AN EXTENSION OF THE BIANCHI-EGNELL STABILITY ESTIMATE TO BAKRY, GENTIL, AND LEDOUX'S GENERALIZATION OF THE SOBOLEV INEQUALITY TO CONTINUOUS DIMENSIONS AND APPLICATIONS

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ABSTRACT OF THE DISSERTATION

An Extension of the Bianchi-Egnell Stability Estimate to Bakry, Gentil, and Ledoux's Generalization of the Sobolev Inequality to Continuous Dimensions and Applications

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The main result of this dissertation is an extension of a stability estimate of the Sobolev Inequality established by Bianchi and Egnell in [BiEg]. Bianchi and Egnell's Stability Estimate answers the question raised by H. Brezis and E. H. Lieb in [BrLi]: "Is there a natural way to bound $\|\nabla\varphi\|_2^2 - C_N^2\|\varphi\|_{\frac{2N}{N-2}}^2$ from below in terms of the 'distance' of φ from the manifold of optimizers in the Sobolev Inequality?" Establishing stability estimates - also known as quantitative versions of sharp inequalities - of other forms of the Sobolev Inequality, as well as other inequalities, is an active topic. See [CiFu], [DoTo], and [FiMa], for stability estimates involving Sobolev inequalities and [CaFi], [DoTo], and [FuMa] for stability estimates on other inequalities. In this dissertation, we extend Bianchi and Egnell's Stability Estimate to a Sobolev Inequality for "continuous dimensions." Bakry, Gentil, and Ledoux have recently proved a sharp extension of the Sobolev Inequality for functions on $\mathbb{R}_+ \times \mathbb{R}^n$, which can be considered as an extension to "continuous dimensions." V. H. Nguyen determined all cases of equality. The dissertation extends the Bianchi-Egnell stability analysis for the Sobolev Inequality to this "continuous dimensional" generalization.

The secondary result of this dissertation is a sketch of the proof of an extension

of a stability estimate of a single case of a sharp Gagliardo-Nirenberg inequality to a whole family of Gagliardo-Nirenberg inequalities, whose sharp constants and extremals were calculated by Del Pino and Dolbeault in [DeDo]. The original stability estimate for the Gagliardo-Nirenberg inequality was stated and proved by E. Carlen and A. Figalli in [CaFi]. The proof for its extension to the entire class of sharp Gagliardo-Nirenberg inequalities of Del Pino and Dolbeault is a direct application of the extension of the Bianchi-Egnell Stability Estimate to Bakry, Gentil, and Ledoux's extension of the Sobolev Inequality to continuous dimensions.

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Chapter 1

Introduction: Functional Inequalities and Stability Estimates

Functional inequalities have been used in a variety of ways in the study of PDEs. In this dissertation, we will state and prove an extension of a stability estimate on a sharp Sobolev Inequality on continuous dimensions and then explain how one could use this result to prove a stability estimate on a family of sharp Gagliardo-Nirenberg inequalities. In this introduction, we will go into considerable detail of the historical context concerning the stability estimates of interest. We begin the a description of the work of the dissertation in subsection 1.4 on page 18. We will begin the introduction with discussion of the Sobolev Inequality, which is of particular interest in this dissertation and to the author.

1.1 The Sobolev Inequality: Early History

The Sobolev Inequality has been a pivotal inequality in the study of PDEs. We state the sharp form of the Sobolev Inequality below:

THEOREM 1.1.1. *Let $N \geq 3$ and $\dot{H}^1(\mathbb{R}^N)$ be the completion of compactly supported functions under the gradient square norm, $\|\nabla \cdot \|_2$. Then,*

$$\|\varphi\|_{2^*} \leq C_N \|\nabla\varphi\|_2\,,\ \forall\varphi \in \dot{H}^1(\mathbb{R}^N)\,, \tag{1.1}$$

where $2^* = \frac{2N}{N-2}$ and C_N is a sharp constant. There is equality if and only if φ is a constant multiple of some

$$F_{t,x_0}(x) = \hat{k} t^{\frac{N-2}{2}} (1 + t|x - x_0|^2)^{\frac{N-2}{2}}\,, \tag{1.2}$$

for $t > 0$, $x_0 \in \mathbb{R}^N$, and $\hat{k}$ a constant such that $\|\nabla F_{1,0}\|_2 = 1$. Consequently,

$$C_N = \|F_{1,0}\|_{2^*} / \|\nabla F_{1,0}\|_2\,.$$

The history of this inequality with its classification of extremals and the calculation of its sharp constant is a bit complicated. Part of the reason for this is that the inequality and its properties have not always been deduced with an eye to application. Instead, the Sobolev Inequality and many of its developments have been proven as objects of interest in themselves, and applications were deduced later. We recount some of this history in the following.

The proof of Theorem 1.1.1 is usually credited to S.L. Sobolev in his 1938 paper, [So], which does not classify extremals or provide a formula for a sharp constant. The Sobolev Inequality was essentially first proved by G.A. Bliss in 1930. To be more precise, a change of variables to the inequality that Bliss derives in [Bl] yields the Sobolev Inequality for radial nondecreasing nonnegative functions. Bliss's result includes formulas for the extremals centered at the origin and the sharp constant of the Sobolev Inequality. If Bliss had knowledge of symmetric decreasing rearrangements, a technique that was used many times later in the analysis of the Sobolev Inequality, he could have used it to prove the Sobolev Inequality with classification of extremals and a formula for the sharp constant. Bliss, however, proved the Sobolev Inequality as a calculus of variations result, without any applications to PDEs in mind. Indeed, Bliss's results got little attention. In fact, Bliss's paper only received a handful of citations up until 1976, when G. Talenti published a paper on the sharp constant of the Sobolev Inequality and classified its extremals.

PDE applications of the Sobolev Inequality did not appear until after Sobolev's 1938 paper, [So]. Moreover, Sobolev's paper was not cited in papers until the early 1950s. Early applications of the Sobolev Inequality to PDEs did not make use of the sharp constant or extremals of the inequality, suggesting that Bliss's work was not generally known. For example, in 1959, S. Agmon published a paper deducing regularity results for solutions of elliptic operators with zero Dirichlet data on balls and hemispheres of balls using the Sobolev Inequality without reference to the sharp constant or extremals, see [Ag] for detail. See also [La], [Fr], and [Bro] for more applications of the Sobolev Inequality to PDEs without use of the sharp constant or extremals.

Without application of symmetric decreasing rearrangement to Bliss's result, and possibly due to a general lack of knowledge of Bliss's work, the sharp constant and extremals for the Sobolev Inequality for $N \geq 3$ was not known for 38 years following Sobolev's proof of the Sobolev Inequality. We discuss some of the major developments leading to the calculation of sharp constants and classification of extremals, and their applications in the next section.

1.2 Search for the Sharp Sobolev Constant, the Extremals, and Some of Applications The Sharp Constant and Extremals

In this section, we chronicle some of the developments leading to the calculation of sharp constants and classification of extremals for the Sobolev Inequality and some of their applications. There have been ground breaking results employing the sharp constant and the extremals. And interestingly, even though Bliss had essentially laid the groundwork for finding the sharp constants and the extremals, it took a while for Mathematicians to calculate the sharp constant and classify the extremals of the Sobolev Inequality. T. Aubin and G. Talenti classified the sharp constants and the extremals of the Sobolev Inequality in 1976, see [Au] and [Ta] for detail. However, estimates of the sharp constant had been used in applications well before this. To our knowledge, the first application of the Sobolev Inequality with an explicit constant is by H. Fujita and then shortly after by Fujita and T. Kato. In his 1961 paper, [Fuj], Fujita used the Sobolev Inequality with an explicit constant to prove existence of weak steady-state solutions of the Navier-Stokes equation provided the boundary data satisfy an explicit smallness condition. In their 1964 paper, [FuKa], Fujita and Kato used the Sobolev Inequality with an explicit constant to prove global existence in time of solutions of the Navier-Stokes equation with suitably small initial conditions. In both papers, the explicit constant in the Sobolev Inequality was used to compute explicit smallness conditions which would guarantee the existence of solutions. In his 1961 paper, Fujita derived an upper bound for the sharp Sobolev constant for $N = 3$ using elementary arguments that we present here in the following.

The Sobolev constant that Fujita derived is given in the following

LEMMA 1.2.1. *Let C_3 denote the sharp constant of the Sobolev Inequality for $N = 3$. Then*

$$C_3 \leq (4/\pi)^{1/3} . \tag{1.3}$$

Proof. We begin by recalling Hardy's Inequality: Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ and set $\hat{\varphi}(x) = \varphi(x)/|x-y|$ for some $y \in \mathbb{R}^3$. Then

$$\|\hat{\varphi}\|_2 \leq 2\|\nabla\varphi\|_2 . \tag{1.4}$$

Having recalled Hardy's inequality, we can now prove Lemma 1.2.1. Let $\varphi \in C_0^\infty(\mathbb{R}^3)$. Let $\psi = \varphi^4$. We have the formula

$$\psi(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\left(\nabla_y \frac{1}{|x-y|}\right)\cdot\nabla_y\psi(y)\mathrm{d}y$$

Thus,

$$\varphi^4(x) = \frac{1}{\pi}\int_{\mathbb{R}^3}\frac{x_i - y_i}{|x-y|}\varphi^3(y)\frac{\partial}{\partial y}\varphi(y)\mathrm{d}y . \tag{1.5}$$

Multiplying (1.5) on both sides by $\varphi^2(x)$ and integrating over $\mathbb{R}^3$ with respect to x, and then changing order of integration and applying (1.4), we obtain

$$\begin{aligned}\|\varphi\|_6^6 \leq & \frac{4}{\pi}\|\nabla\varphi\|_2^2\int_{\mathbb{R}^3}|\varphi^3(y)||\nabla\varphi(y)|\mathrm{d}y\,, \text{ which by Schwartz's Inequality} \\ \leq & \frac{4}{\pi}\|\nabla\varphi\|_2^3\|\varphi\|_6^3 .\end{aligned}$$

□

The first major development in the direction of calculating precise constants and extremals is a 1971 paper by R. Rosen, [Ro], in which it is shown that the sharp constant for $N = 3$ is no bigger than

$$\frac{4^{1/3}}{3^{1/2}\pi^{1/2}} . \tag{1.6}$$

Actually, the constant given by (1.6) is the sharp Sobolev constant for $N = 3$. Indeed, Rosen claims to show that (1.6) is the sharp constant for class C^0 piecewise C^2 functions in $\dot{H}^1(\mathbb{R}^3)$. However, his proof only demonstrates that this is the best constant that is achieved in the Sobolev Inequality for $N = 3$ for class C^0 piecewise C^2 functions, but Rosen's proof does not disprove the possibility that the best constant in the Sobolev

Inequality is not achieved. If Rosen knew that equality in the Sobolev Inequality is achieved, then his proof would have been complete. But, he does not state or prove this fact, and to our knowledge, existence of extremals in the Sobolev Inequality is absent from literature up to this point in history. Nevertheless, Rosen actually computes the extremals of the Sobolev Inequality and uses them to calculate (1.6), which is the sharp constant for the Sobolev Inequality, but fails to prove that these are the extremals and the sharp constant.

Rosen approaches the problem by examining the functional given by

$$R(\varphi) = \|\nabla\varphi\|_2^6/\|\varphi\|_6^6 \,. \tag{1.7}$$

for functions of class C^0 piecewise C^2. In particular, Rosen deduces that if $\hat{\varphi}$ is a critical point of this functional, $\hat{\varphi}$ would satisfy the following PDE:

$$\Delta\hat{\varphi} + \frac{\|\nabla\hat{\varphi}\|_2^2}{\|\hat{\varphi}\|_6^6}\hat{\varphi}^5 = 0 \,. \tag{1.8}$$

Performing a second order Taylor expansion of R with remainder about a solution of (1.8), $\hat{\varphi}$, Rosen obtains

$$F(\hat{\varphi}+\psi) = F(\varphi)\left[1 + 3\|\nabla\hat{\varphi}\|_2^{-2}\int |\nabla\omega|^2 - 5\frac{\|\nabla\hat{\varphi}\|_2^2}{\|\hat{\varphi}\|_6^6}\hat{\varphi}^4\omega^2 \mathrm{d}x + O(\omega^3)\right] , \tag{1.9}$$

where ω is the projection of ψ onto the orthogonal space of $\hat{\varphi}^5$ in L^2. Thus, $\hat{\varphi}$ is a local minimizer of R if

$$I(\omega) \equiv \int |\nabla\omega|^2 - 5\frac{\|\nabla\hat{\varphi}\|_2^2}{\|\hat{\varphi}\|_6^6}\hat{\varphi}^4\omega^2 \mathrm{d}x \tag{1.10}$$

is nonnegative for all ω orthogonal in L^2 to $\hat{\varphi}^5$. Rosen concludes that if for a given solution to (1.8), $\hat{\varphi}$, if the eigenvalue problem

$$\left(-\Delta - 5\frac{\|\nabla\hat{\varphi}\|_2^2}{\|\hat{\varphi}\|_6^6}\right)\omega_n = \lambda_n\omega_n \,, \text{ and } \int \omega_n\hat{\varphi}^5 \mathrm{d}x = 0 \,, \tag{1.11}$$

only admits nonnegative values, then (1.10) will be nonnegative. He concludes by arguing that the only solutions, $\hat{\varphi}$, of (1.8) such that the resulting eigenvalue problem (1.11) admits no negative eigenvalues are the extremals of the Sobolev Inequality. This means that solutions to (1.8) other than the Sobolev extremals are not local minima for the functional R. Thus, the only critical points of the functional R that are of class

C^0 and piecewise C^2 (recall that Rosen restricts himself to this class) that are in fact a minimum of R are the extremals of the Sobolev Inequality. Using this information, Rosen concludes by computing the sharp Sobolev constant. He states that this constant is "the minimum value" for the Sobolev constant for φ "of class C^0 piecewise C^2." It is, of course, by density the Sharp Sobolev constant for all of $\dot{H}^1(\mathbb{R}^3)$.

In a paper published in 1976, [Au], T. Aubin computed the sharp constant for the Sobolev Inequality and classified its extremals. Slightly later in 1976, G. Talenti also published a paper calculating the sharp constant and classifying the extremals. Aubin's work is set in a geometric investigation of the isoperimetric inequality, whereas Talenti's proof is deduced in a purely analytic argument. In fact, the first step of Talenti's proof is to take the symmetric decreasing rearrangement of an arbitrary function in $\dot{H}^1(\mathbb{R}^N)$, show that taking these rearrangements holds the L^{2^*} norm of the function constant while potentially lowering the gradient square norm, and then exhibiting the extremals of the Sobolev Inequality for radial (and in fact all) functions. The nature of the settings and proofs of Aubin and Talenti respectively have greatly influenced how many citations each author received. At this time, Aubin's paper has received about 700 citations, while Talenti's has received about 1,400. That said, Aubin's work led to very important results on a major problem in Geometry, the Yamabe Conjecture. We take some time to recount some of this application of the sharp constant and extremals of the Sobolev Inequality here.

The Sobolev Inequality in its sharp form with extremals was applied in the study of the Yamabe problem, which is summarized as follows: "Given a compact Riemannian manifold (M, g) of dimension $N \geq 3$, find a metric conformal to g with constant scalar curvature." Solving the Yamabe problem is equivalent to solving a nonlinear eigenvalue problem. The Sobolev Inequality has been applied to the analysis of this eigenvalue problem. We will take a moment to explain this connection. Given a Riemannian metric g, $\tilde{g}$ is conformal to g if there is some smooth positive function φ such that $\tilde{g} = \varphi^{2^*-1}g$. Thus, there is a metric, $\tilde{g} = \varphi^{2^*-1}g$, for some smooth positive scalar φ

with constant scalar curvature, λ, if and only if φ satisfies the following PDE:

$$\left(4\frac{N-1}{N-2}\Delta + S\right)\varphi = \lambda\varphi^{2^*-1}\,, \tag{1.12}$$

where S is the scalar curvature of g and Δ is the Laplacian with respect to g. (1.12) is the Euler-Lagrange equation for

$$Q(\tilde{g}) = \frac{\int_M \tilde{S}\mathrm{d}V_{\tilde{g}}}{\left(\int_M \mathrm{d}V_{\tilde{g}}\right)^{(N-2)/N}}\,, \tag{1.13}$$

where $\tilde{S}$ is the scalar curvature of $\tilde{g}$ and $\tilde{g}$ varies over metrics conformally equivalent to g. Note that

$$Q(\tilde{g}) = Q(\varphi^{2^*-1}) = Q_g(\varphi)\,,$$

where

$$Q_{\tilde{g}}(\varphi) = E(\varphi)/\|\varphi\|_{2^*}^2\,,$$

for

$$E(\varphi) = \int_M 4\frac{N-1}{N-2}|\nabla\varphi|^2 + S\varphi^2\mathrm{d}V_g\,,\ \text{and}\ \|\varphi\|_{2^*} = \left(\int_M |\varphi|^{2^*}\right)^{1/2^*}\,.$$

where ∇ is the covariant derivative with respect to g. Solving for the Euler Lagrange equation of (1.13), we deduce that φ is a critical point for Q if and only if it satisfies (1.12) with $\lambda = E(\varphi)/\|\varphi\|_{2^*}^{2^*}$. Having provided this background, we can begin to draw the connection between the Yamabe problem and the Sharp Sobolev Inequality.

The Yamabe invariant, $\lambda(M)$, for (M, g) is given by

$$\begin{aligned}\lambda(M) =& \inf\{Q(\tilde{g}) : \tilde{g} \text{ is conformal to } g\}\\ =& \inf\{Q_g(\varphi) : \varphi \text{ is a smooth, positive function on } M\}\,.\end{aligned} \tag{1.14}$$

The Yamabe invariant is a crucial quantity in the study of the Yamabe problem. The work of Yamabe, Trudinger, and Aubin led to the following

THEOREM 1.2.2. *The Yamabe problem can be solved on any compact manifold M with $\lambda(M) < \lambda(\mathbb{S}^N)$, where $\mathbb{S}^N$ is the sphere with its standard metric.*

It turns out that the Yamabe invariant for the sphere $\mathbb{S}^N$ with the standard metric is a constant multiple of the sharp constant in the Sobolev Inequality. Aubin was able to use the extremals of the Sobolev Inequality to prove that $\lambda(M) \leq \lambda(\mathbb{S}^N)$ for all compact manifolds. The essence of Aubin's proof is to show that for a fixed ball in $\mathbb{R}^N$, the infimum of the ratio $\|\nabla\varphi\|_2/\|\varphi\|_{2^*}$ is the sharp constant, because one can take a Sobolev extremal, and dilate the function so as to squeeze the mass of the function into the ball. This description is vague and imprecise. We will flesh it out in the next subsection in a non-geometric setting, as it is used again by a work by H. Brezis and L. Nirenberg that we will explain, see the outline of the proof of Theorem 1.3.1 provided on pages seven and eight for detail. Relating this squeezing argument to achieve the infimum, we conclude that $\lambda(M) \leq \lambda(\mathbb{S}^N)$ for all compact M.

As shown above, there has been a lot of interplay between PDEs and the sharp Sobolev Inequality and its extremals. The extremals have been used to analyze PDEs, while differential equations have been used to help determine the sharp constant of the Sobolev Inequality and its extremals. Also, as briefly alluded above, the application of symmetric decreasing rearrangement to functions has been very useful in determining the sharp constant and extremals of the Sobolev Inequality. We will now recount a 1983 argument by E. Lieb, [Li] that uses rearrangements and analysis of ODEs to calculate the sharp constant in the Sobolev Inequality and one of its extremals.

Lieb actually proves extremals and calculates the sharp constant for a generalization of the Sobolev Inequality, but modifying this proof to deal exclusively with the Sobolev Inequality is enlightening enough and allows us to keep notation simple. Before preceding, we make the following convention: SD will denote the class of symmetric decreasing functions. Lieb proves the following:

THEOREM 1.2.3. *Let $N \geq 3$, then an extremal of the Sobolev Inequality, $F \in SD$, exists and is given by*

$$F(x) = (1+|x|^2)^{-\frac{N-2}{2}}\,, \text{ and} \tag{1.15}$$

$$C_N = [\pi N(N-2)]^{-1/2}[\Gamma(N)/\Gamma(N/2)]^{1/N}\,. \tag{1.16}$$

Proof. In the course of this proof, Lieb restricts himself to $\varphi \in H^1(\mathbb{R}^N) = \dot{H}^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, but is still able to deduce the extremal F as given above - for $N \in \{3,4\}$, $F \notin H^1(\mathbb{R}^N)$. If $\varphi : \mathbb{R}^N \to \mathbb{R}$ is a function, let φ^* denote the symmetric decreasing rearrangement of φ. To be more precise, φ^* is the unique decreasing nonnegative function such that

$$|\{x \in \mathbb{R}^N : \varphi^*(x) \geq k\}| = |\{x \in \mathbb{R}^N : |\varphi(x)| \geq k\}|\,,\ \forall k > 0\,,$$

where $|\cdot|$ applied to a set denotes Lebesgue measure. The first step of Lieb's proof is to observe that we can focus our attention on functions $\varphi \in SD$, because

$$\|\nabla\varphi^*\|_2 \leq \|\nabla\varphi\|_2\,,\ \text{and}\ \|\varphi^*\|_p = \|\varphi\|_p\,,\ \text{for}\ 1 \leq p \leq \infty.$$

Next, Lieb makes a clever substitution of variables, which allows him to examine the problem of identifying an extremal in $H^1(\mathbb{R}^N)$, for $N \geq 3$, to a new extremal problem in $H^1(\mathbb{R})$. In particular, if we take $\varphi(r)$ to be a the radial representative of $\varphi \in SD$ and make the following change of variables:

$$\Upsilon\left(\frac{N-2}{2}r\right) = e^{\frac{N-2}{2}r}\varphi(e^r)\,. \tag{1.17}$$

Under these change of variables,

$$\left(\frac{2\omega_N}{N-2}\right)^{1/2^*}\|\Upsilon\|_{2^*} = \|\varphi\|_{2^*}\,,\ \text{and}\ \left(\frac{N-2}{2}\omega_N\right)^{1/2}\|\Upsilon' - \Upsilon\|_2 = \|\nabla\varphi\|_2\,, \tag{1.18}$$

where ω_N is the area of the $(N-1)$-sphere. Since, $\varphi \in L^2 \cap SD$,

$$\begin{aligned}
\infty >&\|\varphi\|_2^2 \\
=&\omega_N\int_0^\infty \varphi r^{N-1}\mathrm{d}r \\
\geq&\omega_N\int_0^R \varphi r^{N-1}\mathrm{d}r \\
\geq&\omega_N R^N\varphi(R)^2 \\
\Longrightarrow&\ 0 \leq \varphi(r) \leq Cr^{-\frac{N}{2}}\,.
\end{aligned} \tag{1.19}$$

for some $C > 0$. Thus, $\Upsilon(r) \leq Ce^{-\frac{2}{N-2}r}$.

If we make the further assumption that $\varphi \in L^\infty$, then $\Upsilon(r) \leq Ce^{-|r|/(N-2)}$, which in turn implies that $\int \Upsilon'\Upsilon = 0$, and thus

$$\frac{\|\varphi\|_{2^*}^2}{\|\nabla\varphi\|_2^2} = \omega_N^{-1+2/2^*}\left(\frac{N-2}{2}\right)^{-1-2/2^*}\frac{\|\Upsilon\|_{2^*}^2}{\|\Upsilon'\|_2^2 + \|\Upsilon\|_2^2} =: \omega_N^{-1+2/2^*}\left(\frac{N-2}{2}\right)^{-1-2/2^*} T(\Upsilon)\,. \tag{1.20}$$

Since L^∞ is dense on $\dot{H}^1(\mathbb{R}^N)$, the identity, (1.20), reduces the proof of Theorem 1.2.3 to the proof of the following

THEOREM 1.2.4. *Let*

$$M_{2^*} = \sup\{T(\Upsilon) : \Upsilon \in H^1(\mathbb{R}) \setminus \{0\}\}\,.$$

Then M_{2^*} is finite and achieved by

$$\Upsilon = const.\cosh^{-\frac{N-2}{2}}\left(\frac{2}{N-2}r\right)\,, \quad and \tag{1.21}$$

$$M_{2^*} = \left[\frac{(N-1)\Gamma(N-2)}{\frac{N-2}{2}\Gamma\left(\frac{N-2}{2}\right)^2}\right]^{1-2/2^*}\left(\frac{N-2}{8}\right)^{2/2^*}\frac{2}{N}\,. \tag{1.22}$$

Proof. By the properties of rearrangements, $T(\Upsilon^*) \geq T(\Upsilon)$, so we may assume $\Upsilon \in SD$. Thus, $\Upsilon \in L^\infty$, as $\Upsilon(r) \to 0$ as $r \to -\infty$ and

$$\Upsilon(r)^2 = 2\int_{-\infty}^{r} \Upsilon'(\rho)\Upsilon(\rho)\mathrm{d}\rho \leq 2\|\Upsilon'\|_2\|\Upsilon\|_2\,. \tag{1.23}$$

Let (Υ_i) be a maximizing sequence for T, with $\|\Upsilon_i\|_2^2 + \|\Upsilon_i'\|_2^2 = 1$. A similar argument to the one used to deduce (1.19) allows us to conclude that $\Upsilon_i(r) \leq C|r|^{-1/2}$. Combining this with (1.23), we conclude that

$$\Upsilon_i(r) \leq h(r) \equiv \min(C, C|r|^{-1/2}) \in L^{2^*}\,,$$

because $2^* > 2$. Combining $\|\Upsilon_i\|_2^2 + \|\Upsilon'\|_2^2 = 1$ and (1.23), we deduce that $0 \leq \Upsilon_i \leq \sqrt{2}$ for all i. Thus, by the Helly Selection Principle, we may assume $\Upsilon_i \to \Upsilon \in SD$ pointwise. And so, by the Dominated Convergence Theorem, we know that $\|\Upsilon\|_{2^*} = M_{2^*}$. We can also assume that $\Upsilon_i \rightharpoonup G$ and $\Upsilon_i' \rightharpoonup G'$ in L^2. Since $\Upsilon_i \to \Upsilon$ pointwise, $\Upsilon = G$. Thus,

$$1 = \liminf \|\Upsilon_i'\|_2^2 + \|\Upsilon_i\|_2^2 \geq \|\Upsilon'\|_2^2 + \|\Upsilon\|_2^2\,.$$

Combining this with the fact that $\|\Upsilon\|_{2^*} = M_{2^*}$, we conclude that $T(\Upsilon) = M_{2^*}$.

In order to find a maximizing element, $\Upsilon \in SD$ such that $\|\Upsilon\|_{2^*} = M_{2^*}$ and $\|\Upsilon\|_2^2 + \|\Upsilon'\|_2^2 = 1$, Lieb observes that such an element would be a critical point of T and consequently satisfy the ODE:

$$\Upsilon'' = \Upsilon - \Upsilon^{2^*-1}/M_{2^*}\,, \tag{1.24}$$

in the distributional sense. By standard ODE methods, there is only one solution to (1.24) that vanishes as $|r| \to \infty$. This solution is (1.21) with appropriate constant on the right hand side of (1.21). $\square$

Having proved Theorem 1.2.4, we conclude Theorem 1.2.3. □

1.3 A Bridge from the Sharp Sobolev Inequality to the Sobolev Inequality Stability Estimate

In 1983 H. Brezis and L. Nirenberg wrote a paper, [BrNi], chronicling their investigation of positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. In the process of this investigation, they proved (as a corollary of their main results) an improved Sobolev Inequality. This inequality was derived by Brezis and Nirenberg in an effort to solve the following nonlinear elliptic PDE with critical Sobolev exponent:

$$\begin{aligned} -\Delta\varphi =&\varphi^{2^*-1} + f(x,\varphi)\,, \text{ on } \Omega \\ \varphi >&0\,, \text{ on } \Omega \\ \varphi =&0\,, \text{ on } \partial\Omega\,, \end{aligned} \tag{1.25}$$

for $\Omega \subseteq \mathbb{R}^N$ a bounded domain with $N \geq 3$, $f(x,0) = 0$, and $f(x,\varphi)$ a lower-order perturbation of φ^{2^*-1} in the sense $\lim_{u\to\infty} f(x,\varphi)/\varphi^{2^*-1} = 0$. Solutions of (1.25) are critical points of the functional

$$\Phi(u) = \frac{1}{2}\int |\nabla\varphi|^2 \mathrm{d}x - \frac{1}{2^*}\int \varphi^{2^*}\mathrm{d}x - \int \tilde{F}(x,\varphi)\mathrm{d}x \tag{1.26}$$

where $\tilde{F}(x,\varphi) = \int_0^\varphi f(x,t)\mathrm{d}t$. Brezis and Nirenberg study this problem in stages. They begin by studying (1.25) $f(x,\varphi) = \lambda\varphi$ for $N > 4$ and $N = 3$ respectively (these cases have different proofs). They then turn their attention to (1.25) with general lower order perturbations $f(x,\varphi)$. We will spend some time explaining Brezis and Nirenberg's analysis of (1.25) with $f(x,\varphi) = \lambda\varphi$, as this work illustrates some applications of the Sobolev Inequality and is extremals. A happy byproduct of this work is an improved Sobolev Inequality that opens the way to some radical improvements on the Sobolev Inequality.

Before getting into the details of of Brezis and Nirenberg's analysis for (1.25) with $f(x,\varphi) = \lambda\varphi$, we will give a general overview of their results. First, we rewrite (1.25) in this context:

$$
\begin{aligned}
-\Delta\varphi =&\varphi^{2^*-1}+\lambda\varphi\,,\ \text{on }\Omega\\
\varphi >&0\,,\ \text{on }\Omega\\
\varphi =&0\,,\ \text{on }\partial\Omega\,.
\end{aligned}
\tag{1.27}
$$

The results of their analysis on (1.27) are the following two theorems:

THEOREM 1.3.1. *Let λ_1 be the lowest eigenvalue of $-\Delta$ with zero Dirichlet condition on a bounded domain $\Omega \subseteq \mathbb{R}^N$ with $N \geq 4$. Then for every $\lambda \in (0, \lambda_1)$, there exists a solution of (1.27).*

THEOREM 1.3.2. *Assume $\Omega \subseteq R^3$ is a ball and let λ_1 be the lowest eigenvalue of $-\Delta$ with zero Dirichlet condition on Ω. There exists a solution of (1.27) if and only if $\lambda \in (\frac{1}{4}\lambda_1, \lambda_1)$*

It is easy to show that (1.27) has a solution only if $\lambda < \lambda_1$. To see this, let $\psi_1 > 0$ be an eigenvector of $-\Delta$ with zero Dirichlet condition on Ω and eigenvalue λ_1, and let u be a solution of (1.27) for some λ. Then,

$$
\begin{aligned}
\lambda\int\varphi\psi_1 =&\int(-\Delta\varphi-\varphi^{2^*-1})\psi_1\,,\ \text{which by integration by parts}\\
=&\lambda_1\int\varphi\psi_1-\int\varphi^{2^*-1}\psi_1\\
<&\lambda_1\int\varphi\psi_1\\
\Longrightarrow&\ \lambda > \lambda_1\,.
\end{aligned}
$$

By Pohozaev's Identity, Brezis and Nirenberg show that (1.27) does not have a solution if $\lambda \leq 0$. Although this proof is not difficult, we skip over it in favor of the parts of their paper relating to the Sobolev Inequality and its extremals.

We will only outline the proof of Theorem 1.3.1, i.e. the case when $N \geq 4$ because the proof for Theorem 1.3.2 is similar, but more subtle and not necessary to recount for our purposes. We begin demonstrating the connection of Brezis and Nirenberg's work to the Sobolev Inequality by outlining their proof of Theorem 1.3.1. The heart of their proof is to show that

$$
k_\lambda := \inf_{\varphi\in H_0^1,\|\varphi\|_{2^*}=1}\{\|\nabla\varphi\|_2^2-\lambda\|\varphi\|_{2^*}^2\}
\tag{1.28}
$$

is achieved by some $\varphi \in H_0^1(\Omega)$ for $\lambda > 0$ and that $k_\lambda > 0$ when $0 < \lambda < \lambda_1$ - $H_0^1(\Omega)$ is the closure of compactly supported C^∞ functions under the gradient square norm. Note that $k_0 = C_N^{-2}$. Critical points in H_0^1 (we will often write H_0^1 a shorthand for $H_0^1(\Omega)$ of the functional

$$\Psi(\varphi) = \|\nabla\varphi\|_2^2 - \lambda\|\varphi\|_2^2\,, \tag{1.29}$$

restricted to the sphere $\|\varphi\|_{2^*} = 1$ satisfy the PDE

$$-\Delta\varphi - \lambda\varphi = k_\lambda\varphi^{2^*-1}\,. \tag{1.30}$$

Note that we may assume a minimizer φ of (1.30) is nonnegative. If such a minimizer φ is not nonnegative, we simply take $|\varphi|$ to obtain a nonnegative minimizer. Since φ satisfies (1.30), $\tilde{\varphi} = C_N^{-2/(2^*-1)}\varphi$ will satisfy (1.27); note that $\tilde{\varphi} > 0$ by the strong maximum principle. This proves existence of solutions of (1.27) for $0 < \lambda < \lambda_1$, provided we can show that the infimum k_λ is achieved by Ψ for φ such that $\|\varphi\|_{2^*} = 1$. We explain the proof of this fact in the following paragraphs.

The proof that the desired infimum is achieved hinges on two things. First, Brezis and Nirenberg prove that $k_\lambda < C_N^{-2}$ for $\lambda > 0$. Second, they prove that the infimum, (1.28), is achieved when $k_\lambda < C_N^{-2}$.

The proof that $k_\lambda < C_N^{-2}$ for $\lambda > 0$ was inspired by Aubin's argument in the context of the Yamabe conjecture. The idea of Brezis and Nirenberg's proof is to multiply extremals of the Sobolev Inequality on $\mathbb{R}^N$ by a nonnegative cutoff function in Ω and then vary a corresponding parameter to show that some quantity, Q_λ, given by

$$Q_\lambda(\varphi) = \frac{\|\nabla\varphi\|_2^2 - \lambda\|\varphi\|_2^2}{\|\varphi\|_{2^*}^2}$$

will be less than C_N^{-2}. To be precise, assume $0 \in \Omega$ and fix some nonnegative smooth cutoff function ψ such that $\psi \equiv 1$ in a neighborhood of 0. Then let

$$\varphi_\varepsilon(x) = \psi(x)/(\varepsilon + |x|^2)^{\frac{N-2}{2}}\,,\ \varepsilon > 0\,.$$

Letting $\varepsilon \to 0$ effectively "squeezes" the mass of the optimizer into a neighborhood near 0, so that

$$\|\nabla\varphi_\varepsilon\|_2^2/\|\varphi_\varepsilon\|_{2^*}^2 \to C_N^{-2}\,,$$

and $\|\varphi_\varepsilon\|_2^2$ is large enough relative to $\|\nabla\varphi_\varepsilon\|_2^2$ so that

$$Q_\lambda(\varphi) = \frac{\|\nabla\varphi_\varepsilon\|_2^2 - \lambda\|\varphi_\varepsilon\|_2^2}{\|\varphi_\varepsilon\|_{2^*}^2} < C_N^{-2}\,,$$

for some small ε.

