual tools: dragging, Algebra view, and congruency and similarity of geometric objects. These tools influenced teachers’ understanding of geometrical properties and relations. Our study contributes understanding of the relations between appropriation of digital technologies and the construction of mathematical knowledge. It also indicates that further research is needed to understand how to orchestrate learners’ appropriation of collaborative technological tools in which teachers do not have conventional instructional role in the classroom.
4.1 Introduction

Understanding geometry is important in itself and for understanding other areas of mathematics. It contributes to logical and deductive reasoning about spatial objects and relationships and provides visual representations alongside analytical representation of a mathematical concept (Goldenberg, 1988a; Piez & Voxman, 1997). Pairing geometry learning with technological tools of Web 2.0 enables learners to investigate collaboratively geometric objects, properties, and relations and develop flexible understanding of geometry. In dynamic geometry environments (DGEs), learners can manipulate geometric objects through dragging and discuss different relations among those objects such as dependency. The different tools and affordances that DGEs provide can become part of learners mathematical discourse and used to think about and communicate geometrical ideas (Mariotti, 2000; Sinclair & Yurita, 2008). To be used in one’s mathematical discourse, these tools and affordances have to be appropriated by the learners through engaging in mathematical activities in DGEs. Mathematics teachers and educators need to be aware of this appropriation process and work on supporting learners’ active participation in mathematical activity.

Through active participation in mathematical activities with technological tools, learners can develop mathematical practices so that they “use appropriate tools strategically,” including dynamic geometry environments, which is underscored in the Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2010, p. 7). Using DGEs effectively allows learners to explore mathematical objects, conjecture about them, validate their conjectures, and justify their findings (Arzarello et al., 2002; Olivero & Robutti, 2007). For teachers to achieve this goal and
help their students use technological tools strategically, they must learn how to use these tools themselves by learning with them. This calls for carefully designed professional development experiences that help teachers learn actively and in turn improve their classroom practice (Garet, Porter, Desimone, Birman, & Yoon, 2001).

In addition to allowing students to manipulate geometric objects, integrating dynamic geometry environments (DGEs) into mathematics instruction has the potential to help them develop their geometrical reasoning and mathematical discourse (Mariotti, 2006; Stahl, 2015). In collaborative dynamic-geometry environments, learners can improve their collaborative practices while working with other students (Stahl, 2015). However, little is understood about how learners appropriate technological tools such as DGEs and how this appropriation informs their geometrical knowledge. Careful investigation is required to understand how learners appropriate DGEs and how it shapes their learning of mathematics. To contribute to this understanding, we describe how learners’ collaboratively interacted with online dynamic geometry tools to construct geometric objects, become aware of geometrical relations, and solve geometry problems. Our study responds to this question: Through working on sequences of geometrical tasks for 15 weeks, how do teachers appropriate an online, collaborative dynamic geometry environment and how does this appropriation shape their geometrical knowledge?

4.2 Literature Review

Researchers investigated the use of digital technologies in mathematics education for different purposes. Some researchers focus on investigating effects on teachers of introducing certain technologies in learning mathematics. For example, to investigate changes in teachers’ mathematical knowledge, Hohenwarter et al. (2009) engaged 44
middle and high school mathematics teachers in four workshops to learn about different geometric and algebraic topics in a dynamic geometry environment (DGE), GeoGebra. Similarly, using a different DGE, *The Geometer’s Sketchpad*, Sinclair and Yurita (2008) investigated changes in a teacher’s mathematical discourse shaped by his use of classroom Sketchpad. These studies did not focus primarily on how the teachers interacted with DGE, their main goals were to understand the influence of such technologies on mathematics learning and teaching.

Other researchers investigated how students interact with DGEs. Mariotti (2000, 2001) studied how a DGE, *Cabri*, mediates students’ activities to develop the meaning of proof. She researched high school students’ engagement in construction tasks and their internalization of the artifacts of Cabri’s environment. Through offering a system of signs, the environment played a role of semiotic mediation that allowed students to access the theory of geometry. Concerning proof, using a virtual collaborative environment that includes GeoGebra, Stahl (2015) found that it helped high school students develop geometrical proofs through four dimensions: visual recognition, analysis of geometrical properties, logical ordering, and deductive reasoning. Emphasizing the role of collaboration, his analysis of a small team of students who worked together for eight one-hour sessions, Stahl describes students’ improvement in collaboration, mathematical discourse, and their use of different modes of dragging.

Other studies focused on certain aspects of interacting with DGEs such as dragging geometric objects using a mouse on computer screens or using fingers on touchscreen devices. Arzarello et al. (2002) studied dragging action in DGE and cognitive processes behind each type of dragging. They identified two cognitive modes
linked to dragging: ascending (moving from drawings to theory) and descending processes (moving from theory to drawings). Ascending processes allow users to investigate the drawings freely to look for patterns and invariants. Descending processes are used with a theory in mind to validate or test properties. Within those two cognitive modes, Arzarello et al. (2002) found seven types of dragging: wondering dragging, bound dragging, guided dragging, dummy locus dragging, line dragging, linked dragging, and dragging test. Building on this work, Baccaglini-Frank and Mariotti (2010) developed a model to explain cognitive processes behind different types of dragging. They used four different types of dragging: wondering dragging, maintaining dragging, dragging with trace activated, and dragging test. Wondering dragging is dragging that aims to look for regularities while maintaining dragging is dragging a base point so that the dynamic figure maintain certain properties. Dragging with trace activated is dragging a base point with its trace activated. Drag test is dragging base points to test whether certain properties will meet certain conditions (Baccaglini-Frank & Mariotti, 2010). To understand how dragging relates to students’ geometrical thinking while using touchscreen devices, Arzarello, Bairral, and Danè (2014) investigated the types of screen touching high school students used while solving geometrical problems in a DGE. They identified two domains of manipulations: constructive (tap and hold to construct objects) and relational (including the constructive manipulations, students drag, flick, free or rotate objects). In the relational domain, students were “more focused on their questioning, conceptual understanding and other emergent demands concerning their manipulation of a whole construction” (p. 50).

A final group of studies examined the instrumental transformation of
technological tools into instruments that mediate learners’ activity. For instance, Guin and Trouche’s (1998) investigated the instrumentation process, one of two processes of instrumental genesis, that group of students used to transform graphing and symbolic calculators into mathematical instruments. They concluded that the instrumentation process is complex and slow and that interacting with a technological tool to learning mathematics might not result in transforming the tool into an instrument.

Based on this review, it appears that there is need for longitudinal studies that investigate how users, through the instrumentation process, interact with collaborative DGEs working on coherent sequences of geometrical tasks. Only one longitudinal study (Stahl, 2015) examined learners working on a coherent set of geometrical tasks in a collaborative environment with tools of dynamic mathematics. Stahl’s study was part of our research project and mainly focused on the collaborative aspect of learners’ mathematical activity. Learners’ interactions with DGEs shape their thinking and learning of geometry (Hegedus & Moreno-Armella, 2010; Rabardel & Beguin, 2005), which makes investigating how learners appropriate an online, collaborative dynamic geometry environment important. In our study, collaboration is an essential feature of the dynamic geometry environment. Learners meet synchronously to work in small teams on geometrical tasks. We designed coherent set of tasks for a 15-week professional development course to allow us to investigate longitudinally the development of learners’ interactions and the extension of their geometrical knowledge. We believe that our study can help mathematics educators further understand how learners interact with DGEs and how DGEs shape learners’ geometrical understanding.
4.3 Theoretical Perspective

To understand learners’ appropriation of technological tools, we draw on a Vygotskian perspective about goal-directed, instrument-mediated action and activity. Instrumental genesis (Lonchamp, 2012; Rabardel & Beguin, 2005) theorizes how learners interact with tools that mediate their activity on a task. To appropriate a tool, users (teachers, students, or learners in general) develop their own knowledge of how to use it, which turns the tool into an instrument that mediates an activity between users and a task. The basic concept of the theory is that users engage in an activity in which actions are performed upon an object (matter, reality, object of work…) in order to achieve a goal using an artifact (technical or material component). Rabardel and Beguin (2005) emphasize that the instrument is not just the tool or the artifact, the material device or semiotic construct, it is “a mixed entity, born of both the user and the object: the instrument is a composite entity made up of an artifact component and a scheme component.” (p. 442). An instrument is a two-fold entity, part artifactual and part psychological as utilization schemes. The user acquires a utilization scheme and applies it to the artifact.

Artifacts are subject to two kinds of utilization schemes. The first kind of utilization schemes is usage schemas, which are directly related to the artifact. It constitutes the basic knowledge of how to operate or use the artifact. For example, driving a car for an experienced driver such as changing gears or turning the steering wheel, or being familiar with the components of a digital camera and knowing how to use them. The second kind of utilization schemes is instrument-mediated action schemes, which are more related to the transformations that can be done to the object. These
schemes are concerned with the activity, which can be individual or collective, that will lead the users, using the artifact, to reach a desired goal. For the example of driving a car, the instrument-mediated action scheme will be more focused on the other variables on the road that a driver needs to be aware of and react to their existence to be able to reach the final destination (Lonchamp, 2012; Rabardel & Beguin, 2005).

Just interacting with an artifact is not an instrument-mediated activity. In instrument-mediated activity, instruments mediate users’ activity or action to achieve a certain goal. While engaging in an activity, users monitor consciously the continuous transformation of an object towards their goal. This mediator role that instruments play governs the user-object relations, which might take epistemic or pragmatic forms. The epistemic mediation form focuses on the object and its properties. In this form, the instrument helps the user understands the object and its structure. On the other hand, in the pragmatic mediation form, the user is mainly concerned with the required actions while using the instrument to transform the object into the desired final result (Lonchamp, 2012; Rabardel & Beguin, 2005). The final result is the final transformation of the object, which might not match exactly the initial goal. The user may find certain form of the transformed object satisfying enough and, therefore, end the activity.

During an activity that is mediated by an instrument, it is understandable that the artifact affects the activity; however, users play a major role in shaping the activity. The users’ interactions with an artifact shape the activity. Two different users can approach an artifact differently, develop different utilization schemes, and create two different activities and instruments.
The transformation of an artifact or tool into an instrument, or instrumental genesis, occurs through two important dialectical processes that account for potential changes in the instrument and in the learners, instrumentalization and instrumentation. The instrumentalization process is defined as “the process in which the learner enriches the artifact properties” (Rabardel & Beguin, 2005, p. 444). In this process, the user selects and modifies the properties of the artifact, for example, using a wrench as a hammer. The second process of instrumental genesis is instrumentation. This process is about the development of the learner side of the instrument. The development of the learner is basically the assimilation of an artifact to a scheme and the adaptation of utilization schemes. With the example of using a wrench as a hammer, the learner already had acquired the utilization scheme of a hammer and when a hammer was not available at the time of the action, the learner chose the wrench and associated it to the hammer utilization scheme. This is an example of “direct assimilation of artifact into a utilization scheme” (Rabardel & Beguin, 2005, p. 446), which changes the meaning of the artifact. During the act of assimilation, the learner employs previous utilization schemes to new artifact. In our example, acquiring the hammer utilization scheme led the learner to choose the wrench and not another tool because he is aware of the functions of the hammer and its structure which makes him look for a similar tool that can take the same scheme. In the situation where a new artifact cannot be assimilated to previously acquired utilization scheme, the learner adapts utilization schemes and makes the necessary modifications to it.

Through the two processes of instrumental genesis, instrumentation and instrumentalization, dialectically the tools influence the thinking of the learner and the
learner influences the design of the tools. On the one hand, the structure and functionality of tools shape how the learner uses the tool, which result in shaping the learner’s thinking. On the other hand, the learner’s interactions with the tool also shapes the tool and how is used.

With dynamic geometry environments (DGEs), the feedback that the software gives to the user after manipulating dynamic objects affects the user’s interaction with the software. The environment reacts to the users’ actions through engineered infrastructure that responds to the theory of geometry. This reaction can inform the users’ actions and can shape users’ thinking. Dragging the “hot-spots” of a dynamic figure can change the geometrical properties of the figure and can provide insights into its construction process. Hot-spots are “points that can be used to construct mathematical figures, e.g. join two points with a segment, or construct a piecewise graph, and then used to dynamically change the construction.” (Hegedus & Moreno-Armella, 2010, p. 26).

The relationship between the user and the DGE is a result of co-action between the two (Hegedus & Moreno-Armella, 2010). The notion of co-action has two sides: (a) the user’s action can guide DGE and (b) DGE’s reaction can guide the user. A dynamics software environment allows users to act on it and, in turn, reacts to their actions. As users drag (click, hold, and slide) a hotspot of a geometric figure, the DGE redraws and updates information on the screen, preserving all constructed mathematical relations among objects of the figure. In redrawing, the DGE creates a family of not only visually but also mathematically similar figures. Users may then attend to the reaction of the DGE and experience and understand underlying mathematical relations such as dependencies. DGEs “remember” underlying mathematical relations among various objects of a
construction. For instance, if a point P is the midpoint of a segment AB, then as the length or position of segment AB changes, P’s relationship to AB remains invariant, namely that P is equidistant from the line segment’s endpoints A and B.

Dynamic geometry environments can be seen as a tool that learners can appropriate through the instrumentation process. Learners will need to acquire utilization schemes – usage schemes and instrument-mediated activity schemes – to appropriate the tool. In DGE, the usage schemes include knowledge about the software use and its functionalities. The second level of utilization schemes for a DGE includes knowledge of geometry and dependencies. When learners appropriate a DGE as an instrument, they will be able to use it to demonstrate geometrical concepts and solve geometrical problems. This appropriation may result in knowledge of how to use dynamic geometry software as well as knowledge of geometry. The geometrical knowledge can be a special type of knowledge shaped by DGE. Within DGEs, Straesser (2002) sees that geometry is “lived in differently, broader scope, has a new, more flexible structure, [and] offers easy access to certain heuristic strategies.” (p. 331). Balacheff and Kaput (1996) claim that characteristics of DGEs result in creating new mathematics, a geometry that is different from Euclidian geometry in the plane.

4.4 Methodology

Data come from a project that integrates a cyberlearning environment with digital tools for collaborative geometrical explorations grounded in a pedagogical approach that engages learners in developing significant mathematical discourse. The project investigates learners’ actions as they occur through an iterative coevolution of the technology and curricular resources in the context of engaging, reflective collaborative
learning experiences of significant mathematical discourse by in-service teachers and their students. Our view of mathematics and mathematical discourse drives our design and data generating process. From a psychological perspective, Gattegno (1987) posits that doing mathematics is based on dialog and perception:

No one doubts that mathematics stands by itself, is the clearest of the dialogues of the mind with itself. Mathematics is created by mathematicians conversing first with themselves and with one another. Still, because these dialogues could blend with other dialogues which refer to perceptions of reality taken to exist outside Man…Based on the awareness that relations can be perceived as easily as objects, the dynamics linking different kinds of relationships were extracted by the minds of mathematicians and considered per se. (pp. 13-14)

Mathematics results when a mathematician or any interlocutor talks to herself and to others about specific perceived objects, relations among objects, and dynamics involved with those relations (or relations of relations) (Gattegno, 1987). For dialogue about these relations and dynamics to become something that can be reflected upon, it is important that they not be ephemeral but rather have residence in a material (physical or semiotic) record or inscription. On the one hand, through moment-to-moment discursive interactions, interlocutors can create inscriptions and, during communicative actions, achieve shared meanings of them. Voiced and inscribed mathematical meanings arise through discursive interactions, discussions. Pirie and Schwarzenberger (1988) define a mathematical discussion involving learners this way: “It is purposeful talk on a mathematical subject in which there are genuine pupil contributions and interaction” (p. 461). From a sociocultural perspective, we understand “purposeful talk” as goal-directed discourse and “on a mathematical subject” as about mathematical objects, relations and dynamics of relations. In the setting of our online, collaborative environment with interlocutors—teams of pupils, students, or teachers—collaborating and usually without the contemporaneous presence of a teacher, the discursive contributions and interactions
genuinely emanate from the interlocutors. As such, we define productive mathematical discourse to be goal-directed discursive exchanges about mathematical objects, relations, and dynamics of relations, including questioning, affirming, reasoning, justifying, and generalizing.

To engage participates in a productive mathematical discourse, we design tasks that ask learners to mainly discuss mathematical objects, relations, and relations among those relations. We focus on mathematical knowledge that we believe can be derived through careful investigations. Following Hewitt’s (1999) notion of curriculum, we focus on the ideas or properties that can be derived by attending to and noticing mathematical relations (necessary) rather than focusing on the semiotic conventions such as names, labels, and notations (arbitrary).

In our project, teachers engage in two courses of professional development. The first course, usually in the fall, is designed for middle and high school mathematics teachers to collaborate in an online synchronous DGE to solve open-ended geometrical problems. In the second course, the participating teachers engage their students with the same collaborative DGE. Our professional development design attends to the result of Garet et al. (2001) study, in which they investigated the effects of professional development courses on 1,027 mathematics teachers. They concluded with a set of professional development characteristics that influence teachers’ learning. The characteristics are:

1. Focus on specific content
2. Engage teachers in active learning
3. Consist of coherent set of activities
4. Allow for collaboration among teachers

5. Span over substantial number of hours (Garet et al., 2001).

In our design, we engage teachers with a coherent set of activities about dynamic geometry in the first course. The teachers work in small team (two to four teachers in each team) and meet online for over than 60 hours. We provide weekly feedback for each team to ensure that each participant is actively engaged in the session through manipulating dynamic geometry objects and teams’ discussions. Each team reports in Blackboard to the whole class a summary about their weekly discussion, which allow for second level of collaboration among the teams. In the second professional course, the teachers have the opportunity to collaborate with other teachers and share their challenges while engaging their students with DGEs. This design allows for more effective professional development courses.

4.5 Methods

The data for this paper come from an online professional development course for middle and high school teachers that occurred in 15 consecutive weeks during the second half of 2014. In small teams, seven New Jersey mathematics teachers engaged in interactive, discursive learning of dynamic geometry through collaborating to solve tasks in a computer-supported, collaborative-learning environment: Virtual Math Teams with GeoGebra (VMTwG). Using VMTwG and an online discussion board, they also read and discussed articles about collaboration (Mercer & Sams, 2006; Rowe & Bicknell, 2004), mathematical practices (Common Core State Standards Initiative, 2010), accountable talk (Resnick et al., 2010), technological pedagogical content knowledge (Mishra & Koehler, 2006), implementing technology in mathematics classroom (McGraw & Grant, 2005),
and validating dynamic geometry constructions (Stylianides & Stylianides, 2005). They also analyzed and discussed logs of their VMTwG interactions to negotiate discursive norms and to examine, reflect, and, perhaps, modify their norms and their collaborative and mathematical practices.

