TEACHER LEARNING ABOUT STUDENT MATHEMATICAL REASONING IN A TECHNOLOLGY ENHANCED, COLLABORATIVE COURSE ENVIRONMENT

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ABSTRACT OF THE DISSERTATION

Teacher Learning About Student Mathematical Reasoning in a Technology Enhanced, Collaborative Course Environment

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In recent years knowledge required for effective mathematics teaching has become more defined. A key area is the importance of teachers' attending to student reasoning (e.g., Hiebert et al., 1997). There is increasing evidence that students are capable of constructing "proof-like" forms of reasoning to justify their solutions to tasks (e.g., Maher & Martino, 1996). These and other findings have been influential in shaping national policy by developing Standards for Practice, behaviors that students engage in while doing mathematics, and are best taught in the context of meaningful mathematical activity, including collaboration and discourse (Carpenter et al., 1989). Consequently, there is need for teachers to become aware of the importance of these practices and ways of attending to students mathematical reasoning.

There is extensive work documenting that there is much to be gained by teachers studying episodes of children's learning (e.g., Fennema, et al., 1996). There is also a substantial body of research in mathematics education and the learning sciences suggesting that creating opportunities for people to engage in generative and constructive ways with video has potential to support teacher learning. Research studies in teacher education suggest that both pre and in-service teachers can learn to recognize student reasoning by engaging in collaborative problem solving and then studying videos of children working on the same task (Maher, 2011).

This study describes a course model, designed to examine teacher shifts in knowledge related to recognizing children's mathematical reasoning. The design-based research was carried out over five years in a required course for mathematics education graduate students. The course addresses a review and study of literature in mathematics education research and practice, with special attention to collaborative problem solving, student learning, and emphasis on building knowledge within a designed setting. The results showed both pre (PST) and in-service teachers (IST) grew in their ability to recognize children's reasoning. Analysis of course data provided insight into how teachers developed knowledge about student reasoning and how beliefs shifted in the process. Differences were identified between how PST and IST's situated their experience. Implications of the study include recommendations for PST courses and PD programs.

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Chapter 1: Introduction

1.1 Background of the Problem

In recent years the knowledge required for effective mathematics teaching has become more defined. A key area has been the importance of teachers' attending to student reasoning (Hiebert et al., 1997; Yackel and Hanna, 2003). There is increasing evidence that students are capable of constructing "proof-like" forms of reasoning to justify their solutions to problems (Maher & Martino, 1996; Maher, et al, 2010; Mueller et al, 2012; Yankelewitz et al., 2010), These and other findings have been influential in shaping national policy by developing a set of Standards for Practice (NCTM 2000, NGACBP & CCSO, 2010). The mathematical practices are the behaviors that students engage in while doing mathematics, and thus are best taught in the context of meaningful mathematical activity, including collaboration and discourse (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). Consequently, there is a need for teachers to become aware of the importance of these practices and ways of attending to students mathematical reasoning in their classrooms.

There is extensive work documenting that there is much to be gained by teachers studying episodes of children's learning (e.g., Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996). There is also a substantial body of research in mathematics education and the learning sciences suggesting that creating opportunities for people to engage in generative and constructive ways with video has potential to support teacher learning. Video has become an important resource in teacher preparation and professional development programs as it provides opportunities for teachers to: (1) gain insight into students' learning and ways of reasoning (Zhang, Lundeberg, Koehler, & Eberhardt, 2011; Maher et al., 2010); (2) understand classroom practice (Borko, Jacobs, Eiteljorg, & Pittman, 2008); and (3) engage in analysis and discussion of student learning (Miller & Zhou, 2007). Research studies in teacher education suggest that both pre and in-service teachers can learn to recognize student reasoning by engaging in collaborative problem solving and then studying videos of children's reasoning when working on the same or similar tasks (Maher, 2011).

Critical thinking and reasoning involves making judgments about the reasonableness of arguments. This includes questioning the assumptions as well as the connections that are made in a justification. As indicated in the Common Core State Standards Initiative for Mathematics, a hallmark of mathematical understanding is the ability to justify the correctness of an argument or the meaning behind a rule or procedure (NGACBP & CCSO, 2010). There remains a challenge to make pre and in-service teachers aware of the opportunities to create classroom situations that promote mathematical reasoning for their own students and to understand how that awareness develops. This requires both the knowledge of the variety of forms of reasoning that arise in students' justifications and the judgment to recognize the validity of an argument that may not appear in standard form (McCrory et al., 2012).

1.2 Robert B. Davis Institute for Learning (RBDIL) Video Collection

1.2.1 History of Data Collected Through Longitudinal and Other Studies

The Robert B. Davis Institute for Learning (RBDIL) at Rutgers University houses a unique longitudinal video data set that traces students learning mathematics over extended periods of time. A majority of the videos are situated in classroom or informal settings where learning conditions promote the development of reasoning, problem solving, and justification (Davis, Maher, & Martino, 1992; Maher & Martino, 1996; Maher, 2009; Maher, Powell, & Uptegrove, 2010). The keystone of the video collection is the Rutgers-Kenilworth longitudinal study, which traced development of mathematical thinking and reasoning of a cohort of students from their early elementary grades through secondary schooling and beyond. A cross-sectional phase from an elementary school in Colts Neck also included a yearlong study of critical thinking and reasoning in the fractions strand prior to formal instruction. The strands of tasks developed in those two studies were used to engage urban middle-school students with problem solving in an after-school setting in the Informal Mathematics Learning (IML) project. Another goal of IML was to engage teachers as researchers such that they could observe students' problem solving and attend to students' mathematics reasoning as they justified solutions to tasks (Francisco & Maher, 2011; Maher, Mueller, & Palius, 2010; Mueller & Maher, 2010).

Over the past twenty-five years, the longitudinal and cross-sectional research studies produced more than 4,500 hours of video data, over 40 doctoral dissertations, numerous conference and journal publications, and a methodology for video data analysis (Powell, Francisco, and Maher, 2003). Videos from the RBDIL collection have also been used in professional development, teacher preparation, a six-part teacher professional development workshop through the Annenberg channel called Private Universe Project in Mathematics, a one-hour documentary entitled Surprises in Mind, and in design research on teacher learning about students' mathematical reasoning (Maher, 2011; Maher et al., 2014).

1.2.2 The Video Mosaic Collaborative (VMC)

The VMC, a NSF-funded (Award #DRL-0822204) online video repository, currently stores approximately 400 video clips and also raw videos from the RBDIL video collection showing student reasoning across a variety of content strands such as counting, combinatorics, algebra, and fractions. The videos come from diverse population of students from urban, working class, and suburban communities and span from elementary through secondary. The repository is available for public use at http://www.videomosaic.org. Videos and related metadata in the collection illustrate: (1) the development process of sense making in mathematics as students engage in tasks from different mathematical strands; and (2) the conditions of the learning environment that promote growth in students' reasoning and understanding of mathematical content.

Clips ingested into the VMC are prepared with extensive metadata, which include new ontologies for representations, strands of mathematical problems, and mathematical tools (e.g., calculator). The students present in the video can be traced according to content strands, grade levels, and as individuals. One can also observe the strategies, heuristics, and forms of reasoning that students use to build and justify solutions to problems. The metadata appear as hyperlinks, which provide an alternate means to search for resources that share certain attributes. One benefit of this searching capacity is the ability to find all the videos of particular students, who can be followed from elementary through secondary school as they work on various problem-solving tasks.

1.2.3 The RUAnalytic Tool

The RUAnalytic tool (http://rucore.libraries.rutgers.edu/analytic) allows a user to select portions of video to define events, annotate each segment, and link them together as a multimedia narrative supported by other resources (Figure 1.1). For example, the tool enables the user to annotate video segments selected from existing clips on the VMC and assemble them for a specific reason, such as the development of a lesson plan; the creation of a tool for professional development that traces students' learning and illustrates their reasoning; or to identify examples of the variety of representations and strategies exhibited by students in their problem solving. Once published, the analytics can be shared, and further analyzed. An example VMCAnalytic by Muteb Alqahtani can be found at http://hdl.rutgers.edu/1782.1/Analytic.an.i.41 and is shown in Figure 1.1. In this VMCAnalytic, Alqahtani studies a group of students as they build an understanding of Pascal's Identity during the Night Session, an after-school, evening problem solving dealing with the Addition Rule of Pascal's Identity.

(http://hdl.rutgers.edu/1782.1/rucore00000001201.Video.000065130). The purpose is to show how the students make use of their previous knowledge from problem solving of the Towers and Pizza problems (Appendix B) to build the concept of Pascal's Identity. We see the students make use of a variety of representations and finally apply the formal, symbolic notation to represent their understanding. The video appears on the top with the VMCAnalytic authors' text on the right of it. Below the video there is a timeline of events that play one after another. When one event is finished playing, the next one event automatically plays and the text to the right is updated. The overall description of the analytic is always visible beneath the timeline. A complete list of VMCAnalytics that are currently published can be found on the VMC at: http://www.rbdil.org/analytics.

As has been demonstrated by researchers in the learning sciences, collaborative design with computer tools can foster productive collaborative learning processes (Hmelo, et al., 2001; Kafai, Ching, & Marshall, 1997; Kolodner, et al., 2003; Zahn, Pea, Hesse, & Rosen, 2010). For example, Zahn et al. (2010) demonstrated that new ideas are generated through critical reflection as learners have opportunity to consider their ideas both within and across video clips that they select. In the process of selecting and annotating video clips, learners may initiate discussions about what is important as well as what choices might enable comparisons among different segments, or even different videos. Because of the generative activities required in the video editing tasks, learners can develop initial ideas for comments and then have opportunities, both individually and collectively, to reflect on the ideas proposed and the video clips selected relative to their task goals. Assessment of a student's VMCAnalytic can provide instructors with insight into a student's learning and whether the intended learning outcomes were met. Technology provides a high potential for assessing the process of learning. It also provides an opportunity for monitoring growth in learning (Pellegrino & Quellmalz, 2010). The VMCAnalytic as an assessment tool has great promise to achieve monitoring student learning and measuring learning outcomes.

Figure 1.1: Pascal's Identity, a VMCAnalytic created by Muteb Alqahtani

1.3 Research Questions and Contributions to the Field

The purpose of this study is to trace the attention to student reasoning of pre and in-service teachers enrolled in a graduate level mathematics-education course at a large northeastern University. Using a mixture of qualitative and quantitative methods, the development of teachers' growth in recognizing student reasoning from video, as they engage in a carefully designed intervention will be described. The following questions will guide the research:

1) a) How, if at all, did the participants' beliefs about teaching and learning change from pre-test to post-test and across individual belief items? b) How, if at all, did the participants grow from pre-test to post-test on a Reasoning Assessment?
2) What do pre and in-service teachers attend to when discussing videos of students working on open-ended problem solving tasks that they themselves have worked on?

3) How can a cyber-enabled video annotation tool be used as an assessment tool to gain insight into the creators' knowledge of students reasoning?

Chapter 2: Theories of Learning

Mathematical reasoning involves exploration, collaboration, and communication. The research proposed is influenced by two main learning theories: constructivist and socio-cultural theories of cognition. Following the advice of Sfard (2001), while the two learning theories may appear to be too different to use together, I believe that learning is complex enough that one may be more appropriate to explain a situation. Both learning theories have advantages and disadvantages for learning, which will be described in this section. These theories themselves are not guidelines how to teach, but they do have implications for teaching. This study is also grounded in the idea that teachers require certain levels of content and pedagogical knowledge to be able to recognize students reasoning.

2.1 Constructivism

Constructivist ideas have been known since the 1600s from the writings of Vico and Kant, though learning theories regarding a constructivist approach are sometimes attributed to Piaget from his work during the mid-1900s (Piaget 1964; 1968; 1975). Constructivism has been and still is influential as a theoretical perspective on learning for research in mathematical reasoning (Davis & Maher, 1990) and national policy (NCTM, 1989; 2000; NGACBP & CCSO, 2010). There are various interpretations of constructivism (Moll, 2000). The view for this study builds from the work of Piaget (1964). In the Piagetian view, as learners interpret new knowledge, they restructure and reorganize their knowledge when they process new knowledge (Hatano, 1996). A notable contribution to mathematics education through constructivist learning theories is the research conducted on cognitive obstacles (Confrey, 1990). Studies have shown that cognitive obstacles in mathematics are consistent across various scenarios, are expected as a part of learning, and may arise from individualized or rote instruction (Erlwanger, 1975; Schoenfeld, 1988). This research has had influence on teacher training to focus on attending to cognitive obstacles that students may encounter in learning. Having teachers an understanding of students' reasoning offers teachers an opportunity to diagnose a students' error, as well as avoid instances where cognitive obstacles appear to be masked by the correct numerical solutions that students produce (Nesher, 1987). The idea of cognitive obstacles is important for mathematical reasoning and the design of the course. By focusing on a students' reasoning process and not solely the answer, teachers can attend to diagnosing the process of student learning and work with the students in attending to their cognitive obstacles.

2.2 Socio-cultural Theories of Cognition

For some theorists, social influences are viewed as secondary, although most agree that these influences may still support and constrain an individual's learning experience (Sfard, 2001). Socio-cultural theories of cognition focus on a learner's participation in social practices, roles, and how that participation interacts with an individual's learning process. Vygotsky (1978) posited that the restructuring that occurs in constructivism is not only internal, but is also related to the social influences, such as the communication between students or the teacher and the student. For this research, a socio-cultural framework is important since the design for of the course centered on the building and sharing of ideas during in-class problem solving and online. Their online discussions were driven by attention to the variety of approaches to the problem-solving tasks as well as to the questions posed by this researcher and the course instructor about content, pedagogy, and attention to student learning from video.

2.3 Content and Pedagogical Knowledge

Recent research has focused on the type of knowledge a teacher must possess to successfully lead an inquiry-oriented mathematics classroom. Studies have shown there are different types of knowledge that influence a teacher's ability to lead these type of discussions, the main two being subject matter knowledge (Ball et. al, 2008) and pedagogical content knowledge, (Schulman, 1986). The idea of pedagogical content knowledge was first proposed by Shulman (1986) in the AERA Presidential Address as including:

> For the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations - in a word, the ways of representing and formulating the subject that makes it comprehensible to others. Since there are no single most powerful forms of representation, the teacher must have at hand a veritable armamentarium of alternative forms of representation, some of which derive from research whereas others originate in the wisdom of practice. . . Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult; the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons (page 9).

Pedagogical content knowledge hints at there being more to teaching than just understanding the content involved. One has to also be aware of all the issues students face while learning a concept as well as multiple ways to represent the same concept.

Subject matter knowledge is broken down into three subdomains; specialized content knowledge, common content knowledge, and horizon content knowledge. Specialized content knowledge (SCK) evolved from mathematical knowledge for teaching and does not deal with actually knowing the content you are teaching and instead deals with the mathematical knowledge that a teacher must have in order to participate in successful teaching. Knowledge that would fall into this domain includes understanding what a student is saying, being able to evaluate the correctness of a student's statement, and trying understanding how students reached a solution by following their logic (Hill et al., 2008). For successful guidance of inquiry-oriented classrooms, this knowledge is essential.

Since Schulman coined the idea of pedagogical content knowledge, it has evolved considerably. Recently, Ball et. al (2008) have broken pedagogical content knowledge into three sub-domains; knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum. Knowledge of content and students deals with anticipating potential difficulties that students will face and how to address them. When adding two fractions a common error that pops up is that students will add across the numerator and the denominator and achieve an incorrect answers (e. g. 1 / 3 + 2 / 4 = 3 / 7). A teacher with good knowledge of content and students' learning could anticipate this obstacle beforehand and design activities where they could determine if students hold this misconception and be prepared to to address the issue if it should arise . An example might be involving adding two fractions whose sum is close to one, such as 7 / 8 + 1 / 19. If a student has a misconception about adding numerators and denominators when adding fractions, their sum of of 8 / 27 would offer an opportunity to question the reasonableness of the answer. Knowledge of content and students' cognitive obstacles goes beyond just realizing that a student has an incorrect answer, but involves planning in advance when the obstacle is encountered and being able to diagnose and help a student realize faulty reasoning through questioning.

The final part of pedagogical content knowledge is an individual's knowledge of content and curriculum and how that knowledge is used to decide whtn and how to incorporate mathematical concepts into the lessons. This sub-domain deals with understanding how to best use their own resources as well as those provided by their district, as well as identifying and avoiding the negatives of their implementation.

Chapter 3: Literature Review

3.1 Mathematical Reasoning and Justification

Research on the knowledge for teaching mathematics identify that knowledge of students' mathematical reasoning is essential (Ball, 2003) and so much so, that there is a relationship between following how a student builds his/her knowledge and the student's performance in mathematics (Rowan et al., 1997). An important early finding in research into mathematical reasoning is that in a natural way, children – even young children – show evidence of understanding the idea of mathematical proof (Maher & Martino, 1998; 1996; Maher & Davis, 1995). In justifying their solutions to problems, children provide convincing arguments that take the form of proof by cases, induction, contradiction, and upper and lower bound. Their justifications are driven by an effort to make sense of the problem situation, notice patterns, and pose theories (Mueller, Yankelewitz & Maher 2011; Maher & Martino, 2000). Their solutions are refined through discussions as they negotiate meaning with classmates and structure their investigations (Weber, Maher, Powell & Lee, 2008; Maher, 2005). Students' ability to provide convincing mathematical justifications can help them to understand mathematical proof as a resource to validate mathematical statements (Yackel and Hanna, 2003).

Research conducted in classrooms and informal after-school settings – in urban, suburban and working class environments – show that middle-school aged children rely on their sense making and reasoning to provide convincing arguments (Mueller & Maher, 2010a; 2010b; 2009; Mueller, Yankelewitz & Maher, 2011). Detailed development of students' proof making in solving strands of combinatorics tasks, from the early years through high school, is described in Maher, Powell & Uptegrove, 2010. Some of the tasks in a strand in fractions have been shown to elicit certain forms of reasoning (Yankelewitz, Mueller & Maher, 2010). Across all ages and contexts, formal and informal, certain tasks tend to elicit certain forms of reasoning when students are required to provide a justification for their solutions (Yankelewitz, Mueller & Maher, 2010; Francisco & Maher, 2005).

3.2 Video in Teacher Preparation and Professional Development

Research on video use in mathematics education has demonstrated that there is much to be gained by studying video episodes of children's learning (Borko, Koellner, & Jacobs, 2010; Cobb, Wood & Yackel, 1990; Fennema, Carpenter, Franke, Levi, Jacobs

& Empson, 1996; Jacobs, Borko, & Koellner, 2009; Maher, Landis & Palius, 2010; Maher, Palius & Mueller, 2010; Maher, 2011; Tirosh, 2000). Video recordings allow us to study the subtleties of student behavior. Careful study of videos enables us to trace student cognitive growth in a social setting, and gain insight into how social processes influence personal cognitive development. Videos also have served as a powerful tool for tracing the students' development of mathematical ideas over time (Davis, Maher & Martino, 1992). Videos from longitudinal studies are particularly valuable for research on cognition because they make it possible to follow the same students for several years, learning, in detail, different mathematical content (Maher, 2005; Maher, Powell & Uptegrove, 2010; Schoenfeld, Smith & Arcavi, 1993). Along with the benefits of studying video to trace student learning is the potential of video to impact teacher learning and classroom practices through its use in pre- and in-service teacher education.

Researchers in mathematics teacher education have utilized a range of different pedagogical approaches to using video for learning, which have included lesson study, video clubs, and problem-solving cycles (e.g., Alston, Basu, Morris & Pedrick, 2011; Borko et al., 2008; Maher, Landis & Palius, 2010; Sherin & Han, 2004; Van Es, 2009). Studies have shown that teachers' use of video for learning about their own classroom practices can be insightful and motivating to improve instruction (Sherin & Han, 2004). Studying videos of one's practice allow teachers to think about contextual knowledge related to their own teaching (Goldman, 2007). For example, teachers tend to identify events that are important to them, link those events to their prior knowledge, and use them to evaluate a situation (van Es and Sherin, 2008). In research with pre-service teachers, studying videos provided learning opportunities to see certain teaching practices in action when they were not necessarily visible based on certain field placements (Philip et al., 2007). Further, it is key that selected video episodes align well with the instructional goals of the teacher education context (Seidel, Stürmer, Blomberg, Kobarg & Schwindt, 2011; Zhang, Lundeberg, Koehler & Eberhardt, 2011). Design research studies in teacher education suggest that both pre and in-service teachers can learn to recognize student reasoning by studying videos of children engaged in justifying their solutions to problems (Maher et al, 2014; Maher, 2011). Studies using video-based interventions in professional development have shown accompanying growth in teachers' recognition of student reasoning was significant change in teacher beliefs about student learning (Maher, Palius & Mueller, 2010; Maher, Landis & Palius, 2010). Video clubs have been useful in getting viewers over time to focus on student mathematical thinking and to build trust to promote honest self-reflection (Sherin and van Es, 2009).

Chapter 4: Study Setting and Intervention

4.1 Description of the Course

The setting for this study was five iterations of a fifteen week graduate-level introductory mathematics education course at a large Northeastern university (Appendix A) offered in the fall semester and co-taught by myself and my adviser. The enrolled student population is diverse with a mix of pre and in-service teachers that having varying experience from novice to veteran. It is offered at least once a year, is a required course for master's and doctoral students in mathematics education degree programs, and the enrollment is typically 15-20 students. The frequency of the course offering makes possible design revisions during the study. Due to the timing of the course in the fall semester, the pre-service teachers' did not have any type of field experience in a mathematics classroom.

The course was designed to introduce pre and in-service teachers to the field of mathematics education through a variety of activities that blend in-person, on-campus sessions with interactions done asynchronously online through eCollege, a course management system. The on-campus activities had the teachers working in small groups on mathematical problem-solving tasks, with consideration of how K-12 students might engage with those tasks as they build solutions to tasks. The online course work included reading assignments that introduced participants to theoretical perspectives of learning and research in mathematics education, with guidelines for engaging in reflection and discussion of those readings and consideration of their relevance to teaching practices. Other online course work included studying video clips of children engaged in math problem solving and talking about their mathematical ideas. Through reflection and online discussion the videos will be connected to the readings and hands-on problem solving. The emphasis of the course was on the mathematics, children's learning, and conditions of the learning environment. As a final course project, the teachers worked with video in a variety of contexts. After each iteration of the course I met with the other instructor to discuss changing we would make in the following iteration to improve the course.

4.2 Course Participants

Throughout the five iterations 86 teachers participated in the intervention. The first iteration had 21 teachers, 16 in the second, 14 in the third, 17 in the fourth, and 18 in the fifth. Tables 4.1 and 4.2 break down the teachers by year, program type, and whether they are a pre or in-service teacher.

Table 4.1

Breakdown of teachers by program type.

Iteration	EdM	EdD	PhD	Post-Bac	Total
1	8	3	2	8	21
2	10	2	4	0	16
3	6	4	2	2	14
4	11	1	3	2	17
5	13	1	1	3	18
Total	48	11	12	15	86
Percent	55.81%	12.79%	13.95%	17.44%	100%

Table 4.2

Breakdown of pre-service and in-service teachers by each iteration.

Iteration	Pre-Service	In-Service	Total
1	10	11	21
2	4	12	16
3	6	8	14
4	5	12	17
5	5	13	18
Total	30	56	86
Percent	34.88%	65.12%	100%

4.3 Data Collected

Two assessments, a Belief and a Reasoning Assessment, were administered to all teachers in the class; one before the class began and one at the end. The pre assessments took place before any material was taught in the class. If a teacher did not complete the assessment prior to the first class, they would take the assessment at the beginning while the rest of the class introduced themselves and talked about what they taught. The post assessments were taken during the last week of the course after all the interventions were finished. Both assessments were taken electronically through Sakai, a content management system. Analysis of the data of both assessments are discussed in Chapter 5. The teachers also participated in problem solving in-class, discussions online (Prompts in Appendix H), evaluated student work (Appendix D), completed a final project for the course (Appendix J), and wrote a reflection paper (Appendix I). All the material the teachers produced for the course is also part of the analysis.

4.3.1 Problem solving.

In-class the teachers worked on a set of cognitively challenging mathematical tasks that are open-ended in the sense that there are multiple points of entry in terms of where a learner might begin in working towards a solution, as well as having multiple directions in which the learner might go with strategies and heuristics. The tasks have been tested in a variety of authentic learning environments with learners across a broad range of ages, through longitudinal and cross-sectional research with K-12 students. Synthesizing across these contexts, research has shown that particular tasks tend to elicit certain forms of mathematical reasoning (Yankelewitz, Mueller, & Maher, 2010).

There were ten tasks that were worked on in every iteration are found in Table 4.3 and the wording of the tasks can be found in Appendix B. In the fourth iteration, the teachers worked on a probability task that was related to the Towers task called Guess My Tower. In the fifth iteration, this was replaced with a different task that focused on the law of large numbers entitled Schoolopoly.

Table 4.3

The problem solving tasks worked on during each iteration. Full statement of the tasks can be found in Appendix B.

Problem Solving Task	It. 1	It. 2	It. 3	It. 4	It. 5
Shirts and Pants	x	x	x	x	x
Towers 4-Tall selecting from 2 colors	x	x	x	x	x
Towers N-Tall selecting from 2 colors	x	x	x	x	x
Guess my tower				x	

2-Topping pizzas (Halves)	x	x	x	x	x
4-Topping pizzas (Halves)	x	x	x	x	x
4-Topping pizzas (Whole)	x	x	x	x	x
Ankur’s challenge	x	x	x	x	x
Taxicab	x	x	x	x	x
World series problem	x	x	x	x	x
Problem of points	x	x	x	x	x
Schoolopoly				x	

4.3.2 Belief assessment.

The Belief assessment (Appendix E) contains thirty-four statements about learning and teaching mathematics. The assessment includes items that address what mathematical ideas teachers believe students are capable of doing, what teacher actions in the mathematics classroom evoke learning, and how mathematics should be taught for understanding. Each statement was graded by the teachers on a five-point Likert scale, ranging from strongly disagree to strongly agree. For analysis, the scale was collapsed combining strongly disagree with disagree and strongly agree with agree. An example assessment item reads “Only the most talented students can learn math with understanding” (Question 29).

4.3.3 Reasoning assessment.

In the Reasoning assessment, the teachers watched an edited video of 4th grade students discussing their solutions to the Towers task (Appendix F). In the beginning of the video, students are in a classroom working, in pairs, on solving how many unique towers they could build 5-tall selecting from 2-colors. After several minutes, the video shifts to an interview with four students who are discussing their earlier solutions to towers of varying heights while selecting from two colors. Throughout the video the students make various arguments to support their solution such as: an inductive argument, building towers by building one and then building its opposite (where if the two colors are maroon and yellow, maroon would be the opposite of yellow), using numerical reasoning patterns (e.g., doubling, additive), an argument by contradiction, and two forms of a case argument. The teachers received a prompt that asked them to write an open-ended response and describe: (1) each example of reasoning that a child in the video puts forth, (2) whether or not the reasoning forms a valid argument, (3) whether or not the argument is convincing, and (4) why or why not they find the argument convincing.

4.3.4 Student work modules.

Several of the cycles contained modules with pieces of students' solution from the task they worked on. The student work was chosen to illustrate the variety of representations and arguments produced by the students. The teachers were asked to review the students' representations and work and specifically address: (1) the correctness of the solution provided, (2) description of the strategy used, (3) the validity of the reasoning, and (4) whether or not they find the solution convincing and, if so, why. If they did not find the solution convincing, they were asked to indicate from studying the student work what pedagogical moves they might take to help the student develop a convincing argument.

Table 4.4

Student work modules for each iteration.

Problem Solving Task	It. 1	It. 2	It. 3	It. 4	It. 5

Towers 4-tall selecting from two colors	x	x	x	x
Pizza problem with halves (2 and 4 toppings)	x	x	x	x
Ankur's Challenge	x	x	x	x
Taxicab			x	x
Problem of Points				x
World Series				x

4.3.5 Online discussions.

During the cycles of intervention, the teachers watched and discussed videos, discussed assigned readings, reflected on guest speakers, analyze samples of student written work, and continued the class discussion about problem solving online using eCollege. The videos and student work were stored on the VMC repository. Discussion prompts were provided and can be found in Appendix H. The teachers were randomly grouped online with 3-4 other people with four groups each semester. Occasionally, ideas from one group were posed to the entire class for discussion. The expectation was that the teachers posted an initial response to the prompts halfway through the week and then respond to at least two other people before the week was over. The discussion text was extracted from the course website as PDF files and imported into Dedoose for analysis.

4.3.6 VMCAnalytic project.

The final project for the course varied throughout the study, as the new technology became refined and more videos became available on the VMC. In the first iteration, the teachers worked in small groups with raw unedited videos of students working on counting and combinatorics tasks (Towers and Pizza, See Appendix B.2, B.3, and B.4) in an urban after-school informal mathematics project. Each group produced an overall description of what went on in the video and partitioned the video into smaller clips that highlighted students' reasoning as they worked on the tasks. The partitioned clips were presented to the class along with their reasoning for choosing the clips.

During the second iteration the groups once again work with full raw video, but in addition to partitioning the videos to highlight student reasoning, they needed to use the videos to plan a professional development workshop. The groups were given 60 minutes to deliver their workshop to the rest of the class, as well as submit a paper that outlines their workshop.

In the final three iterations of the course, the RUanalytic tool was used by the teachers to construct a VMCAnalytic to highlight a mathematical concept of interest to them. Examples topics included highlighting examples of teacher questioning, student collaboration and argumentation, illustrating examples of successful student learning, and student reasoning. The teachers were able to work alone or with a partner on their project. In the first year using the RUanalytic tool, the teachers could use any videos they wanted which are hosted on the VMC repository. In the fourth and fifth iteration, the teachers started building their VMCAnalytics using videos that were assigned to watch during the intervention cycles and then they could use other videos on the repository to complete their project. This change was done in response to a preliminary analysis that suggested VMCAnalytic creators who worked with videos they were familiar with tended to produce better VMCAnalytics based on a developed rubric (Hmelo-Silver, Maher, Palius, & Sigley, 2014).

4.3.7 Reflection paper.

At the end of the course the teachers were asked to write a 2-3 page reflection paper about their experience in the course, attending to activities such as readings, in-class problem solving, the videos they studied, and online discussions. The full reflection paper prompt is in Appendix I.

4.4 Teacher Learning about Mathematical Reasoning (TLMR) Instructional Model

For the study an instructional model, Teacher Learning about Mathematical Reasoning (TLMR), was developed. TLMR was designed for teachers to: (a) build knowledge of the various forms of mathematical reasoning the students naturally make use of in their justifying solutions to tasks by engaging in the tasks themselves, (b) attend to the development of students' mathematical reasoning from studying videos and student written work to the tasks they worked on, and (c) learn about the conditions and teacher moves the facilitate student justifications of task solutions. The teachers underwent six cycles of this over a fifteen week period with each cycle last approximately two weeks. Each cycle is explained in detail in this section.

4.4.1 Cycle 1: Introduction to counting and combinatorics.

The goal for the first cycle was to engage the teachers in open-ended problem solving where they need to form arguments and justify them to their group and the entire class. This cycle lasted two weeks with the problem solving occurring in the first week.

