AN AFFINE WEYL GROUP INTERPRETATION OF THE "MOTIVATED PROOFS" OF THE ROGERS-RAMANUJAN AND GORDON-ANDREWS-BRESSOUD IDENTITIES

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ABSTRACT OF THE DISSERTATION

An affine Weyl group interpretation of the "motivated proofs" of the Rogers-Ramanujan and Gordon-Andrews-Bressoud identities

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A motivated proof of the Rogers-Ramanujan identities was given by G. E. Andrews and R. J. Baxter. This proof was generalized to the odd-moduli case of Gordon's identities by J. Lepowsky and M. Zhu, and later to the even-moduli case of the Andrew-Bressoud identities by S. Kanade, Lepowsky, M. C. Russell and A. Sills. We present a reinterpretation of these proofs, with new motivation coming from the affine Weyl group of $\mathfrak{sl}(2)$.

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Dedication

Dedicated to my son, Iggy.

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Chapter 1

Introduction

The classical Rogers-Ramanujan partition identities state that

$$\prod_{\substack{m\geq 1,\\ m\not\equiv 0,\pm 2\,(\mathrm{mod}\,5)}} \frac{1}{1-q^m} = \sum_{n\geq 0} d_1(n)q^n, \tag{1.1}$$

$$\prod_{\substack{m\geq 1,\\ m\not\equiv 0,\pm 1\,(\mathrm{mod}\,5)}} \frac{1}{1-q^m} = \sum_{n\geq 0} d_2(n)q^n, \tag{1.2}$$

where

$d_1(n) =$ the number of partitions of n for which adjacent parts have difference at least 2

and

$d_2(n) =$ the number of partitions of n for which adjacent parts have difference at least 2 and no part is equal to 1

The "product sides" (i.e., the left-hand sides above) of the Rogers-Ramanujan identities enumerate the partitions whose parts obey certain restrictions modulo 5, and the "sum sides" (the right-hand sides) enumerate the partitions with certain difference-two and initial conditions. Generalizations of the Rogers-Ramanujan identities for all odd moduli were discovered by B. Gordon [G1] and G. E. Andrews [A1]. Analogous identities for the even moduli of the form $4k+2$ were discovered by Andrews in [A2] and [A3], and subsequently, for all the even moduli, by D. M. Bressoud in [Br].

In [AB], Andrews and A. Baxter gave an interesting "motivated proof" of these two Rogers-Ramanujan identities, a variant of one of the original proofs by Rogers and Ramanujan and of an earlier proof by Baxter himself. In their proof, they explained the difference-two condition appearing in the sum sides directly from the product sides, and in doing this, they were able to both motivate the expressions on the sum sides and prove the two identities.

Their method was to start by rewriting the product sides of the two identities using the classical Jacobi triple product identity (as in done is most proofs of the Rogers-Ramanujan identities), then to take certain combinations of these series to generate an infinite tower of series. Once notices empirically that these higher series converge to 1 in a suitable sense. They called this assertion the "Empirical Hypothesis", and they were able to prove it by giving closed-formed expressions for the higher series. Then, solving for the base series in terms of these higher series exactly yields the partition conditions present on the sum sides.

Recently, J. Lepowsky together with students and collaborators have given a series of analogous "motivated proofs" for various identities of Rogers-Ramanujan type, in which the "motivation" has been similar in spirit to the motivation in [AB]. Correspondingly, we will use the term "motivated proof" as a technical term. In particular, in [LZ], Lepowsky and M. Zhu gave a "motivated proof" of the Gordon-Andrews generalization of the Rogers-Ramanujan identities, and in [KLRS], S. Kanade, Lepowsky, M. C. Russell and A. Sills gave a "motivated proof" of the Andrews-Bressoud identities.

It is well known that partition identities of Rogers-Ramanujan type are closely related to the representation theory of vertex operator algebras. Early vertex operator theory was in fact motivated by the successful attempt to realize the combinatorial sum sides of identities of this type - exhibiting them as the graded dimensions of vector spaces constructed from certain natural vertex operators. In a series of papers ([LW2]-[LW4]), Lepowsky and R.L. Wilson accomplished this using the theory of "principally twisted Z-operators", built out of the "twisted vertex operators" starting from [LW1]. In fact, the Z-algebraic structure developed in [LW2]-[LW4] gave a vertex-operator-theoretic interpretation of the whole family of Gordon-Andrews-Bressoud identities, as well as a vertex-operator-theoretic proof of the Rogers-Ramanujan identities, in the context of the affine Lie algebra $A_1^{(1)}$. For the cases beyond Rogers-Ramanujan, A. Meurman and M. Primc, in [MP], extended this vertex-operator-theoretic interpretation to a full proof of the higher identities. This Z-algebra viewpoint (or an equivalent formulation) was later used by K. Misra [Mi1]-[Mi4], M. Mandia [Ma], C. Xie [X], S. Capparelli [Cap1]-[Cap2], M. Tamba - Xie [TX], M. Bos - K. Misra [BM], and D. Nandi [N] to give further interpretations and proofs of these same identities, as well as to study new identities, in the context of a wider range of affine Lie algebras. A very different vertex-algebraic approach to the sum sides of the Rogers-Ramanujan and Gordon-Andrews identities was developed in [CLM1], [CLM2], [CalLM1] and [CalLM1] based on "untwisted" intertwining operators.

These connections have been major incentives in seeking out "motivated proofs". However, in all of the "motivated proof" papers referenced above, the proofs have been based entirely on the manipulation of q-series, while all of the vertex-operator-algebraic proofs have made use of deep vertex algebraic theory. This distinction between the two approaches has made it difficult to understand "motivated proofs" from a strictly vertex-operator theoretic viewpoint.

The present work was inspired by our desire to approach this ultimate goal, and we have succeeded in making a significant step in this direction. We choose to view the Jacobi triple product identity as the denominator identity for the affine Lie algebra $A_1^{(1)} = \widehat{\mathfrak{sl}(2)}$, in order to recast the q-series entering into the proofs of the Gordon-Andrews-Bressoud identities in terms of the affine Weyl group of $\mathfrak{sl}(2)$. It turns out that this leads to surprising insights into the nature of these series and into the algebraic and geometric structure underlying the "motivated proofs" of these identities. In particular, most parts of the "motivated proofs" are made shorter and take on a much more natural form, and the nature of certain contributions to the series becomes much more transparent. While the proofs are still based on q-series manipulation, the ad hoc manipulations used previously become motivated transformations that come directly from the Coxeter group. This work should help to bridge the gap between the "motivated proof" papers and the vertex-algebraic papers mentioned above.

We expect that this approach can be generalized to higher rank affine Lie algebras. Applying this analysis to higher-rank affine Lie algebras should lead to many new identities of Rogers-Ramanujan type, as well as to new proofs of known identities.

In this work, first we treat the special "test case" of the Rogers-Ramanujan identities. Once having gone through all the proofs here, we handle the more general case of the Gordon-Andrews identities. Finally, we deal with the Andrews-Bressoud identities, which exhibit some new behavior but follow the same general paradigm.

Chapter 2

Background material

2.1 Affine Lie algebras

We start with recalling basic theory of affine Lie algebras, and specifically information related to the smallest affine Lie algebra

$$A_1^{(1)} = \widehat{\mathfrak{sl}(2)}.$$

This material is covered in many introductory texts - see for example [Ca] or [K]. After reviewing some basic material, we will specify what we will be using in this work.

The general Weyl-Kac denominator formula states that given an affine Lie algebra $\mathfrak{g}$, we have

$$\sum_{w \in W} (-1)^{\ell(w)} e^{w\rho - \rho} = \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}) \tag{2.1}$$

where W is the associated affine Weyl group, Δ^+ is the set of positive roots, and ρ is the fundamental weight.

For $w \in W$, the quantity $w\rho - \rho$ will be extremely important to us. As is well known, this expression is equal to the sum of the positive roots made negative by w (see for example [Ca], Chapter 20). The standard notation for this set of roots is Δ_w, and correspondingly, we will denote the $w\rho = \rho$ by the symbol

$$|\Delta_w|$$

(where the "absolute value" is meant to indicate the sum of the enclosed set). This action can be computed explicitly from the action of the generators (basic reflections): for any root γ,

$$s_i \gamma = \gamma - \langle \alpha_i, \gamma \rangle h_i$$

where α_i, h_i are respectively the root and coroot corresponding to s_i.

Remark 2.1. Although we have written the denominator formula above in its natural generality, in this work, we will be exclusively concerned with the concrete affine Lie algebra $A_1^{(1)}$. Hereafter, whenever W appears, it is to be understood to refer to the Weyl group of this particular affine Lie algebra (i.e., the affine Weyl group of $A_1 = \mathfrak{sl}(2)$).

The group W can be viewed as the group of affine reflections on a one-dimensional lattice. It has generators s_0 and s_1 (corresponding to the two simple roots, α_0 and α_1), with defining relations

$$s_0^2 = s_1^2 = e$$

(and no others).

The sum

$$\delta = \alpha_0 + \alpha_1$$

is the basic imaginary root, and it is well known that there are three strings of positive roots: the imaginary roots

$$k\delta, \quad k > 0$$

and the real roots

$$\alpha_p + k\delta, \quad p = 0, 1, \quad k \geq 0$$

When working in this context, given an element of the root lattice (for example, $|\Delta_w|$), we will denote its coordinates in the α_0, α_1 basis by the subscripts $0, 1$. In particular, we have

$$|\Delta_w| = |\Delta_w|_0 \alpha_0 + |\Delta_w|_1 \alpha_1 \tag{2.2}$$

Remark 2.2. In this work, we will deal extensively with formal power series in a formal variable q (i.e., formal series with non-negative integral powers of q), where the summands are indexed by elements of W. The summand corresponding to $w \in W$ will always be a polynomial in q, and the data making up this polynomial will always be expressed in terms of the $|\Delta_w|$ quantities defined above. Thus, it is of the utmost importance to know the exact dependence of $|\Delta_w|$ on w, and how it changes under certain transformations of W.

We now compute the action of the generators s_0, s_1 on $|\Delta_w|$. First, we recall the basic actions of of the Weyl group on the roots, which follow directly from the formula given above:

$$s_0\alpha_0 = -\alpha_0 \tag{2.3}$$

$$s_0\alpha_1 = 2\alpha_0 + \alpha_1 \tag{2.4}$$

$$s_1\alpha_0 = \alpha_0 + 2\alpha_1 \tag{2.5}$$

$$s_1\alpha_1 = -\alpha_1 \tag{2.6}$$

Applying these equations, we get:

$$|\Delta_{s_0w}|_0 = -|\Delta_w|_0 + 2|\Delta_w|_1 + 1 \tag{2.7}$$

$$|\Delta_{s_0w}|_1 = |\Delta_w|_1 \tag{2.8}$$

$$|\Delta_{s_1w}|_0 = |\Delta_w|_0 \tag{2.9}$$

$$|\Delta_{s_1w}|_1 = 2|\Delta_w|_0 - |\Delta_w|_1 + 1. \tag{2.10}$$

It is also direct from the definition that

$$|\Delta_e| = 0 = 0\alpha_0 + 0\alpha_1$$

Using this initial condition and the above recursions, we see that the values assumed by the $|\Delta_w|$ components are the triangle numbers. Specifically, the points $|\Delta_w|$, $w \in W$ trace out the parabola

$$(|\Delta_w|_0 - |\Delta_w|_1)^2 = |\Delta_w|_0 + |\Delta_w|_1 \tag{2.11}$$

in the $\alpha_0\alpha_1$ plane.

To clarify the above, and for later use, at this point we introduce special notation for Weyl group elements: for $h \geq 0$, let w_h^0 denote the Weyl element of length h whose shortest expression (in the generators s_0, s_1) starts with s_0 and w_h^1 denote the Weyl element of length h whose shortest expression starts with s_1.

Notice that

$$w_0^0 = w_0^1$$

both correspond to the identity element of the group.

Explicitly, for $h \geq 0$, we write

$$w_h^0 = \overbrace{s_0 s_1 s_0 \cdots}^{length\ h} \tag{2.12}$$

$$w_h^1 = \overbrace{s_1 s_0 s_1 \cdots}^{length\ h} \tag{2.13}$$

Then we can restate the conclusions above as

$$|\Delta_{w_h^0}|_0 = \binom{h+1}{2} = |\Delta_{w_h^1}|_1 \tag{2.14}$$

$$|\Delta_{w_h^0}|_1 = \binom{h}{2} = |\Delta_{w_h^1}|_0 \tag{2.15}$$

It follows that

$$|\Delta_{w_h^0}|_0 = |\Delta_{w_{h+1}^0}|_1 \tag{2.16}$$

$$|\Delta_{w_h^1}|_1 = |\Delta_{w_{h+1}^1}|_0 \tag{2.17}$$

Now we have developed enough background to revisit the denominator equation in the context of $A_1^{(1)}$. It becomes

$$\sum_{w\in W}(-1)^{\ell(w)}e^{-|\Delta_w|} = (e^{-\alpha_0}, e^{-\delta})_\infty(e^{-\alpha_1}, e^{-\delta})_\infty(he^{-\delta}, e^{-\delta})_\infty$$

where

$$(a;q)_\infty = \prod_{m\geq 0}(1-aq^m)$$

(the q-Pochhammer symbol).

The technical proofs in this work all deal with the manipulation of formal power series of the form described in Remark 2.2. The series will be different for the different classes of identities we consider, but they will all have certain features in common.

Our basic tools for manipulating these series are the application of certain transformations of W, will which act on the series as "changes of index". Many times, we will need to make use of how these transformations affect the common features of the series. For convenience, at this point we record these results here, and then reference them later as appropriate.

Definition 2.3. *Let $s_1* : W \to W$ be the map of left-composition with the generator s_1. This map is an involution. In terms of the notation introduced above, this map acts on elements of W by interchanging w_h^0 and w_{h+1}^1. Its action on an expression of the form $|\Delta_w|$ is recorded above in (2.9), (2.10).*

Definition 2.4. *Let $f : W \to W$ be the outer automorphism which exchanges the two generators s_0 and s_1. This map is an involution. In terms of the notation introduced above, this map acts on elements of W by interchanging w_h^0 and w_h^1 (in particular, it fixes the identity element $e \in W$). Its action on an expression of the form $|\Delta_w|$ is to switch its components in formula (2.2).*

Notice that s_1* changes the length of each element by ± 1, while $f(\cdot)$ preserves the lengths of elements.

Let w be an element of the affine Weyl group W. Consider the following expressions in the formal variable q:

$$(q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - r}), \quad (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 - r})$$

It follows from the discussion above that the transformation $s_1\cdot$ turns the first expression into

$$(q^{2|\Delta_w|_0 - |\Delta_w|_1 + 1} - q^{|\Delta_w|_0 - r}) = q^{|\Delta_w|_0 - |\Delta_w|_1 + 1}(q^{|\Delta_w|_0} - q^{|\Delta_w|_1 - (r+1)})$$

and the second into

$$(q^{|\Delta_w|_0} - q^{2|\Delta_w|_0 - |\Delta_w|_1 + 1 - r}) = q^{|\Delta_w|_0 - |\Delta_w|_1}(q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - (r-1)})$$

On the other hand, the transformation f will clearly just interchange these two expressions. It follows that:

Corollary 2.5. *Applying the transformations s_1*, $f(\cdot)$ to W gives:*

$$\begin{aligned}
&s_1* : (q^{|\Delta_w|_1} - q^{|\Delta_w|_0})(q^{|\Delta_w|_0} - q^{|\Delta_w|_1 - 1})(q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - 1}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 - h}) \\
&\mapsto (q^{|\Delta_w|_1} - q^{|\Delta_w|_0})(q^{|\Delta_w|_0} - q^{|\Delta_w|_1 - 1})(q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - 1}) \cdots \\
&\quad (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 - h}) q^{h(2|\Delta_w|_0 - 2|\Delta_w|_1 + 1)}
\end{aligned}$$

$$\begin{aligned}
&f(\cdot) : (q^{|\Delta_w|_1} - q^{|\Delta_w|_0})(q^{|\Delta_w|_0} - q^{|\Delta_w|_1 - 1})(q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - 1}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 - h}) \\
&\mapsto -(q^{|\Delta_w|_1} - q^{|\Delta_w|_0})(q^{|\Delta_w|_0} - q^{|\Delta_w|_1 - 1}) \cdots (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - (h-1)})(q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - h})
\end{aligned}$$

$$s_1* : (q^{|\Delta_w|_1} - q^{|\Delta_w|_0})(q^{|\Delta_w|_0} - q^{|\Delta_w|_1-1})(q^{|\Delta_w|_1} - q^{|\Delta_w|_0-1}) \cdots (q^{|\Delta_w|_1} - q^{|\Delta_w|_0-h})$$
$$\mapsto (q^{|\Delta_w|_1} - q^{|\Delta_w|_0})(q^{|\Delta_w|_0} - q^{|\Delta_w|_1-1}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1-h})(q^{|\Delta_w|_0} - q^{|\Delta_w|_1-(h+1)})$$
$$\cdot\, q^{(2h+1)(|\Delta_w|_0 - |\Delta_w|_1)+(h+1)}$$

$$f(\cdot) : (q^{|\Delta_w|_1} - q^{|\Delta_w|_0})(q^{|\Delta_w|_0} - q^{|\Delta_w|_1-1})(q^{|\Delta_w|_1} - q^{|\Delta_w|_0-1}) \cdots (q^{|\Delta_w|_1} - q^{|\Delta_w|_0-h})$$
$$\mapsto -(q^{|\Delta_w|_1} - q^{|\Delta_w|_0})(q^{|\Delta_w|_0} - q^{|\Delta_w|_1-1})(q^{|\Delta_w|_1} - q^{|\Delta_w|_0-1}) \cdots (q^{|\Delta_w|_1} - q^{|\Delta_w|_0-h})$$

Remark 2.6. The result above can be made to look even more natural: when these factors occur in our series, the values $-r$ in the exponents actually arise as the following expressions: $|\Delta_{w_r^1}|_0 - |\Delta_{w_r^1}|_1$ for those factors in parentheses with $|\Delta_w|_0$ first, and $-|\Delta_{w_r^0}|_0 + |\Delta_{w_r^0}|_1$ for those factors in parentheses with $|\Delta_w|_1$ first. Since s_1 acts on W by interchanging $w_r^0 \leftrightarrow w_{r+1}^1$, and the map f interchanges $w_r^0 \leftrightarrow w_r^1$, we see that the transformation of expressions calculated above work even on this level of notation. This is another strong motivation for considering the series in this light.

Remark 2.7. Note in particular that in two of the cases, the expression is only rescaled by a sign and/or power of q. This fact will prove to be vital in the proofs of the “edge-matching” phenomena.

Remark 2.8. As has been indicated above, the series sums we consider will always be over the full Weyl group. Frequently, we visualize the elements of the Weyl group as living on the parabola traced out by the components of their respective $|\Delta_w|$ values. Because of this, it may seem natural to view these sums as “two-sided infinite”, with the identity element (corresponding to the vertex of the parabola) in the middle. However, what turns out to be best is to view the sum as “one-sided” infinite, starting at $w = e$ and continuing in a zig-zag up the parabola. This is akin to the the appearance of the series in [LZ], but with the following important and fundamental difference: in [LZ], the sum was made infinite by “folding” the Weyl group in two and pairing. Essentially, each summand in the series in [LZ] corresponds to the combination of two different Weyl elements. In our approach, we instead introduce a natural linear ordering on the Weyl group: starting with $w_0^0 = e$, we alternatingly apply the two transformations above in the order $s_1\cdot$, $f(\cdot)$. In terms of our notation, the ordering of the Weyl elements is

$$w_0^0, w_1^1, w_1^0, w_2^1, w_2^0, \cdots$$

Remark 2.9. Of course, the analogous transformation s_0* could have also been considered in addition to or instead of s_1*. The fact that we prefer s_1 over s_0 is an arbitrary choice, made to confirm to earlier works on motivated proofs. The origin of this distinction comes from the choice of specializations of formula (2.1) we make to define initial "shelves" of series below - the alternative choices (switching the specialization in each case) would correspond to working with the s_0 transformation throughout.

Remark 2.10. Although we have used the language of the Lie algebra $A_1^{(1)}$in discussing the material above, all of the analysis in this work takes place solely on the level of the affine Weyl group. The relevant underlying theory is that of Coxeter groups, not that of Lie algebras.

2.2 Partitions

When we discuss the combinatorics of the "sum sides" of the identities, we will be using the following terminology concerning partitions:

A **partition** of a non-negative integer n is a finite nonincreasing sequence of positive integers, written as

$$\pi = (\pi_1, \ldots, \pi_t)$$

such that $\pi_1 + \cdots + \pi_t = n$. Each π_s is called a **part** of π. The **length** $\ell(\pi)$ of π is the number of parts in π, and given a positive integer p, the **multiplicity** $m(p)$ of p in π is the number of parts of π equal to p. As is conventional, we say that the integer 0 admits the unique partition into no parts (the empty partition).

Given a sequence (a_n), $n \geq 0$, the corresponding **generating function** is the formal power series

$$\sum_{n \geq 0} a_n q^n$$

in the formal variable q. In all of the identities we consider (and as was already seen in (1.1), (1.2)), we will want to interpret the right-hand sides as generating functions of partitions satisfying certain restrictions. In other words, the coefficient of q^n will be the number of partitions of n obeying the given restrictions.

Chapter 3

Rogers-Ramanujan case - new interpretation

3.1 Introduction

As was mentioned in the introduction, the Rogers-Ramanujan identities were the first identities to be given a "motivated proof" (in [AB]). Here we give a brief summary of their "motivation" and their proof technique.