VMTwG, a product of a collaborative research project among investigators at Rutgers University and Drexel University, is an interactional, synchronous space. It contains support for chat rooms with collaborative tools for mathematical explorations, including a multi-user version of GeoGebra, where team members can define dynamic objects and drag their hotspots around on their screens (see Figure 4.1). VMTwG records users’ chat postings and GeoGebra actions. The research team designed dynamic-geometry tasks that encourage participants to discuss and collaboratively manipulate and construct dynamic-geometry objects, notice dependencies and other relations among the objects, make conjectures, and build justifications.

For this study, we analyze the work of Team 3, which consists of two high school mathematics teachers. Before this course, both teachers did not have any experience with dynamic geometry. The teachers met in VMTwG synchronously for two hours twice a week. We selected this team’s data since its members were expressive of their ideas and object manipulations while working in VMTwG. To understand how teachers interact with VMTwG environment and how the environment shapes their geometrical knowledge, we used the discursive data generated from their work on 55 tasks. The tasks are organized to help the teachers get familiar with the environment in the first three weeks of the semester then shift to involve more mathematical notions. The tasks deal with Euclidian geometry including constructing different triangles, similarity and
congruency in triangles, centers of triangles, transformations, constructing quadrilaterals, relations among quadrilaterals, and open-ended geometrical problems. In general, the tasks ask teachers to explore geometric objects, constructions process, relations among objects, and justify these relations. The tasks are organized in Modules, and the teachers collaborate in each session on one Module, which consist of two to six tasks.

Teachers’ interaction in VMTwG generates discursive data that include the logs of teachers’ chat communications, their GeoGebras interactions, and VMTwG system actions (logging into the system, taking and releasing control of GeoGebras, and switching from one task to another). Using conventional content analysis (Hsieh & Shannon, 2005), we analyzed their discursive data to understand how the teachers interacted with VMTwG and appropriated it. We read all the chat that teachers generated in 15 weeks while working on 55 tasks. We highlighted the moments when teachers were discussing a mathematical idea (not chatting about logistical issues or other technological issues). After highlighting those moments, we looked for ways that teachers, as a team, use to approach the tasks. We were interested in the tools (mathematical ideas and technological aspects of the environment) that helped the teachers solve the task. For example, for some tasks, teachers rely on dragging to understand dependencies among objects. To understand how teachers interacted with VMTwG, we identified the tools that they implement when working on different tasks. We looked at how the team as a unit approaches the tasks.

4.6 Results

Our analysis focuses on understanding how teachers interacted with VMTwG to solve geometrical problems throughout the 15 weeks. Our results show how teachers’
interaction with VMTwG shapes their geometrical understanding. We identified three main categories of actions that the teachers of Team 3 demonstrated while working on geometrical tasks: dragging, Algebra view, and congruency and similarity among objects. The first two categories link directly to the environment. The third category, congruency and similarity, is a conceptual tool linked to affordances of the environment (through checking measurements in Algebra view, dragging, or overlapping objects to check congruency and similarity). To solve open-ended problems and construction tasks, Team 3 relied on affordances of dragging and Algebra view in GeoGebra and congruency and similarity among geometric objects. We present evidence of each category of teachers’ actions and the consequences of appropriating dragging, of using algebra view, and of investigating relations among objects using notions of congruency and similarity.

4.6.1 Appropriating dragging.

In line with an affordance of VMTwG environment, teachers relied heavily on dragging to understand geometrical relations among objects and their construction process. Throughout the semester, teachers showed improvement in their dragging through identifying figures’ hotspots and indications of points’ colors. Their appropriation of dragging allowed them to drag more efficiently (fewer moves while testing objects) and to understand construction process accurately. At the end of the semester, teachers were able to state the figures’ dependencies after looking at the points’ colors and before dragging them. This indicates that teachers’ appropriated dragging through the co-active nature of GeoGebra, hotspots, and the colors of the figures’ points.

To shed a light on teachers’ appropriation of dragging, we present their work on few tasks to show how their understanding of dragging evolved. Their work on Task 8 in
the fourth week of the semester showed how they attended to the hotspots. Task 8 asks teachers to drag objects, to discuss (in the chat window) what they notice about the given figure, and then to construct a similar one (see Figure 4.1). Among other things, as they dragged hotspots, previous tasks engaged them in noticing variances and invariances of objects and relations of figures.

![Figure 4.1 Task 8: Constructing Equilateral Triangle.](image)

This task was intended to extend the teachers’ experience with dragging and geometrical dependencies. Before this session, the teachers have already worked on some basic geometric objects (such as lines, line segments, circles, and circles whose radius is dynamically dependent on a line segment) and were asked to drag and notice relationships among the objects. Those tasks mainly aimed to familiarize the teachers with the functionality of the VMTwG environment and to the cognitive habit of noticing and wondering about the behavior of object and relations among objects (Powell &
Alqahtani, 2015a). That is, the tasks engaged the teachers in becoming aware of co-active relations between their actions and reactions of the VMTwG environment. Below, their chat posting shows that they focused on relationships that were visually apparent. It also shows that the teachers, Gouri and Sophiak, felt the necessity to revisit their understanding of dragging after being instructed to create an equilateral triangle.

<table>
<thead>
<tr>
<th>#</th>
<th>User</th>
<th>Chat Post</th>
</tr>
</thead>
<tbody>
<tr>
<td>26</td>
<td>Sophiak</td>
<td>It seems that point C is fixed but pts A&amp;B are not. I am thinking somehow A&amp;B were used to create the circles which is why the make the circles bigger or smaller.</td>
</tr>
<tr>
<td>27</td>
<td>Sophiak</td>
<td>How about you try to explore now?</td>
</tr>
<tr>
<td>28</td>
<td>Gouri</td>
<td>ok I'll continue on with #2 [the second instruction in Task 8] as well</td>
</tr>
<tr>
<td>29</td>
<td>Sophiak</td>
<td>No, I would like to create the objects as well. I think it is valuable if we both explore</td>
</tr>
<tr>
<td>30</td>
<td>Gouri</td>
<td>C does seem fixed/constrained</td>
</tr>
<tr>
<td>31</td>
<td>Gouri</td>
<td>sure - how about i do it and then you do it as well after?</td>
</tr>
<tr>
<td>32</td>
<td>Sophiak</td>
<td>Sounds good. Please type what you do.</td>
</tr>
<tr>
<td>33</td>
<td>Gouri</td>
<td>So far I created 2 circles</td>
</tr>
<tr>
<td>34</td>
<td>Gouri</td>
<td>and overlapped the D point as the raius point for E</td>
</tr>
<tr>
<td>35</td>
<td>Gouri</td>
<td>one more try</td>
</tr>
<tr>
<td>36</td>
<td>Gouri</td>
<td>ok - i deleted the other circle because i dont need it</td>
</tr>
<tr>
<td>37</td>
<td>Gouri</td>
<td>I somehow thought i could create all 3 points, abc through two circles</td>
</tr>
<tr>
<td>38</td>
<td>Sophiak</td>
<td>How did you create F?</td>
</tr>
<tr>
<td>39</td>
<td>Gouri</td>
<td>I added a point</td>
</tr>
<tr>
<td>40</td>
<td>Gouri</td>
<td>then the polygon tool for the triangle</td>
</tr>
</tbody>
</table>
Sophiak  Did you want to explore your picture to see if it behaves the same way as the original?

Gouri  ok

Gouri  [after dragging for few minutes] I noticed that it's the points that make the circle dynamic

Gouri  and not the circle (in black) itself

Our analysis of this excerpt reveals two aspects of this team’s instrumentation process: collaboration and tool use, which parallels their mathematical understanding.

From a collaboration point of view, the team was trying to establish collaborative norms by starting tasks by exploring the pre-constructed figures and then reproducing those figures. In lines 27 and 29, Sophiak suggests explicitly that Gouri explores before she constructs. This team’s evolving collaborative norm seems to start with each member exploring and sharing noticing and then each member constructing the figures.

Concerning teachers’ actions towards solving the task, the teachers started by stating their noticing of the construction. In line 26, Sophiak mentions that point C is fixed, that points A and B are not, and that points A and B are used to construct the two circles. She states that since dragging points A and B effects the circles then they are used in constructing the circles. It indicates how Sophiak views the relationship between dependency and construction and how she is starting to identify the hotspots of the figure. Her comment at line 26 seems to indicate that she is connecting prior experience (A and B’s independent role) with other tasks to what she notices about the size of the circles.

The co-action—dragging points A and B with the change in the circumference of the circles—provides epistemic mediation since Sophiak acquires knowledge about the relationship between the base points, A and B, and the circumference of the circles. The
second team member, Gouri, takes control, agrees that point C is “fixed/constrained”, and tries to construct a similar figure. She successfully creates a similar figure to the task’s figure. In lines 33 to 40, Gouri describes to Sophiak the process of her construction then, and following Sophiak’s suggestion in line 41, drags and tests Gouri’s construction and the pre-constructed figure. She states after dragging in lines 43 and 45 that “the points that make the circle dynamic and not the circle (in black) itself”. These comments suggest that Gouri was concerned with what is being dragged in a dynamic geometry environment and what makes it dynamic.

This event shows that the teachers are distinguishing between dragging that affects other geometrical properties in addition to the location of an object–dragging the points that relates to the construction–and dragging that only affects the location of an object–dragging the circumference of a circle. The second teacher here is also showing her understanding of the hotspots of the figure. The DGE’s reaction informed the teachers’ dragging. The co-action between the teachers and the environment helped the teachers develop an understanding of the dragging functionality in DGE.

In the seventh week of the semester, teachers worked on Task 20, which deals with constructing perpendicular line (see Figure 4.2). The following excerpt of the teachers’ chat posting shows how teachers started to work on Task 20.

<table>
<thead>
<tr>
<th>#</th>
<th>User</th>
<th>Chat Post</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>Sophiak</td>
<td>I am going to explore first</td>
</tr>
<tr>
<td>13</td>
<td>Gouri</td>
<td>ok</td>
</tr>
<tr>
<td></td>
<td>Sophiak</td>
<td>[drags points A and B in Task 20 ]</td>
</tr>
</tbody>
</table>
As I thought, points D, E, C are all fixed and dependent on points A and B.

I am thinking that points D, E, C were created using the point of intersection tool.

Those are my observations, would you like to play with it before doing the construction?*

[drags points F and G in Task 20]

I realize I left out F and G. Sorry. *

I can try exploring the circle a bit

okay

[drags point A in Task 20]

I agree with your observations so far*

Should we start constructing?

The teachers started this task by dragging the hotspots on the figure, points A and B. The first teacher, Sophiak, immediately dragged only points A and B, which indicates some understanding of the hotspots of the figure. She did not drag a lot (as they used to do at the beginning of the semester) and did not repeat her dragging. She started dragging point A and then dragged point B. At line 14, she states that “points D, E, C are all fixed and dependent on points A and B”. She started her sentence with “As I thought”, which indicates that she has guessed the dependencies in the construction before dragging any objects. Working on a similar task, Task 8, and her understanding of hotspots allowed her to anticipate the dependencies in the construction and minimize her dragging of the objects to test that anticipation. The second teacher, Gouri, asks for control to explore the construction in line 18 then agrees in line 20 with Sophiak’s observations. The team
moves to construct similar figure after their exploration. They successfully create their construction with similar dependencies.

**Figure 4.2** Task 20: Constructing Perpendicular Line.

Later in the same session, the teachers worked on Task 21 to construct perpendicular lines that passes through an arbitrary point. Task 21 presents a line FG that is defined by two independent points, F and G. Another point, point H, is created to move alone the line FG. The task asks the teachers to construct perpendicular line to FG that passes through point H. This team of teachers started the task by discussing the construction dependencies. The following expert shows parts of their discussion.

<table>
<thead>
<tr>
<th>#</th>
<th>User</th>
<th>Chat Post</th>
</tr>
</thead>
<tbody>
<tr>
<td>401</td>
<td>Sophiak</td>
<td>Is G dependent, fixed?</td>
</tr>
</tbody>
</table>
In this excerpt, Sophiak asks in line 401 whether point G is dependent to which Gouri replies with “idk - good question… from the color it's not dependent” in lines 402 and 403. The independent points in GeoGebra usually have blue color. Since point G was blue, Gouri was able to anticipate its dependency. She confirms later on in line 409 that points F and G are independent points.

### 4.6.2 Algebra View.

In addition to the graphical representation of geometric and algebraic objects, GeoGebra provides Algebra view that shows analytical representations of the objects in the graphics area (see Figure 4.3). In this view, users can see more details of the hidden and the visible objects in the graphics view. For example, it shows points locations in the coordinate plane, lines equations, line segments lengths, circles equations, and the areas of polygons. It also indicates the visibility status of each object.
Teachers from Team 3 discovered Algebra view in the third week of the semester. The course instructions did not include exploring the Algebra view or using it while working on the tasks. This discovery influenced the team’s work throughout the semester in many ways. They referred to Algebra view while trying to generate and test conjectures, understand the construction and relations of objects, and to understand the objects’ behavior when dragged. For these purposes, they used the measurements that Algebra view provides and looked for the hidden objects that were used in the construction. We provide few examples of instances where teachers used Algebra view and how that influenced their mathematical understanding.

In the fourth week of the course, the teachers worked on Task 8 (Figure 4.1). They tried to test whether triangle ABC is an equilateral. They dragged points A and B
and constructed another equilateral triangle, DEF, in a similar way with DE as a radius for both circles. While trying to justify why triangles ABC and DEF are equilateral, they had the following discussion:

<table>
<thead>
<tr>
<th>#</th>
<th>User</th>
<th>Chat Post</th>
</tr>
</thead>
<tbody>
<tr>
<td>69</td>
<td>Gouri</td>
<td>It [triangle DEF] is definitely an equilateral triangle like the activity suggests</td>
</tr>
<tr>
<td>70</td>
<td>Sophiak</td>
<td>I don't know what you mean, please clarify</td>
</tr>
<tr>
<td>71</td>
<td>Gouri</td>
<td>sincere all three sides are radii</td>
</tr>
<tr>
<td>72</td>
<td>Sophiak</td>
<td>Can only 1 of us type at a time, I am feeling like our conversation is a bit out of sync.</td>
</tr>
<tr>
<td>73</td>
<td>Sophiak</td>
<td>I pulled the algebra view menu &amp; could see that the lengths &amp; yes this is an equilateral triangle for sure</td>
</tr>
<tr>
<td>74</td>
<td>Sophiak</td>
<td>I am done but I think we need to discuss why it equilateral dynamically. Done typing :)</td>
</tr>
<tr>
<td>75</td>
<td>Sophiak</td>
<td>Referring to your construction, I am thinking since FE, DE, FD are all radii of a 2 circles that are congruent therefore the triangle they make is equilateral.</td>
</tr>
<tr>
<td>76</td>
<td>Gouri</td>
<td>yes the circles are congruent because they share radius DE</td>
</tr>
<tr>
<td>77</td>
<td>Sophiak</td>
<td>Also, when creating the 2 circles it is necessary to create the in a way that makes them dependent on each other &amp; congruent.</td>
</tr>
<tr>
<td>78</td>
<td>Sophiak</td>
<td>To your point, that is why when constructing the 2 circles the 2nd must be done in a way that it shares the same radius with the 1st circles. Does that make sense?</td>
</tr>
<tr>
<td>79</td>
<td>Gouri</td>
<td>yes - spot on!</td>
</tr>
<tr>
<td>80</td>
<td>Gouri</td>
<td>it's enforcing a dependency</td>
</tr>
</tbody>
</table>
In lines 69 and 71 of this excerpt, Gouri justifies why triangle ABC and DEF are equilateral. However, Sophiak initially does not understand Gouri’s justification and, in line 70, asks her to clarify. Before receiving any clarification, Sophiak uses Algebra view to make sure the sides of the triangles are equal (line 73). Even though Gouri stated a justification in line 71, Sophiak states in line 74 that their observation needs a justification. She successfully justifies it in line 75 and Gouri agrees in line 76 and adds that the circles are congruent because they share a radius DE. The Team moves on to discuss the construction of the figure and how that is an important part of having an equilateral triangle. The use of Algebra view allowed Sophiak to verify their observation and move to justify it. Even though Gouri was able to observe and justify the relationships among the triangle side, Sophiak needed to see it herself and investigate why that holds after dragging.

Team 3 used Algebra view to determine the measure of lengths and angles and the slopes of lines. While working on Task 23 (see Figure 4.4) to test different combinations of sides and angles of similar and congruent triangles, Team 3 used Algebra view to find angle measures. They stated that they needed to check the angle measures to decide whether two triangles are congruent. They decided to use Algebra view to check angles and see if they change with dragging. In this task, congruent angle of different triangles have the same Greek letter. In the beginning of doing the task, this naming feature was not clear for the teachers, which is why they used the Algebra view to determine which angles were congruent. Later in Task 23, the teachers stopped using Algebra view and used the angle names (Greek letter) as an indication for angle congruency.
Figure 4.4 Task 23: Triangles Congruency and Similarity.

The last measurement use for Algebra view was to find the slope of lines. In Task 45, the teachers were asked, without using GeoGebra tools for perpendicular and parallel lines, to construct perpendicular and parallel lines. They constructed perpendicular lines using a technique they learned in Task 20. They discussed how to create parallel lines and one teacher stated that they could “use the perpendicular line to construct the parallel line”. Instead of creating perpendicular lines, the teachers used the intersection points of three congruent circles as in Figure 4.5. However, at first, they were not sure whether the lines were parallel. One teacher, Gouri, who did the construction said: “I am wondering if JK in this construction is parallel to IGH”. The other teacher, Sophiak, used Algebra view to check the equations of the lines and stated that “looking at the algebra view lines JK (i in algebra view) and IH (f in algebra view) have the same slope so yes they are parallel”. Afterward, they discussed why their construction creates two parallel lines. They
concluded that they both understand how it is constructed and that the third circle is important for constructing the parallel line. Using Algebra view allowed the teachers to verify their construction and then to discuss the construction process.

