MODULI OF BRIDGELAND STABLE OBJECTS ON AN ENRIQUES SURFACE

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ABSTRACT OF THE DISSERTATION

Moduli of Bridgeland stable objects on an Enriques surface

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We construct projective moduli spaces of semistable objects on an Enriques surface for generic Bridgeland stability condition. On the way, we prove the non-emptiness of $M^s_{H,Y}(v)$, the moduli space of Gieseker stable sheaves on an Enriques surface Y with Mukai vector v of positive rank with respect to a generic polarization H. In the case of a primitive Mukai vector on an unnodal Enriques surface, i.e. one containing no smooth rational curves, we prove irreducibility of $M_{H,Y}(v)$ as well. Using Bayer and Macrì's construction of a natural nef divisor associated to a stability condition, we explore the relation between wall-crossing in the stability manifold and the minimal model program for Bridgeland moduli spaces. We give three applications of our machinery to obtain new information about the classical moduli spaces of Gieseker-stable sheaves:

1) We obtain a region in the ample cone of the moduli space of Gieseker-stable sheaves over Enriques surfaces.

2) We determine the nef cone of the Hilbert scheme of n points on an unnodal Enriques surface in terms of its half-pencils and the Cossec-Dolgachev ϕ-function.

3) We recover some classical results on linear systems on unnodal Enriques surfaces and obtain some new ones about n-very ample line bundles.

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Dedication

To my math teachers, David Doster and Michael Lutz, for opening the door for me to a whole new world.

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Chapter 1

Notation and Conventions

Throughout, we work over $\mathbb{C}$, and X will denote a smooth projective variety over $\mathbb{C}$ unless otherwise specified.

For a (locally-noetherian) scheme (or algebraic space) S, $\mathrm{D}^{\mathrm{b}}(S)$ denotes the bounded derived category of coherent sheaves, $\mathrm{D}_{qc}(S)$ the unbounded derived category of quasi-coherent sheaves, and $\mathrm{D}_{S\text{-perf}}(S\times X)$ the category of S-*perfect complexes*. (An S-perfect complex is a complex of $\mathcal{O}_{S\times X}$-modules which locally, over S, is quasi-isomorphic to a bounded complex of coherent shaves which are flat over S.)

We will abuse notation and denote all derived functors as if they were underived. We denote by p_S and p_X the two projections from $S\times X$ to S and X, respectively. Given $\mathcal{E}\in \mathrm{D}_{qc}(S\times X)$, we denote the Fourier-Mukai functor associated to $\mathcal{E}$ by

$$\Phi_{\mathcal{E}}(\_) := (p_X)_* \left(\mathcal{E}\otimes p_S^*(\_)\right).$$

We define the numerical Grothendieck group of a triangulated category $\mathcal{T}$ by $K_{\mathrm{num}}(\mathcal{T}) := K(\mathcal{T})/\operatorname{Ker}(\chi)$, where $K(\mathcal{T})$ is the Grothendieck K-group and we denote by $\chi(-,-)$ the Euler characteristic: for $E,F\in\mathcal{T}$,

$$\chi(E,F)=\sum_p (-1)^p \operatorname{ext}^p(E,F).$$

In case $\mathcal{T}=\mathrm{D}^{\mathrm{b}}(X)$, we abuse notation and just write $K_{\mathrm{num}}(X) := K_{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}(X))$.

We denote by $\mathrm{NS}(X)$ the Néron-Severi group of X, and write $N^1(X) := \mathrm{NS}(X)\otimes\mathbb{R}$. The ample cone of X and its closure in $N^1(X)$, called the nef cone, are denoted $\mathrm{Amp}(X)$ and $\mathrm{Nef}(X)$, respectively.

We denote the Mukai lattice of X by $H^*_{\mathrm{alg}}(X,\mathbb{Z}) := v(K_{\mathrm{num}}(X))\subset H^*(X,\mathbb{Q})$. Here v denotes the Mukai vector $v(E) := \mathrm{ch}(E).\sqrt{\mathrm{td}(X)}$. For $v=\sum v_i\in H^*_{\mathrm{alg}}(X,\mathbb{Z})$, with

$v_i \in H^{2i}(X, \mathbb{Q})$, define $v^\vee := \sum(-1)^i v_i$, and for $v, w \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$, define $(v, w) := -\int_X v^\vee.w$ so that $\chi(E, F) = -(v(E), v(F))$ [28, Definition and Corollary 6.1.5].

Given a complex $E \in \mathrm{D^b}(X)$, we denote its cohomology sheaves by $\mathcal{H}^*(E)$. The skyscraper sheaf at a point $x \in X$ is denoted by $k(x)$. For a complex number $z \in \mathbb{C}$, we denote its real and imaginary part by $\Re z$ and $\Im z$, respectively.

Chapter 2

Introduction

2.1 Overview

Over the last few decades there has been a great deal of interest in the study of moduli spaces of coherent sheaves on smooth projective varieties, often inspired by mathematical physics and gauge theory. In order to construct such a moduli space, in particular to obtain boundedness, one often restricts one's attention to coherent sheaves satisfying some sense of stability with fixed topological invariants, encoded in the Mukai vector v. The two most ubiquitous definitions of stability are μ_H-stability, or slope stability, and Gieseker-stability, both of which are defined by choosing an ample polarization H on the base variety X. Among the many fascinating aspects of these moduli spaces, other than their uses in physics, is the intimate connection they have with the underlying projective variety.

A particularly tight connection with X is via the choice of an ample polarization H. As the moduli spaces $M_{H,X}(v)$ of Gieseker-semistable sheaves on X with Mukai vector v is constructed as a GIT (Geometric Invariant Theory) quotient with respect to H, varying the polarization H induces a VGIT (Variation of GIT) birational transformation (as defined and studied in [19] and [60]). The corresponding connection with the birational geometry of Gieseker moduli spaces has been studied in [47]. Most important here is that the other birational models obtained from varying the polarization are moduli spaces as well, albeit with a slightly different moduli functor. The so-called Mumford-Thaddeus principle studied in [47] can be seen as an extension of the Hassett-Keel program for $\overline{M}_{g,n}$, where the minimal models of $\overline{M}_{g,n}$ obtained by running the minimal model program are hoped to be modular themselves. For dimensional reasons, however, the natural VGIT approach cannot give all of the birational geometry of these moduli spaces.

A revolutionary approach to this problem came from Bridgeland's definition of the notion of a stability condition on a triangulated category [14], an attempt at a mathematical definition of Douglas's Π-stability [20] for D-branes in string theory. Bridgeland proved that, when nonempty, the set of (full numerical) Bridgeland stability conditions on a triangulated category $\mathcal{T}$ forms a complex manifold of dimension $\operatorname{rk} K_{\mathrm{num}}(\mathcal{T})$.

Possibly the most studied case of these Gieseker moduli spaces is when X is a smooth projective surface, and here a second important connection between the geometry of $M_{H,X}(v)$ and that of X emerges. For example, when X is a projective $K3$ surface and v is primitive, the moduli spaces $M_{H,X}(v)$ of Gieseker-stable sheaves on X of Mukai vector v are projective hyperkähler (i.e. irreducible holomorphic symplectic) manifolds [49]. These are incredibly rare varieties with a beautiful and rigid geometry, and they are quite important as one of the building blocks of varieties with numerically trivial first Chern class c_1 [12]. Along with the related case when v is two times a primitive Mukai vector, these Gieseker moduli spaces (or their smooth resolutions in this non-primitive case) form all but two of the known deformation equivalence classes of such varieties. The other two come instead from constructions involving Gieseker moduli spaces on Abelian surfaces. Similarly, the Gieseker moduli spaces on rational surfaces of various types have been studied and interesting connections with the underlying surface have been unveiled.

Likewise, Bridgeland stability has been most developed for $\mathcal{T} = \mathrm{D}^{\mathrm{b}}(X)$, the bounded derived category of coherent sheaves, when the underlying variety is a smooth projective surface X. Again, the most studied examples have been on $\mathbb{P}^2$ and projective K3 and Abelian surfaces. In each case, the space $\operatorname{Stab}(X)$ of Bridgeland stability conditions has been shown to be nonempty and to admit a wall and chamber decomposition in the following sense: the set of σ-semistable objects with some fixed numerical invariants is constant in each chamber of the decomposition, while crossing a wall necessarily changes the stability of some object in this set.

For the numerical invariants of a Gieseker semistable sheaf, Bridgeland identified in [15] a certain chamber of the corresponding wall and chamber decomposition, the so-called Gieseker chamber, where the set of σ-semistable objects can be identified with the Gieseker semistable sheaves with respect to a generic polarization. He conjectured there that each chamber of $\mathrm{Stab}(X)$ should admit a coarse moduli space of σ-semistable objects and that crossing a wall should induce a birational transformation between the moduli spaces corresponding to adjacent chambers. In this way, Bridgeland forged a new tool for the investigation of the birational geometry of moduli spaces of Gieseker-stable sheaves, expanding the Mumford-Thaddeus principle so that minimal models of a given moduli space are no longer just expected to be Gieseker moduli spaces for a different polarization but are also allowed to parametrize genuine *complexes* of coherent sheaves, stable with respect to some Bridgeland stability condition.

The picture envisioned by Bridgeland has been partially verified. Bertram and Martinez have shown in [13] that on a smooth projective surface birational models obtained by VGIT can be recovered via Bridgeland wall-crossing. More extensive progress has been obtained in examples, first to a certain extent by Arcara and Bertram in [2], but most notably by Arcara, Bertram, Coskun, and Huizenga in [3], for the Hilbert scheme of points on $\mathbb{P}^2$, and then by Bayer and Macrì in [11, 10] for all numerical invariants on K3 surfaces. Following these ground-breaking developments, there has been an explosion of activity surrounding the use of Bridgeland stability conditions to study the minimal model program for moduli spaces of Gieseker-stable sheaves on $\mathbb{P}^2$, Del Pezzo surfaces, K3 surfaces, and Abelian surfaces.

While moduli of both Gieseker stable sheaves and Bridgeland stable objects on $K3$ and Abelian surfaces have arguably received the most attention, the corresponding moduli spaces on Enriques surfaces have been much less studied. Recall that an *Enriques surface* is a smooth projective surface Y with $h^1(\mathcal{O}_Y) = 0$ and canonical bundle $\omega_Y = \mathcal{O}_Y(K_Y)$ a non-trivial 2-torsion element of $\mathrm{Pic}(Y)$. In this dissertation, representing the results of [51, 52], we study moduli spaces of both Gieseker stable sheaves and Bridgeland stable objects on Enriques surfaces and obtain results analogous to those for other classes of projective surfaces.

2.2 Previous results

The investigation of Gieseker moduli spaces on Enriques surfaces was started by Kim in [34], where he proved some general structure results about the locus parametrizing stable locally free sheaves, but in general, the picture is much less complete. Kim himself continued the investigation in [35] with existence results for μ-stable locally free sheaves of rank 2 and a description of some of the geometry of these moduli spaces on an unnodal Enriques surface.[1] Yoshioka successfully computed the Hodge polynomial of $M_{H,Y}(v)$ in [64] for primitive v of positive odd rank and generic polarization H on an unnodal Y, showing that it is equal to the Hodge polynomial of the Hilbert scheme of $\frac{v^2+1}{2}$ points on Y. It follows that these moduli spaces are non-empty, consisting of two isomorphic irreducible components parametrizing sheaves whose determinant line bundles differ by K_Y. Hauzer [26] refined the techniques of Yoshioka to conclude that the Hodge polynomials in even rank are the same as those of a moduli space of sheaves with rank 2 or 4. Moreover, Yamada [62] proved that under certain minor assumptions $M_{H,Y}(v)$ has torsion canonical divisor, like Enriques surfaces themselves. Finally, G. Saccà [57] obtained some beautiful results about the geometry of Gieseker moduli spaces in rank 0. Nevertheless, even non-emptiness and irreducibility for moduli of sheaves of even rank is unknown in general.

Bridgeland stability conditions on Enriques surfaces have been constructed in [45], using a technique called *induction of stability.* For an Enriques surface Y, which is the quotient of a K3 surface $\tilde{Y}$ by a fixed-point free involution ι, it is shown in [45] that $\mathrm{Stab}(Y)$ can be identified with the non-empty closed submanifold of $\mathrm{Stab}(\tilde{Y})$ consisting of stability conditions invariant under the *induced* action of ι^*. Despite the construction of stability conditions on Enriques surfaces, moduli spaces of objects semistable with respect to a given Bridgeland stability condition have been heretofore unexplored.

[1] An Enriques surface is called *nodal* if it contains a smooth rational curve, necessarily of self-intersection -2, and *unnodal* otherwise.

2.3 An Outline and Summary of Main Results

We briefly describe now the contents of this dissertation, emphasizing the original contributions of this thesis to the study of moduli spaces of Gieseker stable sheaves and Bridgeland semistable objects.

We begin by reviewing the necessary background material about Enriques surfaces, moduli spaces of slope and Gieseker stability, and Bridgeland stability in Chapters 3,4, and 5, respectively. See these chapters for definitions and basic properties alluded to here.

Afterward, we discuss in Chapters 6 and 8 the question of constructing moduli spaces of objects in a derived category semistable with respect to a Bridgeland stability condition. Our work grew out of an attempt to understand the techniques of [11, 10] and to generalize them to Enriques surfaces in the hope of obtaining similar results, while at the same time investigating the subtle differences between the geometry of Enriques and K3 surfaces. Our first result is, as one would hope, a direct generalization of [11, Theorem 1.3(a)] about the existence and projectivity of Bridgeland semistable moduli spaces:

Theorem 2.3.1 (Chapter 8)**.** *Let Y be an Enriques surface, $v \in H^*_{\mathrm{alg}}(Y, \mathbb{Z})$ a Mukai vector, and $\sigma \in \mathrm{Stab}^\dagger(Y)$ a stability condition generic with respect to v (i.e. it does not lie on any wall). Then a coarse moduli space $M_{\sigma,Y}(v)$ of σ-semistable objects with Mukai vector v exists and when non-empty is a normal projective variety with torsion canonical divisor.*

To investigate the question of non-emptiness of $M_{\sigma,Y}(v)$, we use the work of Toda [61] to show that a motivic invariant[2] of $M_{\sigma,Y}(v)$ is the same as that of a moduli space of Gieseker semistable sheaves of some (possibly different) Mukai vector u. While non-emptiness is thus established if u is in one of the cases studied previously by Kim or Yoshioka, non-emptiness for arbitrary v (and thus u) does not follow from previous work. We take an aside in Chapter 7 to rectify this problem by proving the non-emptiness

[2] In practice, we use the virtual Hodge polynomial as our motivic invariant, though in theory one may use any other.

and irreducibility of the moduli space $M_{H,Y}(v,L)$ parametrizing Gieseker-semistable sheaves of primitive Mukai vector v of positive rank and determinant L with respect to a generic parametrization H:

Theorem 2.3.2 (Chapter 7)**.** *Let Y be an Enriques surface and v be a primitive Mukai vector of positive rank with $v^2 \geq -1$. For polarization H generic with respect to v, $M_{H,Y}(v,L)$ is non-empty unless $v^2 = 0$, $2|c_1(L)$, and $2 \nmid L + \frac{r}{2}K_Y$. If Y is unnodal, then $M_{H,Y}(v,L)$ is irreducible whenever non-empty.*

We provide various proofs of parts of this theorem to illustrate the strengths and weaknesses of the various techniques available, both classical and modern. In particular, we provide a quick proof of this theorem in Chapter 7 by using our techniques from 6 to reduce all open cases to those already solved. These techniques allow us to avoid some of the technicalities in the use of Fourier-Mukai transforms in [26, 64]. A feature of this method is that we show that the motivic invariant of $M_{H,Y}(v,L)$ is equal to the motivic invariant of a moduli space of Gieseker stable sheaves with a different Mukai vector.

A motivating conjecture behind this investigation suggests that the equality of motivic invariants is actually a byproduct of the two moduli spaces being *birational*. In the appendix to this dissertation, Chapter 11 we confirm this suspicion by using the geometry of elliptic fibrations to reprove non-emptiness and irreducibility when $\gcd(2,c_1)=1$, as well as in the special case $2|c_1$ and $c_1^2=0$. This technique was originally developed by Friedman [23] for rank two sheaves on regular elliptic surfaces and generalized to higher rank sheaves on elliptic K3 surfaces in an influential paper of O'Grady [53]. The idea is to use a special kind of generic polarization for which μ-stability is equivalent to being stable on the generic elliptic fiber. This allows us to directly construct a birational map

$$M_{H,Y}(v,L) \dashrightarrow M_{H,Y}(v-2v(\mathcal{O}_Y), L+K_Y),$$

which geometrically explains the equality of motivic invariants.

We conclude Chapter 7 with the following description of $M_{H,Y}(v)$ for non-primitive v in Section 7.3. In particular, we show that the locus of stable sheaves is always non-empty.

Theorem 2.3.3 (Theorem 8.2.5)**.** *Let $v = mv_0$ be a Mukai vector with v_0 primitive and $m > 0$ with H generic with respect to v.*

(a) The moduli space of Gieseker-semistable sheaves $M_{H,Y}(v)$ is nonempty if and only if $v_0^2 \geq -1$.

(b) Either $\dim M_{H,Y}(v) = v^2 + 1$ and $M^s_{H,Y}(v) \neq \varnothing$, $m = 1, v_0^2 = 0, \dim M_{H,Y}(v) = \dim M^s_{H,Y}(v) = 2$, or $m > 1$ and $v_0^2 \leq 0$.

(c) If $M_{H,Y}(v) \neq M^s_{H,Y}(v)$ and $M^s_{H,Y}(v) \neq \varnothing$, the codimension of the semistable locus is at least 2 if and only if $v_0^2 > 1$ or $m > 2$. Moreover, in this case and the case $M_{H,Y}(v) = M^s_{H,Y}(v)$, $M_{H,Y}(v)$ is normal with torsion canonical divisor.

We return to the realm of Bridgeland stability proper in Chapter 9, where we investigate the relationship between Bridgeland wall-crossing and the birational geometry of $M_{\sigma,Y}(v)$. Suppose for the moment that X is a smooth projective variety admitting coarse moduli spaces of σ-stable objects. To investigate the relationship between wall-crossing on $\mathrm{Stab}(X)$ and the birational geometry of Bridgeland moduli spaces, we make use of the natural nef divisor ℓ_σ on $M_{\sigma,X}(v)$ associated to a stability condition σ in a chamber $\mathcal{C}$ for v [11]. From this construction we get a continuous map $\ell : \mathcal{C} \to \mathrm{Nef}(M_{\sigma,X}(v))$, about which two natural questions arise. The first is whether or not the image of ℓ is contained in the ample cone $\mathrm{Amp}(M_{\sigma,X}(v))$. The closedness of semistability implies that ℓ extends to the closure $\overline{\mathcal{C}}$, and the second question is what happens at the boundary walls of $\mathcal{C}$.

We answer these questions with our next result in the case X is an Enriques surface Y, generalizing [11, Theorem 1.3(b),Theorem 1.4]. To place the result in context, let W be a wall of the chamber decomposition for v, σ_0 a generic point of W (so that it does not lie on any other walls), and $\sigma_\pm$ two nearby stability conditions in adjacent chambers $\mathcal{C}_\pm$ meeting along W with corresponding moduli spaces $M_\pm := M_{\sigma_\pm,Y}(v)$. Then consider the image of σ_0 under the map $\ell_\pm : \mathcal{C}_\pm \to \mathrm{Nef}(M_\pm)$, which we denote by $\ell_{0,\pm}$. Our next main result is this:

Theorem 2.3.4 (Theorems 9.1.3 and 9.2.3)**.** *Let Y be an Enriques surface and $v \in H^*_{\mathrm{alg}}(Y, \mathbb{Z})$ a Mukai vector.*

(a) *Suppose* $\sigma \in \mathrm{Stab}^\dagger(Y)$ *is generic with respect to* v*. Then* ℓ_σ *is ample.*

Now suppose that v *is primitive, then:*

(b) *The divisor classes* $\ell_{0,\pm}$ *are semi-ample (and remain so when restricted to each component of* $M_\pm$*), and they induce contraction morphisms*

$$\pi_\pm : M_\pm \to Z_\pm,$$

where $Z_\pm$ *are normal projective varieties.*

(c) *Suppose that* $M^s_{\sigma_0,Y}(v) \neq \varnothing$ *(so that in particular* $\ell_{0,\pm}$ *is big as well).*

- *If either* $\ell_{0,\pm}$ *is ample, then the other is ample, and the birational map*

$$f_{\sigma_0} : M_+ \dashrightarrow M_-$$

obtained by crossing the wall in σ_0 *extends to an isomorphism.*

- *If* $\ell_{0,\pm}$ *are not ample and the complement of* $M^s_{\sigma_0,Y}(v)$ *has codimension at least 2, then* $f_{\sigma_0} : M_+ \dashrightarrow M_-$ *is the flop induced by* $\ell_{0,+}$*. More precisely, we have a commutative diagram of birational maps*

$$\begin{tikzcd} M_{\sigma_+,Y}(v) \arrow[rr, dashed, "f_{\sigma_0}"] \arrow[dr, "\pi_+"'] & & M_{\sigma_-,Y}(v) \arrow[dl, "\pi_-"] \\ & Z_+ = Z_- & \end{tikzcd} ,$$

and $f^*_{\sigma_0}\ell_{0,-} = \ell_{0,+}$.

We believe that at the wall, $\ell_{\sigma_0,\pm}$ is always big and $\pi_\pm$ always birational, as is the case for K3 surfaces [11], but the powerful hyperkähler methods used in [11] no longer work for Enriques surfaces. Nevertheless, we have ad-hoc arguments in many cases.

We end this thesis with Chapter 10, which consists of three sections on applications. In the first section on the classical moduli spaces of Gieseker stable sheaves, we obtain explicit, effective bounds on the Gieseker chamber and thus, via the Bayer-Macrì map, on the ample cone of these moduli spaces.

In the second section, we use the Bridgeland stability techniques developed in Chapter 9 to describe explicitly $\mathrm{Nef}(Y^{[n]})$, the nef cone of the Hilbert scheme of n points on an Enriques surface Y, in terms of the beautiful geometry of Enriques surfaces and their elliptic pencils. More specifically, denote by $2B$ the divisor parametrizing the locus of non-reduced 0-dimensional subschemes of length n on Y, and for every $H \in \mathrm{Amp}(Y)$, denote by $\tilde{H}$ the locus of 0-dimensional subschemes of length n on Y meeting a member of the linear system $|H|$. Then B and $\langle \tilde{H} | H \in \mathrm{Amp}(Y) \rangle$ generate $\mathrm{Pic}(Y^{[n]})$. We recall the ϕ-function defined in [17, Section 2.7] by

$$\phi(D) = \inf\{|D.F| : F \in \mathrm{Pic}(Y), F^2 = 0\},$$

for $D^2 > 0$. The significance of primitive $F \in \mathrm{Pic}(Y)$ with $F^2 = 0$ is that either F or $-F$ is effective, say F, and $2F$ defines an elliptic pencil on Y with exactly two multiple fibres F and $F + K_Y$, respectively. As suggested by the definition of ϕ, these "half-pencils" govern much of the geometry of Y. We obtain the following result which confirms this overarching theme in the study of Enriques surfaces:

Theorem 2.3.5 (Theorem 10.2.3)**.** *Let Y be an unnodal Enriques surface and $n \geq 2$. Then $\tilde{D} - aB \in \mathrm{Nef}(Y^{[n]})$ if and only if $D \in \mathrm{Nef}(Y)$ and $0 \leq na \leq D.F$ for every $0 < F \in \mathrm{Pic}(Y)$ with $F^2 = 0$, or in other words $0 \leq a \leq \frac{\phi(D)}{n}$. Moreover, the face given by $a = 0$ induces the Hilbert-Chow morphism, and for every ample $H \in \mathrm{Pic}(Y)$, $\tilde{H} - \frac{\phi(H)}{n}B$ induces a flop.*

Recall that the Hilbert-Chow morphism $h : Y^{[n]} \to Y^{(n)}$ sends a 0-dimensional subscheme of length n to its underlying 0-cycle and is a divisorial contraction with exceptional locus $2B$. Moreover, we can describe explicitly the flop induced by $\tilde{H} - \frac{\phi(H)}{n}B$ as follows: for every half-pencil F such that $H.F = \phi(H)$, we get a pair of disjoint codimension n components of the exceptional locus isomorphic to $F^{[n]}$ and $(F + K_Y)^{[n]}$, respectively, where these precisely parametrize the sublocus of n-points on Y contained in F and $F + K_Y$, respectively. On the component corresponding to F, say, the contracted fibers (i.e. the curves of S-equivalent objects) are exactly the fibers of the natural Abel-Jacobi morphism $F^{[n]} \to \mathrm{Jac}^n(F) \cong F$ ($g(F) = 1$) associating to Z the line bundle $\mathcal{O}_F(Z)$, where the objects of $F^{[n]}$ fit into the destabilizing exact sequence

$$0 \to \mathcal{O}(-F) \to I_Z \to \mathcal{O}_F(-Z) \to 0.$$

Upon crossing the wall, we perform a flop, replacing the $\mathbb{P}^{n-1}$-bundle $F^{[n]}$ over the base F with another one parametrizing objects sitting in an exact sequence of the form

$$0 \to \mathcal{O}_F(-Z) \to E \to \mathcal{O}(-F) \to 0,$$

where $E \in \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}(-F), \mathcal{O}_F(-Z)))$.

Theorem 2.3.5 can be seen as giving an alternative definition of the ϕ-function, and we believe that the properties of ϕ can be recovered from the convexity of $\mathrm{Nef}(Y^{[n]})$ and pairing divisors with test curves.

In the final section, we apply Theorem 2.3.5 to recover a weak form of a classical result about linear systems on unnodal Enriques surfaces (see [17, Theorems 4.4.1 and 4.6.1]):

Corollary 2.3.6. *Let Y be an unnodal Enriques surface and $H \in \mathrm{Pic}(Y)$ ample with $H^2 = 2d$. Then*

(a) The linear system $|H|$ is base-point free if and only if $\phi(H) \geq 2$,

(b) If $|H|$ is very ample, then $\phi(H) \geq 3$. Conversely, if $\phi(H) \geq 4$ or $\phi(H) = 3$ and $d = 5$, then $|H|$ is very ample.

(c) The linear system $|2H|$ is base-point free and $|4H|$ is very ample.

Remark 2.3.7. The stronger form of the above result says that for $d \geq 5$, $|H|$ is very ample if and only if $\phi(H) \geq 3$. It follows that for any ample H, even $|3H|$ is very ample. We believe the Bridgeland stability methods we use to prove our weakened version above can be pushed further to prove the full result (see Remark 10.3.3).

We also obtain some new results about n-very ample line bundles on unnodal Enriques surfaces. Recall that a line bundle $\mathcal{O}_X(H)$ on a smooth projective surface X is called n-very ample if the restriction map

$$\mathcal{O}_X(H) \to \mathcal{O}_Z(H)$$

is surjective for every 0-dimensional subscheme Z of length $n+1$. Then we prove the following result:

Corollary 2.3.8. *Let Y be an unnodal Enriques surface and $H \in \operatorname{Pic}(Y)$ ample with $H^2 = 2d$. Then $\mathcal{O}_Y(H)$ is n-very ample provided that*

$$0 \leq n \leq \frac{d \cdot \phi(H)}{2d - \phi(H)} - 1.$$

Both of these results follow from the following vanishing theorem which is a direct consequence of the Bridgeland stability techniques of Theorem 2.3.5:

Proposition 2.3.9 (Proposition 10.3.1)**.** *Let Y be an unnodal Enriques surface and $H \in \operatorname{Pic}(Y)$ ample with $H^2 = 2d$. Then for any $Z \in Y^{[n]}$,*

$$H^i(Y, I_Z(H + K_Y)) = 0, \text{ for } i > 0,$$

provided that

$$1 \leq n < \frac{d \cdot \phi(d)}{2d - \phi(d)}.$$

2.4 Open questions

Some open question about moduli of semistable sheaves on Enriques surfaces persist after the work presented here. Our irreducibility results work for unnodal Enriques surfaces, but irreducibility remains unknown for primitive Mukai vectors on nodal Enriques surfaces. Likewise, irreducibility in the non-primitive case remains open in both cases. Finally, we have focused here on moduli spaces of Gieseker semistable sheaves, but the geometry of moduli spaces of slope-semistable sheaves remains largely unexplored, at least in the non-primitive case (where the moduli spaces can differ significantly). Some work in this direction is contained a forthcoming paper of Yoshioka [66].

A further fundamental question in the connection developed here between wall-crossing and birational geometry is whether or not the divisor $\ell_{\sigma_0,\pm}$ is big in the case of a totally semistable wall, that is $M^s_{\sigma_0,Y}(v) = \varnothing$, which is unknown at the moment. The failure of bigness would give the existence of interesting fibration structures on $M_\pm$ provided by the morphisms $\pi_\pm$. Another open question is whether or not one can classify entirely and in total generality the walls in $\mathrm{Stab}^{\dagger}(Y)$ in terms of the geometry of the morphism $\pi_{\pm}$ for any given primitive Mukai vector v. Even if this is achieved, we wonder if a full Hassett-Keel-type result holds true for Enriques surfaces as shown to be the case for K3 surfaces in [10]. That is, does every minimal model of $M_{\sigma,Y}(v)$ appear after deformation of the stability condition, i.e. as another moduli space of Bridgeland stable objects? We hope to take up both of these questions in the future.

It is natural to wonder what kind of varieties the moduli spaces $M_{\sigma,Y}(v)$ are. For primitive v, these are normal projective varieties with torsion canonical divisors. We suspect, based on examples, that these are always of Calabi-Yau-type, being genuine Calabi-Yau manifolds for v of even rank while only admitting genuine Calabi-Yau étale covers in the case of odd rank.

Chapter 3

Review: Enriques surfaces

We collect here some of the basic definitions and results on Enriques surfaces that we use.

3.1 First properties

Definition 3.1.1. An Enriques surface is a smooth complex projective surface Y with $h^1(\mathcal{O}_Y) = h^2(\mathcal{O}_Y) = 0$ and $2K_Y \sim 0$.

It is well-known (see [7, Lemma VIII.15.1]) that Enriques surfaces have fundamental group $\mathbb{Z}/2\mathbb{Z}$ and universal cover a K3 surface $\tilde{Y}$. We denote the covering map by $\pi : \tilde{Y} \to Y$ and the covering involution by $\iota : \tilde{Y} \to \tilde{Y}$. By Noether's formula, the topological Euler characteristic satisfies $\chi(Y) = 12$. As such, we get

$$h^{1,0}(Y) = h^{0,1}(Y) = h^{2,0}(Y) = h^{0,2}(Y) = 0, h^{1,1}(Y) = 10.$$

Furthermore, it is known that the cohomology of the tangent bundle T_Y is $h^0(T_Y) = h^2(T_Y) = 0$ and $h^1(T_Y) = 10$ [7, Lemma VIII.15.3], so the moduli space of Enriques surfaces is smooth of dimension 10. It is in fact irreducible [25].

Definition 3.1.2. An Enriques surface is called nodal if contains a smooth rational curve and unnodal otherwise.

Nodal Enriques surfaces form a nine-dimensional irreducible divisor in the moduli space of Enriques surfaces, so the condition of being unnodal is open and describes the generic Enriques surface.

3.2 Divisors on Enriques surfaces

Let us recall some facts about divisors on an Enriques surface Y.

To begin with, for any Enriques surface Y, we can write the long exact sequence on cohomology for the exponential exact sequence,

$$0 = H^1(Y, \mathcal{O}_Y) \to \mathrm{Pic}(Y) \to H^2(Y, \mathbb{Z}) \to H^2(Y, \mathcal{O}_Y) = 0,$$

to see that $\mathrm{Pic}(Y) \cong \mathrm{NS}(Y) \cong H^2(Y, \mathbb{Z})$. As the intersection pairing on the torsion-free component $\mathrm{Num}(Y) = H^2(Y, \mathbb{Z})_f$ is unimodular of signature (1,9), it follows that

$$\mathrm{Pic}(Y) \cong U \oplus -E_8 \oplus \langle K_Y \rangle,$$

where $U \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the hyperbolic lattice and $-E_8$ is the even negative-definite lattice obtained by negating the usual even positive-define lattice with Dynkin diagram E_8.

For convenience, we record that for a sheaf E of rank r, Riemann-Roch says that

$$h^0(E) - h^1(E) + h^2(E) = \chi(E) = r + \frac{1}{2}c_1(E)^2 - c_2(E).$$

In particular, for a divisor D we get $\chi(D) = \frac{1}{2}D^2 + 1$.

The following two simple propositions will be especially useful:

Proposition 3.2.1 ([17]). *Let D be a divisor with $D^2 \geq 0$ and $D \neq 0, K_Y$. Then D is effective or $-D$ is effective. If D is effective, then $D + K_Y$ is also effective.*

Proof. As the proof is simple, we include it here. By Riemann-Roch,

$$h^0(D) - h^1(D) + h^2(D) = \frac{1}{2}D^2 + 1 \geq 1,$$

so by Serre duality

$$h^0(D) + h^0(-D + K_Y) = h^0(D) + h^2(D) \geq 1.$$

Thus either D is effective or $-D + K_Y$ is effective, but not both since then K_Y would be effective. Repeating the same argument for $D + K_Y$ gives

$$h^0(D + K_Y) + h^0(-D) \geq 1.$$

Since $D \neq 0$, D and $-D$ cannot be simultaneously effective. Without loss of generality, we may suppose D is effective, so $h^0(-D) = 0$ and effectivity of $D + K_Y$ follows from this last equation. $\square$

Definition 3.2.2. For any divisor D with $D^2 > 0$, we define

$$\phi(D) = \inf\{|D \cdot F||F \in \mathrm{Pic}(Y), F^2 = 0\}.$$

The most important property of ϕ for us is the following [17, Section 2.7]:

Theorem 3.2.3. *$0 < \phi(D)^2 \leq D^2$.*

The importance of effective divisors of square zero in the geometry of Enriques surfaces lies partially in the fact that they correspond to elliptic pencils. Indeed, for an effective divisor F with $F^2 = 0$ and primitive class in the lattice $\mathrm{Num}(Y)$, $2F$ is one of the two non-reduced members of the pencil $|2F|$, whose generic member is a smooth elliptic curve. A pencil whose generic member is a smooth elliptic curve is called an elliptic pencil, and all complete elliptic pencils on an Enriques surface Y arise in this way. Such an F is called an elliptic half-pencil. We will denote by F_A and $F_B = F_A + K_Y$ the two elliptic half-pencils supporting the two double fibers of the elliptic fibration induced by $|2F_A| = |2F_B|$. The following final fact will also be of use to us:

Proposition 3.2.4 ([17])**.** *For every elliptic pencil $|2E|$ on an Enriques surface Y, there exists an elliptic pencil $|2F|$ such that $E.F = 1$.*

Proof. As $\mathrm{Num}(Y)$ is unimodular, we can find $F' \in \mathrm{Pic}(Y)$ such that $E.F' = 1$. Then since $\mathrm{Num}(Y)$ is even, $F := F' - \frac{F'^2}{2}E \in \mathrm{Pic}(Y)$ satisfies $F^2 = 0$ and $F.E = 1$, so $|2F|$ is the required elliptic pencil. $\square$

If Y is an unnodal Enriques surface, then every elliptic half-pencil is irreducible.

3.3 The algebraic Mukai lattice

Let Y be an Enriques surface with universal cover a projective $K3$ surface $\tilde{Y}$ and fixed-point free covering involution ι such that $Y = \tilde{Y}/\langle\iota\rangle$. We recall here a helpful bookkeeping invariant that we will use to describe the topological invariants of sheaves and complexes on Enriques surfaces.

Recall from Chapter 1 that we define the Mukai lattice of Y by

$$H^*_{\mathrm{alg}}(Y,\mathbb{Z}) := v(K_{\mathrm{num}}(Y)) \subset H^*(Y,\mathbb{Q}),$$

where $v(E) = \mathrm{ch}(E).\sqrt{\mathrm{td}(Y)}$ is the Mukai vector. If we denote by ρ_Y the fundamental class, then

$$H^*_{\mathrm{alg}}(Y,\mathbb{Z}) = H^0(Y,\mathbb{Z}) \oplus \mathrm{Num}(Y) \oplus \frac{1}{2}\mathbb{Z}\rho_Y. \tag{3.1}$$

Written according to the decomposition (3.1), we may write the Mukai vector as

$$v(E) = (r(E), c_1(E), \frac{r(E)}{2} + \mathrm{ch}_2(E)).$$

We denote the Mukai pairing $H^*_{\mathrm{alg}}(Y,\mathbb{Z}) \times H^*_{\mathrm{alg}}(Y,\mathbb{Z}) \to \mathbb{Z}$ by $(\_,\_)$; it can be defined by $(v(E), v(F)) := -\chi(E,F)$, where

$$\chi(E,F) = \sum_p (-1)^p \operatorname{ext}^p(E,F)$$

denotes the Euler pairing on $K(Y)$. This becomes non-degenerate when modding out by its kernel to get $K_{\mathrm{num}}(Y)$. According to the decomposition (3.1), we have

$$\big((r,c,s),(r',c',s')\big) = c.c' - rs' - r's,$$

for $(r,c,s),(r',c',s') \in H^*_{\mathrm{alg}}(Y,\mathbb{Z})$.