The proof that the infimum is achieved for $k_\lambda < C_N^{-2}$ follows from a functional analysis argument. We take a moment to explain it here:

LEMMA 1.3.3 (Lieb)**.** *If $k_\lambda < C_N^{-2}$, the infimum, (1.28), is achieved.*

Proof. Let (φ_i) be a minimizing sequence:

$$\|\varphi_i\|_{2^*} = 1 \tag{1.31}$$

$$\|\nabla\varphi_i\|_2^2 - \lambda\|\varphi_i\|_2^2 = k_\lambda + o(1)\,. \tag{1.32}$$

Extract a subsequence (still denoted (φ_i)) such that

$$\begin{aligned}
&\varphi_i \rightharpoonup \varphi\,, \text{ in } H_0^1 \text{ ("$\rightharpoonup$" denotes weak convergence)}\,,\\
&\varphi_i \rightarrow \varphi\,, \text{ in } L^2\,,\\
&\varphi_i \rightarrow \varphi\,, \text{ almost everywhere on } \Omega\,,
\end{aligned}$$

with $\|\varphi\|_{2^*} \le 1$. Set $\psi_i = \varphi_i - \varphi$, such that

$$\begin{aligned}
&\psi_i \rightharpoonup 0\,, \text{ in } H_0^1\,,\\
&\psi_i \rightarrow 0\,, \text{ a.e. on } \Omega\,.
\end{aligned}$$

Note that $\|\nabla\varphi_i\|_2 \ge C_N^{-2}$. Combining this with (1.32), we conclude that

$$\lambda\|\varphi\|_2^2 \ge C_N^{-2} - k_\lambda > 0\,.$$

Thus, $\varphi \not\equiv 0$. Using (1.32) and the fact that $\varphi_i \to \varphi$ in L^2, we conclude that

$$\|\nabla\varphi\|_2^2 + \|\nabla\psi_i\|_2^2 - \lambda\|\varphi\|_2^2 = k_\lambda + o(1)\,. \tag{1.33}$$

On the other hand, applying the Brezis-Lieb Lemma (since $\|\psi_i\|_{2^*}$ is bounded and $\psi_i \to 0$ a.e.)

$$1 = \|\varphi + \psi_i\|_{2^*}^{2^*} = \|\varphi\|_{2^*}^{2^*} + \|\psi_i\|_{2^*}^{2^*} + o(1)\,.$$

Thus,

$$\begin{aligned} 1 &\leq \|\varphi\|_{2^*}^2 + \|\psi_i\|_{2^*}^2 + o(1)\,, \text{ which by the Sobolev Inequality} \\ &\leq \|\varphi\|_{2^*}^2 + C_N^2 \|\nabla\psi_i\|_{2^*}^2 + o(1)\,. \end{aligned} \tag{1.34}$$

We claim that

$$\|\nabla\varphi\|_2^2 - \lambda\|\varphi\|_2^2 \leq k_\lambda \|\varphi\|_{2^*}^2\,, \tag{1.35}$$

which would conclude the proof of Lemma 1.3.3. We prove (1.35) in two cases:

1. $k_\lambda > 0$ (i.e. $0 < \lambda < \lambda_1$)

2. $k_\lambda \leq 0$ (i.e. $\lambda \geq \lambda_1$)

In case 1., (1.34) implies that

$$k_\lambda \leq k_\lambda \|\varphi\|_{2^*}^2 + k_\lambda C_N^2 \|\nabla\psi_i\|_2^2 + o(1)\,. \tag{1.36}$$

Combining (1.33) and (1.36), we conclude (1.35).

In case 2., we have $k_\lambda \leq k_\lambda \|\varphi\|_{2^*}^2$, since $\|\varphi\|_{2^*} \leq 1$. We deduce again (1.35) from (1.33). □

Having proved Theorem 1.3.2, Brezis and Nirenberg use this result to prove the following improvement on the Sobolev Inequality:

COROLLARY 1.3.4. *Assume $\Omega \subseteq \mathbb{R}^3$ is a bounded domain. Then, there exists λ^*, $0 < \lambda^* < \lambda_1$, such that*

$$\|\nabla\varphi\|_2^2 \geq C_3^{-2}\|\varphi\|_6^2 + \lambda^*\|\varphi\|_2^2\,,\ \forall\varphi \in H_0^1(\Omega)\,. \tag{1.37}$$

We may take $\lambda^* = \frac{1}{4}(3|\Omega|/4\pi)^{-2/3}$, where $|\cdot|$ denotes Lebesgue measure. This value is sharp when Ω is a ball.

Proof. Let Ω^* be a ball such that $|\Omega^*| = |\Omega|$ and φ^* denote the symmetric decreasing rearrangement of φ. It is known (see [Li] or [Ta]) that

$$\begin{aligned} &\varphi \in H_0^1(\Omega) \implies \varphi^* \in H_0^1(\Omega^*)\,, \text{ and} \\ &\|\nabla\varphi^*\|_{L^2(\Omega^*)}^2 \leq \|\nabla\varphi\|_{L^2(\Omega)}^2\,. \end{aligned} \tag{1.38}$$

On the other hand, for all $\varphi^* \in H_0^1(\Omega^*)$,

$$\|\nabla\varphi^*\|_{L^2(\Omega^*)}^2 \geq C_3^{-2}\|\varphi^*\|_{L^6(\Omega^*)}^2 + \frac{1}{4}\lambda_1(\Omega^*)\|\varphi^*\|_{L^2(\Omega^*)} . \tag{1.39}$$

This is because there is no solution of 1.27 for $\lambda = \frac{1}{4}\lambda_1(\Omega^*)$, see Theorem 1.3.2. That is, if (1.39) were false, then $k_\lambda < C_3^{-2}$ for $\lambda = \frac{1}{4}\lambda_1(\Omega^*)$, which by Lemma 1.3.3 implies that 1.27 has a solution for $\lambda = \frac{1}{4}\lambda_1(\Omega^*)$, contradicting Theorem 1.3.2. The value of $\lambda^* = \frac{1}{4}(3|\Omega|/4\pi)^{-2/3}$ follows from the fact that $\lambda_1(\Omega^*) = \pi^2/R^2$, where R is the radius of Ω^*. Combining (1.38), (1.39), and the fact that $\|\varphi^*\|_{L^q(\Omega^*)} = \|\varphi\|_{L^q(\Omega)}$ or all q, we obtain (1.37). $\square$

Brezis and Nirenberg observed that (1.37) cannot be extended to $N \geq 4$ with the extra term in the Sobolev Inequality being a constant multiple of $\|\varphi\|_2^2$. However, they deduce the following inequality:

$$\|\nabla\varphi\|_2^2 \geq C_N^{-2}\|\varphi\|_{2^*}^2 + \lambda_p(\Omega)\|\varphi\|_p^2 , \ \forall\varphi \in H_0^1(\Omega) , \tag{1.40}$$

for $\Omega \subseteq \mathbb{R}^N$ a bounded domain, $1 \leq p < 2^*/2$, with $\lambda_p(\Omega)$ a constant depending only on p, N, and Ω and $\lambda_p(\Omega) \to 0$ as $p \to 2^*/2$. In the following year, 1984, Brezis and E. Lieb proved some further improvements upon the Sobolev Inequality for bounded domains. They also posed an important question in the direction of improving the Sobolev Inequality: "Is there a natural way to bound $C_N^2\|\nabla\varphi\|_2^2 - \|\varphi\|_{2^*}^2$ from below in terms of the 'distance' of φ from the set of [extremals given by (1.2)]."

1.4 Motivation for Our Main Results: Deriving Gagliardo-Nirenberg Stability Estimates from Sobolev Stability Estimates

In 1990, G. Bianchi and H. Egnell came up with a strong positive answer to Brezis and Lieb's question, in the form of the following stability estimate:

$$C_N^2\|\nabla\varphi\|_2^2 - \|\varphi\|_{2^*}^2 \geq \alpha d(\varphi, M)^2 , \ \forall\varphi \in \dot{H}^1(\mathbb{R}^N) , \tag{1.41}$$

for some positive constant $\alpha > 0$, all $N \geq 3$, and where $d(\cdot, M)$ is the distance functional given by

$$d(\varphi, M) = \inf_{c\in\mathbb{R}, t>0, x_0\in\mathbb{R}^N} \|\nabla(\varphi - cF_{t,x_0})\|_2 . \tag{1.42}$$

As mentioned above, (1.41) is a stability estimate, which is an inequality that bounds the difference in terms of a sharp inequality from below by the distance of a given function from the extremals of the sharp inequality. All of the work in this dissertation is focused on proving an extension of Bianchi and Egnell's stability estimate to continuous dimensions and applying this result to deduce a further stability estimate on a family of sharp Gagliardo-Nirenberg inequalities whose sharp constant and extremals were deduced by Del Pino and Dolbeault.

An open problem open problem associated with the Bianchi-Egnell Stability Estimate is the calculation of an explicit constant α that satisfies (1.41). In the original stability estimate of Bianchi and Egnell, there is no explicit constant given in the inequality

$$C_N^2\|\nabla\varphi\|_2^2 - \|\varphi\|_{2^*}^2 \geq \alpha d(\varphi, M)^2\,. \tag{1.43}$$

All we know is that there is some $\alpha > 0$ for which the above is true. This is because the process of obtaining the stability estimate involves finding a local stability estimate and then applying concentration compactness to prove that (1.43) must hold for some $\alpha > 0$. Our extension of the Bianchi-Egnell Stability Estimate to continuous dimensions suffers from the same problem. However, following a similar process to the proof of the original Bianchi-Egnell Stability Estimate, we prove a local Bianchi-Egnell Stbaility Estimate, that is in fact more quantitative in nature than Bianchi and Egnell's original local stability estimate. This is because in our proof of our local stability estimate, we use an argument that gets explicit bounds on the remainder of the second order Taylor expansion of the difference of terms in the Sobolev Inequality. Obtaining an explicit constant in our extension of the Bianchi-Egnell Stability Estimate to continuous dimensions would be useful for applications, because it would yield an explicit constant for the full class of sharp Gagliardo-Nirenberg inequalities of Del Pino and Dolbeault. E. Carlen and A. Figalli used a special case of this stability estimate to solve a Keller-Segal Equation. If we had an explicit constant for the stability estimate for the full class of Gagliardo-Nirenberg inequalities of Del Pino and Dolbeault, it could be useful for more PDE applications.

In a recent paper, E. Carlen and A. Figalli applied the Bianchi-Egnell Stability

Estimate to prove a stability estimate for a Gagliardo-Nirenberg inequality and used this result to help solve a Keller-Segal equation. The GN (Gagliardo-Nirenberg) inequality for which Carlen and Figalli prove their stability estimate is a special case of a family of sharp GN inequalities classified by Del Pino and Dolbeault. A natural question is whether one could generalize this stability estimate to the complete family of sharp GN inequalities of Del Pino and Dolbeault. Much of the work in this dissertation is oriented toward solving this problem.

We will present Carlen and Figalli's stability estimate. But, before doing so, we lay some foundation. We begin by stating Del Pino and Dolbeault's statement of the family of sharp GN inequalities: Let $u \in \dot{H}^1(\mathbb{R}^n)$ for $n \geq 2$. Then for $1 \leq s \leq n/(n-2)$ (if $n = 2$, $1 \leq s < \infty$)

$$\|u\|_{2s} \leq A_{n,s} \|\nabla u\|_2^{\gamma} \|u\|_{s+1}^{1-\gamma} \,, \; \gamma = \frac{n(s-1)}{s[2n-(1+s)(n-2)]} \,. \tag{1.44}$$

$A_{n,s}$ is a sharp constant given by

$$A_{n,s} = \frac{\|v\|_{2s}}{\|\nabla v\|_2^{\gamma} \|v\|_{s+1}^{1-\gamma}} \,, \text{ where } v(x) = (1 + |x|^2)^{-1/(s+1)} \,. \tag{1.45}$$

The extremals of (1.44) are the constant multiples of

$$v_{\lambda,x_0}(x) = (1 + \lambda^2 |x - x_0|^2)^{-1/2} \,. \tag{1.46}$$

Carlen and Figalli proved a stability estimate for (1.44) in the special case when $n = 2$ and $s = 3$. Note that $A_{2,3} = \pi^{-1/6}$. When $n = 2$ and $s = 3$, let $\delta_{GN}[\cdot]$ denote the difference of terms in (1.44) given by

$$\delta_{GN}[u] = \|\nabla u\|_2 \|u\|_4^2 - \pi^{1/2} \|u\|_6^3 \,. \tag{1.47}$$

Carlen and Figalli's stability estimate is summarized in the following:

THEOREM 1.4.1. *Let $u \in \dot{H}^1(\mathbb{R}^2)$ be a nonnegative function such that $\|u\|_4 = \|v\|_4$. Then there exist universal constants $K_1, \delta_1 > 0$ such that whenever $\delta_{GN} \leq \delta_1$,*

$$\inf_{\lambda > 0, x_0 \in \mathbb{R}^2} \|u^6 - \lambda^2 v_{\lambda,x_0}\|_1 \leq K_1 \delta_{GN}[u]^{1/2} \,. \tag{1.48}$$

As mentioned before, one would like to generalize the stability estimate of Theorem 1.4.1 to all possible n and s for Del Pino and Dolbeault's family of sharp GN inequalities. To do this via the techniques used by Carlen and Figalli in the proof of Theorem 1.4.1 requires some ingenuity and the extension of Bianchi and Egnell's stability estimate to a continuous dimension setting. To understand why, one must examine the main mechanism in the derivation of Theorem 1.4.1.

The heart of Carlen and Figalli's stability estimate is a link between the Sobolev Inequality and the GN inequalities. This link is recounted in a monograph of D. Bakry, I. Gentil, and M. Ledoux, [BaGe], and was communicated by Bakry to Carlen and Figalli. We will explain the link in the context of Carlen and Figalli's stability estimate, use this link to explain how the Bianchi-Egnell Stability Estimate is connected to Carlen and Figalli's stability estimate, and then give a brief explanation to why the generalization of Carlen and Figalli's estimate requires a continuous dimension extension of the Bianchi-Egnell Stability Estimate.

The heart of Carlen and Figalli's proof of Theorem 1.4.1 is that for φ given by

$$\varphi = (u^{-2}(y) + |z|^2)^{-1}\,,\ y, z \in \mathbb{R}^2\,, \tag{1.49}$$

we have that

$$\sqrt{3}\left(C_4^2\|\nabla\varphi\|_2^2 - \|\varphi\|_4^2\right) = \delta_{GN}[u]\,. \tag{1.50}$$

Applying the Bianchi-Egnell Stability Estimate, we deduce that

$$\begin{aligned} \delta_{GN}[u] \geq & C_0 \inf_{c\in\mathbb{R}, t>0, x_0\in\mathbb{R}^4} \|\nabla(\varphi - cF_{t,x_0})\|_2^2, \text{ which by the Sobolev Inequality} \\ \geq & C_1 \inf_{c\in\mathbb{R}, t>0, x_0\in\mathbb{R}^4} \|\varphi - cF_{t,x_0}\|_4^2\,, \end{aligned} \tag{1.51}$$

for some $x_0 \in \mathbb{R}^4$ and positive constants C_0 and C_1. It takes some work to show that we we can deduce Theorem 1.4.1 from (1.51) and we delay these details until the body of the the dissertation. However, we can now explain why these techniques require the extension of the Bianchi-Egnell Stability Estimate to continuous dimensions.

The fact that Carlen and Figalli could apply the Bianchi-Egnell Stability Estimate to deduce Theorem 1.4.1 with $n = 2$ and $s = 3$ is a happy coincidence. (1.50) is obtained by carefully constructing $\varphi(y, z)$ out of a given $u(y) \in \dot{H}^1(\mathbb{R}^2)$ such that we obtain the relationship given by (1.50). Note that the dimension, $n = 2$, for $u(y)$ corresponds to the y-variable in $\varphi(y, z)$. The fact that the left hand side of (1.50) corresponds to $\delta_{GN}[u]$ for $s = 3$ is a result of the other variable of $\varphi(y, z)$, z, being in two dimensions. If we try to generalize the relationship in (1.50) to a given $u(y) \in \dot{H}^1(\mathbb{R}^N)$ for some carefully chosen $\varphi(y, z)$, for a generalized notion of δ_{GN} over all $n \geq 2$ and $1 \leq s \leq \frac{n}{n-2}$ (or $< \infty$ if $n = 2$), we would run into serious limitations if we restrict ourselves to $\varphi(y, z)$ with z in integer dimensions, i.e. if z is in $\mathbb{R}^m$ with m an integer. To be a little more precise, for a given $u(y) \in \dot{H}^1(\mathbb{R}^N)$ and $\varphi(y, z)$ constructed in a similar fashion to the φ in (1.49), we get a relationship like (1.50) - a constant multiple of the difference in terms of the Sobolev Inequality applied to φ equals the difference in terms in a GN inequality applied to u. The problem is that the value of s that this equality is valid for depends upon the number of dimensions the z-variable of $\varphi(y, z)$ is in. In fact, the only way to obtain this generalized (1.50) for all $n \geq 2$ and $1 \leq s \leq \frac{n}{n-2}$ (or $< \infty$ if $n = 2$), is to consider $\varphi(y, z)$ with the z-variable in continuous dimensions. Consequently, to deduce a stability estimate for the full class of sharp GN inequalities of Del Pino and Dolbeault using Carlen and Figalli's methods, we also need to derive a Bianchi-Egnell stability estimate for $\varphi(y, z)$ with z a continuous dimension variable - we will make this precise in the body of the dissertation.

1.5 Outline of the Body of the Dissertation

Chapter 2: In this chapter we state and prove the main result of the dissertation. This result is a an extension of the Bianchi-Egnell Stability Estimate to continuous dimensions.

Subsection 2.1: This is the introduction for chapter two. We introduce the immediate background to the extension of the Bianchi-Egnell Stability Estimate to continuous dimensions. We also state our main result and a Rellich-Kondrachov type theorem for functions in continuous dimensions. We then outline a proof of the Bianchi-Egnell Stability Estimate using our methods, and then outline the proof of our extension of the Bianchi-Egnell Stability Estimate. We conclude with some examples of applications of he Bianchi-Egnell Stability Estimate.

Subsection 2.2: We present and prove a proposition concerning the second order Taylor Expansion of $\|f + \varepsilon\psi\|_p^2$ at $\varepsilon = 0$ for $p > 2$ and estimate the remainder. This general proposition is useful for our immediate goal of extending the Bianchi-Egnell Stability Estimate, but is also of general interest, as it provides an especially quantitative way to deal with the remainder terms of the calculus of variation argument.

Subsection 2.3: We state a local Bianchi-Egnell Stability Estimate in the continuous dimension setting and outline its proof.

Subsections 2.4 and 2.5: We prove some properties of an operator obtained from the second order coefficient of the Taylor expansion of some $\|\nabla(f + \varepsilon\psi)\|_2^2 - \|f + \varepsilon\psi\|_{2^*}^2$.

Subsection 2.6: We conclude the proof of the Local Bianchi-Egnell Stability Estimate in continuous dimensions.

Subsection 2.7: We make a concentration compactness argument for a sequence of functions that minimize the continuous dimension extension of the Sobolev Inequality. We provide this argument, because no such known argument exists for functions in continuous dimensions.

Subsection 2.8: We prove our Rellich-Kondrachov type theorem for functions on continuous dimensions. Such theorems, to our knowledge, are not currently present in the works other than [Se].

Chapter 3: In this chapter, we state and outline a proof of a stability estimate for a family of sharp Gagliardo-Nirenberg inequalities discovered by Del Pino and Dolbeault.

Subsection 3.1: We give some of the background to our Gagliardo-Nirenberg Stability Estimate and state our result.

Subsection 3.2: We outline a key calculation that provides a bridge from the Sobolev Inequality to the Gagliardo-Nirenberg inequalities of Del Pino and Dolbeault.

Subsection 3.3: We provide an outline of the proof of some special cases of the stability estimate for the sharp Gagliardo-Nirenberg inequalities of Del Pino and Dolbeault using the Sobolev Inequality and the Bianchi-Egnell Stability Estimate. We also explain why the extension to continuous dimensions of the Bianchi-Egnell Stability Estimate proved in chapter 2 is necessary to prove the Gagliardo-Nirenberg Stability Estimate for the whole family of sharp inequalities of Del Pino and Dolbeault, as opposed to just some special cases.

Subsection 3.4: We provide a sketch of the proof of the stability estimate for the full family of sharp Gagliardo-Nirenberg inequalities of Del Pino and Dolbeault. We use the extension of the Bianchi-Egnell Stability Estimate to continuous dimensions in this sketch.

Chapter 2

An Extension of the Bianchi-Egnell Stability Estimate to Bakry, Gentil, and Ledoux's Generalization of the Sobolev Inequality to Continuous Dimensions

2.1 Introduction

This chapter extends a stability estimate of the Sobolev Inequality established by Bianchi and Egnell in [BiEg]. Bianchi and Egnell's Stability Estimate answers the question raised by H. Brezis and E. H. Lieb in [BrLi]: "Is there a natural way to bound $\|\nabla\varphi\|_2^2 - C_N^2\|\varphi\|^2_{\frac{2N}{N-2}}$ from below in terms of the 'distance' of φ from the manifold of optimizers in the Sobolev Inequality?" Establishing stability estimates - also known as quantitative versions of sharp inequalities - of other forms of the Sobolev Inequality, as well as other inequalities, is an active topic. See [CiFu], [DoTo], and [FiMa], for stability estimates involving Sobolev inequalities and [CaFi], [DoTo], and [FuMa] for stability estimates on other inequalities. In this section, we extend Bianchi and Egnell's Stability Estimate to a Sobolev Inequality for "continuous dimensions." Bakry, Gentil, and Ledoux have recently proved a sharp extension of the Sobolev Inequality for functions on $\mathbb{R}_+ \times \mathbb{R}^n$, which can be considered as an extension to "continuous dimensions." V. H. Nguyen determined all cases of equality. The dissertation extends the Bianchi-Egnell stability analysis for the Sobolev Inequality to this "continuous dimensional" generalization.

2.1.1 The Sharp Sobolev Inequality

Let $\dot{H}^1(\mathbb{R}^N)$ be the the completion of the space of smooth real-valued functions with compact support under the norm

$$\|\varphi\|_{\dot{H}^1} := \|\nabla\varphi\|_2 = \left(\int_{\mathbb{R}^N} |\nabla\varphi|^2 \mathrm{d}x\right)^{1/2} ,$$

where for $1 \leq p < \infty$, (and 2 in particular) $\|\varphi\|_p$ denotes the L^p norm of φ, $\|\varphi\|_p = \left(\int_{\mathbb{R}^N} |\varphi|^p \mathrm{d}x\right)^{1/p}$. Define

$$2^* := \frac{2N}{N-2} \ . \tag{2.1}$$

The Sobolev Inequality provides a lower bound for $\|\varphi\|_{\dot{H}^1}$ in terms of $\|\varphi\|_{2^*}$.

THEOREM 2.1.1 (Sharp Sobolev Inequality)**.** *Let $N \geq 3$ be an integer. Then, for all $\varphi \in \dot{H}^1(\mathbb{R}^N) \setminus \{0\}$,*

$$\frac{\|\varphi\|_{2^*}}{\|\varphi\|_{\dot{H}^1}} \leq \frac{\|F_{1,0}\|_{2^*}}{\|F_{1,0}\|_{\dot{H}^1}} =: C_N , \tag{2.2}$$

where

$$F_{t,x_0}(x) := \hat{k}\left(\frac{t}{1+t^2|x-x_0|^2}\right)^{\frac{N-2}{2}} , \tag{2.3}$$

for $t > 0$, $x_0 \in \mathbb{R}^N$, and $\hat{k} > 0$ a constant such that $\|F_{1,0}\|_{\dot{H}^1} = 1$. There is equality if and only if $\varphi = zF_{t,x_0}$ for some $t > 0$, $x_0 \in \mathbb{R}^N$, and some $z \in \mathbb{R} \setminus \{0\}$.

Theorem 2.1.1 in this sharp form, including specification of the cases of equality, was proved by Talenti in [Ta]. The result is also true for complex-valued functions, the only difference being that equality holds for $\varphi = zF_{t,0}$ with $z \in \mathbb{C} \setminus \{0\}$, but for the moment we will restrict our attention to real-valued functions. For another reference on the Sobolev Inequality, see [FrLi].

The fact that the conformal group of $\mathbb{R}^N$ has an action on functions on $\mathbb{R}^N$ that is simultaneously isometric in the L^{2^*} and $\dot{H}^1$ norms determines the sharp constants and optimizers of the inequality. There is a way of using competing symmetries to help deduce the full class of extremals of the Sobolev Inequality for functions on $\mathbb{R}^N$ for $N \geq 3$. This is done as part of a more general setting in a paper by Carlen and Loss, see [CaLo]. Lieb also has a paper, see [Li], in which the Sobolev Inequality is derived and its extremals are deduced via an ODE. In our paper, we deal with a more general setting of the Sobolev Inequality on continuous dimension; we will introduce this in a more precise fashion shortly. The techniques of Carlen and Loss, as well as Lieb, do not appear to have straightforward adaptations to our settings. The conformal subgroup of $\mathbb{R}^N$ that is invariant on $\|\cdot\|_{2^*}$ and $\|\cdot\|_{\dot{H}^1}$ is generated by the following three operations

$$\begin{aligned}
\text{(inversion) } \varphi(x) &\mapsto |x|^{-N+2}\varphi(x/|x|) \\
\text{(dilation) } \varphi(x) &\mapsto \sigma^{\frac{N-2}{2}}\varphi(\sigma x),\ \sigma \in \mathbb{R}_+ \\
\text{(translation) } \varphi(x) &\mapsto \varphi(x-x_0),\ x_0 \in \mathbb{R}^N .
\end{aligned}$$

The extremal functions, $M := \{zF_{t,x_0} | z \in \mathbb{R}, t \in \mathbb{R}_+, x_0 \in \mathbb{R}^N\}$, of (2.2) comprise an $(N+2)$-dimensional manifold in $\dot{H}^1(\mathbb{R}^N)$. We can obtain M by taking the union of the orbits of $zF_{1,0}$ for all $z \in \mathbb{R}$ under the action of conformal group. In fact, M is the union of the orbits of $zF_{1,0}$ for all $z \in \mathbb{R}$ under the subgroup generated by translations and dilations alone.

2.1.2 Bianchi and Egnell's Stability Estimate

A question raised by Brezis and Lieb concerns approximate optimizers of the Sobolev inequality. Suppose for some small $\epsilon > 0$,

$$\frac{\|\varphi\|_{2^*}}{\|\nabla\varphi\|_2} \geq (1-\epsilon)C_N \ .$$

Does it then follow that φ is close, in some metric, to a Sobolev optimizer? A theorem of Bianchi and Egnell gives a strong positive answer to this question. Define the distance between M and a function $\varphi \in \dot{H}^1(\mathbb{R}^N)$ as

$$\delta(\varphi, M) := \inf_{h\in M} \|\nabla(\varphi - h)\|_2 = \inf_{z,t,x_0} \|\nabla(\varphi - zF_{t,x_0})\|_2 \, . \tag{2.4}$$

Bianchi and Egnell's answer to Brezis and Lieb's question is summarized in the following

THEOREM 2.1.2 (Bianchi-Egnell Stability Estimate)**.** *There is a positive constant, α, depending only on the dimension, N, so that*

$$C_N^2\|\nabla\varphi\|_2^2 - \|\varphi\|_{2^*}^2 \geq \alpha\delta(\varphi, M)^2 \, , \tag{2.5}$$

$\forall\varphi \in \dot{H}^1(\mathbb{R}^N)$*. Furthermore, the result is sharp in the sense that it is no longer true if $\delta(\varphi, M)^2$ in (1.5) is replaced with $\delta(\varphi, M)^\beta\|\nabla\varphi\|_2^{2-\beta}$, where $\beta < 2$.*

Recently, Bakry, Gentil, and Ledoux proved a sharp extension of the Sobolev inequality to "fractional dimensions," and showed how it relates to certain optimal Gagliardo-Nirenberg inequalities. V. H. Nguyen has determined all of the extremals in the version of the inequality for real-valued functions. The goal of the present paper is to prove an analogue of Theorem 2.1.2 for this extended Sobolev inequality. Actually, the case we treat is more general, because we generalize Nguyen's classification of extremals from real-valued functions to complex-valued functions. We then prove the analogue of Theorem 2.1.2 for this generalization of Bakry, Gentil, and Ledoux's Theorem with classification of extremals for complex-valued functions. This is notable, because Bianchi and Egnell prove their stability estimate for real-valued functions only, while our stability estimate is for complex-valued functions. This is one of the aspects that make our proof more intricate than Bianchi and Egnell's, but it is hardly the most notable or the most difficult aspect to deal with. Some of the steps in the proof of our extension of the Bianchi-Egnell Stability Estimate are a fairly direct adaptation of steps in Bianchi and Egnell's proof. Others are not. To help highlight these differences, we outline a proof of Theorem 2.1.2 based upon the steps of the proof to our extension of the Bianchi-Egnell Stability Estimate. This outline is provided in subsection 2.1.5. In the outline, we point out where our approach differs from Bianchi and Egnell's, and in particular, which parts require new arguments.

2.1.3 Bakry, Gentil, and Ledoux's Extension of the Sharp Sobolev Inequality with Nguyen's Classification of Extremals

One can generalize the Sharp Sobolev Inequality to continuous dimension, $N > 2$. We can define functions on noninteger dimensions by generalizing the notion of radial functions. To be precise, the L^p-norm of a radial function, φ, on N-dimensional Euclidean space is given by

$$\|\varphi\|_p = \left(\int_{\mathbb{R}_+} |\varphi(\rho)|^p \omega_N \rho^{N-1} \mathrm{d}\rho\right)^{1/p} ,$$

where ω_N is the area of the unit $(N-1)$-sphere given by

$$\omega_N := \frac{2\pi^{N/2}}{\Gamma(\frac{N}{2})} . \tag{2.6}$$

We use this definition to generalize the notion of the area of a unit $(N-1)$-sphere for $N > 0$, possibly noninteger. In order to include the case where N is not an integer, we provide the following definition of $\Gamma(\cdot)$:

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} \mathrm{d}x \,.$$

The L^p-norm of the gradient in radial coordinates is given by

$$\|\varphi\|_{\dot{W}^{1,p}} := \|\nabla\varphi\|_p = \left(\int_{\mathbb{R}_+} |\varphi'(\rho)|^p \omega_N \rho^{N-1} \mathrm{d}\rho \right) \,.$$

Allowing N to take noninteger values larger than 2 gives a generalization of the norms $\|\cdot\|_p$ and $\|\cdot\|_{\dot{W}^{1,p}}$ for noninteger dimensions. In this setting, the analogue of $\dot{H}^1(\mathbb{R}^N)$ from subsection 2.1.1 will be denoted $\dot{W}^{1,p}(\mathbb{R}_+, \omega_N \rho^{N-1} \mathrm{d}\rho)$. $\varphi : [0,\infty) \to \mathbb{C}$ is in $\dot{W}^{1,p}(\mathbb{R}_+, \omega_N \rho^{N-1} \mathrm{d}\rho)$ if and only if $\|\varphi\|_{\dot{W}^{1,p}} < \infty$ and φ is eventually zero in the sense that the measure of $\{|\varphi(\rho)| > \varepsilon\}$ is finite for all $\varepsilon > 0$ with respect to the measure induced by $\omega_N \rho^{N-1} \mathrm{d}\rho$. Having established the appropriate ideas and notation, we state

THEOREM 2.1.3 (Sharp Sobolev Inequality for Radial Functions)**.** *Let $N > 2$, not necessarily an integer, and $p < N$. Then, for all $\varphi \in \dot{W}^{1,p}(\mathbb{R}_+, \omega_N \rho^{N-1} \mathrm{d}\rho) \setminus \{0\}$*

$$\frac{\|\varphi\|_{p^*}}{\|\varphi\|_{\dot{W}^{1,p}}} \leq \frac{\|F_1\|_{p^*}}{\|F_1\|_{\dot{W}^{1,p}}} =: C_N \,, \tag{2.7}$$

where

$$p^* = \frac{pN}{N-p} \,,$$

and

$$F_t(\rho) := \hat{k} \left(\frac{t}{1 + t^{\frac{p}{p-1}} \rho^{\frac{p}{p-1}}} \right)^{\frac{N-p}{p}} \,, \tag{2.8}$$

for $t > 0$ and $\hat{k} > 0$ a constant such that $\|F_1\|_{\dot{W}^{1,p}} = 1$. There is equality if and only if $\varphi = zF_t$ for some $t > 0$ and some non-zero $z \in \mathbb{C}$.

G.A. Bliss proved this in 1930, see [Bl]. In order to deduce Theorem 2.1.3 from Bliss's theorem, one has to make the substitution of variables given by

$$x = \rho^{-\frac{N-p}{p-1}} \,,$$

and set

$$m = p\,, \text{ and } n = p^*\,,$$

where m and n are parameters, and x is the variable as per Theorem 1 of [Bl]. Actually, Bliss only proves Theorem 2.1.3 for nonnegative real-valued functions in $\dot{W}^{1,p}$. But, since replacing φ by $|\varphi|$ preserves the L^p-norm and cannot decrease the $\dot{W}^{1,p}$-norm, Theorem 2.1.3 must hold for functions with both positive and negative values. Once we have this result, it is easy to generalize to complex-valued functions, as seen in an argument given after the statement of Theorem 2.1.4.

Bakry, Gentil, and Ledoux proved an extension of the Sobolev Inequality in p. 322-323 of [2]. This extension is for "cylindrically symmetric" functions on Euclidean space of $m + n$ dimensions, where one of m and n is not necessarily an integer. In this paper, we will take m to be the number that is not necessarily an integer. Our motivation for considering the extension of the Sobolev Inequality to such functions is to extend the Bianchi-Egnell Stability Estimate of the Sobolev Inequality to cylindrically symmetric functions in continuous dimensions. This in turn allows us to generalize a stability estimate, proved by Carlen and Figalli (see Theorem 1.2 of [CaFi]), for a sharp Gagliardo-Nirenberg inequality, to a family of sharp Gagliardo-Nirenberg inequalities established by Del Pino and Dolbeault in [DeDo]. The continuous variable (i.e. noninteger dimension) in the extension of the Bianchi-Egnell Stability Estimate is necessary for this generalization of the Carlen-Figalli Stability Estimate. The extension of the Bianchi-Egnell Stability Estimate, which we state in detail later in this introduction, is proved in this chapter. The generalization of the Carlen-Figalli Stability Estimate, which was one of the original goals of our research, will be sketched in chapter 3.