![Image of the Algebra view](image.png)

**Figure 4.5** Teachers’ work on Task 45.

The second way Team 3 used Algebra view was to determine hidden objects in the construction. Determining hidden objects allowed the teachers to understand the behavior of other objects when dragged (dragging dependencies), construction processes (construction dependencies), and test and justify conjectures such as object properties and relations among objects. For example, while teachers worked on Task 29, where they reflected objects about a line, they had to drag and explore dependencies between two triangles, triangle ABC and XYZ. Triangle XYZ is a reflection of triangle ABC about a hidden line. The following expert is part of their discussion:

# User Chat Post
as I click and drag points abc xyz shift accordingly
Okay, may I explore now?
yes...pls explore.. in algebra view i saw
some hidden objects which included a vector and some other objects that i could not get to show
this may be due to the hint boxes remainig unchecked
It definitely seems like there is a line between the 2 triangles and BAC is the preimage
BAC was reflected over this line to form ZYX
I am willing to try creating our own objects if you are.

In line 73, after dragging the vertices of triangle ABC, Gouri notices that triangle XYZ shifts “accordingly”; that is, it behaves in some relation to triangle ABC. She checks the Algebra view to understand that behavior. The other teacher, Sophiak, says that there is a line between the triangles and that triangle ABC is the preimage. She sees triangle XYZ as an image of triangle ABC. In line 79, Sophiak says “BAC was reflected over this line to form ZYX”. Their reference to Algebra view helped them understand why triangle XYZ moves when triangle ABC is dragged. In another task, Task 36, Team 3 used Algebra view to understand their own construction of the inscribed circle of a triangle. In Task 36, the teachers were asked to construct a triangle and its incenter then create the shortest distance between the incenter and the sides. Afterward, they were ask to create the inscribed circle using the shortest distance as a radius and the incenter as a center. After few attempts, they successfully created the inscribed circle (see Figure 4.6).
To create the inscribed circle, the teachers needed to create the incenter of the triangle, which is the intersecting point of the angle bisectors of the triangle’s angles. This step resulted in creating three lines and their intersection point, point D. When teachers started creating the shortest distance between point D and the triangle’s sides, the lost track of the lines and the points that were created. One teacher, Gouri, said: “I am not sure, what is F and D”. The other teacher, Sophiak, replied that those are intersections points and D is the incenter point. Then she said: “this was making a lot more sense to me in algebra view” which indicates that she has been reading Algebra view to understand the construction. Responding to Gouri’s confusion by saying that Algebra view makes a lot more sense could mean that she was trying to suggest that Gouri should do that same and use Algebra view to understand the construction. In this task, Algebra view has direct influence on teachers’ understanding of the task and the geometrical relations.

Figure 4.6 Teachers’ work on Task 36.
The last purpose for Team 3 to use Algebra view was to test conjectures. The example for this use is Team 3’s work on Task 52. In this task, the teachers create a parallelogram ABCD and its angle bisectors. Using the angle bisectors intersection points, the teachers were asked to create a quadrilateral, EFGH. Then the teachers to drag parallelogram ABCD and explore what happens to quadrilateral EFGH. One task question asks about conditions that cause quadrilateral EFGH to be a square. The teachers successfully created the construction and dragged it vigorously to explore quadrilateral EFGH (see Figure 4.7).

Figure 4.7 Teachers’ work on Task 36.

While trying to explore quadrilateral EFGH, they dragged parallelogram ABCD and determined that quadrilateral EFGH is always a rectangle and the angle bisectors of parallelogram ABCD intersect in one point when two consecutive side of ABCD are equal. The teachers explored the case when quadrilateral EFGH is square. However, they could not state the necessary conditions for EFGH to be a square. One teacher, Sophiak,
said “it seems that in order for efgh to be a square the angle bisectors have to intersect the opposite side at the midpoint”. The second teacher, Gouri, responded with “i'd like to check that using algebra view”. She checked the Algebra view but could not decide whether Sophiak’s conjecture is true. This task was at the end of the course and the teachers were used to using to Algebra view. They would use Algebra view whenever they feel challenged as in Task 52. They found Algebra view informative and helpful to understand constructions and geometrical relations.

4.6.3 Congruency and similarity.

The last category we found about teachers’ interaction with VMTwG is congruency and similarity among geometric objects. From the beginning of the semester, Team 3 was attentive to congruency between line segments and would check their congruency using Algebra view or by dragging and overlapping line segments. Teachers took advantage of the dragging capability of GeoGebra to check whether two line segments or even two triangles are congruent. However, their attention to congruency and similarity among geometric objects intensified after working on Module 8 (Tasks 23-27), Constructing Congruent Triangles, in the eighth week of the semester. Since each task of the course asks implicitly or explicitly about geometrical relations among objects, the teachers found congruency and similarity useful relations to discuss whenever they were asked to explore relations among objects. They would approach tasks looking for congruent objects and similar objects to discuss. Their way of exploring varied between using Algebra view to check line segments' lengths and angles’ measures and using their overlapping technique to check for congruency.
The use of congruency to discuss relations started at the beginning of the semester, the fourth week. The teachers used Algebra view to check line segments’ lengths to talk about geometrical properties and relations. As we saw in Task 8 (Figure 4.1), teachers used Algebra view to determine the length of the sides of the triangle and concluded that they “could see that the lengths & yes this is an equilateral triangle for sure”. Another example comes for the teachers’ work on Task 20 (Figure 4.2). The teachers recreated the figure in the task and used the congruency among line segment to verify that point J is a midpoint of EJ (see Figure 4.8). The following chat expert shows how teachers planned constructing and verifying a midpoint of a line segment.

<table>
<thead>
<tr>
<th>#</th>
<th>User</th>
<th>Chat Post</th>
</tr>
</thead>
<tbody>
<tr>
<td>31</td>
<td>Sophiak</td>
<td>If so, I think the next step is creating points of intersection. *</td>
</tr>
<tr>
<td>32</td>
<td>Gouri</td>
<td>I think it looks good</td>
</tr>
<tr>
<td>33</td>
<td>Gouri</td>
<td>Now it's time to add the points of intersection. I see you have it :)</td>
</tr>
<tr>
<td>34</td>
<td>Sophiak</td>
<td>I am now going create the segments &amp; then their point of intersection.</td>
</tr>
<tr>
<td>35</td>
<td>Sophiak</td>
<td>I am done, will play with it for a minute. *</td>
</tr>
<tr>
<td>36</td>
<td>Gouri</td>
<td>we can also draw segments over EJ and JG and see if their lengths are equivalent using algebra view</td>
</tr>
<tr>
<td>37</td>
<td>Sophiak</td>
<td>I pulled up the algebra view to confirm that we have a midpoint but HJ and HI are both called segment h. Hmmm</td>
</tr>
<tr>
<td>38</td>
<td>Sophiak</td>
<td>*</td>
</tr>
<tr>
<td>39</td>
<td>Sophiak</td>
<td>I realeased control.</td>
</tr>
<tr>
<td>40</td>
<td>Sophiak</td>
<td>*</td>
</tr>
</tbody>
</table>
41 Gouri ok

42 Gouri ok now look at segments i and j

43 Gouri they are equal

44 Sophiak I cant move the algebra view screen...guess I have to be in control

45 Gouri so point J is the midpoint of FG*

Figure 4.8 Teachers’ work on Task 20.

After constructing two congruent circles using FG as a radius, Sophiak says in line 31 that the next step is to create their intersection points. The second teacher, Gouri, agrees with this step in line 33. In line 34 and 35, Sophiak states that she is going to connect the intersection points with line segments and then drag the construction to test it. In line 36 Gouri suggest creating extra line segments on radius FG to be able to see their lengths in Algebra view. In lines 37 and 42-45, both teachers see that the line segments FJ and JG are equal which means that point J is a midpoint of FG. Congruency among line segments played an important role for the teachers to verify their
construction. It helped them see how to construct a midpoint of a line segment even though they did not discuss how to justify this construction mathematically.

Later in the semester, the teachers worked on Module 8, which allowed them to explore congruency and similarity of triangles extensively. Task 23 in Module 8 (see Figure 4.4) provides an opportunity for teachers to explore congruency and similarity among triangles. Team 3 spent about an hour and a half discussing the different combinations of congruent sides and angles between two triangles and their implications for congruency and similarity. They dragged to overlap objects to test congruency between sides and angles (see Figure 4.9).

Figure 4.9 Overlapping triangles in Task 23 to check for congruency.

By clicking on the checkboxes on the left of the screen, teachers choose certain combinations to explore. An arbitrary triangle ABC is visible all the time. After selecting from the table a particular combination, a specific triangle appears under triangle ABC. For example, when teachers selected 0s2a (0s=0 congruent sides and 2a=2 congruent angles), a triangle with two congruent angles to two angles in triangle ABC appeared. In this case, the teachers were able to say 0s02 produced a triangle similar to triangle ABC. To test the congruency of sides and angles, the teachers took advantage of the dynamic
geometry environment and dragged on triangle on top of the other, or as the teachers called it, overlapping. This task helped the teachers strengthening their understanding of congruency and similarity so that almost in every task after this one they discussed some form of congruency and similarity of objects. An example of using triangles similarity is apparent in Team 3’s work on Task 43 (see Figure 4.10).

Figure 4.10 Task 43: Treasure Hunt.

This, created by Thales de Lélis Martins Pereira (Pereira, 2012), a public high school teacher in Juiz de Fora, a city in the state of Minas Gerais, Brazil, reads as the following:

Legend tells of three brothers in Brazil who received the following will from their father: To my oldest son, I leave a pot with gold coins; to my middle son, a pot with silver coins; and to my youngest son, a pot with bronze coins.

The three coins are buried on the farm as follows: Half way between the pot of gold and the pot of bronze, I planted a first tree. Half way between the bronze and silver, a second tree. And half way between the silver and gold, a third and final tree.

Where are the pots of coins?
silver, a second tree. And halfway between the silver and gold, a third and final tree. Where are the pots of coins? (our translation)

Team 3 understood the task and decided to work backwards to find the location of the pots of coins. The team stated that the pots’ locations are at the vertices of a triangle that has each tree as a midpoint of each of its sides. They created the solution in Figure 4.11.

![Figure 4.11 Teachers’ work on Task 43.](image)

Even though the teachers were able to analyze the task correctly, they could not create a solution that has arbitrary locations of the three trees. Their solution fixes the location of the trees. Interestingly, they discussed similarity of the two triangles they created and list the properties that can help them find a general solution. The following excerpt shows part of their discussion about this task.

<table>
<thead>
<tr>
<th>#</th>
<th>User</th>
<th>Chat Post</th>
</tr>
</thead>
<tbody>
<tr>
<td>242</td>
<td>Sophiak</td>
<td>IQK is similar to GHI</td>
</tr>
</tbody>
</table>
Sophiak: Does that make sense?

Sophiak: This is the picture I sketched on paper from reading the textbox [moves tree 1]

Gouri: i see now

Sophiak: yay!

Gouri: you can use similarity to figure out where G and I are

Sophiak: The thing is that

Gouri: so segment IK is parallel to GI?

Gouri: **segment IQ

Gouri: is parallel to GI

Sophiak: IK is parallel to HJ

Gouri: yes

Sophiak: Aso, IK is 1/2 the length of HJ

Gouri: so cool!

Gouri: perfect

In this excerpt, Sophiak states that the two triangles, IQK and GHJ, are similar (it is clear that she meant triangle GHJ not GHI). She tries to make the picture clear for Gouri in line 244. This helps Gouri see that the sides of the two triangles are parallel. In lines 248-253, Sophiak and Gouri state the parallel lines of the two triangles. In line 254, Sophiak states that the sides of the small triangle are half the length of the bigger triangle. Using the fact that each side of triangle IQK, whose vertices are the three trees, is parallel a corresponding side of triangle GHJ, whose vertices are the locations of the three pots of coins, can be used to relocate the locations of the three pots starting from the three trees. After constructing a triangle with the three trees as vertices, a parallel line to each side
can be constructed to pass through the opposite vertex of that side. The three intersection points of the lines are the locations of the pots of coins. Team 3 was able to discuss and plan the solution successfully but did not really create it.

Team 3’s use of dragging, Algebra view, and congruency and similarity was evident throughout 15 weeks of online interactions in VMTwG. The teachers show progress and improvement in efficiency in their use of each of those three tools. Their dragging became more directed to the hotspots of the construction and consists of fewer movements than the beginning of the course. In the first weeks of the course, teachers dragged certain points more than once to be able to understand its behavior. Later on the course, the Team’s dragging is more purposeful and less repetitive. Similarly, using Algebra view changed through the semester. The teachers became aware of more objects in Algebra view and reported less confusion while reading the analytical descriptions of these objects. Lastly, discussing congruency and similarity of objects and triangles became more evident during the second half of the course. This was a result of exposing teachers to Task 23, Triangles Congruency and Similarity, and became a mathematical tool that teachers used to solve the task. These three categories of ways of interacting with VMTwG contributed to teachers’ appropriating the environment and allowed them to use it to discuss mathematical ideas.

4.7 Discussion

In a 15-week long professional development course, a team of two high school teachers was introduced to a collaborative, online, and dynamic geometry environment, called VMTwG. During this course, they interacted in VMTwG to notice variances and invariances of objects and relations in pre-constructed figures or figures that they
constructed and to solve open-ended geometry problems. Our analysis of their interactions allowed us to understand how they appropriated the environment and how this appropriation shaped the extension of their geometrical knowledge. After analyzing their work on 55 geometry tasks, we identified three main categories that describe their interaction with the environment and give insight into their instrumentation process. The categories of team’s actions were dragging geometric objects, using the Algebra view, and employing congruency and similarity among geometric objects. While engaging with the geometrical tasks in VMTwG, teachers appropriated these categories of actions as instruments to further their mathematical understanding.

At the beginning of the course, the teachers appropriated the dragging affordance of the DGE. As they dragged, the teachers attended to variant and invariant characteristics of geometric objects and constructions. Their interactions indicate that they grasped the significance of dragging the hotspots of a construction (Hegedus & Moreno-Armella, 2010). In Task 8, the co-action of the VMTwG environment helped the teachers identify the hotspots as well as to use them to test their construction and become aware of dependencies. The need for more than wondering dragging (Arzarello et al., 2002; Baccaglini-Frank & Mariotti, 2010) in this task motivated the teachers to develop more purposeful dragging. They used maintain dragging to check if triangles in Task 8 maintain their properties and, later, dragged to test the validity of their construction (the drag test).

The team also implemented the use of Algebra view of GeoGebra. It provided an analytical representation of geometric objects. This additional representation contributed to and supported teachers’ learning since it offered different insights into concepts
(Davis, 1992; Davis & Maher, 1997; Goldenberg, 1988a; Piez & Voxman, 1997). With Algebra view, teachers had access to all objects involved in the construction, both hidden and visible objects, which influenced their understanding of construction processes (construction dependencies), objects’ behavior while dragging (dragging dependencies), as well as objects’ properties. This team of teachers used Algebra view to access the length of line segments, angle measures, and slopes of lines to check whether they are parallel. This access to measurements supported teachers’ learning through confirming observations, generating conjectures, validating construction, and understanding relationships. This support is similar to what Olivero and Robutti (2007) describe as measurement modalities that are present in DGEs. Similar to our study, they found that measurements in DGEs support understanding of construction, properties, relationships, and formulating and proving conjectures. Our design of the dynamic geometry tasks did not involve the use of Algebra view. Team 3 was able to move beyond measurements to validate justifications and relations. However, in some cases, Team 3 did not go beyond mentioning the measures of geometric objects in Algebra view. Instead of discussing geometrical properties and logical relations, they used these measures as sufficient evidence of properties and relations. In some of these cases, teachers were concerned that they did not have enough time to finish all the tasks of the session.

The last category concerns the use of congruency and similarity to discuss geometrical explorations. As mentioned above in the Results, one task, Task 23, about triangle congruency and similarity extended teachers’ notion of congruency and similarity among triangles to other geometric shapes. After working on that task, teachers’ implement congruency and similarity while discussing geometric objects and
their relations. This use of congruency and similarity was a tool that helped them work on
dynamic-geometry tasks.

Understanding how instrumentation processes occurs in DGEs informs
implementing DGEs in the learning and teaching of mathematics. Collecting and
analyzing longitudinal data can provide wider and more accurate view of how learners
appropriate DGEs. The appropriation of dragging is an important aspect of transforming
dynamic geometry tools into instruments. The environment’s reaction to dragging helps
users identify independent, partially independent, and constrained geometric figures.
However, users need to be attentive to the results of dragging, noticing whether and
which geometrical properties of figures are maintained. Additionally, the affordance of
the Algebra view can be implemented in task design to help learners gain useful
information about geometric objects and use this information effectively.

The collaboration aspect of our research design plays a significant role in the
team’s appropriation of VMTwG. In our result, few instances showed how collaborative
practices were significant in helping teachers further their mathematical understanding.
However, the intention of this paper was to use one team as a unit of analysis in which
team members work collaboratively to achieve certain goals they set for each tasks.
Future research could investigate how teams’ dynamics and collaborative practices affect
instrumentation process of VMTwG.

Finally, though researchers have developed framework to study teachers’
implementation of technology in classroom and supper their student instrumentation
process (Trouche & Drijvers, 2014), we believe further research is needed to investigate
how teachers orchestrate their classrooms to help their students appropriate collaborative environments such as VMTwG in which the role of teachers differs from traditional classrooms.

The study in the next chapter investigates the mediational role of VMTwG. It uses Rabardel and Beguin’s (2005) categories of instrument mediations in an instrument-mediated activity to categorize how teachers used VMTwG and how their use relates to their mathematical knowledge.
Chapter 5: Mediational Activities in a Dynamic Geometry Environment and Teachers’ Specialized Content Knowledge

Abstract: Dynamic geometry environments can support learning of geometry through meditating learners’ activity. To understand how dynamic geometry environment mediate the activity of mathematics teachers, we used Rabardel’s categories of instrument mediations in an instrument-mediated activity. We analyzed the discursive and inscriptive interactions of 4 mathematics teachers who worked for 15 weeks in a team to construct geometric figures and solve open-ended geometrical problems in a collaborative, dynamic geometry environment. Teachers’ specialized content knowledge was evident when they used the environment epistemically. In addition to Rabardel’s epistemic and pragmatic mediations, we found and coined a third mediation—pedagogic mediation—by which teachers use the environment to help other team members understand particular geometric objects and relations among them. Understanding how teachers use technological tools can inform the design of professional development programs that engage them with such tools to extend their specialized content knowledge.

