OPTIMAL DESIGN OF LIFE TESTING FOR WEIBULL DISTRIBUTION LIFETIME UNITS

by

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ABSTRACT OF THE THESIS

Optimal Design of Life Testing for Weibull Distribution Lifetime Units

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Thesis Director:

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Accelerated life testing of manufactured units is performed in order to speed up testing through either reducing the time required for testing or establishing a predetermined number of failures to stop the test. In this research we develop two cost models for a Type-II censored testing for Weibull distribution life time units. We determine the optimal sample size on test which minimizes the expected total cost of performing the life testing when the parameters of the Weibull distribution are either fixed or unknown. Numerical examples are included to illustrate the cost models based on real data applications.

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Dedication

I would like to dedicate this study to my beloved grandparents, Mrs. Rita de Souza Barcelos and Mr. Osvaldo Barcelos, and eternals Mrs. Edina Cordeiro and Mr. Alcebíades Cordeiro.

To my parents, Mrs. Fátima Cordeiro and Mr. Gelson Cordeiro, and my sister, Mayara Cordeiro.

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Chapter 1

Introduction

The Weibull distribution has an important position in the reliability and life testing fields due to its mobility in fitting time-to-failure distributions. The most general Weibull distribution has the following three parameters: the shape parameter also known as the Weibull slope (β), the scale parameter (θ) and the location parameter (γ). However, the location parameter is rarely used and when set to zero reduces the most common Weibull distribution form to be the two-parameter one. There is also the one-parameter Weibull distribution, known as Exponential distribution, where the value of β is assumed to be known and equal to 1 [1], [14].

In the literature, it can be found various techniques to estimate the parameters of the Weibull distribution, such as the method of moments, the method of maximum likelihood estimation (MLE), and probability plot. For the MLE technique, it is necessary the use of software to solve the equations since it is necessary to solve the likelihood equations numerically [2].

Long testing times are required for highly reliable products before useful failure data becomes available, bringing the interest of manufacturing companies to speed up the required time for testing by a sample of units. Ways of accelerating the life test include: fixing the time that the test will be ended (Type-I censoring), fixing the number of failures as the premise to stop the test (Type-II censoring) or fixing the number of withdrawals at failure time (progressive Type-II censoring) [3]. Regarding to Type-II censored data life test, the user needs to fix the number of failures, r, before testing, which choice will depend on the balance of waiting time to finish the test and the risk of making an error in the product.

The associated cost with performing a Type-II censoring test will depend on how long the test is expected to last, how many samples are going to be placed under test and how unreliable the test is. Thus finding a balance between all those features, such as the total cost is minimized, can be a competitive factor for the companies. In the literature it can be found many studies concerning software testing and its cost models [4], [5], [6], [7], [8], [9], [10].

In this work we consider the two-parameter Weibull distribution for the lifetime product and use the MLE technique to estimate the Weibull parameters considering a Type-II censored life testing. The aim of this research is to develop two different testing cost models for Weibull distribution lifetime units and to determine the optimum sample size on test, considering the fixed number of failures equal 2, for each proposed model. The optimal sample size is given by the number of samples that minimizes the expected total testing cost assuming that the cost of waiting the test end per unit time, the cost of placing a unit on test, the cost of the waiting variance per unit time, the cost to place the testing and the associated cost of test unreliability (risk) are given (this last cost is considered only in the second cost model). In this research, we also develop the optimum sample size policy that minimizes the expected total cost subject to the unknown parameters of the Weibull distribution lifetime where some preliminary failure data is available.

The thesis is structured as follows. In Chapter 2 a general problem definition is presented while in Chapter 3 a description of the first Cost Model for Type II censored data life test is made and the mathematical results for the optimal testing sample size is demonstrate and then illustrate by numerical examples. Chapter 4 describes the second Cost Model and its results for the optimal design of life testing and shows numerical examples in order to illustrate the theoretical results. Finally, the conclusion and future research are presented in Chapter 5.

Chapter 2

Research Objectives

Accelerated life testing has been widely used by manufacturing companies since it reduces the required time for testing by a sample of units. A Type-II censored data life test fixes the number of failures as the premise to stop the test, and its expected total testing time is minimized when the sample size is increased. On the other hand, placing more units to test also increases the total testing cost.

When planning the test, besides the cost of the expected testing time and the cost of placing n units to test, engineers may want to consider some other cost factors such as the cost of the variance on the expected testing time and the cost of placing the test regardless the number of units (setup cost). Since there is a risk of testing units fail before the test time (test unreliability), it is also convenient to consider its associated cost in the total expected testing cost.

Since a Type-II censored data life test has a predetermined number of failures, the sample size of units being tested has a great influence in the total expected testing cost then, determining an optimal sample size n^* to place on test that minimizes this cost is valuable information for manufacturing companies.

In this work we develop two different testing cost models and determine the optimal sample size of Weibull distribution lifetime units that should be placed under test when the number of failures is fixed at 2.

Chapter 3

Cost Model with Considerations of Life Testing Type II Censoring (Model 1)

Assuming that the lifetime t of a product follows a Weibull distribution with parameters β and θ, the probability density function (pdf) $f(t)$ and the probability of an item failing before time t (cumulative distribution function) $F(t)$ are given in Eq. 1 and Eq. 2, respectively. The probability of the product not failing before time t is $1 - F(t)$ and is also called product reliability (Eq. 3) [11].

$$f(t) = \frac{\beta}{\theta^{\beta}} * t^{\beta-1} * e^{-(\frac{t}{\theta})^{\beta}} \tag{1}$$

$$F(t) = 1 - e^{-(\frac{t}{\theta})^{\beta}} \tag{2}$$

$$R(t) = e^{-(\frac{t}{\theta})^{\beta}} \tag{3}$$

A life test on a sample of n units is considered, in which the common underlying distribution of the length of a single unit is given by the Weibull probability density function as given in Eq. 1. The observed failures are denoted by r, where $(1 < r < n)$, and have $t_{i,n}$ defined as the time when the i^{th} failure occurs. The experiment is terminated at time $t_{r,n}$ which means as soon as the r^{th} failure occurs, known as Type-II censored data life test. To estimate the parameters β and θ we performed the MLE method [3] where the following steps were taken:

1) Obtain the likelihood function as it is shown in Eq. 4;
2) Take the derivative of the natural logarithm of the likelihood function (Eq. 5) with respect to β and θ (Eq. 6 and Eq. 7, respectively);
3) Set them equal to 0 and solve for the parameters β and θ (Eq. 8 and Eq. 9, respectively).

$$L(\theta,\beta) = \frac{n!}{(n-r)!} * \left(\frac{\beta}{\theta^{\beta}}\right)^{r} * e^{-\left(\frac{\sum_{i=1}^{r} t_i{}^{\beta}}{\theta^{\beta}}\right)} * \sum_{i=1}^{r} t_i{}^{\beta} * e^{-\left(\frac{t_r}{\theta}\right)^{\beta} *(n-r)} * \prod_{i=1}^{r} t_i{}^{\beta-1} \qquad (4)$$

$$\ln\big(L(\theta,\beta)\big) = const + r\ln(\beta) - r\beta\ln(\theta) + (\beta-1)\sum_{i=1}^{r}\ln(t_i) - \frac{1}{\theta^{\beta}}\left[\sum_{i=1}^{r} t_i{}^{\beta} + (n-r)t_r{}^{\beta}\right] \qquad (5)$$

$$\frac{\partial\ln\big(L(\theta,\beta)\big)}{\partial(\beta)} = \frac{r}{\beta} - r\ln(\theta) + \sum_{i=1}^{r}\ln(t_i) + \frac{1}{\theta^{\beta}}\left[\ln(\theta)\sum_{i=1}^{r} t_i{}^{\beta} - \sum_{i=1}^{r}(t_i{}^{\beta} * \ln(t_i)) + \ln(\theta)\,(n-r)t_r{}^{\beta} - \ln(t_r)(n-r)t_r{}^{\beta}\right] \qquad (6)$$

$$\frac{\partial\ln\big(L(\theta,\beta)\big)}{\partial(\theta)} = -\frac{r\beta}{\theta} + \frac{\beta}{\theta^{\beta+1}}\left[\sum_{i=1}^{r} t_i{}^{\beta} + (n-r)t_r{}^{\beta}\right] \qquad (7)$$

$$\hat{\beta} = \left[\ln(\theta) - \frac{\sum_{i=1}^{r}\ln(t_i)}{r} - \frac{1}{r*\theta^{\beta}}\left[\ln(\theta)\sum_{i=1}^{r} t_i{}^{\beta} - \sum_{i=1}^{r}(t_i{}^{\beta} * \ln(t_i)) + (n-r)t_r{}^{\beta}(\ln(\theta)-\ln(t_r))\right]\right]^{-1} \qquad (8)$$

$$\hat{\theta} = \left[\frac{1}{r}\left(\sum_{i=1}^{r} t_i{}^{\beta} + (n-r)t_r{}^{\beta}\right)\right]^{1/\beta} \qquad (9)$$

Let n be the number of units that are placed on test, the termination of the test be at the r^{th} failure and t_r be the length of the test time until the r^{th} failure occurs. Assuming that the unit lifetime follows a Weibull distribution with parameters β and θ, then the expected and the variance of the test time to end the test are given [12], respectively, as follows:

$$E(t_r) = \theta * \Gamma(\frac{1}{\beta} + 1) * \frac{n!}{(r-1)!(n-r)!} * \sum_{i=0}^{r-1} \frac{(-1)^i * \binom{r-1}{i}}{(n-r+i+1)^{1+\frac{1}{\beta}}} \quad (10)$$

$$V(t_r) = \theta^2 * \Gamma\left(\frac{2}{\beta} + 1\right) * \frac{n!}{(r-1)!(n-r)!} * \sum_{i=0}^{r-1} \frac{(-1)^i * \binom{r-1}{i}}{(n-r+i+1)^{1+\frac{2}{\beta}}} + E(t_r) - [E(t_r)]^2 \quad (11)$$

3.1 Cost Model Formulation ($C_1(n)$)

For the Cost Model 1, here referred as $C_1(n)$, we consider the expected total cost of a Type II censored life testing as the sum of the: (i) expected cost of waiting the test to finish, (ii) cost of all n items placed into test, (iii) fixed cost of setting up the test, and (iv) cost of variance of the expected testing time.