To state Bakry, Gentil, and Ledoux's extension of the Sharp Sobolev Inequality, we define the appropriate norms and spaces. First, we establish some properties of cylindrically symmetric functions. Let $\varphi : [0, \infty) \times \mathbb{R}^n \to \mathbb{C}$ be a cylindrically symmetric function. What we mean when we say that φ is a cylindrically symmetric function is that if we write φ as $\varphi(\rho, x)$, where ρ is a variable with values in $[0, \infty)$ and x is the standard n-tuple on n Cartesian coordinates, that the ρ variable acts as a radial variable in m-dimensions while the x variable represents the other n-dimensions on which φ acts.

If m is an integer, then φ would also have a representation as a function on $\mathbb{R}^{m+n}$. For example,

$$\varphi(\rho,x) = (1+\rho^2+|x|)^{-1}$$

as a cylindrically symmetric function for $m=2$ and $n=2$ has the representation as a function on $\mathbb{R}^4$ given by

$$\varphi(x_1,x_2,x_3,x_4) = \left(1+x_1^2+x_2^2+\sqrt{x_3^2+x_4^2}\right)^{-1},$$

where x_1 and x_2 correspond to the ρ-variable of $\varphi(\rho,x)$ and x_3 and x_4 correspond to the x-variable of $\varphi(\rho,x)$. However, we want m to also possibly be noninteger. Note, that the value of m is not provided when we give the equation for φ. In this paper, the value of m will be determined by the dimensions over which our norms are integrated. To be more precise, the m dimensions of Euclidean space are encoded in the measure of integration corresponding to the ρ variable. This measure is $\omega_m\rho^{m-1}\mathrm{d}\rho$, where ω_m is a generalized notion of the area of the unit $(m-1)$-sphere given by (2.6) - note that this formula is valid for $m>0$. In this case, the L^p-norm of φ is given by

$$\|\varphi\|_p = \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}|\varphi(\rho,x)|^p\omega_m\rho^{m-1}\mathrm{d}\rho\mathrm{d}x\right)^{1/p}.$$

Note that when m is an integer and $\tilde{\varphi}:\mathbb{R}^{m+n}\to\mathbb{C}$ is given by $\tilde{\varphi}(\tilde{x},x)=\varphi(|\tilde{x}|,x)$, then

$$\|\varphi\|_p = \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}|\tilde{\varphi}(\tilde{x},x)|^p\mathrm{d}\tilde{x}\mathrm{d}x\right)^{1/p}.$$

The extension of the gradient square norm, i.e. $\|\nabla\cdot\|_2$, is given by

$$\|\varphi\|_{\dot{H}^1} := \|\nabla_{\rho,x}\varphi\|_2 = \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}(|\varphi_\rho|^2+|\nabla_x\varphi|^2)\omega_m\rho^{m-1}\mathrm{d}\rho\mathrm{d}x\right)^{1/2},$$

where the subscript ρ indicates a partial derivative with respect to ρ. Note that when m is an integer

$$\|\varphi\|_{\dot{H}^1} = \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}|\nabla_{x,\tilde{x}}\tilde{\varphi}(\tilde{x},x)|^2\mathrm{d}\tilde{x}\mathrm{d}x\right)^{1/2}.$$

Let Λ be the measure on $\mathbb{R}_+\times\mathbb{R}^n$ induced by $\omega_m\rho^{m-1}\mathrm{d}\rho\mathrm{d}x$. Accordingly, $\mathrm{d}\Lambda$ is given by

$$\mathrm{d}\Lambda = \omega_m\rho^{m-1}\mathrm{d}\rho\mathrm{d}x\,, \tag{2.9}$$

and for measureable $K \subseteq \mathbb{R}_+ \times \mathbb{R}^n$,

$$\Lambda(K) = \int_K \omega_m \rho^{m-1} \mathrm{d}\rho \mathrm{d}x \,. \tag{2.10}$$

Then, the space, $\dot{H}^1_{\mathbb{C}}(\mathbb{R}_+ \times \mathbb{R}^n, \omega_m \rho^{m-1} \mathrm{d}\rho \mathrm{d}x)$, of complex-valued cylindrically symmetric functions with finite gradient-square norm in continuous dimension will be defined as follows: $\varphi \in \dot{H}^1_{\mathbb{C}}(\mathbb{R}_+ \times \mathbb{R}^n, \omega_m \rho^{m-1} \mathrm{d}\rho \mathrm{d}x)$ if and only if

1. φ is a complex-valued cylindrically symmetric function with a distributional gradient,

2. $\|\nabla\varphi\|_2 < \infty$, and

3. φ is eventually zero in the sense that

   $$\Lambda(\{(\rho, x) \in \mathbb{R}_+ \times \mathbb{R}^n \big| |\varphi(\rho, x)| > \varepsilon\}) < \infty \,,$$

   for all $\varepsilon > 0$.

As a general rule, we will refer to $\dot{H}^1_{\mathbb{C}}(\mathbb{R}_+ \times \mathbb{R}^n, \omega_m \rho^{m-1} \mathrm{d}\rho \mathrm{d}x)$ as $\dot{H}^1_{\mathbb{C}}$. The subspace of real-valued functions in $\dot{H}^1_{\mathbb{C}}$ will be denoted by $\dot{H}^1$. It will often be useful for us to think of $\dot{H}^1_{\mathbb{C}}$ as the direct sum of two copies of $\dot{H}^1$. Also, in this setting, we define

$$2^* := \frac{2(m+n)}{m+n-2} \text{ and } \gamma := \frac{m+n-2}{2} \,. \tag{2.11}$$

Having established this background, we can state Bakry, Gentil, and Ledoux's generalization of the Sobolev Inequality to continuous dimensions with Nguyen's classification of extremals (for reference, see [BaGe] and [Ng]):

THEOREM 2.1.4 (Sobolev Inequality Extension)**.** *Let $m + n > 2$, n an integer, $m > 0$ possibly noninteger. Then, for all $\varphi \in \dot{H}^1_{\mathbb{C}}(\mathbb{R}_+ \times \mathbb{R}^n, \omega_m \rho^{m-1} \mathrm{d}\rho \mathrm{d}x)$*

$$\frac{\|\varphi\|_{2^*}}{\|\varphi\|_{\dot{H}^1}} \leq \frac{\|F_{1,0}\|_{2^*}}{\|F_{1,0}\|_{\dot{H}^1}} =: C_{m,n} \,, \tag{2.12}$$

where

$$zF_{t,x_0}(\rho, x) := \hat{k} z t^{\gamma} (1 + t^2\rho^2 + t^2|x - x_0|^2)^{-\gamma} \,, \tag{2.13}$$

for $x_0 \in \mathbb{R}^n$, $t \in \mathbb{R}_+$, $z \in \mathbb{C}$, and $\hat{k} > 0$ a number such that $\|F_{1,0}\|_{\dot{H}^1} = 1$. (2.12) gives equality if and only if $\varphi = zF_{t,x_0}$ for some $t > 0$, $x_0 \in \mathbb{R}^n$, and nonzero $z \in \mathbb{C}$. The *extremal functions characterized by (2.13) comprise an* $(n+3)$*-dimensional manifold,* $M \subseteq \dot{H}^1_{\mathbb{C}}(\mathbb{R}_+ \times \mathbb{R}^n, \omega_m \rho^{m-1} \mathrm{d}\rho \mathrm{d}x)$.

Bakry, Gentil, and Ledoux derived (2.12). Nguyen classified the extremals for the statement of this theorem for real-valued functions only. However, once one has the Sobolev Inequality with the classification of extremals for real-valued functions, the generalization to complex-valued functions is easy to deduce. To see this, consider $\varphi \in \dot{H}^1_{\mathbb{C}}$ and let

$$\varphi(\rho, x) = R(\rho, x) e^{i\Theta(\rho,x)}\,,$$

where R and Θ are real-valued. Then

$$\begin{aligned} C_{m,n}\|\nabla\varphi\|_{\dot{H}^1} &= C_{m,n}(\|\nabla_{\rho,x}R\|_2^2 + \|R\nabla_{\rho,x}\Theta\|_2^2)^{1/2} \\ &\geq C_{m,n}\|\nabla_{\rho,x}R\|_2 \text{ which by the Sobolev Inequality for real-valued functions} \\ &\geq \|R\|_{2^*} \\ &= \|\varphi\|_{2^*}\,, \end{aligned}$$

i.e. the Sobolev inequality with sharp constant. Moreover, the extremals for complex-valued functions are derived by taking the extremals in the real-valued case and multiplying them by all possible complex numbers. We deduce this by observing that the complex-valued Sobolev Inequality deduced above cannot achieve equality unless $R\nabla_{\rho,x}\Theta$ is zero almost everywhere and R is an extremal.

Bakry, Gentil, and Ledoux derive the generalization of the Sharp Sobolev Inequality by relating a Sobolev Inequality on $\mathbb{S}^{n+1}$ to $(\mathbb{R}_+ \times \mathbb{R}^n, \omega_m \rho^{m-1} \mathrm{d}\rho \mathrm{d}x)$ via stereographic projection, see p. 322-323 of [BaGe] for detail. Bakry's proof is an application of an abstract curvature-dimension condition. Nguyen provides a proof of Theorem 2.1.4 from a mass-transport approach. Nguyen's approach, unlike Bakry's, provides a full classification of extremals (for the Sobolev Theorem Extension for real-valued functions).

2.1.4 The Main Theorem

The main theorem that we prove in this paper is a Bianchi-Egnell Stability Estimate for Theorem 2.1.4. The extremals of Theorem 2.1.4 are given by

$$zF_{t,x_0}(\rho,x) = \hat{k}z\left(\frac{t}{1+t^2\rho^2+t^2|x-x_0|^2}\right)^{\gamma},$$

for $x_0 \in \mathbb{R}^n$, $t \in \mathbb{R}_+$, $z \in \mathbb{C}\setminus\{0\}$, and $\hat{k} > 0$ a number such that $\|F_{1,0}\|_{\dot{H}^1} = 1$. These extremal functions comprise an $(n+3)$-dimensional manifold, $M \subseteq \dot{H}^1_{\mathbb{C}}(\mathbb{R}_+ \times \mathbb{R}^n, \omega_m\rho^{m-1}\mathrm{d}\rho\mathrm{d}x)$. The distance, $\delta(\varphi, M)$, between this manifold and a function $\varphi \in \dot{H}^1_{\mathbb{C}}(\mathbb{R}_+ \times \mathbb{R}^n, \omega_m\rho^{m-1}\mathrm{d}\rho\mathrm{d}x)$ will also be given by (2.4).

The stability estimate we prove here is

THEOREM 2.1.5 (Bianchi-Egnell Extension)**.** *There is a positive constant, α, depending only on the parameters, m and n, $m > 0$ and $n \geq 2$ an integer, so that*

$$C^2_{m,n}\|\varphi\|^2_{\dot{H}^1} - \|\varphi\|^2_{2^*} \geq \alpha\delta(\varphi, M)^2\,, \tag{2.14}$$

$\forall\varphi \in \dot{H}^1_{\mathbb{C}}$. *Furthermore, the result is sharp in the sense that it is no longer true if $\delta(\varphi,M)^2$ in (2.14) is replaced with $\delta(\varphi,M)^{\beta}\|\varphi\|^{2-\beta}_{H^1}$, where $\beta < 2$.*

A notable theorem that we prove in order to prove Theorem 2.1.5 is the following local compactness theorem:

THEOREM 2.1.6. *Let $K \subseteq [0,\infty) \times \mathbb{R}^n$ satisfy the cone property in $\mathbb{R}^{n+1}$, $K \subseteq \{(\rho,x) \in [0,\infty)\times\mathbb{R}^n | \rho_1 < \rho < \rho_2\}$ for some $0 < \rho_1 < \rho_2 < \infty$, and $\Lambda(K) < \infty$, where Λ denotes the measure on $\mathbb{R}_+ \times \mathbb{R}^n$ defined by (2.10). If (φ_j) is bounded in $\dot{H}^1_{\mathbb{C}}$ and U is an open subset of K, then for $1 \leq p < \max\left\{2^*, \frac{2n+2}{n-1}\right\}$, there is some $\varphi \in \dot{H}^1_{\mathbb{C}}$ and some subsequence, (φ_{j_k}), such that $\varphi_{j_k} \to \varphi$ in $L^p_{\mathbb{C}}(U, \omega_m\rho^{m-1}\mathrm{d}\rho\mathrm{d}x)$, where $L^p_{\mathbb{C}}(U,\omega_m\rho^{m-1}\mathrm{d}\rho\mathrm{d}x)$ denotes the space of complex-valued functions on the weighted space $(U, \omega_m\rho^{m-1}\mathrm{d}\rho\mathrm{d}x)$.*

The proof of Theorem 2.1.6 is not too hard and is provided in the final subsection of this chapter.

2.1.5 Outline of a Proof of the Bianchi-Egnell Stability Estimate

In this subsection, we outline a proof of Theorem 2.1.2 based upon the techniques we used to prove Theorem 2.1.5. In this outline, we highlight the differences between proving our Bianchi-Egnell Extension and Bianchi and Egnell's proof of the original Bianchi-Egnell Stability Estimate. There are two key steps to proving Theorem 2.1.2. These key steps in our proof are the same key steps that Bianchi and Egnell's proof are based upon. However, the details for establishing them are quite different at times. The first step is a *Local Bianchi-Egnell Stability Theorem*: If $\varphi \in \dot{H}^1(\mathbb{R}^N)$ is such that $\|\nabla\varphi\|_2 = 1$ and $\delta(\varphi, M) \leq \frac{1}{2}$, then

$$C_N^2\|\nabla\varphi\|_2^2 - \|\varphi\|_{2^*}^2 \geq \alpha_N\delta(\varphi, M)^2 - \kappa_N\delta(\varphi, M)^{\beta_N} , \tag{2.15}$$

where κ_N and β_N are calculable constants, with $\beta_N > 2$ and α_N being the smallest positive eigenvalue of a linear operator with nonnegative discrete spectrum. This allows us to prove Theorem 2.1.2 in a local sense. The second step is a *Concentration Compactness argument* by which we show that if Theorem 2.1.2 is not true "outside" our local region then the Local Bianchi-Egnell Stability Theorem would not be true.

Step 1 - Prove the Local Bianchi-Egnell Stability Theorem:

Part A: Taylor Expand $\|\varphi\|_{2^*}^2$. Since $\delta(\varphi, M) \leq \frac{1}{2} < 1 = \|\varphi\|_{\dot{H}^1}$, there is some $F \in M$ such that

$$\delta(\varphi, M) = \|\varphi - F\|_{\dot{H}^1} .$$

In fact,

$$\varphi = F + \delta(\varphi, M)\psi ,$$

for some $\psi \in \dot{H}^1$ such that $\|\psi\|_{\dot{H}^1} = 1$ and $\psi \perp_{\dot{H}^1} F$. Taylor expanding $\|F + \varepsilon\psi\|_{2^*}^2$ about $\varepsilon = 0$ to the second degree, estimating the remainder, and setting $\varepsilon = \delta(\varphi, M)$, we get that

$$\|\varphi\|_{2^*}^2 = \|F + \delta(\varphi, M)\psi\|_{2^*}^2 \leq \|F\|_{2^*}^2 + \langle\psi, S\psi\rangle_{\dot{H}^1}\delta(\varphi, M)^2 + \kappa_N\delta(\varphi, M)^{\beta_N} ,$$

where $S : \dot{H}^1 \to \dot{H}^1$ is a linear operator, and κ_N and β_N are calculable constants. Our explicit calculation of the term $\kappa_N\delta(\varphi, M)^{\beta_N}$, which is a bound on the remainder term of the second order Taylor expansion, is an improvement upon Bianchi and Egnell's proof. They use the Brezis-Lieb Lemma to conclude that the remainder term in the second order Taylor expansion is $o(\delta(\varphi, M)^2)$.

Part B: Use the Taylor Expansion of $\|\varphi\|_{2^*}^2$ to Deduce the Local Bianchi-Egnell Stability Theorem. Using the last calculation above, and the facts that $\|F\|_{2^*} = C_N\|F\|_{\dot{H}^1}$ and $\psi \perp_{\dot{H}^1} F$, we conclude that

$$C_N^2\|\varphi\|_{\dot{H}^1}^2 - \|\varphi\|_{2^*}^2 \geq \langle\psi, (C_N^2 I - S)\psi\rangle_{\dot{H}^1}\delta(\varphi, M)^2 - \kappa_N\delta(\varphi, M)^{\beta_N} .$$

Next, we assume the following facts: $C_N^2 I - S : \dot{H}^1 \to \dot{H}^1$ has nonnegative, discrete spectrum whose nullspace is spanned by F and $\frac{\mathrm{d}}{\mathrm{d}t}F$ (t is the parameter in the class of Sobolev optimizers corresponding to dilation - refer to (2.3) if necessary) and has a gap at 0. In Bianchi and Egnell's proof, they prove the analogous facts by calculating a few of the lowest eigenvalues and some of the corresponding eigenfunctions of their operator. For Bianchi and Egnell, this is done through separation of variables and analysis of the resulting ODEs. Proving the desired properties of the analogue to $C_N^2 I - S$ in our paper turns out to be more difficult, because the resulting PDE does not separate nicely. So, we delay the proof of these properties of the operator a little bit. Assuming the desired properties, we conclude (2.15), with α_N being the smallest positive eigenvalue of $C_N^2 I - S : \dot{H}^1 \to \dot{H}^1$. Parts C and D are devoted to proving the desired properties of the spectrum and nullspace of $C_N^2 I - S : \dot{H}^1 \to \dot{H}^1$.

Part C: Show That $S : \dot{H}^1 \to \dot{H}^1$ Is a Compact Self-Adjoint Operator. Proving self-adjointness is easy. Proving compactness for the analogue of $S : \dot{H}^1 \to \dot{H}^1$ in our Bianchi-Egnell Extension is difficult, see subsection 2.4 for the precise argument. The heart of the argument for proving compactness in our setting is comparing a part of S to a similar operator for which eigenvalue and eigenfunction analysis is easier to carry out. The comparison to the closely related operator is illuminated by a change of coordinates. The change of coordinates in the current setting would be made by representing φ in terms of its radial and spherical parts and then making a logarithmic substitution, i.e.

$$\begin{aligned}\varphi(x) &= \varphi(r,\zeta),\ r\in[0,\infty) \text{ and } \zeta\in\mathbb{S}^{N-1}\\ &= r^{-\frac{N-2}{2}}\varphi(\ln r,\zeta)\,. \end{aligned} \tag{2.16}$$

We would work with the representative of φ given by $\varphi(u,\zeta)$, $u=\ln r$, as these coordinates make some of our calculations simpler. Bianchi and Egnell only need to make the change from Euclidean coordinates, i.e. $x\in\mathbb{R}^N$, to radial and spherical coordinates, i.e. $(\rho,\zeta)\in\mathbb{R}_+\times\mathbb{S}^{N-1}$, to carry out the analysis of the second order operator that they obtain from their Taylor expansion.

Part D: Show that $C_N^2 I - S : \dot{H}^1 \to \dot{H}^1$ is Positive and Its Nullspace is Spanned by F and $\frac{\mathrm{d}}{\mathrm{d}t}F$. Establishing positivity of $C_N^2 I - S : \dot{H}^1 \to \dot{H}^1$ and showing that F and $\frac{\mathrm{d}}{\mathrm{d}t}F$ are in its nullspace is not hard. In our proof, showing that the analogue of $C_N^2 I - S$ does not have any element in its nullspace that is not a linear combination of F and $\frac{\mathrm{d}}{\mathrm{d}t}F$ is not easy. We reduce the 0-eigenvalue problem to an ODE by showing that any element in the nullspace of $C_N^2 I - S$ that is orthogonal to F in $\dot{H}^1$ must satisfy an ODE. We then show that constant multiples of $\frac{\mathrm{d}}{\mathrm{d}t}F$ are the only solutions off this ODE with finite energy, and consequently the only solutions in $\dot{H}^1$. We conclude by spending some time showing that elements in the nullspace of our analogue to $C_N^2 I - S : \dot{H}^1 \to \dot{H}^1$ must be independent of the variables other than u (keep in mind we are in logarithmic coordinates like those defined in (2.16)). See subsection 2.5 for this argument, in particular, see Proposition 2.5.3 for the ODE argument.

Step 2 - Use Concentration Compactness to conclude the Bianchi-Egnell Stability Estimate:

The local theorem whose proof we just outlined in fact gives Bianchi and Egnell's Stability Theorem in a region around M. We, as well as Bianchi and Egnell, use a Concentration Compactness argument to show that some stability estimate must hold outside this local region also. However, in Bianchi and Egnell's case, since they are working in $\mathbb{R}^N$, they are able to cite this step as a straightforward application of Concentration Compactness as presented by P.L. Lions or M. Struwe. In our proof of the analogue of their stability theorem, the variables and space we work in prevent such a straightforward application. Our treatment of the Concentration Compactness argument in continuous dimensions may be useful, as the argument is delicate and to our knowledge is absent in the current literature.

2.1.6 Outline of Proof of Theorem 2.1.5

In this subsection, we outline the proof of Theorem 2.1.5. Each section of the proof is named, with its name given in italics, and then described.

A Second Order Taylor Expansion of $\|f+\varepsilon\psi\|_p^2$ at $\varepsilon=0$ and an Estimate of the Remainder: In this section, we improve upon the traditional strategy of bounding the remainder of a second order Taylor expansion of the square of the p-norm of a function. In particular, we Taylor expand $\|f+\varepsilon\psi\|_p^2$ around $\varepsilon=0$ to the second degree for $2<p<\infty$ and f real-valued and calculate a precise bound for the remainder term. The previously established strategy for dealing with the remainder term is to apply the Brezis-Lieb Lemma to conclude that the remainder is $o(\varepsilon^2)$.

Statement of a Local Bianchi-Egnell Extension and Outline of Proof: In this section, we restrict our attention to $\varphi\in\dot{H}^1_{\mathbb{C}}$ in a neighborhood of M, the manifold of extremals of the extended Sobolev Inequality. We state a Bianchi-Egnell Stability Estimate in this local setting and then outline its proof. We begin by showing that

$$\varphi = F+\delta(\varphi,M)\psi\,,$$

for some $F\in M$ and then reducing the proof to the case where F is real-valued. Applying the Taylor Expansion result of the previous section and making some additional arguments, we deduce

$$\begin{aligned} C_{m,n}^2\|\varphi\|_{\dot{H}^1}^2-\|\varphi\|_{2^*}^2 &= C_{m,n}^2\|F+\delta(\varphi,M)\psi\|_{\dot{H}^1}^2-\|F+\delta(\varphi,M)\psi\|_{2^*}^2 \\ &\geq \left\langle (C_{m,n}^2 I-S_t)\psi,\psi\right\rangle_{\dot{H}^1_{\mathbb{C}}}\delta(\varphi,M)^2-\kappa_{2^*}\delta(\varphi,M)^{\beta_{2^*}}\,, \end{aligned}$$

for some calculable constants $\kappa_{2^*}>0$ and $\beta_{2^*}>2$, and linear operator $S_t:\dot{H}^1_{\mathbb{C}}\to\dot{H}^1_{\mathbb{C}}$. We then assert that $S_t:\dot{H}^1_{\mathbb{C}}\to\dot{H}^1_{\mathbb{C}}$ is a compact self-adjoint operator and that $C_{m,n}^2 I-S_t:\dot{H}^1_{\mathbb{C}}\to\dot{H}^1_{\mathbb{C}}$ is a positive operator whose nullspace is spanned by $\{(F,0),(\frac{\mathrm{d}}{\mathrm{d}t}F,0),(0,F)\}$ - we use the convention that for $\varphi\in\dot{H}^1_{\mathbb{C}}$, $\varphi=(\xi,\eta)\in\dot{H}^1\oplus\dot{H}^1$ - and is orthogonal in $\dot{H}^1_{\mathbb{C}}$ to ψ. Assuming these facts - their proof is delayed to the next two sections - we deduce that

$$\left\langle (C_{m,n}^2 I - S_t)\psi, \psi \right\rangle_{\dot{H}^1_{\mathbb{C}}} \geq \alpha_{m,n}\,,$$

where $\alpha_{m,n}$ is the smallest positive eigenvalue of $C_{m,n}^2 I - S_t : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$. Combining the last two inequalities above, we conclude that

$$C_{m,n}^2 \|\varphi\|_{\dot{H}^1}^2 - \|\varphi\|_{2^*}^2 \geq \alpha_{m,n}\delta(\varphi, M)^2 - \kappa_{2^*}\delta(\varphi, M)^{\beta_{2^*}}\,,$$

yielding a Local Bianchi-Egnell Stability Estimate when $\delta(\varphi, M)$ is sufficiently small.

$S_t : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ is a Self-Adjoint, Compact Operator: Self-adjointness follows easily. Compactness does not. The gist of the argument is to show that some closely related positive operator is compact and use this fact to show that $S : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ is compact. To do this, we use the kernel of the closely related positive operator to show that some positive even power of this operator is trace class. Hence, by a comparison argument, the original operator will be compact. There is a change to logarithmic variables done in this section, like the one suggested in the paragraph with heading "*Part C*" in subsection 2.4. This change of variables is essential in helping us figure out the precise form of the closely related operator that we use to prove compactness. The argument presented in this section is somewhat lengthy and delicate.

The Nullspace of $C_{m,n}^2 I - S_t : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$: Here we demonstrate that $C_{m,n}^2 I - S_t : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ is positive, its nullspace is spanned by $\{(F, 0), (\frac{\mathrm{d}}{\mathrm{d}t}F, 0), (0, F)\}$, and that all elements in its nullspace are orthogonal in $\dot{H}^1_{\mathbb{C}}$ to ψ. This concludes the proof of the local Bianchi-Egnell Stability Estimate.

Proof of Theorem 2.1.5: Here, we use Concentration Compactness to show that if the Bianchi-Egnell Stability Estimate Extension is not true, then the Local Bianchi-Egnell Stability Estimate that we proved earlier would be untrue. This, of course, is a contradiction. Hence, we conclude the main theorem of this paper. However, Concentration Compactness theorems for cylindrically symmetric functions in continuous dimension are, to our knowledge, absent from literature. Moreover, it was not clear to us that there is an easy way to take preexisting arguments for functions defined on subsets of $\mathbb{R}^N$ to deduce a Concentration Compactness result for cylindrically symmetric functions on continuous dimensions. Thus, we have the next section.

Concentration Compactness: We begin by showing that for $(\varphi_j) \subseteq \dot{H}^1_{\mathbb{C}}$ such that $\|\varphi_j\|_{2^*} = 1$ for all j, and $C^2_{m,n}\|\varphi_j\|^2_{\dot{H}^1} - \|\varphi_j\|^2_{2^*} \to 0$, that dilating the functions appropriately, i.e. taking $(\varphi_j^{\sigma_j})$, where

$$\varphi^{\sigma}(\rho, x) := \sigma^{\gamma}\varphi(\sigma\rho, \sigma x),\ \sigma > 0\,,$$

for appropriate σ_j gives us a subsequence such that

$$\Lambda(\{|\varphi_{j_k}^{\sigma_{j_k}}(x)| > \varepsilon, \rho \leq 4\}) > C\,, \tag{2.17}$$

for some $C > 0$ and $\varepsilon > 0$ and where Λ denotes the measure defined in (2.10). We then apply an analogue of a Concentration Compactness Theorem proved by Lieb to conclude that a translated subsequence of $(\varphi_{j_k}^{\sigma_{j_k}}(x))$ converges weakly in $\dot{H}^1_{\mathbb{C}}$ to a nonzero element φ. The tricky part of proving concentration compactness in this section is proving (2.17). The gist of the argument proving (2.17) is to show that if we dilate the functions in our sequence appropriately and then take their symmetric decreasing rearrangements, then some subsequence of the modified sequence will satisfy the p, q, r-Theorem on a subregion of an annulus. What makes this complicated in our case is that we are working with cylindrically symmetric functions in continuous dimensions. Using some straightforward functional analysis arguments, we conclude that $\varphi \in M$ and $\|\varphi\|_{2^*} = 1$. Noteworthy of these arguments is the use of a Local Compactness Theorem on cylindrically symmetric functions on continuous dimension. This Theorem substitutes for the Rellich-Kondrachov Theorem and can be thought of as a weaker version of the Rellich-Kondrachov Theorem for cylindrically symmetric functions on continuous dimensions. The precise statement of our Local Compactness Theorem is Theorem 2.1.6 in subsection 2.1.4.

Proof of Local Compactness Theorem: Here we prove Theorem 2.1.6.

2.1.7 Applications of Bianchi and Egnell's Stability Analysis and Our Bianchi-Egnell Stability Estimate

We begin this subsection with a discussion of what motivated us to pose and prove the Bianchi-Egnell Extension, Theorem 2.1.5. Bakry, Gentil, and Ledoux showed that their extension of the Sobolev Inequality implies a sharp family of Gagliardo-Nirenberg inequalities that had only recently been proven by Del Pino and Dolbeault, see [DeDo]. Carlen and Figalli explored this connection and used Bakry, Gentil, and Ledoux's techniques to establish a stability estimate for a single case in this family of sharp Gagliardo-Nirenberg inequalities. An essential step to obtaining this stability estimate, is a direct application of the Bianchi-Egnell Stability Estimate. Carlen and Figalli's use of the Bianchi-Egnell Stability Estimate in establishing a stability estimate of the Gagliardo-Nirenberg inequality, raises the question as to whether or not one can generalize their stability estimate to the entire family of Gagliardo-Nirenberg inequalities classified by Del Pino and Dolbaeault. The answer to this question is yes, but using Carlen and Figalli's techniques requires the Bianchi-Egnell Stability Estimate that we prove in this paper. The Bianchi-Egnell Stability Estimate on integer dimensions is not sufficient, because there is an integration step that links the Sobolev Inequality to these sharp Gagliardo-Nirenberg inequalities. This integration step induces a correlation between the dimension of the Sobolev Inequality and a parameter in the Gagliardo-Nirenberg inequalities of Del Pino and Dolbeault. In order to deduce a stability estimate for the Gagliardo-Nirenberg inequalities corresponding to all possible values of this paremeter, one needs a Sobolev Inequality and a Bianchi-Egnell Stability Estimate for cylindrically symmetric functions on continuous dimensions. Thus, we set out to prove the Bianchi-Egnell Stability Estimate Extension in this paper, in order to develop this necessary piece of machinery in proving a stability estimate for the full class of sharp Gagliardo-Nirenberg inequalities of Del Pino and Dolbeault. We will use our Bianchi-Egnell Stability Estimate Extension in a future paper to prove a stability estimate for the full family of Gagliardo-Nirenberg inequalities classified by Del Pino and Dolbeault in [DeDo].

The techniques used to prove the stability estimate of Bianchi and Egnell have been used in solving many partial differential equations, see [AdYa], [MuPi], and [Sm] for some examples. The Gagliardo-Nirenberg stability estimate of Carlen and Figalli, which is a direct application of the Bianchi-Egnell Stability Estimate, was also used to solve a Keller-Segel equation, see [CaFi].

2.2 A Second Order Taylor Expansion of $\|f+\varepsilon\psi\|_p^2$ at $\varepsilon=0$ and an Estimate of the Remainder

In this subsection, we will Taylor expand $\|f+\varepsilon\psi\|_p^2$ at $\varepsilon=0$ to the second degree and estimate the remainder term. This may seem like a simple enough process, but this expansion lies at the heart of Theorem 2.1.5. Moreover, our estimate is an improvement upon the conventional method of dealing with the remainder term. The conventional method is to apply the Brezis-Lieb Lemma to conclude that the remainder term is $o(\varepsilon^2)$. We show that the remainder is bounded by $\kappa_p|\varepsilon|^{\beta_p}$ with $\kappa_p>0$, $\beta_p>2$ calculable constants.