As usual in lattice theory, we call a Mukai vector v *primitive* if it is not divisible in $H^*_{\mathrm{alg}}(Y,\mathbb{Z})$. Note that the covering space map $\pi : \tilde{Y} \to Y$ with covering involution ι induces an embedding

$$\pi^* : H^*_{\mathrm{alg}}(Y,\mathbb{Z}) \hookrightarrow H^*_{\mathrm{alg}}(\tilde{Y},\mathbb{Z}) := H^0(\tilde{Y},\mathbb{Z}) \oplus \mathrm{NS}(\tilde{Y}) \oplus H^4(\tilde{Y},\mathbb{Z})$$

such that $(\pi^* v, \pi^* w) = 2(v,w)$, and it identifies $H^*_{\mathrm{alg}}(Y,\mathbb{Z})$ with an index 2 sublattice of the ι^*-invariant component of $H^*_{\mathrm{alg}}(\tilde{Y},\mathbb{Z})$. However, the embedding of lattices $\pi^* : \mathrm{Num}(Y) \hookrightarrow \mathrm{NS}(\tilde{Y})$ is primitive, i.e. has torsion-free cokernel, and identifies $\mathrm{Num}(Y)$ with the ι^*-invariant part of $\mathrm{NS}(\tilde{Y})$. It follows that for a primitive Mukai vector $v \in H^*_{\mathrm{alg}}(Y,\mathbb{Z})$, $\pi^* v$ is divisible by at most 2. All of this is encapsulated nicely in the following lemma:

Lemma 3.3.1. *A Mukai vector $v = (r, c_1, s) \in H^*_{\mathrm{alg}}(Y, \mathbb{Z})$ is primitive if and only if*

$$\gcd(r, c_1, \frac{r+2s}{2}) = 1.$$

If v is primitive, then $\gcd(r, c_1, 2s) = 1$ or 2, and moreover:

- *if $\gcd(r, c_1, 2s) = 1$, then either r or c_1 is not divisible by 2 (i.e. $\pi^* v$ is primitive);*
- *if $\gcd(r, c_1, 2s) = 2$, then c_2 must be odd and $r + 2s \equiv 2 (\mathrm{mod}\, 4)$ (i.e. $\pi^* v$ is divisible by 2).*

Proof. The conclusion following from v being primitive is precisely the statement of [26, Lemma 2.5], except for the formulation about the primitivity of $\pi^* v$. Let us demonstrate the validity of this reformulation. Suppose that v is primitive but that $\pi^* v$ is divisible by m. Then

$$\pi^* v = (r, \pi^* c_1, 2s) = m(r', c_1', r' + \frac{1}{2}(c_1')^2 - c_2') = (mr', mc', mr' + \frac{m}{2}(c_1')^2 - mc_2'),$$

from which it follows that

$$r = mr', \pi^* c_1 = mc', 2s = mr' + \frac{m}{2}(c')^2 - mc_2'.$$

It follows that $m | \gcd(r, c_1, 2s)$. If $\gcd(r, c_1, 2s) = 1$, then we get $m = 1$, so $\pi^* v$ is primitive. Now let us show that if $\gcd(r, c_1, 2s) = 2$, then $\pi^* v$ is divisible by 2. So let $r = 2r', c_1 = 2c_1'$, and note that s is an integer in this case. As $(c_1')^2$ is an even integer, it follows that $2 | (2r' + (c_1')^2 - 2s)$, and thus we can solve for c_2'. So $\pi^* v$ is indeed divisible by 2 in this case.

Finally, let us show that v is primitive if and only if $\gcd(r, c_1, \frac{r+2s}{2}) = 1$. If v is divisible by m, then as above, $m | r$ and $m | c_1$, so we get $r = mr'$ and $c_1 = mc_1'$. Then from

$$\frac{r}{2} + \frac{1}{2} c_1^2 - c_2 = m(r' + \frac{1}{2}(c_1')^2 - c_2'),$$

we see that $m | c_2$. As $\frac{r+2s}{2} = r + \frac{1}{2} c_1^2 - c_2$, it follows that $m | \frac{r+2s}{2}$. So $\gcd(r, c_1, \frac{r+2s}{2}) = 1$ implies v is primitive. Conversely, if $m | \gcd(r, c_1, \frac{r+2s}{2})$, then $m | c_2$, and

$$v = m\left(\frac{r}{m}, \frac{c_1}{m}, \frac{r/m}{2} + \frac{1}{2}(c_1/m)^2 - \left((c_2/m) - \frac{1-m}{2}(c_1/m)^2\right)\right). \quad \square$$

In particular, for odd rank Mukai vectors or Mukai vectors with c_1 primitive, π^*v is still primitive, while primitive Mukai vectors with $\gcd(r, c_1) = 2$ (and thus necessarily $\gcd(r, c_1, 2s) = 2$) must satisfy $v^2 \equiv 0 (\mathrm{mod}\, 8)$, as can be easily seen.

Chapter 4

Review: Moduli spaces of semistable sheaves

In this chapter, we remind the reader of fundamental definitions and facts about the classical definitions of stability for sheaves. We also recall explicitly the known results about stable sheaves on Enriques surfaces.

4.1 Slope stability

Let $H \in \operatorname{Amp}(X)$ on a smooth projective surface X. We define the slope function μ_H on $\operatorname{Coh} X$ by

$$\mu_H(E) = \begin{cases} \frac{H.c_1(E)}{r(E)} & \text{if } r(E) > 0, \\ +\infty & \text{if } r(E) = 0. \end{cases} \tag{4.1}$$

This gives a notion of slope stability for sheaves, for which Harder-Narasimhan filtrations exist (see [28, Section 1.6]). Recall that a torsion free coherent sheaf E is called slope *semistable* (resp. *stable*) with respect to H if for every $F \subset E$ with $0 < r(F) < r(E)$ we have $\mu_H(F) \leq \mu_H(E)$ (resp. $\mu_H(F) < \mu_H(E)$). We will sometimes use the notation μ-stability, or μ_H-stability if we want to make the dependence on H clear. Also recall that every torsion free coherent sheaf E admits a unique *Harder-Narasimhan filtration*

$$0 = \mathrm{HN}^0(E) \subset \mathrm{HN}^1(E) \subset ... \subset \mathrm{HN}^n(E) = E,$$

with μ-semistable factors $E_i = \mathrm{HN}^i(E)/\operatorname{HN}^{i-1}(E)$ satisfying

$$\mu_H(E_1) > ... > \mu_H(E_n).$$

We can filter a μ-semistable sheaf even further to obtain a *Jordan-Hölder filtration*

$$0 = \mathrm{JH}^0(E) \subset \mathrm{JH}^1(E) \subset ... \subset \mathrm{JH}^n(E) = E,$$

with μ-stable factors $E_i = \mathrm{JH}^i(E)/\mathrm{JH}^{i-1}(E)$ of slope equal to $\mu_H(E)$. Unfortunately, Jordan-Hölder filtrations are not unique, and even the associated graded object $gr_\mu^{\mathrm{JH}}(E) := \oplus_i E_i$ is only uniquely defined in codimension one. On the other hand, the reflexive hull (or double-dual) $E^{**} := gr_\mu^{\mathrm{JH}}(E)^{\vee\vee}$ is uniquely defined [28, Corollary 1.6.10].

4.2 Gieseker stability

Let $H \in \mathrm{Amp}(X)$ on a smooth projective surface X. Recall that the *Hilbert polynomial* is defined by

$$P(E,m) := \chi(E(mH)),$$

for $E \in \mathrm{Coh}(X)$. This polynomial can be uniquely written in the form

$$\sum_{i=0}^{\dim(E)} a_i(E)\frac{m^i}{i!},$$

and we define the *reduced Hilbert polynomial* by

$$p(E,M) := \frac{P(E,m)}{a_{\dim(E)}(E)}.$$

Here the dimension of a coherent sheaf E is the dimension of its support. This gives rise to the notion of H-Gieseker stability for sheaves. We refer to [28, Chapter 1] for basic properties of Gieseker stability, but we just mention that like above, a pure dimensional sheaf E is called *Gieseker semistable* (resp. *Gieseker stable*) if for every proper subsheaf $0 \neq F \subset E$, $p(F,m) \leq p(E,m)$ (resp. $p(F,m) < p(E,m)$) for all $m \gg 0$.[1] Harder-Narasimhan and Jordan-Hölder filtrations are defined analogously as above with p replacing μ_H, except that the associated graded sheaf $gr^{HN}(E)$ is now unique, despite non-uniqueness of the filtration itself.

It is worth pointing out that

$$E \text{ is } \mu\text{-stable} \Rightarrow E \text{ is Gieseker-stable } \Rightarrow E \text{ is Gieseker-semistable } \Rightarrow E \text{ is } \mu\text{-semistable}. \tag{4.2}$$

[1] A sheaf E is *pure dimensional* if it contains no nonzero sheaves of smaller dimension.

4.3 Moduli spaces of semistable sheaves

Let $H \in \mathrm{Amp}(X)$ on a smooth projective surface X. We fix a Mukai vector $v \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$ (or in other words, we fix the topological invariants r, c_1, c_2).

4.3.1 Moduli spaces for Gieseker semistability

We denote by $\mathfrak{M}_{H,X}(v)$ the moduli stack of flat families of H-Gieseker semistable sheaves with Mukai vector v. By [28, Chapter 4] there exists a projective variety $M_{H,X}(v)$ which is a coarse moduli space parameterizing S-equivalence classes of semistable sheaves. Recall that two Gieseker semistable sheaves are called S-equivalent if they have the same graded object associated to their respective Jordan-Hölder filtrations.

Upon considering iterated extensions of stable sheaves, it becomes clear that considering S-equivalence is necessary in order to obtain a coarse moduli space. Indeed, suppose that $\{E_t\}$ is a family of semistable sheaves over $\mathbb{A}^1$ representing $tv \in \mathrm{Ext}^1(E_2, E_1)$, where the E_i are stable sheaves with the same reduced Hilbert polynomial and $0 \neq v \in \mathrm{Ext}^1(E_2, E_1)$. Then in a coarse moduli space $\mathbb{A}^1 - \{0\}$ must be mapped to a single point, as for $t \neq 0$ the E_t are all isomorphic, so we must map $E_0 = E_1 \oplus E_2$ to the same point. That is, a coarse moduli space can only distinguish S-equivalence classes.

The open substack $\mathfrak{M}^s_{H,X}(v) \subseteq \mathfrak{M}_{H,X}(v)$ parameterizing stable sheaves is a $\mathbb{G}_m$-gerbe over the similarly defined open subset $M^s_{H,X}(v) \subseteq M_{H,X}(v)$ of the coarse moduli space. In the sequel, we will suppress the reference to X when X is understood from context.

4.3.2 Moduli spaces for μ-semistability

We denote by $\mathfrak{M}^{\mu ss}_{H,X}(v)$ the moduli stack of flat families of μ_H-semistable sheaves with Mukai vector v. By [28, Section 8.2] there exists a projective variety $M^{\mu ss}_{H,X}(v)$ which gives an algebraic structure to the gauge-theoretic Donaldson-Uhlenbeck compactification of the moduli space of μ-stable locally free sheaves. The open substack $\mathfrak{M}^{\mu s}_{H,X}(v) \subseteq \mathfrak{M}^{\mu ss}_{H,X}(v)$ parameterizing μ-stable sheaves is a $\mathbb{G}_m$-gerbe over the similarly defined open subset $M^{\mu s}_{H,X}(v) \subseteq M^{\mu ss}_{H,X}(v)$. As one would expect, the last implication in (4.2) gives rise to a stack morphism $\mathfrak{M}_{H,X}(v) \to \mathfrak{M}^{\mu ss}_{H,X}(v)$ inducing a morphism $\gamma : M_{H,X}(v) \to M^{\mu ss}_{H,X}(v)$. While $M_{H,X}(v)$ could be described as parametrizing S-equivalence classes of semistable sheaves, the description of $M^{\mu ss}_{H,X}(v)$ as a moduli space is more subtle. A fundamental fact is the following:

Theorem 4.3.1 (Theorem 8.2.11,[28]). *Two μ-semistable sheaves of Mukai vector v define the same point in $M^{\mu ss}_{H,X}(v)$ if and only if $F_1^{**} \cong F_2^{**}$ and $F_i^{**}/gr^{\mathrm{JH}}_{\mu}(F_i)$ define the same 0-cycle in the symmetric product $S^l(X)$.*

A consequence of the proof of Theorem 4.3.1 is that $M^{\mu ss}_{H,X}(v)$ admits a stratification

$$M^{\mu ss}_{H,X}(v) = \bigsqcup_{l \geq 0} M^{\mu-poly}_{H,X}(v) \times S^l(X),$$

where $M^{\mu-poly}_{H,X}(v) \subset M_{H,X}(v)$ denotes the subset of locally free sheaves of Mukai vector v which are the direct sum of μ-stable locally free sheaves of equal slope.

4.4 Wall-and-chamber structure on $\mathrm{Amp}(X)$

We must also recall the definition of a polarization that is *generic* with respect to a given Mukai vector v satisfying $v^2 > -r(v)^2$ (the Bogomolov inequality) and $r(v) \geq 2$ (see [**?**, Section 4.C]). Consider $\xi \in \mathrm{Num}(X)$ with $-\frac{r(v)^2}{4}(v^2 + r(v)^2) \leq \xi^2 < 0$. The *wall* for v corresponding to ξ is the real codimension 1 subcone $\xi^{\perp} \cap \mathrm{Nef}(X) \subset \mathrm{Nef}(X)$. These walls are locally finite. A polarization $H \in \mathrm{Amp}(X)$ is *generic* with respect to v if it does not lie on any of these walls. An important consequence of this is that for a destabilizing subobject F of E, with $v(E)$ satisfying the two inequalities above and H generic with respect to $v(E)$, we must have $v(F) \in \mathbb{R}_{>0} v(E)$. So if v is primitive, any H-Gieseker semistable sheaf E with $v(E) = v$ is Gieseker-stable as well. If, in addition, c_1 is primitive in $\mathrm{Num}(X)$, then any μ_H-semistable sheaf is even μ_H-stable.

4.4.1 Suitable polarizations

For use in Section 11.1, we recall the definition of a v-suitable polarization. We say that a polarization H is suitable with respect to v and an elliptic half-fibre F_A of class f, if for any $\xi \in \mathrm{Num}(X)$ with $-\frac{r(v)^2}{4}(v^2 + r(v)^2) \leq \xi^2 < 0$ either $\xi . f = 0$

or $\xi.f$ and $\xi.H$ have the same sign. For any polarization H, $H+nf$ is suitable for $n \geq \frac{r(v)^2(H.f)}{8}(v^2+r(v)^2)$. This definition is equivalent to H being in (the closure of) the chamber surrounding f. We say that H is a generic suitable polarization if H is in fact in the interior of this chamber. An important consequence of being v-suitable is that if E is μ_H-semistable, then the restriction $E|_F$ is semistable for a generic elliptic fiber $F \in |2F_A|$, and conversely if $E|_F$ is stable, then E is μ_H-stable.

4.5 (Quasi-)universal families

We review one last facet of the general theory of moduli spaces of sheaves. While a coarse moduli space exists, it is not always a fine moduli space, i.e. there does not always exist a universal family of semistable sheaves. To remedy the possible lack of a universal family, Mukai [50] came up with the following substitute, which is usually good enough for most purposes:

Definition 4.5.1. Let T be an algebraic space of finite-type over $\mathbb{C}$ and X a smooth projective variety.

(a) A family $\mathcal{E}$ on $T \times X$ is called a *quasi-family* of objects in $\mathfrak{M}_{H,X}(v)$ if for all closed points $t \in T$, there exists $E \in \mathfrak{M}_{H,X}(v)(\mathbb{C})$ such that $\mathcal{E}_t \cong E^{\oplus \rho}$, where $\rho > 0$ is an integer which is called the *similitude* and is locally constant on T.

(b) Two quasi-families $\mathcal{E}$ and $\mathcal{E}'$ on T, of similitudes ρ and ρ', respectively, are called *equivalent* if there are locally free sheaves $\mathcal{N}$ and $\mathcal{N}'$ on T such that $\mathcal{E} \otimes p_T^*\mathcal{N} \cong \mathcal{E}' \otimes p_T^*\mathcal{N}'$. It follows that the similitudes are related by $\mathrm{rk}\,\mathcal{N} \cdot \rho = \mathrm{rk}\,\mathcal{N}' \cdot \rho'$.

(c) A quasi-family $\mathcal{E}$ is called *quasi-universal* if for every scheme T' and quasi-family $\mathcal{E}'$ on T', there exists a unique morphism $f : T' \to T$ such that $f^*\mathcal{E}$ is equivalent to $\mathcal{E}'$.

The usual techniques (see for example [50, Theorem A.5] or [28, Section 4.6]) show that a quasi-universal family exists on $M^s_{H,X}(v)$ and is unique up to equivalence.

4.6 Moduli of sheaves on Enriques surfaces: What is known?

We close this chapter with a summary of the known results in the Enriques case. From standard results in the deformation theory of sheaves [28, Section 2.A and 4.5], we have for a simple sheaf E, i.e. $\hom(E, E) = 1$,

$$v^2+1 \leq \dim_E M_{H,Y}(v) \leq \dim T_E M_{H,Y}(v) = v^2+1+\mathrm{ext}^2(E,E) = v^2+1+\hom(E, E(K_Y)).$$

Kim's main structure result from [34] is the following:

Theorem 4.6.1. *Let Y be an Enriques surface with K3 cover $\tilde{Y}$.*

(a) $M^s_{H,Y}(v)$ is singular at E if and only if $E \cong E(K_Y)$ except if E belongs to a 0-dimensional component (E is a spherical sheaf and $v^2 = -2$) or a 2-dimensional component ($v^2 = 0$), along which all sheaves are fixed by $- \otimes \mathcal{O}_Y(K_Y)$.

(b) The singular locus of $M^s_{H,Y}$ is a union of the images under π_* of finitely many open subsets $M^{s,\circ}_{H,\tilde{Y}}(w)$ for different $w \in H^*_{\mathrm{alg}}(\tilde{Y})$ such that $\pi_*(w) = v$, where

$$M^{s,\circ}_{H,\tilde{Y}}(w) := \{F \in M^s_{H,\tilde{Y}}(w) | F \ncong \iota^* F\}.$$

The singular locus has even dimension at most $\frac{1}{2}(v^2+4)$. In particular, $M^s_{H,Y}(v)$ is generically smooth and everywhere smooth if $r(v)$ is odd.

(c) The pull-back map $\pi^* : M^s_{H,Y}(v) \to M_{H,\tilde{Y}}(\pi^* v)$ is a double cover onto a Lagrangian subvariety of $M_{H,\tilde{Y}}(\pi^* v)$, the fixed locus of ι^*, and is branched precisely along the locus where $E \cong E(K_Y)$.

We recall from Chapter 3 that an Enriques surface Y is unnodal if it contains no smooth rational curves. One useful consequence of being unnodal is that the ample cone is entirely round, i.e. an effective $D \in \mathrm{Pic}(Y)$ is ample if and only if $D^2 > 0$. Moreover, ι^* acts trivially on $\mathrm{Pic}(\tilde{Y})$ in this case, so $\pi^* \mathrm{Pic}(Y) = \mathrm{Pic}(\tilde{Y})$. An important consequence of this is that for any $E \in M^s_{H,Y}(v)$ with $E \cong E(K_Y)$, $\pi^* v$ must be divisible by 2 in $H^*_{\mathrm{alg}}(\tilde{Y}, \mathbb{Z})$, as follows from the proof of Theorem 4.6.1.

For primitive v, $M^s_{H,Y}(v)$ is thus smooth of dimension $v^2 + 1$, unless $\pi^* v$ is divisible by 2, in which case either $v^2 = 0$ and $M^s_{H,Y}(v)$ contains a smooth exceptional component of dimension 2 (we will see later that this is the entire moduli space) or $M^s_{H,Y}(v)$ is still of dimension v^2+1 with singular locus of codimension at least 2.

Now recall that for a variety X over $\mathbb{C}$, the cohomology with compact support $H^*_c(X,\mathbb{Q})$ has a natural mixed Hodge structure. Let $e^{p,q}(X) := \sum_k (-1)^k h^{p,q}(H^k_c(X))$ and $e(X) := \sum_{p,q} e^{p,q}(X)x^p y^q$ be the virtual Hodge number and Hodge polynomial, respectively. For an Enriques surface Y we recall that the kernel of $\mathrm{NS}(Y) \to \mathrm{Num}(Y)$ is given by $\langle K_Y \rangle$, and thus

$$M_{H,Y}(v) = M_{H,Y}(v, L_1) \coprod M_{H,Y}(v, L_2),$$

where $M_{H,Y}(v, L_i)$ denotes those $E \in M_{H,Y}(v)$ with $\det(E) = L_i$ and $L_2 = L_1(K_Y) \in \mathrm{Pic}(Y)$ so $c_1 = c_1(L_1) = c_2(L_2) \in \mathrm{Num}(Y)$.

The following result is proved in [64]:

Theorem 4.6.2. *Let $v = (r, c_1, s) \in H^*_{\mathrm{alg}}(Y, \mathbb{Z})$ be a primitive Mukai vector with r odd and Y unnodal. Then*

$$e(M_{H,Y}(v, L)) = e(Y^{[\frac{v^2+1}{2}]}),$$

for a generic H, where $L \in \mathrm{Pic}(Y)$ satisfies $c_1(L) = c_1$. In particular,

- *$M_{H,Y}(v) \neq \varnothing$ for a generic H if and only if $v^2 \geq -1$.*
- *$M_{H,Y}(v, L)$ is irreducible for generic H.*

For even rank Mukai vectors, Hauzer proved the following in [26]:

Theorem 4.6.3. *Let Y be an unnodal Enriques surface and $v = (r, c_1, s) \in H^*_{\mathrm{alg}}(Y, \mathbb{Z})$ a primitive Mukai vector with r even. Then for generic polarization H we have*

$$e(M_{H,Y}(v, L)) = e(M_{H,Y}((r', c_1', s'), L')),$$

where r' is 2 or 4.

Non-emptiness of $M_{H,Y}(v)$ with $v^2 \geq 1$ was essentially proved in [35] for the case $r(v) = 2$. He proved irreducibility in half of those cases.

It is worth noting that there are no spherical sheaves (i.e. a stable sheaf E with $E \cong E(K_Y)$ and $\mathrm{ext}^1(E, E) = 0$) on an unnodal Enriques surface Y. Indeed, $E \cong E(K_Y)$

implies that $\pi^*E \cong F \oplus \iota^*F$ for spherical F on $\tilde{Y}$. But then since ι^* now acts trivially on $H^*_{\text{alg}}(\tilde{Y}, \mathbb{Z})$,

$$-8 = (2v(F))^2 = (v(F) + v(\iota^*F))^2 = v(\pi^*E)^2 = -4,$$

a contradiction. So we need only consider stable sheaves with $v^2 \geq -1$ on an unnodal Enriques surface, regardless of the parity of the rank.

On nodal Enriques surfaces, however, we must consider the additional case when $v^2 = -2$ occupied by spherical sheaves. In fact, being nodal is equivalent to carrying a rank two stable spherical bundle [32]. While little work has been done on semistable sheaves on nodal Enriques surfaces in general, there are precise criteria for the existence of spherical bundles:

Theorem 4.6.4 (Theorem 1, [33])**.** *Let $v = (r, c_1(D), s)$ be a positive Mukai vector such that $v^2 = -2$ and ω a generic polarization. Then $M_\omega(v, L) \neq \varnothing$ if and only if $D = N + 2L + \frac{r}{2}K_Y$, where L is some divisor and N is a nodal cycle (i.e. a positive 1-cycle such that $h^1(\mathcal{O}_N) = 0$). In particular, Y must be nodal.*

We would be remiss if we did not at least mention the complete picture that has emerged for stable sheaves on K3 surfaces. The main result in its final form is proved by Yoshioka in [63, Theorems 0.1 & 8.1]. We start by recalling the notion of positive vector, following [63, Definition 0.1].

Definition 4.6.5. Let $v_0 = (r, c, s) \in H^*_{\text{alg}}(\tilde{Y}, \mathbb{Z})$ be a primitive class. We say that v_0 is *positive* if $v_0^2 \geq -2$ and

- either $r > 0$,
- or $r = 0$, c is effective, and $s \neq 0$,
- or $r = c = 0$ and $s > 0$.

Theorem 4.6.6 (Yoshioka)**.** *Let $v \in H^*_{\text{alg}}(\tilde{Y}, \mathbb{Z})$. Assume that $v = mv_0$, with $m \in \mathbb{Z}_{>0}$ and v_0 a primitive positive vector. Then $M_{H,\tilde{Y}}(v)$ is non-empty for all H.*

Remark 4.6.7. We keep the assumptions of Theorem 4.6.6. We further assume that H is *generic* with respect to v so that stable factors of a semistable sheaf E with $v(E) = v$ must have Mukai vector $m'v_0$ for $m' < m$.

(a) By [31], $M_{H,\tilde{Y}}(v)$ is then a normal irreducible projective variety with $\mathbb{Q}$-factorial singularities.

(b) If $m = 1$, then by [63] $M^s_{H,\tilde{Y}}(v) = M_{H,\tilde{Y}}(v)$ is a smooth projective irreducible symplectic manifold of dimension $v^2 + 2$, deformation equivalent to the Hilbert scheme of points on a K3 surface.

Chapter 5

Review: Bridgeland stability

In this chapter, we give a brief review of stability conditions on derived categories in general, as introduced in [14], as well the results we will need for K3 and Enriques surfaces.

5.1 Bridgeland stability conditions

Let X be a smooth projective variety, and denote by $\mathrm{D}^{\mathrm{b}}(X)$ its bounded derived category of coherent sheaves. A *full numerical stability condition* σ on $\mathrm{D}^{\mathrm{b}}(X)$ consists of a pair $(Z, \mathcal{A})$, where $Z\colon K_{\mathrm{num}}(X) \to \mathbb{C}$ is a group homomorphism (called the *central charge*) and $\mathcal{A} \subset \mathrm{D}^{\mathrm{b}}(X)$ is the *heart of a bounded t-structure*, satisfying the following three properties:

(a) For any $0 \neq E \in \mathcal{A}$ the central charge $Z(E)$ lies in the following semi-closed upper half-plane:

$$Z(E) \in \mathbb{H} := \mathcal{H} \cup \mathbb{R}_{<0} = \mathbb{R}_{>0} \cdot e^{(0,1]\cdot i\pi} \tag{5.1}$$

One can think of this condition as two separate positivity conditions: $\Im Z$ defines a rank function on the abelian category $\mathcal{A}$, i.e., a non-negative function $\mathrm{rk}\colon \mathcal{A} \to \mathbb{R}_{\geq 0}$ that is additive on short exact sequences. Similarly, $-\Re Z$ defines a degree function $\deg\colon \mathcal{A} \to \mathbb{R}$, which has the property that $\mathrm{rk}(E) = 0 \Rightarrow \deg(E) > 0$. We can use them to define a notion of slope-stability with respect to Z on the abelian category $\mathcal{A}$ via the slope $\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}$: an object E is called *semistable* (resp. *stable*) if every proper subobject $0 \neq F \subset E$ satisfies $\mu(F) \leq \mu(E)$ (resp. $\mu(F) < \mu(E)$).

(b) With this notion of slope-stability, every object $E \in \mathcal{A}$ has a Harder-Narasimhan filtration $0 = E_0 \hookrightarrow E_1 \hookrightarrow \ldots \hookrightarrow E_n = E$ such that each E_i/E_{i-1} is Z-semistable, with $\mu(E_1/E_0) > \mu(E_2/E_1) > \cdots > \mu(E_n/E_{n-1})$.

(c) There is a constant $C > 0$ such that, for any Z-semistable object $E \in \mathcal{A}$, we have

$$\|E\| \leq C|Z(E)|,$$

where $\|*\|$ is a fixed norm on $K_{\mathrm{num}}(X) \otimes \mathbb{R}$.

This last condition is often called the *support property* and is equivalent to Bridgeland's notion of a *full* stability condition.

Definition 5.1.1. A stability condition is called *algebraic* if its central charge takes values in $\mathbb{Q} \oplus \mathbb{Q}\sqrt{-1}$.

As $K_{\mathrm{num}}(X)$ is finitely generated, for an algebraic stability condition the image of Z is a discrete lattice in $\mathbb{C}$.

Given $(Z, \mathcal{A})$ as above, one can extend the notion of stability to $\mathrm{D}^{\mathrm{b}}(X)$ as follows: for $\phi \in (0, 1]$, we let $\mathcal{P}(\phi) \subset \mathcal{A}$ be the full subcategory of Z-semistable objects with $Z(E) \in \mathbb{R}_{>0}e^{i\phi\pi}$; for general ϕ, it is defined by the compatibility $\mathcal{P}(\phi + n) = \mathcal{P}(\phi)[n]$. Each subcategory $\mathcal{P}(\phi)$ is extension-closed and abelian. Its nonzero objects are called *σ-semistable* of phase ϕ, and its simple objects are called *σ-stable*. Then each object $E \in \mathrm{D}^{\mathrm{b}}(X)$ has a *Harder-Narasimhan filtration*, where the inclusions $E_{i-1} \subset E_i$ are replaced by exact triangles $E_{i-1} \to E_i \to A_i$, and where the A_i's are σ-semistable of decreasing phases ϕ_i. The category $\mathcal{P}(\phi)$ necessarily has finite length. Hence every object in $\mathcal{P}(\phi)$ has a finite Jordan-Hölder filtration, whose filtration quotients are σ-stable objects of the phase ϕ. Two objects $A, B \in \mathcal{P}(\phi)$ are called *S-equivalent* if their Jordan-Hölder factors are the same (up to reordering). We define the *mass* of an object E for a given σ by $m_\sigma(E) = \sum_i |Z_\sigma(A_i)|$, where A_i are the σ-semistable factors of E. Of course, it follows that $|Z(E)| \leq m_\sigma(E)$. We sometimes abuse notation and write $(Z, \mathcal{P})$ in place of $(Z, \mathcal{A})$.

The set of stability conditions will be denoted by $\mathrm{Stab}(X)$. It has a natural metric topology (see [14, Prop. 8.1] for the explicit form of the metric). Bridgeland's main theorem is the following:

Theorem 5.1.2 (Bridgeland)**.** *The map*

$$\mathcal{Z}\colon \operatorname{Stab}(X) \to \operatorname{Hom}(K_{\mathrm{num}}(X), \mathbb{C}), \qquad (Z, \mathcal{A}) \mapsto Z,$$

is a local homeomorphism. In particular, $\operatorname{Stab}(X)$ *is a complex manifold of finite dimension equal to the rank of* $K_{\mathrm{num}}(X)$.

In other words, a stability condition $(Z, \mathcal{A})$ can be deformed uniquely given a small deformation of its central charge Z.

Remark 5.1.3. There are two group actions on $\operatorname{Stab}(X)$, see [14, Lemma 8.2]: the group of autoequivalences $\operatorname{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ acts on the left via $\Pi(Z, \mathcal{A}) = (Z \circ \Pi_*^{-1}, \Pi(\mathcal{A}))$, where $\Pi \in \operatorname{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ and Π_* is the automorphism induced by Π at the level of numerical Grothendieck groups. We will often abuse notation and denote Π_* by Π, when no confusion arises. The universal cover $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ of the group $\mathrm{GL}_2^+(\mathbb{R})$ of matrices with positive determinant acts on the right as a lift of the action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\operatorname{Hom}(K_{\mathrm{num}}(X), \mathbb{C}) \cong \operatorname{Hom}(K_{\mathrm{num}}(X), \mathbb{R}^2)$. We typically only use the action of the subgroup $\mathbb{C} \subset \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ given as the universal cover of $\mathbb{C}^* \subset \mathrm{GL}_2^+(\mathbb{R})$: given $z \in \mathbb{C}$, it acts on $(Z, \mathcal{A})$ by $Z \mapsto e^{2\pi i z} \cdot Z$, and by modifying $\mathcal{A}$ accordingly.

5.2 Stability conditions on K3 and Enriques surfaces

In this section we give a brief review of Bridgeland's results on stability conditions for K3 surfaces in [15], and of results by Toda, Yoshioka and others related to moduli spaces of Bridgeland-stable objects.

5.2.1 Space of stability conditions for a K3 surface

Let $\tilde{Y}$ be a smooth projective K3 surface. Fix $\omega, \beta \in \operatorname{NS}(\tilde{Y})_{\mathbb{Q}}$ with ω ample. Borrowing from the notation of β-twisted slope-semistability, we define the β-twisted ω-slope to be

$$\mu_{\omega,\beta}(E) = \begin{cases} \frac{\omega.(c_1(E) - r(E)\beta)}{r(E)} = \mu_\omega(E) - \omega.\beta & \text{if } r(E) > 0, \\ +\infty & \text{if } r(E) = 0. \end{cases} \tag{5.2}$$

One can define a notion of $\mu_{\omega,\beta}$-stability, but for our purposes, it is easiest to note that this is the same thing as μ_ω-stability with all slopes shifted down by $\omega.\beta$ as suggested by (5.2).

Let $\mathcal{T}(\omega,\beta) \subset \operatorname{Coh}\tilde{Y}$ be the subcategory of torsion sheaves and torsion-free sheaves whose HN-filtrations factors (with respect to slope-stability) have $\mu_{\omega,\beta} > 0$, and $\mathcal{F}(\omega,\beta)$ the subcategory of torsion-free sheaves with HN-filtration factors satisfying $\mu_{\omega,\beta} \leq 0$. Next, consider the abelian category

$$\mathcal{A}(\omega,\beta) := \left\{ E \in \mathrm{D}^{\mathrm{b}}(\tilde{Y}) : \begin{array}{l} \bullet\ \mathcal{H}^p(E) = 0 \text{ for } p \notin \{-1,0\}, \\ \bullet\ \mathcal{H}^{-1}(E) \in \mathcal{F}(\omega,\beta), \\ \bullet\ \mathcal{H}^0(E) \in \mathcal{T}(\omega,\beta) \end{array} \right\}$$

and the $\mathbb{C}$-linear map

$$Z_{\omega,\beta} \colon K_{\mathrm{num}}(\tilde{Y}) \to \mathbb{C}, \qquad E \mapsto (\exp{(\beta + \sqrt{-1}\omega)}, v(E)). \tag{5.3}$$

If $Z_{\omega,\beta}(F) \notin \mathbb{R}_{\leq 0}$ for any spherical sheaf $F \in \operatorname{Coh}(\tilde{Y})$ (e.g., this holds when $\omega^2 > 2$), then by [15, Lemma 6.2, Prop. 7.1], the pair $\sigma_{\omega,\beta} = (Z_{\omega,\beta}, \mathcal{A}(\omega,\beta))$ defines a stability condition. For objects $E \in \mathcal{A}(\omega,\beta)$, we will denote their phase with respect to $\sigma_{\omega,\beta}$ by $\phi_{\omega,\beta}(E) = \phi(Z(E)) \in (0,1]$. By using the support property, as proved in [15, Proposition 10.3], we can extend the above and define stability conditions $\sigma_{\omega,\beta}$, for $\omega, \beta \in \mathrm{NS}(\tilde{Y})_{\mathbb{R}}$. Bridgeland shows further that the $\sigma_{\omega,\beta}$, along with their translates under the action of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$, fill out a connected component $\operatorname{Stab}^\dagger(\tilde{Y})$ of $\operatorname{Stab}(\tilde{Y})$.

5.2.2 Space of stability conditions for an Enriques surface via induction

Let $\pi : \tilde{Y} \to Y$ denote the covering map of an Enriques surface Y by its covering K3 $\tilde{Y}$. Via the fixed-point free covering involution ι, $\operatorname{Coh}(Y)$ is naturally isomorphic to the category of coherent G-sheaves on $\tilde{Y}$, $\operatorname{Coh}_G(\tilde{Y})$, where $G = \langle \iota^* \rangle$, thus giving a natural equivalence of $\mathrm{D}^{\mathrm{b}}(Y)$ with $\mathrm{D}^{\mathrm{b}}_G(\tilde{Y})$. We make this identification implicitly below.

In [45] the authors construct two faithful adjoint functors

$$\mathrm{Forg}_G : \mathrm{D}^{\mathrm{b}}_G(\tilde{Y}) \to \mathrm{D}^{\mathrm{b}}(\tilde{Y}),$$

which forgets the G-sheaf structure, and

$$\mathrm{Inf}_G : \mathrm{D^b}(\tilde{Y}) \to \mathrm{D}^{\mathrm{b}}_G(\tilde{Y}),\ \ \mathrm{Inf}_G(E) := \oplus_{g\in G} g^* E.$$

Under the above identifications we have $\mathrm{Forg}_G = \pi^*$ and $\mathrm{Inf}_G = \pi_*$. Since G acts on $\mathrm{Stab}(\tilde{Y})$ via the natural action of $\mathrm{Aut}(\mathrm{D^b}(\tilde{Y}))$ on $\mathrm{Stab}(\tilde{Y})$, we can define

$$\Gamma_{\tilde{Y}} := \{\sigma \in \mathrm{Stab}(\tilde{Y}) : g^*\sigma = \sigma,\ \text{for all } g \in G\}.$$

These functors induce two continuous maps. The first $(\pi^*)^{-1} : \Gamma_{\tilde{Y}} \to \mathrm{Stab}(\mathrm{D^b}(Y))$ is given by $Z_{(\pi^*)^{-1}(\sigma)} = Z_\sigma \circ \pi^*$ and $\mathcal{P}_{(\pi^*)^{-1}(\sigma)}(\phi) = \{E \in \mathrm{D^b}(Y) : \pi^* E \in \mathcal{P}_\sigma(\phi)\}$, where we use π^* also for the morphism between K-groups. The second $(\pi_*)^{-1} : (\pi^*)^{-1}(\Gamma_{\tilde{Y}}) \to \mathrm{Stab}(\tilde{Y})$ is defined similarly with π^* replaced by π_*.

