ESTIMATES ON NON-DECAYING WHITTAKER FUNCTIONS

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ABSTRACT OF THE DISSERTATION

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Dissertation Director: Stephen Miller

Since the Fourier coefficients of an automorphic form along the nilpotent radical of parabolic subgroup are expressed in terms of Whittaker functions, a better understanding of their growth in every direction would be useful in the study of automorphic forms. Bump and Huntley (1995) used an integral formula which was found by Vinogradov, Takhtadzhyan (1978), and Stade (1988) to obtain precise information of the spherical Whittaker functions $M_{(\nu_1,\nu_2)}(y_1, y_2)$ as both y_1 and $y_2 \to \infty$. To (1995) used a method similar to the characteristic method in the theory of differential equations to compute the leading exponents of asymptotic expansions of a basis of Whittaker functions in the positive Weyl chamber for a split semi-simple Lie group over $\mathbb{R}$, which, in particular, yields a solution to Zuckerman's conjecture for $SL(3, \mathbb{R})$. Templier (2015) has recently used an integral representation by Givental to show To's result: the exponential growth of $M_{(\nu_1,\nu_2)}(y_1, y_2)$ for $y_1, y_2 \geq 1$ as either or both $y_1, y_2 \to \infty$. In this thesis I use a new formula which was derived by Ishii and Stade (2007) to obtain the asymptotic expansions of $M_{(\nu_1,\nu_2)}(t, \frac{1}{t^p})$ and $M_{(\nu_1,\nu_2)}(\frac{1}{t^p}, t)$ as $t \to \infty$ where $2 \leq p \in \frac{1}{2}\mathbb{Z}$, then successfully prove an analog of the Multiplicity One Theorem in these directions, namely that in certain circumstances the moderate growth condition in the theory of automorphic forms is automatic.

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Special thanks are due to my wife, Trang Nguyen, for her continued support and encouragement. Many thanks to my mother, father, and brother who experienced all of the ups and downs of my research.

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Dedication

To Trang Nguyen and my son

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Introduction

Classical automorphic forms are functions on the upper half plane: holomorphic forms with weight, commonly known as modular forms, and the real-analytic forms described by Maass. The forms in my thesis which are generalized on $GL(3)$ are precise analogs of the Maass forms on $GL(2)$.

Let $G = SL(2,\mathbb{R})$, $\Gamma = SL(2,\mathbb{Z})$, and $X \subset G$ be the group of upper triangular, unipotent matrices

$$X = \Big\{ n_x = \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) : x \in \mathbb{R} \Big\}. \tag{1}$$

Also let $Y \subset G$ be the group of diagonal matrices with positive entries:

$$Y = \Big\{ a_y = \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} : y > 0 \Big\}. \tag{2}$$

Now consider the homogeneous space $\mathcal{H} = SL(2,\mathbb{R})/SO(2,\mathbb{R})$, where $SO(2,\mathbb{R})$ is the rotation group. By the Iwasawa decomposition, every $z \in \mathcal{H}$ has a unique representation $z = n_x\, a_y \pmod{SO(2,\mathbb{R})}$ with $n_x \in X$, $a_y \in Y$.

Let $\mathfrak{g}$ be the Lie algebra of G and $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. The center of $U(\mathfrak{g})$ is a polynomial ring in one generator:

$$\Delta = -y^2 \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Big). \tag{3}$$

Definition 0.1 *An* ***automorphic form for*** $SL(2,\mathbb{Z})$ ***of type*** $\nu \in \mathbb{C}$ *is a smooth function* $\phi : \mathcal{H} \to \mathbb{C}$ *satisfying:*

1. $\phi(\gamma\,\tau) = \phi(\tau)$ *for all* $\gamma \in SL(2,\mathbb{Z})$, $\tau \in \mathcal{H}$.

2. $\Delta\phi = \nu(1-\nu)\phi$.

3. *There exists a constant* N *such that* $\phi(iy) = O(y^N)$ *for* y *sufficiently large.*

Since the element $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is in $SL(2,\mathbb{Z})$ it follows that an automorphic form is a periodic function of x and must have a Fourier expansion of type

$$\phi(z) = \sum_{m\in\mathbb{Z}} A_m(y)e^{2\pi imx} \tag{4}$$

Define $W_m(z) = A_m(y)e^{2\pi imx}$, then $W_m(z)$ is a Whittaker function.

Definition 0.2 *A* ***Whittaker function of type*** $\nu \in \mathbb{C}$ *associated to an additive character* $\chi : \mathbb{R} \to \mathbb{S}^1$ *is a smooth nonzero function* $W : \mathcal{H} \to \mathbb{C}$ *which satisfies the following conditions*

$$\Delta W(z) = \nu(1-\nu)W(z), \tag{5}$$

$$W\Big(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) z\Big) = \chi(x)W(z). \tag{6}$$

These Whittaker functions can be classified into three major types according to type ν and additive character $\chi = e^{2\pi imx}$. They are $W(z) = ay^{\nu} + by^{1-\nu}$ if $m = 0$; $e^{2\pi imx}(ae^{-2\pi my}+be^{2\pi my})$ if $\nu = 0,1$; $e^{2\pi imx}\sqrt{2\pi\,|m|\,y}\Big(aK_{\nu-\frac{1}{2}}(2\pi\,|m|\,y)+bI_{\nu-\frac{1}{2}}(2\pi\,|m|\,y)\Big)$ if $m \neq 0, \nu \neq 0,1$, where $a, b \in \mathbb{C}$. Understanding their asymptotic expansions as $y \to \infty$ leads to the multiplicity one theorem for $GL(2,\mathbb{R})$. Especially, only K-Bessel functions appear on the Fourier expansions of *moderate growth* automorphic eigenfunctions on $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R})$.

Theorem 0.3 *For fixed* $x > 0$, $|I_\nu(x)|$ *is strictly decreasing with respect to Re* $(\nu) = \sigma$ *in the right half plane.* $|I_\nu(x)|$ *is strictly decreasing with respect to Im* $(\nu) = t < 0$, *and* $|I_\nu(x)|$ *is strictly increasing with respect to* $t > 0$.

A proof is given in chapter 1.

We have the analog for $GL(3)$. Let $G = SL(3,\mathbb{R})$, $\Gamma = SL(3,\mathbb{Z})$, and $X \subset G$ be the group of upper triangular, unipotent matrices.

$$X = \left\{ n_x = \begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} : x_i \in \mathbb{R} \right\} \tag{7}$$

Also let $Y \subset G$ be the subgroup

$$Y = \left\{ a_y = \begin{pmatrix} y_1^{2/3}y_2^{1/3} & & \\ & y_1^{-1/3}y_2^{1/3} & \\ & & y_1^{-1/3}y_2^{-2/3} \end{pmatrix} : y_1, y_2 > 0 \right\}. \tag{8}$$

Now consider the homogeneous space $\mathcal{H}^3 = G/SO(3,\mathbb{R})$, where $SO(3,\mathbb{R})$ is the rotation group. By the Iwasawa decomposition, every $z \in \mathcal{H}^3$ has a unique representation $z = n_x a_y \pmod{SO(3,\mathbb{R})}$ with $n_x \in X$, $a_y \in Y$.

Let $\mathfrak{g}$ be the Lie algebra of G and $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. The center of $U(\mathfrak{g})$ is a polynomial ring in two generators [3]:

$$\Delta_1 = y_1^2\frac{\partial^2}{\partial y_1^2} + y_2^2\frac{\partial^2}{\partial y_2^2} - y_1y_2\frac{\partial^2}{\partial y_1\partial y_2} + y_1^2\frac{\partial^2}{\partial x_1^2} + y_2^2\frac{\partial^2}{\partial x_2^2} + y_2^2(x_1^2+y_1^2)\frac{\partial^2}{\partial x_3^2} + 2y_2^2x_1\frac{\partial^2}{\partial x_2\partial x_3},$$

$$\begin{aligned}\Delta_2 = &- y_2^2y_1\frac{\partial^3}{\partial y_2^2\partial y_1} + y_2y_1^2\frac{\partial^3}{\partial y_2\partial y_1^2} - y_2^3y_1^2\frac{\partial^3}{\partial x_3^2\partial y_2} + y_2y_1^2\frac{\partial^3}{\partial x_1^2\partial y_2} - 2y_2^2y_1x_1\frac{\partial^3}{\partial x_2\partial x_3\partial y_1}\\ &+ (-x_1^2+y_1^2)y_2^2y_1\frac{\partial^3}{\partial x_3^2\partial y_1} - y_2^2y_1\frac{\partial^3}{\partial x_2^2\partial y_1} + 2y_2^2y_1^2\frac{\partial^3}{\partial x_2\partial x_1\partial x_3} + 2y_2^2y_1x_1\frac{\partial^3}{\partial x_1\partial x_3^2}\\ &+ y_2^2\frac{\partial^2}{\partial y_2^2} - y_1^2\frac{\partial^2}{\partial y_1^2} + 2y_2^2x_1\frac{\partial^2}{\partial x_2 x_3} + (x_1^2+y_1^2)y_2^2\frac{\partial^2}{\partial x_3^2} + y_2^2\frac{\partial^2}{\partial x_2^2} - y_1^2\frac{\partial^2}{\partial x_1^2}.\end{aligned}$$

Definition 0.4 *An* ***automorphic form for*** $SL(3,\mathbb{Z})$ ***of type*** $(\nu_1,\nu_2) \in \mathbb{C}^2$ *is a smooth function* $F : \mathcal{H} \to \mathbb{C}$ *satisfying:*

1. $F(\gamma g) = F(g)$ *for all* $\gamma \in SL(3,\mathbb{Z})$, $g \in \mathcal{H}$,
2. $\Delta_i F(g) = \mu_i(\nu_1,\nu_2)\,F(g)$ *where* $i = 1,2$,
3. *There exists a constant* n_1, n_2 *such that*

$$F\left(\begin{pmatrix} y_1^{2/3}y_2^{1/3} & & \\ & y_1^{-1/3}y_2^{1/3} & \\ & & y_1^{-1/3}y_2^{-2/3}\end{pmatrix}\right) y_1^{n_1}y_2^{n_2}$$

is bounded on the subset of $\mathcal{H}$ *determined by the inequalities* $y_1, y_2 > 1$.