Although the stability estimate that we prove in our paper is for complex-valued functions, we will reduce our calculations to real-valued functions. To this end, we treat $L^p_{\mathbb{C}}$, complex-valued L^p functions, as the direct sum of two copies of the space of real-valued L^p functions. To be more precise, we let $L^p_{\mathbb{C}}=L^p\oplus L^p$, where L^p denotes the space of real-valued L^p functions. When representing elements in $L^p_{\mathbb{C}}$ as a direct sum, the first coordinate will represent the real part of the function and the second coordinate will represent the imaginary part; i.e., if $\psi\in L^p_{\mathbb{C}}$, then $\psi=(\xi,\eta)$ for some $\xi,\eta\in L^p$. The calculation of the Taylor Expansion is summarized in the following

THEOREM 2.2.1. *Let $P_\psi:[-1,1]\to\mathbb{R}$ be given by*

$$P_\psi(\varepsilon)=\|f+\varepsilon\psi\|_p^2\,, \tag{2.18}$$

for $\psi\in L^p_{\mathbb{C}}$, real-valued $f\in L^p_{\mathbb{C}}$, $\|\psi\|_p=\|f\|_p=1$, and $2<p<\infty$. Let $\psi=(\xi,\eta)$. Then,

$$\left|P_\psi(\varepsilon)-1-2\langle f|f|^{p-2},\xi\rangle_{L^2}\varepsilon-\langle\mathcal{L}_{f,p}\psi,\psi\rangle_{L^2\oplus L^2}\varepsilon^2\right|\le\kappa_p|\varepsilon|^{\beta_p}\,, \tag{2.19}$$

where $\mathcal{L}_{f,p}=\mathcal{L}^{Re}_{f,p}\oplus\mathcal{L}^{Im}_{f,p}$ is given by

$$\mathcal{L}^{Re}_{f,p}\xi=-(p-2)\left(\int f|f|^{p-2}\xi\right)f|f|^{p-2}+(p-1)|f|^{p-2}\xi \tag{2.20}$$

$$\mathcal{L}^{Im}_{f,p}\eta=|f|^{p-2}\eta\,, \tag{2.21}$$

and

$$\beta_p = \begin{cases} 3 & \text{if } p \geq 4 \\ 1 + \frac{p}{2} & \text{if } 2 < p \leq 4 \end{cases} \tag{2.22}$$

$$\kappa_p = \begin{cases} \frac{4}{3}(5p^2 - 12p + 7) & \text{if } p \geq 4 \\ \frac{16(p-1)}{p(p+2)} \left[4(p-2) + \left(\frac{3p}{p-2}\right)^{\frac{p}{2}-1} \right] & \text{if } 2 < p \leq 4\,. \end{cases} \tag{2.23}$$

Proof. The proof of Theorem 2.2.1 breaks into three parts. The first is the Taylor Expansion summarized in

LEMMA 2.2.2. *Given the assumptions of Theorem 2.2.1,*

$$P_\psi(\varepsilon) = 1 + 2\langle f|f|^{p-2}, \xi\rangle_{L^2}\varepsilon + \langle \mathcal{L}_{f,p}\psi, \psi\rangle_{L^2 \oplus L^2}\varepsilon^2 + \int_0^\varepsilon \int_0^s P''_\psi(y) - P''_\psi(0)\mathrm{d}y\mathrm{d}s\,. \tag{2.24}$$

Proof. Taylor expanding P_ψ to the second order with remainder yields

$$P_\psi(\varepsilon) = P_\psi(0) + P'_\psi(0)\varepsilon + \frac{1}{2}P''_\psi(0)\varepsilon^2 + \int_0^\varepsilon \int_0^s P''_\psi(y) - P''_\psi(0)\mathrm{d}y\mathrm{d}s\,. \tag{2.25}$$

A straightforward calculation shows that

$$P_\psi(0) = 1\,,\; P'_\psi(0) = 2\langle f|f|^{p-2}\,,\; \xi\rangle_{L^2}\,,\; \frac{1}{2}P''_\psi(0) = \langle \mathcal{L}_{f,p}\psi, \psi\rangle_{L^2\oplus L^2}\,. \tag{2.26}$$

Combining (2.25) and (2.26) yields (2.24). □

The expansion above is a straightforward calculation. Much of the work from here on out is devoted to identifying the behavior of the individual terms in (2.19) as applied to our particular setup. We begin this process by getting an estimate on the remainder term of the right hand side of (2.19). This is a bit subtle. Our proof hinges on a piece of machinery developed by Carlen, Frank, and Lieb in [CaFr]. This machinery is the *duality map*, $\mathcal{D}_p$, on functions from $L^p_{\mathbb{C}}$ to the unit sphere in $L^{p'}_{\mathbb{C}}$ (we take the convention that $\frac{1}{p'} := 1 - \frac{1}{p}$) given by

$$\mathcal{D}_p(g) = \|g\|_p^{1-p}|g|^{p-2}\overline{g}\,.$$

As a consequence of uniform convexity of $L^p_{\mathbb{C}}$ for $1 < p < \infty$, Carlen, Lieb, and Frank deduce Holder continuity of the duality map, see Lemma 3.3 of [CaFr] for detail. This Holder continuity is crucial in bounding the remainder term in (2.19). We set up the application of this Holder continuity of the duality map in the second part of the proof of Theorem 2.2.1 summarized by

LEMMA 2.2.3. *Given the assumptions of Theorem 2.2.1,*

$$\left|P''_\psi(y) - P''_\psi(0)\right| \leq 4(p-2)\left\|\mathcal{D}_p(f^{(y)}) - \mathcal{D}_p(f)\right\|_{p'} + 2(p-1)\left\|\mathcal{D}_{\frac{p}{2}}([f^{(y)}]^2) - \mathcal{D}_{\frac{p}{2}}(f^2)\right\|_{(\frac{p}{2})'}, \tag{2.27}$$

for $f^{(y)} := f + y\psi$.

Proof. We begin with the inequality

$$\begin{aligned}\frac{1}{2(p-2)}\left|P''_\psi(y) - P''_\psi(0)\right| \leq &\left|\|f^{(y)}\|_p^{2-2p}\left(\int |f^{(y)}|^{p-2}[(f+y\xi)\xi + y\eta^2]\right)^2 - \left(\int |f|^{p-2} f\xi\right)^2\right| \\ &+ \left|\|f^{(y)}\|_p^{2-p}\left(\int |f^{(y)}|^{p-4}[(f+y\xi)\xi + y\eta^2]^2\right) - \left(\int |f|^{p-4} f^2\xi^2\right)\right| \\ &+ (p-2)^{-1}\left|\|f^{(y)}\|_p^{2-p}\left(\int |f^{(y)}|^{p-2}|\psi|^2\right) - \left(\int |f|^{p-2}|\psi|^2\right)\right| \\ =:& A_1 + A_2 + A_3\,. \end{aligned} \tag{2.28}$$

We bound A_1, A_2, and A_3 below:

$$A_1 = \left|\|f^{(y)}\|_p^{2-2p}\left(\int |f^{(y)}|^{p-2}[(f+y\xi)\xi + y\eta^2]\right)^2 - \left(\int |f|^{p-2} f\xi\right)^2\right|$$

using the fact that $a^2 - b^2 = (a-b)(a+b)$ and elementary properties of complex numbers

$$\begin{aligned} &\leq \left|\left(\int \mathcal{D}_p(f^{(y)})\psi\right) + \left(\int \mathcal{D}_p(f)\psi\right)\right| \cdot \left|\left(\int \mathcal{D}_p(f^{(y)})\psi\right) - \left(\int \mathcal{D}_p(f)\psi\right)\right| \\ &\leq \left(\frac{\||f^{(y)}|^{p-1}\|_{p'}}{\|f^{(y)}\|_p^{p-1}} + 1\right)\left(\int [\mathcal{D}_p(f^{(y)}) - \mathcal{D}_p(f)]\psi\right) \\ &\leq 2\|\mathcal{D}_p(f^{(y)}) - \mathcal{D}_p(f)\|_{p'}\,, \text{ and} \end{aligned} \tag{2.29}$$

$$\begin{aligned} A_2 &= \left|\|f^{(y)}\|_p^{2-p}\left(\int |f^{(y)}|^{p-4}[(f+y\xi)\xi + y\eta^2]^2\right) - \left(\int |f|^{p-4} f^2\xi^2\right)\right| \\ &\leq \int \left|\|f^{(y)}\|_p^{2-p}|f^{(y)}|^{p-4}[(f+y\xi)\xi + y\eta^2]^2 - |f|^{p-4} f^2\xi^2\right| \end{aligned}$$

using the fact that $a^2 - b^2 = (a-b)(a+b)$ and elementary properties of complex numbers

$$\leq \int \left|\mathcal{D}_{\frac{p}{2}}([f^{(y)}]^2) - \mathcal{D}_{\frac{p}{2}}(f^2)\right| |\psi^2|\,, \text{ which by Holder's Inequality}$$

$$\leq \|\mathcal{D}_{\frac{p}{2}}([f^{(y)}]^2) - \mathcal{D}_{\frac{p}{2}}(f^2)\|_{(\frac{p}{2})'}\,. \tag{2.30}$$

And, a process similar to the one used to bound A_2 shows that

$$A_3 \leq (p-2)^{-1} \left\| \mathcal{D}_{\frac{p}{2}}([f^{(y)}]^2) - \mathcal{D}_{\frac{p}{2}}(f^2) \right\|_{\left(\frac{p}{2}\right)'} . \tag{2.31}$$

Combining (2.28)-(2.31), we conclude (2.27). □

In the third part of the proof of Theorem 2.2.1, we estimate the right hand side of (2.27) via Holder continuity of the duality map, $\mathcal{D}_p$. For completeness, we state this Holder continuity property below:

LEMMA 2.2.4 (Holder Continuity of the Duality Map)**.** *Let $f, g \in L^p(X, \mu)$ for X a measure space and μ its measure. Then*

$$\|\mathcal{D}_p(f) - \mathcal{D}_p(g)\|_{p'} \leq 4(p-1) \frac{\|f-g\|_p}{\|f\|_p + \|g\|_p} \,, \ \text{for } p \geq 2 \tag{2.32}$$

$$\|\mathcal{D}_p(f) - \mathcal{D}_p(g)\|_{p'} \leq 2 \left(p' \frac{\|f-g\|_p}{\|f\|_p + \|g\|_p} \right)^{p-1} \,, \ \text{for } 1 < p \leq 2 \,. \tag{2.33}$$

Applying Lemma 2.2.4 to the right hand side of (2.27) yields

$$\left| P''_\psi(y) - P''_\psi(0) \right| \leq \begin{cases} 16(p-1)(p-2)\frac{\|y\psi\|_p}{\|f^{(y)}\|_p + 1} + 8(p-1)^2 \frac{\|2fy\psi + y^2\psi^2\|_{\frac{p}{2}}}{\|[f^{(y)}]^2\| + \|f^2\|_{\frac{p}{2}}} & \text{if } \frac{p}{2} \geq 2 \\ 16(p-1)(p-2)\frac{\|y\psi\|_p}{\|f^{(y)}\|_p + 1} + 4(p-1) \left[\left(\frac{p}{2}\right)' \frac{\|2fy\psi + y^2\psi^2\|_{\frac{p}{2}}}{\|[f^{(y)}]^2\| + \|f^2\|_{\frac{p}{2}}} \right]^{\frac{p}{2}-1} & \text{if } 1 < \frac{p}{2} \leq 2 \end{cases}$$

which by the Holder and triangle inequalities, and because $y \in [-1, 1]$

$$\leq \begin{cases} 5(9p^2 - 12p + 7)|y| & \text{if } \frac{p}{2} \geq 2 \\ 4(p-1) \left[4(p-2) + \left(\frac{3p}{p-2}\right)^{\frac{p}{2}-1} \right] |y|^{\frac{p}{2}-1} & \text{if } 1 < p \leq 2 \,. \end{cases}$$

Thus,

$$\left| \int_0^\varepsilon \int_0^s P''_\psi(y) - P''_\psi(0) \mathrm{d}y \mathrm{d}s \right| \leq \begin{cases} 5(8p^2 - 12p + 7) \int_0^{|\varepsilon|} \int_0^s y \mathrm{d}y \mathrm{d}s & \text{if } \frac{p}{2} \geq 2 \\ 4(p-1) \left[4(p-2) + \left(\frac{3p}{p-2}\right)^{\frac{p}{2}-1} \right] \int_0^{|\varepsilon|} \int_0^s y^{\frac{p}{2}-1} \mathrm{d}y \mathrm{d}s & \text{if } 1 < \frac{p}{2} \leq 2 \end{cases}$$

$$\leq \kappa_p |\varepsilon|^{\beta_p} \,. \tag{2.34}$$

Combining (2.34) with Lemma 2.2.2 we conclude Theorem 2.2.1. □

2.3 Statement of a Local Bianchi-Egnell Extension and Outline of Proof

In this subsection, we state and outline the proof of the following

THEOREM 2.3.1 (Local Bianchi-Egnell Extension)**.** *Let* $\varphi \in \dot{H}^1_{\mathbb{C}}(\mathbb{R}_+ \times \mathbb{R}^n, \omega_m \rho^{m-1} \mathrm{d}\rho \mathrm{d}x)$ *be such that*

$$\|\varphi\|_{\dot{H}^1} = 1, \text{ and } \delta(\varphi, M) \leq \frac{1}{2} . \tag{2.35}$$

Then

$$C^2_{m,n}\|\varphi\|^2_{\dot{H}^1} - \|\varphi\|^2_{2^*} \geq \alpha_{m,n}\delta(\varphi, M)^2 - \frac{\kappa_{2^*} C^2_{m,n}}{4 \cdot 3^{\frac{\beta_{2^*}}{2}-1}} \delta(\varphi, M)^{\beta_{2^*}} , \tag{2.36}$$

where $\alpha_{m,n}$ *is the smallest positive eigenvalue of the operator* $C^2_{m,n} I - A^{-1}\mathcal{L}_{F_{1,0},2^*} : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ *for* $\mathcal{L}_{f,p}$ *as defined in subsection 2.2 and*

$$A = -\Delta_x - \frac{\partial^2}{\partial \rho^2} - \frac{m-1}{\rho}\frac{\partial}{\partial \rho} .$$

This gives a local version of a Bianchi-Egnell stability estimate for $\varphi \in \dot{H}^1_{\mathbb{C}}$ *such that* $\|\varphi\|_{\dot{H}^1} = 1$*, provided*

$$\delta(\varphi, M) \leq \min\left\{ \left(\frac{\alpha_{m,n}}{2\kappa_{2^*}}\right)^{1/(\beta_{2^*}-2)}, \frac{1}{2} \right\} . \tag{2.37}$$

We begin with the proof of the last sentence of Theorem 2.3.1

Proof. Let $\varphi \in \dot{H}^1_{\mathbb{C}}$ obey (2.37). Then,

$$\begin{aligned} C^2_{m,n}\|\varphi\|^2_{\dot{H}^1} - \|\varphi\|^2_{2^*} &\geq \alpha_{m,n}\delta(\varphi, M)^2 - \kappa_{2^*}\delta(\varphi, M)^{\beta_{2^*}} \\ &= \delta(\varphi, M)^2 \left[\alpha_{m,n} - \kappa_{2^*}\delta(\varphi, M)^{\beta_{2^*}-2}\right] \\ &\geq \frac{\alpha_{m,n}}{2}\delta(\varphi, M)^2, \text{ by (2.37).} \end{aligned}$$

The inequality we deduced above is in fact a Bianchi-Egnell stability estimate as characterized by (2.14) with α in (2.14) equal $\frac{\alpha_{m,n}}{2}$. □

With the last sentence out of the way, we only need to show that (2.35) implies (2.36).

Once we have proved Theorem 2.3.1, we will be able to use it and a Concentration Compactness argument to prove Theorem 2.1.5. About half of the work in this paper is devoted to proving Theorem 2.3.1 - or rather the following reduction of Theorem 2.3.1:

LEMMA 2.3.2. *Let* $\varphi \in \dot{H}^1_{\mathbb{C}}$ *obey (2.35) and be such that*

$$\delta(\varphi, M) = \|\varphi - zF_{t,0}\|_{\dot{H}^1}, \text{ for some } z \in \mathbb{R} \text{ and } t \in \mathbb{R}_+ . \tag{2.38}$$

Then (2.36) holds for φ*.*

It is important to note that we stipulated that $z \in \mathbb{R}$, not $\mathbb{C}$, because this is half of the reduction. The other half of the reduction is that a minimizing element of φ is $zF_{t,0}$ as opposed to a more general zF_{t,x_0}.

In order to prove Theorem 2.3.1, we would like to use Theorem 2.2.1. This requires φ to be in the form $f + \delta(\varphi, M)\psi$. φ is in fact in such a form due to

LEMMA 2.3.3. *Let $\varphi \in \dot{H}^1_{\mathbb{C}}$ be such that*

$$\delta(\varphi, M) < \|\varphi\|_{\dot{H}^1} . \tag{2.39}$$

Then, $\exists zF_{t,x_0} \in M$ such that

$$\delta(\varphi, M) = \|\varphi - zF_{t,x_0}\|_{\dot{H}^1} . \tag{2.40}$$

Proof. M can be viewed as a continuous imbedding of $\mathbb{C} \times \mathbb{R}_+ \times \mathbb{R}^n$ into $\dot{H}^1_{\mathbb{C}}$ by the map

$$(z, t, x_0) \mapsto zF_{t,x_0} .$$

Thus, the existence of an element zF_{t,x_0} satisfying (2.40) is a consequence of the continuity of the map from $\mathbb{C} \times \mathbb{R}_+ \times \mathbb{R}$ to $\mathbb{R}$ given by

$$(z, t, x_0) \mapsto \|\varphi - zF_{t,x_0}\|_{\dot{H}^1}$$

and the fact that (2.39) implies that any such minimizing triple (z, t, x_0) must occur on a set away from the origin and infinity in $\mathbb{C} \times \mathbb{R}_+ \times \mathbb{R}^n$. Thus, a minimizing element zF_{t,x_0} exists by lower semicontinuity on bounded sets in Euclidean space. This is an adaptation of a proof to an analogous statement in [BiEg], see Lemma 1 in [BiEg] for more detail. $\square$

Applying Lemma 2.3.3, we conclude that φ obeying the assumptions of Theorem 2.3.1 has the form

$$\varphi = zF_{t,x_0} + \delta(\varphi, M)\psi ,$$

for some $\psi \in \dot{H}^1_{\mathbb{C}}$ such that $\|\psi\|_{\dot{H}^1} = 1$. If we multiply φ by $\overline{z}/|z|$ and translate $(0, x_0)$ to the origin - both operations are invariant on $\|\cdot\|_{\dot{H}^1}$, $\|\cdot\|_{2^*}$, $\delta(\cdot, M)$ - then we end up with some $\tilde{\varphi} = |z|F_{t,0} + \delta(\tilde{\varphi}, M)\psi$ whose relevant norms and distances are the same as

φ. Thus, if (2.36) holds for functions obeying (2.35) and (2.38), then they hold for all functions obeying (2.35), i.e. Theorem 2.3.1 is a corollary of Lemma 2.3.2.

Now that we have shown that Theorem 2.3.1 is a corollary of Lemma 2.3.2, we will use Theorem 2.2.1 to begin to prove Lemma 2.3.2. To be precise, we will prove the following

LEMMA 2.3.4. *Let $\varphi \in \dot{H}^1_{\mathbb{C}}$ satisfy the assumptions of Lemma 2.3.2. Then*

$$C_{m,n}^2\|\varphi\|_{\dot{H}^1}^2 - \|\varphi\|_{2^*}^2 \geq \left\langle (C_{m,n}^2 I - A^{-1}\mathcal{L}_{C_{m,n}^{-1}F_{t,0},2^*})\psi, \psi \right\rangle_{\dot{H}^1_{\mathbb{C}}} \delta(\varphi, M)^2 - \frac{\kappa_{2^*}C_{m,n}^2}{4\cdot 3^{\frac{\beta_{2^*}}{2}-1}}\delta(\varphi, M)^{\beta_{2^*}}\,, \tag{2.41}$$

where $\psi \in \dot{H}^1_{\mathbb{C}}$ is such that

$$\varphi = zF_{t,0} + \delta(\varphi, M)\psi, \text{ and } \|\psi\|_{\dot{H}^1} = 1\,. \tag{2.42}$$

Proof. φ satisfies (2.42) and $\psi \perp_{\dot{H}^1_{\mathbb{C}}} F_{t,0}$ as a result of (2.38). Consistent with the notation of subsection 2.2, we let $\psi = (\xi, \eta) \in \dot{H}^1 \oplus \dot{H}^1$. Applying Theorem 2.2.1 to $\|\varphi\|_{2^*}^2$ yields

$$\begin{aligned}\|\varphi\|_{2^*}^2 =& z^2C_{m,n}^2\left\|\frac{F_{t,0}}{C_{m,n}} + \frac{\delta(\varphi,M)\|\psi\|_{2^*}}{zC_{m,n}}\cdot\frac{\psi}{\|\psi\|_{2^*}}\right\|_{2^*}^2 \\ \leq& z^2C_{m,n}^2 + 2zC_{m,n}^{2-2^*}\langle F_{t,0}|F_{t,0}|^{2^*-2}, \xi\rangle_{L^2}\delta(\varphi, M) + \left\langle \mathcal{L}_{C_{m,n}^{-1}F_{t,0},2^*}\psi, \psi\right\rangle_{L^2\oplus L^2}\delta(\varphi, M)^2 \\ &+ \frac{\kappa_{2^*}\|\psi\|_{2^*}^{\beta_{2^*}}}{(zC_{m,n})^{\beta_{2^*}-2}}\delta(\varphi, M)^{\beta_{2^*}}\,.\end{aligned} \tag{2.43}$$

We claim that the coefficient of first order in the right hand side of (2.43) equals zero. To see this, we consider the function $R_\psi : \mathbb{R} \to \mathbb{R}$ given by

$$R_\psi(\varepsilon) = \frac{\|F_{t,0} + \varepsilon\psi\|_{2^*}^2}{\|F_{t,0} + \varepsilon\psi\|_{\dot{H}^1}^2}\,.$$

Since $F_{t,0}$ is an extremal of the Sobolev Inequality

$$\begin{aligned}&0 = R'_\psi(0) = 2\|F_{t,0}\|_{2^*}^{2-2^*}\left\langle F_{t,0}|F_{t,0}|^{2^*-2}, \xi\right\rangle_{L^2} - 2\|F_{t,0}\|_{2^*}^2\langle F_{t,0}, \xi\rangle_{\dot{H}^1} \\ &\implies \left\langle F_{t,0}|F_{t,0}|^{2^*-2}, \xi\right\rangle_{L^2} = \|F_{t,0}\|_{2^*}^{2^*}\langle F_{t,0}, \xi\rangle_{\dot{H}^1} = 0, \text{ as } \psi \perp_{\dot{H}^1_{\mathbb{C}}} F_{t,0}\,.\end{aligned} \tag{2.44}$$

Thus, by (2.43), (2.44), and the fact that $\psi \perp_{\dot{H}^1_{\mathbb{C}}} F_{t,0}$

$$
\begin{aligned}
C_{m,n}^2\|\varphi\|_{\dot{H}^1}^2 - \|\varphi\|_{2^*}^2 \geq & C_{m,n}^2\left[\|zF_{t,0}\|_{\dot{H}^1}^2 + \|\psi\|_{\dot{H}^1}^2\delta(\varphi,M)^2\right] \\
& - \left[C_{m,n}^2 z^2 + \left\langle \mathcal{L}_{C_{m,n}^{-1}F_{t,0},2^*}\psi,\psi\right\rangle_{L^2\oplus L^2}\delta(\varphi,M)^2 + \frac{\kappa_{2^*}\|\psi\|_{2^*}^{\beta_{2^*}}}{(zC_{m,n})^{\beta_{2^*}-2}}\delta(\varphi,M)^{\beta_{2^*}}\right] \\
& \text{and since } \langle A\varphi_1,\varphi_2\rangle_{L^2\oplus L^2} = \langle\varphi_1,\varphi_2\rangle_{\dot{H}^1_{\mathbb{C}}} \text{ for all } \varphi_1,\varphi_2 \in \dot{H}^1_{\mathbb{C}} \\
= & \left\langle (C_{m,n}^2 I - A^{-1}\mathcal{L}_{C_{m,n}^{-1}F_{t,0},2^*})\psi,\psi\right\rangle_{\dot{H}^1_{\mathbb{C}}}\delta(\varphi,M)^2 - \frac{\kappa_{2^*}\|\psi\|_{2^*}^{\beta_{2^*}}}{(zC_{m,n})^{\beta_{2^*}-2}}\delta(\varphi,M)^{\beta_{2^*}}\,.
\end{aligned}
\tag{2.45}
$$

(2.35), (2.42), and the Sobolev Inequality imply

$$
|z| \geq \frac{\sqrt{3}}{2}, \text{ and } \|\psi\|_{2^*} \leq C_{m,n}/2\,. \tag{2.46}
$$

(2.45) and (2.46) allow us to conclude (2.41). □

Having proved that under the assumptions of Lemma 2.3.2 that (2.41) holds, we only need to show that

$$
\left\langle \left(C_{m,n}^2 I - A^{-1}\mathcal{L}_{C_{m,n}^{-1}F_{t,0},0}\right)\psi,\psi\right\rangle_{\dot{H}^1_{\mathbb{C}}} \geq \alpha_{m,n}\,, \tag{2.47}
$$

in order to prove Lemma 2.3.2, which in turn proves Theorem 2.3.1. In order to simplify notation, we let

$$
S_t = A^{-1}\mathcal{L}_{C_{m,n}^{-1}F_{t,0},2^*}\,.
$$

We prove (2.47) by proving the following

THEOREM 2.3.5. $C_{m,n}^2 I - S_t : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ *has a nonnegative, bounded, discrete spectrum, whose eigenvalues are independent of the value of the parameter* t. *This spectrum has at most one accumulation point, which if it exists, is at* $C_{m,n}^2$. *Let* λ_i, $i = 0,1,2,\ldots$, *(with this list possibly finite) denote the eigenvalues of* $C_{m,n}^2 I - S_t : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ *whose value are less than* $C_{m,n}^2$, *with* λ_i *listed in increasing order. Then,* $\lambda_0 = 0$ *and its corresponding eigenspace is spanned by* $\{(F_{t,0},0),(\frac{\mathrm{d}}{\mathrm{d}t}F_{t,0},0),(0,F_{t,0})\}$. *Finally,* $\{(F_{t,0},0),(\frac{\mathrm{d}}{\mathrm{d}t}F_{t,0},0),(0,F_{t,0})\} \perp_{\dot{H}^1} \psi$.

We split the proof of Theorem 2.3.5 into the proof of two smaller theorems and a brief argument establishing independence of eigenvalues from the value of the parameter t. We state these theorems below

THEOREM 2.3.6. *$S_t : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ is a self-adjoint compact operator.*

THEOREM 2.3.7. *$C^2_{m,n}I - S_t : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ is a positive operator, its nullspace is spanned by $\{(F_{t,0}, 0), (\frac{\mathrm{d}}{\mathrm{d}t}F_{t,0}, 0), (0, F_{t,0})\}$, and $\psi \perp_{\dot{H}^1_{\mathbb{C}}} \{(F_{t,0}, 0), (\frac{\mathrm{d}}{\mathrm{d}t}F_{t,0}, 0), (0, F_{t,0})\}$.*

Once we have proved theorems 2.3.6 and 2.3.7, all of Theorem 2.3.5, except for the independence of eigenvalues from the value of t, follows via Fredholm Theory. The proofs of theorems 2.3.6 and 2.3.7 are somewhat difficult and are presented in sections four and five respectively. We prove the independence of eigenvalues of $C^2_{m,n}I - S_t : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ from the value of t here; a change of coordinates makes this proof more readily apparent. We obtain the appropriate coordinate system through several changes. First we change to (w, θ, ζ)-coordinates, $(w, \theta, \zeta) \in [0, \infty) \times [0, \pi/2] \times \mathbb{S}^{n-1}$, where

$$\varphi(w, \theta, \zeta) = \varphi(\rho, x), \text{ for } \rho = w\cos\theta, x = (w\sin\theta, \zeta)\,. \tag{2.48}$$

And then, we change to (u, θ, ζ)-coordinates, $(u, \theta, \zeta) \in \mathbb{R} \times [0, \pi/2] \times \mathbb{S}^{n-1}$, given by

$$\varphi(u, \theta, \zeta) = w^{\gamma}\varphi(w, \theta, \zeta), \text{ for } u = \ln w \text{ and } \gamma \text{ given by (2.11)}\,. \tag{2.49}$$

In (u, θ, ζ)-coordinates

$$F_{t,0}(u, \theta, \zeta) = k_0 2^{-\gamma}\cosh^{-\gamma}(u + \ln t)\,. \tag{2.50}$$

Thus, $F_{t,0}$ and $F_{t',0}$ are related by a translation of in u-coordinates by $\ln t' - \ln t$. This fact combined with the explicit formula of $\mathcal{L}$ as per Theorem 2.2.1 and that $C^2_{m,n}I - S_t = C^2_{m,n}I - A^{-1}\mathcal{L}_{C^{-1}_{m,n}F_{t,0},0}$ allows us to conclude that the eigenvalues of $C^2_{m,n}I - S_t : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ are independent of t. Combining this with Theorem 2.3.5, we conclude (2.47) with $\alpha_{m,n} = \lambda_1$, which is the definition of $\alpha_{m,n}$ as per Theorem 2.3.1. Thus, we have shown Lemma 2.3.2, which in turn proves Theorem 2.3.1.

2.4 $S_t : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ is a Self-Adjoint, Compact Operator

In this section, we prove Theorem 2.3.6. We begin with the following

LEMMA 2.4.1. *$S_t : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ is self-adjoint.*

Proof. Let $\varphi_1, \varphi_2 \in \dot{H}^1_{\mathbb{C}}$. Then,

$$\langle \varphi_1, S_t \varphi_2 \rangle_{\dot{H}^1_{\mathbb{C}}} = \left\langle \varphi_1, A^{-1} \mathcal{L}_{C^{-1}_{m,n} F_{t,0}, 2^*} \varphi_2 \right\rangle_{\dot{H}^1_{\mathbb{C}}} = \left\langle \varphi_1, \mathcal{L}_{C^{-1}_{m,n} F_{t,0}, 2^*} \varphi_2 \right\rangle_{L^2 \oplus L^2}.$$

It is clear from the explicit form of $\mathcal{L}_{C^{-1}_{m,n} F_{t,0}, 2^*}$ provided in Theorem 2.2.1 that $\mathcal{L}_{C^{-1}_{m,n} F_{t,0}, 2^*} : L^2 \oplus L^2 \to L^2 \oplus L^2$ is self-adjoint. Thus, $S_t : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ is self-adjoint. □

Next, we prove the following

LEMMA 2.4.2. *$S_t : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ is compact.*

Proof. This proof is quite involved. We use this paragraph to outline the proof and then carry out the proof in mini-sections headed by phrases in italics. First, we reduce proving compactness of $S_t : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ to proving compactness of $A^{-1} \mathcal{L}^{Im}_{C^{-1}_{m,n} F_{t,0}, 2^*} : \dot{H}^1 \to \dot{H}^1$ (we will omit the subscripts $C^{-1}_{m,n} F_{t,0}, 2^*$ henceforth). Next, we use the fact that $A^{1/2} : \dot{H}^1 \to L^2$, the square root of A, is an isometry to reduce proving compactness of $A^{-1} \mathcal{L}^{Im} : \dot{H}^1 \to \dot{H}^1$ to proving compactness of $A^{-1/2} \mathcal{L}^{Im} A^{-1/2} : L^2 \to L^2$. This follows due to commutativity of the following diagram

$$\begin{array}{ccc} \dot{H}^1 & \overset{A^{-1}\mathcal{L}^{Im}}{\rightarrow} & \dot{H}^1 \\ \downarrow_{A^{1/2}} & & \uparrow_{A^{-1/2}} \\ L^2 & \overset{A^{-1/2}\mathcal{L}^{Im}A^{-1/2}}{\rightarrow} & L^2 \,. \end{array} \tag{2.51}$$

This reduction is crucial, because it reduces the proof of compactness over $\dot{H}^1$ to L^2, where verification of compactness is much easier to do directly. Next, we change coordinates and reduce showing compactness of $A^{-1/2} \mathcal{L}^{Im} A^{-1/2} : L^2 \to L^2$ to a closely related operator, $(\mathcal{L}^{Im})^{1/2} \hat{A}^{-1} (\mathcal{L}^{Im})^{1/2} : L^2 \to L^2$. The change of coordinates preceding this reduction is also crucial, because it helps illuminate the route that we take to verify compactness of $A^{-1/2} \mathcal{L}^{Im} A^{-1/2} : L^2 \to L^2$. At this point, we have reduced the compactness problem to a more manageable situation. We proceed by endeavoring to show that $(\mathcal{L}^{Im})^{1/2} \hat{A}^{-1} (\mathcal{L}^{Im})^{1/2} : L^2 \to L^2$ has arbitrarily good finite rank approximation in the operator norm. In particular, we calculate the Green's function of $\hat{A}$ and use this calculation to show that the trace of $[(\mathcal{L}^{Im})^{1/2} \hat{A}^{-1} (\mathcal{L}^{Im})^{1/2}]^d$ is finite for some suitably large even value of d, i.e. $[(\mathcal{L}^{Im})^{1/2} \hat{A}^{-1} (\mathcal{L}^{Im})^{1/2}]^d$ is trace class. Since

$[(\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2}]^d : L^2 \to L^2$ is trace class, it is compact and its eigenvalues converge to zero. By standard spectral theory, the eigenvalues of $[(\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2}]^d$ are the d-th power of the eigenvalues of $(\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2}$. Thus, the eigenvalues of $(\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2}$ also converge to zero, which implies that $(\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2} : L^2 \to L^2$ has arbitrarily good finite rank approximation in the operator norm. Thus, to show that $(\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2} : L^2 \to L^2$ is compact, we only need to show that $[(\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2}]^d$ is trace class. One should note that in the course of the proof, we show that $d > \frac{n+1}{2}$ is sufficient, and it does not appear that $[(\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2}]^d$ is necessarily trace class for smaller d.

Mini-Section 1: Compactness of $S_t : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ is implied by compactness of $A^{-1}\mathcal{L}^{Im} : \dot{H}^1 \to \dot{H}^1$. Note that

$$S_t = A^{-1}\mathcal{L} = A^{-1}\mathcal{L}^{Re} \oplus A^{-1}\mathcal{L}^{Im}\,,$$

as per Theorem 2.2.1. Thus, to show that $S_t : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ is compact, it suffices to show that $A^{-1}\mathcal{L}^{Re}, A^{-1}\mathcal{L}^{Im} : \dot{H}^1 \to \dot{H}^1$ are compact. Also,

$$\begin{aligned} A^{-1}\mathcal{L}^{Re}\xi &= A^{-1}\left[-(2^*-2)C_{m,n}^{2-2\cdot 2^*}\left(\int F_{t,0}^{2^*-1}\xi\right)F_{t,0}^{2^*-1}\right] + A^{-1}\left[C_{m,n}^{2-2^*}F_{t,0}^{2^*-2}(2^*-1)\xi\right] \\ &=: P\xi + (2^*-1)A^{-1}\mathcal{L}^{Im}\xi\,. \end{aligned} \tag{2.52}$$

Note that the calculations used to obtain (2.44) could also be used to show that

$$\left\langle F_{t,0}^{2^*-1}, \xi\right\rangle_{L^2} = \|F_{t,0}\|_{2^*}^{2^*}\left\langle AF_{t,0}, \xi\right\rangle_{L^2}, \forall \xi \in \dot{H}^1\,.$$

Since F is of class $\mathcal{C}^\infty$, this implies that

$$F_{t,0}^{2^*-1} = C_{m,n}^{2^*}AF_{t,0}, \text{ because } \|F\|_{2^*} = C_{m,n}\,. \tag{2.53}$$

This in turn implies that

$$P\xi = -(2^*-2)C_{m,n}^{2^*}\left\langle F_{t,0}, \xi\right\rangle_{\dot{H}^1} F_{t,0}\,. \tag{2.54}$$

Thus, $P : \dot{H}^1 \to \dot{H}^1$ is a projection operator onto $F_{t,0}$, which implies that $P : \dot{H}^1 \to \dot{H}^1$ is compact. Combining this fact with (2.52), we only need to show that $A^{-1}\mathcal{L}^{Im} : \dot{H}^1 \to \dot{H}^1$ is compact in order to conclude that $S_t : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ is compact.