We consider the connected component $\mathrm{Stab}^\dagger(\tilde{Y}) \subset \mathrm{Stab}(\tilde{Y})$ described in the section above. The following result is relevant to us:

Theorem 5.2.1 (Proposition 3.1, [45])**.** *The non-empty subset* $\Sigma(Y) := (\pi^*)^{-1}(\Gamma_{\tilde{Y}} \cap \mathrm{Stab}^\dagger(\tilde{Y}))$ *is open and closed in* $\mathrm{Stab}(Y)$*, and it is embedded into* $\mathrm{Stab}^\dagger(\tilde{Y})$ *as a closed submanifold via the functor* $(\pi_*)^{-1}$*. Moreover, the diagram*

$$\begin{tikzcd}
\Gamma_{\tilde{Y}} \cap \mathrm{Stab}^\dagger(\tilde{Y}) \arrow[r, "(\pi^*)^{-1}"] \arrow[d] & \Sigma(Y) \arrow[r, "(\pi_*)^{-1}"] \arrow[d, "\mathcal{Z}"] & \Gamma_{\tilde{Y}} \cap \mathrm{Stab}^\dagger(\tilde{Y}) \arrow[d] \\
(K_{\mathrm{num}}(\tilde{Y})_{\mathbb{C}})^\vee_G \arrow[r, "(\pi^*)^\vee"] & K_{\mathrm{num}}(Y)^\vee_{\mathbb{C}} \arrow[r, "\pi_*^\vee"] & (K_{\mathrm{num}}(\tilde{Y})_{\mathbb{C}})^\vee_G
\end{tikzcd}$$

commutes.

5.3 The Wall-and-Chamber structure

A key ingredient in the connection between the stability manifold and the birational geometry of Bridgeland moduli spaces is the existence of a wall-and-chamber structure on $\mathrm{Stab}(X)$. For a fixed $\sigma \in \mathrm{Stab}(X)$, we say a subset $\mathcal{S} \subset \mathrm{D^b}(X)$ has *bounded mass* if there exists $m > 0$ such that $m_\sigma(E) \leq m$ for all $E \in \mathcal{S}$. It follows from the definition of the metric topology on $\mathrm{Stab}(X)$ that being of bounded mass is independent of the specific initial stability condition σ and depends only on the connected component it lies on. We have the following general result:

Proposition 5.3.1 (Proposition 2.8, [61])**.** *Let X be a smooth projective variety. Assume that for any bounded mass subset $\mathcal{S} \subset \mathrm{D}^{\mathrm{b}}(X)$ the set of numerical classes*

$$\{[E] \in K_{\mathrm{num}}(X) | E \in \mathcal{S}\}$$

is finite. Then for any compact subset $B \subset \mathrm{Stab}^*(X)$ (an arbitrary connected component of $\mathrm{Stab}(X)$), there exists a finite number of real codimension one submanifolds $\{W_\gamma | \gamma \in \Gamma\}$ on $\mathrm{Stab}^*(X)$ such that if Γ' is a subset of Γ and

$$\mathcal{C} \subset \bigcap_{\gamma \in \Gamma'} (B \cap W_\gamma) \backslash \bigcup_{\gamma \notin \Gamma'} W_\gamma$$

is one of the connected components, then if $E \in \mathcal{S}$ is semistable for some $\sigma \in \mathcal{C}$, then it is semistable for all $\sigma \in \mathcal{C}$.

We now verify the assumption of the proposition when $X = Y$ is an Enriques surface:

Lemma 5.3.2. *Suppose the subset $\mathcal{S} \subset \mathrm{D}^{\mathrm{b}}(Y)$ has bounded mass in $\mathrm{Stab}^\dagger(Y)$. Then the set of numerical classes $\{[E] | E \in \mathcal{S}\}$ is finite.*

Proof. Since the conclusion is true for bounded mass subsets $\mathcal{S}' \subset \mathrm{D}^{\mathrm{b}}(\tilde{Y})$ for the covering K3 surface $\tilde{Y}$ above, we first show that $\pi^*(\mathcal{S})$ has bounded mass. Indeed, let $\sigma' \in \mathrm{Stab}^\dagger(\tilde{Y})$ induce $\sigma \in \mathrm{Stab}^\dagger(Y)$. Then by our assumption on $\mathcal{S}$, there exists $m > 0$ such that $m_\sigma(E) \leq m$ for any $E \in \mathcal{S}$, and the proof of [45, Lemma 2.8] shows that the HN-filtration of π^*E with respect to σ' is the image via π^* of the HN-filtration of E with respect to σ. Then $m_\sigma(E) = m_{\sigma'}(\pi^*E)$ from this and the definition of the induction of stability conditions. This shows that $\mathcal{S}' = \pi^*(\mathcal{S})$ is of bounded mass.

It follows that $\{[F] \in K_{\mathrm{num}}(\tilde{Y}) | F \in \mathcal{S}'\}$ is a finite set. But if $F = \pi^*E$, then $[F] = \pi^*[E]$, and π^* is an isomorphism onto $K_{\mathrm{num}}(\tilde{Y})_G$, so the set $\{[E] \in K_{\mathrm{num}}(Y) | E \in \mathcal{S}\}$ is finite. $\square$

For the remainder of this section, we let X denote any smooth projective variety satisfying the assumption of Proposition 5.3.1, though for our purposes $X = Y$ or $\tilde{Y}$. It is worthwhile to point out the following fact which is crucial in considering the stability of objects as σ varies:

Lemma 5.3.3 (Proposition 9.3,[15]). *Given a subset of $\mathcal{S} \subset \mathrm{D}^{\mathrm{b}}(X)$ of bounded mass and a compact subset B, then*

$$\{v(E)|E \in \mathcal{S} \textit{ or is a (semi)stable factor of some } E' \in \mathcal{S} \textit{ for some } \sigma \in B\}$$

is a finite set.

The most important consequence of the construction of W_γ in Proposition 5.3.1 is that when $\sigma \in \mathcal{C}$, the only numerical classes with the same phase as some $E \in \mathcal{S}$ must lie on the ray $\mathbb{R}_{>0}[E]$. Following [44], we call the codimension one submanifolds from Proposition 5.3.1 *pseudo-walls* for the bounded mass subset $\mathcal{S}$.

Let us now fix a class $v \in K_{\mathrm{num}}(X)$, and consider the set $\mathcal{S}$ of σ-semistable objects $E \in \mathrm{D}^{\mathrm{b}}(X)$ of class v as σ varies. This is by definition bounded. Consider the corresponding finite set from Lemma 5.3.3 and the resulting wall-and-chamber decomposition. By throwing out those pseudo-walls which do not actually correspond to subobjects of some $E \in \mathcal{S}$, we arrive at the following useful wall-and-chamber decomposition:

Proposition 5.3.4. *There exists a locally finite set of* walls *(pseudo-walls corresponding to genuine subobjects of semistable objects with Mukai vector v) in* $\mathrm{Stab}(X)$*, depending only on v, with the following properties:*

(a) When σ varies within a chamber, the sets of σ-semistable and σ-stable objects of class v do not change.

(b) When σ lies on a single wall $W \subset \mathrm{Stab}(X)$, then there is a σ-semistable object that is unstable in one of the adjacent chambers, and semistable in the other adjacent chamber.

(c) When we restrict to an intersection of finitely many walls $W_1, \ldots, W_k$, we obtain a wall-and-chamber decomposition on $W_1 \cap \cdots \cap W_k$ with the same properties, where the walls are given by the intersections $W \cap W_1 \cap \cdots \cap W_k$ for any of the walls $W \subset \mathrm{Stab}(X)$ with respect to v.

If v is primitive, then from the proof of [15, Proposition 9.4] σ lies on a wall if and only if there exists a strictly σ-semistable object of class v. From the above constructions, the Jordan-Hölder filtrations of σ-semistable objects do not change when σ varies within a chamber.

Definition 5.3.5. Let $v \in K_{\mathrm{num}}(X)$. A stability condition is called *generic* with respect to v if it does not lie on a wall in the sense of Proposition 5.3.4.

Chapter 6

Moduli stacks of semistable objects

We begin in this chapter to present our own original research contributions. We prove here that Enriques surfaces admit moduli Artin stacks of objects in the derived category that are semistable with respect to a Bridgeland stability condition.

6.1 Basic properties of semistable objects

We first collect here some fundamental facts about Bridgeland semistable objects on Enriques surfaces. The central technique is comparison with Bridgeland stability on the covering K3 surface. The first step in that direction is the following general result:

Lemma 6.1.1 (Proposition 2.5,[16])**.** *Let Y be an Enriques surface and $\tilde{Y}$ its K3 universal cover.*

(a) Let $F \in \mathrm{D^b}(\tilde{Y})$. Then there is an object $E \in \mathrm{D^b}(Y)$ such that $\pi^*E \cong F$ if and only if $\iota^*F \cong F$.

(b) Let $E \in \mathrm{D^b}(Y)$. Then there is an object $F \in \mathrm{D^b}(\tilde{Y})$ such that $\pi_*F \cong E$ if and only if $E \otimes \omega_Y \cong E$.

Focusing now on Bridgeland stability, we set some notation for the rest of the section. Fix a Mukai vector v, and for any $\sigma \in \mathrm{Stab}^\dagger(Y)$, we will denote by $\sigma' \in \Gamma_{\tilde{Y}} \cap \mathrm{Stab}^\dagger(\tilde{Y})$ a stability condition such that $(\pi^*)^{-1}(\sigma') = \sigma$. To compare stable objects on Y and $\tilde{Y}$ we first make the following observation:

Lemma 6.1.2. *If $E, F \in M_{\sigma,Y}(v)$ are σ-stable and $\pi^*E \cong \pi^*F$, then $E \cong F$ or $E \cong F \otimes \omega_Y$.*

Proof. Indeed, pushing forward implies that

$$E \oplus (E \otimes \omega_Y) \cong \pi_* \pi^* E \cong \pi_* \pi^* F \cong F \oplus (F \otimes \omega_Y).$$

Taking Hom's gives that either

$$\mathrm{Hom}(E, F) \neq 0 \text{ or } \mathrm{Hom}(E, F \otimes \omega_Y) \neq 0.$$

But since E and F ($F \otimes \omega_Y$ respectively) are both σ-stable of the same phase, any non-zero homomorphism must be an isomorphism. $\square$

We will often need to exclude one of these possibilities:

Lemma 6.1.3. *If E is σ-stable of phase ϕ, then $\pi^* E$ is σ'-stable of the same phase, unless $E \cong E \otimes \omega_Y$, in which case $\pi^* E \cong F \oplus \iota^* F$, with $F \ncong \iota^* F$ σ'-stable objects of phase ϕ, and thus not stable. Moreover, in this case $E \cong \pi_*(F) \cong \pi_*(\iota^* F)$.*

Proof. By definition $\pi^* E$ is σ'-semistable, so suppose that it is strictly semistable. Let $F \subset \pi^* E$ be a proper nontrivial σ'-stable subobject of the same phase ϕ. If $F \cong \iota^* F$, then there is a proper nontrivial σ-stable object $E' \subset E$ of phase ϕ, contradicting stability of E.

Otherwise, $F \ncong \iota^* F$ and $\iota^* F \subset \pi^* E$ is also σ'-stable of phase ϕ. Consider the short exact sequence

$$0 \to F \cap \iota^* F \to F \oplus \iota^* F \to F + \iota^* F \to 0,$$

which gives

$$2Z(F) = Z(F \oplus \iota^* F) = Z(F \cap \iota^* F) + Z(F + \iota^* F),$$

where we write $Z(-) = Z_{\sigma'}(-), \phi(-) = \phi_{\sigma'}(-)$ to be concise. By the see-saw principle and semistability of $F \oplus \iota^* F$, we must have either

$$\phi(F \cap \iota^* F) < \phi < \phi(F + \iota^* F), \text{ or } \phi(F \cap \iota^* F) = \phi = \phi(F + \iota^* F).$$

Since $F + \iota^* F \subset \pi^* E$, semistability implies that we must have equality everywhere. But then $F \cap \iota^* F \subset F$ of the same phase, so either

$$F \cap \iota^* F = 0 \text{ or } F,$$

by the stability of F. We assumed $F \ncong \iota^*F$, so we must be in the first case. Thus $F \oplus \iota^*F \cong F + \iota^*F$ is an ι^*-invariant nontrivial subobject of π^*E of phase ϕ. It must thus come from a nontrivial subobject $F' \subset E$ of phase ϕ. If F' is proper, equivalently $F \oplus \iota^*F$ is proper, then this contradicts the stability of E. Thus we must have $F \oplus \iota^*F \cong \pi^*E$. Pushing forward gives that

$$E \oplus E \otimes \omega_Y \cong \pi_*(F) \oplus \pi_*(\iota^*F) \cong \pi_*(F)^{\oplus 2}.$$

From this and adjunction we deduce that

$$\begin{aligned} \operatorname{Hom}(E,E) \oplus \operatorname{Hom}(E, E \otimes \omega_Y) &= \operatorname{Hom}(E, \pi_*(F))^{\oplus 2} = \operatorname{Hom}(\pi^*E, F)^{\oplus 2} \\ &= \operatorname{Hom}(F,F)^{\oplus 2} \oplus \operatorname{Hom}(F, \iota^*F)^{\oplus 2}. \end{aligned}$$

Since F and ι^*F are non-isomorphic σ'-stable objects of the same phase, $\operatorname{Hom}(F, \iota^*F) = 0$, while $\operatorname{Hom}(E,E) = \operatorname{Hom}(F,F) = \mathbb{C}$. Thus $\operatorname{Hom}(E, E \otimes \omega_Y) = \mathbb{C}$ which implies $E \cong E \otimes \omega_Y$ by stability. Similar considerations show that $E \cong \pi_*(F)$.

For the converse, we have by adjunction that

$$\begin{aligned} \operatorname{Hom}(\pi^*E, \pi^*E) &\cong \operatorname{Hom}(E, \pi_*\pi^*E) \\ &\cong \operatorname{Hom}(E, E \oplus (E \otimes \omega_Y)) \cong \operatorname{Hom}(E,E)^{\oplus 2}. \end{aligned}$$

Thus $\mathbb{C}^{\oplus 2} = \operatorname{Hom}(\pi^*E, \pi^*E)$ implies π^*E cannot be stable. □

We finish this section with a necessary condition on a Mukai vector for the existence of Bridgeland stable objects:

Lemma 6.1.4. *If E is σ-stable, then $v(E)^2 \geq -1$, unless $v(E)^2 = -2$ which occurs precisely when E is spherical.*

Proof. From Serre duality and the definition of the Mukai pairing

$$v(E)^2 = \operatorname{ext}^1(E,E) - \hom(E,E) - \operatorname{ext}^2(E,E) = \operatorname{ext}^1(E,E) - \hom(E,E) - \hom(E, E \otimes \omega_Y).$$

By stability $\hom(E,E) = 1$ and $\hom(E, E \otimes \omega_Y) = 0$ or 1. In the first case,

$$v(E)^2 + 1 = \operatorname{ext}^1(E,E) \geq 0.$$

In the latter case,

$$v(E)^2 + 2 = \mathrm{ext}^1(E,E) \geq 0,$$

and $v(E)^2 < -1$ implies $v(E)^2 = -2$, so E is spherical as $\mathrm{ext}^2(E,E) = 1$ implies $E \cong E \otimes \omega_Y$. $\square$

6.2 Moduli stacks of Bridgeland semistable objects

We would like to use the results of [61] to construct for each stability condition $\sigma \in \mathrm{Stab}^\dagger(Y)$ a moduli stack of σ-semistable objects which is an Artin stack of finite type over $\mathbb{C}$.

Fix a smooth projective surface X (to be either Y or $\tilde{Y}$ as above). Let $\mathfrak{M}_X$ be the 2-functor

$$\mathfrak{M}_X \colon (\mathrm{Sch}/\mathbb{C}) \to (\mathrm{groupoids}),$$

which sends a $\mathbb{C}$-scheme S to the groupoid $\mathfrak{M}_X(S)$ whose objects consist of $\mathcal{E} \in \mathrm{D}_{S\text{-perf}}(S \times X)$ satisfying

$$\mathrm{Ext}^i(\mathcal{E}_s, \mathcal{E}_s) = 0, \text{ for all } i < 0 \text{ and } s \in S.$$

Lieblich proved the following theorem:

Theorem 6.2.1 ([43])**.** *The 2-functor $\mathfrak{M}_X$ is an Artin stack of locally finite type over $\mathbb{C}$.*

Fix $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(X), \phi \in \mathbb{R}$, and $v \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$. Then any object $E \in \mathcal{P}(\phi)$ satisfies

$$\mathrm{Ext}^i(E,E) = 0, \text{ for all } i < 0.$$

Indeed $\mathrm{Ext}^i(E,E) = \mathrm{Hom}(E, E[i])$, and $E \in \mathcal{P}(\phi)$ implies $E[i] \in \mathcal{P}(\phi + i)$. Since $i < 0$, $\phi + i < \phi$, and from the definition of a stability condition, we must then have $\mathrm{Hom}(E, E[i]) = 0$.

Definition 6.2.2. Define $M_{\sigma,X}(v,\phi)$ to be the set of σ-semistable objects of phase ϕ and Mukai vector v, and

$$\mathfrak{M}_{\sigma,X}(v,\phi) \subset \mathfrak{M}_X,$$

to be the substack parametrizing families of objects in $M_{\sigma,X}(v,\phi)$. As ϕ is determined mod $\mathbb{Z}$ by v and σ, we will drop it from the notation and assume henceforth that it is in $(0,1]$.

Remark 6.2.3. By [61, Lemma 2.9] and Remark 5.1.3 above, we may in fact assume that $\phi = 1$, $Z(v) = -1$, and σ is algebraic. We will make explicit when we are assuming this.

Toda proved the following helpful result:

Lemma 6.2.4 ([61])**.** *Assume $M_{\sigma,X}(v)$ is bounded and $\mathfrak{M}_{\sigma,X}(v) \subset \mathfrak{M}_X$ is an open substack. Then $\mathfrak{M}_{\sigma,X}(v)$ is an Artin stack of finite type over $\mathbb{C}$.*

This is the essential ingredient we need to prove the main theorem of this section:

Theorem 6.2.5. *Let Y be an Enriques surface. For any $v \in H^*_{\mathrm{alg}}(Y,\mathbb{Z})$ and $\sigma \in$ $\mathrm{Stab}^\dagger(Y)$, $\mathfrak{M}_{\sigma,Y}(v)$ is an Artin stack of finite type over $\mathbb{C}$.*

We can easily prove the openness of σ-stability on Y:

Proposition 6.2.6. *For any $v \in H^*_{\mathrm{alg}}(Y,\mathbb{Z})$ and $\sigma \in \mathrm{Stab}^\dagger(Y)$, $\mathfrak{M}_{\sigma,Y}(v)$ is an open substack of $\mathfrak{M}_Y$.*

Proof. By [61, Lemma 3.6] this reduces to proving that for any smooth quasi-projective variety S and $\mathcal{E} \in \mathfrak{M}_Y(S)$ such that the locus

$$S^\circ = \{s \in S | \mathcal{E}_s \in M_{\sigma,Y}(v)\},$$

is not empty, there is an open subset U of S contained in S°. Of course, $(1 \times \pi)^*\mathcal{E} \in \mathfrak{M}_{\tilde{Y}}(S)$, and by definition of induced stability conditions, the corresponding set S° for $(1 \times \pi)^*\mathcal{E}$ and $M_{\sigma',\tilde{Y}}(\pi^* v)$ remains the same. By [61, Section 4] there is an open set U of S contained in S° so the result follows. $\square$

By Lemma 6.2.4, all that remains is to prove the boundedness of $M_{\sigma,Y}(v)$. We begin with boundedness of stable objects:

Proposition 6.2.7. *Denote by $M^s_{\sigma,Y}(v) \subset M_{\sigma,Y}(v)$ the subset of σ-stable objects. Then $M^s_{\sigma,Y}(v)$ is bounded.*

Proof. From Lemma 6.1.3 we know that for any $E \in M^s_{\sigma,Y}(v)$, π^*E is σ'-stable unless $E \cong E \otimes \omega_Y$ in which case $\pi^*E \cong F \oplus \iota^*F$ for σ'-stable objects $F \ncong \iota^*F$ of the same phase.

Let us the consider the first case. Then by boundedness of $M_{\sigma',\tilde{Y}}(\pi^*v,\phi)$ [61, Theorem 14.2], there exists a scheme Q of finite type over $\mathbb{C}$ and $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(Q \times \tilde{Y})$ such that every $F \in M_{\sigma',\tilde{Y}}(\pi^*v,\phi)$ is equal to $\mathcal{F}_q$ for some closed point $q \in Q$. Consider the locally closed subscheme

$$T := \{q \in Q | \iota^*\mathcal{F}_q \cong \mathcal{F}_q, \mathcal{F}_q \in M^s_{\sigma',\tilde{Y}}(\pi^*v)\},$$

which is still of finite type over $\mathbb{C}$, and the restriction $\mathcal{F}_T$. Then from [16, Proposition 2.5] it follows that there exists $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(T \times Y)$ such that $(1 \times \pi)^*(\mathcal{E}) \cong \mathcal{F}_T$. Consider the disjoint union of two copies of T, which is still of finite type over $\mathbb{C}$, with $\mathcal{E}$ on the first copy of T and $\mathcal{E} \otimes p_Y^*\omega_Y$ on the second. Then by Lemma 6.1.2 and the definition of induced stability conditions, it follows that $M^s_{\sigma,Y}(v)$ is bounded.

In the second case, consider $u \in H^*_{\mathrm{alg}}(\tilde{Y},\mathbb{Z})$ such that $\pi_*(u) = v$. Note that by Lemma 5.3.3 only finitely many of these Mukai vectors appear as $v(F)$ for decompositions $E \cong F \oplus \iota^*F$. Then by boundedness of $M_{(\pi_*)^{-1}(\sigma),\tilde{Y}}(u)$, we have a scheme W of finite type over $\mathbb{C}$ and $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}(W \times \tilde{Y})$ representing every element $M_{(\pi_*)^{-1}(\sigma),\tilde{Y}}(u)$. Now consider the open set

$$V := \{w \in W | \iota^*\mathcal{G}_w \ncong \mathcal{G}_w, \mathcal{G}_w \text{ is stable}\}$$

and $(1 \times \pi)_*(\mathcal{G}|_V) \in \mathrm{D}^{\mathrm{b}}(V \times Y)$. Then V is still of finite type. Taking the finite union over the relevant u's represents every member of $M^s_{\sigma,Y}(v)$.

Together these prove the claim. □

To prove boundedness in general, let us recall the following simple result:

Lemma 6.2.8 ([61, Lemma 3.16])**.** *Let X be a smooth projective variety and subsets $\mathcal{S}_i \subset \mathrm{D}^{\mathrm{b}}(X)$, $1 \le i \le 3$, with $\mathcal{S}_i$ bounded for $i = 1,2$. Suppose that any $E_3 \in \mathcal{S}_3$ sits in a distinguished triangle,*

$$E_1 \to E_3 \to E_2,$$

with $E_i \in \mathcal{S}_i$ for $i = 1,2$. Then $\mathcal{S}_3$ is also bounded.

Proposition 6.2.9. *$M_{\sigma,Y}(v)$ is bounded for any σ and v.*

Proof. By Lemma 5.3.3, the number of Mukai vectors of possible stable factors for $E \in M_{\sigma,Y}(v)$ is finite. By induction on the number of stable factors, we see that the claim follows from Proposition 6.2.7 and Lemma 6.2.8 above. $\square$

As above we denote by $\mathfrak{M}^s_{\sigma,Y}(v) \subset \mathfrak{M}_{\sigma,Y}(v)$ the open substack parametrizing stable objects (and analogously for the corresponding sets of objects). Inaba proved in [29] that $\mathfrak{M}^s_{\sigma,Y}(v)$ is a $\mathbb{G}_m$-gerbe over a separated algebraic space that we denote by $M^s_{\sigma,Y}(v)$. We have the following further result:

Lemma 6.2.10. *Fix $v \in H^*_{\mathrm{alg}}(Y,\mathbb{Z})$.*

(a) The moduli stack $\mathfrak{M}_{\sigma,Y}(v)$ satisfies the valuative criterion of closedness.

(b) Assume that $\mathfrak{M}_{\sigma,Y}(v) = \mathfrak{M}^s_{\sigma,Y}(v)$. Then the course moduli space $M_{\sigma,Y}(v)$ is a proper algebraic space.

Proof. By Remark 6.2.3, we may assume that $Z(v) = -1$ and that σ is algebraic so that $\mathcal{P}(1)$ is Noetherian. But then [1, Theorem 4.1.1] implies the lemma. $\square$

We finish this section by proving that certain coarse invariants of these moduli spaces are the same as those of appropriate moduli spaces of sheaves. To do this we use derived auto-equivalences. While only nodal Enriques surfaces admit spherical objects [45, Lemma 3.17] and their corresponding spherical (Seidel-Thomas) twists, all Enriques surfaces have closely related derived auto-equivalences corresponding to exceptional objects. These are objects $E \in \mathrm{D}^{\mathrm{b}}(Y)$ with $\mathrm{ext}^i(E,E) = 0$ for $i \neq 0$ and $\hom(E,E) = 1$, so that in particular $E \ncong E \otimes \omega_Y$ and $v(E)^2 = -1$. It follows immediately that π^*E is a spherical object on $\tilde{Y}$. We have the following result about the associated spherical twist $\mathrm{ST}_{\pi^*E}(-)$:

Proposition 6.2.11. *Let $E \in \mathrm{D}^{\mathrm{b}}(Y)$ be an exceptional object and $\mathrm{ST}_{\pi^*E}(-)$ the spherical twist associated to π^*E, i.e. the derived auto-equivalence defined by the exact triangle*

$$\mathrm{Hom}^\bullet(\pi^*E, F) \otimes \pi^*E \to F \to \mathrm{ST}_{\pi^*E}(F)$$

for every $F \in \mathrm{D^b}(Y)$. Then ST_{π^*E} preserves $\mathrm{D^b}(\tilde{Y})_G \cong \mathrm{D^b}(Y)$ and thus descends to an auto-equivalence on $\mathrm{D^b}(Y)$. The effect on cohomology is the map

$$v(F) \mapsto v + 2(v(F), v(E))v(E).$$

Proof. The important observation here is that both π^*E and $F \in \mathrm{D^b}(\tilde{Y})_G$, so $\mathrm{Hom}^\bullet(\pi^*E, F)$ is G-invariant and thus the first morphism is in $\mathrm{D^b}(\tilde{Y})_G$. Completing it to an exact triangle stays inside $\mathrm{D^b}(\tilde{Y})_G$, and thus we see that $\mathrm{ST}_{\pi^*E}(F)$ is G-invariant as well. It follows that ST_{π^*E} descends to an auto-equivalence on $\mathrm{D^b}(Y)$.

The statement about the action on cohomology follows from the above description and [27, Lemma 8.12]. □

These auto-equivalences were referred to as Fourier-Mukai transforms associated to (-1)-reflection in [64] and modular reflections in [67], but the above interpretation strengthens and elucidates the connection with the covering spherical twist on the K3 surface $\tilde{Y}$. For brevity we call these *weakly-spherical twists*. Now we are ready for our theorem.

Theorem 6.2.12. *The motivic invariant of $\mathfrak{M}_{\sigma,Y}(v)$ for all $\sigma \in \mathrm{Stab}^\dagger(Y)$ is the same as that of $M_{H,Y}(w)$ for a generic polarization with respect to a positive Mukai vector w in the same orbit as v under the action of $\mathrm{Aut}(\mathrm{D^b}(Y))$. In particular, if $M_{H,Y}(w)$ is nonempty, then $\mathfrak{M}_{\sigma,Y}(v)(\mathbb{C}) \neq \varnothing$.*

Proof. The construction of the Joyce invariant $J(v)$ of [61, Section 5] is quite general, and Lemma 5.12 there applies. Likewise the analogous algebra $A(\mathcal{A}_\phi, \Lambda, \chi)$ is still commutative since ω_Y is numerically trivial and thus the Mukai pairing is commutative. This and the results above show that [61, Theorem 5.24 and Corollary 5.26] still apply. In particular, $J(v)$ is the motivic invariant of the proper coarse moduli space $M_{\sigma,Y}(v)$, should it be known to exist, and is invariant under autoequivalences and changes in σ.

We can thus assume that v is positive. Indeed, if $r \neq 0$, then we can shift by 1, i.e. $E \mapsto E[1]$, to make $r > 0$ if necessary. If $r = 0$ but $s \neq 0$, then we can apply the weakly-spherical twist through $\mathcal{O}_Y$ and a shift, if necessary, to make v positive. Finally, we are reduced to the case $v = (0, C, 0)$. We can tensor with $\mathcal{O}(D)$ for any $D \in \mathrm{Pic}(Y)$

such that $D.C \neq 0$, and then apply the weakly-spherical twist through $\mathcal{O}_Y$ and a shift to make v positive.

Choose a polarization $H \in \mathrm{Amp}(Y)$ that is generic with respect to v and $\beta \in \mathrm{NS}(Y)_{\mathbb{Q}}$ such that $\mu_{H,\beta}(v) > 0$. Setting $\omega = tH$ and $\sigma_t := (\pi^*)^{-1}(\sigma_{\pi*\omega,\pi^*\beta})$, [15, Proposition 14.2] shows that for $t \gg 0$, $\mathfrak{M}_{\sigma_t,Y}(v) = \mathfrak{M}^{\beta}_{H,Y}(v)$, i.e. we may choose σ such that the moduli stack $\mathfrak{M}_{\sigma,Y}(v)$ is the same as the moduli stack $\mathfrak{M}_{H,Y}(v)$ of (H,β)-twised Gieseker semistable sheaves on Y with Mukai vector v for a generic polarization H. This stack therefore admits a projective coarse moduli space and $J(v)$ is the motivic invariant of $M_{H,Y}(v)$. $\square$

In Chapter 7 we will use this result to deduce the nonemptiness of $\mathfrak{M}_{\sigma,Y}(v)(\mathbb{C})$ as well as to obtain new results about classical moduli of sheaves.

6.3 The Geometry of the Morphism π^*

We begin here our investigation of the relationship between the geometry of the moduli spaces $\mathfrak{M}_{\sigma,Y}(v)$ and $\mathfrak{M}_{\sigma',\tilde{Y}}(\pi^*v)$, where again $\sigma = (\pi^*)^{-1}(\sigma')$ for an invariant stability condition σ'. Notice that π^* induces a morphism of stacks

$$\pi^* : \mathfrak{M}_{\sigma,Y}(v,\phi) \to \mathfrak{M}_{\sigma',\tilde{Y}}(\pi^*v,\phi).$$

Since ι induces an autoequivalence of $\mathrm{D^b}(\tilde{Y})$, and we've chosen $\sigma' \in \Gamma_{\tilde{Y}}$, ι induces an involution ι^* on $\mathfrak{M}_{\sigma',\tilde{Y}}(\pi^*v)$. It follows that π^* factors through the fixed point substack $\mathrm{Fix}(\iota)$, a closed substack, to give a morphism

$$\pi^* : \mathfrak{M}_{\sigma,Y}(v) \to \mathrm{Fix}(\iota),$$

which we still denote by π^*[1].

[1]To avoid too many stack-theoretic complications, we will define the fixed point stack as $\mathrm{Fix}(\iota) := M_{\sigma',\tilde{Y}}(\pi^*v)^G \times_{M_{\sigma',\tilde{Y}}(\pi^*v)} \mathfrak{M}_{\sigma',\tilde{Y}}(\pi^*v)$, where $M_{\sigma',\tilde{Y}}(\pi^*v)^G \subset M_{\sigma',\tilde{Y}}(\pi^*v)$ is the fixed-point subscheme of the coarse moduli space. This ensures that the fixed point substack is in fact a closed substack, smooth if the ambient stack is. It also ensures that the morphism to the fixed point substack descends through Inaba's rigidification by $\mathbb{G}_m$. For a more general and thorough discussion of these issues, see [56]

As usual, we start by considering the stable locus and generalize the results and arguments of [34],[57] to the case of Bridgeland moduli spaces. First we note the following fact:

Lemma 6.3.1. $\mathrm{Fix}(\iota)\cap M^s_{\sigma',\tilde{Y}}(\pi^*v)$ *is a union of isotropic algebraic subspaces of* $M^s_{\sigma',\tilde{Y}}(\pi^*v)$.

Proof. In [30, Theorem 3.3] Inaba generalized the by-now classical result from [49] that the moduli space of stable sheaves on a K3 surface $\tilde{Y}$ carries a non-degenerate symplectic form. Recall that for $F\in M_{\sigma',\tilde{Y}}(\pi^*v)$ Inaba defined the sympletic form ω on the smooth algebraic space $M^s_{\sigma',\tilde{Y}}(\pi^*v)$ by considering the composition

$$\mathrm{Ext}^1(F,F)\times\mathrm{Ext}^1(F,F)\to\mathrm{Ext}^2(F,F)\to H^2(\tilde{Y},\mathcal{O}_{\tilde{Y}})=H^2(\tilde{Y},\omega_{\tilde{Y}})\cong\mathbb{C}$$
$$(e,f)\longmapsto e\cup f\ \longmapsto\mathrm{tr}(e\cup f),$$

where the identification of $H^2(\tilde{Y},\omega_{\tilde{Y}})$ with $\mathbb{C}$ is dual to the isomorphism between $H^0(\tilde{Y},\omega_{\tilde{Y}})$ and $\mathbb{C}$, where the former is generated by the unique holomorphic 2-form α up to scaling. Since ι^* sends α to $-\alpha$, it follows that ω is anti-sympletic, i.e. $\omega(\iota^*e,\iota^*f)=-\omega(e,f)$.

Moreover, as $M^s_{\sigma',\tilde{Y}}(\pi^*v)$ is a smooth algebraic space, $\mathrm{Fix}(\iota)\cap M^s_{\sigma',\tilde{Y}}(\pi^*v)$ is the union of smooth subspaces. The fact that it is isotropic follows from the fact that ι^* is anti-symplectic. □

Proposition 6.3.2. *The morphism of stacks*

$$\pi^*:\mathfrak{M}_{\sigma,Y}(v)\to\mathrm{Fix}(\iota)\subset\mathfrak{M}_{\sigma',\tilde{Y}}(\pi^*v)$$

is onto. The induced morphism

$$\pi^{*,s}:\mathfrak{M}^s_{\sigma,Y}(v)\to\mathrm{Fix}(\iota)$$

is a 2-to-1 cover onto its image, étale away from those points with $E\cong E\otimes\omega_Y$.

Proof. First we show that π^* is surjective onto $\mathrm{Fix}(\iota)$. Indeed, suppose $F\in M_{\sigma',\tilde{Y}}(\pi^*v)$ is invariant under ι^*. From [16, Proposition 2.5], there exists an object $E\in\mathrm{D}^{\mathrm{b}}(Y)$ such that $\pi^*E\cong F$. From the definition of induced stability conditions, it follows that $E\in M_{\sigma,Y}(v)$. Moreover, it clearly follows that if F is stable, then so is E.

Now we prove that $\pi^{*,s}$ is unramified. Lieblich and Inaba (in [43] and [29], respectively) generalized the well-known results about the deformation theory of coherent sheaves to complexes of such. In particular, for $E \in M^s_{\sigma,Y}(v)$ and $F = \pi^*E$, the tangent spaces are

$$T_E\mathfrak{M}_{\sigma,Y}(v) \cong \mathrm{Ext}^1(E,E), \text{ and } T_F\mathfrak{M}_{\sigma',\tilde{Y}}(\pi^*v) \cong \mathrm{Ext}^1(F,F),$$

and the differential is just the natural map

$$d\pi^* : \mathrm{Ext}^1(E,E) \to \mathrm{Ext}^1(F,F).$$

Note that it follows from Riemann-Roch that if E and $F = \pi^*E$ are both stable, $\mathfrak{M}_{\sigma,Y}(v)$ is smooth at E (since the obstruction space vanishes because $\mathrm{Ext}^2(E,E) = 0$) of dimension $\dim T_E = v^2+1$ while $\mathfrak{M}_{\sigma',\tilde{Y}}(\pi^*v)$ is smooth at F of dimension $(\pi^*v)^2+2 = 2\dim\mathfrak{M}_{\sigma,Y}(v)$.