The theory of automorphic forms on $GL(3)$ was greatly advanced by the work of Jacquet, Piatetski-Shapiro and Shalika, who proved the Fourier expansion of the form

$$\begin{aligned} F(g) &= \sum_{k\in\mathbb{Z}}[P^{k,0,0}F](g) + \sum_{\ell=1}^{\infty}\sum_{\gamma\in\Gamma_\infty^{(2)}\backslash\Gamma^{(2)}}\sum_{k\in\mathbb{Z}}[P^{k,0,\ell}F]\left(\left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right)g\right) \\ &= \sum_{\ell\in\mathbb{Z}}[P^{0,0,\ell}F](g) + \sum_{k=1}^{\infty}\sum_{\gamma\in\Gamma_\infty^{(2)}\backslash\Gamma^{(2)}}\sum_{\ell\in\mathbb{Z}}[P^{k,0,\ell}F]\left(\left(\begin{smallmatrix}1 & \\ & \gamma\end{smallmatrix}\right)g\right), \end{aligned} \tag{9}$$

where $\Gamma^{(2)} = SL(2,\mathbb{Z})$, $\Gamma^{(2)}_\infty$ is its subgroup of unit upper triangular matrices, and the coefficients $P^{k,0,\ell}F$ are defined by

$$[P^{k,0,\ell}F](g) \quad := \quad \int_{(\mathbb{Z}\backslash\mathbb{R})^3} F\left(\left(\begin{smallmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{smallmatrix}\right) g\right) e^{-2\pi i(kx+\ell y)}\, dx\, dy\, dz\,. \tag{10}$$

$[P^{k,0,\ell}F](g)$ is therefore a Whittaker function.

Definition 0.5 *A* ***Whittaker function of type*** (ν_1,ν_2) *associated to an additive character* $\chi : \mathbb{R}^2 \to \mathbb{S}^1$ *is a smooth nonzero function* $W : \mathcal{H}^3 \to \mathbb{C}$ *which satisfies the following conditions*

$$\Delta_i W(g) = \mu_i(\nu_1,\nu_2)W(g) \textit{ for } i = 1,2; \tag{11}$$

$$W\left(\left(\begin{smallmatrix}1 & x_1 & x_3\\ & 1 & x_2\\ & & 1\end{smallmatrix}\right) g\right) = \chi(x)W(g). \tag{12}$$

These Whittaker functions can be classified into three major types according to type (ν_1,ν_2) and the additive character χ. They include:

- Polynomials $y_1^{2\nu_1+\nu_2} y_2^{\nu_1+2\nu_2}$
- Decaying functions $y_1^{\frac{\nu_1}{2}+\nu_2+\frac{1}{2}} y_2^{\nu_1+2\nu_2} K_{\frac{3\nu_1-1}{2}}(2\pi y_1)$, $y_1^{2\nu_1+\nu_2} y_2^{\nu_1+\frac{\nu_2}{2}+\frac{1}{2}} K_{\frac{3\nu_2-1}{2}}(2\pi y_2)$,

$$W_{(\nu_1,\nu_2)}(y_1,y_2) = \frac{4\pi^2 y_1^{1+\frac{\nu_1}{2}-\frac{\nu_2}{2}}}{y_2^{-1+\frac{\nu_1}{2}-\frac{\nu_2}{2}}} \int_0^\infty K_{\frac{3\nu_1+3\nu_2-2}{2}}(2\pi y_1\sqrt{1+x}) K_{\frac{3\nu_1+3\nu_2-2}{2}}(2\pi y_2\sqrt{1+x^{-1}}) x^{\frac{3\nu_1-3\nu_2}{4}} \frac{dx}{x}$$

- Non-decaying functions $y_1^{\frac{\nu_1}{2}+\nu_2+\frac{1}{2}} y_2^{\nu_1+2\nu_2} I_{\frac{3\nu_1-1}{2}}(2\pi y_1)$, $y_1^{2\nu_1+\nu_2} y_2^{\nu_1+\frac{\nu_2}{2}+\frac{1}{2}} I_{\frac{3\nu_2-1}{2}}(2\pi y_2)$,

$$M_{(\nu_1,\nu_2)}(y_1,y_2) = \sum_{k_1,k_2=0}^{\infty} \frac{\Gamma(k_1+k_2+\frac{3\nu_1+3\nu_2}{2})(\pi y_1)^{2k_1+2\nu_1+\nu_2}(\pi y_2)^{2k_2+\nu_1+2\nu_2}}{k_1!k_2!\Gamma(k_1+\frac{3\nu_1+1}{2})\Gamma(k_2+\frac{3\nu_2+1}{2})\Gamma(k_1+\frac{3\nu_1+3\nu_2}{2})\Gamma(k_2+\frac{3\nu_1+3\nu_2}{2})}$$

Since the Fourier coefficients of an automorphic form along the nilpotent radical of parabolic subgroup are expressed in terms of Whittaker functions, a better understanding of their growth in every direction would be useful in the study of automorphic forms. Bump and Huntley (1995) used an integral formula which was found by Vinogradov, Takhtadzhyan (1978), and Stade (1988) to obtain precise information of the spherical Whittaker functions $M_{(\nu_1,\nu_2)}(y_1,y_2)$ as both y_1 and $y_2 \to \infty$. To (1995) used a method similar to the characteristic method in the theory of differential equations to compute the leading exponents of asymptotic expansions of a basis of Whittaker functions in the positive Weyl chamber for a split semi-simple Lie group over $\mathbb{R}$, which, in particular, yields a solution to Zuckerman's conjecture for $SL(3,\mathbb{R})$. Templier (2015) has recently used an integral representation by Givental to show To's result: the exponential growth of $M_{(\nu_1,\nu_2)}(y_1,y_2)$ for $y_1,y_2 \geq 1$ as either or both $y_1,y_2 \to \infty$. I use a new formula which was derived by Ishii and Stade (2007)

$$\mathcal{M}_{(a_1,a_2,a_3)}(y_1,y_2) = Cy_1y_2 \sum_{k=0}^{\infty} \frac{(\pi y_1)^{k-\frac{a_1}{2}} (\pi y_2)^{k+\frac{a_3}{2}}}{k!\,\Gamma(k+\frac{a_3-a_1}{2}+1)} I_{k+\frac{a_3-a_2}{2}}(2\pi y_1)\, I_{k+\frac{a_2-a_1}{2}}(2\pi y_2) \tag{13}$$

where $C = \Gamma(\frac{a_3-a_2}{2}+1)\,\Gamma(\frac{a_3-a_1}{2}+1)\,\Gamma(\frac{a_2-a_1}{2}+1)$, then obtain the asymptotic expansion of $\mathcal{M}_{(a_1,a_2,a_3)}(t,\frac{1}{t^p})$ as $t\to\infty$, where $2 \leq p \in \frac{1}{2}\mathbb{Z}$ in Chapter 2. Note that the relation of $M_{(\nu_1,\nu_2)}(y_1,y_2)$ and $\mathcal{M}_{(a_1,a_2,a_3)}(y_1,y_2)$ is defined by (2.20).

Theorem 0.6 *Let $p \geq 2$ and $p \in \frac{1}{2}\mathbb{Z}$. For any $(a_1,a_2,a_3)\in\mathbb{C}^3$, the asymptotics of $\mathcal{M}$-Whittaker functions $\mathcal{M}_{(a_1,a_2,a_3)}(t,\frac{1}{t^p})$ and $\mathcal{M}_{(a_1,a_2,a_3)}(\frac{1}{t^p},t)$ for $SL(3,\mathbb{Z})$ with $t\to+\infty$ are*

$$\mathcal{M}_{(a_1,a_2,a_3)}(t,\frac{1}{t^p}) \sim \frac{\pi^{-\frac{3a_1}{2}-1}\Gamma(\frac{a_3-a_2}{2}+1)}{2}\,\frac{e^{2\pi t}}{t^{(p-\frac{1}{2})(1-a_1)}} \tag{14}$$

$$\mathcal{M}_{(a_1,a_2,a_3)}(\frac{1}{t^p},t) \sim \frac{\pi^{\frac{3a_3}{2}-1}\Gamma(\frac{a_2-a_1}{2}+1)}{2}\,\frac{e^{2\pi t}}{t^{(p-\frac{1}{2})(a_3+1)}}$$

If $\nu_1,\nu_2 \neq \frac{1}{3}$ and $\nu_1+\nu_2\neq\frac{2}{3}$ then $\{M_{\omega_k(\nu_1,\nu_2)}(y_1,y_2)\}_{k=1}^{6}$ is a basis for Whittaker space on $SL(3,\mathbb{R})$, where $\omega_k(\nu_1,\nu_2)$ is the Weyl group action on $\mathbb{C}^2$ (Bump, 1984, Page 24). Based on the previous asymptotic expansions, I proved the following multiplicity one theorem in the directions $(\frac{1}{t^p},t)$ and $(t,\frac{1}{t^p})$, where $t\to+\infty$.

Theorem 0.7 (Coroot Multiplicity One) *Assume $2\leq p\in\frac{1}{2}\mathbb{Z}$, $\nu_1,\nu_2\neq\frac{1}{3}$, and $\nu_1+\nu_2\neq\frac{2}{3}$. The unique combination (up to constants) $\sum_{k=1}^{6}\alpha_k M_{\omega_k(\nu_1,\nu_2)}$ which is not of exponential growth at $(t,\frac{1}{t^p})$ and $(\frac{1}{t^p},t)$ as $t\to\infty$ is $\alpha_0=\alpha_4=\alpha_5=1$ and $\alpha_1=\alpha_2=\alpha_3=-1$.*

Moreover, the only non-growing combination of M-Whittaker functions $M_{\omega_k(\nu_1,\nu_2)}$ is the W-Whittaker function:

$$M_{\omega_0(\nu_1,\nu_2)} - M_{\omega_1(\nu_1,\nu_2)} - M_{\omega_2(\nu_1,\nu_2)} - M_{\omega_3(\nu_1,\nu_2)} + M_{\omega_4(\nu_1,\nu_2)} + M_{\omega_5(\nu_1,\nu_2)} = W_{(\nu_1,\nu_2)} \tag{15}$$

The coroot multiplicity one theorem is the key to understand the growth of Fourier coefficients of automorphic eigenfunction on $SL(3,\mathbb{R})$. It leads to my joint work with Stephen Miller [13] addressing the absence of non-decaying Whittaker functions in the Piatetski-Shapiro/Shalika Fourier expansion of automorphic forms on $SL(3,R)$. This confirms part of a conjecture of Miatello and Wallach, who assert the moderate growth condition is automatically satisfied for automorphic eigenfunctions on semisimple groups of split rank greater than 1. In particular, the condition (3) in the definition 0.4 is redundant.