Let c_1 be the cost of waiting per unit time until the test is ended, c_2 be the cost of placing an item under test, c_3 be the cost to set up the testing and c_4 be the cost of expected testing time variance. The expected total cost proposed in this work is adapted from [6], [7] and [8], and it can be written as follows:

$$C_1(n) = c_1 E(t_r) + c_2 n + c_3 + c_4 V(t_r) \quad (12)$$

where $E(t_r)$, and $V(t_r)$ are given in Eq. 10 and Eq. 11, respectively.

3.2 Modeling Results

In this section, we present the mathematical solution for the optimal sample size that minimizes the total expected cost of testing ($C_1(n)$), with a fixed number of failures at $r = 2$, considering both the cases where the Weibull parameters are given and where they are unknown and some preliminary failure data is available.

3.2.1 Known Weibull Parameters

Assume that all the costs and Weibull parameters are given, and that $r = 2$. We can determine the optimal value n, say n^*, that minimizes the total expected cost $C_1(n)$ as shown in the following theorem.

Theorem 1: For given values $\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3$ and c_4, and r equal 2, there exists the optimum sample size n on test, say n^*, that minimizes the expected total cost, $C_1(n)$, subject to the known parameters β and θ of the Weibull distribution lifetime and

$$n^* = \inf\{n: G(n) \leq c_2\} \tag{13}$$

where

$$G(n) = (c_1 + c_4)\theta\Gamma\left(\frac{1}{\beta} + 1\right)\left[\frac{n}{(n+1)^{\frac{1}{\beta}}} + \frac{n}{(n-1)^{\frac{1}{\beta}}} - \frac{2n}{(n)^{\frac{1}{\beta}}}\right] + c_4\theta^2\Gamma\left(\frac{2}{\beta} + 1\right)\left[\frac{n}{(n+1)^{\frac{2}{\beta}}} + \frac{n}{(n-1)^{\frac{2}{\beta}}} - \frac{2n}{(n)^{\frac{2}{\beta}}}\right] + c_4\theta^2\Gamma\left(\frac{1}{\beta} + 1\right)^2\left[\left(\frac{n}{(n-1)^{\frac{1}{\beta}}} - \frac{n-1}{(n)^{\frac{1}{\beta}}}\right)^2 - \left(\frac{n+1}{(n)^{\frac{1}{\beta}}} - \frac{n}{(n+1)^{\frac{1}{\beta}}}\right)^2\right] \tag{14}$$

Proof: Define $\Delta C_1(n) = C_1(n+1) - C_1(n)$. From Eq. 12 we obtain:

$$\Delta C_1(n) = c_2 + (c_1 + c_4)\theta\Gamma\left(\frac{1}{\beta} + 1\right)\left[\frac{2n}{(n)^{\frac{1}{\beta}}} - \frac{n}{(n+1)^{\frac{1}{\beta}}} - \frac{n}{(n-1)^{\frac{1}{\beta}}}\right] + c_4\theta^2\Gamma\left(\frac{2}{\beta} + 1\right)\left[\frac{2n}{(n)^{\frac{2}{\beta}}} - \frac{n}{(n+1)^{\frac{2}{\beta}}} - \frac{n}{(n-1)^{\frac{2}{\beta}}}\right] + c_4\theta^2\Gamma\left(\frac{1}{\beta} + 1\right)^2\left[\left(\frac{n+1}{(n)^{\frac{1}{\beta}}} - \frac{n}{(n+1)^{\frac{1}{\beta}}}\right)^2 - \left(\frac{n}{(n-1)^{\frac{1}{\beta}}} - \frac{n-1}{(n)^{\frac{1}{\beta}}}\right)^2\right] \tag{15}$$

Eq. 15 can be simplified as follows:

$$\Delta C_1(n) = c_2 - G(n) \tag{16}$$

where $G(n)$ is as Eq. 14.

One has $\Delta C_1(n) \geq 0$, if and only if, $G(n) \leq c_2$. The function $G(n)$ can be empirically shown to be decreasing in n for all $n \geq r$, so there exists a value n^* such that $G(n) \leq c_2$. This implies that

$$n^* = \inf\{n: G(n) \leq c_2\}$$

3.2.2 Unknown Weibull Parameters

Assume that all the costs are given, that r is equal 2, and that preliminary failure time data is available. We can determine the optimal value n, say n^*, that minimizes the total expected cost $C_1(n)$ as shown in the following theorem.

Theorem 2: For given values $\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3$ and c_4, and $r = 2$, there exists the optimum sample size n on test, say n^*, that minimizes the expected total cost, $C_1(n)$, subject to the unknown parameters β and θ of the Weibull distribution lifetime and

$$n^* = \inf\{n: H(n) \leq c_2\} \tag{17}$$

where

$$H(n) = a_1\left(\frac{n}{(n+1)^{\frac{1}{\widehat{\beta_{n+1}}}}} - \frac{n+1}{(n)^{\frac{1}{\widehat{\beta_{n+1}}}}}\right)\left(c_1 + c_4\left(1 - a_1\left(\frac{n+1}{(n)^{\frac{1}{\widehat{\beta_{n+1}}}}} - \frac{n}{(n+1)^{\frac{1}{\widehat{\beta_{n+1}}}}}\right)\right)\right) + a_0\left(\frac{n-1}{(n)^{\frac{1}{\widehat{\beta_n}}}} - \frac{n}{(n-1)^{\frac{1}{\widehat{\beta_n}}}}\right)\left(-c_1 - c_4\left(1 - a_0\left(\frac{n}{(n-1)^{\frac{1}{\widehat{\beta_n}}}} - \frac{n-1}{(n)^{\frac{1}{\widehat{\beta_n}}}}\right)\right)\right) + c_4\left(b_0\left(\frac{n}{(n-1)^{\frac{2}{\widehat{\beta_n}}}} - \frac{n-1}{(n)^{\frac{2}{\widehat{\beta_n}}}}\right) - b_1\left(\frac{n+1}{(n)^{\frac{2}{\widehat{\beta_{n+1}}}}} - \frac{n}{(n+1)^{\frac{2}{\widehat{\beta_{n+1}}}}}\right)\right) \tag{18}$$

and

$$a_0 = \hat{\theta}_n \Gamma\left(\frac{1}{\widehat{\beta_n}} + 1\right) \tag{19}$$

$$a_1 = \hat{\theta}_{n+1} \Gamma\left(\frac{1}{\widehat{\beta_{n+1}}} + 1\right) \tag{20}$$

$$b_0 = \hat{\theta}_n{}^2 \left(\Gamma\left(\frac{2}{\widehat{\beta_n}} + 1\right)\right) \tag{21}$$

$$b_1 = \hat{\theta}_{n+1}{}^2 \left(\Gamma\left(\frac{2}{\widehat{\beta_{n+1}}} + 1\right)\right) \tag{22}$$

Proof: Define $\Delta C_1(n) = C_1(n+1) - C_1(n)$. From Eq. 12 we obtain:

$$\Delta C_1(n) = c_2 + a_1\left(\frac{n+1}{(n)^{\frac{1}{\widehat{\beta_{n+1}}}}} - \frac{n}{(n+1)^{\frac{1}{\widehat{\beta_{n+1}}}}}\right)\left(c_1 + c_4\left(1 - a_1\left(\frac{n+1}{(n)^{\frac{1}{\widehat{\beta_{n+1}}}}} - \frac{n}{(n+1)^{\frac{1}{\widehat{\beta_{n+1}}}}}\right)\right)\right) + a_0\left(\frac{n}{(n-1)^{\frac{1}{\widehat{\beta_n}}}} - \frac{n-1}{(n)^{\frac{1}{\widehat{\beta_n}}}}\right)\left(-c_1 - c_4\left(1 - a_0\left(\frac{n}{(n-1)^{\frac{1}{\widehat{\beta_n}}}} - \frac{n-1}{(n)^{\frac{1}{\widehat{\beta_n}}}}\right)\right)\right) + c_4\left(b_1\left(\frac{n+1}{(n)^{\frac{2}{\widehat{\beta_{n+1}}}}} - \frac{n}{(n+1)^{\frac{2}{\widehat{\beta_{n+1}}}}}\right) - b_0\left(\frac{n}{(n-1)^{\frac{2}{\widehat{\beta_n}}}} - \frac{n-1}{(n)^{\frac{2}{\widehat{\beta_n}}}}\right)\right) \tag{23}$$

where a_0, a_1, b_0 and $b_1$are given in Eq. 19, Eq. 20, Eq. 21, and Eq. 22, respectively.