We claim that the differential must be injective for $E \not\cong E\otimes\omega_Y$. Indeed, suppose $E' \in \mathrm{Ext}^1(E,E)$, i.e. E' is an extension

$$0 \to E \to E' \to E \to 0,$$

in $\mathcal{P}_\sigma(\phi)$. Notice from applying $\mathrm{Hom}(-,E\otimes\omega_Y)$ and noting that E and $E\otimes\omega_Y$ are nonisomorphic and stable of the same phase so that $\mathrm{Hom}(E,E\otimes\omega_Y) = 0$, we must have $\mathrm{Hom}(E',E\otimes\omega_Y) = 0$. Suppose that $\pi^*E' = 0 \in \mathrm{Ext}^1(F,F)$, i.e. the short exact sequence

$$0 \to F \to \pi^*E' \to F \to 0$$

in $\mathcal{P}_{\sigma'}(\phi)$ splits. But then so does the short exact sequence

$$0 \to \pi_*(F) \to \pi_*(\pi^*E') \to \pi_*(F) \to 0.$$

But this is precisely the sequence

$$0 \to E\oplus(E\otimes\omega_Y) \to E'\oplus(E'\otimes\omega_Y) \to E\oplus(E\otimes\omega_Y) \to 0.$$

Since $\mathrm{Hom}(E',E\otimes\omega_Y) = \mathrm{Hom}(E'\otimes\omega_Y,E) = 0$, it follows that any morphism

$$E'\oplus(E'\otimes\omega_Y) \to E\oplus(E\otimes\omega_Y)$$

must be component wise, and thus any splitting of this short exact sequence induces a splitting of the original exact sequence

$$0 \to E \to E' \to E \to 0,$$

proving injectivity.

Finally, note that $d\pi^*$ factors through $T_F \operatorname{Fix}(\iota)$. Since $\operatorname{Fix}(\iota) \cap \mathfrak{M}^s_{\sigma',\tilde{Y}}(\pi^* v)$ is smooth and isotropic by Lemma 6.3.1, it follows that $d\pi^*$ is isomorphic onto $T_F \operatorname{Fix}(\iota)$, so π^* is étale at E. That it is 2-to-1 follows from Lemma 6.1.2. $\square$

Remark 6.3.3. If $\operatorname{Fix}(\iota) \cap \mathfrak{M}^s_{\sigma',\tilde{Y}}(\pi^* v)$ is nonempty, then it follows that every component is a Lagrangian substack from the above proposition.

6.4 Singularities of Bridgeland moduli spaces and their canonoical divisor

We describe here the structure of the singularities of the algebraic space parametrizing Bridgeland stable objects:

Theorem 6.4.1. *Let Y be an Enriques surface, $v \in H^*_{\mathrm{alg}}(Y, \mathbb{Z})$, and $\sigma \in \operatorname{Stab}^\dagger(Y)$ (not necessarily generic). Then the algebraic space $M^s_{\sigma,Y}(v)$ is singular at E if and only if $E \cong E \otimes \omega_Y$ and E lies on a component of dimension $v^2 + 1$. The singular locus of $M^s_{\sigma,Y}(v)$ is the union of the images under π_* of finitely many components of the algebraic spaces*

$$M^s_{\sigma',\tilde{Y}}(w)^\circ = \{F \in M^s_{\sigma',\tilde{Y}}(w) | F \ncong \iota^* F\},$$

as $w \in H^*_{\mathrm{alg}}(\tilde{Y}, \mathbb{Z})$ ranges over classes such that $\pi_*(w) = v$. Consequently,

$$\dim \operatorname{Sing}(M^s_{\sigma,Y}(v)) \leq \frac{1}{2}(\dim M^s_{\sigma,Y}(v) + 3)$$

so that $M^s_{\sigma,Y}(v)$ is generically smooth. It is possible that $M^s_{\sigma,Y}(v)$ has irreducible components of dimension 0 and 2, which are necessarily smooth, if $E \cong E \otimes \omega_Y$ and

$$\dim_E M^s_{\sigma,Y}(v) = v^2 + 2,$$

i.e. $v^2 = -2$ or 0.

Proof. The proof of the main theorem in [34] generalizes without change. □

While Theorem 6.4.1 gives a global description of the singular locus, we now address the local nature of these singularities and the canonical bundle of Bridgeland moduli spaces:

Theorem 6.4.2. *Suppose that Y is an Enriques surface, $v \in H^*_{\mathrm{alg}}(Y,\mathbb{Z})$, and $\sigma \in \mathrm{Stab}^\dagger(Y)$. Suppose that the fixed locus of $-\otimes\omega_Y$ has codimension at least 2. Then $M^s_{\sigma,Y}(v)$ is normal and Gorenstein with only canonical l.c.i. singularities. Furthermore, $\omega_{M^s_{\sigma,Y}(v)}$ is torsion in $\mathrm{Pic}(M^s_{\sigma,Y}(v))$.*

Proof. The first statement follows directly as in [62], while the statement about the canonical divisor is shown precisely as in [28, Proposition 8.3.1]. □

Remark 6.4.3. If v^2 is odd, then the rank of v is odd and by Theorem 6.4.1 the fixed locus of $-\otimes\omega_Y$ is empty, so the hypothesis of the theorem above is certainly satisfied. Furthermore, the codimension of this fixed locus is at least 2 if $v^2 \geq 5$ by Theorem 6.4.1 since it must be of even dimension at most $\frac{1}{2}(\dim M^s_{\sigma,Y}(v)+3)$. Moreover, we have equality in this dimension estimate only if $c_1(F)=\iota^*c_1(F)$, in which case $2 \mid \pi^*v$. From Section 3.3 we know that if π^*v is divisible by 2 for primitive v then $v^2 \equiv 0 \pmod 8$. So if $v^2=4$, then the fixed locus has codimension at least 2 if v is primitive, but if $v^2=2$, it is possible that the fixed locus is a divisor. If Y is unnodal, however, we automatically have $c_1(F)=\iota^*c_1(F)$ so this is impossible since π^*v is primitive in this case. Finally, let us note that on exceptional components of dimension 2, i.e. $v^2=0$ and $E\cong E\otimes\omega_Y$, $\mathcal{E}xt^2_p(\mathcal{E},\mathcal{E})\cong\mathcal{O}_{M^s_{\sigma,Y}(v)}$, so the conclusion of the theorem continues to hold.

Chapter 7

Interlude: New results on moduli of stable sheaves on Enriques surfaces

We have now established the fundamental theory of moduli stacks of Bridgeland semistable objects, and we use it to settle the foundational issue of non-emptiness and irreducibility of Gieseker moduli spaces on Enriques surfaces. We begin with proving Theorem 2.3.2 in the case of a primitive Mukai vector on an unnodal Enriques surface. Recall from Chapter 4 that Yoshioka has shown that in odd rank, $M_{H,Y}(v, L)$ is irreducible of dimension $v^2 + 1$, while Hauzer's result Theorem 4.6.3 reduces the even rank case to ranks two and four. We consider first the case of rank four.

7.1 Classification of chern classes

To make our investigation easier, in this section we use the lattice theoretic techniques employed by Hauzer in [26] to reduce the study of rank 4 sheaves R to the case $\chi(R) = 1$ and R has degree 2 or 4 on the generic fibre of some elliptic fibration of Y, depending on the divisibility of $c_1(R)$. In particular, we prove the following helpful reduction result:

Theorem 7.1.1. *Let $v = (4, c_1, s)$ be a primitive Mukai vector on an Enriques surface Y. Then we can find divisor D such that for any coherent sheaf R with $v(R) = v$, $\chi(R(D)) = 1$ and $c_1(R(D)).f = \pm 1$ if $2 \nmid c_1$ or $c_1(R(D)).f = 2$ if $2 \mid c_1$, where F_A is an elliptic half-pencil with $c_1(F_A) = f$.*

Proof. To begin with, denote by σ, f the canonical basis of U so that

$$\mathrm{Num}(Y) = \mathrm{Pic}(Y)/\langle K_Y \rangle = U \oplus -E_8,$$

where we can assume that σ and f represent effective elliptic half-pencils G_A and F_A, respectively.

Write $c_1 = d_1\sigma + d_2 f + \xi$ with $\xi \in -E_8$. Then for any R with $v(R) = v$, we have

$$v(R \otimes \mathcal{O}_Y(D)) = v\,\mathrm{ch}(\mathcal{O}_Y(D)) = (4, c_1 + 4D, 2D^2 + c_1.D + s),$$

so choosing $D = kG_A + jF_A$ appropriately we can assume $-2 < d_i \leq 2$. Letting $l = \gcd(4, \xi)$, we can choose $\xi_1 \in -E_8$ such that $(\xi + 4\xi_1)/l$ is primitive by [26, Lemma 2.1]. Then by replacing v by $v\,\mathrm{ch}(\xi_1)$ we can assume that ξ/l is primitive. Suppose $d_1 = 0, d_2 = 2b$ for $b = 0, 1$ with the case $d_1 = 2b, d_2 = 0$ being dealt with by switching σ and f. First suppose that $b = 1$. Then clearly, $l = 1, 2, 4$. If $l \neq 1$, then by Lemma 3.3.1, $\chi(R) = \frac{r+2s}{2}$ is odd. So $\chi(R \otimes \mathcal{O}_Y((\frac{1-\chi(R)}{2})G_A)) = \chi(R) + 2\frac{1-\chi(R)}{2} = 1$. Furthermore, note that $c_1(R \otimes \mathcal{O}_Y((\frac{1-\chi(R)}{2})G_A)).\sigma = 2$. If $l = 1$, then ξ is primitive so we may choose $\eta \in -E_8$ such that $\xi.\eta = 1$. Then σ and $f' := -\frac{\eta^2}{2}\sigma + f + \eta$ span a hyperbolic plane, and the coordinates of c_1 in this new basis are $c_1.f' = 1 - \eta^2$, which is odd, and $c_1.\sigma = 2$, respectively. Then by tensoring with an appropriate multiple of $\mathcal{O}_Y(G_A)$ we can assume we are in the case where one of the $d_i = \pm 1$, to be dealt with momentarily.

Finally, suppose that $b = 0$, i.e. $c_1 \in -E_8$. Then $l = 1$ or 2, and since ξ/l is primitive, we choose $\eta \in -E_8$ such that $\frac{\xi}{l}.\eta = 1$ and let $\sigma' = \sigma - \frac{\eta^2}{2}f + \eta$. Again σ' and f span a hyperbolic plane and the coordinates of the hyperbolic part of c_1 in this new basis are $c_1.f = 0$ and $c_1.\sigma' = \xi.\eta = l$. So we are reduced to cases $(d_1, d_2) = (0, 1)$ or $(0, 2)$. The latter we dealt with above, so let us treat the remaining cases.

We are left with the two cases where either $(d_1, d_2) = (2, 2)$ or one of $d_i = \pm 1$. In the first case, $l = \gcd(4, \xi) = 1, 2, 4$ and ξ/l is assumed to be primitive. If $l = 4$, let $f' = \sigma + f + \eta$ for $\eta \in -E_8$ satisfying $\eta^2 = -2$. Then σ and f' span a hyperbolic plane, and the hyperbolic coordinates of c_1 in this new basis are $c_1.f' = 4 + \xi.\eta$, which is divisible by 4, and $c_1.\sigma = 2$. By tensoring by $\mathcal{O}_Y(\frac{c_1.f'}{4}G_A)$ we reduce to the case $(d_1, d_2) = (0, 2)$ considered above. Note that this also works if $\xi = 0$. If $l = 2$, then choose $\eta \in -E_8$ such that $(\xi/2 - \eta)^2 \equiv (\xi/2)^2 + 2 (\mathrm{mod}\, 4)$. To see that such an η exists, choose any η' such that $\frac{\xi}{2}.\eta' = 1$. If $\eta'^2 \equiv 0 (\mathrm{mod}\, 4)$, then there is nothing to be shown as we let $\eta = \eta'$. Now suppose $\eta'^2 \equiv 2 (\mathrm{mod}\, 4)$. If $(\frac{\xi}{2})^2 \equiv 2 (\mathrm{mod}\, 4)$, then let $\eta = \frac{\xi}{2} + 2\eta'$, while if $(\frac{\xi}{2})^2 \equiv 0 (\mathrm{mod}\, 4)$, then let $\eta = \frac{\xi}{2} + \eta'$. Then in either case we that $\sigma' := \sigma - \frac{\eta^2}{2}f + \eta$ and f span a hyperbolic plane in which the coordinates of c_1 are

$c_1.f = 2$ and $c_1.\sigma' = 2(1 - \frac{\eta^2}{2}) + 2\frac{\xi'}{2}.\eta \equiv 0 (\mathrm{mod}\, 4)$. Thus we reduce to the previously considered case of $(d_1, d_2) = (2, 0)$. If $l = 1$, then for $\eta \in -E_8$, $\sigma' := \sigma - \frac{\eta^2}{2} f + \eta$ and f span a hyperbolic plane. Since ξ is primitive, we choose $\eta' \in -E_8$ such that $\xi.\eta' = 1$. If $\eta'^2 \equiv 0 (\mathrm{mod}\, 4)$, then let $\eta = -\eta'$, and if $\eta'^2 \equiv 2 (\mathrm{mod}\, 4)$, then let $\eta = \eta'$. In either case, we find that the coordinates of c_1 in these new hyperbolic coordinates are $c_1.f = 2$ and $c_1.\sigma' = 2(1 - \frac{\eta^2}{2}) + \xi.\eta \equiv 1 (\mathrm{mod}\, 4)$, so we reduce to the case where one of the $d_i = 1$, which we consider now.

If, say $d_1 = \pm 1$, then $\chi(R \otimes \mathcal{O}_Y((1 - \chi(R))|d_1|))F_A) = 1$ and $c_1(R \otimes \mathcal{O}_Y((1 - \chi(R))|d_1|F_A)).f = d_i = \pm 1$. This gives the result. $\square$

Now we conclude this section by using the same techniques as in the lemma above to prove irreducibility in the case left unresolved by Kim in [35], namely when $c_2 = \frac{1}{2}c_1^2$. We do this by simply showing that this case may be reduced to Kim's first case, i.e. when $c_2 = \frac{1}{2}c_1^2 + 1$, where he was able to prove irreducibility.

Theorem 7.1.2. *All moduli spaces $M_H((2, c_1, s), L)$ such that $u := (2, c_1, s)$ is primitive and $u^2 = c_1^2 - 4s \geq 0$ are non-empty and irreducible for general H unless $u^2 = 0$ and $2 \mid c_1$ in which case the result is true if and only if $2 \nmid L$.*

Proof. As above, we can write $c_1 = d_1\sigma + d_2 f + \xi$ with $\xi \in -E_8$. Let us first suppose that $2 \mid c_1$. Note that either L or $L + K_Y$ is divisible by 2. So we may twist any P with $v(P) = u$ by $\frac{L}{2}$ or $\frac{L+K_Y}{2}$, allowing us to assume that $\det(P) = \mathcal{O}_Y$ or $\mathcal{O}_Y(K_Y)$. Taking G_A such that $c_1(G_A) = \sigma$ and twisting by $\mathcal{O}_Y(G_A)$, we can assume that $c_1(P) = 2\sigma$. As all of these operations thus far are isometries, we must have that $s(P) \leq 0$ and is an even integer. Then $c_1(P(-\frac{s(P)}{2}F_A)) = 2\sigma - s(P)f$ and $\chi(P(-\frac{s(P)}{2}F_A)) = 1$. This is equivalent to $s(P(-\frac{s(P)}{2}F_A)) = 0$, i.e. $c_2(P(-\frac{s(P)}{2}F_A)) = \frac{1}{2}c_1(P(-\frac{s(P)}{2}F_A))^2 + 1$. If $u^2 > 0$, then $s(P) \neq 0$, so $c_1(P(-\frac{s(P)}{2}F_A))$ is ample in this case and the result follows from [35, Theorem I].

So we may assume for the moment that $2 \nmid c_1$ and that furthermore $d_i = 0, 1$ after twisting by $\mathcal{O}_Y(kG_A + jF_A)$ if necessary. If $d_1 = 1$, say, then twisting by $\mathcal{O}_Y(-s(P)F_A)$ allows us to assume that $s(P) = 0$ and $c_1.f = 1$, and analogously, if $d_2 = 1$. If $d_1 = d_2 = 0$, then we may reduce to the case when one of the $d_i = 1$. Indeed, as we are assuming $2 \nmid c_1$, then it follows from [26, Lemma 2.1] that we may find $\xi_1 \in -E_8$ such that $\xi + 2\xi_1$ is primitive. Twisting by $\mathcal{O}_Y(D)$ such that $c_1(D) = \xi_1$, we may assume that $c_1(P) = \xi$ is primitive with $\xi \in -E_8$. Then as $-E_8$ is unimodular, we may find η such that $\xi.\eta = 1$. Replacing σ by $\sigma' := \sigma - \frac{\eta^2}{2} f + \eta$, we get σ', f span a hyperbolic plane with $\sigma'.c_1 = \xi.\eta = 1$ and $f.c_1 = 0$ so that the coordinate of f in c_1 in these new coordinates is 1, so we are in the previous case. As we can assume $s(P) = 0$, we get that $u^2 \geq 0$ implies $c_1^2 \geq 0$. After dualizing if necessary, we may assume that c_1 is the class of an effective divisor. Then again the theorem follows from [35] if $u^2 > 0$ and [26] if $u^2 = 0$.

Finally, let us assume $2 \mid c_1$ and $u^2 = 0$. Then as above we may assume that $\det(P) = \mathcal{O}_Y$ or $\mathcal{O}_Y(K_Y)$ and that $s(P) = 0$. It was shown in [64] that $M_H((2,0,0), K_Y) \cong Y$, while it will follow from Remark 7.2.3 that $M_H((2,0,0), \mathcal{O}_Y) = \varnothing$. $\square$

7.2 A quick proof of non-emptiness and irreducibility using Bridgeland stability and derived category techniques

In this section we provide a quick-and-dirty proof of Theorem 2.3.2 using the techniques we developed in the previous chapter, namely moduli spaces of Bridgeland stable objects and wall-crossing.

Let us summarize what we have so far. We have shown that in studying rank four moduli spaces we need only consider $M_H((4, c_1, -1), L)$, where $c_1^2 \geq -8$ and there exists an elliptic half-pencil F_A with $c_1(F_A) = f$ such that $c_1.f = \pm 1, 2$. With these notations set, we prove the following theorem:

Theorem 7.2.1. *Let $v = (4, c_1, -1)$ be a primitive Mukai vector, $L \in \operatorname{Pic}(Y)$ such that $c_1(L) = c_1$, and H a generic polarization with respect to v. Then $e(M_H(v, L)) = e(M_H(v - 2v(\mathcal{O}_Y), L + K_Y))$. In particular,*

- *$M_H(v, L)$ is non-empty and irreducible if $\gcd(2, c_1) = 1$ and $v^2 \geq 0$;*
- *$M_H(v, L)$ is non-empty and irreducible if $\gcd(2, c_1) = 2$ and $v^2 > 0$;*
- *$M_H(v, L)$ is non-empty and irreducible if $\gcd(2, c_1) = 2$, $v^2 = 0$, and $2 \mid L$;*

- $M_H(v, L) = \varnothing$ *if* $\gcd(2, c_1) = 2, v^2 = 0$, *and* $2 \nmid L$.

Proof. Let $\sigma \in \mathrm{Stab}^\dagger(Y)$ be a generic stability condition in the Gieseker chamber for H, i.e. the moduli space of σ-stable objects $M_\sigma(v) \cong M_H(v)$. Applying a special Fourier-Mukai transform $\mathcal{H}$, called the (-1)-reflection through $\mathcal{O}_Y$, we get that $M_\sigma(v) \cong M_{\mathcal{H}_*(\sigma)}(\mathcal{H}_*(v))$, where $\mathcal{H}_*(\sigma)$ is the stability condition obtained by applying $\mathcal{H}$. The action of $\mathcal{H}$ on the Mukai lattice, $\mathcal{H}_*$, takes a Mukai vector v to $v + 2\,(v, v(\mathcal{O}_Y))\, v(\mathcal{O}_Y)$ and changes the determinant line bundle by $L \mapsto L + \chi(v)K_Y$. In our case, this means that $\mathcal{H}_*(v) = v - 2v(\mathcal{O}_Y)$, as $(v, v(\mathcal{O}_Y)) = -1$, and L becomes $L + K_Y$. Thus $M_\sigma(v, L) \cong M_{\mathcal{H}_*(\sigma)}(v - 2v(\mathcal{O}_Y), L + K_Y)$. By Theorem 6.2.12, the Hodge polynomial does not change as we change the stability condition. Moving the stability condition $\sigma' := \mathcal{H}_*(\sigma)$ to the Gieseker chamber for $u := v - 2v(\mathcal{O}_Y)$ gives the claimed equality of Hodge polynomials.

As $\chi(u) = -1$, we see that for any P with $v(P) = u$, $\chi(P(\kappa F_A)) = 1$ for $\kappa = 2, -2$, or 1 if $c_1.f$ is 1,-1, or 2, respectively. Then $c_1(P(\kappa F_A)) = c_1 + 2\kappa f$ and $c_1(P(\kappa F_A))^2 = c_1^2 + 4\kappa c_1.f = c_1^2 + 8 = v^2 = u^2$ in all cases. Then for $v^2 > 0$, non-emptiness and irreducibility follow from [35, Theorem 1], as up to dualizing $c_1(P(\kappa F_A))$ may be taken to be ample. The case of $v^2 = 0, \gcd(2, c_1) = 1$ follows from [26]. The case when $v^2 = 0, \gcd(2, c_1) = 2$ follows from the end of the proof of Theorem 7.1.2 above. $\square$

We can also use derived category techniques to prove directly that given a primitive Mukai vector $v = (r, c_1, s)$ with $\gcd(2, c_1) = 2$ and $v^2 = 0$, then stable sheaves with Mukai vector v form an irreducible family and can only have one of the two possible determinants. In fact, we show a bit more:

Theorem 7.2.2. *Let v be a primitive Mukai vector on an Enriques surface Y with $v^2 = 0$ and $\pi^* v$ divisible by 2, i.e.* $\gcd(2, c_1) = 2$. *Let H be a polarization that is generic with respect to v. Then $M_H(v)$ is a smooth irreducible surface, isomorphic to Y itself.*

Proof. Since v is primitive and H generic, $M := M_H(v)$ is projective and parametrizes only stable sheaves. Let $w = \frac{1}{2}\pi^* v \in H^*_{\mathrm{alg}}(\tilde{Y}, \mathbb{Z})$. Then from [63], $M_{\pi^* H, \tilde{Y}}(w)$ is a smooth K3 surface which parametrizes only stable sheaves. From the proof of Theorem 4.6.1 in [34], $\pi_* : M_{\pi^* H, \tilde{Y}}(w) \to M$ is an étale double cover onto its image. Thus M has a smooth two dimensional component $M_1 := \pi_*(M_{\pi^* H, \tilde{Y}}(w))$ which is fixed pointwise by tensoring with $\mathcal{O}_Y(K_Y)$. Moreover, as the quotient of a K3 surface by a fixed-point free involution, M_1 is an Enriques surface.

Let us fix a quasi-universal family $\mathcal{E}$ on M_1 of similitude ρ. Then for $[F] \notin M_1$, $\operatorname{Ext}^i(F, \mathcal{E}_t) = 0$ for all i and $t \in M_1$ as $v^2 = 0$ and any G with $[G] \in M_1$ satisfies $G \cong G(K_Y)$. Thus $\mathcal{E}xt^i_p(q^*F, \mathcal{E}) = 0$ for all i.

If $[F] \in M_1$, then the situation is more delicate. By [6] there exists a complex $\mathcal{P}^\bullet$ of locally free sheaves $\mathcal{P}^i$ of finite rank such that the i-th cohomology $\mathcal{H}^i(\mathcal{P}^\bullet) \cong \mathcal{E}xt^i_p(q^*F, \mathcal{E})$ and $\mathcal{H}^i(\mathcal{P}^\bullet_t) \cong \operatorname{Ext}^i(F, \mathcal{E}_t)$. Moreover, this complex is bounded from above and we may assume $\mathcal{P}^i = 0$ for $i < 0$. Of course, $\mathcal{E}xt^0_p(q^*F, \mathcal{E})$ is a skyscraper sheaf concentrated at $t_0 = [F]$. Since it's a subsheaf of the locally free sheaf $\mathcal{P}^0$, it must be 0. Furthermore, we note that $\ker(d^i)$ is always locally free as the kernel of a surjection from the locally free $\mathcal{P}^i$ to the torsion-free sheaf $\operatorname{im}(d^i)$ on the smooth surface M_1. As $\mathcal{P}^\bullet$ is exact at $\mathcal{P}^i$ for $i > 2$, since $\operatorname{Ext}^i(F, \mathcal{E}_t) = 0$ for $i > 0$, and $\mathcal{P}^i = 0$ for large i, we can work backwards using exactness and replace $\mathcal{P}^2$ by $\ker(d^2)$ to get

$$0 \to \operatorname{im}(d^1) \to \mathcal{P}^2 \to \mathcal{H}^2(\mathcal{P}^\bullet) \to 0.$$

It follows that

$$\mathcal{H}^2(\mathcal{P}^\bullet)(t) = \mathcal{P}^2(t)/\operatorname{im}(d^1)(t) \cong \mathcal{H}^2(\mathcal{P}^\bullet(t)) \cong \operatorname{Ext}^2(F, \mathcal{E}_t)$$

for any closed point t. Thus $\mathcal{E}xt^2_p(q^*F, \mathcal{E})$ is a torsion sheaf concentrated at $t_0 = [F]$ with length ρ. Finally, we notice that since $\ker(d^1)$ is locally free and contains the locally free $\mathcal{P}^0$ with quotient supported in dimension zero, it follows that the quotient, $\mathcal{E}xt^1_p(q^*F, \mathcal{E})$, is 0.

By the Grothendieck-Riemann-Roch formula,

$$a := \operatorname{ch}([\mathcal{E}xt^0_p(q^*F, \mathcal{E})] - [\mathcal{E}xt^1_p(q^*F, \mathcal{E})] + [\mathcal{E}xt^2_p(q^*F, \mathcal{E})])$$

depends only on $\operatorname{ch}(F)$ and $\operatorname{ch}(\mathcal{E})$. Since $\operatorname{ch}(F)$ is constant for all $[F] \in M$, so is a. But for $F \notin M_1$, $a = 0$, while for $F \in M_1$ $a = -\rho \operatorname{ch}(\mathbb{C}(t_0))$. But $0 \neq -\rho\chi(\mathbb{C}(t_0)) =$

$\langle a \cdot td(M), [M_1]\rangle$, which is a contradiction unless M is irreducible and equal to M_1. Thus $M = M_1$ is an Enriques surface.

To prove that M is in fact isomorphic to Y itself we consider the Fourier-Mukai transform $\Phi_{\mathcal{E}} : \mathrm{D^b}(M) \to \mathrm{D^b}(Y)$ induced by the universal sheaf $\mathcal{E}$.[1] Then $\Phi_{\mathcal{E}}$ is fully faithful by [27, Corollary 7.5], since $\mathrm{Hom}(\mathcal{E}_x, \mathcal{E}_x) = \mathbb{C}$, while $\mathrm{Ext}^i(\mathcal{E}_x, \mathcal{E}_y) = 0$ for all i and $x \neq y$. For all $x \in M$ we have

$$\Phi_{\mathcal{E}}(\mathbb{C}(x)) \otimes \mathcal{O}_Y(K_Y) = \mathcal{E}_x(K_Y) \cong \mathcal{E}_x = \Phi_{\mathcal{E}}(\mathbb{C}(x)),$$

so $\Phi_{\mathcal{E}}$ is moreover an equivalence by a result of Bridgeland [27, Proposition 7.11]. Thus we have two Enriques surfaces, M and Y, which are derived equivalent. They must in fact be isomorphic by a result of Bridgeland and Maciocia [27, Proposition 12.20]. $\square$

Remark 7.2.3. Note that the above proof shows that if $M_H(v, L) \neq \varnothing$ with v as above, then $M_H(v, L+K_Y) = \varnothing$. By keeping track of the determinant in the proof of Theorem 4.6.3, one sees that in fact $M_H((r, c_1, s), L) \neq \varnothing$ if and only if $2 \mid L + (s + \frac{r}{2})K_Y$ if $\gcd(2, c_1) = 2$ and $v^2 = 0$.

7.3 The existence of stable sheaves in the non-primitive case

After dealing exclusively with primitive Mukai vectors above, we conclude the chapter with a result about stable sheaves in the non-primitive case. Let $v = mv_0$ with v_0 a primitive Mukai vector such that $v_0^2 \geq -1$, and H generic with respect to v. Then for $m > 1$, any destabilizing subsheaf of a sheaf in $M_H(v)$ must have Mukai vector $m'v_0$ for $m' < m$. We generalize standard arguments to show the following result:

Theorem 7.3.1. *Let $v = mv_0$ be a Mukai vector with v_0 primitive and $m > 0$ with H generic with respect to v on an unnodal Enriques surface Y.*

(a) The moduli space of Gieseker-semistable sheaves $M_H(v) \neq \varnothing$ if and only if $v_0^2 \geq -1$.

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[1] We can drop the prefix quasi by applying [28, Theorem 4.6.5] with B in their notation running through the sheaves $\mathcal{O}_Y, \mathbb{C}([\mathrm{pt}])$ and $\mathcal{O}_Y(D)$ for D such that $D.c_1(v)$ is minimal. Indeed, using the fact that $\gcd(r, c_1, 2s) = 2$ and $r + 2s \equiv 2 (\mathrm{mod}\, 4)$, one sees that $\gcd((v, v(B))) = 1$ as B runs through these sheaves.

(b) *Either* $\dim M_H(v) = v^2 + 1$ *and* $M^s_H(v) \neq \varnothing$, $m = 1, v_0^2 = 0, \dim M_H(v) = \dim M^s_H(v) = 2$, *or* $m > 1$ *and* $v_0^2 \leq 0$.

(c) *If* $M_H(v) \neq M^s_H(v)$ *and* $M^s_H(v) \neq \varnothing$, *the codimension of the semistable locus is at least 2 if and only if* $v_0^2 > 1$ *or* $m > 2$. *Moreover, in this case and the case* $M_H(v) = M^s_H(v)$, $M_H(v)$ *is normal with torsion canonical divisor.*

Proof. If $v_0^2 \geq -1$, then part (a) follows from Theorem 2.3.2 above. For the converse, note that any stable factor of an element of $M_H(v) \neq \varnothing$ would have to have Mukai vector $m'v_0$ for $m' < m$ by genericity of H. But then $m'^2 v_0^2 = (m'v_0)^2 \geq -1$, so $v_0^2 \geq -1$.

For (b), again notice that the genericity of H means that any stable factor of an object of $M_H(v)$ must have Mukai vector $m'v_0$ for $m' < m$, which implies that the strictly semistable locus is the image of the natural map

$$\mathrm{SSL} : \coprod_{m_1+m_2=m, m_i>0} M_H(m_1v_0) \times M_H(m_2v_0) \to M_H(v).$$

Assume $v_0^2 > 0$. Then for $m = 1$, $M_H(v) = M^s_H(v)$, and we have noted already that $\dim M_H(v) = v^2 + 1$. If $m > 1$, then by induction, we deduce that the image of the map SSL has dimension equal to the maximum of $(m_1^2 + m_2^2)v_0^2 + 2$ for $m_1 + m_2 = m, m_i > 0$. This is strictly less than $v^2 + 1$.

Furthermore, we can construct a semistable sheaf E' with Mukai vector v which is also Schur, i.e. $\mathrm{Hom}(E', E') = \mathbb{C}$. By the inductive assumption, we can consider $E \in M^s_H((m-1)v_0)$, and let $F \in M_H(v_0)$. Now $\chi(F, E) = -(v(F), v(E)) = -(m-1)v_0^2 < 0$, so $\mathrm{Ext}^1(F, E) \neq 0$. Take E' to be a nontrivial extension

$$0 \to E \to E' \to F \to 0.$$

Then any endomorphism of E' gives rise to a homomorphism $E \to F$, of which there are none since these are both stable of the same phase and have different Mukai vectors (or can be chosen to be non-isomorphic if $m = 2$). Thus any endomorphism of E' induces an endomorphism of E, and the kernel of this induced map $\mathrm{Hom}(E', E') \to \mathrm{Hom}(E, E) = \mathbb{C}$ is precisely $\mathrm{Hom}(F, E')$, which vanishes since the extension is non-trivial. Thus $\mathrm{Hom}(E', E') = \mathbb{C}$.

We can deduce non-emptiness of $M_H^s(v)$ from a dimension estimate as follows. Since E' is Schur, we get

$$v^2+1 \le \dim_{E'} M_H(v) \le \dim T_{E'} M_H(v) = v^2+1+\hom(E', E'\otimes \mathcal{O}_Y(K_Y)).$$

As we mentioned above, the strictly semistable locus must have dimension smaller than v^2+1. So even though E' is not stable, it lies on a component which must contain stable objects. From Theorem 4.6.1, (smooth) components of the stable locus of dimension greater than v^2+1 can occur only if $v_0^2=0$, so $v_0^2>0$ implies that the locus of points fixed by $-\otimes\mathcal{O}_Y(K_Y)$ has positive codimension. Then we may choose $E\in M_H^s((m-1)v_0)$ so that $E\not\cong E\otimes\mathcal{O}_Y(K_Y)$. Stability of E and F and a diagram chase then show that $\mathrm{Hom}(E', E'\otimes\mathcal{O}_Y(K_Y))=0$, so $M_H(v)$ is smooth at E' of dimension v^2+1 as claimed.

Furthermore, observe that the strictly semistable locus has codimension

$$v^2+1-(m_1^2v_0^2+m_2^2v_0^2+2) = (m_1+m_2)^2v_0^2+1-(m_1^2v_0^2+m_2^2v_0^2+2) = 2m_1m_2v_0^2-1,$$

for some choice of $m=m_1+m_2, m_i>0$. This is at least 2, if $v_0^2>1$ or $m>2$, but equal to 1 when $m=2$ and $v_0^2=1$, hence the first part of (c). For the second part of (c), notice that the singularities of $M_H^s(v)$, i.e. where $\mathrm{ext}^2(E,E)=1$, are all hypersurface singularities, so normality follows from the dimension estimates in Theorem 4.6.1 and the large codimension of the strictly semistable locus. K-triviality follows from these considerations and the proof of [28, Proposition 8.3.1].

If $v_0^2\le 0$, then it is easily seen that stable sheaves occur only if $m=1$ or 2, with this second case possible only if $v_0^2=0$ and π^*v_0 primitive (see Lemmas 8.2.2 and 8.2.3 below). Finally, by considering the map SSL it follows that

$$\dim M_H(v) = \left\{\begin{array}{l} 0 \text{ if } v_0^2=-1 \\ m \text{ if } v_0^2=0, \pi^*v_0 \text{ primitive} \\ 2m \text{ if } v_0^2=0, \pi^*v_0 \text{ divisible by 2}\end{array}\right\} \neq v^2+1,$$

if $m>1$. $\square$

7.4 Extending to nodal Enriques surfaces via deformation theory

We have dealt with semistable sheaves exclusively on unnodal Enriques surfaces up to this point, but we may extend the above existence result to nodal Enriques surfaces via a deformation argument. This argument appears in a paper of Yoshioka [65] which came out shortly after the preprint appearance of [51] and provides an alternative proof to the proofs of Theorem 2.3.2 that we have reproduced here from [51].

Theorem 7.4.1. *Suppose that v is a positive rank primitive Mukai vector such that $v^2 \geq -2$ on a nodal Enriques surface Y. If $v^2 = 0$ and $\gcd(r, c_1) = 2$, suppose in addtion that $L \equiv \frac{r}{2} K_Y \bmod 2$, and if $v^2 = -2$, suppose that $L \equiv N + \frac{r}{2} K_Y \bmod 2$. Then for generic polarization H, $M_{H,Y}(v, L) \neq \varnothing$, and conversely.*

Proof. It is clear that $v^2 \geq -2$ is necessary for the existence of a stable sheaf.

Suppose first that $v^2 \geq -1$. We saw in Chapter 3 that $H^1(Y, \mathcal{T}_Y) = \mathbb{C}^{10}$ and $H^2(Y, \mathcal{T}_Y) = H^2(Y, \mathcal{O}_Y) = 0$, so the deformations of a polarized Enriques surface (Y, H) are unobstructed [59, Section 3.3.3]. In particular, given a nodal polarized Enriques surface $(Y, H) := (Y_0, H_0)$, we may choose a generic deformation of the pair so that for the resulting one-parameter family of Enriques surfaces $\psi : \mathcal{Y} \to B$ and family of polarizations $\mathcal{H}$, the generic fiber Y_t is unnodal and H_t is generic with respect to v. By [28, Theorem 4.3.7], we may take the relative moduli space of semistable sheaves $\psi_M : M_{\mathcal{H}, \mathcal{Y}/B}(v, L) \to B$ with ψ_M proper. Now, the generic fiber of ψ_M is non-empty by the unnodal case of Theorem 2.3.2, so the image of ψ_M is dense in B. Thus ψ_M is surjective by the properness of ψ_M. Thus the fiber over $0 \in B$, $M_{H_0, Y_0}(v, L)$, is non-empty.

It follows from Theorem 7.2.2 that $M_{H,Y}(v, L) = \varnothing$ in case $v^2 = 0, \gcd(r, c_1) = 2$, and $2 \nmid L + \frac{r}{2} K_Y$.

The case $v^2 = -2$ follows from Theorem 4.6.4. ∎

Chapter 8

Projectivity of Coarse Moduli Spaces

Having taken a detour to use our results on Bridgeland moduli to provide precise criteria for non-emptiness of moduli spaces of Gieseker semistable sheaves (and irreducibility in the case of unnodal Enriques surfaces), we return in this chapter to develop the theory of Bridgeland moduli further. We begin with proving nonemptiness of Bridgeland moduli and then show that these moduli stacks admit projective coarse moduli spaces for generic stability condition.

