CANTOR MINIMAL SYSTEMS FROM A DESCRIPTIVE PERSPECTIVE

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ABSTRACT OF THE DISSERTATION

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In recent years, the study of the Borel complexity of naturally occurring classification problems has been a major focus in descriptive set theory. This thesis is a contribution to the project of analyzing the Borel complexity of the topological conjugacy relation on various Cantor minimal systems.

We prove that the topological conjugacy relation on pointed Cantor minimal systems is Borel bireducible with the Borel equivalence relation $\Delta_{\mathbb{R}}^{+}$. As a byproduct of our analysis, we also show that $\Delta_{\mathbb{R}}^{+}$ is a lower bound for the Borel complexity of the topological conjugacy relation on Cantor minimal systems.

The other main result of this thesis concerns the topological conjugacy relation on Toeplitz subshifts. We prove that the topological conjugacy relation on Toeplitz subshifts with separated holes is a hyperfinite Borel equivalence relation. This result provides a partial affirmative answer to a question asked by Sabok and Tsankov.

As pointed Cantor minimal systems are represented by properly ordered Bratteli diagrams, we also establish that the Borel complexity of equivalence of properly ordered Bratteli diagrams is $\Delta_{\mathbb{R}}^{+}$.

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Table of Contents

Chapter 1

Introduction

Classification problems have undoubtedly been a central theme in mathematics throughout history. However, not all classification problems have satisfactory solutions. Consequently, it is natural to ask whether or not one can measure *how difficult* a classification problem is. In other words, is it possible to rigorously compare the relative complexity of the complete invariants of classification problems?

Over the last couple of decades, this question has been successfully and affirmatively answered by descriptive set theory. This thesis is a contribution to the analysis of some classification problems in topological dynamics from the point of view of descriptive set theory.

1.1 Background from descriptive set theory

Descriptive set theory classically studies the behavior of the "definable" subsets of *Polish spaces*, i.e. completely metrizable separable topological spaces. By the definable subsets of Polish spaces, one usually means the *projective sets*, i.e. those sets that can be obtained from the Borel subsets of Polish spaces by finitely many applications of projection and complementation.

In this thesis, we shall only be concerned with Borel sets and analytic sets. Recall that a subset of a Polish space X is said to be *analytic* if it is the projection of a Borel set $B \subseteq X \times Y$ for some Polish space Y.

For the framework which we will introduce, we wish to consider Polish spaces not with their topologies but rather with their Borel σ-algebra structures. A measurable space $(X, \mathcal{B})$ is called a *standard Borel space* if $\mathcal{B}$ is the Borel σ-algebra of some Polish topology on X. Examples of standard Borel spaces include finite sets, $\mathbb{N}$, $\mathbb{R}^n$, the Baire space $\mathbb{N}^{\mathbb{N}}$, and the Cantor space $2^{\mathbb{N}}$ together with the Borel σ-algebras arising from their usual topologies.

Let $(X, \mathcal{B})$ and $(Y, \mathcal{B}')$ be standard Borel spaces. A map $f : X \to Y$ is called *Borel* if $f^{-1}[B] \in \mathcal{B}$ for all $B \in \mathcal{B}'$. Equivalently, f is Borel if and only if its graph is a Borel subset of $X \times Y$, where the product $X \times Y$ is endowed with the product σ-algebra. We note that endowing $X \times Y$ with the product σ-algebra is equivalent to endowing it with the Borel σ-algebra of the product topology $\tau \times \tau'$ for some Polish topologies τ and τ' generating $\mathcal{B}$ and $\mathcal{B}'$ respectively.

Two standard Borel spaces $(X, \mathcal{B})$ and $(Y, \mathcal{B}')$ are said to be *(Borel) isomorphic* if there exists a bijection $f : X \to Y$ such that both f and f^{-1} are Borel. The following is a classical result of Kuratowski [Kec95, Theorem 15.6].

Theorem 1.1.1 (The Borel Isomorphism Theorem)**.** *Any two uncountable standard Borel spaces are isomorphic.*

A subset of a Polish space with the subspace topology is usually not Polish. In fact, it is well-known that those subsets that form Polish spaces with the subspace topology are exactly the G_δ subsets. In contrast to this, *every* Borel subset of a standard Borel space can be regarded as a standard Borel space in its own right. More precisely, if $A \subseteq X$ is a Borel subset of a standard Borel space $(X, \mathcal{B})$, then $(A, \mathcal{B} \upharpoonright A)$ is also a standard Borel space where

$$\mathcal{B} \upharpoonright A = \{A \cap B : B \in \mathcal{B}\}$$

For general background in descriptive set theory, we refer the reader to [Kec95]. From now on, while denoting a standard Borel space by $(X, \mathcal{B})$, we shall usually drop the collection of measurable sets $\mathcal{B}$ and refer to X as a standard Borel space if the standard Borel structure is understood from the context.

1.2 Classification problems and definable equivalence relations

Under appropriate coding and identification, various collections of mathematical structures can be naturally regarded as standard Borel spaces; and it turns out that many classification problems on these structures can be regarded as definable equivalence relations on the corresponding standard Borel spaces.

For instance, consider the problem of classifying countable graphs up to graph isomorphism. Since we only wish to consider countable graphs up to isomorphism, we may assume without loss of generality that the underlying vertex sets of the graphs we will classify are $\mathbb{N}$. Identifying each graph $(\mathbb{N}, E)$ with the characteristic function of its symmetric irreflexive edge relation $E \subseteq \mathbb{N} \times \mathbb{N}$, we can regard each countable graph as an element of the Polish space $2^{\mathbb{N}\times\mathbb{N}}$. Conversely, each element of $2^{\mathbb{N}\times\mathbb{N}}$ which is the characteristic function of a symmetric irreflexive binary relation on $\mathbb{N}$ can be regarded as the edge relation of some countable graph. It is easily checked that the subset

$$\{\chi_E \in 2^{\mathbb{N}\times\mathbb{N}} : E \text{ is symmetric and irreflexive}\}$$

is a Borel subset of $2^{\mathbb{N}\times\mathbb{N}}$ and hence is a standard Borel space. This example is a special case of a more general construction.

Let $\mathcal{L}$ be a countable language. Since constants and functions may be interpreted as relations, we may suppose without loss of generality that $\mathcal{L} = \{R_i\}_{i\in I}$ where each R_i is an n_i-ary relation symbol. Then the Polish space

$$\mathrm{Mod}_{\mathcal{L}} := \prod_{i\in I} 2^{\mathbb{N}^{n_i}}$$

may be viewed as the space of $\mathcal{L}$-structures with underlying universe $\mathbb{N}$. More specifically, each element $x \in \prod_{i\in I} 2^{\mathbb{N}^{n_i}}$ codes the $\mathcal{L}$-structure $\mathcal{M}_x = (\mathbb{N}, \{R_i^{\mathcal{M}_x}\}_{i\in I})$ where

$$R_i^{\mathcal{M}_x}(k_1, \ldots, k_{n_i}) \Leftrightarrow x_i(k_1, \ldots, k_{n_i}) = 1$$

for every $i \in I$ and $(k_1, \ldots, k_{n_i}) \in \mathbb{N}^{n_i}$. It is routine to check that for any sentence φ in the infinitary logic $\mathcal{L}_{\omega_1,\omega}$, the set

$$\mathrm{Mod}_{\mathcal{L}}(\varphi) = \{x \in \mathrm{Mod}_{\mathcal{L}} : \mathcal{M}_x \models \varphi\}$$

is an isomorphism-invariant Borel subset of $\mathrm{Mod}_{\mathcal{L}}$ and thus is a standard Borel space. Moreover, the isomorphism relation $\cong_\varphi$ on $\mathrm{Mod}_{\mathcal{L}}(\varphi)$ is the orbit equivalence relation of the *logic action* of the infinite symmetric group S_∞ defined by

$$\pi \cdot x = y \Leftrightarrow R_i^{\mathcal{M}_y}(k_1, \ldots, k_{n_i}) = R_i^{\mathcal{M}_x}(\pi^{-1}(k_1), \ldots, \pi^{-1}(k_{n_i}))$$

for any $\pi \in S_\infty$, $i \in I$ and $(k_1, \ldots, k_{n_i}) \in \mathbb{N}^{n_i}$. It is easily checked that this action is continuous with respect to the Polish topology on S_∞ induced from the topology of the Baire space $\mathbb{N}^\mathbb{N}$, which turns S_∞ into a Polish group. It follows that $\cong_\varphi$ is an analytic subset of the product space $\mathrm{Mod}_\mathcal{L}(\varphi) \times \mathrm{Mod}_\mathcal{L}(\varphi)$.

Although this construction provides a plethora of examples, it is not the only way to regard naturally occurring classification problems as definable equivalence relations on Polish spaces. It turns out that one can perform similar constructions for many other classification problems as long as the objects we wish to classify are determined by a countable amount of data.

1.3 Borel reducibility

In our investigation of the relative complexity of classification problems, we shall use the notion of *Borel reducibility*, introduced by Friedman and Stanley [FS89] to measure the relative complexity of the corresponding definable equivalence relations.

An equivalence relation $E \subseteq X \times X$ on a standard Borel space X is called a *Borel equivalence relation* (respectively, an *analytic equivalence relation*) if it is a Borel subset (respectively, an analytic subset) of $X \times X$. Given two analytic equivalence relations E and F on standard Borel spaces X and Y respectively, a Borel map $f : X \to Y$ is called a *Borel reduction* from E to F if for all $x, y \in X$,

$$x \mathrel{E} y \Longleftrightarrow f(x) \mathrel{F} f(y)$$

We say that E is *Borel reducible* to F, written $E \leq_B F$, if there exists a Borel reduction from E to F. Two analytic equivalence relations E and F are said to be *Borel bireducible*, written $E \sim_B F$, if both $E \leq_B F$ and $F \leq_B E$. Clearly $\sim_B$ defines an equivalence relation on the class of analytic equivalence relations. The equivalence class $[E]_{\sim_B}$ will be referred to as the *Borel complexity* of E. Finally, we will write $E <_B F$ if both $E \leq_B F$ and $F \nleq_B E$.

The intuition behind the requirement that reductions be Borel is the following. Assume for the moment that we allow reductions to be arbitrary maps. Then for any equivalence relation E on a standard Borel space X we can construct a reduction from

E to the identity relation $\Delta_{\mathbb{R}}$ using well-orderings of X/E and $\mathbb{R}$. However, having been constructed by the axiom of choice, these reductions would most likely be pathological sets and cannot be described in a reasonable manner.[1] We wish to exclude such maps and thus require that our reductions be Borel.

Intuitively speaking, a Borel reduction from E to F may be regarded as an "explicit" computation which allows us to obtain a set of complete invariants for the classification problem associated with E using a set of complete invariants for the classification problem associated with F. Thus, if E is Borel reducible to F, then the classification problem associated with E is at most as complex as the classification problem associated with F.

Notice that the notion of Borel reducibility can be defined for arbitrary classes of equivalence relations. Nevertheless, the equivalence relations that are relevant to the analysis of most classification problems turn out to be the analytic equivalence relations.

The relation $\leq_B$ defines a quasi-order on the collection of equivalence relations on standard Borel spaces. The quasi-order structures of the analytic and Borel equivalence relations under $\leq_B$ have been well-studied and are interesting in their own right, regardless of their connections with classification problems. In the following sections, we will recall some basic results that are used throughout this thesis. For general background, we refer the reader to [Kan08, Gao09].

1.4 Universal relations

Let $\mathcal{H}$ be a class of equivalence relations. Then an equivalence relation E is said to be *universal* for the class $\mathcal{H}$ if $E \in \mathcal{H}$ and $F \leq_B E$ for every $F \in \mathcal{H}$. Using universal analytic sets, Becker and Kechris [BK96] showed that the class of analytic equivalence relations admits universal elements. Examples of universal analytic equivalence relations include isomorphism of separable Banach spaces [FLR09], isometric bi-embeddability of Polish metric spaces [LR05], and bi-embeddability of countable graphs [LR05].

[1] It is well-known that no well-ordering of $\mathbb{R}$ can be analytic or co-analytic. However, it is consistent with ZFC that there exists a Δ^1_2 well-ordering of $\mathbb{R}$. On the other hand, if sufficiently large cardinals exist, then there does not exist a projective well-ordering of $\mathbb{R}$.

In contrast to analytic equivalence relations, the class of Borel equivalence relations does not admit a universal element. Let E be a Borel equivalence relation on a standard Borel space X. Consider the Borel equivalence relation E^+ on the space $X^{\mathbb{N}}$ defined by

$$xE^+y \Leftrightarrow \{[x_n]_E : n \in \mathbb{N}\} = \{[y_n]_E : n \in \mathbb{N}\}$$

The operation $E \mapsto E^+$ is called the *Friedman-Stanley jump*. That $E \mapsto E^+$ is indeed a jump operation for non-trivial Borel equivalence relations is a result of Friedman and Stanley [FS89].

Theorem 1.4.1 (Friedman-Stanley)**.** *If E is a Borel equivalence relation on a standard Borel space X with more than one equivalence class, then $E <_B E^+$.*

Another class of analytic equivalence relations that admits universal elements is the class of equivalence relations that are classifiable by countable structures. Here, an equivalence relation E on a standard Borel space X is said to be *classifiable by countable structures* if $E \leq_B \cong_\varphi$ for some sentence φ in the infinitary logic $\mathcal{L}_{\omega_1,\omega}$ over a countable relational language $\mathcal{L}$. Since they are Borel reducible to analytic equivalence relations, equivalence relations that are classifiable by countable structures are necessarily analytic. It is worth noting that up to Borel bireducibility, these are *exactly* the orbit equivalence relations of Borel actions of closed subgroups of S_∞ on standard Borel spaces [BK96].

An equivalence relation E is said to be *Borel complete* if it is universal for the class of equivalence relations that are classifiable by countable structures. Although the existence of Borel complete relations does not immediately follow from the definition, many natural classification problems have been proven to be Borel complete. Examples of Borel complete relations include isometry of Polish ultrametric spaces [GK03], and the isomorphism relations on countable groups [Mek81], countable linear orders [FS89], and countable Boolean algebras [CG01].

1.5 The structure of Borel equivalence relations under $\leq_B$ at low levels

In this section, we will focus our attention on the low levels of the $\leq_B$-hierarchy of Borel equivalence relations. Let Δ_X denote the identity relation on the standard Borel space X. Up to Borel bireducibility, the $\leq_B$-hierarchy of the Borel equivalence relations starts with the initial segment

$$\Delta_1 <_B \Delta_2 <_B \cdots <_B \Delta_{\mathbb{N}}$$

consisting of the Borel equivalence relations with at most countably many equivalence classes. An example of a naturally occuring classification problem with Borel complexity $\Delta_{\mathbb{N}}$ is the isomorphism relation on finitely generated abelian groups.

The following remarkable theorem, which is a special case of a more general result of Silver [Sil80] predating the subject, shows that every Borel equivalence relation with uncountably many classes has perfectly many classes and that $\Delta_{\mathbb{R}}$ is the immediate $\leq_B$-successor to $\Delta_{\mathbb{N}}$.

Theorem 1.5.1 (Silver dichotomy)**.** *Let E be a Borel equivalence relation on a standard Borel space X. Then either $E \leq_B \Delta_{\mathbb{N}}$ or $\Delta_{\mathbb{R}} \leq_B E$.*

A Borel equivalence relation E is said to be *smooth* (or *concretely classifiable*) if $E \leq_B \Delta_X$ for some (equivalently, every) uncountable standard Borel space X. For example, if a Borel equivalence relation E on a standard Borel space X admits a *Borel transversal*, i.e. a Borel subset $T \subseteq X$ which intersects every E-class exactly at a single point, then E is smooth. (The converse does not hold for arbitrary Borel equivalence relations. However, it holds for certain natural classes of Borel equivalence relations such as the orbit equivalence relations of Borel actions of Polish groups [Gao09, Corollary 5.4.12].)

A Borel equivalence relation E is said to be *finite* if all E-classes are finite. It is not difficult to check that finite Borel equivalence relations admit Borel transversals and hence are smooth [Kan08, Proposition 7.2.1].

Examples of smooth Borel equivalence relations among classification problems include isomorphism of countable divisible abelian groups [TV99, Example 1], isomorphism of Bernoulli shifts [Orn70], isomorphism of finitely splitting rooted trees on $\mathbb{N}$ [Gao09, Theorem 13.2.3], and isometric classification of compact metric spaces [Gao09, Theorem 14.2.1].

Not all Borel equivalence relations are smooth. For example, the Borel equivalence relation E_0 on $2^{\mathbb{N}}$ defined by

$$xE_0y \ \Leftrightarrow \exists m \forall n \geq m \ x_n = y_n$$

is not smooth [Gao09, Proposition 6.1.7]. Examples of classification problems with Borel complexity E_0 include isomorphism of torsion-free abelian groups of rank 1 [TV99, Example 2] and isometric classification of Heine-Borel ultrametric spaces [GK03].

Quite remarkably, E_0 turns out to be the immediate $\leq_B$-successor of $\Delta_{\mathbb{R}}$. Harrington, Kechris, and Louveau [HKL90] proved the following result, generalizing an earlier result of Glimm and Effros.

Theorem 1.5.2 (Harrington-Kechris-Louveau dichotomy)**.** *Let E be a Borel equivalence relation on a standard Borel space X. Then either $E \leq_B \Delta_{\mathbb{R}}$ or $E_0 \leq_B E$.*

Hence, up to Borel bireducibility, we have the following initial segment

$$\Delta_1 <_B \Delta_2 <_B \cdots <_B \Delta_{\mathbb{N}} <_B \Delta_{\mathbb{R}} <_B E_0$$

of the $\leq_B$-hierarchy of Borel equivalence relations. Unfortunately, the linearity breaks down at E_0, as we shall see later.

1.5.1 Countable Borel equivalence relations

A Borel equivalence relation E is said to be *countable* if all E-classes are countable. Among the Borel equivalence relations, the class of countable Borel equivalence relations is of particular interest and is perhaps the most studied. In this section, we will recall some basic results from the theory of countable Borel equivalence relations. For general background, we refer the reader to [JKL02].

In practice, most countable Borel equivalence relations appear as the orbit equivalence relations of Borel actions of countable groups. For example, E_0 can be realized as the orbit equivalence relation of the bit-wise addition action of the group $\bigoplus_{i \in \mathbb{N}} \mathbb{Z}_2$ on $2^{\mathbb{N}}$, where $\mathbb{Z}_2$ denotes the cyclic group of order 2. Let G be a countable discrete group with a Borel action $\Gamma : G \times X \to X$ on a standard Borel space X. Then the orbit equivalence relation E_G^X given by

$$x \; E_G^X \; y \Leftrightarrow \exists g \in G \; g \cdot x = y$$

is a countable Borel equivalence relation. The following remarkable theorem of Feldman and Moore [FM77] shows that *every* countable Borel equivalence relation arises in this fashion.

Theorem 1.5.3 (Feldman-Moore)**.** *Let E be a countable Borel equivalence relation on a standard Borel space X. Then there exist a countable discrete group G and a Borel action $\Gamma : G \times X \to X$ such that $E = E_G^X$.*

As a corollary to the Feldman-Moore theorem, Dougherty, Jackson, and Kechris [DJK94] proved that the class of countable Borel equivalence relations admits a universal element.

Theorem 1.5.4 (Dougherty-Jackson-Kechris)**.** *There exists a countable Borel equivalence relation E such that $F \leq_B E$ for all countable Borel equivalence relations F.*

More specifically, consider the orbit equivalence relation E_∞ of the shift action of the free group $\mathbb{F}_2$ on two generators on $2^{\mathbb{F}_2}$ given by

$$(\gamma \cdot f)(\alpha) = f(\gamma^{-1}\alpha)$$

for all $\gamma, \alpha \in \mathbb{F}_2$ and $f \in 2^{\mathbb{F}_2}$. Then E_∞ is a universal countable Borel equivalence relation. Examples of classification problems with Borel complexity E_∞ include conformal equivalence of Riemann surfaces [HK00], isomorphism of countable locally finite trees [JKL02], isomorphism of finitely generated groups [TV99], and arithmetic equivalence of subsets of natural numbers [MSS16].

It is well-known that $E_0 <_B E_\infty$ [Gao09, Theorem 7.4.10]. One may ask whether or not there are intermediate countable Borel equivalence relations strictly between E_0 and E_∞. Adams and Kechris [AK00] proved the following theorem, which shows that there are uncountably many $\leq_B$-incomparable countable Borel equivalence relations and that the quasi-order of countable Borel equivalence relations under $\leq_B$ is extremely complicated.

Theorem 1.5.5 (Adams-Kechris)**.** *There exists a map $A \mapsto E_A$ that assigns a countable Borel equivalence relation E_A to each Borel subset of $2^{\mathbb{N}}$ such that*

$$E_A \leq_B E_B \Leftrightarrow A \subseteq B$$

1.5.2 Hyperfinite and hypersmooth Borel equivalence relations

A Borel equivalence relation E on a standard Borel space X is said to be *hyperfinite* (respectively, *hypersmooth*) if there exists an increasing sequence $F_0 \subseteq F_1 \subseteq \cdots$ of finite (respectively, smooth) Borel equivalence relations such that $E = \bigcup_{i \in \mathbb{N}} F_i$. Clearly every hyperfinite Borel equivalence relation is hypersmooth. We shall see in a moment that the converse also holds for countable Borel equivalence relations.

It is easily seen that E_0 is hyperfinite. Moreover, E_0 is universal for the class of hyperfinite Borel equivalence relations. Dougherty, Jackson, and Kechris [DJK94] proved the following theorem based on earlier work by Weiss [Wei84] and Slaman-Steel [SS88].

Theorem 1.5.6 (Dougherty-Jackson-Kechris)**.** *Let E be a countable Borel equivalence relation on a standard Borel space X. Then the following are equivalent.*

- *E is hyperfinite.*
- *E is hypersmooth.*
- *$E \leq_B E_0$.*
- *There exists a Borel action of $\mathbb{Z}$ on X such that $E = E_{\mathbb{Z}}^X$.*
- *There exists a Borel assignment $C \mapsto <_C$ giving for each equivalence class C a linear order $<_C$ of C of order type finite or $\mathbb{Z}$. (Here that $C \mapsto <_C$ is Borel means that the relation $R(x, y, z) \Leftrightarrow x <_{[y]_E} z$ is Borel.)*

For an example of a hypersmooth Borel equivalence relation which is not countable, consider the Borel equivalence relation E_1 on $(2^{\mathbb{N}})^{\mathbb{N}}$ defined by

$$x \; E_1 \; y \Leftrightarrow \exists m \forall n \geq m \; x_n = y_n$$

Then it is easily checked that E_1 is hypersmooth. Moreover, E_1 is universal for the class of hypersmooth Borel equivalence relations [KL97, Proposition 1.3].

That a Borel equivalence relation is not countable does not exclude the possibility that it may be *essentially countable*, i.e. Borel reducible to a countable Borel equivalence relation. However, E_1 is known to be not essentially countable. Indeed, we have the stronger result that E_1 is not Borel reducible to any orbit equivalence relation of a Borel action of a Polish group [KL97]. Moreover, the following remarkable theorem of Kechris and Louveau [KL97] shows that there are no $\leq_B$-intermediate Borel equivalence relations strictly between E_0 and E_1.

Theorem 1.5.7 (Kechris-Louveau dichotomy)**.** *Let E be a Borel equivalence relation on a standard Borel space X such that $E \leq E_1$. Then either $E \leq_B E_0$ or $E \sim_B E_1$.*

Combined with the previous results, this theorem completely determines the structure of hypersmooth Borel equivalence relations.

1.5.3 The Borel equivalence relation $\Delta_{\mathbb{R}}^+$

We will now turn our attention to another direction in the $\leq_B$-hierarchy. Recall that $\Delta_{\mathbb{R}}^+$ denotes the Friedman-Stanley jump of the identity relation $\Delta_{\mathbb{R}}$. An example of a classification problem with Borel complexity $\Delta_{\mathbb{R}}^+$ is the conjugacy relation on ergodic measure preserving transformations with discrete spectrum [For00, Theorem 65].

By the Borel isomorphism theorem, $\Delta_{\mathbb{R}}^+$ is Borel bireducible with Δ_X^+ for any uncountable standard Borel space X. It follows from the Feldman-Moore theorem that for any countable Borel equivalence relation E we have that $E \leq_B \Delta_{\mathbb{R}}^+$. To see this, let E be a countable Borel equivalence relation on a standard Borel space X and let G be a countable discrete group with a Borel action on X such that $E = E_G^X$. Fix an enumeration $\{g_i\}_{i\in\mathbb{N}}$ of the group G. Then it is easily checked that the map $x \mapsto (g_i \cdot x)_{i\in\mathbb{N}}$ is a Borel reduction from E to Δ_X^+. On the other hand, it is well-known that $\Delta_{\mathbb{R}}^+$ is not essentially countable [Kan08, Theorem 17.1.3, Section 17.2]. Thus we have that $E_\infty <_B \Delta_{\mathbb{R}}^+$.

1.6 Background from topological dynamics

Topological dynamics in the broadest sense is the study of the behavior of pairs (X, G, Λ) where X is a topological space, G is a topological semigroup and $\Lambda : G \times X \to X$ is a continuous action. From now on, any such pair will be referred to as a *flow*.

In this thesis, we shall only be concerned with those flows where the *phase space* X is compact metrizable and G is the additive group of integers $\mathbb{Z}$ endowed with the discrete topology, in which case we can replace (X, G, Λ) by (X, φ) where $\varphi : X \to X$ is the homeomorphism defined by the action of the generator $1 \in \mathbb{Z}$. We remark that the notions that will be introduced in this section are defined in many sources for pairs (X, φ) where φ is an arbitrary continuous map on X. That being said, we shall take the liberty of restricting all relevant definitions to the case where φ is a homeomorphism.