Eq. 23 can be simplified as follows:

$$\Delta C_1(n) = c_2 - H(n) \tag{24}$$

where $H(n)$ is as Eq. 18.

One has $\Delta C_1(n) \geq 0$, if and only if, $H(n) \leq c_2$. The function $H(n)$ can be empirically shown to be decreasing in n for all $n \geq r$, so there exists a value n^* such that $H(n) \leq c_2$. This implies that

$$n^* = \inf\{n: H(n) \leq c_2\}$$

3.3 Numerical Examples

To illustrate the results of the theorems presented in subsections 3.2.1 and 3.2.2, in this section we show numerical examples where the optimal sample size of units, that should be placed under Type II censored data life test, is solved. All the calculations were made using MATLAB.

For all the examples we will assume $\mathrm{c}_1 = 25$ per unit hour, $\mathrm{c}_2 = 500$ per item, $\mathrm{c}_3 = 100$ per testing and $\mathrm{c}_4 = 15$ per unit hour.

3.3.1 Given Weibull parameters

In this subsection, we present two different examples to illustrate the mathematical solution of the optimum sample size assuming that the Weibull parameters are given as showed in *Theorem 1*.

Example 1

In this first example we assume that the Weibull parameters β and θ are known and their respective values are 1.646 and 162.2. The parameters and the cost values were used to obtain the values of the $G(n)$ function (Eq. 14) in MATLAB. The obtained results are shown in Table I.

Table I: Example 1 obtained results of $\boldsymbol{G(n)}$ and $\boldsymbol{C_1(n)}$

n	$G(n)$	$C_1(n)$
12	881.5184	16235.98
13	741.5568	15854.46
14	632.1418	15612.91
15	545.0369	15480.77
16	474.5974	15435.73
17	416.8516	15461.13
18	368.9402	15544.28
19	328.7632	15675.34

According to *Theorem* 1, and since $c_2 = 500$, we will search for the first value of n where its respective $G(n)$ value is less than or equal to 500. Thus, the optimum sample size is $n^* = 16$ units with a total expected testing cost of $\$15{,}435.73$.

Fig 3.1. graphically shows that $G(n)$ is non-increasing in n, while Fig. 3.2 shows a 3-D plot of $G(n)$, $\boldsymbol{C_1(n)}$ and n.

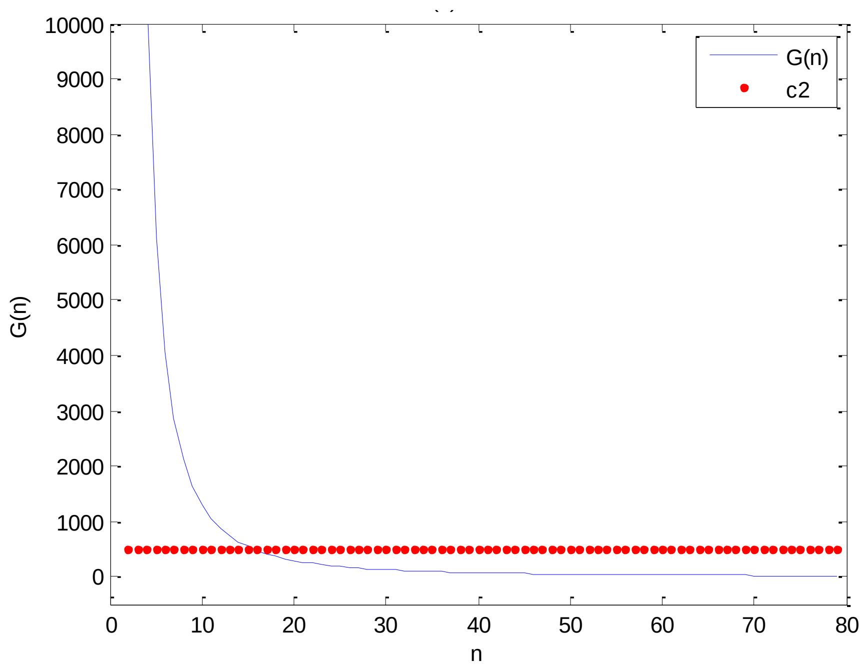

Figure 3.1 – $\boldsymbol{G(n)}$ function of example 1

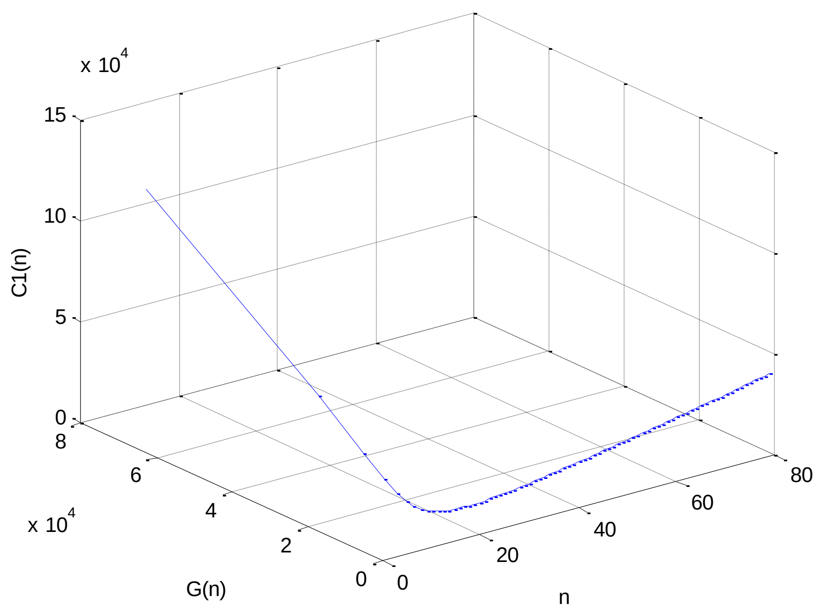

Figure 3.2 – 3-D plot ($\boldsymbol{G(n)}$ x $\boldsymbol{C_1(n)}$ x $\boldsymbol{n}$) of example 1

Example 2

In this example we consider different Weibull parameter values, which are $\beta = 1.044$ and $\theta = 362.2$. The obtained results can be seen in Table II.

Since we assume $c_2 = 500$, we will search again for the first value of n that satisfies $G(n) \leq 500$. As result, we can say that the optimum sample size is $n^* = 27$ units and the total expected testing cost is $\$21{,}091.90$.

Table II: Example 2 obtained results of $\boldsymbol{G(n)}$ and $\boldsymbol{C_1(n)}$

n	$G(n)$	$C_1(n)$
23	746.08	21,613.89
24	660.94	21,367.81
25	588.5	21,206.87
26	526.46	21,118.37
27	473.01	21,091.90
28	426.69	21,118.90
29	386.34	21,192.21
30	351.02	21,305.87

In Fig 3.3. it can be seen that $G(n)$ is non-increasing in n, while Fig. 3.4 shows a 3-D plot of $G(n)$, $\boldsymbol{C_1(n)}$ and n.