<u>*Mini-Section 2:*</u> *Compactness of* $A^{-1}\mathcal{L}^{Im} : \dot{H}^1 \to \dot{H}^1$ *is implied by compactness of* $(\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2} : \dot{H}^1 \to \dot{H}^1$. We already explained in the above via the commutative diagram, (2.51), that $A^{-1}\mathcal{L}^{Im} : \dot{H}^1 \to \dot{H}^1$ is compact if and only if $A^{-1/2}\mathcal{L}^{Im}A^{-1/2} : L^2 \to L^2$ is compact. This reduces the problem of proving compactness of an operator from $\dot{H}^1$ to $\dot{H}^1$ to an operator from L^2 to L^2, making direct verification of compactness easier. However, the explicit form of $A^{-1/2}\mathcal{L}^{Im}A^{-1/2}$ does not seem to suggest any easy way to verify the desired compactness. But, changing from (ρ, x)-coordinates to (u, v, ζ)-coordinates, $(u, v, \zeta) \in \mathbb{R} \times [-1, 1] \times \mathbb{S}^{n-1}$, provides a set of coordinates for which this verification is easier. We obtain (u, v, ζ)-coordinates by making a change of variables from (u, θ, ζ)-coordinates (see (2.48) and (2.49) for reference) with respect to the angular coordinate, θ, given by

$$v = 2\cos^2\theta - 1\,.$$

In these coordinates, A has the explicit form (γ is as defined in (2.11))

$$A = \gamma^2 I - \frac{\partial^2}{\partial u^2} - 4(1 - v^2)\frac{\partial^2}{\partial v^2} - 4\left(\frac{m-n}{2} - \frac{m+n}{2}v\right)\frac{\partial}{\partial v} - \frac{2}{v+1}\Delta_{\mathbb{S}^{n-1}(\zeta)}\,. \quad (2.55)$$

(2.55) is almost a nice formula of A for which we can write the Green's function of A and use this to prove the desired compactness. However, the last term in the right hand side of (2.55) is nonlinear and makes figuring out the Green's function of A difficult. Thus, we use the closely related operator $\hat{A}$ given by

$$\hat{A} = \gamma^2 I - \frac{\partial^2}{\partial u^2} - 4(1 - v^2)\frac{\partial^2}{\partial v^2} - 4\left(\frac{m-n}{2} - \frac{m+n}{2}v\right)\frac{\partial}{\partial v} - \Delta_{\mathbb{S}^{n-1}(\zeta)}\,, \quad (2.56)$$

to help us show the desired compactness. More precisely, we show that $A^{-1/2}\mathcal{L}^{Im}A^{-1/2} : L^2 \to L^2$ is compact by showing that $(\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2} : L^2 \to L^2$ is compact. This last reduction is justified as follows: if $(\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2} : L^2 \to L^2$ is compact, then $(\mathcal{L}^{Im})^{1/2}A^{-1}(\mathcal{L}^{Im})^{1/2} : L^2 \to L^2$ is compact. This is because

$$A \geq \hat{A} \geq 0 \implies (\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2} \geq (\mathcal{L}^{Im})^{1/2}A^{-1}(\mathcal{L}^{Im})^{1/2} \geq 0\,.$$

Next, let

$$B := A^{-1/2}(\mathcal{L}^{Im})^{1/2}\,.$$

If $(\mathcal{L}^{Im})^{1/2}A^{-1}(\mathcal{L}^{Im})^{1/2} : L^2 \to L^2$ is compact, then

$$\begin{aligned}
& B^*B = (\mathcal{L}^{Im})^{1/2}A^{-1}(\mathcal{L}^{Im})^{1/2} : L^2 \to L^2 \text{ is compact} \\
\Longrightarrow\quad & B, B^* : L^2 \to L^2 \text{ are bounded} \\
\Longrightarrow\quad & B\mathcal{L}^{Im}A^{-1}(\mathcal{L}^{Im})^{1/2}B^* = (A^{-1/2}\mathcal{L}^{Im}A^{-1/2})^2 : L^2 \to L^2 \text{ is compact} \\
\Longrightarrow\quad & A^{-1/2}\mathcal{L}^{Im}A^{-1/2} : L^2 \to L^2 \text{ is compact}\,.
\end{aligned}$$

Thus, compactness of $(\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2} : L^2 \to L^2$ implies compactness of $A^{-1/2}\mathcal{L}^{Im}A^{-1/2} : L^2 \to L^2$.

Mini-Section 3: Calculating the Green's function of $\hat{A}$. We can write $\hat{A}$ as

$$\hat{A} = U + V + W\,,$$

where

$$\begin{aligned}
U &= \gamma^2 I - \frac{\partial^2}{\partial u^2} \\
V &= -4(1-v^2)\frac{\partial^2}{\partial v^2} - 4[\hat{\alpha} - \hat{\beta} - (\hat{\alpha} + \hat{\beta} + 2)v]\frac{\partial}{\partial v}, \text{ and} \\
W &= -\Delta_{\mathbb{S}^{n-1}(\zeta)}\,,
\end{aligned}$$

for $\hat{\alpha} = \frac{m-2}{2}$ and $\hat{\beta} = \frac{n-2}{2}$. We can build the Green's function for $\hat{A}$ out of the eigenfunctions and eigenvalues of U, V, and W. The eigenfunctions of U are $q_k(u) := e^{-iku}$, with corresponding eigenvalues $\gamma^2 + k^2$. The eigenfunctions of V are the Jacobi Polynomials, $p_j^{\hat{\alpha},\hat{\beta}}(v)$, with corresponding eigenvalues $\sigma_j = 4j(j + \frac{m+n}{2} - 1)$; for more detail, see p. 60 of [Sz]. The eigenfunctions of W are the spherical harmonics of $\mathbb{S}^{n-1}$. We will let $g_l(\zeta)$ denote the spherical harmonics, arranged in such a fashion that their corresponding eigenvalues, τ_l, for W are nonincreasing (all of the eigenvalues will be nonnegative). Thus, the Green's function of $\hat{A}$ is

$$\begin{aligned}
G(u, v, \zeta|\tilde{u}, \tilde{v}, \tilde{\zeta}) &= \sum_{j,l\geq 0} \int_{\mathbb{R}} \frac{1}{\gamma^2 + k^2 + \tau_l + \sigma_j} q_k(u)p_j^{\hat{\alpha},\hat{\beta}}(v)g_l(\zeta)\bar{q}_k(\tilde{u})\bar{p}_j^{\hat{\alpha},\hat{\beta}}(\tilde{v})\bar{g}_l(\tilde{\zeta})\mathrm{d}k \\
&= \sum_{j,l\geq 0} \pi(\gamma^2 + \tau_l + \sigma_j)^{-1/2} e^{-|u-\tilde{u}|\sqrt{\gamma^2+\tau_l+\sigma_j}} p_j^{\hat{\alpha},\hat{\beta}}(v)g_l(\zeta)\bar{p}_j^{\hat{\alpha},\hat{\beta}}(\tilde{v})\bar{g}_l(\tilde{\zeta})\,.
\end{aligned} \tag{2.57}$$

Using the Green's function above, we will show that $(\mathcal{L}^{Im})^{1/2}\hat{A}^{-1/2}(\mathcal{L}^{Im})^{1/2} : L^2 \to L^2$ is compact.

Mini-Section 4: *$(\mathcal{L}^{Im})^{1/2}\hat{A}^{-1/2}(\mathcal{L}^{Im})^{1/2} : L^2 \to L^2$ has arbitrarily good finite rank approximation in the operator norm.* We will show that $(\mathcal{L}^{Im})^{1/2}\hat{A}^{-1/2}(\mathcal{L}^{Im})^{1/2} : L^2 \to L^2$ is compact by showing that for $d > \frac{n+1}{2}$,

$$\mathrm{Tr}\left[((\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2})^d\right] < \infty\,, \tag{2.58}$$

i.e. $((\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2})^d$ is trace class. At the end of the first paragraph of the proof of Lemma 2.4.2, we showed that if we prove (2.58), then we can conclude that $(\mathcal{L}^{Im})^{1/2}\hat{A}^{-1/2}(\mathcal{L}^{Im})^{1/2} : L^2 \to L^2$ has arbitrarily good finite rank approximation in the operator norm, and so is compact. Thus, to show that $(\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2} : L^2 \to L^2$ is compact, we only need to prove (2.58).

In order to prove (2.58), we derive the kernel for $((\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2})^d$. Using (2.57), we calculate that the kernel for $(\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2}$ is

$$K(u,v,\zeta|\tilde{u},\tilde{v},\tilde{\zeta}) = \hat{F}(u)\hat{F}(\tilde{u})\sum_{j,l}\frac{\pi}{\xi_{j,l}}e^{-|u-v|\xi_{j,l}}p_j^{\alpha,\beta}(v)\bar{p}_j^{\alpha,\beta}(\tilde{v})g_l(\zeta)\bar{g}_l(\tilde{\zeta})\,,$$

where $\hat{F} = (C_{m,n}^{-1}F_{t,0})^{(2^*-2)/2}$ and $\xi_{j,l} = \sqrt{\gamma^2+\tau_l+\sigma_j}$. Using this, we calculate the kernel, K_d, for $((\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2})^d$. Before doing so, we make the following convention to simplify notation:

$$\int \cdot\ \mathrm{d}\Lambda(u,v,\zeta)$$

denotes the integral over $\mathbb{R}\times[-1,1]\times\mathbb{S}^{n-1}$ with $\mathrm{d}\Lambda(u,v,\zeta)$ representing $\mathrm{d}\Lambda$ as defined in (2.9) corresponding to the change of coordinates from (ρ,x) to (u,v,ζ). Thus,

$$\begin{aligned}
&K_d(u_1,v_1,\zeta_1|u_{d+1},v_{d+1},\zeta_{d+1})\\
=\;& \int\int\cdots\int\prod_{i=1}^{d}K(u_i,v_i,\zeta_i|u_{i+1},v_{i+1},\zeta_{i+1})\mathrm{d}\Lambda(u_2,v_2,\zeta_2)\mathrm{d}\Lambda(u_3,v_3,\zeta_3)\cdots\mathrm{d}\Lambda(u_d,v_d,\zeta_d)\\
=\;& \hat{F}(u_1)\hat{F}(u_{d+1})\sum_{j,l\geq 0}\frac{\pi^d}{\xi_{j,l}^d}p_j^{\hat{\alpha},\hat{\beta}}(v_1)\bar{p}_j^{\hat{\alpha},\hat{\beta}}(v_{d+1})g_l(\zeta_1)\bar{g}(\zeta_{d+1})\\
&\int\int\cdots\int\prod_{i=2}^{d}\hat{F}^2(u_i)|p_j^{\hat{\alpha},\hat{\beta}}(v_i)|^2|g_l(\zeta_i)|^2\prod_{a=1}^{d}e^{-\xi_{j,l}|u_{a+1}-u_a|}\mathrm{d}\Lambda(u_2,v_2,\zeta_2)\mathrm{d}\Lambda(u_3,v_3,\zeta_3)\cdots\mathrm{d}\Lambda(u_d,v_d,\zeta_d)\\
=\;& \hat{F}(u_1)\hat{F}(u_{d+1})\sum_{j,l\geq 0}\frac{\pi^d}{\xi_{j,l}^d}p_j^{\hat{\alpha},\hat{\beta}}(v_1)\bar{p}_j^{\hat{\alpha},\hat{\beta}}(v_{d+1})g_l(\zeta_1)\bar{g}(\zeta_{d+1})\int_{\mathbb{R}^{d-1}}\prod_{i=2}^{d}\hat{F}^2(u_i)\prod_{a=1}^{d}e^{-\xi_{j,l}|u_{a+1}-u_a|}\mathrm{d}u_2\mathrm{d}u_3\cdots\mathrm{d}u_d
\end{aligned}$$

Thus,

$$
\begin{aligned}
\operatorname{Tr}\left[((\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2})^d\right] &= \|K_d(u_1,v_1,\zeta_1|u_1,v_1,\zeta_1)\|_1 \\
&= \int\int_{\mathbb{R}^{d-1}} \sum_{j,l\geq 0} \frac{\pi^d}{\xi_{j,l}^d} |p_j^{\hat{\alpha},\hat{\beta}}(v_1)|^2 |g_l(\zeta_1)|^2 \hat{F}^2(u_1) \\
&\quad \prod_{i=1}^{d} \hat{F}^2(u_i) \left(\prod_{a=1}^{d-1} e^{-\xi_{j,l}|u_{a+1}-u_1|}\right) e^{-\xi_{j,l}|u_1-u_d|} \mathrm{d}u_2\mathrm{d}u_3\cdots\mathrm{d}u_d\mathrm{d}\Lambda(u_1,v_1,\zeta_1) \\
&= \sum_{j,l\geq 0} \frac{\pi^d}{\xi_{j,l}^d} \int_{\mathbb{R}^d} \left(\prod_{i=1}^{d} \hat{F}^2(u_i)\right) e^{-\xi_{j,l}|u_1-u_d|} \prod_{a=1}^{d-1} e^{-\xi_{j,l}|u_{a+1}-u_a|} \mathrm{d}u_1\mathrm{d}u_2\cdots\mathrm{d}u_d\,.
\end{aligned} \tag{2.59}
$$

We can apply the Generalized Young's Inequality to each of the integrals in the right hand side of (2.59). The version of the Generalized Young's Inequality we use in this setting is

Generalized Young's Inequality: *Let R_i be a real $1\times d$ matrix and $h_i:\mathbb{R}\to\mathbb{R}$ be a function for $i=1,2,\ldots,2d$. Also, let $p_1,p_2,\ldots,p_{2d}$ be such that*

$$\sum_{i=1}^{2d}\frac{1}{p_i} = d\,.$$

Then,

$$\int_{\mathbb{R}^d}\prod_{i=1}^{2d} h_i(R_i\cdot\vec{u})\mathrm{d}u_1\mathrm{d}u_2\cdots\mathrm{d}u_d \leq \hat{C}_{d,p_1,p_2,\ldots,p_{2d}}\prod_{i=1}^{2d}\|h_i\|_{L^{p_i}(\mathbb{R})}\,,$$

for some finite constant $\hat{C}_{d,p_1,p_2,\ldots,p_{2d}}$ provided for any $J\subseteq\{1,2,\ldots,2d\}$ such that $\operatorname{card}(J)\leq d$ *has the property that*

$$\dim(\operatorname{span}_{i\in J}\{R_i\}) \geq \sum_{i\in J}\frac{1}{p_i}\,. \tag{2.60}$$

In order to apply the Generalized Young's Inequality to (2.59), we prove

LEMMA 2.4.3. *The expression on the right hand side of (2.59) satisfies the conditions of the Generalized Young's Inequality as stated above. In particular, (2.60) is satisfied.*

Proof. In each integral in the sum of the right hand side of (2.59), we can take

$$\begin{aligned}
p_i &= 2 && \text{for } i = 1, 2, \ldots, 2d; \\
R_i &= \hat{e}_i && \text{for } i = 1, \ldots, d; \\
h_i(u) &= \hat{F}^2(u) && \text{for } i = 1, \ldots, d; \\
R_i &= \hat{e}_{i-d+1} - \hat{e}_{i-d} && \text{for } i = d+1, \ldots, 2d-1; \\
R_{2d} &= \hat{e}_1 - \hat{e}_d && \text{and} \\
h_i(u) &= e^{-\xi_{j,l}|u|} && \text{for } i = d+1, \ldots, 2d\,.
\end{aligned}$$

Since $p_i = 2$, (2.60) reduces to showing that

$$\dim(\operatorname{span}_{i\in J}\{R_i\}) \geq \frac{\operatorname{card}(J)}{2}\,. \tag{2.61}$$

If at least Card$(J)/2$ of the elements in J are a subset of $\{1, \ldots, d\}$, then the corresponding $R_i = \hat{e}_1$ and (2.61) is satisfied. Next, we observe that any proper subset of

$$\{R_{d+1} = \hat{e}_2 - \hat{e}_1, R_{d+2} = \hat{e}_3 - \hat{e}_2, \ldots, R_{2d-1} = \hat{e}_d = \hat{e}_{d-1}, R_{2d} = \hat{e}_1 - \hat{e}_d\}$$

is linearly independent. Thus, in the case where at least card$(J)/2$ of the elements in J are a subset of $\{d+1, \ldots, 2d\}$, the corresponding R_i will be linearly independent, unless $J = \{d+1, \ldots, 2d\}$, in which case their span will have $d-1$ dimensions, while card$(J) = d$, where $d \geq 2$. In both these cases, (2.61) is satisfied. $\square$

Applying the Generalized Young's Inequality to (2.59), we get that

$$\begin{aligned}
\operatorname{Tr}\left[((\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2})^d\right] \leq & \hat{C}_{d,2,2,\ldots,2}\sum_{j,l}\frac{\pi^d}{\xi_{j,l}^d}\|\hat{F}^2\|_{L^2(\mathbb{R})}^d\|e^{-\xi_{j,l}}\|_{L^2(\mathbb{R})}^d \\
= & \hat{C}_{d,2,2,\ldots,2}\pi^d\|\hat{F}^2\|_{L^2(\mathbb{R})}^d\sum_{j,l}(\gamma^2+\tau_l+\sigma_j)^{-d}
\end{aligned} \tag{2.62}$$

Since the eigenvalues of the spherical harmonics on $\mathbb{S}^{n-1}$ are $\tau_l = l(l+n-2)$ with corresponding multiplicity $\binom{n+l-1}{l} - \binom{n+l-2}{l-1}$, see [St] for reference, and $\sigma_j = 4j(j+$

$\frac{m+n}{2}-1)$, the right hand side of (2.62) equals

$$\begin{aligned}\hat{C}_d\sum_{j,l\geq 0}\frac{\binom{n+l-1}{l}-\binom{n+l-2}{l-1}}{[\gamma^2+4j(j+\frac{m+n}{2}-1)+l(l+n-2)]^d} =& \hat{C}_d\sum_{0\leq j,l<n-1}\frac{\binom{n+l-1}{l}-\binom{n+l-2}{l-1}}{[\gamma^2+4j(j+\frac{m+n}{2}-1)+l(l+n-2)]^d}\\ &+\hat{C}_d\sum_{j,l\geq n-1}\frac{\binom{n+l-1}{l}-\binom{n+l-2}{l-1}}{[\gamma^2+4j(j+\frac{m+n}{2}-1)+l(l+n-2)]^d}\,,\end{aligned} \tag{2.63}$$

where $\hat{C}_d=\hat{C}_{d,2,2,\ldots,2}\pi^d\|\hat{F}^2\|^d_{L^2(\mathbb{R}^d)}$. Now,

$$\begin{aligned}&\binom{n+l-1}{l}-\binom{n+l-2}{l-1}\\ =&\ \frac{1}{(n-1)!}[(n+l-1)(n+l-2)\cdots(l+1)-(n+l-2)(n+l-3)\cdots l],\text{ which for } l\geq n-1\\ \leq&\ (2l)^{n-1}\,.\end{aligned}$$

Thus, if $d>\frac{n+1}{2}$,

$$\begin{aligned}\sum_{j,l\geq n-1}\frac{\binom{n+l-1}{l}-\binom{n+l-2}{l-1}}{[\gamma^2+4j(j+\frac{m+n}{2}-1)+l(l+n-2)]^d} &\leq 2^{n-1}\sum_{j,l\geq n-1}\frac{l^{n-1}}{[\gamma^2+j^2+l^2]^d}\\ &\leq 2^{n-1}\sum_{j,l\geq n-1}\frac{1}{[\gamma^2+j^2+l^2]^{(d-\frac{n-1}{2})}}\\ &< \infty\,.\end{aligned}$$

We can conclude by the above that (2.63) is finite when $d>\frac{n+1}{2}$. Hence, (2.58) holds for $d>\frac{n+1}{2}$. Thus, $(\mathcal{L}^{Im})^{1/2}\hat{A}^{-1}(\mathcal{L}^{Im})^{1/2}:L^2\to L^2$, and ultimately $S_t:\dot{H}^1_{\mathbb{C}}\to\dot{H}^1_{\mathbb{C}}$, is compact. $\square$

Combining lemmas 2.4.1 and 2.4.2 we conclude Theorem 2.3.6.

2.5 The Nullspace of $C^2_{m,n}I-S_t:\dot{H}^1_{\mathbb{C}}\to\dot{H}^1_{\mathbb{C}}$

In this section, we prove Theorem 2.3.7. We will use (u,θ,ζ)-coordinates (see (2.48) and (2.49) for reference). To simplify notation, we will often omit the subscripts $t,0$ from $F_{t,0}$, t from S_t, and the subscripts $C^{-1}_{m,n}F_{t,0},2^*$ from the operators $\mathcal{L}_{C^{-1}_{m,n}F_{t,0},2^*}$, $\mathcal{L}^{Re}_{C^{-1}_{m,n}F_{t,0},2^*}$, and $\mathcal{L}^{Im}_{C^{-1}_{m,n}F_{t,0},2^*}$. Also to simplify notation, $\mathrm{Null}(C^2_{m,n}I-S)$ will denote the nullspace of $C^2_{m,n}I-S:\dot{H}^1_{\mathbb{C}}\to\dot{H}^1_{\mathbb{C}}$.

We prove Theorem 2.3.7 in a series of three lemmas. In the first lemma, we show that $\{(F,0),(\frac{\mathrm{d}}{\mathrm{d}t}F,0),(0,F)\} \subseteq \mathrm{Null}(C_{m,n}^2 I - S)$ and $C_{m,n}^2 I - S : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ is positive. In the second lemma, we show that no element of $\mathrm{Null}(C_{m,n}^2 I - S)$ is linearly independent of $\{(F,0),(\frac{\mathrm{d}}{\mathrm{d}t}F,0),(0,F)\}$. Proving this lemma is a bit tricky. The proof breaks into four steps, each of which is headed by a phrase in italics. In the first step, we reduce the proof of the lemma to showing that the space of zeroes of $C_{m,n}^2 A - \mathcal{L}^{Re}$ in $\{F\}^{\perp_{\dot{H}^1}}$, functions in $\dot{H}^1$ perpendicular to F, is spanned by $\frac{\mathrm{d}}{\mathrm{d}t}F$. In the second step, we show that the zeroes of $C_{m,n}^2 A - \mathcal{L}^{Re}$ in $\{F\}^{\perp_{\dot{H}^1}}$ that are independent of θ and ζ are constant multiples of $\frac{\mathrm{d}F}{\mathrm{d}t}$. The proof of this fact boils down to showing that any zero of $C_{m,n}^2 A - \mathcal{L}^{Re}$ that is linearly independent of $\frac{\mathrm{d}F}{\mathrm{d}t}$ would have infinite energy. The precise proof of this fact is clever, and perhaps the most interesting proof in this section. In the third and fourth steps, we show that zeroes of $C_{m,n}^2 A - \mathcal{L}^{Re}$ in $\{F\}^{\perp_{\dot{H}^1}}$ are independent of θ and ζ. We conclude the section with the third lemma, in which we show that $\psi \perp_{\dot{H}^1_{\mathbb{C}}} \mathrm{Null}(C_{m,n}^2 I - S)$.

We begin with the following

LEMMA 2.5.1. $\{(F,0),(\frac{\mathrm{d}}{\mathrm{d}t}F,0),(0,F)\} \subseteq \mathrm{Null}(C_{m,n}^2 I - S)$ *and* $C_{m,n}^2 I - S : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ *is a positive operator.*

Proof. We begin by observing that

$$\begin{aligned} 0 &= \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\Big|_{\varepsilon=0} \frac{\|F+\varepsilon F\|_{2^*}^2}{\|F+\varepsilon F\|_{\dot{H}^1}^2} \\ &= \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\Big|_{\varepsilon=0} \|F+\varepsilon F\|_{2^*}^2 - 2C_{m,n}^2 \\ &= 2\left(\langle \mathcal{L}F, F\rangle_{L^2\oplus L^2} - C_{m,n}^2\right) \\ &= 2\left\langle (S - C_{m,n}^2 I)F, F\right\rangle_{\dot{H}^1_{\mathbb{C}}} . \end{aligned} \tag{2.64}$$

Thus, $(F,0) \in \mathrm{Null}(C_{m,n}^2 I - S)$. A similar calculation shows that $(0,F) \in \mathrm{Null}(C_{m,n}^2 I - S)$. In a similar manner, differentiating $\|F_{t+\varepsilon,0}\|_{2^*}^2 / \|F_{t+\varepsilon,0}\|_{\dot{H}^1}^2$ to the second order and evaluating at $\varepsilon = 0$ shows that $(\frac{\mathrm{d}}{\mathrm{d}t}F, 0) \in \mathrm{Null}(C_{m,n}^2 I - S)$.

Let $\tilde{\psi} \in \dot{H}^1_{\mathbb{C}}$ be such that $\|\tilde{\psi}\|_{\dot{H}^1} = 1$ and $\tilde{\psi} \perp_{\dot{H}^1_{\mathbb{C}}} F$, and $\varepsilon \in [-1, 1]$, then

$$\begin{aligned}
0 \leq & C_{m,n}^2 \|F + \varepsilon\tilde{\psi}\|_{\dot{H}^1}^2 - \|F + \varepsilon\tilde{\psi}\|_{2^*}^2, \text{ which by Theorem 2.2.1 and the fact that } \tilde{\psi} \perp_{\dot{H}^1_{\mathbb{C}}} F \\
\leq & C_{m,n}^2 \left(\|F\|_{\dot{H}^1}^2 + \varepsilon^2 \|\tilde{\psi}\|_{\dot{H}^1}^2 \right) - \left(\|F\|_{2^*}^2 + \langle S\tilde{\psi}, \tilde{\psi}\rangle_{\dot{H}_{\mathbb{C}}} \varepsilon^2 - \frac{\kappa_{2^*} C_{m,n}^2}{4 \cdot 3^{\frac{\beta_{2^*}}{2} - 1}} \varepsilon^{\beta_{2^*}} \right) \\
= & \left\langle (C_{m,n}^2 I - S)\tilde{\psi}, \tilde{\psi} \right\rangle_{\dot{H}^1_{\mathbb{C}}} \varepsilon^2 + \frac{\kappa_{2^*} C_{m,n}^2}{4 \cdot 2^{\frac{\beta_{2^*}}{2} - 1}} |\varepsilon|^{\beta_{2^*}} \\
\implies & \left\langle (C_{m,n}^2 I - S)\tilde{\psi}, \tilde{\psi} \right\rangle_{\dot{H}^1_{\mathbb{C}}} \geq - \frac{\kappa_{2^*} C_{m,n}^2}{4 \cdot 2^{\frac{\beta_{2^*}}{2} - 1}} |\varepsilon|^{\beta_{2^*} - 2}, \ \forall \varepsilon \in [-1, 1]. \text{ And, since } \beta_{2^*} > 2, \text{ this} \\
\implies & \left\langle (C_{m,n}^2 I - S)\tilde{\psi}, \tilde{\psi} \right\rangle_{\dot{H}^1_{\mathbb{C}}} \geq 0, \text{ for } \tilde{\psi} \perp_{\dot{H}^1_{\mathbb{C}}} F. \qquad (2.65)
\end{aligned}$$

Combining (2.65) with the fact that $(F, 0) \in \text{Null}(C_{m,n}^2 I - S)$ concludes the proof of Lemma 2.5.1. ☐

At this point, we have proved that $\{(F, 0), (\frac{\mathrm{d}}{\mathrm{d}t}F, 0), (0, F)\} \subseteq \text{Null}(C_{m,n}^2 I - S)$. So, to prove that $\{(F, 0), (\frac{\mathrm{d}}{\mathrm{d}t}F, 0), (0, F)\}$ spans $\text{Null}(C_{m,n}^2 I - S)$, we show the following

LEMMA 2.5.2. *No element in* $\text{Null}(C_{m,n}^2 I - S)$ *is linearly independent of* $\{(F, 0), (\frac{\mathrm{d}}{\mathrm{d}t}F, 0), (0, F)\}$.

Proof. <u>*Step 1:*</u> *Reduce the proof to showing that the space of zeroes of $C_{m,n}^2 A - \mathcal{L}^{Re}$ in $\{F\}^{\perp_{\dot{H}^1}}$ is spanned by $\frac{\mathrm{d}}{\mathrm{d}t}F$.* We begin by proving

$$C_{m,n}^2 I - A^{-1}\mathcal{L}^{Im} > 0 \text{ on } \{F\}^{\perp_{\dot{H}^1}}, \qquad (2.66)$$

where

$$\{F\}^{\perp_{\dot{H}^1}} := \left\{ \xi \in \dot{H}^1 \middle| \xi \perp_{\dot{H}^1} F \right\}.$$

Combining (2.52) and (2.54) yields

$$A^{-1}\mathcal{L}^{Re} = (2^* - 1)A^{-1}\mathcal{L}^{Im} \text{ on } \{F\}^{\perp_{\dot{H}^1}}.$$

Thus,

$$C_{m,n}^2 I - A^{-1}\mathcal{L}^{Im} = \frac{2^* - 2}{2^* - 1} C_{m,n}^2 I + \frac{1}{2^* - 1}(C_{m,n}^2 I - A^{-1}\mathcal{L}^{Re}) \text{ on } \{F\}^{\perp_{\dot{H}^1}}.$$

This combined with the fact that $C_{m,n}^2 I - S : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ is positive (and so $C_{m,n}^2 I - A^{-1}\mathcal{L}^{Re} : \dot{H}^1 \to \dot{H}^1$ is positive), allows us to conclude (2.66). Thus, if $\varphi = (\xi, \eta)$ is in

$\mathrm{Null}(C_{m,n}^2 I - S)$ with $\xi, \eta \in \{F\}^{\perp_{\dot{H}^1}}$, then it is of the form $(\xi, 0)$ for some $\xi \in \{F\}^{\perp_{\dot{H}^1}}$. Note that

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{1}{t}\frac{\partial F}{\partial u}\,, \tag{2.67}$$

and integration by parts in the u-variable shows that $\frac{\partial F}{\partial u} \perp_{\dot{H}^1} F$. Thus, $\frac{\mathrm{d}F}{\mathrm{d}t} \in \{F\}^{\perp_{\dot{H}^1}}$. And so, if we can show that

$$(C_{m,n}^2 A - \mathcal{L}^{Re})\xi = 0 \text{ and } \xi \in \{F\}^{\perp_{\dot{H}^1}} \implies \xi = c\frac{\mathrm{d}F}{\mathrm{d}t}, \text{ for some } c \in \mathbb{R}\,, \tag{2.68}$$

then we have proved Lemma 2.5.2. We could try to prove (2.68) by thinking of

$$(C_{m,n}^2 A - \mathcal{L}^{Re})\xi = 0 \tag{2.69}$$

as a differential equation and trying to find all of its solutions. However, this would be tricky, as A does not separate nicely. Also, not all solutions of (2.69) are in $\dot{H}^1$. So we would need to identify which solutions of (2.69) are in $\dot{H}^1$. What we do instead is show that any solution of (2.69) dependent upon the u-variable only and linearly independent of $\frac{\mathrm{d}}{\mathrm{d}t}F$ must have infinite energy. And then, we show that solutions of (2.69) must be independent of the θ and ζ variables.

Step 2: *The zeroes of $C_{m,n}^2 A - \mathcal{L}^{Re}$ that are linearly independent of $\frac{\mathrm{d}}{\mathrm{d}t}F$ have infinite energy.* The zeroes of $C_{m,n}^2 A - \mathcal{L}^{Re}$ in $\{F\}^{\perp_{\dot{H}^1}}$ are independent of θ and ζ, and so are radial - we delay the proof of this fact to steps three and four. Hence, if $(C_{m,n}^2 A - \mathcal{L}^{Re})\xi = 0$ for some $\xi \in \{F\}^{\perp_{\dot{H}^1}}$, then

$$X\xi := C_{m,n}^2(\gamma^2\xi - \frac{\partial^2}{\partial u^2}\xi) - (2^* - 1)C_{m,n}^{2-2^*}F^{2^*-2}\xi = 0\,, \tag{2.70}$$

where ξ is independent of θ and ζ. Since the functions satisfying (2.70) are radial, when solving (2.70), we will treat X as if it were a differential operator on $\mathbb{R}$ in the variable u. However, we will be interested in whether or not these solutions are in $\dot{H}^1$. Thus, to verify (2.68), we prove the following

PROPOSITION 2.5.3. *Consider (2.70) as a differential equation in the variable u only. Let $\tilde{\phi}$ be a solution of (2.70) that is linearly independent of $\frac{\mathrm{d}}{\mathrm{d}t}F$. Then, ϕ given by $\phi(u, \theta, \zeta) = \tilde{\phi}(u)$ is not in $\dot{H}^1$, because*

$$\|\phi\|_{\dot{H}^1} = \infty\,. \tag{2.71}$$

Proof. In the following, since we are working with functions on $\mathbb{R}$ in the u variable, we will take F as such a function since it is independent of θ and ζ. By (2.67), it suffices to show that Proposition 2.5.3 holds if we replace $\frac{\mathrm{d}F}{\mathrm{d}t}$ with $\frac{\partial F}{\partial u}$. Using $\frac{\partial F}{\partial u}$ instead of $\frac{\mathrm{d}F}{\mathrm{d}t}$ makes some of our calculations in this proof easier. Thus, we will prove Proposition 2.5.3 for $\frac{\partial F}{\partial u}$ instead of $\frac{\mathrm{d}F}{\mathrm{d}t}$. We write F_u, F_{uu}, F_{uuu}, etc. to denote u derivatives of F. Considering solutions of (2.70), we may view linear independence with respect to F_u in terms of the initial conditions $\tilde{\xi}(0)$, $\tilde{\xi}'(0)$. To be more precise, we will consider (2.70) with initial conditions

$$\tilde{\xi}(0) = \tilde{\alpha}, \tilde{\xi}'(0) = \tilde{\beta}\,, \tag{2.72}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are constants. Since (2.70) is a second order linear equation with continuous coefficients, (2.70) combined with (2.72) determines a unique solution. F_u is the solution of (2.70) satisfying (2.72) with $\tilde{\alpha} = 0$ and $\tilde{\beta} = F_{uu}(0) \neq 0$. All solutions of (2.70) that are linearly independent of F_u satisfy (2.72) with some $\tilde{\alpha} \neq 0$. If such a solution, call it $\tilde{\phi}$, were to have the property that ϕ given by $\phi(u, \theta, \zeta) = \tilde{\phi}(u)$, is in $\dot{H}^1$, then $\tilde{\tilde{\xi}} = c_1 F_u + c_2 \tilde{\phi}$ for appropriate $c_1, c_2 \in \mathbb{R}$ would satisfy (2.70) and (2.72) for $\tilde{\alpha} = 1$ and $\tilde{\beta} = 0$. Moreover, since $F_u, \tilde{\phi} \in \dot{H}^1$, ξ given by $\xi(u, \theta, \zeta) = \tilde{\tilde{\xi}}(u)$, would be an element of $\dot{H}^1$. We will prove Proposition 2.5.3 by showing that ξ is not in $\dot{H}^1$.