A *topological dynamical system* is a pair (X, φ) where X is a compact metrizable topological space and $\varphi : X \to X$ is a homeomorphism. If (X, φ) and (Y, ψ) are topological dynamical systems, then (Y, ψ) is called a *factor* of (X, φ) if there exists a continuous surjection $\pi : X \to Y$ such that

$$\pi \circ \varphi = \psi \circ \pi$$

If the factor map $\pi : X \to Y$ is a homeomorphism, then (X, φ) and (Y, ψ) are said to be *topologically conjugate* and π is called a *topological conjugacy*. Similarly, we define the class of *pointed topological dynamical systems* as the class of triples of the form (X, φ, x) where (X, φ) is a topological dynamical system and $x \in X$. Two pointed systems (X, φ, x) and (Y, ψ, y) are said to be *(pointed) topologically conjugate* if there exists a topological conjugacy $\pi : X \to Y$ between (X, φ) and (Y, ψ) such that $\pi(x) = y$.

A topological dynamical system (X, φ) is said to be *equicontinuous* if the family of functions $\{\varphi^i : i \in \mathbb{Z}\}$ is uniformly equicontinuous. It is well-known that every topological dynamical system (X, φ) admits a *maximal equicontinuous factor* (Y, ψ) in the sense that (Y, ψ) is an equicontinuous factor of (X, φ) and every equicontinuous factor of (X, φ) is a factor of (Y, ψ) through a factor map which makes the corresponding diagram commute. Moreover, the maximal equicontinuous factor of a topological dynamical system is unique up to topological conjugacy [Kůr03, Theorem 2.44].

Given a topological dynamical system (X, φ), a subset $Y \subseteq X$ is said to be *φ-invariant* if $\varphi[Y] = Y$. For notational convenience, we will write *invariant* if the map φ is clear from the context. A *subsystem* of (X, φ) is a topological dynamical system of the form (Y, φ) where Y is a non-empty closed invariant subset of X.

1.6.1 Minimality

The "irreducible" objects among topological dynamical systems are the minimal dynamical systems, where a topological dynamical system (X, φ) is said to be *minimal* if (X, φ) has no proper subsystems. Equivalently, (X, φ) is minimal if for every invariant closed subset $Y \subseteq X$ we have either $Y = \emptyset$ or $Y = X$.

Given a point $x \in X$ in a topological dynamical system (X, φ), the *orbit* of x under φ is defined to be the set

$$\mathrm{Orb}(x) = \{\varphi^i(x) : i \in \mathbb{Z}\}$$

For any $U \subseteq X$, the set of *return times* of x to the subset U is the set

$$Ret_U(X, \varphi, x) := \{i \in \mathbb{Z} : \varphi^i(x) \in U\}$$

The point $x \in X$ is said to be an *almost periodic point* of (X, φ) if for every open neighborhood U of x, the set of return times $Ret_U(X, \varphi, x)$ is *syndetic*, i.e. there exists an integer $k \geq 1$ such that $Ret_U(X, \varphi, x) \cap \{i, i+1, \ldots, i+k\} \neq \emptyset$ for all $i \in \mathbb{Z}$. Minimality has various equivalent characterizations. For completeness, we include the proof of the equivalences of these characterizations.

Theorem 1.6.1. *[Kůr03] Let (X, φ) be a topological dynamical system. Then the following are equivalent.*

a. (X, φ) is minimal.

b. For all $x \in X$, the orbit $Orb(x)$ is dense in X.

c. For all non-empty open $U \subseteq X$, there exists $k \in \mathbb{N}^+$ such that $\bigcup_{i=-k}^{k} \varphi^i[U] = X$.

d. For all $x \in X$ and for all non-empty open $U \subseteq X$, the set $Ret_U(X, \varphi, x)$ is syndetic.

e. For some $x \in X$, the orbit $Orb(x)$ is dense in X and x is an almost periodic point.

Proof. [a $\Rightarrow$ b]: Assume that (X, φ) is minimal. For every $x \in X$, the orbit closure $\overline{\mathrm{Orb}}(x)$ is a closed non-empty invariant subset and hence equals X by minimality.

[b $\Rightarrow$ a]: Assume that every orbit is dense. Let $Y \subseteq X$ be a closed non-empty invariant set. Then $\overline{\mathrm{Orb}}(x) \subseteq Y$ for any $x \in Y$. By the assumption, this implies that $Y = X$.

[a $\Rightarrow$ c]: Assume that (X, φ) is minimal. Let $U \subseteq X$ be a non-empty open set. Then $Y = X \setminus \bigcup_{i=-\infty}^{\infty} \varphi^i[U]$ is a proper closed invariant subset. By minimality, $Y = \emptyset$ and hence $\bigcup_{i=-\infty}^{\infty} \varphi^i[U]$ is an open cover of X. By compactness, there exists $k \in \mathbb{N}^+$ such that $\bigcup_{i=-k}^{k} \varphi^i[U] = X$.

[c $\Rightarrow$ d]: Assume (c) and let $x \in X$ and $U \subseteq X$ be a non-empty open set. By the assumption, there exists $k \in \mathbb{N}^+$ such that $\bigcup_{i=-k}^{+k} \varphi^i[U] = X$. But this means that for all $i \in \mathbb{Z}$ there exists $-k \leq p \leq k$ such that $\varphi^{i-p}(x) \in U$. Equivalently, $Ret_U(X, \varphi, x)$ is syndetic with gaps bounded by at most $2k$.

[d $\Rightarrow$ e]: Assume (d) and let $x \in X$ be any point. Then for any non-empty open set U and non-empty open neighborhood V of x, $Ret_U(X, \varphi, x)$ is non-empty and $Ret_V(X, \varphi, x)$ is syndetic.

[e $\Rightarrow$ a]: Assume towards a contradiction that (X, φ) is not minimal and that X is the closure of the orbit of some almost periodic point $x \in X$. Let $\emptyset \subsetneq W \subsetneq X$ be a proper non-empty closed invariant subset of X. If $x \in W$, then $\overline{\mathrm{Orb}}(x) \subseteq W$ since W is closed and invariant. But this contradicts the assumption that $\overline{\mathrm{Orb}}(x) = X$. Thus $x \notin W$. It follows that there exists a closed set V such that $x \in Int(V)$ and $V \cap W = \emptyset$. By almost periodicity, there exists $k \in \mathbb{N}^+$ such that for any $i \in \mathbb{Z}$ there exists $0 \leq p \leq k$ with $\varphi^{i+p}(x) \in V$. Since W is invariant, the closed set $\bigcup_{i=0}^{k} \varphi^{-i}[V]$ does not intersect W. Thus $X \setminus \bigcup_{i=0}^{k} \varphi^{-i}[V]$ is a non-empty open set. Since the orbit $\mathrm{Orb}(x)$ is dense by assumption, there exists $n \in \mathbb{Z}$ such that $\varphi^n(x) \in X \setminus \bigcup_{i=0}^{k} \varphi^{-i}[V]$. But this implies that $\varphi^{n+p}(x) \notin V$ for any $0 \leq p \leq k$, which is a contradiction. $\square$

1.6.2 Cantor dynamical systems

Recall that a topological space is said to be a *Cantor space* if it is perfect, compact, totally disconnected and metrizable. It is well-known that there exists a unique Cantor space up to homeomorphism. From now on, when we say *the* Cantor space, we mean the topological space $2^{\mathbb{N}}$ with its usual product topology.

A *Cantor dynamical system* is a topological dynamical system (X, φ) where X is a Cantor space. We will refer to minimal Cantor dynamical systems as *Cantor minimal systems*.

Example 1.6.2 (Odometers)**.** Let $(u_i)_{i\in\mathbb{N}}$ be a sequence of natural numbers such that $(u_i)_{i\in\mathbb{N}}$ is not eventually constant, $u_i > 1$ and $u_i|u_{i+1}$ for all $i \in \mathbb{N}$. Consider the sequence of canonical group homomorphisms

$$\mathbb{Z}_{u_0} \longleftarrow \mathbb{Z}_{u_1} \longleftarrow \mathbb{Z}_{u_2} \cdots$$

where $\mathbb{Z}_{u_i}$ denotes the cyclic group of order u_i. Let $\mathtt{Odo}((u_i)_{i\in\mathbb{N}})$ be the inverse limit group

$$\mathtt{Odo}((u_i)_{i\in\mathbb{N}}) := \varprojlim \mathbb{Z}_{u_i} = \{(m_i) \in \prod_{i\in\mathbb{N}} \mathbb{Z}_{u_i} : m_j \equiv m_i \ (mod\ u_i) \text{ for } j > i\}$$

with the induced topology. Then the pair $(\mathtt{Odo}((u_i)_{i\in\mathbb{N}}), \eta)$ is a Cantor dynamical system, where $\eta(h) = h + \hat{1}$ and $\hat{1} = (1, 1, 1, \dots)$. Moreover, it is easily checked that $\hat{0} = (0, 0, \dots)$ is an almost periodic point with dense orbit. Hence, $(\mathtt{Odo}((u_i)_{i\in\mathbb{N}}), \eta)$ is a Cantor minimal system by Theorem 1.6.1. The Cantor minimal system $(\mathtt{Odo}((u_i)_{i\in\mathbb{N}}), \eta)$ is called the *odometer* (or the *adding machine*) associated with $(u_i)_{i\in\mathbb{N}}$.

The classification problem for odometers is central to some proofs in this thesis. We will next recall some basic results regarding the classification of odometers up to topological conjugacy. For a detailed survey of odometers, we refer the reader to [Dow05].

A *supernatural number* is a formal product $\prod_{i\in\mathbb{N}^+} \mathrm{p}_i^{k_i}$ where p_i is the i-th prime number and $k_i \in \mathbb{N} \cup \{\infty\}$ for all $i \in \mathbb{N}$. For each sequence $(u_i)_{i\in\mathbb{N}}$ of positive integers, define $\mathrm{lcm}(u_i)_{i\in\mathbb{N}}$ to be the supernatural number $\mathbf{u} = \prod_{i\in\mathbb{N}^+} \mathrm{p}_i^{k_i}$ where

$$k_i = sup\{j \in \mathbb{N} : \exists m \in \mathbb{N}\ \mathrm{p}_i^j | u_m\}$$

Given a supernatural number $\mathbf{u}$, any sequence $(u_i)_{i\in\mathbb{N}}$ of positive integers such that $\mathrm{lcm}(u_i)_{i\in\mathbb{N}} = \mathbf{u}$ and $u_i | u_{i+1}$ for all $i \in \mathbb{N}$ will be called a *factorization* of $\mathbf{u}$; and any positive integer q dividing some u_i will be called a *factor* of $\mathbf{u}$.

It turns out that the set of supernatural numbers is a complete set of invariants for topological conjugacy of odometers [BS95] and hence the topological conjugacy problem for odometers is smooth.

Theorem 1.6.3. *The odometers* $(\mathtt{Odo}((u_i)_{i\in\mathbb{N}}), \eta)$ *and* $(\mathtt{Odo}((v_i)_{i\in\mathbb{N}}), \eta)$ *are topologically conjugate if and only if* $\mathrm{lcm}(u_i)_{i\in\mathbb{N}} = \mathrm{lcm}(v_i)_{i\in\mathbb{N}}$.

1.6.3 Properly ordered Bratteli diagrams

It is well-known that pointed Cantor minimal systems can be represented by infinite directed multigraphs known as properly ordered Bratteli diagrams. In this section, following [HPS92, Dur10], we shall give a brief overview of the correspondence between properly ordered Bratteli diagrams and pointed Cantor minimal systems.

An *unordered Bratteli diagram* (or simply, a *Bratteli diagram*) is a pair (V, E) consisting of a vertex set V and an edge set E which can be partitioned into non-empty finite sets $V = \bigsqcup_{n=0}^{\infty} V_n$ and $E = \bigsqcup_{n=1}^{\infty} E_n$ such that the following conditions hold:

1. $V_0 = \{v_0\}$ is a singleton.

2. There exist a range map $r : E \to V$ and a source map $s : E \to V$ such that $r[E_n] \subseteq V_n$ and $s[E_n] \subseteq V_{n-1}$ for all $n \in \mathbb{N}^+$. Moreover, $s^{-1}(v) \neq \emptyset$ for all $v \in V$ and $r^{-1}(v) \neq \emptyset$ for all $v \in V - V_0$.

Bratteli diagrams are often given diagrammatic representations as directed graphs consisting of the vertices V_n at (horizontal) level n and the edges E_n connecting the vertices at level $n-1$ with the vertices at level n. (For an example, see Figure 1, where the orientation is taken to be in the downward direction.)

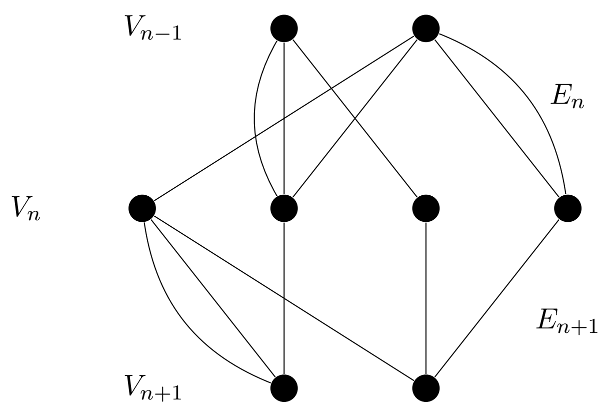

Figure 1

If we fix a linear order on V_n for each $n \in \mathbb{N}$, then the edge set E_n determines a $|V_n| \times |V_{n-1}|$ incidence matrix $M_n = (m_{ij})$ defined by

$$m_{ij} = |\{e \in E_n : r(e) = u_i \ \wedge \ s(e) = w_j\}|$$

where u_i is the i-th vertex in V_n and w_j is the j-th vertex in V_{n-1}. For example, if we order the vertices at each level in Figure 1 from left to right, then the corresponding incidence matrices M_n and M_{n+1} are

$$M_n = \begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } M_{n+1} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Given a Bratteli diagram (V, E) and $k, l \in \mathbb{N}$ with $k < l$, define $E_{k+1} \circ ... \circ E_l$ to be the set of paths from V_k to V_l. More specifically, $E_{k+1} \circ ... \circ E_l$ is the set

$$\{(e_{k+1}, ..., e_l) : r(e_i) = s(e_{i+1}) \ i = k+1, ..., l-1 \ \wedge \ e_i \in E_i \ i = k+1, ..., l\}$$

The corresponding range and source maps are defined by $r(e_{k+1}, ..., e_l) := r(e_l)$ and $s(e_{k+1}, ..., e_l) := s(e_{k+1})$ respectively. Observe that the product matrix $M_l \cdot ... \cdot M_{k+1}$ is the incidence matrix of the edge set $E_{k+1} \circ ... \circ E_l$.

For any sequence $0 = m_0 < m_1 < m_2 < ...$ of natural numbers, we define the *telescoping* of (V, E) with respect to $(m_i)_{i \in \mathbb{N}}$ to be the Bratteli diagram (V', E') where

$V'_n = V_{m_n}$, $E'_n = E_{m_{n-1}+1} \circ ... \circ E_{m_n}$ and the range and source maps are defined as above. For example, if we telescope the diagram in Figure 1 to the levels $n-1$ and $n+1$, then we get the diagram in Figure 2.

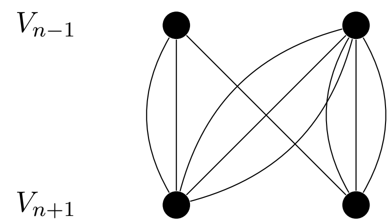

Figure 2

A Bratteli diagram (V, E) is called *simple* if there exists a telescoping (V', E') of (V, E) such that all the incidence matrices of (V', E') have only non-zero entries, i.e. every vertex of (V', E') at any level is connected to every vertex at the next level. It is easily checked that (V, E) is simple if and only if for every $n \in \mathbb{N}$ there exists an integer $m > n$ such that there is a path from each vertex in V_n to each vertex in V_m.

Two Bratteli diagrams (V, E) and (V', E') are said to be *isomorphic* if there exist bijections $f : V \to V'$ and $g : E \to E'$ which preserve the gradings and intertwine the respective source and range maps, i.e. $s' \circ g = f \circ s$ and $r' \circ g = f \circ r$. From now on, the equivalence relation on Bratteli diagrams generated by isomorphism and telescoping will be denoted by $\sim$.

An *ordered Bratteli diagram* is a triple of the form $(V, E, \preccurlyeq)$ where (V, E) is a Bratteli diagram and $\preccurlyeq$ is a partial order on E such that for all edges $e, e' \in E$, e and e' are $\preccurlyeq$-comparable if and only if $r(e) = r(e')$. Let $B = (V, E, \preccurlyeq)$ be an ordered Bratteli diagram. We define the *Bratteli compactum* associated with $B = (V, E, \preccurlyeq)$ to be the space of infinite paths

$$X_B = \{(e_i)_{i \in \mathbb{N}^+} : \forall i \in \mathbb{N}^+ \; e_i \in E_i \;\wedge\; r(e_i) = s(e_{i+1})\}$$

endowed with the topology generated by the basic clopen sets of the form

$$[e_1, e_2, \ldots, e_k]_B = \{(f_i)_{i \in \mathbb{N}^+} \in X_B : (\forall 1 \le i \le k) \; e_i = f_i\}$$

It is straightforward to verify that the metric d_B on X_B defined by

$$d_B((e_i)_{i\in\mathbb{N}^+},(f_i)_{i\in\mathbb{N}^+}) = 2^{-k}$$

where $k = min\{i : e_i \neq f_i\}$ induces the same topology. We remark that the topological space X_B is determined solely by (V, E). Moreover, it is easily checked that if (V, E) is a simple Bratteli diagram and X_B is infinite, then X_B is a Cantor space.

Given an ordered Bratteli diagram $(V, E, \preccurlyeq)$ and $k < l$ in $\mathbb{N}$, the set of paths $E_{k+1} \circ \cdots \circ E_l$ from V_k to V_l can be given an induced lexicographic order defined by

$$(f_{k+1}, f_{k+2}, \ldots, f_l) \prec (e_{k+1}, e_{k+2}, \ldots, e_l)$$

if and only if for some i with $k + 1 \leq i \leq l$ we have $f_i \prec e_i$ and $f_j = e_j$ for all $i < j \leq l$. One readily checks that if $(V, E, \preccurlyeq)$ is an ordered Bratteli diagram, (V', E') is a telescoping of (V, E), and $\preccurlyeq'$ is the corresponding lexicographic order, then $(V', E', \preccurlyeq')$ is an ordered Bratteli diagram. In this case, $(V', E', \preccurlyeq')$ is called a telescoping of $(V, E, \preccurlyeq)$. Two ordered Bratteli diagrams are said to be isomorphic if and only if there is an isomorphism of underlying unordered Bratteli diagrams which respects the partial order structure on edges. Let $\approx$ denote the equivalence relation on ordered Bratteli diagrams generated by telescoping and isomorphism.

Given an ordered Bratteli diagram $(V, E, \preccurlyeq)$, let E_{max} and E_{min} denote the sets of maximal and minimal elements of E respectively. $(V, E, \preccurlyeq)$ is said to be *properly ordered* if

- X_B is infinite.
- (V, E) is a simple Bratteli diagram.
- There exists a unique path $x_{min} = (e_i)_{i\in\mathbb{N}^+}$ such that $e_i \in E_{min}$ for all $i \in \mathbb{N}^+$ and there exists a unique path $x_{max} = (f_i)_{i\in\mathbb{N}^+}$ such that $f_i \in E_{max}$ for all $i \in \mathbb{N}^+$.

In this case, x_{min} and x_{max} are called the minimal and maximal paths respectively. (We remark that some authors require the space X_B of infinite paths to be infinite as a part of the definition of an ordered Bratteli diagram to exclude Bratteli compacta which are finite.)

For every properly ordered Bratteli diagram $B = (V, E, \preccurlyeq)$, we can define a homeomorphism $\lambda_B : X_B \to X_B$, called the *Vershik map*, as follows:

- $\lambda_B(x_{max}) = x_{min}$
- $\lambda_B(e_1, e_2, \ldots, e_k, e_{k+1}, \ldots) = (f_1, f_2, \ldots, f_k, e_{k+1}, \ldots)$ where k is the least integer such that $e_k \notin E_{max}$, f_k is the successor of e_k in E, and $(f_1, f_2, ..., f_{k-1})$ is the unique minimal path in $E_1 \circ E_2 \circ \cdots \circ E_{k-1}$ with range equal to the source of f_k.

It is routine to check that $(X_B, \lambda_B, x_{max})$ is a pointed Cantor minimal system [HPS92, Section 3]. Any such dynamical system is called a *Bratteli-Vershik dynamical system.* It turns out that every Cantor minimal system is topologically conjugate to a Bratteli-Vershik dynamical system.

Theorem 1.6.4. *[HPS92] For any pointed Cantor minimal system (X, φ, x) there exists a properly ordered Bratteli diagram $B = (V, E, \preccurlyeq)$ such that (X, φ, x) is (pointed) topologically conjugate to $(X_B, \lambda_B, x_{max})$. Moreover, if (X_i, φ_i, x_i) corresponds to the properly ordered Bratteli diagram $B^i = (V^i, E^i, \preccurlyeq^i)$ for $i = 0, 1$, then (X_0, φ_0, x_0) is (pointed) topologically conjugate to (X_1, φ_1, x_1) if and only if $B^0 \approx B^1$.*

Given a pointed Cantor minimal system (X, φ, x), any properly ordered Bratteli diagram $B = (V, E, \preccurlyeq)$ such that (X, φ, x) is topologically conjugate to $(X_B, \lambda_B, x_{max})$ will be referred to as a *Bratteli-Vershik representation* of (X, φ, x). In Chapter 7, we will discuss in detail how to construct a Bratteli-Vershik representation of a given pointed Cantor minimal system.

1.6.4 Subshifts

For the purposes of this thesis, an *alphabet* is a finite set with at least two elements. Given an alphabet $\mathfrak{a}$, endow $\mathfrak{a}$ with the discrete topology and consider the topological space $\mathfrak{a}^{\mathbb{Z}}$ together with the left-shift map $\sigma : \mathfrak{a}^{\mathbb{Z}} \to \mathfrak{a}^{\mathbb{Z}}$ defined by

$$(\sigma(\alpha))(i) = \alpha(i + 1)$$

for all $i \in \mathbb{Z}$ and $\alpha \in \mathfrak{a}^{\mathbb{Z}}$. It is easily checked that $(\mathfrak{a}^{\mathbb{Z}}, \sigma)$ is a Cantor dynamical system. Any subsystem (O, σ) of $(\mathfrak{a}^{\mathbb{Z}}, \sigma)$ is called a *subshift* over the alphabet $\mathfrak{a}$. For notational convenience, we shall often drop the left-shift map σ and refer to O as a subshift. Thus, a subshift over an alphabet $\mathfrak{a}$ is simply a closed σ-invariant subset of $\mathfrak{a}^{\mathbb{Z}}$. For any sequence $\alpha \in \mathfrak{a}^{\mathbb{Z}}$, we define the *subshift generated by* α to be the closure of its orbit $\mathrm{Orb}(\alpha)$. The following well-known theorem [LM95, Theorem 6.2.9] characterizes the factor maps between subshifts over an alphabet $\mathfrak{a}$.

Theorem 1.6.5 (Curtis-Hedlund-Lyndon). *Let $X, Y \subseteq \mathfrak{a}^{\mathbb{Z}}$ be subshifts and let*

$$\pi : X \to Y$$

be a continuous map from X to Y commuting with σ. Then there exist $i \in \mathbb{N}$ and a block code, i.e. a function $C : \mathfrak{a}^{2i+1} \to \mathfrak{a}$, such that

$$(\pi(\alpha))(k) = C(\alpha[k-i, k+i])$$

for all $k \in \mathbb{Z}$ and $\alpha \in X$, where $\alpha[k, l]$ denotes the subblock

$$(\alpha(k), \alpha(k+1), \ldots, \alpha(l))$$

of the bi-infinite sequence α.

For any block code $C : \mathfrak{a}^{2i+1} \to \mathfrak{a}$, the natural number i called the *length* of the block code C and is denoted by $|C|$. For any topological conjugacy $\pi : X \to Y$ between subshifts X and Y, we define the *length* of π to be the natural number

$$|\pi| = max\{min\{|C| : C \text{ induces } \pi\}, min\{|C| : C \text{ induces } \pi^{-1}\}\}$$

A subshift $O \subseteq \mathfrak{a}^{\mathbb{Z}}$ is said to be *minimal* if the topological dynamical system (O, σ) is minimal. Being a closed subspace of a Cantor space, any subshift is totally disconnected, compact, and metrizable. If it is also minimal and infinite, then it has no isolated points and hence is a Cantor space itself. Thus, infinite minimal subshifts are Cantor minimal systems. Finite minimal subshifts are obviously classified up to topological conjugacy by their cardinalities. From now on, we shall exclude these trivial cases and assume that minimal subshifts are infinite.

Recently, minimal subshifts have been a particular focus of study due to the structure of their topological full groups. Here the *topological full group* $[[\varphi]]$ of a Cantor dynamical system (X, φ) is the group of homeomorphisms $\psi : X \to X$ such that there exists a continuous function $n : X \to \mathbb{Z}$ with $\psi(x) = \varphi^{n(x)}(x)$ for all $x \in X$. Matui [Mat06] showed that the commutator subgroup of the topological full group of an infinite minimal subshift is finitely generated, infinite and simple; and Juschenko and Monod [JM13] proved that these groups are also amenable. This result provided the first examples of finitely generated simple amenable infinite groups.

It is known by the work of Giordano, Putnam and Skau [GPS99] that the commutator subgroups of the topological full groups of two Cantor minimal systems (X, φ) and (Y, ψ) are isomorphic if and only if (X, φ) and (Y, ψ) are *flip conjugate*, i.e. (X, φ) is topologically conjugate to (Y, ψ) or (Y, ψ^{-1}). Using this fact, Thomas [Tho13] constructed a Borel reduction from flip conjugacy of minimal subshifts over a finite alphabet to isomorphism of finitely generated simple amenable groups and proved that the isomorphism relation on such groups is not smooth. This result provides a strong motivation for studying conjugacy relations on minimal subshifts. If one could show that flip conjugacy of minimal subshifts is a universal countable Borel equivalence relation, then one would obtain that isomorphism of finitely generated simple amenable groups is as complex as it possibly could be, i.e. as complex as isomorphism of *arbitrary* finitely generated groups.