8.1 Non-emptiness

We may apply Theorem 6.2.12 and the results of the last chapter to obtain the following result:

Theorem 8.1.1. *Let $v = mv_0, m > 0$, where $v_0 = (r, c_1(L), s)$ is a primitive Mukai vector with $v_0^2 \geq -2$. If $v_0^2 = -2$, then we also assume that Y is nodal and $c_1(L) \equiv N \bmod 2$ for a nodal cycle N. Then for any $\sigma \in \mathrm{Stab}^\dagger(Y)$, $\mathfrak{M}_{\sigma,Y}(v)(\mathbb{C}) \neq \varnothing$.*

Proof. Since we are interested at the moment in semistable objects, it suffices to consider the case when $m = 1$, i.e. $v = v_0$ is primitive. Indeed, if $E_0 \in \mathfrak{M}_{\sigma,Y}(v_0)(\mathbb{C})$, then $E = E_0^{\oplus m} \in \mathfrak{M}_{\sigma,Y}(v)(\mathbb{C})$. By the remarks preceeding [14, Lemma 8.2], semistability is a closed condition, so it also suffices to suppose that σ is generic with respect to v so that every σ-semistable object of class v is stable.

By Theorem 6.2.12, non-emptiness of $\mathfrak{M}_{\sigma,Y}(v)(\mathbb{C})$ follows from the non-emptiness of $M_{H,Y}(w)$ for a positive w in the orbit of v under $\mathrm{Aut}(\mathrm{D^b}(Y))$. Thus the result follows immediately from Theorem 2.3.2 if $v^2 \geq -1$. So suppose that Y is nodal, $v^2 = -2$, and $c_1(L) \equiv N \bmod 2$ for a nodal cycle N. The result follows from Theorem 7.4.1

once we observe that in the steps we used to reach a positive Mukai vector in the proof of Theorem 6.2.12 we at most change $c_1(L)$ to $-c_1(L)$, if at all, and $c_1(L) \equiv -c_1(L) \bmod 2$. $\square$

8.2 The unnodal case

Now we address the existence of a projective coarse moduli space for these stacks. Let us first consider the case of unnodal Enriques surfaces, where we can provide a complete picture. As a fundamental first step, we consider the case of a primitive Mukai vector v. By Lemma 6.1.4, for Y unnodal we must assume $v^2 \geq -1$.

Corollary 8.2.1. *Suppose that Y is unnodal, $v \in H^*_{\mathrm{alg}}(Y,\mathbb{Z})$ is primitive with $v^2 \geq -1$, and $\sigma \in \mathrm{Stab}^\dagger(Y)$ generic with respect to v. Then there is a projective coarse moduli space $M_{\sigma,Y}(v)$ parametrizing only stable objects. $M_{\sigma,Y}(v)$ is a normal Gorenstein projective variety (smooth if $2 \nmid \pi^*v$) with torsion canonical divisor and two irreducible components of dimension v^2+1, unless $v^2 = 0$ and $2 \mid \pi^*v$, in which case $M_{\sigma,Y}(v) \cong Y$ has dimension $2 = v^2 + 2$.*

Proof. Since v is primitive and σ generic, $M^s_{\sigma,Y}(v) = M_{\sigma,Y}(v)$ is a proper algebraic space, nonempty by Theorem 6.2.12. As Y is unnodal, it follows that $\mathrm{Stab}^\dagger(Y) = \mathrm{Stab}^\dagger(\tilde{Y})$ [45, Lemma 3.10, Proposition 3.12, and Lemma 3.14] so that we may take σ to be generic for both v and π^*v. Proposition 6.3.2 shows that $M_{\sigma,Y}(v)$ is a finite cover of $\mathrm{Fix}(\iota)$, a (smooth) closed subvariety of the projective coarse moduli space $M_{\sigma',\tilde{Y}}(\pi^*v)$ (which exists and is projective by [11, Theorem 1.3]). Thus $M_{\sigma,Y}(v)$ is projective as well.

The statements about the dimension, singularities, and canonical divisor follow from Theorems 6.4.1, 6.4.2. That $M_{\sigma,Y}(v) \cong Y$ when $v^2 = 0$ and $2 \mid \pi^*v$ follows precisely as in [51, Theorem 5.2]. Finally, from the proof of Theorem 6.2.12, we see that the motivic invariant of $M_{\sigma,Y}(v)$ is the same as that of a related moduli space of stable sheaves with respect to a generic polarization. But then Theorem 2.3.2 shows that $M_{\sigma,Y}(v)$ consists of two irreducible components $M_{\sigma,Y}(v, L_1)$ and $M_{\sigma,Y}(L_2)$ parametrizing $E \in M_{\sigma,Y}(v)$ with $\det(E) = L_1$ (resp. $\det(E) = L_2$), where $L_1 - L_2 = K_Y$. $\square$

Now consider the general case, $v = mv_0$ where $m \in \mathbb{Z}_{>0}$ and $v_0 \in H^*_{\text{alg}}(Y, \mathbb{Z})$ is primitive with $v_0^2 \geq -1$. As a warm-up, we begin with the cases $v_0^2 = -1, 0$.

Lemma 8.2.2. *Assume $v_0^2 = -1$. Then for all $\sigma \in \mathrm{Stab}^\dagger(Y)$ generic with respect to v, $\mathfrak{M}_{\sigma,Y}(v)$ admits a projective coarse moduli space $M_{\sigma,Y}(v)$ consisting of $m+1$ points.*

Proof. The proof of [11, Lemma 7.1] shows that for $m = 1$ the stack $\mathfrak{M}_{\sigma',\tilde{Y}}(\pi^* v)$ is a $\mathbb{G}_m$-gerbe over a point representing a single object F_0, which must be spherical and fixed by ι^*. Thus it descends to an object E_0 in $M^s_{\sigma,Y}(v) = M_{\sigma,Y}(v)$. By Lemma 6.1.3 we must have $\mathrm{Hom}(E_0, E_0 \otimes \omega_Y)^\vee = \mathrm{Ext}^2(E_0, E_0) = 0$ since F_0 is stable. Thus $\mathfrak{M}_{\sigma,Y}(v)$ is smooth at E_0. It follows that $M_{\sigma,Y}(v)$ consists of two reduced points E_0 and $E_0 \otimes \omega_Y$.

If $m > 1$, then the argument in [11, Lemma 7.1] shows that every σ'-semistable object with Mukai vector $\pi^* v$ must be of the form $F_0^{\oplus m}$. We notice that

$$\mathrm{ext}^1(E_0, E_0) = \mathrm{ext}^1(E_0 \otimes \omega_Y, E_0 \otimes \omega_Y) = \mathrm{ext}^1(E_0, E_0 \otimes \omega_Y) = \mathrm{ext}^1(E_0 \otimes \omega_Y, E_0) = 0.$$

Indeed

$$\begin{aligned} -1 = v_0^2 &= (v(E_0), v(E_0 \otimes \omega_Y)) \\ &= \mathrm{ext}^1(E_0, E_0 \otimes \omega_Y) - \mathrm{hom}(E_0, E_0 \otimes \omega_Y) - \mathrm{hom}(E_0, E_0) \\ &= \mathrm{ext}^1(E_0, E_0 \otimes \omega_Y) - 1. \end{aligned}$$

By genericity of σ, all stable factors of an element of $M_{\sigma,Y}(v)$ must have Mukai vector $m' v_0$ for $m' < m$, so by induction we conclude that the only σ-semistable objects with Mukai vector v are precisely $E_0^{\oplus m}, E_0^{\oplus m-1} \oplus (E_0 \otimes \omega_Y), ..., E_0 \oplus (E_0 \otimes \omega_Y)^{\oplus m-1}, (E_0 \otimes \omega_Y)^{\oplus m}$. □

Lemma 8.2.3. *Assume that $v_0^2 = 0$. Let σ be generic with respect to v. Then:*

(a) for $m = 1$,

- *if $\pi^* v$ is primitive, then $M_{\sigma,Y}(v)$ is the disjoint union of two smooth irreducible elliptic curves.*
- *if $\pi^* v$ is divisible by 2, $M_{\sigma,Y}(v)$ is isomorphic to Y itself.*

(b) for $m > 1$,

- *if $\pi^* v$ is primitive, then a projective coarse moduli space $M_{\sigma,Y}(v)$ exists and*

$$M_{\sigma,Y}(v) \cong \coprod_{2m_1+m_2=m} \mathrm{Sym}^{m_1}(M^s_{\sigma,Y}(2v_0)) \times \mathrm{Sym}^{m_2}(M_{\sigma,Y}(v_0)).$$

- *if $\pi^* v$ is divisible by 2, then a projective coarse moduli space $M_{\sigma,Y}(v)$ exists and*

$$M_{\sigma,Y}(v) \cong \mathrm{Sym}^m(M_{\sigma,Y}(v_0)).$$

Proof. Part (a) follows from Corollary 8.2.1. We only need to observe that if the canonical divisor of curve is torsion, then it must be trivial so that both components of $M_{\sigma,Y}(v)$ are elliptic curves.

For the proof of (b), recall that $M_{\sigma',\tilde{Y}}(\pi^* v) \cong \mathrm{Sym}^m(M_{\sigma',\tilde{Y}}(\pi^* v_0))$ [11, Lemma 7.2(b)]. It follows that the stable locus $M^s_{\sigma',\tilde{Y}}(\pi^* v) = \varnothing$ since on the one hand it would be a dense open subset of $M_{\sigma',\tilde{Y}}(\pi^* v)$ which has dimension $2m$, and on the other hand it would also have to be smooth of dimension $(\pi^* m v_0)^2 + 2 = 2$, which is impossible for $m > 1$. Thus any $E \in M^s_{\sigma,Y}(v)$ would have to be in the exceptional case of Lemma 6.1.3, i.e. $E \cong E \otimes \omega_Y$ and $\pi^* E \cong F \oplus \iota^* F$ where F is stable of Mukai vector $\frac{m}{2}\pi^* v_0$ and $F \not\cong \iota^* F$. As noted above, for stable objects to exist on $\tilde{Y}$ we must have $m/2 = 1$, i.e. $m = 2$. As for the semistable locus, it follows from the genericity of σ that any stable factor of a semistable object $E \in M_{\sigma,Y}(v)$ must have Mukai vector $m' v_0$ for $m' < m$. Repeating the above argument inductively, we find that if $\pi^* v$ is primitive, a canonical representative of the S-equivalence class of an object $E \in M_{\sigma,Y}(v)$ is a direct sum of objects in $M^s_{\sigma,Y}(2v_0)$ and objects in $M_{\sigma,Y}(v_0)$. Thus the coarse moduli space parametrizing S-equivalence classes is

$$\coprod_{2m_1+m_2=m} \mathrm{Sym}^{m_1}(M^s_{\sigma,Y}(2v_0)) \times \mathrm{Sym}^{m_2}(M_{\sigma,Y}(v_0)).$$

Since the morphism π^* from $M_{\sigma,Y}(v)$ to $M_{\sigma',\tilde{Y}}(\pi^* v)$ is quasi-finite (as follows from the above decomposition) and proper (as the two Artin stacks themselves were proper), we find that the morphism of coarse moduli spaces is finite. Thus $M_{\sigma,Y}(v)$ is projective.

Now consider the second case in (b). As usual, from the genericity of σ it follows that any stable factors of an object in $M_{\sigma,Y}(v)$ must be of the form $m' v_0$ for $m' < m$.

By the arguments above, $M^s_{\sigma,Y}(v) = \varnothing$ for $m > 1$, so it follows that the S-equivalence classes of the objects in $M_{\sigma,Y}(v)$ are represented by $\mathrm{Sym}^m(M_{\sigma,Y}(v_0))$. $\square$

We can now generalize the above argument to show projectivity in general:

Theorem 8.2.4. *Let $v = mv_0$, $m > 0$, be a Mukai vector with v_0 primitive and $v_0^2 > 0$. Then for generic $\sigma \in \mathrm{Stab}^\dagger(Y)$, a projective coarse moduli space $M_{\sigma,Y}(v)$ exists.*

Proof. As usual we notice that the genericity of σ means that any stable factor of an object of $M_{\sigma,Y}(v)$ must have Mukai vector $m'v_0$ for $m' < m$, which implies that the strictly semistable locus is the image of the natural map

$$\mathrm{SSL} : \coprod_{m_1+m_2=m, m_i>0} M_{\sigma,Y}(m_1v_0) \times M_{\sigma,Y}(m_2v_0) \to M_{\sigma,Y}(v).$$

Now if for two strictly semistable objects E and E', π^*E and π^*E' are S-equivalent, then the stable factors coincide and appear with the same multiplicities in the graded object. But a stable factor of E (or E') remains stable after pull-back unless it is fixed under $-\otimes\omega_Y$. For such a stable factor S we have stable $Q \in \mathrm{D^b}(\tilde{Y})$ such that $S \cong \pi_*(Q)$, or equivalently $\pi^*S = Q \oplus \iota^*Q$ with $Q \not\cong \iota^*Q$. All of this implies that E and E' have the same stable factors that are fixed under $-\otimes\omega_Y$, with the same multiplicites, and their stable factors that are not invariant under $-\otimes\omega_Y$ can only differ by tensoring with ω_Y. Thus again the proper morphism

$$\pi^* : M_{\sigma,Y}(v) \to M_{\sigma',\tilde{Y}}(\pi^*v)$$

between coarse moduli spaces is quasi-finite and thus finite. Since the latter is projective by [11, Theorem 1.3], $M_{\sigma,Y}(v)$ must be projective as well. $\square$

v_0^2	$\pi^* v_0$	m	$M^s_{\sigma,Y}(v)$	$\dim M_{\sigma,Y}(v)$	$\operatorname{codim} M^{ss}_{\sigma,Y}(v)$
-1	–	1	$\neq \varnothing$	$0 = v^2 + 1$	∞
-1	–	> 1	$\varnothing$	$0 \neq v^2 + 1$	0
0	primitive	1	$\neq \varnothing$	$1 = v^2 + 1$	∞
0	primitive	2	$\neq \varnothing$	$2 \neq v^2 + 1$	0
0	primitive	> 2	$\varnothing$	$m \neq v^2 + 1$	0
0	non-primitive	1	$\neq \varnothing$	$2 \neq v^2 + 1$	∞
0	non-primitive	> 1	$\varnothing$	$2m \neq v^2 + 1$	0
1	–	$1, > 2$	$\neq \varnothing$	$v^2 + 1$	$\infty, > 1$
1	–	2	$\neq \varnothing$	$v^2 + 1$	1
> 1	–	$m \geq 1$	$\neq \varnothing$	$v^2 + 1$	> 1

Inspired by [10, Theorem 2.15], we can use the above technique to determine the dimension of $M_{\sigma,Y}(v)$ and of its semistable locus, as well as to ensure the existence of stable objects.

Theorem 8.2.5. *Suppose that Y is unnodal, and let $v = mv_0$ be a Mukai vector with v_0 primitive, and $m \in \mathbb{Z}_{>0}$. For $\sigma \in \operatorname{Stab}^\dagger(Y)$ generic with respect to v,*

(a) The coarse moduli space $M_{\sigma,Y}(v) \neq \varnothing$ if and only if $v_0^2 \geq -1$.

(b) The dimension and codimension of $M_{\sigma,Y}(v)$ and $M^{ss}_{\sigma,Y}(v)$, respectively, follow the above table.

Proof. If $v_0^2 \geq -1$, then part (a) follows from Theorem 6.2.12 above. For the converse, note that any stable factor of an element of $M_{\sigma,Y}(v) \neq \varnothing$ would have to have Mukai vector $m'v_0$ for $m' < m$ by genericity of σ. But then $m'^2 v_0^2 = (m'v_0)^2 \geq -1$, so $v_0^2 \geq -1$.

For (b), again notice that the genericity of σ means that any stable factor of an object of $M_{\sigma,Y}(v)$ must have Mukai vector $m'v_0$ for $m' < m$, which implies that the strictly semistable locus is the image of the natural map

$$\mathrm{SSL} : \coprod_{m_1+m_2=m, m_i>0} M_{\sigma,Y}(m_1 v_0) \times M_{\sigma,Y}(m_2 v_0) \to M_{\sigma,Y}(v).$$

Assume $v_0^2 > 0$. Then for $m = 1$, $M_{\sigma,Y}(v) = M^s_{\sigma,Y}(v)$, and we have seen already that $\dim M_{\sigma,Y}(v) = v^2 + 1$. If $m > 1$, then by the induction, we deduce that the image of the map SSL has dimension equal to the maximum of $(m_1^2 + m_2^2)v_0^2 + 2$ for $m_1 + m_2 = m, m_i > 0$.

We can construct a semistable object E' with Mukai vector v which is also Schur, i.e. $\mathrm{Hom}(E', E') = \mathbb{C}$. By the inductive assumption, we can consider $E \in M^s_{Y,\sigma}((m-1)v_0)$, and let $F \in M_{Y,\sigma}(v_0)$. Now $\chi(F, E) = -(v(F), v(E)) = -(m-1)v_0^2 < 0$, so $\mathrm{Ext}^1(F, E) \neq 0$. Take E' to be a nontrivial extension

$$0 \to E \to E' \to F \to 0.$$

Then any endomorphism of E' gives rise to a homomorphism $E \to F$, of which there are none since these are both stable of the same phase and have different Mukai vectors (or can be chosen to be non-isomorphic if $m = 2$). Thus any endomorphism of E' induces an endomorphism of E, and the kernel of this induced map $\mathrm{Hom}(E', E') \to \mathrm{Hom}(E, E) = \mathbb{C}$ is precisely $\mathrm{Hom}(F, E')$, which vanishes since the extension is non-trivial. Thus $\mathrm{Hom}(E', E') = \mathbb{C}$.

We can deduce non-emptiness of $M^s_{\sigma,Y}(v)$ from a dimension estimate as follows. Since E' is Schur, we get

$$v^2 + 1 \leq \dim_{E'} M_{\sigma,Y}(v) \leq \dim T_{E'} M_{\sigma,Y}(v) = v^2 + 1 + \hom(E', E' \otimes \omega_Y).$$

Notice that the strictly semistable locus must have dimension smaller than $v^2 + 1$. So even though E' is not stable, it lies on a component which must contain stable objects. Moreover, as we will see in the next section, (smooth) components of the stable locus of dimension greater than $v^2 + 1$ can occur only if $v_0^2 = 0$, so in the current situation the locus of points fixed by $- \otimes \omega_Y$ has positive codimension. Then we may choose $E \in M^s_{Y,\sigma}((m-1)v_0)$ such that $E \not\cong E \otimes \omega_Y$ and F such that $F \not\cong F \otimes \omega_Y$ (and such that $F \not\cong E \otimes \omega_Y$ if $m = 2$). Stability of E and F and a diagram chase then show that $\mathrm{Hom}(E', E' \otimes \omega_Y) = 0$, so $M_{\sigma,Y}(v)$ is smooth at E' of dimension $v^2 + 1$ as claimed.

Furthermore, observe that the strictly semistable locus has codimension

$$v^2 + 1 - (m_1^2 v_0^2 + m_2^2 v_0^2 + 2) = (m_1 + m_2)^2 v_0^2 + 1 - (m_1^2 v_0^2 + m_2^2 v_0^2 + 2) = 2m_1 m_2 v_0^2 - 1 \geq 2,$$

if $v_0^2 > 1$ or $m > 2$, hence part (c).

The cases with $v_0^2 \leq 0$ have already been covered in Lemmas 8.2.2 and 8.2.3. $\square$

Let us conclude this section by summarizing the consequences of the above theorem in the unnodal case.

Corollary 8.2.6. *Let Y be an unnodal Enriques surface, $v = mv_0 \in H^*_{\mathrm{alg}}(Y, \mathbb{Z})$ with $m \in \mathbb{Z}_{>0}$, $v_0^2 > 0$ and v_0 primitive. Furthermore, let $\sigma \in \mathrm{Stab}^{\dagger}(Y)$ be generic with respect to v. The projective variety $M_{\sigma,Y}(v)$ is normal of dimension $v^2 + 1$ with torsion $K_{M_{\sigma,Y}(v)}$ with possible exception when $v_0^2 = 1$ and $m = 2$.*

Proof. Except for the case $v_0^2 = 1$ and $m = 2$, the semistable locus has high codimension by Theorem 8.2.5, and by Theorem 6.4.1 and Remark 6.4.3, so does the fixed locus of $-\otimes \omega_Y$ on $M^s_{\sigma,Y}(v)$. Thus the class of $K_{M_{\sigma,Y}(v)}$ is determined by its restriction to $M^s_{\sigma,Y}(v)$, so the result follows from Theorem 6.4.2. $\square$

8.3 The nodal case

When Y is a nodal Enriques surface, $\mathrm{Stab}^{\dagger}(Y) \cong \Gamma_{\tilde{Y}} \cap \mathrm{Stab}^{\dagger}(\tilde{Y})$ is a proper submanifold of $\mathrm{Stab}^{\dagger}(\tilde{Y})$, and it is possible that an entire chamber for v is contained in a wall for $\pi^* v$ and the results of [11] no longer produce a projective coarse moduli space $M_{\sigma',\tilde{Y}}(\pi^* v)$. Nevertheless, we can still move forward in the majority of cases of a primitive Mukai vector:

Theorem 8.3.1. *Suppose that Y is a nodal Enriques surface, and let $v \in H^*_{\mathrm{alg}}(Y, \mathbb{Z})$ be a primitive Mukai vector such that $v^2 \geq -2$ and $\sigma \in \mathrm{Stab}^{\dagger}(Y)$ generic with respect to v. If $\pi^* v$ is primitive (i.e. $\gcd(r, c_1) = 1$), then a projective coarse moduli space $M_{\sigma,Y}(v)$ exists.*

Proof. By Theorem 6.2.12 $M_{\sigma,Y}(v)$ is nonempty. If σ' is in a chamber for $\pi^* v$, then the result follows as in Corollary 8.2.1. So we may assume that σ' is on a wall. In Theorem 6.4.1, we saw that $M_{\sigma,Y}(v) = M^s_{\sigma,Y}(v)$ is a proper algebraic space with singularities along the fixed locus of $-\otimes \omega_Y$, which is precisely the locus of σ-stable objects which pull back to strictly σ'-semistable objects. If we deform σ' to a sufficiently nearby stability condition σ'_+ in an adjacent chamber for π^*v, then for every $E \in M_{\sigma,Y}(v) - \mathrm{Sing}(M_{\sigma,Y}(v))$, π^*E is σ'_+-stable by openness of stability.

On the other hand, each irreducible component of $\mathrm{Sing}(M_{\sigma,Y}(v))$ is $\pi_*(M^s_{\sigma',\tilde{Y}}(w)^\circ)$ for some $w \in H^*_{\mathrm{alg}}(\tilde{Y},\mathbb{Z})$ such that $\pi_*(w) = v$ and parametrizes E such that $\pi^*E = F \oplus \iota^*F$ for $F \in M^s_{\sigma',\tilde{Y}}(w)$. As π^*v is primitive, so is w, and w cannot be invariant under ι^* so there are no objects in $M^s_{\sigma',\tilde{Y}}(w)$ invariant under ι^* (i.e. $M^s_{\sigma',\tilde{Y}}(w)^\circ = M^s_{\sigma',\tilde{Y}}(w)$). By openness of stability, it follows that both F and ι^*F remain σ'_+-stable, with say $\phi_{\sigma'_+}(F) < \phi_{\sigma'_+}(\iota^*F)$. Then by [58, Lemma 3.10],[9, Lemma 5.9], all non-trivial extensions

$$0 \to F \to \overline{E} \to \iota^*F \to 0$$

are σ'_+-stable with Mukai vector π^*v.

As π^*v is primitive, it follows from [11, Proposition 8.1] that some irreducible components of $\mathfrak{M}_{\sigma,Y}(v)$ admit normal projective coarse moduli spaces, even though σ' is not general, and one of these coarse moduli spaces is obtained by contracting all curves in $M_{\sigma'_+,\tilde{Y}}(\pi^*v)$ of objects that become S-equivalent with respect to σ'. The argument above shows that $\pi^* : \mathfrak{M}_{\sigma,Y}(v) \to \mathfrak{M}_{\sigma',\tilde{Y}}(\pi^*v)$ lands in this component. Then the usual argument from Corollary 8.2.1 shows that $M_{\sigma,Y}(v)$ is a projective variety. □

There are two non-primitive cases we can also deal with easily. The first follows directly from the preceeding result since coarse moduli spaces on $\tilde{Y}$ exist for π^*v and non-generic σ' in this case (known as O'Grady's example, see [48]):

Corollary 8.3.2. *Let $v = 2v_0$, where v_0 is a primitive Mukai vector on Y with $v_0^2 = 1$ and $\sigma \in \mathrm{Stab}^\dagger(Y)$ generic with respect to v. Then a projective coarse moduli space $M_{\sigma,Y}(v)$ exists.*

The second case concerns direct sums of spherical objects:

Proposition 8.3.3. *Let Y be a nodal Enriques surface and v_0 primitive with $v_0^2 = -2$ satisfying the condition of Theorem 4.6.4. Then for $m \in \mathbb{Z}_{>0}$, $v = mv_0$, and σ a generic stability condition with respect to v, $\mathfrak{M}_{\sigma,Y}(v)$ admits a coarse moduli space $M_{\sigma,Y}(v)$ consisting of a single point.*

Proof. Theorem 6.2.12 shows that $\mathfrak{M}_{\sigma,Y}(v_0) = \mathfrak{M}^s_{\sigma,Y}(v_0) \neq \varnothing$ is a $\mathbb{G}_m$-gerbe over a point. Indeed, we know there is some object E_0, and if there were another E, then $\chi(E_0, E) = -v_0^2 = 2$, so we'd have to have $\hom(E_0, E) \geq 1$ or $\hom(E, E_0) \geq 1$. But this forces $E \cong E_0$ by stability. Then E_0 is spherical and has no self-extensions. Moreover, if $m > 1$ then $v_0^2 < -2$, so there can be no stable objects. Thus $M_{\sigma,Y}(v) = \{E_0^{\oplus m}\}$. $\square$

Chapter 9

Bridgeland wall-crossing and birational geometry

Now that we have established the existence of the projective coarse moduli spaces $M_{\sigma,Y}(v)$ for stability conditions $\sigma \in \mathcal{C} \subset \mathrm{Stab}^\dagger(Y)$, a chamber with respect to v, we explore in this chapter what happens when we let σ approach a wall of $\mathcal{C}$ and the relationship between Bridgeland wall-crossing and the birational geometry of $M_{\sigma,Y}(v)$. We introduce our main tool, the Bayer-Macrì map, in the first section and show that crossing certain walls induces a flop.

9.1 A Natural Nef Divisor

To investigate the connection between wall-crossing in $\mathrm{Stab}^\dagger(Y)$ and the birational geometry of $M_{\sigma,Y}(v)$, we will make use of a construction due to Bayer and Macrì [11]. By Remark 6.2.3, we may assume that σ is algebraic and $Z(v) = -1$. Then given a *quasi-universal* family on $M_{\sigma,Y}(v)$ (see [50, 28, 11] for definitions and details), the natural numerical divisor $\ell_{\sigma,\mathcal{E}}$ is defined by

$$\ell_\sigma.C := \Im Z(p_{Y,*}(\mathcal{E} \otimes p^*_{M_{\sigma,Y}(v)}\mathcal{O}_C))$$

for every projective curve $C \subset M_{\sigma,Y}(v)$. By [11, Proposition and Definition 3.2] this association gives a well-defined nef divisor class. As quasi-universal families are unique up to a certain notion of equivalence, for a quasi-universal family $\mathcal{E}$ of similitude ρ (i.e. for any closed point $s = [E] \in M_{\sigma,Y}(v)$, $\mathcal{E}_s \cong E^\rho$) we define $\ell_\sigma := \frac{1}{\rho}\ell_{\sigma,\mathcal{E}}$, which removes the dependence on the specific quasi-universal family [28, Lemma 8.1.2]. The usual techniques (see for example [50, Theorem A.5] or [28, Section 4.6]) show that a quasi-universal family always exists on $M^s_{\sigma,X}(v)$ and is unique up to equivalence. In particular, if σ is generic and v primitive, then we get a well-defined nef divisor class on $M_{\sigma,X}(v)$. The most important property of ℓ_σ is the following positivity result:

Lemma 9.1.1 (Theorem 4.1, [11]). *$\ell_{\sigma,\mathcal{E}}.C > 0$ if and only if for two general closed points $c, c' \in C$, the corresponding objects $\mathcal{E}_c, \mathcal{E}_{c'} \in \mathrm{D}^{\mathrm{b}}(X)$ are not S-equivalent.*

To go further in the case of Enriques surfaces, we must note the following general result relating this divisor class to the pull-back of the corresponding divisor class on the inducing variety:

Proposition 9.1.2. *Suppose G acts fixed-point-freely on a smooth projective variety X with $Y = X/G$ and projection π. Set $\sigma = (\pi^*)^{-1}(\sigma')$ with corresponding pull-back morphism of stacks*

$$\pi^* : \mathfrak{M}_{\sigma,Y}(v) \to \mathfrak{M}_{\sigma',X}(\pi^* v).$$

Then

$$(\pi^*)^* \ell_{\sigma'} = \ell_\sigma.$$

Proof. From the definition, it suffices to check that

$$(\pi^*)^* \ell_{\sigma'}.C = \ell_\sigma.C$$

for every projective curve C with a morphism $C \to \mathfrak{M}_{\sigma,Y}(v)$. For such a curve C we get a universal object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(C \times Y)$, and the universal object corresponding to the composition with π^* is $\mathcal{F} = (1 \times \pi)^*(\mathcal{E}) \in \mathrm{D}^{\mathrm{b}}(C \times X)$. Consider the following commutative diagram

$$\begin{array}{ccccc}
C & \xleftarrow{p_C} & C \times X & \xrightarrow{p_X} & X \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1\times\pi} & & \downarrow{\scriptstyle \pi} \\
C & \xleftarrow{p_C} & C \times Y & \xrightarrow{p_Y} & Y
\end{array}$$

of schemes.

Then we have

$$\begin{aligned}(\pi^*)^*\ell_{\sigma'}.C &= \ell_{\sigma'}.(\pi^*)_*(C) = \Im Z_{\sigma'}(\Phi_{(1\times\pi)^*\mathcal{E}}(\mathcal{O}_C)) = \Im Z_{\sigma'}((p_X)_*(\mathcal{F}\otimes(p_C)^*(\mathcal{O}_C)))\\
&= \Im Z_{\sigma'}((p_X)_*(\mathcal{F}\otimes(1\times\pi)^*(p_C)^*(\mathcal{O}_C))))\\
&= \Im Z_{\sigma'}((p_X)_*((1\times\pi)^*(\mathcal{E}\otimes(p_C)^*(\mathcal{O}_C))))\\
&= \Im Z_{\sigma'}(\pi^*((p_Y)_*(\mathcal{E}\otimes(p_C)^*\mathcal{O}_C))))\\
&= \Im Z_\sigma(\Phi_{\mathcal{E}}(\mathcal{O}_C)) = \ell_\sigma.C,\end{aligned}$$

as required. □

Finally, we use this proposition to deduce the ampleness of ℓ_σ for σ in the interior of a chamber:

Theorem 9.1.3. *Let $v = mv_0 \in H^*_{\mathrm{alg}}(Y,\mathbb{Z})$, and $\sigma \in \mathrm{Stab}^\dagger(Y)$ generic with respect to v, where v_0 is primitive, $v_0^2 > 0$. If Y is unnodal and $m \in \mathbb{Z}_{>0}$ or Y is nodal and either $m = 1, 2 \nmid \pi^*v$ or $m = 2, v_0^2 = 1$, then ℓ_σ is ample on the projective variety $M_{\sigma,Y}(v)$.*

Proof. In case Y is unnodal, then we may assume σ' is generic as well, so by [11, Corollary 7.5] and the discussion after it, $\ell_{\sigma'}$ is ample on $M_{\sigma',\tilde{Y}}(\pi^*v)$. In the cases considered for Y nodal, the same is true for $\ell_{\sigma'}$ on $M_{\sigma',\tilde{Y}}(\pi^*v)$ by [11, Proposition 8.1] and the discussion preceding it. By Theorem 8.2.4 the morphism π^* of Proposition 9.1.2 is a finite morphism, and thus $\ell_\sigma = (\pi^*)^*\ell_{\sigma'}$ is ample. □

9.2 Flops via Wall-Crossing

For the rest of the section, assume that v is a primitive Mukai vector v with $v^2 \geq 1$ (with the assumption $2 \nmid \pi^*v$ in case Y is nodal). With the above preparations, we can now investigate the birational operation induced by crossing a wall W in $\mathrm{Stab}^\dagger(Y)$ associated to v.

Let $\sigma_0 = (W_0, \mathcal{A}_0) \in W$ be a generic point on the wall. Let $\sigma_+ = (Z_+, \mathcal{A}_+), \sigma_- = (Z_-, \mathcal{A}_-)$ be two algebraic stability conditions in the two adjacent chambers on each side of W. From Section 6.4, the two moduli spaces $M_\pm := M_{\sigma_\pm,Y}(v)$ are non-empty numerically K-trivial normal projective varieties, smooth outside of codimension two, and parametrize only stable objects. Choosing (quasi-)universal families $\mathcal{E}_\pm$ on $M_\pm$ of $\sigma_\pm$-stable objects, we obtain (quasi-)families of σ_0-semistable objects from the closedness of semistability. By [11, Theorem 4.1], these two families give two nef divisor classes $\ell_{0,\pm} := \ell_{\sigma_0,\mathcal{E}_\pm}$ on $M_\pm$.

Following [11], we enumerate four different possible phenomena at the wall W depending on the codimension of the locus of strictly σ_0-semistable objects and the existence of curves $C \subset M_\pm$ with $\ell_{0,\pm}.C = 0$, i.e. curves parametrizing S-equivalent objects. We call the wall W

(a) a *fake wall* if there are no curves in $M_\pm$ of objects that are S-equivalent to each other with respect to σ_0,

(b) a *totally semistable wall*, if $M^s_{\sigma_0}(v) = \varnothing$,

(c) a *flopping wall*, if W is not a fake wall and $M^s_{\sigma_0}(v) \subset M_\pm$ has complement of codimension at least two,

(d) a *bouncing wall*, if there is an isomorphism $M_+ \cong M_-$ that maps $\ell_{0,+}$ to $\ell_{0,-}$, and there are divisors $D_\pm \subset M_\pm$ that are covered by curves of objects that are S-equivalent to each other with respect to σ_0.

We may assume that σ_0 is algebraic, $W_0(v) = -1$, and $\phi = 1$. Then $\ell_{0,\pm}$ is the pull-back by the finite-morphism π^* of a semi-ample divisor [11, Section 8]. Thus $\ell_{0,\pm}$ is itself semi-ample (as are its restrictions to each irreducible component of $M_\pm$).

We thus get induced contraction morphisms [42, Theorem 2.1.27]

$$\pi_{\sigma_\pm} : M_\pm \to Z_\pm,$$

where $Z_\pm$ are normal projective varieties. We denote the induced ample divisor class on $Z_\pm$ by ℓ_0, i.e., the ample divisor pulling back to $\ell_{0,\pm}$. If $M^s_{\sigma_0,Y}(v) \neq \varnothing$, then by the openness of stability for primitive Mukai vectors [15, Proposition 9.4] these objects remain $\sigma_\pm$-stable, and we denote by $f_{\sigma_0} : M_+ \dashrightarrow M_-$ the induced birational map.

As observed in [11], $\pi_{\sigma_\pm}$ is an isomorphism if and only if the wall W is a fake wall, a divisorial contraction if W is a bouncing wall, and a flopping contraction if W is a flopping wall.

The proof of [11, Proposition 8.1] carries through unchanged to yield the following result which shows that $Z_\pm$ is a union of components of a coarse moduli space of σ_0-semistable objects up to a finite cover:

Proposition 9.2.1. *The space $Z_\pm$ has the following universal property: For any proper irreducible scheme $S \in \mathrm{Sch}_\mathbb{C}$, and for any family $\mathcal{E} \in \mathfrak{M}_{\sigma_0,Y}(v)(S)$ such that there exists a closed point $s \in S$ such that $\mathcal{E}_s \in \mathfrak{M}_{\sigma_\pm,Y}(v)(\mathbb{C})$, there exists a finite morphism $q : T \to S$ and a natural morphism $f_{q^*\mathcal{E}} : T \to Z_\pm$.*

Now, as $K_{M_\pm}$ is torsion, the existence of a σ_0-stable object of class v inducing the birational map f_{σ_0} can be extended to an isomorphism away from a locus of codimension at least 2 (see for example [40, Proposition 3.52(2)]). The proof of [11, Lemma 10.10] then carries through unchanged to give the following identification of $\ell_{0,+}$ with $\ell_{0,-}$:

Lemma 9.2.2. *Assume that there exists a σ_0-stable object of class v and identify the Néron-Severi groups of $M_\pm(v)$ by extending the common open subset $M^s_{\sigma_0,Y}(v)$ to an isomorphism outside of codimension two. Under this identification, $\ell_{0,+} = \ell_{0,-}$.*

Finally, we have enough preparation to prove our main result about the relationship between wall-crossing and birational geometry:

Theorem 9.2.3. *(a) The divisor classes $\ell_{0,\pm}$ are semiample (and remain so when restricted to each component of $M_\pm$), and they induce contraction morphisms*

$$\pi_\pm : M_\pm \to Z_\pm,$$

where $Z_\pm$ are normal projective varieties.