4.4.1.1 Cycle 1: The Shirts and Pants and Towers series of tasks.

The task the teachers engage in had them constructing towers of varying heights (Appendix B.2) selecting from 2 colors using Unifix cubes (Figure 4.1). They first had to build towers for 4-tall selecting from 2 colors, justify their solution, and then conjecture about 3-tall selecting from 2 colors. After checking their conjecture they then had to figure out for n-tall selecting from 2 colors. The common strategies that used were opposites (where if the two colors selected are blue and yellow, the blue would be "opposite" of the yellow; Figure 4.2), a case argument based on how many of each color were in the tower (Figure 4.3), and an inductive argument where they start with towers 1-tall selecting from 2 colors and show that you could either add a blue or yellow to the top of each one (Figure 4.4).

Figure 4.1: Two colors of Unifix cubes.

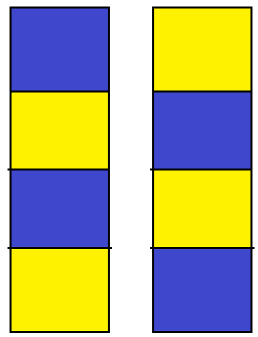


Figure 4.2: An example of "opposite" Towers where yellow is the "opposite" of blue.

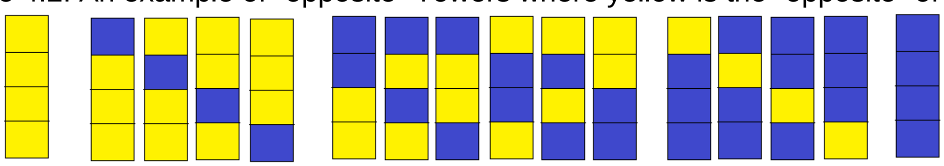


Figure 4.3: A Case Argument for Towers 4-Tall selecting from two colors. The cases are arranged into all yellow and no blue, three yellow and one blue, two yellow and two blue, one yellow and three blue, and all blue.

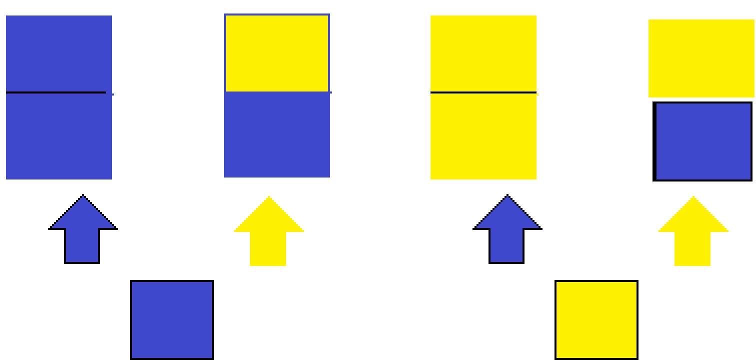


Figure 4.4: An argument by induction for why the amount of Towers double in the Towers problems selecting from two colors.

4.4.1.2 Cycle 1: The videos.

The teachers were assigned four videos of students reasoning about combinatorics tasks over three weeks. All the videos featured students working on a task, discussing their solution to their group or an interviewer, and justifying and responding to questions about their solution. During the first week, they watched a group of three students work on the Shirts and Pants (Appendix B.1) task. The students first initially worked on the task in second grade and produced two answers; 5 and 6. The incorrect answer of 5 was due to the suggestion that one outfit shouldn't be counted because it didn't match. When two of the students (Stephanie and Dana) revisited the task in third grade they quickly solved it to produce an answer of 6. In the second week they watched a series of videos of students working on the Towers task of varying heights selecting from 2 colors. In the first, Stephanie and Dana from the Shirts and Pants video, are justifying their solution to towers 4, 3, and 5 tall. The second is an interview with Stephanie where she explains her thought process as she solved the Towers task. The last clip was an interview with a student, Meredith, who solved the 4-tall selecting from 2 colors Towers task and then was asked to make a conjecture for how many towers there would be 3-tall selecting from 2 colors. She at first claims there would be the same amount, but when building them she changes her answer to 8 because when you pull the top off you will create duplicates.

Table 4.5

Videos used during Cycle 1 by each iteration.

Video Clip Title	It. 1	It. 2	It. 3	It. 4	It. 5
Shirts and Pants (PUP Math version)	x	x	x	x	x
Stephanie and Dana work on Towers	x	x	x	x	x

Stephanie towers interview	x	x	x	x	x
Meredith removes the top cube	x	x	x	x	x

4.4.1.3 Cycle 1: The readings.

Readings for the first cycle were selected to highlight the benefits of attending to students mathematical reasoning instead of just focusing on the answer. The first week had the teachers reading an article about a student named Benny who was involved in an individualized type of instruction called IPI (Erlwanger, 1973). While Benny was able to achieve 85% mastery on the content based on test scores, when the author talked to Benny the conversations revealed that he did not understand what he was doing and instead was randomly applying rules he had invented that occasionally worked. This paper was paired with an article from Skemp (1976) about the difference between relational and instrumental understanding. In later years, the teachers also read an article about Khan Academy which touted a student as brilliant for being able to complete approximately 650 inverse trigonometry problems in the fifth grade, but there was no indication that the student actually understood the material. In the second part of the cycle, the readings focused on the reasoning of the students that were portrayed in the videos they watched.

Table 4.6

Readings used during Cycle 1 by each iteration.

Paper author(s) and year	It. 1	It. 2	It. 3	It. 4	It. 5
Erlwanger (1973)	x	x	x	x	x
Skemp (1976)				x	x

Maher (2010)	x	x	x	x	x
Maher & Uptegrove (2010)	x	x	x	x	x
Maher (2009)	x	x	x	x	x
Maher & Yankelewitz (2010)	x	x	x		
Maher, Sran, & Yankelewitz (2010a)	x	x	x		
Maher, Sran, & Yankelewitz (2010b)		x	x		
Maher & Martino (1992)	x	x			
Maher & Weber (2010)	x	x			

4.4.1.4 Cycle 1: Student work module.

The teachers were presented with four pieces of student work from the Towers 4-tall selecting from two colors task. All the examples of the work had the correct solution of 16 towers, but their reasoning varied from incorrect to partially correct to fully correct. The examples were specifically chosen to see if the teachers are starting to pay attention to the reasoning provided by the students in their work instead of just looking to see if they achieved the correct answer.

4.4.1.5 Cycle 1: Online discussions.

The discussions in this cycle focused around the teacher's own problem solving in relation to that of the students in the video, discuss elements of the students in the videos problem solving, eliciting their beliefs about what is important for students to be able to do in the mathematics classroom, and to make connections among the ideas presented in the Skemp (1976) paper with what they are reading about Benny and Khan Academy.

4.4.2 Cycle 2: Making pizzas.

The goals for this cycle were to have the teachers continue engage in mathematical reasoning with open-ended problem solving tasks, have them realize the limitations offered by some argument types, focus on mathematical notation, and have them start to attend to the structure of mathematical tasks.

4.4.2.1 Cycle 2: The tasks.

During this cycle, the teachers worked on three variations of pizza task: Pizza with Halves, selecting from 2 toppings (Appendix B.4); Whole Pizzas, selecting from 4 toppings (Appendix B.3); Pizzas with Halves, selecting from 4 toppings (Appendix B.4). The whole pizza task has a similar structure to the Towers task from the previous cycle and the solution can be expressed as 2^n where n is the number of toppings. The two in the formula relates to the choice of the topping being on or off of the pizza. A common strategy used by the teachers on the whole pizza task involved creating cases (Figure 4.5) where they categorized the pizzas based on the number of toppings with 0 toppings being an all cheese pizza. The strategies used for the half pizza task varied based on the number of toppings. When working with two toppings, the teachers used various arguments based on cases to produce an answer of 10. Some chose to separate the cases into pizzas without halves and pizzas with halves, based on the number of toppings used, the total number of toppings on the pizza including cheese as a topping, and grouping them by whole, half, and mixed (Figure 4.6). Those strategies tended to be inefficient when working on the 4-toppings with halves due to there being 136 combinations. Some teachers did solve the task by making a relationship between the whole pizzas and halves by drawing lines connecting one combination to another where each end of the line represented half of the pizza (Figure 4.7).

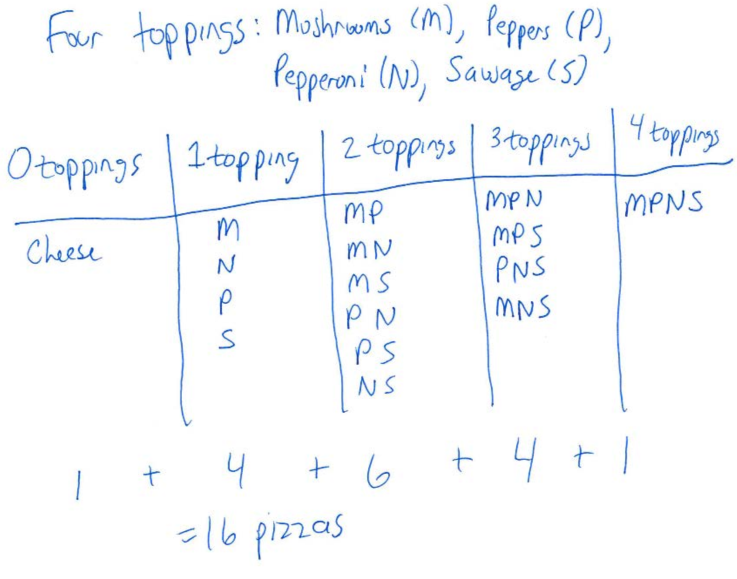


Figure 4.5: A Cases Argument for whole pizzas selecting from four toppings.

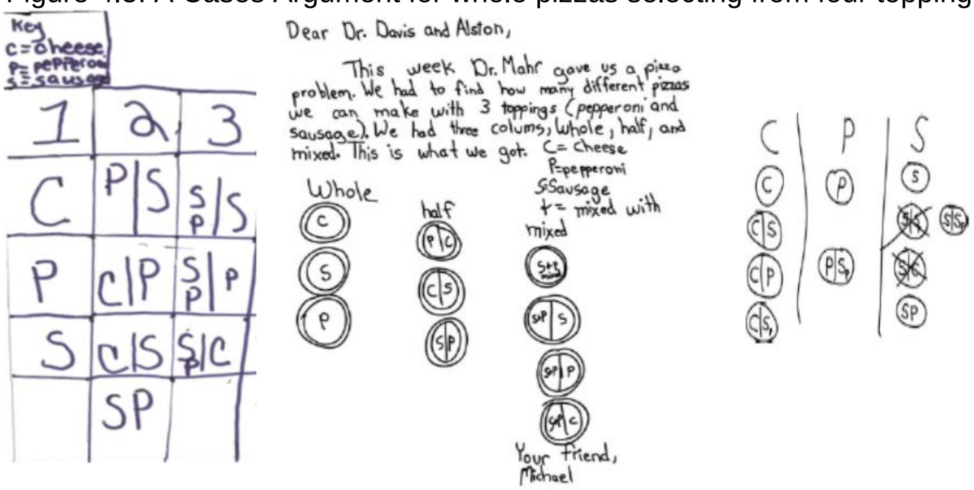


Figure 4.6: Three different case arguments for making pizzas with halves and selecting from two toppings.

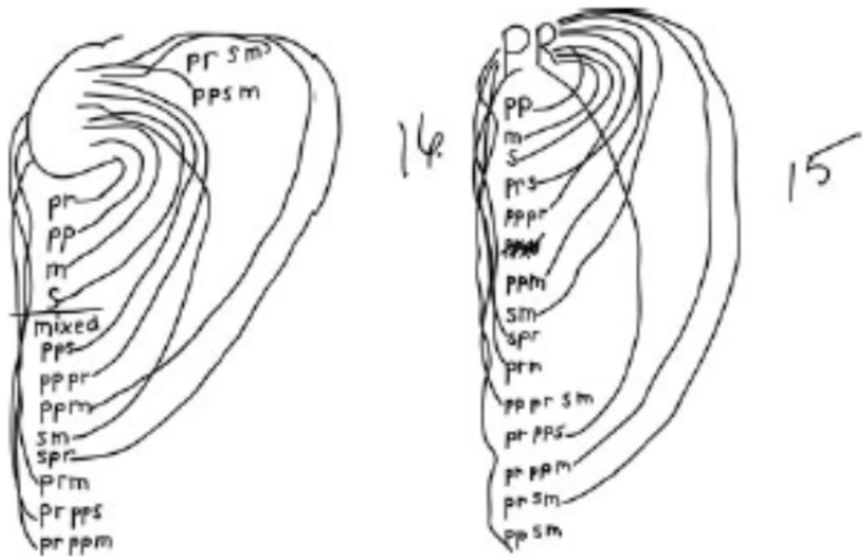


Figure 4.7: A recursive solution to the pizza with halves selecting from 4 toppings problem.

4.4.2.2 Cycle 2: The videos.

After working on the task, the teachers were then assigned to study three video clips online. The first video, http://dx.doi.org/doi:10.7282/T3HM57PQ, followed twelve fifth-grade students across two class periods as they worked on the Pizza task with Halves selecting from two toppings. The second video, http://dx.doi.org/doi:10.7282/T3HM57PQ, focused on the same students from the previous video as they worked on the Whole Pizza task with 4 toppings and the Pizza task with Halves selecting from four toppings. The third clip, http://dx.doi.org/doi:10.7282/T3VX0FRD, was a task-based interview with a fourth grade student named Brandon. It shows Brandon explaining his solution to the Whole Pizza task with four toppings. After explaining his solution, Brandon is asked whether this task reminded him of any other tasks that he had worked on and he remarked that it reminded him of the Towers task. After resolving the Towers 4-tall selecting from 2 colors task, Brandon makes a connection between the similarity in structure of the two tasks, recognizing that the two choices for a pizza topping (represented by a 1 or 0) for being on or off the pizza is similar to the two choices for the color of a particular block of the tower (e.g., red or yellow).

Table 4.7

Videos used during Cycle 2 by each iteration.

Video Clip Title	It. 1	It. 2	It. 3	It. 4	It. 5
Brandon interview (PUP Math)	x	x	x	x	x
Pizza – 2 toppings with halves (PUP Math)	x	x	x	x	x
Pizza – 4 toppings whole and with halves (PUP Math)	x	x	x	x	x

4.4.2.3 Cycle 2: The readings.

The first reading gave background to the Brandon interview that they watched (Maher and Martino, 1998). It situated the Brandon video as a part of a longer study and included details that preceded the interview as well a textual analysis of Brandon's problem solving. The second reading dealt with the topic of isomorphisms in mathematics education (Greer and Harel, 1998). This paper made reference to Brandon as an example of a nine-year old student having an insight in recognizing an isomorphism similar to the mathematician Poincare.

Table 4.8

Readings used during Cycle 2 by each iteration.

Paper author(s) and year	It. 1	It. 2	It. 3	It. 4	It. 5
Maher & Martino (1998)	x	x	x	x	x
Greer & Harel (1998)				x	x

Maher & Martino (1996)	x	x	x	x	x
Maher, Sran, & Yankelewitz (2010c)	x	x	x		

4.4.2.4 Cycle 2: Student work module.

The student work module for this cycle contained four pieces of student solutions from the Pizza with Halves task selecting from two toppings. These were chosen to illustrate the variety of representations and case arguments produced by the students.

4.4.2.5 Cycle 2: Online discussions.

For this module, the guiding questions focused on the notation that Brandon used in his problem solving. Teachers were asked to discuss how, if at all, Brandon's choice of notation was helpful to him in recognizing the relationship between the Pizza and Towers selecting from 2 colors tasks. They were also asked to discuss the forms of reasoning displayed by Brandon in the video, and the role of isomorphisms in mathematical cognition. Finally, they were asked to compare their own problem solving with that of the students in the videos.

4.4.3 Cycle 3: Extension of the Towers task.

The goal of this cycle was to re-visit the Towers task solving task from the first cycle by engaging in an extension of the task. The extension to the task was posed by a student named Ankur and one of the topics for discussion is about having students pose problems. While the task is not isomorphic to the original Towers selecting from 2 colors task, the problem solving strategies used (e.g., case argument) may be modified to fit this task. This cycle lasted two weeks.

4.4.3.1 Cycle 3: Ankur's Challenge task.

The new Towers task was named Ankur's challenge after the student who created the task and involves building towers 4-tall selecting from 3-colors with the restriction that one of each color must appear in each tower. One way of solving the task is to split the task into three cases (assuming the 3 colors are black, yellow, and blue), all the towers 4-tall with; (a) 2 blue, 1 yellow, and 1 black, (b) 2 yellow, 1 blue, and 1 black, and (c) 2 black, 1 yellow, and 1 blue. The task would be solved for one of the cases yielding an answer of 12 which can then be multiplied by 3 to obtain an answer of 36, since the number of towers in case (a) is equivalent to the number in (b) and (c) (Figure 4.8).

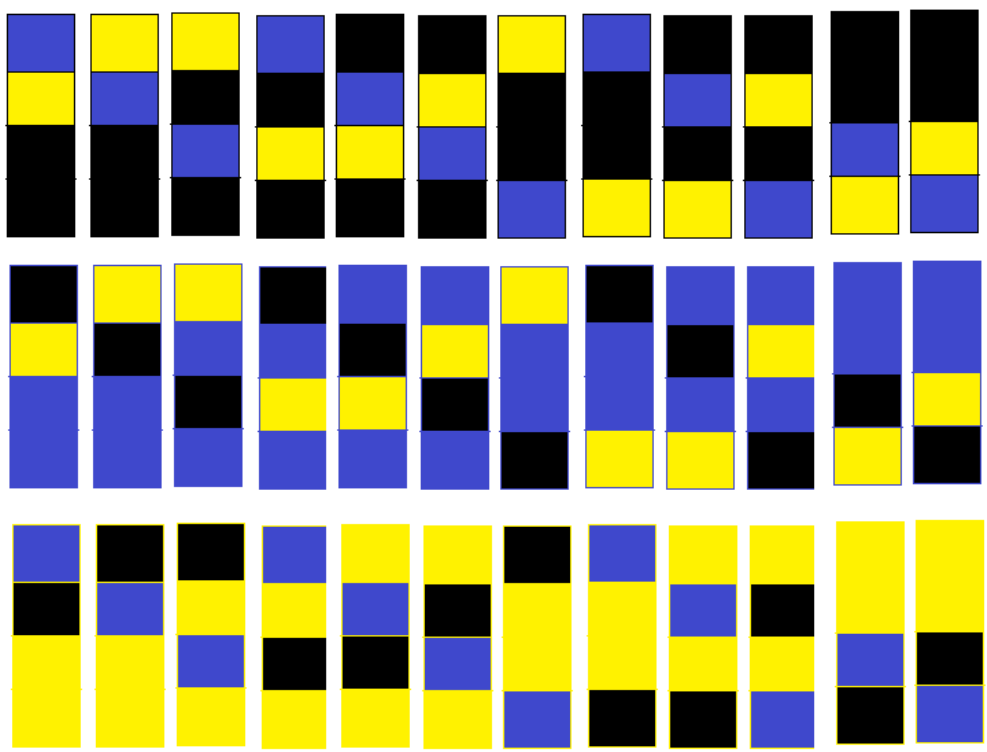


Figure 4.8: A solution to Ankur's Challenge where the towers are grouped in cases of two of one color and one of the other two colors.

4.4.3.2 Cycle 3: The videos.

There is one video in this cycle entitled Romina's Proof to Ankur's Challenge which was used in every iteration of the study. In the video, Romina is working on Ankur's Challenge with several of her classmates (including Ankur). She provides a solution where she focuses on one color, say blue, and constructs six towers containing two blue. For the other spaces in the tower, she writes "xo" vertically indicating that one of two colors can go there (Figure 4.9). She then concludes that for each of the six with two of one color you can have “an o and an x and a x and an o, so you have to multiply each of these six by two” which gives her twelve towers for two of just one color. Those twelve would then “be multiplied by three for the three different colors and you get 36”.

Figure 4.9: A screenshot from the video Romina's Proof to Ankur's Challenge where she shows her solution to the task.

4.4.3.3 Cycle 3: The readings.

The first week of readings explored the connection between mathematical reasoning and formal mathematical proof and how explaining and justifying contribute to learning mathematics. The second week has them explore deeper about what does it mean to understand something and consider how far can they actually go with working on a task like Towers.

Table 4.9

Readings used during Cycle 3 by each iteration.

Paper author(s) and year	It. 1	It. 2	It. 3	It. 4	It. 5
Maher & Muter (2010)	x	x	x	x	x
Yackel & Hanna (2003)	x	x	x	x	x
Davis (1992)	x	x	x	x	x

Speiser (2010)	x	x	x
Maher (1998)		x	

4.4.3.4 Cycle 3: Student work module.

Three pieces of student work were analyzed by the teachers of students solving the Ankur's challenge task. All three of the pieces of work take a strategy that was used to solve the Towers task from the first cycle and try to adapt them to fit the new parameters of the task. Two of the approaches are successful and the third produces an incorrect solution.

4.4.3.5 Cycle 3: Online discussions.

In addition to discussing their own problem solving, the discussion for this module had them consider the differences between mathematical reasoning and what constitutes a mathematical proof. The teachers also compared the idea of mathematical understanding presented by Davis (1992) with that of the idea proposed by Skemp (1976) and how understanding of a mathematical concept can take on multiple meanings.

4.4.4 Cycle 4: Taxicab geometry.

The Taxicab task (Appendix B.8) is another combinatorics task which is isomorphic to the Towers and Whole Pizza tasks. The task is presented differently in that one must work with a rectangular grid in considering how to move between two points. The goal of this two-week cycle is to continue the problem solving and start to watch longer un-edited videos to prepare for their final project where they will work with longer video.

4.4.4.1 Cycle 4: The task.

The statement of the task has the teachers determine the number of different shortest paths from a static Taxi Stand and three endpoints on a rectangular grid. First, the teachers need to realize that to obtain the shortest path, they can only go two directions on the grid (down and to the right). The two options for moving make the task similar to Towers selecting from 2 colors and Pizza without Halves since in all of them you have two choices (down and to the right, yellow or blue, topping on or off). Once they determine that they can only go two directions, they can use that knowledge to find the total number of shortest routes, usually by tracing the paths on the grids which ends up being harder as each endpoint is farther away from the Taxi Stand. A solution can be found by considering how many ways one can get to the point directly above and to the left. Adding those two sums gives you the total number of ways to get to the point since you are limited to only traveling down and to the right. Using this method one can work their way from the Taxi Stand to every point on the rectangular grid revealing Pascal's Triangle (Figure 4.10).

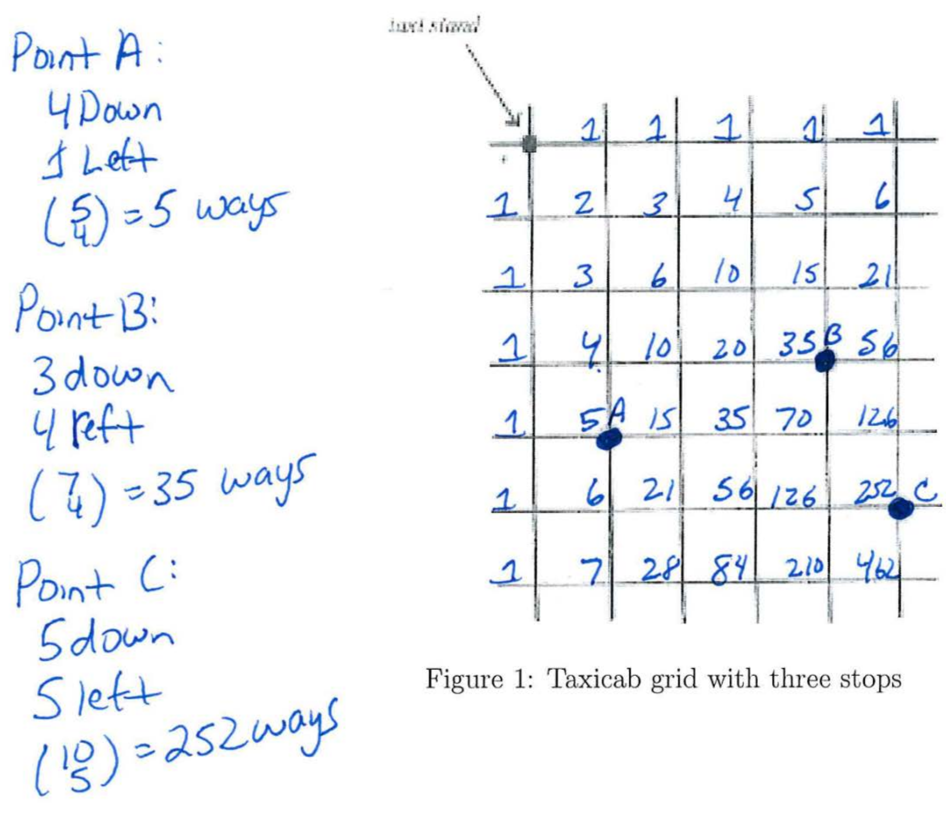


Figure 4.10: Sample solution to the Taxicab problem which reveals Pascal's Triangle.

4.4.4.2 Cycle 4: The videos.

In the previous cycles, all the videos were carefully edited and most contained some sort of voice over to situate the problem solving. In this cycle instead the teachers worked with long, un-edited video clips of high school students working on the Taxicab task. This was done to get the teachers used to working with longer videos which they will use for their final project in the course as well as to see what reasoning they can extract from the video which has a lot more going on in it. The video was not available in the first iteration, so it was only used in the other four.

Table 4.10

Videos used during Cycle 4 by each iteration.

Video Clip Title	It. 1	It. 2	It. 3	It. 4	It. 5
Taxicab problem, clip 1 of 5: The shortest distance between two points.		x	x	x	x
Taxicab problem, clip 2 of 5: Investigating the number of shortest paths		x	x	x	x
Taxicab problem, clip 3 of 5: It's Pascal's triangle! But Why?		x	x	x	x
Taxicab Problem, Clip 4 of 5: Explaining the Taxicab and Towers Isomorphism		x	x	x	x
Taxicab problem, clip 5 of 5: Extending the taxicab correspondence to pizza with toppings and binary notation		x	x	x	x

4.4.4.3 Cycle 4: The readings.

A chapter by Powell (2010) based on his dissertation was used to help the teachers situate the Taxicab problem solving session they watched in the video. In the chapter, Powell describes how the session came about, the problem solving strategies of the students, and how the Taxicab task tied into the bigger picture for the students.

4.4.4.4 Cycle 4: Student work module.

Three examples of student work were used for the Taxicab module. All the solutions were correct, but had different arguments for how they arrived at their answer. One contained a written narrative by a student for how they solved the task where he made connection between lining people up against a wall and the paths the taxi could take. The second student work was a tree diagram used by a student to solve simpler cases of the task which they then tried to generalize. The final piece of student work was pictures of red and yellow towers where the red was moving downward and the yellow was moving to the right. The number of shortest paths were expressed as towers of certain heights (e.g., to reach point A on the Taxicab grid one had to move 5 spaces, 4 down and 1 to the right, therefore their solution was the number of towers 5 tall containing 4 red and 1 yellow).

4.4.4.5 Cycle 4: Online discussion.

As in previous weeks, the discussion revolved around the teachers comparing their own problem solving to that of the children in the videos.

4.4.5 Cycle 5: Introduction to probability.

The goal of this cycle was to have the teachers consider ideas in probability building off of the skills they developed in solving combinatorics tasks. Both the World Series (Appendix B.6) and the Problem of Points (Appendix B.7) have the problem solver generating a list of combinations, but the difference is that the probability of these events happening are not equally weighted, so they need to be adjusted based on how likely they are.

4.4.5.1 Cycle 5: The World Series and Problem of Points tasks.

In the World Series task, the teachers had to consider how likely it is that a best of seven series of games between two evenly matched teams (i.e., both have a 50% chance to win each game) would end in 4, 5, 6, and 7 games. An example of the series ending in 4 games would be Team A winning all four or Team B winning all four which is 2 ways. How likely that is to happen needs to be considered though and is 1/2*1/2*1/2*1/2 or 1/16th. Therefore, there is a 2/16th chance that the World Series would end in four games. This idea can be repeated by figuring out how many ways the game would end in 5, 6, or 7 games and then multiply that by the probability of that event occurring (e.g., 0.5^5 for 5 games). A common incorrect solution that is produced is a result of treating all the events equally likely. There are 2 ways for the Series to end in 4 games, 8 for 5 games, 20 for 6 games, and 40 for 7 games. All those combinations are added up to produce a sample space of 70 and then the probabilities end up being 2/70 for 4 games, 8/70 for 5, 20/70 for 6, and 40/70 for 7.

The Problem of Points is a famous task that was posed by Chevalier de Mere to Blaire Pascal and was the first example of the concept of expected value (Katz, 1993). The structure is very similar to the World Series task in that the number of combinations need to be weighted differently based on the probability that they occur. The difference is that instead of both players starting from 0, one player has 8 points and the other has 7, and they are playing the first to 10 points. This would be similar to figuring out the solution to the World Series task where one team already has a 1-0 lead. An extra step is involved in solving the Problem of Points has the problem solver using the probability to calculate how 100 francs should be divided among the two players, which is each player’s expected value based on the situation when the game ended.

Way world Series ends in <u>four games</u>

Team A - AAAA

Team B - BBBB

<u>Five games</u>

assume team A wins last game. Then of the first four games they must have 3 wins and 1 loss or $\binom{4}{3} = 4$ ways. Same reasoning gives 4 ways for team B

<u>Six games</u>

Same reasoning, win 6th and in first 5 win 3, lose 2 or $\binom{5}{3}$ ways $= 10$ for each team, 20 total

<u>Seven Games</u>

Win 7th, of first 6 win 3, lose 3

$\binom{6}{3} = 20$ for each team, 40 total

# of games	ways to end	Chance event occurs
4	2	$(1/2)^4 = 1/16$
5	8	$(1/2)^5 = 1/32$
6	20	$(1/2)^6 = 1/64$
7	40	$(1/2)^7 = 1/128$

$P(\text{ends in 4 games}) = 2(1/16) = 1/8 = 12.5\%$

$P(\text{ends in 5 games}) = 8(1/32) = 8/32 = 25\%$

$P(\text{ends in 6 games}) = 20(1/64) = 20/64 = 31.25\%$

$P(\text{ends in 7 games}) = 40(1/128) = 40/128 = 31.25\%$

Figure 4.11: A correct solution to the World Series problem.

Incorrect way:

# of games	ways to end
4	2
5	8
6	20
7	40

total: 70

so $P(\text{end in 4 games}) = 2/70$

$P(\text{end in 5 games}) = 8/70$

$P(\text{end in 6 games}) = 20/70$

$P(\text{end in 7 games}) = 40/70$

Figure 4.12: A common incorrect solution to the World Series problem where the events are treated as equally likely.