1.7 Precise statements of the main results of this thesis

In the following subsections, we shall recall some earlier results on the Borel complexity of the topological conjugacy relation on various dynamical systems and present the main results of this thesis.

1.7.1 The topological conjugacy relation on minimal subshifts

Among the various classes of minimal dynamical systems that we have introduced, the topological conjugacy relation is most well-studied on the class of subshifts.

Clemens [Cle09] proved that the topological conjugacy relation on subshifts over a finite alphabet is a universal countable Borel equivalence relation. Gao, Jackson, and

Seward [GJS15] analyzed topological conjugacy of *generalized G-subshifts*, i.e. subsystems of the flow $(\mathfrak{a}^G, G, \Lambda)$ where $\mathfrak{a}$ is an alphabet, G is a countably infinite discrete group, and the action Λ of G on $\mathfrak{a}^G$ is given by

$$(g \cdot \alpha)(h) = \alpha(g^{-1}h)$$

for all $g, h \in G$ and $\alpha \in \mathfrak{a}^G$. Two such G-subshifts are said to be topologically conjugate if there exists a homeomorphism between them which commutes with the action of G. Gao, Jackson, and, Seward showed that topological conjugacy of G-subshifts is Borel bireducible with E_0 when G is locally finite; and that topological conjugacy of G-subshifts is a universal countable Borel equivalence relation when G is not locally finite. They also proved that topological conjugacy of *minimal* subshifts over a finite alphabet is not smooth and posed the question of determining the Borel complexity of the topological conjugacy relation on minimal subshifts.

Since then, the project of analyzing the Borel complexity of the topological conjugacy relation for restricted classes of minimal subshifts has been pursued in different directions. For example, Gao and Hill [GH] have shown that topological conjugacy of minimal rank-1 systems is Borel bireducible with E_0. Thomas [Tho13] proved that the topological conjugacy relation is not smooth for the class of *Toeplitz subshifts*, i.e. minimal subshifts that contain bi-infinite sequences in which every subblock appears periodically.

Subsequent to Thomas' result on Toeplitz subshifts, Sabok and Tsankov [ST15] analyzed topological conjugacy of generalized Toeplitz G-subshifts for residually finite groups G over a finite alphabet. They proved that topological conjugacy of generalized Toeplitz G-subshifts is not hyperfinite if G is residually finite and non-amenable; and that topological conjugacy of Toeplitz subshifts with separated holes is 1-amenable in the following sense.

A countable Borel equivalence relation E on a standard Borel space X is said to be 1-*amenable* if there exists a sequence of positive Borel functions $f_n : E \to \mathbb{R}^+$ such that letting $f_n^x(y) = f_n(x, y)$ we have that

- For all $x \in X$, $f_n^x \in \ell_1([x]_E)$ and $\|f_n^x\|_1 = 1$,

- For all $x, y \in X$ such that xEy, we have that $\lim_{n\to\infty} \|f_n^x - f_n^y\|_1 = 0$

It is well-known that hyperfiniteness implies 1-amenability [JKL02, Proposition 2.13] and that 1-amenable relations are hyperfinite μ-almost everywhere for every Borel probability measure μ on X [KM04, Corollary 10.2]. On the other hand, it is still open whether 1-amenability implies hyperfiniteness.

Sabok and Tsankov asked whether or not topological conjugacy of Toeplitz subshifts is hyperfinite. One of the main results of this thesis is the following partial affirmative answer.

Theorem A. *The topological conjugacy relation on Toeplitz subshifts with separated holes over a finite alphabet is hyperfinite.*

Indeed, we will prove that the topological conjugacy relation is hyperfinite on a larger class of Toeplitz subshifts which we shall call Toeplitz subshifts *with growing blocks*. On the other hand, the question of whether topological conjugacy of all Toeplitz subshifts over a finite alphabet is hyperfinite remains open.

1.7.2 The topological conjugacy relation on Cantor minimal systems

The other main results of this thesis concern the topological conjugacy relation on Cantor minimal systems. More specifically, we shall determine the Borel complexity of the topological conjugacy relation on pointed Cantor minimal systems, and provide a lower bound for the Borel complexity of the topological conjugacy relation on Cantor minimal systems.

As far as the author knows, the Borel complexities of these relations have not been previously studied in this generality. On the other hand, the Borel complexity of the topological conjugacy relation on *arbitrary* Cantor dynamical systems has been determined by Camerlo and Gao [CG01].

Since we wish to classify Cantor dynamical systems up to topological conjugacy, we may assume without loss of generality that the underlying topological spaces of the Cantor dynamical systems that we study are $2^{\mathbb{N}}$. Then each Cantor dynamical system $(2^{\mathbb{N}}, \varphi)$ can be identified with the corresponding homeomorphism φ and the Polish group $H(2^{\mathbb{N}})$ consisting of homeomorphisms of the Cantor space $2^{\mathbb{N}}$ can be regarded as the standard Borel space of Cantor dynamical systems. Moreover, the topological conjugacy relation on $H(2^{\mathbb{N}})$ is precisely the orbit equivalence relation of the continuous action of $H(2^{\mathbb{N}})$ on itself by conjugation.

It is well-known that $H(2^{\mathbb{N}})$ is isomorphic to a closed subgroup of the infinite symmetric group S_∞ and thus the topological conjugacy relation on Cantor dynamical systems is classifiable by countable structures. Camerlo and Gao proved that this relation is as complicated as it could possibly be, i.e. it is Borel complete [CG01, Theorem 5]. Unfortunately, the Cantor dynamical systems constructed in their proof are far from minimal and do not give any non-trivial bounds for the Borel complexity of the topological conjugacy relation on Cantor minimal systems. In this thesis, we shall prove the following theorem.

Theorem B. $\Delta^+_{\mathbb{R}}$ *is Borel reducible to the topological conjugacy relation on Cantor minimal systems.*

Theorem B will be obtained as a byproduct of our analysis of topological conjugacy of *pointed* Cantor minimal systems, which is the main focus of this thesis. Using Stone duality, we shall show that the set of countable atomless Boolean subalgebras of $\mathcal{P}(\mathbb{Z})$ which are closed under the map $A \mapsto A - 1$ and whose non-empty elements are syndetic sets is a complete set of invariants for topological conjugacy of pointed Cantor minimal systems. This will enable us to prove the following theorem.

Theorem C. $\Delta^+_{\mathbb{R}}$ *is Borel bireducible with the topological conjugacy relation on pointed Cantor minimal systems.*

Combining Theorem C and Theorem 1.6.4, we will show that the Borel complexity of $\approx$-equivalence of properly ordered Bratteli diagrams is exactly $\Delta^+_{\mathbb{R}}$. As an application of this result, we will prove a non-uniformity theorem regarding whether or not one can assign proper orderings to simple Bratteli diagrams in a uniform $\approx$-invariant Borel way.

1.8 Organization of this thesis

The remainder of this thesis is organized as follows. In Chapter 2, we will use Stone duality to represent pointed Cantor minimal systems by certain Boolean subalgebras of the Boolean algebra $\mathcal{P}(\mathbb{Z})$ of subsets of $\mathbb{Z}$. We will also characterize minimal subshifts in terms of the generating sets of their associated Boolean algebras. In Chapter 3, we will construct the standard Borel spaces of Cantor minimal systems, minimal subshifts, and properly ordered Bratteli diagrams. In Chapter 4, we will prove Theorem B and Theorem C. In Chapter 5, we will first give an overview of Toeplitz subshifts and construct the standard Borel spaces of various subclasses of Toeplitz subshifts. Then we shall analyze topological conjugacy of Toeplitz subshifts. In Chapter 6, we will prove Theorem A. In Chapter 7, we will first establish the correspondence between pointed Cantor minimal systems and properly ordered Bratteli diagrams. Then we will interpret our results in terms of properly ordered Bratteli diagrams. Finally, we will discuss some applications of our results and some possible further research directions.

1.9 Remarks on notation

In the remainder of this thesis, we will retain most of the notation introduced in this chapter. We would particularly like to emphasize that the symbols σ and ξ will be reserved for shift maps throughout this thesis. The symbol σ will always denote the left-shift map on $\mathfrak{a}^{\mathbb{Z}}$ defined by

$$(\sigma(\alpha))(i) = \alpha(i+1) \text{ for all } i \in \mathbb{Z} \text{ and } \alpha \in \mathfrak{a}^{\mathbb{Z}}$$

for every alphabet $\mathfrak{a}$. The symbol ξ will always denote the shift map on $\mathcal{P}(\mathbb{Z})$ given by

$$\xi(A) := \{a - 1 : a \in A\}$$

for all $A \in \mathcal{P}(\mathbb{Z})$. We will often need to use Boolean algebras of sets, i.e. Boolean algebras of the form $(\mathcal{A}, \cup, \cap, \cdot^C, \emptyset, X)$ where $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra of subsets of X for some set X. For notational convenience, we shall simply refer to the underlying collection of sets $\mathcal{A}$ as a Boolean algebra and never write the corresponding six-tuple.

Chapter 2

Stone duality and Cantor minimal systems

2.1 Stone duality

It is a classical result of Stone [Sto36, Sto37] that there exists a natural correspondence between Boolean algebras and *Boolean spaces*, i.e. zero-dimensional compact Hausdorff spaces. In more detail, given a Boolean algebra $\mathcal{A}$, consider the set $S(\mathcal{A})$ of ultrafilters on $\mathcal{A}$ endowed with the topology generated by the basic open sets of the form

$$\{U \in S(\mathcal{A}) : A \in U\}$$

for some $A \in \mathcal{A}$. Then the resulting topological space $S(\mathcal{A})$ is a Boolean space, which is called the Stone space of $\mathcal{A}$. Moreover, $\mathcal{A}$ is isomorphic to the Boolean algebra of clopen subsets of its Stone space via the map

$$A \mapsto \{U \in S(\mathcal{A}) : A \in U\}$$

Conversely, every Boolean space X is homeomorphic to the Stone space of the Boolean algebra $\mathbb{B}_X$ of its clopen subsets via the homeomorphism

$$w \mapsto \{U \in \mathbb{B}_X : w \in U\}$$

This correspondence extends to a duality between homomorphisms of Boolean algebras and continuous maps of Boolean spaces. Before we formulate this duality, we recall some basic definitions.

For every homomorphism $f : \mathcal{A} \to \mathcal{B}$ of Boolean algebras, the *dual* is the map $f_* : S(\mathcal{B}) \to S(\mathcal{A})$ defined by $f_*(w) = f^{-1}[w]$ for every $w \in S(\mathcal{B})$. For every continuous map $\psi : X \to Y$ of Boolean spaces, the *dual* is the map $\psi_* : \mathbb{B}_Y \to \mathbb{B}_X$ defined by $\psi_*(U) = \psi^{-1}[U]$ for every $U \in \mathbb{B}_Y$.

Theorem 2.1.1 (Stone duality)**.** *[Kop89] Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ be homomorphisms of Boolean algebras and let $\phi : X \to Y$ and $\psi : Y \to Z$ be continuous maps of Boolean algebras. Then*

- *$f_* : S(\mathcal{B}) \to S(\mathcal{A})$ is continuous and $\phi_* : \mathbb{B}_Y \to \mathbb{B}_X$ is a homomorphism.*
- *f is injective (respectively, surjective) if and only if f_* is surjective (respectively, injective).*
- *ϕ is injective (respectively, surjective) if and only if ϕ_* is surjective (respectively, injective).*
- *$(Id_{\mathcal{A}})_* = Id_{S(\mathcal{A})}$ and $(Id_X)_* = Id_{\mathbb{B}_X}$.*
- *$(g \circ f)_* = f_* \circ g_*$ and $(\psi \circ \phi)_* = \phi_* \circ \psi_*$.*

A Boolean subalgebra of $\mathcal{P}(\mathbb{Z})$ is said to be a $\mathbb{Z}$-*syndetic algebra* if its non-empty elements are syndetic sets and it is closed under both the shift map ξ and ξ^{-1}. In this chapter, we shall use Stone duality to show that the set of countable atomless $\mathbb{Z}$-syndetic algebras is a set of complete invariants for topological conjugacy of pointed Cantor minimal systems. We will need the following easy but useful proposition.

Proposition 2.1.2. *Let (X, φ, x) be a pointed topological dynamical system. Then the map $U \mapsto Ret_U(X, \varphi, x) = \{k \in \mathbb{Z} : \varphi^k(x) \in U\}$ is a Boolean algebra homomorphism from $\mathbb{B}_X$ to $\mathcal{P}(\mathbb{Z})$.*

Proof. Let $U, V \in \mathbb{B}_X$ be clopen sets. Then we have

$$\begin{aligned}
Ret_{U\cup V}(X, \varphi, x) &= \{k \in \mathbb{Z} : \varphi^k(x) \in U \cup V\} \\
&= \{k \in \mathbb{Z} : \varphi^k(x) \in U\} \cup \{k \in \mathbb{Z} : \varphi^k(x) \in V\} \\
&= Ret_U(X, \varphi, x) \cup Ret_V(X, \varphi, x) \\
Ret_{U\cap V}(X, \varphi, x) &= \{k \in \mathbb{Z} : \varphi^k(x) \in U \cap V\} \\
&= \{k \in \mathbb{Z} : \varphi^k(x) \in U\} \cap \{k \in \mathbb{Z} : \varphi^k(x) \in V\} \\
&= Ret_U(X, \varphi, x) \cap Ret_V(X, \varphi, x)
\end{aligned}$$

Clearly $Ret_{\emptyset}(X,\varphi,x) = \emptyset$ and $Ret_X(X,\varphi,x) = \mathbb{Z}$. Thus $U \mapsto Ret_U(X,\varphi,x)$ is a Boolean algebra homomorphism. $\square$

2.2 Representing pointed Cantor minimal systems by countable atomless $\mathbb{Z}$-syndetic algebras

Given a pointed Cantor minimal system (X,φ,x), we define its *return times algebra* $Ret(X,\varphi,x)$ to be the collection

$$Ret(X,\varphi,x) := \{Ret_U(X,\varphi,x) : U \in \mathbb{B}_X\}$$

By Proposition 2.1.2, $Ret(X,\varphi,x)$ is a Boolean subalgebra of $\mathcal{P}(\mathbb{Z})$. Moreover, by Theorem 1.6.1, the minimality of (X,φ,x) implies that the map $U \mapsto Ret_U(X,\varphi,x)$ is injective and that $Ret(X,\varphi,x)$ is a countable atomless $\mathbb{Z}$-syndetic algebra. From now on, any countable atomless $\mathbb{Z}$-syndetic algebra will be referred to as a *return times algebra*. Our choice of terminology is justified by the following lemma.

Lemma 2.2.1. *If $\mathcal{A}$ is a return times algebra, then there exists a pointed Cantor minimal system (X,φ,x) such that $\mathcal{A} = Ret(X,\varphi,x)$.*

Proof. It is well-known that there exists a unique countable atomless Boolean algebra up to isomorphism [Kop89, Corollary 5.16]. Thus $\mathcal{A}$ is isomorphic to the Boolean algebra of clopen subsets of $2^{\mathbb{N}}$ and it follows from Stone duality that $S(\mathcal{A})$ is homeomorphic to $2^{\mathbb{N}}$.

Let $\xi_* : S(\mathcal{A}) \to S(\mathcal{A})$ be the dual homeomorphism of the automorphism ξ of $\mathcal{A}$ and let $x_{\mathcal{A}} \in S(\mathcal{A})$ be the ultrafilter $\{A \in \mathcal{A} : 0 \in A\}$. We claim that $(S(\mathcal{A}), \xi_*, x_{\mathcal{A}})$ is a pointed Cantor minimal system such that $\mathcal{A} = Ret(S(\mathcal{A}), \xi_*, x_{\mathcal{A}})$. For each $A \in \mathcal{A}$, the set of return times of $x_{\mathcal{A}}$ to the clopen set $U = \{w \in S(\mathcal{A}) : A \in w\}$ is

$$\begin{aligned} Ret_U(S(\mathcal{A}), \xi_*, x_{\mathcal{A}}) &= \{k \in \mathbb{Z} : \xi_*^k(x_{\mathcal{A}}) \in U\} \\ &= \{k \in \mathbb{Z} : A \in \xi_*^k(x_{\mathcal{A}})\} \\ &= \{k \in \mathbb{Z} : k \in A\} \\ &= A \end{aligned}$$

It follows that $Ret(S(\mathcal{A}), \xi_*, x_{\mathcal{A}}) = \mathcal{A}$ and that $x_{\mathcal{A}}$ is an almost periodic point. Furthermore, the orbit of $x_{\mathcal{A}}$ meets every non-empty clopen set and hence is dense in $S(\mathcal{A})$. Therefore $(S(\mathcal{A}), \xi_*, x_{\mathcal{A}})$ is a pointed Cantor minimal system by Theorem 1.6.1. □

We shall refer to $(S(\mathcal{A}), \xi_*, x_{\mathcal{A}})$ as the *ultrafilter dynamical system* associated with the return times algebra $\mathcal{A}$. The following lemma shows that every pointed Cantor minimal system can be represented as the ultrafilter dynamical system associated with its return times algebra. Consequently, the collection of return time algebras is a set of complete invariants for topological conjugacy of pointed Cantor minimal systems.

Lemma 2.2.2. *Let (X, φ, x) be a pointed Cantor minimal system and let $\mathcal{A}$ be its return times algebra $Ret(X, \varphi, x)$. Then (X, φ, x) is topologically conjugate to $(S(\mathcal{A}), \xi_*, x_{\mathcal{A}})$.*

Proof. Recall that the map $\rho : \mathcal{A} \to \mathbb{B}_X$ given by $Ret_U(X, \varphi, x) \mapsto U$ is an isomorphism of Boolean algebras. Let $\rho_* : S(\mathbb{B}_X) \to S(\mathcal{A})$ be its dual homeomorphism. By Stone's theorem, we know that the map $\theta : X \mapsto S(\mathbb{B}_X)$ given by $w \mapsto \{U \in \mathbb{B}_X : w \in U\}$ is a homeomorphism. We claim that the homeomorphism $\rho_* \circ \theta$ is a topological conjugacy between (X, φ, x) and $(S(\mathcal{A}), \xi_*, x_{\mathcal{A}})$. Obviously, $(\rho_* \circ \theta)(x) = x_{\mathcal{A}}$. Moreover, for all $w \in X$, we have that

$$\begin{aligned}
((\rho_* \circ \theta) \circ \varphi)(w) &= \rho_*(\{U \in \mathbb{B}_X : \varphi(w) \in U\}) \\
&= \rho_*(\{\varphi[U] \in \mathbb{B}_X : w \in U\}) \\
&= \{(\rho^{-1} \circ \varphi)[U] \in \mathbb{B}_X : w \in U\} \\
&= \{Ret_{\varphi[U]}(X, \varphi, x) \in \mathcal{A} : w \in U\} \\
&= \xi_*(\{Ret_U(X, \varphi, x) \in \mathcal{A} : w \in U\}) \\
&= \xi_*(\{\rho^{-1}[U] \in \mathbb{B}_X : w \in U\}) \\
&= \xi_*(\rho_*(\{U \in \mathbb{B}_X : w \in U\})) \\
&= (\xi_* \circ (\rho_* \circ \theta))(w)
\end{aligned}$$

□

Corollary 2.2.3. *Two pointed Cantor minimal systems (X, φ, x) and (Y, ψ, y) are topologically conjugate if and only if $Ret(X, \varphi, x) = Ret(Y, \psi, y)$.*

Proof. Assume that (X, φ, x) and (Y, ψ, y) are topologically conjugate via the homeomorphism $\pi : X \to Y$. Since π_* is an isomorphism between $\mathbb{B}_Y$ and $\mathbb{B}_X$, we have that

$$\begin{aligned} Ret(X, \varphi, x) &= \{Ret_U(X, \varphi, x) : U \in \mathbb{B}_X\} \\ &= \{Ret_{\pi[U]}(Y, \psi, y) : U \in \mathbb{B}_X\} \\ &= \{Ret_V(Y, \psi, y) : V \in \mathbb{B}_Y\} = Ret(Y, \psi, y) \end{aligned}$$

For the converse direction, assume that $Ret(X, \varphi, x) = \mathcal{A} = Ret(Y, \psi, y)$. Then it follows from Lemma 2.2.2 that (X, φ, x) and (Y, ψ, y) are both topologically conjugate to $(S(\mathcal{A}), \xi_*, x_{\mathcal{A}})$. $\square$

Stone duality provides a way of rephrasing topological questions about Boolean spaces in terms of algebraic questions about Boolean algebras and vice versa. In general, this translation does not necessarily simplify the questions. However, in many cases one approach is much easier than the other. For example, consider the classification problem for odometers up to topological conjugacy.

Example 2.2.4. Let $(s_k)_{k\in\mathbb{N}}$ be a sequence of natural numbers such that $(s_k)_{k\in\mathbb{N}}$ is not eventually constant, $s_k > 1$ and $s_k | s_{k+1}$ for all $k \in \mathbb{N}$ and consider the odometer $\mathtt{Odo}((s_k)_{k\in\mathbb{N}})$ corresponding to the sequence $(s_k)_{k\in\mathbb{N}}$. Recall that the topology of

$$\mathtt{Odo}((s_k)_{k\in\mathbb{N}}) = \{w \in \prod_{k\in\mathbb{N}} \mathbb{Z}_{s_k} : \forall j > i \ w_j \equiv w_i \ (mod\ s_i)\}$$

is induced by the product topology on $\prod_{k\in\mathbb{N}} \mathbb{Z}_{s_k}$ where each component has the discrete topology. Thus the Boolean algebra of clopen subsets of $\mathtt{Odo}((s_k)_{k\in\mathbb{N}})$ is generated by the collection

$$\{\{w \in \mathtt{Odo}(s_k)_{k\in\mathbb{N}} : w(i) = j\} : i \in \mathbb{N} \ \wedge \ 0 \leq j < s_i\}$$

under the Boolean operations. It easily follows that a non-empty subset of the integers is in the set of return times $Ret(\mathtt{Odo}((s_k)_{k\in\mathbb{N}}), \eta, \mathbf{0})$ if and only if it is a finite union of infinite arithmetic progressions of the form $p+q\mathbb{Z}$ where q is a factor of the supernatural number $\operatorname{lcm}(s_k)_{k\in\mathbb{N}}$. This observation immediately gives a simple proof of the left-to-right direction of Theorem 1.6.3 as follows.

Assume that $\operatorname{lcm}(s_k)_{k\in\mathbb{N}} \neq \operatorname{lcm}(r_k)_{k\in\mathbb{N}}$. We can assume without loss of generality that there exists a factor q of $\operatorname{lcm}(s_k)_{k\in\mathbb{N}}$ which is not a factor of $\operatorname{lcm}(r_k)_{k\in\mathbb{N}}$. Then

$$q\mathbb{Z} \in \mathit{Ret}(\mathtt{Odo}((s_k)_{k\in\mathbb{N}}), \eta, \mathbf{0})$$

but

$$q\mathbb{Z} \notin \mathit{Ret}(\mathtt{Odo}((r_k)_{k\in\mathbb{N}}), \eta, \mathbf{0})$$

Hence, $(\mathtt{Odo}((s_k)_{k\in\mathbb{N}}), \eta, \mathbf{0})$ and $(\mathtt{Odo}((r_k)_{k\in\mathbb{N}}), \eta, \mathbf{0})$ are not (pointed) topologically conjugate. On the other hand, since odometers are topological groups, if $(\mathtt{Odo}((s_k)_{k\in\mathbb{N}}), \eta)$ and $(\mathtt{Odo}(rs_k)_{k\in\mathbb{N}}), \eta)$ were topologically conjugate via a homeomorphism π, then the pointed systems $(\mathtt{Odo}((s_k)_{k\in\mathbb{N}}), \eta, \mathbf{0})$ and $(\mathtt{Odo}((r_k)_{k\in\mathbb{N}}), \eta, \mathbf{0})$ would be (pointed) topologically conjugate via the homeomorphism $w \mapsto \pi(w) - \pi(\mathbf{0})$. Hence, odometers corresponding to different supernatural numbers are not topologically conjugate.

The advantage of studying return times algebra representations of pointed Cantor minimal systems rather than the minimal homeomorphisms themselves will become more apparent when we present the proof of Theorem C.

2.3 Representing pointed minimal subshifts by finitely generated atomless $\mathbb{Z}$-syndetic algebras

In the rest of this thesis, we will often need to regard subsets of integers of the form $\mathit{Ret}_U(X, \varphi, x)$ as elements of $2^{\mathbb{Z}}$. From now on, the corresponding characteristic function $\chi_{\mathit{Ret}_U(X,\varphi,x)} : \mathbb{Z} \to 2$ of $\mathit{Ret}_U(X, \varphi, x)$ will be denoted by $\mathit{ret}_U(X, \varphi, x)$.

In this section, we shall characterize the Cantor minimal systems that are topologically conjugate to minimal subshifts over finite alphabets in terms of the generating sets of their return times algebras. We begin by noting the following trivial but useful observation.