Observe that,

$$\begin{aligned}
0 &= \int_0^{u_0} \left[(2^* - 1)C_{m,n}^{-2^*} F^{2^*-2} - \gamma^2\right] \tilde{\tilde{\xi}} F_u - \tilde{\tilde{\xi}} \left[(2^* - 1)C_{m,n}^{-2^*} F^{2^*-2} - \gamma^2\right] F_u \mathrm{d}u, \text{ since } F_u \text{ and } \tilde{\tilde{\xi}} \text{ satisfy (2.70)} \\
&= \int_0^{u_0} -\tilde{\tilde{\xi}}'' F_u + \tilde{\tilde{\xi}} F_{uuu} \mathrm{d}u \\
&= -\tilde{\tilde{\xi}}' F_u|_0^{u_0} + \tilde{\tilde{\xi}} F_{uu}|_0^{u_0} - \int_0^{u_0} -\tilde{\tilde{\xi}}' F_{uu} + \tilde{\tilde{\xi}}' F_{uu} \mathrm{d}u \\
&= -\tilde{\tilde{\xi}}'(u_0) F_u(u_0) + \tilde{\tilde{\xi}}'(0) F_u(0) + \tilde{\tilde{\xi}}(u_0) F_{uu}(u_0) - \tilde{\tilde{\xi}}(0) F_{uu}(0)\,.
\end{aligned}$$

Recall that $\tilde{\tilde{\xi}}'(0) = F_u(0) = 0$ and $\tilde{\tilde{\xi}}(0) = 1$. Thus, by the above, we have that

$$\tilde{\tilde{\xi}}(u_0) F_{uu}(u_0) - \tilde{\tilde{\xi}}'(u_0) F_u(u_0) = \tilde{\tilde{\xi}}(0) F_{uu}(0) = F_{uu}(0) \neq 0\,.$$

Next, fix some $\varepsilon > 0$. Since $F_u(u), F_{uu}(u) \to 0$ uniformly as $|u| \to \infty$ (refer to (2.50) for the formula of F), there is some δ such that for $|u| > \delta$

$$\begin{aligned}|F_{uu}(0)| &= |\tilde{\tilde{\xi}}(u)F_{uu}(u) - \tilde{\tilde{\xi}}'(u)F_u(u)| \\ &\leq |\tilde{\tilde{\xi}}(u)F_{uu}(u)| + |\tilde{\tilde{\xi}}'(u)F_u(u)|, \text{ which by Cauchy-Schwarz} \\ &\leq (|\tilde{\tilde{\xi}}(u)|^2 + |\tilde{\tilde{\xi}}'(u)|^2)^{1/2}(|F_u(u)|^2 + |F_{uu}|^2)^{1/2} \\ &\leq \sqrt{2}\varepsilon(|\tilde{\tilde{\xi}}(u)|^2 + |\tilde{\tilde{\xi}}'(u)|^2)^{1/2}\,. \end{aligned} \tag{2.73}$$

If $\xi \in \dot{H}^1$, then since we are in (u, θ, ζ)-coordinates

$$\|\xi\|^2_{\dot{H}^1} = \omega_m \int_{\mathbb{S}^{n-1}} \int_0^{\pi/2} \int_{\mathbb{R}} \gamma^2|\xi|^2 + |\xi_u|^2 \mathrm{d}u \cos^{m-1}\theta \sin^{n-1}\theta \mathrm{d}\theta \mathrm{d}\Omega(\zeta) < \infty \tag{2.74}$$

(Ω denotes the uniform probability norm on $\mathbb{S}^{n-1}$). (2.73) and (2.74) imply that $F_{uu}(0) = 0$, contradicting the fact that $F_{uu}(0) \neq 0$. Thus, $\xi \notin \dot{H}^1$, and by the argument in the first paragraph of this proof, Proposition 2.5.3 must hold. $\square$

Since we know that $(\frac{\mathrm{d}}{\mathrm{d}t}F, 0) \in \mathrm{Null}(C^2_{m,n}I - S)$, Proposition 2.5.3 allows us to conclude (2.68).

Step 3: *Reduce proving that the zeroes of $C^2_{m,n}A - \mathcal{L}^{Re}$ in $\{F\}^{\perp_{\dot{H}^1}}$ are independent of θ and ζ to proving that the zeroes of $C^2_{m,n}\hat{A} - \mathcal{L}^{Re}$ in $\{F\}^{\perp_{\dot{H}^1}}$ are independent of θ and ζ.* Recall that $\hat{A}$ (refer to (2.56) for reference) is an operator that is closely related to A such that $\hat{A} \leq A$ in L^2. Thus,

$$C^2_{m,n}\hat{A} - \mathcal{L}^{Re} \leq C^2_{m,n}A - \mathcal{L}^{Re} \text{ in } L^2\,.$$

If we can show that

$$\left\langle \xi, (C^2_{m,n}\hat{A} - \mathcal{L}^{Re})\xi \right\rangle_{L^2} \geq 0, \text{ for } \xi \in \{F\}^{\perp_{\dot{H}^1}}\,, \tag{2.75}$$

then we only need to show that the zeroes of $C^2_{m,n}\hat{A} - \mathcal{L}$ in $\{F\}^{\perp_{\dot{H}^1}}$ are independent of θ and ζ in order to prove that the zeroes of $C^2_{m,n}A - \mathcal{L}$ in $\{F\}^{\perp_{\dot{H}^1}}$ are independent of θ and ζ. Thus, we prove the following

PROPOSITION 2.5.4. *Let $\hat{A}$ be as defined in (2.56). Then (2.75) holds.*

Proof. We begin by verifying (2.75). First we observe that in (u, θ, ζ)-coordinates

$$\hat{A} = \gamma^2 I - \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial \theta^2} - ([n-1]\cot\theta - [m-1]\tan\theta)\frac{\partial}{\partial\theta} - \Delta_{\mathbb{S}^{n-1}(\zeta)}\,. \tag{2.76}$$

Then, we define

$$\begin{aligned} X &:= C_{m,n}^2\left(\gamma^2 I - \frac{\partial^2}{\partial u^2}\right) - (2^*-1)C_{m,n}^{2-2^*}F^{2^*-2} \\ Y &:= C_{m,n}^2\left[\frac{\partial^2}{\partial\theta^2} - ([n-1]\cot\theta - [m-1]\tan\theta)\frac{\partial}{\partial\theta}\right], \text{ and} \\ Z &:= -C_{m,n}^2\Delta_{\mathbb{S}^{n-1}(\zeta)}, \end{aligned}$$

so that

$$C_{m,n}^2\hat{A} - \mathcal{L}^{Re} = X + Y + Z\,, \text{ on } \{F\}^{\perp_{\dot{H}^1}}\,. \tag{2.77}$$

Y and Z are positive complete operators in $L^2([0,\pi/2], \cos^{m-1}\theta\sin^{n-1}\theta \mathrm{d}\theta)$ and $L^2(\mathbb{S}^{n-1}, \mathrm{d}\Omega(\zeta))$ respectively. X is also a complete operator for $L^2(\mathbb{R}, \mathrm{d}u)$; this is a result of standard spectral theory. Moreover, X is positive for functions in $L^2(\mathbb{R}, \mathrm{d}u)$ that are in $\{F\}^{\perp_{\dot{H}^1}}$ when considered as functions in $L^2(\mathbb{R}\times[0,\pi/2]\times\mathbb{S}^{n-1}, \omega_m \mathrm{d}u\cos^{m-1}\theta\sin^{n-1}\theta\mathrm{d}\theta\mathrm{d}(\zeta))$. This is because if $\xi \in \{F\}^{\perp_{\dot{H}^1}}$ is independent of θ and ζ, then

$$0 \le \langle \xi, (C_{m,n}^2 I - S)\xi\rangle_{\dot{H}^1} = \langle \xi, (C_{m,n}^2 A - \mathcal{L}^{Re})\xi\rangle_{L^2} = \langle \xi, X\xi\rangle_{L^2}\,.$$

Combining the properties of X, Y, and Z deduced above with (2.77), we conclude that (2.75) holds. □

Step 4: Show that zeroes of $C_{m,n}^2\hat{A} - \mathcal{L}^{Re}$ in $\{F\}^{\perp_{\dot{H}^1}}$ are independent of θ and ζ. We start by establishing independence from θ by proving the following

PROPOSITION 2.5.5. *The space of eigenfunctions of Y with Neumann boundary conditions and exactly one zero on the interval $[0,\pi/2]$ is spanned by*

$$g(\theta) = \frac{n-m}{m+n} + \cos(2\theta)\,, \tag{2.78}$$

with eigenvalue

$$\lambda = 2(m+n)C_{m,n}^2\,. \tag{2.79}$$

Proof. The key to proving this proposition is rewriting the coefficient of the $\frac{\partial}{\partial\theta}$ term of Y in terms of $\cos(2\theta)$ and $\sin(2\theta)$. More precisely,

$$(n-1)\cot\theta - (m-1)\tan\theta = \frac{n-m}{\sin(2\theta)} + (m+n-2)\frac{\cos(2\theta)}{\sin(2\theta)}\,.$$

Thus,

$$\begin{aligned} C_{m,n}^{-2} Y g &= -g'' - \left[\frac{n-m}{\sin(2\theta)} + (m+n-2)\frac{\cos(2\theta)}{\sin(2\theta)}\right] g' \\ &= 4\cos(2\theta) + 2\left[\frac{n-m}{\sin(2\theta)} + (m+n-2)\frac{\cos(2\theta)}{\sin(2\theta)}\right]\sin(2\theta) \\ &= 2[(n-m) + (m+n)\cos(2\theta)] \\ &= 2(m+n)g\,. \end{aligned}$$

The fact that there are no more linearly independent eigenfunctions of Y with exactly one zero in $[0, \pi/2]$ follows from standard Sturm-Liouville Theory. $\square$

The only eigenfunctions of Y without any zeroes satisfying the Von Neumann conditions are the constant functions. Any eigenfunctions of Y satisfying the Von Neumann conditions, excluding the constant functions and $g(\theta)$, will have eigenvalue more than $2(m+n)C_{m,n}^2$ - this is also a consequence of standard Sturm-Liouville Theory. Recalling that $C_{m,n}^2 \hat{A} - \mathcal{L}^{Re} = X + Y + Z$ on $\{F\}^{\perp_{\dot{H}^1}}$, where X depends on u only, Y depends on θ only, Z depends on ζ only, and that $\langle \xi, (C_{m,n}^2 \hat{A} - \mathcal{L}^{Re})\xi\rangle_{L^2} \geq 0$ for $\xi \in \{F\}^{\perp_{\dot{H}^1}}$, our analysis of Y allows us to conclude that the 0-modes of $C_{m,n}^2 \hat{A} - \mathcal{L}^{Re}$ must be independent of θ.

Next, we establish independence from ζ. The eigenvalues of Z are nonnegative and discrete. The smallest eigenvalue is $\sigma_0 = 0$ and the corresponding space of eigenfunctions are the constant functions. The second smallest eigenvalue is $\sigma_1 = (n-1)C_{m,n}^2$, see [20] for reference. Thus, the zeroes of $C_{m,n}^2 \hat{A} - \mathcal{L}^{Re}$ in $\{F\}^{\perp_{\dot{H}^1}}$ must be independent of ζ. This concludes step 4.

Combining the results of steps 1-4 allows us to conclude Lemma 2.5.2. $\square$

The last thing we need to prove in order to conclude Theorem 2.3.7 is that $\psi \perp_{\dot{H}^1_{\mathbb{C}}} \{(F, 0), (\frac{\mathrm{d}}{\mathrm{d}t}F, 0), (0, F)\}$. We already showed that $\psi \perp_{\dot{H}^1_{\mathbb{C}}} (F, 0)$ in Lemma 2.3.4. Thus, it suffices to prove the following

LEMMA 2.5.6. $\psi \perp_{\dot{H}^1_{\mathbb{C}}} \{(\frac{\mathrm{d}}{\mathrm{d}t}F, 0), (0, F)\}$.

Proof. First we prove that $\psi \perp_{\dot{H}^1_{\mathbb{C}}} (\frac{\mathrm{d}}{\mathrm{d}t}F, 0)$. We begin by observing that

$$\delta(\varphi, M) = \|\varphi - zF_{t,0}\|_{\dot{H}^1}$$

implies that

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t} \|\varphi - zF_{s,0}\|^2_{\dot{H}^1} \,. \tag{2.80}$$

Exploiting (2.80), we get that (in the following subscripts denote partial derivatives)

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t} \|\varphi - zF_{s,0}\|^2_{\dot{H}^1} \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t} \int \gamma^2|\varphi - zF_{s,0}|^2 + |(\varphi - zF_{s,0})_u|^2 + |(\varphi - zF_{s,0})_\theta|^2 + \csc^2\theta |\nabla_{\mathbb{S}^{n-1}}(\varphi - zF_{s,0})|^2 \mathrm{d}\Lambda \\
&= 2\int \gamma^2(-z\frac{\mathrm{d}}{\mathrm{d}t}F)\operatorname{Re}(\varphi - zF) + (-z\frac{\mathrm{d}}{\mathrm{d}t}F)_u \operatorname{Re}[(\varphi - zF)_u] + (-z\frac{\mathrm{d}}{\mathrm{d}t}F)_\theta \operatorname{Re}[(\varphi - zF)_\theta] \\
&\quad +2\csc^2\theta[\nabla_{\mathbb{S}^{n-1}}(-z\frac{\mathrm{d}}{\mathrm{d}t}F)]\cdot[\nabla_{\mathbb{S}^{n-1}}\operatorname{Re}(\varphi - zF)]\mathrm{d}\Lambda \\
&= -2z^2\left\langle \delta(\varphi, M)\psi, (\frac{\mathrm{d}}{\mathrm{d}t}F, 0)\right\rangle_{\dot{H}^1_{\mathbb{C}}} ,
\end{aligned}$$

i.e. $\psi \perp_{\dot{H}^1_{\mathbb{C}}} (\frac{\mathrm{d}}{\mathrm{d}t}F, 0)$.

Next, we show that $\psi \perp_{\dot{H}^1_{\mathbb{C}}} (0, F)$. Recall that $(0, F)$ is an eigenfunction of $S_t : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$, which is self-adjoint and compact. Thus, if ψ were not perpendicular to $(0, F)$, then since $\|(0, F)\|_{\dot{H}^1} = 1$,

$$\left\|\psi - \langle\psi, (0, F)\rangle_{\dot{H}^1_{\mathbb{C}}}(0, F)\right\|_{\dot{H}^1} < \|\psi\|_{\dot{H}^1} = 1 \,. \tag{2.81}$$

Letting $\varepsilon = \langle\psi, (0, F)\rangle_{\dot{H}^1_{\mathbb{C}}}$, we deduce that

$$\begin{aligned}
\|\varphi - (zF, \delta(\varphi, M)\varepsilon F)\|_{\dot{H}^1} &= \|\delta(\varphi, M)\psi - \delta(\varphi, M)\varepsilon(0, F)\|_{\dot{H}^1} \\
&= \delta(\varphi, M)\|\psi - \varepsilon(0, F)\|_{\dot{H}^1}, \text{ which by (2.81)} \\
&< \delta(\varphi, M) \,.
\end{aligned}$$

This contradicts the assumption (2.38) made in Lemma 2.3.2 that

$$\delta(\varphi, M) = \|\varphi - zF\|_{\dot{H}^1} \,,$$

because $(zF, \delta(\varphi, M)\varepsilon F) \in M$. Thus, $\psi \perp_{\dot{H}^1_{\mathbb{C}}} (0, F)$. □

Combining lemmas 2.5.1, 2.5.2, and 2.5.6 we conclude Theorem 2.3.7. Thus, we have proven theorems 2.3.6 and 2.3.7. Combining these theorems with the outline of the proof of Theorem 2.3.1 provided in subsection 2.3, we conclude Theorem 2.3.1.

2.6 Proof of Theorem 2.1.5

We begin by proving the sharpness statement. Let $\varphi \in \dot{H}^1_{\mathbb{C}}$ satisfy the assumptions of Lemma 2.3.2. Applying the results of the second order Taylor Expansion with the remainder bound, a calculation similar to the one used to obtain (2.45) yields that

$$\begin{aligned} C_{m,n}^2 \|\varphi\|_{\dot{H}^1}^2 - \|\varphi\|_{2^*}^2 &\leq \left\langle (C_{m,n}^2 I - S_t)\psi, \psi \right\rangle_{\dot{H}^1_{\mathbb{C}}} \delta(\varphi, M)^2 + \frac{\kappa_{2^*} C_{m,n}^2}{4 \cdot 3^{\frac{\beta_{2^*}}{2} - 1}} \delta(\varphi, M)^{\beta_{2^*}} \\ &\leq \tilde{C} \delta(\varphi, M)^2, \text{ for some } \tilde{C} > 0 \,, \end{aligned} \tag{2.82}$$

because $S_t : \dot{H}^1_{\mathbb{C}} \to \dot{H}^1_{\mathbb{C}}$ has a bounded spectrum, $\delta(\varphi, M) \leq 1$, and $\beta_{2^*} > 2$. If the sharpness statement at the end of Theorem 2.1.5 were false, then we could find some $\tilde{\alpha} > 0$ such that

$$C_{m,n}^2 \|\varphi\|_{\dot{H}^1}^2 - \|\varphi\|_{2^*}^2 \geq \tilde{\alpha} \delta(\varphi, M)^\beta \,, \tag{2.83}$$

for some $\beta < 2$ and all $\varphi \in \dot{H}^1_{\mathbb{C}}$. However, for φ obeying the conditions of Lemma 2.3.2, (2.82) and (2.83) would imply

$$\tilde{C}/\tilde{\alpha} \geq \delta(\varphi, M)^{-(2-\beta)}, \text{ with } \beta < 2 \,,$$

which is clearly a contradiction for $\delta(\varphi, M)$ sufficiently small. This proves the sharpness statement.

The rest of the proof of Theorem 2.1.5 follows by contradiction. Assume Theorem 2.1.5 is false. Then, there is some $(\varphi_j) \subseteq \dot{H}^1_{\mathbb{C}}$ such that

$$\frac{C_{m,n}^2 \|\varphi_j\|_{\dot{H}^1}^2 - \|\varphi_j\|_{2^*}^2}{\delta(\varphi_j, M)^2} \to 0 \,. \tag{2.84}$$

We can assume that $\|\varphi_j\|_{\dot{H}^1} = 1$ for all j, because replacing φ_j with $c\varphi_j$, c a nonzero constant, does not change the value of the left hand side of (2.84). In this case, $(\delta(\varphi_j, M)) \in [0,1]$ for all j, and some subsequence, $(\delta(\varphi_{j_k}, M))$, converges to some $B \in [0,1]$. If $B = 0$, then (2.84) contradicts Theorem 2.3.1, specifically condition (2.36) of Theorem 2.3.1.

Next, we show that B must equal 0. A fortiori, (2.84) implies

$$C_{m,n}^2 \|\varphi_{j_k}\|_{\dot{H}^1}^2 - \|\varphi_{j_k}\|_{2^*}^2 \to 0 \,. \tag{2.85}$$

We will use (2.85) in a Concentration Compactness argument to show that a subsequence, say $(\delta(\hat{\varphi}_{k_l}, M))$, of $(\delta(\hat{\varphi}_k, M))$ converges to zero, where $\hat{\varphi}_k$ is given by

$$\hat{\varphi}_k(\rho, x) = \sigma_k^{\gamma} \varphi_{j_k}(\sigma_k \rho, \sigma_k(x - x_k)),$$

for some $(\sigma_k) \subseteq \mathbb{R}_+$ and $(x_k) \subseteq \mathbb{R}^n$. Since $\delta(\cdot, M)$ is conformally invariant,

$$\delta(\varphi_{j_{k_l}}, M) = \delta(\hat{\varphi}_{k_l}, M) \to 0,$$

from which we conclude that B must in fact be 0. Thus, we have reduced the proof of Theorem 2.1.5 to illustrating the Concentration Compactness argument for cylindrically symmetric functions in continuous dimension. We do this in detail in the next and final section.

2.7 Concentration Compactness

In the following, we will assume that $\|\varphi_j\|_{2^*} = 1$, instead of $\|\varphi_j\|_{\dot{H}^1} = 1$. This does not change any key properties. More precisely, (2.84) will still hold and we can replace the assumption that $\delta(\varphi_{j_k}, M) \to B \neq 0$ with $\delta(\varphi_{j_k}, M) \to B/C_{m,n}$, which is also nonzero. We also establish some notation. Let φ be a cylindrically symmetric function and $\sigma > 0$. Then φ^σ is given by (recall by (2.11) that $\gamma = \frac{m+n-2}{2}$)

$$\varphi^\sigma(\rho, x) = \sigma^\gamma \varphi(\sigma\rho, \sigma x).$$

In this section, we will prove

THEOREM 2.7.1. *Let $(\varphi_j) \subseteq \dot{H}^1_{\mathbb{C}}$ be such that*

$$C_{m,n}^2 \|\varphi_j\|_{\dot{H}^1}^2 - \|\varphi_j\|_{2^*}^2 \to 0 \text{ and } \|\varphi_j\|_{2^*} = 1, \forall j. \tag{2.86}$$

Then there is some $(\sigma_j) \subseteq \mathbb{R}_+$ and $(x_j) \subseteq \mathbb{R}^n$ such that $(\hat{\varphi}_j)$ given by

$$\hat{\varphi}_j(\rho, x) = \varphi_j^{\sigma_j}(\rho, x + x_j), \tag{2.87}$$

has a subsequence, $(\hat{\varphi}_{j_k})$, that converges strongly in $\dot{H}^1_{\mathbb{C}}$ to some $F \in M$ such that $\|F\|_{2^*} = 1$.

The proof of this theorem breaks into three parts. In the first part, we prove the following

LEMMA 2.7.2. *Let $(\varphi_j) \subseteq \dot{H}^1_{\mathbb{C}}$ satisfy condition (2.86). Then, there is some $(\sigma_j) \subseteq \mathbb{R}_+$ such that $(\varphi_j^{\sigma_j})$ has a subsequence, $(\varphi_{j_k}^{\sigma_{j_k}})$, and some $\varepsilon, C > 0$ such that*

$$\Lambda(\{|\varphi_{j_k}^{\sigma_{j_k}}(\rho, x)| > \varepsilon, \rho \leq 4\}) > C\,, \tag{2.88}$$

where Λ denotes the measure defined in (2.10).

Once we have proved Lemma 2.7.2, we can apply an analogue of Lieb's Concentration Compactness Theorem, Theorem 8.10 on page 215 of [LiLo], to the subsequence $(\varphi_{j_k}^{\sigma_{j_k}})$ that satisfies (2.88). We state this analogue below:

THEOREM 2.7.3. *Let (φ_j) be a bounded sequence of functions in $\dot{H}^1_{\mathbb{C}}$. Suppose there exist $\varepsilon > 0$ and $R < \infty$ such that $E_j := \{|\varphi_j(\rho, x)| > \varepsilon, \rho \leq R\}$ has measure $\Lambda(E_j) \geq \delta > 0$ for some δ and for all j. Then, there exists $(x_j) \subseteq \mathbb{R}^n$ such that $\varphi_j^T(\rho, x) := \varphi_j(\rho, x + x_j)$ has a subsequence that converges weakly in $\dot{H}^1_{\mathbb{C}}$ to a nonzero function.*

We delay the proof of this theorem to the end of this section, because it is a relatively straightforward adaptation of Lieb's proof of his original Concentration Compactness Theorem. After applying Theorem 2.7.3 and relabeling indices, we deduce some $(\hat{\varphi}_j)$, as given in (2.87), such that some subsequence, $(\hat{\varphi}_{j_k})$, converges weakly in $\dot{H}^1_{\mathbb{C}}$ to a nonzero element, φ. The second part of the proof of Theorem 2.7.1 involves some relatively straightforward functional analysis arguments that allow us to show that $(\hat{\varphi}_{j_k})$ converges strongly in $\dot{H}^1_{\mathbb{C}}$ to some $\varphi \in M$ such that $\|\varphi\|_{2^*} = 1$. The third part of the proof of Theorem 2.7.1 is the delayed proof of Theorem 2.7.3.

Part 1 of proof of Theorem 2.7.1 - Proving Lemma 2.7.2: Most of the hard work in proving Theorem 2.7.1 is devoted to proving Lemma 2.7.2. We break the proof of Lemma 2.7.2 into three steps, which we outline in this paragraph. In the first step, we reduce proving Lemma 2.7.2 to proving that (2.88) holds for a modified subsequence of (φ_j). More precisely, we take the sequence (φ_j) and dilate its elements to obtain a new sequence $(\varphi_j^{\sigma_j})$ such that the symmetric decreasing rearrangments, $\tilde{\varphi}_j$, in the x-variable of the $\varphi_j^{\sigma_j}$ have the property that

$$\|\chi_{\{(\rho^2+|x|^2)^{1/2}\leq 1\}}\tilde{\varphi}_j\|_{2^*}^{2^*} = 1/2, \forall j\,. \tag{2.89}$$

We then conclude that to prove Lemma 2.7.2, it suffices to show that (2.88) holds for some subsequence of the modified sequence, $(\tilde{\varphi}_j)$. In the second and third steps, we prove that (2.88) holds for a subsequence of $(\tilde{\varphi}_j)$. Actually, we show that (2.88) holds not just for a subsequence of $(\tilde{\varphi}_j)$, but for a subsequence of $(\chi\tilde{\varphi}_j)$, where χ is a nicely behaved cutoff function. In the second step, we show that a subsequence of $(\|\chi_{\{1\leq w\leq 2\}}\tilde{\varphi}_j\|_{2^*}^{2^*})$ converges to a positive constant; the result of this step is summarized in Proposition 2.7.7. In the third step, we leverage the result of the second step to show the $\tilde{\varphi}_j$ times one of two possible cutoff functions yields a sequence that satisfies the p,q,r-Theorem; the result of this step is summarized in Proposition 2.7.8. From this, we deduce that there must be some cutoff, $\chi\leq 1$, such that a subsequence of $(\chi\tilde{\varphi}_j)$ satisfies (2.88). This concludes the proof of Lemma (2.7.2).

Step 1: Reduce the proof of Lemma 2.7.2 to its analogue for a modified sequence, $(\tilde{\varphi}_j)$. For a cylindrically symmetric function, φ, let φ^* denote the symmetric decreasing rearrangement of φ in the x-variable. To be more precise, φ^* is the nonnegative function obtained by the following: Fix $\rho\in(0,\infty)$. Then, $\varphi^*(\rho,x)$ is the symmetric decreasing function in x such that for all $\varepsilon>0$

$$\left|\{x\in\mathbb{R}^n \big| |\varphi^*(\rho,x)|>\varepsilon\}\right| = \left|\{x\in\mathbb{R}^n \big| |\varphi(\rho,x)|>\varepsilon\}\right|\,, \tag{2.90}$$

where $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}^n$.

We will choose $(\sigma_j)\subseteq\mathbb{R}_+$ such that $\tilde{\varphi}_j := (\varphi_j^{\sigma_j})^*$ satisfies (2.89). We can choose such σ_j, because of

LEMMA 2.7.4. *Let $\varphi\in\dot{H}^1_{\mathbb{C}}$ be a function such that*

$$\|\varphi\|_{2^*} = 1\,. \tag{2.91}$$

Then, for any $c\in(0,1)$, there is some $\sigma\in\mathbb{R}_+$ such that

$$\|\chi_{\{(\rho^2+|x|^2)^{1/2}\leq 1\}}(\varphi^\sigma)^*\|_{2^*}^{2^*} = c\,. \tag{2.92}$$

Proof. We will use (w, θ, ζ)-coordinates instead of (ρ, x)-coordinates (refer to (2.48) for reference). We prove this lemma by first showing that we can pick some $\sigma_2 > \sigma_1 > 0$ such that

$$\|\chi_{\{w\leq 1\}}(\varphi^{\sigma_1})^*\|_{2^*}^{2^*} \leq c/2, \text{ and} \tag{2.93}$$

$$\|\chi_{\{w\leq 1\}}(\varphi^{\sigma_2})^*\|_{2^*}^{2^*} \geq (c+1)/2\,, \tag{2.94}$$

and then proving that the map,

$$\sigma \mapsto \|\chi_{\{w\leq 1\}}(\varphi^{\sigma})^*\|_{2^*},$$

is continuous. Thus, by the Intermediate Value Theorem, there is some $\sigma \in \mathbb{R}_+$ for which (2.92) holds.

First, we show (2.93). By the Dominated Convergence Theorem, we can pick some $\tilde{M}$ such that

$$E_{\tilde{M}} = \{(w, \theta, \zeta) \in \mathbb{R}_+ \times [0, \pi/2] \times \mathbb{S}^{n-1} \big| |\varphi(w, \theta, \zeta)| \leq \tilde{M}\}\,,$$

has the property that

$$\|\chi_{E_{\tilde{M}}}\varphi\|_{2^*}^{2^*} \geq 1 - \frac{c}{4}\,. \tag{2.95}$$

If we pick σ_1 small enough such that

$$\sigma_1^{m+n}\tilde{M}^{2^*} \leq \frac{c}{4\Lambda(\{w \leq 1\})}\,, \tag{2.96}$$

and take $E_{\tilde{M}}^C$ to be the complement of $E_{\tilde{M}}$, then

$$\begin{aligned}
\|\chi_{\{w\leq 1\}}(\varphi^{\sigma_1})^*\|_{2^*}^{2^*} &= \|\chi_{\{w\leq 1\}}([\chi_{E_{\tilde{M}}}\varphi]^{\sigma_1} + [\chi_{E_{\tilde{M}}^C}\varphi]^{\sigma_2})^*\|_{2^*}^{2^*}, \text{ because } E_{\tilde{M}} \cap E_{\tilde{M}}^C = \emptyset \\
&\leq \|\chi_{\{w\leq 1\}}([\chi_{E_{\tilde{M}}}\varphi]^{\sigma_1})^*\|_{2^*}^{2^*} + \|\chi_{\{w\leq 1\}}([\chi_{E_{\tilde{M}}^C}\varphi]^{\sigma_1})^*\|_{2^*}^{2^*} \\
&\quad \text{because } \{(w,\theta,\zeta)\big|(\chi_{E_{\tilde{M}}}\varphi)^{\sigma_1} > 0\} \cap \{(w,\theta,\zeta)\big|(\chi_{E_{\tilde{M}}^C}\varphi)^{\sigma_1} > 0\} = \emptyset \\
&\leq \Lambda(\{w \leq 1\})\|[\chi_{E_{\tilde{M}}}\varphi]^{\sigma_1}\|_{\infty}^{2^*} + \|\chi_{E_{\tilde{M}}^C}\varphi\|_{2^*}^{2^*}, \text{ which by (2.91), (2.95), and (2.96)} \\
&\leq c/2\,.
\end{aligned}$$

This proves (2.93).

Next, we show (2.94). By the Dominated Convergence Theorem, there is some $R < \infty$ such that

$$\|\chi_{\{w\leq R\}}\varphi\|_{2^*}^{2^*} \geq (c+1)/2\,.$$

Taking $\sigma_2 = R$ yields (2.94).

Finally, we will show that

$$\sigma \mapsto \|\chi_{\{w \le 1\}} (\varphi^\sigma)^*\|_{2^*} \tag{2.97}$$

is continuous. Fix some $\varepsilon > 0$ and $\sigma_0 \in \mathbb{R}_+$. Take some $\Psi \in C_C^\infty([0,\infty) \times [0,\pi/2] \times \mathbb{S}^{n-1})$ such that

$$\|\varphi - \Psi\|_{2^*} < \varepsilon/3\,.$$

Then,

$$\begin{aligned}
\big| \|\chi_{\{w\le 1\}}(\varphi^\sigma)^*\|_{2^*} - \|\chi_{\{w\le 1\}}(\varphi^{\sigma_0})^*\|_{2^*} \big| \le& \|(\varphi^\sigma)^* - (\varphi^{\sigma_0})^*\|_{2^*} \\
\le& \|\varphi^\sigma - \varphi^{\sigma_0}\|_{2^*} \\
\le& \|\varphi^\sigma - \Psi^\sigma\|_{2^*} + \|\Psi^\sigma - \Psi^{\sigma_0}\|_{2^*} + \|\Psi^{\sigma_0} - \varphi^{\sigma_0}\|_{2^*} \\
=& \|(\varphi - \Psi)^\sigma\|_{2^*} + \|\Psi^\sigma - \Psi^{\sigma_0}\|_{2^*} + \|(\varphi-\Psi)^{\sigma_0}\|_{2^*} \\
<& 2\varepsilon/3 + \|\Psi^\sigma - \Psi^{\sigma_0}\|_{2^*}\,.
\end{aligned} \tag{2.98}$$

Thus, we only need to show that the map given by

$$\sigma \mapsto \Psi^\sigma \tag{2.99}$$

is continuous at σ_0 to prove continuity of the map given by (2.97). We do this by proving sequential continuity. Let $(\sigma_j) \subseteq \mathbb{R}_+$ be a sequence such that $\sigma_j \to \sigma_0$. Then, (σ_j) has a finite supremum, k_1, and a positive infimum, k_2. Also, since $\Psi \in C_C([0,\infty) \times \mathbb{R}^n \times \mathbb{S}^n)$, there is some $N < \infty$ that bounds Ψ from above and some $R < \infty$ such that $\operatorname{supp}(\Psi) \subseteq \{w \le R\}$. Combining these facts with the definition of the dilation operation given by $\Psi \mapsto \Psi^\sigma$, we conclude that (Ψ^{σ_j}) has a ceiling function, $\Xi \in L^{2^*}$, given by

$$\Xi(w,\theta,\zeta) = k_1^{(m+n)/2^*} N \chi_{\{w \le R/k_2\}}\,.$$

Also, $\Psi^{\sigma_j} \to \Psi^{\sigma_0}$ pointwise, because Ψ is continuous. Combining the existence of the L^{2^*} ceiling function, Ξ, for (Ψ_{σ_j}) with the pointwise convergence of Ψ^{σ_j} to Ψ^{σ_0}, we apply the Dominated Convergence Theorem and conclude that

$$\lim_{j\to\infty} \|\Psi^{\sigma_j} - \Psi^{\sigma_0}\|_{2^*} = 0\,.$$

Thus, the map given by (2.99) is continuous. Combining this with (2.98), we conclude that

$$\big|\|\chi_{\{w\leq 1\}}(\varphi^{\sigma})^*\|_{2^*} - \|\chi_{\{w\leq 1\}}(\varphi^{\sigma_0})^*\|_{2^*}\big| < \varepsilon\,,$$

for σ sufficiently close to σ_0. Hence, the map given by (2.97) is continuous. $\square$

The definition of φ^* ensures that if a subsequence of $(\tilde{\varphi}_j)$ satisfies (2.88), in the sense that

$$\Lambda(\{|\tilde{\varphi}_{j_k}| > \varepsilon, \rho \leq 4\}) > C\,,$$

for some $\varepsilon, C > 0$, then the corresponding subsequence, $(\varphi_{j_k}^{\sigma_{j_k}})$, will satisfy (2.88). Thus, in order to prove Lemma 2.7.2, it suffices to prove

LEMMA 2.7.5. *The sequence of functions, $(\tilde{\varphi}_j)$, has a subsequence that satisfies (2.88).*

Steps two and three are devoted to proving Lemma 2.7.5. Each step contains a proposition that helps us to prove Lemma 2.7.5.