Our first result in the joint paper shows that the presence of a non-decaying Whittaker function implies that the terms in (9) are not bounded:

Theorem. [13] *Let $F \in C^\infty(SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3,\mathbb{R}))$ be an eigenfunction of the full ring of bi-invariant differential operators on $SL(3,\mathbb{R})$ which does not satisfy the moderate growth. Suppose that some $[P^{k,0,\ell}F](g)$ does not have moderate growth. Then one of the two Fourier expansions in (9) must contain unbounded large terms, and in particular is not absolutely convergent.*

Moreover, an absolutely convergent Fourier expansion containing only decaying Whittaker functions must have moderate growth, it implies the following strengthening:

Corollary. [13] *The Miatello-Wallach conjecture is true for eigenfunctions $F \in C^\infty(SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3,\mathbb{R}))$ of the full ring of bi-invariant differential operators on $SL(3,\mathbb{R})$ for which*

$$[P^{k,0,\ell}F]\Big(\left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right) g\Big) \text{ and } [P^{k,0,\ell}F]\Big(\left(\begin{smallmatrix}1 & \\ & \gamma\end{smallmatrix}\right) g\Big) \text{ for } k \in \mathbb{Z}, \ell > 0, \gamma \in \Gamma^{(2)}_\infty\backslash\Gamma^{(2)}$$

are bounded for any fixed $g \in SL(3,\mathbb{R})$.

Finally, we show there are no analogs of eigenfunctions on $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R})$ that have both exponential growth and growing Whittaker functions. Put differently, an exponential bound is sufficient to rule out non-decaying Whittaker functions.

Theorem. [13] *Let $F \in C^\infty(SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3,\mathbb{R}))$ be an eigenfunction of the full ring of bi-invariant differential operators on $SL(3,\mathbb{R})$, and assume that*

$$\left|F\left(\left(\begin{smallmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}a_1 & 0 & 0\\ 0 & a_2 & 0\\ 0 & 0 & a_3\end{smallmatrix}\right)\right)\right| \quad \leq \quad C\,\exp(K(\frac{a_1}{a_2}+\frac{a_2}{a_3}))\,, \quad a_1 \geq \frac{\sqrt{3}}{2}a_2 \geq \frac{3}{4}a_3\,, \qquad (16)$$

for some positive constants C and K. Then F's Fourier expansion cannot contain non-decaying Whittaker functions.

Chapter 1

MODIFIED BESSEL FUNCTIONS

Let us consider the second-order modified Bessel differential equation

$$x^2y''(x) + xy'(x) - (x^2 + \nu^2)y(x) = 0. \tag{1.1}$$

Its two linearly independent solutions are called *modified Bessel functions of the first* and *second kinds of order* ν, denoted by I_ν and K_ν, respectively. It is well known that the modified Bessel function of the first kind I_ν can be represented as the infinite series:

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k} \tag{1.2}$$

where $x \in \mathbb{C}$, since this series converges absolutely everywhere by the ratio test. Note that for fixed $x \neq 0$, $I_\nu(x)$ is an entire function of ν. Moreover, $I_\nu(x)$ is real and positive when $\nu > 0$ and $x > 0$. The asymptotic expansions of $I_\nu(x)$ as $x \to \infty$ is

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}}\left[1 + \frac{1-4\nu^2}{8x} + \frac{(1^2-4\nu^2)(3^2-4\nu^2)}{2!\,(8x)^2} + \cdots\right] \qquad (\arg x < \frac{\pi}{2}) \tag{1.3}$$

The modified Bessel function of the second kind K_ν is defined by

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi} \tag{1.4}$$

where the right hand side of this equation is replaced by its limiting value if ν is an integer or zero. The asymptotic expansions of $K_\nu(x)$ as $x \to \infty$ is

$$K_\nu(x) \sim \frac{\pi e^{-x}}{\sqrt{2\pi x}}\left[1 + \frac{1-4\nu^2}{8x} + \frac{(1^2-4\nu^2)(3^2-4\nu^2)}{2!\,(8x)^2} + \cdots\right] \qquad (\arg x < \frac{\pi}{2}) \tag{1.5}$$

Many inequalities and monotonicity properties for the functions I_ν and K_ν and their several combinations have been deduced by many authors, motivated by various problems that arise in wave mechanics, fluid mechanics, electrical engineering, quantum billiards, biophysics, mathematical physics, finite elasticity, probability and statistics, special relativity, etc.

Reudink [12] in 1968 established the inequality $\frac{\partial}{\partial\nu} I_\nu(x) < 0$ for all $x, \nu > 0$.

Lemma 1.1 [12] *For any $x > 0$ and $\nu > 0$, we have $\frac{\partial}{\partial\nu} I_\nu(x) < 0$.*

Proof. The modified Bessel function $K_\nu(x)$ has the integral representation

$$K_\nu(x) = \int_0^\infty e^{-x\cosh(t)} \cosh(\nu t)\, dt, \tag{1.6}$$

and its derivative with respect to ν

$$\frac{\partial}{\partial\nu} K_\nu(x) = \int_0^\infty e^{-x\cosh(t)} \sinh(\nu t)\, t\, dt \tag{1.7}$$

is positive when x and ν are positive.

Consider the following integral

$$I_\nu(x)K_\nu(x) = \frac{2}{\pi^2} \int_0^\infty \frac{\lambda\sinh(\pi\lambda)}{\lambda^2+\nu^2}\, K_{i\lambda}^2(x)\, d\lambda \qquad \text{where } Re(\nu) > 0. \tag{1.8}$$

Differentiate the above formula with respect to ν to obtain an expression for $\frac{\partial}{\partial\nu} I_\nu(x)$,

$$\frac{\partial}{\partial\nu} I_\nu(x) = -\frac{1}{K_\nu(x)} \Big[I_\nu(x) \frac{\partial K_\nu(x)}{\partial\nu} + \frac{4\nu}{\pi^2} \int_0^\infty \frac{\lambda\sinh(\pi\lambda)}{(\lambda^2+\nu^2)^2}\, K_{i\lambda}^2(x)\, d\lambda \Big]. \tag{1.9}$$

Using (1.6), we obtain that $K_{i\lambda}(x)$ is real when λ is real and $x > 0$; hence for $\nu > 0$, the integral

$$\int_0^\infty \frac{\lambda\sinh(\pi\lambda)}{(\lambda^2+\nu^2)^2}\, K_{i\lambda}^2(x)\, d\lambda \geq 0. \tag{1.10}$$

Therefore, since $K_\nu(x), \frac{\partial}{\partial\nu} K_\nu(x), I_\nu(x)$ are positive for $\nu > 0$ and $x > 0$, it follows immediately that

$$\frac{\partial}{\partial\nu} I_\nu(x) < 0.$$

$\square$

Lemma 1.2 [20] *For $Re(\mu+\nu) > -1$, we have*

$$I_\mu(x)\, I_\nu(x) = \frac{2}{\pi} \int_0^{\pi/2} I_{\mu+\nu}(2x\cos\theta)\, \cos(\mu-\nu)\theta\; d\theta \tag{1.11}$$

Proof. The coefficient of $\left(\frac{x}{2}\right)^{\mu+\nu+2m}$ in the product of the two absolutely convergent series

$$I_\mu(x)\, I_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{\mu+2k}}{k!\,\Gamma(\mu+k+1)} \times \sum_{l=0}^{\infty} \frac{(x/2)^{\nu+2l}}{l!\,\Gamma(\nu+l+1)} \tag{1.12}$$

is

$$\begin{aligned}
&\sum_{k=0}^{m} \frac{1}{k!\,\Gamma(\nu+k+1)\,(m-k)!\,\Gamma(\mu+m-k+1)} \\
&= \frac{(-1)^m}{m!\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)} \sum_{k=0}^{m} C_k^m\,(-\nu-m)_{m-k}\,(-\mu-m)_k \\
&= \frac{(-1)^m\,(-\mu-\nu-2m)_m}{m!\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)} \\
&= \frac{(\mu+\nu+m+1)_m}{m!\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)}
\end{aligned}$$

Vandermonde's theorem is used to sum the finite series: $(a+b)_n = \sum_{j=0}^{n} C_j^n\,(a)_{n-j}(b)_j$ where $(a)_n = \dfrac{\Gamma(a+1)}{\Gamma(a+1-n)}$.
Hence, for all values of μ and ν,

$$I_\mu(x)\, I_\nu(x) = \sum_{m=0}^{\infty} \frac{(\frac{x}{2})^{\mu+\nu+2m}(\mu+\nu+m+1)_m}{m!\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)} \tag{1.13}$$

Applying formula

$$\int_0^{\pi/2} \cos^{\mu+\nu+2m}\theta\;\cos(\mu-\nu)\theta\;d\theta = \frac{\pi\,\Gamma(\mu+\nu+2m+1)}{2^{\mu+\nu+2m+1}\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)} \tag{1.14}$$

to (1.13) , provided that $Re(\mu+\nu) > -1$, we obtain

$$\begin{aligned}
I_\mu(x) I_\nu(x) &= \frac{2}{\pi} \sum_{m=0}^{\infty} \int_0^{\pi/2} \frac{x^{\mu+\nu+2m}\;\cos^{\mu+\nu+2m}\theta}{m!\,\Gamma(\mu+\nu+m+1)}\;\cos(\mu-\nu)\theta\;d\theta \\
&= \frac{2}{\pi} \int_0^{\pi/2} I_{\mu+\nu}(2x\cos\theta)\;\cos(\mu-\nu)\theta\;d\theta.
\end{aligned}$$

□

Lemma 1.1 now can be extended to the case when ν is in the right half plane to compute the asymptotic expansion of M-Whittaker functions in the next chapter.