Proposition 2.3.1. *Let (X, φ) be a topological dynamical system and let U be a clopen subset of X. Then the map $r_U : X \to 2^{\mathbb{Z}}$ defined by $x \mapsto ret_U(X, \varphi, x)$ is continuous. Moreover, $r_U \circ \varphi = \sigma \circ r_U$.*

Proof. Since U is clopen, the characteristic function $\chi_U(x) : X \to 2$ is continuous and hence $r_U(x) = (\chi_U(\varphi^n(x)))_{n\in\mathbb{Z}}$ is continuous. It follows from the definition of r_U that $r_U \circ \varphi = \sigma \circ r_U$. □

Fix a Cantor minimal system (X, φ). For each non-empty subset $F \subseteq \mathbb{B}_X$ consider the map $ret_F : X \to (2^{\mathbb{Z}})^F$ given by $x \mapsto (ret_U(X, \varphi, x))_{U\in F}$. The map ret_F is continuous on each component by Proposition 2.3.1 and hence is continuous on the space $(2^{\mathbb{Z}})^F$ endowed with the product topology. Moreover, $ret_F \circ \varphi = \lambda_F \circ ret_F$ where λ_F is the componentwise shift map defined by $\lambda_F(w) = (\sigma(w(U)))_{U\in F}$. Consider the space $(2^F)^{\mathbb{Z}}$ endowed with the product topology where each component 2^F has the discrete topology. Let η_F be the map from $(2^{\mathbb{Z}})^F$ to $(2^F)^{\mathbb{Z}}$ given by

$$(\eta_F(w)(k))(U) = (w(U))(k)$$

for all $w \in (2^{\mathbb{Z}})^F$, $U \in F$ and $k \in \mathbb{Z}$. Clearly η_F is a bijection and $\sigma \circ \eta_F = \eta_F \circ \lambda_F$.

Proposition 2.3.2. *η_F is continuous whenever F is finite.*

Proof. Let F be finite. For each $f \in 2^F$ and $i \in \mathbb{Z}$, consider the basic open set $W_{i,f} = \{v \in (2^F)^{\mathbb{Z}} : v(i) = f\}$. Then we have

$$\eta_F^{-1}[W_{i,f}] = \bigcap_{U\in F} \{w \in (2^{\mathbb{Z}})^F : w(U)(i) = f(U)\}$$

Since the sets $\{w \in (2^{\mathbb{Z}})^F : w(U)(i) = f(U)\}$ are open in $(2^{\mathbb{Z}})^F$ and F is finite, $\eta_F^{-1}[W_{i,f}]$ is open and hence η_F is continuous. □

It follows that if there exists a finite $F \subseteq \mathbb{B}_X$ such that ret_F is injective, then $\eta_F \circ ret_F$ is a topological conjugacy from (X, φ) onto a minimal subshift over the alphabet 2^F. In order for ret_F to be injective, it is sufficient for F to generate $\mathbb{B}_X$ under φ and the Boolean operations, since $\mathbb{B}_X$ separates the points of X. On the other hand, for each $x \in X$, the Boolean algebras $\mathbb{B}_X$ and $Ret(X, \varphi, x)$ are isomorphic via the map

$U \mapsto Ret_U(X, \varphi, x)$. Hence, $\mathbb{B}_X$ is generated by finitely many elements under φ and the Boolean operations if and only if $Ret(X, \varphi, x)$ is generated by finitely many elements under ξ and the Boolean operations for some (equivalently, every) $x \in X$.

These observations suggest the following definition. A return times algebra $\mathcal{A}$ is said to be *finitely generated* if there exists a finite subset $F \subseteq \mathcal{A}$ such that $\mathcal{A}$ is the Boolean algebra generated by the collection $\{\xi^k(A) : A \in F \wedge k \in \mathbb{Z}\}$. In this case, the subset $F \subseteq \mathcal{A}$ is called a *generating set* of $\mathcal{A}$. We are now ready to present the main theorem of this section.

Theorem 2.3.3. *Let (X, φ, x) be a pointed Cantor minimal system. Then (X, φ, x) is topologically conjugate to a pointed minimal subshift over some finite alphabet if and only if $Ret(X, \varphi, x)$ is finitely generated.*

Proof. Assume that (X, φ, x) is topologically conjugate to a pointed minimal subshift (O, σ, w) over some finite alphabet $\mathfrak{a}$. Then by Corollary 2.2.3,

$$Ret(X, \varphi, x) = Ret(O, \sigma, w)$$

On the other hand, since the topology of O is induced by the topology of $\mathfrak{a}^{\mathbb{Z}}$, the return times algebra $Ret(O, \sigma, w)$ is generated by the finite generating set

$$\{Ret_{U_s}(O, \sigma, w) : s \in \mathfrak{a}\}$$

where U_s is the basic clopen set $\{v \in \mathfrak{a}^{\mathbb{Z}} : v(0) = s\}$. For the converse direction, assume that $Ret(X, \varphi, x)$ is finitely generated with a finite generating set F'. Let F be the preimage of F' under the map $U \mapsto Ret_U(X, \varphi, x)$. Then it follows from the previous discussion that $\eta_F \circ ret_F$ is a topological conjugacy from (X, φ, x) onto a pointed minimal subshift over the alphabet 2^F. $\square$

Chapter 3

The construction of various standard Borel spaces

In order to discuss the Borel complexity of an equivalence relation on a class of structures, we need to code these structures as elements of a standard Borel space. In general, there may be more than one way to do this. For example, the class of finitely generated groups can be identified with a subspace of the Polish space $2^{\mathbb{N}^3}$ consisting of elements satisfying an $\mathcal{L}_{\omega_1\omega}$-sentence defining the ternary relations on $\mathbb{N}$ coding finitely generated groups [TV99] as well as with the space of normal subgroups of the free group $\mathbb{F}_\omega$ on ω generators which contain all but finitely many of a distinguished set of generators [Wil12, §3.2].

In practice, whenever there are different natural codings of the same class of structures as standard Borel spaces, these codings turn out to be equivalent in the sense that there exist Borel maps between the corresponding standard Borel spaces which map codes of structures to codes of equivalent structures. The following principle, which first appeared in [Gao09] in a slightly different form, seems to be true.

> For any class $\mathcal{H}$ of mathematical structures, if (X_1, Ω_1) and (X_2, Ω_2) are two standard Borel spaces naturally coding elements of $\mathcal{H}$, then there exists a Borel map $f : X_1 \to X_2$ such that $f(x)$ and x are isomorphic as mathematical structures for every $x \in \mathcal{H}$.

In other words, there is essentially one way to code a class of structures as a standard Borel space, if there exists any at all. Even though each instance of this principle is a mathematical statement, the general principle is a philosophical statement since the notion of "natural coding" cannot be mathematically defined. This principle may be considered as an analogue of the Church-Turing thesis. For a more detailed discussion, we refer the reader to [Gao09, §14.1].

In the rest of this chapter, we shall construct the various standard Borel spaces that are used throughout this thesis. For the concrete instances of the aforementioned principle, we will sketch how the equivalence of different codings can be proved.

3.1 The standard Borel space of Cantor minimal systems

For any Cantor minimal system (X, φ), after choosing a clopen basis for the topology of X, one can find a homeomorphism from X to $2^{\mathbb{N}}$ and construct a topologically conjugate system $(2^{\mathbb{N}}, \psi)$. Therefore, it is sufficient to code those Cantor minimal systems which have $2^{\mathbb{N}}$ as their underlying topological spaces.

Let $\mathbb{B}$ be the countable atomless Boolean algebra of clopen subsets of $2^{\mathbb{N}}$. Recall that by Stone duality the homeomorphism group $H(2^{\mathbb{N}})$ of the Cantor space $2^{\mathbb{N}}$ is isomorphic to the automorphism group $Aut(\mathbb{B})$ of the Boolean algebra $\mathbb{B}$ via the isomorphism

$$\varphi \mapsto \varphi_*^{-1}$$

Thus, we can identify $H(2^{\mathbb{N}})$ with the subspace $Aut(\mathbb{B})$ of the Polish space $\mathbb{B}^{\mathbb{B}}$. A function $f \in \mathbb{B}^{\mathbb{B}}$ is an automorphism of $\mathbb{B}$ if f satisfies the following conditions.

a. $\forall U, V \in \mathbb{B}\ U = V \ \vee f(U) \neq f(V)$

b. $\forall V \in \mathbb{B}\ \exists U \in \mathbb{B}\ f(U) = V$

c. $f(\emptyset) = \emptyset$

d. $f(2^{\mathbb{N}}) = 2^{\mathbb{N}}$

e. $\forall U, V \in \mathbb{B}\ f(U \cup V) = f(U) \cup f(V) \ \wedge\ f(U \cap V) = f(U) \cap f(V)$

It is easily checked that these conditions define a G_δ subset of $\mathbb{B}^{\mathbb{B}}$. Hence, $H(2^{\mathbb{N}})$ is a Polish space with the induced topology. Indeed, it is a closed subgroup of the Polish group $Sym(\mathbb{B})$.

Recall by Theorem 1.6.1 that a Cantor dynamical system $(2^{\mathbb{N}}, \varphi)$ is minimal if and only if

$$\forall U \in \mathbb{B}\ \exists k \in \mathbb{N}\ 2^{\mathbb{N}} = \bigcup_{i=-k}^{k} \varphi_*^{-i}(U)$$

Observe that the quantifier-free part of this sentence defines a Borel subset of $\mathbb{B}^{\mathbb{B}}$ for each $k \in \mathbb{N}$ and $U \in \mathbb{B}$. It follows that the set

$$\mathcal{M}_{2^{\mathbb{N}}} := \{\varphi_*^{-1} \in H(2^{\mathbb{N}}) : (2^{\mathbb{N}}, \varphi) \text{ is minimal}\}$$

is a Borel subset of $H(2^{\mathbb{N}})$ and hence is a standard Borel space. The standard Borel space of pointed Cantor minimal systems is simply

$$\mathcal{M}^*_{2^{\mathbb{N}}} := \mathcal{M}_{2^{\mathbb{N}}} \times 2^{\mathbb{N}}$$

Consider the action $\Lambda : H(2^{\mathbb{N}}) \times \mathcal{M}_{2^{\mathbb{N}}} \to \mathcal{M}_{2^{\mathbb{N}}}$ given by

$$\varphi_* \cdot \psi_* = \varphi_* \circ \psi_* \circ \varphi_*^{-1}$$

for all $\varphi_*, \psi_* \in H(2^{\mathbb{N}})$. It is easily checked that the action Λ is Borel. Moreover, the orbit equivalence relation of Λ is the topological conjugacy relation $\cong_{tc}$ on $\mathcal{M}_{2^{\mathbb{N}}}$. Similarly, the topological conjugacy relation $\cong^*_{tc}$ on $\mathcal{M}^*_{2^{\mathbb{N}}}$ is the orbit equivalence relation of the Borel action of $H(2^{\mathbb{N}})$ on $\mathcal{M}^*_{2^{\mathbb{N}}}$ given by

$$\varphi_* \cdot (\psi_*, w) = (\varphi_* \circ \psi_* \circ \varphi_*^{-1}, \varphi^{-1}(w))$$

for all $\varphi_*, \psi_* \in H(2^{\mathbb{N}})$ and $w \in 2^{\mathbb{N}}$. Hence the equivalence relations $\cong_{tc}$ and $\cong^*_{tc}$ are both analytic equivalence relations since they are the orbit equivalence relations of Borel actions of a Polish group.

We will next discuss how the space of minimal subshifts over finite alphabets can be constructed as a subspace of $\mathcal{M}_{2^{\mathbb{N}}}$. We will first argue that whether or not a pointed Cantor minimal system has a finitely generated return times algebra can be expressed with a Borel condition.

Observe that the map from $\mathcal{M}^*_{2^{\mathbb{N}}}$ to $(2^{\mathbb{Z}})^{\mathbb{B}}$ given by $(\varphi_*, w) \mapsto (ret_U(2^{\mathbb{N}}, \varphi^{-1}, w))_{U \in \mathbb{B}}$ is Borel since whether or not $w \in V$ can be checked in a Borel way for every $V \in \mathbb{B}$ and

$$(ret_U(2^{\mathbb{N}}, \varphi^{-1}, w))(k) = 1 \Leftrightarrow w \in (\varphi_*)^{-k}(U)$$

Identifying elements of $\mathcal{P}(\mathbb{Z})$ with their characteristic functions in $2^{\mathbb{Z}}$, we can construct a Borel map that sends each sequence $\mathbf{w}$ in $(2^{\mathbb{Z}})^{\mathbb{B}}$ to a sequence in $(2^{\mathbb{Z}})^{\mathbb{B}}$ that lists the elements of the Boolean algebra generated by $\{\sigma^k(\mathbf{w}(i)) : i \in \mathbb{B} \ \wedge \ k \in \mathbb{Z}\}$, possibly with repetitions. (To list the elements of this Boolean algebra, we can use a fixed enumeration of the free Boolean algebra on the symbol set $\{\gamma_i^k : i \in \mathbb{B} \ \wedge k \in \mathbb{Z}\}$.)

It follows that whether or not a return times algebra is finitely generated can be expressed with a Borel condition. Hence, the subset of $\mathcal{M}^*_{2^{\mathbb{N}}}$ consisting of elements that have finitely generated return times algebras is Borel. By Theorem 2.3.3, these are precisely those pointed Cantor minimal systems that are topologically conjugate to pointed minimal subshifts over finite alphabets. Since whether or not the return times algebra of a pointed Cantor minimal system is finitely generated is independent of the distinguished point, the subset of $\mathcal{M}_{2^{\mathbb{N}}}$ consisting of Cantor minimal systems that are topologically conjugate to minimal subshifts over finite alphabets is Borel and hence is a standard Borel space.

It turns out that the standard Borel space of minimal subshifts over finite alphabets which we described above is not the most convenient space for our purposes. In the next section, instead of working with objects coding minimal subshifts, we will take a more direct approach and construct the space of minimal subshifts over a finite alphabet $\mathfrak{a}$ as the *set* of minimal subshifts over the alphabet $\mathfrak{a}$ endowed with an appropriate standard Borel structure. This will allow us to regard the topological conjugacy relation on minimal subshifts as a countable Borel equivalence relation rather than an essentially countable Borel equivalence relation.

3.2 The standard Borel space of minimal subshifts

Recall that a subshift over a finite alphabet $\mathfrak{a}$ is a topological dynamical system (O, σ) where O is a closed σ-invariant subset of the compact space $\mathfrak{a}^{\mathbb{Z}}$.

It is well-known that for any metric space (X, d), the space $K(X)$ of non-empty compact subsets of X endowed with the topology induced by the Hausdorff metric

$$\delta_d(C_1, C_2) = \max\{\max_{x \in C1} d(x, C_2), \max_{y \in C_2} d(y, C_1)\}$$

is a Polish space. It is easily checked that the map $C \mapsto \sigma[C]$ on $K(\mathfrak{a}^{\mathbb{Z}})$ is continuous and hence the set

$$\mathcal{S}_{\sigma,\mathfrak{a}} := \{O \in K(\mathfrak{a}^{\mathbb{Z}}) : \sigma[O] = O\}$$

of subshifts over the alphabet $\mathfrak{a}$ is a closed subset of $K(\mathfrak{a}^{\mathbb{Z}})$ [Cle09, Lemma 3]. It is well-known that there exists a sequence of Borel functions $g_k : K(\mathfrak{a}^{\mathbb{Z}}) \to \mathfrak{a}^{\mathbb{Z}}$ such that for every $C \in K(\mathfrak{a}^{\mathbb{Z}})$ the set $\{f_k(C)\}_{k\in\mathbb{Z}}$ is dense in C [Kec95, Theorem 12.23]. Then for every $O \in \mathcal{S}_{\sigma,\mathfrak{a}}$ and $w \in O$ we have

$$\overline{\mathrm{Orb}}(w) = O \Leftrightarrow \forall k \ \forall i \ \exists j \ g_i(O)[-k,k] = w[j-k, j+k]$$

Observe that $w \in O$ is an almost periodic point of (O,σ) if and only if each subblocks of w appears along w with bounded gaps, which can be expressed with a Borel condition. Thus the set

$$\mathcal{M}_{\sigma,\mathfrak{a}} := \{O \in \mathcal{S}_{\sigma,\mathfrak{a}} : O \text{ is infinite } \wedge \ g_0(O) \text{ is almost periodic } \wedge \ \overline{\mathrm{Orb}}(g_0(O)) = O\}$$

is a Borel subset of $\mathcal{S}_{\sigma,\mathfrak{a}}$ and hence is a standard Borel space. By Theorem 1.6.1, the standard Borel space $\mathcal{M}_{\sigma,\mathfrak{a}}$ is precisely the standard Borel space of minimal subshifts over the alphabet $\mathfrak{a}$.

It easily follows from the Curtis-Hedlund-Lyndon theorem that the topological conjugacy relation on $\mathcal{S}_{\sigma,\mathfrak{a}}$ is a countable Borel equivalence relation [Cle09, Lemma 9]. Thus, the topological conjugacy relation is a countable Borel equivalence relation on any Borel subspace of $\mathcal{S}_{\sigma,\mathfrak{a}}$. In particular, it is a countable Borel equivalence relation on $\mathcal{M}_{\sigma,\mathfrak{a}}$.

To construct the standard Borel space of minimal subshifts over finite alphabets, we can suppose without loss of generality that the finite alphabets which we use are finite ordinals. Then the standard Borel space of minimal subshifts over finite alphabets is simply

$$\mathcal{M}_{\sigma} = \bigcup_{n\geq 2} \mathcal{M}_{\sigma,n}$$

We have constructed two different standard Borel spaces for the class of minimal subshifts over finite alphabets. We will now sketch how the equivalence of these constructions can be proved.

Given $\varphi_* \in \mathcal{M}_{2^{\mathbb{N}}}$ coding a Cantor minimal system that is topologically conjugate to a minimal subshift, we can first exhaustively search for some generating set F of

$$Ret(2^{\mathbb{N}}, \varphi^{-1}, (0,0,0,...))$$

using the maps we have constructed in §3.1. We then use the proof of Theorem 2.3.3 to construct a minimal subshift over the alphabet $2^{|F|}$ which is topologically conjugate to $(2^{\mathbb{N}}, \varphi^{-1})$. Conversely, given a minimal subshift O in $\mathcal{M}_\sigma$, we first choose a point $w \in O$ and construct the return times algebra $\mathcal{A} = Ret(O, \sigma, w)$. We then apply a back-and-forth argument to construct an isomorphism of Boolean algebras between $\mathcal{A}$ and $\mathbb{B}$. Since (O, σ, w) is topologically conjugate to $(S(\mathcal{A}), \xi_*, x_{\mathcal{A}})$, the isomorphism we constructed between $\mathcal{A}$ and $\mathbb{B}$ can be used to find an automorphism of $\mathbb{B}$ coding (O, σ). We skip the routine details of checking that these procedures define Borel maps between the corresponding standard Borel spaces.

3.3 The standard Borel space of properly ordered Bratteli diagrams

In this section, we shall construct the standard Borel spaces of simple Bratteli diagrams and properly ordered Bratteli diagrams.

Given a Bratteli diagram (V, E), we can suppose without loss of generality that the vertex set and the edge set are fixed countably infinite sets $\mathbf{V}$ and $\mathbf{E}$ respectively. We then code (V, E) by the triple $(f, g, h) \in \mathbb{N}^{\mathbf{V}} \times \mathbb{N}^{\mathbf{E}} \times (\mathbf{V} \times \mathbf{V})^{\mathbf{E}}$ such that $f(v) = n$ for each $v \in V_n$, $g(e) = n$ for each $e \in E_n$, and $h(e) = (s(e), r(e))$ for each $e \in E$ where s and r are the corresponding source and range maps.

Let $\mathcal{SBD}$ be the subset of the Polish space $\mathbb{N}^{\mathbf{V}} \times \mathbb{N}^{\mathbf{E}} \times (\mathbf{V} \times \mathbf{V})^{\mathbf{E}}$ consisting of elements (f, g, h) satisfying the following conditions.

a. For all $i \in \mathbb{N}$, $\{v \in \mathbf{V} : f(v) = i\}$ is non-empty and finite.

b. For all $v, w \in \mathbf{V}$, if $f(v) = f(w) = 0$, then $v = w$.

c. For all non-zero $i \in \mathbb{N}$, $\{e \in \mathbf{E} : g(e) = i\}$ is non-empty and finite.

d. For all $e \in \mathbf{E}$, $g(e) \neq 0$.

e. For all $v, w \in \mathbf{V}$ and $e \in \mathbf{E}$, $g(e) = f(w) = f(v) + 1$ whenever $h(e) = (v, w)$.

f. For all $i \in \mathbb{N}$, there exists $j > i$ such that for all $v, w \in \mathbf{V}$ with $f(v) = i$ and $f(w) = j$, there exist $e_{i+1}, ..., e_j \in \mathbf{E}$ such that $(e_{i+1}, ..., e_j)$ is a path from v to w.

Observe that the set $\mathcal{SBD}$ is precisely the set of triples coding simple Bratteli diagrams. Moreover, conditions (a)-(f) define a Borel subset of $\mathbb{N}^{\mathbf{V}} \times \mathbb{N}^{\mathbf{E}} \times (\mathbf{V} \times \mathbf{V})^{\mathbf{E}}$ and hence $\mathcal{SBD}$ is a standard Borel space.

In order to code ordered Bratteli diagrams, we need to incorporate the partial order relations on the edge set $\mathbf{E}$. Identifying binary relations on $\mathbf{E}$ with their characteristic functions in the Polish space $2^{\mathbf{E} \times \mathbf{E}}$, we can regard the space of ordered Bratteli diagrams as the Borel subset of $\mathbb{N}^{\mathbf{V}} \times \mathbb{N}^{\mathbf{E}} \times (\mathbf{V} \times \mathbf{V})^{\mathbf{E}} \times 2^{\mathbf{E} \times \mathbf{E}}$ consisting of quadruples (f, g, h, p) satisfying conditions (a)-(e) and the following conditions

g. For all $e \in \mathbf{E}$, $p(e, e) = 1$.

h. For all $e, e' \in \mathbf{E}$, $p(e, e') = 1$ and $p(e', e) = 1$ implies $e = e'$.

i. For all $e, e', e'' \in \mathbf{E}$, $p(e, e') = 1$ and $p(e', e'') = 1$ implies $p(e, e'') = 1$.

j. For all $e, e' \in \mathbf{E}$, $p(e, e') = 1$ or $p(e', e) = 1$ if and only if $\pi_2(h(e)) = \pi_2(h(e'))$ where π_2 is the projection map onto the second coordinate.

Recall that an ordered Bratteli diagram is properly ordered if it is simple, its space of infinite paths is infinite, and there exist unique maximal and minimal paths. The first property is expressed by the Borel condition (f). For the second property, consider the Borel condition

k. The set $\{i \in \mathbb{N} : \exists e, e' \in \mathbf{E}\ e \neq e'\ \wedge g(e) = g(e') = i\}$ is infinite

It is easily checked that if B is a simple Bratteli diagram, then the space of infinite paths X_B is infinite if and only if the condition (k) holds. However, the last property seems to require an existential quantification over the Polish space $\mathbf{E}^{\mathbb{N}}$ and hence is an analytic condition. We will show that it is equivalent to a Borel condition for the quadruples coding ordered Bratteli diagrams.

Given an ordered Bratteli diagram $(\mathbf{V}, \mathbf{E}, \preccurlyeq)$, for each vertex $v \in \mathbf{V}$ there exists a unique path from the root v_0 to v each element of which is in $\mathbf{E}_{min}$. It follows that if we "mark" the minimal edges in the diagrammatic representation of $(\mathbf{V}, \mathbf{E})$ together with the vertices which they connect, then we obtain a tree $\mathbf{T}_{min}$ whose edge set is exactly $\mathbf{E}_{min}$. Since $\mathbf{T}_{min}$ is finitely branching, König's lemma implies that the following are equivalent

- There is a unique infinite branch in $\mathbf{T}_{min}$.
- For every vertex $v \in \mathbf{T}_{min}$, there exists a unique successor v^+ of v in $\mathbf{T}_{min}$ such that there exist infinitely many $w \in \mathbf{T}_{min}$ above v^+.

Similarly, one can argue that having a unique maximal path can be expressed with a condition that quantifies over countable sets. It follows that the subset $\mathcal{POBD}$ of $\mathbb{N}^{\mathbf{V}} \times \mathbb{N}^{\mathbf{E}} \times (\mathbf{V} \times \mathbf{V})^{\mathbf{E}} \times 2^{\mathbf{E} \times \mathbf{E}}$ coding properly ordered Bratteli diagrams is Borel and hence is a standard Borel space.

Let $\sim$ and $\approx$ denote equivalence of simple Bratteli diagrams and properly ordered Bratteli diagrams on the standard Borel spaces $\mathcal{SBD}$ and $\mathcal{POBD}$ respectively. One can show that $\sim$ and $\approx$ are both analytic equivalence relations by reducing these relations to isomorphism relations on other mathematical structures represented by Bratteli diagrams. For example, it is proved in [Ell10, §2.3] that $\sim$ is Borel bireducible with the isomorphism relation on countable simple locally finite groups of strongly diagonal type, which is an analytic equivalence relation. In §7.1, we will prove that $\approx$ and $\cong^*_{tc}$ are Borel bireducible by constructing Borel maps between $\mathcal{M}^*_{2^{\mathbb{N}}}$ and $\mathcal{POBD}$ mapping pointed Cantor minimal systems to their Bratteli-Vershik representations and vice versa.

Chapter 4

Proofs of Theorem B and Theorem C

In this chapter, we shall prove Theorem B and Theorem C. We begin by deducing that $\cong_{tc}^* \leq_B \Delta_{\mathbb{R}}^+$ from the fact that the set of return times algebras is a set of complete invariants for topological conjugacy of pointed Cantor minimal systems.

Lemma 4.1. $\cong_{tc}^* \leq_B \Delta_{\mathbb{R}}^+$.