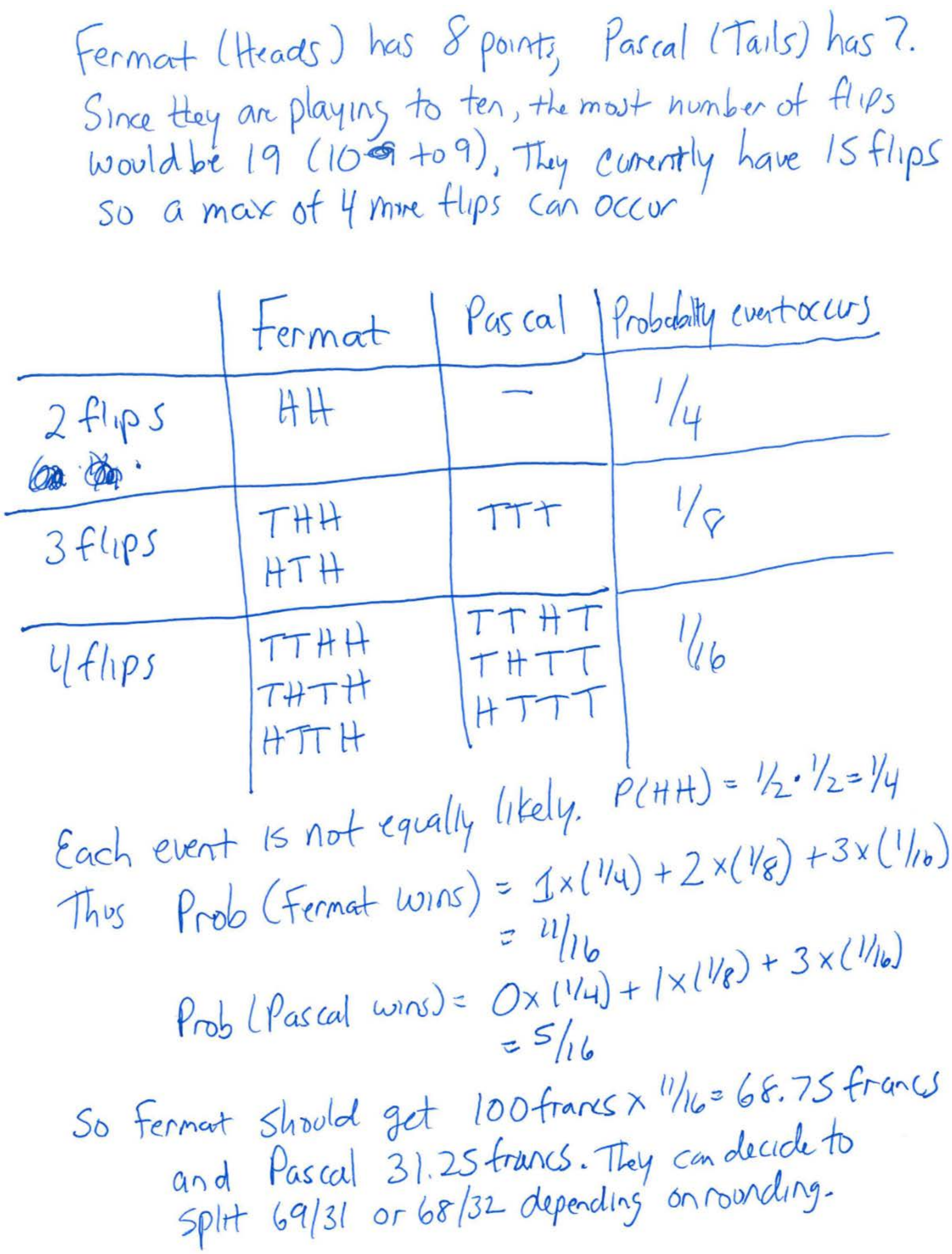


Figure 4.13: A sample correct solution to the Problem of Points.

4.4.5.2 Cycle 5: The videos.

The teachers watched an edited video of five students working on the World Series task over three sessions. Four students produced the correct answer and one had the common incorrect solution. Between the first and second session, a group of graduate students who solved the task themselves and produced the common incorrect solution, wrote the students a letter explaining why they are wrong. The students investigated the graduate students claim and found out that the graduate students were wrong and wrote them a letter explaining why. During the process of writing the letter, the student who produced the incorrect solution changed his mind and realized why his solution is incorrect.

4.4.5.3 Cycle 5: The readings.

There was only one reading for this cycle and it was a series of letters that were written by Pascal and Fermat (1654) where they discussed the Problem of Points that the teachers worked on. In the later iterations, they also looked at the letters between the group from the video they watched and the graduate students who incorrectly said that the students were wrong.

4.4.5.4 Cycle 5: Student work module.

Two examples of student work from the World Series and three examples of the Problem of Points were used. The World Series work contain one example of a student using the entire sample space of 128 to solve each part of the task and the other incorrectly treats the events with the same probability of occurring. The three Problem of Point examples highlight various incorrect strategies that students may use to reason about the task.

4.4.5.5 Cycle 5: Online discussion.

Questions for this cycle had the teachers consider the connection between combinatorics and probability, to consider an extension of the task where the teams are not evenly matched (e.g., Team A is 60% to win and Team B is 40%), to discuss about why the probability of winning 6 and 7 games is the same, and the relationship of this task with previous ones they have worked on.

4.4.6 Cycle 6: Reflecting.

This cycle contained no new problem solving component, but was designed to have the teachers reflect on what they did in the previous cycles in two ways: (a) by listening to and readings the reflections of the students who engaged in the problem solving and how it shaped their mathematical thinking and (b) by tying all the mathematical ideas together to explore Pascal's Triangle, Pascal's Pyramid, Pascal's Identity, and the binomial expansion. This cycle lasted from two to three weeks depending on the year.

4.4.6.1 Cycle 6: The tasks.

During this cycle, all the previous tasks were re-visited to make connections to mathematical topics such as Pascal's Triangle (Figure 4.14). An example would be the Tower's task. Each individual row corresponds to the total number of towers that one could build of a certain height when selecting from two colors. The row that contains 1,4,6,4, and 1 would map to the towers 4-tall selecting from 2 colors. Each individual entry on that row also maps to a specific case of the towers. There are 4 towers 4-tall selecting from 2 colors that contain exactly one blue and three yellow as well as three blue and one yellow. Six towers can be built with exactly two blue and two yellow. The ones on the outer end correspond to towers with all yellow or all blue. When expanding to 3-colors instead of Pascal's Triangle, one can map the entries of Pascal's Pyramid to the towers. Pascal's Identity can be shown through the inductive argument with towers (Figure 4.15). A similar argument can be produced for the pizza without halves task and taxicab. These ideas were explored by the class in a full class discussion environment.

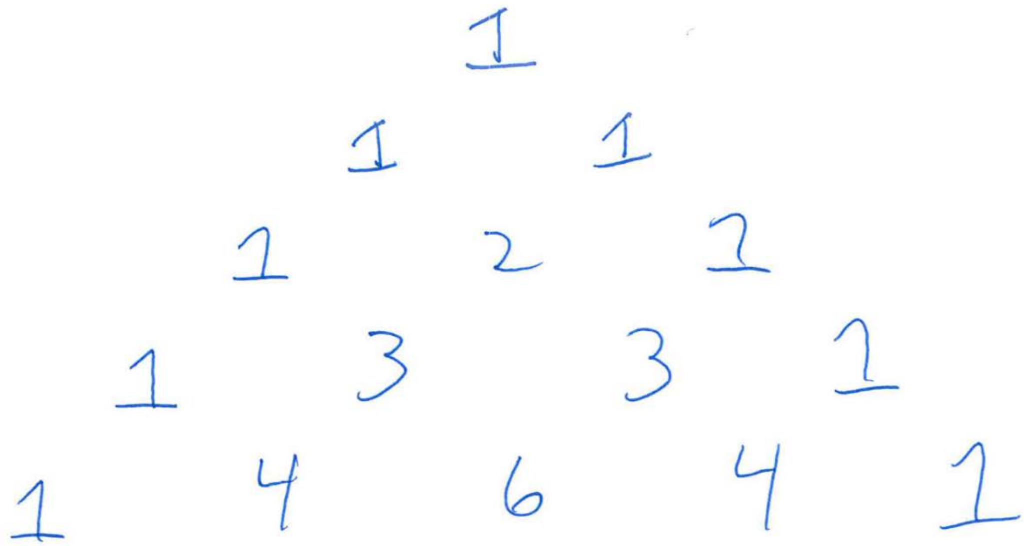


Figure 4.14: The first five rows of Pascal's Triangle.

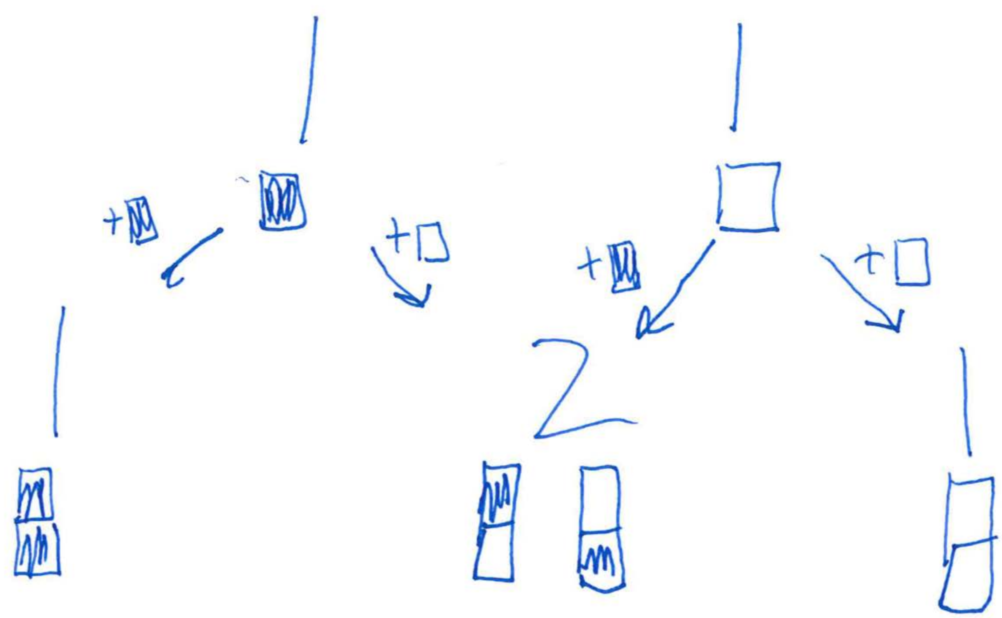


Figure 4.15: The relationship of Pascal's Triangle to the Towers tasks.

4.4.6.2 Cycle 6: The videos.

The first video, entitled the Night Session, focused on a group of 11th grade students as they worked to map their solutions to the towers and pizza tasks to the formal notation that is found in textbooks. While constructing the formal notation, the students start connecting the variables in the equation to what type of tower or pizza that would be. Not only were the students able to successfully produced Pascal's Identity and the formula for the binomial coefficient (n choose k), but they were able to make sense of it by giving examples of what the formula would mean in relation to towers and pizzas.

The second group of videos contained the voices of some of the students they watched throughout the semester as they discussed their experiences in working on open-ended problem solving tasks. One of the videos followed a single student, Romina, throughout the years, piecing together video of her working in the classroom on tasks and interviews that occurred 11th grade, sophomore year of college, and three times after she had graduated college where she reflected on her problem solving experience and how it has impacted her life. The second video contains comments from eight students about their experiences interspersed between footage of them graduating from high school.

Table 4.11

Videos used during Cycle 6 by each iteration.

Video Clip Title	It. 1	It. 2	It. 3	It. 4	It. 5
Night Session (PUP Math)	x	x	x	x	x
Students reflecting on their experience	x	x	x	x	x
Romina's Story	x	x	x	x	x

4.4.6.3 Cycle 6: The readings.

The papers focused on providing insights on how to go about promoting mathematical reasoning in the classroom by using examples from the longitudinal study. A paper added in later iterations (Francisco, 2013) supplemented the video of the Kenilworth students reflecting on their experience by situating it in the literature and providing more examples.

Table 4.12

Readings used during Cycle 6 by each iteration.

Paper author(s) and year	It. 1	It. 2	It. 3	It. 4	It. 5
Uptegrove (2010)	x	x	x	x	x
Maher (2005)	x	x	x		
Maher & Speiser (1997)	x	x	x		
Francisco & Maher (2005)	x	x	x	x	x
Francisco (2013)				x	x

4.4.6.4 Cycle 6: Student work module.

This cycle contained no student work module.

4.4.6.5 Cycle 6: Online discussion.

For the final cycle, the discussion focused on the conditions in the mathematics classroom that might be required into make mathematics a meaningful subject for all students all through high school. The goal was to tie together the teachers experiences and how it has shaped their thinking and compare it with the reflections provided by the students in the videos and the readings.

Chapter 5: Studying the Change in Beliefs and Attending to Reasoning

All teachers in the study took two assessments as a pretest before the course started and as a posttest after they engaged in the six cycles. All the teachers in the study took the pretest before engaging in any course activities. Three students did not finish the course and thus did not take the post test. These students were not included in the analyses.

The first assessment, a Beliefs Assessment, consisted of a series of statements about learning and teaching mathematics that called for a rating on a 5-point Likert scale from strongly disagree to strongly agree. The second assessment, a Reasoning Assessment, required the teachers to watch a ten minute video of 9-10 year old students sharing their justifications of their solution to the 3-tall Towers task, first selecting from two colors, and then of varying heights. In an open-ended response the teachers were asked to note any reasoning they observed from the students' arguments in the video and whether or not they found that reasoning convincing. Both assessments results are being compared to three separate comparison groups who contain comparable populations as the teachers in the study.

5.1 The comparison group

Four courses were identified for data collection as a comparison group. The courses contained similar populations (i.e., majority in-service with some pre-service teachers) as the teachers in the study. Two of the comparison groups participated in the same course a different semester; the other group of were from a course that focused on high-school mathematics learning. During the comparison groups courses, the teachers in the class worked on open-ended problem solving tasks and the focus was on students' mathematical reasoning. What differed was the set of tasks and videos that were used as well as the course structure. The teachers in the comparison group did not take the course where the study was conducted at any point in their program (i.e., no teacher was involved in both the comparison and experimental group at any time).

5.2 Beliefs Assessment

Before the semester started, the teachers in the course received an invitation to take the Beliefs Assessment using a course management system, Sakai. They were presented with thirty-four questions about mathematics teaching and learning which they had to rate on a 5-point Likert scale from strongly disagree to strongly agree (Appendix E). The statements were created to align with what the 2000 NCTM Standards considered as what is required for effective mathematics teaching and were either aligned or not aligned with the Standards. For analysis, a subset of the questions relevant to the study was chosen for tracking their beliefs. The questions and their intended response appear in Table 5.1.

Table 5.1

Belief assessment questions used in the study and their alignment to the NCTM Standards.

Belief Question	Alignment with Standards Answer
Learners generally understand more mathematics than their teachers or parents expect.	Disagree
It's helpful to encourage student-to-student talking during math activities.	Agree
Math is primarily about learning the procedures.	Disagree

Students will get confused if you show them more than one way to solve a problem.	Disagree
All students are capable of working on complex math tasks.	Agree
If students learn math concepts before they learn the procedures, they are more likely to understand the concepts.	Agree
Manipulatives should only be used with students who don’t learn from the textbook.	Disagree
Young children must master math facts before starting to solve problems.	Disagree
Learners generally have more flexible solution strategies than their teachers or parents expect.	Agree
Manipulatives cannot be used to justify a solution to a problem.	Disagree
Learners can solve problems in novel ways before being taught to solve such problems.	Agree
Understanding math concepts is more powerful than memorizing procedures.	Agree
If students learn math concepts before procedures, they are more likely to understand the procedures when they learn them.	Agree

Collaborative learning is effective only for those students who actually talk during group work.	Disagree
Only the most talented students can learn math with understanding.	Disagree
The idea that students are responsible for their own learning does not work in practice.	Disagree
Teachers need to adjust math instruction to accommodate a range of student abilities.	Agree

5.2.1 Belief assessment scoring methodology.

For analysis based on the relatively small sample size, the 5-point Likert scale was collapsed to merge strongly disagree with disagree and strongly agree with agree making the scale agree (A), undecided (U), and disagree (D). For analysis purposes each agree and disagree beliefs response is labeled as high (H) or low (L) depending on whether the frequency of the response is the desired response (H) or not (L). For example, consider the question “Only the most talented students can learn math with understanding”. For that question, the Standards aligned response would be “disagree”. If a teacher selected disagree, that response would be coded as H since it is in line with a recommendation from the Standards. A selection of undecided would be coded as U and “agree” would be L.

Based on the responses to the Beliefs Assessment, seventeen pre and post beliefs were assigned to each teacher with one of nine possible pre and post transitions. A transition of H->H would signify that on the pretest the teacher belief was aligned for the specific question and also on the posttest. There are nine possible transitions from pre to post, namely: H->H, H->U, H->L, U->H, U->U, U->L, L->H, L->U, L->L.

The analysis focused on the transition from pre to post test for each individual question (in contrast to analyzing all the questions as a whole data set, on average). This approach was chosen since multiple learning topics in mathematics education composed the assessment and attention could be given to each topic.. By conducting a question by question analysis, topics that could be identified with greater influence could be identified as a consequence of the TLMR intervention model.

5.2.2 Question by question analysis.

The analyses examined the transition from pre to post assessment for each question for each teacher. The data were analyzed to examine if the teachers moved towards beliefs that were aligned with the Standards from pre to post-tests. The rationale for the analysis is to determine which beliefs, if any, were impacted by the TLMR model. Each question was analyzed using a Wilcoxon rank-sum test, a nonparametric alternative to the t-test. A shift is described as significant when a p-value < 0.05. The effect sizes were calculated using an estimator suggested by Grissom and Kim (2012) which takes the U statistic generated by the test and then divide it by the product of the two sample sizes which will estimate that a score randomly drawn from one population will be greater than the other. This methodology was chosen over using Cohen's *d* due to the smaller sample size and non-normality of the data.

Initially, differences between each iteration of the data were examined. With no a significant difference in both pre-test scores and shifts between all five iterations, the data was lumped together giving a sample size of 86 with 30 pre-service teachers (PST) and 56 in-service teachers (IST). A similar result was found for the three control groups that were combined to produce a sample size of 42 with 17 PSTs and 25 ISTs.

5.2.2.1 Belief assessment: Experimental pre to post scores.

A first analysis was to study a shift in beliefs from the experimental cohort from pre to post assessments. The hypothesis underlying this analysis is that due to the intervention, the teachers' beliefs would remain aligned to the Standards if they were already aligned and if they were not by the end of the intervention would be more closely aligned. The results for each question are found in Table 5.2. The analysis indicates that out of the 17 questions, teacher responses of 13 of the questions showed significant positive shifts from pre to post assessments. Two of the questions that did not meet the criteria were questions that were about the use of manipulatives in problem solving. The other two were pedagogical strategies related to showing multiple strategies to students and adjusting instruction to meet the range of student abilities.

Table 5.2

Shift in beliefs from pre to post towards alignment with the NCTM Standards for the experimental group.

Belief Question	p-value	Effect Size
Q1 - Learners generally understand more mathematics than their teachers or parents expect	6.18E-07	0.71
Q4 - It's helpful to encourage student-to-student talking during math activities.	0.0071	0.51
Q5 - Math is primarily about learning the procedures.	3.01E-03	0.43
Q6 - Students will get confused if you show them more than one way to solve a problem.	0.512	0.48
Q7 - All students are capable of working on complex math tasks.	0.0024	0.56

Q9 - If students learn math concepts before they learn the procedures, they are more likely to understand the concepts.	0.0046	0.59
Q10 - Manipulatives should only be used with students who don't learn from the textbook.	0.1236	0.512
Q11 - Young children must master math facts before starting to solve problems.	0.0057	0.398
Q15 - Learners generally have more flexible solution strategies than their teachers or parents expect.	0.0335	0.573
Q17 - Manipulatives cannot be used to justify a solution to a problem.	0.064	0.473
Q18 - Learners can solve problems in novel ways before being taught to solve such problems.	0.0482	0.517
Q19 - Understanding math concepts is more powerful than memorizing procedures.	0.0182	0.523
Q21 - If students learn math concepts before procedures, they are more likely to understand the procedures when they learn them.	0.03363	0.528
Q23 - Collaborative learning is effective only for those students who actually talk during group work.	0.0017	0.419

Q29 - Only the most talented students can learn math with understanding.	0.0018	0.494
Q30 - The idea that students are responsible for their own learning does not work in practice.	0.04608	0.444
Q31 - Teachers need to adjust math instruction to accommodate a range of student abilities.	0.3122	0.48702

5.2.2.2 Comparing shifts between the experimental and comparison groups. Next, positive shifts between the experimental and comparison group were examined. Since results were very similar for both groups on their pre-test values and the experimental group showed significant shifts, it was decided to compare the post-test values between the two groups. The p-values for each question appear in Table 5.3. All the mathematical ideas addressed in these questions were introduced in both the comparison and experimental group. However, responses to 11 of the 17 questions on the post-test results were significantly different. The experimental group shifted in responses to key topics such as Q1, Q4, Q11, and Q19 which are highly emphasized in the Standards.

Table 5.3

Table of p-values for individual belief questions comparing the experimental group to the comparison groups shift in beliefs.

Belief Question	p-value	Effect Size
Q1 - Learners generally understand more mathematics than their teachers or parents expect	0.0007	0.639

Q4 - It's helpful to encourage student-to-student talking during math activities.	0.0026	0.565
Q5 - Math is primarily about learning the procedures.	0.0528	0.441
Q6 - Students will get confused if you show them more than one way to solve a problem.	0.0087	0.399
Q7 - All students are capable of working on complex math tasks.	0.0067	0.608
Q9 - If students learn math concepts before they learn the procedures, they are more likely to understand the concepts.	0.0164	0.592
Q10 - Manipulatives should only be used with students who don't learn from the textbook.	0.1007	0.453
Q11 - Young children must master math facts before starting to solve problems.	4.36E-08	0.245
Q15 - Learners generally have more flexible solution strategies than their teachers or parents expect.	0.0002	0.635
Q17 - Manipulatives cannot be used to justify a solution to a problem.	0.0922	0.446
Q18 - Learners can solve problems in novel ways before being taught to solve such problems.	0.1783	0.529

Q19 - Understanding math concepts is more powerful than memorizing procedures.	0.00116	0.59
Q21 - If students learn math concepts before procedures, they are more likely to understand the procedures when they learn them.	0.0243	0.53
Q23 - Collaborative learning is effective only for those students who actually talk during group work.	0.02667	0.427
Q29 - Only the most talented students can learn math with understanding.	0.0434	0.476
Q30 - The idea that students are responsible for their own learning does not work in practice.	0.0776	0.423
Q31 - Teachers need to adjust math instruction to accommodate a range of student abilities.	0.6112	0.516

5.2.2.3 Comparing pre-service to in-service teachers' shift in beliefs

The questions that did not show significant positive shift from pre to post when comparing the experimental group to the comparison were more aligned with pedagogical techniques than theories about how students learn or what they are capable of learning. As a result, it was decided to compare the results of the pre-service teachers with that of the in-service teachers to see if identified differences might explain some of the results. Table 5.4 contains the p-values of comparing the two populations. Two of the questions that were aligned with pedagogical techniques (Q10 and Q31), and were not significant when looking at the whole experimental population, were significantly different between the two populations with the ISTs having more responses aligned with the Standards. Several other questions that had some pedagogical elements underlying the learning theory (e.g., Q4 about student-to-student talking compared to theory about collaboration) also were significantly different with the ISTs showing greater positive shifts.

Table 5.4

Shift in beliefs from pre to post towards alignment with the NCTM standards in the beliefs assessment comparing pre-service teachers to in-service teachers.

Belief Question	p-value	Effect Size
Q1 - Learners generally understand more mathematics than their teachers or parents expect	0.7821	0.489
Q4 - It's helpful to encourage student-to-student talking during math activities.	0.0496	0.517
Q5 - Math is primarily about learning the procedures.	0.0458	0.563
Q6 - Students will get confused if you show them more than one way to solve a problem.	0.1637	0.555
Q7 - All students are capable of working on complex math tasks.	0.9831	0.501
Q9 - If students learn math concepts before they learn the procedures, they are more likely to understand the concepts.	0.5126	0.526

Q10 - Manipulatives should only be used with students who don't learn from the textbook.	0.0113	0.426
Q11 - Young children must master math facts before starting to solve problems.	0.5785	0.526
Q15 - Learners generally have more flexible solution strategies than their teachers or parents expect.	0.2554	0.464
Q17 - Manipulatives cannot be used to justify a solution to a problem.	0.3927	0.471
Q18 - Learners can solve problems in novel ways before being taught to solve such problems.	0.9432	0.498
Q19 - Understanding math concepts is more powerful than memorizing procedures.	0.9773	0.499
Q21 - If students learn math concepts before procedures, they are more likely to understand the procedures when they learn them.	0.5097	0.482
Q23 - Collaborative learning is effective only for those students who actually talk during group work.	0.8361	0.492
Q29 - Only the most talented students can learn math with understanding.	0.0399	0.5

Q30 - The idea that students are responsible for their own learning does not work in practice.	0.3984	0.542
Q31 - Teachers need to adjust math instruction to accommodate a range of student abilities.	0.045	0.561

5.2.2.4 Individual questions that show the greatest shifts

Overall, across all the analysis questions that focused on what students are capable of learning (Q1, Q7, Q15, Q29, Q30) and the what students are capable of in mathematics (Q5, Q9, Q11, Q19, Q21) showed significant change for all the teachers who participated in the TLMR model. Questions that were more pedagogically based such, as the use of manipulatives in learning mathematics (Q10, Q17) and accommodating instruction for multiple student abilities (Q31) did not show significant change.

5.3 The Reasoning Assessment

Similar to the Beliefs Assessment, the teachers also completed the Reasoning Assessment for the pre and posttest; the pretest took place before the semester started using the online Sakai site. In the Reasoning Assessment, the teachers watched an edited video of 4th grade students discussing their solutions to the Towers task (Appendix F). In the beginning of the video, students are in a classroom working, in pairs, on solving how many unique towers they could build 5-tall selecting from 2-colors. After several minutes, the video shifts to an interview with four students who are discussing their earlier solutions to towers of varying heights while selecting from two colors. Throughout the video the students make various arguments to support their solution such as: an inductive argument, an argument by contradiction, and two forms of a case argument. The teachers received a prompt that asked them to write an open-ended response and describe: (1) each example of reasoning that a child in the video puts forth, (2) whether or not the reasoning forms a valid argument, (3) whether or not the argument is convincing, and (4) why or why not they find the argument convincing.

5.3.1 Scoring methodology.

Each open ended response was scored independently by a group of three graduate students using a rubric that is found in Appendix F. Initially, the group worked with responses from a pilot study for training and to establish reliability. Once a reliability greater than 90% was obtained, 25% of the total data were graded by this researcher. The other two scorers would double score about 20% of the data scored by this researcher (5% of the total data) with some overlap between the other two scorers. The scorers met to compare reliability between and among each other. This process was repeated three more times until 100% of the data was scored. This approach was taken to ensure that reliability was obtained multiple times (in contrast to establishing reliability on a small subset of the data and then scoring the rest independently). It was judged to be a more robust way to approach the analysis, with every cycle of scoring producing inter-rater reliability greater than 90%.

For the analysis, the focus was on the complete description for four of the argument types: cases, the alternate cases, induction, and contradiction. Each argument had multiple features to it and the data were coded for whether or not teachers did not notice any of the argument, parts of the argument, or the complete argument. All the arguments are related to the Towers problem selecting from two colors with varying heights. A positive shift in recognizing an argument would be defined as noticing more features of an argument on the post-test than the pre-test. For example, the inductive argument has three parts to it; (a) realizing that there are two options for Towers selecting from two colors, (b) for a tower of a certain height, all the possible towers there could have one of those two colors added to it so the total would double as the height increases by one, and (c) something that refers to being able to generalize the argument such as this pattern holds and as a result for Towers n-tall selecting from two colors would be represented by 2^n. A teacher may notice just parts (a) and (b) in their pre-test. If teachers noticed (c) on the post-test, it was coded as growth.

Similar to the Belief Assessment data, the pre-test scores between the comparison and experimental group were similar at the onset. The growth between iterations was not significantly different except for the contradiction argument between two of the iterations (the third and fourth, $p = 0.048$). As a result, the data from all five iterations were combined into one experimental group since it was determined sensible to compare the experimental and comparison since they both started with similar scores. The same subgroups (Experimental Pre vs. Experimental Post, Comparison Post vs. Experimental Post, PST Post vs IST Post, IST Pre vs IST Post, PST Pre vs. PST Post) analyzed for the Belief Assessment are also analyzed for the Reasoning.

5.3.2 The cases argument.

In the video, the cases argument was proposed by several of the students. A complete arguments includes mentioning the following things: (a) Using all blue or no red cubes to build one tower, (b) one blue cubes and two red cubes resulting in three unique towers, (c) two red cubes and one cube resulting in three unique towers, and (d) Using all red or no blue cubes to build one tower. A teacher might notice either nothing (missing), one to three parts of the argument (partial), or all four features of the argument (complete).

5.3.2.1 Growth in attending to the cases argument.

The results of the analysis appear in Table 5.5. When comparing the experimental and comparison groups post-test scores, the experimental group grew significantly in their attending to the cases argument. The group that participated in the TLMR intervention model also grew significantly, while the comparison group did not. The ISTs also grew much more compared to the PST.

Table 5.5

Growth in attending to the cases argument in the assessment across multiple sub-populations.

Teacher Population	p-value	Effect Size
Experimental Post vs Comparison Post	1.99E-08	0.248
Experimental Pre vs. Experimental Post	1.31E-05	0.351
IST Post vs PST Post	0.002	0.441
PST Pre vs. Post	0.0178	0.29
IST Pre vs. Post	0.0005	0.39

5.3.2.2 Percent with the complete cases argument.

About 77% of the teachers in the experimental group attended to the complete cases argument on the post-test compared to only 47% on the pre-test. One common theme between both the experimental and comparison groups is fewer numbers of teachers attending to only part of the argument. The majority (> 90%) either missed the argument completely or explained the complete argument.

Table 5.6

Percent of experimental and comparison teachers who noticed none, parts of, or the complete cases argument on the pre and post-test.

Teacher Population	% Missing	% Partial	% Complete
Experimental Pre-Test	50	2.3	47.7

Experimental Post-Test	21.6	1.1	77.3
Comparison Pre-Test	61.2	6.1	32.7
Comparison Post-Test	57.1	8.2	34.7

5.3.3 The alternate cases argument.

This argument is similar to the cases argument in 5.3.2, but includes noticing an additional feature that was proposed by a student named Stephanie in the video. When constructing towers with two blue and one red cube, she separated the case into sub groups: -one where the blues were “stuck together” (i.e., BBR, RBB) and “stuck apart” or “took apart” (i.e., BRB). Since there is overlap with argument 5.3.2, for analysis here, it was decided to only examine whether or not teachers noticed this one feature since the growth for (a) through (d) in 5.3.2 would be covered by that analysis. There is only one feature to this argument, so either it is missing or complete.

5.3.3.1 Growth in attending to the alternate cases argument.

The results of the analysis appear in Table 5.7. When comparing the experimental and comparison group post test scores, the experimental group grew significantly in their attending to the alternate cases argument. The group who participates in the TLMR model intervention also grew significantly, while the comparison group did not. The ISTs also grew much more compared to the PST.

Table 5.7

Growth in attending to the alternate cases argument in the assessment across multiple sub-populations.