Starting with the product sides of the identities (1.1), (1.2), one subtracts the second series (which they denoted $G_2(q)$) from the first one (denoted $G_1(q)$), and divided by q to obtain a new formal series $G_3(q)$. Next, one forms $G_4(q) = (G_2(q) - G_3(q))/q^2$. One repeats this process, giving $G_i(q) = (G_{i-2}(q) - G_{i-1}(q))/q^{i-2}$ for all $i \geq 3$, and notices *empirically* that for each $i \geq 1$, $G_i(q)$ is a power series in the formal variable q, it has constant term 1, and $G_i(q) - 1$ is divisible by q^i. This is the "Empirical Hypothesis" of Andrews-Baxter, and its truth easily leads to a proof the two Rogers-Ramanujan identities.

To prove the Empirical Hypothesis directly from the product sides, one starts by transforming the two initial Rogers-Ramanujan product sides into alternating sums using the Jacobi triple product identity. By running these alternating sums through the recursion given above, one obtains alternating sum formulas for the next several G_i series, and can conjecture and prove closed-formed formulas for all of the higher series. However, the calculations necessary to establish these formulas (formal manipulation and reindexing of q-series) are essentially ad-hoc.

We will be following [LZ] and the appendices to [CKLMQRS], which refined this argument. These papers incorporate concepts not used in [AB], such as placing series on "shelves", as we will review. While the basic logic of the proof below is the same as in these earlier papers, we will go through the motivated proof "from scratch" using our new perspective, highlighting the differences between these prior works and our new approach, and giving new motivation for each step. Our approach leans heavily on the observation first made by Lepowsky and S. Milne in [LM], that the Rogers-Ramanujan series arise as graded dimensions of the level-3 modules for the affine Lie algebra $A_1^{(1)}$.

3.2 Definitions and background

By specializing the quantities $e^{-\alpha_0}, e^{-\alpha_1}$ in (2.1) to appropriate powers of a single variable q, we obtain the product sides of the Rogers-Ramanujan identities (up to a factor), and can use the identity above to express them in alternating-sum form.

Specifically, specializing $e^{-\alpha_0} \mapsto q^3$, $e^{-\alpha_1} \mapsto q^2$, we get

$$\begin{aligned}(q^2,q^5)_\infty(q^3,q^5)_\infty(q^5,q^5)_\infty &= (q,q)_\infty(q,q^5)_\infty^{-1}(q^4,q^5)_\infty^{-1} \\ &= \sum_{w\in W}(-1)^{\ell(w)}q^{3|\Delta_w|_0+2|\Delta_w|_0}\end{aligned} \tag{3.1}$$

Specializing $e^{-\alpha_0} \mapsto q^4$, $e^{-\alpha_1} \mapsto q$, we get

$$\begin{aligned}(q,q^5)_\infty(q^4,q^5)_\infty(q^5,q^5)_\infty &= (q,q)_\infty(q^2,q^5)_\infty^{-1}(q^3,q^5)_\infty^{-1} \\ &= \sum_{w\in W}(-1)^{\ell(w)}q^{4|\Delta_w|_0+|\Delta_w|_0}\end{aligned} \tag{3.2}$$

Denote by $F(q)$ the series

$$(q,q)_\infty = \prod_{n\geq 1}(1-q^n)$$

Dividing these equations through by $F(q)$, we recognize on the left-hand sides the Rogers-Ramanujan product expressions from (1.1), (1.2). In the proofs, we will work with the right-hand sides, which we denote as

$$R_1(q) = \frac{1}{F(q)}\sum_{w\in W}(-1)^{\ell(w)}q^{3|\Delta_w|_0+2|\Delta_w|_1}$$

$$R_2(q) = \frac{1}{F(q)}\sum_{w\in W}(-1)^{\ell(w)}q^{4|\Delta_w|_0+|\Delta_w|_1}$$

respectively (we choose the symbol R instead of G for these series to distinguish between these and the more general Gordon-Andrews series of the next chapter).

We emphasize that these two expressions are our definition of a "zeroth shelf" of series, which we will subsequently use to generate infinitely many higher shelves of series.

Starting with these two series, we inductively define an infinite sequence of series by the equation

$$R_{i+2}(q) = \frac{R_i(q) - R_{i+1}(q)}{q^i} \tag{3.3}$$

As mentioned above, this recursion was originally motivated (in [AB]) by the empirically evident fact that the resulting R_i series are always of the form $1 + q^i + \cdots$, with all positive coefficients. A new feature seen here for the first time is that this recursion also aligns naturally to the recursion for the $|\Delta_w|$ terms. This provides a new motivation coming directly from the Coxeter group structure.

We arrange these series into shelves of two series each. The two series on shelf j are R_{j+1} and R_{j+2}. In particular, the zeroth shelf consists of the series R_1 and R_2 given above, and the second series on each shelf is the same as the first series on the next shelf.

In order to use the recursion, it is necessary that all the series on a given shelf must share a similar form. We demonstrate how this works for the first few shelves:

The zeroth shelf consists of the two series

$$R_{0+1}(q) = \frac{1}{F(q)} \sum_{w \in W} (-1)^{\ell(w)} q^{3|\Delta_w|_0 + 2|\Delta_w|_1}$$

$$R_{0+2}(q) = \frac{1}{F(q)} \sum_{w \in W} (-1)^{\ell(w)} q^{4|\Delta_w|_0 + |\Delta_w|_1}$$

Substituting these series into the recursion yields

$$R_{1+2}(q) = \frac{R_{0+1}(q) - R_{0+2}(q)}{q} = \frac{1}{qF(q)} \sum_{w \in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) q^{3|\Delta_w|_0 + |\Delta_w|_1}$$

To complete shelf $j = 1$, we need a similarly-shaped formula for the R_2 series. The natural thing to try is to have the same factor in parentheses and ensure that after distributing the parentheses, the positive part is identical with our previous R_2 formula. This approach yields:

$$R_{1+1}(q) = \frac{1}{F(q)} \sum_{w \in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) q^{4|\Delta_w|_0}$$

In order for this formula to be valid, we need the difference $R_{0+2} - R_{1+1}$ to be 0. This difference is the series

$$\frac{1}{F(q)} \sum_{w \in W} (-1)^{\ell(w)} q^{5|\Delta_w|_0}$$

which is in fact 0, as can be seen by applying the "change of index" $w \mapsto s_1 w$. This has the effect of changing the length by 1, but has no effect on the value of $|\Delta_w|_0$. Hence the overall effect is to negate each term of the series, which in turn implies that the whole sum is 0.

Next, we recursively compute

$$R_{2+2}(q) = \frac{R_{1+1}(q) - R_{1+2}(q)}{q^2} = \frac{1}{q^2F(q)} \sum_{w\in W} (-1)^{\ell(w)}(q^{|\Delta_w|_1} - q^{|\Delta_w|_0})(q^{|\Delta_w|_0} - q^{|\Delta_w|_1 - 1})q^{3|\Delta_w|_0}$$

To complete shelf $j = 2$, we need a similarly-shaped formula for the R_3 series. Again, we attempt to match the parentheses, which gives

$$R_{2+1}(q) = \frac{1}{qF(q)} \sum_{w\in W} (-1)^{\ell(w)}(q^{|\Delta_w|_1} - q^{|\Delta_w|_0})(q^{|\Delta_w|_0} - q^{|\Delta_w|_1 - 1})q^{2|\Delta_w|_0 + |\Delta_w|_1}$$

As before, in order to verify that this series expression is identical to the previous expression for $R_3(q)$, we need to consider the difference $R_{1+2} - R_{2+1}$. This is given by

$$\frac{1}{q^2F(q)} \sum_{w\in W} (-1)^{\ell(w)}(q^{|\Delta_w|_1} - q^{|\Delta_w|_0})q^{2|\Delta_w|_0 + 2|\Delta_w|_1}$$

This time, we apply the change of index $w \mapsto f(w)$. Clearly the length of a Weyl group element is invariant under this transformation, and its effect on $|\Delta_w|$ is to interchange the components $|\Delta_w|_0$ and $|\Delta_w|_1$.

Hence the transformed expression is

$$\frac{1}{q^2F(q)} \sum_{f(w)\in W} (-1)^{\ell(w)}(q^{|\Delta_w|_0} - q^{|\Delta_w|_1})q^{2|\Delta_w|_1 + 2|\Delta_w|_0}$$

which is once again precisely the negation of the original difference series. As above, this implies that the sum is 0, and that the two forms of the R_3 series are both valid.

Notice that in this last computation, the bijection $w \leftrightarrow f(w)$ fixes the identity element of the Weyl group, $w = e$. However, this does not cause any problems: the summand corresponding to the identity element is 0 because the factor $(q^{|\Delta_e|_0} - q^{|\Delta_e|_1})$ is $(q^0 - q^0) = 0$.

Although it is not readily apparent yet at this early stage, it is also the case that the powers of q appearing in the denominator before the sum can be nicely expressed in terms of Weyl data. This will be made explicit below in the general series formulas.

3.3 Closed formulas

We have the following remarkable closed-form formulas for these series:

Theorem 3.1. *Let $j \geq 0$ and $i = 1, 2$. If $j = 2h$ is even, then*

$$R_{2h+i}(q) = \frac{1}{F(q)} q^{-2|\Delta_{w_h^0}|_0 (i-1) - \left[|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1\right](2-i)}$$
$$\cdot \sum_{w \in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w_h^1}|_0 - |\Delta_{w_h^1}|_1})$$
$$q^{(2+i-h)|\Delta_w|_0 + (3-i-h)|\Delta_w|_1} \tag{3.4}$$

If $j = 2h+1$ is odd, then

$$R_{(2h+1)+i}(q) = \frac{1}{F(q)} q^{-\left[|\Delta_{w_{h+1}^1}|_0 + |\Delta_{w_{h+1}^1}|_1\right](i-1) - 2|\Delta_{w_{h+1}^1}|_0 (2-i)}$$
$$\cdot \sum_{w \in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1})$$
$$q^{(5-i-h)|\Delta_w|_0 + (i-1-h)|\Delta_w|_1} \tag{3.5}$$

Remark 3.2. We mentioned in Remark 2.8 that one way in which our formulas differ from those in earlier works is that ours are “unfolded”. Now that we have stated this theorem, we can justify this claim by comparing our formulas (3.4), (3.5) to the corresponding formulas from Theorem 2.1 in [LZ] (in the special case $k = 2$, which gives the Rogers-Ramanujan series). Each summand of the series there corresponds to *two* terms of our series as given above. In fact, the pairings of terms on even and odd shelves (values of j) correspond exactly to the involutions defined in Definitions 2.3, 2.4 respectively. However, this disparity between even and odd shelves is invisible in [LZ].

Remark 3.3. By replacing some of the expressions in terms of $|\Delta_w|$ data with their numerical values, we can write the above formulas in a much more compact form:

$$R_{2h+i}(q) = \frac{1}{q^{h(h+1)(i-1)+h^2(2-i)} F(q)}$$
$$\cdot \sum_{w \in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0})(q^{|\Delta_w|_0} - q^{|\Delta_w|_1 - 1}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 - h})$$
$$q^{(2+i-h)|\Delta_w|_0 + (3-i-h)|\Delta_w|_1} \tag{3.6}$$

$$R_{(2h+1)+i}(q) = \frac{1}{q^{(h+1)^2(i-1)+h(h+1)(2-i)} F(q)}$$
$$\cdot \sum_{w \in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0})(q^{|\Delta_w|_0} - q^{|\Delta_w|_1 - 1}) \cdots (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - h})$$
$$q^{(5-i-h)|\Delta_w|_0 + (i-1-h)|\Delta_w|_1} \tag{3.7}$$

In the statement of the theorem, we have chosen to write out the expanded version for several reasons: first of all, this is the form in which the exponents actually arise out of the recursions. It also makes it clearer how the transformations $s_1\cdot$, $f(\cdot)$ act on these factors. Finally, in the proof of the Empirical Hypothesis in the next section, this formulation makes it evident how cancellations arise in the exponents for the smallest non-zero contribution. However, choosing to write the formulas this way does necessitate an extra step in the proofs of the edge-matching, invoking certain identities of the $|\Delta_w|$ components in order to properly compare the two different series. In this section we will point out these shifts whenever they occur, but in later sections we will suppress these reminders.

Proof. The expressions above give two different formulas for the "edge" cases — i.e., $i = 1$ on shelf $j \geq 1$ and $i = 2$ on shelf $j - 1$ — so we first prove that they are compatible.

Let $\overline{R}_{j,i}(q)$ denote the right-hand side of the formulas above. We will verify that the difference

$$\overline{R}_{j,2}(q) - \overline{R}_{j+1,1}(q) = 0$$

We do this by performing a suitable "change of index" on the sum, namely, the two transformations defined above in Definitions 2.3 and 2.4. The calculation will depend on the parities of the shelves involved.

Let $j = 2h$ (so that $j + 1 = 2h + 1$). Then we have:

$$\begin{aligned}
&\overline{R}_{2h,2}(q) - \overline{R}_{2h+1,1}(q)\\
&= \frac{1}{F(q)} q^{-2|\Delta_{w_h^0}|_0} \sum_{w\in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w_h^1}|_0 - |\Delta_{w_h^1}|_1})\\
&\qquad\qquad \cdot q^{(4-h)|\Delta_w|_0 + (1-h)|\Delta_w|_1}\\
&- \frac{1}{F(q)} q^{-2|\Delta_{w_{h+1}^1}|_0} \sum_{w\in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w_h^1}|_0 - |\Delta_{w_h^1}|_1})\\
&\qquad\qquad \cdot (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1}) \cdot q^{(4-h)|\Delta_w|_0 - h|\Delta_w|_1}\\
&= \frac{1}{F(q)} q^{-3|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1} \sum_{w\in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w_h^1}|_0 - |\Delta_{w_h^1}|_1})\\
&\qquad\qquad \cdot q^{(5-h)|\Delta_w|_0 - h|\Delta_w|_1}
\end{aligned}$$

Above, we have implicitly made the identification

$$|\Delta_{w_h^0}|_0 = |\Delta_{w_{h+1}^1}|_0$$

for the leading exponents.

We now apply the transformation s_1* (from Definition 2.3) to the Weyl group. This is a bijection of the Weyl group, so must leave the above sum invariant. Recall from (2.7) that the effect of this transformation on the components $|\Delta_w|_0$ and $|\Delta_w|_1$:

$$|\Delta_w| = |\Delta_w|_0\alpha_0 + |\Delta_w|_1\alpha_1 \mapsto |\Delta_{s_1 w}| = |\Delta_w|_0\alpha_0 + (2|\Delta_w|_0 - |\Delta_w|_1 + 1)\alpha_1$$

Also recall that it changes the length of each Weyl element by one. We have already computed above in Corollary 2.5 the effect of this transformation on the factors in parentheses. Hence the above sum is transformed into:

$$\begin{aligned}
&\overline{R}_{2h,2}(q) - \overline{R}_{2h+1,1}(q)\\
&= \frac{1}{F(q)} q^{-3|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1} \sum_{s_1 w \in W} (-1)^{\ell(s_1 w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w_h^1}|_0 - |\Delta_{w_h^1}|_1})\\
&\qquad \cdot q^{h(2|\Delta_w|_0 - 2|\Delta_w|_1 + 1) + (5-h)|\Delta_w|_0 - h(2|\Delta_w|_0 - |\Delta_w|_1 + 1)}\\
&= -\frac{1}{F(q)} q^{-3|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1} \sum_{s_1 w \in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w_h^1}|_0 - |\Delta_{w_h^1}|_1})\\
&\qquad \cdot q^{(5-h)|\Delta_w|_0 - h|\Delta_w|_1}
\end{aligned}$$

This final expression is exactly the negation of the starting series. The anti-commutativity of this expression proves that it must be 0.

Next, let $j = 2h+1$ (so that $j+1 = 2(h+1)$). Then we have:

$$\begin{aligned}
&\overline{R}_{2h+1,2}(q) - \overline{R}_{2(h+1),1}(q)\\
&= \frac{1}{F(q)} q^{-\left[|\Delta_{w_{h+1}^1}|_0 + |\Delta_{w_{h+1}^1}|_1\right]} \sum_{w \in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1})\\
&\qquad \cdot q^{(3-h)|\Delta_w|_0 + (1-h)|\Delta_w|_1}\\
&- \frac{1}{F(q)} q^{-\left[|\Delta_{w_{h+1}^0}|_0 + |\Delta_{w_{h+1}^0}|_1\right]} \sum_{w \in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1})\\
&\qquad \cdot (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w_{h+1}^1}|_0 - |\Delta_{w_{h+1}^1}|_1}) q^{(2-h)|\Delta_w|_0 + (1-h)|\Delta_w|_1}\\
&= \frac{1}{F(q)} q^{-2|\Delta_{w_{h+1}^1}|_1} \sum_{w \in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1})\\
&\qquad \cdot q^{(2-h)|\Delta_w|_0 + (2-h)|\Delta_w|_1}
\end{aligned}$$

Above, we have implicitly made the identification

$$|\Delta_{w^0_{h+1}}|_0 + |\Delta_{w^0_{h+1}}|_1 = |\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1$$

for the leading exponents.

We now apply to the Weyl group the transformation f, defined in Definition 2.4. This is a bijection of the Weyl group, so must leave the above sum invariant. Recall that the effect of this transformation is to interchange the components $|\Delta_w|_0$ and $|\Delta_w|_1$, and that it leaves invariant the length of the Weyl element. Invoking Corollary 2.5, we see that the above series is transformed into:

$$\begin{aligned}
&\overline{R}_{2h+1,2}(q) - \overline{R}_{2(h+1),1}(q) \\
&= -\frac{1}{F(q)} q^{-2|\Delta_{w^1_{h+1}}|_1} \sum_{f(w)\in W} (-1)^{\ell(f(w))} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w^0_h}|_0 + |\Delta_{w^0_h}|_1}) \\
&\qquad\qquad\qquad\qquad \cdot q^{(2-h)|\Delta_w|_1 + (2-h)|\Delta_w|_0}
\end{aligned}$$

Again the overall effect of this transformation is to negate the sum, and the anti-commutativity of this expression proves that it must be 0.

Remark 3.4. The calculations in this proof are simpler and more philosophically satisfying than the corresponding steps in the proof in [LZ] (or [CKLMQRS] for the edge-matching, which was suppressed in [LZ]). Because the series there were "folded" (see Remark 3.2), it was necessary in the proofs to break up each series and reindex different halves separately in order to match up terms appropriately. Here, the sums are always over the full Weyl group W and terms always match up naturally. Moreover, the "reindexing" now comes from a natural transformation of the indexing group W, instead of an ad hoc shifting.

Now that we have verified the edge-matching, it remains to show that these formulas satisfy the recursion. We handle this in two cases once more. In both cases, we take the difference of the two shelf j series formulas to get the $i = 2$ entry on shelf $j + 1$.

First, let $j = 2h$, $j+1 = 2h+1$:

$$\frac{R_{2h+1}(q) - R_{2h+2}(q)}{q^{2h+1}}$$

$$= \frac{1}{q^{2h+1}}\Bigg[\frac{1}{F(q)} q^{-|\Delta_{w_h^0}|_0 - |\Delta_{w_h^0}|_1} \sum_{w\in W}(-1)^{\ell(w)}(q^{|\Delta_w|_1} - q^{|\Delta_w|_0})\cdots(q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w_h^1}|_0 - |\Delta_{w_h^1}|_1}) \cdot q^{(3-h)|\Delta_w|_0 + (2-h)|\Delta_w|_1}$$

$$- \frac{1}{F(q)} q^{-2|\Delta_{w_h^0}|_0} \sum_{w\in W}(-1)^{\ell(w)}(q^{|\Delta_w|_1} - q^{|\Delta_w|_0})\cdots(q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w_h^1}|_0 - |\Delta_{w_h^1}|_1}) \cdot q^{(4-h)|\Delta_w|_0 + (1-h)|\Delta_w|_1}\Bigg]$$

$$= \frac{1}{F(q)} q^{-|\Delta_{w_h^0}|_0 - |\Delta_{w_h^0}|_1 - (2h+1)} \sum_{w\in W}(-1)^{\ell(w)}(q^{|\Delta_w|_1} - q^{|\Delta_w|_0})\cdots(q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w_h^1}|_0 - |\Delta_{w_h^1}|_1}) \cdot (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1}) q^{(3-h)|\Delta_w|_0 + (1-h)|\Delta_w|_1}$$

$$= R_{(2h+1)+2}(q)$$

Secondly, let $j = 2h+1$, $j+1 = 2(h+1)$:

$$\frac{R_{(2h+1)+1}(q) - R_{(2h+1)+2}(q)}{q^{2h+2}}$$

$$= \frac{1}{q^{2h+2}}\Bigg[\frac{1}{F(q)} q^{-2|\Delta_{w_{h+1}^1}|_0} \sum_{w\in W}(-1)^{\ell(w)}(q^{|\Delta_w|_1} - q^{|\Delta_w|_0})\cdots(q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1}) \cdot q^{(4-h)|\Delta_w|_0 - h|\Delta_w|_1}$$

$$- \frac{1}{F(q)} q^{-|\Delta_{w_{h+1}^1}|_0 - |\Delta_{w_{h+1}^1}|_1} \sum_{w\in W}(-1)^{\ell(w)}(q^{|\Delta_w|_1} - q^{|\Delta_w|_0})\cdots(q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1}) \cdot q^{(3-h)|\Delta_w|_0 + (1-h)|\Delta_w|_1}\Bigg]$$

$$= \frac{1}{F(q)} q^{-2|\Delta_{w_{h+1}^1}|_0} \sum_{w\in W}(-1)^{\ell(w)}(q^{|\Delta_w|_1} - q^{|\Delta_w|_0})\cdots(q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1}) \cdot (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w_{h+1}^1}|_0 - |\Delta_{w_{h+1}^1}|_1}) q^{(3-h)|\Delta_w|_0 - h|\Delta_w|_1}$$

$$= R_{(2h+2)+2}(q)$$

$\square$

Remark 3.5. From these calculations, we see that the most natural way to express the denominators of the recursions (3.3) is as

$$q^{2h+1} = q^{(|\Delta_{w_{h+1}^1}|_0 + |\Delta_{w_{h+1}^1}|_1) - (|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1)} = q^{|\Delta_{w_{h+1}^1}|_1 - |\Delta_{w_h^0}|_1}$$

when $j = 2h$ and

$$q^{2(h+1)} = q^{2[|\Delta_{w^0_{h+1}}|_0 - |\Delta_{w^1_{h+1}}|_0]}$$

when $j = 2h + 1$ (indeed, we have already implicitly made use of these expressions to obtain the final lines of the equations above).