Proof. Recall that $\Delta_{2^{\mathbb{Z}}}^+$ and $\Delta_{\mathbb{R}}^+$ are Borel bireducible since any two uncountable standard Borel spaces are isomorphic. Thus it is sufficient to prove that $\cong_{tc}^* \leq_B \Delta_{2^{\mathbb{Z}}}^+$. Let $f : \mathcal{M}_{2^{\mathbb{N}}}^* \to (2^{\mathbb{Z}})^{\mathbb{N}}$ be the map given by

$$f((\varphi_*, w)) = (ret_{g(i)}(2^{\mathbb{N}}, \varphi^{-1}, w))_{i \in \mathbb{N}}$$

where $g : \mathbb{N} \to \mathbb{B}$ is a fixed enumeration of the clopen subsets of $2^{\mathbb{N}}$. It follows from the discussion in §3.1 that f is a Borel map. By Corollary 2.2.3, f is a Borel reduction from $\cong_{tc}^*$ to $\Delta_{2^{\mathbb{Z}}}^+$. □

To show that $\Delta_{\mathbb{R}}^+ \leq_B \cong_{tc}^*$, it is enough to injectively assign a return times algebra to each non-empty countable subset of $\mathbb{R}$. In order to construct these return times algebras, we will need a rich collection of syndetic subsets of $\mathbb{Z}$ and these will be obtained from a non-Cantor minimal system. Fix an irrational number $\gamma \in (0, 1)$ and consider the irrational rotation $T_\gamma : [0, 1) \to [0, 1)$ defined by $x \mapsto x + \gamma \ (mod\ 1)$ where $[0, 1)$ is identified with the quotient $\mathbb{R}/\mathbb{Z}$. It is well-known that the topological dynamical system $([0, 1), T_\gamma)$ is minimal [Kůr03, Proposition 1.32].

Our collection of syndetic sets will be constructed in a manner similar to the construction of *Sturmian words*. A Sturmian word is a 0-1 sequence of the form $ret_{[0,\gamma)}([0, 1), T_\gamma, x)$ for some $x \in [0, 1)$. The main difference will be that we do not insist that the endpoint of the half open interval be the same as the rotation angle.

Let $I \subseteq (0,1)$ be a non-empty countable set and let $\mathcal{A}^I$ denote the Boolean algebra consisting of the subsets of $[0,1)$ generated by the collection

$$\mathbb{G}^I = \{T_\gamma^k[[0,\alpha)] : k \in \mathbb{Z} \ \wedge \ \alpha \in I\}$$

Proposition 4.2. *$\mathcal{A}^I$ is a countable atomless Boolean subalgebra of $\mathcal{P}([0,1))$ whose non-empty elements are finite unions of half open intervals and which is closed under both T_γ and T_γ^{-1}.*

Proof. Observe that complements and intersections of finite unions of half open intervals in $[0,1)$ are also finite unions of half open intervals. Since $\mathbb{G}^I$ is a countable subcollection of $\mathcal{P}([0,1))$ consisting of finite unions of half open intervals which is closed under both T_γ and T_γ^{-1}, the same is true of the Boolean algebra $\mathcal{A}^I$ generated by $\mathbb{G}^I$. To see that $\mathcal{A}^I$ is atomless, assume to the contrary that there exists an atom $\emptyset \neq A \subsetneq [0,1)$ in $\mathcal{A}^I$. Recall that the T_γ-orbit of every point is dense by the minimality of $([0,1),T_\gamma)$. It follows that there exists $k \in \mathbb{Z} \setminus \{0\}$ such that $A \cap T_\gamma^k[A] \neq \emptyset$. Note that $k\gamma$ is also irrational and hence $([0,1),T_{k\gamma})$ is also minimal. Since A is an atom in $\mathcal{A}^I$, we have that $A \cap T_\gamma^k[A] = A$. But then $\overline{\{T_\gamma^{ki}(x) : i \in \mathbb{Z}\}} \subseteq \overline{A}$ for any $x \in A$ and hence $\{T_\gamma^{ki}(x) : i \in \mathbb{Z}\}$ is not dense in $[0,1)$ for any $x \in A$, which contradicts the minimality of $([0,1),T_{k\gamma})$. $\square$

Let $\mathcal{A}_I$ be the image of $\mathcal{A}^I$ under the Boolean algebra homomorphism

$$U \mapsto Ret_U([0,1),T_\gamma,0)$$

It follows from Proposition 4.2 that $\mathcal{A}_I$ is a countable atomless subalgebra of $\mathcal{P}(\mathbb{Z})$ which is closed under both ξ and ξ^{-1}. By the minimality of $([0,1),T_\gamma)$, since each $U \in \mathcal{A}^I$ contains an open interval, the set $Ret_U([0,1),T_\gamma,0)$ is a syndetic subset of $\mathbb{Z}$ for every $U \in \mathcal{A}^I$. Hence $\mathcal{A}_I$ is a return times algebra.

4.1 Asymptotic densities of the return times sets

Recall that the *asymptotic density* of a subset A of $\mathbb{Z}$ is defined to be the limit

$$\mathrm{Dens}(A) := \lim_{n\to\infty} \frac{|A \cap [-n,n]|}{2n+1}$$

whenever it exists. Identifying $\mathcal{P}(\mathbb{Z})$ with $2^{\mathbb{Z}}$, we can similarly define the asymptotic density of an element $\alpha \in 2^{\mathbb{Z}}$ to be the limit

$$\mathrm{Dens}(\alpha) := \lim_{n\to\infty} \frac{|\{k \in \mathbb{Z} : \alpha(k) = 1\} \cap [-n, n]|}{2n+1}$$

whenever it exists. In this section, we shall show that the set of asymptotic densities of elements of $\mathcal{A}_I$ is a topological conjugacy invariant for the collection of Cantor minimal systems of the form $(S(\mathcal{A}_I), \xi_*)$. We will need the following well-known theorem.

Theorem 4.1.1. *[EW11] Let $\gamma \in [0,1)$ be an irrational number. Then for any $x \in [0,1)$ the sequence $(T_\gamma^i(x))_{i\in\mathbb{N}}$ is equidistributed in $[0,1)$ in the sense that for any $a, b \in [0,1)$ with $0 \leq a \leq b < 1$ we have that*

$$\lim_{n\to\infty} \frac{|\{j : 0 \leq j < n, x_j \in [a,b]\}|}{n} = b - a$$

By applying Theorem 4.1.1 to the irrational rotations T_γ and $T_{1-\gamma}$, it is easily checked that for every $0 \leq a \leq b < 1$ and $x \in [0,1)$, we have that

$$\mathrm{Dens}(Ret_{[a,b)}([0,1), T_\gamma, x)) = b - a$$

Since each element of $\mathcal{A}^I$ is a finite union of half open intervals, this implies that

$$\mathrm{Dens}(Ret_U([0,1), T_\gamma, 0)) = \mu(U)$$

for every $U \in \mathcal{A}^I$ where μ is the usual Lebesgue measure on $[0,1)$. Having shown that elements of $\mathcal{A}_I$ have well-defined asymptotic densities, we define the density set of $\mathcal{A}_I$ to be the collection

$$\begin{aligned}\mathrm{Dens}(\mathcal{A}_I) &:= \{\mathrm{Dens}(A) : A \in \mathcal{A}_I\} \\ &= \{\mathrm{Dens}(Ret_U([0,1), T_\gamma, 0)) : U \in \mathcal{A}^I\} \\ &= \{\mu(U) : U \in \mathcal{A}^I\}\end{aligned}$$

In order to prove that $\mathrm{Dens}(\mathcal{A}_I)$ is an invariant of the topological conjugacy class of $(S(\mathcal{A}_I), \xi_*)$, we will need the following technical lemma.

Lemma 4.1.2. *Let $U \subseteq [0,1)$ be a finite union of half open intervals. Then for every α in the subshift of $2^{\mathbb{Z}}$ generated by $ret_U([0,1), T_\gamma, 0)$ we have that $Dens(\alpha) = \mu(U)$.*

Proof. Let α be in the subshift generated by $ret_U([0,1), T_\gamma, 0)$. It is sufficient to find some $v \in [0,1)$ such that $\alpha = ret_U([0,1), T_\gamma, v)$ since we know that

$$\text{Dens}(ret_U([0,1), T_\gamma, v)) = \mu(U)$$

by the previous discussion. As α is in the subshift generated by $ret_U([0,1), T_\gamma, 0)$, there exists a sequence $(n_k)_{k\in\mathbb{N}}$ of integers such that $\alpha = \lim_{k\to\infty} \sigma^{n_k}(ret_U([0,1), T_\gamma, 0))$. Notice that

$$\alpha = \lim_{k\to\infty} \sigma^{n_k}(ret_U([0,1), T_\gamma, 0)) = \lim_{k\to\infty}(ret_U([0,1), T_\gamma, T_\gamma^{n_k}(0)))$$

Hence, our target point $v \in [0,1)$ *should* be the limit of the sequence $T_\gamma^{n_k}(0)$ in $[0,1)$. However, there is no reason that this sequence should converge. Nevertheless, the sequential compactness of $[0,1)$ implies that there exists some subsequence $(n_{k_i})_{i\in\mathbb{N}}$ such that $T_\gamma^{n_{k_i}}(0)$ is convergent, say with the limit $v = \lim_{i\to\infty} T_\gamma^{n_{k_i}}(0)$. We would like to move the limit operation inside so that

$$\lim_{i\to\infty} ret_U([0,1), T_\gamma, T_\gamma^{n_{k_i}}(0)) = ret_U([0,1), T_\gamma, \lim_{i\to\infty} T_\gamma^{n_{k_i}}(0)) = ret_U([0,1), T_\gamma, v)$$

If the function $ret_U([0,1), T_\gamma, \cdot)$ were continuous, then this step would be justified. However, Proposition 2.3.1 may fail if U is not clopen and $ret_U(X, \varphi, \cdot)$ need not be continuous in general. Even though $ret_U([0,1), T_\gamma, v)$ is not necessarily α, we will next prove that these sequences can differ at only finitely many indices.

Let B_v be the set of indices $\{j \in \mathbb{Z} : T_\gamma^j(v) \in \partial U\}$ where ∂U denotes the boundary of U. Note that ∂U is finite and hence B_v is also finite. Otherwise, v would be a periodic point of $([0,1), T_\gamma)$, which contradicts the minimality of $([0,1), T_\gamma)$. We will show that

$$ret_U([0,1), T_\gamma, v) \restriction (\mathbb{Z} - B_v) = \lim_{i\to\infty}(ret_U([0,1), T_\gamma, T_\gamma^{n_{k_i}}(0)) \restriction (\mathbb{Z} - B_v))$$

where the limit is taken in the topological space $2^{\mathbb{Z}-B_v}$. For each $k \geq 1$, choose $\delta_k > 0$ such that

$$\delta_k < min\{d(T_\gamma^j(v), y) : y \in \partial U \wedge\ -k \leq j \leq k\ \wedge j \notin B_v\}$$

where d is the usual metric on $\mathbb{R}/\mathbb{Z} \cong [0,1)$. Since T_γ is an isometry with respect to d, it follows from the choice of δ_k that for any v' in the open ball $B_d(v, \delta_k)$ and for any $-k \leq j \leq k$ with $j \notin B_v$, we have that $T_\gamma^j(v) \in Int(U) \Leftrightarrow T_\gamma^j(v') \in Int(U)$

where $Int(U)$ is the interior of U. In other words, for any $v' \in B_d(v, \delta_k)$ and for any $-k \leq j \leq k$ with $j \notin B_v$, we have that

$$ret_U([0,1), T_\gamma, v)(j) = ret_U([0,1), T_\gamma, v')(j)$$

Since $v = \lim_{i\to\infty} T_\gamma^{n_{k_i}}(0)$, we know that for any $k \geq 1$, there exists $m \geq 0$ such that for all $i \geq m$ we have $|v - T_\gamma^{n_{k_i}}(0)| < \delta_k$. It follows that

$$\begin{aligned}
ret_U([0,1), T_\gamma, v) \upharpoonright (\mathbb{Z} - B_v) &= \lim_{i\to\infty} (ret_U([0,1), T_\gamma, T_\gamma^{n_{k_i}}(0)) \upharpoonright (\mathbb{Z} - B_v)) \\
&= (\lim_{i\to\infty} ret_U([0,1), T_\gamma, T_\gamma^{n_{k_i}}(0)) \upharpoonright (\mathbb{Z} - B_v) \\
&= \alpha \upharpoonright (\mathbb{Z} - B_v)
\end{aligned}$$

This implies that α and $ret_U([0,1), T_\gamma, v)$ have the same asymptotic density $\mu(U)$. $\square$

We are now ready to prove the main result of this section.

Corollary 4.1.3. *For every non-empty countable $I, J \subseteq [0,1)$, Dens$(\mathcal{A}_I) =$ Dens$(\mathcal{A}_J)$ whenever $(S(\mathcal{A}_I), \xi_*)$ and $(S(\mathcal{A}_J), \xi_*)$ are topologically conjugate.*

Proof. Assume that $(S(\mathcal{A}_I), \xi)$ and $(S(\mathcal{A}_J), \xi)$ are topologically conjugate via the homeomorphism $\pi : S(\mathcal{A}_I) \to S(\mathcal{A}_J)$. Let $r \in \mathrm{Dens}(\mathcal{A}_I)$. Since $Ret(S(\mathcal{A}_I), \xi_*, x_{\mathcal{A}_I})) = \mathcal{A}_I$, there exists a clopen subset W of $S(\mathcal{A}_I)$ such that

$$r = \mathrm{Dens}(Ret_W(S(\mathcal{A}_I), \xi_*, x_{\mathcal{A}_I})) = \mathrm{Dens}(Ret_{\pi[W]}(S(\mathcal{A}_J), \xi_*, \pi(x_{\mathcal{A}_I})))$$

It follows from Proposition 2.3.1 that the image of $S(\mathcal{A}_J)$ under the map

$$w \mapsto ret_{\pi[W]}(S(\mathcal{A}_J), \xi_*, w)$$

is a subshift. This subshift is minimal since it is the factor of a minimal dynamical system. Moreover, we know that

$$ret_W(S(\mathcal{A}_I), \xi_*, x_{\mathcal{A}_I}) = ret_U([0,1), T_\gamma, 0)$$

for some $U \in \mathcal{A}^I$; and every sequence in the subshift generated by $ret_U([0,1), T_\gamma, 0)$ has the same asymptotic density by Lemma 4.1.2. In particular,

$$r = \mathrm{Dens}(Ret_{\pi[W]}(S(\mathcal{A}_J), \xi_*, \pi(x_{\mathcal{A}_I}))) = \mathrm{Dens}(Ret_{\pi[W]}(S(\mathcal{A}_J), \xi_*, x_{\mathcal{A}_J}))$$

and hence $r \in \mathrm{Dens}(\mathcal{A}_J)$. Carrying out this argument symmetrically, we have that

$$\mathrm{Dens}(\mathcal{A}_I) = \mathrm{Dens}(\mathcal{A}_J)$$

$\square$

4.2 Reducing $\Delta^+_{\mathbb{R}}$ to $\cong_{tc}$ and $\cong^*_{tc}$

In this section, we will construct Borel reductions from $\Delta^+_{\mathbb{R}}$ to both $\cong_{tc}$ and $\cong^*_{tc}$. This will prove Theorem B and complete the proof of Theorem C.

Recall that $\Delta^+_{\mathcal{I}} \sim_B \Delta^+_{\mathbb{R}}$ for any uncountable Borel subset $\mathcal{I}$ of $\mathbb{R}$. Thus it is sufficient to show that $\Delta^+_{\mathcal{I}}$ is Borel reducible to both $\cong_{tc}$ and $\cong^*_{tc}$ for some appropriately chosen Borel subset $\mathcal{I} \subseteq (0, 1)$ of size continuum.

The key observation to construct such Borel reductions is the following. Taking unions, intersections, and complements introduce no new boundary points as we generate $\mathcal{A}^I$ from $\mathbb{G}^I$. Hence, the set of boundary points of elements of $\mathcal{A}^I$ is exactly the set of boundary points of elements of $\mathbb{G}^I$ which is contained in the $\mathbb{Q}$-span of $\{1, \gamma\} \cup I$. Thus the density set $\mathrm{Dens}(\mathcal{A}_I)$ is contained in the $\mathbb{Q}$-span of $\{1, \gamma\} \cup I$ since

$$\mathrm{Dens}(\mathcal{A}_I) = \{\mu(U) : U \in \mathcal{A}^I\}$$

Lemma 4.2.1. *There exist an irrational number $\gamma \in (0, 1)$ and a Borel subset $\mathcal{I} \subseteq (0, 1)$ of size continuum such that $\mathcal{I} \cap \{1, \gamma\} = \emptyset$ and $\mathcal{I} \cup \{1, \gamma\}$ is $\mathbb{Q}$-linearly independent.*

Proof. Fix a labeling of the vertices of the full binary tree of height ω by $\mathbb{N}$. For any infinite path $\alpha \in 2^{\mathbb{N}}$, let $A_\alpha \subseteq \mathbb{N}$ be the set of labels of the vertices that α passes through. Observe that intersection of any two such sets is finite. We claim[1] that if we let $r_\alpha = \sum_{i=0}^{\infty} \chi_{A_\alpha}(i) \cdot 2^{-(i+1)^2}$ for each $\alpha \in 2^{\mathbb{N}}$, then the set $\{r_\alpha : \alpha \in 2^{\mathbb{N}}\}$ is a $\mathbb{Q}$-linearly independent subset of $(0, 1)$ of size continuum, where χ_{A_α} denotes the characteristic function of A_α. To see this, assume to the contrary that there exist $\{\alpha_i\}_{i=0}^{k} \subseteq 2^{\mathbb{N}}$ and $\{a_{\alpha_i}\}_{i=0}^{k} \subseteq \mathbb{Q} \setminus \{0\}$ such that $\sum_{i=0}^{k} a_{\alpha_i} r_{\alpha_i} = 0$. By multiplying both

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[1] The author learned this trick from a MathOverflow post [Gow] by Sir William Timothy Gowers that is posted under the username "gowers" (http://mathoverflow.net/users/1459/gowers).

sides by an appropriate integer, we may assume without loss of generality that the coefficients a_{α_i} are integers. Since the pairwise intersections of the sets A_{α_i} are finite, we can find $n \in A_{\alpha_0}$ such that $n \geq 2\sum_{i=0}^{k} |a_{\alpha_i}|$ and each $m \geq n$ belongs to at most one of the sets A_{α_i}. Then the binary expansion of $\sum_{i=0}^{k} a_{\alpha_i} r_{\alpha_i}$ necessarily contains the binary digit 1 between its $(n-1)^2$-th binary digit and $(n+1)^2$-th binary digit, which is a contradiction.

Let $\gamma \in \{r_\alpha : \alpha \in 2^{\mathbb{N}}\}$ and set $\mathcal{I} := \{r_\alpha : \alpha \in 2^{\mathbb{N}}\} \backslash \{\gamma\}$. Then $\mathcal{I}$ and γ satisfy our requirements. $\square$

Theorem 4.2.2. *$\Delta_{\mathbb{R}}^{+}$ is Borel reducible to both $\cong_{tc}$ and $\cong_{tc}^{*}$.*

Proof. Fix some irrational number $\gamma \in (0,1)$ and a Borel subset $\mathcal{I} \subseteq (0,1)$ as in Lemma 4.2.1. Given any $\mathbf{S} \in \mathcal{I}^{\mathbb{N}}$, let $f(\mathbf{S})$ and $g(\mathbf{S})$ be elements of $\mathcal{M}_{2^{\mathbb{N}}}$ and $\mathcal{M}_{2^{\mathbb{N}}}^{*}$ which code $(S(\mathcal{A}_S), \xi_*)$ and $(S(\mathcal{A}_S), \xi_*, x_{\mathcal{A}_S})$ respectively, where $S = \{\mathbf{S}_i \in \mathcal{I} : i \in \mathbb{N}\}$ and $\mathcal{A}_S$ is computed using the irrational rotation by γ. We will show that f and g are Borel reductions from $\Delta_{\mathcal{I}}^{+}$ to $\cong_{tc}$ and $\cong_{tc}^{*}$ respectively.

We first argue that f and g are Borel maps from $\mathcal{I}^{\mathbb{N}}$ to $\mathcal{M}_{2^{\mathbb{N}}}$ and $\mathcal{M}_{2^{\mathbb{N}}}^{*}$. Notice that given any $\mathbf{S} \in \mathcal{I}^{\mathbb{N}}$, we can construct the sequence

$$(ret_{[0,\mathbf{S}_i)}([0,1), T_\gamma, 0))_{i \in \mathbb{N}}$$

in a Borel way. It follows from the discussions in §3.1 that there exists a Borel map sending this sequence to some sequence $\underline{\mathcal{A}_S}$ in $(2^{\mathbb{Z}})^{\mathbb{N}}$ that lists the Boolean algebra generated by

$$\{\sigma^k(ret_{[0,\mathbf{S}_i)}([0,1), T_\gamma, 0)) : i \in \mathbb{N} \ \wedge \ k \in \mathbb{Z}\}$$

The sequence $\underline{\mathcal{A}_S}$ enumerates the elements of $\mathcal{A}_S$, possibly with repetitions. Assume for the moment that we have a Boolean algebra isomorphism $i_S : \mathcal{A}_S \to \mathbb{B}$. Then the corresponding elements of $\mathcal{M}_{2^{\mathbb{N}}}$ and $\mathcal{M}_{2^{\mathbb{N}}}^{*}$ which code $(S(\mathcal{A}_S), \xi_*)$ and $(S(\mathcal{A}_S), \xi_*, x_{\mathcal{A}_S})$ are given by

$$f(\mathbf{S})(U) = V \Leftrightarrow i_S(\sigma^{-1}(i_S^{-1}(U))) = V$$

and

$$g(\mathbf{S}) = (f(\mathbf{S}), \bigcap\{U \in \mathbb{B} : 0 \in i_S^{-1}(U)\})$$

respectively. We now describe how the sequence $(i_S(\underline{\mathcal{A}_S}(k)))_{k\in\mathbb{N}}$ in $\mathbb{B}^{\mathbb{N}}$ can be obtained from $\underline{\mathcal{A}_S}$ in a Borel way. Fix an enumeration of $\mathbb{B}$ and the set of finite partial bijections from $\mathbb{N}$ into $\mathbb{N}$. Using these enumerations, given $\underline{\mathcal{A}_S}$, we can construct a sequence (β_n) of Boolean algebra isomorphisms from finite Boolean subalgebras of $\mathcal{A}_S$ to finite Boolean subalgebras of $\mathbb{B}$ such that $\beta_n \subseteq \beta_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} \beta_n$ is an isomorphism between $\mathcal{A}_S$ and $\mathbb{B}$. (This is a standard back-and-forth argument. At stage $2n$ and $2n+1$, we make sure that n-th elements of $\underline{\mathcal{A}_S}$ and $\mathbb{B}$ are already in the domain and the range of β_{2n} and β_{2n+1} respectively.)

It is easily checked that the procedure above can be defined in a Borel way. Hence f and g are Borel maps from $\mathcal{I}^{\mathbb{N}}$ to $\mathcal{M}_{2^{\mathbb{N}}}$ and $\mathcal{M}^*_{2^{\mathbb{N}}}$ respectively. We now check that f and g are reductions from $\Delta^+_{\mathcal{I}}$ to $\cong_{tc}$ and $\cong^*_{tc}$ respectively. Pick $\mathbf{S}, \mathbf{S}' \in \mathcal{I}^{\mathbb{N}}$ such that $\mathbf{S}$ is $\Delta^+_{\mathcal{I}}$-equivalent to $\mathbf{S}'$. Then clearly

$$Ret(S(\mathcal{A}_S), \xi_*, x_{\mathcal{A}_S}) = \mathcal{A}_S = \mathcal{A}_{S'} = Ret(S(\mathcal{A}_{S'}), \xi_*, x_{\mathcal{A}_{S'}})$$

It follows from Corollary 2.2.3 that $g(\mathbf{S}) \cong^*_{tc} g(\mathbf{S}')$ and hence $f(\mathbf{S}) \cong_{tc} f(\mathbf{S}')$. Now pick $\mathbf{S}, \mathbf{S}' \in \mathcal{I}^{\mathbb{N}}$ such that $\mathbf{S}$ is not $\Delta^+_{\mathcal{I}}$-equivalent to $\mathbf{S}'$. Recall that $\mathrm{Dens}(\mathcal{A}_S)$ and $\mathrm{Dens}(\mathcal{A}_{S'})$ are contained in the $\mathbb{Q}$-spans of $\{1,\gamma\} \cup S$ and $\{1,\gamma\} \cup S'$ respectively. Moreover, we know that $S \subseteq \mathrm{Dens}(\mathcal{A}_S)$ and $S' \subseteq \mathrm{Dens}(\mathcal{A}_{S'})$. Since $\mathcal{I} \cup \{1,\gamma\}$ is $\mathbb{Q}$-linearly independent, we have that $\mathrm{Dens}(\mathcal{A}_S) \neq \mathrm{Dens}(\mathcal{A}_{S'})$. Then it follows from Corollary 4.1.3 that $f(\mathcal{A}_S) \ncong_{tc} f(\mathcal{A}_{S'})$ and hence $g(\mathcal{A}_S) \ncong^*_{tc} g(\mathcal{A}_{S'})$. $\square$

This proves Theorem B and completes the proof of Theorem C.

Chapter 5

Basic structure of Toeplitz subshifts

In this chapter, following [Wil84, Dow05], we will give a detailed overview of the structure of Toeplitz sequences and Toeplitz subshifts. For the next two chapters, fix a finite alphabet $\mathfrak{n} \in \mathbb{N}$.

5.1 Toeplitz sequences and their scales

A bi-infinite sequence $\alpha \in \mathfrak{n}^{\mathbb{Z}}$ is called a *Toeplitz sequence* over the alphabet $\mathfrak{n}$ if for all $i \in \mathbb{Z}$ there exists $j \in \mathbb{N}^{+}$ such that $\alpha(i+kj) = \alpha(i)$ for all $k \in \mathbb{Z}$. Equivalently, Toeplitz sequences are those in which every subblock appears periodically. Periodic sequences are obviously Toeplitz. However, we shall exclude these since we are interested in infinite subshifts generated by Toeplitz sequences and periodic sequences have finite orbits under σ. From now on, all Toeplitz sequences are assumed to be non-periodic unless stated otherwise.

Example 5.1.1. Let $\square$ denote the blank symbol. At stage 0 of our construction, we start with the two-sided constant sequence of blank symbols. For $i \geq 1$,

- At stage $2i-1$, we choose the index j corresponding to the leftmost blank symbol in the interval $[0, 2^i)$ not yet filled at the previous stages and replace each blank symbol at position $j + k2^{2i-1}$ with the symbol 0.