5.1 Introduction

Advances in technology, specifically, digital technologies, influence how people interact with their environment and with each other. In particular, technologies developed for teaching and learning mathematics have the potential to allow learners to visualize and explore mathematical ideas. Technologies also allow learners to interact with each other and collaborate to build mathematical knowledge. One significant factor of successful use of technology in mathematics classroom is to understand how technology
influences teachers and students’ social interactions and shapes their mathematical
to Vygotsky (1978) emphasizes the role of tools and signs for cognitive
development. He argues that intellectual development occurs through engagement in
tool-mediated activities that allow for social interactions. In other words, interacting with
artifacts available in someone’s environment and with other individuals allows for human
development and knowledge construction. In mathematics learning, learners construct
mathematical knowledge through engaging in activities that are mediated by tools such as
dynamic geometry software and with other individuals such as classmates and teachers.
Understanding how mathematical activity contributes to individuals’ construction of
mathematical knowledge involves careful investigation of how they interact with
mathematical tools involved in their activity and with other learners. Several studies use
Vygotsky’s notion of mediation to explain learners’ interactions with technological tools
in mathematical activities (for example, Barcelos, Batista, & Passerino, 2011; Hoyles &
Noss, 2009; Laborde, 2007; Mariotti, 2000). Technological tools mediate learners’
activity and provide additional tools and signs that can support students’ mathematical
discourse and building of meaning. Helping students construct mathematical meaning
while interacting with mathematical tools requires teachers to carefully plan and
implement how their students engage in mathematical activities.

To implement mathematical tools effectively in mathematics classrooms, teachers
need to learn how to use the tools. Their understanding of how to use the tools and how
tools relate to mathematical ideas can be seen as part of their mathematical knowledge for
teaching (Ball et al., 2008; Hill, Ball, & Schilling, 2008), which has a direct influence on
students’ achievement (Ball et al., 2005; Hill et al., 2005; Rowan et al., 1997). This
creates a need for investigating how teachers learn to use digital technological tools, how these tools mediate teachers’ activities, and how teachers’ knowledge evolves while interacting with technological tools.

To understand the mediation of technological tools in teachers’ activity and the formation of their mathematical knowledge, we investigated mediational activities in a collaborative, dynamic geometry environment called Virtual Math Teams with GeoGebra (VMTwG)\(^3\) as teams of mathematics teachers worked on mathematical tasks. Our goal was to understand how teachers interact with technological tools while solving mathematical tasks that promote productive mathematical discourse (Powell & Alqahtani, 2015a) and how this interaction influences their mathematical knowledge. Here, we present the results of our analysis of the discursive and inscriptive interactions of four middle and high school teachers working in one team. We identify different meditations in teachers’ activities and associated mathematical ideas with which teachers engaged. In the following sections, we discuss how previous studies investigated learners’ interaction with technological tools and how these studies relate to our theoretical perspective. In particular, we present findings in the literature about how dynamic geometry environments mediate mathematical activity. Then we describe our research methods and results and finally reflect on the implications of our results in light of related literature.

\(^3\) The environment, Virtual Math Teams (VMT), has been the focus of years of development by a team led by Gerry Stahl, Drexel University, and Stephen Weimar, The Math Forum @ the National Council of Teachers of Mathematics, and the target of considerable research (see, for example, Stahl, 2008; Stahl, 2009). Recent research has been conducted on an updated environment, VMTwG, that encompasses a multiuser version of the dynamic geometry environment, GeoGebra, (Grisi-Dicker, Powell, Silverman, & Fetter, 2012; Powell, Grisi-Dicker, & Alqahtani, 2013; Stahl, 2013, 2015).
5.2 Related Literature and Theoretical Perspective

To understand how technological tools mediate teachers’ activity and shape their mathematical knowledge, we draw on the theories of instrumental genesis (Lonchamp, 2012; Rabardel & Beguin, 2005) and mathematical knowledge for teaching (MKT) (Ball, Lubienski, & Mewborn, 2001; Ball et al., 2008; Chinnappan & Lawson, 2005; Hill et al., 2008). Instrumental genesis allows us to investigate how VMTwG mediates teachers’ activity through analyzing the relationships among users, instrument, and objects. We use MKT to investigate how teachers’ tool-mediated activity in VMTwG extends their mathematical knowledge. In this section, we discuss instrumental genesis and instrument-mediated activity then discuss how we coordinate it to aspects of mathematical knowledge for teaching.

5.2.1 Mediation activities in dynamic geometry.

Instrumental genesis offers a perspective to understand how a technological environment such as VMTwG, including its dynamic geometry tools, can mediate mathematical activity. It is rooted in Vygotsky’s perspective on the role of cultural signs and tools in human development. Based on Marx’s theory of historical materialism, Vygotsky believed that material tools developed historically influence human’s cognitive development and behavior and further extended the idea to include the role of signs (for example, written and spoken language, number systems) (Vygotsky, 1978). Tools and signs mediate human activity differently. Tools are externally oriented while signs are internally oriented. Tools guide individual’s actions on the object and results in changing the object. In contrast, signs are psychological tools that guide cognitive activity and do not aim to change the object directly. Signs are social means that are used to
communicate with others before becoming psychological tools that influence one’s cognitive activity (Wertsch, 1985). Even though tools and signs seem to count for most aspects of mediated activity, it is important to mention that “cognitive activity is not limited to the use of tools or signs” (Vygotsky, 1978, p. 55). Human development occurs through a process of internalization in which individuals transform external activities that are linked to tools into internal activities that are linked to signs (Mariotti, 2000; Vygotsky, 1978). Since tools and signs are products of surrounding environment, human development results in an internalization of sociocultural practices, “transformation of social phenomena into psychological phenomena” (Wertsch, 1985, p. 63). The social environment and available tools are important for cognitive development.

Building on these ideas about mediated activity and the role of tools and signs in human development, Rabardel and Beguin (2005) theorize how individuals appropriate tools and use them as instruments to act upon objects, which results in changes in both the individuals and the instrument. In their theory of instrumental genesis, users (teachers, students, or learners in general) appropriate a tool or an artifact through developing their own knowledge of how to use it, which turns the tool into an instrument that mediates activity between users and a task. Appropriating a tool happens through assimilating or adapting a utilization scheme. This process is called instrumentation and it involves changes in how users use and assign meanings to tools. When users change the tools and their use, the process of instrumentalization occurs. It this process, users modify the tools’ functions by adding new functions or modify existing ones to perform different actions in the instrument-mediated activity. Instruments mediate users’ activity to achieve a certain goal. Users engage in an activity in which actions are performed upon an object...
(matter, reality, object of work…) to achieve a goal using an artifact (technical or material component). This mediation role that instruments play governs the user-object relations, which might take epistemic or pragmatic meditational forms. Epistemic mediation focuses on the object and its properties, it is “oriented towards getting acquainted with the object and its properties” (Rabardel & Beguin, 2005, p. 433). The instrument helps the user understand the object and its structure. In pragmatic mediation, the user is mainly concerned with actions required to transform the object into a desired final form (Lonchamp, 2012; Rabardel & Beguin, 2005).

Each mediational form has aspects that can inform users’ activity. Rabardel and Beguin (2005) describe two dimensions of mediations: reflexive mediation and interpersonal mediation. Reflexive mediations “concern the relations the subject weaves with himself… to manage himself through the instrument” (p. 433). This reflection allows users to internally organize the activity. For example, while solving geometrical tasks in VMTwG, one teacher colors her final desired geometric object to highlight it from other objects involved in the construction. Inter-personal mediations “concern mediated relationships to others” (p. 433) in a collective activity. The organization of the activity in this type of mediations depends on the users’ collective actions on the object (Rabardel & Beguin, 2005). For instance, in the chat panel in VMTwG, a team of teachers uses an asterisk at the end of chat postings to indicate to others when they have finished expressing their idea. Epistemic and pragmatic mediations can be reflexive or inter-personal mediation depending on how the instrument is informing user’s understanding: about oneself or in relation with others (Lonchamp, 2012). In the case of dynamic geometry environments (DGEs), users can employ the environment epistemically or
pragmatically and at the same time reflect on their activity. However, inter-personal mediator is only possible when more than one user is participating in an activity such as an activity in the VMTwG environment.

In DGEs, mathematical activities can be mediated in multiple ways. Users act on geometric objects and DGEs react to their actions in a manner that corresponds to engineered infrastructure that responds to the theory of geometry (Hegedus & Moreno-Armella, 2010; Moreno-Armella & Hegedus, 2009). This co-active relationship between the environment and users allows users to monitor and reflect on their activity. In a mediational activity in DGE, the environment provides instantaneous feedback to users’ actions. This feedback can inform the users’ actions and shape their thinking. One significant action in DGEs is dragging the “hot-spots” of a dynamic figure since it can change the geometrical properties of the figure and can provide insights into its construction. Hot-spots are “points that can be used to construct mathematical figures, e.g. join two points with a segment, or construct a piecewise graph, and then used to dynamically change the construction.” (Hegedus & Moreno-Armella, 2010, p. 26).

Additionally, DGEs provide a group of objects and actions that users engage in their activity. These objects and actions become associated with signs that learners use to communicate with others and are the basis to build meaning when internalized (Mariotti, 2000). For example, dragging changes properties of geometric objects in DGEs and can be viewed as externally oriented that guide actions on the objects. When internalized, dragging becomes a sign that can be used to test the validity of a construction and to communicate with other when discussing the construction (Mariotti, 2000). Similarly, dragging and the trace tool in DGEs offer actions and signs that learners can internalize
and use to build different mathematical meanings in their activity (Falcade, Laborde, & Mariotti, 2007). Meaning in tool-mediated activities has been studied through analyzing the relationship between the user and the tools (Bussi & Mariotti, 2008). Mediated activities produce signs that are related to the artifacts involved in the activity, which allows individuals to negotiate and assign meanings to those signs. Personal meanings in this case is related to the activity, but also mathematical meaning can be shaped by the activity (Bussi & Mariotti, 2008).

The DGE signs and tools that learners internalize affect their geometrical discourse and activity (Sinclair & Yurita, 2008). These internalized signs and tools can transform into static geometry activity and inform its discourse. DGEs support students search for variants and invariants and allow students to engage in more rigorous activities through communicating to investigate mathematical relationships (Hoyles & Noss, 2009; Laborde, 2007).

These studies have investigated learners’ interactions with DGEs from a sociocultural perspective. Within this perspective, many researchers have used and contributed to our understanding of how learners appropriate tools and signs of DGEs. Studies indicate that DGEs can support learning of mathematics through providing tools and signs that learners use as they interact with each other when engaged with mathematical ideas. Nevertheless, there is a need to investigate the mediational roles that tools and signs play as learners engage in mathematical activity (Rabardel & Beguin, 2005). Through analyzing mediational roles of DGEs in learners’ activity, we provide additional lens to understand how collaborative DGEs mediate mathematical activities and influence mathematical understanding.
5.2.2 Teachers’ knowledge.

To understand interactions between teachers’ knowledge and their use of DGE, we draw on the notion of mathematical knowledge for teaching (MKT). Ball et al. (2008) posit that MKT consists of two major domains: subject matter knowledge and pedagogical content knowledge, each divided into three subdomains (see Figure 5.1).

The first subdomain of subject matter knowledge is common content knowledge (CCK) and is defined as the mathematical knowledge that is needed by anyone to solve a mathematical problem such as to perform calculations or to solve equations. The second subdomain is specialized content knowledge (SCK), which is the mathematical knowledge and skills that are needed for teaching mathematics and includes knowing how to present mathematical ideas, choose or modify representations for specific purposes, and modify mathematical tasks to change its difficulty level (Ball et al., 2008). The last subdomain is horizon content knowledge (HCK), which is knowledge of the larger landscape of mathematics as a curricular discipline in which a specific mathematical idea is situated and helps teachers respond to students’ ideas (Ball & Bass, 2009; Ball et al., 2008). Other researchers have further theorized this in-action work of teaching, which connects a particular mathematical idea to curricular ideas that students have or will meet as a perspective of advanced mathematics (Zazkis & Mamolo, 2011) that among other things informs teachers’ planned classroom practices (Wasserman & Stockton, 2013).

The second major domain of MKT is pedagogical content knowledge with three subdomains: knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum. Teachers’ KCS informs the
materials they choose and how they use them, tasks they assign, and examples they present. This subdomain of knowledge includes teachers’ understanding of how their students will respond to instructional activities and requires teachers to be attentive to aspects of students’ mathematical thinking such as common errors (Hill et al., 2008). The subdomain of KCT combines knowing about teaching and knowing about mathematics and, for example, is evidenced when teachers decide on an instructional sequence for a topic (Ball et al., 2008). The last subdomain of pedagogical content knowledge is knowledge of content and curriculum, which includes knowledge about content of school curriculum and of curriculum materials (Ball et al., 2008).

Figure 5.1 Domains of mathematical knowledge for teaching (Ball et al., 2008, p. 403).

Among the six subdomains of MKT, we focus in our study on teachers’ specialized content knowledge (SCK). This is justified since our data concern practicing teachers working on tasks aimed to extend their knowledge of dynamic geometry. Their
work occurs in the same a collaborative, virtual environment in which, subsequent to the professional development course, they will engage their students and with tasks that are either the same or similar to ones they will use with their students. Teachers’ SCK is critical to their classroom practice since it influences how they attend to students’ thinking and recognize how mathematical ideas can be represented in multiple ways. We do not discuss teachers’ PCK since our data do not include observations or discussions of teachers’ implementation of VMTwG with their students. Nevertheless, there are moments when teachers reveal aspects of their PCK when, during their VMTwG interactions, they talk about how their students would use the environment or respond to certain tasks. Yet, those moments are not relevant to our current study.

We are aware that Ball and her colleagues’ theoretical perspective on specialized content knowledge has been criticized. Flores, Escudero, and Carrillo (2013) question how Ball et al. (2008) define SCK as the mathematical knowledge that is needed for teaching mathematics and “is not needed–or even desirable–in settings other than teaching” (Ball et al., 2008, p. 400). Flores et al. (2013) suggest that this view of SCK requires an understanding what mathematical knowledge is involved in settings other than teaching. Furthermore, they question whether the mathematical tasks of teaching (see, Ball et al., 2008, p. 400) involve knowledge of mathematics that is indeed unique to teaching, not “shared by other professions” (Flores et al., 2013, p. 3057). As to SCK’s relationship to CCK, they argue that it is not clear how the two knowledge subdomains relate to each other and wonder whether SCK is a “deeper or amplified kind of CCK” (p. 3057).
For us, the mathematical knowledge that enables teachers to engage the mathematical tasks of teaching may not be unique to teaching but it is crucial. The mathematical knowledge that makes SCK special is not its uniqueness for teaching but rather its importance to enable teachers to engage tasks of mathematics teaching such as those elaborated by Ball et al. (2008). Further, however CCK may be defined, teachers not only need it but also need mathematical knowledge beyond it, and as such CCK as a subset of SCK. That is, teachers need to know what they want students to learn, CCK, and how to engage the mathematical tasks of teaching such as evaluating the plausibility of students’ claims and connecting a current topic to previous or future topics.

5.3 Methods

We investigate how a digital tool mediates mathematical activity to contribute to the wider understanding of how learners interact with mathematical tools and how the production of mathematical knowledge occurs. Our study is part of a larger project that aims to investigate how collaborative dynamic environment supports the extension of learners’ productive mathematical discourse. Viewing mathematics as a dialogic subject (Gattegno, 1987; Pimm, 1987), collaborators from Drexel University and Rutgers University designed a collaborative learning environment called Virtual Math Teams with GeoGebra (VMTwG). This synchronous environment enables users online to communicate chat messaging and to solve collaboratively mathematical problems. In addition to messaging technology, it also contains a shared dynamic geometry space, a multi-user version of GeoGebra, in which users in the session can construct and manipulate geometric and algebraic objects. The Drexel-Rutgers research team designed a sequence of connected geometrical tasks that aim to engage participants in productive
mathematical discourse (Powell & Alqahtani, 2015a). The research group conducted several iterations of task design and engaging teachers and their students in VMTwG to work on their collaborative mathematical problem solving through productive mathematical discourse.

The data for this study come from an online professional development course for middle and high school teachers that occurred in the semester of fall 2013. In small teams, 13 New Jersey mathematics teachers engaged discursively to learn dynamic geometry in VMTwG. For 15 weeks, they work on 65 tasks collaboratively and individually. Forty-nine of the tasks engaged the teachers collaboratively to solve problems in Euclidean geometry. In addition to their mathematical work, the teachers also read and, using VMTwG, synchronously and, using an online forum, asynchronously discussed several articles. These involved topics about collaboration (Mercer & Sams, 2006; Rowe & Bicknell, 2004), mathematical practices (Common Core State Standards Initiative, 2010), accountable talk (Resnick et al., 2010), technological pedagogical content knowledge (Mishra & Koehler, 2006), lesson structures for technology integration in mathematics classroom that maximize learning opportunities (McGraw & Grant, 2005), and validating dynamic geometry constructions (Stylianides & Stylianides, 2005). Furthermore, the teachers analyzed recordings of their VMTwG interactions to examine, reflect, and modify their collaborative and mathematical practices.

The 13 teachers worked in four small teams consisting of one team with four members and the other three teams with three members each. In the VMTwG environment, team members can define dynamic objects and, on their computer screens, drag constituent parts around (see Figure 5.2). VMTwG records users’ chat postings and
GeoGebra actions. The research team elaborated a systematic process (Powell & Alqahtani, 2015a) to design dynamic-geometry tasks that encourage participants to discuss and collaboratively manipulate and construct dynamic-geometry objects, notice dependencies and other relations among the objects, make conjectures, and build justifications. The team also provided feedback to participants that to encourage them to notice, make sense of, wonder about, or reconsider mathematical ideas evident in their sessions.

For this study, we present analysis of the work of one team, Team 1, which consists of four middle and high school mathematics teachers with usernames bhupinder_k, carrigsb44, ceder, and sunny blaze. Before the course, the teachers have little experience with dynamic geometry. They met in VMTwG synchronously twice a week for two hours each time. The focus of our analysis is to understand interactions between mediational activities in VMTwG and the extension of teachers’ specialized content knowledge. We analyzed the discursive and inscriptive data generated from Team 1’s interactions in VMTwG. The tasks on which the team worked were designed to engage teachers in productive mathematical discourse around topics of Euclidian geometry. Some tasks involved investigating relationships among objects, properties of objects, and building justifications for constructed or pre-constructed figures. For each of the 15 weeks, three to six related tasks were grouped in what we called a module.