Teacher Population	p-value	Effect Size
Experimental Post vs Comparison Post	6.22E-07	0.277
Experimental Post vs. Experimental Post	4.37E-07	0.32
IST Post vs PST Post	0.0385	0.455
PST Pre vs. Post	0.0024	0.26
IST Pre vs. Post	9.49E-05	0.355

5.3.3.2 Percentage with complete alternate cases argument.

Two-thirds of the experimental teachers attended to the alternate cases argument on the post-test compared to only 30% on the pre-test. The comparison group showed a slight increase from 14% to 22%.

Table 5.8

Percent of experimental and comparison teachers who noticed none or the complete alternate cases argument on the pre and post-test.

Teacher Population	% Missing	% Complete
Experimental Pre-Test	69.3	30.7
Experimental Post-Test	3.3	6.7
Comparison Pre-Test	85.7	14.3

Comparison Post-Test	77.6	22.4

5.3.4 The argument by induction.

The inductive argument was proposed by a student named Milin in the video. The three features coded for were: (a) realizing that there are two options for Towers selecting from two colors, (b) for a tower of a certain height, all the possible towers there could have one of those two colors added to it so the total would double as the height increases by one, and (c) something that refers to being able to generalize the argument such as this pattern holds and as a result for Towers n-tall selecting from two colors would be represented by 2^n. Since there are three features to this argument, it can either be missing, partial, or complete.

5.3.4.1 Growth in attending to the argument by induction.

The results of the analysis appear in Table 5.9. When comparing the experimental and comparison group post test scores, the experimental group grew significantly in their attending to the inductive argument. The group who participated in the TLMR intervention model also grew significantly, while the comparison group did not. The ISTs also grew much more compared to the PST.

Table 5.9

Growth in attending to the inductive argument in the assessment across multiple sub-populations.

Teacher Population	p-value	Effect Size
Experimental Post vs Comparison Post	0.0002	0.324
Experimental Post vs. Experimental Post	8.00E-07	0.317
IST Post vs PST Post	0.0283	0.437

PST Pre vs. Post	0.019	0.344
IST Pre vs. Post	5.21E-05	0.297

5.3.4.2 Percentage with complete argument by induction.

Only 22.7% of the experimental teachers attended to the full complete argument on the post-test, whereas there was a slight increase in attending to parts of the argument. Both the experimental and comparison referenced parts (a) and (b) with their partial descriptions, but did not include the generalization part.

Table 5.10

Percent of experimental and comparison teachers who noticed none, parts of, or the complete inductive argument on the pre and post-test.

Teacher Population	% Missing	% Partial	% Complete
Experimental Pre-Test	55.7	37.5	6.8
Experimental Post-Test	23.9	53.4	22.7
Comparison Pre-Test	55.1	40.8	4.1
Comparison Post-Test	53.1	36.7	10.2

5.3.5 The argument by contradiction.

The argument by contradiction was proposed by Stephanie when she was explaining her cases argument to another student named Jeff. When explaining her towers for one blue with two reds, she moves the blue throughout the towers in an elevator pattern (Figure 5.1). After her explanation, she claims that she can't put a blue cube in another position since that would result in a tower 4 cubes high, contradicting a given condition that towers were to be 3-tall. Hence, the three towers she built were the only ones she could build three tall with one blue and two reds. There is only one feature to this argument, so either it is missing or complete.

Figure 5.1: Stephanie's elevator pattern showing all the Towers 3-tall selecting from two colors with exactly one blue.

5.3.5.1 Growth in attending to the argument by contradiction.

The results of the analysis appear in Table 5.11. When comparing the experimental and comparison group post test scores, the experimental group grew significantly in their attending to the argument by contradiction. The group who participates in the TLMR model also grew significantly, while the comparison group did not. The ISTs also grew much more compared to the PST, but less so compared to the other argument types.

Table 5.11

Growth in attending to the argument by contradiction in the assessment across multiple sub-populations.

Teacher Population	p-value	Effect Size

Experimental Post vs Comparison Post	0.0114	0.402
Experimental Post vs. Experimental Post	1.24E-05	0.364
IST Post vs PST Post	0.0484	0.537
PST Pre vs. Post	0.0662	0.38
IST Pre vs. Post	0.0001	0.355

5.3.5.2 Percentage with the complete argument by contradiction.

While the experimental group grew significantly in attending to the argument by contradiction, only 31.8% of the teachers picked up on the argument on the post-test. Not one comparison teacher who was missing the argument on the pre-test picked it up on the post-test.

Table 5.12

Percent of experimental and comparison teachers who noticed none or the complete argument by contradiction on the pre and post-test.

Teacher Population	% Missing	% Complete
Experimental Pre-Test	95.5	4.5
Experimental Post-Test	68.2	31.8
Comparison Pre-Test	87.8	12.2
Comparison Post-Test	87.8	12.2

Chapter 6: Tracing the Shifts in Reasoning and Beliefs

In addition to the pre and post assessments that were analyzed in Chapter 5, the teachers participated in weekly online discussions and produced a final project that involved annotating video. These data were analyzed to gain insight into how their attending to student reasoning changed throughout the semester. Also it was of interest to identify common behaviors among the teachers who performed better on the pre and post-tests. Understanding what the successful teachers attended to during their discussions might provide guidelines for orchestrating productive discussions online to encourage teachers to attend to student reasoning.

6.1 Online Discussions

During the cycles of intervention, teachers watched and discussed videos, discussed assigned readings, reflected on presentations of guest speakers, analyzed samples of student written work, and continued the online class discussion about problem solving using eCollege. They had access to videos and student work that were stored on the VMC repository. Discussion prompts were provided for the online component and these questions can be found in Appendix H. For each course iteration, the teachers were randomly assigned to small groups of three to four members to encourage discussions online. There were four discussion groups each semester. Occasionally, ideas from one group were posed to the other groups for whole class discussions. The expectation was that teachers posted an initial response to the prompts halfway through the week and then respond to at least two other people before the week was over. The discussion text was extracted from the course website as PDF files for analysis. Each cycle generated approximately 160 pages of text. With 6 cycles each semester, each iteration produced between 900 and 1000 pages of text to analyze. The modules that were related to student work **are not** a part of this analysis.

6.1.1 Methodology.

For each cycle, a random iteration was chosen for the initial analysis (e.g., Cycle 1 was taken from Iteration 2, Cycle 2 from Iteration 4). The data were open coded by three graduate students and then discussed when needed to resolve differences. After the discussions, a set of codes were developed which were then used to code the entire data set. The codes were grouped into three categories; codes related to the videos, codes related to the readings, and a general code about mathematics education. The data were split among the three coders with 20% overlap. Again, coders met regularly to discuss their results and compare their results to the overlapped data. Two running totals were kept - the frequency of a code appearing and the percentage of teachers during that cycle who mentioned the code. Since the teachers made multiple posts and the posts could contain multiple instances of a code, the frequency of a code appearing could exceed the number of teachers in the class. A second tally of the percentage of teachers who mentioned the code was maintained in the event a teacher mentioned a code multiple times. It was important to avoid skewing the results by making it appear as if many of the teachers were talking about a subject when they actually were not.

6.1.1.1 Codes related to the videos.

Eight codes emerged related to videos heading. Three of the codes had to do with relating what the teacher saw on the video to either their own problem solving, a teacher in the class's problem solving, or to a student from a previous video they had watched in the class. Examples of these codes were along the lines of "*The strategy Brandon used in the video is similar to the way I thought about the problem with the Tower colors being analogous to the pizza topping being on or off".* The relationship could be positive (such as described previously) or negative (e.g., My strategy was more efficient than the strategy used by the students). Two of the codes were based on whether or not the teacher compared what went on in the readings to what they saw in the video (e.g., *In* *the reading they talked about the benefits of peer to peer questioning and you could see examples of that when Milin asked Stephanie about her answer*) and relating the video to some sort of teaching practice (e.g., *The questions asked by the researcher were open-ended which allowed Stephanie to explain her reasoning fully)*.

A second group of codes had to do with the problem solving exhibited by the students in the videos they watched. The discussions were coded to look for acknowledgement of the reasoning displayed by the students and the representations they used. For the reasoning code, the teachers could either identify the reasoning correctly (e.g., *Stephanie separated her argument into multiple cases - those with 0 blue and 3 red, 1 blue and 2 red, 2 blue stuck together and 1 red, 2 blue stuck apart and 1 red, and 3 red and 0 blue)* or incorrectly (e.g., *When Jeff was calculating the number of wins for the World Series he took the total number of combinations and divided them by each other).* The teachers who also made faulty inferences or who incorrectly interpreted student behaviors by making claims that were false (e.g. attributing to students statements that were not made) were also coded at incorrect. Codes were also applied when the teachers mentioned a representation constructed by the students. Examples of representations were use of certain notations, drawings, or tables.

6.1.1.2 Codes related to the readings.

Similar to the video codes, the reading codes contained three that related something in the readings to either the teachers own, others' in the class, or students' from the videos problem solving. The other two codes were relating the readings to something that occurred in the video or to the practice of teaching (e.g., *The article talked about the studnts [sic] difficulty in producing proofs, and when I teach I have the same issues in my classroom).*

6.1.1.3 General codes.

Only one code fell under this category and that had to do with referring to arguments and justifications as a social activity. The reason this code was separated from the video and reading categories is that when coding the data it was hard to discern whether or not the teacher was referring to the social aspect as a result of watching the video, reading the papers, previously held beliefs, or engaging in the problem solving themselves. An example would be "*The need for justification is what leads students to think deeper about a topic and fully understand the specific problem. Being able to explain and justify their solutions and convince others of their solution is a very powerful tool and perhaps is where the most learning takes place. Conceptual learning and understanding arises when there is reason to support it. This is what made Romina's proof so convincing.".* While this post referenced the video of *Romina's Proof to Ankur's Challenge* there is no indication that the video was the cause for the thought or whether the teacher had the thought and used the video to support the argument. Originally, when tying this code to either video or the readings, it produced a lot of variability among the coders. The idea of referring to arguments and justifications as a social activity is an important concept for mathematical reasoning and to not lose the code, it was kept but just not in relation to the video or readings and instead stood on its own.

6.1.2 Coding an example online cycle.

This section will describe one cycle of data from one iteration. The data in this cycle took place in the fifth week of the course during the first iteration and lasted two weeks. During the in-class problem solving, the teachers worked in small groups on the Ankur's Challenge (Appendix B.X) task. After presenting their solutions to the class and discussing them, their follow-up assignment was to study a video, *Romina's Proof to Ankur's Challenge*, of five 10th grade students who worked on the same problem-solving activity. In addition to the video, they were to read a paper by Erna Yackel and Gila Hanna (2003) about the importance of Reasoning and Proof. Specifically, the posted assignment was:

This week's assignment for online work involves a video and three readings, with threaded discussion, that follows class work on problem solving for the Ankur's Challenge task. The following are intended to guide discussion in your small groups:

(1) Describe Romina's strategy for solving the "Ankur's challenge" problem.

(2) In your opinion, is this solution a convincing one? Why or why not?

(3) According to the Yackel & Hanna chapter, both von Glaserfeld and Thompson equate reasoning with learning (p. 227). From this perspective, in what ways do explaining and justifying contribute to learning mathematics?

Pooling the data analyzed across the four groups of teachers, a total of 71 posts were made in the discussion threads. Individual posts tended to be rich in the scope of commentary that the teacher offered in their respective group's discussion thread. That is, a single post often commented on more than one aspect of teacher reflection on problem solving within context of the study. An example of the code "referred to arguments/justifications as a social activity" was a post by Mike (all names are pseudonyms) that said "*I think asking the students to convince their peers is what makes this study special, the solution is not very important. When trying to convince someone the students really deepen their understanding as Beth says reformulate, reorganize, rethink, and restate their argument.*" By considering the video episode in its broader context of the research study from which it came, Mike is noticing how certain features of the learning environment, namely peer evaluation of arguments, contribute to learning mathematics with understanding. A post by a student, Tom, that was coded for both correctly mentioning the reasoning of a student in the video and acknowledging representations they used read "*I agree with you that "explanation and justification help the student rethink about his/her ideas and push him to make sense of their findings before making them public". This is why Romina was successful in her proof. Every time she explained herself, she was able to make more sense of it and therefore justify her findings better. At first, she had 2's next to each of the 6 towers and then got confused herself why they put them there, then erased them and rewrote her solution in a more clear, convincing way. If Romina did not have to explain herself and justify her solution, she may have never came up with the nice, elegant solution that she did*". All the posts were coded based on the scheme described in 6.1.1.

Table 6.1 summarizes the codes for the cycle. For this cycle, a major point of discussion was the video of Romina's Proof to Ankur's Challenge where 52 of the 71 posts contained at least one reference to the video. Though the reading was mentioned 11 times in 71 posts, only one teacher posted about the connection between the reading and their own problem solving. The same student then made a connection between what Romina did in the video to the reading, as well as how to teach proof in the classroom. Another common theme in the discussions was this idea of learning as a social activity. This can be seen in the excerpt of Tom's post above where he talks about how explaining and justifying and how it made Romina successful in her proof. Eighty-four percent of the teachers across the four groups mentioned this topic at least once in their postings.

Table 6.1

Distribution of codes from a sample online discussion cycle.

Theme of reflective commentary	Frequency	% of participants who mentioned topic
Mentioned video at all	52	100%
Related video(s) to own problem solving	23	95.2%
Related video(s) to others' (in-class) problem solving	14	57.1%
Related video(s) to others' (students from videos) problem solving	9	42.85%
Mentioned (correctly) the reasoning of a student from video	17	61.9%
Mentioned (incorrectly) the reasoning of a student from video	1	4.7%
Acknowledged the representations used by a student in a video	6	23.8%
Related video(s) to the practice of teaching	16	57.1%

Mentioned reading at all	11	23.8%
Related reading(s) to own problem solving	1	4.7%
Related reading(s) to others' (in-class) problem solving	0	0%
Related reading(s) to others' (students from videos) problem solving	1	4.7%
Related idea from reading(s) to video(s)	1	4.7%
Related idea from reading(s) to the practice of teaching	2	4.7%
Referred to arguments / justification as a social activity	21	85.7%

6.1.3 Online discussion analysis.

Three types of analysis were conducted on the online discussion data. The first analysis compared the type of posts the teachers made across each cycle during each iteration. The second analysis compared the ratio of posts about video to posts about the reading and posts about the classroom environment. The third analysis looked at the growth of attending to representations across each iteration.

6.1.3.1 Comparing types of posts across cycles.

Figures 6.1 – 6.5 look at the distribution of codes for each cycle separated by each iteration. Several themes emerged across all the iterations.

Note: On Figure 6.1 – 6.5 the legend across the X-axis from left to right reads: Related video(s) to own problem solving, Related video(s) to others' (in-class) problem solving, Related video(s) to others' (students from other videos) problem solving, Mentioned (correctly) the reasoning of a student from the video(s), Mentioned incorrectly the reasoning of a student from the video(s), Acknowledged the representations used by a student in the video(s), Related video(s) to the practice of teaching, Related reading(s) to own problem solving, Related reading(s) to others' (in-class) problem solving, Related reading(s) to others' (students from videos) problem solving, Related idea from reading(s) to video(s), Related idea from reading(s) to the practice of teaching, Referred to arguments / justification as a social activity.

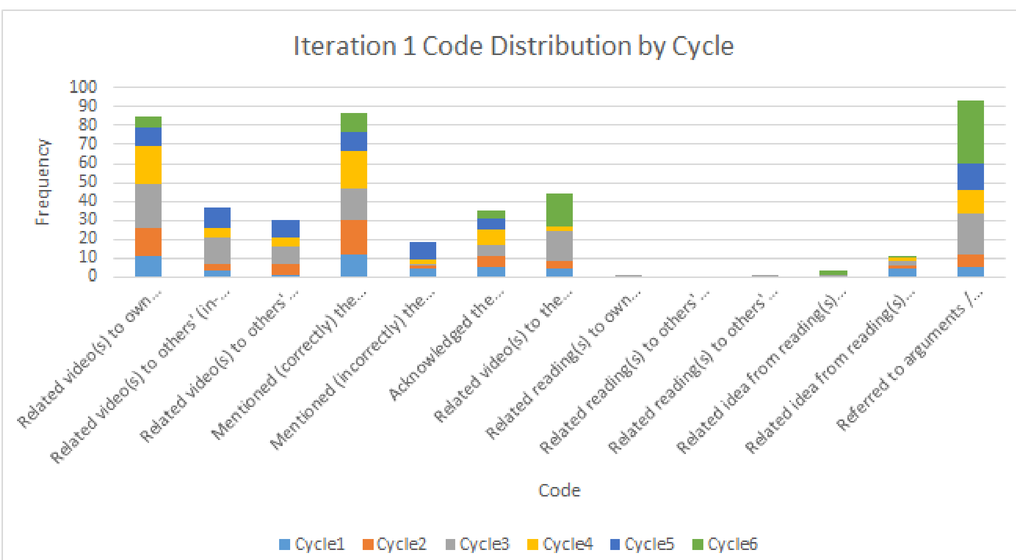


Figure 6.1: Distribution of codes by cycle for Iteration 1.

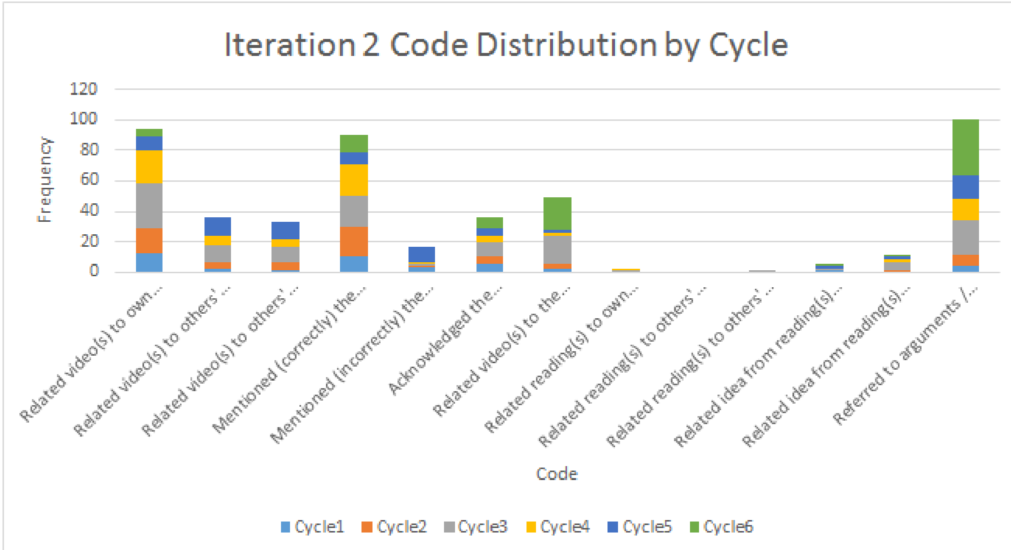


Figure 6.2: Distribution of codes by cycle for Iteration 2.

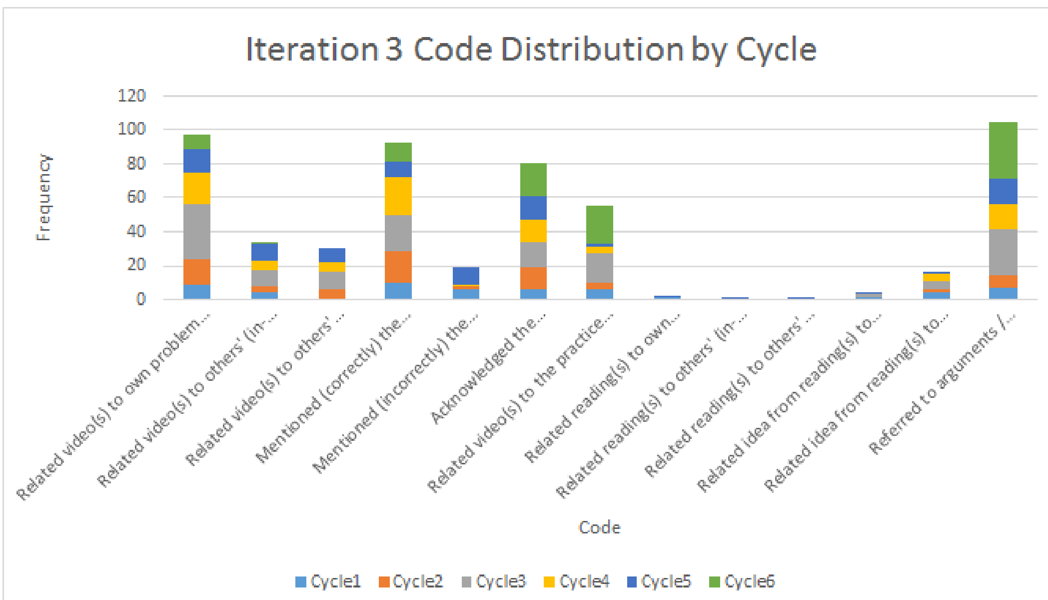


Figure 6.3: Distribution of codes by cycle for Iteration 3.

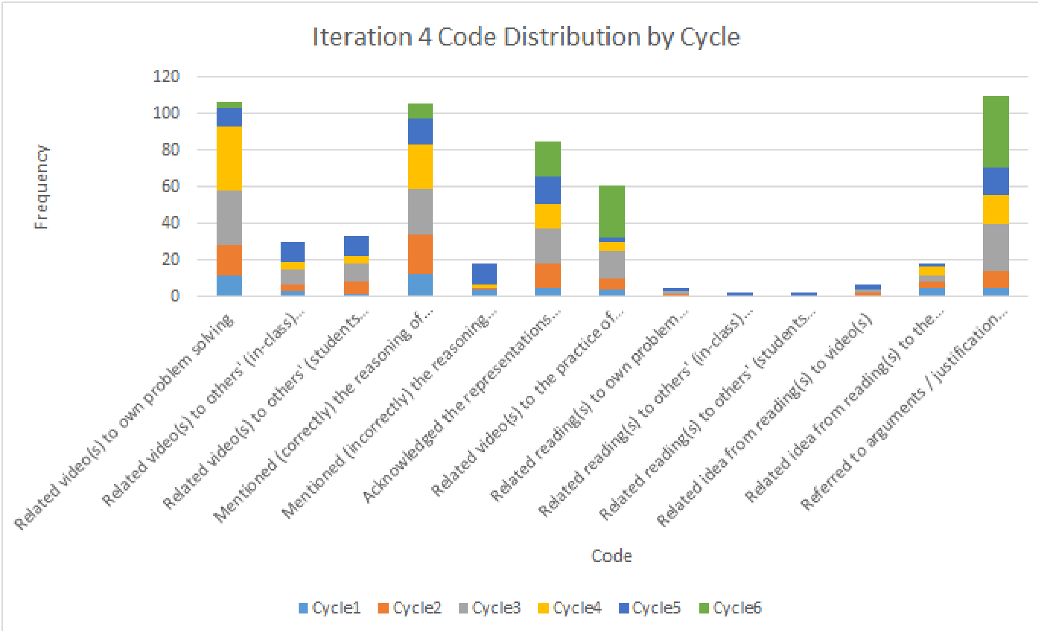


Figure 6.4: Distribution of codes by cycle for Iteration 4.

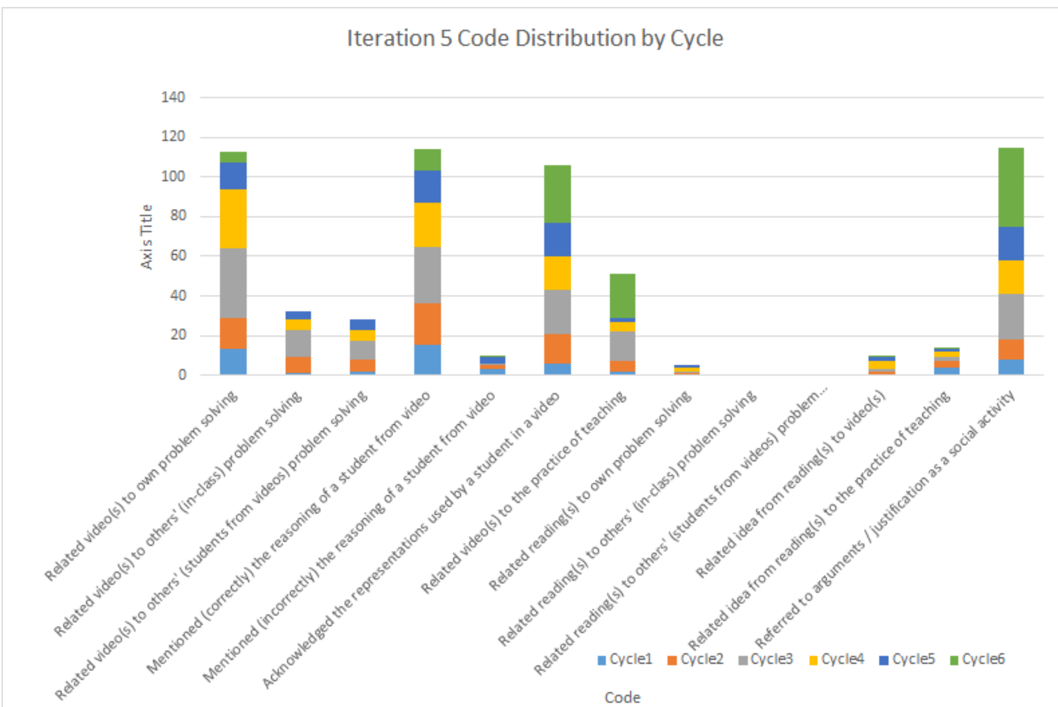


Figure 6.5: Distribution of codes by cycle for Iteration 5.

6.1.3.1.1 Relating video(s) to problem solving.

Across all iterations, there is steady growth between the number of references to relating video(s) to own and others' (both in-class and students from videos) problem solving from cycle 1 to 4. There was a decrease during cycle 5 which was about probability. The videos in cycle 6 had little to do with problem solving and were more about reflecting, so the drop in reference to videos there makes sense since there was no problem solving component to the cycle.

6.1.3.1.2 Increase in attending to correct reasoning and decrease in incorrect reasoning.

Similar to the results about relating video(s) to problem solving, attending to the correct reasoning of the students in the video increased from cycle 1 to 4 while attending to incorrect reasoning decreased. The probability cycle showed a reverse in the trends as fewer posts noted the correct reasoning displayed in the video and there was an increase in attending to incorrect reasoning.

6.1.3.1.3 Lack of posts relating readings.

Across all cycles, the teachers' posts about the readings rarely tied into theirs and others' problem solving. Few made a connection between the readings and what they watched in the videos or the practice of teaching. There is evidence that the teachers read the papers as they did post about them, but the posts tended to be summaries of what they read instead of relating it to other ideas. Although less common, some teachers also posted summaries of the videos without attending to any of the topics.

6.1.3.1.4 Classroom environment.

Throughout each cycle and iteration there was an increase in the number of posts that referenced the idea that mathematical reasoning and justification are best suited in an environment that promotes collaboration among students. During the last cycle, which was a reflection cycle, the majority of posts focused on this concept as the teachers attended to the features of the classroom that were similar across all the videos they watched throughout the semester.

6.1.3.1.5 Relation to practice.

In two of the cycles (3 and 6) the teachers related the videos more often to the practice of teaching. Neither cycle had many teachers relating the ideas presented in the readings to their practice. Not captured in the figures above is the difference between PST and ISTs who tie the video to practice. This topic is explored in section 6.1.4.

6.1.3.2 Ratio of posts about video to other types of posts

This analysis looked at the percentage of the posts which were about videos, readings, and other topics (i.e., the classroom environment). Figures 6.6 – 6.10 show that across all cycles and iterations, the majority of the posts referred to the videos. The first cycle contained the most posts the referred the readings, but the amount stayed relatively low throughout each cycle. During the last cycle there was an increase in the amount of posts that attended to the learning environment.

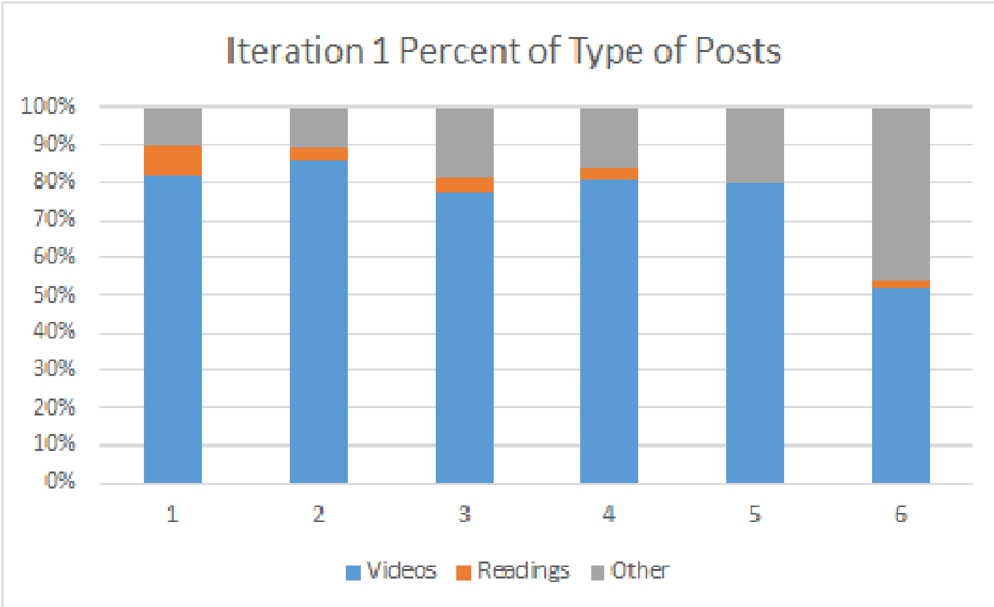


Figure 6.6: Comparison of percent of posts about videos, readings, and other (i.e., argument and justifications as a social activity) by cycle for Iteration 1.

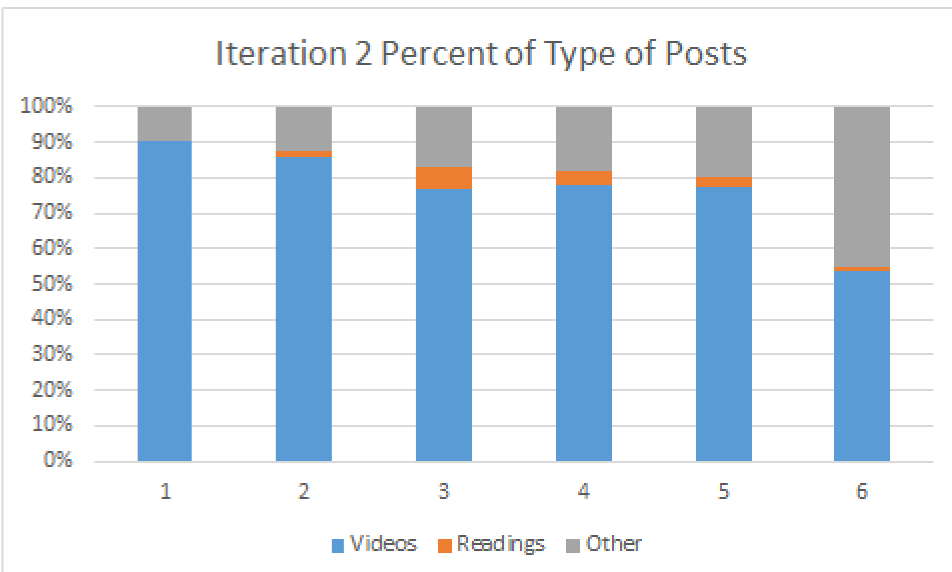


Figure 6.7: Comparison of percent of posts about videos, readings, and other (i.e., argument and justifications as a social activity) by cycle for Iteration 2.