- At stage $2i$, we choose the index j corresponding to the rightmost blank symbol in the interval $[0, 2^{i+1})$ not yet filled at the previous stages and replace each blank symbol at position $j + k2^{2i}$ with the symbol 1.

We illustrate the first three stages of this construction below.

$$\ldots\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\ldots$$

$$\ldots 0\Box 0\Box 0\Box 0\Box 0\Box 0\Box 0\Box 0\Box 0\Box 0\Box 0\Box 0\Box 0\Box 0\Box 0\Box 0\Box\ldots$$

$$\ldots 0\Box 0\,1\,0\Box 0\,1\,0\Box 0\,1\,0\Box 0\,1\,0\Box 0\,1\,0\Box 0\,1\,0\Box 0\,1\,0\Box 0\,1\ldots$$

$$\ldots 0\,0\,0\,1\,0\Box 0\,1\,0\,0\,0\,1\,0\Box 0\,1\,0\,0\,0\,1\,0\Box 0\,1\,0\,0\,0\,1\,0\Box 0\,1\ldots$$

where the index of the leftmost slot is 0. Let α_i be the two-sided sequence over the alphabet $2 \cup \{\Box\}$ obtained at the i-th stage of this construction. It is easily checked that the sequence $\alpha \in 2^{\mathbb{Z}}$ defined by

$$\alpha(j) = \lim_{i \to \infty} \alpha_i(j)$$

is a Toeplitz sequence over the alphabet 2.

We shall see later that every Toeplitz sequence can be obtained by such a recursive construction. In order to carry out such an analysis, we will need the following objects associated to each sequence $\alpha \in \mathfrak{n}^{\mathbb{Z}}$ for each $p \in \mathbb{N}^+$.

- The *p-periodic parts* of α is defined to be the set of indices

  $$Per_p(\alpha) := \bigcup_{a \in \mathfrak{n}} Per_p(\alpha, a)$$

  where $Per_p(\alpha, a) := \{i \in \mathbb{Z} : \forall k \in \mathbb{Z}\ \alpha(i + pk) = a\}$ for each symbol $a \in \mathfrak{n}$. In other words,

  $$Per_p(\alpha) = \{i \in \mathbb{Z} : \forall k \in \mathbb{Z}\ \alpha(i) = \alpha(i + pk)\}$$

  p is called a *period* of α if $Per_p(\alpha) \neq \emptyset$. It follows from the definitions that the sequence α is a Toeplitz sequence if and only if $\bigcup_{p \in \mathbb{N}^+} Per_p(\alpha) = \mathbb{Z}$.

- The sequence obtained from α by replacing $\alpha(i)$ with the blank symbol $\Box$ for each $i \notin Per_p(\alpha)$ will be called the *p-skeleton* of α. The p-skeleton of α will be denoted by $Skel(\alpha, p)$.

- Any subblock of the p-skeleton of α which consists of non-blank symbols and which is preceded and followed by a blank symbol will be called a *filled p-block* of the p-skeleton of α.

- The indices of the p-skeleton of α containing the blank symbol will be called the *p-holes* of α.
- The set of *p-symbols* of α is the set of words $W_p(\alpha) = \{\alpha[kp, (k+1)p) : k \in \mathbb{Z}\}$.

For example, consider the Toeplitz sequence α constructed in Example 5.1.1. The set of essential periods of α is $\{2^i : i \in \mathbb{N}^+\}$. For each $i \in \mathbb{N}^+$, the 2^i-skeleton of α is the sequence α_i that is obtained at the i-th stage of the construction.

Proposition 5.1.2. *Let $\alpha \in \mathfrak{n}^{\mathbb{Z}}$ be a sequence and let $p, q \in \mathbb{N}^+$ be periods of α.*

a. If $p|q$, then $Per_p(\alpha) \subseteq Per_q(\alpha)$.

b. If $Per_p(\alpha) \subseteq Per_q(\alpha)$, then $Per_{gcd(p,q)}(\alpha) = Per_p(\alpha)$.

Proof. (a) Assume that $q = pk$ for some $k \in \mathbb{N}^+$. If $i \in Per_p(\alpha)$, then $\alpha(i) = \alpha(i+pkl)$ for all $l \in \mathbb{Z}$ and hence $i \in Per_{pk}(\alpha) = Per_q(\alpha)$. Thus $Per_p(\alpha) \subseteq Per_q(\alpha)$.

(b) Assume that $Per_p(\alpha) \subseteq Per_q(\alpha)$. By Bézout's identity, there exists $k_1, k_2 \in \mathbb{Z}$ such that $gcd(p, q) = k_1 p + k_2 q$. If $i \in Per_p(\alpha) \subseteq Per_q(\alpha)$, then

$$\alpha(i) = \alpha(i + kk_1 p) = \alpha(i + kk_1 p + kk_2 q) = \alpha(i + kgcd(p, q))$$

for any $k \in \mathbb{Z}$ and hence $i \in Per_{gcd(p,q)}(\alpha)$. Thus $Per_p(\alpha) \subseteq Per_{gcd(p,q)}(\alpha)$. The converse inclusion follows from part (a). □

Let $\alpha \in \mathfrak{n}^{\mathbb{Z}}$. On the one hand, any multiple of a period of α is a period of α by Proposition 5.1.2.a. On the other hand, we do not want to take "irrelevant" periods into consideration. Using Proposition 5.1.2.b, we can restrict our attention to those periods that are minimal among the periods that give the same p-skeletons. A positive integer $p \in \mathbb{N}^+$ is called an *essential period* of α if p is a period of α and $Per_q(\alpha) \neq Per_p(\alpha)$ for all $1 \leq q < p$. Equivalently, p is an essential period of α if and only if the p-skeleton of α is not periodic with any smaller period.

Proposition 5.1.3. *If $p, q \in \mathbb{N}^+$ are essential periods of a sequence α then so is $lcm(p, q)$.*

Proof. Let p and q be essential periods of α. Assume to the contrary that $lcm(p,q)$ is not an essential period. Then there exists a period r with $1 \leq r < lcm(p,q)$ such that

$$Per_r(\alpha) = Per_{lcm(p,q)}(\alpha) \supseteq Per_p(\alpha), Per_q(\alpha)$$

By Proposition 5.1.2, we have that

$$Per_p(\alpha) = Per_{gcd(p,r)}(\alpha) \text{ and } Per_q(\alpha) = Per_{gcd(q,r)}(\alpha)$$

Since p and q are essential periods, $gcd(p,r) = p$ and $gcd(q,r) = q$ and hence $lcm(p,q)|r$ contradicting the assumption that $r < lcm(p,q)$. $\square$

Thus we can associate a supernatural number to each sequence by taking the least common multiple of its essential periods. The *scale* of a Toeplitz sequence α is the supernatural number $\mathbf{u}_\alpha = \mathrm{lcm}(u_i)_{i\in\mathbb{N}}$ where u_i is an enumeration of the essential periods of α.

5.2 Toeplitz subshifts and their maximal equicontinuous factors

Every subblock of a Toeplitz sequence α appears periodically along α and hence the return times of α to any basic clopen subset of its shift orbit closure $\overline{\mathrm{Orb}}(\alpha)$ contains an infinite progression of the form $p + q\mathbb{Z}$. It follows that α is an almost periodic point of $(\overline{\mathrm{Orb}}(\alpha), \sigma)$ and hence $(\overline{\mathrm{Orb}}(\alpha), \sigma)$ is a minimal subshift by Theorem 1.6.1. From now on, such minimal subshifts will be called *Toeplitz subshifts*.

In this section, we will prove that the maximal equicontinuous factor of a Toeplitz subshift $\overline{\mathrm{Orb}}(\alpha)$ is the odometer associated to the supernatural number $\mathbf{u}_\alpha$. All results in this section are originally due to Williams [Wil84]. However, the statements of the following results are slightly more general than Williams' original results and hence we include the proofs for completeness. For the rest of this section, fix a Toeplitz sequence $\alpha \in \mathfrak{n}^{\mathbb{Z}}$.

Lemma 5.2.1. *[Wil84] Let $p \in \mathbb{N}^+$ and for each $0 \leq k < p$, define*

$$A(\alpha, p, k) := \{\sigma^i(\alpha) : i \equiv k \ (mod\ p)\}$$

Then each element of $\overline{A(\alpha,p,k)}$ *has the same* p*-skeleton as* $\sigma^k(\alpha)$*, i.e. for each symbol* $a \in \mathfrak{n}$*, we have that* $Per_p(\sigma^k(\alpha),a) = Per_p(\gamma,a)$ *for all* $\gamma \in \overline{A(\alpha,p,k)}$*.*

Proof. Let $\gamma \in \mathfrak{n}^{\mathbb{Z}}$ be in $\overline{A(\alpha,p,k)}$. Then there exists a sequence $\sigma^{pm_i+k}(\alpha)$ converging to γ. Since $Per_p(\sigma^{pm_i+k}(\alpha),a) = Per_p(\sigma^k(\alpha),a)$ for all $i \in \mathbb{N}$, we have that $Per_p(\sigma^k(\alpha),a) \subseteq Per_p(\gamma,a)$ for each $a \in \mathfrak{n}$.

Now assume that for some symbol $a \in \mathfrak{n}$ there exists $m \in Per_p(\gamma,a) - Per_p(\sigma^k(\alpha),a)$. Since $m \notin Per_p(\sigma^k(\alpha),a)$, there exist another symbol $b \neq a$ and an integer m' such that $m' \equiv m \ (mod\ p)$ and $\sigma^k(\alpha)(m') = b$. But since $\sigma^k(\alpha)$ is a Toeplitz sequence, we can find $q \in \mathbb{N}^+$ such that $m' \in Per_q(\sigma^k(\alpha),b) \subseteq Per_{pq}(\sigma^k(\alpha),b)$. This implies that for any $\gamma' \in A(\alpha,p,k)$ there exists $0 \leq m'' < pq$ with $m'' \equiv m' \equiv m \ (mod\ p)$ such that $\gamma'(m'') = b$. However, $\gamma(m'') = a$ for all $m'' \equiv m \ (mod\ p)$ by assumption and hence no sequence of elements of $A_k^{r_i}$ can converge to γ which contradicts our assumption. Thus $Per_p(\sigma^k(\alpha),a) \supseteq Per_p(\gamma,a)$ for all $a \in \mathfrak{n}$, which completes the proof. □

Lemma 5.2.2. *[Wil84] Let* $(r_i)_{i\in\mathbb{N}}$ *be a factorization of* $\mathbf{u}_\alpha$ *and let* $A(\alpha,r_i,k)$ *be defined as in Lemma 5.2.1. For each* $i \in \mathbb{N}$*, we have that*

a. $\{\overline{A(\alpha,r_i,k)} : 0 \leq k < r_i\}$ *is a partition of* $\overline{Orb}(\alpha)$*.*

b. $\overline{A(\alpha,r_i,k)} \subseteq \overline{A(\alpha,r_j,l)}$ *for all* $j < i$ *and* $k \equiv l \ (mod\ r_j)$*.*

c. $\sigma[\overline{A(\alpha,r_i,k)}] = \overline{A(\alpha,r_i,k+1)}$ *for* $k < r_i - 1$ *and* $\sigma[\overline{A(\alpha,r_i,r_i-1)}] = \overline{A(\alpha,r_i,0)}$*.*

Proof. By the definition of $\overline{\mathrm{Orb}}(\alpha)$, we have that $\bigcup\{\overline{A(\alpha,r_i,k)} : 0 \leq k < r_i\} \subseteq \overline{\mathrm{Orb}}(\alpha)$. Let $i \in \mathbb{N}$ and let p be an essential period of α such that $r_i | p$, say $p = q.r_i$. We will first prove that $A(\alpha,p,k)$ and $A(\alpha,p,l)$ are at a positive distance apart from each other if $k \not\equiv l \ (mod\ p)$. Let k and l be integers such that $k \not\equiv l \ (mod\ p)$. Since p is an essential period of α, $\sigma^k(\alpha)$ and $\sigma^l(\alpha)$ must have different p-skeletons, which implies that there exists $a \in \mathfrak{n}$ such that $m \in Per_p(\sigma^k(\alpha),a) - Per_p(\sigma^l(\alpha),a)$ for some integer m. Then there exists $m' \equiv m \ (mod\ p)$ such that $\sigma^l(\alpha)(m') = b$ for some symbol $b \neq a$. Let q' be a period of $\sigma^l(\alpha)$ such that $p|q'$ and $m' \in Per_{q'}(\sigma^l(\alpha),b)$. Observe that for any $\gamma \in A(\alpha,p,l)$ there exists $0 \leq m'' < q'$ such that $m'' \equiv m' \equiv m \ (mod\ p)$ and $\gamma(m'') = b$.

However, we know that $\gamma'(m'') = a$ for all $\gamma' \in A(\alpha, p, k)$ and for all $m'' \equiv m \ (mod\ p)$. Hence the distance between γ and γ' is greater or equal to $2^{-(q'+1)}$ for any $\gamma \in A(\alpha, p, l)$ and $\gamma' \in A(\alpha, p, k)$.

For any $0 \leq k < r_i$, we have that $A(\alpha, r_i, k) = \bigsqcup_{j=0}^{q-1} A(\alpha, p, jr_i + k)$. Since the sets $A(\alpha, p, jr_i + k)$ are all at a positive distance apart from each other, it follows that the sets $\overline{A(\alpha, p, jr_i + k)}$ are disjoint and that

$$\overline{A(\alpha, r_i, k)} = \overline{\bigsqcup_{j=0}^{q-1} A(\alpha, p, jr_i + k)} = \bigsqcup_{j=0}^{q-1} \overline{A(\alpha, p, jr_i + k)}$$

Hence the sets $\overline{A(\alpha, r_i, k)}$ are disjoint. Similarly, we have that

$$\overline{\mathrm{Orb}}(\alpha) = \overline{\bigsqcup_{k=0}^{r_i-1} \bigsqcup_{j=0}^{q-1} A(\alpha, p, jr_i + k)} = \bigsqcup_{k=0}^{r_i-1} \overline{A(\alpha, r_i, k)} = \bigcup \{\overline{A(\alpha, r_i, k)} : 0 \leq k < r_i\}$$

which completes the proof of (a). Statements (b) and (c) easily follow from the definitions. $\square$

Before we construct the maximal equicontinuous factor of $\overline{\mathrm{Orb}}(\alpha)$, we will mention an important consequence of these lemmas. By Lemma 5.2.1 and Lemma 5.2.2.a, any essential period of α is an essential period of any $\gamma \in \overline{\mathrm{Orb}}(\alpha)$ and vice versa. Therefore, it makes sense to define the *scale* of a Toeplitz subshift O to be the supernatural number that is the least common multiple of all essential periods of some (equivalently, every) point of O.

Consider the map $\psi : \overline{\mathrm{Orb}}(\alpha) \to \mathtt{Odo}(u_i)_{i\in\mathbb{N}}$ given by

$$\psi(x) = (m_i)_{i\in\mathbb{N}}$$

where $x \in \overline{A(\alpha, u_i, m_i)}$ and the sequence $(u_i)_{i\in\mathbb{N}}$ is a factorization of $\mathbf{u}_\alpha$. The map ψ is continuous since the inverse images of the basic clopen sets of $\mathtt{Odo}(u_i)_{i\in\mathbb{N}}$ are clopen in $\overline{\mathrm{Orb}}(\alpha)$. (This follows from the fact that the sets in the partitions in Lemma 5.2.2.a are clopen in the relative topology.) Moreover, we have that $\psi \circ \sigma = \eta \circ \psi$ and hence ψ is a factor map. In order to show that $\mathtt{Odo}(u_i)_{i\in\mathbb{N}}$ is the maximal equicontinuous factor of $\overline{\mathrm{Orb}}(\alpha)$, it is sufficient to prove that $\psi^{-1}[\psi(\alpha)] = \{\alpha\}$. (For example, see [Pau76, Proposition 1.1].)

Recall by Lemma 5.2.1 that the u_i-skeletons of the sequences in the set $\overline{A(\alpha, u_i, m_i)}$ are the same. Hence two sequences $\beta, \beta' \in \overline{\mathrm{Orb}}(\alpha)$ have the same u_k-skeleton whenever $\psi(\beta) \upharpoonright k+1 = \psi(\beta') \upharpoonright k+1$. This implies that ψ is one to one on the set of Toeplitz sequences since every subblock of a Toeplitz sequence eventually appears in some u_k-skeleton. In particular, we have that $\psi^{-1}[\psi(\alpha)] = \{\alpha\}$, which completes the proof that $\mathtt{Odo}(u_i)_{i\in\mathbb{N}}$ is the maximal equicontinuous factor of $\overline{\mathrm{Orb}}(\alpha)$.

Since the maximal equicontinuous factor of a topological dynamical system is unique up to topological conjugacy, it follows from Theorem 1.6.3 that topologically conjugate Toeplitz subshifts have the same scale.

5.3 Various subclasses of Toeplitz subshifts

Given a Toeplitz sequence α and a factorization $(u_i)_{i\in\mathbb{N}}$ of its scale $\mathbf{u}_\alpha$, we can imagine α being obtained by a recursive construction where we start the construction with the two-sided constant sequence of blank symbols and replace the blank symbols corresponding to the indices $Per_{u_i}(\alpha)$ periodically with the appropriate symbols at the i-th stage. This way of understanding Toeplitz sequences from their constructions allows us to isolate some special types of Toeplitz sequences as considered by Downarowicz [Dow05, Section 9].

Of particular interest in this thesis will be the class of Toeplitz subshifts with separated holes. A Toeplitz subshift O is said to have *separated holes* with respect to $(u_i)_{i\in\mathbb{N}}$ if the minimal distance between the u_i-holes in the u_i-skeleton of every (equivalently, some) element of O grows to infinity with i, where $(u_i)_{i\in\mathbb{N}}$ is a factorization of the scale of O. It turns out that whether or not a Toeplitz subshift has separated holes is independent of the particular factorization $(u_i)_{i\in\mathbb{N}}$.

Proposition 5.3.1. *Let $(u_i)_{i\in\mathbb{N}}$ and $(v_i)_{i\in\mathbb{N}}$ be two factorizations of the same supernatural number $\mathbf{u}$ and let O be a Toeplitz subshift with scale $\mathbf{u}$. Then O has separated holes with respect to $(u_i)_{i\in\mathbb{N}}$ if and only if O has separated holes with respect to $(v_i)_{i\in\mathbb{N}}$.*

Proof. Let O be a Toeplitz subshift with scale $\mathbf{u}$ and let α be some sequence in O.

Assume that O has separated holes with respect to $(u_i)_{i\in\mathbb{N}}$. It follows from

$$\text{lcm}(u_i)_{i\in\mathbb{N}} = \text{lcm}(v_i)_{i\in\mathbb{N}}$$

that for any $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $u_i | v_j$ and hence $Per_{u_i}(\alpha) \subseteq Per_{v_j}(\alpha)$ by Proposition 5.1.2. It follows that O has separated holes with respect to some subsequence $(v_{j_k})_{k\in\mathbb{N}}$. But the minimal distance between v_i-holes in the v_i-skeleton of α cannot decrease as a function of i since $Per_{v_i}(\alpha) \subseteq Per_{v_{i+1}}(\alpha)$. Hence O has separated holes with respect to $(v_i)_{i\in\mathbb{N}}$. Carrying out this argument symmetrically, we have that if O has separated holes with respect to $(v_i)_{i\in\mathbb{N}}$, then it also has separated holes with respect to $(u_i)_{i\in\mathbb{N}}$. $\square$

In this section, we will define a property that generalizes the property of having separated holes. Given a Toeplitz subshift O and a Toeplitz sequence $\alpha \in O$, let $A(\alpha, p, k)$ be defined as in Lemma 5.2.1. Notice that for any $\beta \in \overline{\text{Orb}}(\alpha)$, regardless of whether or not β is a Toeplitz sequence, we have that

$$\{\overline{A(\beta, u_i, k)} : 0 \leq k < u_i\} = \{\overline{A(\alpha, u_i, k)} : 0 \leq k < u_i\}$$

since the orbit of β is dense in $\overline{\text{Orb}}(\alpha)$ by minimality. Therefore, this partition only depends on u_i and it will be denoted by $Parts(\overline{\text{Orb}}(\alpha), u_i)$.

By Lemma 5.2.1, every element of $\overline{A(\alpha, u_i, k)}$ has the same u_i-skeleton. Consequently, for each $W \in Parts(\overline{\text{Orb}}(\alpha), u_i)$, we can define the *$u_i$-skeleton* of W to be the u_i-skeleton of some (equivalently, every) element of W and denote it by $Skel(W, u_i)$. Define $Parts_*(O, u_i)$ to be the set

$$\{W \in Parts(O, u_i) : Skel(W, u_i)(0) \neq \square \ \wedge \ Skel(W, u_i)(-1) = \square\}$$

If $Parts_*(O, u_i)$ is non-empty, then for each $W \in Parts_*(O, u_i)$, let $length(W)$ be the smallest positive integer such that $Skel(u_i, W)(length(W)) = \square$. In other words, $length(W)$ is the length of the filled u_i-block of the u_i-skeleton of W whose first non-blank symbol is positioned at index 0. We note that the set $Parts_*(O, u_i)$ is non-empty for all but finitely many i. The Toeplitz subshift O is said to have *growing blocks* with respect to $(u_i)_{i\in\mathbb{N}}$ if

$$\lim_{i\to\infty} min\{length(W) : W \in Parts_*(O, u_i)\} = +\infty$$

i.e., O has growing blocks with respect to $(u_i)_{i \in \mathbb{N}}$ if the minimal length of filled u_i-blocks grows to infinity with i.

If a Toeplitz subshift O has separated holes with respect to a factorization of its scale $\mathbf{u}$, then it has separated holes with respect to any factorization of $\mathbf{u}$ by Proposition 5.3.1 and hence it has growing blocks with respect to any factorization of $\mathbf{u}$.

Unfortunately, unlike having separated holes, having growing blocks is not independent of the factorization $(u_i)_{i \in \mathbb{N}}$. Consider the Toeplitz sequence whose (2^k5)-skeletons restricted to the interval $[0, 2^k5)$ are given by

$$0\square\square\square0$$

$$0\square 1 \square 00\square\square\square0$$

$$0\,0\,1\,0\,00\square\square\square00111001010$$

$$0\,0\,1\,0\,00\square 1 \square 00111001010001000\square\square\square00111001010$$

$$\cdots$$

for each $k \in \mathbb{N}$. We initially start with the 5-skeleton consisting of the repeated blocks $0\square\square\square0$. At every odd stage k, we fill the hole in the middle of the leftmost $\square\square\square$ block along each interval $[j2^k5, (j+1)2^k5)$ with the symbol 1. At every even stage k, along each interval $[j2^k5, (j+1)2^k5)$, we fill the first two single holes with the symbol 0, the remaining single holes with the symbol 1, and replace the rightmost $\square\square\square$ block by the block 101. It is easily checked that the Toeplitz subshift generated by this Toeplitz sequence does not have growing blocks with respect to $(2^k5)_{k \in \mathbb{N}}$. However, it does have growing blocks with respect to $(4^k5)_{k \in \mathbb{N}}$.

5.4 The standard Borel spaces of various subclasses of Toeplitz subshifts

In this section, we construct the standard Borel spaces of various subclasses of Toeplitz subshifts over the alphabet $\mathfrak{n}$ as subspaces of the space of minimal subshifts over the alphabet $\mathfrak{n}$ constructed in §3.2. We will need the following theorem regarding the Borel definability of Baire category notions.

Theorem 5.4.1. *[ST15] Let X be a Polish space and let $F(X)$ be the Effros Borel space $F(X)$ consisting of closed subsets of X. Then for any Borel subset $A \subseteq X$, the set*

$$\{F \in F(X) : (\exists^* x \in F)\ x \in A\}$$

is Borel, where the quantifier $\exists^* x \in F$ stands for "For non-meagerly many x in F".

Recall that minimal subshifts form a Borel subset of $K(\mathfrak{n}^{\mathbb{Z}})$, which is identical to $F(\mathfrak{n}^{\mathbb{Z}})$ by the compactness of $\mathfrak{n}^{\mathbb{Z}}$. Since the Toeplitz sequences of a Toeplitz subshift form a dense G_δ subset [Dow05, Theorem 5.1] and the set of Toeplitz sequences is a Borel subset of $\mathfrak{n}^{\mathbb{Z}}$, it follows from Theorem 5.4.1 that the set

$$\mathcal{T}_{\mathfrak{n}} := \{O \in K(\mathfrak{n}^{\mathbb{Z}}) : O \text{ is a Toeplitz subshift}\}$$

is a Borel subset of $K(\mathfrak{n}^{\mathbb{Z}})$ and hence is a standard Borel space.

Next we will construct the standard Borel spaces of Toeplitz subshifts with growing blocks and separated holes. However, since having growing blocks is not independent of the factorization we use for each supernatural number, in order to talk about the standard Borel space of Toeplitz subshifts with growing blocks, we need to fix a map that assigns a factorization to each supernatural number. Moreover, we want to express the property of having growing blocks with a Borel condition and hence the factorization map we will use should be Borel when considered as a function from $(\mathbb{N} \cup \{\infty\})^{\mathbb{N}}$ to $(\mathbb{N}^+)^{\mathbb{N}}$. Given a supernatural number $\mathbf{r} = \prod_{i \in \mathbb{N}^+} \mathrm{p}_i^{k_i}$, let

$$\dot{r}_t = \prod_{1 \le i \le t+1} \mathrm{p}_i^{min\{k_i, t+1\}}$$

and define the natural factorization $(r_t)_{t \in \mathbb{N}}$ of $\mathbf{r}$ to be the sequence obtained from the sequence $(\dot{r}_t)_{t \in \mathbb{N}}$ by deleting all 1's and the repeated terms. We note that all results in this thesis hold for any Borel factorization of supernatural numbers.