Theorem 1.1 *For fixed $x > 0$, $|I_\nu(x)|$ is strictly decreasing with respect to Re $(\nu) = \sigma$ in the right half plane. $|I_\nu(x)|$ is strictly decreasing with respect to Im $(\nu) = t < 0$, and $|I_\nu(x)|$ is strictly increasing with respect to $t > 0$.*

Proof. Apply Lemma 1.2 with noting that $\overline{I_{\sigma+it}(x)} = I_{\sigma-it}(x)$ for arbitrary $x, \sigma, t > 0$, we have

$$|I_{\sigma+it}(x)|^2 = I_{\sigma+it}(x) I_{\sigma-it}(x) = \frac{2}{\pi}\int_0^{\pi/2} I_{2\sigma}(2x\cos\theta)\ \cosh(2t\theta)\ d\theta. \tag{1.15}$$

Differentiate the above formula with respect to σ then use Lemma 1.1

$$\frac{d}{d\sigma}|I_{\sigma+it}(x)|^2 = \frac{2}{\pi}\int_0^{\pi/2} \frac{\partial I_{2\sigma}(2x\cos\theta)}{\partial\sigma}\ \cosh(2t\theta)\ d\theta < 0, \tag{1.16}$$

Differentiate (1.15) with respect to t

$$\frac{d}{dt}|I_{\sigma+it}(x)|^2 = \frac{2}{\pi}\int_0^{\pi/2} I_{2\sigma}(2x\cos\theta)\ \sinh(2t\theta)\ 2\theta\ d\theta < 0 \qquad \text{if } t < 0,$$

$$\frac{d}{dt}|I_{\sigma+it}(x)|^2 = \frac{2}{\pi}\int_0^{\pi/2} I_{2\sigma}(2x\cos\theta)\ \sinh(2t\theta)\ 2\theta\ d\theta > 0 \qquad \text{if } t > 0.$$

□

Corollary 1.2 *For any $x > 0$ and Re $(\nu) > 0$, we have $|I_{\nu+1}(x)| < |I_\nu(x)|$.*

Corollary 1.3 *For any $x > 0$ and Re $(\nu) = \sigma > 0$, both $I_\nu(x)$ and $\dfrac{\partial}{\partial\nu} I_\nu(x)$ are nonzero.*

Chapter 2

WHITTAKER FUNCTIONS FOR $SL(3,\mathbb{R})$

Let $G = SL(3,\mathbb{R})$, $\Gamma = SL(3,\mathbb{Z})$, and $X \subset G$ be the group of upper triangular, unipotent matrices,

$$X = \left\{ n_x = \begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} : x_i \in \mathbb{R} \right\}. \tag{2.1}$$

Also let $Y \subset G$ be the subgroup,

$$Y = \left\{ a_y = \begin{pmatrix} y_1^{2/3} y_2^{1/3} & & \\ & y_1^{-1/3} y_2^{1/3} & \\ & & y_1^{-1/3} y_2^{-2/3} \end{pmatrix} : y_1, y_2 > 0 \right\}. \tag{2.2}$$

We have $y_1(a_y) = \dfrac{y_1^{2/3} y_2^{1/3}}{y_1^{-1/3} y_2^{1/3}}, y_2(a_y) = \dfrac{y_1^{-1/3} y_2^{1/3}}{y_1^{-1/3} y_2^{-2/3}}$ are roots of $SL(3,\mathbb{R})$.
Now consider the homogeneous space

$$\mathcal{H}^3 = SL(3,\mathbb{R})/SO(3,\mathbb{R})$$

(the "generalized upper half-plane"), where $SO(3,\mathbb{R})$ is the rotation group. By the Iwasawa decomposition, every $z \in \mathcal{H}^3$ has a unique representation

$$z \equiv n_x a_y \pmod{SO(3,\mathbb{R})}$$

with $x \in X$, $y \in Y$. That is,

$$z = \begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} \begin{pmatrix} y_1^{2/3} y_2^{1/3} & & \\ & y_1^{-1/3} y_2^{1/3} & \\ & & y_1^{-1/3} y_2^{-2/3} \end{pmatrix} = \begin{pmatrix} y_1^{2/3} y_2^{1/3} & y_1^{-1/3} y_2^{1/3} x_1 & y_1^{-1/3} y_2^{-2/3} x_3 \\ & y_1^{-1/3} y_2^{1/3} & y_1^{-1/3} y_2^{-2/3} x_2 \\ & & y_1^{-1/3} y_2^{-2/3} \end{pmatrix}. \tag{2.3}$$

We have an action of G on $\mathcal{H}^3$ by left matrix multiplication. A function on $\mathcal{H}^3$ will always be identified with the corresponding function on G obtained by composition with the canonical map $G \to \mathcal{H}^3$.

We will need the following facts about Lie algebras. Let $\mathfrak{g}$ be the Lie algebra of G. If $A \in \mathfrak{g}$, then A acts on $C^\infty(G)$ by

$$(Af)(u) = \tfrac{d}{dt} f(u \cdot \exp(tA)) \mid_{t=0} \tag{2.4}$$

$(u \in G, f \in C^\infty(G)\,)$. Also let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$; $U(\mathfrak{g})$ may be identified with the ring of differential operators on G generated by all $A \in \mathfrak{g}$. It may be shown that the center of $U(\mathfrak{g})$ acts as an algebra $\mathcal{D}$ of differential operators on $\mathcal{H}^3$. Then $\mathcal{D}$ is commutative; in fact, we have that $\mathcal{D}$ is a polynomial ring in 2 generators. Moreover, $\mathcal{D}$ commutes with the action of G on $\mathcal{H}^3$. That is, if $d \in \mathcal{D}$ and $f \in C^\infty(\mathcal{H}^3)$, then

$$d(f \circ \gamma)(z) = (df \circ \gamma)(z) \tag{2.5}$$

for all $z \in \mathcal{H}^3, \gamma \in G$. Bump showed that the algebra $\mathcal{D}$ of $GL(3,\mathbb{R})$–invariant differential operators on $\mathcal{H}^3$ has generators ([3], page 33, 34)

$$\begin{aligned}
\Delta_1 =& y_1^2 \frac{\partial^2}{\partial y_1^2} + y_2^2 \frac{\partial^2}{\partial y_2^2} - y_1 y_2 \frac{\partial^2}{\partial y_1 \partial y_2} + y_1^2 \frac{\partial^2}{\partial x_1^2} + y_2^2 \frac{\partial^2}{\partial x_2^2} + y_2^2(x_1^2 + y_1^2)\frac{\partial^2}{\partial x_3^2} + 2y_2^2 x_1 \frac{\partial^2}{\partial x_2 \partial x_3}, \\
\Delta_2 =& - y_2^2 y_1 \frac{\partial^3}{\partial y_2^2 \partial y_1} + y_2 y_1^2 \frac{\partial^3}{\partial y_2 \partial y_1^2} - y_2^3 y_1^2 \frac{\partial^3}{\partial x_3^2 \partial y_2} + y_2 y_1^2 \frac{\partial^3}{\partial x_1^2 \partial y_2} - 2y_2^2 y_1 x_1 \frac{\partial^3}{\partial x_2 \partial x_3 \partial y_1} \\
&+ (-x_1^2 + y_1^2) y_2^2 y_1 \frac{\partial^3}{\partial x_3^2 \partial y_1} - y_2^2 y_1 \frac{\partial^3}{\partial x_2^2 \partial y_1} + 2y_2^2 y_1^2 \frac{\partial^3}{\partial x_2 \partial x_1 \partial x_3} + 2y_2^2 y_1 x_1 \frac{\partial^3}{\partial x_1 \partial x_3^2} \\
&+ y_2^2 \frac{\partial^2}{\partial y_2^2} - y_1^2 \frac{\partial^2}{\partial y_1^2} + 2y_2^2 x_1 \frac{\partial^2}{\partial x_2 x_3} + (x_1^2 + y_1^2) y_2^2 \frac{\partial^2}{\partial x_3^2} + y_2^2 \frac{\partial^2}{\partial x_2^2} - y_1^2 \frac{\partial^2}{\partial x_1^2}.
\end{aligned}$$

Let $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$ (and $z \equiv xy$ as above), we define

$$H_\nu : \mathcal{H}^3 \to \mathbb{C}$$

by

$$H_\nu(z) = H_{(\nu_1,\nu_2)}(z) = y_1^{\nu_1 + 2\nu_2} y_2^{2\nu_1 + \nu_2}. \tag{2.6}$$

It may be shown that H_ν is an eigenfunction of D ([3], page 33, 34). That is,

$$\Delta_1 H_\nu = \mu_\nu(\Delta_1) H_\nu,$$

$$\Delta_2 H_\nu = \mu_\nu(\Delta_2) H_\nu.$$

Let us identify the Weyl group $\mathcal{W}$ of $GL(3,\mathbb{R})$ with the set of matrices $\{\omega_i \,|\, i = 0,1,\dots 5\}$, where

$$\omega_0 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \qquad \omega_1 = \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix},$$

$$\omega_2 = \begin{pmatrix} & -1 & \\ -1 & & \\ & & -1 \end{pmatrix}, \qquad \omega_3 = \begin{pmatrix} -1 & & \\ & & -1 \\ & -1 & \end{pmatrix},$$

$$\omega_4 = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}, \qquad \omega_5 = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}.$$

We define an action of $\mathcal{W}$ on $\mathbb{C}^2$ by requiring, for each $\omega \in \mathcal{W}$, that

$$H_{(\nu_1-\frac{1}{3},\, \nu_2-\frac{1}{3})}(y) = H_{(\mu_1-\frac{1}{3},\, \mu_2-\frac{1}{3})}(\omega y) \tag{2.7}$$

if $(\mu_1,\mu_2) = \omega(\nu_1,\nu_2)$. One then computes that ([3], page 20)

$$\begin{aligned} \omega_0(\nu_1,\nu_2) &= (\nu_1,\nu_2), \\ \omega_1(\nu_1,\nu_2) &= (\frac{2}{3}-\nu_2, \frac{2}{3}-\nu_1), \\ \omega_2(\nu_1,\nu_2) &= (\nu_1+\nu_2-\frac{1}{3}, \frac{2}{3}-\nu_2), \\ \omega_3(\nu_1,\nu_2) &= (\frac{2}{3}-\nu_1, \nu_1+\nu_2-\frac{1}{3}), \\ \omega_4(\nu_1,\nu_2) &= (1-\nu_1-\nu_2, \nu_1), \\ \omega_5(\nu_1,\nu_2) &= (\nu_2, 1-\nu_1-\nu_2). \end{aligned} \tag{2.8}$$

It is convenient to introduce the three auxiliary parameters ([3], page 20)

$$\begin{aligned} a_1 &= -\nu_1 - 2\nu_2 + 1, \\ a_2 &= -\nu_1 + \nu_2, \\ a_3 &= 2\nu_1 + \nu_2 - 1, \end{aligned} \tag{2.9}$$

note that $a_1 + a_2 + a_3 = 0$. Further,

$$H_\nu(z) = y_1^{1-a_1}\, y_2^{1+a_3}.$$

The action of $\mathcal{W}$ on $\mathbb{C}^2$ then permutes the indeterminates a_1, a_2, a_3 ([15], page 703):

$$
\begin{aligned}
\omega_0(a_1, a_2, a_3) &= (a_1, a_2, a_3), \\
\omega_1(a_1, a_2, a_3) &= (a_3, a_2, a_1), \\
\omega_2(a_1, a_2, a_3) &= (a_2, a_1, a_3), \\
\omega_3(a_1, a_2, a_3) &= (a_1, a_3, a_2), \\
\omega_4(a_1, a_2, a_3) &= (a_2, a_3, a_1), \\
\omega_5(a_1, a_2, a_3) &= (a_3, a_1, a_2).
\end{aligned} \tag{2.10}
$$

If we let

$$
\begin{aligned}
\mu_1 &= -1 - a_1 a_2 - a_2 a_3 - a_1 a_3, \\
\mu_2 &= -a_1 a_2 a_3,
\end{aligned} \tag{2.11}
$$

then one may show that $\mu_1 = \mu_\nu(\Delta_1)$ and $\mu_2 = \mu_\nu(\Delta_2)$; that is,

$$\Delta_1 H_\nu = \mu_1 H_\nu,$$

$$\Delta_2 H_\nu = \mu_2 H_\nu.$$

We now wish to discuss $GL(3, \mathbb{R})$-Whittaker functions.