The discursive and inscriptive data that were generated from teachers’ work in VMTwG include the logs of teachers’ chat communications, their GeoGebra actions, and VMTwG system actions (logging into the system, taking and releasing control of GeoGebra, and switching from one task to another). Using conventional content analysis
(Hsieh & Shannon, 2005), we analyzed the data to understand interactions between mediational activities in VMTwG and the extension of teachers’ specialized content knowledge. For each session, we examined all team members’ chat postings and highlighted the mathematical moments of the session. Here to code these discursive mathematical moments, we used Gattegno’s (1987) dialogic notion of what constitutes mathematics: teachers discussing ideas about geometric objects, relations among objects, and relations among relations. After highlighting those discursive moments, we examined how teachers used the environment in each of those moments and for what purpose.

Using Rabardel and Beguin’s (2005) mediation roles, we categorized teachers’ use of VMTwG into epistemic and pragmatic use. With each mediational use, we attended to the mathematical ideas that teachers were discussing and looked for what we believed to be specialized content knowledge. In the results section, we present our findings of how teachers used VMTwG epistemically and pragmatically. From our analysis, we noticed a third mediational activity in VMTwG and for that coined the term, pedagogic mediation. It occurs when teachers demonstrate an idea or procedure to others with the intention or effect of bringing something to their team members’ notice. After that, we present moments where teachers’ SCK was evident in their work and relate it to mediational activities in the environment.

5.4 Results

Our analysis of Team 1’s interactions in VMTwG during a 15-week professional development course allowed us to categorize the mediational roles of VMTwG in teachers’ activities while investigating given and their own constructed geometric figures as well as solving geometrical problems. Following Rabardel’s (2005) mediational
categories, we identified moments when teachers used VMTwG predominately as an epistemic or pragmatic mediator. In addition, in our analysis, we found a third mediational role that we call pedagogic mediation. We identified that our participants used VMTwG to teach each other, convey ideas, and illustrated how it could be used to teach certain topics. In the following sections, we provide examples of Team 1’s interactions and the three mediational activities of VMTwG: epistemic, pragmatic, and pedagogic. Afterward, we present examples of extensions of teachers’ specialized content knowledge while interacting within the virtual environment. We argue that extensions of teachers’ specialized content knowledge is particularly visible in epistemic mediational activities.

5.4.1 Epistemic mediation.

In this mediation role, the virtual environment is used to explore properties of mathematical objects and relations among them. The teachers took advantage of affordances of the environment (chat and GeoGebra) to investigate properties of objects and relations among these objects. Teachers use this knowledge of objects to apply appropriate transformations on the objects to reach a desirable form. We present two examples of teachers’ work on Tasks 6.06 and 13.02 that demonstrate prominently this mediational role of the environment.

5.4.1.1 Task 6.06.

This task engages teachers with properties and construction of a triangle’s orthocenter. In this task, the teachers of Team 1 were asked to create a triangle and its orthocenter then to drag the vertices of the triangle and discuss what they notice about the
behavior of its orthocenter. In the tasks that preceded this one, the teachers explored a
triangle’s centroid, circumcenter, and incenter. At the end of Task 6.06, the teachers had
to create the centroid, circumcenter, incenter, and orthocenter of one triangle and
challenged to discuss relationships that they notice among the triangle centers. Team 1
constructed the figure shown in Figure 5.2.

![Figure 5.2 Team 1’s construction at the end of Task 6.06.](image)

After constructing the four different centers of a triangle, members of Team 1 dragged
the vertices of the triangle and explored the behavior of the centers. The following
excerpt is part of their discussion:

<table>
<thead>
<tr>
<th>#</th>
<th>User</th>
<th>Chat Post</th>
</tr>
</thead>
<tbody>
<tr>
<td>606</td>
<td>ceder</td>
<td>we have to rename them though</td>
</tr>
<tr>
<td>607</td>
<td>bhupinder_k</td>
<td>K - incenter, L - circumcenter, J - centroid, M- orthocenter</td>
</tr>
<tr>
<td>608</td>
<td>sunny blaze</td>
<td>i thought i saw something</td>
</tr>
<tr>
<td>609</td>
<td>sunny blaze</td>
<td>wait</td>
</tr>
</tbody>
</table>
After creating the four different centers of their triangle, the teachers started to drag the vertices of the triangle and to state what they notice. While bhupinder_k dragged the vertices, sunny blaze noticed that some of the points lie on the same line (lines 608-610). In line 612, she then decides that the points sometimes lie on the same line. In line 613,
bhupinder_k mentions that sunny blaze’s idea leads to Euler’s line. The others state that they do not know about Euler’s line. After that, sunny blaze took control and constructed Euler’s line. The construction of the line helped the others see that only three points lie on same line. As sunny blaze stated, “everything but the incenter is on the line” (line 635). Afterward, in line 637, ceder asked sunny blaze about how she thought to construct the line. In line 638, sunny blaze responded that she dragged the vertices and noticed that only three points lie on same line and that helped her construct Euler’s line. She does not stop at that point, she continues to think about the order of the points on Euler’ line (line 640) and later in the session she wonders about the “the distance between the points”.

5.4.1.2 Task 13.02.

This task invited the teachers to explore by dragging the vertices of eight different-looking quadrilaterals and, for each, to identify relations among its sides such as constraints and dependencies. The first quadrilateral, labeled “poly1”, was constructed by connecting four independent points. The teachers dragged its vertices and decided that poly1 does not have any constraints. While dragging its vertices, they move one vertex to cross one side (see Figure 5.3). Figure 5.3 contains the resulting figure, where they dragged point D to cross line segment BC. They wondered whether polygon ABCD can still be considered a quadrilateral. They stated that now they have six sides and wondered about the type of polygon they made.
Figure 5.3 Team 1’s work on Task 13.02.

In the task, the second polygon, poly2, is a kite. After dragging its vertices, the teachers mentioned that its diagonals are always perpendicular. Then they dragged the kite’s vertices to make a concave kite and discussed whether its diagonals remained perpendicular. They concluded that figure is still a kite and that its diagonals remain perpendicular. They arrived at this since one team member carefully dragged the poly2’s vertices to convince the others that quadrilateral’s diagonals are perpendicular even as concave kite.

In these two tasks, 6.06 and 13.02, Team 1 investigated properties and relations of geometric objects. Through constructing and manipulating geometric objects and discussing what team members noticed, they engaged with important mathematical ideas. In Task 6.06, teachers dragged the vertices of the triangle and observed the behavior of its centers. They used the VMTwG environment to see the relations among these points, which can be considered an epistemic use. The reaction that the environment offered by moving the centers according to the location of the triangle vertices enabled teachers to
notice that three points were collinear. The teacher, sunny blaze, who first mentioned that relation though that all points were collinear but dragged the vertices more and revised her statement to say only three points were collinear. To assist her teammates to see this relation, she constructed the line segment that connects those three centers. Her actions in GeoGebra, dragging and constructing the line, are examples of how DGE can be used as an epistemic mediation to gain insights into the properties and relations of geometric objects. Later, we discuss further Team 1’s work on Task 6.06 to indicate its relevance to another mediational role of the VMTwG environment.

Similarly, in Task 13.02, teachers dragged vertices of quadrilaterals to understand their properties. They went beyond identifying the constraints of the quadrilaterals to exploring possible shapes formed by dragging the vertices of quadrilaterals. Their investigations of the first polygon, poly 1, and discussion about whether the shape they obtained in Figure 5.3 is still a quadrilateral evidence the epistemic mediational role of their activity with quadrilaterals formed by dragging. In addition, further evidence of the epistemic mediational role of their activity is their exploration of the kite, poly 2. They dragged vertices of the kite and investigated concave kites. Their discussion about the kite’s diagonals, in both the concave and convex cases, was possible because of affordances of the environment and emerged from epistemic mediation.

### 5.4.2 Pragmatic mediation.

In pragmatic mediation, the environment mediated the teachers’ actions on objects to transform them to certain desirable forms. Teachers monitored continuously the transformation on the objects and were concerned with their final forms as they responded to tasks’ instructions. The tasks that we designed and invited teachers to work
on ask them to manipulate and construct dynamic figures as well as discuss and justify mathematical relations they perceive. The majority of the tasks involved manipulation and construction. Because of the nature of our tasks and the design of the multimodal virtual environment (chat and dynamic geometry), the mediational role of VMTwG in the teachers’ activity was mostly pragmatic. However, to respond to the tasks’ instructions and apply certain transformations on objects, the teachers needed to investigate and discuss objects’ properties and relations, which we view as an epistemic use of the environment. There is a need to shift continually between pragmatic and epistemic use of the environment to respond successfully to the tasks. In the following two examples of teachers working on Tasks 10.01 and 14.04 can demonstrate the pragmatic mediation role that VMTwG played.

5.4.2.1 Task 10.01.

This Task invites teachers to drag and discuss the construction of an equilateral triangle inscribed in another equilateral triangle (see Figure 5.4). In particular, the team of teachers needs to understand relations of constraints and dependencies between the two triangles among particular objects (points and line segments).
Figure 5.4 Task 10.01: Inscribed Triangles.

In triangle ABC, points A and B are independent points. For the inner triangle, DEF, point D is constrained to side AC and the other points, E and F, are fully restricted and cannot be moved. The teachers of Team 1 dragged points A, B, and D and stated that triangles ABC and DEF are each equilateral. They also stated the dependencies in the construction. One teacher said: “C is dependent on A and B” and another stated that points “E and F are dependent on D.” They successfully created an outer equilateral triangle, GHI (see Figure 5.5). They struggled to construct the inscribed one. They dragged the given figure and described relations among the points, sides, and angles. They also discussed what GeoGebra tools can be used to reproduce the given construction such as midpoint, perpendicular line, circle, angle bisector, line segments with fixed length, and the compass tool. They evaluated the use of each tool and tried some of them. Eventually, one teacher tried the compass tool and stated “the distance from G to J is the same as I to the point on IH and H to the point on Hg” (see Figure 5.5).
The teacher explained to the other team members her solution saying: “I used the compass to make GJ the radius and then set I as the center and H as the center… then the intersection points which makes them dependent then connect!!!” The other team members agreed with this solution and saw how it worked.

5.4.2.2 Task 14.04.

In Task 14.04, the teachers constructed a parallelogram and its angle bisectors and then connected the intersection points of the angle bisectors to form a quadrilateral. The task also invited the teachers to drag the vertices of the parallelogram and notice the type of quadrilateral formed by the connected intersection points of the angle bisectors. In Figure 5.6, the teachers constructed parallelogram ABCD and its angle bisectors, which intersected in points E, F, G, and H. Those points constitute the vertices of the inner quadrilateral. They also hid parallelogram ABCD’s angle bisectors. After spending time dragging the vertices of ABCD, the teachers concluded that quadrilateral EFGH is always
a rectangle. They explored more and stated that “it seems that the only time we will get a square is if we have a rectangle on the outside.” By “on the outside” they meant quadrilateral ABCD. They also noticed that when ABCD is a square or a rhombus, EFGH is a single point. Their observations encouraged them to construct the angle bisectors of quadrilateral EFGH and explore the quadrilateral that results from connecting those angle bisectors (see Figure 5.6).

The Team’s solution of Task 14.04.

The teachers dragged points A, B, and C and confirmed their findings. Since EFGH is always rectangle, the inner quadrilateral, IJKL, is always a square. One teacher summarized their findings saying: “ABCD is a parallelogram, rectangle, square, EFGH is a rectangle square or point, and IJKL is a square or point, so one less option each time.” The task also asked the teachers to justify their findings. One teacher stated that “the angle bisectors (the ones that are opposite of each other) are always parallel... and because we started with a parallelogram, they always intersect with the other set of parallel angle bisectors perpendicularly.”
The two examples above demonstrate the pragmatic mediation role of the environment in which teachers focus on the required transformations on objects to reach a desired goal. Tasks 10.01 and 14.04 show teachers’ activity being pragmatically mediated by VMTwG as they construct geometric figures and manipulate objects. In the first example, teachers were asked to reconstruct the figure presented to them. They started by dragging its points and trying to understand its structure. They stated the dependencies involved in the constructions and the relationships among the objects then used that to plan and reconstruct the figure. Their investigation of the figure and its properties can be considered as epistemic use or VMTwG. The construction of an equilateral triangle, dragging and testing construction, and applying different tools of GeoGebra can be viewed as pragmatic use because the teachers were using the environment to construct the two triangles. They kept trying different tools and discussing different ideas, which indicate their preoccupation with the final construction.

When pragmatic mediational role is predominant, we found that pragmatic and epistemic mediational roles co-occur in the teachers’ activity. This co-occurrence is of the pragmatic and epistemic mediations is evident in the second example above. The instructions of the task ask the teachers to construct a parallelogram and its angle bisectors. They used the appropriate tools to respond to the task. They dragged the independent vertices of the parallelogram sufficiently to explore all possible cases of the types of quadrilaterals constructed by connecting the intersection points of the angle bisectors. The teachers extended their construction and constructed the angle bisectors of the second quadrilateral (rectangle) to confirm that the intersection points of the angle bisectors of a rectangle creates a square. Throughout their work on this task, teachers
focused on the desired form of the construction and how to respond to the task’s instructions. Pragmatic use of DGEs is expected for tasks that were particularly designed for this type of environment. However, if tasks were not intended to take advantage of DGEs’ affordances such as dragging, learners might not feel the need to use the environment to solve the tasks.

5.4.3 Pedagogic mediation.

In addition to epistemic and pragmatic mediations, as we analyzed our data, we noticed a third mediational role of VMTwG in the teachers’ activity and call it pedagogic. This is an *inter-personal* mediation as it relates to how users interact with each other. It may have been particularly visible in our data since our participants were middle and high school teachers and they have an interest in using the environment to help others understand mathematical concepts. From the beginning of the professional development course, the teachers used VMTwG to help other team members engage with certain geometrical properties or relations among geometric objects. For instance, they took care to organize the graphical space of GeoGebra and to label geometric objects. There were moments when the teachers talked about how their students might respond to the environment, showing concern for the pedagogical use of dynamic geometry environments. We view moments when teachers’ use of the environment to mediate others noticing or understanding properties or relations as pedagogic mediation. In these moments, the environment is used as a mediator not to make transformations on the objects or explore objects’ properties and relations, but rather to mediate others understanding of transformations, properties, and relations. An example of these moments is what sunny blaze did to assist her teammates in Task 6.06 above. She
constructed a line segment to show how three centers of the triangle—orthocenter, centroid, and circumcenter—lie on one line, Euler’s line. Also, in the following two examples, we show how VMTwG pedagogic mediational role in the teachers’ activity. The first example is teachers’ activity on a task in which they justify Thales’ theorem. The second example shows teachers’ activity on a different task where they explore one case of triangle congruency.

5.4.3.1 Task 5.01.

After working individually on different visualizations of Thales and Pythagoras’ theorems, in Task 5.01, the teachers were encouraged to discuss their resulting insights and wonderings about the theorems. In Task 4.01, the teachers worked individually to construct a dynamic triangle inscribed in a semicircle and to notice the measures of the inscribed angles. Then they were invited to develop a proof for Thales’ theorem. In Task 5.01, teachers discussed what they noticed about Thales theorem. One teacher said that she never heard of Thales’ theorem and others teachers said they were not familiar with its proof. One teacher, ceder, said she was able to prove the theorem using a hint that was given in Task 4.01 about constructing a radius to form isosceles triangles. Another teacher, sunny blaze, said: “so as the points were dragged, there were 2 isosceles triangles inside the right triangle... did anyone see this?” She then said: “i wish we had the tab in front of us to confirm”. This comments encouraged ceder to offer to do the construction. The others agreed and asked her to make the construction (see Figure 5.7).
Figure 5.7 For Task 5.01, the construction of a member of Team 1.

After ceder constructed the figure in Figure 5.7, sunny blaze stated that she was not sure why there are two isosceles triangles. Part of their discussion follows:

# User Chat Post
59 sunny blaze (besides the fact that I'm not 100% sure what the theorem actually is), if D and B were not connected by a line segment, then how would you be able to see that there are two isosceles triangles inside the right triangle? Is the idea to see that there are 2 isosceles triangles no matter what vertex of the right triangle is dragged or is the idea to see that you can always form a right triangle inside a circle

60 ceder the point is that there is always a right angle at D

61 ceder and he used the fact that isosceles triangles have two angles that are equal and that the sum of interior angles of a triangle are 180 since he had previously proven that

62 ceder so A=ADB

63 ceder and C=CDB

64 ceder because of the isosceles triangles he say
sunny blaze: see i was way off, i wasn't sure what to look for because I didn't read Topic 4, i just went in and viewed the tabs. i really wish i would have spent more time on this last night

ceder: yeah, I find that document to be a big help.

sunny blaze: 

ceder: so we know those angles are equal because of the two isosceles triangles

vec: then...

ceder: A+ADB+C+CDB=180

ceder: is that ok so far?

bhupinder_k: Right

ceder: I know its hard with all the letters

In line 59, sunny blaze inquires about the theorem and her team member’s, ceder’s, construction. She wonders about the purpose of the construction of line segment BD in relation to the theorem: is it to show that no matter which vertex is dragged the formation of two isosceles triangles is invariant or is it to show that an inscribed triangle in a circle is always right? In response, in lines 60 and 61, ceder explains that the theorem says that angle D is always a right angle and that to prove his theorem Thales used the isosceles triangles and the fact that the sum of the interior angles of a triangle is 180. She lists the congruent angles that she will use to prove the theorem later. In lines 79 to 84, ceder states that the sum of the interior angles of triangle ABC is 180 then she asks if everyone is following her argument. The session continues after these lines and ceder shows the
team her proof of Thales’ theorem. During her demonstration, she was concerned with whether her team members were following her argument (see lines 82 and 84). She also dragged point D to show that the right angle was invariant. Later in the session, other team members took control and dragged different points vigorously to explore the invariance of the right angle and the isosceles triangles.

5.4.3.2 Task 9.02.

The aim of Task 9.02 is to help teachers explore triangle congruency, specifically the side-side-angle. The task presents a triangle and three line segments that are connected as in Figure 5.8. To engage teachers in exploring the side-side-angle case of triangle congruency, two line segments were congruent to two sides in the triangle (DE is congruent to BC and EF is congruent to AC) and angle KDE is congruent to angle ABC in the triangle. The teachers were asked to drag different points and, by connecting points F and K, to investigate how many possible triangles they can make.

