RELATIVE RIPS MACHINE AND THIN TYPE COMPONENTS OF BAND COMPLEXES

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Abstract

Relative Rips Machine

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The Rips machine is a method of studying the action of groups on real trees. Roughly speaking, the Rips machine is an algorithm that takes as input a finite 2-complex equipped with a transversely measured lamination, namely a band complex, and puts it in a "normal form", which is the disjoint union of finitely many sub-laminations. Each component of this normal form belongs to one of the four types: *simplicial, surface, toral* and *thin*. The earlier three types are well-studied, whereas thin type does not have a standard model. Building on the work of [BF95], the first part of this thesis provides an additional structure for thin type components of band complexes. The second part of this paper develops a version of the Rips machine which studies pairs of band complexes. The goal of this machine is to convert pairs of band complexes into standard forms which can be further used to study sub-laminations and subgroup actions.

To our little growing E-tree.

Contents

1 Introduction

The *Rips machine* was introduced by Eliyahu Rips in about 1991 to study the action of groups on real trees. In [BF95], Bestvina and Feighn described the machine as processes made of geometric moves on a special class of 2-complexes equipped with measured laminations, called *band complexes*, which is also interpreted as a system consisting a finite number of isometries between compact intervals in [GLP94].

A band complex is constructed in the following way (See more details in definition 2.9). A band B is a space of the form $b \times I$ where b is an arc of the real line and $I = [1, 0]$. $b \times \{0\}$ and $b \times \{1\}$ are called *bases* of B. A subset of the form $\{point\} \times I$ is called a *vertical fiber* of B. A union of bands Y is a space obtained from a multi-interval Γ by attaching a finite collection of bands through length-preserving homeomorphisms from their bases into Γ. Each band is assigned a *weight* according to the attaching map. B is a weight 0 band if it is an (foliated) annulus. B is a weight $\frac{1}{2}$ band if it is a Mobius band. Otherwise B is a weight 1 band. A *leaf* of Y is an equivalence class of the equivalence relation on Y generated by $x \sim x'$ if $\{x, x'\}$ is a subset of some vertical fiber of a band in Y. Y is then naturally *foliated* by its leaves. A band complex X is a CW 2-complex based on a union of bands Y with 0, 1, and 2-cells attached in such way that the image of all attaching maps is disjoint from the "interior" of Y (the measured lamination on Y). The intersections between these attached cells and Y are called *attaching regions*.

The Rips machine takes as input a band complex X with its underlying union of bands Y and converts Y into a normal form, namely a finite disjoint union

of components each of which has one of the following four types: *simplicial* (every leaf is compact), *surface, toral* and *thin* (every leaf is dense). The latter three are also called the *minimal* components. Roughly speaking, a surface component is a compact hyperbolic surface with geodesic lamination and a toral component is the 2-skeleton of a n-torus with the lamination induced by irrational planes of codimension 1. The first part of this paper will provide some additional structure for thin type. The main feature of thin type is that arbitrarily *thin bands* (bases with small measure) will be created as the Rips machine is applied and each thin band is a *naked band* (it is disjoint from attaching regions of cells). Building on the work of [BF95], in section 3, we show:

Proposition 1.1. *Let X be a band complex with its underlying union of bands Y. Suppose Y consists a single component of thin type. Let $X = X_0, X_1, \ldots$ be an infinite sequence of band complexes formed by the Rips machine. Denote the underlying union of bands for X_i by Y_i. Then the following holds.*

- *There exist integers $A > 0$, $N > 0$, $m \geq 0$ such that for any $n > N$, $\overline{Y}_n$ [1]contains exactly m generalized bands[2] of length less than A, such bands are called **short bands**. As a consequence[3], $\overline{Y}_n$ contains a bounded number of generalized bands whose lengths are greater than A. These bands are called **long bands**.*

- *For $n > N$, there is a natural bijection between short bands $\{\mathbf{S}_n^i\}$ in Y_n*

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[1]$\overline{Y}_n$ is Y_n omitting weight 0 bands

[2]See definition 3.1.