Definition 2.1 *A **Whittaker function of type** (ν_1, ν_2) associated to an additive character $\psi : \mathbb{R}^2 \to \mathbb{S}^1$ is a smooth nonzero function $W : \mathcal{H}^3 \to \mathbb{C}$ which satisfies the following three conditions*

- $\Delta_1 W(g) = \mu_1 W(g)$,
- $\Delta_2 W(g) = \mu_2 W(g)$,
- $W\left(\left(\begin{smallmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{smallmatrix} \right) g \right) = \psi(x_1, x_2)\, W(g)$.

A Whittaker function $W(z)$ of type ν associated to an additive character $\psi(x_1, x_2)$ can always be written in the formula

$$W(z) = \psi(x_1, x_2)\, a_\nu(y),$$

where $a_\nu(y)$ is a function of y only.

Case $\psi(x_1,x_2)=1$

Since $\dfrac{\partial a_\nu(y)}{\partial x_i}=0$, the function $a_\nu(y)$ must then satisfy

$$\begin{aligned}\left[y_1^2\frac{\partial^2}{\partial y_1^2}+y_2^2\frac{\partial^2}{\partial y_2^2}-y_1y_2\frac{\partial^2}{\partial y_1y_2}\right]a_\nu(y)=\mu_1 a_\nu(y);\\ \left[-y_1^2y_2\frac{\partial^3}{\partial y_1^2\partial y_2}+y_1y_2^2\frac{\partial^3}{\partial y_1\partial y_2^2}+y_1^2\frac{\partial^2}{\partial y_1^2}-y_2^2\frac{\partial^2}{\partial y_2^2}\right]a_\nu(y)=\mu_2 a_\nu(y).\end{aligned} \tag{2.12}$$

Fixing $\lambda=(a_1,a_2,a_3)$, the space of solutions to the differential equations (2.12) is generated by the six functions $H_{\omega_i(a_1,a_2,a_3)}(y_1,y_2)$, where $\omega_i\in\mathcal{W}$ defined by

$$\begin{aligned}H_{\omega_0(a_1,a_2,a_3)}(y_1,y_2)&=y_1^{1-a_1}\,y_2^{1+a_3},\\ H_{\omega_1(a_1,a_2,a_3)}(y_1,y_2)&=y_1^{1-a_3}\,y_2^{1+a_1},\\ H_{\omega_2(a_1,a_2,a_3)}(y_1,y_2)&=y_1^{1-a_1}\,y_2^{1+a_2},\\ H_{\omega_3(a_1,a_2,a_3)}(y_1,y_2)&=y_1^{1-a_2}\,y_2^{1+a_3},\\ H_{\omega_4(a_1,a_2,a_3)}(y_1,y_2)&=y_1^{1-a_3}\,y_2^{1+a_2},\\ H_{\omega_5(a_1,a_2,a_3)}(y_1,y_2)&=y_1^{1-a_2}\,y_2^{1+a_1}.\end{aligned} \tag{2.13}$$

Case $\psi(x_1,x_2)=e^{2\pi i x_1}$

Since $\dfrac{\partial a_\nu(y)}{\partial x_i}=0$, the function $a_\nu(y)$ must then satisfy

$$\begin{aligned}&\left[y_1^2\frac{\partial^2}{\partial y_1^2}+y_2^2\frac{\partial^2}{\partial y_2^2}-y_1y_2\frac{\partial^2}{\partial y_1y_2}-4\pi^2\,y_1^2\right]a_\nu(y)=\mu_1 a_\nu(y);\\ &\left[-y_1^2y_2\frac{\partial^3}{\partial y_1^2\partial y_2}+y_1y_2^2\frac{\partial^3}{\partial y_1\partial y_2^2}+4\pi^2y_1^2y_2\frac{\partial}{\partial y_2}\right.\\ &\qquad\left.+y_1^2\frac{\partial^2}{\partial y_1^2}-y_2^2\frac{\partial^2}{\partial y_2^2}-4\pi^2y_1^2\right]a_\nu(y)=\mu_2 a_\nu(y).\end{aligned} \tag{2.14}$$

Fixing $\lambda = (a_1, a_2, a_3)$, the space of solutions to the differential equations (2.14) is generated by the six functions defined by

$$\begin{aligned}
\mathcal{M}^{\alpha_1}_{\text{degen, }\lambda}(y_1, y_2) &= y_1^{1-\frac{a_1}{2}}\, y_2^{1-a_1}\, I_{\frac{a_3-a_2}{2}}(2\pi y_1),\\
\mathcal{W}^{\alpha_1}_{\text{degen, }\lambda}(y_1, y_2) &= y_1^{1-\frac{a_1}{2}}\, y_2^{1-a_1}\, K_{\frac{a_3-a_2}{2}}(2\pi y_1),\\
\mathcal{M}^{\alpha_1}_{\text{degen, }(123)\lambda}(y_1, y_2) &= y_1^{1-\frac{a_2}{2}}\, y_2^{1-a_2}\, I_{\frac{a_1-a_3}{2}}(2\pi y_1),\\
\mathcal{W}^{\alpha_1}_{\text{degen, }(123)\lambda}(y_1, y_2) &= y_1^{1-\frac{a_2}{2}}\, y_2^{1-a_2}\, K_{\frac{a_1-a_3}{2}}(2\pi y_1),\\
\mathcal{M}^{\alpha_1}_{\text{degen, }(321)\lambda}(y_1, y_2) &= y_1^{1-\frac{a_3}{2}}\, y_2^{1-a_3}\, I_{\frac{a_2-a_1}{2}}(2\pi y_1),\\
\mathcal{W}^{\alpha_1}_{\text{degen, }(321)\lambda}(y_1, y_2) &= y_1^{1-\frac{a_3}{2}}\, y_2^{1-a_3}\, K_{\frac{a_2-a_1}{2}}(2\pi y_1).
\end{aligned} \tag{2.15}$$

Case $\psi(x_1, x_2) = e^{2\pi i x_2}$

Since $\dfrac{\partial a_\nu(y)}{\partial x_i} = 0$, the function $a_\nu(y)$ must then satisfy

$$\begin{aligned}
&\left[y_1^2\frac{\partial^2}{\partial y_1^2} + y_2^2\frac{\partial^2}{\partial y_2^2} - y_1y_2\frac{\partial^2}{\partial y_1 y_2} - 4\pi^2\, y_2^2\right]a_\nu(y) = \mu_1 a_\nu(y);\\
&\left[-y_1^2y_2\frac{\partial^3}{\partial y_1^2\partial y_2} + y_1y_2^2\frac{\partial^3}{\partial y_1\partial y_2^2} - 4\pi^2 y_2^2 y_1\frac{\partial}{\partial y_1}\right.\\
&\qquad\qquad \left. + y_1^2\frac{\partial^2}{\partial y_1^2} - y_2^2\frac{\partial^2}{\partial y_2^2} + 4\pi^2 y_2^2\right]a_\nu(y) = \mu_2 a_\nu(y).
\end{aligned} \tag{2.16}$$

Fixing $\lambda = (a_1, a_2, a_3)$, the space of solutions to the differential equations (2.16) is generated by the six functions defined by

$$\begin{aligned}
\mathcal{M}^{\alpha_2}_{\text{degen, }\lambda}(y_1, y_2) &= y_1^{1+a_3}\, y_2^{1+\frac{a_3}{2}}\, I_{\frac{a_2-a_1}{2}}(2\pi y_2),\\
\mathcal{W}^{\alpha_2}_{\text{degen, }\lambda}(y_1, y_2) &= y_1^{1+a_3}\, y_2^{1+\frac{a_3}{2}}\, K_{\frac{a_2-a_1}{2}}(2\pi y_2),\\
\mathcal{M}^{\alpha_2}_{\text{degen, }(123)\lambda}(y_1, y_2) &= y_1^{1+a_2}\, y_2^{1+\frac{a_2}{2}}\, I_{\frac{a_1-a_3}{2}}(2\pi y_2),\\
\mathcal{W}^{\alpha_2}_{\text{degen, }(123)\lambda}(y_1, y_2) &= y_1^{1+a_2}\, y_2^{1+\frac{a_2}{2}}\, K_{\frac{a_1-a_3}{2}}(2\pi y_2),\\
\mathcal{M}^{\alpha_2}_{\text{degen, }(321)\lambda}(y_1, y_2) &= y_1^{1+a_1}\, y_2^{1+\frac{a_1}{2}}\, I_{\frac{a_3-a_2}{2}}(2\pi y_2),\\
\mathcal{W}^{\alpha_2}_{\text{degen, }(321)\lambda}(y_1, y_2) &= y_1^{1+a_1}\, y_2^{1+\frac{a_1}{2}}\, K_{\frac{a_3-a_2}{2}}(2\pi y_2).
\end{aligned} \tag{2.17}$$

Case $\psi(x_1, x_2) = e^{2\pi i(x_1+x_2)}$

The function $a_\nu(y)$ must then satisfy

$$\left[y_1^2\frac{\partial^2}{\partial y_1^2} + y_2^2\frac{\partial^2}{\partial y_2^2} - y_1y_2\frac{\partial^2}{\partial y_1 y_2} - 4\pi^2(y_1^2+y_2^2)\right]a_\nu(y) = \mu_1 a_\nu(y);$$
$$\left[-y_1^2y_2\frac{\partial^3}{\partial y_1^2\partial y_2} + y_1y_2^2\frac{\partial^3}{\partial y_1\partial y_2^2} + 4\pi^2y_1^2y_2\frac{\partial}{\partial y_2} - 4\pi^2y_1y_2^2\frac{\partial}{\partial y_1}\right. \tag{2.18}$$
$$\left. + y_1^2\frac{\partial^2}{\partial y_1^2} - y_2^2\frac{\partial^2}{\partial y_2^2} - 4\pi^2y_1^2 + 4\pi^2y_2^2\right]a_\nu(y) = \mu_2 a_\nu(y).$$