(b) Suppose that $M^s_{\sigma_0,Y}(v) \neq \varnothing$.

- *If either $\ell_{0,\pm}$ is ample, then the other is ample, and the birational map*

$$f_{\sigma_0} : M_+ \dashrightarrow M_-$$

obtained by crossing the wall in σ_0 extends to an isomorphism.

- *If $\ell_{0,\pm}$ are not ample and the complement of $M^s_{\sigma_0,Y}(v)$ has codimension at least 2, then $f_{\sigma_0} : M_+ \dashrightarrow M_-$ is the flop induced by $\ell_{0,+}$. More precisely, we have a commutative diagram of birational maps*

$$\begin{tikzcd} M_{\sigma_+,Y}(v) \arrow[rr, dashed, "f_{\sigma_0}"] \arrow[dr, "\pi_+"'] & & M_{\sigma_-,Y}(v) \arrow[dl, "\pi_-"] \\ & Z_+ = Z_- & \end{tikzcd},$$

and $f^*_{\sigma_0}\ell_{0,-} = \ell_{0,+}$.

Proof. It only remains to prove part (b). The first case of (b) follows from Lemma 9.2.2 and the theorem of Matsusaka and Mumford [41, Exercise 5.6].

For the second case, note that by the discussion preceeding Lemma 9.2.2, we need not specify in which moduli space we assume the complement of the σ_0-stable locus to have codimension 2. From this codimension condition and projectivity of these moduli spaces, it follows that numerical divisor classes are determined by their intersection numbers with curves contained in $M^s_{\sigma_0,Y}(v)$. By Lemma 9.2.2 we have $f^*_{\sigma_0}\ell_{0,-} = \ell_{0,+}$. Since $\ell_{0,+}$ is not ample, f_{σ_0} does not extend to an isomorphism. The identification of $\ell_{0,\pm}$ and the codimension condition force $Z_+ = Z_-$ from their construction in [42, Proposition 2.1.27]. This gives the claimed commutativity of the diagram and thus the description of the birational map as a flop. □

Remark 9.2.4. We believe the semiample divisors $\ell_{0,\pm}$ are big as well. If they weren't, then every irreducible component of $M_\pm$ would fiber over a component of $Z_\pm$ with positive dimensional fibers, i.e. $\dim Z_\pm < \dim M_\pm$. From the description of semistable objects as extensions of their stable factors, we suspect that it would follow that $M_\pm$ are then covered by a family of rational curves contracted by $\pi_\pm$. By [18, Remark 4.2 (4)], $M_{\sigma_\pm}$ would then be uniruled, which is impossible it has vanishing Kodaira dimension [18, Corollary 4.12]. This applies to each component, so the same argument shows that the restriction to each component is big as well.

Chapter 10

Applications

We demonstrate in this chapter some of the applications of the machinery we have developed to other areas of algebraic geometry not directly related to derived categories. We begin with describing an explicit region of the ample cone of moduli of stable sheaves, and then move on to explicitly and completely describe the nef cone of the Hilbert scheme of points on an unnodal Enriques surface. We apply these results in the last section to prove a vanishing theorem useful for studying linear sustems on Enriques surfaces.

10.1 Moduli of stable sheaves

We now use the above work to set up the investigation of the birational geometry of the classical moduli spaces of sheaves on an Enriques surface. We define the *Mukai homomorphism* $\theta_v : v^\perp \to N^1(M_{\sigma,Y}(v))$ by

$$\theta_v(w).C := \frac{1}{\rho}(w, \Phi_{\mathcal{E}}(\mathcal{O}_C)), \text{ for every projective integral curve } C \subset M_{\sigma,Y}(v),$$

where $\mathcal{E}$ is a quasi-universal family of similitude ρ and $\Phi_{\mathcal{E}}$ is the associated Fourier-Mukai transform. We relate θ_v and ℓ_σ explicitly in the following result, whose proof is identical to [11, Lemma 9.2]:

Lemma 10.1.1. *Let Y be an Enriques surface, $v = (r, c, s)$ a primitive Mukai vector with $v^2 \geq -1$, and let $\sigma = \sigma_{\omega,\beta} \in \mathrm{Stab}^\dagger(Y)$ be a generic stability condition with respect to v. Then the divisor class $\ell_\sigma \in N^1(M_{\sigma,Y}(v))$ is a positive multiple of $\theta_v(w_{\sigma_{\omega,\beta}})$, where $w_{\sigma_{\omega,\beta}} = (R_{\omega,\beta}, C_{\omega,\beta}, S_{\omega,\beta})$ is given by*

$$R_{\omega,\beta} = c.\omega - r\beta.\omega,$$

$$C_{\omega,\beta} = (s - \beta.c + r\frac{\beta^2 - \omega^2}{2})\omega + (c.\omega - r\beta.\omega)\beta, \ \ and$$
$$S_{\omega,\beta} = c.\omega\frac{\beta^2 - \omega^2}{2} + s\beta.\omega - (c.\beta)(\beta.\omega).$$

We give a bound now for the walls of the "Gieseker chamber" for any (primitive) Mukai vector v, i.e. the chamber for which Bridgeland stability of objects of class v is equivalent to β-twisted Gieseker stability. Fix a class $\beta \in \mathrm{NS}(Y)_{\mathbb{Q}}$, and let ω vary on a ray in the ample cone. Given v with positive rank and slope, we saw above that for $\omega \gg 0$ stable objects of class v are exactly the β-twisted Gieseker stable sheaves. Below we give explicit bounds for the Gieseker chamber that depend only on ω^2, β, and v.

Definition 10.1.2. Given divisor classes β and $\omega = tH$ with $H \in \mathrm{Pic}(Y)$ ample, and given a class $v = (r, c, s)$ with $v^2 \geq -1$, we write $(r, c_\beta, s_\beta) = e^{-\beta}(r, c, s)$ so that $c_\beta = c - r\beta, s_\beta = r\frac{\beta^2}{2} - c.\beta + s$. Define the β-twised slope and discrepancy of v with respect to ω by

$$\mu_{\omega,\beta}(v) = \frac{\omega.c_\beta}{r}, \text{ and } \delta_{\omega,\beta}(v) = -\frac{s_\beta}{r} + 1 + \frac{1}{2}\frac{\mu_{\omega,\beta}(v)^2}{\omega^2},$$

respectively.

Note that scaling ω rescales $\mu_{\omega,\beta}$ by the same factor while leaving $\delta_{\omega,\beta}$ invariant. Recall that a torsion-free coherent sheaf F is called β-twisted Gieseker stable with respect to ω if for every proper subsheaf $0 \neq G \subset F$ we have

$$\mu_{\omega,\beta}(G) \leq \mu_{\omega,\beta}(F) \text{ and } \delta_{\omega,\beta}(G) > \delta_{\omega,\beta}(F), \text{ if } \mu_{\omega,\beta}(G) = \mu_{\omega,\beta}(F).$$

This is an unravelling of the usual definition via reduced β-twisted Hilbert polynomials. It follows that the definition of twisted Gieseker stability only depends on H and not t. Also, notice that the Hodge Index theorem implies that $\omega^2(c_\beta)^2 \leq (\omega.c_\beta)^2$, so

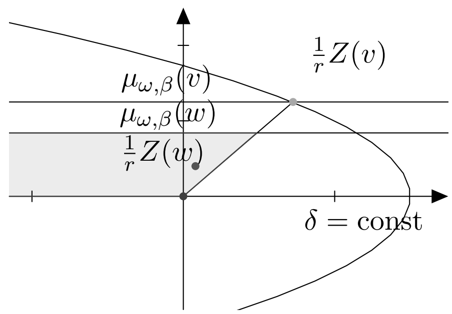
$$\delta_{\omega,\beta}(v) \geq -\frac{s_\beta}{r} + 1 + \frac{c_\beta^2}{2r^2} = \frac{v^2}{2r^2} + 1 \geq 1 - \frac{1}{2r^2} \geq \frac{1}{2} > 0, \tag{10.1}$$

where the second to last inequality follows from the assumption that $v^2 \geq -1$. Using this notation, we can write the central charge $Z_{\omega,\beta}(v)$ in terms of the slope and the discrepancy as in [11]:

$$\frac{1}{r}Z_{\omega,\beta}(v) = i\mu_{\omega,\beta}(v) + \frac{\omega^2}{2} - 1 - \frac{\mu_{\omega,\beta}(v)^2}{2\omega^2} + \delta_{\omega,\beta}(v). \tag{10.2}$$

For now, we fix a Mukai vector $v = (r, c, s)$ with $r > 0$ and $\mu_{\omega,\beta}(v) > 0$. We have the following lemma whose proof is the same as in [11, Lemma 9.10] and essentially follows from Figure 10.1 and equation (10.2):

Lemma 10.1.3. *Assume $\omega^2 > 1$ so that $\sigma_{\omega,\beta}$ is gauranteed to be a stability condition. Then any Mukai vector $w \in H^*_{\mathrm{alg}}(Y)$ with $r(w) > 0$ and $0 < \mu_{\omega,\beta}(w) < \mu_{\omega,\beta}(v)$ such that the phase of $Z_{\omega,\beta}(w)$ is bigger than or equal to the phase of $Z_{\omega,\beta}(v)$ satisfies $\delta_{\omega,\beta}(w) < \delta_{\omega,\beta}(v)$, as long as $\sigma_{\omega,\beta}$ is a stability condition.*



We define an analogue of the set defined in [11, Definition 9.11]. Let D_v be the subset of the lattice $H^*_{\mathrm{alg}}(Y, \mathbb{Z})$ defined by

$$\{w : 0 < r(w) \leq r(v), w^2 \geq -1, 0 < \mu_{\omega,\beta}(w) < \mu_{\omega,\beta}(v), \delta_{\omega,\beta}(w) < \delta_{\omega,\beta}(v)\}.$$

The discussion there extends to show that D_v is finite and depends on H but not t. Furthermore, they define

$$\mu^{\max}(v) := \max(\{\mu_{\omega,\beta}(w) : w \in D_v\} \cup \{\frac{r(v)}{r(v)+1}\mu_{\omega,\beta}(v)\}).$$

We can use this definition obtain an effective lower bound for the Gieseker chamber:

Lemma 10.1.4. *Let E be a β-twisted Gieseker-stable sheaf with $v(E) = v$. If*

$$\omega^2 > 1 + \frac{\mu^{\max}(v)}{\mu_{\omega,\beta} - \mu^{\max}(v)}\delta_{\omega,\beta}(v) + \sqrt{\left(1 + \frac{\mu^{\max}(v)}{\mu_{\omega,\beta} - \mu^{\max}(v)}\delta_{\omega,\beta}(v)\right)^2 - \mu^{\max}(v)\mu_{\omega,\beta}(v)}$$

then E is $Z_{\omega,\beta}$-stable.

Proof. Although the proof only differs slightly from that of [11, Lemma 9.13], we explain it in full.

Consider a destabilizing short exact sequence $A \hookrightarrow E \twoheadrightarrow B$ in $\mathcal{A}(\omega,\beta)$ with $\phi_{\omega,\beta}(A) \geq \phi_{\omega,\beta}(E)$. From the long exact sequence on cohomology, it follows that A is a sheaf. Consider the HN-filtration of A with respect to $\mu_{\omega,\beta}$-slope stability in $\operatorname{Coh} X$,

$$0 = \mathrm{HN}^0(A) \subset \mathrm{HN}^1(A) \subset \ldots \subset \mathrm{HN}^n(A) = A,$$

and let $A_i = \mathrm{HN}^i / \mathrm{HN}^{i-1}$ be its HN-filtration factors. From the definition of $\mathcal{A}(\omega,\beta)$ it follows that $\mu_{\omega,\beta}(A_i) > 0$ for all i. Since the kernel of $A \to E$, $\mathcal{H}^{-1}(B)$, lies in $\mathcal{F}(\omega,\beta)$, we see that $\mu_{\omega,\beta}(A_i) \leq \mu_{\omega,\beta}(A_1) \leq \mu_{\omega,\beta}(v)$. Indeed, if i is minimal such that $\mathrm{HN}^i(A)$ has nonzero image in E, then A_i admits a nontrivial morphism to E and thus $\mu_{\omega,\beta}(A_i) \leq \mu_{\omega,\beta}(E)$. Suppose $i > 1$, then $\mathrm{HN}^1(A) = A_1$ maps to zero in E and thus is contained in $\mathcal{H}^{-1}(B)$. If j is minimal such that $A_1 \subset \mathrm{HN}^j(\mathcal{H}^{-1}(B))$, then it has nonzero image in the j-th HN-filtration factor of $\mathcal{H}^{-1}(B)$, a contradiction since this has $\mu_{\omega,\beta} \leq 0$ from the definition of $\mathcal{F}(\omega,\beta)$.

Since $\phi_{\omega,\beta}(A) \geq \phi_{\omega,\beta}(E)$, we can choose some i such that $\phi_{\omega,\beta}(A_i) \geq \phi_{\omega,\beta}(E)$ by the see-saw property. We show that $\mu_{\omega,\beta}(A_i) < \mu_{\omega,\beta}(E)$. If not, then $\mu_{\omega,\beta}(A_i) = \mu_{\omega,\beta}(E)$, so $i = 1$. Consider the composition $A_1 \hookrightarrow A \to E$ with kernel K. Then if $K \neq 0$, $\mu_{\omega,\beta}(K) = \mu_{\omega,\beta}(E)$. But $K \subset \mathcal{H}^{-1}(B)$, so as above we get a contradiction to the fact that $\mathcal{H}^{-1}(B) \in \mathcal{F}(\omega,\beta)$, and thus $K = 0$. But then $\mu_{\omega,\beta}(A_1) = \mu_{\omega,\beta}(E)$ and $\phi_{\omega,\beta}(A_1) \geq \phi_{\omega,\beta}(E)$ imply that $\Re Z(A_1) \leq \Re Z(E)$, so from equation (10.2) we see that $\delta_{\omega,\beta}(A_1) \leq \delta_{\omega,\beta}(E)$, contradicting the β-twisted Gieseker stability of E.

Let w be the primitive generator of the positive ray spanned by $v(A_i)$. Then $\mu_{\omega,\beta}(w) = \mu_{\omega,\beta}(A_i)$, and from Lemma 10.1.3 and the definition of D_v it follows that if $r(w) \leq r(v)$ we have $\mu_{\omega,\beta}(w) \leq \mu^{\max}(v)$. If $r(w) \geq r(v)+1$, notice that

$$\omega.c_\beta(w) \leq \omega.c_\beta(A_i) = \Im Z(A_i) \leq \Im Z(A) \leq \Im Z(E) = \omega.c_\beta(v),$$

so $\mu_{\omega,\beta}(w) \leq \frac{r(v)}{r(v)+1}\mu_{\omega,\beta}(v) \leq \mu^{\max}(v)$ in this case as well.

Consider the complex number

$$z := i\mu^{\max}(v) + \frac{\omega^2}{2} - 1 - \frac{\mu^{\max}(v)^2}{2\omega^2}.$$

Then it follows that $\phi_{\omega,\beta}(w) \leq \phi(z)$ from Lemma 10.1.3 with $Z(v)$ replaced with z. Now $\Im \frac{\mu_{\omega,\beta}(v)}{\mu^{\max}(v)} z = \Im \frac{1}{r(v)} Z(v)$, and it is easy to see that $\Re \frac{\mu_{\omega,\beta}(v)}{\mu^{\max}(v)} z > \Re \frac{1}{r(v)} Z(v)$ for ω with ω^2 as in the hypothesis. Thus $\phi(z) < \phi_{\omega,\beta}(v)$. This gives the contradiction

$$\phi_{\omega,\beta}(E) \leq \phi_{\omega,\beta}(A_i) \leq \phi(z) < \phi_{\omega,\beta}(E).$$

□

Remark 10.1.5. As observed right before equation 10.2, $\delta_{\omega,\beta}(w) \geq \frac{1}{2}$, so we can in fact replace the complex number z in the proof above by

$$z := i\mu^{\max}(v) + \frac{\omega^2}{2} - 1 - \frac{\mu^{\max}(v)^2}{2\omega^2} + \frac{1}{2}$$

to obtain a sharper bound. We leave the necessary modifications to the reader as the bound above is usually an over estimate and in individual applications one can do better.

Nevertheless, the significance of this result is that it gives a lower bound on the t required to ensure that $M_{\sigma_{tH,\beta},Y}(v) \cong M_H^\beta(v)$, the moduli space of ($\beta$-twisted) Gieseker stable sheaves. Since we have an ample divisor on this moduli space, given by $\ell_{\sigma_{tH,\beta}}$, we get an explicit line segment of the ample cone. This argument gives us the

Corollary 10.1.6. *Let $v \in H^*_{\text{alg}}(Y, \mathbb{Z})$ be primitive of positive rank with $v^2 \geq -1$. Let $\omega, \beta \in \mathrm{NS}(Y)_{\mathbb{Q}}$ be generic with respect to v such that $\omega.c_\beta(v) > 0$. If*

$$\omega^2 > 1 + \frac{\mu^{\max}(v)}{\mu_{\omega,\beta} - \mu^{\max}(v)}\delta_{\omega,\beta}(v) + \sqrt{\left(1 + \frac{\mu^{\max}(v)}{\mu_{\omega,\beta} - \mu^{\max}(v)}\delta_{\omega,\beta}(v)\right)^2 - \mu^{\max}(v)\mu_{\omega,\beta}(v)},$$

then

$$\theta_v(w_{\omega,\beta}) \subset \mathrm{Amp}(M_H^\beta(v)).$$

Since all Enriques surfaces have the same lattice, this gives a universal bound for all Enriques surfaces.

10.2 Hilbert Schemes of points on an Enriques Surface

In this section we apply the techniques developed above to determine the nef cone of $Y^{[n]}$ for unnodal Y. Let $v = (1, 0, \frac{1}{2} - n)$. Then as above we consider an ample divisor

$H \in \operatorname{Pic}(Y)$ on an unnodal Enriques surface Y with K3 cover $\tilde{Y}$, and let $\omega = tH$ and $\beta \in \mathrm{NS}(Y)_{\mathbb{Q}}$ with $t > 0$. We remark that since Y is unnodal, so is $\tilde{Y}$, and thus the ample cone of Y is the connected component of the round cone $D^2 > 0$ containing an ample divisor. It follows that the nef cone (and consequently the effective cone since Y is unnodal) is the closure of this cone, given by $D^2 \geq 0$. For $t \gg 0$ and $\beta.H < 0$, $M_{t,\beta}(v) := M_{\sigma_{\omega,\beta},Y}(v) = Y^{[n]}$, the Hilbert scheme of n points on Y. It is well known that $Y^{[n]}$ is a smooth irreducible projective variety of dimension $2n$, and the Hilbert-Chow morphism $h : Y^{[n]} \to Y^{(n)}$ to the n-th symmetric product of Y is a crepant resolution of singularities [21], and since Y is a regular surface, i.e. $H^1(Y, \mathcal{O}_Y) = 0$, $\operatorname{Pic}(Y^{[n]}) \cong \operatorname{Pic}(Y) \times \mathbb{Z}$ [22, Corollary 6.3]. This identification can be described explicitly as follows: let $L^{(n)} = \psi_*(\otimes_{i=1}^n pr_i^*(L))^{S_n}$, where $\psi : Y^n \to Y^{(n)}$ is the quotient map and $pr_i : Y^n \to Y$ is the i-th projection. Then $\operatorname{Pic}(Y^{(n)}) \cong \operatorname{Pic}(Y)$ via this identification, and thus $\operatorname{Pic}(Y^{[n]})$ is generated by $h^* \operatorname{Pic}(Y^{(n)})$ and the divisor class B where $2B$ is the exceptional divisor parametrizing non-reduced 0-dimensional subschemes of length n in Y. For an ample divisor $H \in \operatorname{Pic}(Y)$ denote by $\tilde{H}$ the corresponding divisor on $Y^{[n]}$, which is nef and big but not ample, as the pull-back of an ample divisor under the projective birational morphism h. Thus $\widetilde{\operatorname{Nef}(Y)}$ forms an entire face of the nef cone of $Y^{[n]}$.

We'd like to apply the results and techniques of the preceeding sections to study the birational geometry of $Y^{[n]}$. We first have the following easy result (see [11, Example 9.1] for the corresponding discussion for K3 surfaces):

Proposition 10.2.1. *The (closure of the) wall consisting of stability conditions $\sigma_{tH,\beta}$, as H ranges in $\operatorname{Amp}(Y)$ and $\beta \in H^{\perp}$ for each fixed H, is one wall of the Gieseker chamber $\mathcal{C}_G$. Moreover, it is a bouncing wall sent by $\ell : \overline{\mathcal{C}}_G \to \operatorname{Nef}(Y^{[n]})$ to the wall $\widetilde{\operatorname{Nef}(Y)}$ above.*

Proof. Indeed, we have

$$Z_{tH,\beta}(I_Z) = (e^{\beta+itH}, 1 + (\frac{1}{2} - n)[pt]) = (n - \frac{1}{2}) + t^2\frac{H^2}{2} - \beta^2 \in \mathbb{R}_{>0},$$

for any 0-dimensional subscheme Z of length n since $\beta.H = 0$ implies $\beta^2 \leq 0$ by the Hodge Index Theorem, so $I_Z \notin \mathcal{A}(\omega, \beta)$ but $I_Z[1]$ is. We have the following short exact sequence in $\mathcal{P}(1)$,

$$0 \to \mathcal{O}_Z \to I_Z[1] \to \mathcal{O}_Y[1] \to 0$$

which makes I_Y strictly semistable. Notice that by filtering $\mathcal{O}_Z$ by structure sheaves of closed points, we see that I_Z is S-equivalent to $\mathcal{O}_Y[1] \oplus \bigoplus_{i=1}^r (k(p_i))^{\oplus m_i}$, where p_i are the closed points appearing in the support of Z with multiplicities m_i. It follows that I_Z and $I_{Z'}$ are S-equivalent if and only if Z, Z' get mapped to the same point by h. Thus $\ell_{tH,\beta}$ contracts precisely the fibers of $h : Y^{[n]} \to Y^{(n)}$. It follows that $\ell_{tH,\beta} = h^*A$ for some ample divisor A on $Y^{(n)}$. That $\ell_{tH,\beta}$ is in fact $\tilde{H}$ (or at least $\sim_{\mathbb{R}^+} \tilde{H}$) follows from [28, Examples 8.2.1 and 8.2.9] and Lemma 10.1.1.

Crossing this wall, i.e. taking β such $1 \gg \beta.H > 0$, does not change the moduli space, i.e. $M_{t,\beta}(v) \cong Y^{[n]}$, but it does change the universal family by replacing I_Z with its derived dual $\mathbf{R}\mathcal{H}om(I_Z, \mathcal{O}_Y)[1]$ (see [46, Theorem 3.1, Lemma 3.2]). Following a path from a point such that $\beta.H < 0$ to one with $\beta.H > 0$ causes the nef divisor $\ell_{tH,\beta}$ to hit the wall $\widetilde{\mathrm{Nef}(Y)}$ and bounce back into the interior of the ample cone. ☐

The above behavior is common for a wall inducing a divisorial contraction, hence the name "bouncing wall." Before we describe further wall-crossing behavior, let us first point out a simple fact that is very helpful:

Lemma 10.2.2. *Let $0 \to E \to I_Z \to Q \to 0$ be a non-trivial short exact sequence in $\mathcal{A}_{t,b}$. Then E is a forsion free sheaf, $\mathcal{H}^0(Q)$ is a quotient of I_Z of rank 0, and the kernel of $I_Z \to \mathcal{H}^0(Q)$ is an ideal sheaf $I_{Z'}(-D)$ for some effective curve D and some zero-dimensional scheme Z'.*

Proof. As above, we consider the long exact sequence in cohomology to see that E must be a sheaf fitting into the exact sequence

$$0 \to \mathcal{H}^{-1}(Q) \to E \to I_Z \to \mathcal{H}^0(Q) \to 0.$$

If $\mathcal{H}^0(Q)$ had rank one, then it would have to be equal to I_Z, as it's torsion-free, and then we'd get that $\mathcal{H}^{-1}(Q) = E$, which is only possible if they are both 0, since $\mathcal{H}^{-1}(Q) \in \mathcal{F}(\omega, \beta)$ while $E \in \mathcal{T}(\omega, \beta)$, contrary to the assumption of non-triviality. Thus $\mathcal{H}^0(Q)$ is a quotient of I_Z of rank 0, so its kernel must be of the form claimed in the lemma. Since $I_{Z'}(-D)$ and $\mathcal{H}^{-1}(Q)$ are both torsion-free, E must also be torsion-free. ☐

Above we fixed t and H and varied β across the hyperplane $H^\perp$ in $N^1(Y)$. Now fix H with $H^2 = 2d$ and $k \in \mathbb{Z}_{>0}$ such that $k^2 \leq 2d$. To simplify matters we consider stability conditions $\sigma_{t,b} := \sigma_{tH,bH}$ in the real 2-dimensional slice of $\mathrm{Stab}^\dagger(Y)$ represented by the upper half-plane $\{(b,t)|b \in \mathbb{R}, t \in \mathbb{R}_{>0}\}$. It is well-known (see [44, Section 2]) that pseudo-walls corresponding to possibly destabilizing subobjects intersect this plane in nested semi-circles with centers along the b-axis. Recall that on an Enriques surface Y one defines for any $D \in \mathrm{Pic}(Y)$ with $D^2 > 0$,

$$\phi(D) = \inf\{|D.F| : F \in \mathrm{Pic}(Y), F^2 = 0, F \neq 0\},$$

as in [17, Section 2.7], where it is shown that $\phi(D) \leq \sqrt{D^2}$. Now we are ready to prove our main theorem about $\mathrm{Nef}(Y^{[n]})$:

Theorem 10.2.3. *Let Y be an unnodal Enriques surface and $n \geq 2$. Then $\tilde{D} - aB \in \mathrm{Nef}(Y^{[n]})$ if and only if $D \in \mathrm{Nef}(Y)$ and $0 \leq na \leq D.F$ for every $0 < F \in \mathrm{Pic}(Y)$ with $F^2 = 0$, or in other words $0 \leq a \leq \frac{\phi(D)}{n}$. Moreover, the face given by $a = 0$ induces the Hilbert-Chow morphism, and for every ample $H \in \mathrm{Pic}(Y)$, $\tilde{H} - \frac{\phi(H)}{n}B$ induces a flop.*

Proof. Consider $0 < F \in \mathrm{Pic}(Y)$ with $F^2 = 0$ and $H.F = k$, so in particular $k \geq \phi(H)$. Set $b = -\frac{k+\epsilon}{2d}$ for $0 < \epsilon \ll 1$ and irrational so that there exists no exceptional or spherical object S such that $\Im Z_{t,b}(S) = 0$. It follows that $\sigma_{t,b}$ is a stability condition for all $t > 0$. By considering the equation of the pseudo-wall corresponding to $\mathcal{O}(-F)$, we see that I_Z and $\mathcal{O}(-F)$ have the same phase for $Z_{t_0,b}$ where $t_0 := \frac{1}{2d}\sqrt{2d-k^2}+O(\epsilon)$. Moreover, $\phi_{t,b}(\mathcal{O}(-F)) < \phi_{t,b}(I_Z)$ for $t > t_0$.

We claim that any I_Z is stable for $t > t_0$. As usual, consider a destabilizing subobject E as in Lemma 10.2.2, and as in the proof of Lemma 10.1.4, we consider the semistable factors appearing in its HN-filtration with respect to $\mu_{t,b}$-slope stability. Take one of them, say A_i, with $v(A_i) = (r, C, s)$ and $r > 0$. Then

$$0 < t(H.C + r(k+\epsilon)) = \Im Z_{t,b}(A_i) < \Im Z_{t,b}(I_Z) = t(k+\epsilon),$$

from which it follows that

$$-rk < H.C < -rk + k,$$

and thus

$$0 < H.C + rk < k.$$

The stable factors of A_i have $v^2 \geq -1$ and rank at least one. Thus $\delta_{t,b}(A_i) \geq \frac{1}{2}$ from (10.1). From (10.2) we see that

$$\Re Z_{t,b}(A_i) \geq rdt^2 - \frac{r}{2} - \frac{k^2}{4dr} + O(\epsilon),$$

so that

$$\phi_{t,b}(A_i) < \frac{t(k + r\epsilon)}{rdt^2 - \frac{r}{2} - \frac{k^2}{4dr} + O(\epsilon)} =: \phi_0(t).$$

One can easily check that $\phi_0(t) < \phi_{t,b}(I_Z)$ for $t > t_0$.

Thus we are reduced to considering the objects $\mathcal{O}(-F)$ for effective $F \in \operatorname{Pic}(Y)$ with $F^2 = 0$ and $H.F = k$ and $k \geq \phi(H)$. The largest such value of t_0 occurs for k minimal, i.e. $k = \phi(H)$, so we assume this to be the case. For those Z admitting a morphism $\mathcal{O}(-F) \to I_Z$, we see that the exact sequence

$$0 \to \mathcal{O}(-F) \to I_Z \to \mathcal{O}_F(-Z) \to 0,$$

destabilizes I_Z at t_0. The locus of such Z is isomorphic to $F^{[n]}$ of dimension n and can be described as the Brill-Noether locus consisting of those Z such that $h^0(I_Z(F)) > 0$ (and thus necessarily equal to 1). We get another (disconnected) component of the strictly semistable locus by considering the adjoint half-pencil $F + K_Y$ (i.e. the other double fiber of the elliptic fibration induced by $|2F|$) and those Z with $h^0(I_Z(F + K_Y)) \neq 0$ which get destabilized by the corresponding exact sequence

$$0 \to \mathcal{O}(-F - K_Y) \to I_Z \to \mathcal{O}_{F+K_Y}(-Z) \to 0.$$

Similarly, for any of the other finitely many half-pencils F' such that $H.F' = k$ we get two additional components of the strictly semistable locus which must necessarily intersect the above components. Indeed, for two half-pencils F and F' with $H.F = H.F' = k$, we see that $-2F.F' = (F - F')^2 \leq 0$, and we get strict inequality unless

$F' = F$ or $F + K_Y$. Choosing the n points to lie on this non-empty intersection $F \cap F'$ (with multiplicities if necessary), we see that the corresponding Brill-Noether loci intersect.

To see what the contracted curves are, we first observe that $v(\mathcal{O}_F(-Z)) = (0, F, -n)$ is primitive since F is, and one can show that $\mathcal{O}_F(-Z)$ is stable. Line bundles are always stable on an unnodal surface by [4, Theorem 5.4 and Proposition 6.3], so $\mathcal{O}(-F)$ is also stable in our case. Then it follows that for $Z, Z' \in F^{[n]}$, I_Z and $I_{Z'}$ are S-equivalent if and only if $\mathcal{O}_F(-Z) \cong \mathcal{O}_F(-Z')$, i.e. Z and Z' are linearly equivalent divisors on F. Thus π_+ in Theorem 9.2.3 contracts precisely the fibers of the classical Abel-Jacobi morphism $F^{[n]} \to \mathrm{Jac}^n(F) \cong F$ (since F is a smooth elliptic curve) which associates to an effective divisor of degree n on F its associated line bundle. Crossing the wall then induces a flop with exceptional locus of codimension n. As $\mathcal{O}(-F)$ and $\mathcal{O}_F(-Z)$ are stable of the same phase along this wall, we see that

$$\mathrm{Ext}^1(\mathcal{O}(-F), \mathcal{O}_F(-Z)) = ((1, -F, \frac{1}{2}), (0, F, -n)) = n,$$

so by [46, Section 4] it follows that the objects I_Z with $Z \in F^{[n]}$ are replaced by non-trivial extensions

$$0 \to \mathcal{O}_F(-Z) \to E \to \mathcal{O}(-F) \to 0$$

after crossing the wall.

Now we can plug $\omega = t_0 H, \beta = -\frac{k+\epsilon}{2d}, r = 1, c = 0, s = \frac{1}{2} - n$ into the formulas from Lemma 10.1.1, and letting $\epsilon \to 0$ we see that $w_{t_0 \cdot H, -1/2d} \sim_{\mathbb{R}_+} (1, -nH, n - \frac{1}{2})$. As $\theta_v(1, 0, n - \frac{1}{2}) = -B$ and $\theta_v(0, -H, 0) = \tilde{H}$, we get $\theta_v(w_{t_0 \cdot H, -1/2d}) \sim_{\mathbb{R}_+} \tilde{H} - \frac{\phi(H)}{n} B$ and $\theta_v(w_{\infty \cdot H, -1/2d}) \sim_{\mathbb{R}_+} \tilde{H}$. From the above discussion we see that both of these are extremal in the nef cone, with the first ray corresponding to a flop and the second ray to the Hilbert-Chow morphism.

The statement of the theorem follows from the above discussion and the density of rational rays in $N^1(Y^{[n]})$. $\square$

Remark 10.2.4. One direction of the above theorem is more elementary and can be obtained directly as follows. For any effective curve F, fix $p_1, ..., p_{n-1}$ distinct points not on F and consider the curve

$$C_F(n) = \{Z \in Y^{[n]} | Z = \{p_1, ..., p_{n-1}, p\}\} \subset Y^{[n]},$$

where the point p varies along F. Then $\tilde{D}.C_F(n) = D.F$ and $B.C_F(n) = 0$, so $\tilde{D} - aB \in \mathrm{Nef}(Y^{[n]})$ implies D is nef by pairing with $C_F(n)$ for all effective F. Denote by $C(n)$ the generic fiber of the Hilbert-Chow morphism. Then $\tilde{D}.C(n) = 0$ and $B.C(n) = -1$, so we see that $\tilde{D} - aB \in \mathrm{Nef}(Y^{[n]})$ implies $a \geq 0$. Finally, for any half-pencil F, consider a pencil of degree n effective divisors on F and the corresponding $g^1_n : F \to \mathbb{P}^1$. ${g^1_n}^* \mathcal{O}_{\mathbb{P}^1}(-x)$ for $x \in \mathbb{P}^1$ gives a curve $R_F(n)$ on $Y^{[n]}$ consisting of those objects sitting in short exact sequences

$$0 \to \mathcal{O}(-F) \to I_Z \to {g^1_n}^*(-x) \to 0.$$

Riemann-Hurwitz gives that the ramification divisor of g^1_n has degree $2n$. This is precisely $2B.R_F(n)$, and $\tilde{D}.R_F(n) = D.F$. Then $\tilde{D} - aB \in \mathrm{Nef}(Y^{[n]})$ implies that $na \leq D.F$.

The reverse direction is the more surprising one. Indeed, the fact that the nef cone is not strictly smaller than the above upper bounds is a beautiful manifestation of the fact that the half-pencils control the geometry of Enriques surfaces.

10.3 Applications to linear systems

Following an idea from [2], we can apply the above theorem to study linear systems on Y itself. The first result in this direction is the following:

Proposition 10.3.1. *Let Y be an unnodal surface and $H \in \mathrm{Pic}(Y)$ ample with $H^2 = 2d$. Then for any $Z \in Y^{[n]}$,*

$$H^i(Y, I_Z(H + K_Y)) = 0, \text{ for } i > 0,$$

provided that

$$1 \leq n < \frac{d \cdot \phi(d)}{2d - \phi(d)}.$$

Proof. First notice that

$$\begin{aligned} H^i(Y, I_Z(H+K_Y)) &\cong \mathrm{Ext}^i(\mathcal{O}_Y, I_Z(H+K_Y)) \\ &\cong \mathrm{Ext}^{i+1}_{\mathrm{D^b}(Y)}(\mathcal{O}_Y[1], I_Z(H+K_Y)) \cong \mathrm{Ext}^{1-i}_{\mathcal{A}(\omega,\beta)}(I_Z(H), \mathcal{O}_Y[1])^{\vee}, \end{aligned}$$

as long as both $I_Z(H)$ and $\mathcal{O}_Y[1]$ are both in $\mathcal{A}(\omega,\beta)$. Consider again the upper half (b,t)-plane representing stability conditions with $\omega = tH, \beta = bH$. Then for $0 \le b < 1$ and $t > 0$, $I_Z(H)$ and $\mathcal{O}_Y[1]$ are both in $\mathcal{A}_{t,b}$ so we automatically get $H^2(Y, I_Z(H+K_Y)) = 0$. Furthermore, we get $H^1(Y, I_Z(H+K_Y))^{\vee}$ is identified with $\mathrm{Hom}_{\mathcal{A}_{t,b}}(I_Z(H), \mathcal{O}_Y[1])$, and this vanishes if we can choose $b \in [0,1)$ and $t > 0$ such that both $I_Z(H)$ and $\mathcal{O}_Y[1]$ are $\sigma_{t,b}$-stable and $\phi_{t,b}(I_Z(H)) \ge \phi_{t,b}(\mathcal{O}_Y[1])$. Of course, $\mathcal{O}_Y[1]$ is always stable for $b, t > 0$ by [4, Proposition 6.3]. From the proof of Theorem 2.3.5, we know that I_Z is stable above the wall corresponding to the destabilizing subobject $\mathcal{O}(-F)$, for a half-pencil F with $H.F = \phi(H)$. From the formulas given for pseudo-walls on arbitrary surfaces in [44, Section 2], this wall is given by

$$\left(b + \frac{n}{\phi(H)}\right)^2 + t^2 - \frac{1-2n}{2d} - \frac{n^2}{\phi(H)^2} = 0, t > 0$$

in the (b,t)-plane. By [4, Section 3], this means that $I_Z(H)$ is stable above the wall given by

$$\left(b - 1 + \frac{n}{\phi(H)}\right)^2 + t^2 - \frac{1-2n}{2d} - \frac{n^2}{\phi(H)^2} = 0, t > 0.$$

Now consider the pseudo-wall corresponding to when $\phi_{t,b}(I_Z(H)) = \phi_{t,b}(\mathcal{O}_Y[1])$, given by the equation

$$1 - 2b^2 d + 2b(d-n) - 2dt^2 = 0, t > 0.$$

One can easily see that these semi-circles are either nested, coincide, or disjoint, and they intersect the line $b = \frac{1}{2}$ for

$$t_0 = \frac{\sqrt{2dn + \phi(H) - \frac{d\phi(H)}{2} - 2n\phi(H)}}{\sqrt{2}\sqrt{d}\sqrt{\phi(H)}}, t_1 = \frac{\sqrt{1 + \frac{d}{2} - n}}{\sqrt{2}\sqrt{d}},$$

respectively. Then $t_1 \ge t_0 \ge 0$ or $t_1 > 0$ and $t_0 \notin \mathbb{R}$ guarantee that both $I_Z(H)$ and $\mathcal{O}_Y[1]$ are $\sigma_{t_1,1/2}$-stable and $\phi_{t_1,1/2}(I_Z(H)) = \phi_{t_1,1/2}(\mathcal{O}_Y[1])$, as required. These conditions combined are equivalent to

$$1 \leq n < \frac{d \cdot \phi(H)}{2d - \phi(H)},$$

as required. □

This immediately allows us to recover some classical results about linear systems on unnodal Enriques surfaces (see [17, Theorems 4.4.1 and 4.6.1]):

Corollary 10.3.2. *Let Y be an unnodal Enriques surface and $H \in \mathrm{Pic}(Y)$ ample with $H^2 = 2d$. Then*

(a) The linear system $|H|$ is base-point free if and only if $\phi(H) \geq 2$,

(b) If $|H|$ is very ample, then $\phi(H) \geq 3$. Conversely, if $\phi(H) \geq 4$ or $\phi(H) = 3$ and $H^2 = 10$, then $|H|$ is very ample.