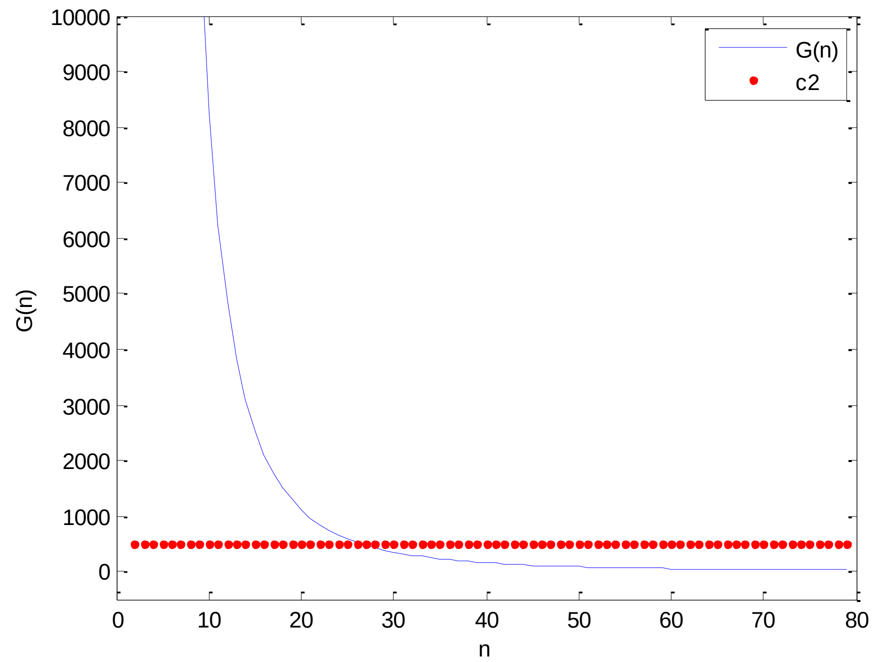

Figure 3.3 – $G(n)$ function of example 2

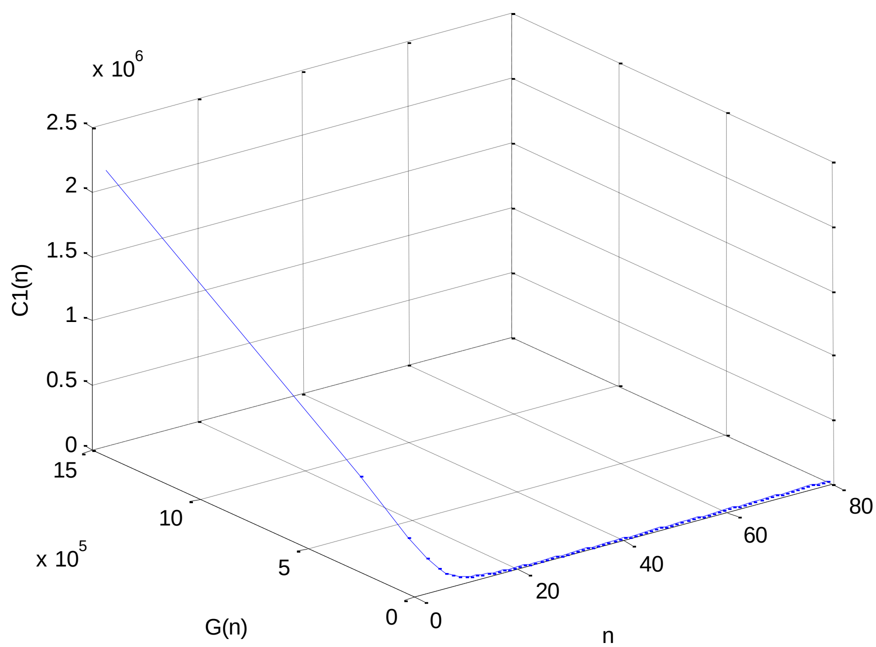

Figure 3.4 – 3-D plot ($G(n)$ x $C_1(n)$ x n) of example 2

3.3.2 Unknown Weibull parameters

In this subsection we show two different numerical examples in order to illustrate the mathematical solution of the optimum sample size for the case were the Weibull parameters are unknown (*Theorem 2*) and some preliminary failure data is available. This preliminary data is used to estimate the Weibull parameters $\hat{\beta}$ and $\hat{\theta}$ since they will vary for each different possible sample size n as it can be seen in Eq. 8 and Eq. 9, respectively.

Example 3

Here we will use preliminary failure time data from a literature example [3], where the two first failure events occur at $t_1 = 12.5$ and $t_2 = 24.4$. The failure times t_1 and t_2 are first used to obtain the n combinations of Weibull parameter estimators $\hat{\beta}$ and $\hat{\theta}$. The parameters and the cost values were used to obtain the values of the $H(n)$ function (Eq. 18). Thus, a sample of the obtained results of $\hat{\beta}$, $\hat{\theta}$ and $H(n)$ for each n are shown on Table III.

According to *Theorem* 2, and since $\mathrm{c}_2 = 500$, we will search for the first value of n that satisfies $H(n) \leq 500$. Thus, we can say that the optimum sample size is $n^* = 13$ units and the total expected testing cost is $\$18{,}450.78$.

Table III: Example 3 obtained results of $\hat{\beta}$, $\hat{\theta}$ and $H(n)$

n	$\hat{\beta}$	$\hat{\theta}$	$H(n)$
10	3.0763	182.42	692.62
11	3.068	182.7	601.61
12	3.0612	182.94	528.39
13	3.0555	183.16	468.52
14	3.0506	183.36	418.87
15	3.0464	183.54	377.18
16	3.0427	183.7	341.79
17	3.0395	183.86	311.47

In Fig 3.5. it can be graphically seen that $H(n)$ is non-increasing in n, while Fig. 3.6 shows a 3-D plot of $H(n)$ x $\hat{\beta}$ x n and $H(n)$ x $\hat{\theta}$ x n.

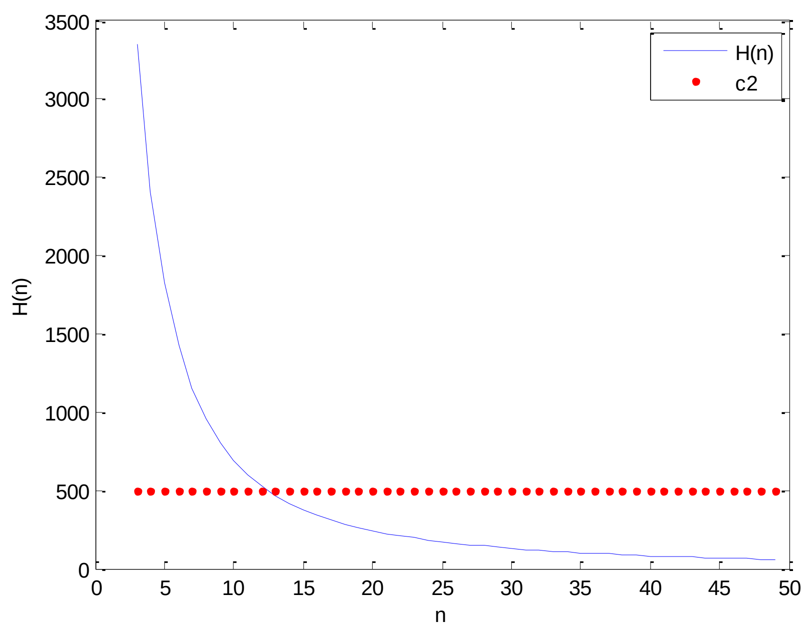

Figure 3.5 – $H(n)$ function of example 3

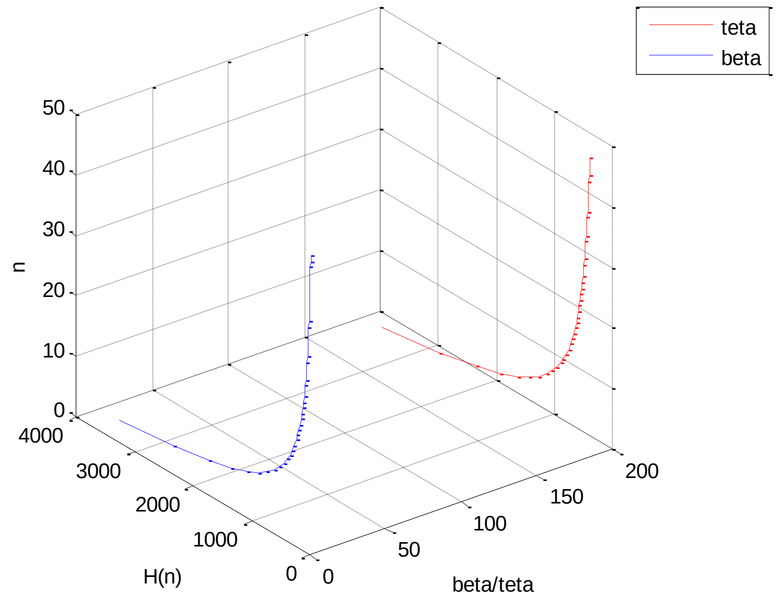

Figure 3.6 – 3-D plot ($H(n)$x$\hat{\beta}$xn and $H(n)$x$\hat{\theta}$x n) of example 3

Example 4

In this example we use real data as the preliminary failure time data [13] where the time of the first 2 failure events are $t_1 = 10$ and $t_2 = 16$. As in the previous example, the failures times (t_1, t_2) are first used to obtain the n combinations of Weibull parameter estimators $\hat{\beta}$ and $\hat{\theta}$. The estimators and the cost values are then used to obtain the $H(n)$ values. The obtained results can be seen on Table IV.

Considering that we assumed the same cost $\mathrm{c}_2 = 500$ for all examples, we look for the first n that satisfies $H(n) \leq c_2$. $H(51) = 491.99$ and $H(52) = 505.97$, we can say that the optimum sample size is $n^* = 51$ units and the total expected testing cost is $\$86{,}411.51$.

Table IV: Example 4 obtained results of $\hat{\beta}$, $\hat{\theta}$ and $H(n)$

n	$\hat{\beta}$	$\hat{\theta}$	$H(n)$
48	4.2796	712.55	536.025
49	4.2791	712.58	520.628
50	4.2786	712.61	505.965
51	4.2781	712.64	491.988
52	4.2777	712.67	478.651
53	4.2773	712.7	465.915
54	4.2769	712.73	453.742
55	4.2765	712.75	442.098

In Fig 3.7. it can be graphically seen that $H(n)$ is non-increasing in n, while in Fig. 3.8 it can be seen a 3-D plot of $H(n)$ x $\hat{\beta}$ x n and $H(n)$ x $\hat{\theta}$ x n.