Theorem 2.2 [3] *Assume $\nu_1 \neq \frac{1}{3}$, $\nu_2 \neq \frac{1}{3}$, and $\nu_1+\nu_2 \neq \frac{2}{3}$ the space S_ν of solutions to the differential equations (2.18) is generated by the six linearly independent functions $M_{\omega_i(\nu_1,\nu_2)}(y_1,y_2)$, where $\omega_i \in \mathcal{W}$ defined by*

$$M_{\omega_0(\nu_1,\nu_2)}(y_1,y_2) = \sum_{k_1,k_2=0}^{\infty} \frac{\Gamma(k_1+k_2+\frac{3\nu_1+3\nu_2}{2})\,(\pi y_1)^{2k_1+\nu_1+2\nu_2}\,(\pi y_2)^{2k_2+2\nu_1+\nu_2}}{k_1!\,k_2!\,\Gamma(k_1+\frac{3\nu_1+1}{2})\,\Gamma(k_2+\frac{3\nu_2+1}{2})\,\Gamma(k_1+\frac{3\nu_1+3\nu_2}{2})\,\Gamma(k_2+\frac{3\nu_1+3\nu_2}{2})} \tag{2.19}$$

and $M_{\omega_i(\nu_1,\nu_2)}(y_1,y_2)$ is obtained from $M_{(\nu_1,\nu_2)}(y_1,y_2)$ by letting $\mathcal{W}$ act on (ν_1,ν_2).

Proof. [3], p. 24 □

Using three auxiliary parameters a_1, a_2, a_3, we define

$$\mathcal{M}_{(a_1,a_2,a_3)}(y_1,y_2) = \frac{\Gamma(\frac{a_3-a_2}{2}+1)\Gamma(\frac{a_3-a_1}{2}+1)\Gamma(\frac{a_2-a_1}{2}+1)}{\pi^2}M_{(\nu_1,\nu_2)}(y_1,y_2). \tag{2.20}$$

Theorem 2.3 [9] *The above function $\mathcal{M}_{(a_1,a_2,a_3)}(y_1,y_2)$ can be written as follow*

$$\mathcal{M}_{(a_1,a_2,a_3)}(y_1,y_2) = Cy_1y_2\sum_{k=0}^{\infty}\frac{(\pi y_1)^{k-\frac{a_1}{2}}\,(\pi y_2)^{k+\frac{a_3}{2}}}{k!\;\Gamma(k+\frac{a_3-a_1}{2}+1)}\,I_{k+\frac{a_3-a_2}{2}}(2\pi y_1)\,I_{k+\frac{a_2-a_1}{2}}(2\pi y_2) \tag{2.21}$$

where $C = \Gamma(\frac{a_3-a_2}{2}+1)\,\Gamma(\frac{a_3-a_1}{2}+1)\,\Gamma(\frac{a_2-a_1}{2}+1)$.

Proof. [9], (22), p. 298 □

An asymptotic expansion of a finite sum is the sum of asymptotic expansion. However, an asymptotic expansion of an *infinite* sum in general is not the sum of *infinite* asymptotic expansions. It does hold with some additional uniformity assumptions of the following lemma.

Lemma 2.1 (Infinite sum of asymptotic expansions) *Let p be a positive real number. Assume that for every $k \in \mathbb{N}$, functions $f_k(t)$ satisfy the following two conditions:*

- $f_k(t) \sim \sum_{l=k}^{\infty} \dfrac{a_{kl}}{t^l}$ *as $t \to \infty$ where $a_{kl} \in \mathbb{C}$,*
- $|f_{k+1}(t)| \leq \dfrac{|f_k(t)|}{t^p}$ *for every $t > 0$.*

Then

$$\sum_{k=0}^{\infty} f_k(t) \sim \sum_{k=0}^{\infty} \frac{\sum_{m=0}^{k} a_{mk}}{t^k} \qquad \text{as } t \to \infty.$$

Proof. I need to prove for each $N \in \mathbb{N}$,

$$\lim_{t\to\infty} t^N \left(\sum_{k=0}^{\infty} f_k(t) - \sum_{k=0}^{N} \frac{\sum_{m=0}^{k} a_{mk}}{t^k} \right) = 0$$

Fix an arbitrary $\epsilon > 0$. Since $f_k(t) \sim \sum_{l=k}^{\infty} \dfrac{a_{kl}}{t^l}$ as $t \to \infty$ for each $0 \leq k \leq N$ there is T_k such that for all $t \geq T_k$,

$$|t^N \left(f_k(t) - \sum_{l=k}^{N} \frac{a_{kl}}{t^l} \right)| < \frac{\epsilon}{2^{k+1}}. \tag{2.22}$$

Since $f_{N+1}(t) \sim \sum_{l=N+1}^{\infty} \dfrac{a_{(N+1)l}}{t^l}$ there is T_{N+1} such that for all $t \geq T_{N+1}$,

$$|t^N f_{N+1}(t)| < \frac{\epsilon}{2^{N+2}}. \tag{2.23}$$

The inequality $|f_{k+1}(t)| \leq \dfrac{|f_k(t)|}{t^p}$ show that for each $l \geq 2$, and $t \geq \max(T_{N+1}, 2^{1/p})$,

$$|t^N f_{N+l}(t)| < |t^N f_{N+1}(t) \frac{1}{t^{p(l-1)}}| \leq \frac{\epsilon}{2^{N+l+1}}. \tag{2.24}$$

Summing them together, we have for all $t \geq \max(T_0, \cdots, T_{N+1}, 2^{1/p})$,

$$|t^N \left(\sum_{k=0}^{\infty} f_k(t) - \sum_{k=0}^{N} \frac{\sum_{m=0}^{k} a_{mk}}{t^k} \right)| < \epsilon. \tag{2.25}$$

The lemma, therefore, is proved. $\square$

Theorem 2.4 *Let $p \geq 2$ and $p \in \frac{1}{2}\mathbb{Z}$. For any $(a_1, a_2, a_3) \in \mathbb{C}^3$, the asymptotic expansion of $\mathcal{M}$-Whittaker functions $\mathcal{M}_{(a_1,a_2,a_3)}(t, \frac{1}{t^p})$ for $SL(3, \mathbb{Z})$ with $t \to +\infty$ is*

$$\mathcal{M}_{(a_1,a_2,a_3)}(t, \frac{1}{t^p}) \sim \frac{\pi^{-\frac{3a_1}{2}-1}\Gamma(\frac{a_3-a_2}{2}+1)}{2} \frac{e^{2\pi t}}{t^{(p-\frac{1}{2})(1-a_1)}} \cdot \left(1 + \frac{1-(a_3-a_2)^2}{16\pi\, t} + \frac{[1-(a_3-a_2)^2]\ [9-(a_3-a_2)^2]}{512\pi^2\, t^2} + \cdots \right). \tag{2.26}$$

Proof. By (2.21), we obtain

$$\mathcal{M}_{(a_1,a_2,a_3)}(t,\frac{1}{t^p}) = C\frac{1}{t^{p-1}}\sum_{k=0}^{\infty}\frac{\pi^{2k+\frac{a_3-a_1}{2}}\, I_{k+\frac{a_3-a_2}{2}}(2\pi t)\, I_{k+\frac{a_2-a_1}{2}}(\frac{2\pi}{t^p})}{k!\,\Gamma(k+\frac{a_3-a_1}{2}+1)\; t^{(p-1)k+\frac{pa_3}{2}+\frac{a_1}{2}}} \tag{2.27}$$

Denote

$$\theta_k(t) := C\frac{1}{t^{p-1}}\frac{\pi^{2k+\frac{a_3-a_1}{2}}\, I_{k+\frac{a_3-a_2}{2}}(2\pi t)\, I_{k+\frac{a_2-a_1}{2}}(\frac{2\pi}{t^p})}{k!\,\Gamma(k+\frac{a_3-a_1}{2}+1)\; t^{(p-1)k+\frac{pa_3}{2}+\frac{a_1}{2}}} \tag{2.28}$$

The asymptotic series expansion of $\theta_k(t)$ is

$$\begin{aligned}
\theta_k(t) &\sim \frac{C}{t^{(p-1)k+\frac{pa_3}{2}+\frac{a_1}{2}+p-1}}\frac{\pi^{2k+\frac{a_3-a_1}{2}}}{k!\,\Gamma(k+\frac{a_3-a_1}{2}+1)}\left[\sum_{m=0}^{\infty}\frac{\left(\frac{\pi}{t^p}\right)^{2m+k+\frac{a_2-a_1}{2}}}{m!\,\Gamma(m+1+k+\frac{a_2-a_1}{2})}\right] \\
&\cdot\frac{e^{2\pi t}}{2\pi\sqrt{t}}\left(1+\frac{1-(2k+a_3-a_2)^2}{16\pi t}+\frac{[1-(2k+a_3-a_2)^2]\,[9-(2k+a_3-a_2)^2]}{512\pi^2t^2}+\cdots\right) \\
&\sim \frac{C\pi^{3k+\frac{3a_3+3a_2}{2}-1}\, e^{2\pi t}}{2.k!\,\Gamma(k+\frac{a_3-a_1}{2}+1)t^{(p-\frac{1}{2})(2k+a_3+a_2+1)}}\left(\frac{1}{\Gamma(k+\frac{a_2-a_1}{2}+1)}+\right. \\
&\left.+\frac{1-(2k+a_3-a_2)^2}{16\pi\,\Gamma(k+\frac{a_2-a_1}{2}+1)}\frac{1}{t}+\frac{[1-(2k+a_3-a_2)^2]\,[9-(2k+a_3-a_2)^2]}{512\pi^2\,\Gamma(k+\frac{a_2-a_1}{2}+1)}\frac{1}{t^2}+\cdots\right)
\end{aligned}$$

Theorem 1.1 established

$$|I_{\nu+1}(t)| < |I_\nu(t)| \qquad \text{where } Re(\nu)>0 \text{ and } t>0.$$

For k very large, we have

$$\begin{aligned}
|\theta_{k+1}(t)| &= |\frac{C}{t^{p-1}}\frac{\pi^{2k+\frac{a_3-a_1}{2}+2}}{k!\Gamma(k+\frac{a_3-a_1}{2}+2)}\frac{I_{k+\frac{a_3-a_2}{2}+1}(2\pi t)I_{k+\frac{a_2-a_1}{2}+1}(\frac{2\pi}{t^p})}{t^{(p-1)(k+1)+\frac{pa_3}{2}+\frac{a_1}{2}}}| \\
&< \frac{\pi^2}{|(k+1)(k+\frac{a_3-a_1}{2}+1)|}\frac{|\theta_k(t)|}{t^{p-1}}
\end{aligned}$$