Step 2: Show that a subsequence of $(\|\chi_{\{1\leq w\leq 2\}}\tilde{\varphi}_j\|_{2^*}^{2^*})$ converges to a positive constant. Before stating and proving the main proposition in this step, we lay some foundation. First, we collect a couple of inequalities that we will use later.

PROPOSITION 2.7.6. *Let $h_1, h_2 \in L^p$ be such that $0 < \|h_1\|_p \leq \|h_2\|_p$. Then*

$$\|h_1+h_2\|_p \leq \|h_1\|_p + \|h_2\|_p - \frac{(p-1)\|h_1\|_p}{4}\left\|\frac{h_1}{\|h_1\|_p} - \frac{h_2}{\|h_2\|_p}\right\|_p^2, \text{ if } 1 < p \leq 2 \quad (2.100)$$

$$\|h_1+h_2\|_p \leq \|h_1\|_p + \|h_2\|_p - \frac{\|h_1\|_p}{p2^{p-1}}\left\|\frac{h_1}{\|h_1\|_p} - \frac{h_2}{\|h_2\|_p}\right\|_p^p, \text{ if } 2 \leq p < \infty\,. \quad (2.101)$$

Proof. If $1 < p \leq 2$, then by (3.3) of [CaFr]

$$\|\hat{h}_1 + \hat{h}_2\|_p \leq 2 - \frac{p-1}{4}\|\hat{h}_1 - \hat{h}_2\|_p\,. \quad (2.102)$$

for $\hat{h}_1, \hat{h}_2 \in L^p$ such that $\|\hat{h}_1\|_p = \|\hat{h}_2\|_p = 1$. Thus,

$$\begin{aligned}
\|h_1+h_2\|_p &\leq \left\|\frac{\|h_2\|_p - \|h_1\|_p}{\|h_2\|_p}h_2\right\|_p + \left\|\frac{\|h_1\|_p}{\|h_2\|_p}h_2 + h_1\right\|_p \\
&= \|h_2\|_p - \|h_1\|_p + \|h_1\|_p\left\|\frac{h_2}{\|h_2\|_p} + \frac{h_1}{\|h_1\|_p}\right\|_p, \text{ which by (2.102)} \\
&\leq \|h_2\|_p + \|h_1\|_p - \frac{(p-1)\|h_1\|_p}{4}\left\|\frac{h_1}{\|h_1\|_p} - \frac{h_2}{\|h_2\|_p}\right\|_p.
\end{aligned}$$

The proof of (2.101) is similar to that of (2.100), except instead of using (2.102) we use: If $2 \leq p < \infty$, then by (3.4) of [CaFr],

$$\|\hat{h}_1 + \hat{h}_2\|_p \leq 2 - \frac{1}{p2^{p-1}}\|\hat{h}_1 - \hat{h}_2\|_p^p\,,$$

for $\hat{h}_1, \hat{h}_2 \in L^p$ such that $\|\hat{h}_1\|_p = \|\hat{h}_2\|_p = 1$. □

Next, we observe that

$$C_{m,n}^2\|\tilde{\varphi}_j\|_{\dot{H}^1}^2 - \|\tilde{\varphi}_j\|_{2^*}^2 \to 0 \text{ and } \|\tilde{\varphi}_j\|_{2^*}^{2^*} = 1, \forall j\,. \tag{2.103}$$

Moreover, $\tilde{\varphi}_j$, is independent of ζ for all j, because each $\tilde{\varphi}_j$ is rearranged in x. Letting

$$a_j = \|\chi_{\{1<w<2\}}\tilde{\varphi}_j\|_{2^*}^{2^*} \text{ and } b_j = \|\chi_{\{w\geq 2\}}\tilde{\varphi}_j\|_{2^*}^{2^*}\,,$$

and passing to a subsequence, if necessary, we have that

$$a_j \to a \in [0,1/2] \text{ and } b_j \to b \in [0,1/2]\,,$$

where, due to (2.89), $a_j + b_j = a + b = 1/2$. This brings us to the main proposition in this step:

PROPOSITION 2.7.7. $a \neq 0$.

Proof. Assume $a = 0$. Then, $b = 1/2$. Let, $\chi_1, \chi_2 \in C^\infty([0,\infty) \times [0,\pi/2] \times \mathbb{S}^{n-1})$ be such that

$$0 \leq \chi_1, \chi_2 \leq 1$$

$$\chi_1 \text{ and } \chi_2 \text{ are independent of } \theta \text{ and } \zeta$$

$$\chi_1 = 1 \text{ for } w \leq 1 \text{ and } \chi_1 = 0 \text{ for } w \geq 2 \tag{2.104}$$

$$\chi_2 = 0 \text{ for } w \leq 1 \text{ and } \chi_2 = 1 \text{ for } w \geq 2, \text{ and} \tag{2.105}$$

$$\chi_1^2 + \chi_2^2 = 1\,. \tag{2.106}$$

Then, for $\varphi \in \dot{H}^1$ and $i = 1, 2$,

$$\begin{aligned}
\|\nabla_{w,\theta,\zeta}(\chi_i\varphi)\|_2^2 &= \int [(\chi_i)_w\varphi + \chi_i\varphi_w]^2 + w^{-2}[\chi_i\varphi_\theta]^2 + w^{-2}\csc^2\theta|\chi_i\nabla_{\mathbb{S}^{n-1}}\varphi|^2 \mathrm{d}\Lambda \\
&= \int \varphi^2(\chi_i)_w^2 \mathrm{d}\Lambda + 2\int \chi_i\varphi(\chi_i)_w\varphi_w \mathrm{d}\Lambda + \int \chi_i^2(\varphi_w^2 + w^{-2}\varphi_\theta^2 + w^{-2}\csc^2\theta|\nabla_{\mathbb{S}^{n-1}}\varphi|^2)\mathrm{d}\Lambda \\
&\leq C_1\|\chi_{\{1\leq w\leq 2\}}\varphi^2\|_{2^*/2}\|\chi_{\{1\leq w\leq 2\}}\|_{(2^*/2)'} + C_2\|\chi_{\{1\leq w\leq 2\}}\varphi^2\|_{2^*/2}^{1/2}\|\chi_{\{1\leq w\leq 2\}}\|_{(2^*/2)'}^{1/2}\|\nabla_{w,\theta,\zeta}\varphi\|_2 \\
&\quad + \|\chi_i^2\nabla_{w,\theta,\zeta}\varphi\|_2^2, \text{ for some } C_1, C_2 > 0\,.
\end{aligned}$$

The last inequality is obtained by applying Holder's Inequality (twice to obtain the middle term on the right hand side) and the fact that $(\chi_i)_w \le \tilde{C}\chi_{\{1\le w\le 2\}}$ for some finite constant $\tilde{C}$ and for $i=1,2$. We can rewrite the inequality above as

$$\|\nabla_{w,\theta,\zeta}(\chi_i\varphi)\|_2^2 \le \tilde{C}_1\|\chi_{\{1\le w\le 2\}}\varphi\|_{2^*}^2 + \tilde{C}_2\|\chi_{\{1\le w\le 2\}}\varphi\|_{2^*} + \|\chi_i\nabla_{w,\theta,\zeta}\varphi\|_2^2\,, \tag{2.107}$$

for appropriate constants $\tilde{C}_1$ and $\tilde{C}_2$. Let $\varphi_{i,j} := \chi_i\tilde{\varphi}_j$. Then

$$\|\tilde{\varphi}_j\|_{2^*}^2 = \|\varphi_{1,j}^2 + \varphi_{2,j}^2\|_{2^*/2}, \text{ which by Proposition 2.7.6}$$

$$\le \begin{cases} \|\varphi_{1,j}^2\|_{2^*/2} + \|\varphi_{2,j}^2\|_{2^*/2} - \frac{(2/2^*-1)\|\varphi_{2,j}^2\|_{2^*/2}}{4}\left\|\frac{\varphi_{1,j}^2}{\|\varphi_{1,j}^2\|_{2^*/2}} - \frac{\varphi_{2,j}^2}{\|\varphi_{2,j}^2\|_{2^*/2}}\right\|_{2^*/2} & \text{if } 1 < \frac{2^*}{2} \le 2 \\ \|\varphi_{1,j}^2\|_{2^*/2} + \|\varphi_{2,j}^2\|_{2^*/2} - \frac{\|\varphi_{2,j}^2\|_{2^*/2}}{(2^*/2)2^{2/2^*-1}}\left\|\frac{\varphi_{1,j}^2}{\|\varphi_{1,j}^2\|_{2^*/2}} - \frac{\varphi_{2,j}^2}{\|\varphi_{2,j}^2\|_{2^*/2}}\right\|_{2^*/2} & \text{if } 2 \le \frac{2^*}{2} < \infty \end{cases}$$

$$\le \begin{cases} \|\varphi_{1,j}\|_{2^*}^2 + \|\varphi_{2,j}\|_{2^*}^2 - \frac{2/2^*-1}{4}\left(\frac{1}{4}\right)^{4/2^*} & \text{if } 1 < \frac{2^*}{2} \le 2 \\ \|\varphi_{1,j}\|_{2^*}^2 + \|\varphi_{2,j}\|_{2^*}^2 - \frac{2}{2^* 2^{2^*/2-1}}\left(\frac{1}{4}\right)^{4/2^*} & \text{if } 2 \le \frac{2^*}{2} < \infty\,. \end{cases} \tag{2.108}$$

We deduce the last inequality, because

$$\|\varphi_{2,j}^2\|_{2^*/2} = \|\chi_2\tilde{\varphi}_j\|_{2^*}^2 \ge \left(\frac{1}{4}\right)^{2/2^*}, \text{ for large } j, \text{ by (2.105) and because } b_j \to b = 1/2\,,$$

and

$$\begin{aligned} & \left\|\frac{\varphi_{1,j}^2}{\|\varphi_{1,j}^2\|_{2^*/2}} - \frac{\varphi_{2,j}^2}{\|\varphi_{2,j}^2\|_{2^*/2}}\right\|_{2^*/2} \\ \ge\ & \left\|\frac{\chi_{\{w\ge 2\}}(\tilde{\varphi}_j)^2}{\|(\chi_2\tilde{\varphi}_j)^2\|_{2^*/2}}\right\|_{2^*/2}, \text{ by (2.104), (2.105), and the definition of } \varphi_{i,j},\ i=1,2 \\ \ge\ & \|\chi_{\{w\ge 2\}}(\tilde{\varphi}_j)^2\|_{2^*/2}\,, \text{ because } \|(\chi_2\tilde{\varphi}_j)^2\|_{2^*/2} \le \|\tilde{\varphi}_j\|_{2^*}^2 = 1 \\ \ge\ & \left(\frac{1}{4}\right)^{2/2^*}, \text{ for large } j, \text{ because } b = 1/2\,. \end{aligned}$$

Since

$$\|\varphi_{1,j}\|_{2^*}^2, \|\varphi_{2,j}\|_{2^*}^2 \le \|\tilde{\varphi}_j\|_{2^*}^2 = 1\,,$$

we can conclude by (2.108) that there is some constant $d_{2^*} < 1$ dependent only on the value of 2^* such that

$$
\begin{aligned}
\|\tilde{\varphi}_j\|_{2^*}^2 \leq & d_{2^*}(\|\varphi_{1,j}\|_{2^*}^2 + \|\varphi_{2,j}\|_{2^*}^2)\,, \text{ which by the Sobolev Inequality} \\
\leq & d_{2^*} C_{m,n}^2 (\|\nabla_{w,\theta,\zeta}\varphi_{1,j}\|_2^2 + \|\nabla_{w,\theta,\zeta}\varphi_{2,j}\|_2^2)\,, \text{ which by (2.107)} \\
\leq & d_{2^*} C_{m,n}^2 \sum_{i=1}^{2} (\tilde{C}_1 \|\chi_{\{1\leq w\leq 2\}}\tilde{\varphi}_j\|_{2^*}^2 + \tilde{C}_2 \|\chi_{\{1\leq w\leq 2\}}\tilde{\varphi}_j\|_{2^*} + \|\chi_i \nabla_{w,\theta,\zeta}\tilde{\varphi}_j\|_2^2) \\
\leq & d_{2^*} C_{m,n}^2 (\varepsilon + \|\chi_1 \nabla_{w,\theta,\zeta}\tilde{\varphi}_j\|_2^2 + \|\chi_2 \nabla_{w,\theta,\zeta}\tilde{\varphi}_j\|_2^2) \\
& \text{for any } \varepsilon > 0 \text{ and sufficiently large } j, \text{ because we assumed that } a = 0 \\
= & d_{2^*} C_{m,n}^2 (\varepsilon + \|\tilde{\varphi}_j\|_{\dot{H}^1}^2)\,, \text{ by (2.106)}\,. \qquad (2.109)
\end{aligned}
$$

Since $d_{2^*} < 1$; $\|\tilde{\varphi}_j\|_{2^*} = 1$, for all j; $\frac{\|\tilde{\varphi}_j\|_{2^*}^2}{\|\tilde{\varphi}_j\|_{\dot{H}^1}^2} \to C_{m,n}^2$; and we can pick ε to be arbitrarily small, (2.109) contradicts the Sobolev Inequality. Recall that we arrived at (2.109) by assuming that $a = 0$. Hence, we conclude that $a \neq 0$. □

<u>Step 3:</u> *Show the $\tilde{\varphi}_j$ times one of two possible cutoff functions yields a sequence that satisfies the p, q, r-Theorem.* Let, $\chi_3 \in C^\infty([0,\infty) \times [0,\pi/2] \times \mathbb{S}^{n-1})$ be such that

$$
\begin{aligned}
& 0 \leq \chi_3 \leq 1 && (2.110) \\
& \chi_3 \text{ is independent of } \zeta \\
& \chi_3 = 1 \text{ on } \{1 \leq w \leq 2\} \cap \{0 \leq \theta \leq \pi/4\} \\
& \chi_3 = 0 \text{ on } \{w \leq 1/2\} \cup \{w \geq 4\} \cup \{\theta \geq \pi/3\}\,, && (2.111)
\end{aligned}
$$

and χ_3^R be the cutoff function given by

$$\chi_3^R(w,\theta,\zeta) = \chi_3(w, \frac{\pi}{2} - \theta, \zeta)\,.$$

Next, let

$$\varphi_{3,j} := \chi_3 \tilde{\varphi}_j \text{ and } \varphi_{3,j}^R := \chi_3^R \tilde{\varphi}_j\,. \qquad (2.112)$$

This brings us to the main proposition in this step:

PROPOSITION 2.7.8. *A subsequence of either $(\varphi_{3,j})$ or $(\varphi_{3,j}^R)$ satisfies the p,q,r-Theorem with*

$$p = 1, q = 2^*, r = \frac{2(m+n-\delta)}{m+n-\delta-2}\,,$$

for some $\delta > 0$.

Proof. The definitions of $\varphi_{3,j}$ and $\varphi_{3,j}^R$ imply that

$$\|\varphi_{3,j}\|_{2^*}^{2^*} + \|\varphi_{3,j}^R\|_{2^*}^{2^*} \geq \|\chi_{\{1<w<2\}}\tilde{\varphi}_j\|_{2^*}^{2^*} = a_j\,.$$

Thus, passing to a subsequence, if necessary, either $(\|\varphi_{3,j}\|_{2^*})$ or $(\|\varphi_{3,j}^R\|_{2^*})$ is bounded below by $(a/3)^{1/2^*}$. Whichever sequence is bounded below by $(a/3)^{1/2^*}$ will satisfy the p, q, r-Theorem. The sequence that is bounded below will satify the "q" part of the p, q, r-Theorem as posed in the proposition (i.e. bounded below in the L^{2^*} norm). It will also satisfy the "p" part as posed in the proposition (i.e. bounded above in the L^1 norm). We show this below for $(\varphi_{3,j})$ - the proof for $(\varphi_{3,j}^R)$ being identical.

$$\begin{aligned}\|\varphi_{3,j}\|_1 =&\|\chi_3\tilde{\varphi}_j\|_1, \text{ which by the definition of } \chi_3 \text{ and Holder's Inequality}\\ &\leq\|\tilde{\varphi}_j\|_{2^*}\Lambda(\{1/2 \leq w \leq 4\})^{1/(2^*)'}\\ &=\Lambda(\{1/2 \leq w \leq 4\})^{1/(2^*)'}\,. \end{aligned} \tag{2.113}$$

At this point, we only need to show the "r" part of the p, q, r-Theorem is satisfied. We deal with the case when $(\|\varphi_{3,j}\|_{2^*})$ is bounded below by $(a/3)^{1/2^*}$ and the case when $(\|\varphi_{3,j}^R\|_{2^*})$ is bounded below by $(a/3)^{1/2^*}$ separately.

$(\|\varphi_{3,j}\|_{2^*})$ *is bounded below by* $(a/3)^{1/2^*}$: We begin with the following identity and definitions:

$$\begin{aligned}
\mathrm{d}\Lambda &= \omega_m w^{m+n-1}\cos^{m-1}\theta\sin^{n-1}\theta \mathrm{d}w\mathrm{d}\theta\mathrm{d}\Omega(\zeta)\\
\mathrm{d}\Lambda_{m/2,n} &:= \omega_{m/2} w^{\frac{m}{2}+n-1}\cos^{\frac{m}{2}-1}\theta\sin^{n-1}\theta \mathrm{d}w\mathrm{d}\theta\mathrm{d}\Omega(\zeta)\\
\|\varphi\|_{\dot{H}^1;(m/2,n)} &:= \left(\int \varphi_w^2 + w^{-2}\varphi_\theta^2 + w^{-2}\csc^2\theta|\nabla_{\mathbb{S}^{n-1}}\varphi|^2 \mathrm{d}\Lambda_{m/2,n}\right)^{1/2}\\
\hat{2}^* &:= \frac{2(m/2+n)}{m/2+n-2}\\
\|\varphi\|_{\hat{2}^*;(m/2,n)} &:= \left(\int |\varphi|^{\hat{2}^*}\mathrm{d}\Lambda_{m/2,n}\right)^{1/\hat{2}^*}\,.
\end{aligned}$$

Then,

$$\begin{aligned}
\|\varphi_{3,j}\|_{\hat{2}^*} &= \left(\int |\varphi_{3,j}|^{\hat{2}^*} \mathrm{d}\Lambda\right)^{1/\hat{2}^*}, \text{ which by (2.110), (2.111), and (2.112)} \\
&\leq \left(\frac{4^{m/2}\omega_m}{\omega_{m/2}}\right)^{1/\hat{2}^*} \left(\int |\varphi_{3,j}|^{\hat{2}^*} \mathrm{d}\Lambda_{m/2,n}\right)^{1/\hat{2}^*}, \text{ which by the Sobolev Inequality} \\
&\leq C_{m/2,n} \left(\frac{4^{m/2}\omega_m}{\omega_{m/2}}\right)^{1/\hat{2}^*} \|\varphi_{3,j}\|_{\dot{H}^1;(m/2,n)} \\
&= C_{m/2,n} \left(\frac{4^{m/2}\omega_m}{\omega_{m/2}}\right)^{1/2^*} \left(\int [\varphi_{3,j}]_w^2 + w^{-2}[\varphi_{3,j}]_\theta^2 + w^{-2}\csc^2\theta |\nabla_{\mathbb{S}^{n-1}}\varphi_{3,j}|^2 \mathrm{d}\Lambda_{m/2,n}\right)^{1/2} \\
&\quad \text{which by (2.110), (2.111), and (2.112)} \\
&\leq C_{m/2,n} 4^{(1/\hat{2}^*+1/2)\frac{m}{2}} \left(\frac{\omega_{m/2}}{\omega_m}\right)^{1/2-1/\hat{2}^*} \|\varphi_{3,j}\|_{\dot{H}^1} \\
&< M_1, \forall j\,,
\end{aligned}$$

for some finite M_1, because $(\|\varphi_{3,j}\|_{\dot{H}^1})$ must be bounded due to (2.103).

$(\|\varphi^R_{3,j}\|_{2^*})$ *is bounded below by* $(a/3)^{1/2^*}$: Choose some $\delta > 0$ such that

$$m + n - \delta > 2 \text{ and } \delta < n\,.$$

Let

$$\tilde{2}^* := \frac{2(m+n-\delta)}{m+n-\delta-2}\,.$$

In this case, it is crucial to note that since we assumed that χ_3 is independent of ζ,

$$\varphi^R_{3,j} = \chi^R_3 \tilde{\varphi}^*_j \implies \varphi^R_{3,j} \text{ is independent of } \zeta\,.$$

Thus,

$$\begin{aligned}
\|\varphi_{3,j}^R\|_{\tilde{2}^*} &= \left(\int_{\mathbb{S}^{n-1}}\int_0^{\pi/2}\int_0^\infty |\varphi_{3,j}^R|^{\tilde{2}^*}\omega_m w^{m+n-1}\mathrm{d}w\cos^{m-1}\theta\sin^{n-1}\theta\mathrm{d}\theta\mathrm{d}\Omega(\zeta)\right)^{1/2^*}\\
&= \left(\int_0^{\pi/2}\int_0^\infty |\varphi_{3,j}^R|^{\tilde{2}^*}\omega_m\omega_n w^{m+n-1}\mathrm{d}w\cos^{m-1}\theta\sin^{n-1}\theta\mathrm{d}\theta\right)^{1/\tilde{2}^*}, \text{ which by (2.110), (2.111), and (2}\\
&\leq \left(\frac{4^\delta\omega_n}{\omega_{n-\delta}}\right)^{1/\tilde{2}^*} C_{m,n-\delta}\left(\int_0^{\pi/2}\int_0^\infty |\varphi_{3,j}^R|^{\tilde{2}^*}\omega_m\omega_{n-\delta}w^{m+n-\delta-1}\mathrm{d}w\cos^{m-1}\theta\sin^{n-\delta-1}\theta\mathrm{d}\theta\right)^{1/2}\\
&\quad\text{which by the Sobolev Inequality}\\
&\leq \left(\frac{4^\delta\omega_n}{\omega_{n-\delta}}\right)^{1/\tilde{2}^*} C_{m,n-\delta}\left(\int_0^{\pi/2}\int_0^\infty ([\varphi_{3,j}]_w^2+w^{-2}[\varphi_{3,j}]_\theta^2)w^{m+n-\delta-1}\mathrm{d}w\cos^{m-1}\theta\sin^{n-\delta-1}\theta\mathrm{d}\theta\right)^{1/2}\\
&\quad\text{which by (2.110), (2.111), and (2.112)}\\
&\leq \left(\frac{4^\delta\omega_n}{\omega_{n-\delta}}\right)^{1/2^*}\left(\frac{4^\delta\omega_{n-\delta}}{\omega_n}\right)^{1/2}\|\varphi_{3,j}^R\|_{\dot{H}^1}\\
&< M_2, \forall j\,,
\end{aligned}$$

for some finite M_2, because $(\|\varphi_{3,j}^R\|_{\dot{H}^1})$ must be bounded due to (2.103). □

We can now apply the result of Proposition 2.7.8 to prove Lemma 2.7.2.

Proof of Lemma 2.7.2. Since $\mathrm{supp}(\chi_3), \mathrm{supp}\ (\chi_3^R) \subseteq \{w \leq 4\}$ and $0 \leq \chi_3 \leq 1$, Proposition 2.7.8 shows that some sequence, $(\Lambda(\{|\tilde{\varphi}_{j_k}(w,\theta,\zeta)| > \varepsilon, w \leq 4\}))$, is bounded below by a positive constant, C. Since $\{w \leq 4\} \subseteq \{\rho \leq 4\}$, Lemma 2.7.5 is true a fortiori. Thus, by the reduction in step one, Lemma 2.7.2 holds. □

At this point, we relabel indices and apply Theorem 2.7.3 - we have not proved Theorem 2.7.3 yet, but will prove it at the end of this section - to conclude that there exists $(x_j) \subseteq \mathbb{R}^n$ such that

$$\hat{\varphi}_j(\rho, x) = \varphi_j^{\sigma_j}(\rho, x + x_j)$$

has a subsequence that converges to some nonzero φ in $\dot{H}^1_{\mathbb{C}}$. We will show that this convergence is in fact strong convergence in $\dot{H}^1_{\mathbb{C}}$ and that φ is an extremal of the Sobolev Inequality with L^{2^*} norm equal one.

Part 2 of proof of Theorem 2.7.1 - Conclusion of proof using Functional Analysis arguments: The title of this part is self-explanatory. A notable feature of this part is an application of the local compactness theorem, Theorem 2.1.6. This theorem is a Rellich-Kondrachov type Theorem for cylindrically symmetric functions in continuous dimension and anything of its type is, to our knowledge, absent from literature. We prove Theorem 2.1.6 in the next and final section.

Applying Theorem 2.1.6, passing to a subsequence if necessary, we may assume that $\hat{\varphi}_j$ converges to φ almost everywhere. We now argue that $\hat{\varphi}_j$ converges strongly in $\dot{H}^1_{\mathbb{C}}$ and φ is an extremal of the Sobolev Inequality. Weak convergence in $\dot{H}^1_{\mathbb{C}}$ implies that

$$\|\hat{\varphi}_j\|^2_{\dot{H}^1} = \|\varphi\|^2_{\dot{H}^1} + \|\hat{\varphi}_j - \varphi\|^2_{\dot{H}^1} + o(1)\,. \tag{2.114}$$

Next, we observe that almost everywhere convergence and the Brezis-Lieb Lemma imply that

$$\|\hat{\varphi}_j\|^{2^*}_{2^*} = \|\varphi\|^{2^*}_{2^*} + \|\hat{\varphi}_j - \varphi\|^{2^*}_{2^*} + o(1)\,. \tag{2.115}$$

Combining this with the concavity of $y \mapsto y^{2/2^*}$, and passing to a subsequence if necessary, we deduce that

$$\lim \|\hat{\varphi}_j\|^2_{2^*} \leq \|\varphi\|^2_{2^*} + \lim \|\hat{\varphi}_j - \varphi\|^2_{2^*}\,.$$

Thus,

$$\begin{aligned} 1 =& \lim \|\hat{\varphi}_j\|^2_{2^*}, \text{ by assumption} \\ \leq& \|\varphi\|^2_{2^*} + \lim \|\hat{\varphi}_j - \varphi\|^2_{2^*}, \text{ which by the Sobolev Inequality} \\ \leq& C^2_{m,n}(\|\varphi\|^2_{\dot{H}^1} + \lim \|\hat{\varphi}_j - \varphi\|^2_{\dot{H}^1}), \text{ which by (2.84) and (2.114)} \\ =& 1\,. \end{aligned} \tag{2.116}$$

(2.116) implies that

$$C^2_{m,n}\|\varphi\|^2_{\dot{H}^1} - \|\varphi\|^2_{2^*} + \lim(C^2_{m,n}\|\hat{\varphi}_j - \varphi\|^2_{\dot{H}^1} - \|\hat{\varphi}_j - \varphi\|^2_{2^*}) = 0\,. \tag{2.117}$$

The Sobolev Inequality implies that

$$C^2_{m,n}\|\hat{\varphi}_j - \varphi\|^2_{\dot{H}^1} - \|\hat{\varphi}_j - \varphi\|^2_{2^*},\, C^2_{m,n}\|\varphi\|^2_{\dot{H}^1} - \|\varphi\|^2_{2^*} \geq 0\,.$$

Combining this with (2.117), we conclude that

$$C^2_{m,n}\|\varphi\|^2_{\dot{H}^1} - \|\varphi\|^2_{2^*} = 0\,,$$

i.e. φ is an extremal.

(2.116) also implies that

$$1 = \|\varphi\|_{2^*}^2 + \lim \|\hat{\varphi}_j - \varphi\|_{2^*}^2 \, . \tag{2.118}$$

Since $y \mapsto y^{2/2^*}$ is strictly concave and φ is nonzero, (2.115); $\|\varphi_j\|_{2^*} = 1$, $\forall j$; and, (2.118) allow us to conclude that $\|\varphi\|_{2^*}^2 = 1$. Thus, $\hat{\varphi}_j$ converges to φ in norm and weakly. These two characteristics imply that $\hat{\varphi}_j$ converges to φ strongly in $\dot{H}^1_{\mathbb{C}}$. This would conclude the proof of Theorem 2.7.1, except we have not yet proved Theorem 2.7.3. We conclude this section by proving Theorem 2.7.3.

Part 3 of proof of Theorem 2.7.1 - *Proof of Theorem 2.7.3:*. Let B_y denote the ball of unit radius in $\mathbb{R}^n$ centered at $y \in \mathbb{R}^n$. By Theorem 2.1.6 and the Banach-Alaoglu Theorem, it suffices to prove that we can find x_j and $\delta > 0$ such that $\Lambda\left(\{\rho \leq R+1, x \in B_{x_j}\} \cap \{|\varphi_j(\rho,x)| \geq \varepsilon/2\}\right.$ δ, for all j, for then $\int_{\{\rho \leq R+1, x \in B_0\}} |\varphi_j^T| \mathrm{d}\Lambda \geq \delta\varepsilon/2$, and so no weak limit can vanish. Without loss of generality, we may assume $\varphi_j \geq 0$, for all j. Thus, we henceforth assume that $E_j := \{\varphi_j(\rho,x) > \varepsilon, \rho \leq R\}$ - refer back to Theorem 2.7.3 for the original definition of E_j.

Let $\Psi_j = \chi_4(\varphi_j - \varepsilon/2)_+$, where $\chi_4 \in C^\infty([0,\infty) \times \mathbb{R}^n)$ is such that

$$\begin{aligned} &0 \leq \chi_4 \leq 1 \\ &\chi_4 = 1 \text{ for } \rho \leq R \\ &\chi_4 = 0 \text{ for } \rho \geq R+1 \, . \end{aligned}$$

Note that $\Lambda(\mathrm{supp}[(\varphi_j - \varepsilon/2)_+]) \leq C < \infty$ for all j and some C, because $(\|\varphi_j\|_{2^*})$ is uniformly bounded. More precisely, if $C_* < \infty$ bounds $(\|\varphi_j\|_{2^*})$ above, then

$$C_* \geq \left(\frac{\varepsilon}{2}\right)^{2^*} \Lambda(\{\varphi_j \geq \varepsilon/2\}) \implies \Lambda(\mathrm{supp}[(\varphi_j - \varepsilon/2)_+]) \leq C_* \left(\frac{2}{\varepsilon}\right)^{2^*} =: J \, . \tag{2.119}$$

Thus, $\Psi_j \in L^2$, for all j. Also, $\Psi_j \geq \varepsilon/2$ on E_j. Thus,

$$\begin{aligned} \frac{\int |\nabla_{\rho,x}\Psi_j|^2 \mathrm{d}\Lambda}{\int |\Psi_j|^2 \mathrm{d}\Lambda} \quad &\leq \quad \frac{2\int |\nabla_{\rho,x}\chi_4|^2 |(\varphi_j - \varepsilon/2)_+|^2 + |\chi_4|^2 |\nabla_{\rho,x}(\varphi - \varepsilon/2)_+|^2 \mathrm{d}\Lambda}{\delta(\varepsilon/2)^2} \\ &\quad\quad \text{which by Holder's Inequality and (2.119)} \\ &\leq \quad \frac{2(\|\nabla_{\rho,x}\chi_4\|_\infty J^{1/(2^*/2)'} \|(\varphi - \varepsilon/2)_+\|_{2^*}^2 + \|\nabla_{\rho,x}(\varphi_2 - \varepsilon/2)\|_2)}{\delta(\varepsilon/2)^2} \\ &=: \quad W \, . \end{aligned}$$

Let G be a nonzero $C^\infty_C(\mathbb{R}^n)$ function supported on B_0 and let $G^y(x) = G(x-y)$. Define $\lambda := \int_{\mathbb{R}^n} |\nabla_x G|^2 \mathrm{d}x / \int_{\mathbb{R}^n} |G|^2 \mathrm{d}x$, where ∇_x denotes the gradient over the x variable.

Let $\Theta^y_j(\rho, x) = G^y(x)\Psi_j(\rho, x)$. Then,

$$|\nabla_{\rho,x}\Theta^y_j|^2 \leq 2(|\nabla_x G^y|^2|\Psi_j|^2 + |G^y|^2|\nabla_{\rho,x}\Psi_j|^2)\,.$$

Consider

$$\begin{aligned} T^y_j :=& \int |\nabla_{\rho,x}\Theta^y_j|^2 - 4(W+\lambda)|\Theta^y_j|^2 \mathrm{d}\Lambda \\ \leq& 2\int |\nabla_x G^y|^2|\Psi_j|^2 + |G^y|^2|\nabla_{\rho,x}\Psi_j|^2 - 2(W+\lambda)|G^y_j|^2|\Psi_j|^2 \mathrm{d}\Lambda\,. \end{aligned} \tag{2.120}$$

Thus,

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}^n} T^y_j \mathrm{d}y \leq& \int |\nabla_x G|^2 \mathrm{d}x \int |\Psi_j|^2 \mathrm{d}\Lambda + \int |G|^2 \mathrm{d}x \int |\nabla_{\rho,x}\Psi_j|^2 \mathrm{d}\Lambda - 2(W+\lambda)\int |G|^2 \mathrm{d}x \int |\Psi_j|^2 \mathrm{d}\Lambda \\ <& 0\,. \end{aligned} \tag{2.121}$$

Combining (2.120) and (2.121), we conclude that for each j there is some x_j such that $\|\Theta^{x_j}_j\|_2 > 0$ and

$$\|\nabla_{\rho,x}\Theta^{x_j}_j\|_2 / \|\Theta^{x_j}_j\|_2 < 2(W+\lambda)\,. \tag{2.122}$$

We will use this fact to prove that $(\Lambda(\mathrm{supp}(\Theta^{x_j}_j)))$ is uniformly bounded below by a positive constant.