Remark 3.6. In previous works on motivated proofs, the sum was always reindexed at each stage of the recursion to ensure that all summands were nonzero. This was natural, because there was no visible reason for preserving the information of having terms equal to zero at the beginning of the sum. However, our sums are never "reindexed" (they are always over the full Weyl group), so we do end up with a finite number of summands equal to zero at the start of every R series past the first shelf.

Remark 3.7. It is important to note that the common factor $F(q)$ plays no role in the proof here, except for the identification with the original $R_i(q)$, $i = 1, \ldots, k$. The factor $F(q)$ could have been replaced with any nonzero formal power series in q, and every step of the proof would have been identical (beyond the identification with the original $R_i(q)$, $i = 1, \ldots, k$), and equivalent to the existing step. However, $F(q)$ is crucial for the "Empirical Hypothesis," which in fact, as we shall see, uniquely determines this factor.

3.4 Empirical Hypothesis

As a consequence of Theorem 3.1, we are now in a position to formulate and prove the Empirical Hypothesis. This is the main ingredient needed to complete the motivated proof.

Theorem 3.8 (Empirical Hypothesis)**.** *For any $j \geq 0$ and $i = 1, 2$,*

$$R_{j+i}(q) = 1 + q^{j+1}\gamma(q)$$

for some

$$\gamma(q) \in \mathbb{C}[[q]].$$

Remark 3.9. Note that since $R_{j+2}(q) = R_{(j+1)+1}(q)$, Theorem 3.8 implies that we can write $R_{j+2}(q) = 1 + q^{j+2}\gamma(q)$ where $\gamma(q)$ is some formal power series.

Remark 3.10. The proof of this theorem is a more involved argument than for the corresponding result in [LZ] and [CKLMQRS], because it is necessary to "translate" the data of the series from Weyl notation to integers in order to compare the term of the series to the shape of the proposed Empirical Hypothesis.

An overview of the proof: Recall the linear order on W introduced above, in Remark 2.8. In general, for each R series, the first few summands in this ordering will be equal to 0. The first term which is nonzero will contribute the initial $1+q^{j+1}$ in the Empirical Hypothesis, and in general, all other contributions from it and all subsequent terms will involve only greater powers of q. In the special case $i = 2$, a closer analysis shows that the next term after the least nonzero contributor contributes exactly what is needed to make the series of the form $1 + q^{j+2}$.

Proof. First, we consider the even-shelf series, which from Theorem 3.1 are of the form

$$R_{2h+i}(q) = \frac{1}{F(q)} q^{-2|\Delta_{w_h^0}|_0(i-1) - \left[|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1\right](2-i)}$$
$$\cdot \sum_{w\in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w_h^1}|_0 - |\Delta_{w_h^1}|_1}) \cdot q^{(2+i-h)|\Delta_w|_0 + (3-i-h)|\Delta_w|_1}$$

First suppose that w is one of the first $2h$ elements (with respect to the linear order)- i.e., w is either of the form w_r^0 for $r < h$, or w_r^1 for $r \leq h$. It follows from (2.14), (2.15) that

$$|\Delta_{w_r^0}|_0 - |\Delta_{w_r^0}|_1 = r = |\Delta_{w_r^1}|_1 - |\Delta_{w_r^1}|_0 \tag{3.8}$$

Considering the corresponding term, we see that exactly one of these factors inside the parentheses will be zero in this case. Specifically, if $w = w_r^0$ for some for $r < h$, then

$$(q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w_r^0}|_0 + |\Delta_{w_r^0}|_1}) = (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - r}) = 0 \tag{3.9}$$

while if $w = w_r^1$ for some $r \leq h$, then

$$(q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w_r^1}|_0 - |\Delta_{w_r^1}|_1}) = (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 - r}) = 0 \tag{3.10}$$

We claim that the first nonzero contribution to the series is from the term $w = w_h^0$, and that this contribution is of the shape $1 + q^{2h+1} + O(q^{2h+2})$. This claim is justified by the following sequence of computations, starting with the $w = w_h^0$ term from the series referenced above:

$$\frac{1}{F(q)} q^{-2|\Delta_{w_h^0}|_0 (i-1) - \left[|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1\right](2-i)}$$

$$(-1)^{\ell(w_h^0)} (q^{|\Delta_{w_h^0}|_1} - q^{|\Delta_{w_h^0}|_0}) \cdots (q^{|\Delta_{w_h^0}|_0} - q^{|\Delta_{w_h^0}|_1 + |\Delta_{w_h^1}|_0 - |\Delta_{w_h^1}|_1}) \cdot q^{(2+i-h)|\Delta_{w_h^0}|_0 + (3-i-h)|\Delta_{w_h^0}|_1}$$

$$= \frac{1}{F(q)} q^{-2|\Delta_{w_h^0}|_0 (i-1) - \left[|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1\right](2-i)}$$

$$(-1)^h \{q^{|\Delta_{w_h^0}|_1} (1 - q^{|\Delta_{w_h^0}|_0 - |\Delta_{w_h^0}|_1})\} \cdots \{-q^{|\Delta_{w_h^0}|_1 - h} (1 - q^{|\Delta_{w_h^0}|_0 - |\Delta_{w_h^0}|_1 + h})\}$$

$$\cdot q^{(2+i-h)|\Delta_{w_h^0}|_0 + (3-i-h)|\Delta_{w_h^0}|_1}$$

$$= \frac{1}{F(q)} (-1)^{2h} (1-q) \cdots (1-q^{2h})$$

$$q^{-2|\Delta_{w_h^0}|_0 (i-1) - \left[|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1\right](2-i) + 2h|\Delta_{w_h^0}|_1 - |\Delta_{w_h^0}|_0 + (2+i-h)|\Delta_{w_h^0}|_0 + (3-i-h)|\Delta_{w_h^0}|_1}$$

In the middle of the calculation, we used the fact that in the factors $-q^{|\Delta_w|_1 - r}$ for $r = 1, \ldots, h$, each r is really $|\Delta_{w_r^1}|_1 - |\Delta_{w_r^1}|_0$. Then the relations (2.14) - (2.17) allow us to collapse the sum $1 + \cdots + h$ into a telescoping sum which yields the singleton $-|\Delta_{w_h^0}|_0$ in the exponent on the next line.

Looking more closely at the exponent of q, we can regroup it as

$$\begin{aligned}
&2\Big[(|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1) - (|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1)\Big] \\
&+ i\Big[(|\Delta_{w_h^0}|_0 - |\Delta_{w_h^0}|_1) - (|\Delta_{w_h^0}|_0 - |\Delta_{w_h^0}|_1)\Big] \\
&+ |\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1 - h(|\Delta_{w_h^0}|_0 - |\Delta_{w_h^0}|_1) \\
&= 0 + 0 + h^2 - h^2 \\
&= 0
\end{aligned}$$

Moreover, expanding

$$F(q) = (1-q)(1-q^2)(1-q^3)\cdots$$

we see that we can cancel the first $2h$ factors of $F(q)$ with the remaining factors in the term. Hence this term is in fact equal to

$$(q^{2h}, q)_\infty^{-1} = 1 + q^{2h+1} + \cdots$$

(as can be seen by using the standard expansion $(1-q)^{-1} = 1 + q + q^2 + q^3 + \cdots$)

To verify that all subsequent contributions are $O(q^{2h+1})$, we will once again make use of the two bijective transformations of the Weyl group using Corollary 2.5.

The proof here is essentially by induction: starting with w_h^0 (for which we have explicitly calculated the contribution), we compute how the minimal power of q in the next (with respect to the linear order) term is related to the powers present in the previous term. We will perform two calculations, one for each of our the transformations (and corresponding respectively to those w of each of the two forms w_r^0, w_r^1). For these calculations, the part of the data that is the same across all terms (i.e., the leading factors written in terms of $F(q)$ and the components of $|\Delta_{w_h^0}|$) will not be affected, and hence can be neglected in the computations.

First, suppose that w is of the form w_r^0 for some $r \geq h$. Using Definition 2.3 and Corollary 2.5, we are able to express the term corresponding to $s_1 w = s_1 w_r^0 = w_{r+1}^1$ in terms of the w_r^0 data. The overall effect is to scale the term by a certain power of q. By combining the factor from Corollary 2.5 and taking the difference of the original and transformed ending exponents, we calculate that this power of q is equal to

$$\begin{aligned}
&h\Big(2|\Delta_w|_0 - 2|\Delta_w|_1 + 1\Big)\\
&+ (2+i-h)|\Delta_w|_0 + (3-i-h)(2|\Delta_w|_0 - |\Delta_w|_1 + 1)\\
&- (2+i-h)|\Delta_w|_0 + (3-i-h)|\Delta_w|_1\\
&= h\Big(2|\Delta_w|_0 - 2|\Delta_w|_1 + 1\Big)\\
&+ (3-i-h)(2|\Delta_w|_0 - 2|\Delta_w|_1 + 1)\\
&= (3-i)(2|\Delta_w|_0 - 2|\Delta_w|_1 + 1)\\
&\geq 2h+1
\end{aligned}$$

where for the last inequality holds, we have used the facts that $i \leq 2$ and that for elements of the form w_r^0 with $r \geq h$, we have $|\Delta_w|_0 - |\Delta_w|_1 \geq h$. (Notice that in particular, we get equality in the last line only when $i = 2$ and $r = h$, and in this case the least contribution from this term is $-q^{2h+1}$, which precisely cancels out the q^{2h+1} from the previous term, giving us a series of the form $1 + q^{2h+2} + \cdots$. This shows the agreement between the edge-matching and the Empirical Hypothesis). In general, this confirms that all higher contributions are $O(q^{2h+1})$.

Next, suppose that $w = w_r^1$ for some $r \geq h+1$. Here the next Weyl element in the linear order is $f(w_r^1) = w_r^0$. From Definition 2.4 and Corollary 2.5, the relevant parts of the terms that we need to compare are

$$(q^{|\Delta_w|_0} - q^{|\Delta_w|_1 - h}) \cdot q^{(2+i-h)|\Delta_w|_0 + (3-i-h)|\Delta_w|_1}$$
$$= q^{(3+i-h)|\Delta_w|_0 + (3-i-h)|\Delta_w|_1} - q^{(2+i-h)|\Delta_w|_0 + (4-i-h)|\Delta_w|_1 - h}$$

for the w_r^1 term and

$$-(q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - h})q^{(2+i-h)|\Delta_w|_1 + (3-i-h)|\Delta_w|_0}$$
$$q^{(3+i-h)|\Delta_w|_0 + (3-i-h)|\Delta_w|_1} - q^{(2+i-h)|\Delta_w|_0 + (4-i-h)|\Delta_w|_1 - h}$$

The ratios between the parts of these expressions is

$$q^{(-2i+1)|\Delta_w|_0 + (2i-1)|\Delta_w|_1 - h}$$

for the positive parts and

$$q^{(-2i+1)|\Delta_w|_0 + (2i-1)|\Delta_w|_1 + h}$$

for the negative parts. The ratio between positive parts is the smaller of the two, and we can bound its exponent as

$$(2i-1)\left[|\Delta_w|_1 - |\Delta_w|_0\right] - h \geq (2i-1)(h+1) - h \geq 1$$

Hence the power of q must increase, and here, too, the contribution is $O(q^{2h+1})$.

Combining these results, we have verified the Empirical Hypothesis for even shelves.

For odd shelves, from Theorem 3.1, the series are of the form

$$R_{(2h+1)+i}(q) = \frac{1}{F(q)} q^{-\left[|\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1\right](i-1) - 2|\Delta_{w^1_{h+1}}|_0 (2-i)}$$
$$\sum_{w \in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w^0_h}|_0 + |\Delta_{w^0_h}|_1})$$
$$\cdot q^{(5-i-h)|\Delta_w|_0 + (i-1-h)|\Delta_w|_1}$$

First suppose that w is one of the first $2h+1$ elements (with respect to the linear order) - i.e., that w is either of the form w_r^0 for $r \leq h$, or w_r^1 for $r \leq h$. Then reasoning as above, it follows from the formulas (3.8), (3.9), and (3.10) that the terms corresponding to these elements are 0.

We claim that the first nonzero contribution to the series is from the term $w = w^1_{h+1}$, and that this contribution is of the shape $1 + q^{2h+2} + O(q^{2h+3})$. This claim is justified by the following sequence of computations, starting with the $w = w^1_{h+1}$ term from the series referenced above:

$$\frac{1}{F(q)} q^{-\left[|\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1\right](i-1) - 2|\Delta_{w^0_{h+1}}|_0 (2-i)}$$
$$(-1)^{\ell(w^1_{h+1})}(q^{|\Delta_{w^1_{h+1}}|_1} - q^{|\Delta_{w^1_{h+1}}|_0}) \cdots (q^{|\Delta_{w^1_{h+1}}|_1} - q^{|\Delta_{w^1_{h+1}}|_0 - |\Delta_{w^0_h}|_0 + |\Delta_{w^0_h}|_1})$$
$$\cdot\, q^{(5-i-h)|\Delta_{w^1_{h+1}}|_0 + (i-1-h)|\Delta_{w^1_{h+1}}|_1}$$
$$= \frac{1}{F(q)} q^{-\left[|\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1\right](i-1) - 2|\Delta_{w^0_{h+1}}|_0 (2-i)}$$
$$(-1)^{h+1}\{-q^{|\Delta_{w^1_{h+1}}|_0}(1 - q^{|\Delta_{w^1_{h+1}}|_1 - |\Delta_{w^1_{h+1}}|_0})\} \cdots \{-q^{|\Delta_{w^1_{h+1}}|_0 - h}(1 - q^{|\Delta_{w^1_{h+1}}|_1 - |\Delta_{w^1_{h+1}}|_0 + h})\}$$
$$\cdot\, q^{(5-i-h)|\Delta_{w^0_h}|_0 + (i-1-h)|\Delta_{w^0_h}|_1}$$
$$= \frac{1}{F(q)}(-1)^{2h+2}(1-q)\cdots(1-q^{2h+1})$$
$$q^{-\left[|\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1\right](i-1) - 2|\Delta_{w^0_{h+1}}|_0 (2-i) + (2h+1)|\Delta_{w^0_h}|_1 - |\Delta_{w^1_{h+1}}|_0 + (5-i-h)|\Delta_{w^0_h}|_0 + (i-1-h)|\Delta_{w^0_h}|_1}$$

As above, the relations (2.14) - (2.17) allow us to collapse the telescoping sum $-(1 + \cdots + h)$ into the singleton $-|\Delta_{w^1_{h+1}}|_0$ in the exponent on the next line.

Looking more closely at the exponent of q, we can regroup it as

$$\begin{aligned}
&4\left[|\Delta_{w^1_{h+1}}|_0 - |\Delta_{w^1_{h+1}}|_0\right] \\
&+ i\left[(|\Delta_{w^1_{h+1}}|_1 - |\Delta_{w^1_{h+1}}|_0) - (|\Delta_{w^1_{h+1}}|_1 - |\Delta_{w^1_{h+1}}|_0)\right] \\
&+ |\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1 - (h+1)(|\Delta_{w^1_{h+1}}|_1 - |\Delta_{w^1_{h+1}}|_0) \\
&= 0 + 0 + (h+1)^2 - (h+1)^2 \\
&= 0
\end{aligned}$$

Moreover, expanding $F(q) = (1-q)(1-q^2)(1-q^3)\cdots$, we see that we can cancel the first $2h+1$ factors of $F(q)$ with the remaining factors in the term. Hence this term is in fact equal to

$$(q^{2h+1}, q)^{-1}_\infty = 1 + q^{2h+2} + \cdots$$

To verify that all subsequent contributions are $O(q^{2h+2})$, we use the same technique as for the even shelves: applying the two transformations to terms coming from the above series and verifying that exponents always increase.

First suppose w is of the form w_r^1 for some $r \geq h+1$, so that the next Weyl element in the linear order is $f(w_r^1) = w_r^0$. By Definition 2.4 and Corollary 2.5, the $f(\cdot)$ transformation acts as negation on the factors in parentheses, and its effect on the ending exponents results in rescaling by a power of q, whose exponent is:

$$\begin{aligned}
&(i-1-h)|\Delta_w|_0 + (5-i-h)|\Delta_w|_1 \\
&- (5-i-h)|\Delta_w|_0 - (i-1-h)|\Delta_w|_1 \\
&= (2i-6)|\Delta_w|_0 + (-2i+6)|\Delta_w|_1 \\
&= 2(3-i)[|\Delta_w|_1 - |\Delta_w|_0] \\
&\geq 2h+2
\end{aligned}$$

with equality only when $i = 2$ and $w = w_{h+1}^1$. Since we have made the assumption that the preceding term is at least $1 + O(q^{2h+2})$, and the minimal exponent here is at least $2h+2$ higher, we conclude that the contribution here is $O(q^{2h+2})$.

Next suppose w is of the form w_r^0 for some $r \geq h+1$, so that the next Weyl element is $s_1 w_r^0 = w_{r+1}^1$. Using Definition 2.3 and Corollary 2.5, the relevant difference in the two corresponding terms is the replacement of

$$\begin{aligned}
&(q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - h}) q^{(5-i-h)|\Delta_w|_0 + (i-1-h)|\Delta_w|_1} \\
&= q^{(5-i-h)|\Delta_w|_0 + (i-h)|\Delta_w|_1} - q^{(6-i-h)|\Delta_w|_0 + (i-1-h)|\Delta_w|_1 - h}
\end{aligned}$$

from the w_r^0 term with

$$\begin{aligned}
&- (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 - (h+1)}) q^{(2h+1)(|\Delta_w|_0 - |\Delta_w|_1) + (h+1) + (5-i-h)|\Delta_w|_0 + (i-1-h)(2|\Delta_w|_0 - |\Delta_w|_1 + 1)} \\
&= q^{(4+i-h)|\Delta_w|_0 + (-i+1-h)|\Delta_w|_1 - h} - q^{(5+i-h)|\Delta_w|_0 + (-i-h)|\Delta_w|_1}
\end{aligned}$$

for the w_{r+1}^1 term.

The ratios of the positive and negative parts here are respectively

$$q^{(2i-1)|\Delta_w|_0 + (-2i+1)|\Delta_w|_1 + i - h}, q^{(2i-1)|\Delta_w|_0 + (-2i+1)|\Delta_w|_1 + i + h}$$

The lower ratio is between the positive parts, with exponent

$$(2i-1)\left[|\Delta_w|_0 - |\Delta_w|_1\right] + i - h \geq i+1$$

(using the fact that $|\Delta_w|_0 - |\Delta_w|_1 \geq h+1$ for w in the range we're considering). Therefore it is also true in this situation that the power of q increases, and the contribution to the series remains $O(q^{2h+2})$.

Having done both cases, we see that all higher contributions are $O(q^{2h+2})$, and we have verified the Empiricial Hypothesis for odd shelves.

Remark 3.11. Our proof of the Empirical Hypothesis provides justification for the linear order we have defined on W (see Remark 2.8). Also, compared to the proof of the corresponding Empirical Hypothesis in [LZ], here we get additional information as to the contributions to the series from each term, in terms of its Weyl index. This was hidden before because of how the series were written without leading 0s (see Remark 3.6).

$\square$

3.5 Combinatorics

To complete the motivated proof, we now recall Theorem 2.2 from [LZ].

Remark 3.12. The proof we give here (which is in the spirit of [LZ], Remark 4.1) is not the most satisfying philosophically because it requires us to have "known in advance" the shape of the sum sides. In [LZ], the primary proof of the theorem is much more detailed and interesting. In particular, the polynomials ${}_ih_l^{(j)}$ below are computed explicitly, and the combinatorics of the sum sides are derived directly from just the R series we have been working with. However, since our new viewpoint in this work has nothing to add to this argument, we just give the shorter proof here for the sake of convenience.

Theorem 3.13. *For each $j = 0, 1, \ldots,$ $R_{j+1}(q)$ is the generating function of partitions in which subsequent parts differ by at least 2, such that the smallest part is greater than or equal to $j+1$.*

Proof. Suppose $J_1, J_2, \ldots$ is an infinite sequence of formal power series in q which satisfy the recursions (3.3) (with J in place of R) and the Empirical Hypothesis. By rewriting (3.3) to solve for the lowest-indexed series and applying this formula iteratively, we see that for each $i = 1, \ldots, 2$, we have expressions

$$J_i(q) = \sum_{p=1}^{2} {}_ih_p^{(j)}(q) J_{j+p}(q)$$

for some polynomials ${}_ih_p^{(j)}(q) \in \mathbb{C}[q]$. Notice that the coefficients ${}_ih_p^{(j)}(q)$ of these combinations depend only on the recursions, not directly on the Js. It follows from the Empirical Hypothesis that the series $J_1, \ldots, J_k$ are uniquely determined (considering the combination at shelf j determines the first j terms of the series J_i just in terms of the ${}_ih_p^{(j)}(q)$). Hence, the whole sequence $J_1, J_2, \ldots$ is uniquely determined.