Team 1 worked on this task for few minutes and decided that only one unique triangle can be made and that it is congruent to triangle ABC. In feedback subsequent to their session, we invited the teachers to revisit this task and to observe carefully as they dragged points F and K. Team 1 met again and revisited the task. Since one member, ceder, did not attend the first session, she asked the team to let her try this task first. For some time, she dragged different points. While she dragged point F, another teacher commented that line segment “EF moves like the radius of a circle” and said she was wondering “if that has anything to do with it”. Two seconds later, the teacher in control said: “I think there are two triangles… K will hit the circumference in two places”. The way this teacher dragged point F allowed other teachers to see that point F is restricted to
a circle with center E with EF as a radius. The line segment EF is congruent to AB and that is the reason why point F has a constant distance from point E. Her use of “circumference” to describe the situation enabled other members see how one can make the two triangles.

![Image of GeoGebra interface with a circle and two triangles](image.png)

**Figure 5.8 Task 9.02: Exploring different relationships.**

To make her observations more visible to others, ceder dragged points F and K carefully to demonstrate the two triangles. Other teachers asked her to remake the second triangle (non-congruent). After showing the two triangles, she stated that it is not always the case that there will be two triangles and tried different cases to see whether they can make two triangles again.

The teachers’ work in these examples shows the pedagogic mediational role of the environment in the teachers’ activity as one team member helps others understand mathematical ideas. In Task 5.01, one teacher used GeoGebra to show how she proves Thales’ theorem. She was attentive to her construction, dragging and representing objects
in GeoGebra, as she tried to make it clear and readable. At that moment, she used the environment to help others understand the theorem and her proof of it. She used her knowledge about how to organize objects and to drag them to improve her demonstration. In a similar way, she used GeoGebra in task 9.02 first to understand how to the construction behaves and how many triangles she can make. Then she showed others that there are two different triangles possible. She dragged certain points and communicated carefully with her teammates about her observations.

5.4.4 Specialized content knowledge and instrument-mediated activity.

As a response to the critique that SCK received, we used parts of Ball and her colleagues’ definition (2008) to operationalize SCK and illustrate how teachers’ engaged SCK in their activity within VMTwG. Our operational definition of teachers’ SCK includes moments when teachers ask “why” questions, wonder about different ways to represent mathematical ideas, look for “different interpretations” of ideas (Ball et al., 2008, p. 401), categorize mathematical ideas in different ways, or investigate the structure of mathematical objects and their relations. In those moments, teachers were not concerned particularly with solving the particular geometrical problem. In some cases, they had already solved the problem or constructed a desired figure. Instead, similar to the epistemic mediational moments, their concern is about properties and relations of mathematical objects. In addition to the moments of epistemic mediation presented above, we present three examples to suggest how teachers’ SCK can be part of instrument-mediated activity with dynamic geometry. These examples show that teachers’ SCK is more visible when they use the environment epistemically to investigate properties and relationships among mathematical objects.
5.4.4.1 Task 3.05.

Task 3.05 invited the teachers to drag base points of six different pre-constructed triangles and identify dependencies involved in each construction. This task comes after four tasks that dealt with the construction of equilateral, isosceles, right, and right isosceles triangles. Teachers worked on this task in the second synchronous meeting of the professional development sessions, so they were still trying to understand tools and functions of dynamic geometry. The first triangle in this task was constructed by connecting three independent points. The second triangle is an isosceles triangle, and teachers spent time identifying dependencies involved in its construction. The third triangle is equilateral. The teachers were able to identify its dependencies since, in Task 3.02, they had constructed an equilateral triangle. The fourth figure, JKL, with right angle at K, is a right triangle. One teacher organized her observation in one sentence. One teacher took some time dragging its vertices and then commented: “in poly 4, when I move L, J and K do not move. when I move J, L moves. when I move K, L moves. so L is dependent on J and K”. Before stating the dependencies in the triangle, she described the consequences of dragging each vertex. Her categorization of dragging consequences included two cases: points that are not affected by dragging another point and points that are affected by dragging other points. In the second case, she called those points dependent on the points that are being dragged. In her example, point L is dependent on points J and K.

We view this organization of dragging and dependency in an early session of the professional development sessions as a moment that engaged teachers’ specialized content knowledge since it involves teachers’ analysis and summarization of the structure
of a mathematical idea. At this moment, the mediational role of the environment was an epistemic since it was concerned understating the properties of the objects. The teacher in this moment used the environment her to build a link between dragging and dependencies. This link serves to extend her understanding of mathematical dependency to include relations among geometric objects.

5.4.4.2 Task 5.01.

As mentioned earlier, Task 5.01 invited the teachers to discuss their insights into Thales and Pythagorean theorems and their proofs. In the week previous to working on this task, they explored individually different representations of Pythagorean theorem. The representations included extensions of the theorem to create different regular polygons on the sides of a right triangle as well as creating circles whose diameters are the sides of the triangle (see Figure 5.9).

![Figure 5.9 Task 4.07: Visualization of Pythagorean’s Theorem.](image)
Besides discussing elaborately the theorem of Thales in Task 5.01, which we also see as moments that involves teachers’ SCK, the teachers discussed different visualizations and extensions of Pythagorean’s theorem. In their synchronous session, each teacher reported on which ones they found particularly helpful to visualize the theorem and how the extensions of the theorem were new to them. In particular, they spent most of their session discussing relations among the areas of three circles whose diameters are the sides of a right triangle. One teacher reported that she tried the construction on her own, which encouraged the team to discuss the dependencies and the construction of Task 4.07 (see Figure 5.9). They started by stating that “all the midpoints are dependent on the vertices” and “and the vertices are dependent on the perpendicular lines to keep the right angle”. Another teacher stated, “the area of the circle with the center that was the midpoint of the hypotenuse was dependent on the other circles” to which another teacher responded, “all the circles are dependent on the midpoints and vertices”. This discussion of the dependency of the areas of the circles helped the teacher reconsider her comment about the areas, she said later: “it was definitely related, not sure about dependent”. Another teacher reported that if they have enough time, she would like “to discuss whether the area of the large circle is or isnt dependent”.

Teachers’ use of the environment in this task and their investigation of its construction involved their SCK as they were looking for different ways to represent and interpret mathematical ideas. They were trying to investigate an extension of Pythagorean Theorem in details and understand how to construct its visualization. Their use of the environment at that moment was mostly epistemic. This shows how the epistemic mediational role of the environment can engage teachers’ SCK.
5.4.4.3 Task 13.03.

This task invites the teachers to construct squares in different ways without using measurements. In Task 13.01, teachers constructed perpendicular and parallel lines, and this work was meant to provide insight into how to construct quadrilaterals containing parallel lines or right angles. After one teacher took control of the GeoGebra window to construct squares, another team member suggested using the methods they used for constructing perpendicular and parallel lines (see Figure 5.10).

![Figure 5.10 Team 1’s work on Task 13.01.](image)

The teacher in control replied to his teammate’s suggestion saying “i cn use a different one... ill try something different”. She created a line segment, rotated it by 90 degrees around each of its two endpoints, and then connected the endpoints of the new line segments to create the last line segment of the square. She also was exposed to another solution that was presented in Task 13.01 but wanted to try transformations. She dragged to test her construction to see whether it had the properties she wanted. That is, to construct a square, she switched between pragmatic use and epistemic use of the
environment. Moreover, her decision to try different method to construct a square can be seen as a moment that involves teacher’s SCK since it engages teachers with multiple ways to construct and investigate squares.

The three examples that we have presented illustrate how, while working in DGE, teachers engage their SCK. The examples show teachers categorizing dependencies among objects based on their behavior when dragged, investigating how and why objects relate to each other in the Pythagorean Theorem, and trying different ways of constructing a square. These moments involved an epistemic use of the environment. The first example involved teachers’ investigating dependencies involved in constructing different types of triangles. One teacher dragged and monitored the environment’s reactions to her dragging. Her systematic dragging and monitoring allowed her to classify her observations. She stated the dependencies involved in the construction along with her justifications for her decisions. The second example showed teachers investigating why and how circles whose diameters are the sides of a right triangle relate to each other. They also constructed a right triangle and circles. The last example illustrated how one teacher tried different ways to construct squares. She did not follow a method that she used to construct perpendicular and parallel lines to construct a square. She insisted on trying a new way to construct a square and this can be seen as a moment that involves teachers’ SCK.

5.5 Discussion

Doing mathematics is complex social and cultural practice. Central to the practice are mediational tools. We combine the mediating tools of virtual chat and GeoGebra to comprise the multimodal digital, collaborative mediational tool of VMTwG. In the light
of the theory of instrumental geniuses (Lonchamp, 2012; Rabardel & Beguin, 2005),
what mediational roles does VMTwG play in the extension of a teachers’ knowledge of
dynamic geometry and their consequent appropriation and internalization of the cultural
tools of VMTwG required to do geometry in a dynamic, digital environment? In
particular, we found it important to understand the mediational role that digital
technologies such as VMTwG can play in the development of teachers’ specialize content
knowledge. Analyzing the mediational interactions of a team of four middle and high
school teachers working with digital collaborative and dynamic visualization
technologies, VMTwG, allowed us to extend Rabardel’s mediational roles of instruments
(Lonchamp, 2012; Rabardel & Beguin, 2005). The mediational role of VMTwG in
mathematics teachers’ activity can take three forms: epistemic, pragmatic, and pedagogic.
In epistemic mediation, teachers interacted with VMTwG to investigate objects
properties and their relations. Pragmatic mediation occurred when teachers used the
environment to transform objects into certain forms. Pedagogic mediation evidenced
when teachers engaged the environment to help others learn about mathematical objects
and their transformations and relations among objects. This mediational role was visible
in the teachers’ activity but it could also be visible in students’ mathematical activity.
Understanding the mediational of VMTwG in learning mathematics can inform our
understanding of how learners interact with other technological tools for doing
mathematics and how learners build their knowledge using the tools.

Attending to the knowledge that teachers engaged while interacting with VMTwG
informed our analysis of teachers’ knowledge. We argue that teachers’ specialized
knowledge is visible when instruments mediate epistemically their mathematical activity.
We viewed specialized content knowledge as the knowledge that involves multiple ways of investigating, representing, and organizing mathematical objects. This knowledge is not necessarily limited to teachers; others might have an interest in this mathematical knowledge. However, what is special about this knowledge is that it is important for mathematics teaching. It goes beyond just solving or finding an answer to a problem. It supports teachers’ understanding of different ways to think about mathematical ideas. Defining what can be considered a specialized knowledge may not be an easy task. It depends on the goals of the mathematics lesson, teachers’ view of acceptable outcomes, and the students’ level of mathematical knowledge. When using mathematical tools, teachers’ specialized knowledge is evident when they use the tools to investigate mathematical relations and properties (i.e. using tools epistemically) for different purposes. This epistemic interaction with tools can inform teachers’ pragmatic use and help them transform objects into desired forms. One of these two mediational roles “may be predominant, though they usually interact constantly” (Rabardel & Beguin, 2005, p. 433) to inform teachers’ mathematical activity. Similarly, pedagogic mediational role can be predominant while at the same time one or both of the other mediational roles may be at play.

Understanding how instruments mediate mathematical activities allows teachers and mathematics educators to be aware of what kind of mathematical knowledge that students might engage with in their activity. It informs how teachers and mathematics educators design tasks collaborative environments to engage learners epistemically, pragmatically, and pedagogically in mathematical activity and productive mathematical discourse. After appropriating the mathematical tool, learners can it use to investigate
properties and relations among mathematical objects, transform mathematical objects, and help others understand those properties and transformation. When users use the tool pedagogically and try to help others understand the tool and the mathematical objects, they may use existing functions of the environment differently. Their modification of the environment’s function can be considered as instrumentalization.

Pedagogic mediation adds to our understanding of how teachers interact with mathematical tools. Understanding how teachers interact with DGE and with other technological tools in general informs professional development design and implementation. It also informs task design in DGEs for teachers and students. Further research is needed to investigate the mediation of technological tools not only in professional development setting but also in mathematics classrooms.
Chapter 6: Conclusion

Dynamic geometry environments provide valuable tools that can support the learning and teaching of mathematics (Baccaglini-Frank & Mariotti, 2010; Chan & Leung, 2014; De Villiers, 2004; Falcade et al., 2007; Hölzl, 1996; Jones, 2000; Laborde, 2000; Mariotti, 2000; Stahl, 2015). To implement dynamic geometry in mathematics instruction successfully, teachers need to understand the affordances such environments provide and how one can use these affordances to explore mathematical ideas. Helping teachers understand dynamic geometry environments requires engaging them for a substantial number of hours in an active learning that involves this technology (Garet et al., 2001). Existing research on teachers’ use of dynamic geometry tends to investigate their use of dynamic geometry in a short period of time (e.g., Hohenwarter et al., 2009; Koyuncu et al., 2014; Sinclair & Yurita, 2008). In contrast, this dissertation study provides longitudinal analysis of teachers’ work in a dynamic geometry environment. The main purpose of this study was to understand how teachers appropriate technological tools and how they discursively develop their geometrical knowledge collaborating in an online dynamic geometry environment.

6.1 Findings and Significance

The findings of this qualitative investigation highlight significant aspects of mathematics teachers’ interactions with a collaborative dynamic geometry environment and the development of their geometrical knowledge. The three studies presented in Chapters 3, 4, and 5 provide different insights into teachers’ appropriation of the dynamic environment and their geometrical understanding. In Chapter 3, I presented the result of a case study that investigated the dragging affordance of dynamic geometry environment.
Mathematics education researchers agree that the defining feature of DGEs is the ability to drag geometric objects (Arzarello et al., 2002; Hölzl, 1996; Ruthven et al., 2008; Scher, 2000; Sinclair & Yurita, 2008). This study focused on examining how teachers appropriate dragging and how their appropriation influences the discursive development of their understanding of dependency among geometric objects. Dependency is the salient mathematical idea behind dragging. The results showed that teachers relied on the co-active relation between their actions and the DGE’s reactions to their dragging of hotspots (Hegedus & Moreno-Armella, 2010; Moreno-Armella & Hegedus, 2009). They realized that in addition to its location dragging the hotspots of a dynamic figure may change the properties of that figure. Their appropriation of dragging was also evident through discussing the GeoGebra-generated colors of points their behavior. They became aware of the different colors for independent, partially-dependent, and dependent points and that allowed them to identify dependent relations among geometric objects. It also enabled them improve their use of dragging. Their dragging became more purposeful and efficient. Different dragging modalities (Arzarello et al., 1998; Arzarello et al., 2002; Baccaglini-Frank & Mariotti, 2010) were evident in teachers’ interactions with geometrical tasks in the environment. Teachers used wondering dragging early in their instrumentation to look for regularities in dynamic figures and moved to more purposeful dragging such as maintaining dragging, dragging with trace, and dragging test. Appropriating dragging influenced teachers’ mathematical activities significantly. It helped them understand objects properties and relations among different geometric objects through engaging in meaningful experiments (Hölzl, 1996).

The second study, presented in Chapter 4, investigated longitudinally how
teachers appropriated collaborative, dynamic geometry environment and how this appropriation shapes their actions while solving geometrical tasks. It examined the work of two high school mathematics teachers. The teachers collaborated as a team in a professional development course for 15 weeks to solve 55 geometrical tasks. Similar to what Guin and Trouche (1998) found, this study showed that appropriating mathematical tools and using them as instruments is a complex, slow process. The analysis of this team of teachers’ work showed that they used three technological and cognitive tools while engaging with the tasks: dragging, Algebra view, and congruency and similarity among geometric objects. As mentioned above, appropriating dragging relied on the co-active nature of DGE and identifying the hotspots of dynamic figures (Hegedus & Moreno-Armella, 2010; Moreno-Armella & Hegedus, 2009). Teachers’ use of Algebra View aligned with the work of Olivero and Robutti (2007) who investigated how learners use measurements in DGEs. The teachers used Algebra View to identify objects properties and relations among different objects through examining measurements of segment lengths, angle measures, and the slopes of lines. They used their examination of these measurements to conjecture and verify their conjectures, and then, to justify them. Teachers’ appropriation of these tools influenced their understanding of geometrical properties and relations. The longitudinal discursive and inscriptive data of teachers’ interactions in VMTwG revealed that teachers’ appropriation of technological tools co-emerged with their mathematical ideas. Teachers used dynamic geometry tools instrumentally while working on geometrical tasks and engaging with mathematical ideas. Their instrumental use of tools also evolved along the professional development course. Analyzing longitudinal interactions of how learners interact with DGEs provides
deeper insights into how they come to understand the environment and how they develop their mathematical knowledge (Stahl, 2015). While Stahl’s study (2015) is the first to offer a longitudinal analysis of students’ collaboration modes in collaborative DGE and the geometrical ideas in which they engaged, our investigation is the first longitudinal examination of teachers’ instrumental genesis and the extension of their geometrical knowledge. The present longitudinal study is important since it shows how teachers’ emergent mathematical ideas and actions early in the 15-week professional development course influence later ideas and actions. This investigation contributes to understanding the complex, slow process that within a collaborative virtual environment teachers experience appropriating new technological and cognitive tools.

The last study, presented in Chapter 5, used Rabardel and Beguin’s (2005) categories of instrument genesis in an instrument-mediated activity to examine the mediational roles of the VMTwG environment throughout 15 weeks of teachers working on geometrical tasks and the tasks’ relations with teachers’ mathematical knowledge. The data for this study were the discursive and inscriptive interactions of four middle and high school mathematics teachers who worked in VMTwG as a team to construct geometric figures and solve open-ended geometrical problems. The results showed that teachers used the environment epistemically to investigate geometrical properties and relations and pragmatically to transform geometric objects into desired forms. These uses aligned with Rabardel’s epistemic and pragmatic mediations (Lonchamp, 2012; Rabardel & Beguin, 2005). Other researchers (Bussi & Mariotti, 2008; Falcade et al., 2007; Mariotti, 2000) also identified these types of mediation of DGEs. However, in our investigation, teachers’ interactions in VMTwG revealed a third type of mediation, what we call
'pedagogic mediation’. This mediational role is one where teachers use the environment to help other team members understand particular geometric objects and relations among them. This category of mediation contributes to the theory of instrumental genesis.