Recall that there exists a Borel map which chooses a point from each element in $K(\mathfrak{n}^{\mathbb{Z}})$. Now fix a Borel map that chooses a point from each element of $\mathcal{T}_{\mathfrak{n}}$. Since all points in a Toeplitz subshift have the same essential periods, we can construct a Borel map from $\tau : \mathcal{T}_{\mathfrak{n}} \to (\mathbb{N}^+)^{\mathbb{N}}$ that sends each Toeplitz subshift to the natural factorization of its scale.

By Lemma 5.2.1 and Lemma 5.2.2.a, the p-skeleton structures of all points in a Toeplitz subshift are the same, up to shifting. Moreover, both having separated holes and growing blocks with respect to the natural factorization can be expressed by Borel conditions. Thus both

$$\mathcal{T}_{\mathfrak{n}}^{*} := \{O \in \mathcal{T}_{\mathfrak{n}} : O \text{ has separated holes}\}$$

and

$$\mathcal{T}_{\mathfrak{n}}^{**} := \{O \in \mathcal{T}_{\mathfrak{n}} : O \text{ has growing blocks with respect to } \tau(O)\}$$

are Borel subsets of $\mathcal{T}_{\mathfrak{n}}$ and hence are standard Borel spaces. The restriction of the topological conjugacy relation to $\mathcal{T}_{\mathfrak{n}}$, $\mathcal{T}_{\mathfrak{n}}^{*}$, and $\mathcal{T}_{\mathfrak{n}}^{**}$ are clearly countable Borel equivalence relations. Moreover, it follows from the work of Thomas [Tho13] that the topological conjugacy relation on $\mathcal{T}_{\mathfrak{n}}^{*}$ is not smooth.

5.5 Topological conjugacy of Toeplitz subshifts

In [DKL95], Downarowicz, Kwiatkowski, and Lacroix found a criterion for Toeplitz subshifts to be topologically conjugate. In the proof of Theorem A, we will need this criterion in a slightly more general form than it was originally formulated. In this section, we will include these more general statements with their proofs. We note that all results in this section are extracted from [DKL95, Theorem 1].

Lemma 5.5.1. *Let (O, σ, α) and (O', σ, β) be pointed Toeplitz subshifts and let π be a topological conjugacy from (O, σ, α) to (O', σ, β). Then for any $p \in \mathbb{N}^+$ such that $[-|\pi|, |\pi|] \subseteq Per_p(\alpha), Per_p(\beta)$ there exists $\phi \in Sym(\mathfrak{n}^p)$ such that*

$$\phi(\alpha[kp, (k+1)p)) = \beta[kp, (k+1)p) \text{ for all } k \in \mathbb{Z}$$

where $|\pi|$ denotes the length of the topological conjugacy π as defined in §1.6.4.

Proof. Let $p \in \mathbb{N}^+$ be such that $[-|\pi|, |\pi|] \subseteq Per_p(\alpha), Per_p(\beta)$. Consider the relation $\Gamma : W_p(\alpha) \to W_p(\beta)$ given by

$$\Gamma(\alpha[kp, (k+1)p)) = \beta[kp, (k+1)p)$$

for each $k \in \mathbb{Z}$. We want to prove that Γ is well-defined and one to one. Pick $k, k' \in \mathbb{Z}$ such that $\alpha[kp,(k+1)p) = \alpha[k'p,(k'+1)p)$. Since $[-|\pi|,|\pi|] \subseteq Per_p(\alpha), Per_p(\beta)$ we have that

$$\alpha[kp-|\pi|,(k+1)p+|\pi|] = \alpha[k'p-|\pi|,(k'+1)p+|\pi|]$$

By the definition of $|\pi|$, there exists some block code C inducing π such that $|C| \leq |\pi|$. Then we have that

$$\begin{aligned}
\beta(kp+u) &= (\pi(\alpha))(kp+u) \\
&= C(\alpha[kp+u-|C|,kp+u+|C|]) \\
&= C(\alpha[k'p+u-|C|,k'p+u+|C|]) \\
&= (\pi(\alpha))(k'p+u) \\
&= \beta(k'p+u)
\end{aligned}$$

for any $0 \leq u < p$ and hence $\beta[kp,(k+1)p) = \beta[k'p,(k'+1)p)$. This proves that Γ is well-defined. Since there exists a block code C' inducing π^{-1} such that $|C'| \leq |\pi|$, a symmetrical argument shows that Γ is one to one. It follows that Γ is a bijection and hence we can choose $\phi \in Sym(\mathfrak{n}^p)$ to be any permutation extending Γ. $\square$

Lemma 5.5.2. *Let O and O' be Toeplitz subshifts with the same scale $\boldsymbol{r}$. Assume that there exist a factor p of $\boldsymbol{r}$ and $\phi \in Sym(\mathfrak{n}^p)$ such that*

$$\phi(\alpha[kp,(k+1)p)) = \beta[kp,(k+1)p) \text{ for all } k \in \mathbb{Z}$$

for some points $\alpha \in O$ and $\beta \in O'$. Then (O,σ,α) and (O',σ,β) are pointed topologically conjugate.

Proof. Observe that ϕ induces a homeomorphism $\widehat{\phi}$ of $\mathfrak{n}^{\mathbb{Z}}$ defined by

$$\widehat{\phi}(\gamma)[kp,(k+1)p) = \phi(\gamma[kp,(k+1)p))$$

for all $k \in \mathbb{Z}$ and $\gamma \in \mathfrak{n}^{\mathbb{Z}}$. Obviously $\widehat{\phi}(\sigma^{pk}(\alpha)) = \sigma^{pk}(\beta)$ for any $k \in \mathbb{Z}$. Let

$$A(\alpha,p,0) = \{\sigma^i(\alpha) : i \equiv 0 \ (mod\ p)\}$$

and

$$A(\beta,p,0) = \{\sigma^i(\beta) : i \equiv 0 \ (mod\ p)\}$$

Since $\widehat{\phi}$ is a homeomorphism and $(\widehat{\phi})^{-1} = \widehat{\phi^{-1}}$, it easily follows that

$$\widehat{\phi}[\overline{A(\alpha, p, 0)}] = \overline{A(\beta, p, 0)}$$

Recall by Lemma 5.2.2 that $\{\overline{A(\alpha, p, k)} : 0 \leq k < p\}$ and $\{\overline{A(\beta, p, k)} : 0 \leq k < p\}$ are partitions of O and O' respectively. Let π be the map from O to O' given by

$$\pi(\gamma) = \sigma^i(\widehat{\phi}(\sigma^{-i}(\gamma))) \text{ if } \gamma \in \overline{A(\alpha, p, i)}$$

Obviously π is a bijection between O and O'. Moreover, it is continuous on each $\overline{A(\alpha, p, i)}$. Since the sets $\overline{A(\alpha, p, i)}$ are at a positive distant apart from each other, it follows that π is continuous on O and hence is a homeomorphism between O and O'. We want to show that π is shift preserving. For any $0 \leq i < p-2$ and for any $\gamma \in \overline{A(\alpha, p, i)}$, we have that

$$\pi(\sigma(\gamma)) = \sigma^{i+1}(\widehat{\phi}(\sigma^{-(i+1)}(\sigma(\gamma)))) = \sigma(\sigma^i(\widehat{\phi}(\sigma^{-i}(\gamma)))) = \sigma(\pi(\gamma))$$

Since $\widehat{\phi}$ commutes with σ^p, for any $\gamma \in \overline{A(\alpha, p, p-1)}$ we have that

$$\begin{aligned}
\sigma(\pi(\gamma)) &= \sigma(\sigma^{(p-1)}(\widehat{\phi}(\sigma^{-(p-1)}(\gamma)))) \\
&= \sigma^p(\widehat{\phi}(\sigma^{-(p-1)}(\gamma))) \\
&= \widehat{\phi}(\sigma^p(\sigma^{-(p-1)}(\gamma))) \\
&= \widehat{\phi}(\sigma(\gamma)) \\
&= \pi(\sigma(\gamma))
\end{aligned}$$

Therefore, π is a topological conjugacy between O and O' sending α to β. $\square$

We remark that the proofs of Lemma 5.5.1 and Lemma 5.5.2 together imply that if O and O' are topologically conjugate Toeplitz subshifts, then some elements of the partition $Parts(O, p)$ are mapped onto some elements of the partition $Parts(O', p)$ under the natural action of $Sym(n^p)$ for a sufficiently large factor p of the common scale.

Chapter 6

Proof of Theorem A

In this chapter, we shall prove Theorem A. Indeed, we will prove the stronger result that the topological conjugacy relation on the standard Borel space $\mathcal{T}_{\mathfrak{n}}^{**}$ of Toeplitz subshifts with growing blocks is hyperfinite.

Recall that the set $K(\mathfrak{n}^{\mathbb{Z}})$ of non-empty compact subsets of $\mathfrak{n}^{\mathbb{Z}}$ is a Polish space endowed with the topology induced by the Hausdorff metric defined in §3.2. For each $p \in \mathbb{N}^{+}$, consider the action of the symmetric group $Sym(\mathfrak{n}^{p})$ on $K(\mathfrak{n}^{\mathbb{Z}})$ defined by

$$\phi \cdot K \mapsto \widehat{\phi}[K]$$

where $\widehat{\phi}$ is the homeomorphism of $\mathfrak{n}^{\mathbb{Z}}$ given by

$$\widehat{\phi}(\gamma)[kp,(k+1)p) = \phi(\gamma[kp,(k+1)p)) \text{ for all } k \in \mathbb{Z} \text{ and for all } \gamma \in \mathfrak{n}^{\mathbb{Z}}.$$

Since there exists a sequence of Borel functions that choose a dense set of points from each element of $K(\mathfrak{n}^{\mathbb{Z}})$ [Kec95, Theorem 12.23], given $K, K' \in K(\mathfrak{n}^{\mathbb{Z}})$, we can check in a Borel way whether or not a dense subset of K is mapped onto a dense subset K' under $\widehat{\phi}$. By the continuity of $\widehat{\phi}$ and the closedness of K and K', this is equivalent to $\widehat{\phi}[K] = K'$. It follows that the action of $Sym(\mathfrak{n}^{p})$ on $K(\mathfrak{n}^{\mathbb{Z}})$ defined above is Borel. Let $\mathtt{D}_p$ denote the orbit equivalence relation of this Borel action. Clearly $\mathtt{D}_p$ is a finite Borel equivalence relation since it is the orbit equivalence relation of a Borel action of a finite group on a standard Borel space.

Before we present the proof of the main result of this chapter, we will prove the following easy but useful proposition which shows that if there exists a p-hole in the p-skeleton of a sequence α, then there exists a p-hole in the p-skeleton of its image under a block code, which is no further from the p-hole in the sequence α than the length of the block code.

Proposition 6.1. *Let (O, σ, α) and (O', σ, β) be pointed Toeplitz subshifts and let π be a factor map from O onto O' such that $\pi(\alpha) = \beta$. Assume that $m \in \mathbb{N}^+$ is the length of some block code C inducing π. Then for all $p \in \mathbb{N}^+$ and $k \in \mathbb{Z}$, we have that $k \in Per_p(\beta)$ whenever $[k-m, k+m] \subseteq Per_p(\alpha)$.*

Proof. For all $p \in \mathbb{N}^+$ and $k \in \mathbb{Z}$, if $[k-m, k+m] \subseteq Per_p(\alpha)$, then for all $l \in \mathbb{Z}$

$$\begin{aligned}\beta(k+pl) = (\pi(\alpha))(k+pl) &= C(\alpha[k+pl-m, k+pl+m]) \\ &= C(\alpha[k-m, k+m]) \\ &= (\pi(\alpha))(k) = \beta(k)\end{aligned}$$

which implies that $k \in Per_p(\beta)$. □

We are now ready to present the main theorem of this chapter.

Theorem 6.2. *The topological conjugacy relation on $\mathcal{T}_{\mathfrak{n}}^{**}$ is Borel reducible to E_1.*

Proof. It is not difficult to check that the set

$$\mathrm{Fin}(K(\mathfrak{n}^{\mathbb{Z}})) := \{F \subseteq K(\mathfrak{n}^{\mathbb{Z}}) : F \text{ is finite and non-empty}\}$$

is a Borel subset of $K(K(\mathfrak{n}^{\mathbb{Z}}))$ and hence is a standard Borel space. Let $\mathtt{D}_p^{\mathrm{fin}}$ be the equivalence relation on $\mathrm{Fin}(K(\mathfrak{n}^{\mathbb{Z}}))$ given by

$$(F, F') \in \mathtt{D}_p^{\mathrm{fin}} \Leftrightarrow \{[W]_{\mathtt{D}_p} : W \in F\} = \{[W]_{\mathtt{D}_p} : W \in F'\}$$

for each $p \in \mathbb{N}^+$. Even though $\mathtt{D}_p^{\mathrm{fin}}$ is not a subrelation of $\mathtt{D}_p^+$, we can think of $\mathtt{D}_p^{\mathrm{fin}}$ as the restriction of the Friedman-Stanley jump of $\mathtt{D}_p$ to the finite subsets of $K(\mathfrak{n}^{\mathbb{Z}})$.

Let $\backsim_{\mathfrak{n}}$ be the equivalence relation on $(\mathbb{N}^+)^{\mathbb{N}} \times (\mathrm{Fin}(K(\mathfrak{n}^{\mathbb{Z}})))^{\mathbb{N}}$ defined by

$$(r, (F_i)_{i\in\mathbb{N}}) \backsim_{\mathfrak{n}} (s, (F'_i)_{i\in\mathbb{N}}) \Longleftrightarrow r = s \ \wedge \ \exists j \ \forall i \geq j \ (F_i, F'_i) \in \mathtt{D}_{r_i}^{\mathrm{fin}}$$

Since each $\mathtt{D}_p$ is a finite Borel equivalence relation, each $\mathtt{D}_p^{\mathrm{fin}}$ is a finite Borel equivalence relation and hence is smooth. Fix a Borel isomorphism $h_0 : (\mathbb{N}^+)^{\mathbb{N}} \to 2^{\mathbb{N}}$ and for each $p \in \mathbb{N}^+$, fix some Borel reduction $h_p : \mathrm{Fin}(K(\mathfrak{n}^{\mathbb{Z}}))) \to 2^{\mathbb{N}}$ from $\mathtt{D}_p^{\mathrm{fin}}$ to $\Delta_{2^{\mathbb{N}}}$. Then it is easily checked that the map $h : (\mathbb{N}^+)^{\mathbb{N}} \times (\mathrm{Fin}(K(\mathfrak{n}^{\mathbb{Z}})))^{\mathbb{N}} \to (2^{\mathbb{N}})^{\mathbb{N}}$ given by

$$(r, (F_i)_{i\in\mathbb{N}}) \mapsto (h_0(r), h_{r_0}(F_0), h_0(r), h_{r_1}(F_1), \dots)$$

is a Borel reduction from $\backsim_{\mathfrak{n}}$ to E_1. Thus it is sufficient to prove that the topological conjugacy relation on $\mathcal{T}_{\mathfrak{n}}^{**}$ is Borel reducible to $\backsim_{\mathfrak{n}}$. Let $f : \mathcal{T}_{\mathfrak{n}}^{**} \to (\mathbb{N}^{+})^{\mathbb{N}} \times (\mathrm{Fin}(K(\mathfrak{n}^{\mathbb{Z}})))^{\mathbb{N}}$ be the map given by

$$f(O) = (\tau(O), \chi(O))$$

where $\tau(O)$ is the natural factorization of the scale of O defined in §5.4 and the sequence $\chi(O)$ is defined as follows. For each $i \in \mathbb{N}$, if $Parts_*(O, \tau(O)_i) \neq \emptyset$, then we define

$$\chi(O)_i = \{\sigma^{\lfloor j/2 \rfloor}[W] : W \in Parts_*(O, \tau(O)_i) \ \wedge \ length(W) = j\}$$

Otherwise, if $Parts_*(O, \tau(O)_i) = \emptyset$, then we set $\chi(O)_i = \{\mathfrak{n}^{\mathbb{Z}}\}$. In other words, possibly except for finitely many i for which $Parts_*(O, \tau(O)_i) = \emptyset$, we define $\chi(O)_i$ to be the subset of $Parts(O, \tau(O)_i)$ consisting of those elements which position the midpoints of the filled $\tau(O)_i$-blocks in the $\tau(O)_i$-skeleton of O at index 0. (If such a block has even length, then its "midpoint" is defined to be the index which cuts the block in such a way that there is one more non-blank symbol on its left than on its right.)

We claim that f is a Borel reduction from the topological conjugacy relation on $\mathcal{T}_{\mathfrak{n}}^{**}$ to the equivalence relation $\backsim_{\mathfrak{n}}$. It is straightforward to check that f is Borel and we will skip the tedious details.

To see that f is a reduction, pick $O, O' \in \mathcal{T}_{\mathfrak{n}}^{**}$ such that O and O' are topologically conjugate and let $\pi : O \to O'$ be a topological conjugacy. Recall that topologically conjugate Toeplitz subshifts have the same scale and hence $\tau(O) = \tau(O')$. Let $(r_i)_{i \in \mathbb{N}}$ be the sequence $\tau(O)$. Since O and O' both have growing blocks with respect to $(r_i)_{i \in \mathbb{N}}$, there exists n_0 such that the minimal lengths of the filled r_i-blocks of O and O' are both greater than $4|\pi| + 6$ for all $i \geq n_0$.

We claim that $(\chi(O)_i, \chi(O')_i) \in \mathtt{D}_{r_i}^{\mathrm{fin}}$ for all $i \geq n_0$, which implies that $f(O) \backsim_{\mathfrak{n}} f(O')$. Let $i \in \mathbb{N}$ be such that $i \geq n_0$. We want to show that

$$\{[W]_{\mathtt{D}_{r_i}} : W \in \chi(O)_i\} = \{[W]_{\mathtt{D}_{r_i}} : W \in \chi(O')_i\}$$

Pick $W \in \chi(O)_i$. By the definition of $\chi(O)$, W is of the form $\sigma^{\lfloor n/2 \rfloor}[Z]$ for some set $Z \in Parts_*(O, r_i)$ with $length(Z) = n$. Choose $\alpha \in W$ and set $\beta = \pi(\alpha)$.

Let $k = \lfloor n/2 \rfloor - |\pi| - 2$ and $k' = \lfloor n/2 \rfloor + |\pi| + 2$. By the choice of i, we have that $n \geq 4|\pi| + 6$ and hence $k \geq |\pi| + 1$. Since

$$[-k - |\pi| - 1, k + |\pi| + 1] \subseteq Per_{r_i}(\alpha)$$

it follows from Proposition 6.1 that $[-k, k] \subseteq Per_{r_i}(\beta)$ and hence the subblock $\beta[-k, k]$ is a part of some filled r_i-block of $Skel(\beta, r_i)$. Similarly, it follows from Proposition 6.1 that there are at least two r_i-holes in $Skel(\beta, r_i)$ along the interval $[-k', k']$ since $Skel(\alpha, r_i)$ has two r_i-holes at the indices $-1 - \lfloor n/2 \rfloor$ and $n - \lfloor n/2 \rfloor$. Let $q' < 0 < q$ be the r_i-holes in the skeleton $Skel(\beta, r_i)$ such that

$$Skel(\beta, r_i)(q'') \neq \square$$

for all $q' < q'' < q$. Clearly we have that $-k' \leq q' < -k < k < q \leq k'$. Set $j = \lceil (q+q')/2 \rceil$. Notice that the filled r_i-block to which $\beta[-k, k]$ belongs is $\beta[q'+1, q-1]$ and the midpoint of this filled r_i-block is j. Hence $\sigma^j[\pi[W]] \in \chi(O')_i$.

By the choice of i, we know that the minimal lengths of filled-r_i-blocks of α and $\sigma^j(\beta)$ are both greater than $4|\pi|+6$. Since W and $\sigma^j[\pi[W]]$ both position the midpoints of the corresponding filled r_i-blocks at 0, we have that

$$[-2|\pi| - 2, 2|\pi| + 2] \subseteq Per_{r_i}(\alpha), Per_{r_i}(\sigma^j(\beta))$$

On the other hand, it follows from the previous inequalities that

$$j = \lceil (q + q')/2 \rceil \leq \lceil (k' - k)/2 \rceil \leq |\pi| + 2$$

and hence the topological conjugacy $\sigma^j \circ \pi$ sending α to $\sigma^j(\beta)$ and its inverse can be given by some block codes of length at most $2|\pi| + 2$. Consequently, Lemma 5.5.1 implies that there exists $\phi \in Sym(\mathfrak{n}^{r_i})$ such that

$$\phi(\alpha[lr_i, (l+1)r_i)) = (\sigma^j(\beta))[lr_i, (l+1)r_i)$$

for all $l \in \mathbb{Z}$. Then it easily follows from the proof of Lemma 5.5.2 that the induced homeomorphism $\widehat{\phi}$ bijectively maps W onto $\sigma^j[\pi[W]]$. Therefore, W and $\sigma^j[\pi[W]]$ are $\mathtt{D}_{r_i}$-equivalent which shows that

$$\{[W]_{\mathtt{D}_{r_i}} : W \in \chi(O)_i\} \subseteq \{[W]_{\mathtt{D}_{r_i}} : W \in \chi(O')_i\}$$

Carrying out this argument symmetrically, we can easily obtain $(\chi(O)_i, \chi(O')_i) \in \mathtt{D}^{\mathrm{fin}}_{r_i}$. Hence, $f(O) \backsim_{\mathfrak{n}} f(O')$ whenever O and O' are topologically conjugate.

Now pick $O, O' \in \mathcal{T}^{**}_{\mathfrak{n}}$ and assume that $f(O) \backsim_{\mathfrak{n}} f(O')$. Then $\tau(O) = \tau(O')$; and for some sufficiently large i, there exists $W \in Parts(O, \tau(O)_i)$ which is bijectively mapped onto some $W' \in Parts(O', \tau(O)_i)$ via a homeomorphism $\widehat{\phi}$ induced by a permutation $\phi \in Sym(\mathfrak{n}^{\tau(O)_i})$. It follows from Lemma 5.5.2 that O and O' are topologically conjugate. $\square$

Proof of Theorem A. It follows from Theorem 6.2 that the topological conjugacy relation on $\mathcal{T}^{**}_{\mathfrak{n}}$ is hypersmooth. By Theorem 1.5.6, this relation is hyperfinite since it is a hypersmooth countable Borel equivalence relation. Consequently, its restrictions onto Borel subsets of $\mathcal{T}^{**}_{\mathfrak{n}}$ are hyperfinite. In particular, the topological conjugacy relation on $\mathcal{T}^{*}_{\mathfrak{n}}$ is hyperfinite. $\square$

Chapter 7

Equivalence of properly ordered Bratteli diagrams

In this chapter, we shall analyze the Borel complexity of equivalence of properly ordered Bratelli diagrams. In particular, we will show that $\approx$ and $\cong_{tc}^*$ are Borel bireducible. Then we will then restrict the equivalence relation $\approx$ to the class of properly ordered Bratteli diagrams of finite rank and analyze its Borel complexity. Finally, we will discuss some applications of our results and some possible further research directions.

7.1 Pointed Cantor minimal systems and properly ordered Bratteli diagrams

In this section, we will describe how one can construct Borel maps between $\mathcal{M}_{2^{\mathbb{N}}}^*$ and $\mathcal{POBD}$ sending pointed Cantor minimal systems to their Bratteli-Vershik representations and vice versa. In our treatment of the correspondence between pointed Cantor minimal systems and properly ordered Bratteli diagrams, we shall follow [Dur10, HPS92, Ska00].

We begin by introducing *Kakutani-Rohlin* partitions of Cantor minimal systems. A *clopen partition* $\mathcal{P}$ of a topological space X is a partition consisting of clopen subsets of X. Notice that a clopen partition of a compact space is necessarily finite.

Definition 7.1.1. *A Kakutani-Rohlin partition of a Cantor minimal system (X, φ) is a clopen partition $\mathcal{Q}$ of X of the form*

$$\mathcal{Q} = \{\varphi^j[B_k] : 1 \leq k \leq t \ \wedge \ 0 \leq j < h_k\}$$

where $t, h_k \in \mathbb{N}^+$ and B_k is a clopen subset of X for each $0 \leq k \leq i$. For each $1 \leq k \leq t$, the collection $\{\varphi^j[B_k] : 0 \leq j < h_k\}$ is called the k-th tower of $\mathcal{Q}$ and the set B_k is *said to be the base of the k-th tower. The height of the k-th tower is defined to be the number h_k.*

Let (X, φ, x) be a pointed Cantor minimal system and assume for the moment that there exists a sequence $(\mathcal{Q}_i)_{i\in\mathbb{N}}$ of Kakutani-Rohlin partitions with

$$\mathcal{Q}_i = \{\varphi^j[B_k^i] : 1 \le k \le t_i \ \wedge\ 0 \le j < h_{i,k}\}$$

such that

a. $\mathcal{Q}_0 = \{X\}$, $t_0 = 1$, $h_{0,1} = 1$, and $B_1^0 = X$.

b. $\bigcap_{i\in\mathbb{N}} \bigcup_{1\le k\le t_i} B_k^i = \{x\}$,

c. $\mathcal{Q}_{i+1}$ is finer than $\mathcal{Q}_i$ for all $i \in \mathbb{N}$, and

d. $\bigcup_{i\in\mathbb{N}} \mathcal{Q}_i$ generates the topology of X.