By our work in the earlier sections (the definition of the R_i series in (3.3) and Theorem 3.8), the series R_i above satisfy these conditions. Let S_i denote the generating functions of the classes of partitions described in the statement of the theorem. By uniqueness, it is now enough to check that the S_i also satisfy the recursions and Empirical Hypothesis.

The Empirical Hypothesis (that $S_i = 1 + q^{i+1} + \cdots$) follows directly from the definition. (Recall from earlier that the empty partition of 0 is valid, and vacuously satisfies the conditions).

To check the recursions, we consider the series

$$\frac{S_i(q) - S_{i+1}(q)}{q^i}$$

The series in the numerator counts partitions satisfying the difference 2 condition, and for which the smallest part is exactly i. The denominator has the effect of deleting this part, and the next smallest part must be no less than $i+2$. Hence this is the generating function $S_{i+2}(q)$, and we have verified the recursions. $\square$

Chapter 4

Gordon-Andrews case - new interpretation

4.1 Introduction

The Rogers-Ramanujan identities admit a natural generalization, the Gordon-Andrews odd-modulus identities. In explicit power-series form, the identities state that for any $k \geq 2$, $i = 1, \cdots, k$

$$\prod_{\substack{m \geq 1, \\ m \not\equiv 0, \pm(k-i+1) \pmod{2k+1}}} \frac{1}{1-q^m} = \sum_{n \geq 0} d_{k,i}(n) q^n, \tag{4.1}$$

$d_{k,i}(n) =$ the number of partitions of n for which parts at distance $k-1$ have difference at least 2, and 1 appears as a part at most $k-i$ times

It is clear that when $k = 2$, these identities reduce to the Rogers-Ramanujan identities.

From our perspective, this generalization comes from a different choice of specialization of the same denominator identity. Ultimately, it corresponds to the graded dimensions of the $A_1^{(1)}$modules at odd level, extending the special case of Rogers-Ramanujan at level 3.

As mentioned in the introduction, in [LZ], Lepowsky and Zhu gave a motivated proof of these identities. Concepts developed in later works on motivated proofs (in particular, the shelf picture) were not explicitly present in this work, but in the appendices to [CKLMQRS], the appropriate structure was made explicit. We largely follow that appendix here, but use our new viewpoint and notation to highlight the underlying algebraic structure at play. In anticipation of this section, much of the work in the Rogers-Ramanujan special case we did out above was written in a form that allows for straightforward extension to this level of generality.

Although we will not stress these points in this chapter, remarks analogous to Remarks 3.2, 3.3, 3.4, 3.6, 3.10, and 3.11 are true in this setting as well.

4.2 Definitions and background

Fix an integer $k \geq 2$, and allow the parameter i to range over the values $1, \cdots, k$. Consider the following specializations of (2.1): $e^{-\alpha_0} \mapsto q^{k+i}$, $e^{-\alpha_1} \mapsto q^{k+1-i}$

These yield

$$\begin{aligned}(q^{k+1-i}, q^{2k+1})_\infty (q^{k+i}, q^5)_\infty (q^{2k+1}, q^{2k+1})_\infty &= (q,q)_\infty \prod_{\substack{m=1,\cdots,2k+1, \\ m \not\equiv k+i, k+1-i, 2k+1 \mod 2k+1}} (q^m, q^{2k+1})_\infty^{-1} \\ &= \sum_{w \in W} (-1)^{\ell(w)} q^{(k+i)|\Delta_w|_0 + (k+1-i)|\Delta_w|_1}\end{aligned} \tag{4.2}$$

Let $F(q)$ denote the series $(q,q)_\infty = \prod_{n=1}^{\infty}(1-q^n)$, as above. Again we divide through these equations by $F(q)$, and recognize on the product sides of the Gordon-Andrews identities (4.1) on the left-hand sides. Taking the right-hand sides, we define a zeroth shelf of series by

$$G_i(q) = \frac{1}{F(q)} \sum_{w \in W} (-1)^{\ell(w)} q^{(k+i)|\Delta_w|_0 + (k+1-i)|\Delta_w|_1}$$

After defining this zeroth shelf, we recursively generate higher shelves of series according to the following rules: given the k series on the jth shelf, $G_{(k-1)j+i}(q), i = 1, \cdots, k$ we tautologically have

$$G_{(k-1)(j+1)+1} = G_{(k-1)j+k}$$

and for $i = 2, \cdots, k$, we define

$$G_{(k-1)(j+1)+i}(q) = \frac{G_{(k-1)j+(k-i+1)}(q) - G_{(k-1)j+(k-i+2)}}{q^{(j+1)(i-1)}} \tag{4.3}$$

We write out these formulas explicitly for the first shelf, for expository purposes. For $i = 2, \cdots, k$:

$$G_{(k-1)+i}(q) = \frac{1}{F(q)} \sum_{w \in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) q^{(2k-i+1)|\Delta_w|_0 + (i-1)|\Delta_w|_1} \tag{4.4}$$

Remark 4.1. If we allow $i = 1$ in the expression (4.4), it stands to reason that the series we get should be $G_{(k-1)+1}$, and hence the same as $G_k(q) = G_{(k-1)0+k}$ from the zeroth shelf. In fact, this is the case - this is an example of the edge-matching phenomenon which will be expounded upon in the next section.

Let us see how this equality comes about in this instance: the zeroth-shelf formula for $G_k(q)$ is

$$G_k(q) = \frac{1}{F(q)} \sum_{w \in W} (-1)^{\ell(w)} q^{2k|\Delta_w|_0 + |\Delta_w|_1}$$

while the extrapolated first-shelf formula is

$$G_{(k-1)+1}(q) \tag{4.5}$$

$$= \frac{1}{F(q)} \sum_{w \in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) q^{2k|\Delta_w|_0} \tag{4.6}$$

$$= \frac{1}{F(q)} \sum_{w \in W} (-1)^{\ell(w)} [q^{2k|\Delta_w|_0 + |\Delta_w|_1} - q^{(2k+1)|\Delta_w|_0}] \tag{4.7}$$

The difference between these two expressions is thus

$$\frac{1}{F(q)} \sum_{w \in W} (-1)^{\ell(w)} q^{(2k+1)|\Delta_w|_0}$$

However, we know that in the affine Weyl group W, the effect of left-composing with the fundamental reflection s_1 will not change the value of $|\Delta_w|$ but will change the length by 1. Hence this sum is zero because the terms cancel in pairs: $w, s_1 w$.

4.3 Closed formulas

We have the following closed-form formulas for these series:

Theorem 4.2. *Let $j \geq 0$ and $i = 1, \cdots, k$. If $j = 2h$ is even, then*

$$G_{(k-1)2h+i}(q) = \frac{1}{F(q)} q^{-2|\Delta_{w_h^0}|_0 (i-1) - \left[|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1\right](k-i)}$$
$$\sum_{w \in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w_h^1}|_0 - |\Delta_{w_h^1}|_1}) \cdot q^{(k+i-h)|\Delta_w|_0 + (k-i+1-h)|\Delta_w|_1} \tag{4.8}$$

If $j = 2h+1$ is odd, then

$$G_{(k-1)(2h+1)+i}(q) = \frac{1}{F(q)} q^{-\left[|\Delta_{w_{h+1}^1}|_0 + |\Delta_{w_{h+1}^1}|_1\right](i-1) + 2|\Delta_{w_{h+1}^1}|_0 (k-i)}$$
$$\sum_{w \in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1}) \cdot q^{(2k-i+1-h)|\Delta_w|_0 + (i-1-h)|\Delta_w|_1} \tag{4.9}$$

Proof. The expressions above give two different formulas for the "edge" cases — $i = k$ for shelf $j \geq 0$ and $i = 1$ for shelf $j + 1$ — so we first prove that they are compatible.

Let $\overline{G}_{j,i}(q)$ denote the right-hand side of the formulas above. We will verify that the difference

$$\overline{G}_{j,k}(q) - \overline{G}_{j+1,1}(q) = 0$$

This works in the same way as in the Rogers-Ramanujan case: the involutions s_1*, $f(\cdot)$ (from Definitions 2.3, 2.4) of the Weyl group are applied to the series as "changes of basis", and these have the effect of negating the difference series.

Let $j = 2h$ (so that $j + 1 = 2h + 1$). Then we have:

$$\begin{aligned}
&\overline{G}_{2h,k}(q) - \overline{G}_{2h+1,1}(q)\\
&= \frac{1}{F(q)} q^{-2|\Delta_{w_h^0}|_0 (k-1)} \sum_{w \in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w_h^1}|_0 - |\Delta_{w_h^1}|_1})\\
&\qquad\qquad \cdot q^{(2k-h)|\Delta_w|_0 + (1-h)|\Delta_w|_1}\\
&- \frac{1}{F(q)} q^{-2|\Delta_{w_{h+1}^1}|_0 (k-1)} \sum_{w \in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w_h^1}|_0 - |\Delta_{w_h^1}|_1})\\
&\qquad\qquad \cdot (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1}) \cdot q^{(2k-h)|\Delta_w|_0 - h|\Delta_w|_1}\\
&= \frac{1}{F(q)} q^{-|\Delta_{w_h^0}|_0 (2k-1) + |\Delta_{w_h^0}|_1} \sum_{w \in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w_h^1}|_0 - |\Delta_{w_h^1}|_1})\\
&\qquad\qquad \cdot q^{(2k+1-h)|\Delta_w|_0 - h|\Delta_w|_1}
\end{aligned}$$

We now apply the transformation s_1* to the Weyl group. This is a bijection of the Weyl group, so must leave the above sum invariant. Using Corollary 2.5, we get:

$$\begin{aligned}
&\frac{1}{F(q)} q^{-|\Delta_{w_h^0}|_0 (2k-1) + |\Delta_{w_h^0}|_1} \sum_{s_1 w \in W} (-1)^{\ell(s_1 w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w_h^1}|_0 - |\Delta_{w_h^1}|_1})\\
&\qquad\qquad \cdot q^{(2h+1)(|\Delta_w|_0 - |\Delta_w|_1) + (h+1) + (2k+1-h)|\Delta_w|_0 - h(2|\Delta_w|_0 - |\Delta_w|_1 + 1)}\\
&= -\frac{1}{F(q)} q^{-|\Delta_{w_h^0}|_0 (2k-1) + |\Delta_{w_h^0}|_1} \sum_{s_1 w \in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w_h^1}|_0 - |\Delta_{w_h^1}|_1})\\
&\qquad\qquad \cdot q^{(2k+1-h)|\Delta_w|_0 - h|\Delta_w|_1}
\end{aligned}$$

This final expression is exactly the negation of the starting sum. The anti-commutativity of this expression proves that it must be 0.

Next, let $j = 2h+1$ (so that $j+1 = 2(h+1)$). Then we have:

$$
\begin{aligned}
&\overline{G}_{2h+1,k}(q) - \overline{G}_{2(h+1),1}(q)\\
&= \frac{1}{F(q)} q^{-\left[|\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1\right](k-1)}\\
&\qquad \sum_{w\in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w^0_h}|_0 + |\Delta_{w^0_h}|_1}) \cdot q^{(k+1-h)|\Delta_w|_0 + (k-1-h)|\Delta_w|_1}\\
&- \frac{1}{F(q)} q^{-\left[|\Delta_{w^0_{h+1}}|_0 + |\Delta_{w^0_{h+1}}|_1\right](k-1)}\\
&\qquad \sum_{w\in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w^1_{h+1}}|_0 - |\Delta_{w^1_{h+1}}|_1}) \cdot q^{(k-h)|\Delta_w|_0 + (k-1-h)|\Delta_w|_1}\\
&= \frac{1}{F(q)} q^{-|\Delta_{w^1_{h+1}}|_0 (k-2) - |\Delta_{w^1_{h+1}}|_1 k} \sum_{w\in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w^0_{h+1}}|_0 + |\Delta_{w^0_{h+1}}|_1})\\
&\qquad\qquad \cdot q^{(k-h)|\Delta_w|_0 + (k-h)|\Delta_w|_1}
\end{aligned}
$$

We now apply to the Weyl group the outer automorphism $f(\cdot)$. This is a bijection of the Weyl group, so must leave the above sum invariant. Using Corollary 2.5, we get

$$
\begin{aligned}
&\frac{1}{F(q)} q^{-|\Delta_{w^1_{h+1}}|_0 (k-2) - |\Delta_{w^1_{h+1}}|_1 k}\\
&\qquad - \sum_{w\in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w^0_{h+1}}|_0 + |\Delta_{w^0_{h+1}}|_1}) \cdot q^{(k-h)|\Delta_w|_1 + (k-h)|\Delta_w|_0}
\end{aligned}
$$

Again the overall effect of this transformation is to negate the sum, and the anti-commutativity of this expression proves that it must be 0.

Once we have verified the edge-matching, it remains to show that these closed formulas satisfy the recursion. We do this in two cases once more. In both cases, we take the difference of two shelf j series formulas to get the ith entry on shelf $j+1$. As in Remark 3.5, we express the denominators of the recursions in terms of appropriate Weyl data.

First, let $j = 2h$, $j+1 = 2h+1$:

$$\frac{G_{2h(k-1)+(k-i+1)}(q) - G_{2h(k-1)+(k-i+2)}(q)}{q^{(|\Delta_{w^1_{h+1}}|_1 - |\Delta_{w^0_h}|_1)(i-1)}}$$

$$= q^{-\left[|\Delta_{w^1_{h+1}}|_1 - |\Delta_{w^0_h}|_1\right](i-1)}$$

$$\left[\frac{1}{F(q)} q^{-2|\Delta_{w^0_h}|_0 (k-i) - \left[|\Delta_{w^0_h}|_0 + |\Delta_{w^0_h}|_1\right](i-1)}\right.$$

$$\sum_{w\in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w^1_h}|_0 - |\Delta_{w^1_h}|_1}) \cdot q^{(2k-i+1-h)|\Delta_w|_0 + (i-h)|\Delta_w|_1}$$

$$- \frac{1}{F(q)} q^{-2|\Delta_{w^0_h}|_0 (k-i+1) + \left[|\Delta_{w^0_h}|_0 + |\Delta_{w^0_h}|_1\right](i-2)}$$

$$\left.\sum_{w\in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w^1_h}|_0 - |\Delta_{w^1_h}|_1}) \cdot q^{(2k-i+2-h)|\Delta_w|_0 + (i-1-h)|\Delta_w|_1}\right]$$

$$= \frac{1}{F(q)} q^{-2|\Delta_{w^0_h}|_0 (k-i) - \left[|\Delta_{w^0_h}|_0 + |\Delta_{w^1_{h+1}}|_1\right](i-1)}$$

$$\sum_{w\in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w^1_h}|_0 - |\Delta_{w^1_h}|_1}) \cdot (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w^0_h}|_0 + |\Delta_{w^0_h}|_1})$$

$$\cdot\, q^{(2k-i+1-h)|\Delta_w|_0 + (i-1-h)|\Delta_w|_1}$$

$$= \frac{1}{F(q)} q^{-2|\Delta_{w^1_{h+1}}|_0 (k-i) - \left[|\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1\right](i-1)}$$

$$\sum_{w\in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w^1_h}|_0 - |\Delta_{w^1_h}|_1}) \cdot (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w^0_h}|_0 + |\Delta_{w^0_h}|_1})$$

$$\cdot\, q^{(2k-i+1-h)|\Delta_w|_0 + (i-1-h)|\Delta_w|_1}$$

$$= G_{(2h+1)(k-1)+i}(q)$$

Next, let $j = 2h+1$, $j+1 = 2(h+1)$:

$$\frac{G_{(2h+1)(k-1)+(k-i+1)}(q) - G_{(2h+1)(k-1)+(k-i+2)}(q)}{q^{2(|\Delta_{w^0_{h+1}}|_0 - |\Delta_{w^1_{h+1}}|_0)(i-1)}}$$

$$= q^{-2\left[|\Delta_{w^0_{h+1}}|_0 - |\Delta_{w^1_{h+1}}|_0\right](i-1)}$$

$$\Bigg[\frac{1}{F(q)} q^{-\left[|\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1\right](k-i) - 2|\Delta_{w^1_{h+1}}|_0 (i-1)}$$

$$\sum_{w\in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w^0_h}|_0 + |\Delta_{w^0_h}|_1}) \cdot q^{(k+i-h)|\Delta_w|_0 + (k-i-h)|\Delta_w|_1}$$

$$- \frac{1}{F(q)} q^{-\left[|\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1\right](k-i+1) - 2|\Delta_{w^1_{h+1}}|_0 (i-2)}$$

$$\sum_{w\in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w^0_h}|_0 + |\Delta_{w^0_h}|_1}) \cdot q^{(k+i-1-h)|\Delta_w|_0 + (k-i+1-h)|\Delta_w|_1}\Bigg]$$

$$= \frac{1}{F(q)} q^{-2|\Delta_{w^0_{h+1}}|_0 (i-1) - \left[|\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1\right](k-i)}$$

$$\sum_{w\in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w^0_h}|_0 + |\Delta_{w^0_h}|_1})$$

$$\cdot (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 - |\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1}) q^{(k+i-1-h)|\Delta_w|_0 + (k-i-h)|\Delta_w|_1}$$

$$= \frac{1}{F(q)} q^{-2|\Delta_{w^0_{h+1}}|_0 (i-1) - \left[|\Delta_{w^0_{h+1}}|_0 + |\Delta_{w^0_{h+1}}|_1\right](k-i)}$$

$$\sum_{w\in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w^0_h}|_0 + |\Delta_{w^0_h}|_1})$$

$$\cdot (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 - |\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1}) q^{(k+i-1-h)|\Delta_w|_0 + (k-i-h)|\Delta_w|_1}$$

$$= G_{(2h+2)(k-1)+i}(q)$$

□

4.4 Empirical Hypothesis

As a consequence of Theorem 4.2, we are now in a position to formulate and prove the Empirical Hypothesis. This is the main ingredient needed to complete the motivated proof.

Theorem 4.3 (Empirical Hypothesis)**.** *For any $j \geq 0$ and $i = 1, \cdots, k$,*

$$G_{(k-1)j+i}(q) = 1 + q^{j+1}\gamma(q)$$

for some

$$\gamma(q) \in \mathbb{C}[[q]].$$

Remark 4.4. Note that since $G_{(k-1)j+k}(q) = G_{(k-1)(j+1)+1}(q)$, Theorem 4.3 implies that we can write $G_{(k-1)j+k}(q) = 1 + q^{j+2}\gamma(q)$ where $\gamma(q)$ is some formal power series.

The structure of the proof is largely unchanged from the Rogers-Ramanujan special case. In general, for each G series, the first few summands in the linear ordering will be equal to 0. The first term which is nonzero will contribute the initial $1 + q^{j+1}$ in the Empirical Hypothesis, and all other contributions from it and all subsequent terms will contribute only higher powers of q. In the special case $i = k$, a closer analysis shows that the next term after the least nonzero contributor contributes exactly what is needed to push the Empirical Hypothesis up to the shape it needs to be for the next shelf.

Unless otherwise noted, the justification for individual steps in the calculations are identical to what they were for the Rogers-Ramanujan case. Hence we will largely suppress explanatory comments for these proofs (but see Remark 3.3 for more details).

Proof. First, we consider the even-shelf series, which from Theorem 4.2 are of the form

$$\begin{aligned} G_{2h(k-1)+i}(q) = \frac{1}{F(q)} q^{-2|\Delta_{w_h^0}|_0(i-1) - \left[|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1\right](k-i)} \\ \sum_{w\in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_0} - q^{|\Delta_w|_1 + |\Delta_{w_h^1}|_0 - |\Delta_{w_h^1}|_1}) \\ \cdot\, q^{(k+i-h)|\Delta_w|_0 + (k-i+1-h)|\Delta_w|_1} \end{aligned}$$

First suppose that w is one of the first $2h$ elements (with respect to the linear order) - i.e., w is either of the form w_r^0 for $r < h$, or w_r^1 for $r \leq h$. Then as before, it follows from the formulas (3.8), (3.9), (3.10) that the corresponding terms of the series are 0.

We claim that the first nonzero contribution to the series is from the term $w = w_h^0$, and that this contribution is of the shape $1 + q^{2h+1} + O(q^{2h+2})$. This claim is justified by the following sequence of computations, starting with the $w = w_h^0$ term from the series referenced above:

$$
\begin{aligned}
&\frac{1}{F(q)} q^{-2|\Delta_{w_h^0}|_0 (i-1) - \left[|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1 \right](k-i)} \\
&\qquad (-1)^{\ell(w_h^0)} (q^{|\Delta_{w_h^0}|_1} - q^{|\Delta_{w_h^0}|_0}) \cdots (q^{|\Delta_{w_h^0}|_0} - q^{|\Delta_{w_h^0}|_1 + |\Delta_{w_h^1}|_0 - |\Delta_{w_h^1}|_1}) \\
&\qquad\qquad \cdot q^{(k+i-h)|\Delta_{w_h^0}|_0 + (k-i+1-h)|\Delta_{w_h^0}|_1} \\
&= \frac{1}{F(q)} q^{-2|\Delta_{w_h^0}|_0 (i-1) - \left[|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1 \right](k-i)} \\
&\qquad (-1)^h \{ q^{|\Delta_{w_h^0}|_1} (1 - q^{|\Delta_{w_h^0}|_0 - |\Delta_{w_h^0}|_1}) \} \cdots \{ -q^{|\Delta_{w_h^0}|_1 - h} (1 - q^{|\Delta_{w_h^0}|_0 - |\Delta_{w_h^0}|_1 + h}) \} \\
&\qquad\quad \cdot q^{(k+i-h)|\Delta_{w_h^0}|_0 + (k-i+1-h)|\Delta_{w_h^0}|_1} \\
&= \frac{1}{F(q)} (-1)^{2h} (1-q) \cdots (1-q^{2h}) \\
&\qquad q^{-2|\Delta_{w_h^0}|_0 (i-1) - \left[|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1 \right](k-i) + 2h|\Delta_{w_h^0}|_1 - |\Delta_{w_h^0}|_0 + (k+i-h)|\Delta_{w_h^0}|_0 + (k-i+1-h)|\Delta_{w_h^0}|_1}
\end{aligned}
$$

As in the corresponding part of the proof in the Rogers-Ramanujan case, the singleton $|\Delta_{w_h^0}|_0$ component of the exponent on the ending line comes from the sum $1 + \cdots + h$ (interpreted appropriately) on the previous line.