(c) The linear system $|2H|$ is base-point free and $|4H|$ is very ample.

Proof. Base-point freeness and very ampleness are equivalent to the surjectivity of the restriction map

$$H^0(Y, \mathcal{O}_Y(H)) \to H^0(Z, \mathcal{O}_Z(H))$$

as Z ranges over all 0-dimensional subschemes of length 1 and 2, respectively. Since H ample implies the vanishing of $H^1(Y, \mathcal{O}_Y(H))$ by [17, Theorem 1.3.1], this is equivalent to the vanishing of $H^1(Y, I_Z(H))$ in each case. Although the easy directions of both (a) and (b) are classical and elementary (see [17, Theorem 4.4.1(i) and Lemma 4.6.1]), we include their proofs here for completeness.

For (a), suppose $\phi(H) = 1$, and let F be a half-pencil such that $H.F = 1$ and Z be the reduced point of intersection $H \cap F$. Then $|H|$ restricted to F is a degree 1 linear system on the elliptic curve F, so $h^0(H|_F) = 1$. Thus Z is a base-point of $|H|$, showing the easy direction of (a). The converse follows from Proposition 10.3.1 as

$$\frac{d \cdot \phi(d)}{2d - \phi(d)} > 1$$

if $\phi(d) \geq 2$.

The easy direction above also shows that $|H|$ cannot be very ample if $\phi(H) = 1$. To finish the easy direction of (b), suppose that $\phi(H) = 2$. If $H^2 = 4$, then $h^0(H) = 3$, so $|H|$ induces a morphism of degree 4 onto $\mathbb{P}^2$, which is clearly not an embedding. If $H^2 \geq 6$, then we may choose a half-pencil F with $H.F = 2$ so that $(H - F)^2 = H^2 - 4 > 0$. Since $(H - F).F = 2 > 0$, it follows that $H - F$ is also ample and thus that $H^1(Y, \mathcal{O}_Y(H - F)) = 0$. From the exact sequence

$$0 \to \mathcal{O}_Y(H - F) \to \mathcal{O}_Y(H) \to \mathcal{O}_F(H) \to 0,$$

we see that

$$h^0(\mathcal{O}_Y(H - F)) = h^0(\mathcal{O}_Y(H)) - h^0(\mathcal{O}_F(H)) = h^0(\mathcal{O}_Y(H)) - 2,$$

since $|H|_F|$ is a degree 2 linear system on the elliptic curve F. It follows that $|H|$ induces a degree 2 map of F onto a line, so $|H|$ is not very ample. The converse again follows directly from Proposition 10.3.1

Part (c) follows immediately from parts (a) and (b). □

Remark 10.3.3. It is important to note that Corollary 10.3.2 is a weakening of the classical theory of linear systems on unnodal Enriques surfaces. Indeed, [17, Theorem 4.4.1] states that $|H|$ is very ample for all H with $\phi(H) \geq 3$ without the degree restriction we impose above. It thus follows that even $|3H|$ is very ample. We believe the Bridgeland stability techniques employed above can be used to recover the remaining cases. All that is required is a more careful analysis of what occurs at and beyond the first wall of the nef cone of $Y^{[n]}$. At the wall, Theorem 10.2.3 describes what destabilizes I_Z, so it is possible in theory to determine precisely for what strictly semistable I_Z, if any, $\mathrm{Hom}_{\mathcal{A}_{t,b}}(I_Z(H), \mathcal{O}_Y[1])$ fails to vanish.

We can also obtain some new results about n-very ample line bundles. Recall that a line bundle $\mathcal{O}_Y(H)$ is called n-very ample if the restriction map

$$\mathcal{O}_Y(H) \to \mathcal{O}_Z(H)$$

is surjective for every 0-dimensional subscheme Z of length $n + 1$. Proposition 10.3.1 immediately gives the following result:

Corollary 10.3.4. *Let Y be an unnodal Enriques surface and $H \in \operatorname{Pic}(Y)$ ample with $H^2 = 2d$. Then $\mathcal{O}_Y(H)$ is n-very ample provided that*

$$0 \leq n \leq \frac{d \cdot \phi(H)}{2d - \phi(H)} - 1.$$

While this result is not as strong as the complete characterization of n-very ampleness given by Knutsen [36, 37, 38, 39], we believe that a more careful application of the above techniques can prove the same result.

Chapter 11

Appendix: Some neo-classical proofs of non-emptiness and irreducibility for moduli of sheaves

In this final chapter, which has a different flavor and thrust than the main body of this dissertation, we provide another proof of Theorem 7.2.1 without the use of Bridgeland stability. The proof is longer and more involved, but it does provide a bit more. Namely, this second proof establishes a birational map $M_{H,Y}(v, L) \dashrightarrow M_{H,Y}(v - 2v(\mathcal{O}_Y), L + K_Y)$ which explains the equality of Hodge-polynomials in Theorem 7.2.1 by a result of Batyrev [8].

Recall from Section 7.2 that we had reduced showing the non-emptiness and irreducibility of Giesker stable moduli to proving the same for $M_H((4, c_1, -1), L)$, where $c_1^2 \geq -8$ and there exists an elliptic half-pencil F_A with $c_1(F_A) = f$ such that $c_1.f = \pm 1, 2$.

11.1 Non-emptiness and irreducibility of moduli spaces in rank 4 when $c_1.f = \pm 1$.

In this section, we make use of the notion of *suitable* polarizations H for an elliptic fibration $p : Y \to \mathbb{P}^1$. Recall from Section 4.4.1 that for a suitable polarization H and primitive c_1, μ_H-stability of a torsion-free sheaf R is equivalent to the stability of its restriction $R|_F$ to a generic elliptic fibre. As stable vector bundles on elliptic curves are well understood from the work of Atiyah [5], we are armed with a powerful tool to study stable sheaves on Y. The general strategy will be to use the fact that $h^0(R) \geq \chi(R) = 1$ to write $R \in M_H(v, L)$ as an extension

$$0 \to \mathcal{O}_Y \to R \to Q \to 0,$$

with Q μ-stable. This will give us a rational map $M_H(v, L) \dashrightarrow M_H(w, L)$, where $w := v - v(\mathcal{O}_Y)$. As $w^2 = v^2 + 1$, we cannot hope to get too much information from this map itself, but as the above exact sequence forces $h^0(Q(K_Y)) = h^0(R(K_Y)) > 0$, which is larger than expected since $\chi(Q) = 0$, we can iterate the process and try to write

$$0 \to \mathcal{O}_Y \to Q(K_Y) \to P \to 0,$$

with $P \in M_H(u, L + K_Y)$ where $u := v - 2v(\mathcal{O}_Y)$. It will indeed turn out that this process induces a birational map

$$M_H(v, L) \dashrightarrow M_H(u, L + K_Y),$$

allowing us to deduce non-emptiness and irreducibility from Theorem 7.1.2. This construction in fact produces μ-stable vector bundles and explains the equality of Hodge polynomials in Theorem 7.2.1.

Now let us proceed to the details. We begin with the special case when $\gcd(2, c_1) = 2$ and $c_1^2 = 0$. As noted above, we may assume that $v = (4, c_1, -1)$ with c_1 pairing to 2 with some elliptic half-pencil, and thus from $c_1^2 = 0$, it follows that c_1 represents the numerical class of an effective divisor. In fact, for any coherent sheaf R with $v(R) = v$, $\det(R) = 2F_A$ or $2F_A + K_Y = F_A + F_B$, for an elliptic half-pencil F_A with conjugate half-pencil F_B and $c_1(F_A) = f$. Take $H = F_A + G_A + nF_A$ where G_A is another elliptic half-pencil such that $G_A.F_A = 1$ and $n \geq 47$. Let us write $c_1(G_A) = \sigma$. Then H is ample and v-suitable for the elliptic fibration $|2F_A|$ [28, Remark 5.3.6]. With these preparations, we prove the following result.

Theorem 11.1.1. *Let $v = (4, c_1, -1)$ with $c_1 = 2f$ and H' a generic v-suitable polarization. Then $M_{H'}(v, 2F_A)$ is a non-empty, irreducible 9-fold, singular along a 6-dimensional generalized Enriques manifold. Similarly, $M_{H'}(v, 2F_A + K_Y)$ is a non-empty, smooth irreducible 9-fold. In either case, $M_{H'}(v, \det(R))$ generically parametrizes μ-stable vector bundles and is birational to $M_{H'}(v - 2v(\mathcal{O}_Y), \det(R) + K_Y)$.*

Proof. The statement about singularities follows from Theorem 4.6.1 above. Indeed, a singular point represents a Gieseker stable sheaf $R \cong \pi_*(E)$ for a rank 2 π^*H'-stable sheaf on $\tilde{Y}$. Since Y is unnodal, $v(E) = v(\iota^*E)$, so $v(E) = \frac{1}{2}\pi^*v$. As H' is generic (and thus so is π^*H') and $v(E)$ is primitive, $M_{\pi^*H'}(v(E))$ is a non-empty, irreducible smooth projective hyperkähler manifold. Since v itself is primitive, we see that the action of ι^* on this moduli space can have no fixed points as these would correspond to stable sheaves on Y with Mukai vector $\frac{1}{2}v$. Thus the singular locus consists of one 6-dimensional component which is the quotient of the hyperkähler manifold $M_{\pi^*H'}(v(E))$, so it is a generalized Enriques manifold in the sense of Oguiso and Schröer [54]. Note also that $\det(R)=\det(\pi_*(E))$ is divisible by 2 in $\mathrm{Pic}(Y)$, so the singular locus must lie solely in $M_{H'}(v,2F_A)$, and moreover this already shows that a component of this moduli space contains μ-stable locally free sheaves, but we will not make use of this observation.

For the remaining statements of the theorem, let us first consider stability with respect to H. As $s(v)=-1$, for any μ-stable $R\in M_H(v)$ we have $h^0(R)+h^2(R)\geq\chi(R)=s+\frac{r}{2}=1$. Moreover, as $h^2(R)=h^0(R^\vee(K_Y))$, we see that $h^2(R)=0$ since $R^\vee(K_Y)$ is a locally free μ-stable sheaf with $\mu=-\frac{c_1.H}{4}=-\frac{1}{2}<0$. So $h^0(R)\geq 1$, and taking the saturation of a section gives

$$0\to I_Z(D)\to R\to Q\to 0$$

with Z 0-dimensional, D effective, and Q torsion-free. By μ-stability of R, we get $0\leq D.H<\frac{c_1.H}{4}=\frac{1}{2}$. Thus $D.H=0$ so that $D=0$ from the ampleness of H. As $\mathcal{O}_Y\to R$ then factors through I_Z, we see that $Z=\varnothing$. So we may write

$$0\to\mathcal{O}_Y\to R\to Q\to 0,$$

with Q torsion-free with $w=v(Q)=(3,c_1,-\frac{3}{2})$. Let us show that Q is μ-stable.

Suppose to the contrary that Q had a destabilizing quotient G of rank $r'=1,2$. Then as G is also a quotient of R, we see that $\mu(R)<\mu(G)<\mu(Q)$. As H is w-suitable as well, we see that

$$0=\frac{c_1.f}{4}\leq\frac{c_1(G).f}{r'}\leq\frac{c_1.f}{3}=0,$$

so $c_1(G).f=0$. From the definition of H, we get that the strict inequalities of μ become

$$\frac{1}{2}=\frac{c_1.H}{4}<\frac{c_1(G).g}{r'}<\frac{c_1.H}{3}=\frac{2}{3},$$

which gives an immediate contradiction. Thus Q is μ_H-semistable. Of course, we cannot have $\mu(G) = \mu(Q)$ either as then $c_1(G).H = \frac{2}{3}$ or $\frac{4}{3}$, which is absurd. Thus indeed Q is μ_H-stable.

Tensoring by K_Y, we find that $h^0(Q(K_Y)) = h^0(R(K_Y)) \geq 1$ (for the same reason as above). Moreover, the set of R fitting into an exact sequence as above are parametrized by $\mathbb{P}\operatorname{Ext}^1(Q, \mathcal{O}_Y)$, which by Serre duality has dimension $h^1(Q(K_Y))-1 = h^0(Q(K_Y))-1$ as $\chi(Q(K_Y)) = 0$. Proceeding similarly with $Q(K_Y)$, we get that a section of $Q(K_Y)$ factors through $I_Z(D)$ as above to give

$$0 \to I_Z(D) \to Q(K_Y) \to P \to 0,$$

with P torsion-free. As above, μ-stability of $Q(K_Y)$ implies that $0 \leq D.H < \mu_H(Q(K_Y)) = \frac{2}{3}$. Thus again $D = 0$ and $Z = \varnothing$. So we may write

$$0 \to \mathcal{O}_Y \to Q(K_Y) \to P \to 0,$$

with P torsion-free, $v(P) = (2, c_1, -2)$, and $N := \det(P) = \det(R) + K_Y$. Let us again show that P is μ-stable.

Suppose to the contrary that P had a destabilizing quotient $I_W(L)$, with W a 0-dimensional subscheme. Then as $I_W(L)$ is also a quotient of $Q(K_Y)$, we get

$$\frac{2}{3} = \mu(Q) < \mu(I_W(L)) = L.H \leq \mu(P) = 1.$$

The only possibility is that $L.H = 1$, i.e. $\mu(I_W(L)) = \mu(P)$. So P is certainly μ-semistable. If H does not lie on a wall for $u := v - 2v(\mathcal{O}_Y)$, then we may take $H' = H$ which is already generic. Otherwise, H lies on a wall for u for all $n \geq 0$. Choose a very near polarization H' in one of the adjacent chambers with respect to both u and v. Then the strict inequalities above will remain valid, as will the stability of R and Q. As there are only finitely many such L, we can choose H' such that now $\mu(I_W(L)) = L.H > \mu(P)$. We may assume then that P is μ-stable in either case, unless $c_1(L)$ is proportional to $c_1 = 2f$, i.e. $c_1(L) = f$ so that $L = F_A$ or F_B, in which case $\mu(I_W(L)) = \mu(P)$ for any choice of H'.

Without loss of generality, we may write P in this case as an extension

$$0 \to I_{W_0}(F_A) \to P \to I_{W_1}(F_A + \epsilon K_Y) \to 0,$$

where the W_i are 0-dimensional subschemes such that $3 = c_2(P) = l(W_0) + l(W_1)$ and $\epsilon = 0, 1$. Then either one of the $W_i = \varnothing$ while the other has length 3 or one has length 2 while the other has length 1. Let us separate into two cases.

First suppose that $\det(R) = 2F_A$. Then $N = 2F_A + K_Y$ so that $\epsilon = 1$, and we have

$$0 \to I_{W_0}(F_A) \to P \to I_{W_1}(F_B) \to 0.$$

If $l(W_0) = 1, l(W_1) = 2$, then it is not difficult to see that $\mathrm{ext}^1(I_{W_1}(F_B), I_{W_0}(F_A)) = 2$, so the space of such P's is at most 7. The same holds true if $l(W_0) = 2, l(W_1) = 1$. If $W_0 = \varnothing, l(W_1) = 3$, then $\mathrm{ext}^1(I_{W_1}(F_B), \mathcal{O}_Y(F_A)) = h^1(I_{W_1}) = 2$ while if $l(W_0) = 3, l(W_1) = 0$, then $\mathrm{ext}^1(\mathcal{O}_Y(F_B), I_{W_0}(F_A)) = 3$. Thus we see that the space of such P's is at most 7 or 8. The second case of $\det(R) = 2F_A + K_Y$ is similar with the space of such P's at most 8.

Now let us point out that the choice of R and a point in $\mathbb{P}H^0(R)$ determines Q and a point in $\mathbb{P}\,\mathrm{Ext}^1(Q, \mathcal{O}_Y) \cong \mathbb{P}H^0(Q(K_Y))$ which determines P and a point in $\mathbb{P}\,\mathrm{Ext}^1(P, \mathcal{O}_Y) \cong \mathbb{P}H^1(P(K_Y))$, and vice-versa. Moreover, fixing R and varying the section varies Q and similarly for $Q(K_Y)$ and P. Finally, we see that $h^0(R) = h^1(P(K_Y))$. All together, this argument shows that the dimension of the locus of those R's which give strictly semistable P's is the same as the dimension of those P's, which is at most 8. As all elements of $M_{H'}(v, N + K_Y)$ are Gieseker stable, every irreducible component of $M_{H'}(v, N + K_Y)$ has dimension $v^2 + 1 = 9 > 8$.

Moreover, the sublocus of $M_{H'}(v, N + K_Y)$ parametrizing non-locally free sheaves has dimension at most 6. Indeed, $v(R^{\vee\vee}) = (4, N + K_Y, l - 1)$ for $l = l(R^{\vee\vee}/R) \geq 0$, and $R^{\vee\vee}$ is a Gieseker stable locally free sheaf. Then from $0 \leq v(R^{\vee\vee})^2 = 8 - 8l$, we see that $l \leq 1$. Thus every non-locally free R determines a (non-uniquely determined) point in $\mathrm{Quot}(R^{\vee\vee}, 1)$ which is irreducible of dimension 5 [28, Theorem 6.A.1]. As the $R^{\vee\vee}$ vary in the 1-dimensional moduli space $M_{H'}((4, 2f, 0), N + K_Y)$, we see that indeed the dimension of this sublocus is at most 6.

So every irreducible component of $M_{H'}(v, N + K_Y)$ generically parametrizes locally free sheaves. Let us finally show that the sublocus of $M_{H'}(v, N + K_Y)$ parametrizing strictly μ-semistable sheaves is proper. From the above paragraph, we may assume R

is locally free. Then R fits into an extension,

$$0 \to E \to R \to E' \to 0,$$

with E, E' μ-stable, $c_1(E) = c_1(E') = f$, and E locally free since R is. As R is Gieseker stable we must have $\frac{\chi(E)}{2} < \frac{\chi(R)}{4} = \frac{1}{4}$, so

$$2 - c_2(E) = s(E) + \frac{r(E)}{2} = \chi(E) \leq 0,$$

so that $c_2(E) \geq 2$. From $0 \leq v(E')^2 = -4(1 - c_2(E'))$, we get $c_2(E') \geq 1$. As their sum must be $c_2(R) = 3$, we see that $c_2(E) = 2, c_2(E') = 1$. Now $p(E') > p(E)$ so $\hom(E', E) = 0$. Furthermore, it is easily seen that E' is locally free since $v(E')^2 = 0$, the minimum value. Thus $\operatorname{ext}^2(E', E) = \hom(E, E'(K_Y)) = 0$ since E and $E'(K_Y)$ are μ-stable locally free sheaves of the same slope. It follows that $\operatorname{ext}^1(E', E) = \langle v(E'), v(E)\rangle = 2$. As the E's move in a 5 dimensional family and the E''s move along a curve, we see that the sublocus of strictly μ-semistable sheaves has dimension at most 7.

It follows from the above discussion that the generic element of any component of $M_{H'}(v, N + K_Y)$ is locally free, μ-stable, and gives P as above which is also μ-stable. Moreover, we have shown that if $V^i_j := \{R | h^(R) = i, h^0(R(K_Y) = j\} \subset M_{H'}(v, N + K_Y)^{\mu-st,l.f.}$, then $\dim V^i_j \leq \dim U^{j-1}_{i-1}$, where $U^{j-1}_{i-1} := \{P | h^0(P) = i - 1, h^0(P(K_Y)) = j-1\} \subset M_{H'}(u, N)$. We also note that $P(G_A)$ satisfies $c_2(P(G_A)) = \frac{1}{2}c_1(P(G_A))^2+1$, so by [35, Theorem 1] $M_{H'}(u, N) \cong M_{H'}(v(P(G_A)), N+2G_A)$ is irreducible and generically parametrizes μ-stable, locally free sheaves such that $h^0(P(G_A)) = h^0(P(G_A+K_Y)) = 1$ with isolated section. It follows that we may write

$$0 \to \mathcal{O}_Y \to P(G_A) \to I_Z(N + 2G_A) \to 0,$$

for generic $P \in M_{H'}(u, N)$, where $l(Z) = 5$. It follows that $h^0(I_Z(N + 2G_A)) = 0, h^0(I_Z(N + 2G_A + K_Y)) = 1$, and twisting by $\mathcal{O}_Y(-G_A)$, we get that

$$h^0(P) = h^0(I_Z(N + G_A)), h^0(P(K_Y)) = h^0(I_Z(N + G_A + K_Y))$$

since $h^1(-G_A) = h^1(-G_B) = h^1(-G_A + K_Y) = 0$. Now $h^0(I_Z(N + G_A)) \leq h^0(I_Z(N + 2G_A)) = 0$, and if $h^0(I_Z(N + G_A + K_Y)) > 0$, then we'd get

$$0 \to \mathcal{O}_Y(-N - G_A + K_Y) \to I_Z \to \mathcal{O}_C(-Z) \to 0,$$

for some $C \in |N + G_A + K_Y|$. Then twisting by $\mathcal{O}_Y(N + 2G_A)$ we'd have

$$0 \to \mathcal{O}_Y(G_A + K_Y) \to I_Z(N + 2G_A) \to \mathcal{O}_C(N|_C + (2G_A)|_C - Z) \to 0,$$

which is absurd as $h^0(I_Z(N + 2G_A)) = 0$. Thus the generic element of $M_{H'}(u, N)$ is a μ-stable locally free sheaf in U_0^0, which is an open irreducible subset of dimension $u^2 + 1 = 9$. It follows that the generic element of $M_{H'}(v, N + K_Y)$ is in V_1^1. In that case we get that no choices were involved in determining Q and P, and vice-versa. Thus $M_{H'}(v, N + K_Y)$, if non-empty, is irreducible and birational to $M_{H'}(u, N)$. It just remains to prove non-emptiness.

By a result of Qin [55, Theorem 2], $M_{H'}(u, N)$ and $M_H(u, N)$ are birational. So let us take a μ-stable locally free sheaf in U_0^0 which is also μ_H-stable. Then $\operatorname{ext}^1(P, \mathcal{O}_Y) = h^1(P(K_Y)) = 1$, so there is a unique non-split extension

$$0 \to \mathcal{O}_Y \to Q(K_Y) \to P \to 0,$$

which is locally free. We claim that $Q(K_Y)$ is μ_H-stable. If not, then let G be a saturated μ_H-stable destabilizing subsheaf of rank $r' = 1$ or 2, necessarily locally free. Then as $\mu_H(G) > \mu_H(Q(K_Y)) > 0$, G cannot be contained in $\mathcal{O}_Y$ so that we get a non-trivial homomorphism $G \to P$. It follows that $\mu_H(G) \leq \mu_H(P) = 1$. If $\mu_H(G) = 1$, then $G \cong P$ as they are both μ-stable vector bundles of the same slope, splitting the extension. Thus

$$\frac{2}{3} = \mu_H(Q(K_Y)) < \mu(G) < \mu(P) = 1,$$

which is impossible for a sheaf of rank 1 or 2. Thus $Q(K_Y)$ is μ_H-semistable, and because $\gcd(3, c_1) = 1$, it must be μ_H-stable as well. Furthermore, we have $h^1(Q(K_Y)) = h^0(Q(K_Y)) = 1, h^0(Q) = 0$. So consider the unique non-split extension in $\operatorname{Ext}^1(Q, \mathcal{O}_Y)$,

$$0 \to \mathcal{O}_Y \to R \to Q \to 0,$$

which is again locally free. We claim that R is in fact μ_H-stable and thus $\mu_{H'}$-stable by openness of stability. Suppose on the contrary that R had a saturated μ_H-stable destabilizing subsheaf G of rank $0 < r' < 4$, necessarily locally free. Then arguing as before we get

$$\frac{1}{2} = \mu_H(R) < \mu_H(G) < \mu_H(Q) = \frac{2}{3}.$$

This is clearly impossible from the numerics, so we get that R is μ_H-semistable. If H lies on a wall for v, then as before we see that R remains $\mu_{H'}$-semistable, and is only strictly semistable if it has μ-stable subobjects E of rank 2 and $c_1(E)=f$. It is not difficult to see that E maps injectively into Q and that further $E(K_Y)$ injects into P. Thus $h^0(E(K_Y))=0$, so that $\chi(E)=\chi(E(K_Y))\leq 0$ since $h^2(E)=h^2(E(K_Y))=0$ by stability. Thus R is Gieseker stable, and it follows that $M_{H'}(v,N+K_Y)^{l.f.}\neq\varnothing$, and since $M_{H'}(v,N+K_Y)$ is irreducible and consists generically of μ-stable vector bundles, we can deform R to obtain a locally free μ-stable sheaf. Alternatively, we have seen explicitly above what these strictly μ-semistable sheaves look like and that the space of such R, equivalently the space of the corresponding P's, is a proper subset, so choosing P outside of this sublocus gives a μ-stable R.

The theorem is thus proved for H' generic but very close to H. The result follows for H' in an arbitrary chamber by using the invariance of motivic invariants from Theorem 6.2.12. $\square$

The proof above contains the essential idea behind our approach in the case $\gcd(2,c_1)=1$, except that the divisibility of c_1 by 2 above forced us to deal with the more complicated issue of μ-semistable sheaves. We will not encounter such issues when c_1 is primitive, and the technique will be to use the fact that $h^0(R)>0$ to compare $M_H(v,L)$ to moduli spaces of stable sheaves of lower rank, where we ensure the stability of quotients by using the suitability of H. Recall that we may assume that $v=(4,c_1,-1)$ with the class of a half-pencil f such that $c_1.f=\pm 1$. Denote by $F_A, F_B=F_A+K_Y$ the actual effective divisors with $c_1(F_A)=f$, as above. Then our main result is the following:

Theorem 11.1.2. *Let H be a generic polarization that is suitable for the Mukai vectors $v, w:=v-v(\mathcal{O}_Y)$, and $u:=v-2v(\mathcal{O}_Y)$ with respect to the elliptic fibration $p:Y\to\mathbb{P}^1$ induced by $|2F_A|$, where v is a Mukai vector as above such that $c_1^2+8=v^2\geq 0$. Then $M_H(v,L)$ is a nonempty, smooth, irreducible projective manifold of dimension v^2+1 parametrizing μ-stable sheaves, birational to $M_H(u,L+K_Y)$.*

Proof. As noted above, the fact that $\gcd(2,c_1)=1$ will preclude the existence of any properly μ-semistable sheaves. If $R \in M_H(v, L)$ is not locally-free, then $R^{\vee\vee}$ is a μ-stable locally free sheaf with $v(R^{\vee\vee}) = (4, c_1, l-1)$ for $l = l(R^{\vee\vee}/R) \leq \frac{v^2}{8}$. Then for each choice of l, R determines a (non-unique) point in $\mathrm{Quot}(R^{\vee\vee}, l)$ so the locus of R such that $l(R^{\vee\vee}/R) = l$ has dimension at most

$$\dim M_H(v(R^{\vee\vee}), L) + \dim \mathrm{Quot}(R^{\vee\vee}, l) = v(R^{\vee\vee})^2 + 1 + 5l = v^2 + 1 - 3l = \dim M_H(v, L) - 3l.$$

Thus every component of $M_H(v, L)$ consists generically of locally free sheaves.

For any $R \in M_H(v, L)$, $h^0(R) + h^2(R) = \chi(R) + h^1(F) \geq 1$, so either $h^0(R) > 0$ or $h^0(R^\vee(K_Y)) = h^2(R) > 0$, and by stability only one of these can happen. Since each irreducible component of $M_H(v, L)$ contains locally free sheaves, we see that dualizing and twisting by $\mathcal{O}_Y(K_Y)$ induces a bijection between the set of irreducible components of $M_H((4, c_1, -1), L)$ and the set of irreducible components of $M_H((4, -c_1, -1), -L + K_Y)$. Thus if the theorem is true for those c_1 such that $M_H(v, L)$ parametrizes sheaves that have sections, then the theorem is true in general. So we may assume that R is locally free with $h^0(R) > 0$, from which it follows that $c_1.f = 1$.

Then we may write

$$0 \to \mathcal{O}_Y(D_R) \to R \to Q \to 0,$$

with Q torsion-free and D_R effective. From the v-suitability of H it follows that $c_1(D_R).f \leq \frac{c_1.f}{4} = \frac{1}{4}$, and thus we see that D_R is supported in the fibers of p. Thus $D_R = n_R F_A + \epsilon_R K_Y$, where $\epsilon_R = 0, 1$ and $n_R \geq 0$. Let

$$V^i_{D_R,j} := \left\{ R |\ R(-D_R) \text{ has an regular section, } h^0(R(-D_R)) = i, h^0(R(-D_R + K_Y)) = j \right\}.$$

Then R and the choice of a section of $R(-D_R)$ determine Q, and for fixed Q, such R are parametrized by $\mathbb{P}\,\mathrm{Ext}^1(Q(-D_R), \mathcal{O}_Y)$ which has dimension $h^1(Q(-D_R + K_Y)) - 1$. As $\chi(Q(-D_R)) = -n_R$, it follows that

$$\dim V^i_{D_R,j} \leq \dim W^{i-1}_{D_R,j} + j - i + n_R,$$

where $W^{i-1}_{D_R,j} := \left\{ Q | h^0(Q(-D_R)) = i - 1, h^0(Q(-D_R + K_Y)) = j \right\}$. Here we would like to show that $W^{i-1}_{D_R,j} \subset M_H(v - v(\mathcal{O}_Y(D_R)), L - D_R)$, i.e. that Q is μ-stable. Suppose not, and take a destabilizing quotient G of rank $r' = 1, 2$. Then as G is also a quotient of R, we see that

$$\mu_H(R) < \mu_H(G) < \mu_H(Q),$$

and from v-suitability of H we get that

$$\frac{1}{4} = \frac{c_1.f}{4} \leq \frac{c_1(G).f}{r'} \leq \frac{c_1.f}{3} = \frac{1}{3},$$

a contradiction. Thus Q is μ-semistable and thus μ-stable since $\gcd(3, c_1) = 1$.

If $j = 0$, then as $\dim M_H(v - v(\mathcal{O}_Y(D_R)), L - D_R) = c_1^2 + 10 - 2n_R = \dim M_H(v, L) + 1 - 2n_R$ and $i \geq 1$, we see that

$$\dim V^i_{D_R,j} \leq \dim M_H(v, L) + 1 - i - n_R < \dim M_H(v, L),$$

if $i > 1$ or $n_R > 0$.

So suppose that $j > 0$. Then we can write

$$0 \to I_W(D_Q) \to Q(-D_R + K_Y) \to P \to 0,$$

with P torsion-free and D_Q effective. By the suitability of H, we see again that D_Q is supported in fibers so that $D_Q = n_Q F_A + \epsilon_Q K_Y$, with $n_Q \geq 0$ and $\epsilon_Q = 0, 1$.

By taking reflexive duals one obtains the following commutative diagram in which the rows and columns are exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I_W(D_Q) \arrow[r] \arrow[d] & Q(-D_R + K_Y) \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}(D_Q) \arrow[r] \arrow[d] & Q^{\vee\vee}(-D_R + K_Y) \arrow[r] \arrow[d] & \tilde{P} \arrow[r] \arrow[d] & 0, \\
0 \arrow[r] & \mathcal{O}_W \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $\mathcal{O}_Y(D_Q)$ is saturated in $Q^{\vee\vee}(-D_R + K_Y)$ with quotient $\tilde{P}$. The locus in $M_H(v(Q))$ of those Q with the same reflexive-dual $Q^{\vee\vee}$ and value of $l = l(Q^{\vee\vee}/Q)$ is isomorphic to the Hilbert scheme of l-points $Y^{[l]}$.[1] We are only concerned with those $T = Q^{\vee\vee}/Q$ whose support contains a 0-dimensional scheme W such that $h^0(I_W(D_Q)) > 0$. The locus of such W has dimension at most

$$l(W) + \dim |D_Q| = l(W) + \left\lfloor \frac{n_Q - \epsilon_Q}{2} \right\rfloor,$$

by Lemma 11.2.1. The remaining points of the support of T are unrestricted, so for fixed locally free $Q^{\vee\vee}$, the dimension of the locus in $M_H(v(Q))$ that we are interested in is at most

$$l(W) + \left\lfloor \frac{n_Q - \epsilon_Q}{2} \right\rfloor + 2l(H) = l(T) + l(H) + \left\lfloor \frac{n_Q - \epsilon_Q}{2} \right\rfloor. \tag{11.1}$$

Also for any torsion-free sheaf F,

$$h^0(F) \leq h^0(F^{\vee\vee}) \leq h^0(F) + l(F^{\vee\vee}/F). \tag{11.2}$$

With these observations, we continue along our way to describing the generic sheaf on each component of $M_H(v, L)$. As above, we see that $Q^{\vee\vee}$ and the choice of a section of $Q^{\vee\vee}(-D_R - D_Q + K_Y)$ determine $\tilde{P}(-D_Q)$ (up to scaling), and for a fixed $\tilde{P}$ such $Q^{\vee\vee}(-D_R - D_Q + K_Y)$'s are parametrized by $\mathbb{P}\operatorname{Ext}^1(\tilde{P}(-D_Q), \mathcal{O}_Y)$ of dimension $h^1(\tilde{P}(-D_Q + K_Y)) - 1$. So let

$$W^{i-1,a}_{D_R,D_Q,j,b} := \left\{ Q \in W^{i-1}_{D_R,j} : \begin{array}{l} \bullet\ Q^{\vee\vee}(-D_R - D_Q + K_Y) \text{ has a regular section,} \\ \bullet\ h^0(Q^{\vee\vee}(-D_R - D_Q)) = a \\ \bullet\ h^0(Q^{\vee\vee}(-D_R - D_Q + K_Y)) = b \end{array} \right\},$$

where $b \geq 1$. Moreover, define $\widetilde{W^{i-1,a}_{D_R,D_Q,j,b}} := \left\{ Q^{\vee\vee} | Q \in W^{i-1,a}_{D_R,D_Q,j,b} \right\}$. Then as above we see that

$$\dim \widetilde{W^{i-1,a}_{D_R,D_Q,j,b}} \leq \dim U^{b-1}_{D_R,D_Q,a} - b + 1 + a + n_R + n_Q - l(T),$$

since $\chi(\tilde{P}(-D_Q + K_Y)) = -1 - n_R - n_Q + l(T)$. Moreover, from (11.1) we get that

$$\dim W^{i-1,a}_{D_R,D_Q,j,b} \leq \dim U^{b-1}_{D_R,D_Q,a} - b + 1 + a + n_R + n_Q + \left\lfloor \frac{n_Q - \epsilon_Q}{2} \right\rfloor + l(H).$$

---

[1] Since μ-stability and μ-semistability are equivalent in our case, the fibers of the morphism $M \to M^{\mu ss}$, in the notation of [28, Chapter 8], are precisely the fibers of the Hilbert-Chow morphism $Y^{[l]} \to Y^{(l)}$ Thus the usual stratification of the Donaldson-Uhlenbeck compactification applies to M with the symmetric product replaced by the Hilbert scheme of points. See [28] for a more detailed discussion of what sheaves are identified in $M^{\mu ss}$.