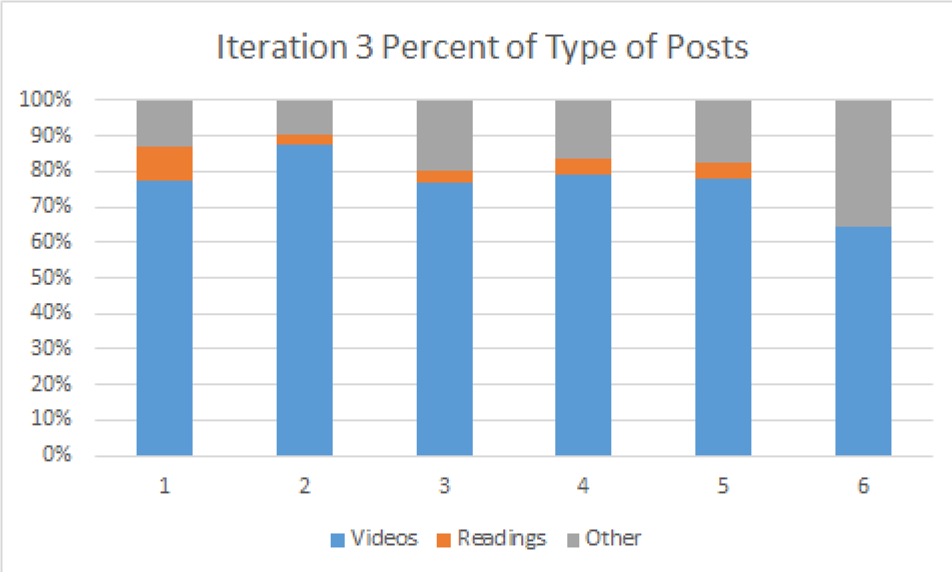


Figure 6.8: Comparison of percent of posts about videos, readings, and other (i.e., argument and justifications as a social activity) by cycle for Iteration 3.

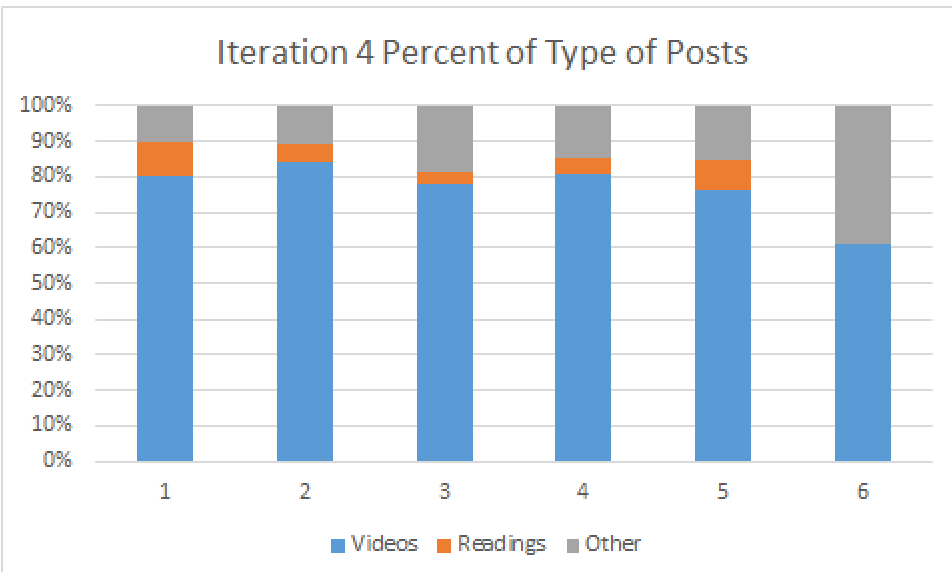


Figure 6.9: Comparison of percent of posts about videos, readings, and other (i.e., argument and justifications as a social activity) by cycle for Iteration 4.

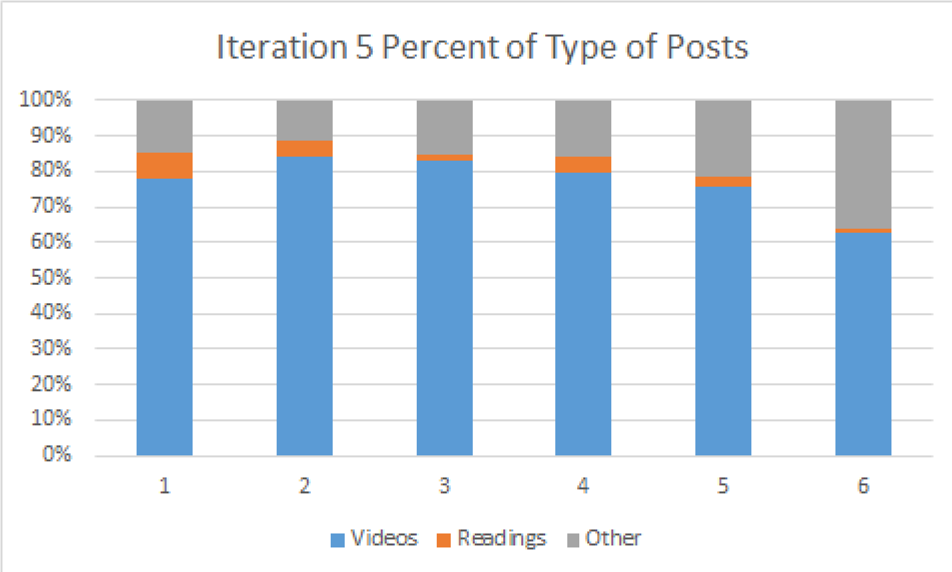


Figure 6.10: Comparison of percent of posts about videos, readings, and other (i.e., argument and justifications as a social activity) by cycle for Iteration 5.

6.1.3.3 Growth of attending to representations.

During the second Iteration, student work modules started to be included into the TLMR model. During Iteration 2 and 3 there were three modules of student work, a fourth module was added in Iteration 4, and a fifth and sixth were added during Iteration 5. In the intervention, the first student work module was always introduced during Cycle 2.

Figure 6.11 maps the frequency of the teachers attending to the representations the students created in the videos during their online discussion (Note: This analysis does not include their discussion inside the student work module; these posts only reference the general discussion). In cycle 1, there is not much change between each iteration. In the other five cycles, there is not much change between Iteration 1 and 2 (when there was no student work module). However, between iterations 2, 3, 4, and 5 there is a significant change in the amount of discussion about the representations the students created in the videos. For example, in Iteration 1 during cycle 3 there were only six mentions about the student representations, but in Iteration 5 there were 22. Cycle 6 saw a big shift from only 4 posts mentioning representations during Iteration 1 compared to 29 in Iteration 5.

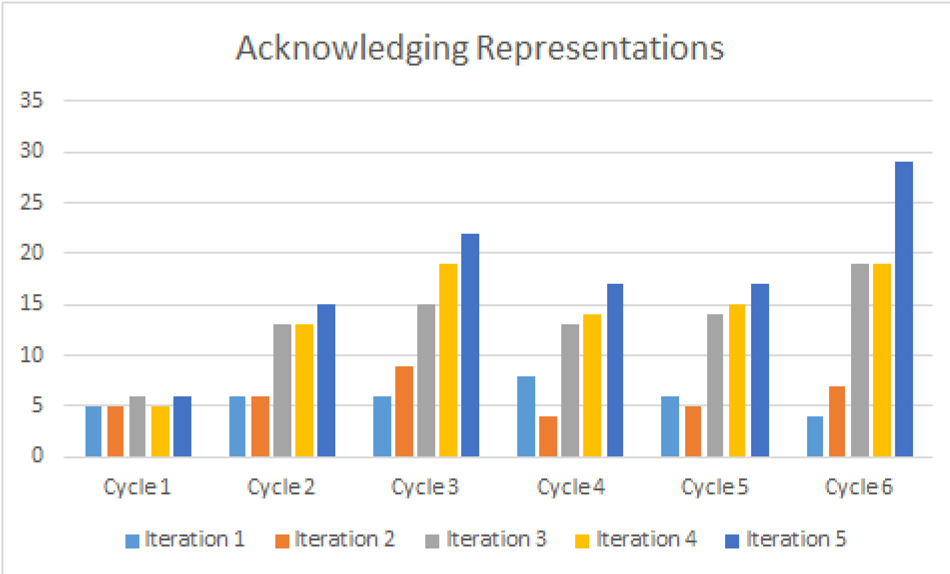


Figure 6.11: Change of acknowledging representations in each cycle by each iteration.

6.2 VMCAnalytic Topics

During the third, fourth, and fifth years of the study, the teachers used a video annotation tool, the RUanalytic, to construct multimedia narratives using the videos on the Video Mosaic website. The assignment required that they edit and annotate video clips to use as evidence to highlight a topic they wanted to explore. Sample topics were tracing a student's understanding of a mathematical topic over several problem sessions and focusing on the questions a teacher/researcher asks when facilitating a problem-solving session.

The teachers participated in training sessions to learn how to use the tool; they were provided text and video tutorials that covered the technical aspects of using the tool. As a component of the intervention, they also watched several VMCAnalytics which were published (i.e., peer reviewed and ingested into the VMC repository as permanent objects that could be shared). They also had several opportunities throughout the intervention to present their preliminary work to other teachers and the instructors to receive feedback. The teachers were grouped with two others from the class to work on their project; they were given responsibility for providing feedback to each other using the commenting feature of the tool.

6.2.1 Methodology.

A feature of the RUanalytic tool is the capability to export all the text into a word document (Figure 6.12). All the teachers who participated in iterations three, four, or five had their analytics exported and coded for analysis by a team of three coders. Forty-three VMCAnalytics were produced in the three iterations.

Each VMCAnalytic has two parts; (a) an overall description where the teacher sets up their argument and describes what their VMCAnalytic is about and (b) a series of events that contain segments of a video clip and text that is displayed alongside the clip while it is playing. Initially, the text from both (a) and (b) was coded openly and discussed among the coders. The codes were then categorized into two groups, one that focused on the students and their thinking and the other that attended to teacher/researcher pedagogical moves. Examples of codes that were categorized as focusing on the student included topics such as the following: tracing a student's problem solving behavior for a particular task over several sessions; identifying the representations that were used by the students; following the strategies or reasoning used by a student while working on a problem; diagnosing the difficulty students had with a particular math topic; and making connections between what the students were doing and relevant literature. Examples of topics that were categorized under classroom environment are included the following: attention to teacher questioning; promoting positive affect in students; and teacher/researcher pedagogical moves.

In addition to incorporating their own text for the overall description and event descriptions and choosing the video clips for each event, the teachers were asked to identify a purpose for their VMCAnalytic. To do this, they were asked to choose from a list of topics that were provided and select at least one from the list to describe the purpose of their analytic The topic choices were: (1) effective teaching, (2) homework activity, (3) lesson activity, (4) professional development activity, (5) sudent collaboration, (6) student elaboration, (7) student engagement, (8) student model building, (9) seasoning, and (10) representations. Along with the two other coders, concensus of the meaning of these terms evolved. Then, a secondary analysis on the VMCAnalytics was conducted in order to determine if the identified topics matched the content of the final analytics.

Stoughton.Rowland, T. (2002a). Generic proofs in number theory. In S. Campbell and R. Zazkis (Eds.) Learning and teaching number theory: Research in cognition and instruction. (pp. 157-184). Westport, CT: Ablex Publishing.Rowland, T. (2002b). Proofs in number theory: History and heresy. In Proceedings of the Twenty-Sixth Annual Meeting of the International Group for the Psychology of Mathematics Education, (Vol. I, pp. 230-235). Norwich, England.Yankelewitz, D. (2009). The development of mathematical reasoning in elementary school studentsâ€™ exploration of fraction ideas. Unpublished doctoral dissertation, Rutgers, The State University of New Jersey.

Purpose(s) Effective teaching; Lesson activity; Professional development activity; Student model building; Reasoning; Representation

Activity

[redacted]

Events

Event: 1

Title Andrew presents his model and uses generic reasoning

Description In this event, Andrew presents a model to compare 1/2 and 1/4 using a purple rod to represent 1. He lines up a train of two red rods and another train of four white rods. He reasons directly that 1/2 (the red rod) was larger than 1/4 (one white rod) by 1/4 (one white rod). He then says that he thinks the solution will always be 1/4, regardless of the model used. Researcher Maher asks him why he thinks that. Andrew says that all the models that had been built and shared during the session â€œalways had the room for one more fourth, and I think that because usually the fourths, or two of â€˜em are equal up to the half, so then it would be a fourth.â€□ As he is speaking, he shows, on his model, that two white rods equal the length of the red rod. With this statement, Andrew uses generic reasoning to justify why 1/2 is always larger than 1/4 by 1/4, regardless of the model chosen to represent these fractions.

Resource https://rucore.libraries.rutgers.edu/rutgers-lib/40579/MOV/1/play/

Timecode 00:01:56 - 00:03:53

Activity [redacted]

Content Type video

Event: 2

Figure 6.12: Example of an exported VMCAnalytic.

6.2.2 Analysis of focus of VMCAnalytics

Based on the coding of teachers' overall description, each of the VMCAnalytics was grouped as either focusing on the student or focusing on the classroom environment. Overall, 25 out of the 43 (58.13%) VMCAnalytics were categorized as focusing on the students; 18 out of the 43 (41.87%) were categorized as focusing on the classroom environment. Table 6.2 shows a breakdown by iteration and shows that in iterations 3 and 5, there was a 2:1 ratio of focus on students compared to classroom environment whereas Iteration 4 was nearly even.

Table 6.2

Codes applied to teacher generated VMCAnalytics by each category.

Iteration	Focus on Students	Focus on Classroom Environment
3	8	4
4	6	7
5	12	6

Based on the differences observed in Chapter 5 between PSTs and ISTs, it was decided to partition the data into those two groups to look for any differences. The results are found below in Table 6.3. Of the 25 VMCAnalytics that focused on students, 22 (88%) of them came from ISTs and only 3 (12%) were from PSTs. Out of the 30 ISTs who created a VMCAnalytic in the last three iterations, 73.3% of them focused on the student, and 26.7% focused on the classroom environment. Of the 13 PSTs, only 3 (23.07%) focused on the students and the other ten focused on the classroom environment (76.93%).

Table 6.3

Codes applied to teacher generated VMCAnalytics by each category comparing pre-service teachers to in-service teachers.

Sub-population	Focus on Students	Focus on Classroom Environment
PST	3	10
IST	22	8
Overall	25	18

6.2.3 Analysis comparing their intended outcomes to what they showed.

The teachers were able to add up to ten tags to identify the purpose of the VMCAnalytic. All forty-three teachers added at least one tag, with most teachers adding three. Table 6.4 shows the breakdown of the tags applied. These are separated into either PST of IST groups. For both PSTs and IST, the most common tags they used were effective teaching, student elaboration, and reasoning.

Table 6.4

Topics selected by the teacher for what their VMCAnalytics showed.

Topics	PST	IST	Total
Effective Teaching	6	10	16
Homework Activity	0	0	0
Lesson Activity	1	4	5
Professional Development Activity	4	6	10
Student collaboration	4	6	10
Student elaboration	6	7	13

Student engagement	3	5	8
Student model building	3	5	8
Reasoning	8	10	18
Representations	2	5	7

Each VMCAnalytic was blindly coded using the categories described above along with an interpretation within that category. This researcher and the two other coders achieved IRR > 90% and came to agreement about the codes in which there was initial disagreement. These codes were then compared to what the teachers claimed to be the purpose of their VMCAnalytic.

Table 6.5 shows the results of the comparison. A difference between this researcher's coding and the tags applied by the teachers appears with regards to PSTs coding of the reasoning purpose. Eight of the PSTs said the purpose of their analytic was to highlight student reasoning. However, of those eight teachers, only two actually referred to reasoning as a topic, whereas for ISTs 9 of the 10 that claimed to show student reasoning produced VMCanalytics that actually displayed student reasoning. For the ISTs, 10 teachers claimed that the purpose of their VMCAnalytic was to show effective teaching. However, only 3 teachers mentioned teaching at all. .

Table 6.5

Topics selected by the teacher for what their VMCAnalytics showed compared to what was coded by the author.

Topic	PST	My coding PST	IST	My coding IST

Effective Teaching	6	5	10	3
Homework Activity	0	0	0	0
Lesson Activity	1	1	4	4
Professional Development Activity	4	2	6	5
Student collaboration	4	2	6	6
Student elaboration	6	3	7	6
Student engagement	3	3	5	5
Student model building	3	3	5	5
Reasoning	8	2	10	9
Representations	2	2	5	5

Chapter 7: Discussion

7.1 Introduction

Five iterations of a graduate level mathematics education course taken by in-service (IST) and pre-service teachers (PST) were examined to study what teachers learned while undergoing an intervention model that was designed to promote their attending to student reasoning. Several themes emerged tracing how the teachers attended to student reasoning. Also studied were differences in recognizing student reasoning between subsets of teachers. This chapter will summarize the findings presented in the previous chapters.

7.2 Validation of the Teacher Learning about Mathematical Reasoning (TLMR) Model

A major finding of the study is the validation of the TLMR model where teachers' engaged in open-ended problem solving, discussed their solutions, watched videos of students engaging in the same or similar tasks, and read about related work. When the results were analyzed in contrast to a comparable population, the teachers who engaged with the TLMR showed a significant positive shift in attending to videos of student reasoning and a shift in their beliefs towards what is outlined in the NCTM Standards. This shift was observed for both ISTs and PSTs. Considering the current environment of mathematics education with emphasis on the new Common Core Standards and the associated assessments that focus on mathematical reasoning, the results are important.

While there was some shift in teachers attending to the argument by contradiction from the pre-test to the post-test, it was only a small percentage of the teachers who recognized the argument by contradiction. Several studies (e.g., Thompson, 1996) have showed that students struggle with the idea of argument by contradiction with some not recognizing it as a valid form of arguing. Others dismiss the argument preferring to focus on more direct forms of argument. Future research is needed to understand why only about one third of the teachers' attended to this argument compared to more than 70% for the other argument forms and also what intervention may help teachers attend to this valid form of reasoning. One might speculate that the lack of understanding might be explained by different emphasis in curriculum, with less attention to logic and the equivalence of a statement and its contrapositive.

7.3 Videos as Anchors for Discussions about Students Mathematical Reasoning

For the online discussions, teachers had three sources they could draw upon: their own experiences, the videos, and the readings. Across all cycles and iterations, discussion about a video in which the teachers had to construct their own interpretation of events drove the discussion although textual resources were provided that explained topics in detail. Providing the teachers the opportunity to engage in the same problem solving as the students they observed on videos, and reflect on their own problem solving led to discussions that provided some insight into what the teachers were thinking as compared to the discussions about the readings which tended to focus on a summary of the main ideas of the reading without insight into ideas that the teachers may hold. Using the combination of problem solving with studying video provided an opportunity for teachers to engage and reflect with the ideas in new ways that were not available with only textual resources.

7.4 Student Work and the Connection with Referencing Representations

While during the first iteration of the study the teachers were attending to students reasoning in the videos, their posts were mainly referring to the verbal reasoning from the students as they talked with each other. Instances where the student representations were described tended to attribute wrong ideas to the students or misrepresent their representations. In the practice of teaching it is important for teachers to attend to both the verbal and written reasoning of children. Also, the PSTs may not have had much opportunity to attend to student written work, while the ISTs regularly are expected to evaluate homework and tests.

After the first iteration, student work modules were introduced into the TLMR model. After participating in the modules, the teachers started to correctly reference the representations created by the students in the video more often and the variety of representations became a focal point for the discussion. The references to the representations did not take away from referencing the students' verbal explanations. Instead, the teachers were now noticing both forms of reasoning and comparing and contrasting them with each other. Other studies on teacher noticing have only focused on video captured from a teacher's own classroom (e.g., Sherin & van Es, 2009), video from a video repository (e.g., Maher et. al, 2014), or examining student work with writing assignments without using video (Warshauer, Strickland, Namakshi, Hickman, & Bhattacharyya, 2015). The results from this study show that it is possible for PSTs and ISTs to attend to student reasoning in both written and verbal form while making connections between them.

7.5 Difficulty with Probability

The fifth cycle of the intervention introduced the topic of probability through two problem tasks, the World Series problem and the Problem of Points. Previous to this cycle there was a steady increase in identifying the correct reasoning of the students in the videos and discussion about the students problem solving relative to their own and others. All the problems in the previous cycle were in the counting and combinatorics domain. However, during this cycle there was a consistent drop across each iteration with regards to those topics as well as an increase in faulty reasoning by the teachers in the study even after class discussions took place.

While there is a relationship between the topics of combinatorics and probability, the jump from one to the other caused issues. There is extensive literature that shows both children and adults struggle with probability (e.g., Fischbein, 1975). In the fifth Iteration a simpler probability problem was added before the World Series problem to see if that would be helpful, but the differences were no significant. Future research is needed to understand the difficulties that students face as they connect the ideas in combinatorics and probability and how, if at all, the TLMR can help bridge the gap.

7.6 Recommendations for PST training and PD Programs

While both the PSTs and ISTs grew in attending to student reasoning, there were differences between the two populations. Comparing the two populations, the ISTs shifted more towards the beliefs expressed in the Standards and attended to more forms of reasoning than the PSTs. It should be emphasized that the PSTs in this study had little to no classroom experience. They did not, for example, participate in a practicum experience where they would observe a classroom for a set amount of hours per semester, and it was not a requirement for the course that they have a classroom experience.

One possible explanation for the difference may be explained through their discussion posts. When discussing the students reasoning from the videos, the PSTs tended to give the credit to the learning environment instead of the reasoning of the student. When referencing the reasoning of a student, their discussion tended to be framed on how the teacher [note: In the videos the facilitators were researchers and not teachers, but both ISTs and PSTs tended to refer to them as teachers] promoted student reasoning through questioning or how the design of the tasks elicited the reasoning. This trend re-appeared in the topics the PSTs chose for the VMCAnalytic project as they overwhelmingly chose to construct narratives that were about features of the classroom environment to highlight what they considered “good” teaching. While the PSTs claimed that their VMCAnalytics highlighted student reasoning, very few actually did.

ISTs also attended to the features of the learning environment, but they also attributed agency to the student in developing their reasoning. Their posts and VMCAnalytics tended to reflect more of an interplay between the two whereas the PSTs also tended to look at it more like a script that you follow to get students to reason. This view is consistent with those expressed by Flores (2009) in that PSTs tend to first approach teaching through the eyes of the teacher and try to mimic teaching styles. Perspectives chosen by the PSTs seem to reflect their imminent concerns about pedagogy with transitioning to the new role of classroom teacher, whereas ISTs have the foundation of their prior experiences to focus on the details of students' learning as a primary aspect of pedagogy.

Going forward, when working with PSTs and the TLMR model, the activities should be embedded in some sort of practicum or classroom experience for the PSTs. By giving them the opportunity to try the same or similar tasks with actual children and share their experiences with other PSTs and ISTs, they may be able to see the interplay between the environment and the students instead of just focusing on one of these factors. If possible, pairing the PSTs with ISTs in the same class would be ideal since they both will have a common experience going through the TLMR to discuss about and help each other make sense of their experiences instead of putting the PST into a more traditional type classroom which may cause confusion due to lack of support from the teacher.

7.7 Limitations

This study contributes to the literature about mathematical reasoning in a positive way. It reveals design practices that build pre and in-service teachers’ own reasoning as well as recognizing the growth of reasoning in students. It is reasonable to expect significant growth with certain conditions in place (e.g., duration of project, access to rich tools and relevant videos, opportunities to collaborate and exchange ideas). Knowledge about student reasoning can be used to help better equip teachers to teach mathematics as outlined in the mathematical Standards for Practice. The study revealed potential benefits of using video in teacher preparation and professional development with support from the resources available on the VMC. Studying participant use of the VMCAnalytic provides another assessment tool that can uncover their understanding of how students learn. Further, the study offered important new knowledge to modify course design so that student learning is optimized.

The results of this study may be generalizable for other students in a similar course setting. However, it is not expected that the results would generalize to teachers in courses whose designs are not comparable. Another limitation is the idea of scalability. The instructors of the course have experience working with teachers and the TLMR model. Future studies are needed to determine whether the model can be used by other instructors with the same success rate. However, these limitations are outweighed by the insight gained into the development of teachers recognizing and critiquing students' reasoning and may guide future design and interventions.

7.8 Future Studies

7.8.1 Reasoning as a general skill.

A future study can focus on determining whether learning to attend to student reasoning is embedded in a specific mathematical context or whether a teacher develop general knolwdge across content domains in attending to reasoning skill. This study took place predominantly within the context of counting and combinatorics tasks.. It would be interesting to explore whether the knowledge the teachers gained transfer successfully to studying reasoning in another domain, such as fractions and rational numbers without directly experiencing working on the tasks and studying videos of students' reasoning about these tasks. Introducing the probability module in cycle 5 provided mixed results. Might that be due to the difficulty in learning probability ideas, or understanding probabilistic reasoning, or because the teachers did not have the knowledge to transfer to a new domain?

7.8.2 Classroom applications.

While both PSTs and ISTs grew in their ability to notice student reasoning, there needs to be work to understand how they take that knowledge to the classroom when they plan and enact their own lessons with their students. Developing the skill of attending to reasoning is crucial, but teachers also need to know how to engage with the student reasoning and use the reasoning in the lessons they teach. Future studies are suggested that follow teachers implementing of lessons in their own classrooms.

Appendix A: Example Class Syllabus

Introduction to Mathematics Education, **Fall 2010** (15:254:540 Sec. 01)

HYBRID COURSE (Index # 01143)

On-Campus Meeting Dates: 9/13, 9/20, 10/4, 10/18, 11/1, 11/8, 12/13, 12/20

Mondays, 4:50-7:30, GSE Room 30

Graduate School of Education, 10 Seminary Place

Professor Carolyn A. Maher

Chapter 8: CONTACT INFO

Instructor	Carolyn Maher
Assistant Instructor	Marjory Palius
Course Web Support	Robert Sigley

OFFICE HOURS

Mondays (on-campus dates only), 3:30-4:30 and by appointment

OBJECTIVES

This course is designed to introduce participants to the field of mathematics education through a variety of activities that blend in-person, on-campus sessions with interactions done asynchronously online through a course web site. The on-campus activities will be to work in small groups on mathematical problem-solving tasks, with consideration of how K-12 students might engage with those tasks as they build solutions to problems. The online course work will include reading assignments that introduce participants to theoretical perspectives on learning and research in math education, with guidelines for engaging in reflection and discussion of those readings and considerations of their relevance to teaching practices. Other online course work will include studying video clips of children engaged in math problem solving and talking about their mathematical ideas; through reflection and online discussion the videos will be connected to the readings and hands-on problem solving. Emphasis will be on the mathematics, children's learning, and conditions of the learning environment. We will focus on the content strand of counting and combinatorics, from early years through high school, and consider implications drawn from research for instruction in light of NCTM Standards.

This course also is designed as a site for examining how teachers can learn about students' mathematical reasoning through studying videos that feature children doing thoughtful mathematics. Part of that process entails first engaging as a learner with cognitively challenging tasks by working with a partner or small group, and then attentively viewing videos of students who engage with those same tasks. You will complete assessments (pre and post) for measuring the impact of course activities in the focal mathematical strand on what you notice and how you describe what you observe in an example video and on beliefs about learning and teaching math. You will be given a consent form about whether your assessments can be among those analyzed for ongoing research. Completing the assessments is not optional; it is a course requirement.

COURSE REQUIREMENTS

You are invited to be an active participant in the class through small group work in the classroom and though web-based discussions, projects, lectures and writing. Successful completion of the course requires that you engage in all activities and submit all assignments. You are required to:

1. Complete all pre- and post-assessments.

2. Attend all on-campus sessions and lectures.

3. Actively participate in online discussions as you engage with assignments (readings and videos) and respond to guiding questions as posted on the eCompanion course website. You are required to make at least one original posting and respond to at least two group member postings per week.

4. Be knowledgeable of all the assigned readings and video clip viewings.

5. Prepare a teaching philosophy statement. Note that this is a GSE requirement for the Introduction to Mathematics Education course. More information will be provided to you with ePortfolio instructions, requirements, and rubric for evaluating your statement. It will be due by November 15. This is a personal philosophy of teaching, which should include the following elements:

a. Age group for which you are considering this personal statement of teaching

b. What you think the purposes of schooling should be

c. Your position on the questions or problems central to your discipline

d. How students learn

e. How you will teach

f. Why and how you will respond to differences in ability, interest and background of your students

6. Complete an *Individual / Group Research Project*. You will work within small groups to do research about students learning mathematics using videos from the Davis Institute's collection that were collected during the Informal Mathematics Learning study with urban middle school students. Each group member will transcribe about 20-25 minutes of video, and then partner with another group member to verify each other's transcript. In this manner the groups will produce a verified transcript for the whole session to analyze collaboratively. Together, you will study the video and transcript to describe the emergence of particular mathematical ideas and ways of reasoning that are expressed by students as they engage in problem solving tasks that you will have done for yourselves in this course. You will be supported in your analysis, which will identify clips of video that illustrate how mathematical ideas and reasoning emerge during the session. Your group also will prepare short descriptions

of the video clips that you identify. PowerPoint™ presentations of findings will be presented to the entire class. Also you will provide an individual final report of your analysis of the session that should draw from the group's collaborative work. Formats for the presentations will be provided.

7. Complete a short (1-2 pages) *reflection paper* about your work in this course. This will be the final assignment and due on December 20. You should reflect on your knowledge of the mathematics, research on how students learn, and implications for teaching with regard to NCTM Standards. You may review your postings on the course web site and notes from problem solving and sharing of solutions as you develop your reflective assessment, which should be about one to two pages in length.

You will be evaluated on your work products for individual / group research project, reflection paper, and teaching philosophy statement, as well as your participation in person and on line.

COURSE OUTLINE AND ASSIGNMENTS

9/8/2010 ONLINE ASSIGNMENT	**Assignments – pre-assessments, readings, and respond online to guiding questions:** Complete pre-assessments using the eCompanion course web site. These assessments must be completed prior to the on-campus class session on Sept. 13.