Results also showed that teachers’ specialized content knowledge (Ball et al., 2008) was evident when teachers used the environment epistemically. Working on tasks that required teachers to investigate geometric objects and relations and use the environment epistemically allowed teachers to extend their specialized content knowledge.

This dissertation study highlights the complex interactions among tools, instruments, and mathematical knowledge. Dynamic geometry environments can support the learning of geometry through meditating learners’ activity (Bussi & Mariotti, 2008; Stahl, 2015). For learners to appropriate tools of dynamic geometry and use them as instruments requires carefully designed tasks that support learners’ instrumentation (Trouche & Drijvers, 2014). During instrumentation, learners engage their mathematical knowledge and extend it. Instrumentation and learners’ mathematical knowledge in a collaborative dynamic geometry environment co-emerge while engaging in mathematical activities in such environment.

6.2 Limitations

This dissertation investigated teachers’ geometrical knowledge and appropriation of a collaborative dynamic geometry environment. However, it used the teams of teachers as the unit of analysis. It did not account deeply for how teachers interacted within their teams. It viewed teams’ products as a group product without attending to the group dynamics. The collaborative aspect is an important aspect that could be studied in the future. Another limitation of this study is that it did not link teachers’ work in the
professional development courses to their classroom practice. Even though studies showed that teachers’ engagement with dynamic geometry influences their classroom practice (Alqahtani & Powell, 2015; Sinclair & Yurita, 2008), careful examination of how teachers’ knowledge with dynamic geometry can transfer to their classroom practices is needed. Additionally, as Powell and Hanna (2006, 2010) argue, teachers’ mathematical knowledge can effectively be observed through their practice, which can be seen as the significant knowledge that teachers bring to their classrooms as a result of going through professional development. This knowledge can be similar to the knowledge that teachers demonstrate in the professional development programs, but is some cases it could be a modification of the knowledge they engage with in the professional development. An example of how teachers’ knowledge and practice may differ is the study of Viseu and Ponte (2012). They investigated a pre-service teacher’s lesson plans and tasks before teaching and observed how he modified his lessons and tasks in his teaching practice.

Lastly, this qualitative investigation relied on the theory of instrumental genesis to understand teachers’ appropriation of the dynamic geometry environment. Different lenses might provide other insights into how teachers use such environments and how that use links and influences to their mathematical knowledge.

6.3 Implications and Future Research

Centering teachers’ mathematical activity on exploring the properties of geometrical objects and relations among the objects as well as on using dragging to investigate the objects and relations evidenced to be valuable for developing teachers’ mathematical knowledge. Teachers’ also appropriated other affordances of the dynamic
geometry environment and used them while engaging in mathematical tasks. Appropriating these affordances allowed teachers to use DGE to investigate epistemically the mathematical objects and their relations. It also allowed them to use the environment pragmatically and transform geometric objects and constructions to certain form. These actions in the environment allowed teachers to engage their mathematical knowledge, specifically their subject matter knowledge. Moreover, working in a collaborative setting allowed teachers to use the DGE pedagogically to help other team members visualize certain properties and relations among geometric objects. It is important to support teachers to use dynamic geometry environments pedagogically in their classrooms. This issue is one direction for future research that focuses on supporting teachers’ implementations of collaborative dynamic geometry environments, especially when the role of teachers in the classroom is more virtual than presential (Powell & Alqahtani, 2015b).

A second implication of this dissertation study concerns how task should be designed in collaborative dynamic geometry environments. For such an environment, findings of this study recommend task design and suggest creating coherent sequences of tasks. Chapter 3 and 4 showed that teachers’ appropriating of dragging was significant for developing their knowledge of dependencies among geometric objects. This confirms the importance of highlighting the dragging affordance of dynamic geometry environments in mathematical tasks. Analysis of the longitudinal data in Chapters 4 and 5 provides recommendations for designing coherent sequences of tasks that can be implemented over time in collaborative dynamic geometry environments. Findings in Chapter 4 highlight the importance of certain tools in dynamic geometry environment such as
dragging and Algebra View as well as cognitive tools like congruency and similarity among geometric objects. Task designers should attend to supporting learners’ instrumentation of these tools in earlier tasks to enable them to use these tools instrumentally with later tasks.

The design of this dissertation study can support the investigation of mathematics teachers noticing. It attends to helping teachers focus their mathematical activity on noticing properties of mathematical objects and relations among these objects. It also encourages teachers to attend each team members’ work and contributions. This can help teachers build deeper understanding of how students respond to mathematical tasks and how to support their productive noticing and mathematical discourse. Teachers can support students’ mathematical discourse by attending to students’ reasoning and problem-solving strategies (Jacobs, Lamb, & Philipp, 2010), which requires teachers to be aware of multiple ways of reasoning and different strategies that students may use. Research has shown that having teachers observe students solve mathematical problems can support their noticing of students’ thinking (Sherin & van Es, 2005; Star & Strickland, 2008; Van Es & Sherin, 2008). Nevertheless, it is essential to provide opportunities for teachers to engage in problem-solving tasks and develop their own reasoning and strategies so that they can better notice and appreciate students’ reasoning and strategies.

Future research can fruitfully include aspects of collaboration in dynamic geometry environments to understand how individuals appropriate and develop their mathematical knowledge within teams. Investigating the co-emergence of learners’ instrumentation and mathematical knowledge in areas other than geometry such as
algebra and calculus is another direction for future research. Dynamic geometry environments provide tools that support learning and teaching of many mathematical subjects including some advanced topics such as differential geometry and complex analysis. Investigating these topics can support understanding of teachers’ specialized content knowledge in different areas of mathematics. Additionally, the pedagogic mediational role of technological tools can provide insights into teachers’ pedagogical content knowledge and its influence on their classroom practice.

Finally, this dissertation study can be extended to examine the appropriation of different technologies in learning and teaching mathematics and investigate how learners’ appropriation of these technologies influences the development of their mathematical knowledge. Research is needed to investigate how technologies mediate learners’ mathematical activities and what kind of mathematics students construct with different technologies.
Appendix A: Tasks in VMTwG

Module 1: Before Constructing Together: Individual Activities

On your own, do these warm-up tasks. Do them on the computer that you will be using with your team.

When you are finished with one task, move on to the next one. You can always go back to a room for an old task if you get more ideas or want to work more on that task.

Task 1: Welcome

Read and follow the instructions. Write in the chat window what you notice about the behavior of the shown objects and relationships among them.

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This is the most recent version of the tasks that we used in the professional development course in the semester of fall 2015. This set of tasks is a result of four years of revising and testing tasks in VMTwG.
Next, create your own similar geometric objects using the buttons in the Tool Bar. Drag them around and notice if and how they change. To create a triangle or any other polygon, use the polygon tool and click to make points to define the polygon and then click again on the first point to complete it.

**Task 2: Helpful Hints**

In this tab are some hints for taking advantage of the limited space on your computer.

Make your VMTwG window as big as possible by dragging its lower right corner.

There are a number of Zoom Tools, which can be pulled down from the Move Graphic Tool (the crossed arrows on the right end of the Tool Bar).
If you are using a touchpad on your computer, you can zoom with a two-finger touchpad gesture. **Caution:** It is easy to move things around without wanting to as your fingers move on the touchpad. Things can quickly zoom out of sight. This can happen even when you do not have control of construction. Then you will not see the same instructions and figures as others on your team.

**Note:** Most Zoom Tools will only effect what **you** see on your computer screen, **not** what your teammates see on theirs. It is possible for you to create points on an area of your screen that your teammates do not see. If this happens, ask everyone if they would like for you to adjust everyone’s screen to the same zoom level. If they do, then select the menu item “GeoGebra” | “Share Your View.”

When you work in a team, it is important to have everyone agree before changes are made that affect everyone. If you want to delete a point or other object – especially one that you did not create – be sure that everyone agrees it should be deleted. **Note:** Undo does **NOT** work in VMTwG.

**Caution:** Be very careful when **deleting** points or lines. If any objects are dependent on them, those objects will also be deleted. It is easy to unintentionally delete a lot of the group’s work.

Instead of deleting objects, you should usually **hide** them. Then the objects that are dependent on them will still be there and the dependencies will still be in effect. Use control-click (on a Mac) or right-click (in Windows) with the cursor on a point or line to bring up the context-menu. Select “Show Object” to unclick that option and hide the object. You can also use this context-menu to hide the object’s label, rename it, etc.
You may want to **save** the current state of your GeoGebra tab to a .ggb file on your desktop sometimes so you can load it back if things are deleted. Use the menu “File” | “Save” to save your work periodically. Use the menu “File” | “Open” to load the latest saved version back into the current tab. Check with everyone in your team because this will change the content of the tab for everyone.

If the instructions in a tab are somehow erased, you can look them up in this document. You can also scroll back in the history of the tab to see what it looked like in the past. Finally, you can have your whole team go to a different room if one is available in the VMT Lobby that is not assigned to another team.

If you have **technical problems** with the chat or the figures in the tabs not showing properly, you should probably close your VMT window and go back to the VMT Lobby to open the room again.

### Module 2: Messing Around with Dynamic Geometry

In this topic, you will practice some basic skills in dynamic geometry with your teammates.

To fully take advantage of this course, we have found it particularly helpful to develop a particular set of cognitive tools or habits of the mind.

As you and your teammates engage the Modules, in the chat, we encourage you to discuss explicitly three geometrical perceptions: (1) what you **notice**, (2) what it **means** to you, and (3) what you **wonder** about. In the chat window, you should articulate clearly what **quantities** or **qualities** (objects) and what **relationships** (relations among the objects and even relations among the relations) you notice. Afterward, list what you wonder
about these objects and relations. When you discuss the things that you notice, talk about not only what you see but also underlying geometrical ideas and relationships. Detailed discussions of these perceptions, quantities, qualities, and relationships should become a habit of how you and your teammates interact in VMTwG.

In this course, we are interested how your team discusses what you notice, what geometrical ideas you develop, and what geometrical reasoning you use. Right answers are not the main goal of the tasks in which you will engage, so use only the ideas that your teammates and you jointly developed, not ideas, for instance, from the Internet.

**Working Collaboratively**

Since the goal of this course is for you to think together with your teammates about mathematical objects and relations and to communicate effectively, you need to collaborate productively. As you interact in VMTwG, keep in mind the following general guidelines for collaborative work:

<table>
<thead>
<tr>
<th>READ CHAT POSTINGS TO</th>
<th>WRITE CHAT POSTINGS TO</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. be prepared to refer to and connect to someone else’s ideas.</td>
<td>1. make your thinking available for the group to use.</td>
</tr>
<tr>
<td>2. get thoughts on open questions.</td>
<td>2. develop your thinking.</td>
</tr>
<tr>
<td>3. get new perspective on your thoughts.</td>
<td>3. get feedback on your ideas.</td>
</tr>
<tr>
<td>4. give feedback to others.</td>
<td></td>
</tr>
</tbody>
</table>

In general, try to include in your chat statements like these: (a) what I think should be done, (b) what I am doing and (c) the significance of what I/you did. Take steps as a group, rather than just trying to do the whole thing by yourself.

**Task 3: Dynamic Points, Lines and Circles**

Everything in geometry is built up from simple **points**. In dynamic geometry, a point can be dragged from its current position to any other location. For instance, a line **segment** is made up of all the points (the “**locus**”) along the shortest (direct, straight) path between two points (the **endpoints** of the segment).

A **circle** is all the points (“circumference,” “locus”) that are a certain distance (“**radius**”) from one point (“**center**”). Therefore, any line segment from the center point of a circle to its circumference is a radius of the circle and is necessarily the same length as every other radius of that circle. Even if you drag the circle and change its size and the length of its radius, every radius will again be the same length as every other radius of that circle.

In this activity, create some basic dynamic-geometry objects and drag them to observe their behavior. Do not forget that you have to press the “Take Control” button to take actions in GeoGebra.
Task 4: Constructing Stick Figures

You can be creative in GeoGebra. This activity shows a Stick Woman constructed in GeoGebra. Make other stick people and move them around.

Notice that some points or lines are **dependent on** other points or lines. Dependencies like this are very important in dynamic geometry. We will explore how to analyze, construct and discuss dependencies in the following tasks.
After this and every other Module, post a summary in the Discussion Board in Blackboard of the results of your team’s VMTwG activities that responds to the questions in the text box below. Your summary will be available course instructors and the other teams.

In the summary in Blackboard, list your team’s responses to these questions:

What did your team notice about geometric objects and relations among the objects?

What geometric meaning did your team give to what it noticed?

What did your team wonder about the geometric objects and relations it noticed?

Module 3: Constructing Dynamic-Geometry Objects

In this section, you will practice some basic skills in dynamic geometry.

As your teammates and you work on the tasks in this and all other modules, remember to
**discuss explicitly in the chat** these three questions:

- What you *notice* about geometric objects and relations among the objects?
- What geometric *meaning* do you give to what you noticed?
- What do you wonder about the geometric objects and relations you noticed?

These **key discussion questions** are cognitive tools that you want to ask habitually as your mind engages with tasks in this course.

The main goal of these tasks is not to for you to get the right answer but to explore and develop your own geometric ideas. So, use only the ideas that your teammates and you jointly develop, not ideas from the Internet or elsewhere.

**Discuss what you are doing in the chat.** This way you have a record of your ideas. In general, try to say in the chat what you plan to do before you do it in GeoGebra. Then, say what you did in GeoGebra and how and why you did it. Let other teammates try to do it, too. Finally, chat together about the significance of what you all did.

Take turns. Work together as a team, rather than just trying to figure things out by yourself. Here are more suggestions that may help your team have successful collaboration:

1. Discuss things and ask questions.
2. Include everyone’s ideas.
3. Ask what your team members
4. Discuss things and ask questions.
5. Include everyone’s ideas.
6. Ask what your team members
7. Discuss things and ask questions.
8. Include everyone’s ideas.
9. Voice all doubts, questions, and critiques.
10. Ensure everyone’s contributions
think and what their reasons are. are valued.

4. Cooperate to work together. 11. Decide what to focus on, have

5. Listen to each other. ways of keeping track of and

6. Agree before deciding. returning to other ideas and

7. Make sure all of the ideas are on questions, and use multiple

the table. approaches.

8. Consider all ideas put forth, no 12. Be sure that each team member

matter how promising or relevant. knows how to construct figures in
each task.

Recall that for the effective discursive and mathematical functioning of your team, for
each task, it is important that each team member can eventually construct specific
g geometric figures. For this reason, the course Modules provide opportunities for you to construct individually as well as collaboratively specific geometric figures that
embody particular geometric properties or relations such as constraints and dependencies.

| Task 5: The Drag Test |

When you construct a point to be on a line (or on a segment, or ray, or circle) in dynamic
gometry, it is constrained to stay on that line; its location is dependent upon the location
of that line, which can be dragged to a new location.

Use the “drag test” to check if a point really is constrained to the line: select the Move
tool (the first tool in the Tool Bar, with the arrow icon), click on your point and try to
drag it; see if it stays on the line. Drag an endpoint of the line. Drag the line. What
happens to the point? Make sure that everyone in your team can drag objects around and notices what happens.

![Diagram of geometric figures]

**Task 6: Dependent Geometric Figures**

Take control and construct some lines and segments with some points on them, like in the example shown in the activity. You’ll notice the following:

- Some points can be dragged **freely**.

- Some can only be dragged in certain ways, and we say they are partially **constrained**.

- Others cannot be dragged directly at all, and we say they are fully **dependent**.
In the summary in Blackboard, list your team’s responses to these questions:

What did your team notice about geometric objects and relations among the objects?

What geometric meaning did your team give to what it noticed?

What did your team wonder about the geometric objects and relations it noticed?

Module 4: Constructing an Equilateral Triangle

The construction of an **equilateral triangle** illustrates some important ideas in dynamic geometry. With the tasks in this Module, you will explore that construction.

**Task 7: Constructing an Equilateral Triangle**
This simple example shows important features of dynamic geometry. Using just a few points, segments and circles (strategically related), it constructs a triangle whose sides are always equal no matter how the points, segments or circles are dragged. Using this construction, the triangle **must** be equilateral (without having to measure the sides or the angles).

Everyone on the team should construct an equilateral triangle. First, drag the one that is there to see how it works. Take turns controlling the GeoGebra tools.

**Task 8: Finding Dynamic Triangles**

There are many types of triangles. A generic or arbitrary triangle has no special constraints on its sides or angles. In dynamic geometry, such a triangle may be dragged into special cases; it can look like, for instance, a right triangle or an equilateral triangle.
However, it will not have the constraints of a right angle vertex or have equal sides built into it by its construction, so it will not necessarily retain the special-case characteristics of a right or equilateral triangle when it is dragged again.

Triangles can be looked at in terms of sides. A scalene (Greek: skalenos - "uneven, unequal") triangle has no two sides of equal length. An isosceles triangle has two sides of equal length. An equilateral triangle has three sides of equal length.

Triangles also can be classified in terms of angles: acute, obtuse, and right triangle.

In the geometric figure constructed in this task, many triangles can be found. Explore the relationships that are created among line segments and angles. If any segments or angles are equal, how did the construction of the figure make them equal? Can you justify why triangle ABC is always equilateral? Justify other types of triangles you identified in the figure.

When you drag point F, what happens to triangle ABF or triangle AEF? In some positions, it can *look* like a different kind of figure, but it always has certain relationships. What kinds of angles can you find? Are there right angles? Are there lines perpendicular to other lines? Are they always that way? Do they have to be? Can you explain why they are?

In this task, we introduce a Checkbox for providing hints. Work as team on the task without peeping at the hints unless your team and you jointly decided that you have exhausted your ideas.
Task 9: Constructing Triangles

Constructing triangles can be done in many ways. In this task, you will explore some of them.

In the summary in Blackboard, list your team’s responses to these questions:

What did your team notice about geometric objects and relations among the objects?
What geometric meaning did your team give to what it noticed?

What did your team wonder about the geometric objects and relations it noticed?

Module 5: Constructing using the Compass

In this Module, you will explore new tools to construct different geometric objects.

Task 10: Copying a Length

After Euclid constructed an equilateral triangle, he used that construction to show how to copy a segment length to another location. In dynamic geometry, this means making the length of a second segment (CH) dependent on the length of the first segment (AB).

![Diagram showing construction process]

Task 11: Using the Compass Tool
First, get a good introduction to using the GeoGebra compass tool from this YouTube clip: [http://www.youtube.com/watch?v=AdBNfEOEVco](http://www.youtube.com/watch?v=AdBNfEOEVco).