Looking more closely at the exponent of q, we can regroup it as

$$
\begin{aligned}
& k\left[(|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1) - (|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1) \right] \\
& + i\left[(|\Delta_{w_h^0}|_0 - |\Delta_{w_h^0}|_1) - (|\Delta_{w_h^0}|_0 - |\Delta_{w_h^0}|_1) \right] \\
& + |\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1 - h(|\Delta_{w_h^0}|_0 - |\Delta_{w_h^0}|_1) \\
& = k * 0 + i * 0 + h^2 - h^2 \\
& = 0
\end{aligned}
$$

Moreover, expanding $F(q) = (1-q)(1-q^2)(1-q^3)\cdots$, we see that we can cancel the first $2h$ factors of $F(q)$ with the remaining factors in the term. Hence this term is in fact equal to

$$
(q^{2h}, q)_\infty^{-1} = 1 + q^{2h+1} + \cdots
$$

To verify that all subsequent contributions are $O(q^{2h+1})$, we argue as in the previous chapter.

First, suppose that w is of the form w_r^0 for some $r \geq h$, so that the next Weyl element in the linear order is $s_1 w_r^0 = w_{r+1}^1$. As was the case earlier, the effect of applying the transformation $s_1 *$ is to rescale the term by a power of q, whose exponent can be computed by summing the contributions from Definition 2.3 and Corollary 2.5. We get:

$$\begin{aligned}
&h\Big(2|\Delta_w|_0 - 2|\Delta_w|_1 + 1\Big)\\
&+ (k+i-h)|\Delta_w|_0 + (k-i+1-h)(2|\Delta_w|_0 - |\Delta_w|_1 + 1)\\
&- (k+i-h)|\Delta_w|_0 + (k-i+1-h)|\Delta_w|_1\\
&= h\Big(2|\Delta_w|_0 - 2|\Delta_w|_1 + 1\Big)\\
&+ (k-i+1-h)(2|\Delta_w|_0 - 2|\Delta_w|_1 + 1)\\
&= (k-i+1)(2|\Delta_w|_0 - 2|\Delta_w|_1 + 1)\\
&\geq 2h+1
\end{aligned}$$

For the last inequality, we have used the facts that $i \leq k$ and that for elements of the form w_r^0 with $r \geq h$, we have $|\Delta_w|_0 - |\Delta_w|_1 \geq h$. (Notice that in particular, we get equality in the last line only when $i = k$ and $r = h$, and in this case the least contribution from this term is $-q^{2h+1}$, which precisely cancels out the q^{2h+1} from the previous term, giving us a series of the form $1 + q^{2h+2} + \cdots$. This shows the agreement between the edge-matching and the Empirical Hypothesis). In general, this confirms that all higher contributions are $O(q^{2h+1})$.

Next, suppose w is of the form w_r^1 for some $r \geq h+1$, so that the next Weyl element is $f(w_r^1) = w_r^0$. From Definition 2.4 and Corollary 2.5, the factors that are different between these two terms are

$$\begin{aligned}
&(q^{|\Delta_w|_0} - q^{|\Delta_w|_1 - h})q^{(k+i-h)|\Delta_w|_0 + (k-i+1-h)|\Delta_w|_1}\\
&= q^{(k+i+1-h)|\Delta_w|_0 + (k-i-1-h)|\Delta_w|_1} - q^{(k+i-h)|\Delta_w|_0 + (k-i+2-h)|\Delta_w|_1 - h}
\end{aligned}$$

for the w_r^1 term versus

$$\begin{aligned}
&(q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - h}) \cdot q^{(k+i-h)|\Delta_w|_1 + (k-i+1-h)|\Delta_w|_0}\\
&= q^{(k-i+2-h)|\Delta_w|_0 + (k+i-h)|\Delta_w|_1 - h} - q^{(k-i+1-h)|\Delta_w|_0 + (k+i+1-h)|\Delta_w|_1 - h}
\end{aligned}$$

for the w_r^0 term.

The ratios between the parts of these expressions is

$$q^{(-2i+1)|\Delta_w|_0 + (2i-1)|\Delta_w|_1 - h}$$

for the positive parts and

$$q^{(-2i+1)|\Delta_w|_0+(2i-1)|\Delta_w|_1+h}$$

for the negative parts. The ratio between positive parts is the smaller of the two, and we can bound its exponent as

$$(2i-1)\Big[|\Delta_w|_1 - |\Delta_w|_0\Big] - h \geq (2i-1)(h+1) - h \geq 1$$

Hence the power of q must increase, and here, too, the contribution is $O(q^{2h+1})$.

Combining these results, we have verified the Empirical Hypothesis for even shelves.

For odd shelves, from Theorem 4.2 the series are of the form

$$\begin{aligned} G_{(2h+1)(k-1)+i}(q) = \frac{1}{F(q)} q^{-\Big[|\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1\Big](i-1) - 2|\Delta_{w^1_{h+1}}|_0(k-i)} \\ \sum_{w\in W} (-1)^{\ell(w)} (q^{|\Delta_w|_1} - q^{|\Delta_w|_0}) \cdots (q^{|\Delta_w|_1} - q^{|\Delta_w|_0 - |\Delta_{w^0_h}|_0 + |\Delta_{w^0_h}|_1}) \\ \cdot q^{(2k-i+1-h)|\Delta_w|_0 + (i-1-h)|\Delta_w|_1} \end{aligned}$$

First suppose that w is one of the first $2h+1$ elements (with respect to the linear order) - i.e., w is either of the form w^0_r for $r \leq h$, or w^1_r for $r \leq h$. Then as before, it follows from formulas (3.8), (3.9), (3.10) that the corresponding terms of the series are 0.

We claim that the first nonzero contribution to the series is from the term $w = w^1_{h+1}$, and that this contribution is of the shape $1 + q^{2h+2} + O(q^{2h+3})$. This claim is justified by the following sequence of computations, starting with the $w = w^1_{h+1}$ term from the series referenced above:

$$\frac{1}{F(q)}q^{-\left[|\Delta_{w^1_{h+1}}|_0+|\Delta_{w^1_{h+1}}|_1\right](i-1)-2|\Delta_{w^0_{h+1}}|_0(k-i)}$$
$$(-1)^{\ell(w^1_{h+1})}(q^{|\Delta_{w^1_{h+1}}|_1}-q^{|\Delta_{w^1_{h+1}}|_0})\cdots(q^{|\Delta_{w^1_{h+1}}|_1}-q^{|\Delta_{w^1_{h+1}}|_0-|\Delta_{w^0_h}|_0+|\Delta_{w^0_h}|_1})$$
$$\cdot\, q^{(2k-i+1-h)|\Delta_{w^1_{h+1}}|_0+(i-1-h)|\Delta_{w^1_{h+1}}|_1}$$
$$=\frac{1}{F(q)}q^{-\left[|\Delta_{w^1_{h+1}}|_0+|\Delta_{w^1_{h+1}}|_1\right](i-1)-2|\Delta_{w^0_{h+1}}|_0(k-i)}$$
$$(-1)^{h+1}\{-q^{|\Delta_{w^1_{h+1}}|_0}(1-q^{|\Delta_{w^1_{h+1}}|_1-|\Delta_{w^1_{h+1}}|_0})\}\cdots\{-q^{|\Delta_{w^1_{h+1}}|_0-h}(1-q^{|\Delta_{w^1_{h+1}}|_1-|\Delta_{w^1_{h+1}}|_0+h})\}$$
$$\cdot\, q^{(2k-i+1-h)|\Delta_{w^0_h}|_0+(i-1-h)|\Delta_{w^0_h}|_1}$$
$$=\frac{1}{F(q)}(-1)^{2h+2}(1-q)\cdots(1-q^{2h+1})$$
$$q^{-\left[|\Delta_{w^1_{h+1}}|_0+|\Delta_{w^1_{h+1}}|_1\right](i-1)-2|\Delta_{w^0_{h+1}}|_0(k-i)+(2h+1)|\Delta_{w^0_h}|_1-|\Delta_{w^1_{h+1}}|_0+(2k-i+1-h)|\Delta_{w^0_h}|_0+(i-1-h)|\Delta_{w^0_h}|_1}$$

Looking more closely at the exponent of q, we can regroup it as

$$\begin{aligned}
&2k\left[|\Delta_{w^1_{h+1}}|_0-|\Delta_{w^1_{h+1}}|_0\right]\\
&+i\left[(|\Delta_{w^1_{h+1}}|_1-|\Delta_{w^1_{h+1}}|_0)-(|\Delta_{w^1_{h+1}}|_1-|\Delta_{w^1_{h+1}}|_0)\right]\\
&+|\Delta_{w^1_{h+1}}|_0+|\Delta_{w^1_{h+1}}|_1-(h+1)(|\Delta_{w^1_{h+1}}|_1-|\Delta_{w^1_{h+1}}|_0)\\
&=2k*0+i*0+(h+1)^2-(h+1)^2\\
&=0
\end{aligned}$$

Moreover, expanding $F(q)=(1-q)(1-q^2)(1-q^3)\cdots$, we see that we can cancel the first $2h+1$ factors of $F(q)$ with the remaining factors in the term. Hence this term is in fact equal to

$$(q^{2h+1},q)^{-1}_\infty=1+q^{2h+2}+\cdots$$

To verify that all subsequent contributions are $O(q^{2h+2})$, we argue as before.

First, suppose w is of the form w^1_r for some $r\geq h+1$, so that the next Weyl element is $f(w^1_r)=w^0_r$. By Corollary 2.5, the action of this transformation on the w^1_r term is by negation on the group of factors in parentheses. Hence to study the change in the power of q, it is enough to just consider the change in the ending exponent, which is

$$\begin{aligned}&(i-1-h)|\Delta_w|_0 + (2k-i+1-h)|\Delta_w|_1\\&-(2k-i+1-h)|\Delta_w|_0 - (i-1-h)|\Delta_w|_1\\&=(-2k+2i-2)|\Delta_w|_0 + (2k-2i+2)|\Delta_w|_1\\&=2(k-i+1)\big[|\Delta_w|_1 - |\Delta_w|_0\big]\\&\geq 2h+2\end{aligned}$$

with equality only when $i = k$ and $w = w^1_{h+1}$. Since we have made the assumption that the preceding term is at least $1 + O(q^{2h+2})$, and the minimal exponent here is at least $2h + 2$ higher, we conclude that the contribution here is $O(q^{2h+2})$.

Now assume w is of the form w^0_r for $r \geq h+1$, so that the next Weyl element is $s_1 w^0_r = w^1_{r+1}$. By Corollary 2.5, the overall change in factors when applying s_1* to the w^0_r term is the shift from

$$\begin{aligned}&(q^{|\Delta_w|_1} - q^{|\Delta_w|_0-h})q^{(2k-i+1-h)|\Delta_w|_0+(i-1-h)|\Delta_w|_1}\\&= q^{(2k-i+1-h)|\Delta_w|_0+(i-h)|\Delta_w|_1} - q^{(2k-i+2-h)|\Delta_w|_0+(i-1-h)|\Delta_w|_1-h}\end{aligned}$$

in the w^0_r term to

$$\begin{aligned}&-(q^{|\Delta_w|_0} - q^{|\Delta_w|_1-(h+1)})q^{(2h+1)(|\Delta_w|_0-|\Delta_w|_1)+(h+1)+(2k-i+1-h)|\Delta_w|_0+(i-1-h)(2|\Delta_w|_0-|\Delta_w|_1+1)}\\&= q^{(2k+i-h)|\Delta_w|_0+(-i+1-h)|\Delta_w|_1-h} - q^{(2k+i+1-h)|\Delta_w|_0+(-i-h)|\Delta_w|_1}\end{aligned}$$

in the w^1_{r+1} term.

The ratios between the positive and negative parts are respectively

$$q^{(2i-1)|\Delta_w|_0+(-2i+1)|\Delta_w|_1+i-h}, \quad q^{(2i-1)|\Delta_w|_0+(-2i+1)|\Delta_w|_1+i+h}$$

The lower ratio is that between the positive parts, and there the exponent is

$$(2i-1)\Big[|\Delta_w|_0 - |\Delta_w|_1\Big] + i - h \geq i+1$$

(using the fact that $|\Delta_w|_0 - |\Delta_w|_1 \geq h+1$ in the region specified). Therefore in this situation too the power of q increases, and the contribution to the series remains $O(q^{2h+2})$.

Having done both cases, we see that all higher contributions are $O(q^{2h+2})$, and we have verified the Empiricial Hypothesis for odd shelves. $\square$

4.5 Combinatorics

To complete the motivated proof, we now recall Theorem 2.2 from [LZ].

Remark 4.5. As in Remark 3.12, the proof we give here is much simpler than the main proof in [LZ], but not as deep because it requires us to "know in advance" the combinatorics. Essentially, we are performing a verification rather than a proof from scratch. Again, see Remark 4.1 in [LZ] for more details.

Theorem 4.6. *For each $i = 1, \dots, k$, $j = 0, 1, 2, \dots$, $G_{(k-1)j+i}(q)$ is the generating function of partitions satisfying*

1. *difference at least 2 at distance $k-1$*

2. *smallest part is at least $j+1$*

3. *$j+1$ appears as a part at most $k-i$ times*

Proof. Suppose $K_1, K_2, \dots$ is an infinite sequence of formal power series in q which satisfy the recursions (4.3) (with K in place of G) and the Empirical Hypothesis. By rewriting (4.3) to solve for the lowest-indexed series and applying this formula iteratively, we see that for each $i = 1, \cdots, k$, we have expressions

$$K_i(q) = \sum_{p=1}^{k} {}_ih_p^{(j)}(q) K_{(k-1)j+p}(q)$$

for some polynomials ${}_ih_p^{(j)}(q) \in \mathbb{C}[q]$. Notice that the coefficients ${}_ih_p^{(j)}(q)$ of these combinations depend only on the recursions, not directly on the Ks. It follows from the Empirical Hypothesis that the series $K_1, \dots, K_k$ are uniquely determined (for example, considering the combination at shelf j determines the first j terms of the series K_i just in terms of the ${}_ih_p^{(j)}(q)$). Hence, the whole sequence $K_1, K_2, \dots$ is uniquely determined.

By our work in the earlier sections (the definition of the G_i series in (4.3) and Theorem 4.3), the series G_i above satisfy these conditions. Let H_i denote the generating functions of the classes of partitions described in the statement of the theorem. By uniqueness, it is now enough to check that the H_i also satisfy the recursions and Empirical Hypothesis.

The Empirical Hypothesis (that $H_{(k-1)j+i} = 1 + q^{j+1} + \cdots$) follows directly from the definition.

To check the recursions, we consider the series

$$\frac{H_{(k-1)j+(k-i+1)}(q) - H_{(k-1)j+(k-i+2)}(q)}{q^{(j+1)(i-1)}}$$

The series in the numerator counts partitions satisfying the first two conditions of the theorem, and for which the part $j+1$ appears with multiplicity $i-1$. The denominator has the effect of deleting all of these parts. In order to satisfy condition 1, the there can be at most $k-i$ parts equal to $j+2$. Hence this is the generating function $H_{(k-1)(j+1)+i}(q)$, and we have verified the recursions. $\square$

Chapter 5

Andrews-Bressoud case - new interpretation

5.1 Introduction

Another set of identities closely related to the Rogers-Ramanujan identities are the Andrews-Bressoud even-modulus identities, whose product sides correspond to the graded dimensions of modules for $A_1^{(1)}$ at even level.

These identities state that for every $k \geq 2$ and $i = 1, \cdots, k$:

$$\frac{\prod_{m\geq 1}(1-q^{2km})(1-q^{2km-k+(i-1)})(1-q^{2km-k-(i-1)})}{\prod_{m\geq 1}(1-q^m)} = \sum_{n\geq 0} b_{k,i}(n)q^n, \tag{5.1}$$

where $b_{k,i}(n)$ is the number of partitions $\pi = (\pi_1, \cdots, \pi_s)$ of n (with $\pi_t \geq \pi_{t+1}$) satisfying

1. difference at least 2 at distance $k-1$
2. $\pi_t - \pi_{t+k-2} \leq 1$ only if $\pi_t + \cdots + \pi_{t+k-2} \equiv k+i \mod 2$
3. at most $k-i$ parts equal to 1

In other words, the partitions satisfy difference (at least) 2 at distance $k-1$, and in fact difference 2 at distance $k-2$ unless a certain parity condition is met.

As in the odd-modulus case, it is possible to define higher shelves of B series by recursion. However, if we do this just in terms of the B_is, we are forced to encounter the undesirable behavior of having non-pure powers of q in the denominators of the recursions, which is not "motivated" and would not admit a description from our viewpoint. In [KLRS], which we follow below, this difficulty was dealt with by introducing intermediate "ghost shelves" of series. Combining the true and ghost series lead to recursions which do involve only pure powers of q. In this section, we reinterpret the proof from [KLRS] using our new viewpoint and Weyl-theoretic notation, and see how it fits the paradigm we have established.

As in the previous chapter, remarks analogous to those given in the Rogers-Ramanujan case are true here as well.

5.2 Definitions and background

Fix an integer $k \geq 2$, and allow the parameter i to range over the values $1, \cdots, k$. Consider the following specializations of (2.1): $e^{-\alpha_0} \mapsto q^{k+i-1}$, $e^{-\alpha_1} \mapsto q^{k-i+1}$. This yields

$$\begin{aligned}(q^{k-i+1}, q^{2k})_\infty (q^{k+i-1}, q^{2k})_\infty (q^{2k}, q^{2k})_\infty &= (q, q)_\infty \prod_{\substack{m=1,\cdots,2k+1, \\ m \not\equiv k+i-1,\ k+-i+1,\ 2k \mod 2k}} (q^m, q^{2k})_\infty^{-1} \\ &= \sum_{w \in W} (-1)^{\ell(w)} q^{(k+i-1)|\Delta_w|_0 + (k+1-i)|\Delta_w|_1}\end{aligned} \tag{5.2}$$

As above, let $F(q)$ denote the series

$$(q, q)_\infty = \prod_{n=1}^{\infty} (1 - q^n)$$

Once again we divide through these equations by $F(q)$, and recognize the Andrews-Bressoud product sides from (5.1) on the left. We define a zeroth shelf of series to be the right-hand sides:

$$B_i(q) = \frac{1}{F(q)} \sum_{w \in W} (-1)^{\ell(w)} q^{(k+i)|\Delta_w|_0 + (k+1-i)|\Delta_w|_1}$$

for $i = 1, \cdots, k$.

Let us see how appropriate recursions (from [KLRS]) simultaneously define the the zeroth ghost shelf and the true first shelf. Define:

$$B_{k+1}(q) = \frac{B_{k-1} - \widetilde{B}_k}{q} = \widetilde{B}_k \tag{5.3}$$

$$B_{k+p} = \frac{B_{k-p} - \widetilde{B}_{k-p+1}}{q^p} = \frac{\widetilde{B}_{k-p+1} - B_{k-p+2}}{q^{p-1}} \tag{5.4}$$

for $p = 2, \cdots, k-1$ In these recursions, the right-hand equality is used to define the ghost series $\widetilde{B}_i$ for $i = 2, \cdots, k$, and then either equality serves to define the next shelf of true series $B_{(k-1)+i}$.

Solving these equations yield the definitions of the zeroth shelf of ghost series:

$$\widetilde{B}_k = \frac{1}{(1+q)F(q)} \sum_{w \in W} (-1)^{\ell(w)} q^{(2k-2)|\Delta_w|_0 + 2|\Delta_w|_1} \tag{5.5}$$

$$\widetilde{B}_i = \frac{1}{(1+q)F(q)} \sum_{w \in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0 + 1}) q^{(k+i-2)|\Delta_w|_0 + (k-i)|\Delta_w|_1} \tag{5.6}$$

for $i = 2, \cdots, k-1$.

Remark 5.1. Notice that allowing $i = k$ in the second expression gives

$$\widetilde{B}_k = \frac{1}{(1+q)F(q)} \sum_{w \in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0+1}) q^{(2k-2)|\Delta_w|_0}$$

which is valid because its difference from the defined $\widetilde{B_k}$ is

$$\frac{q}{(1+q)F(q)} \sum_{w \in W} (-1)^{\ell(w)} q^{2k|\Delta_w|_0}$$

which cancels to 0 under the pairing of Weyl elements $w, s_1 w$.