	<u>After completing the assessments</u>, the reading assignment is: **(1)** Erlwanger, S. H. (1973). Benny's Conception of Rules and Answers in IPI Mathematics. *The Journal of Children's Mathematical Behavior* 1(2), 7-26.
9/13/2010 ON-CAMPUS	**Class Activities:** Introduction to the course; Engage in problem-solving task from early grades counting strand with problem extensions and focused discussion about representations; Review syllabus and discuss course requirements. **Assignments – readings, videos and respond online to guiding questions:** **(1)** Maher, C. A. & Martino, A. M. (1992). Teachers building on students' thinking. *Arithmetic Teacher, 39*(7), 32-37. **(2)** Maher, C. A. & Weber, K. (2010). Representation Systems and Constructing Conceptual Understanding. Special Issue of the *Mediterranean Journal for Research in Mathematics Education 9*(1), 91-106. **(3)** Combinatorics book (Maher, Powell & Uptegrove, Eds.)*, chapters 1 & 2 **(4) Videos:** Clips to be posted on course web site (Shirts/Pants, Grades 2 & 3)

9/20/2010 ON-CAMPUS	**Class Activities:** Engage in task for building towers 5/4/3/n-tall; Extend problem solving to "Guess My Tower" task; Discuss heuristics and problem solving for early tasks in counting strand, and ideas from assigned readings. Introduce Projects **Assignments – readings, videos and respond online to guiding questions:** **(1)** Maher, C. A. & Martino, A. M. (1996). The development of the idea of mathematical proof: A 5-year case study. In F. Lester (Ed.), *Journal for Research in Mathematics Education, 27* (2), 194-214. **(2)** Maher, C. A. (2009). Children's reasoning: Discovering the idea of mathematical proof. In M. Blanton, D. Stylianou and E. Knuth (Eds.), *Teaching and learning proof across the K-16 curriculum* (pp. 120-132). New Jersey: Taylor Francis - Routledge. **(3)** Combinatorics book, Chapters 3, 4 and 5 **(4) Videos:** Building towers clips from grades 4 & 5; Interview with Meredith; Manjit's clips
9/27/2010 ONLINE ASSIGNMENT	**Online Activities:** Respond to the guiding questions to be posted online for engagement in threaded discussion about the various towers problem-solving tasks and related videos and readings. **Assigned readings:**

	(1) JRME Monograph #4**, Introduction and Chapters 1 and 2
10/4/2010 ON-CAMPUS	**Class Activities:** Engage in a pizza problem task: pizzas with halves, selecting from 4 toppings; pizzas, selecting from 4 toppings; pizzas, selecting from n-toppings. Share how solutions were found and examine representations used in problem solving; Consider whether justifications offered are convincing and why / why not; Watch Brandon video and share observations / impressions; Discuss readings; Support for projects: status of transcription and verification **Assignments – readings, videos and respond online to guiding questions:** **(1)** Maher, C. A. & Martino, A. (1998). "Brandon's Proof and Isomorphism". In C. A. Maher, *Can teachers help children make convincing arguments? A glimpse into the process.* Rio de Janeiro, Brazil: Universidade Santa Ursula. **(2)** Combinatorics book, Chapters 6 **(3) Videos:** PUP-Math Pizza clips
10/11/2010	**Online Activities:** Respond to the guiding questions to be posted online; Consider video clips viewed thus far and what they indicate about students' learning of

ONLINE ASSIGNMENT	mathematics and the conditions of the learning environment; Share ideas and themes raised in the assigned readings. **Assigned Readings: (1)** Davis, R. B. (1992). Understanding 'understanding' (1992). *The Journal of Mathematical Behavior, 11*, 225-241. **(2)** JRME Monograph #4, Chapters 3 and 5
10/18/2010 ON-CAMPUS	**Class Activities:** Introduce "Ankur's Challenge" as problem-solving task; Support for Projects: Discuss expectations for research projects in greater detail, specifically identification of clips based on where critical events occur. **Assignments – readings, videos and respond online to guiding questions:** **(1)** Combinatorics book, Chapter 8 **(2)** Yackel, E. & Hanna, G. (2003). Reasoning and proof. In J. Kilpatrick, G. W. Martin, and D. Schifter, (Eds.), *A Research Companion to Principles and Standards for School Mathematics* (pp. 227-236). Reston, VA: National Council of Teachers of Mathematics. House E. 1980. *Evaluating with validity.* Sage Press, Beverly Hills. **(3) Videos:** PUP-Math Romina's proof

10/25/2010 ONLINE ASSIGNMENT	**Online Activities:** Respond to the guiding questions to be posted online for engagement in threaded discussion about ideas from recent problem solving activities and related videos and readings, with focus on reasoning and proof in mathematics. **Assigned Readings: (1)** Tsamir, P., Tirosh, D., Dreyfus, T., Barkai, R. & Tabach, M. (2009). Should proofs be minimal? Ms T's evaluation of secondary students' proofs. *The Journal of Mathematical Behavior 28*, 58-67.
11/1/2010 ON-CAMPUS	**Class Activities:** Engage in exploration of binomial expansion; Discuss readings. **Guest Lecture:** Professor Gunnar Gjone, *What is Quality in Mathematics Teacher Education?- and How Can We Improve It?* **Assignments – readings, videos and respond online to guiding questions:** **(1)** Maher, C. A. & Speiser, R. (1997). How far can you go with block towers? Stephanie's Intellectual Development. *The Journal of Mathematical Behavior, 16*(2), 125-132. **(2)** Combinatorics book, Chapter 7 **(3) Videos:** Clips from Eman's dissertation research or ??

11/8/2010 ON-CAMPUS	**Class Activities:** Exploration of Pascal's Triangle; Investigation of meaning behind symbolic notation – Why does the addition rule work? Discussion of readings. Engage in the Taxicab Problem and share ideas that emerge from the task. Provide support for projects: preparation of descriptions for the groups' selected video clips and their relation to input forms for placing clips into the VMC **Assignments – readings, videos and respond online to guiding questions:** **(1)** Combinatorics book, Chapters 12 and 13 **(2) Videos:** Night Session clips and Taxicab clips
11/15/2010 ONLINE ASSIGNMENT	**Online Activities:** Respond to the guiding questions to be posted online for engagement in threaded discussion about ideas from recent problem solving activities and related videos and readings, Guidelines to be posted for preparing for presentations in December ***Teaching Philosophy Statement DUE – Submit via Sakai per instructions*** **Assignments – readings, videos and respond online to guiding questions:**

	(1) Maher, C. A. (2005). How students structure their investigations and learn mathematics: Insights from a long-term study. *The Journal of Mathematical Behavior, 24*(1) 1-14. **(2)** Francisco, J. M. & Maher, C. A. (2005). Conditions for promoting reasoning in problem solving: Insights from a longitudinal study. Special Issue: Mathematical problem solving: What we know and where we are going (Guest Editors: Cai, J, Mamona-Downs, J. & Weber, K.) *The Journal of Mathematical Behavior, 24(*3-4), 361-372. **(3) Videos:** clips from KW student interviews reflecting on their experiences
11/29/2010 ONLINE ASSIGNMENT	**Online Activities:** Respond to the guiding questions to be posted online for engagement in threaded discussion about the *STRAND* of counting-combinatorics tasks and their impact on students in the KW longitudinal study, with focus on epistemology and the relationship to KW students' reflections on their experiences. **Assignments:** Continue work on research projects. Do the post-assessments, which should be completed by end of November.
12/6/2010	**Online Activities:** Complete post-assessments using the eCompanion course web site. After these have been submitted, you may begin work on the reflective

ONLINE ASSIGNMENT	assessment. Work with your group to finalize presentations for research project reports.
12/13/2010 ON-CAMPUS	**Class Activity:** Research Project Reports
12/20/2010 ON-CAMPUS	**Class Activity:** Research Project Reports *Reflection Paper DUE*

Notes about reading assignments:

Assigned readings will be made available through the eCompanion site for this course.

* Maher, C. A., Powell, A. B. & Uptegrove, E. (Eds.), (in press). *Combinatorics and reasoning: Representing, justifying and building isomorphisms.* Springer Publishers.

Readings from the above-listed book are being made available, however the book is still in press and must not be cited.

The JRME Monograph #4 is:

** Davis, R. B., Maher, C. A. & Noddings, N. (Eds.). (1990). Constructivist views on the teaching and learning of mathematics: Journal for Research in Mathematics Education, Monograph No. 4. Reston, VA: National Council of Teachers of Mathematics.

As a general guideline for engaging in online discussions, we offer a few words on "Netiquete." This is drawn from Palloff, R. M., & Pratt, K. (1999). *Building learning communities in cyberspace*. San Francisco: Jossey-Bass, p. 101.

- Check the discussion frequently and respond appropriately and on the subject
- Focus on one subject per message and use pertinent, informative, and not-too-long subject titles
- Capitalize words only to highlight a point or for titles. Capitalizing otherwise is generally viewed as SHOUTING
- Be professional and careful with your online interaction
- Cite all quotes, references, and sources
- When posting a long message, it is generally considered courteous to warn readers at the beginning of the message that is a lengthy post
- It is inappropriate to forward someone else's message(s) without their permission

- Use humor carefully. The absence of face-to-face cues can cause humor to be misinterpreted as criticism or flaming (angry, antagonistic criticism). Feel free to use emoticons such as :-) or ;-) to let others know that you’re being humorou

Appendix B: Statement of Problems

B.1 Shirts and Pants

Stephen has a white shirt, a blue shirt, and a yellow shirt. He has a pair of blue pants and a pair of white pants. How many different outfits can he make?

B.2 The Towers Problem

You have two colors of Unifix cubes available with which to build towers. Make as many different looking towers as is possible, each exactly four cubes high selecting from those two colors. Find a way to convince yourself and others that you have found all possible towers four cubes high and that you have no duplicates. The problem can be expanded later to looking at towers of height: 5, 3, n and to more colors (e.g., 3 - yellow, red, and blue).

B.3 A Four-Topping Pizza Problem

A pizza shop offers a basic cheese pizza with tomato sauce (no halves). A customer can then select from the following toppings to add to the whole basic pizza: peppers, sausage, mushrooms, and pepperoni. How many different choices for pizza does a customer have? List all the possible different selections. Find a way to convince each other that you have accounted for all possibilities.

B.4 A Pizza Problem with Halves (Two or four toppings)

A local pizza shop has asked us to help them keep track of certain pizza sales. Their standard ”plain” pizza contains cheese. On this cheese pizza, one or two toppings can be added to either half of the plain pie or the whole pie. How many possible choices for pizza do customers have if they can select from two different toppings (sausage and pepperoni) that could be placed on either the whole cheese pizza or half a cheese pizza? List all the possible different selections. Find a way to convince each other that you have accounted for all possibilities. This problem can be expanded to using four toppings.

B.5 Ankur's Challenge (Towers problem)

Find as many towers as possible that are 4-cubes tall if you can select from three colors and there must be at least one of each color in each tower. Show that you have found all the possibilities.

B.6 The World Series Problem

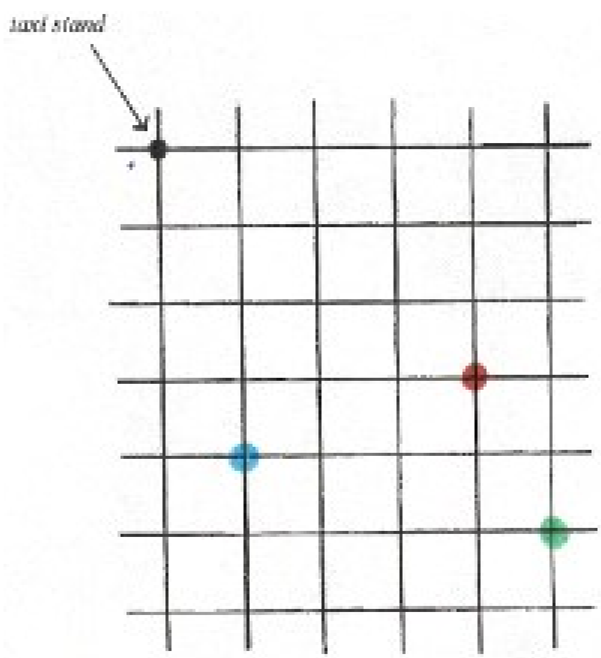
In a World Series, two teams play each other in at least four and at most seven games. The first team to win four games is the winner of the World Series. Assuming that the teams are equally matched, what is the probability that a World Series will be won: (a) in four games? (b) In five games? (c) In six games? (d) In seven games?

B.7 The Problem of Points

Pascal and Fermat are sitting in a cafe in Paris and decide to play a game of flipping a coin. If the coin comes up heads, Fermat gets a point. If it comes up tails, Pascal gets a point. The first to get ten points wins. They each ante up fifty francs, making the total pot worth one hundred francs. They are, of course, playing ”winner takes all.” But then a strange thing happens. Fermat is winning, 8 points to 7, when he receives an urgent message that his child is sick and he must rush to his home in Toulouse. The carriage man who delivered the message offers to take him, but only if they leave immediately. Of course, Pascal understands, but later, in correspondence, the problem arises: how should the 100 Francs be divided? Justify your solution.

B.8 The Taxicab Problem

A taxi driver is given a specific territory of a town, represented by the grid in the diagram below. All trips originate at the taxi stand, the point in the top left corner of the grid. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated by the other points on the grid. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route. What is the shortest route from the taxi stand to each point? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answers.



Appendix C – Citations for Readings Used in the Course

Agnew, G., Mills, C. M., & Maher, C. A. (2010). VMCAnalytic: Developing a collaborative video analysis tool for education faculty and practicing educators. In R. H. Sprague, Jr. (Ed.), *Proceedings of the 43rd Annual Hawaii International Conference on System Sciences (HICCS-43): Abstracts and CD-ROM of Full Papers.* IEEE Computer Society, Conference Publishing Services: Los Alamitos, CA.

Bartell, T. (2011). Caring, race, culture, and power: A research synthesis toward supporting mathematics teachers in caring with awareness. *Journal of Urban Mathematics Education*, *4*(1), 50-74.

Battey, D. (2013). "Good" mathematics teaching for students of color and those in poverty: the importance of relational interactions within instruction. *Educational Studies in mathematics*, *82*(1), 125-144.

Davis, R. B. (1992). Understanding 'understanding' (1992). *The Journal of Mathematical Behavior, 11*, 225-241.

Davis, R. B. & Maher, C. A. (1990). What do we do when we do mathematics? In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), Constructivist Views on the Teaching and Learning of Mathematics*, Journal for Research in Mathematics Education Monograph No.* 4, 65-78. Reston, VA: National Council of Teachers of Mathematics.

Davis, R. B., Maher, C. A., & Noddings, N. (1990). Constructivist Views on the Teaching and Learning of Mathematics. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), Constructivist Views on the Teaching and Learning of Mathematics*, Journal for Research in Mathematics Education Monograph No.* 4, 1-3. Reston, VA: National Council of Teachers of Mathematics.

Eberbach, C., & Crowley, K. (2009). From everyday to scientific observation: How children learn to observe the biologist's world. *Review of Educational Research*, *79*(1), 39-68.

Erlwanger, S. H. (1973). Benny's Conception of Rules and Answers in IPI Mathematics. *The Journal of Children's Mathematical Behavior*, 1(2), 7-26.

Ewing, J. (2011). Mathematical intimidation: Driven by the data. *Notices of the AMS*, *58*(5), 667-673.

Francisco, J. M. (2013). Learning in collaborative settings: students building on each other's ideas to promote their mathematical understanding. *Educational Studies in Mathematics*, *82*(3), 417-438.

Francisco, J. M. & Maher, C. A. (2005). Conditions for promoting reasoning in problem solving: Insights from a longitudinal study. Special Issue: Mathematical problem solving: What we know and where we are going (Guest Editors: Cai, J, Mamona-Downs, J. & Weber, K.). *The Journal of Mathematical Behavior, 24(*3-4), 361-372.

Fried, M. N. (2007). Didactics and history of mathematics: Knowledge and self-knowledge. *Educational Studies in Mathematics*, *66*(2), 203-223.

Fried, M. N. (2001). Can mathematics education and history of mathematics coexist?. *Science & Education*, *10*(4), 391-408.

Greer, B., & Harel, G. (1998). The role of isomorphisms in mathematical cognition. *The Journal of Mathematical Behavior*, *17*(1), 5-24.

Iannone, P., Inglis, M., Mejía-Ramos, J. P., Simpson, A., & Weber, K. (2011). Does generating examples aid proof production? *Educational Studies in Mathematics*, *77*(1), 1-14.

Klein, D. (2003). A brief history of American K-12 mathematics education in the 20th century. *Mathematical cognition*, 175-259.

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Appendix D – Student Work Assessment

For each module the teachers were asked to comment on:

a) correctness of their solution.

b) insight into their strategy

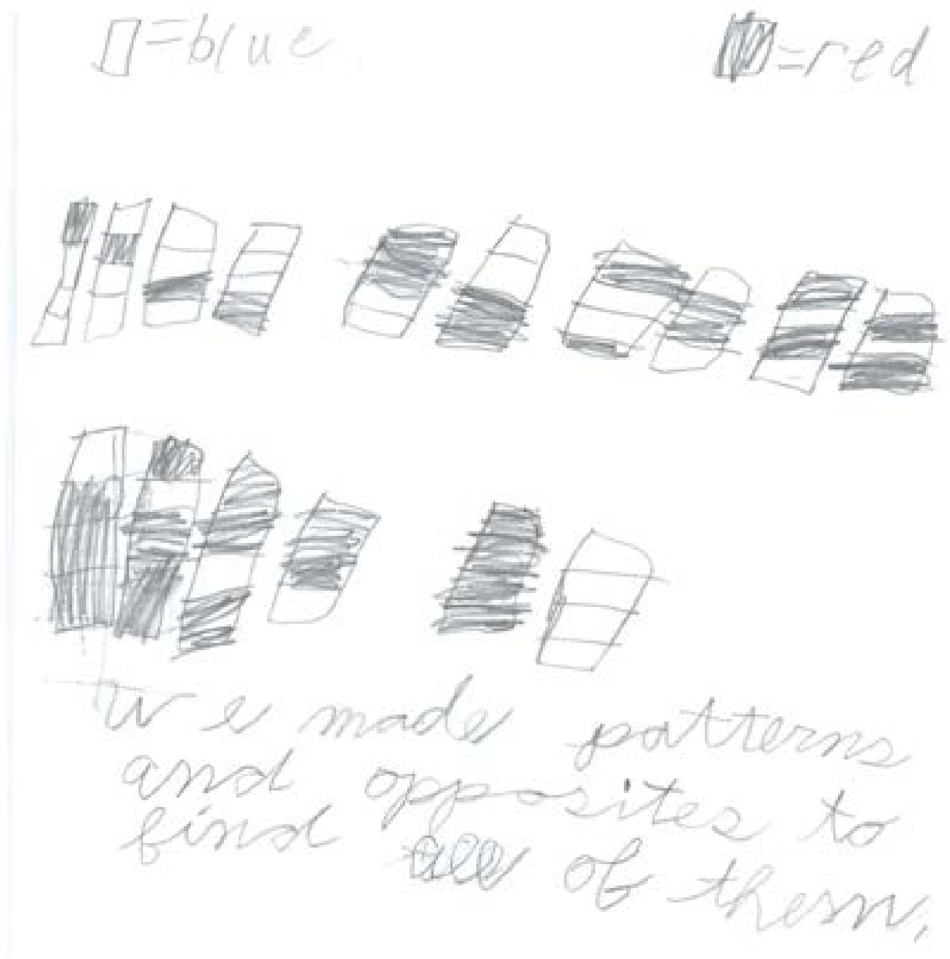
c) validity of their reasoning

d) whether or not you find the solution convincing?

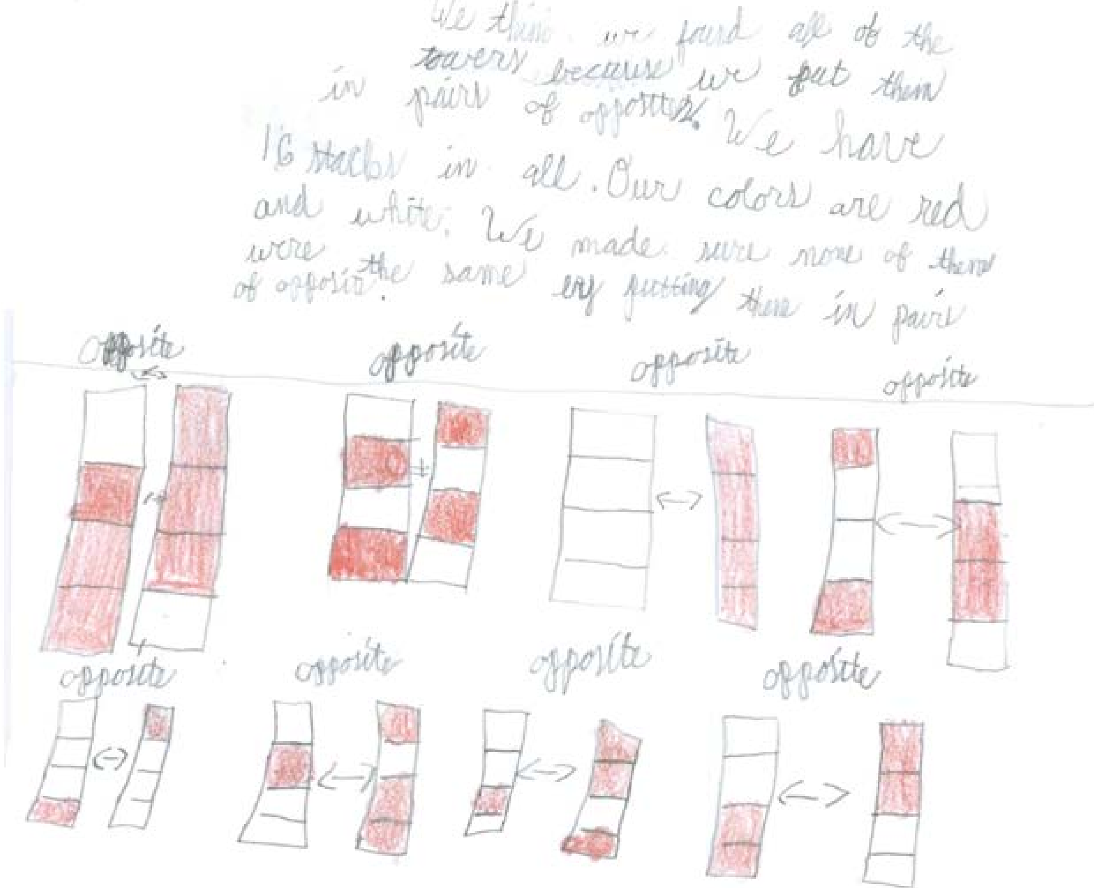
If you do NOT find the solution convincing, indicate what aspect of their work you might ask them to say more about to help them move forward in developing a convincing argument?

Towers Selecting From Two Colors Module

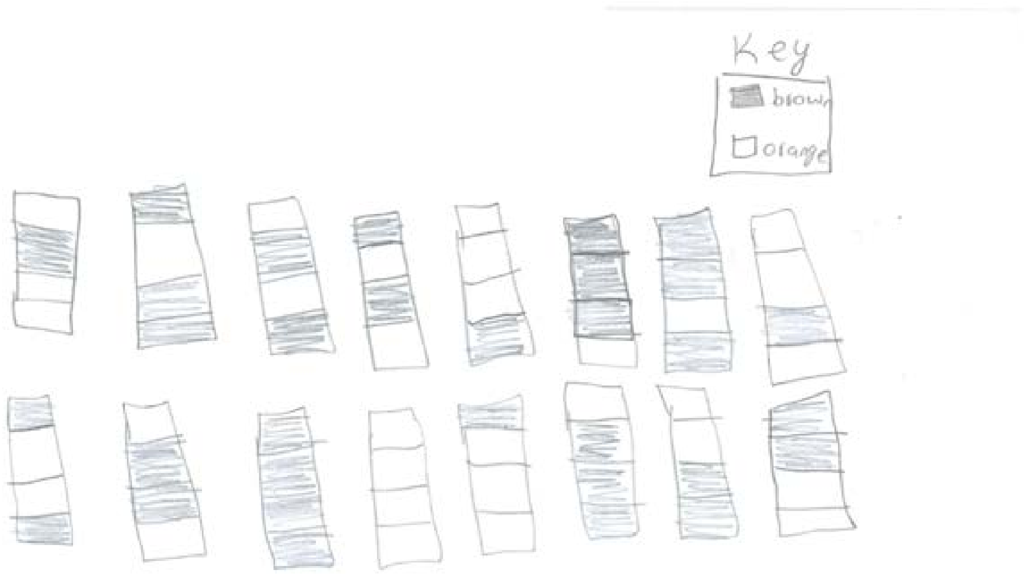
1) Student work by Tony, a 3rd grader, in response to the two-color towers problem.



2) Student work by Kelly, a 3rd grader, in response to the two-color towers problem

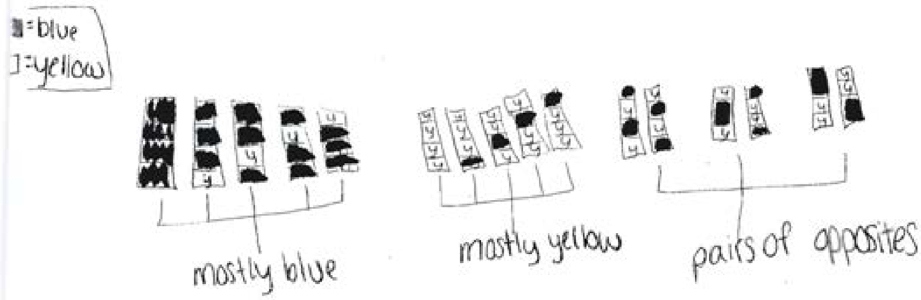


3) Student work by Jaime, a 7th grader, in response to the two-color towers problem



We think that 16 is all the possibiliti
for the blocks because we had 2
colors, and $2 \times 2 = 4$ and $4 \times 4 = 16$,
and 4 is the number of blocks we
needed to have in each tower.

4) Student work by Cathy, a 6th grader, on the two-color towers problem



What I mean by mostly is that there is either all blue cubes or 3 blue cubes and one yellow cube. I know they are all there because we tried every possible place to put the one yellow and if we put in in another spot, it would be a duplicate.

What I mean by mostly yellow is that in this group there is either all yellow or 3 yellow and one blue cube. I know they are all here because when we put 3 yellow and one blue, we put the blue in each of the 4 spots. If we put it anywhere else, it would be a duplicate.

In the towers in this group, there are two blue cubes and two yellow cubes. I know we have them all because I checked by doing this. If the blue is on top, we have the other one right underneath, two spots underneath, and on the bottom. We also have the opposite for each of those. We did the same thing for one blue in each other spot, and made sure the were no duplicates.

Pizza Student Work Module

1)

March5

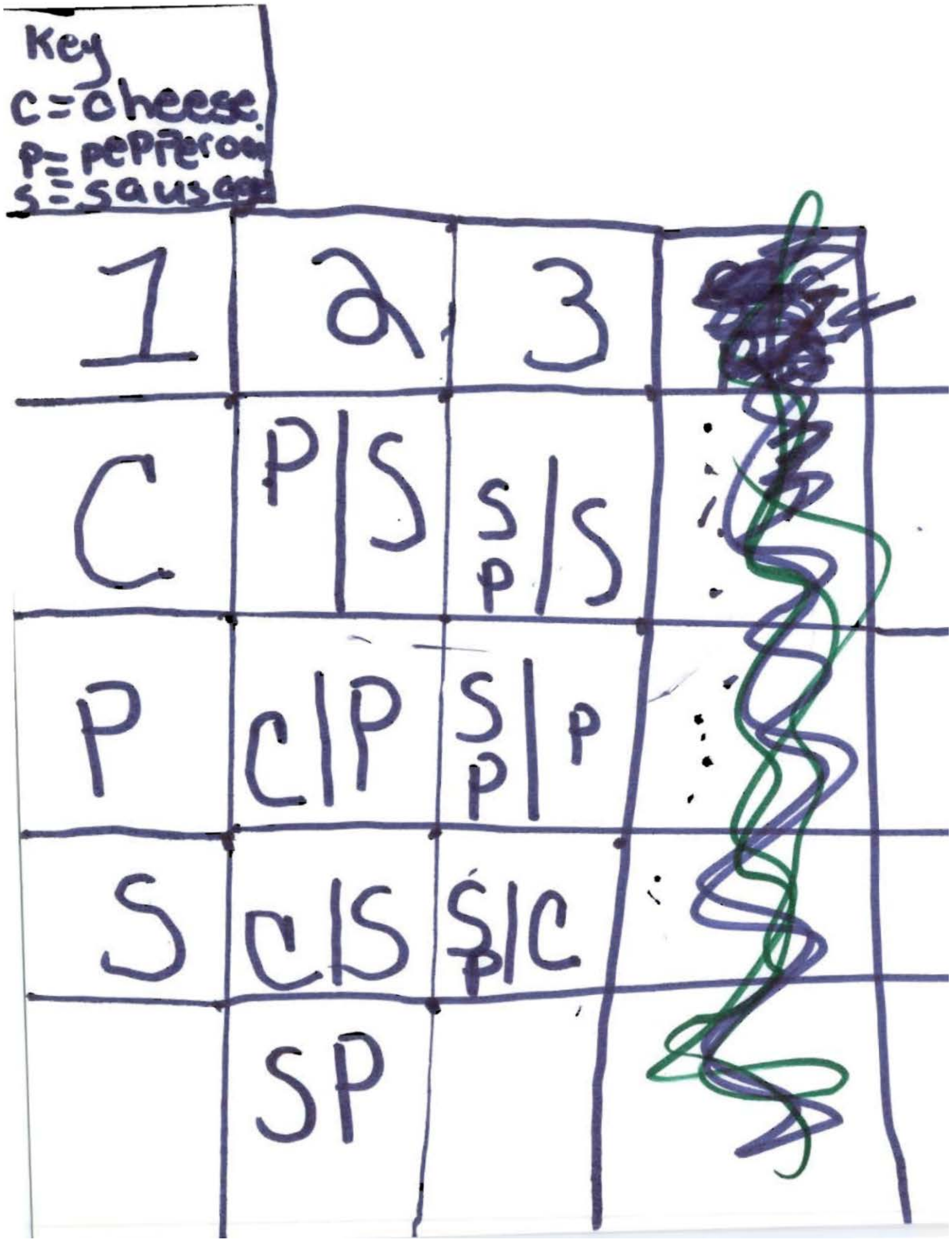
Dear Dr Davis and Alston

This week Dr. Mahr gave us a challengi problem. We were drawing pictures to figure out the problem. The three toppings ~~wh~~ we had to choose from cheese, sausage and pepperoni. We tried all combinations. At the end we came up with 10 combination I'm going to draw apicture to show you how wef igured it out.