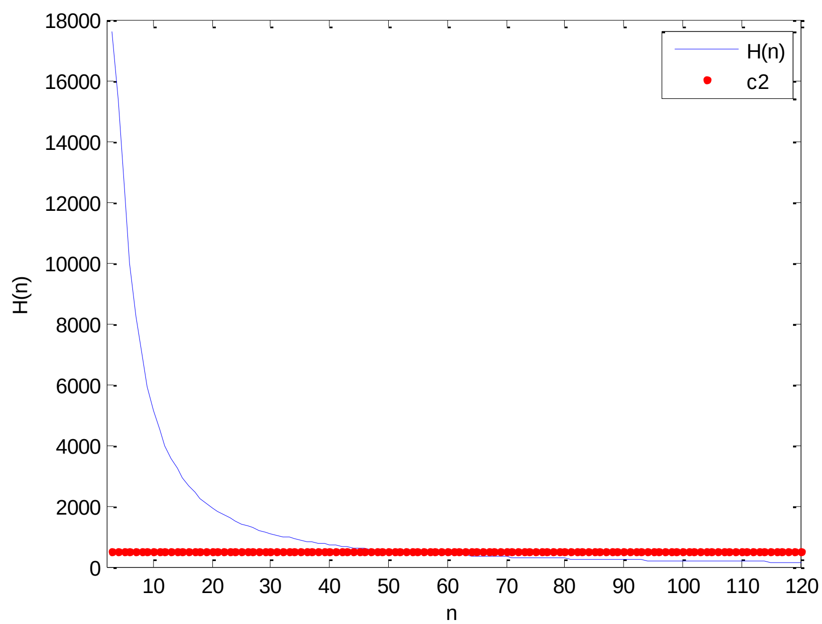

Figure 3.7 – $H(n)$ function of example 4

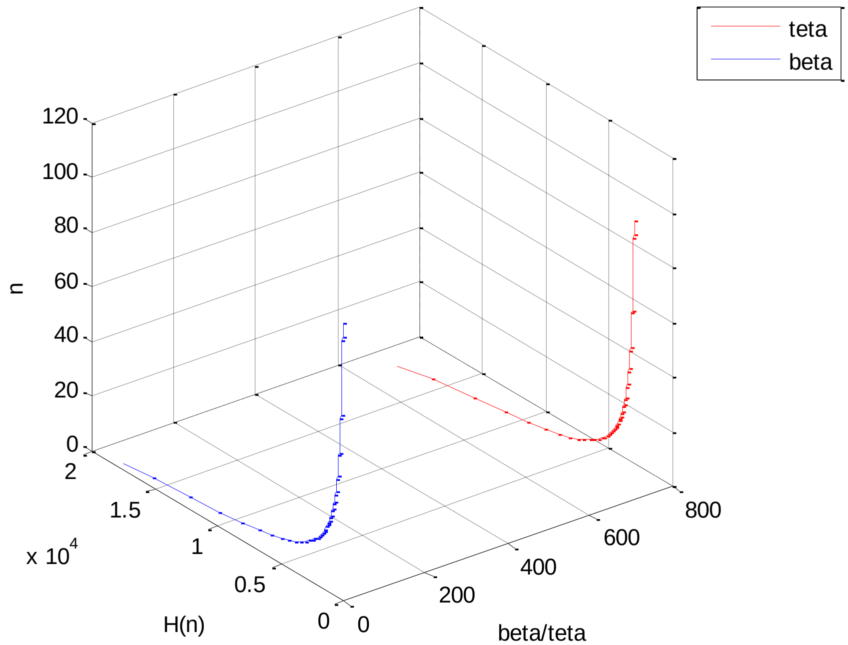

Figure 3.8 – 3-D plot of ($H(n)$x$\hat{\beta}$xn and $H(n)$x$\hat{\theta}$x n) of example 4

Chapter 4

Cost Model with Considerations of Life Testing Type II Censoring and Test Risk (Model 2)

In this section we are going to present the second Cost Model where the cost associated to test unreliability is also taken into account for a Life Testing Type II Censoring, and also show its mathematical optimum testing sample size and numerical examples.

4.1 Cost Model Formulation ($C_2(n)$)

In Cost Model 2, denoted as $C_2(n)$, we consider the expected total cost of a Type II censored life testing as the sum of the expected cost of waiting the test to finish (c_1), the cost of all items placed into test (c_2), the fixed cost of setting up the test (c_3), the cost of variance of the expected testing time (c_4) and the cost of the risk of testing units fail before the expected test time (c_5). The expected total cost model proposed is adapted from [6], [7] and [8], and can be written as follows:

$$C_2(n) = c_1 E(t_r) + c_2 n + c_3 + c_4 V(t_r) + c_5 (1 - R(E(t_r))) \quad (25)$$

where $R(E(t_r))$, $E(t_r)$, $V(t_r)$ are given in Eq. 3, Eq. 10 and Eq. 11, respectively.

4.2 Modeling Results

In this section we present the mathematical solutions of the optimal sample size that should be placed under test for the case where the Weibull parameters are given and for the case where they are unknown but some preliminary failure data is available.

4.2.1 Known Weibull parameters

Assuming that all the costs and the Weibull parameters are given, and that r is fixed at 2, we can determine the optimal value n, say n^*, which minimizes the total expected cost $C_2(n)$ as shown in Theorem 3.

Theorem 3: For given values $\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3, \mathrm{c}_4, \mathrm{c}_5$ and $r = 2$ there exists the optimum sample size n on test, say n^*, that minimizes the expected total cost, $C_2(n)$, subject to the known parameters β and θ of the Weibull distribution lifetime and

$$n^* = \inf\{n: I(n) \leq c_2\} \qquad (26)$$

where

$$I(n) = (c_1 + c_4)\theta\Gamma\left(\frac{1}{\beta}+1\right)\left[\frac{n}{(n+1)^{\frac{1}{\beta}}} + \frac{n}{(n-1)^{\frac{1}{\beta}}} - \frac{2n}{(n)^{\frac{1}{\beta}}}\right] + c_4\theta^2\Gamma\left(\frac{2}{\beta}+1\right)\left[\frac{n}{(n+1)^{\frac{2}{\beta}}} + \frac{n}{(n-1)^{\frac{2}{\beta}}} - \frac{2n}{(n)^{\frac{2}{\beta}}}\right] + c_4\theta^2\Gamma\left(\frac{1}{\beta}+1\right)^2\left[\left(\frac{n}{(n-1)^{\frac{1}{\beta}}} - \frac{n-1}{(n)^{\frac{1}{\beta}}}\right)^2 - \left(\frac{n+1}{(n)^{\frac{1}{\beta}}} - \frac{n}{(n+1)^{\frac{1}{\beta}}}\right)^2 + c_5\left(e^{-\left(\frac{E(t_{r,n+1})}{\theta}\right)^{\beta}} - e^{-\left(\frac{E(t_{r,n})}{\theta}\right)^{\beta}}\right)\right] \tag{27}$$

Proof: Define $\Delta C_2(n) = C_2(n+1) - C_2(n)$. From Eq. 25 we obtain:

$$\Delta C_2(n) = c_2 + (c_1 + c_4)\theta\Gamma\left(\frac{1}{\beta}+1\right)\left[\frac{2n}{(n)^{\frac{1}{\beta}}} - \frac{n}{(n+1)^{\frac{1}{\beta}}} - \frac{n}{(n-1)^{\frac{1}{\beta}}}\right] + c_4\theta^2\Gamma\left(\frac{2}{\beta}+1\right)\left[\frac{2n}{(n)^{\frac{2}{\beta}}} - \frac{n}{(n+1)^{\frac{2}{\beta}}} - \frac{n}{(n-1)^{\frac{2}{\beta}}}\right] + c_4\theta^2\Gamma\left(\frac{1}{\beta}+1\right)^2\left[\left(\frac{n+1}{(n)^{\frac{1}{\beta}}} - \frac{n}{(n+1)^{\frac{1}{\beta}}}\right)^2 - \left(\frac{n}{(n-1)^{\frac{1}{\beta}}} - \frac{n-1}{(n)^{\frac{1}{\beta}}}\right)^2\right] + c_5\left(e^{-\left(\frac{E(t_{r,n})}{\theta}\right)^{\beta}} - e^{-\left(\frac{E(t_{r,n+1})}{\theta}\right)^{\beta}}\right) \tag{28}$$

Eq. 28 can be simplified as follows:

$$\Delta C_2(n) = c_2 - I(n) \tag{29}$$

where $I(n)$ is as Eq. 27.

One has $\Delta C_2(n) \geq 0$, if and only if, $I(n) \leq c_2$. The function $I(n)$ can be empirically shown to be decreasing in n for all $n \geq r$, so there exists a value n^* such that $I(n) \leq c_2$. This implies that

$$n^* = \inf\{n: I(n) \leq c_2\}$$

4.2.2 Unknown Weibull parameters

Assuming that all the costs are given, and that r is fixed at 2, we can determine the optimal value n, say n^*, that minimizes the total expected cost $C_2(n)$ as shown in the following theorem.