Everyone on the team should have a chance to try the compass tool.

Task 12: Making Dependent Segments

In this task, you will compare copying a segment length with *copy-and-paste* to copying the same segment using the *compass* tool.

Task 13: Adding Segment Lengths
In dynamic geometry, you can construct figures that have complicated dependencies of some objects on other objects. Here you will construct one segment whose length is dependent on the length of two other segments.

Task 14: Copying vs. Constructing a Congruent Triangle

Two segments are called congruent if they are equal in length.

Two triangles are called congruent if all their corresponding sides are congruent.

In this task, a triangle ABC is copied two ways: with copy-and-paste to create A₁B₁C₁ and by copying each side length with the compass tool to construct triangle DEF. Your construction should look similar to what you see below.

Which triangles do you think will stay congruent with the drag test?
Task 15: Constructing a Congruent Angle

Two angles are called congruent if they are equal in measure. Two triangles are called similar if all their corresponding angles are congruent.

When are two similar triangles congruent?

There is no tool in GeoGebra for copying an angle, but you can use the compass tool to do it in a couple of steps. Watch this YouTube clip:

https://www.youtube.com/watch?v=EiceoNtQhaE. It shows how to copy an angle in GeoGebra. Chat about why the technique for copying an angle works.

When you drag point A, what happens to the sides of triangle DEN compared to the corresponding sides of triangle ABC? What happens to the angles of triangle DEN compared to corresponding angles of triangle ABC?
In the summary in Blackboard, list your team’s responses to these questions:

What did your team notice about geometric objects and relations among the objects?

What geometric meaning did your team give to what it noticed?

What did your team wonder about the geometric objects and relations it noticed?

Module 6: Constructing Other Triangles

A triangle seems to be a relatively simple geometric construction: simply join three segments at their endpoints. Yet, there are many surprising and complex relationships possible in triangles with different dependencies designed into them.
Task 16: Triangles with Dependencies

Objects in dynamic geometry can change their shapes. A triangle might appear to look isosceles or right or even equilateral, but then change its visual appearance when it is dragged. It is only isosceles or right or equilateral if it was constructed to be that way with constraints or dependencies constructed into it. Here are some triangles that were constructed with hidden dependencies.

Task 17: An Isosceles Triangle

An isosceles triangle has two sides that are always the same length.
**Task 18: A Right Triangle**

A triangle with two sides perpendicular to each other is a right triangle. Here, without using measurements, you will construct a right triangle so that one side is always perpendicular to another, but the sides can be any lengths.

In this task, we introduce a Checkbox for providing challenge tasks. After you work as team on the original task, you may jointly decide to tackle the challenge task.
Task 19: An Isosceles-Right Triangle

How can you combine the construction methods for an isosceles triangle and for a right triangle to design a triangle that has a right angle and two equal sides?

Be sure to take turns controlling the GeoGebra tools and chat about what you are doing. Work together—do not just try to solve something yourself and then explain it to your teammates. Talk it out in the chat:

- Make sure everyone in the team understands what was done and can do it.
- Discuss what you notice and what you wonder about the construction.
- Discuss why it worked: why is the triangle you constructed always isosceles and always right-angled no matter how you drag it?

In the summary in Blackboard, list your team’s responses to these questions:

What did your team notice about geometric objects and relations among the objects?
What geometric meaning did your team give to what it noticed?
What did your team wonder about the geometric objects and relations it noticed?

Module 7: Constructing Objects and Relations Using Circles

Circles are very useful for constructing figures with dependencies. Because all radii of a circle are the same length, you can make the length of one segment be dependent on the length of another segment by constructing both segments to be radii of the same circle. This is what Euclid did to construct an equilateral triangle. In this Module, you and your team will use circles to construct midpoints of segments, perpendicular lines, and parallel lines.

Task 20: Constructing the Midpoint

As you already saw, the construction process for an equilateral triangle creates a number of interesting relationships among different points and segments.

Note: You can change the “properties” of a dynamic-geometry object by first Taking Control and then control-clicking (on a Mac computer: hold down the “control” key and click) or right-clicking (on a Windows computer) on the object. You will get a pop-up menu. You can hide the object (but its constraints still remain), show/hide its label information, change its name or alter its other properties (like color and line style).
Task 21: Constructing a Perpendicular Line

For many geometry constructions (like constructing a right angle, a right triangle or an altitude), it is necessary to construct a new line perpendicular to an existing line. It may also be necessary to construct a perpendicular line to a segment through a certain point, which is not the midpoint of the segment. Without using the Perpendicular Line tool, how can you construct a perpendicular line to another line that goes through a certain point? Drag to test whether your line is always perpendicular and if so, why?
Task 22: Constructing a Parallel Line

In geometry, once you learn how to do one construction, you can use that as one step in a larger construction. Without using the Parallel Line tool, how can you construct a parallel line to another line?

Plan and construct the perpendicular bisector of line segment AB.

Plan and construct a perpendicular line to the perpendicular line you created at an arbitrary point C on the line.

Discuss what you notice about your final construction.

Discuss and justify whether what you notice is always or only sometimes true.
In the summary in Blackboard, list your team’s responses to these questions:

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What did your team wonder about the geometric objects and relations it noticed?

Module 8: Constructing Congruent Triangles

Two triangles are called “congruent” if all their six corresponding parts—angles and sides—are equal. Dynamic geometry can help to visualize the different combinations of their corresponding parts needed to make two triangles congruent.

Task 23: Combinations of Sides and Angles of Congruent Triangles

To have a triangle with a certain shape or size, what necessary sufficient conditions are needed on its sides and angles? Given a triangle with a certain shape and size, what are necessary and sufficient conditions are needed to create a triangle congruent to the given one? Two congruent triangles have six equal corresponding parts (three sides and three angles).

However, to guarantee congruency between the two triangles, is it necessary to constrain all six parts to be equal? For instance, as you have seen in Task 15, two triangles with their corresponding 3 angles equal are called “similar” but they may not be “congruent”. Are there other combinations of corresponding sides and angles that will make two triangles similar?

If two triangles have their corresponding angles equal and then you constrain two pairs of corresponding sides to be equal, will the triangles necessarily be congruent? Suppose you
only constrain one pair of the corresponding sides to be equal, will the triangles necessarily be congruent?

Given an arbitrary triangle ABC, the table below shows all possible combinations of constraints on the number of sides and angles of a second triangle that can be congruent to triangle ABC. For example, 1s1a represents a situation in which one side and one angle of the second triangle is congruent to the corresponding side and angle in triangle ABC. Similarly, 3s2a represents a situation in which three sides and two angles of the second triangle are congruent to corresponding sides and angles in triangle ABC. What does 2s3a represent?

Examine the combinations in the table and discuss what kind of triangle (congruent, similar, or neither congruent nor similar) each combination will produce in relation to a given triangle, triangle ABC. Which combinations always guarantee that a second triangle is congruent to triangle ABC? Sometimes congruent? Which combinations represent minimal conditions to guarantee congruency?

<table>
<thead>
<tr>
<th>Number of Sides</th>
<th>0a</th>
<th>1a</th>
<th>2a</th>
<th>3a</th>
</tr>
</thead>
<tbody>
<tr>
<td>0s</td>
<td>0s0a</td>
<td>0s1a</td>
<td>0s2a</td>
<td>0s3a</td>
</tr>
<tr>
<td>1s</td>
<td>1s0a</td>
<td>1s1a</td>
<td>1s2a</td>
<td>1s3a</td>
</tr>
<tr>
<td>2s</td>
<td>2s0a</td>
<td>2s1a</td>
<td>2s2a</td>
<td>2s3a</td>
</tr>
<tr>
<td>3s</td>
<td>3s0a</td>
<td>3s1a</td>
<td>3s2a</td>
<td>3s3a</td>
</tr>
</tbody>
</table>
In this Task’s GeoGebra file, the hash marks represent congruency between the sides of triangle ABC and DEF. The angles with same names ($\alpha$, $\beta$, $\gamma$) have the same measures. See Figure 8.1.

![Figure 8.1](image)

**Task 24: Side-Side-Side (SSS)**

If all three sides of one triangle are equal to the corresponding sides of another triangle, then the two triangles are congruent. This is called the “Side-Side-Side” (or “SSS”) rule.

In dynamic geometry, you can make the 3 side lengths of one triangle be dependent on
the corresponding 3 side lengths of another triangle, ensuring that the two triangles will always be congruent.

**Task 25: Side-Angle-Side (SAS)**

In this Task, you will construct a congruent triangle to a given triangle by duplicating two sides and the angle between them.

To remember how to duplicate an angle, you may want to review Task 15 and watch this YouTube clip: [https://www.youtube.com/watch?v=EiceoNtQhaE](https://www.youtube.com/watch?v=EiceoNtQhaE).
Task 26: Angle-Side-Angle (ASA)

In this Task, you will construct congruent triangles by duplicating two angles and the side between them.

Task 27: Side-Side-Angle (SSA)

What if two corresponding sides and an angle are equal, but it is not the angle included between the two sides?
In the summary in Blackboard, list your team’s responses to these questions:

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Module 9: Constructing Transformations

There are several rigid transformations in dynamic geometry, which move an object around without changing its size or shape. See the tool menu for GeoGebra transformation tools.
* **Translate by Vector** -- creates a copy of the object at a distance and in a direction determined by a vector (a segment pointing in a direction).

* **Reflect about Line** -- creates a copy of the object flipped across a line.

* **Rotate around Point** -- creates a copy of the object rotated around a point.

**Task 28: Translating by a Vector**

In GeoGebra, a transformation of a figure like triangle ABC creates a congruent figure that is dependent on the original figure. In a translation, the new figure is moved or displaced in the distance and direction indicated by a vector. A vector is like a segment except that it has a particular direction as well as a length.

In this Task, triangle ABC is translated to triangle XYZ. Construct your own figure and vector. Translate your figure a couple of times and do a drag test to see if the figures stay congruent and what difference different vectors make.
Task 29: Reflecting About a Line

Take turns to drag the figures in this Task and chat about what you notice.

Task 30: Rotating Around a Point

Click on the hints. Chat about what they show. Create your own rotated figures.
Task 31: Combining Transformations

You can combine transformations. For instance, you can first translate and then rotate an image. Do you get the same result if you rotate first and then translate?

When you combine two or more different kinds of transformations, which transformations have to be done in a certain order and which combinations are the same in any order?

Task 32: Transformations Game

Rigid transformations have been used to move triangle A to the location of triangle B.

Work as a team to transform triangle A to triangle B without moving the triangle outside the shaded area. All transformations must take place within the shaded area.
In the summary in Blackboard, list your team’s responses to these questions:

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**Module 10: Exploring Advanced Constructions**

There are a number of ways to define the “center” of a triangle, each with its own special properties. In this Module, you will explore and construct different centers of triangles.

**Task 33: Perpendicular Bisectors of the Sides of a Triangle**
This Module begins with an exploration of the perpendicular bisectors of the sides of triangles.

**Task 34: The Circumcenter of a Triangle**

Make sure every member of your team participates in the discussion and in taking control of GeoGebra.
Why do you think the point of intersection of the perpendicular bisectors of the sides of a triangle is called the circumcenter of the triangle?

**Task 35: The Incenter of a Triangle**

In the previous two Tasks, you explored the circumcenter of a triangle. In this Task, you will extend this investigation to explore the intersection point of the angle bisectors of a triangle, called the incenter of the triangle.

Why do you think the point of intersection of the angle bisectors of the angles of a triangle is called the incenter of the triangle? What is it a center of?

**Task 36: The Inscribed Circle of a Triangle**
The point of intersection of the perpendicular bisectors of the sides of a triangle, the circumcenter, is the center of the circle that circumscribed the triangle. In similar manner, the point of intersection of the angle bisectors of the angles of a triangle, the incenter, is the center of the circle that is inscribed in that triangle.

**Task 37: The Centroid of a Triangle**

You have explored properties of circumcenters and incenters. You may have also explored relationships between the circumcenter and incenter of a triangle. In this Task, you will explore another intersection point, that of the medians of a triangle, called the centroid.
The centroid of a triangle is its geometric center. It is considered to be the balance point of the mass of a triangle.

**Task 38: The Orthocenter of a Triangle**

The “orthocenter” of a triangle is the meeting point of the three altitudes of the triangle. An “altitude” of a triangle is the segment that is perpendicular to a side and goes to the opposite vertex.
Another way of defining the orthocenter is to say that it is the intersection point of the orthogonal projections of the vertices.

In the summary in Blackboard, list your team’s responses to these questions:

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Module 11: Exploring Relationships among Centers of Triangles (Individual Activities)

Work on this topic on your own.

Task 39: The Euler Segment of a Triangle

In this Task, you will explore relationships among the four triangle centers you constructed in the previous module.
Like the Swiss mathematician named Euler, you probably discovered a relationship among three of a triangle’s centers. He discovered that in a triangle, the centroid always lies on a line segment between the circumcenter and the orthocenter. He did this in the 1700s—without dynamic geometry tools. In his honor, we call that segment between the circumcenter and the orthocenter, Euler’s segment. Euler’s work renewed interest in geometry and led to many discoveries beyond Euclid.

Task 40: The Nine-Point Circle of a Triangle

In a triangle, you can construct a circle that passes through a number of special points. In this task, you will explore relationships among different elements of a triangle. A number of centers and related points of a triangle are all closely related by Euler’s Segment.
Module 12: Solving Construction Problems

Here is a set of challenge problems for your team. Select three problems to start with and if you still have time, work on the last Task.

If the team does not solve them during its session, try to solve them on your own and report your findings in the next team session.

**Task 41: Reflections on Euler’s Segment and Nine-point Circle**

Discuss in the chat with your teammates your ideas and answers to the questions in Module 11.
Task 42: Justifying Perpendicular Lines

In this Task, work as a team on proving that a relationship between line segments of intersecting circles.

Explore the relationship between the line segment joining the centers of two intersecting circles and the line segment joining their two points of intersection for these three cases: 1) the two circles are congruent and the center of each is on the circumference of the other, 2) the two circles are congruent but the center of each is not on the circumference of the other, and 3) the two circles are not congruent.

Construct a convincing argument that justifies whether the line segment joining the centers of the two intersecting circles is perpendicular to the line segment joining their two points of intersection.

Task 43: Treasure Hunt

Can you discover the pot of gold in this tale told by Thales de Lélis Martins Pereira, a high school teacher in Brazil? You might want to construct some extra lines in the activity.
Task 44: Square and Circle

Determine the radius of the circle.

In the summary in Blackboard, list your team’s responses to these questions:

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Module 13: Constructing Quadrilaterals

In Latin, the term “quadrilateral” means “four-sided”. In geometry, a polygon with four sides forms a quadrilateral. In dynamic geometry, a polygon with four sides that remains four-sided when dragged is a quadrilateral. There are many different kinds of quadrilaterals.

Task 45: Construction Perpendicular and Parallel lines

Practice constructing perpendicular and parallel lines. You may wish to review Tasks 21 and 22 to remind yourself how to construct perpendicular and parallel lines.

Task 46: Dragging Different Quadrilaterals

There are many different quadrilaterals with specific construction characteristics and dynamic behaviors.
Identify the constraints on each of the quadrilaterals in this Task.

Task 47: Constructing Different Quadrilaterals

Construct the different kinds of quadrilaterals pictured in Task 46.

In the summary in Blackboard, list your team’s responses to these questions:

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**Module 14: Relations among Quadrilaterals**

*(Note: This is a long topic. Do not spend too much time on any one tab. Continue to work on the final tabs after your team session.)*

Quadrilaterals have special relations among their parts.

**Task 48: Hierarchy of Quadrilaterals**

Construct a hierarchy of types of quadrilaterals. How many types can you include? You can use the descriptions of constraints, use the common names like rhombus and kite or use a combination of these. You may wish to refer to Task 46 to remind yourself of the relations you identified among quadrilaterals.

**Task 49: Connecting the Midpoints of a Quadrilateral’s Sides**

Construct a quadrilateral and connect the midpoints of its sides to form another quadrilateral. Investigate your figure. Develop conjectures of what is always true about it. Justify your conjectures. What do you wonder about?
Task 50: Angle Bisectors of Quadrilaterals

Follow the instructions to construct the desired figure and discuss with your teammates the relationships you notice.

Task 51: Exploring the Angle Bisectors of a Parallelogram

In this Task, you will explore the angle bisectors of special quadrilaterals, conjecture about relations among these angle bisectors, and justify your conjectures.
In the summary in Blackboard, list your team’s responses to these questions:

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**Module 15: Construct Inscribed Polygons**

This section presents challenging problems for your team to construct.

**Task 52: Tangent Circles**

In this task, you will explore different relations among geometric objects and justify them.
Task 53: The Inscribed Triangles Challenge Problem

Construct a pair of inscribed triangles. First, explore the given figure. Note the dependencies in the figure. Then construct your own pair of inscribed triangles that behaves the same way. Triangle DEF is called “inscribed” in triangle ABC if its vertices D, E and F are on the sides of triangle ABC.

Task 54: The Inscribed Squares Challenge Problem
Try the same problem with inscribed squares.

Task 55: Proving Inscribed Triangles

Constructing figures in dynamic geometry—like the inscribed triangles—requires thinking about dependencies among points, segments and circles. You can talk about these dependencies in the form of proofs, which explain why the relationships among the points, segments and circles are always, necessarily true, even when any of the points are dragged around.

Many proofs in geometry involve showing that some triangles are congruent to others. Chat about what you can prove and how you know that certain relationships are necessarily true in your figure. Explain your proof to your team.

If you have not studied congruent triangles yet, then you may not be able to complete the proof. Come back to this after you study the activities on congruent triangles.
In the summary in Blackboard, list your team’s responses to these questions:

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1. Triangle ABC was constructed with point C at the intersection of two circles with radius AB and centers at A and B.

2. So you can prove that AC=AB and that BC=AB, because they are radii of the same circles. Therefore, triangle ABC is equilateral. Its sides are all equal and its angles are all equal.

3. If segments AD, CF and BE are constructed to be equal, can you prove that CD=BF=AE?

4. Can you prove the 3 small triangles are congruent? How?

5. What can you prove about triangle DEF? How?
References


Kiray, S. Alan (Eds.), *Proceedings of the International Conference on Education in Mathematics, Science & Technology* (pp. 84-94). Antalya, Turkey.