Using these expressions, the next shelf of true series is defined as:

$$B_{k+p} = \frac{1}{(1+q)q^{p-1}F(q)} \sum_{w \in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0}) q^{(2k-p-1)|\Delta_w|_0 + (p-1)|\Delta_w|_1}$$

The general recursions are as follows:

$$B_{(k-1)(j+1)+2}(q) = \frac{B_{(k-1)j+(k-1)} - \widetilde{B}_{(k-1)j+k}}{q^{j+1}} = \widetilde{B}_{(k-1)j+k} \tag{5.7}$$

$$B_{(k-1)(j+1)+i} = \frac{B_{(k-1)j+(k-i+1)} - \widetilde{B}_{(k-1)j+(k-i+2)}}{q^{(j+1)(i-1)}} = \frac{\widetilde{B}_{(k-1)j+(k-i+2)} - B_{(k-1)j+(k-i+3)}}{q^{(j+1)(i-2)}} \tag{5.8}$$

for $i = 3, \cdots, k$.

Solving these recursions for the ghost series leads to the following definitions: For $i = 2, \cdots, k-1$,

$$\widetilde{B}_{(k-1)j+i}(q) = \frac{B_{(k-1)j+(i-1)} + q^{j+1} B_{(k-1)j+(i+1)}}{1+q^{j+1}} \tag{5.9}$$

For $i = k$,

$$\widetilde{B}_{(k-1)j+k}(q) = \frac{B_{(k-1)j+(k-1)}}{1+q^{j+1}} \tag{5.10}$$

5.3 Closed formulas

We have the following closed-form formulas for these series:

Theorem 5.2. *Let $j \geq 0$ and $i = 1, \cdots, k$.*

If $j = 2h$ is even, then

$$B_{(k-1)2h+i}(q) = \frac{1}{(1+q)\cdots(1+q^{2h})F(q)} q^{-2|\Delta_{w_h^0}|_0 (i-1) - \left[|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1 \right](k-i) + h(2h+1)}$$
$$\sum_{w \in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0}) \cdots (q^{2|\Delta_w|_0} - q^{2|\Delta_w|_1 + 2|\Delta_{w_h^1}|_0 - 2|\Delta_{w_h^1}|_1}) \cdot q^{(k+i-1-2h)|\Delta_w|_0 + (k-i+1-2h)|\Delta_w|_1} \tag{5.11}$$

If $j = 2h+1$ is odd, then

$$B_{(k-1)(2h+1)+i}(q) = \frac{1}{(1+q)\cdots(1+q^{2h+1})F(q)} q^{-\left[|\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1\right](i-1) - 2|\Delta_{w^1_{h+1}}|_0(k-i) + (h+1)(2h+1)}$$
$$\sum_{w\in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0}) \cdots (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0 - 2|\Delta_{w^0_h}|_0 + 2|\Delta_{w^0_h}|_1}) \cdot q^{(2k-i-2h)|\Delta_w|_0 + (i-2-2h)|\Delta_w|_1} \tag{5.12}$$

Theorem 5.3. *Let $j \geq 0$ and $i = 2, \cdots, k$.*

If $j = 2h$ is even, then

$$\widetilde{B}_{(k-1)2h+i}(q) = \frac{1}{(1+q)\cdots(1+q^{2h+1})F(q)} q^{-2|\Delta_{w^0_h}|_0(i-1) - \left[|\Delta_{w^0_h}|_0 + |\Delta_{w^0_h}|_1\right](k-i) + 2h(h+1)}$$
$$\sum_{w\in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0}) \cdots (q^{2|\Delta_w|_0} - q^{2|\Delta_w|_1 + 2|\Delta_{w^1_h}|_0 - 2|\Delta_{w^1_h}|_1})$$
$$\cdot (q^{2|\Delta_w|_1} + q^{2|\Delta_w|_0 + 1}) q^{(k+i-2-2h)|\Delta_w|_0 + (k-i-2h)|\Delta_w|_1} \tag{5.13}$$

If $j = 2h+1$ is odd, then

$$\widetilde{B}_{(k-1)(2h+1)+i}(q) = \frac{1}{(1+q)\cdots(1+q^{2h+2})F(q)} q^{-\left[|\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1\right](i-1) - 2|\Delta_{w^1_{h+1}}|_0(k-i) + 2(h+1)^2}$$
$$\sum_{w\in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0}) \cdots (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0 - 2|\Delta_{w^0_h}|_0 + 2|\Delta_{w^0_h}|_1})$$
$$\cdot (q^{2|\Delta_w|_0} + q^{2|\Delta_w|_1}) q^{(2k-i-1-2h)|\Delta_w|_0 + (i-3-2h)|\Delta_w|_1} \tag{5.14}$$

Remark 5.4. There is a subtle point here that bears mentioning: for the Gordon-Andrews series, the edge-matching was logically needed for the generation of new shelves of series. Specifically, in order to obtain the formula for $G_{(k-1)(j+1)+k}(q)$, one needs the "shelf j" formulas for both $G_{(k-1)j+1}(q)$ and $G_{(k-1)j+2}(q)$. Since the series $G_{(k-1)j+1}(q)$ was originally defined as $G_{(k-1)(j-1)+k}(q)$ (i.e., given in its shelf $j-1$ form), this requires edge-matching.

For the Andrews-Bressoud identities, the only time the ghost series appear in the recursions, they are always in the form $\widetilde{B}_{(k-1)j+i}(q)$ for $i = 2, \cdots, k$. Hence there is no need to prove an edge-matching result for these series (although it certainly does hold!). Edge-matching is still required for the true series, however, for the same reason as in the Gordon-Andrews case.

Proof. We will once again have to check that these formulas obey edge-matching and the recursions.

Let $\overline{B}_{j,i}(q)$ denote the right-hand sides of the formulas for the true series above. We will verify that the differences

$$\overline{B}_{j,k}(q) - \overline{B}_{j+1,1}(q) = 0$$

We do this by performing suitable "changes of index" on the sum, namely, the transformations s_1*, $f(\cdot)$.

Let $j = 2h$ (so that $j+1 = 2h+1$). Then we have:

$$\begin{aligned}
&\overline{B}_{2h,k}(q) - \overline{B}_{2h+1,1}(q)\\
&= \frac{1}{(1+q)\cdots(1+q^{2h})F(q)} q^{-2|\Delta_{w_h^0}|_0 (k-1)+h(2h+1)}\\
&\sum_{w\in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0}) \cdots (q^{2|\Delta_w|_0} - q^{2|\Delta_w|_1+2|\Delta_{w_h^1}|_0 - 2|\Delta_{w_h^1}|_1}) \cdot q^{(2k-1-2h)|\Delta_w|_0+(1-2h)|\Delta_w|_1}\\
&- \frac{1}{(1+q)\cdots(1+q^{2h+1})F(q)} q^{-2|\Delta_{w_{h+1}^1}|_0 (k-1)+(h+1)(2h+1)}\\
&\sum_{w\in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0}) \cdots (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0 - 2|\Delta_{w_h^0}|_0 + 2|\Delta_{w_h^0}|_1}) \cdot q^{(2k-1-2h)|\Delta_w|_0+(-1-2h)|\Delta_w|_1}\\
&= \frac{1}{(1+q)\cdots(1+q^{2h+1})F(q)} q^{-2|\Delta_{w_h^0}|_0 (k-1)+h(2h+1)}\\
&\sum_{w\in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0}) \cdots (q^{2|\Delta_w|_0} - q^{2|\Delta_w|_1+2|\Delta_{w_h^1}|_0 - 2|\Delta_{w_h^1}|_1})\\
&\cdot [(1+q^{2h+1}) q^{(2k-1-2h)|\Delta_w|_0+(1-2h)|\Delta_w|_1} - q^{2h+1}(q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0 - 2h}) q^{(2k-1-2h)|\Delta_w|_0+(-1-2h)|\Delta_w|_1}]\\
&= \frac{1}{(1+q)\cdots(1+q^{2h+1})F(q)} q^{-2|\Delta_{w_h^0}|_0 (k-1)+h(2h+1)}\\
&\sum_{w\in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0}) \cdots (q^{2|\Delta_w|_0} - q^{2|\Delta_w|_1+2|\Delta_{w_h^1}|_0 - 2|\Delta_{w_h^1}|_1})\\
&\cdot [q^{(2k-1-2h)|\Delta_w|_0+(1-2h)|\Delta_w|_1} + q^{(2k+1-2h)|\Delta_w|_0+(-1-2h)|\Delta_w|_1+1}]\\
&= \frac{1}{(1+q)\cdots(1+q^{2h+1})F(q)} q^{-2|\Delta_{w_h^0}|_0 (k-1)+h(2h+1)}\\
&\sum_{w\in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0}) \cdots (q^{2|\Delta_w|_0} - q^{2|\Delta_w|_1+2|\Delta_{w_h^1}|_0 - 2|\Delta_{w_h^1}|_1})\\
&\cdot (q^{2|\Delta_w|_1} + q^{2|\Delta_w|_0+1}) q^{(2k-1-2h)|\Delta_w|_0+(-1-2h)|\Delta_w|_1}
\end{aligned}$$

When we apply the transformation s_1* and use Corollary 2.5 (with q^2 in place of q), the series becomes:

$$\frac{1}{(1+q)\cdots(1+q^{2h+1})F(q)}q^{-2|\Delta_{w_h^0}|_0(k-1)+h(2h+1)}$$
$$\sum_{s_1w\in W}(-1)^{\ell(s_1w)}q^{2h(2|\Delta_w|_0-2|\Delta_w|_1+1)}(q^{2|\Delta_w|_1}-q^{2|\Delta_w|_0})\cdots(q^{2|\Delta_w|_0}-q^{2|\Delta_w|_1+2|\Delta_{w_h^1}|_0-2|\Delta_{w_h^1}|_1})$$
$$\cdot(q^{2(2|\Delta_w|_0-|\Delta_w|_1+1)}+q^{2|\Delta_w|_0+1})q^{(2k-1-2h)|\Delta_w|_0+(-1-2h)(2|\Delta_w|_0-|\Delta_w|_1+1)}$$
$$\frac{1}{(1+q)\cdots(1+q^{2h+1})F(q)}q^{-2|\Delta_{w_h^0}|_0(k-1)+h(2h+1)}$$
$$\sum_{s_1w\in W}-(-1)^{\ell(w)}(q^{2|\Delta_w|_1}-q^{2|\Delta_w|_0})\cdots(q^{2|\Delta_w|_0}-q^{2|\Delta_w|_1+2|\Delta_{w_h^1}|_0-2|\Delta_{w_h^1}|_1})(q^{2|\Delta_w|_0+1}+q^{2|\Delta_w|_1})$$
$$\cdot\, q^{2h(2|\Delta_w|_0-2|\Delta_w|_1+1)+(2|\Delta_w|_0-2|\Delta_w|_1+1)+(2k-1-2h)|\Delta_w|_0+(-1-2h)(2|\Delta_w|_0-|\Delta_w|_1+1)}$$
$$=-\frac{1}{(1+q)\cdots(1+q^{2h+1})F(q)}q^{-2|\Delta_{w_h^0}|_0(k-1)+h(2h+1)}$$
$$\sum_{s_1w\in W}(-1)^{\ell(w)}(q^{2|\Delta_w|_1}-q^{2|\Delta_w|_0})\cdots(q^{2|\Delta_w|_0}-q^{2|\Delta_w|_1+2|\Delta_{w_h^1}|_0-2|\Delta_{w_h^1}|_1})$$
$$\cdot(q^{2|\Delta_w|_1}+q^{2|\Delta_w|_0+1})q^{(2k-1-2h)|\Delta_w|_0+(-1-2h)|\Delta_w|_1}$$

Comparing this to the last line of the series of equations above, we see that this transformation has exactly negated the expression. This anti-commutativity proves that the difference is 0.

Next, let $j = 2h+1$, $j+1 = 2(h+1)$. Then:

$$\overline{B}_{2h+1,k}(q) - \overline{B}_{2(h+1),1}(q)$$

$$= \frac{1}{(1+q)\cdots(1+q^{2h+1})F(q)} q^{-\left[|\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1\right](k-1)+(h+1)(2h+1)}$$

$$\sum_{w\in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0}) \cdots (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0 - 2|\Delta_{w^0_h}|_0 + 2|\Delta_{w^0_h}|_1}) \cdot q^{(k-2h)|\Delta_w|_0 + (k-2-2h)|\Delta_w|_1}$$

$$- \frac{1}{(1+q)\cdots(1+q^{2(h+1)})F(q)} q^{-\left[|\Delta_{w^0_{h+1}}|_0 + |\Delta_{w^0_{h+1}}|_1\right](k-1)+(h+1)(2h+3)}$$

$$\sum_{w\in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0}) \cdots (q^{2|\Delta_w|_0} - q^{2|\Delta_w|_1 + 2|\Delta_{w^1_{h+1}}|_0 - 2|\Delta_{w^1_{h+1}}|_1})$$

$$\cdot\, q^{(k-2-2h)|\Delta_w|_0 + (k-2-2h)|\Delta_w|_1}$$

$$= \frac{1}{(1+q)\cdots(1+q^{2(h+1)})F(q)} q^{-\left[|\Delta_{w^0_{h+1}}|_0 + |\Delta_{w^0_{h+1}}|_1\right](k-1)+(h+1)(2h+1)}$$

$$\sum_{w\in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0}) \cdots (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0 - 2|\Delta_{w^0_h}|_0 + 2|\Delta_{w^0_h}|_1})$$

$$\cdot \Big[(1 + q^{2(h+1)}) q^{(k-2h)|\Delta_w|_0 + (k-2-2h)|\Delta_w|_1}$$

$$- q^{2(h+1)} (q^{2|\Delta_w|_0} - q^{2|\Delta_w|_1 - 2(h+1)}) q^{(k-2-2h)|\Delta_w|_0 + (k-2-2h)|\Delta_w|_1}\Big]$$

$$= \frac{1}{(1+q)\cdots(1+q^{2(h+1)})F(q)} q^{-\left[|\Delta_{w^0_{h+1}}|_0 + |\Delta_{w^0_{h+1}}|_1\right](k-1)+(h+1)(2h+1)}$$

$$\sum_{w\in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0}) \cdots (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0 - 2|\Delta_{w^0_h}|_0 + 2|\Delta_{w^0_h}|_1})$$

$$\cdot\, [q^{(k-2h)|\Delta_w|_0 + (k-2-2h)|\Delta_w|_1} + q^{(k-2-2h)|\Delta_w|_0 + (k-2h)|\Delta_w|_1}]$$

$$= \frac{1}{(1+q)\cdots(1+q^{2(h+1)})F(q)} q^{-\left[|\Delta_{w^0_{h+1}}|_0 + |\Delta_{w^0_{h+1}}|_1\right](k-1)+(h+1)(2h+1)}$$

$$\sum_{w\in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0}) \cdots (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0 - 2|\Delta_{w^0_h}|_0 + 2|\Delta_{w^0_h}|_1})$$

$$\cdot\, (q^{2|\Delta_w|_0} + q^{2|\Delta_w|_1}) q^{(k-2(h+1))|\Delta_w|_0 + (k-2(h+1))|\Delta_w|_1}$$

It is simple to see that the transformation $f(\cdot)$ negates this sum (Corollary 2.5, with q^2 in place of q, describes how it acts on the factors in parentheses, and it is clear that it leaves the exponent at the end invariant). Hence, as above, the difference is 0.

This finishes the proof of the edge-matching.

Now, we verify that these formulas obey the required recursions. First, we deal with the ghost series.

For the ghost series on shelf $j = 2h$, $i = 2, \cdots, k-1$:

$$\frac{B_{(k-1)2h+(i-1)} + q^{2h+1}B_{(k-1)2h+(i+1)}}{1+q^{2h+1}}$$

$$= \frac{1}{(1+q)\cdots(1+q^{2h+1})F(q)} q^{-2|\Delta_{w_h^0}|_0(i-1) - \left[|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1\right](k-i)+h(2h+1)}$$

$$\sum_{w\in W}(-1)^{\ell(w)}(q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0})\cdots(q^{2|\Delta_w|_0} - q^{2|\Delta_w|_1+2|\Delta_{w_h^1}|_0-2|\Delta_{w_h^1}|_1})$$

$$\cdot\left[q^{2|\Delta_{w_h^0}|_0 - [|\Delta_{w_h^0}|_0+|\Delta_{w_h^0}|_1]+(k+i-2-2h)|\Delta_w|_0+(k-i+2-2h)|\Delta_w|_1}\right.$$

$$\left. + q^{(2h+1)-2|\Delta_{w_h^0}|_0+[|\Delta_{w_h^0}|_0+|\Delta_{w_h^0}|_1]+(k+i-2h)|\Delta_w|_0+(k-i-2h)|\Delta_w|_1}\right]$$

$$= \frac{1}{(1+q)\cdots(1+q^{2h+1})F(q)} q^{-2|\Delta_{w_h^0}|_0(i-1) - \left[|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1\right](k-i)+2h(h+1)}$$

$$\sum_{w\in W}(-1)^{\ell(w)}(q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0})\cdots(q^{2|\Delta_w|_0} - q^{2|\Delta_w|_1+2|\Delta_{w_h^1}|_0-2|\Delta_{w_h^1}|_1})$$

$$\cdot(q^{2|\Delta_w|_1} + q^{2|\Delta_w|_0+1})q^{(k+i-2-2h)|\Delta_w|_0+(k-i-2h)|\Delta_w|_1}$$

$$= \widetilde{B}_{(k-1)2h+i}(q)$$

When $i = k$, the recursive definition of the new series takes a slightly different form, and we instead compute:

$$\frac{B_{(k-1)2h+(k-1)}}{1+q^{2h+1}}$$

$$= \frac{1}{(1+q)\cdots(1+q^{2h+1})F(q)} q^{-2|\Delta_{w_h^0}|_0(k-1) - \left[|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1\right]+h(2h+1)}$$

$$\sum_{w\in W}(-1)^{\ell(w)}(q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0})\cdots(q^{2|\Delta_w|_0} - q^{2|\Delta_w|_1+2|\Delta_{w_h^1}|_0-2|\Delta_{w_h^1}|_1})$$

$$\cdot q^{(2k-2-2h)|\Delta_w|_0+(2-2h)|\Delta_w|_1}$$

This is not identical to the result of substituting $i = k$ into Theorem 5.3, but the difference is equal to

$$\frac{1}{(1+q)\cdots(1+q^{2h+1})F(q)}q^{-2|\Delta_{w_h^0}|_0(k-1)-\left[|\Delta_{w_h^0}|_0+|\Delta_{w_h^0}|_1\right]+h(2h+1)}$$
$$\sum_{w\in W}(-1)^{\ell(w)}(q^{2|\Delta_w|_1}-q^{2|\Delta_w|_0})\cdots(q^{2|\Delta_w|_0}-q^{2|\Delta_w|_1+2|\Delta_{w_h^1}|_0-2|\Delta_{w_h^1}|_1})$$
$$\cdot\, q^{(2k-2h)|\Delta_w|_0+(-2h)|\Delta_w|_1+1}$$

Applying Corollary 2.5, we see that the s_1* transformation exactly negates this last expression, which forces it to be zero. This, in turn, verifies the above formula for $\widetilde{B}_{(k-1)2h+k}(q)$.

For the ghost series on shelf $j=2h+1$, $i=2,\cdots,k-1$:

$$\frac{B_{(k-1)(2h+1)+(i-1)}+q^{2h+2}B_{(k-1)(2h+1)+(i+1)}}{1+q^{2h+2}}$$
$$=\frac{1}{(1+q)\cdots(1+q^{2h+2})F(q)}q^{-\left[|\Delta_{w_{h+1}^1}|_0+|\Delta_{w_{h+1}^1}|_1\right](i-1)-2|\Delta_{w_{h+1}^1}|_0(k-i)+(h+1)(2h+1)}$$
$$\sum_{w\in W}(-1)^{\ell(w)}(q^{2|\Delta_w|_1}-q^{2|\Delta_w|_0})\cdots(q^{2|\Delta_w|_1}-q^{2|\Delta_w|_0-2|\Delta_{w_h^0}|_0+2|\Delta_{w_h^0}|_1})$$
$$\cdot\Big[q^{[|\Delta_{w_{h+1}^1}|_0+|\Delta_{w_{h+1}^1}|_]-2|\Delta_{w_{h+1}^1}|_+(2k-i+1-2h)|\Delta_w|_0+(i-3-2h)|\Delta_w|_1}$$
$$+\,q^{2h+2-[|\Delta_{w_{h+1}^1}|_0+|\Delta_{w_{h+1}^1}|_1]+2|\Delta_{w_{h+1}^1}|_+(2k-i-1-2h)|\Delta_w|_0+(i-1-2h)|\Delta_w|_1}\Big]$$
$$=\frac{1}{(1+q)\cdots(1+q^{2h+2})F(q)}q^{-\left[|\Delta_{w_{h+1}^1}|_0+|\Delta_{w_{h+1}^1}|_1\right](i-1)-2|\Delta_{w_{h+1}^1}|_0(k-i)+2(h+1)^2}$$
$$\sum_{w\in W}(-1)^{\ell(w)}(q^{2|\Delta_w|_1}-q^{2|\Delta_w|_0})\cdots(q^{2|\Delta_w|_1}-q^{2|\Delta_w|_0-2|\Delta_{w_h^0}|_0+2|\Delta_{w_h^0}|_1})$$
$$\cdot\,(q^{2|\Delta_w|_0}+q^{2|\Delta_w|_1})q^{(2k-i-1-2h)|\Delta_w|_0+(i-3-2h)|\Delta_w|_1}$$
$$=\widetilde{B}_{(k-1)(2h+1)+i}(q)$$

When $i=k$, the recursive definition of the new series takes a slightly different form, and we instead compute:

$$
\begin{aligned}
&\frac{B_{(k-1)(2h+1)+(k-1)}}{1+q^{2h+2}}\\
&= \frac{1}{(1+q)\cdots(1+q^{2h+2})F(q)} q^{-\left[|\Delta_{w^1_{h+1}}|_0+|\Delta_{w^1_{h+1}}|_1\right](k-2)-2|\Delta_{w^1_{h+1}}|_0+(h+1)(2h+1)}\\
&\sum_{w\in W}(-1)^{\ell(w)}(q^{2|\Delta_w|_1}-q^{2|\Delta_w|_0})\cdots(q^{2|\Delta_w|_1}-q^{2|\Delta_w|_0-2|\Delta_{w^0_h}|_0+2|\Delta_{w^0_h}|_1})\\
&\qquad\qquad \cdot q^{(k-1-2h)|\Delta_w|_0+(k-3-2h)|\Delta_w|_1}
\end{aligned}
$$

This is not identical to the result of substituting $i=k$ into Theorem 5.3, but the difference is equal to

$$
\begin{aligned}
&\frac{1}{(1+q)\cdots(1+q^{2h+2})F(q)} q^{-\left[|\Delta_{w^1_{h+1}}|_0+|\Delta_{w^1_{h+1}}|_1\right](k-2)-2|\Delta_{w^1_{h+1}}|_0+(h+1)(2h+1)}\\
&\sum_{w\in W}(-1)^{\ell(w)}(q^{2|\Delta_w|_1}-q^{2|\Delta_w|_0})\cdots(q^{2|\Delta_w|_1}-q^{2|\Delta_w|_0-2|\Delta_{w^0_h}|_0+2|\Delta_{w^0_h}|_1})\\
&\qquad\qquad \cdot q^{(k-1-2h)|\Delta_w|_0+(k-1-2h)|\Delta_w|_1}
\end{aligned}
$$

Applying Corollary 2.5, we see that the $f(\cdot)$ transformation exactly negates this last expression, which forces it to be zero. This, in turn, verifies the above formula for $\widetilde{B}_{(k-1)(2h+1)+k}(q)$.