Let $\varphi \in \dot{H}^1_{\mathbb{C}}$ be such that $\Lambda(\mathrm{supp}(\varphi)) < \infty$. Then

$$\begin{aligned} \|\varphi\|_2^2 \leq& \Lambda(\mathrm{supp}\ (\varphi))^{1/(2^*/2)'}\|\varphi\|_{2^*}^2 \\ \leq& C_{m,n}\Lambda(\mathrm{supp}(\varphi))^{1/(2^*/2)'}\|\nabla_{\rho,x}\varphi\|_2^2 \\ \implies& \|\nabla_{\rho,x}\varphi\|_2^2/\|\varphi\|_2^2 \geq C_{m,n}^{-1}\Lambda(\mathrm{supp}(\varphi))^{-1/(2^*/2)'}\,. \end{aligned} \tag{2.123}$$

(2.122) and (2.123) imply that $\Lambda(\mathrm{supp}(\Theta^{x_j}_j)) \geq \delta > 0$, for all j, for some δ. Combining this with the fact that $\mathrm{supp}(\Theta^{x_j}_j) \subseteq \{\rho \leq R+1, x \in B_{x_j}\}$, we conclude that $\Lambda(\{\rho \leq R+1, x \in B_{x_j}\} \cap \{\varphi_j(\rho, x) \geq \varepsilon/2\}) \geq \delta$, for all j. □

2.8 Proof of Rellich-Kondrachov Type Theorem

In the following, we prove Theorem 2.1.6. We restate this theorem below:

Theorem 2.1.6. *Let $K \subseteq [0,\infty) \times \mathbb{R}^n$ satisfy the cone property in $\mathbb{R}^{n+1}$, $K \subseteq \{(\rho, x) \in [0,\infty) \times \mathbb{R}^n | \rho_1 < \rho < \rho_2\}$ for some $0 < \rho_1 < \rho_2 < \infty$, and $\Lambda(K) < \infty$, where Λ denotes the measure on $\mathbb{R}_+ \times \mathbb{R}^n$ defined by (2.10). If (φ_j) is bounded in $\dot{H}^1_{\mathbb{C}}$ and U is an open subset of K, then for $1 \leq p < \max\left\{2^*, \frac{2n+2}{n-1}\right\}$, there is some $\varphi \in \dot{H}^1_{\mathbb{C}}$ and some subsequence, (φ_{j_k}), such that $\varphi_{j_k} \to \varphi$ in $L^p_{\mathbb{C}}(U, \omega_m \rho^{m-1} \mathrm{d}\rho \mathrm{d}x)$.*

Proof. First, we note that, taking a subsequence if necessary, $\varphi_j \rightharpoonup \varphi$ in $\dot{H}^1_{\mathbb{C}}$ for some φ. Next, we show that (φ_j) is bounded in $H^1(K)$. To this end, we show that $L^q_{\mathbb{C}}(V, \mathrm{d}\rho\mathrm{d}x)$ and $L^q_{\mathbb{C}}(V, \omega_m\rho^{m-1}\mathrm{d}\rho\mathrm{d}x)$ are equivalent norms for $1 \leq q < \infty$, when

$$V \subseteq \left\{(\rho, x) \in [0,\infty) \times \mathbb{R}^n \middle| \rho_1 < \rho < \rho_2\right\}, \tag{2.124}$$

because (2.124) implies that

$$\omega_m^{-1/q}\rho_2^{-(m-1)/q}\|\cdot\|_{L^q(V,\omega_m\rho^{m-1}\mathrm{d}\rho\mathrm{d}x)} \leq \|\cdot\|_{L^q(V,\mathrm{d}\rho\mathrm{d}x)} \leq \omega_m^{-1/q}\rho_1^{-(m-1)/q}\rho_1^{-(m-1)/q}\|\cdot\|_{L^q(V,\omega_m\rho^{m-1}\mathrm{d}\rho\mathrm{d}x)}\,.$$

Thus,

$$\|\nabla\varphi_j\|_{L^2(K,\mathrm{d}\rho\mathrm{d}x)} \leq \omega_m^{-1/2}\rho_2^{-(m-1)/2}\|\nabla\varphi_j\|_{L^2(K,\omega_m\rho^{m-1}\mathrm{d}\rho\mathrm{d}x)}\,, \tag{2.125}$$

and

$$\begin{aligned}\|\varphi_j\|_{L^2(K,\mathrm{d}\rho\mathrm{d}x)} \leq &\omega_m^{-1/2}\rho_2^{-(m-1)/2}\|\varphi_j\|_{L^2(K,\omega_m\rho^{m-1}\mathrm{d}\rho\mathrm{d}x)}\\ \leq &\omega_m^{-1/2}\rho_2^{-(m-1)/2}\Lambda(K)^{1/(m+n)}\|\varphi_j\|^2_{L^{2^*}(K,\omega_m\rho^{m-1}\mathrm{d}\rho\mathrm{d}x)}\,,\ \text{which by Theorem 2.1.4}\\ \leq &\omega_m^{-1/2}\rho_2^{-(m-1)/2}\Lambda(K)^{1/(m+n)}C^2_{m,n}\|\nabla\varphi_j\|^2_{L^2(K,\omega_m\rho^{m-1}\mathrm{d}\rho\mathrm{d}x)}\,.\end{aligned} \tag{2.126}$$

Combining (2.125) and (2.126), we conclude that (φ_j) is bounded $H^1(K)$. Applying the Rellich-Kondrachov Theorem, we conclude that if $1 \leq p < \frac{2n+2}{n-1}$, then there is some $\Psi \in H^1(K)$ and some subsequence, (φ_{j_k}), such that for $U \subseteq K$, U open,

$$\varphi_{j_k} \to \Psi \text{ in } L^p(U, \mathrm{d}\rho\mathrm{d}x)\,.$$

Since $\|\cdot\|_{L^p(U,\mathrm{d}\rho\mathrm{d}x)}$ and $\|\cdot\|_{L^p(U,\omega_m\rho^{m-1}\mathrm{d}\rho\mathrm{d}x)}$ are equivalent norms, we conclude that $\varphi_{j_k} \to \Psi$ in $L^p_{\mathbb{C}}(U, \omega_m\rho^{m-1}\mathrm{d}\rho\mathrm{d}x)$. Since $\varphi_j \rightharpoonup \varphi$ in $\dot{H}^1_{\mathbb{C}}$, we conclude that $\Psi = \varphi$.

We conclude by showing that if $\frac{2n+2}{n-1} < 2^*$, then for $\frac{2n+2}{n-1} \leq p < 2^*$, there is some (φ_{j_k}) such that $\varphi_{j_k} \to \varphi$ in $L^p_{\mathbb{C}}(U, \omega_m \rho^{m-1} \mathrm{d}\rho \mathrm{d}x)$. By the Holder Inequality,

$$\|\varphi_j - \varphi\|_p^p \leq \|\varphi_j - \varphi\|^{\alpha}_{(p-q)\frac{r}{r-1}} \|\varphi_j - \varphi\|^{\beta}_{qr} \tag{2.127}$$

for some $\alpha, \beta > 0$, $1 < q < p$, and $1 < r < \infty$. If we choose q and r such that

$$qr = 2^*\,,$$

then

$$(p-q)\frac{r}{r-1} = (p-q)\frac{2^*/q}{(2^*/q)-1}\,. \tag{2.128}$$

Note that

$$q < p < 2^* \implies \frac{2^*/q}{(2^*/q)-1} < \frac{2^*/p}{(2^*/p)-1} < \infty\,. \tag{2.129}$$

Combining (2.128) and (2.129), we conclude that if q is close enough to p, then

$$(p-q)\frac{r}{r-1} < p\,. \tag{2.130}$$

Choosing a value for q for which (2.130) holds and $(p-q)\frac{q}{q-1} \geq 1$, and a corresponding subsequence, (φ_{j_k}), such that $\varphi_{j_k} \to \varphi$ in $L_{\mathbb{C}}^{(p-q)\frac{r}{r-1}}(U, \omega_m \rho^{m-1} \mathrm{d}\rho \mathrm{d}x)$, (2.127) yields

$$\begin{aligned}
\lim_{k\to\infty} \|\varphi_{j_k} - \varphi\|_p^p &\leq \lim_{k\to\infty} \|\varphi_{j_k} - \varphi\|^{\alpha}_{(p-q)\frac{r}{r-1}} \|\varphi_{j_k} - \varphi\|^{\beta}_{2^*} \\
&\quad \text{and since } \varphi_j \text{ is bounded in } \dot{H}^1_{\mathbb{C}} \text{ and } \varphi_{j_k} \to \varphi \text{ in } L_{\mathbb{C}}^{(p-q)\frac{r}{r-1}}(U, \omega_m \rho^{m-1} \mathrm{d}\rho \mathrm{d}x) \\
&= 0\,.
\end{aligned}$$

□

Chapter 3

A Stability Result for the Del Pino and Dolbeault's Family of Sharp Gagliardo-Nirenberg Inequalities

3.1 Main Result

In this chapter, we use the stability estimate, Theorem 2.1.5, on Bakry, Gentil, and Ledoux's continuous dimension extension of the Sobolev Inequality to derive a stability estimate for a class of sharp GN (Gagliardo-Nirenberg) inequalities - the sharp constants and extremals of these inequalities were made explicit in [DeDo] by Del Pino and Dolbeault. More precisely, we establish a stability estimate for the one parameter family of GN inequalities for functions in $\dot{H}^1(\mathbb{R}^n)$, $n \geq 2$, and parameter $1 \leq s \leq \frac{n}{n-2}$ (if $n = 2$, $1 \leq s < \infty$) given by

$$\|u\|_{2s} \leq A_{n,s} \|\nabla u\|_2^{\mu} \|u\|_{s+1}^{1-\mu}\,, \;\mu = \frac{n(s-1)}{s[2n-(s+1)(n-2)]}\,, \tag{3.1}$$

where $A_{n,s}$ is a sharp constant depending on n and s. This family of sharp inequalities is one of only two GN inequalities for which the sharp constant and extremals are known, the other being the Nash Inequality. One should note that in (3.1) when $s = 1$, $\mu = 0$, and (3.1) is a trivial inequality; and when $s = \frac{n}{n-2}$, $\mu = 1$, and (3.1) is the Sobolev Inequality. Also, μ varies continuously between 0 and 1 as s varies between 1 and $\frac{n}{n-2}$ (or ∞ if $n = 2$).

Roughly speaking, the stability estimate tells us how far away a given function is from the manifold of optimizers for the GN (Gagliardo-Nirenberg) inequalities in terms of its GN deficit, denoted $\delta_{GN}[u]$, which is given by

$$\delta_{GN}[u] := A_{n,s}^{4s/2^*} \|\nabla u\|_2^{\mu 4s/2^*} \|u\|_{s+1}^{(1-\mu)4s/2^*} - \|u\|_{2s}^{4s/2^*}\,. \tag{3.2}$$

The complete set of extremals of the one-parameter set of GN inequalities given in (3.2) is the constant multiples of the functions given by

$$v_{\lambda,x_0}(x) = \lambda^{\frac{n}{2s}}(1+\lambda^2|x-x_0|^2)^{-1/(s-1)}\,,\ \lambda > 0\,, x_0 \in \mathbb{R}^n\,. \tag{3.3}$$

These extremals differ from the extremals in the Sobolev Inequality only in their exponent, suggesting a potentially deep link between the two inequalities. Indeed, there is a deep link that we will make precise in the following subsections. It is this link that allows us to obtain a stability estimate for the GN inequalities from the stability estimate on the continuous dimension Sobolev Inequality. For convenience, we define the following function:

$$v(x) := v_{1,0}(x)\,. \tag{3.4}$$

Note that the above implies that

$$A_{n,s} = \frac{\|v\|_{2s}}{\|\nabla v\|_2^{\mu}\|v\|_{s+1}^{1-\mu}}\,. \tag{3.5}$$

The precise statement of the stability estimate that we prove for (3.1) is

THEOREM 3.1.1. *Let $u \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative function such that $\|u\|_{2s} = \|v\|_{2s}$. Then there exist positive constants $K_1 := K_1(n,s)$ and $\delta_1 := \delta_1(n,s)$, depending upon n and s, such that whenever $\delta_{GN}[u] \le \delta_1$,*

$$\inf_{\lambda>0, x_0\in\mathbb{R}^n} \|u^{2s} - v_{\lambda,x_0}^{2s}\|_1 \le K_1\delta_{GN}[u]^{1/2}\,. \tag{3.6}$$

Remark 3.1.2. Note that in (3.6), the GN deficit, $\delta_{GN}[u]^{1/2}$, bounds the L^1 distance of u^{2s}, provided $\|u\|_{2s} = \|v\|_{2s}$, from v_{λ,x_0}^{2s} for all $\lambda > 0, x_0 \in \mathbb{R}^n$. This is weaker than the Bianchi-Egnell Stability Estimate, where the Sobolev Deficit, which we define as

$$\delta_{Sob}[\varphi] := C_N^2\|\nabla\varphi\|_2^2 - \|\varphi\|_{2^*}^2\,,$$

bounds the distance of φ from the manifold of extremals in the Sobolev Inequality with respect to the gradient square norm. The Bianchi-Egnell Stability Estimate is stronger in the sense that we can obtain a bound on the L^1 distance of φ^{2^*} from $\hat{c}F_{t,x_0}$ for all $t > 0, x_0 \in \mathbb{R}^n$, from the Bianchi-Egnell Stability Estimate provided $\hat{c}$ and φ are nonnegative and $\|\varphi\|_{2^*} = \|\hat{c}F_{t,x_0}\|_{2^*} = 1$. We demonstrate this below:

$$\begin{aligned}\delta_{Sob}[\varphi]^{1/2} &\geq \alpha \inf_{t>0,x_0\in\mathbb{R}^n} \|\varphi - \hat{c}F_{t,x_0}\|_{\dot{H}^1}\ , \text{ by the Bianchi-Egnell Stability Estimate}\\ &\geq C_N^{-1}\alpha \inf_{t>0,x_0\in\mathbb{R}^n} \|\varphi - \hat{c}F_{t,x_0}\|_{2^*}\ , \text{ by the Sobolev Inequality}\\ &\geq 2^{-(2^*-1)}C_N^{-1}\alpha \inf_{t>0,x_0\in\mathbb{R}^n} \|\varphi^{2^*} - (\hat{c}F_{t,x_0})^{2^*}\|_1\ ,\end{aligned}$$

by an application of the Mean Value Theorem and Holder's Inequality and because we assumed that φ and $\hat{c}$ are nonnegative and $\|\varphi\|_{2^*} = \|\hat{c}F_{t,x_0}\|_{2^*} = 1$. Although the stability estimate for the GN inequalities of Del Pino and Dolbeault is weaker than the Bianchi-Egnell Stability Estimate, it is still appropriate for applications. In fact, the type of bound provided by Theorem 3.1.1 is exactly what was needed in its original application by Carlen and Figalli in [CaFi] in which they proved the special case of Theorem 3.1.1 for $n = 2$ and $s = 3$ in order to solve a Keller-Segal equation. This is because in this application, Carlen and Figalli needed stability on u^{2s} as a measure.

Theorem 3.1.1 is a generalization of Carlen and Figalli's Theorem 1.2 in [CaFi], which is the special case of Theorem 3.1.1 for $n = 2$ and $s = 3$. In the next subsection, we will state and prove a special case of Theorem 3.1.1 that corresponds to the cases of Theorem 3.1.1 that can be proved using the Sobolev Inequality and the Bianchi-Egnell Stability Estimate for integer dimensions only. This will illustrate the connection between the Sobolev Inequality and the GN inequalities as well as their stability estimates. Once we establish these connections, we can better explain the need for Theorem 2.1.5 in proving Theorem 3.1.1 - indeed, we explain this in detail in subsection 3.3.

3.2 Deriving the Sharp GN Inequalitis of Del Pino and Dolbeault from the Sobolev Inequality and its Continuous Dimension Extension

The key to our method of deriving a stability estimate on a sharp GN inequality is a striking observation connecting the Sobolev Inequality to the GN inequalities. Let $\varphi : \mathbb{R}^{m+n} \to \mathbb{R}$ be given by

$$\varphi(y,z) = [f(y) + |z|^2]^{-(m+n-2)/2}\ , \tag{3.7}$$

for $f \geq 0$. Then

$$|\nabla_{y,z}\varphi|^2 = \left(\frac{m+n-2}{2}\right)^2 [f+|z|^2]^{-(m+n)}(|\nabla_y f|^2 + 4|z|^2)\,, \text{ and}$$
$$|\varphi|^{2^*} = [f+|z|^2]^{-(m+n)}\,.$$

Thus, the integrated Sobolev Inequality takes the form

$$\left(\int_{\mathbb{R}^n} f^{-\frac{m+2n}{2}}\mathrm{d}y\right) \leq c_1 \int_{\mathbb{R}^n} f^{-\frac{m+2n}{2}}|\nabla_y f|^2 \mathrm{d}y + c_2 \int_{\mathbb{R}^n} f^{-\frac{m+2n-2}{2}}\mathrm{d}y\,, \tag{3.8}$$

for constants c_1 and c_2 depending upon m and n. Since we obtained (3.8) by integrating the Sobolev Inequality applied to (3.7) in the z-variable, (3.8) yields equality if $f(y) = 1 + |y|^2$, because this f makes φ given by (3.7) into an extremal for the Sobolev Inequality. Replacing f with $u^{-\frac{4}{m+n-2}}$, (3.8) becomes

$$\|u\|_{2s}^{4s/2^*} \leq c_3\|\nabla u\|_2^2 + c_4\|u\|_{s+1}^{s+1}\,, \tag{3.9}$$

for further constants c_3 and c_4, and s given by

$$s = \frac{m+2n}{m+2n-4}\,.$$

If we replace u with u_λ given by

$$u_\lambda(y) = \lambda^{n/2s}u(\lambda y)\,,$$

in (3.9) and then optimize with respect to λ, we find that for u such that

$$\frac{\|\nabla u\|_2^2}{\|u\|_{s+1}^{s+1}} = \frac{\|\nabla v\|_2^2}{\|v\|_{s+1}^{s+1}}$$

(3.9) becomes (3.1), because

$$c_3\|\nabla u\|_2^2 + c_4\|u\|_{s+1}^{s+1} = A_{n,s}^{4s/2^*}\|\nabla u\|_2^{4s/2^*\mu}\|\nabla u\|_{s+1}^{4s/2^*(1-\mu)}\,.$$

3.3 GN Stability Estimate Using the Integer Dimension Sobolev Inequality and Bianchi-Egnell Stability Estimate

In this subsection, we will outline the proof of the cases of Theorem 3.1.1 using the integer dimension Sobolev Inequality and Bianchi-Egnell Stability Estimate. We begin by observing that by restricting ourselves to the Sobolev Inequality and the Bianchi-Egnell Stability Estimate on integer dimensions, we can only prove Theorem 3.1.1 for s given by (3.11) for $m \in \mathbb{N}$.

Remark 3.3.1. The restriction $m \in \mathbb{N}$ is a consequence of our applying the Sobolev Inequality and the Bianchi-Egnell Stability Estimate for integer dimensions. The identity, (3.11), and the restriction $m \in \mathbb{N}$ imply that if we only relied on the Sobolev Inequality and the Bianchi-Egnell Stability Estimate for integer dimensions, that we would only be able to prove Theorem 3.1.1, for some rational values of s. This would leave us well short of the full range of values of s in Del Pino and Dolbeault's family of sharp GN inequalities. Employing Bakry, Gentil, and Ledoux's generalization of the Sobolev Inequality and proving the Bianchi-Egnell Stability Estimate given by Theorem 2.1.5 lets us employ all values of $m > 0$ and consequently prove Theorem 3.1.1 for the full range of values for s, i.e. $1 \leq s \leq \frac{n}{n-2}$ when $n > 2$ or $1 \leq s \leq \infty$ when $n = 2$.

The heart of the proof of is the link between the Sobolev Inequality and the GN Inequality summarized in the following

PROPOSITION 3.3.2. *Let $m \in \mathbb{N}$. Also, let $u \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative function such that*

$$\frac{\|u\|_{s+1}^{s+1}}{\|\nabla u\|_2^2} = \frac{\|v\|_{s+1}^{s+1}}{\|\nabla v\|_2^2}\,, \tag{3.10}$$

where

$$s = \frac{m+2n}{m+2n-4}\,. \tag{3.11}$$

Let $\varphi_u : \mathbb{R}^{m+n} \to \mathbb{R}$ be given by

$$\varphi_u(y,z) = [w_u(y) + |z|^2]^{-\frac{m+n-2}{2}}\,, \tag{3.12}$$

where

$$w_u(y) = u^{-\frac{4}{m+2n-4}}(y)\,,\ \textit{and}\ y \in \mathbb{R}^n, z \in \mathbb{R}^m\,. \tag{3.13}$$

Then

$$\tilde{C}_1^{-1}(C_{m+n}^2\|\nabla\varphi_u\|_2^2 - \|\varphi_u\|_{2^*}^2) = A_{n,s}^{4s/2^*}\|\nabla u\|_2^{\mu 4s/2^*}\|u\|_{s+1}^{(1-\mu)4s/2^*} - \|u\|_{2s}^{4s/2^*} = \delta_{GN}[u]\,, \tag{3.14}$$

where

$$\tilde{C}_1 = \left(\int_{\mathbb{R}^m} [1+|\zeta|^2]^{-(m+n)} \mathrm{d}\zeta\right)^{2/2^*} . \tag{3.15}$$

Remark 3.3.3. Note that (3.14) asserts that

$$C_1^{-1}\delta_{Sob}[\varphi_u] = \delta_{GN}[u]\,,$$

where $\delta_{Sob}[\cdot]$ is the difference of terms in the Sobolev Inequality given by

$$\delta_{Sob}[\varphi] = C_N^2\|\nabla\varphi\|_2^2 - \|\varphi\|_{2^*}^2\,.$$

The reason Proposition 3.3.2 is the key to proving Theorem 3.1.1 for $m \in \mathbb{N}$ is that we can use (3.14) and leverage the Bianchi-Egnell Stability Estimate to get a stability estimate for the GN inequalities with $m \in \mathbb{N}$. Obtaining Theorem 3.1.1 for all possible values of s is a direct application of Theorem 2.1.5, which is the main result of chapter 2, because once we obtain the analogue of Proposition 3.3.2 for φ_u in continuous dimensions, we can apply Theorem 2.1.5 to obtain a stability estimate for the full class of sharp GN inequalituies of Del Pino and Dolbeault.

Proof. We begin by observing that (recall that $\gamma = \frac{m+n-2}{2}$)

$$\begin{aligned}
&\gamma^{-2}\|\nabla\varphi_u\|_2^2\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} |\nabla_y w_u(y)|^2[w_u(y)+|z|^2]^{-(m+n)}\mathrm{d}z\mathrm{d}y + \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} 4|z|^2[w_u(y)+|z|^2]^{-(m+n)}\mathrm{d}z\mathrm{d}y\\
&\quad \text{taking } \zeta = w_u^{1/2}z\\
&= \int_{\mathbb{R}^n} |\nabla_y w_u|^2 w_u^{-\frac{m+2n}{2}}\mathrm{d}y\int_{\mathbb{R}^m}[1+|\zeta|^2]^{-(m+n)}\mathrm{d}\zeta + \int_{\mathbb{R}^n} 4w_u^{-\frac{m+2n-2}{2}}\mathrm{d}y\int_{\mathbb{R}^m}|\zeta|^2[1+|\zeta|^2]^{-(m+n)}\mathrm{d}\zeta\\
&\quad \text{which by (3.13)}\\
&\quad = \left(\frac{4}{m+2n-4}\right)\int_{\mathbb{R}^n}|\nabla_y u|^2\mathrm{d}y\int_{\mathbb{R}^m}[1+|\zeta|^2]^{-(m+n)}\mathrm{d}\zeta + 4\int_{\mathbb{R}^n} u^{\frac{2(m+2n-2)}{m+2n-4}}\mathrm{d}y\int_{\mathbb{R}^m}|\zeta|^2[1+|\zeta|^2]^{-(m+n)}\mathrm{d}\zeta
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
\|\varphi_u\|_{2^*}^2 &= \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}[w_u(y)+|z|^2]^{-(m+n)}\mathrm{d}z\mathrm{d}y\right)^{2/2^*}, \text{ taking } \zeta = w_u^{1/2}z\\
&= \left(\int_{\mathbb{R}^n} w_u^{-(m+2n)/2}\mathrm{d}y\int_{\mathbb{R}^m}[1+|\zeta|^2]^{-(m+n)}\mathrm{d}\zeta\right)^{2/2^*}, \text{ which by (3.13)}\\
&= \left(\int_{\mathbb{R}^n} u^{\frac{2(m+2n)}{m+2n-4}}\mathrm{d}y\int_{\mathbb{R}^m}[1+|\zeta|^2]^{-(m+n)}\mathrm{d}\zeta\right)^{2/2^*} .
\end{aligned} \tag{3.17}$$

We will use (3.16) and (3.17) to derive the GN deficit with $\left(\|\nabla u\|_2^{\mu}\|u\|_{s+1}^{1-\mu}\right)^{4s/2^*}$ coming from (3.16) and (3.10), and $\|u\|_{2s}^{4s/2^*}$ coming from (3.17).

Combining (3.16) and (3.17), we have that

$$C_{m,n}^2\|\nabla\varphi_u\|_2^2 - \|\varphi_u\|_{2^*}^2 = \tilde{C}_1(\tilde{C}_2\|\nabla u\|_2^2 + \tilde{C}_3\|u\|_{s+1}^{s+1} - \|u\|_{2s}^{4s/2^*})\,, \tag{3.18}$$

where

$$\begin{aligned}\tilde{C}_1 &= \left(\int_{\mathbb{R}^m}[1+|\zeta|^2]^{-(m+n)}\right)^{2/2^*}\\ \tilde{C}_2 &= C_{m,n}^2\gamma^2\left(\frac{4}{m+2n-4}\right)^2\tilde{C}_1^{2^*/2-1}\\ \tilde{C}_3 &= 4C_{m,n}^2\gamma^2\left(\frac{4}{m+2n-4}\right)^2\tilde{C}_1^{-1}\int_{\mathbb{R}^m}|\zeta|^2[1+|\zeta|^2]^{-(m+n)}\mathrm{d}\zeta\,.\end{aligned}$$

If we take $u = v$, then $\varphi_u = F_{\hat{k}^{-1},0}$, and (3.18) gives

$$\begin{aligned}\tilde{C}_1(\tilde{C}_2\|\nabla v\|_2^2 + \tilde{C}_3\|v\|_{s+1}^{s+1} - \|v\|_{2s}^{4s/2^*}) &= C_{m,n}^2\|F_{\hat{k}^{-1},0}\|_{\dot{H}^1}^2 - \|F_{\hat{k}^{-1},0}\|_{2^*}^2\\ &= 0\\ &= \tilde{C}_1(A_{n,s}^{4s/2^*}\|\nabla v\|_2^{\mu 4s/2^*}\|v\|_{s+1}^{(1-\mu)4s/2^*} - \|v\|_{2s}^{4s/2^*})\end{aligned}$$

Thus,

$$\tilde{C}_2\|\nabla v\|_2^2 + \tilde{C}_3\|v\|_{s+1}^{s+1} = A_{n,s}^{4s/2^*}\|\nabla v\|_2^{\mu 4s/2^*}\|v\|_{s+1}^{(1-\mu)4s/2^*} - \|v\|_{2s}^{4s/2^*}\,. \tag{3.19}$$

We claim that (3.19) holds if we replace v with u, provided u satisfies (3.10). We verify this claim by observing that (3.19) is equivalent to

$$\tilde{C}_2 + \tilde{C}_3\frac{\|v\|_{s+1}^{s+1}}{\|\nabla v\|_2^2} = A_{n,s}^{4s/2^*}\frac{(\|v\|_{s+1}^{s+1})^{(1-\mu)4s/2^*(s+1)}}{(\|\nabla v\|_2^2)^{1-\mu 2s/2^*}}\,, \tag{3.20}$$

and that

$$\frac{(1-\mu)4s}{2^*(s+1)} = 1 - \frac{\mu 2s}{2^*} = \frac{m}{m+n}\,, \tag{3.21}$$

which can be verified directly by writing μ, s and 2^* in terms of their formulas (3.1), (3.11), and (2.11). (3.21) implies that the exponents in the numerator and denominator of the right hand side of (3.20) are the same. Hence, if u obeys (3.10), then (3.20) holds if we replace v with u. This verifies our claim, because (3.19) is equivalent to (3.20).

The results of the last paragraph allow us to conclude that if u satisfies (3.10), then

$$\begin{aligned}C_{m,n}^2\|\nabla\varphi_u\|_2^2 - \|\varphi_u\|_{2^*}^2 =& \tilde{C}_1(\tilde{C}_2\|\nabla u\|_2^2 + \tilde{C}_3\|u\|_{s+1}^{s+1} - \|u\|_{2s}^{4s/2^*})\\ =&\tilde{C}_1(A_{n,s}^{4s/2^*}\|\nabla u\|_2^{\mu 4s/2^*}\|u\|_{s+1}^{(1-\mu)4s/2^*} - \|u\|_{2s}^{4s/2^*})\,,\end{aligned}$$

which is equivalent to (3.14). □

Having proved Proposition 3.3.2, we apply the Bianchi-Egnell Stability Estimate and the Sobolev Inequality to help deduce Theorem 3.1.1 for s as per (3.11) and $m \in \mathbb{N}$. To be more precise, by Proposition 3.3.2, we have for all nonnegative $u \in \dot{H}^1(\mathbb{R}^n)$ obeying (3.10) and φ_u given by (3.12)

$$\begin{aligned}\delta_{GN}[u] =&\tilde{C}_1^{-1}\left(C_{m,n}^2\|\nabla\varphi_u\|_2^2 - \|\varphi_u\|_{2^*}^2\right)\,,\ \text{which by the Bianchi-Egnell Stability Estimate}\\ \geq& C\inf_{c\in\mathbb{R},t>0,x_0\in\mathbb{R}^{m+n}}\|\nabla(\varphi_u - cF_{t,x_0})\|_2\,,\ \text{which by the Sobolev Inequality}\\ \geq& C'\inf_{c\in\mathbb{R},t>0,x_0\in\mathbb{R}^{m+n}}\|\varphi_u - cF_{t,x_0}\|_{2^*}\,,\end{aligned}\tag{3.22}$$

for some $C, C' > 0$. The remainder of the argument is to show that if $\delta_{GN}[u] \leq \delta_1$ for some appropriately small $\delta_1 > 0$ and for some $y_0 \in \mathbb{R}^n$, then

$$\begin{aligned}K_1\|\varphi_u - cF_{t,x_0}\|_{2^*} \geq& \|u^{2s} - v^{2s}(\cdot - y_0)\|_1\,,\ \text{which for some appropriate } \lambda > 0\\ =& \|u^{2s} - \lambda^n v^{2s}(\lambda\cdot - y_0)\|_1\,,\end{aligned}\tag{3.23}$$

which proves Theorem 3.1.1 for s as per (3.11) and $m \in \mathbb{N}$. We skip the details used to deduce (3.23) in favor of concentrating on the arguments for deducing Proposition 3.3.2 and (3.22), because these are the arguments we need to understand in order to understand the necessity of deriving the continuous dimension extension of the Bianchi-Egnell Stability Estimate in proving Theorem 3.1.1. Having set the stage, we explain this necessity in the next subsection.

3.4 Application of Continuous Dimension Extensions of the Sobolev Inequality and the Bianchi-Egnell Stability Estimate in Proving Theorem 3.1.1

In this subsection, we outline the proof of Theorem 3.1.1. This outline will generally parallel the outline of the proof outlined in the previous subsection - the differences in the outlines should hopefully clarify the necessity and application of Bakry, Gentil, and Ledoux's extension of the Sobolev Inequality and the associated stability estimate, Theorem 2.1.5.

Like the proof in the previous subsection, the key to the proof of Theorem 3.1.1 is

PROPOSITION 3.4.1. *Proposition 3.3.2 holds with the following changes:*

1. *We introduce continuous dimensions by taking*

$$m > 0\,,$$

2. *$\varphi_u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is given by*

$$\varphi_u(y,\rho) = [w_u(y) + \rho^2]^{-\frac{m+n-2}{2}}\,, \tag{3.24}$$

3. *and*

$$\tilde{C}_1 = \left(\int_{\mathbb{R}_+} [1+\theta^2]^{-(m+n)} \omega_m \theta^{m-1} \mathrm{d}\theta\right)^{2/2^*}\,.$$

Remark 3.4.2. Note that the above is a continuous dimension generalization of Proposition 3.3.2, in the sense that if m is an integer, $\varphi_u(y,\rho)$ in Proposition 3.4.1 is simply the representative of $\varphi_u(y,z)$ with z written in radial coordinates. On the same note, $\tilde{C}_1$ in Proposition 3.4.1 is the same as $\tilde{C}_1$ given by (3.15), indeed, the integrand in the formula for $\tilde{C}_1$ in Proposition 3.4.1 is simply the radial version of the integrand in (3.15). Moreover, the equality given by (3.14) in the continuous dimension setting, $m > 0$, equates the difference in terms of Bakry, Gentil, and Ledoux's extension of the Sobolev Inequality to continuous dimensions to the GN deficit. This is the first place in the proof of Theorem 3.1.1 in which the use of continuous dimension functional inequalities occurs.

Having proved Proposition 3.4.1, we apply the extension to continuous dimensions of the Bianchi-Egnell Stability Estimate, Theorem 2.1.5, and Bakry, Gentil, and Ledoux's continuous dimension extension of the Sobolev Inequality to help deduce Theorem 3.1.1. To be more precise, by Proposition 3.4.1, we have for all nonnegative $u \in \dot{H}^1(\mathbb{R}^n)$

obeying (3.10) and φ_u given by (3.24)

$$\begin{aligned}
\delta_{GN}[u] =& \tilde{C}_1^{-1}\left(C_{m,n}^2\|\nabla\varphi_u\|_2^2 - \|\varphi_u\|_{2^*}^2\right)\,,\ \text{which by Theorem 2.1.5}\\
\geq& C \inf_{c\in\mathbb{R},t>0,x_0\in\mathbb{R}^n}\|\nabla(\varphi_u - cF_{t,x_0})\|_2\,,\ \text{which by Bakry, Gentil, and Ledoux's Theorem}\\
\geq& C' \inf_{c\in\mathbb{R},t>0,x_0\in\mathbb{R}^n}\|\varphi_u - cF_{t,x_0}\|_{2^*}\,,
\end{aligned}$$

for some $C, C' > 0$. Just like in the proof in the previous subsection, the remainder of the argument is to show that if $\delta_{GN}[u] \leq \delta_1$ for some appropriately small $\delta_1 > 0$ and for some $y_0 \in \mathbb{R}^n$, then

$$\begin{aligned}
K_1\|\varphi_u - cF_{t,x_0}\|_{2^*} &\geq \|u^{2s} - v^{2s}(\cdot - y_0)\|_1\,,\ \text{which for some appropriate } \lambda > 0\\
&= \|u^{2s} - \lambda^n v^{2s}(\lambda\cdot - y_0)\|_1\,,
\end{aligned}$$

which proves Theorem 3.1.1. This last part of the proof is independent of Bakry, Gentil, and Ledoux's extension of the Sobolev Inequality and Theorem 2.1.5.

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