Apply Lemma 2.1, we get

$$\begin{aligned}
\mathcal{M}_{(a_1,a_2,a_3)}(t,\frac{1}{t^p}) &\sim \frac{\pi^{-\frac{3a_1}{2}-1}\,\Gamma(\frac{a_3-a_2}{2}+1)}{2}\frac{e^{2\pi t}}{t^{(p-\frac{1}{2})(-a_1+1)}} \\
&\cdot\left(1+\frac{1-(a_3-a_2)^2}{16\pi}\frac{1}{t}+\frac{[1-(a_3-a_2)^2]\,[9-(a_3-a_2)^2]}{512\pi^2}\frac{1}{t^2}+\cdots\right).
\end{aligned}$$

□

Similarity, we have

Theorem 2.5 *Let $p \geq 2$ and $p \in \dfrac{1}{2}\mathbb{Z}$. For any $(a_1, a_2, a_3) \in \mathbb{C}^3$, the asymptotic expansion of $\mathcal{M}$-Whittaker functions $\mathcal{M}_{(a_1,a_2,a_3)}(\frac{1}{t^p}, t)$ for $SL(3,\mathbb{Z})$ with $t \to +\infty$ is*

$$\mathcal{M}_{(a_1,a_2,a_3)}(\frac{1}{t^p}, t) \sim \frac{\pi^{\frac{3a_3}{2}-1}\Gamma(\frac{a_2-a_1}{2}+1)}{2} \frac{e^{2\pi t}}{t^{(p-\frac{1}{2})(a_3+1)}} \cdot \left(1 + \frac{1-(a_2-a_1)^2}{16\pi}\frac{1}{t} + \frac{[1-(a_2-a_1)^2][9-(a_2-a_1)^2]}{512\pi^2}\frac{1}{t^2} + \cdots\right).$$

Proof. By (2.21), we obtain

$$\mathcal{M}_{(a_1,a_2,a_3)}(\frac{1}{t^p}, t) = C\frac{1}{t^{p-1}}\sum_{k=0}^{\infty} \frac{\pi^{2k+\frac{a_3-a_1}{2}}}{k!\,\Gamma(k+\frac{a_3-a_1}{2}+1)} \frac{I_{k+\frac{a_3-a_2}{2}}(\frac{2\pi}{t^p})\, I_{k+\frac{a_2-a_1}{2}}(2\pi t)}{t^{(p-1)k-\frac{pa_1}{2}-\frac{a_3}{2}}}. \tag{2.29}$$

Denote

$$\theta_k(t) := \frac{C}{t^{p-1}} \frac{\pi^{2k+\frac{a_3-a_1}{2}}}{k!\,\Gamma(k+\frac{a_3-a_1}{2}+1)} \frac{I_{k+\frac{a_3-a_2}{2}}(\frac{2\pi}{t^p})\, I_{k+\frac{a_2-a_1}{2}}(2\pi t)}{t^{(p-1)k-\frac{pa_1}{2}-\frac{a_3}{2}}}. \tag{2.30}$$

The asymptotic series expansion of $\theta_k(t)$ is

$$\begin{aligned}
\theta_k(t) &\sim \frac{C}{t^{(p-1)k-\frac{pa_1}{2}-\frac{a_3}{2}+p-1}} \frac{\pi^{2k+\frac{a_3-a_1}{2}}}{k!\,\Gamma(k+\frac{a_3-a_1}{2}+1)} \left[\sum_{m=0}^{\infty} \frac{\left(\frac{\pi}{t^p}\right)^{2m+k+\frac{a_3-a_2}{2}}}{m!\,\Gamma(m+1+k+\frac{a_3-a_2}{2})}\right] \\
&\cdot \frac{e^{2\pi t}}{2\pi\sqrt{t}}\left(1 + \frac{1-(2k+a_2-a_1)^2}{16\pi\, t} + \frac{[1-(2k+a_2-a_1)^2]\,[9-(2k+a_2-a_1)^2]}{512\pi^2\, t^2} + \cdots\right) \\
&\sim \frac{C\,\pi^{3k+\frac{3a_3}{2}-1}\, e^{2\pi t}}{2.k!\,\Gamma(k+\frac{a_3-a_1}{2}+1)t^{(p-\frac{1}{2})(2k+a_3+1)}}\left(\frac{1}{\Gamma(k+\frac{a_3-a_2}{2}+1)} + \right. \\
&\left. + \frac{1-(k+a_2-a_1)^2}{16\pi\,\Gamma(k+\frac{a_3-a_2}{2}+1)}\frac{1}{t} + \frac{[1-(2k+a_2-a_1)^2]\,[9-(2k+a_2-a_1)^2]}{512\pi^2\,\Gamma(k+\frac{a_3-a_2}{2}+1)}\frac{1}{t^2} + \cdots\right)
\end{aligned}$$

Theorem 1.1 established

$$|I_{\nu+1}(t)| < |I_\nu(t)| \qquad \text{where } Re(\nu) > 0 \text{ and } x > 0.$$

For k very large, we have

$$\begin{aligned}
|\theta_{k+1}(t)| &= |\frac{C}{t^{p-1}} \frac{\pi^{2k+\frac{a_3-a_1}{2}+2}}{k!\,\Gamma(k+\frac{a_3-a_1}{2}+2)} \frac{I_{k+\frac{a_3-a_2}{2}+1}(2\pi t)\, I_{k+\frac{a_2-a_1}{2}+1}(\frac{2\pi}{t^p})}{t^{(p-1)(k+1)-\frac{pa_1}{2}-\frac{a_3}{2}}}| \\
&< \frac{\pi^2}{|(k+1)(k+\frac{a_3-a_1}{2}+1)|}\frac{|\theta_k(t)|}{t^{p-1}}
\end{aligned}$$

Apply Lemma 2.1, we get

$$\mathcal{M}_{(a_1,a_2,a_3)}(\frac{1}{t^p},t) \sim \frac{\pi^{\frac{3a_3}{2}-1}\Gamma(\frac{a_2-a_1}{2}+1)}{2}\frac{e^{2\pi t}}{t^{(p-\frac{1}{2})(a_3+1)}}$$
$$\cdot\left(1+\frac{1-(a_2-a_1)^2}{16\pi}\frac{1}{t}+\frac{[1-(a_2-a_1)^2]\,[9-(a_2-a_1)^2]}{512\pi^2}\frac{1}{t^2}+\cdots\right).$$

□

We now wish to find a $GL(3,\mathbb{Z})$-Whittaker function

$$W_{(\nu_1,\nu_2)}(z) = \mathrm{e}(x_1+x_2)W_{(\nu_1,\nu_2)}(y_1,y_2)$$

that grows at most polynomially in the $y_i{}'s$. Bump shows that the function we are looking for may be given by

$$W_\lambda(y_1,y_2) = \frac{1}{4}\frac{1}{(2\pi i)^2}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\int_{\sigma_2-i\infty}^{\sigma_2+i\infty} V_{(\nu_1,\nu_2)}(s_1,s_2)\ (\pi y_1)^{1-s_1}(\pi y_2)^{1-s_2}\,ds_1\,ds_2 \tag{2.31}$$

($\sigma_i = Re(s_i)$), where

$$V_\lambda(s_1,s_2) = \frac{\Gamma(\frac{s_1+a_1}{2})\,\Gamma(\frac{s_1+a_2}{2})\,\Gamma(\frac{s_1+a_3}{2})\ \Gamma(\frac{s_2-a_1}{2})\,\Gamma(\frac{s_2-a_2}{2})\,\Gamma(\frac{s_2-a_3}{2})}{\Gamma(\frac{s_1+s_2}{2})}.$$

Vinogradov-Takhtajan [19] and Stade [16] proved the integral formula

$$\begin{aligned} W_\lambda(y_1,y_2) &= 4\,y_1^{1-a_2/2}y_2^{1+a_2/2}\,\times \\ &\int_0^\infty K_{(a_1-a_3)/2}(2\pi y_1\sqrt{1+x})K_{(a_1-a_3)/2}(2\pi y_2\sqrt{1+x^{-1}})\,x^{-3a_2/4}\,\frac{dx}{x} \end{aligned} \tag{2.32}$$

It follows from this integral and the inequalities $|K_\nu(x)| \le K_{\mathrm{Re}(\nu)}(x)$, $K_{\mathrm{Re}(\nu)}(x) > 0$ that

$$|W_\lambda(y_1,y_2)| \quad\le\quad W_{\mathrm{Re}(\lambda)}(y_1,y_2) \tag{2.33}$$

where the righthand side is in fact positive.

Lemma 2.2 [4] *There exists an integer N (depending on λ) such that*

$$|\,W_\lambda(y_1,y_2)\,| \quad\ll\quad (y_1y_2)^{-N}\,e^{-\pi(y_1+y_2)} \tag{2.34}$$

for any $y_1,y_2>0$, where the implied constant depends only on λ.

Proof. I am explaining the proof in [13].

Since the terms outside the integral in (2.32) can be absorbed into the constant and polynomial factors, it suffices to estimate the integral

$$V = \int_0^\infty K_{(a_1-a_3)/2}(2\pi y_1\sqrt{1+x})\, K_{(a_1-a_3)/2}(2\pi y_2\sqrt{1+x^{-1}})\, x^{-3a_2/4-1}\, dx\,, \qquad (2.35)$$

Note that formula (1.5) shows that $e^u K_\nu(u)$ is bounded for large u, and formula (1.4) shows that $K_\nu(u)$ is bounded by $|u|^{-q}$ for some $q > 0$ as $u \to 0$.