Next, for the true series on shelf $j = 2h$:

$$\frac{B_{(k-1)2h+(k-i+1)} - \widetilde{B}_{(k-1)2h+(k-i+1)}}{q^{(2h+1)(i-1)}}$$

$$= \frac{1}{(1+q)\cdots(1+q^{2h+1})F(q)} q^{-2|\Delta_{w_h^0}|_0(k-i) - \left[|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1\right](i-1)+h(2h+1)}$$

$$\sum_{w\in W}(-1)^{\ell(w)}(q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0})\cdots(q^{2|\Delta_w|_0} - q^{2|\Delta_w|_1+2|\Delta_{w_h^1}|_0-2|\Delta_{w_h^1}|_1})$$

$$\cdot\Big[(1+q^{2h+1})q^{(2k-i-2h)|\Delta_w|_0+(i-2h)|\Delta_w|_1}$$

$$-(q^{2|\Delta_w|_1} + q^{2|\Delta_w|_0+1})q^{-2|\Delta_{w_h^0}|_0+[|\Delta_{w_h^0}|_0+|\Delta_{w_h^0}|_1]+(h+1)+(2k-i-2h)|\Delta_w|_0+(i-2-2h)|\Delta_w|_1}\Big]$$

$$= \frac{1}{(1+q)\cdots(1+q^{2h+1})F(q)} q^{-\left[|\Delta_{w_{h+1}^1}|_0 + |\Delta_{w_{h+1}^1}|_1\right](i-1)-2|\Delta_{w_{h+1}^1}|_0(k-i)+(h+1)(2h+1)}$$

$$\sum_{w\in W}(-1)^{\ell(w)}(q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0})\cdots(q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0-2|\Delta_{w_h^0}|_0+2|\Delta_{w_h^0}|_1})$$

$$\cdot\, q^{(2k-i-2h)|\Delta_w|_0+(i-2-2h)|\Delta_w|_1}$$

$$= B_{(k-1)(2h+1)+i}(q)$$

For the true series on shelf $j = 2h+1$:

$$\frac{B_{(k-1)(2h+1)+(k-i+1)} - \widetilde{B}_{(k-1)(2h+1)+(k-i+2)}}{q^{2(h+1)(i-1)}}$$

$$= \frac{1}{(1+q)\cdots(1+q^{2h+2})F(q)} q^{-\left[|\Delta_{w_{h+1}^1}|_0 + |\Delta_{w_{h+1}^1}|_1\right](k-i)-2|\Delta_{w_{h+1}^1}|_0(i-1)+(h+1)(2h+1)}$$

$$\sum_{w\in W}(-1)^{\ell(w)}(q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0})\cdots(q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0-2|\Delta_{w_h^0}|_0+2|\Delta_{w_h^0}|_1})$$

$$\cdot\Big[(1+q^{2h+2})q^{(k+i-1-2h)|\Delta_w|_0+(k-i-1-2h)|\Delta_w|_1}$$

$$-(q^{2|\Delta_w|_0} + q^{2|\Delta_w|_1})q^{-[|\Delta_{w_{h+1}^1}|_0+|\Delta_{w_{h+1}^1}|_1]+2|\Delta_{w_{h+1}^1}|_0+(h+1)+(k+i-3-2h)|\Delta_w|_0+(k-i-1-2h)|\Delta_w|_1}\Big]$$

$$= \frac{1}{(1+q)\cdots(1+q^{2(h+1)})F(q)} q^{-2|\Delta_{w_{h+1}^0}|_0(i-1) - \left[|\Delta_{w_{h+1}^0}|_0 + |\Delta_{w_{h+1}^0}|_1\right](k-i)+(h+1)(2h+3)}$$

$$\sum_{w\in W}(-1)^{\ell(w)}(q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0})\cdots(q^{2|\Delta_w|_0} - q^{2|\Delta_w|_1+2|\Delta_{w_{h+1}^1}|_0-2|\Delta_{w_{h+1}^1}|_1})$$

$$\cdot\, q^{(k+i-1-2(h+1))|\Delta_w|_0+(k-i+1-2(h+1))|\Delta_w|_1}$$

$$= B_{(k-1)(2(h+1))+i}(q)$$

This finishes the proof. $\square$

5.4 Empirical Hypothesis

The only Empirical Hypothesis logically needed to complete the proof is that for the true series. However, the ghost series also obey a Empirical Hypothesis, with an entirely analogous statement and proof. We include it here for completeness, and because we use it to simplify the proof of the combinatorics in the next section.

As in the other Empirical Hypotheses proved earlier, the behavior of the terms here will follow the same pattern: finitely many terms at the beginning of the order contribute 0, the first nonzero contribution is what provides the first two elements of the series (giving the proper "shape" to the series as predicted by the Empirical Hypothesis), and all subsequent terms only contribute higher powers. However, we do encounter a new phenomenon here not seen in the Gordon-Andrews proofs: the minimal power of q coming from each term are no longer always strictly increasing with respect to the linear order. Instead, when $i = 1$, it is possible for there to be a decrease instead, just not so great a one as to disturb the shape of the Empirical Hypothesis.

Theorem 5.5 (Empirical Hypothesis). *For any $j \geq 0$ and $i = 1, \cdots, k$,*

$$B_{(k-1)j+i}(q) = 1 + q^{j+1}\gamma(q)$$

for some

$$\gamma(q) \in \mathbb{C}[[q]].$$

Remark 5.6. The proof of this theorem actually establishes a stronger result:

$$B_{(k-1)j+i}(q) = 1 + q^{j+1} + \cdots$$

for $i = 1, \cdots, k-1$ and

$$B_{(k-1)j+k}(q) = 1 + q^{j+2} + \cdots$$

Proof. Consider

$$B_{(k-1)2h+i}(q) = \frac{1}{(1+q)\cdots(1+q^{2h})F(q)} q^{-2|\Delta_{w_h^0}|_0 (i-1) + -\left[|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1\right](k-i) + h(2h+1)}$$
$$\sum_{w\in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0}) \cdots (q^{2|\Delta_w|_0} - q^{2|\Delta_w|_1 + 2|\Delta_{w_h^1}|_0 + 2|\Delta_{w_h^1}|_1})$$
$$\cdot\, q^{(k+i-1-2h)|\Delta_w|_0 + (k-i+1-2h)|\Delta_w|_1}$$

First suppose that w is one of the first $2h$ elements (with respect to the linear order) - i.e., w is either of the form w_r^0 for $r < h$, or w_r^1 for $r \leq h$. Then as in the earlier proofs, formulas (3.8), (3.9), and (3.10) (with q^2 in place of q) prove that the corresponding terms of the series are 0.

The first nonzero contribution to the series comes at the term $w = w_h^0$, and that this contribution is of the shape $1+q^{2h+1}+O(q^{2h+2})$. This claim is justified by the following sequence of computations, starting with the $w = w_h^0$ term from the series referenced above:

$$
\begin{aligned}
&\frac{1}{(1+q)\cdots(1+q^{2h})F(q)} q^{-2|\Delta_{w_h^0}|_0 (i-1)+-\left[|\Delta_{w_h^0}|_0+|\Delta_{w_h^0}|_1\right](k-i)+h(2h+1)} \\
&(-1)^{\ell(w_h^0)}\{q^{2|\Delta_{w_h^0}|_1}(1-q^{2(|\Delta_{w_h^0}|_0-|\Delta_{w_h^0}|_1)})\}\cdots\{-q^{2|\Delta_{w_h^0}|_1-2h}(1-q^{2(|\Delta_{w_h^0}|_0-2|\Delta_{w_h^0}|_1+h)})\} \\
&\qquad \cdot q^{(k+i-1-2h)|\Delta_{w_h^0}|_0+(k-i+1-2h)|\Delta_{w_h^0}|_1} \\
&= \frac{(-1)^{2h}}{(1+q)\cdots(1+q^{2h})F(q)} \\
&(1-q^2)\cdots(1-q^{4h})q^{4h|\Delta_{w_h^0}|_1-2|\Delta_{w_h^0}|_0-2|\Delta_{w_h^0}|_0(i-1)-[|\Delta_{w_h^0}|_0+|\Delta_{w_h^0}|_1](k-i)+h(2h+1)} \\
&\qquad\qquad \cdot q^{(k+i-1-2h)|\Delta_w|_0+(k-i+1-2h)|\Delta_w|_1}
\end{aligned}
$$

Looking more closely at the ending exponent of q, we can regroup it as

$$
\begin{aligned}
&k\left[(|\Delta_{w_h^0}|_0+|\Delta_{w_h^0}|_1)-(|\Delta_{w_h^0}|_0+|\Delta_{w_h^0}|_1)\right] \\
&+i\left[-2|\Delta_{w_h^0}|_0+|\Delta_{w_h^0}|_0+|\Delta_{w_h^0}|_1+|\Delta_{w_h^0}|_0-|\Delta_{w_h^0}|_1\right] \\
&+h(2h+1)-(2h+1)(|\Delta_{w_h^0}|_0-|\Delta_{w_h^0}|_1) \\
&= 0+0+h(2h+1)-(2h+1)h \\
&= 0
\end{aligned}
$$

Moreover, writing $F(q) = (1-q)\cdots(1-q^{2h})(q^{2h},q)_\infty$, we see that by combining all the factors in parentheses with the first $2h$ factors of $F(q)$, we can cancel all these factors leaving just

$$(q^{2h},q)_\infty^{-1} = 1+q^{2h+1}+\cdots$$

To see that all higher contributions fall within the specified range, we again use the techniques of applying our two transformations to individual terms and proving inductively that the resulting expression involves sufficiently high powers of q. (As mentioned earlier, there are some cases here in which the power of q actually does decrease when applying these transformations).

First, assume w is of the form w_r^0 for some $r \geq h$, so that the next Weyl element is $s_1 w_r^0 = w_{r+1}^1$. Corollary 2.5 (with q^2 in place of q) tells us the effect of the transformation s_1* on this term is to rescale it by a power of q, and that the exponent of this scaling power is:

$$
\begin{aligned}
&2h[|\Delta_w|_0 - |\Delta_w|_1 + 1] + 2h[|\Delta_w|_0 - |\Delta_w|_1] + (k+i-1-2h)|\Delta_w|_0 \\
&+ (k-i+1-2h)[2|\Delta_w|_0 - |\Delta_w|_1 + 1] - (k+i-1-2h)|\Delta_w|_0 + (k-i+1-2h)|\Delta_w|_1 \\
&= 2h[2|\Delta_w|_0 - 2|\Delta_w|_1 + 1] + (k-i+1-2h)[2|\Delta_w|_0 - 2|\Delta_w|_1 + 1] \\
&= (k-i+1)[2|\Delta_w|_0 - 2|\Delta_w|_1 + 1] \\
&\geq 2h+1
\end{aligned}
$$

with equality only when $i = k$, $w = w_h^0$. This is large enough to make this contribution $O(q^{2h+1})$ (and is exactly what is needed to match the more precise Empirical Hypothesis in the edge case).

Next, assume w is of the form w_r^1 for some $r \geq h+1$, so that the next Weyl element is $f(w_r^1) = w_r^0$. From Corollary 2.5 (with q^2 in place of q), when applying the transformation $f(\cdot)$ to this term, the overall difference is the replacement of the factor

$$
\begin{aligned}
&(q^{2|\Delta_w|_0} - q^{2|\Delta_w|_1 - 2h})q^{(k+i-1-2h)|\Delta_w|_0 + (k-i+1-2h)|\Delta_w|_1} \\
&= q^{(k+i+1-2h)|\Delta_w|_0 + (k-i+1-2h)|\Delta_w|_1} - q^{(k+i-1-2h)|\Delta_w|_0 + (k-i+3-2h)|\Delta_w|_1 - 2h}
\end{aligned}
$$

from the original term with

$$
\begin{aligned}
&-(q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0 - 2h})q^{(k-i+1-2h)|\Delta_w|_0 + (k+i-1-2h)|\Delta_w|_1} \\
&= q^{(k-i+3-2h)|\Delta_w|_0 + (k+i-1-2h)|\Delta_w|_1 - 2h} - q^{(k-i+1-2h)|\Delta_w|_0 + (k+i+1-2h)|\Delta_w|_1}
\end{aligned}
$$

from the transformed term.

The ratios between the positive and negative expressions here are respectively

$$q^{(-2i+2)|\Delta_w|_0 + (2i-2)|\Delta_w|_1 - 2h} \text{ and } q^{(-2i+2)|\Delta_w|_0 + (2i-2)|\Delta_w|_1 + 2h}$$

which means that the minimum change in the power of q is

$$2[(i-1)(|\Delta_w|_1 - |\Delta_w|_0) - h] \geq 2h$$

with equality only when $i = 1$. If $i > 1$, this is ≥ 2 because we are assuming $w = w_r^1$ for some $r \geq h+1$, so certainly the result holds for all of these i. However, looking above, when $i = 1$ the minimal power of q appearing in the original w term is at least

$$k(2h+1) \geq 4h+2$$

so even in the $i = 1$ case the exponent will still be $O(q^{2h+1})$.

Now consider

$$B_{(k-1)(2h+1)+i}(q) = \frac{1}{(1+q)\cdots(1+q^{2h+1})F(q)} q^{-\left[|\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1\right](i-1) - 2|\Delta_{w^1_{h+1}}|_0 (k-i) + (h+1)(2h+1)}$$
$$\sum_{w \in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0}) \cdots (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0 - 2|\Delta_{w^0_h}|_0 + 2|\Delta_{w^0_h}|_1}) \cdot q^{(2k-i-2h)|\Delta_w|_0 + (i-2-2h)|\Delta_w|_1}$$

Suppose that w is one of the first $2h+1$ elements (with respect to the linear order) - i.e., w is either of the form w^0_r or w^1_r for $r \leq h$. Once again, formulas (3.8), (3.9), and (3.10) (with q^2 in place of q) prove that the corresponding terms of the series are 0.

The first nonzero contribution to the series comes at the term $w = w^1_{h+1}$, and that this contribution is of the shape $1 + q^{2h+2} + O(q^{2h+3})$. This claim is justified by the following sequence of computations, starting with the $w = w^1_{h+1}$ term from the series referenced above:

$$\frac{1}{(1+q)\cdots(1+q^{2h+1})F(q)} q^{-\left[|\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1\right](i-1) - 2|\Delta_{w^1_{h+1}}|_0 (k-i) + (h+1)(2h+1)}$$
$$(-1)^{\ell(w^1_{h+1})} \{-q^{2|\Delta_{w^1_{h+1}}|_0} (1 - q^{2(|\Delta_{w^1_{h+1}}|_1 - |\Delta_{w^1_{h+1}}|_0)})\} \cdots \{-q^{2|\Delta_{w^1_{h+1}}|_0 - 2h} (1 - q^{2(|\Delta_{w^1_{h+1}}|_1 - 2|\Delta_{w^1_{h+1}}|_0 + h)})\}$$
$$\cdot q^{(2k-i-2h)|\Delta_{w^1_{h+1}}|_0 + (i-2-2h)|\Delta_{w^1_{h+1}}|_1}$$
$$= \frac{(-1)^{2h+2}}{(1+q)\cdots(1+q^{2h+1})F(q)}$$
$$(1-q^2)\cdots(1-q^{4h+2}) q^{(4h+2)|\Delta_{w^1_{h+1}}|_0 - 2|\Delta_{w^1_{h+1}}|_0 - [|\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1](i-1) - 2|\Delta_{w^1_{h+1}}|_0 (k-i) + (h+1)(2h+1)}$$
$$\cdot q^{(2k-i-2h)|\Delta_{w^1_{h+1}}|_0 + (i-2-2h)|\Delta_{w^1_{h+1}}|_1}$$

Looking more closely at the ending exponent of q, we can regroup it as

$$\begin{aligned}
&2k\left[|\Delta_{w^1_{h+1}}|_0 - |\Delta_{w^1_{h+1}}|_0\right] \\
&+ i\left[(|\Delta_{w^1_{h+1}}|_0 - |\Delta_{w^1_{h+1}}|_1) - (|\Delta_{w^1_{h+1}}|_0 - |\Delta_{w^1_{h+1}}|_1)\right] \\
&+ (h+1)(2h+1) + (2h+1)(|\Delta_{w^1_{h+1}}|_0 - |\Delta_{w^1_{h+1}}|_1) \\
&= 0 + 0 + (h+1)(2h+1) - (2h+1)(h+1) \\
&= 0
\end{aligned}$$

Moreover, writing $F(q) = (1-q)\cdots(1-q^{2h+1})(q^{2h+1}, q)_\infty$, we see that by combining all the factors in parentheses with the first $2h+1$ factors of $F(q)$, we can cancel all these factors leaving just

$$(q^{2h+1}, q)_\infty^{-1} = 1 + q^{2h+2} + \cdots$$

Now we apply our same two transformations to individual terms to prove inductively that the remaining contributions all involve only sufficiently high powers of q.

First, assume w is of the form w_r^1 for some $r \geq h+1$, so that the next Weyl element is $f(w_r^1) = w_r^0$. By Corollary 2.5, the overall effect of the $f(\cdot)$ transformation on this term is to negate the factors in parentheses, and rescale the ending power of q. We compute that the exponent of this scaling power is

$$\begin{aligned}
&[(i-3)-(2k-i-1)]|\Delta_w|_0 + [(2k-i-1)-(i-3)]|\Delta_w|_1 \\
&= 2(k-i+1)(|\Delta_w|_1 - |\Delta_w|_0) \\
&\geq 2h+2
\end{aligned}$$

with equality only when $i = k$, $w = w_{h+1}^1$. This is large enough to make this contribution $O(q^{2h+2})$ (and is exactly what is needed to match the more precise Empirical Hypothesis in the edge case).

Next, assume w is of the form w_r^0 for some $r \geq h+1$, so that the next Weyl element is $s_1 w_r^0 = w_{r+1}^1$. By Corollary 2.5, applying the s_1 transformation, the difference between the original and transformed terms is the replacement of

$$\begin{aligned}
&(q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0 - 2h})q^{(2k-i-2h)|\Delta_w|_0 + (i-2-2h)|\Delta_w|_1} \\
&= q^{(2k-i-2h)|\Delta_w|_0 + (i-2h)|\Delta_w|_1} - q^{(2k-i+2-2h)|\Delta_w|_0 + (i-2-2h)|\Delta_w|_1 - 2h}
\end{aligned}$$

from the original term with

$$\begin{aligned}
&-(q^{2|\Delta_w|_0} - q^{2|\Delta_w|_1 - 2(h+1)}) \\
&\cdot q^{2(h+1)(|\Delta_w|_0 - |\Delta_w|_1 + 1) + 2h(|\Delta_w|_0 - |\Delta_w|_1) + (2k-i-2h)|\Delta_w|_0 + (i-2-2h)(|\Delta_w|_0 - |\Delta_w|_1 + 1)} \\
&= (q^{2|\Delta_w|_1 - 2(h+1)} - q^{2|\Delta_w|_0})q^{(2k-i-2-2h)|\Delta_w|_0 + (-i-2h)|\Delta_w|_1 + i} \\
&= q^{(2k-i-2-2h)|\Delta_w|_0 + (-i+2-2h)|\Delta_w|_1 + i - 2(h+1)} - q^{(2k-i-2h)|\Delta_w|_0 + (-i-2h)|\Delta_w|_1 + i}
\end{aligned}$$

in the transformed term.