Theorem 4: For given values $\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3, \mathrm{c}_4, \mathrm{c}_5$ and $r = 2$ there exists the optimum sample size n on test, say n^*, that minimizes the expected total cost, $C_2(n)$, subject to the unknown parameters β and θ of the Weibull distribution lifetime and

$$n^* = \inf\{n: J(n) \leq c_2\} \tag{30}$$

where

$$J(n) =$$

$$a_1\left(\frac{n}{(n+1)^{\frac{1}{\widehat{\beta n+1}}}} - \frac{n+1}{(n)^{\frac{1}{\widehat{\beta n+1}}}}\right)\left(c_1 + c_4\left(1 - a_1\left(\frac{n+1}{(n)^{\frac{1}{\widehat{\beta n+1}}}} - \frac{n}{(n+1)^{\frac{1}{\widehat{\beta n+1}}}}\right)\right)\right) +$$

$$a_0\left(\frac{n-1}{(n)^{\frac{1}{\widehat{\beta n}}}} - \frac{n}{(n-1)^{\frac{1}{\widehat{\beta n}}}}\right)\left(-c_1 - c_4\left(1 - a_0\left(\frac{n}{(n-1)^{\frac{1}{\widehat{\beta n}}}} - \frac{n-1}{(n)^{\frac{1}{\widehat{\beta n}}}}\right)\right)\right) +$$

$$c_4\left(b_0\left(\frac{n}{(n-1)^{\frac{2}{\widehat{\beta_n}}}}-\frac{n-1}{(n)^{\frac{2}{\widehat{\beta_n}}}}\right)-b_1\left(\frac{n+1}{(n)^{\frac{2}{\widehat{\beta_{n+1}}}}}-\frac{n}{(n+1)^{\frac{2}{\widehat{\beta_{n+1}}}}}\right)\right)+$$

$$c_5\left(e^{-\left(\frac{E(t_{r,n+1})}{\hat{\theta}_{n+1}}\right)^{\widehat{\beta_{n+1}}}}-e^{-\left(\frac{E(t_{r,n})}{\hat{\theta}_n}\right)^{\widehat{\beta_n}}}\right) \tag{31}$$

and

$$a_0=\hat{\theta}_n\Gamma\left(\frac{1}{\widehat{\beta_n}}+1\right) \tag{32}$$

$$a_1=\hat{\theta}_{n+1}\Gamma\left(\frac{1}{\widehat{\beta_{n+1}}}+1\right) \tag{33}$$

$$b_0=\hat{\theta}_n{}^2\left(\Gamma\left(\frac{2}{\widehat{\beta_n}}+1\right)\right) \tag{34}$$

$$b_1=\hat{\theta}_{n+1}{}^2\left(\Gamma\left(\frac{2}{\widehat{\beta_{n+1}}}+1\right)\right) \tag{35}$$

Proof: Define $\Delta C_2(n)=C_2(n+1)-C_2(n)$. From Eq. 25 we obtain:

$$\Delta C_2(n)=c_2+a_1\left(\frac{n+1}{(n)^{\frac{1}{\widehat{\beta_{n+1}}}}}-\frac{n}{(n+1)^{\frac{1}{\widehat{\beta_{n+1}}}}}\right)\left(c_1+c_4\left(1-a_1\left(\frac{n+1}{(n)^{\frac{1}{\widehat{\beta_{n+1}}}}}-\right.\right.\right.$$

$$\left.\left.\left.\frac{n}{(n+1)^{\frac{1}{\widehat{\beta_{n+1}}}}}\right)\right)\right)+a_0\left(\frac{n}{(n-1)^{\frac{1}{\widehat{\beta_n}}}}-\frac{n-1}{(n)^{\frac{1}{\widehat{\beta_n}}}}\right)\left(-c_1-c_4\left(1-a_0\left(\frac{n}{(n-1)^{\frac{1}{\widehat{\beta_n}}}}-\frac{n-1}{(n)^{\frac{1}{\widehat{\beta_n}}}}\right)\right)\right)+$$

$$c_4\left(b_1\left(\frac{n+1}{(n)^{\frac{2}{\widehat{\beta_{n+1}}}}}-\frac{n}{(n+1)^{\frac{2}{\widehat{\beta_{n+1}}}}}\right)-b_0\left(\frac{n}{(n-1)^{\frac{2}{\widehat{\beta_n}}}}-\frac{n-1}{(n)^{\frac{2}{\widehat{\beta_n}}}}\right)\right)+c_5\left(e^{-\left(\frac{E(t_{r,n})}{\widehat{\theta}_n}\right)^{\widehat{\beta_n}}}-e^{-\left(\frac{E(t_{r,n+1})}{\widehat{\theta}_{n+1}}\right)^{\widehat{\beta_{n+1}}}}\right) \quad (36)$$

where a_0, a_1, b_0 and $b_1$are given in Eq. 32, Eq. 33, Eq. 34, and Eq. 35, respectively.

Eq. 36 can be simplified as follows:

$$\Delta C_2(n) = c_2 - J(n) \quad (37)$$

where $J(n)$ is as Eq. 31.

One has $\Delta C_2(n) \geq 0$, if and only if, $J(n) \leq c_2$. The function $J(n)$ can be empirically shown to be decreasing in n for all $n \geq r$, so there exists a value n^* such that $J(n) \leq c_2$. This implies that

$$n^* = \inf\{n: J(n) \leq c_2\}$$

4.3 Numerical Examples

To illustrate the results of the theorems 3 and 4, as showed in subsections 4.2.1 and 4.2.2, respectively, we provide numerical examples where the optimal sample size is solved. All the calculations were made using MATLAB programs.

For all the examples we will assume $\mathrm{c}_1 = 25$ per unit hour, $\mathrm{c}_2 = 500$ per item, $\mathrm{c}_3 = 100$ per testing, $\mathrm{c}_4 = 15$ per unit hour and $\mathrm{c}_5 = 1000$.

4.3.1 Given Weibull Parameters

In this subsection we show two different numerical examples in order to illustrate *Theorem 3* that shows the mathematical solution for the optimal sample size of a Type II censored life test with known Weibull parameters.

Example 5

In this example we assume that the Weibull parameter values are $\beta = 1.646$ and $\theta = 162.2$. The values of the costs and of the parameters are used in order to obtain the results of the function $I(n)$ as in Eq. 27. Thus, the obtained results are shown in Table V.

Considering that $c_2 = 500$, according to *Theorem* 3, we look for the first value of n that satisfies $I(n) \leq 500$. Thus, we can say that the optimum sample size is $n^* = 16$ units and the total expected testing cost is $\$15{,}435.73$.

Table V: Example 5 obtained results of $I(n)$ and $C_2(n)$

n	$I(n)$	$C_2(n)$
12	881.518413	16235.98
13	741.556767	15854.46
14	632.141762	15612.91
15	545.036887	15480.77
16	474.597413	15435.73
17	416.851595	15461.13
18	368.940204	15544.28
19	328.763245	15675.34

Fig 4.1. graphically shows that $I(n)$ is non-increasing in n and Fig. 4.2 shows a 3-D plot of $I(n)$, $C_2(n)$ and n.

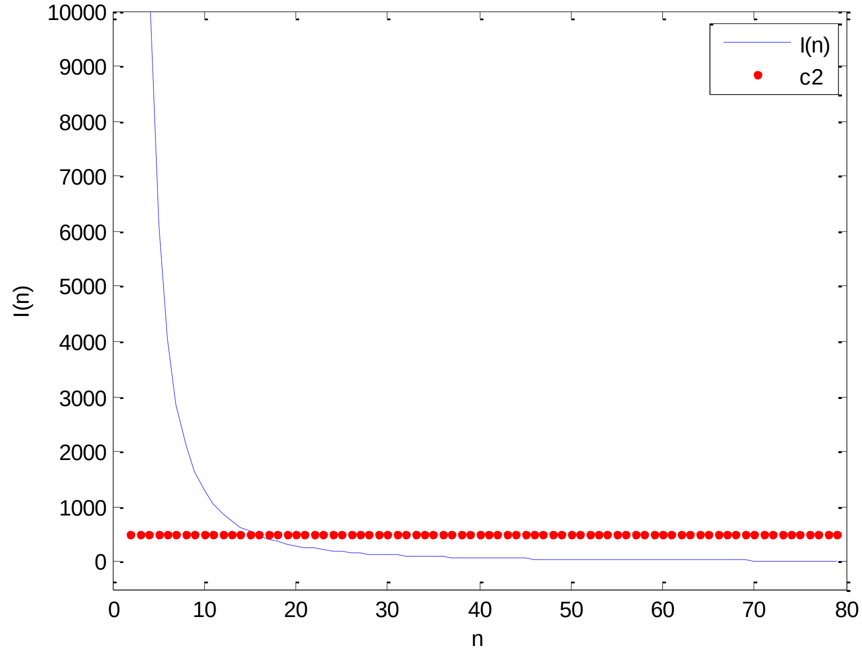

Figure 4.1 – $I(n)$ function of example 5

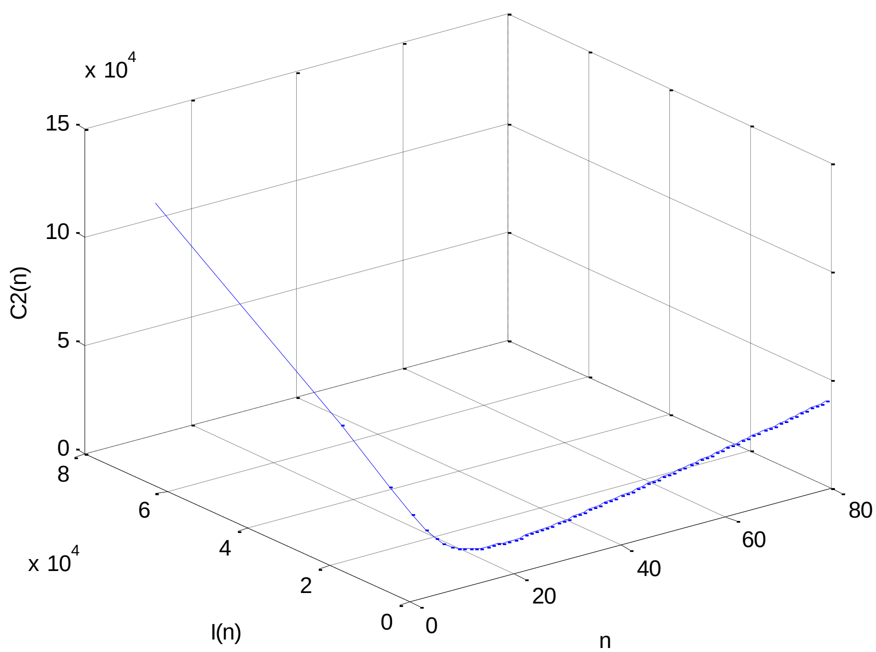

Figure 4.2 – 3-D plot ($I(n)$ x $C_2(n)$ x n) of example 5

Example 6

In this example we consider different Weibull parameter values: $\beta = 1.044$ and $\theta = 362.2$. The obtained results of $I(n)$ and $C_2(n)$ can be seen on Table VI.