We now split the range of integration into three pieces: $0 < x < 1/2$, $1/2 \le x \le 2$, and $2 < x < \infty$.

$$\begin{aligned} V_2 &= \int_{1/2}^2 K_{(a_1-a_3)/2}(2\pi y_1\sqrt{1+x})\, K_{(a_1-a_3)/2}(2\pi y_2\sqrt{1+x^{-1}})\, x^{-3a_2/4-1}\, dx\,, \\ &\ll \int_{1/2}^2 \frac{e^{-2\pi y_1\sqrt{1+x}}}{\sqrt{y_1}}\, \frac{e^{-2\pi y_2\sqrt{1+x^{-1}}}}{\sqrt{y_2}}\, x^{-3a_2/4-1}\, dx \ll e^{-2\pi(y_1+y_2)}\, (y_1y_2)^{-1/2} \end{aligned} \qquad (2.36)$$

$$\begin{aligned} V_3 &= \int_2^\infty K_{(a_1-a_3)/2}(2\pi y_1\sqrt{1+x})\, K_{(a_1-a_3)/2}(2\pi y_2\sqrt{1+x^{-1}})\, x^{-3a_2/4-1}\, dx\,, \\ &\ll \int_2^\infty \frac{e^{-2\pi y_1\sqrt{1+x}}}{\sqrt{y_1}}\, \frac{e^{-2\pi y_2\sqrt{1+x^{-1}}}}{\sqrt{y_2}}\, x^{-3a_2/4-1}\, dx \\ &\ll (y_1y_2)^{-1/2} \int_2^\infty e^{-2\pi y_1\sqrt{1+x}}\, e^{-2\pi y_2}\, x^{-3a_2/4-1}\, dx \qquad \text{now let } x = u^2+2u \\ &= e^{-2\pi(y_1+y_2)}\, (y_1y_2)^{-1/2} \int_{\sqrt{3}-1}^\infty e^{-2\pi y_1 u}\, (u^2+2u)^{-3a_2/4-1}\, (2u+2)du \\ &\ll e^{-2\pi(y_1+y_2)}\, (y_1y_2)^{-1/2} \int_{\sqrt{3}-1}^\infty e^{-2\pi y_1 u}\, u^{-3a_2/2}\, du \ll e^{-2\pi(y_1+y_2)}\, (y_1y_2)^{-N} \end{aligned} \qquad (2.37)$$

for some N and an implied constant depends only on λ.

Similarity,

$$\begin{aligned} V_1 &= \int_0^{1/2} K_{(a_1-a_3)/2}(2\pi y_1\sqrt{1+x})\, K_{(a_1-a_3)/2}(2\pi y_2\sqrt{1+x^{-1}})\, x^{-3a_2/4-1}\, dx\,, \\ &= \int_2^\infty K_{(a_1-a_3)/2}(2\pi y_1\sqrt{1+x^{-1}})\, K_{(a_1-a_3)/2}(2\pi y_2\sqrt{1+x})\, x^{3a_2/4-1}\, dx \\ &\ll e^{-2\pi(y_1+y_2)}\, (y_1y_2)^{-N} \end{aligned} \qquad (2.38)$$

Sum up these estimates, the lemma therefore is proved. □

Theorem 2.6 (Coroot Multiplicity One) *Assume $p \ge 2$, $p \in \frac{1}{2}\mathbb{Z}$, $\nu_1, \nu_2 \ne \frac{1}{3}$, and $\nu_1 + \nu_2 \ne \frac{2}{3}$. The unique combination (up to constants) $S = \sum_{k=0}^5 \alpha_k M_{\omega_k(\nu_1,\nu_2)}$ which* *is not of exponential growth at* $(t, \frac{1}{t^p})$ *and* $(\frac{1}{t^p}, t)$ *as* $t \to \infty$ *is* $\alpha_0 = \alpha_4 = \alpha_5 = 1$ *and* $\alpha_1 = \alpha_2 = \alpha_3 = -1$.

Moreover, the only non-growing combination of M-Whittaker functions $M_{\omega_k(\nu_1,\nu_2)}$ *is the W-Whittaker function:*

$$M_{\omega_0(\nu_1,\nu_2)} - M_{\omega_1(\nu_1,\nu_2)} - M_{\omega_2(\nu_1,\nu_2)} - M_{\omega_3(\nu_1,\nu_2)} + M_{\omega_4(\nu_1,\nu_2)} + M_{\omega_5(\nu_1,\nu_2)} = W_{(\nu_1,\nu_2)}.$$

Proof. By (2.20) and Theorem 2.4, we have:

$$M_{\omega_0(\nu_1,\nu_2)}(t, \frac{1}{t^p}) \sim \frac{\pi^{-\frac{a_1}{2}-1}}{2\,\Gamma(\frac{a_3-a_1}{2}+1)\,\Gamma(\frac{a_2-a_1}{2}+1)}\ \frac{e^{2\pi t}}{t^{(p-\frac{1}{2})(-a_1+1)}}$$

$$M_{\omega_2(\nu_1,\nu_2)}(t, \frac{1}{t^p}) \sim \frac{\pi^{-\frac{a_1}{2}-1}}{2\,\Gamma(\frac{a_3-a_1}{2}+1)\,\Gamma(\frac{a_2-a_1}{2}+1)}\ \frac{e^{2\pi t}}{t^{(p-\frac{1}{2})(-a_1+1)}}$$

$$M_{\omega_1(\nu_1,\nu_2)}(t, \frac{1}{t^p}) \sim \frac{\pi^{-\frac{a_3}{2}-1}}{2\,\Gamma(\frac{a_1-a_3}{2}+1)\,\Gamma(\frac{a_2-a_3}{2}+1)}\ \frac{e^{2\pi t}}{t^{(p-\frac{1}{2})(-a_3+1)}}$$

$$M_{\omega_4(\nu_1,\nu_2)}(t, \frac{1}{t^p}) \sim \frac{\pi^{-\frac{a_3}{2}-1}}{2\,\Gamma(\frac{a_1-a_3}{2}+1)\,\Gamma(\frac{a_2-a_3}{2}+1)}\ \frac{e^{2\pi t}}{t^{(p-\frac{1}{2})(-a_3+1)}}$$

$$M_{\omega_3(\nu_1,\nu_2)}(t, \frac{1}{t^p}) \sim \frac{\pi^{-\frac{a_2}{2}-1}}{2\,\Gamma(\frac{a_3-a_2}{2}+1)\,\Gamma(\frac{a_1-a_2}{2}+1)}\ \frac{e^{2\pi t}}{t^{(p-\frac{1}{2})(-a_2+1)}}$$

$$M_{\omega_5(\nu_1,\nu_2)}(t, \frac{1}{t^p}) \sim \frac{\pi^{-\frac{a_2}{2}-1}}{2\,\Gamma(\frac{a_3-a_2}{2}+1)\,\Gamma(\frac{a_1-a_2}{2}+1)}\ \frac{e^{2\pi t}}{t^{(p-\frac{1}{2})(-a_2+1)}}$$

By the asumption, $\nu_1, \nu_2 \neq \dfrac{1}{3}$, and $\nu_1+\nu_2 \neq \dfrac{2}{3}$, we have a_1, a_2, a_3 are distinct. Therefore, M-Whittaker functions $M_{\omega_0(\nu_1,\nu_2)}(t, \frac{1}{t^p})$, $M_{\omega_1(\nu_1,\nu_2)}(t, \frac{1}{t^p})$, $M_{\omega_2(\nu_1,\nu_2)}(t, \frac{1}{t^p})$ do not have the same asymptotics. As $t \to \infty$ if

$$\sum_{k=0}^{5} \alpha_k M_{\omega_k(\nu_1,\nu_2)}](t, \frac{1}{t^p}) \sim 0 \tag{2.39}$$

then $\qquad \alpha_0 = -\alpha_2, \quad \alpha_1 = -\alpha_4, \quad \alpha_3 = -\alpha_5.$

Similarity,

$$M_{\omega_0(\nu_1,\nu_2)}(\frac{1}{t^p}, t) \sim \frac{\pi^{\frac{3a_3}{2}+1}}{2\,\Gamma(\frac{a_3-a_2}{2}+1)\,\Gamma(\frac{a_3-a_1}{2}+1)}\frac{e^{2\pi t}}{t^{(p-\frac{1}{2})(a_3+1)}}$$

$$M_{\omega_3(\nu_1,\nu_2)}(\frac{1}{t^p}, t) \sim \frac{\pi^{\frac{3a_3}{2}+1}}{2\,\Gamma(\frac{a_3-a_2}{2}+1)\,\Gamma(\frac{a_3-a_1}{2}+1)}\frac{e^{2\pi t}}{t^{(p-\frac{1}{2})(a_3+1)}}$$

$$M_{\omega_2(\nu_1,\nu_2)}(\frac{1}{t^p}, t) \sim \frac{\pi^{\frac{3a_2}{2}+1}}{2\,\Gamma(\frac{a_2-a_3}{2}+1)\,\Gamma(\frac{a_2-a_1}{2}+1)}\frac{e^{2\pi t}}{t^{(p-\frac{1}{2})(a_2+1)}}$$

$$M_{\omega_4(\nu_1,\nu_2)}(\frac{1}{t^p},t) \sim \frac{\pi^{\frac{3a_2}{2}+1}}{2\,\Gamma(\frac{a_2-a_3}{2}+1)\,\Gamma(\frac{a_2-a_1}{2}+1)} \frac{e^{2\pi t}}{t^{(p-\frac{1}{2})(a_2+1)}}$$

$$M_{\omega_1(\nu_1,\nu_2)}(\frac{1}{t^p},t) \sim \frac{\pi^{\frac{3a_1}{2}+1}}{2\,\Gamma(\frac{a_1-a_3}{2}+1)\,\Gamma(\frac{a_1-a_2}{2}+1)} \frac{e^{2\pi t}}{t^{(p-\frac{1}{2})(a_1+1)}}$$

$$M_{\omega_5(\nu_1,\nu_2)}(\frac{1}{t^p},t) \sim \frac{\pi^{\frac{3a_1}{2}+1}}{2\,\Gamma(\frac{a_1-a_3}{2}+1)\,\Gamma(\frac{a_1-a_2}{2}+1)} \frac{e^{2\pi t}}{t^{(p-\frac{1}{2})(a_1+1)}}$$

Since a_1, a_2, a_3 are distinct, $M_{\omega_0(\nu_1,\nu_2)}(t, \frac{1}{t^p})$, $M_{\omega_1(\nu_1,\nu_2)}(t, \frac{1}{t^p})$, $M_{\omega_3(\nu_1,\nu_2)}(t, \frac{1}{t^p})$ do not have the same asymptotics. As $t \to \infty$ if

$$\sum_{k=0}^{5} \alpha_k M_{\omega_k(\nu_1,\nu_2)}](t, \frac{1}{t^p}) \sim 0 \tag{2.40}$$

then $\alpha_0 = -\alpha_3, \quad \alpha_2 = -\alpha_4, \quad \alpha_1 = -\alpha_5.$

Therefore, up to constants $\alpha_0 = \alpha_4 = \alpha_5 = 1$ and $\alpha_1 = \alpha_2 = \alpha_3 = -1$. □

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