The ratios between the positive and negative expressions here are respectively

$$q^{(2i-2)|\Delta_w|_0+(-2i+2)|\Delta_w|_1-2(h+1)+i} \text{ and } q^{(2i-2)|\Delta_w|_0+(-2i+2)|\Delta_w|_1-2(h+1)+i+2h}$$

which means that the minimum change in the power of q is

$$2[(i-1)(|\Delta_w|_0 - |\Delta_w|_1) - (h+1)] + i$$

This is greater than or equal to $-(2h+1)$, with equality only when $i = 1$. If $i > 1$, this is at least 2 because we are assuming $w = w_r^0$ for some $r \geq h+1$, so certainly the result holds for all of these i. However, looking above, when $i = 1$ the minimal power of q appearing in the original w term is at least

$$2k(h+1) \geq 4h+4$$

so even in the $i = 1$ case the exponent will still be $O(q^{2h+2})$.

This finishes the proof of the Empirical Hypothesis for the true series. $\square$

Theorem 5.7 (Empirical Hypothesis). *For any $j \geq 0$ and $i = 2, \cdots, k$,*

$$\widetilde{B}_{(k-1)j+i}(q) = 1 + q^{j+1}\gamma(q)$$

for some

$$\gamma(q) \in \mathbb{C}[[q]].$$

Remark 5.8. The proof of this theorem actually establishes a stronger result:

$$\widetilde{B}_{(k-1)j+i}(q) = 1 + q^{j+1} + \cdots$$

for $i = 2, \cdots, k-1$ and

$$\widetilde{B}_{(k-1)j+k}(q) = 1 + q^{j+2} + \cdots$$

Proof. Consider

$$\widetilde{B}_{(k-1)2h+i}(q) = \frac{1}{(1+q)\cdots(1+q^{2h+1})F(q)} q^{-2|\Delta_{w_h^0}|_0(i-1) - \left[|\Delta_{w_h^0}|_0 + |\Delta_{w_h^0}|_1\right](k-i) + 2h(h+1)}$$
$$\sum_{w \in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0}) \cdots (q^{2|\Delta_w|_0} - q^{2|\Delta_w|_1 + 2|\Delta_{w_h^1}|_0 - 2|\Delta_{w_h^1}|_1})$$
$$\cdot (q^{2|\Delta_w|_1} + q^{|\Delta_w|_0+1}) q^{(k+i-2-2h)|\Delta_w|_0 + (k-i-2h)|\Delta_w|_1}$$

First suppose that w is one of the first $2h$ elements (with respect to the linear order) - i.e., w is either of the form w_r^0 for $r < h$, or w_r^1 for $r \leq h$. Then formulas (3.8), (3.9), and (3.10) force these terms to be 0.

The first nonzero contribution to the series comes at the term $w = w_h^0$, and this contribution is of the shape $1+q^{2h+1}+O(q^{2h+2})$. This claim is justified by the following sequence of computations, starting with the $w = w_h^0$ term from the series referenced above:

$$\begin{aligned}
&\frac{1}{(1+q)\cdots(1+q^{2h+1})F(q)} q^{-2|\Delta_{w_h^0}|_0(i-1)-\left[|\Delta_{w_h^0}|_0+|\Delta_{w_h^0}|_1\right](k-i)+2h(h+1)} \\
&(-1)^{\ell(w_h^0)}\{q^{2|\Delta_{w_h^0}|_1}(1-q^{2(|\Delta_{w_h^0}|_0-|\Delta_{w_h^0}|_1)})\}\cdots\{-q^{2|\Delta_{w_h^0}|_1-2h}(1-q^{2(|\Delta_{w_h^0}|_0-2|\Delta_{w_h^0}|_1+h)})\} \\
&\qquad \cdot \{q^{2|\Delta_w|_1}(1-q^{2(|\Delta_{w_h^0}|_0-|\Delta_{w_h^0}|_1)+1})\}q^{(k+i-2-2h)|\Delta_{w_h^0}|_0+(k-i-2h)|\Delta_{w_h^0}|_1} \\
&= \frac{(-1)^{2h}}{(1+q)\cdots(1+q^{2h+1})F(q)} \\
&(1-q^2)\cdots(1-q^{4h})(1+q^{2h+1})q^{2(2h+1)|\Delta_{w_h^0}|_1-2|\Delta_{w_h^0}|_0-2|\Delta_{w_h^0}|_0(i-1)-[|\Delta_{w_h^0}|_0+|\Delta_{w_h^0}|_1](k-i)+2h(h+1)} \\
&\qquad\qquad \cdot q^{(k+i-1-2h)|\Delta_w|_0+(k-i+1-2h)|\Delta_w|_1}
\end{aligned}$$

Looking more closely at the ending exponent of q, we can regroup it as

$$\begin{aligned}
&k\left[(|\Delta_{w_h^0}|_0+|\Delta_{w_h^0}|_1)-(|\Delta_{w_h^0}|_0+|\Delta_{w_h^0}|_1)\right] \\
&+i\left[(|\Delta_{w_h^0}|_0-|\Delta_{w_h^0}|_1)-(|\Delta_{w_h^0}|_0-|\Delta_{w_h^0}|_1)\right] \\
&+2h(h+1)-2(h+1)(|\Delta_{w_h^0}|_0-|\Delta_{w_h^0}|_1) \\
&=0+0+2h(h+1)-2h(h+1) \\
&=0
\end{aligned}$$

Moreover, writing $F(q) = (1-q)\cdots(1-q^{2h})(q^{2h},q)_\infty$, we see that by combining all the factors in parentheses with the first $2h$ factors of $F(q)$, we can cancel all these factors leaving just $(q^{2h},q)_\infty^{-1}$. The factor $(1+q^{2h+1})$ is cancelled by the same factor appearing explicitly in the term. Hence this term is in fact equal to

$$(q^{2h},q)_\infty^{-1} = 1+q^{2h+1}+\cdots$$

To see that all higher contributions fall within the specified range, we again use the techniques of applying our two transformations to individual terms and proving inductively that the resulting expression involves sufficiently high powers of q. (As mentioned earlier, there are some cases here in which the power of q actually does decrease when applying these transformations).

First, assume w is of the form w_r^0 for $r \geq h$, so that the next Weyl element is $s_1 w_r^0 = wr + 1^1$. By Corollary 2.5 (with q^2 in place of q), the effect of the s_1* transformation on this term is to negate it and rescale by a power of q whose exponent is

$$\begin{aligned}
&(2h+1)(2|\Delta_w|_0 - |\Delta_w|_1 + 1) + (k-i-2h)(2|\Delta_w|_0 - |\Delta_w|_1 + 1) \\
&= (k-i+1)[2|\Delta_w|_0 - 2|\Delta_w|_1 + 1] \\
&\geq 2h+1
\end{aligned}$$

with equality only when $i = k$, $w = w_h^0$. This is large enough to make this contribution $O(q^{2h+1})$ (and is exactly what is needed to match the more precise Empirical Hypothesis in the edge case).

Next, assume w is of the form w_r^1 for some $r \geq h+1$, so that the next Weyl element is $f(w_r^1) = w_r^0$. From Corollary 2.5, the transformation $f(\cdot)$ has the effect of negating the term and replacing

$$\begin{aligned}
&(q^{2|\Delta_w|_0} - q^{2|\Delta_w|_1 - 2h})(q^{2|\Delta_w|_1} + q^{2|\Delta_w|_0+1})q^{(k+i-2-2h)|\Delta_w|_0 + (k-i-2h)|\Delta_w|_1} \\
&= (q^{2|\Delta_w|_0 + 2|\Delta_w|_1} + q^{4|\Delta_w|_0+1} - q^{4|\Delta_w|_1 - 2h} - q^{2|\Delta_w|_0 + 2|\Delta_w|_1 - 2h+1})q^{(k+i-2-2h)|\Delta_w|_0 + (k-i-2h)|\Delta_w|_1}
\end{aligned}$$

from the original term with

$$\begin{aligned}
&-(q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0 - 2h})(q^{2|\Delta_w|_0} + q^{2|\Delta_w|_1+1})q^{(k-i-2h)|\Delta_w|_0 + (k+i-2-2h)|\Delta_w|_1} \\
&= -(q^{2|\Delta_w|_0 + 2|\Delta_w|_1} + q^{4|\Delta_w|_1+1} - q^{4|\Delta_w|_0 - 2h} - q^{2|\Delta_w|_0 + 2|\Delta_w|_1 - 2h+1})q^{(k-i-2h)|\Delta_w|_0 + (k+i-2-2h)|\Delta_w|_1}
\end{aligned}$$

from the transformed term.

The ratios between corresponding parts here are respectively

$$q^{(-2i+2)|\Delta_w|_0 + (2i-2)|\Delta_w|_1}, \quad q^{(-2i+2)|\Delta_w|_0 + (2i-2)|\Delta_w|_1 - (2h+1)},$$

$$q^{(-2i+2)|\Delta_w|_0 + (2i-2)|\Delta_w|_1 + (2h+1)}, \quad q^{(-2i+2)|\Delta_w|_0 + (2i-2)|\Delta_w|_1}$$

which means that the minimum change in the power of q is

$$2[(i-1)(|\Delta_w|_1 - |\Delta_w|_0)] - (2h+1)$$

This is at least $-(2h+1)$, with equality only when $i=1$. If $i>1$, this is at least 1 because we are assuming $w = w_r^1$ for some $r \geq h+1$, so certainly the result holds for all of these i. However, looking above, when $i=1$ the minimal power of q appearing in the original w term is at least

$$k(2h+1) \geq 4h+2$$

which outweighs this potential negative shift, so even in the $i=1$ case the exponent will still be $O(q^{2h+1})$.

Now consider

$$\widetilde{B}_{(k-1)(2h+1)+i}(q) = \frac{1}{(1+q)\cdots(1+q^{2h+2})F(q)} q^{-\left[|\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1\right](i-1) - 2|\Delta_{w^1_{h+1}}|_0 (k-i) + 2(h+1)^2}$$
$$\sum_{w\in W} (-1)^{\ell(w)} (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0}) \cdots (q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0 - 2|\Delta_{w^0_h}|_0 + 2|\Delta_{w^0_h}|_1})$$
$$\cdot (q^{2|\Delta_w|_0} + q^{2|\Delta_w|_1}) q^{(2k-i-1-2h)|\Delta_w|_0 + (i-3-2h)|\Delta_w|_1}$$

Suppose that w is one of the first $2h+1$ elements (with respect to the linear order) - i.e., w is either of the form w^0_r or w^1_r for $r \leq h$. Once again, formulas (3.8), (3.9), and (3.10) (with q^2 in place of q) prove that the corresponding terms of the series are 0.

The first nonzero contribution to the series comes at the term $w = w^1_{h+1}$, and that this contribution is of the shape $1 + q^{2h+2} + O(q^{2h+3})$. This claim is justified by the following sequence of computations, starting with the $w = w^1_{h+1}$ term from the series referenced above:

$$\frac{1}{(1+q)\cdots(1+q^{2h+2})F(q)} q^{-\left[|\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1\right](i-1) - 2|\Delta_{w^1_{h+1}}|_0 (k-i) + 2(h+1)^2}$$
$$(-1)^{\ell(w^1_{h+1})} \{-q^{2|\Delta_{w^1_{h+1}}|_0} (1 - q^{2(|\Delta_{w^1_{h+1}}|_1 - |\Delta_{w^1_{h+1}}|_0)})\} \cdots$$
$$\{-q^{2|\Delta_{w^1_{h+1}}|_0 - 2h} (1 - q^{2(|\Delta_{w^1_{h+1}}|_1 - 2|\Delta_{w^1_{h+1}}|_0 + h)})\}$$
$$\cdot \{q^{2|\Delta_{w^1_{h+1}}|_0} (1 - q^{2(|\Delta_{w^1_{h+1}}|_1 - |\Delta_{w^1_{h+1}}|_0)})\} q^{(2k-i-1-2h)|\Delta_{w^1_{h+1}}|_0 + (i-3-2h)|\Delta_{w^1_{h+1}}|_1}$$
$$= \frac{(-1)^{2h+2}}{(1+q)\cdots(1+q^{2h+2})F(q)}$$
$$(1-q^2)\cdots(1-q^{4h+2})(1+q^{2h+2}) q^{2(2h+1)|\Delta_{w^1_{h+1}}|_0 - [|\Delta_{w^1_{h+1}}|_0 + |\Delta_{w^1_{h+1}}|_1](i-1) - 2|\Delta_{w^1_{h+1}}|_0 (k-i) + 2(h+1)^2}$$
$$\cdot q^{(2k-i-1-2h)|\Delta_{w^1_{h+1}}|_0 + (i-3-2h)|\Delta_{w^1_{h+1}}|_1}$$

Looking more closely at the ending exponent of q, we can regroup it as

$$\begin{aligned}
&2k\left[\left|\Delta_{w^1_{h+1}}\right|_0 - \left|\Delta_{w^1_{h+1}}\right|_0\right]\\
&+ i\left[(\left|\Delta_{w^1_{h+1}}\right|_1 - \left|\Delta_{w^1_{h+1}}\right|_0) - (\left|\Delta_{w^1_{h+1}}\right|_1 - \left|\Delta_{w^1_{h+1}}\right|_0)\right]\\
&+ 2(h+1)^2 + 2(h+1)(\left|\Delta_{w^1_{h+1}}\right|_1 - \left|\Delta_{w^1_{h+1}}\right|_0)\\
&= 0 + 0 + 2(h+1)^2 - 2(h+1)^2\\
&= 0
\end{aligned}$$

Moreover, writing $F(q) = (1-q)\cdots(1-q^{2h+1})(q^{2h+1}, q)_\infty$, we see that by combining all the factors in parentheses with the first $2h+1$ factors of $F(q)$ together with the other factors appearing in the denominator, we can cancel all these factors. Moreover, the remaining factor of $(1+q^{2h+2})$ exactly cancels the remaining factor in the denominator. Hence this term is in fact equal to

$$(q^{2h+1}, q)_\infty^{-1} = 1 + q^{2h+2} + \cdots$$

Now we apply our same two transformations to individual terms to prove inductively that the remaining contributions all involve only sufficiently high powers of q.

First, assume w is of the form w^1_r for some $r \geq h+1$, so that the next Weyl element is $f(w^1_r) = w^0_r$. From Corollary 2.5, the transformation $f(\cdot)$ has the effect of rescaling the term by a power of q whose exponent is

$$\begin{aligned}
&[(i-3) - (2k-i-1)]\left|\Delta_w\right|_0 + [(2k-i-1) - (i-3)]\left|\Delta_w\right|_1\\
&= 2(k-i+1)(\left|\Delta_w\right|_1 - \left|\Delta_w\right|_0)\\
&\geq 2h+2
\end{aligned}$$

with equality only when $i = k$, $w = w^1_{h+1}$. This is large enough to make this contribution $O(q^{2h+2})$ (and is exactly what is needed to match the more precise Empirical Hypothesis in the edge case).

Next, assume w is of the form w^0_r for $r \geq h$, so that the next Weyl element is $s_1 w^0_r = wr + 1^1$. By Corollary 2.5, the s_1* transformation has the overall effect of replacing the factors

$$(q^{2|\Delta_w|_1} - q^{2|\Delta_w|_0 - 2h})(q^{2|\Delta_w|_0} + q^{2|\Delta_w|_1})q^{(2k-i-1-2h)|\Delta_w|_0 + (i-3-2h)|\Delta_w|_1}$$

from the original term with

$$-\left(q^{2|\Delta_w|_0}-q^{2|\Delta_w|_1-2(h+1)}\right)\left(q^{2|\Delta_w|_1}+q^{2|\Delta_w|_0+2}\right)$$
$$q^{4(h+1)(|\Delta_w|_0-|\Delta_w|_1)+2(h+1)+(2k+i-7-6h)|\Delta_w|_0+(-i+3+2h)|\Delta_w|_1+(i-3-2h)}$$

for the transformed term.

As above, when we distribute the factors in parentheses and compare like terms, we find there are three possible ratios between corresponding parts:

$$q^{(2i-2)|\Delta_w|_0+(-2i+2)|\Delta_w|_1+i-1\pm 2(h+1)}, \quad q^{(2i-2)|\Delta_w|_0+(-2i+2)|\Delta_w|_1+i-1}$$

Thus, the minimum change in the power of q is

$$2[(i-1)(|\Delta_w|_0-|\Delta_w|_1)-(h+1)]+i-1$$

This is at least $-2(h+1)$, with equality only when $i=1$. If $i>1$, this is at least 1 because we are assuming $w=w_r^0$ for some $r\geq h+1$, so certainly the result holds for all of these i. However, looking above, when $i=1$ the minimal power of q appearing in the original w term is at least

$$2k(h+1)\geq 4h+4$$

so even in the $i=1$ case the exponent will still be $O(q^{2h+2})$.

This finishes the proof of the Empirical Hypothesis for the ghost series.

□

5.5 Combinatorics

To finish the proof of the Andrews-Bressoud identities, we recall Theorem 7.3 from [KLRS].

Remark 5.9. As in for the Gordon-Andrews series, the proof we give here is not the main proof from [KLRS]. Instead, it is in the spirit of Remark 7.6 there.

Theorem 5.10. *For each $i=1,\dots,k$, $j=0,1,2,\dots$, $B_{(k-1)j+i}(q)$ is the generating function of partitions $\pi=(\pi_1,\cdots,\pi_s)$ satisfying*

1. *difference at least 2 at distance $k-1$*

2. $\pi_t - \pi_{t+k-2} \leq 1$ *only if* $\pi_t + \cdots + \pi_{t+k-2} \equiv (k-1)j + i + k \mod 2$

3. *smallest part is at least* $j+1$

4. $j+1$ *appears as a part at most* $k-i$ *times*

and $\widetilde{B}_{(k-1)j+i}(q)$ *is the generating function of partitions satisfying*

1. *difference at least* 2 *at distance* $k-1$

2. $\pi_t - \pi_{t+k-2} \leq 1$ *only if* $\pi_t + \cdots + \pi_{t+k-2} \equiv (k-1)j + i + k + 1 \mod 2$

3. *smallest part is at least* $j+1$

4. $j+1$ *appears as a part at most* $k-i$ *times*

Proof. Suppose $L_1, L_2, \ldots$ is an infinite sequence of formal power series in q which satisfy the recursions (5.7) (with L in place of B) and the Empirical Hypothesis. By rewriting (5.7) to eliminate the ghost series and to solve for the lowest-indexed series and applying this formula iteratively, we see that for each $i = 1, \ldots, k$, we have expressions

$$L_i(q) = \sum_{p=1}^{k} {}_ih_p^{(j)}(q) L_{(k-1)j+p}(q)$$

for some polynomials ${}_ih_p^{(j)}(q) \in \mathbb{C}[q]$. Notice that the coefficients ${}_ih_p^{(j)}(q)$ of these combinations depend only on the recursions, not directly on the Ls. It follows from the Empirical Hypothesis that the series $L_1, \ldots, L_k$ are uniquely determined (for example, considering the combination at shelf j determines the first j terms of the series J_i just in terms of the ${}_ih_p^{(j)}(q)$). Hence, the whole sequence $L_1, L_2, \ldots$ is uniquely determined.

By our work in the earlier sections (the definition of the B_i and $\widetilde{B}_i$ series in (5.7) and Theorems 5.5, 5.7), the series B_i, $\widetilde{B}_i$ above satisfy these conditions. Let D_i, $\widetilde{D}_i$ denote the generating functions of the classes of partitions described in the statement of the current theorem. By uniqueness, it is now enough to check that the D_i and $\widetilde{D}_i$ also satisfy the recursions and Empirical Hypothesis.

The Empirical Hypotheses for the D_i and $\widetilde{D}_i$ follow directly from the definitions.

To check the recursions, first we consider

$$\frac{D_{(k-1)j+(k-1)} - \widetilde{D}_{(k-1)j+k}}{q^{j+1}}$$

The indices of the two series in the numerator differ by 1, but because the parity condition is offset for the true series versus the ghosts, they agree on the parity condition. Hence, the numerator is the generating function of partitions satisfying the first three conditions above (with parity $(k-1)j+1$), and such that the part $j+1$ has multiplicity one. Once that part is eliminated by the denominator, the next smallest part must be at least $j+2$. It can occur no more than $k-2$ times to satisfy the first (difference at distance) condition, and this is allowed by the second (parity) condition because

$$(j+1)+(k-2)(j+2)=(k-1)(j+1)+(k-2)\equiv(k-1)j+(k-1)+k \mod 2$$

Hence expression is identical to the series $D_{(k-1)(j+1)+2}(q)$. A similar argument shows that it also agrees with $\widetilde{D}_{(k-1)j+k}$ (in this case, the parity condition is violated, so we end up with the smaller upper bound of $k-2$ for the multiplicity of $j+2$).

Next, we consider

$$\frac{D_{(k-1)j+(k-i+1)}-\widetilde{D}_{(k-1)j+(k-i+2)}}{q^{(j+1)(i-1)}}$$

Again the parity conditions match, so the numerator counts partitions satisfying conditions 1 to 3 above and such that the part $j+1$ appears with multiplicity $i-1$. By condition 1, such parititions can have at most $k-i$ parts equal to $j+2$, and the maximum number is allowed by condition 3 because

$$(i-1)(j+1)+(k-i)(j+2)=(k-1)(j+1)+(k-i)\equiv(k-1)(j+1)+i+k \mod 2$$

Hence, once we have shifted these partitions by the effect of the denominator, we are left with the series $D_{(k-1)(j+1)+i}(q)$ A similar argument shows that we get the same series by considering

$$\frac{\widetilde{D}_{(k-1)j+(k-i+2)}-D_{(k-1)j+(k-i+3)}}{q^{(j+1)(i-2)}}$$

(as in the second case above, the parity condition forces a lower upper bound). □

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