Table VI: Example 6 obtained results of $I(n)$ and $C_2(n)$

n	$I(n)$	$C_2(n)$
22	846.8607	21960.75
23	746.0766	21613.89
24	660.9395	21367.81
25	588.5027	21206.87
26	526.4644	21118.37
27	473.0079	21091.90
28	426.6865	21118.90
29	386.338	21192.21

According to *Theorem 3*, and since $\mathrm{c}_2 = 500$, we look for the first n value such that $I(n) \leq c_2$ is satisfied. We can say that the optimum sample size is $n^* = 27$ units with a total expected cost of $\$21{,}091.90$.

In Fig 4.3. it can be graphically seen that $I(n)$ is non-increasing in n, while Fig. 4.4 shows a 3-D plot of $I(n)$, $C_2(n)$ and n.

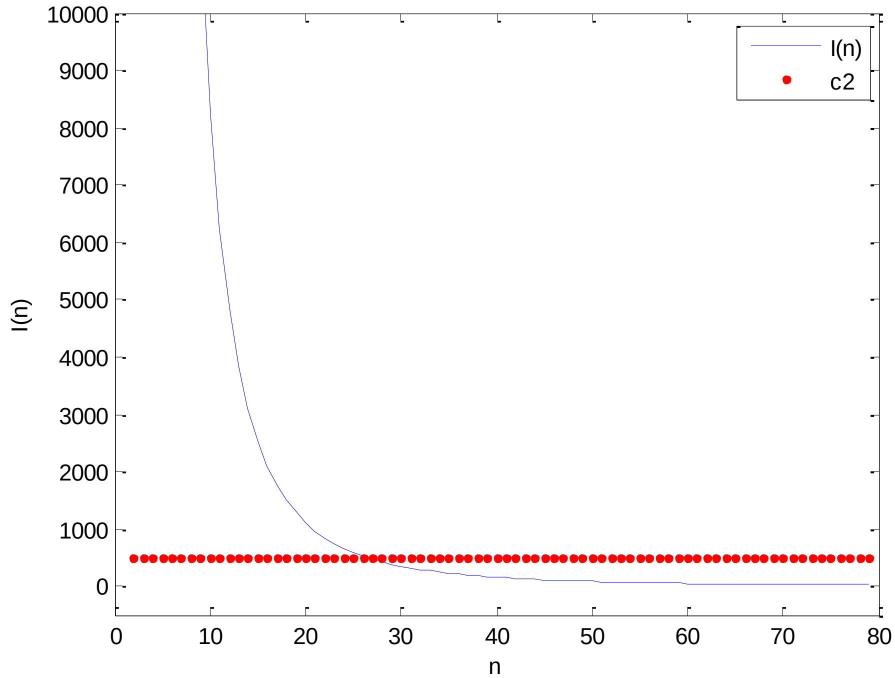

Figure 4.3 – $I(n)$ function of example 6

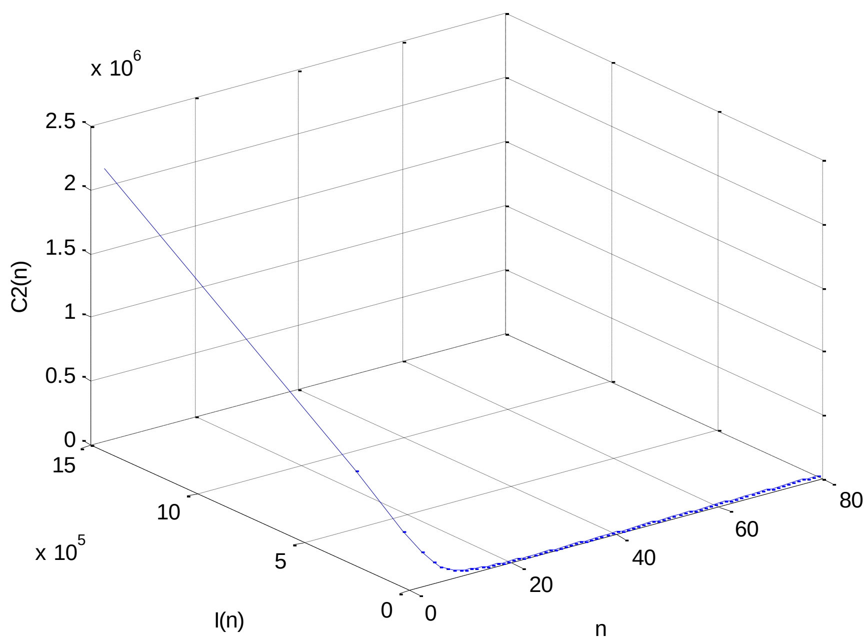

Figure 4.4 – 3-D plot ($I(n)$ x $C_2(n)$ x n) of example 6

4.3.2 Unknown Weibull parameters

In this subsection we show two different numerical examples in order to illustrate the mathematical solution of the optimum sample size, as in *Theorem 4*, for the case where the Weibull parameters are unknown but some preliminary failure data is available. The failure times are required to estimate the Weibull parameters $\hat{\beta}$ and $\hat{\theta}$ for each n as in Eq. 8 and Eq. 9, respectively.

Example 7

In this example we use the literature data [3], $t_1 = 12.5$ and $t_2 = 24.4$, as preliminary failure time data. The times t_1 and t_2 are first used to obtain the n combinations of the Weibull parameter estimators $\hat{\beta}$ and $\hat{\theta}$. The cost values and the estimators are then used to obtain the values of the function $J(n)$ (Eq. 31). Thus, the obtained results for $\hat{\beta}$, $\hat{\theta}$ and $J(n)$ are shown on Table VII.

Table VII: Example 7 obtained results of $\hat{\beta}$, $\hat{\theta}$ and $J(n)$

n	$\hat{\beta}$	$\hat{\theta}$	$J(n)$
10	3.0763	182.42	692.62
11	3.068	182.7	601.61
12	3.0612	182.94	528.4
13	3.0555	183.16	468.52
14	3.0506	183.36	418.87
15	3.0464	183.54	377.18
16	3.0427	183.7	341.79

Considering that $c_2 = 500$ and according to *Theorem 4*, we look for the first value of n where $J(n) \leq 500$ is satisfied. Thus, we can say that the optimum sample size is $n^* = 13$ with a total expected testing cost is $\$18{,}452.89$.

In Fig 4.5. it can be graphically seen that $J(n)$ is non-increasing in n, while in Fig. 4.6 it can be seen a 3-D plot of $J(n)$ x $\hat{\beta}$ x n and $G(n)$ x $\hat{\theta}$ x n.