Using the sequence $(\mathcal{Q}_i)_{i\in\mathbb{N}}$, we will define a properly ordered Bratteli diagram whose associated Bratteli-Vershik dynamical system is topologically conjugate to (X, φ, x). The i-th vertex set of our properly ordered Bratteli diagram will have one vertex for each tower in the i-th Kakutani-Rohlin partition $\mathcal{Q}_i$. More specifically, for each $i \in \mathbb{N}$, let V_i be the i-th vertex set

$$V_i := \{(i, 1), \dots, (i, t_i)\}$$

Now, for each $i \in \mathbb{N}^+$, let E_i be the i-th edge set

$$E_i := \{(i, k, k', j) : \varphi^j[B_{k'}^i] \subseteq B_k^{i-1},\ 1 \le k \le t_{i-1},\ 1 \le k' \le t_i,\ 0 \le j \le h_{i,k'}\}$$

The source and range maps of these edges are given by

$$s(i, k, k', j) := (i-1, k) \text{ and } r(i, k, k', j) := (i, k')$$

respectively. Finally, let $\preccurlyeq$ be the partial order on the edge set given by

$$(i_1, k_1, k_1', j_1) \preccurlyeq (i_2, k_2, k_2', j_2) \Leftrightarrow i_1 = i_2 \ \wedge\ k_1' = k_2' \ \wedge\ j_1 \le j_2$$

It is well-known that $B = (V, E, \preccurlyeq)$ is a properly ordered Bratteli diagram and that (X, φ, x) is topologically conjugate to $(X_B, \lambda_B, x_{max})$ [HPS92, Dur10].

We shall next prove that such sequences of Kakutani-Rohlin partitions always exist. We will need the following technical lemma.

Lemma 7.1.1. *[HPS92, Put89] Let $\mathcal{Q}$ be a clopen partition of X and C be a clopen subset of X. Then there exists a clopen partition $C_1, \ldots, C_t$ of C and positive integers $\{h_i : 1 \leq i \leq t\}$ such that*

$$\mathcal{Q}' = \{\varphi^j[C_i] : 1 \leq i \leq t \ \wedge \ 0 \leq j < h_i\}$$

is a Kakutani-Rohlin partition of X which is finer than $\mathcal{Q}$.

Proof. Consider the first entrance map $r : C \to \mathbb{N}^+$ given by

$$r(x) = inf\{i \in \mathbb{N}^+ : \varphi^i(x) \in C\}$$

Since (X, φ) is minimal and C is clopen, the map r is well-defined and continuous. It follows that $r[C]$ is compact and hence is finite. Let $r[C] = \{h_1, \ldots, h_{t'}\}$ and set $C'_i = r^{-1}[\{h_i\}]$. It is straightforward to check that the collection

$$\{C'_i : 1 \leq i \leq t'\}$$

is a partition of C and that the collection

$$\mathcal{Q}' = \{\varphi^j[C'_i] : 1 \leq i \leq t' \ \wedge \ 0 \leq j < h_i\}$$

is a partition of X. The partition $\mathcal{Q}'$ is not necessarily finer than $\mathcal{Q}$. This can easily be fixed as follows. For each $Z \in \mathcal{Q}$, if Z intersects some element $\varphi^k[C'_{i_0}]$ of the i_0-th tower but does not contain it, then we split the i_0-th tower of $\mathcal{Q}'$ into the following towers

$$\{\varphi^{j-k}[\varphi^k[C'_{i_0}] \cap Z] : 0 \leq j < h_{i_0}\}$$

$$\{\varphi^{j-k}[\varphi^k[C'_{i_0} \setminus Z]] : 0 \leq j < h_{i_0}\}$$

and then replace $\mathcal{Q}'$ by the partition

$$\{\varphi^j[C'_i] : 1 \leq i \leq t' \ \wedge \ 0 \leq j < h_i \ \wedge \ i \neq i_0\} \bigcup$$

$$\{\varphi^{j-k}[\varphi^k[C'_{i_0}] \cap Z], \varphi^{j-k}[\varphi^k[C'_{i_0} \setminus Z]] : 0 \leq j < h_{i_0}\}$$

We repeat this procedure for the remaining $Z \in \mathcal{Q}$ and refine the partition $\mathcal{Q}'$ at each stage if necessary. After finitely many steps, we exhaust all the elements of $\mathcal{Q}$ and the resulting partition $\mathcal{Q}'$ satisfies the requirements. $\square$

Theorem 7.1.2. *[HPS92] There exists a sequence $(\mathcal{Q}_i)_{i\in\mathbb{N}}$ of Kakutani-Rohlin partitions*

$$\mathcal{Q}_i = \{\varphi^j[B_k^i] : 1 \le k \le t_i \ \wedge \ 0 \le j < h_{i,k}\}$$

such that

a. $\mathcal{Q}_0 = \{X\}$, $t_0 = 1$, $h_{0,1} = 1$, *and* $B_1^0 = X$.

b. $\bigcap_{i\in\mathbb{N}} \bigcup_{1\le k\le t_i} B_k^i = \{x\}$,

c. $\mathcal{Q}_{i+1}$ *is finer than* $\mathcal{Q}_i$ *for all* $i \in \mathbb{N}$, *and*

d. $\bigcup_{i\in\mathbb{N}} \mathcal{Q}_i$ *generates the topology of* X.

Proof. Fix a decreasing sequence of clopen sets $(C_i)_{i\in\mathbb{N}^+}$ with $\bigcap_{i\in\mathbb{N}^+} C_i = \{x\}$ and a sequence of clopen partitions $(\mathcal{W}_i)_{i\in\mathbb{N}^+}$ such that $\mathcal{W}_{i+1}$ is finer than $\mathcal{W}_i$ for all $i \in \mathbb{N}^+$ and the collection $\bigcup_{i\in\mathbb{N}^+} \mathcal{W}_i$ generates the topology of X. By Lemma 7.1.1, there exists a Kakutani-Rohlin partition $\mathcal{Q}_1$ such that $\mathcal{Q}_1$ is finer than $\mathcal{W}_1$ and the bases of towers of $\mathcal{Q}_1$ form a partition of C_1. For each $i \ge 2$, we obtain $\mathcal{Q}_i$ by applying Lemma 7.1.1 inductively to the clopen set C_i and the clopen partition

$$\mathcal{W}_i \vee \mathcal{Q}_{i-1} := \{W \cap Q : W \in \mathcal{W}_i \ \wedge \ Q \in \mathcal{Q}_{i-1} \ \wedge \ W \cap Q \neq \emptyset\}$$

It is routine to check that the resulting sequence $(\mathcal{Q}_i)_{i\in\mathbb{N}}$ of Kakutani-Rohlin partitions satisfies properties (a)-(d). $\square$

Given an element of $\mathcal{M}^*_{2^\mathbb{N}}$, by applying the constructions in the proofs of Theorem 7.1.2 and Lemma 7.1.1, we can construct a sequence of Kakutani-Rohlin partitions satisfying properties (a)-(d) in a Borel way. (The only step of the proofs that seems to require a quantification over an uncountable set is finding $r[C]$ in the proof of Lemma 7.1.1. However, since X has no isolated points and $r[C]$ is discrete, in order to find

$r[C]$, it is sufficient to evaluate r on a dense countable subset of C, which can be done in a Borel way using a fixed countable dense subset of X.)

It is routine to check that the map that sends each element of $\mathcal{M}^*_{2^{\mathbb{N}}}$ to the properly ordered Bratteli diagram obtained by the construction given at the beginning of this section which uses the sequence of Kakutani-Rohlin partitions constructed by the above procedure is Borel. This Borel map sends each pointed Cantor minimal system to one of its Bratteli-Vershik representations and it follows from Theorem 1.6.4 that $\cong^*_{tc} \leq_B \approx$.

We will now sketch how one can construct a Borel map from $\mathcal{POBD}$ to $\mathcal{M}^*_{2^{\mathbb{N}}}$ which sends each properly ordered Bratteli diagram to an automorphism of $\mathbb{B}$ coding the Bratteli-Vershik dynamical system it represents.

Recall from the proof of Theorem 4.2.2 that there exists a Borel map from the space of sequences in $(2^{\mathbb{Z}})^{\mathbb{N}}$ that enumerate return times algebras to the space $\mathcal{M}^*_{2^{\mathbb{N}}}$ such that each return times algebra is mapped to an automorphism of $\mathbb{B}$ coding the corresponding ultrafilter dynamical system. A straightforward but tedious analysis shows that the map which sends each properly ordered Bratteli diagram in $\mathcal{POBD}$ to an enumeration of the return times algebra of its associated Bratteli-Vershik dynamical system is Borel. Composing these maps, we obtain a Borel reduction witnessing that $\approx \leq_B \cong^*_{tc}$.

Having determined the Borel complexity of $\approx$, the next step would be to analyze how the Borel complexity changes when we restrict our attention to various subclasses of properly ordered Bratteli diagrams. In this thesis, we shall only consider equivalence of properly ordered Bratteli diagrams of finite rank.

7.2 Properly ordered Bratteli diagrams of finite rank

A Bratteli diagram (V, E) is said to be of *finite rank* if there exists $n \in \mathbb{N}$ such that $|V_k| \leq n$ for all $k \in \mathbb{N}$. Downarowicz and Maass [DM08] proved that the Bratteli-Vershik dynamical system associated to a properly ordered Bratteli diagram of finite rank is topologically conjugate to either an odometer or a minimal subshift over a finite alphabet.

Since topological conjugacy of odometers is smooth and topological conjugacy of minimal subshifts over finite alphabets is a countable Borel equivalence relation, equivalence of properly ordered Bratteli diagrams of finite rank is an essentially countable Borel equivalence relation and hence is Borel reducible to E_∞.

Theorem 2.3.3 implies that the return times algebra of a Bratteli-Vershik dynamical system arising from a finite rank properly ordered Bratteli diagram is finitely generated, unless the system is topologically conjugate to an odometer.

Since the set $\mathcal{I}$ in the proof of Theorem 4.2.2 was chosen to be $\mathbb{Q}$-linearly independent, the return times algebra of the pointed Cantor minimal system $(S(\mathcal{A}_S), \xi_*, x_{\mathcal{A}_S})$ constructed in that proof is not finitely generated unless the corresponding countable non-empty subset S of $\mathcal{I}$ is finite. Hence, the properly ordered Bratteli diagrams corresponding to the pointed Cantor minimal system $(S(\mathcal{A}_S), \xi_*, x_{\mathcal{A}_S})$ are of infinite rank for any countably infinite $S \subseteq \mathcal{I}$.

It is routine to check that for any uncountable standard Borel space X, the equivalence relation Δ_X^+ is Borel bireducible with its restriction to the Borel subset

$$\{x \in X^{\mathbb{N}} : \{x_n : n \in \mathbb{N}\} \text{ is infinite}\}$$

Consequently, equivalence of properly ordered Bratteli diagrams of infinite rank is Borel bireducible with $\Delta_{\mathbb{R}}^+$. Combining these observations with the fact that $E_\infty <_B \Delta_{\mathbb{R}}^+$, we obtain the following corollary.

Corollary 7.2.1. *Equivalence of properly ordered Bratteli diagrams of finite rank is strictly less complex than equivalence of properly ordered Bratteli diagrams of infinite rank.*

7.3 A non-uniformity theorem

In this section, as an application of Theorem C, we will prove a non-uniformity theorem regarding assigning proper orderings to simple Bratteli diagrams.

Assume that we are given an unordered Bratteli diagram (V, E) such that the incidence matrices have only positive entries at each level. Then we can easily attach a partial order $\preccurlyeq$ to (V, E) as follows so that $(V, E, \preccurlyeq)$ is a properly ordered Bratteli diagram [Ska91, Section 1]. Fix a linear order $\leq^*$ on E and a linear order $\leq_n$ on V_n for each $n \in \mathbb{N}$. Given $e, e' \in E_{n+1}$ with $r(e) = r(e')$, define $e \preccurlyeq e'$ if and only if either $s(e) <_n s(e')$ or, $s(e) = s(e')$ and $e <^* e'$. It is not difficult to see that the sources of the minimal (respectively, maximal) edges are the same at every level and hence there is a unique minimal (respectively, maximal) path.

Therefore, given a simple unordered Bratteli diagram B, we can explicitly attach a partial order to the edges and obtain a properly ordered Bratteli diagram B^*, possibly after telescoping B. Carrying out this procedure on the relevant standard Borel spaces, it is not difficult to prove that there exists a Borel map $f : \mathcal{SBD} \to \mathcal{POBD}$ such that for every $B \in \mathcal{SBD}$ we have $f(B) \sim B$ as unordered Bratteli diagrams. On the other hand, this map is not "uniform" in the sense that $B_1 \sim B_2$ does not necessarily imply $f(B_1) \approx f(B_2)$.

One can ask whether or not such a uniform map exists. If we do not insist that f be well-behaved, then we can use the axiom of choice to choose a representative from each $\sim$-class and map each $\sim$-class to the properly ordered Bratteli diagram obtained from the corresponding representative. We will prove that there does not exist such a uniform Borel map. We first need to understand the complexity of $\sim$-equivalence of simple Bratteli diagrams.

Hjorth [Hjo02] has proved that the isomorphism relation $\cong_{\mathcal{TFA}}$ on the standard Borel space of countable torsion-free abelian groups is not Borel. Ellis showed that $\cong_{\mathcal{TFA}}$ is Borel reducible to the isomorphism relation for simple dimension groups [Ell10, Proposition 6.2] and it essentially follows from the work of Effros, Handelman, and Shen [EHS80] that the isomorphism relation for simple dimension groups is Borel reducible to the equivalence relation $\sim$ on the space of simple Bratteli diagrams. (For a detailed discussion of this construction, we refer the reader to [Eff81, Chapter 3].)

On the one hand, $\sim$ is not Borel since $\cong_{\mathcal{TFA}}$ is Borel reducible to it. On the other hand, $\approx$ is Borel since we have proved that it is Borel bireducible with $\cong^*_{tc}$ and hence is Borel bireducible with $\Delta^+_{\mathbb{R}}$. These observations immediately imply the following result.

Theorem 7.3.1. *There exists no Borel map $f : \mathcal{SBD} \to \mathcal{POBD}$ such that*

- *$f(B) \sim B$ as unordered Bratteli diagrams and*
- *$f(B) \approx f(B')$ whenever $B \sim B'$*

for all $B, B' \in \mathcal{SBD}$.

Proof. Assume towards a contradiction that there exists such a Borel map f. Then f is a Borel reduction from $\sim$ to $\approx$. This implies that $\sim$ is Borel, which is a contradiction. $\square$

7.4 Further research directions

In this concluding section, we will discuss some open problems regarding the Borel complexity of the topological conjugacy relation on various topological dynamical systems.

Even though we have provided a lower bound for the Borel complexity of the topological conjugacy relation on Cantor minimal systems, we do not know any non-trivial upper bounds. The techniques used in this thesis are designed to analyze topological conjugacy of pointed Cantor minimal systems and it is not clear to us whether or not they can be used to find any upper bounds for topological conjugacy of unpointed Cantor minimal systems.

Open Question 1. *What is the Borel complexity of the topological conjugacy relation on Cantor minimal systems? In particular, is this relation even Borel?*

We have observed that equivalence of properly ordered Bratteli diagrams of finite rank is essentially countable and hence is Borel reducible to E_∞. Moreover, E_0 is a lower bound for the Borel complexity of this relation. To see this, let (O, σ, α) be a Toeplitz subshift over the alphabet 2, where α is a Toeplitz sequence such that the scale of α is 2^∞ and the 2^k-skeleton of α contains a single hole along every interval of length 2^k for each $k \in \mathbb{N}^+$. It follows from [Dow05, Theorem 13.1] that there exists a σ-invariant Borel probability measure on the space

$$X_O = \{\beta \in 2^{\mathbb{Z}} : \beta \in O \ \wedge \ \beta \text{ is a Toeplitz sequence}\}$$

Consequently, the orbit equivalence relation of the left-shift action of $\mathbb{Z}$ on X_O cannot have a Borel transversal and hence is not smooth. It follows that the equivalence relation E on X_O defined by

$$\beta \ E \ \beta' \Leftrightarrow (O, \sigma, \beta) \text{ and } (O, \sigma, \beta') \text{ are topologically conjugate}$$

is not smooth since it is a countable Borel equivalence relation which contains a non-smooth countable Borel equivalence relation [Tho13, Proposition 2.1]. An analysis of the construction of Bratteli-Vershik representations of pointed Toeplitz subshifts with Toeplitz points described in [GJ00, Theorem 8] shows that there exists a Borel map from X_O to the space of properly ordered Bratteli diagrams sending each β to a Bratteli-Vershik representation of (O, σ, β) of *finite rank*. It follows that $E \sim_B E_0$ is Borel reducible to equivalence of properly ordered Bratteli diagrams of finite rank. As far as the author knows, E_0 is the best known lower bound for the Borel complexity of this relation.

Open Question 2. *What is the Borel complexity of equivalence of properly ordered Bratteli diagrams of finite rank? More generally, what is the Borel complexity of topological conjugacy of pointed minimal subshifts over a finite alphabet?*

Finally, we would like to point out that the question of Sabok and Tsankov regarding topological conjugacy of Toeplitz subshifts in its full generality remains open. We believe that the answer to this question is affirmative and that this relation is hyperfinite.

Recall that if O and O' are topologically conjugate Toeplitz subshifts, then some elements of the partition $Parts(O, p)$ are mapped onto some elements of the partition $Parts(O', p)$ under the natural action of $Sym(n^p)$ on $K(\mathfrak{n}^{\mathbb{Z}})$ for a sufficiently large factor p of the common scale. The proof of Theorem 6.2 relies on the fact that we can eventually identify the "correct" subset of each partition $Parts(O, p)$ in a Borel way for Toeplitz subshifts with growing blocks. We believe that it might be possible to find such a Borel choice for arbitrary Toeplitz subshifts and construct a Borel reduction from topological conjugacy of Toeplitz subshifts to E_1, which would imply the hyperfiniteness of the former equivalence relation.

Appendix A

Restricting the Friedman-Stanley jump to finite subsets

Given a Polish space X, the set $K(X)$ of non-empty compact subsets of X is a Polish space endowed with the topology induced by the Hausdorff metric defined by

$$\delta_d(C_1, C_2) = \max\{\max_{x \in C1} d(x, C_2), \max_{y \in C_2} d(y, C_1)\}$$

It is easily checked that the set

$$\mathrm{Fin}(X) := \{F \subseteq X : F \text{ is finite and non-empty}\}$$

is an F_σ subset of $K(X)$ and hence is a standard Borel space. Given a Borel equivalence relation E on X, let E^{fin} be the equivalence relation on $\mathrm{Fin}(X)$ defined by

$$u\ E^{\mathrm{fin}}\ v \Leftrightarrow \{[x]_E : x \in u\} = \{[x]_E : x \in v\}$$

It is well-known that there exists a sequence of Borel functions $f_k : K(X) \to X$ such that $\{f_k(C)\}_{k \in \mathbb{N}}$ is dense in C for all $C \in K(X)$ [Kec95, Theorem 12.23]. It follows that

$$u\ E^{\mathrm{fin}}\ v \Leftrightarrow (\forall i\ \exists j\ (f_i(u), f_j(v)) \in E)\ \wedge\ (\forall i\ \exists j\ (f_i(v), f_j(u)) \in E)$$

Thus E^{fin} is a Borel equivalence relation. Even though E^{fin} is not a subrelation of E^+, we can think of E^{fin} as the restriction of the Friedman-Stanley jump to the finite subsets of X. (It is straightforward to check that E^{fin} is Borel bireducible with the restriction of E^+ to the Borel subset of $X^{\mathbb{N}}$ consisting of sequences in which only finitely many elements of X appear.) We will now explore some basic properties of the map $E \mapsto E^{\mathrm{fin}}$.

Proposition A.0.1. *Let E be a Borel equivalence relation on a standard Borel space X. Then $E \leq_B E^{fin}$.*

Proof. The Borel map $x \mapsto \{x\}$ is a Borel reduction from E to E^{fin}. $\square$

Proposition A.0.2. *Let E and F be Borel equivalence relations on standard Borel spaces X and Y respectively. If $E \leq_B F$, then $E^{fin} \leq_B F^{fin}$.*

Proof. Assume that $E \leq_B F$. Let $g : X \to Y$ be a Borel reduction from E to F. Then it is easily checked that the map from $\mathrm{Fin}(X)$ to $\mathrm{Fin}(Y)$ given by $u \mapsto \{g(x) : x \in u\}$ is a Borel reduction from E^{fin} to F^{fin}. □

Proposition A.0.3. *Let $E_0 \subseteq E_1 \subseteq \ldots$ be an increasing sequence of Borel equivalence relations on a standard Borel space X. Then $E_0^{fin} \subseteq E_1^{fin} \subseteq \ldots$ is an increasing sequence of Borel equivalence relations on $Fin(X)$ and $\bigcup_{i\in\mathbb{N}} E_i^{fin} = E^{fin}$ where $E = \bigcup_{i\in\mathbb{N}} E_i$.*

Proof. It is clear that $E_i^{\mathrm{fin}} \subseteq E_{i+1}^{\mathrm{fin}}$ for all $i \in \mathbb{N}$ since $E_i \subseteq E_{i+1}$. Similarly, we have that $\bigcup_{i\in\mathbb{N}} E_i^{\mathrm{fin}} \subseteq E^{\mathrm{fin}}$ since $E_i \subseteq E$. For the converse inclusion, let $u, v \in \mathrm{Fin}(X)$ and assume that $(u, v) \in E^{\mathrm{fin}}$. Then for all $x \in u$ there exists $y \in v$ such that $(x, y) \in E$ and vice versa. Since $E = \bigcup_{i\in\mathbb{N}} E_i$ and both u and v are finite, these finitely many equivalences are witnessed by E_n for some sufficiently large n. But then $(u, v) \in E_n^{\mathrm{fin}}$ and hence $(u, v) \in \bigcup_{i\in\mathbb{N}} E_i^{fin}$. □

Corollary A.0.4. *Let E be a Borel equivalence relation on a standard Borel space X. Then*

a. If E has finite (respectively, countable) equivalence classes, then so does E^{fin}.

b. If E is smooth, then so is E^{fin}.

c. If E is hyperfinite, then so is E^{fin}.

d. If E is hypersmooth, then so is E^{fin}.

Proof. [a.] Choose $u \in E^{\mathrm{fin}}$, say $u = \{x_0, \ldots, x_k\}$ for some $k \in \mathbb{N}$. Notice that any representative of the equivalence class $[u]_{E^{\mathrm{fin}}}$ is a non-empty finite subset of $\bigcup_{i=0}^{k}[x_i]_E$. Thus there are at most $|\mathcal{P}_{\mathrm{fin}}(\bigcup_{i=0}^{k}[x_i]_E)|$-many representatives of the equivalence class $[u]_{E^{\mathrm{fin}}}$ where $\mathcal{P}_{\mathrm{fin}}(A)$ is the set of finite subsets of A. It follows that if E has finite (respectively, countable) equivalence classes, then so does E^{fin}.

[b.] This follows from Proposition A.0.2 and the fact that $\Delta_{\mathbb{R}}^{\mathrm{fin}} = \Delta_{\mathrm{Fin}(\mathbb{R})}$.

[c.] Assume that E is hyperfinite, say $E = \bigcup_{n\in\mathbb{N}} E_n$ for some increasing sequence $(E_n)_{n\in\mathbb{N}}$ of finite Borel equivalence relations. Then each E_n^{fin} has finite equivalence classes by part (a) and it follows from Proposition A.0.3 that $E^{\text{fin}} = \bigcup_{n\in\mathbb{N}} E_n^{\text{fin}}$ is hyperfinite.

[d.] Assume that E is hypersmooth, say $E = \bigcup_{n\in\mathbb{N}} E_n$ for some increasing sequence $(E_n)_{n\in\mathbb{N}}$ of smooth Borel equivalence relations. Then each E_n^{fin} is smooth by part (b) and it follows from Proposition A.0.3 that $E^{\text{fin}} = \bigcup_{n\in\mathbb{N}} E_n^{\text{fin}}$ is hypersmooth. □

By Corollary A.0.4, the Borel equivalence relations $\Delta_{\mathbb{N}}$, $\Delta_{\mathbb{R}}$, E_0, E_1, and E_∞ are fixed points of the map $E \mapsto E^{\text{fin}}$ up to Borel bireducibility. Based on this observation, one might conjecture that $E \sim_B E^{\text{fin}}$ for all Borel equivalence relations E with infinitely many E-classes. However, this naive conjecture turns out to be false. As we shall see later, for every countable Borel equivalence relation E, the equivalence relation E^{fin} behaves like a universal finite index extension of E; and not every countable Borel equivalence relation is Borel bireducible with all of its finite index extensions. We first need to recall some basic definitions.

Let $E \subseteq F$ be countable Borel equivalence relations on a standard Borel space X. Then F is called a *finite index extension* of E if every F-class consists of finitely many E-classes. We will write $[F : E] < \infty$ to denote that F is a finite index extension of E.

Theorem A.0.5. *Let E be a countable Borel equivalence relation on a standard Borel space X. Then for every countable Borel equivalence relation F on X with $[F : E] < \infty$, we have that $F \leq_B E^{fin}$.*

Proof. By the Feldman-Moore theorem, there exists a countable discrete group G such that $F = E_G^X$ for some Borel action of G on X. Let $(g_i)_{i\in\mathbb{N}}$ be a fixed enumeration of elements of G and consider the map $h : X \to \text{Fin}(X)$ defined by

$$x \mapsto \{g_i \cdot x : i \leq j_x\}$$

where j_x is the least natural number such that

$$\forall k\ \exists i \leq j_x\ (g_k \cdot x, g_i \cdot x) \in E$$

It is easily checked that h is a Borel map. Notice that h maps every x to a finite subset of X that contains representatives of each E-class contained in $[x]_F$. Hence $xFy \Leftrightarrow h(x)E^{\text{fin}}h(y)$ for all $x, y \in X$. $\square$

It follows that if $E \subseteq F$ is a pair of countable Borel equivalence relations on a standard Borel space X such that $[F : E] < \infty$ and $F \nleq_B E$, then $E^{\text{fin}} \nleq_B E$ and hence $E <_B E^{\text{fin}}$. It is well-known that such pairs of countable Borel equivalence relations exist [Ada02]. One may ask whether or not the only obstacle for a countable Borel equivalence relation E to satisfy $E \sim_B E^{\text{fin}}$ is the existence of such a finite index extension.

Open Question 3. *Let E be a countable Borel equivalence relation on a standard Borel space X such that $E^{fin} \nleq_B E$. Does there necessarily exist a countable Borel equivalence relation F such that $[F : E] < \infty$ and $F \nleq_B E$?*

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