Whole	Half	Mixed
Pepper oni Sausage Cheese	Half plain Half pepperoni Half plain	Sausage + Pepperoni (mixed)
		Half Sausage +pepperoni half Sausage
		Half Sausage +pepperoni Half plain
		Half sausage + pepperoni half pepperoni

Your Friend
[redacted]

2)


Key
c=cheese
P= pePiferon
s=sausag
1
2
3
C
P|S
S P|S
P
C|P
S P|P
S
C|S
S P|C
SP

3)

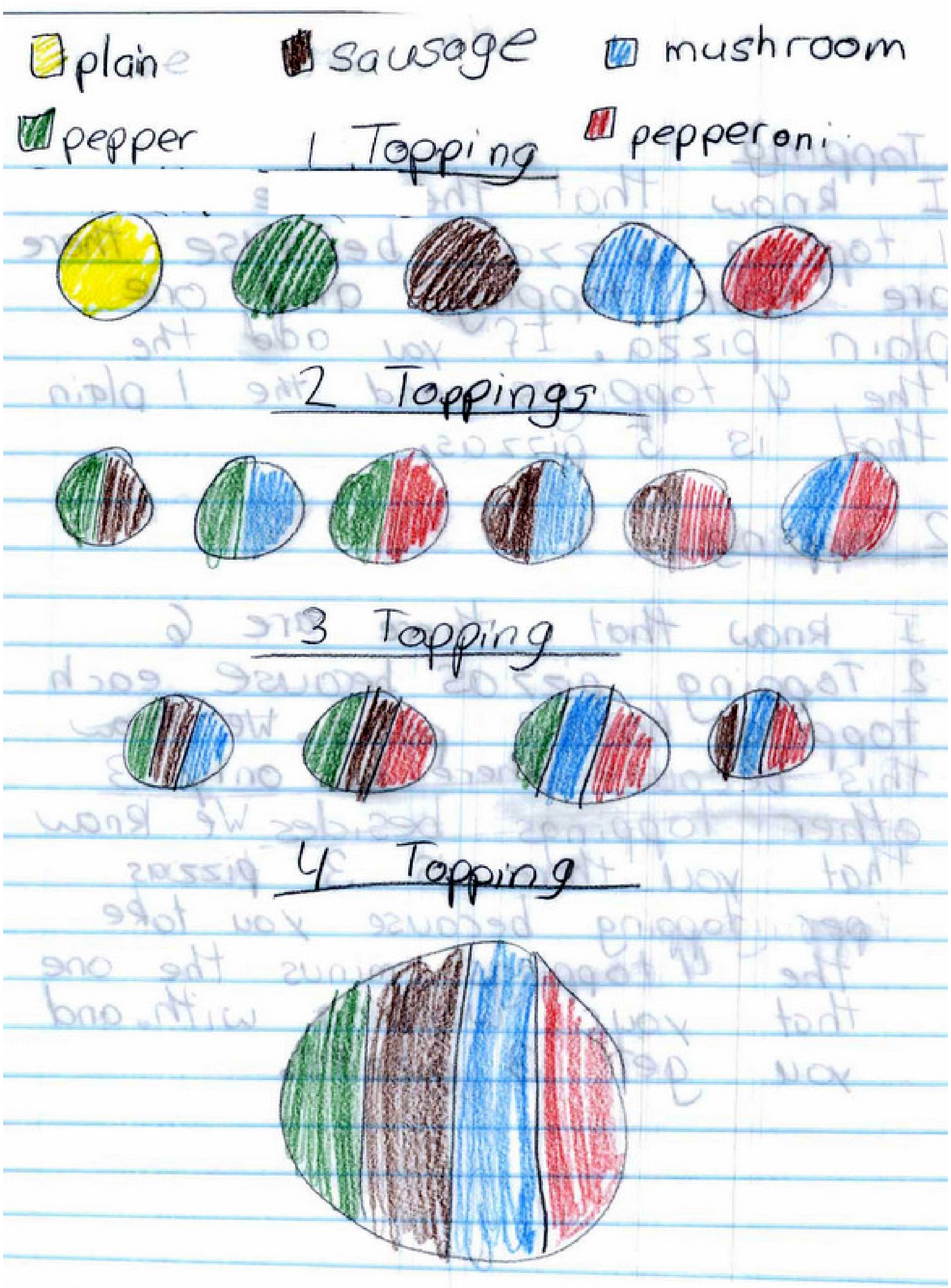
	Peppers	sausage	mushrooms	pepperoni
√1.	1	1	1	1
√2.	1	0	1	1
√3.	1	0	0	1
√4.	1	0	0	0
√5.	0	1	1	1
√6.	0	1	1	0
√7.	0	1	0	0
√8.	0	0	1	1
9.	0	0	1	0
√10.	0	0	0	0
√11.	1	0	1	0
√12	0	1	0	1
√13.	1	1	0	1
√14.	0	0	0	1
√15.	1	1	0	0
√16.	1	1	1	0

To get out answer it has many steps. First we found all the combinations of 4 which is 1. then we found all the combinations of 1 which is 4

We have noticed that each topping is in 8 combos. Eight is half of 16 which is the total number of combos. Half of eight is 4 which is the amount of toppings. There are 16 combinations in all. There are 4 combos of 1 and 1 combo of 4. There are 6 combos of 2 and 4 combos of 3. There are eight different combinantions and eight symmetrical combinations. The amount of combos for 3 is the same as the combos for 1 because for each combination of 1 there is a combination of 3.

This reminds me of the building block problem. It reminds me because we used the same method to find the answer.

4)


plain
sausage
mushroom
pepper
1 Topping
pepperoni
2 Toppings
3 Topping
4 Topping

1 Topping
I know that there are 5
1 topping pizza's because there
are four toppings and one
plain pizza. If you add the
the 4 toppings and the 1 plain
that is 5 pizzas.

2 Topping

I know that there are 6
2 Topping pizzas because each
topping has 3 pairs. ~~We know
this because there is only 3
other toppings besides~~ We know
that you there is 3 pizzas
per topping because you take
the 4 toppings minus the one
that your working with and
you get 3 pizzas.

3 topping

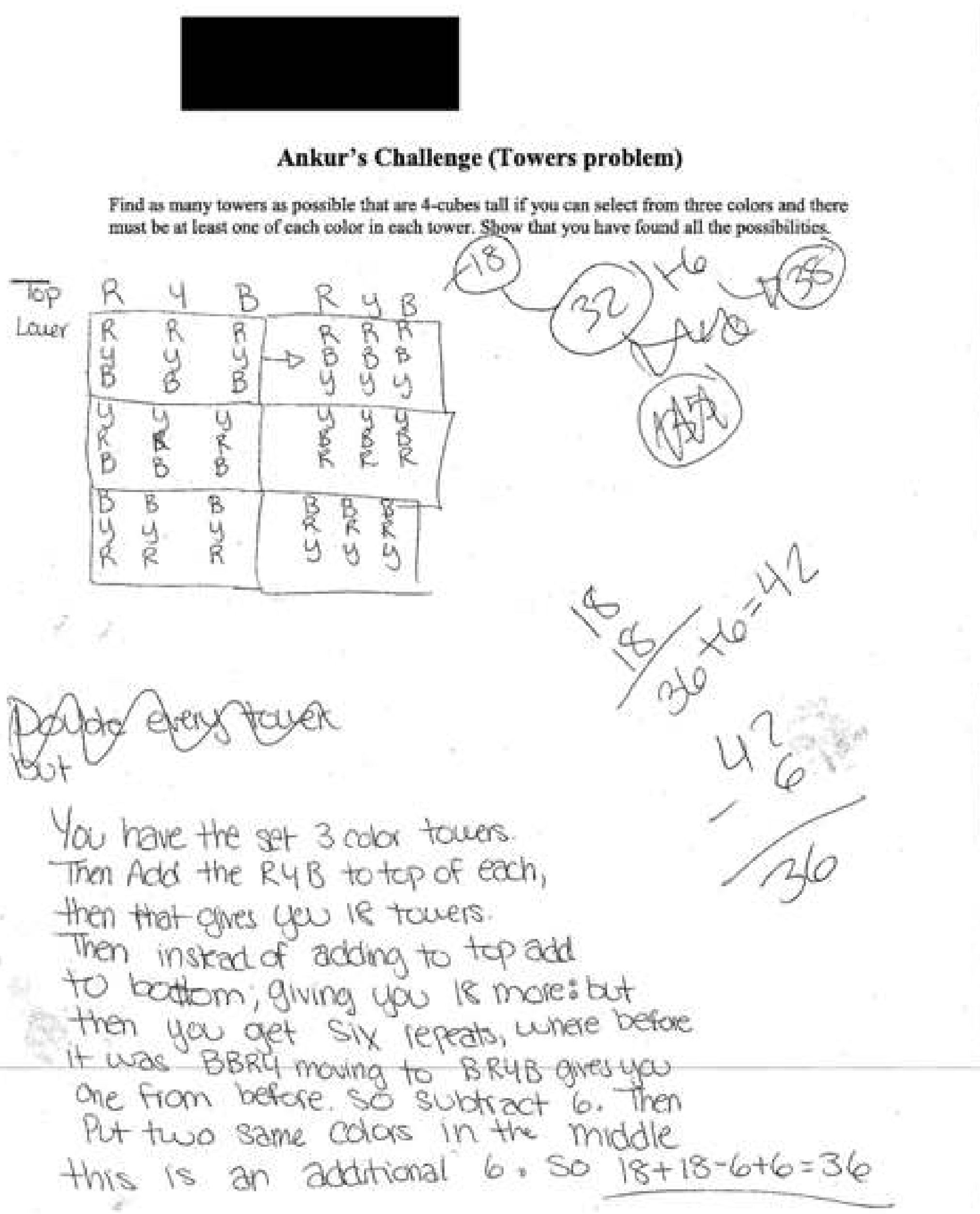
I know that there are 4
3 topping pizzas because
there is more toppings. We
know that there are
3 pizza per topping
because you take 4
toppings minus the one
that your working with
and you get 3 pizzas
per topping

4 topping

All the toppings on 1 pizza

Ankur's Challenge Student Work Module

1)



[redacted]

Ankur's Challenge (Towers problem)

Find as many towers as possible that are 4-cubes tall if you can select from three colors and there must be at least one of each color in each tower. Show that you have found all the possibilities.

Top / Lower	R	Y	B	R	Y	B
	R Y B	R Y B	R Y B	R B Y	R B Y	R B Y
	Y R B	Y R B	Y R B	Y B R	Y B R	Y B R
	B Y R	B Y R	B Y R	B R Y	B R Y	B R Y

18 32 +6 38

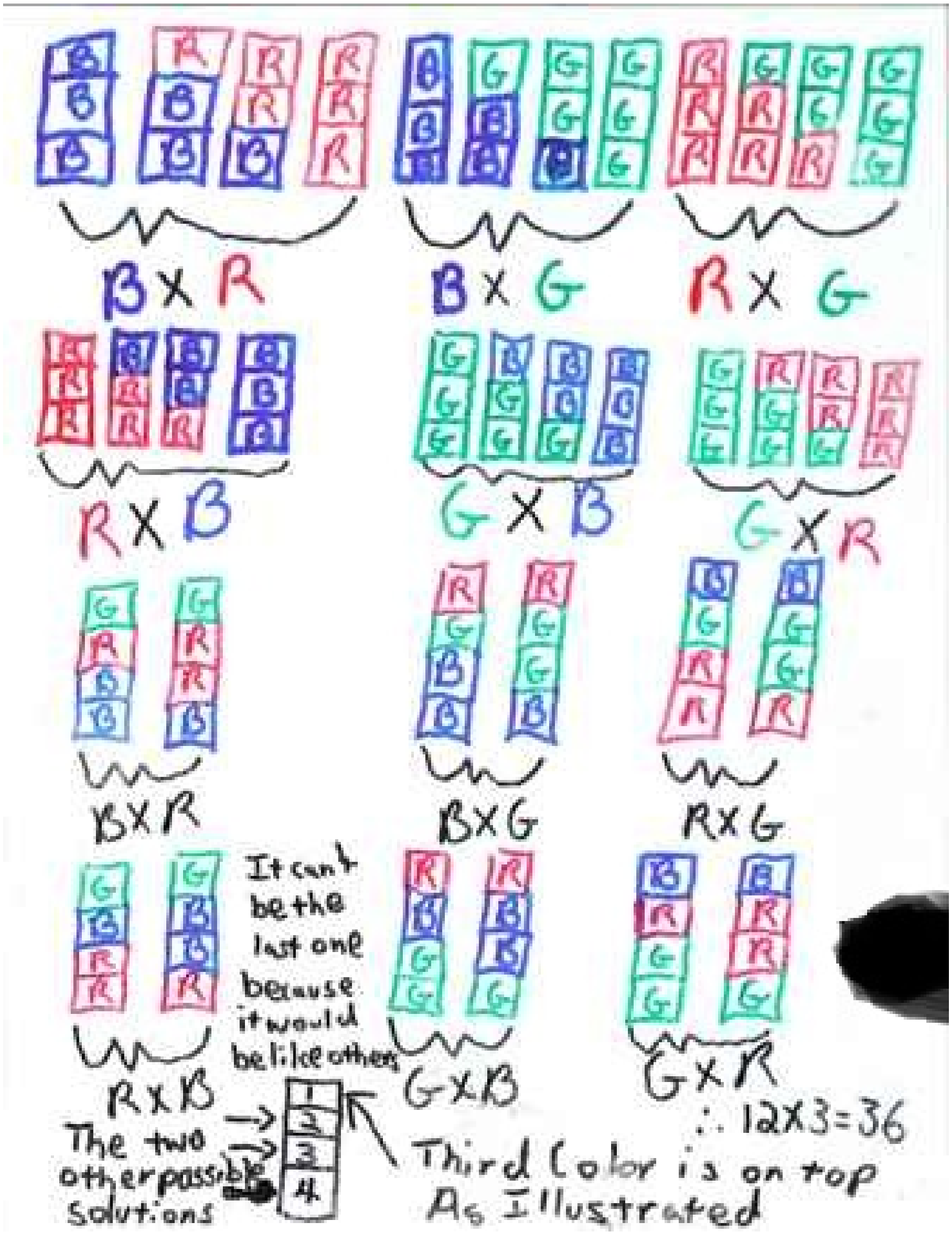
18
18
36 +6 = 42

42
−6
36

~~Double every tower~~
But

You have the set 3 color towers.
Then Add the RYB to top of each,
then that gives you 18 towers.
Then instead of adding to top add
to bottom, giving you 18 more: but
then you get six repeats, where before
it was BBRY moving to BRYB gives you
one from before. So subtract 6. Then
Put two same colors in the middle
this is an additional 6. So 18+18−6+6=36

2)


B x R
B x G
R x G
R x B
G x B
G x R
B x R
B x G
R x G
It can't be the last one because it would be like others
R x B
G x B
G x R
The two other possible solutions
∴ 12x3=36
Third Color is on top As Illustrated

3)

Originally, what I did was try to find all the unique towers three blocks tall that contained one of each color. What I came up with was:

R R Y Y B B
B Y R B Y R
Y B B R R Y

I then realized that by simply adding one more block of any color on top of any of these 6 towers, I would have 18 unique towers, each 4 tall.

RBY RBY RBY RBY RBY RBY

R R Y Y B B
B Y R B Y R
Y B B R R Y

I could also have added the blocks of each color to the bottom but then I realized I would have created some duplicates. The reason this happens is because the top block and bottom block are the same color. For example:

[redacted]

$$\begin{array}{c} Y \\ + \\ \begin{bmatrix} R \\ B \\ Y \end{bmatrix} \end{array} = \begin{array}{c} \begin{bmatrix} Y \\ R \\ B \end{bmatrix} \\ + \\ Y \end{array}$$

Therefore, I only get 12 unique towers in addition to the original 18 I came up with:

R	R	Y	Y	B	B
B	Y	R	B	Y	R
Y	B	B	R	R	Y
^	^	^	^	^	^
B Y	B Y	B R	B R	Y R	Y R

There are now two remaining places I could put an additional block into my original 3-block towers
1) between the 2nd and 3rd blocks
2) between the 1st and 2nd blocks

HOWEVER, it should be noted that if I put any color block that is NOT identical to the "old" 2nd block, I create duplicates:

[redacted]

Original: R B Y [redacted]

R		R		R		R
.		Y	AND LIKEWISE	.		|
Y	=	B		R	=	R
.		|		.		B
B		Y		B		Y
Y				Y		

Therefore, I can only add a blue block, in which case adding it between the 1st and 2nd blocks is identical to adding it between the 2nd and 3rd blocks, and I get the following 6 remaining towers:

R	R	Y	Y	B	B
B	Y	R	B	Y	R
B	Y	R	B	Y	R
Y	B	B	R	R	Y

Since there are no more possible positions to introduce a new block to my original 3-block towers, I computed:

$18 + 12 + 6 = \underline{36 \text{ towers}}$

Taxicab Student Work Module

1) I assumed that the taxi stand was at point (0,0) on the grid (imagine the paper was turned 90 degrees counterclockwise). And therefore the blue point is (4,1), the green point is (5, 5) and the red point is (3,4). Since we don't know distance, I am going to call each distance along one grid square, one unit.

The shortest route between any two points is a straight line. However, we will assume that the taxi has to go on streets along the grid paths. Therefore, the shortest distance to each point is to move the minimum in each direction, x and y (down and right on paper if held upright), to get to the point. We can prove it for the blue point.

For the blue point, we know the shortest distance as a bird flies would be the square root of 17. This is calculated using the Pythagorean Theorem with a right triangle of base lengths 4 and 1. However, if the taxi driver is moving along grid lines, she must move a distance in whole units (integers) as each pickup point is at an intersection of grid lines. The next greatest integer after the square root of 17 is 5. Therefore, 5 is the absolute minimum distance she could travel, we need to see if that can be achieved.

Sure enough, this can be achieved by moving 4 units to the right and 1 unit up. So we know that the shortest distance is 5 units. And she is only going to want to go 4 times to the right and 1 unit up. Any more would be a waste of time and be longer.

However, the next question is how many ways can she move 4 units to the right and 1 unit up? There is no restriction to the order of her moving in each direction. She has to move 5 units, but there are 5 different ways for her to get there. She could move UP at five different times. Either UP first, after going R once, after going R twice, going R three times or going all four rights.

In order: URRRR, RURRR, RRURR, RRRUR or RRRRU. 5 ways.

Another way to think of this is how many ways could we re-arrange the letters URRRR? While it may be easy to write them out like above, it gets harder as we move to the green and red dots. So it will be easier to get a general solution first.

So first lets think about how many ways we can write out five terms where order matters. For instance if we had 5 people, how many ways could they line up against a wall? There are 5 people that could be selected for the first spot, after choosing one, there are only 4 for the second, then 3 for the third spot, 2 left for the fourth spot and only 1 for the last spot. So 5*4*3*2*1 or 5! ways of lining them up (120). So there are 120 ways or re-arranging 5 items. This doesn't equate with our answer of 5. Why? Because we can't see all the repeated orders.

We cannot distinguish between the different R's when we re-arrange them. So imagine that for the five people that line up, four were identical male clones and one female. We may be selecting clone 1 then clone 2 or clone 2 and then clone 1 for the first two spots. However, an observer wouldn't be able to tell the difference between the two arrangements. So in this list of 120 arrangements, we would have U R R R R and another U R R R R where the R's moved. We just can't tell the difference (and don't want to care).

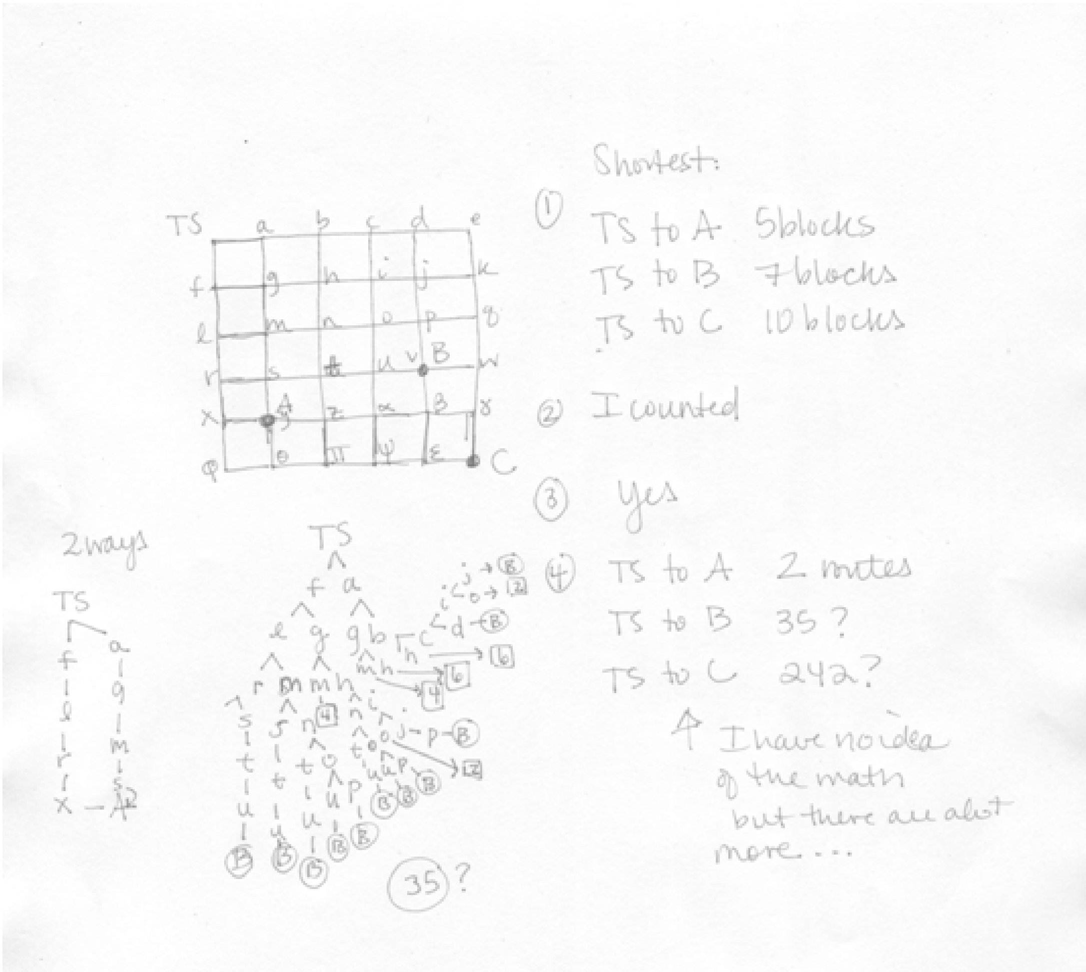
So how many ways were those 4 R's rearranged? That's the same method as above with the five people lining up. Its 4*3*2*1 or 4! (24). So every time we see the four Rs, there are actually really 24 different ways they were arranged that we cannot distinguish. We don't want them counted and therefore that must be divided from our 5!. The U can only be written one way (1!), but I am including it to help us for the next two points.

So our answer of number of routes to the blue point is 5! / (4! * 1!). or 120/24 = 5.

Using this general solution, I think we can find the number of ways of getting to the red point (3, 4) which is 7 units from the stand by using 7! / (4! * 3!) or 35. We are really saying to get to the red point we need to go R R R U U U U in some order. The 7! / (4! * 3!) is the number of distinguishable orders of writing that out.

For the green point I got 252 different ways to go the distance of 10 units.

2)


Shortest:
① TS to A 5blocks
TS to B 7blocks
TS to C 10 blocks
② I counted
③ Yes
④ TS to A 2 routes
TS to B 35 ?
TS to C 242?
I have no idea
of the math
but there are alot
more . . .
2ways
35?

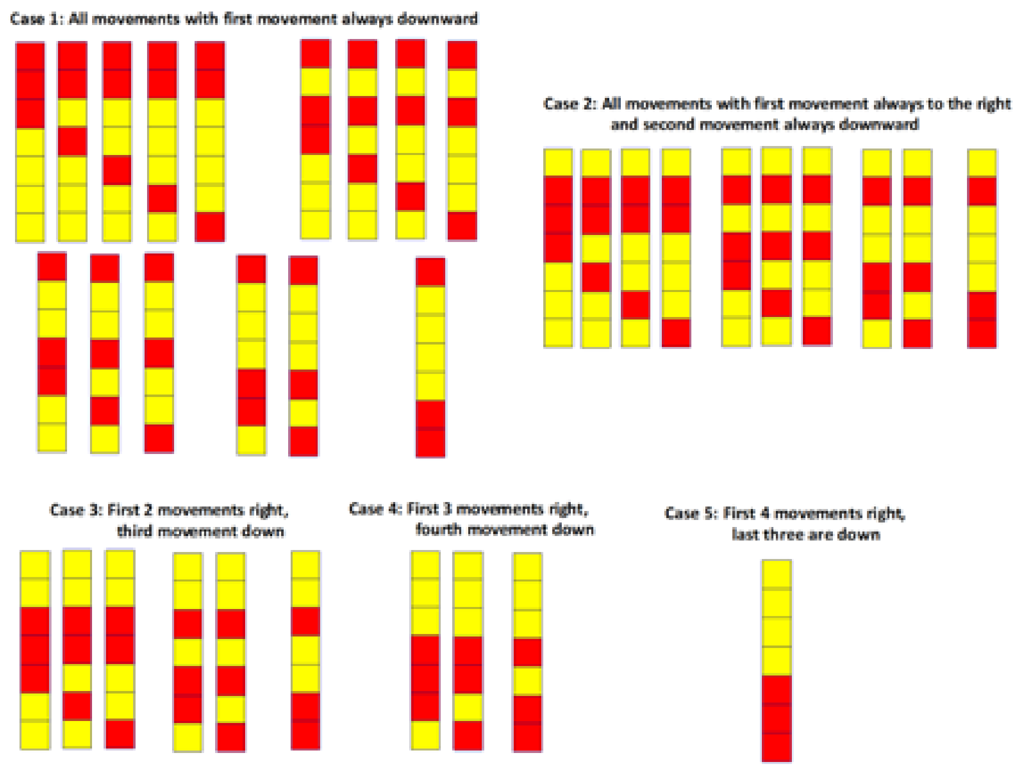

TS to B
20
35
15
TS to C
242 ?

3)

Argument by Cases

Movement Downward   Movement to the Right

Each tower represents a downward movement or a right movement from the taxi stand to the red dot stop. Since you have seven movements altogether, that makes a tower of seven. Since you can at most move down three times and at most move to the right four times - all the towers will have three red and four yellow blocks. Here are all the cases to represent all the possible movements from taxi stand to red dot stop



Total of 35 towers, therefore a total of 35 possible shortest routes from the taxi stand to the red dot stop. This can be done from the other stops as well.

You can compare each tower to the grid to see how each would represent a possible route. Also compare to Travis's drawings on possible routes - same difference, I just turned them into towers.

World Series Problem Student Work Module

1)

Matt

The World Series Problem

In a World Series, two teams play each other in at least four and at most seven games. The first team to win four games is the winner of the World Series. Assuming that the teams are equally matched, what is the probability that a World Series will be won: (a) in four games? (b) in five games? (c) in six games? (d) in seven games?

Teams = A & B, A denotes team A wins, B denotes B wins.

S = Sample Space: set of all 7 letter words using As and Bs.

$|S| = 2\ 2\ 2\ 2\ 2\ 2\ 2 = 2^7$

# of outcomes where A wins in 4 $= \underset{A}{1}\ \underset{A}{1}\ \underset{A}{1}\ \underset{A}{1}\ 2\ 2\ 2 = 2^3 = 8$

# of outcomes where A wins in 5 $= \underset{B}{1}\ \underset{A}{1}\ \underset{A}{1}\ \underset{A}{1}\ \underset{A}{1}\ 2\ 2 = 2^2 \times 4 = 16$

4 places for B. fixed

# of outcomes where A wins in 6 $= \underset{B}{1}\ \underset{B}{1}\ \underset{A}{1}\ \underset{A}{1}\ \underset{A}{1}\ \underset{A}{1}\ 2 = 2 \times 10 = 20.$

5 choose 2 places for the Bs. fixed.

# of outcomes where A wins in 7 $= \underset{B}{1}\ \underset{B}{1}\ \underset{B}{1}\ \underset{A}{1}\ \underset{A}{1}\ \underset{A}{1}\ \underset{A}{1} = 1 \times 20 = 20.$

6 choose 3 places for the Bs. fixed

Next →

Ends in 4 $= \frac{8 \times 2}{128} = \frac{16}{128}$

in 5 $= \frac{16 \times 2}{128} = \frac{32}{128}$

in 6 $= \frac{20 \times 2}{128} = \frac{40}{128}$

in 7 $= \frac{20 \times 2}{128} = \frac{40}{128}$

Check: $\frac{16 + 32 + 40 + 40}{128} = \frac{128}{128} = 1.$

World Series Problem

First I tried to figure out the total amount of outcomes that is possible. My initial thought was 2*2*2*2*2*2*2 or 2^7 where each 2 represented either team winning. But then I realized that not all games are possible because once a team wins at least 4 games, they will not continue to play so that eliminates some choices. I then thought of each type of outcome. I started with the outcome of winning the World Series in 4 games. That could happen where team A or team B wins all 4 in a row. So, there are 2 ways that could happen. $\binom{4}{0} + \binom{4}{4}$ represents Team A not winning any (or B winning) and Team A winning all 4 (or B losing). $\binom{4}{0} + \binom{4}{4} = 1 + 1 = 2$. So next I thought about the outcome of winning the World Series in 5 games, which happens when Team A or Team B wins by having some combination of 1 loss and 4 wins. $\binom{5}{1} + \binom{5}{4}$ represents Team A winning 1 out of the 5 games and Team A winning 4 games out of the 5 games. $\binom{5}{1} + \binom{5}{4} = 5 + 5 = 10$, so there are 10 outcomes for either time winning in 5 games, which includes if either won in 4 games. After I thought about when a team wins in 6 games, which happens when Team A or Team B wins by having some combination of 2 losses and 4 wins, again including a team winning in 4 or 5 games. $\binom{6}{2} + \binom{6}{4} = 15 + 15 = 30$, so there are 30 outcomes for either time winning in 6 games. Lastly, since they must win in 7 games, the probability that they will win in at least 7 games is 100%. To find the outcomes, you can use $\binom{7}{3} + \binom{7}{4} = 35 + 35 = 70$, so there are 70 outcomes for either time winning in 7 games. I knew since the probability of winning in 7 games has to be 100% which meant that the total number of outcomes had to be the same as the outcome of winning in 7 times or in other words both 70 because 70/70 gives you 100%.

So then I thought about how I was convinced that 70 is the total number of outcomes which is necessary to find each of the probabilities.

This brought me to row 7 of Pascal's triangle.

1	7	21	35	35	21	7	1
Team A won 7 times	Team A won 6 times	Team A won 5 times	Team A won 4 times	Team A won 3 times	Team A won 2 times	Team A won 1 time	Team A won 0 times
Team B won 0 times	Team B won 1 time	Team B won 2 times	Team B won 3 times	Team B won 4 times	Team B won 5 times	Team B won 6 times	Team B won 7 times

The only cases that are possible are the cases of A winning 4 times and B winning 3 times or A winning 3 times and B winning 4 times. This includes the cases of when a team wins within 4, 5, 6, or 7 games. I convinced myself of that by looking at the simpler cases. I started with winning in 4 games:

AAAA~~BBB~~ If a letter is crossed out, they wouldn't actually play that game.

BBBB~~AAA~~

2)

So there are 2 cases of winning in 4 games, as stated above but they both are considered $\binom{7}{3}$ and $\binom{7}{4}$.

Winning in 5 games:

AAAA~~BBB~~ : already represented in winning in 4 games

AAABA~~BB~~

AABAA~~BB~~

ABAAA~~BB~~
BAAAA~~BB~~

Same idea for B winning in 5 games, you would just replace the B's with A's and A's with B's.

If I continued, you would see for winning in 6 games, there were 2 cases of winning in 4 games, and 8 cases of winning in 5 games mixed within winning in 6 games. So, I convinced myself that the 70 cases represented at least 7 games which included winning in 4, 5, and 6 games.

Therefore,

The probability of winning a world series in 4 games is $\frac{2}{70}$.

The probability of winning a world series in 5 games is $\frac{10}{70}$.

The probability of winning a world series in 6 games is $\frac{30}{70}$.

The probability of winning a world series in 7 games is $\frac{70}{70}$.

Problem of Points Student Work Module

1) There will be at most 7 flips of the coin. so 100/7 equals 14 with a remainder of 2. For each flip of the coin the winner gets 14 francs. So in this case the person with 2 I think that was fermat, he gets 28 francs. Pascal who had one win gets 14. That leaves a difference of 58 francs left in the pot. Divide that evenly among them so that Fermat gets a total of 57 francs and pascal gets a total of 43 francs.