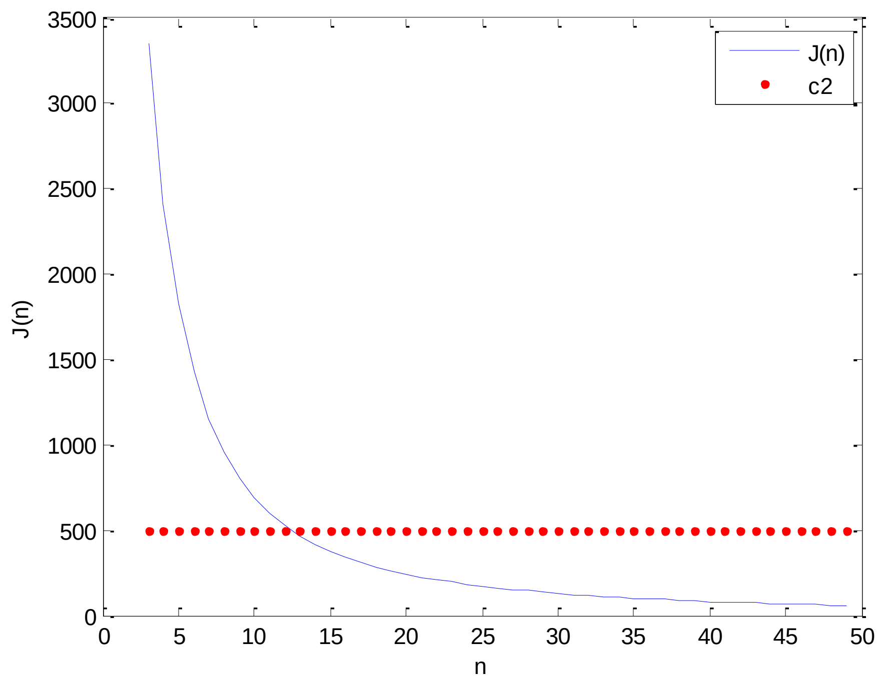

Figure 4.5 – $J(n)$ function of example 7

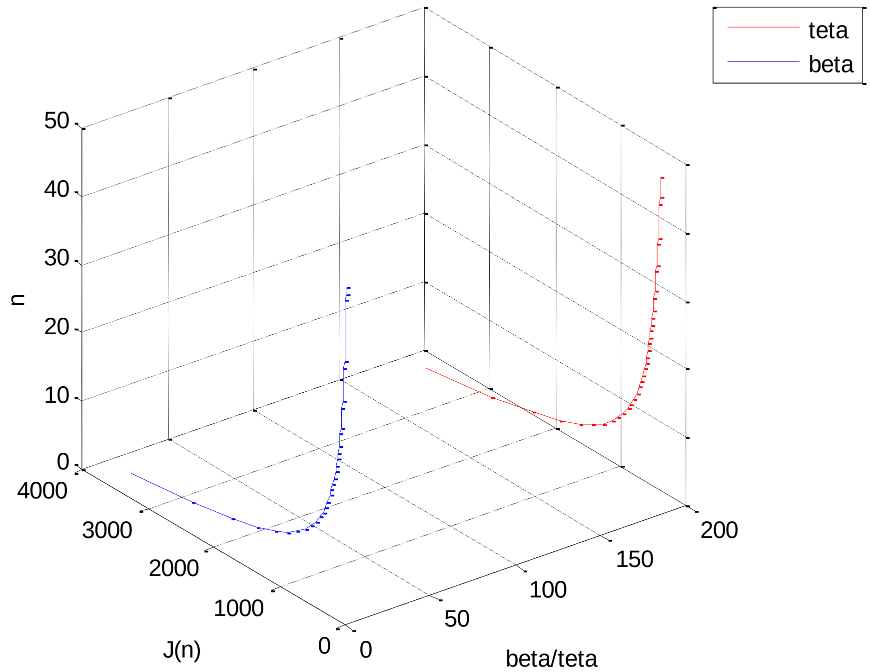

Figure 4.6 – 3-D plot ($J(n)$x$\hat{\beta}$xn and $J(n)$x$\hat{\theta}$xn) of example 7

Example 8

In this example we use real failure time data [13] where the first 2 failure events occur at $t_1 = 10$ and $t_2 = 16$. The times t_1 and t_2 are first used to obtain the n combinations of the Weibull parameter estimators $\hat{\beta}$ and $\hat{\theta}$. Then, the parameter and cost values are used to obtain the values of $J(n)$. The obtained results can be seen on Table VIII.

According to *Theorem* 4, we look for the first value of n that satisfies $J(n) \leq c_2$. Since $c_2 = 500$, $J(50) = 505.965$ and $J(51) = 491.988$, we can say that the optimum sample size is $n^* = 51$ units with a total expected cost of $\$86{,}411.51$.

Table VIII: Example 8 obtained results of $\hat{\beta}$, $\hat{\theta}$ and $J(n)$

n	$\hat{\beta}$	$\hat{\theta}$	$J(n)$
48	4.2796	712.55	536.025
49	4.2791	712.58	520.628
50	4.2786	712.61	505.965
51	4.2781	712.64	491.988
52	4.2777	712.67	478.651
53	4.2773	712.7	465.915
54	4.2769	712.73	453.742
55	4.2765	712.75	442.098

Fig 4.7. graphically shows that $J(n)$ is non-increasing in n, while Fig. 4.8 shows a 3-D plot of $J(n)$ x $\hat{\beta}$ x n and $J(n)$ x $\hat{\theta}$ x n.

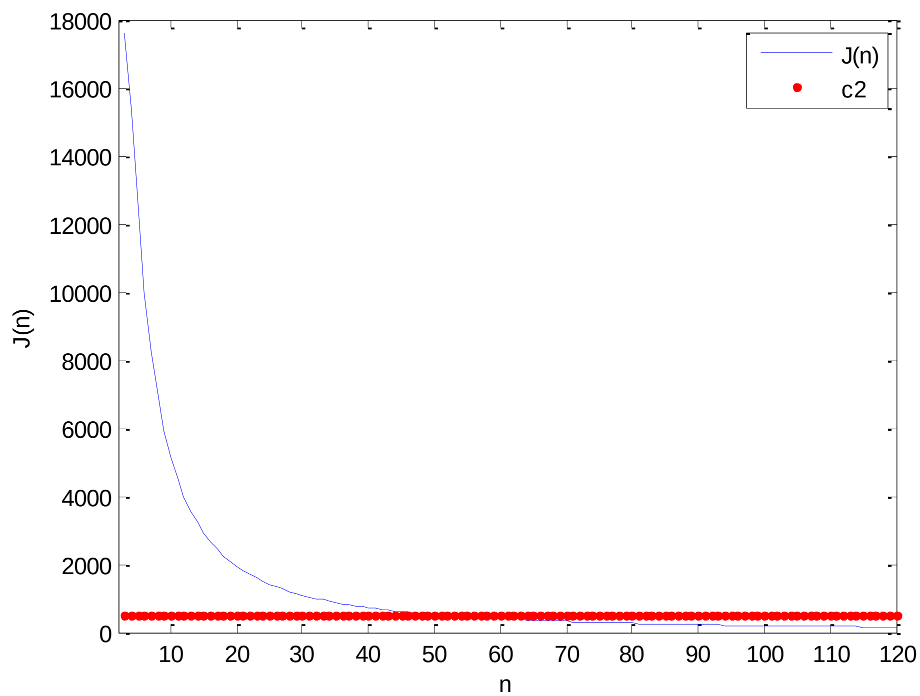

Figure 4.7 – $J(n)$ function of example 8

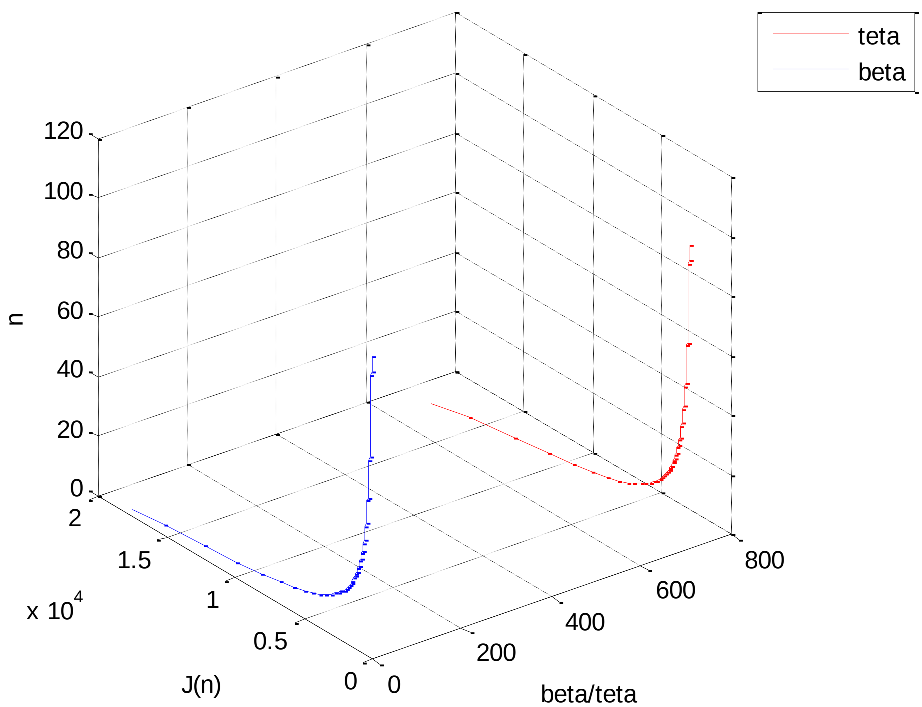

Figure 4.8 – 3-D plot ($J(n)$x$\hat{\beta}$xn and $J(n)$x$\hat{\theta}$xn) of example 8

Chapter 5

Conclusion and Future Research

The Weibull distribution has great importance to the reliability and life testing fields since it is versatile when fitting time-to-failure distributions.

Long testing times are required for highly reliable products before useful failure data becomes available, bringing the interest of manufacturing companies to speed up the testing. Reducing the time of testing can be achieved through the widely used accelerated life testing methods namely: Type-I censoring, Type-II censoring, and progressive Type-II censoring.

There is an associated cost when performing life testing, which will depend on the cost model's factors such as: total expected time of testing, number of units placed under test, setup cost, and a cost associated to the risk of testing units fail before the testing time.

In this work we developed two testing cost models for a Type-II censoring testing for units under Weibull distribution life time, and determined the optimal sample size on test, considering a fixed number of failures equal 2, that minimizes the total expected cost for both cost models considering two scenarios: when the Weibull parameters are given or predetermined and when they are unknown but a preliminary failure data is available.

This research can be extended to the cases where the optimum Weibull lifetime units sample size, that are placed under test, can be obtained for any predetermined number of failures to terminate the test.

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