Here, of course,

$$U^{b-1}_{D_R,D_Q,a} := \left\{\tilde{P} | h^0(\tilde{P}(-D_Q)) = b-1, h^0(\tilde{P}(-D_Q+K_Y)) = a\right\},$$

and we would like to show that

$$U^{b-1}_{D_R,D_Q,a} \subset M_H((2, c_1 - 4D_R - D_Q, -2-n_R), L - 4D_R - D_Q + K_Y),$$

i.e. that $\tilde{P}$ is μ-stable. So suppose that $\tilde{P}$ had a destabilizing quotient $I_{W'}(E)$. Then as $I_{W'}(E)$ is also a quotient of $Q(-D_R+K_Y)$, we get that $\mu_H(Q(-D_R+K_Y)) < \mu_H(E) < \mu_H(\tilde{P})$, and thus

$$\frac{1}{3} = \frac{(c_1 - 4n_R f).f}{3} \leq c_1(E).f \leq \frac{(c_1 - 4n_R f - n_Q f).f}{2} = \frac{1}{2},$$

which is absurd. Therefore $\tilde{P}$ is μ-semistable and thus μ-stable since c_1 is primitive.

Suppose $a > 0$, then as $h^0(\tilde{P}(-D_Q)) = b-1, h^0(\tilde{P}(-D_Q+K_Y)) = a$, we can write

$$0 \to I_{Z'}(D_P) \to \tilde{P}(-D_Q+K_Y) \to I_Z(L - 4D_R - 3D_Q - D_P + K_Y) \to 0,$$

for effective D_P and 0-schemes Z, Z'. We see as above that $D_P = n_P F_A + \epsilon_P K_Y, n_P \geq 0, \epsilon_P = 0, 1$. As above we obtain a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_{Z'}(D_P) & \longrightarrow & \tilde{P}(-D_Q+K_Y) & \longrightarrow & I_Z(B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(D_P) & \longrightarrow & P^{\vee\vee}(-D_P+K_Y) & \longrightarrow & I_{\tilde{Z}}(B) & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{Z'} & \longrightarrow & T' & \longrightarrow & H' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $B := L - 4D_R - 3D_Q - D_P + K_Y$, and we note that $\tilde{P}^{\vee\vee} = P^{\vee\vee}$ and $\mathcal{O}_Y(D_P)$ is saturated in $P^{\vee\vee}(-D_R - D_Q + K_Y)$ with quotient $I_{\tilde{Z}}(B)$. Furthermore, note that

$$h^0(\tilde{P}) \leq h^0(P^{\vee\vee}) \leq h^0(\tilde{P}) + l(T'), l(T') + l(H) = l(P^{\vee\vee}/P), \tag{11.3}$$

and $h^0(I_{\tilde{Z}}(B-D_P)=h^0(P^{\vee\vee}(-D_Q-D_P+K_Y))-1$ and $h^0(I_{\tilde{Z}}(B-D_P+K_Y))=h^0(P^{\vee\vee}(-D_Q-D_P))$. So let

$$U^{b-1,m}_{D_R,D_Q,D_P,a,n} := \left\{ \tilde{P} \in U^{b-1}_{D_R,D_Q,a} : \begin{array}{l} \bullet\, P^{\vee\vee}(-D_Q-D_P+K_Y) \text{ has a regular section,} \\ \bullet\, h^0(P^{\vee\vee}(-D_Q-D_P))=m \\ \bullet\, h^0(P^{\vee\vee}(-D_Q-D_P+K_Y))=n \end{array} \right\},$$

$$\widetilde{U^{b-1,m}_{D_R,D_Q,D_P,a,n}} := \left\{ P^{\vee\vee} | \tilde{P} \in U^{b-1,m}_{D_R,D_Q,D_P,a,n} \right\},$$

and define

$$S^m_{n-1} := \left\{ \tilde{Z} | h^0(I_{\tilde{Z}}(B-D_P+K_Y))=m, h^0(I_{\tilde{Z}}(B-D_P))=n-1 \right\}.$$

Here $S^m_{n-1} \subset Y^{(l(\tilde{Z}))}$ with $l(\tilde{Z})=\frac{1}{2}c_1^2+3-3n_R-2n_Q-n_P-l(T)-l(T')$. Then following the usual explanation, we see that

$$\dim \widetilde{U^{b-1,m}_{D_R,D_Q,D_P,a,n}} \le \dim S^m_{n-1} - n + m + 2 + n_R + n_Q + n_P - l(T) - l(T').$$

From the explanation leading to (11.1) we get that

$$\dim U^{b-1,m}_{D_R,D_Q,D_P,a,n} \le \dim S^m_{n-1} - n + m + 2 + n_R + n_Q + n_P + \left\lfloor \frac{n_P-\epsilon_P}{2} \right\rfloor + l(H') - l(T).$$

Tracing through the dimension estimates so far, we see that the dimension d of the corresponding locus in $M_H(v,L)$ satisfies

$$\begin{aligned} d \le{}& \dim S^m_{n-1} + j + a + m + 3 - i - b - n - l(T) && (11.4)\\ &+ 3n_R + 2n_Q + n_P + \left\lfloor \frac{n_P-\epsilon_P}{2} \right\rfloor + \left\lfloor \frac{n_Q-\epsilon_Q}{2} \right\rfloor + l(H') + l(H). && (11.5) \end{aligned}$$

Ultimately, we would like to show that $d < \dim M_H(v,L)$ when $n_R > 0$ or $(i-1)(j-1) > 0$. To do so, we must relate a,b to i,j. We consider 3 cases. First suppose that $2 \mid n_Q$ and $\epsilon_Q = 0$. Then we may assume that D_Q is a union of $\frac{n_Q}{2}$ disjoint generic elliptic fibers. Tensoring the exact sequence

$$0 \to \mathcal{O}_Y(-D_Q) \to \mathcal{O}_Y \to \bigoplus \mathcal{O}_{F_i} \to 0$$

with $Q^{\vee\vee}(-D_R+K_Y)$, we obtain

$$0 \to Q^{\vee\vee}(-D_R-D_Q+K_Y) \to Q^{\vee\vee}(-D_R+K_Y) \to \bigoplus Q^{\vee\vee}(-D_R+K_Y)|_{F_i} \to 0,$$

as $Q^{\vee\vee}$ is locally free. As $Q^{\vee\vee}(-D_R+K_Y)|_{F_i}$ is a stable vector bundle of degree 2 on the smooth elliptic curve F_i, $h^0(Q^{\vee\vee}(-D_R+K_Y)=2$ from [5]. It follows from the exact sequence above that

$$j=h^0(Q(-D_R+K_Y))\leq h^0(Q^{\vee\vee}(-D_R+K_Y))\leq h^0(Q^{\vee\vee}(-D_R-D_Q+K_Y))+n_Q=b+n_Q. \tag{11.6}$$

From (11.2) we get

$$a=h^0(Q^{\vee\vee}(-D_R-D_Q))\leq h^0(Q^{\vee\vee}(-D_R))\leq h^0(Q(-D_R))+l(Q^{\vee\vee}/Q)=i-1+l(T). \tag{11.7}$$

Plugging these estimates into (11.4), we get

$$d\leq S^m_{n-1}+2+m+3n_R+\frac{7}{2}n_Q+n_P+\left\lfloor\frac{n_P-\epsilon_P}{2}\right\rfloor+l(H)+l(H')-n.$$

When $2\mid n_Q$ and $\epsilon_Q=1$, then D_Q+K_Y can be assumed to be a union of $\frac{n_Q}{2}$ generic elliptic fibers F_i, so we tensor

$$0\to\mathcal{O}_Y(-D_Q+K_Y)\to\mathcal{O}_Y\to\bigoplus\mathcal{O}_{F_i}\to 0$$

with $Q^{\vee\vee}(-D_R+K_Y)$, to get that

$$j=h^0(Q(-D_R+K_Y))\leq h^0(Q^{\vee\vee}(-D_R+K_Y))\leq h^0(Q^{\vee\vee}(-D_R-D_Q))+n_Q,$$

as above. We also see that $h^0(Q^{\vee\vee}(-D_R))\leq b+n_Q$ upon twisting by K_Y. Tensoring the exact sequence

$$0\to\mathcal{O}_Y(-D_Q)\to\mathcal{O}_Y\to\mathcal{O}_{D_Q}\to 0$$

by $Q^{\vee\vee}(-D_R)$ shows that

$$h^0(Q^{\vee\vee}(-D_R-D_Q)\leq h^0(Q^{\vee\vee}(-D_R)),$$

so that putting it all together we get that

$$j\leq b+2n_Q, a\leq h^0(Q^{\vee\vee}(-D_R))\leq i-1+l(Q^{\vee\vee}/Q). \tag{11.8}$$

Plugging these estimates into (11.4) gives

$$d\leq\dim S^m_{n-1}+1+m+3n_R+\frac{9}{2}n_Q+n_P+\left\lfloor\frac{n_P-\epsilon_P}{2}\right\rfloor+l(H)+l(H')-n.$$

When $2 \nmid n_Q$, we may twist by F_A or $F_A + K_Y$ to put ourselves in either of the situations above (albeit with $\frac{n_Q+1}{2}$ smooth elliptic fibers). In particular, let

$$F := Q(-D_R + K_Y) \otimes \mathcal{O}_Y(F_A), F' := F(K_Y) = Q(-D_R + K_Y) \otimes \mathcal{O}_Y(F_A + K_Y),$$

where F (resp. F') contains $\mathcal{O}_Y(D_Q + F_A)$ (resp. $\mathcal{O}_Y(D_Q + F_A + K_Y)$) as a saturated subsheaf. Then without loss of generality suppose that F is in the first situation above, i.e. $\epsilon_Q = 0$ so that $D_Q + F_A$ is divisible by 2. Then

$$h^0(F) \leq h^0(F^{\vee\vee}) \leq h^0(F^{\vee\vee}(-D_Q - F_A)) + n_Q + 1,$$

while F' is in the second situation so

$$h^0(F') \leq h^0(F'^{\vee\vee}) \leq h^0(F'^{\vee\vee}(-D_Q - F_A)) + n_Q + 1.$$

But $h^0(Q^{\vee\vee}(-D_R + K_Y) \leq h^0(F^{\vee\vee})$ and $h^0(Q^{\vee\vee}(-D_R + K_Y)) \leq h^0(F'^{\vee\vee})$, as F_A and $F_A + K_Y$ are both effective. Furthermore,

$$F^{\vee\vee}(-D_Q - F_A) = Q^{\vee\vee}(-D_R + F_A + K_Y - D_Q - F_A) = Q^{\vee\vee}(-D_R - D_Q + K_Y), \text{ and}$$

$$F'^{\vee\vee}(-D_Q - F_A) = Q^{\vee\vee}(-D_R - D_Q).$$

Thus we get

$$j \leq \min\{b + n_Q + 1, a + n_Q + 1\}, a \leq i - 1 + l(T). \tag{11.9}$$

Plugging the first estimate for j and the estimate for a into (11.4), we find that

$$d \leq \dim S^m_{n-1} + 3 + m + 3n_R + \frac{7}{2}n_Q + n_P + \left\lfloor \frac{n_P - \epsilon_P}{2} \right\rfloor - n - \frac{1}{2} + l(H) + l(H').$$

We will use the second estimate for j in the case $2 \nmid n_Q$ later.

To summarize, we see that

$$d \leq \dim S^m_{n-1} + m + 3n_R + n_P + \left\lfloor \frac{n_P - \epsilon_P}{2} \right\rfloor - n + l(H) + l(H') + \left\{ \begin{array}{l} \frac{5}{2} + \frac{7}{2}n_Q, \text{ if } 2 \nmid n_Q \\ 2 + \frac{7}{2}n_Q, \text{ if } 2 \mid D_Q \\ 1 + \frac{9}{2}n_Q, \text{ if } 2 \mid D_Q + K_Y \end{array} \right\}. \tag{11.10}$$

Suppose first that $m > 0$. Then by Lemma 11.2.1

$$\begin{aligned}\dim S^m_{n-1} &\leq \frac{1}{2}(c_1 - (4n_R + 3n_Q + 2n_P)f)^2 + 1 + \\ &\quad \frac{1}{2}c_1^2 + 3 - 3n_R - 2n_Q - n_P - l(T) - l(T') - m \\ &= c_1^2 + 4 - 7n_R - 5n_Q - 3n_P - l(T) - l(T') - m.\end{aligned}$$

Plugging this into (11.10) we get

$$\begin{aligned}&d \leq \\ &\left\{\begin{array}{l} c_1^2 + \frac{13}{2} - 4n_R - \frac{3}{2}n_Q - 2n_P + \left\lfloor \frac{n_P - \epsilon_P}{2} \right\rfloor - n, \text{ if } 2 \nmid n_Q \\ c_1^2 + 6 - 4n_R - \frac{3}{2}n_Q - 2n_P + \left\lfloor \frac{n_P - \epsilon_P}{2} \right\rfloor - n, \text{ if } 2|D_Q \\ c_1^2 + 5 - 4n_R - \frac{1}{2}n_Q - 2n_P + \left\lfloor \frac{n_P - \epsilon_P}{2} \right\rfloor - n, \text{ if } 2|D_Q + K_Y \end{array}\right\} \\ &- (l(T) - l(H)) - (l(T') - l(H')).\end{aligned}$$

As $l(H) \leq l(T)$ and $l(H') \leq l(T')$, we find in each case that $d < c_1^2 + 9 = \dim M_H(v, L)$. Thus we may assume that $m = 0$. If $n - 1 > 0$, then from Lemma 11.2.1 we see that

$$\begin{aligned}\dim S^0_{n-1} &\leq \frac{1}{2}(c_1 - (4n_R + 3n_Q + 2n_P)f)^2 + l(\tilde{Z}) - (n-2) \\ &= c_1^2 + 5 - 7n_R - 5n_Q - 3n_P - l(T) - l(T') - n.\end{aligned}$$

Plugging this into (11.10), we see that

$$\begin{aligned}&d \leq \\ &\left\{\begin{array}{l} c_1^2 + \frac{15}{2} - 4n_R - \frac{3}{2}n_Q - 2n_P + \left\lfloor \frac{n_P - \epsilon_P}{2} \right\rfloor - 2n, \text{ if } 2 \nmid n_Q \\ c_1^2 + 7 - 4n_R - \frac{3}{2}n_Q - 2n_P + \left\lfloor \frac{n_P - \epsilon_P}{2} \right\rfloor - 2n, \text{ if } 2 \mid D_Q \\ c_1^2 + 6 - 4n_R - \frac{1}{2}n_Q - 2n_P + \left\lfloor \frac{n_P - \epsilon_P}{2} \right\rfloor - 2n, \text{ if } 2 \mid D_Q + K_Y \end{array}\right\} \\ &- (l(T) - l(H)) - (l(T') - l(H')).\end{aligned}$$

In all three cases, we again see that $d < M_H(v, L)$. Finally, we must deal with the case $m = 0, n - 1 = 0$, i.e. S^0_0. This is an open dense subset of $Y^{[l(\tilde{Z})]}$, so plugging $\dim Y^{[l(\tilde{Z})]} = 2l(\tilde{Z}) = 2(\frac{1}{2}c_1^2 + 3 - 3n_R - 2n_Q - n_P - l(T) - l(T'))$ into (11.10) again gives a sublocus of $\dim M_H(v, L)$ of strictly smaller dimension in all cases.

Let us summarize the above considerations. Having assumed that $a = h^0(\tilde{P}(-D_Q + K_Y)) > 0$, we found that the corresponding locus in $M_H(v, L)$ had strictly smaller dimension. It follows that the generic element of $M_H(v, L)$ must lie in the preimage of the $U^{b-1}_{D_R,D_Q,0}$ as D_R, D_Q, b are allowed to vary. Again letting d denote the dimension of this preimage for any such choice, we find that

$$d \le \dim W^{i-1,0}_{D_R,D_Q,j,b} + j - i + n_R \le \dim U^{b-1}_{D_R,D_Q,0} + 2n_R + n_Q + \left\lfloor \frac{n_Q - \epsilon_Q}{2} \right\rfloor + j + 1 - i - b + l(H). \tag{11.11}$$

Generically, we would expect

$$h^0(\tilde{P}(-D_Q)) = b - 1 = 0$$

for $\tilde{P} \in M_H(v(\tilde{P}))$ as $\chi(\tilde{P}(-D_Q)) < 0$. So suppose that $b > 1$, and thus we can write

$$0 \to I_{Z'}(D_P) \to \tilde{P}(-D_Q) \to I_Z(L - 4D_R - 3D_Q - D_P + K_Y) \to 0,$$

for effective D_P and 0-cycles Z, Z' as before. Similarly, let

$$U^{b-1,m}_{D_R,D_Q,D_P,0,n} := \left\{ \tilde{P} \in U^{b-1}_{D_R,D_Q,0} : \begin{array}{l} \bullet\ P^{\vee\vee}(-D_Q - D_P) \text{ has a regular section,} \\ \bullet\ h^0(P^{\vee\vee}(-D_Q - D_P)) = m \\ \bullet\ h^0(P^{\vee\vee}(-D_Q - D_P + K_Y)) = n \end{array} \right\},$$

$$\widetilde{U^{b-1,m}_{D_R,D_Q,D_P,0,n}} := \left\{ P^{\vee\vee} | \tilde{P} \in U^{b-1,m}_{D_R,D_Q,D_P,0,n} \right\},$$

and define

$$S^{m-1}_n := \left\{ \tilde{Z} | h^0(I_{\tilde{Z}}(B - D_P + K_Y)) = m - 1, h^0(I_{\tilde{Z}}(B - D_P)) = n \right\}.$$

Here again we have $S^{m-1}_n \subset Y^{[l(\tilde{Z})]}$ with $l(\tilde{Z}) = \frac{1}{2}c_1^2 + 3 - 3n_R - 2n_Q - n_P - l(T) - l(T')$, and likewise we see that

$$\dim \widetilde{U^{b-1,m}_{D_R,D_Q,D_P,0,n}} \le \dim S^{m-1}_n - m + n + 2 + n_R + n_Q + n_P - l(T) - l(T').$$

From the argument leading up to (11.1) we get that

$$\dim U^{b-1,m}_{D_R,D_Q,D_P,0,n} \le \dim S^{m-1}_n - m + n + 2 + n_R + n_Q + n_P + \left\lfloor \frac{n_P - \epsilon_P}{2} \right\rfloor + l(H') - l(T).$$

Plugging this into (11.11) we find that

$$\begin{aligned} d \le \dim S^{m-1}_n + 3 + 3n_R + 2n_Q + n_P + \left\lfloor \frac{n_P - \epsilon_P}{2} \right\rfloor + \left\lfloor \frac{n_Q - \epsilon_Q}{2} \right\rfloor \\ + n - m - l(T) + l(H') + l(H) + j - i - b. \end{aligned}$$

Recall the estimates from (11.6,11.7,11.9),

$$j \leq \left\{\begin{array}{l} n_Q + 1 + b, \text{ if } n_Q \text{ is odd} \\ n_Q + b, \text{ if } n_Q \text{ is even, and } \epsilon_Q = 0 \\ a + n_Q, \text{ if } n_Q \text{ is even, and } \epsilon_Q = 1 \end{array}\right\}, \tag{11.12}$$

where the final case was shown on the way to proving (11.8). As we are currently assuming $a = 0$, it follows that

$$d \leq \dim S_n^{m-1} + 3n_R + 3n_Q + n_P + \left\lfloor \frac{n_P - \epsilon_P}{2} \right\rfloor + \left\lfloor \frac{n_Q - \epsilon_Q}{2} \right\rfloor \tag{11.13}$$

$$+ n - m - (l(T) - l(H)) + l(H') - i + \left\{\begin{array}{l} 4, \text{ if } 2 \nmid n_Q \\ 3, \text{ if } 2 \mid D_Q \\ 3 - b, \text{ if } 2 \mid D_Q + K_Y \end{array}\right\} \tag{11.14}$$

$$\leq \dim S_n^{m-1} + 3n_R + 3n_Q + n_P + n - m + l(H') - i + \tag{11.15}$$

$$\left\lfloor \frac{n_P - \epsilon_P}{2} \right\rfloor + \left\lfloor \frac{n_Q - \epsilon_Q}{2} \right\rfloor + \left\{\begin{array}{l} 4, \text{ if } 2 \nmid n_Q \\ 3, \text{ if } 2 \mid D_Q \\ 3 - b, \text{ if } 2 \mid D_Q + K_Y \end{array}\right\}, \tag{11.16}$$

since $l(H) \leq l(T)$.

From Lemma 11.2.1, it follows that if $n > 0$ then

$$\begin{aligned} \dim S_n^{m-1} \leq & \frac{1}{2}(c_1 - (4n_R + 3n_Q + 2n_P)f)^2 + 1 + \\ & \frac{1}{2}c_1^2 + 3 - 3n_R - 2n_Q - n_P - l(T) - l(T') - n \\ & = c_1^2 + 4 - 7n_R - 5n_Q - 3n_P - l(T) - l(T') - n. \end{aligned}$$

Combining these two inequalities we obtain

$$d \leq c_1^2 - 4n_R - 2n_Q - 2n_P + \left\lfloor \frac{n_P - \epsilon_P}{2} \right\rfloor + \left\lfloor \frac{n_Q - \epsilon_Q}{2} \right\rfloor - m - l(T) - (l(T') - l(H')) - i$$

$$+ \left\{\begin{array}{l} 8, \text{ if } 2 \nmid n_Q \\ 7, \text{ if } 2 \mid D_Q \\ 7 - b, \text{ if } 2 \mid D_Q + K_Y \end{array}\right\} < \dim M_H(v, L)$$

in all cases since again $l(H') \leq l(T')$. Thus we may assume $n = 0$.

If $m-1>0$, then

$$\begin{aligned}\dim S_0^{m-1} \leq \frac{1}{2}&(c_1-(4n_R+3n_Q+2n_P)f)^2+1+\\ &\frac{1}{2}c_1^2+3-3n_R-2n_Q-n_P-l(T)-l(T')-(m-1)\\ &=c_1^2+5-7n_R-5n_Q-3n_P-l(T)-l(T')-m,\end{aligned}$$

and similar to above we get

$$d \leq c_1^2-4n_R-2n_Q-2n_P+\left\lfloor\frac{n_P-\epsilon_P}{2}\right\rfloor+\left\lfloor\frac{n_Q-\epsilon_Q}{2}\right\rfloor-2m-l(T)-(l(T')-l(H'))-i$$
$$+\left\{\begin{array}{l}9, \text{ if } 2\nmid n_Q\\ 8, \text{ if } 2\mid D_Q\\ 8-b, \text{ if } 2\mid D_Q+K_Y\end{array}\right\}<\dim M_H(v,L)$$

in all cases since $l(H')\leq l(T'), m>1, i\geq 1$.

We are left with the case $m=1, n=0$, i.e. S_0^0, which is a dense open subset of $Y^{[l(\tilde{Z})]}$ of dimension

$$\begin{aligned}2l(\tilde{Z}) &= 2(\frac{1}{2}c_1^2+3-3n_R-2n_Q-n_P-l(T)-l(T'))\\ &=c_1^2+6-6n_R-4n_Q-2n_P-2l(T)-2l(T').\end{aligned}$$

Plugging this into (11.13), we obtain that

$$\begin{aligned}&d \leq c_1^2-3n_R-n_Q-n_P-2l(T)-l(T')-(l(T')-l(H'))-i+\\ &\left\lfloor\frac{n_P-\epsilon_P}{2}\right\rfloor+\left\lfloor\frac{n_Q-\epsilon_Q}{2}\right\rfloor+\left\{\begin{array}{l}9, \text{ if } 2\nmid n_Q\\ 8, \text{ if } 2\mid D_Q\\ 8-b, \text{ if } 2\mid D_Q+K_Y\end{array}\right\}<\dim M_H(v,L).\end{aligned}$$

It follows that the generic element of any component of $M_H(v,L)$ must lie in the preimage of $U^0_{D_R,D_Q,0}$, i.e. $b=1$, which we have shown to be an open dense subset of $M_H(v(\tilde{P}))$, which is of dimension $v(\tilde{P})^2+1=c_1^2+9-4n_R-2n_Q-4l(T)$. Then plugging $a=0, b=1$ and this dimension into (11.11), we get

$$d \leq \dim U^0_{D_R,D_Q,0}+2n_R+n_Q+\left\lfloor\frac{n_Q-\epsilon_Q}{2}\right\rfloor+j-i+l(H) \tag{11.17}$$
$$=c_1^2+9-2n_R-n_Q-4l(T)+\left\lfloor\frac{n_Q-\epsilon_Q}{2}\right\rfloor+j-i+l(H) \tag{11.18}$$
$$\leq c_1^2+9-2n_R-n_Q-3l(T)+\left\lfloor\frac{n_Q-\epsilon_Q}{2}\right\rfloor+j-i. \tag{11.19}$$

Now we note that from the exact sequence

$$0 \to \mathcal{O}_Y(D_Q + K_Y) \to Q^{\vee\vee}(-D_R) \to \tilde{P}(K_Y) \to 0,$$

it follows that $h^0(Q^{\vee\vee}(-D_R) \geq h^0(\mathcal{O}_Y(D_Q + K_Y)) = \left\lceil \frac{n_Q+\epsilon_Q}{2} \right\rceil$, and from

$$0 \to Q(-D_R) \to Q^{\vee\vee}(-D_R) \to T \to 0,$$

we get that $h^0(Q^{\vee\vee}(-D_R) \leq h^0(Q(-D_R)) + l(T)$, i.e.

$$\left\lceil \frac{n_Q + \epsilon_Q}{2} \right\rceil \leq i - 1 + l(T).$$

Plugging this into (11.17) we get

$$d \leq c_1^2 + 8 - 2n_R - n_Q - 2l(T) - \epsilon_Q + j,$$

and using (11.12)[2] and the fact that $a = 0, b = 1$, we get that $d < \dim M_H(v, L)$ unless $l(Q^{\vee\vee}/Q) = n_R = \epsilon_Q = 0$. Assuming this to be the case, we see that $D_R = 0$ and $Q = Q^{\vee\vee}$, so

$$d \leq \dim U^0_{0,D_Q,0} + n_Q + j - i = c_1^2 + 9 - n_Q + j - i.$$

As $\epsilon_Q = 0$, we get from (11.12) (and the estimate $j \leq n_Q + 1 + a$ if n_Q is odd), that in fact

$$d \leq c_1^2 + 10 - i.$$

Note that if $D_Q \neq 0$, then $D_Q + K_Y$ is still effective, so we get

$$0 \to \mathcal{O}_Y(D_Q + K_Y) \to Q \to P(K_Y) \to 0,$$

and thus $i - 1 = h^0(Q) > 0$, which implies that $d < \dim M_H(v, L)$. Thus we must have $D_Q = 0$. It follows that $h^0(Q) = a = 0, h^0(Q(K_Y)) = b = 1$.

Finally, we can conclude from all of the above that the generic R on every component of $M_H(v, L)$ lies in $V^1_{0,1}$ and as such determines uniquely and is uniquely determined by $Q \in W^0_{0,0,1} \subset M_H(v - v(\mathcal{O}_Y), L)$. Likewise, Q determines $Q(K_Y)$ which determines uniquely and is uniquely determined by $P \in U^0_{0,0,0} \subset M_H(v - 2v(\mathcal{O}_Y), L + K_Y)$. In

[2]When n_Q is odd we instead use the estimate $j \leq n_Q + 1 + a$ as shown above in (11.9).

the course of the work above, we've shown that $U^0_{0,0,0}$ is open in $M_H(u, L + K_Y)$. By Theorem 7.1.2, $M_H(u, L+K_Y)$ is non-empty and irreducible. From the uniqueness of everything in this construction, we see that $M_H(v, L)$ is irreducible and birational to $M_H(v - 2v(\mathcal{O}_Y), L + K_Y)$, with an isomorphism between $V^1_{0,1}$ and $U^0_{0,0,0}$, as long as $M_H(v, L)$ is non-empty.

To show this non-emptiness, take a locally free $P \in U^0_{0,0,0}$. Then as $\chi(P) = -1$, we get that there is a unique non-trivial extension

$$0 \to \mathcal{O}_Y \to Q(K_Y) \to P \to 0.$$

We show that $Q(K_Y)$ is μ-stable. Suppose that $Q(K_Y)$ had a stable destabilizing saturated subsheaf G of rank $r' = 1, 2$. Then G is locally free with $\mu_H(G) > \mu_H(Q(K_Y)) > 0$ so that there is a non-zero morphism $G \to P$ and thus $\mu_H(G) \leq \mu_H(P)$. If $\mu_H(G) = \mu_H(P)$ then $G \to P$ would be an isomorphism as a non-zero morphism between stable locally free sheaves of the same slope [24, Proposition 4.7], which would contradict the non-triviality of the extension. Thus $\mu_H(Q(K_Y)) < \mu_H(G) < \mu_H(P)$ which implies that

$$\frac{1}{3} = \frac{c_1(Q(K_Y)).f}{3} \leq \frac{c_1(G).f}{r'} \leq \frac{c_1.f}{2} = \frac{1}{2}.$$

The only possibility is that $r' = 2$ and $c_1(G).f = 1$.

Now let $F \in |2F_A|$ be a generic smooth elliptic fibre. Upon restricting to F, we get a non-zero morphism $\psi|_F$ between $G|_F$ and $P|_F$, which are both semistable rank 2 vector bundles on F of the same degree, namely 2. Note that semistability follows from the suitability of H. It follows that ψ_F is an isomorphism, splitting the restriction of the extension, and thus that the natural restriction map

$$\rho : \mathrm{Ext}^1_Y(P, \mathcal{O}_Y) \to \mathrm{Ext}^1(P|_F, \mathcal{O}_F)$$

is zero. We will show that this is not the case. The map ρ is dual to the natural coboundary map

$$\rho^\vee : H^0(F, P|_F) \to H^1(Y, P(K_Y))$$

coming from the long exact sequence on cohomology of the short exact sequence

$$0 \to P(K_Y) \to P(F + K_Y) \to P|_F \to 0.$$

If $\rho^\vee = 0$, then we have a short exact sequence

$$0 \to H^0(P(K_Y)) \to H^0(P(F + K_Y)) \to H^0(P|_F) \to 0,$$

and since $P|_F$ is semistable of degree 2 on the elliptic curve F, it follows that $h^0(F, P|_F) = 2$ [5]. Thus $h^0(P(F + K_Y)) = h^0(P(K_Y)) + 2 = 2$ for $P \in U^0_{0,0,0}$, a contradiction. So $Q(K_Y)$ is μ-semistable. As c_1 is primitive, $Q(K_Y)$ is μ-stable as well. Furthermore, it is clear that $h^0(Q(K_Y)) = 1$ and $h^0(Q) = 0$. As $\chi(Q) = \chi(Q(K_Y)) = 0$, we also get $h^1(Q(K_Y)) = 1$.

We consider the unique non-split extension R in $\mathrm{Ext}^1(Q, \mathcal{O}_Y) \cong H^1(Q(K_Y))^\vee$, fitting into the exact sequence

$$0 \to \mathcal{O}_Y \to R \to Q \to 0.$$

We would like to show that R is μ-stable. If R were unstable, then there would be a saturated stable destabilizing sheaf of rank $r = 1, 2, 3$, which we may presume to be locally free as R is. Then $\mu(G) > \mu(R) > 0$, and as a consequence we get a nonzero morphism $\psi : G \to Q$ so that $\mu(G) \leq \mu(Q)$. If $\mu(G) = \mu(Q)$, then as above, ψ would have to be an isomorphism contradicting non-splitness of the extension. Thus

$$\mu(R) < \mu(G) < \mu(Q),$$

which implies that

$$\frac{1}{4} = \frac{c_1.f}{4} \leq \frac{c_1(G).f}{r} \leq \frac{c_1.f}{3} = \frac{1}{3},$$

from the suitability of H. We are left with the possibility that $r = 3$ and $c_1(M).f = 1$. Then restricting to a generic smooth elliptic fibre $F \in |2F_A|$, we get $\psi|_F : G|_F \to Q|_F$ is a non-zero morphism between stable vector bundles of the same slope, which is thus an isomorphism. As this splits the restricted extension, we see that the restriction map

$$\rho : \mathrm{Ext}^1(Q, \mathcal{O}_Y) \to \mathrm{Ext}^1(Q|_F, \mathcal{O}_F)$$

is zero. We will again see that this is impossible. Consider the dual map

$$\rho^\vee : H^0(F, Q|_F) \to H^1(Y, Q(K_Y)),$$

which again comes from the long exact sequence of cohomology for the short exact sequence

$$0 \to Q(K_Y) \to Q(F + K_Y) \to Q|_F \to 0.$$

As $Q|_F$ is a stable vector bundle of degree 2 on the elliptic curve F, $h^0(Q|_F) = 2$, so if $\rho^\vee = 0$, then $h^0(Q(F + K_Y)) = h^0(Q(K_Y)) + h^0(Q|_F) = 1 + 2 = 3$. We show that this cannot be the case.

Pushing forward the exact sequence defining $Q(K_Y)$,

$$0 \to \mathcal{O}_Y \to Q(K_Y) \to P \to 0,$$

we get the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to p_*Q(K_Y) \to p_*P \to \mathcal{O}_{\mathbb{P}^1}(-1) \to R^1p_*Q(K_Y) \to R^1p_*P, \tag{11.20}$$

where $R^1p_*\mathcal{O}_Y \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ follows from [17, Proposition 5.1.3 and Corollary 5.1.1]. We first note that as $Q(K_Y)$ and P are torsion free, so are $p_*Q(K_Y)$ and p_*P. It follows that they are locally free of rank 2 since their restrictions to the generic fibre have two dimensional spaces of sections. Thus $p_*Q(K_Y) \cong \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2), p_*P \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$. Since $h^0(Q(K_Y)) = 1$, we must have $d_2 = 0, d_1 < 0$, say. Similarly, since $h^0(P) = 0$ but $h^0(P(F)) = 1$, we have $a < b = -1$. On the reduced fibers F, which are smooth elliptic curves, $h^1(P|_F) = h^1(Q(K_Y)|_F) = 0$ by [5]. So $R^1p_*(P) = R^1p_*Q(K_Y) = 0$ away from the two points under the double fibers. From the Leray spectral sequence, we get

$$0 \to H^1(p_*G) \to H^1(G) \to H^0(R^1p_*G) \to 0,$$

for any coherent sheaf G on Y. Applying this to P, we get that since

$$h^1(p_*P) = h^1(\mathcal{O}_{\mathbb{P}^1}(a)) = h^0(\mathcal{O}_{\mathbb{P}^1}(-a-2)) \geq 1,$$

as $-a \geq 2$, so $h^1(P) = 1$ implies that $R^1p_*P = 0$ and $a = -2$. Similarly, we see that $h^1(p_*Q(K_Y)) = h^1(\mathcal{O}_{\mathbb{P}^1}(d_1)) = h^0(-d_1 - 2)$, and this quantity is at most 1 from the Leray spectral sequence and the fact that $h^1(Q(K_Y)) = 1$. Thus $d_1 = -1$ or -2.

Let $F' \in |2F_A|$ be any fibre, possibly a double fiber. Then as $Q(K_Y)|_{F'} \cong Q|_{F'}$, we may take cohomology of the exact sequence

$$0 \to Q(-F') \to Q \to Q|_{F'} \to 0,$$

to calculate the cohomology of $Q(K_Y)|_{F'}$ using the fact that $h^i(Q) = 0$ for all i. For generic fibre F, we get $h^1(Q(-F)) = 2$ and $h^0(Q(-F)) = h^2(Q(-F)) = 0$ since the restriction of Q to F is stable of degree 2 and thus has $h^0(Q|_F) = 2, h^1(Q|_F) = 0$. For any other fibre F', $Q(-F') \cong Q(-F)$, so we see that $h^0(Q|_{F'}) = 2, h^1(Q|_{F'}) = 0$. Thus $R^1p_*Q(K_Y) = 0$ since the restriction to any fibre has $h^1 = 0$. Taking degrees in (11.20), we see that $d_1 = -2$. Thus $p_*Q(K_Y) \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}$, from which it follows by the projection formula that

$$p_*Q(F + K_Y) \cong p_*Q(K_Y) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$

Thus $h^0(Q(F + K_Y)) = 2$, a contradiction. Thus R is μ-semistable. As c_1 is primitive, genericity of H implies that R is μ-stable as well. □

11.2 Appendix: Dimension estimates for Brill-Noether loci on Hilbert schemes of points

We prove in this appendix the dimension estimate we used in the body of the paper. The result concerns bounding the dimension of the locus of 0-cycles with given cohomology with respect to a linear system.

Lemma 11.2.1. *Let L be an effective divisor and $S^i := \{Z | h^0(I_Z(L)) = i\} \subset Y^{(l)}$. Then for $i > 0$,*

$$\dim S^i \leq \dim |L| + l(Z) - (i - 1).$$

In particular, if L is ample, then $\dim |L| = \frac{1}{2}L^2$, so

$$\dim S^i \leq \frac{1}{2}L^2 + 1 + l(Z) - i.$$

Proof. Denote by $S := \{(Z, \mathbb{C}v) | \mathbb{C}v \in |I_Z(L)|\} \subset Y^{(l)} \times |L|$. Then the second projection

$$p_2 : S \to |L|,$$

is surjective with fibers of dimension $l(Z)$. Indeed, for any $C \in |L|$ the fiber over C is $C^{(l)}$. Thus $\dim S = \dim |L| + l(Z)$. The image of S under the first projection is

$$p_1(S) = \bigcup_{i>0} S^i,$$

where the fiber over $Z \in Y^{(l)}$ is $|I_Z(L)|$. Thus $p_1^{-1}(S^i)$ is a $\mathbb{P}^{i-1}$ bundle over S^i, from which it follows that

$$\dim S^i \leq \dim |L| + l(Z) - (i-1).$$

□

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