2) We already know that the first 3 coin tosses were 2 heads and 1 tail (not necessarily in that order). In order to figure out who has a greater probability of winning, we need to look at the different ways for each player to win and the probability that each case happens. Fermat needs 2 more heads to come up before 3 tails do, in order to win. So, the different ways for Fermat to win will have 2 heads and 0, 1, or 2 tails. Also, a heads must be the last flip because it will end the game. These are the possible options for the remaining flips that will result in the Fermat winning:

0 tails: HH

1 tail: HTH, THH

2 tails: HTTH, THTH, TTHH

We can figure out the probabilities of each of these cases because we know that the probability that a head or a tail will occur is ½.

P(HH)= (½)(½) = ¼

P(HTH)= (½)(½)(½) = 1/8

P(THH)= (½)(½)(½) = 1/8

P(HTTH)= (½)(½)(½)(½) = 1/16

P(THTH)= (½)(½)(½)(½) = 1/16

P(TTHH)= (½)(½)(½)(½) = 1/16

By adding these probabilities, we find that the probability that Fermat will win the game is 11/16. In order for Pascal to win, he needs 3 tails to come up before 2 heads do. So, the different ways for Pascal to win will have 3 tails and 0 or 1 heads. Also, a tails must be the last flip, ending the game. Here are Pascal's winning possibilites:

0 heads: TTT

1 head: HTTT, THTT, TTHT

The probabilities are calculated the same way as before:

P(TTT)= (½)(½)(½) = 1/8

P(HTTT) = (½)(½)(½)(½) = 1/16

P(THTT) = (½)(½)(½)(½) = 1/16

P(TTHT) = (½)(½)(½)(½) = 1/16

Adding up these probabilities, we see that Pascal's chance of winning the game is 5/16. So, Fermat should get 11/16 of the francs, which is 68.75 francs. Perhaps he is a generous friend, giving Pascal the benefit of the doubt and can round down to 68 francs. Then, Pascal should receive 32 francs (a little more than 5/16 of the francs). If Fermat is greedy, he'll round up to 69 francs, giving Pascal 31 francs.

3) I thought the best way to divide up the 100 francs would be to find the probabilities of Fermat and Pascal winning and use those results to divide the francs. I hope this is the proper strategy.

Since Fermat is ahead two points to one, we need to calculate how many ways there are for Fermat to win (had they continued playing). Since they have tossed the coin three times, and there is a maximum of 7 tosses before a winner is declared, there is a maximum of four possible coin flips left. There are three possibilities of the outcomes thus far: HHT, HTH, THH (since Fermat gets a point for heads, H, and Pascal gets a point for tails, T).

Since there are four new coin tosses that could happen if they continued the game, let's calculate the probability that either Fermat or Pascal would win. Fermat needs to get two more heads to win. Since there are four spaces to fill, that means that Pascal could get two tails. To find the number of ways to arrange the heads and tails we would do 4!/(2!*2!) = 6. Therefore there are six ways to rearrange the four possible positions left to play for Fermat to win.

Since there are three scenarios: HHT, HTH, and THH, we can add the six ways that Fermat could win to either of these three possibilities (HHT, HTH, THH). Thus, we have 3 * 6 = 18 possible ways for Fermat to win.

Now let's calculate the probability that Pascal were to win. Since he is down one to two, he needs three tails in order to win. There are only four positions left to play (if they had continued to play), and one of those positions would have to be a H by Fermat. Since there are again four spaces to fill, three of them must be tails and one must be heads, we have 4!/(3!1!) = 4. Therefore there are four ways to rearrange the four possible positions left to play for Pascal to win.

Since there are three scenarios, we can add the four ways that Pascal could win to either of these three possibilities. Thus we have 3 * 4 = 12 possible ways for Pascal to win.

Since there are 18 ways for Fermat to win, and 12 ways for Pascal to win, there are 30 possible ways for either of them to win. Taking the probability for Fermat to win, would be 18/30 = 3/5 and the probability for Pascal to win, would be 12/30 = 2/5.

With these probabilities, we can divide up the 100 francs. 100* (3/5) = 60 and 100*(2/5) = 40. 60 + 40 = 100 francs, so all francs are accounted for.

Appendix E: Belief Assessment

1. Learners generally understand more mathematics than their teachers or parents expect.

1	2	3	4	5
Strongly Agree				Strongly Disagree

2. Teachers should make sure that students know the correct procedure for solving a problem.

1	2	3	4	5
Strongly Agree				Strongly Disagree

3. Calculators can help students learn math facts.

1	2	3	4	5
Strongly Agree				Strongly Disagree

4. It’s helpful to encourage student-to-student talking during math activities.

1 2 3 4 5

Strongly Agree Strongly Disagree

5. Math is primarily about learning the procedures.

1 2 3 4 5

Strongly Agree Strongly Disagree

Students will get confused if you show them more than one way to solve a problem.

1 2 3 4 5

Strongly Agree Strongly Disagree

All students are capable of working on complex math tasks.

1 2 3 4 5

Strongly Agree Strongly Disagree

Math is primarily about identifying patterns.

1 2 3 4 5

Strongly Agree Strongly Disagree

6. If students learn math concepts before they learn the procedures, they are more likely to understand the concepts.

1 2 3 4 5

Strongly Agree Strongly Disagree

Manipulatives should only be used with students who don't learn from the textbook.

1 2 3 4 5

Strongly Agree Strongly Disagree

7. Young children must master math facts before starting to solve problems.

1 2 3 4 5

Strongly Agree Strongly Disagree

Teachers should show students multiple ways of solving a problem.

1 2 3 4 5

Strongly Agree Strongly Disagree

8. Only really smart students are capable of working on complex math tasks.

1 2 3 4 5

Strongly Agree Strongly Disagree

9. Calculators should be introduced only after students learn math facts.

1 2 3 4 5

Strongly Agree Strongly Disagree

10. Learners generally have more flexible solution strategies than their teachers or parents expect.

1 2 3 4 5

Strongly Agree Strongly Disagree

11. Math is primarily about communication.

1 2 3 4 5

Strongly Agree Strongly Disagree

12. Manipulatives cannot be used to justify a solution to a problem.

1 2 3 4 5

Strongly Agree Strongly Disagree

13. Learners can solve problems in novel ways before being taught to solve such problems.

1 2 3 4 5

Strongly Agree Strongly Disagree

14. Understanding math concepts is more powerful than memorizing procedures.

1 2 3 4 5

Strongly Agree Strongly Disagree

15. Diagrams are not to be accepted as justifications for procedures.

1 2 3 4 5

Strongly Agree Strongly Disagree

16. If students learn math concepts before procedures, they are more likely to understand the procedures when they learn them.

1 2 3 4 5

Strongly Agree Strongly Disagree

17. Students are able to tell when their teacher does not like mathematics.

1 2 3 4 5

Strongly Agree Strongly Disagree

18. Collaborative learning is effective only for those students who actually talk during group work.

1 2 3 4 5

Strongly Agree Strongly Disagree

19. Students should be corrected by the teacher if their answers are incorrect.

1 2 3 4 5

Strongly Agree Strongly Disagree

20. Mixed ability groups are effective organizations for stronger students to help slower learners.

1 2 3 4 5

Strongly Agree Strongly Disagree

21. Collaborative groups work best if students are grouped according to like abilities.

1 2 3 4 5

Strongly Agree Strongly Disagree

22. Conflicts in learning arise if teachers facilitate multiple solutions.

1 2 3 4 5

Strongly Agree Strongly Disagree

23. Learning a step-by-step approach is helpful for slow learners.

1 2 3 4 5

Strongly Agree Strongly Disagree

24. Only the most talented students can learn math with understanding.

1 2 3 4 5

Strongly Agree Strongly Disagree

25. The idea that students are responsible for their own learning does not work in practice.

1 2 3 4 5

Strongly Agree Strongly Disagree

26. Teachers need to adjust math instruction to accommodate a range of student abilities.

1 2 3 4 5

Strongly Agree Strongly Disagree

27. Teacher questioning of students' solutions tends to undermine students' confidence.

1 2 3 4 5

Strongly Agree Strongly Disagree

28. Teachers should intervene as little as possible when students are working on open-ended mathematics problems.

1 2 3 4 5

Strongly Agree Strongly Disagree

29. Students should not be penalized for making a computational error when they use the correct procedures for solving a problem.

1 2 3 4 5

Strongly Agree Strongly Disagree

Appendix F: Reasoning Assessment

Title: Gang of Four

Context: This episode is an assessment interview with four 4th grade students, Milin, Michelle, Jeff and Stephanie, for building all possible different towers of a particular height when selecting from two colors of unifix cubes. The children, working in pairs, had built towers four and five cubes tall during class sessions. Each of the children was subsequently interviewed individually and asked to describe how he or she had approached the tasks and to justify any solutions that had been constructed. In this group interview, the students are sharing their ideas about the towers problems, explaining and justifying their solutions to each other. While they consider towers of various heights during the session, they specifically reason about towers that are three cubes tall. Although unifix cubes were available, the children chose not to use them during the interview. The segment begins with short clips from the 4th grade classroom session to provide a background context of the students' building and organizing their towers with unifix cubes.

After viewing the video of the children explaining and justifying their approaches to the problems, please describe as **completely** as you can: (1) each example of reasoning that a child puts forth; (2) whether or not the reasoning forms a valid argument; (3) whether or not the argument is convincing; and (4) why or why not you are convinced. Give evidence from the interview to support any claims that you make. You may refer to the attached transcript as needed.

Each response will be evaluated according to the following criteria:

- Recognition of children's arguments
- Your assessment of the validity or not of children's reasoning
- Evidence to support your claims
- Whether the warrants you give are partial or complete

Video is located at:

Part 1 - https://www.youtube.com/v/_nlgGLDxMWQ

Part 2 - http://www.youtube.com/v/GeCSMI7xOcE

Appendix G: Reasoning Grading Rubric

Instructions for completing the on-line Rubric for scoring a participant response to the Gang of Four assessment video:

Enter the ID number where indicated on the video assessment form. As you scroll through the rubric, mark the appropriate box to indicate the presence or absence of each item in the rubric relative to the participant's description of the children's activity in the video.

A. The first section of the rubric deals with items referring to the mathematical ideas in the task that may be identified by participants and includes the following four categories: Problem Tasks, Representations, Mathematical Reasoning and Arguments.

B. The second section of the rubric deals with whether or not the participant considered the student(s)' reasoning and arguments to be convincing. As each response is scored, a list gets generated of those items identified by the scorer in the first section as reasoning that was noted by the participant. For each of these items, the scorer notes whether the participant indicated that he/she found the reasoning to be convincing or not convincing.

Note that a participant's remarks about a child's reasoning, argument or behavior should only be scored as "convincing" or "not convincing" if

the participant specifically indicates that to be the case somewhere in the response.

S**coring Holistically**

Study participants watch a video clip from the "Gang of Four" interview with researcher and four 4th graders: Milin, Michelle, Jeff and Stephanie. In an open-ended format, participants respond to a prompt that asks them to describe as completely as they can: (1) each example of reasoning that a child puts forth; (2) whether or not the reasoning forms a valid argument; (3) whether or not the argument is convincing; and (4) why or why not you are convinced. They are asked to give evidence from the interview to support any claims that they make; and they are provided with copy of transcript for the video clip.

Always begin scoring of an assessment by reading the participant's response in its entirety to get a sense of its scope. Then review it again more carefully to look for written evidence that support scoring of particular rubric items. Because the response format is totally open ended, the participant has freedom to express response in any desired organization. The entire response must be considered, since a participant may respond to one part of the assessment instructions in detail and not repeat this detail in response to the other parts. Indication of convincingness may occur in any portion of the participant's response to the assessment.

Note that the scoring focus is on mathematical reasoning, and of less importance is the language used to express that reasoning. A sophisticated response may name argument type and discuss it only in general form. Other responses may use very informal language. What someone says in his or her response matters more than how it is expressed. Examples can be helpful. Thus, we will use a Wiki to post illustrative (but not exhaustive) examples from participant response data that was scored in previous efforts.

Scorer's Guide to use in responding to the rubric concerning Problem Tasks, Mathematical Representations, Reasoning and Arguments (Questions 1 through 8):

The following is a list of Problem Tasks, Representations, Mathematical Reasoning and Arguments referred to by the children during the video. Items have been identified by the research team from studying the transcript as well as the video.

1. Problem Tasks identified

a. Towers of height 3-cubes with two colors
b. Towers of height 2-cubes with two colors
c. Towers of height 4-cubes with two colors
d. Towers of height 5-cubes with two colors
e. Towers of height 10-cubes with two colors
f. Towers of any height (height "n") with two colors

Examples:

- a. Towers of height 1-cube with two colors

”how many patterns they could make from towers of 1 block, 2 blocks, 3 blocks, etc.”

“Since there can only be 2 towers for tower of 1”

- b. Towers of height 2-cubes with two colors

”how many patterns they could make from towers of 1 block, 2 blocks, 3 blocks, etc.”

“…and for towers of 2, they saw that…”

- c. Towers of height 3-cubes with two colors

”how many patterns they could make from towers of 1 block, 2 blocks, 3 blocks, etc.”

“So for 3 high, build towers of all red…”

- d. Towers of height 4-cubes with two colors

“…worked together to figure out how many different four and five block combinations a person can make using two different colored unifix cubes.”

“It also led Michelle to the incorrect conclusion that there are 12 towers with a height of four blocks.”

- e. Towers of height 5-cubes with two colors

“…worked together to figure out how many different four and five block combinations a person can make using two different colored unifix cubes.”

“Jeff was able to use the pattern ‘times 2’ to justify the towers of 5 question…”

- f. Towers of height 10-cubes with two colors

“led Stephanie to finally say a ten block tower had 1,024…”

“…at the end, Stephanie ‘figured it out…’ towers of 10 = 1,024.”

- g. Towers of any height (height "n") with two colors

"They realized that for every one tower of blocks of n height…"

"In his own words, he explained why the pattern requires you to multiply 2 as n increases by 1."

2. Representations constructed or referenced

a. Descriptions (verbal) of towers and how they are built

"First she starts with a solid red tower, 2 red blocks. Then does the towers that have one blue so blue/red/red, red/blue/red, red/red/blue

"…example of reasoning was making opposites. …the student used all of one color (blue) and then all of the other color (red). Next they went onto the top color different than the rest (red, blue, blue) and the opposite of that (blue, red, red)"

b. Diagrams or charts of towers

"She also used a diagram to support her answer."

"Once they start drawing the patterns down it was easier to see what they were saying."

c. Numbers or letters used as symbols to represent cubes or towers

"Stephanie lists arrangements like r/r/r r/b/b etc."

3. Numerical Reasoning Patterns identified

Patterns mentioned by a participant may include only parts of the patterns listed below, but the scorer may be able to infer which pattern is being mentioned. a. Additive (2, 4, 6, 8 …..)

b. Doubling or "times two" (2, 4, 8, 16 …)

"…you just multiply by two."

"They continue to link the numbers of new towers to two times the previous number."

c. Base squared (1, 4, 9, 16, 25 ….)

d. Alternative or combined (2, 4, 8, 12 …) EXAMPLE (MFP): WHEN PARTICIPANT MENTIONS AT LEAST 8 AND 12, THEN CHECK 3d.

"It also leads Michelle to the incorrect conclusion that there are 12 towers with a height of four blocks." "*Michelle*: 'for this three high you would have eight towers and four high, you would have twelve towers and then you keep doing it like that…'."

4. Spatial Reasoning Patterns identified

a. The term "pattern" referring to arrangement of colored cubes within a tower.

"Showed pattern."

“She followed a pattern while constructing her diagram: no blues (3 red), one blue…”

b. Opposites (two towers with corresponding positions having alternate colors)

“Jeff says that ‘everything is opposites’. He is using the pattern of switching colors.”

“Jeff: ‘They’re all opposites’.”

c. Identifying towers, or groups of towers, by the “pattern” of how colored cubes are placed (e.g., a “staircase” pattern of a single cube of one color in consecutively lower – or higher – positions in each tower; or towers with patterns of alternating colored cubes; or more than one cube of one color together in consecutively lower or higher positions).

“She then puts 1 blue cube at the top of the 2 red blocks and moves the blue’s position ‘down the stairs.’ Next, Step shows all possibilities with 2 blues…”

“They also argued it would be easier to (when drawing or building these towers) go by the order of how many blocks of each color they are using.”

5. Other Reasoning Features noted

- a. Direct Answers (unexplained answers for number of towers for certain heights)

NOTE (MFP): 5a is only for those direct answers that do not connect to patterns (3 and 4) or other reasoning (6, 7, and 8).

“Stephanie ‘figured it out…’ towers of 10 = 1,024.”

b. Guessing

Milin says this is guessing.”

c. Randomly building towers and checking for duplicates

“Finally a student randomly picked combinations until they thought they had exhausted all of them.”

6. Inductive Argument (note that a participant may refer to it as recursive or including recursion). This argument may be expressed with reference to towers of a specific height, as in features (a) and (b) below. It also may be expressed in general form, as in features (c) and (d) below.

a. When building towers selecting from two colors, there are exactly two unique towers of height one. With a single position in the tower, the one cube can be (say) either red or blue.

“Since there can be only 2 towers for tower of 1…”

“…because you know the most basic number of towers for one height which is two…”

b. For the two unique towers, one cube in height, cubes of one of the colors can be placed on top of each tower producing two unique towers, 2 cubes high. Cubes of the other color can be placed on top of a duplicate pair of towers one cube high producing two more unique towers. The resulting four towers, 2 cubes high, will contain no duplicates since the two unique pairs differ from each other in the top cube.

"…and for towers of 2, they saw that they can add either a blue or a red to each of the two towers. So blue + red and blue + blue possible for the first one and red + blue and red + red possible for the second one, so they got 4 towers of 2."

c. Towers of any height, "n", when selecting from two colors, can be generated similarly by taking all the towers with height, "n-1" – that are known to be unique with no duplicates because they were generated recursively from towers one cube high. Cubes of one of the colors (say, red) can be placed on top of each of the towers, producing unique towers "n" cubes high. Cubes of the other color (say, blue) can be placed on top of a duplicate set of towers, "n-1" cubes high, producing a second set of unique towers, "n" cubes high.

"Milin's argument of adding 'one more color for each one.' He then clarifies it is actually two colors, blue or red which gives two towers per each tower of the previous height."

d. The resulting total set of towers, "n" cubes high, will contain no duplicates since the two generated sets (each of which contained no duplicates) differ from each other in the top cube. This resulting set of towers, "n" cubes high, will always include two times the number of towers as the "n-1" high set.

"…you could add 1 of each color to the one end for a total of two more combinations. Which would double your answer. So each time you add one more block to the tower, your total number of different combinations doubles."

7. Stephanie's cases argument for towers 3 cubes high selecting from two colors (blue and red) results in a set of 8 unique towers. A complete argument includes the following cases with the justification for each case. Note that written responses by study participants may well be fragmentary and use much less precise language than the following. Also note that an argument can only be considering a cases argument, rather than the use of a pattern, if the participant clearly defines one or more of the cases; in other words, what it is a case of:

a. All blue cubes or no red cubes resulting in only one tower.
Justification – Any other 3 cube high tower that is all blue would be a duplicate of this one.

"Stephanie's use of patterns 'one blue, two blues' continuing to three blues."

Justification "..starting with all one color…"

b. One blue cube and two red cubes resulting in three unique (different) towers. Justification - No more towers can be created with one blue cube and two red cubes because there are only three positions in the tower for the blue cube to occupy. Another position – allowing another tower – would result in a tower 4 cubes high.

"Then does the towers that have one blue so blue/red/red, red/blue/red, red/red/blue."

Justification "…and then putting in one of the other color in as many different places as possible."

c. Two blue cubes stuck together and one red cube resulting in two unique towers. Justification – No more towers can be created of two blue cubes "stuck together" and one red cube in the third position because the two together must be in positions one and two or two and three of the three possible positions in the tower.

"She says that she is doing it with the blues stuck together."

Justification “Once that option was exhausted, the students went on to two of the other color stuck together as in many different places as possible.”

d. No blue cubes or all red cubes results in one tower.

Justification – Any other 3 cube high tower that is all red would be a duplicate of this one and there can be no more single color towers because there are only two colors.

“First she starts with a solid red tower, 3 red blocks.”

Justification “Once this was exhausted the student went to three of the other color.”

e. Two blue “stuck apart” or separated by one red cube results in one tower.

Justification - No more towers can be created by two blue cubes “stuck apart” or separated by the red cube, because, with only three positions, position 2 is the only one that can be considered “in-between” the other two.

“ …until she got caught up in the issue of whether the 2-blue blocks were stuck together or apart.”

Justification “Finally, the student split up the two of the other color to make her final variation.”

8. An alternate cases argument for towers 3 cubes high selecting from two colors (blue and red) proposed by several of the children. Several of the cases overlap completely with the ones articulated by Stephanie and those should be

scored in Item 7. Only the portion of cases argument that is different from Stephanie's is to be scored in Item 8.

f. One red cube and two blue cubes resulting in three unique (different) towers. Justification - No more towers can be created with one red cube and two blue cubes because there are only three positions in the tower for the red cube to occupy. Another position – allowing another tower – would result in a tower 4 cubes high.

** Participants may describe argument 8f. as better (preferred, more elegant, etc.) than the way Stephanie organized her cases, which bifurcated 8f. into 7c. and 7e.*

"So for 3 high, build towers of all red, one red…"

Justification "There's red… blue/red/red and you can't make any more in this, so you go on to the next one…"

Scorer's Guide to use in responding to the rubric concerning whether or not the participant considered the student(s)' mathematical reasoning and arguments to be convincing or not convincing (Questions 9 through 12):

9. For Question 9, the online rubric is programmed to generate a list including each of the items that the scorer marked positively for Questions 2 through 8. For each of these items, the scorer is to note whether the participant indicated that this particular mathematical reasoning and/or argument by one or more of the children was convincing. The absence of a positive (convincing)

response for any item does not necessarily mean that the participant considered this particular item to be NOT convincing.

10. The scorer will only mark Question 10 positively if the participant indicates that the children's mathematical reasoning was convincing but gives no specific details about which item of reasoning or piece of an argument was convincing.

11. For Question 11, the scorer will consider an identical list of the items marked as present in the participant's description in Questions 2 through 8. However, this time the scorer will only mark an item as present if the participant specifically indicates that it was NOT convincing.

12. The scorer will only mark Question 12 as present if the participant indicates that the children's mathematical reasoning was NOT convincing but gives no specific details about which item of reasoning or piece of an argument was convincing.

Appendix H: Discussion Prompts

Cycle 1

Week 1:

We have a few assignments for the week. You are reading three articles: (1) from Skemp about relational and instrumental understanding; (2) from Erlwanger about a student named

Benny, and (3) a chapter from Maher on the background of the longitudinal study. As you read these articles think about connections, if any, between/among them.

Also, consider the following questions from each paper to guide your group discussion.

Maher: Longitudinal study chapter:

Consider the historical context of the study. What, if anything, do you find striking? Are there aspects of the study that are relevant/plausible today? If so, what might they be?

Skemp: Relational/instrumental understanding

-Give an example of “rule without reason”? and provide a rationale.

Erlwanger: Benny article

(a) In IPI, a proficiency level for students of 85% was established for all tests to monitor and diagnose student progress. While this seems like a reasonable goal, provide some explanations as to how the tracking of Benny's progress failed.

(b) Benny relied on recognizing patterns to solve problems. Discuss the pros and cons of this approach. How, if at all, is Benny's approach different than the patterns approach used by the students in the Gang of Four video?

(c) Generalizations are an important objective in mathematical learning. It appeared that Benny liked to generalize. Provide some explanations as to why it did not work for Benny.

(d) Discuss the pros and cons of instructional programs such as IPI and the restrictions/advantages, if any, for teachers.

(e) How do you think Benny's mathematics thinking might have been different if he were involved in a classroom environment similar to the one during the longitudinal study?

Week 2:

1) For the videos you watched last week - What mathematical ideas were students building? How did they represent the ideas? What did you notice about how they worked together?

2) Meredith was eight years old when working on building towers.

(a) Describe what evidence, if any, is there of knowledge building.

(b) Describe the researcher's intervention and how, if at all, you might have intervened otherwise.

3) Discuss how an examination of students' representations might provide insight into their conceptual understanding.

4) What understanding might be gleaned from Stephanie, Dana and Michael's representations in solving the outfit problem you watched last week? What about from the towers videos?

Cycle 2

Week 1:

For this week we will read two articles about the concept of an isomorphism. There are also two videos: one about a student named Brandon and one of the Kenilworth students working on the pizza w/2 toppings with halves. For the video and readings, please discuss in your groups:

1. Discuss the notation that Brandon used. In what ways was it helpful in relating his solution of the pizza problem (selecting from 4 toppings) to the 4-tall towers problem (selecting from two colors)?

2. Greer and Harel discuss the role of isomorphisms in mathematical cognition. What is an isomorphism? How is it related to mathematical cognition?

3. Does Brandon control for variables in justifying his solution? If yes, describe. Do you notice any other forms of reasoning used by Brandon?

4. Contrast your engagement in pizza problem solving with that of the children in the video.

Week 2:

1) Meredith and Jackie were eight years old when working on building towers.

(a) Describe what evidence, if any, is there of knowledge building.

(b) Describe the researcher's intervention and how, if at all, you might have intervened otherwise.

2) Discuss the complexity of organizing instruction so that ideas might travel in a classroom. Draw from your participation in Monday's class as well as what you see in the video of students working on pizza problems last week.

3) Discuss the interplay of concrete experience in building mathematical knowledge.

(a) Describe merits/obstacles

(b) Give examples

4) Maher claims that sense making and reasoning in problem solving come naturally to students. How, if at all, does this claim fit your experience in learning and teaching.

Cycle 3

Week 1:

This week's assignment for online work involves a video and three readings, with threaded discussion, that follows class work on problem solving for the Ankur's Challange task. The following are intended to guide discussion in your small groups:

(1) Describe Romina's strategy for solving the "Ankur's challenge" problem.

(2) In your opinion, is this solution a convincing one? Why or why not?

(3) According to the Yackel & Hanna chapter, both von Glaserfeld and Thompson equate reasoning with learning (p. 227). From this perspective, in what ways do explaining and justifying contribute to learning mathematics?

Week 2:

Read the article "Understanding Understanding" and respond to the following question:

As a goal in mathematical learning, the idea of “understanding” is widely accepted. Both Skemp and Davis offer their views about what it means to “understand”. In this week’s reading, Davis further challenges us to understand ‘understanding’. Discuss

a. How “understanding” of a mathematical concept can take on multiple meanings, and

b. The complexity of understanding 'understanding' from Davis’ view.

Cycle 4

Week 1:

For the Taxicab materials - comment on the problem solving strategies of the students. What did you like/dislike about their problem solving? Compare what went on in the videos to your own problem solving from class.

For the Pedemonte paper:

One of the ideas in the paper is that students produce incorrect proofs because they are not able to transform the structure of argumentation into deductive structure for proof. a) what do you think its hard for people to do this, b) do you notice this in your students, c) do you notice this in yourself?

The results in the paper are limited to the domain of geometry - is it possible to extend the results to other mathematical domains? Why or why not?

Cycle 5

For this week, please post your World Series solution to your group. Discuss others solutions. Also read CH12 from the combinatorics books and watch the Night Session video. Comment on:

a. How the students attribute meaning to the symbols; and

b. How they develop the "addition rule" for Pascal's triangle

8.1

Cycle 6

Week 1:

Week 2:

Read and discuss: Conditions for promoting reasoning in problem solving: Insights from a longitudinal study

Consider, in your discussion of this chapter what relationships, if any, there might be between problem solving and mathematical reasoning.

Appendix I: Reflection Prompt

Reflection Paper

Introduction to Mathematics Education 15:254:540:01

In your reflection about your experiences in the course this past semester, discuss

1. The role of understanding in the following contexts:

(a) your own mathematical learning,

(b) the mathematical learning of students, and

(c) implications, if any, for your own teaching.

2. Discuss the value of content and format of the course, with particular attention to the following with at least two specific examples for each:

(a) problem tasks

(b) collaborative problem solving

(c) readings

(d) video viewing (cite at least two)

(f) online discussion

(g) what you learned by making an Analytic

(h) evaluating student work

3. We welcome suggestions for course improvement and would like you to comment on the balance among the following: in person meetings, laboratory work, online opportunities for discussion, guest speaker, as well as the time allocations for the various assignments, including the final Analytic project. What might you suggest be modified for subsequent course offerings?

4. We invite other comments and feedback. In particular, if this course were not required, would you recommend it for others to take? Why or why not?

5. Do you think the course could be offered in a fully on line format? Why or why not?

Submit your reflection as a word document in the dropbox

Appendix J: Videos Used In the Study

Cycle 1:

Meredith Removes the Top Cube [video]. Retrieved from https://www.youtube.com/watch?v=iqW05xcKHCA

PUP Math Shirts and Pants [video]. Retrieved from http://dx.doi.org/doi:10.7282/T3MC8Z77

Stephanie Grade 3 Towers interview excerpts [video]. Retrieved from http://dx.doi.org/doi:10.7282/T3FJ2F7X

Stephanie problem solving excerpts from the four and three-tall towers problem [video]. Retrieved from http://dx.doi.org/doi:10.7282/T39S1PGR

Cycle 2:

PUP Math Brandon interview [video]. Retrieved from http://dx.doi.org/doi:10.7282/T3VX0FRD

PUP Math Pizza, Clip 1 of 2: Pizza halves with two toppings [video]. Retrieved from http://dx.doi.org/doi:10.7282/T3HM57PQ

PUP Math pizza, Clip 2 of 2: Whole and Half Pizzas with Four Toppings [video]. Retrieved from http://dx.doi.org/doi:10.7282/T3NC60FW

Cycle 3:

PUP Math Romina's proof to Ankur's challenge [video]. Retrieved from http://dx.doi.org/doi:10.7282/T30P0Z85

Cycle 4:

Taxicab problem, clip 1 of 5: the shortest distance between two points. [video]. Retrieved from http://dx.doi.org/doi:10.7282/T39W0FBQ

Taxicab problem, clip 2 of 5: investigating the number of shortest paths. [video]. Retrieved from http://dx.doi.org/doi:10.7282/T3FJ2GNQ

Taxicab problem, clip 3 of 5: It's Pascal's triangle! But Why? [video]. Retrieved from http://dx.doi.org/doi:10.7282/T3K937CF

Taxicab Problem, Clip 4 of 5: Explaining the Taxicab and Towers Isomorphism [video]. Retrieved from http://dx.doi.org/doi:10.7282/T3Q2402Q

Taxicab problem, clip 5 of 5: extending the taxicab correspondence to pizza with toppings and binary notation [video]. Retrieved from http://dx.doi.org/doi:10.7282/T3TT4QSB

Cycle 5:

PUP Math World series [video]. Retrieved from http://dx.doi.org/doi:10.7282/T3CV4H0V

Cycle 6:

PUP Math Night session [video]. Retrieved from http://dx.doi.org/doi:10.7282/T34F1Q0W

Romina’s Story [video]. Retrieved from http://rbdil.org/rominasstory.html

Students Reflecting on Their Experience [video]. Retrieved from

https://www.youtube.com/watch?v=UTGiDp_Q6RE

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