[3]It is shown in [BF95] that all bands in $\overline{Y}_n$ can be organized into a bounded number of generalized bands.

and short bands $\{\mathbf{S}_{n+1}^i\}$ Y_{n+1}. More precisely, every short band $\mathbf{S}_{Y_{n+1}}$ in Y_{n+1} is a sub-generalized band of a short band $\mathbf{S}_{Y_n}$ in Y_n (induced by the inclusion $Y_{n+1} \hookrightarrow Y_n$). In particular, $\mathbf{S}_{Y_{n+1}}$ and $\mathbf{S}_{Y_n}$ have the same length.

- An ***island*** is either a connected component of the union of short bands or a component of Γ_i. For $n > N$ (see definition 3.7). There is a natural bijection f_n between islands $\{\mathcal{I}_n^i\}$ in Y_n and islands $\{\mathcal{I}_{n+1}^i\}$ in Y_{n+1}. Moreover, up to passing to a subsequence, $f_n : \mathcal{I}_n^i \to \mathcal{I}_{n+1}^i$ is a homeomorphism for $\forall i$. We say that the homeomorphism type of each island stabilizes. See figure 1.

- Each base of a long band is contained in an island. Thus, every long band travels from one island to another island (maybe the same one).

Considering each island as a vertex and each long band as an edge, a thin component then can be viewed as a "graph of spaces", see figure 2.

It is proved in [BF09] that every long band is disjoint from the attaching regions of cells. Thus each cell of X_i attaches to Y_i along some island. Moreover, we will show that $(*)$ f_n in above propostion can be construced in such a way that it is the indentity on no-trivial (not null-homotopic) attaching regions of cells. Based on $(*)$, in analogy to the Sela's shortening argument for surfaces [Sel01], we will show in section 3.3 that thin component can also be shortened.
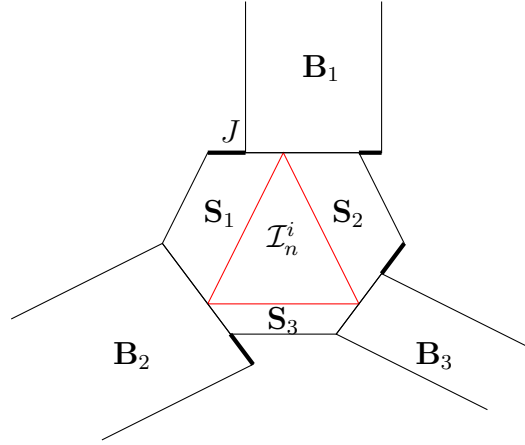
Figure 1: This is a local picture in Y_n of an island $\mathcal{I}_n^i$ formed by the union of short bands $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$ (determined by the union of *short fibers*, see definition 3.5, showing in red). $\mathbf{B}_1, \mathbf{B}_2$ and $\mathbf{B}_3$ are long bands, one of whose bases is contained in $\mathcal{I}_n^i$. Y_{n+1} is obtained from Y_n by collasping a *free subarc* (see definition 4), say J. This move preserves the homeomprhism type of $\mathcal{I}_n^i$.
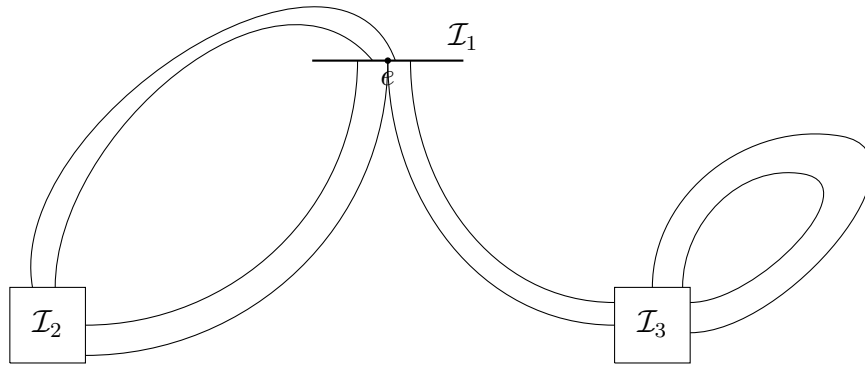
Figure 2: A "graph of spaces" like structure for a thin component. $\mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_3$ are islands. In particular, $\mathcal{I}_1$ is a *type II* island (see definition 3.7) determined by e. Bands travel between islands are long bands.

Given a group acting on a real tree, we may ask questions about how its subgroups act on their corresponding minimal subtrees. The main part of this paper (section 4 - section 6) is a version of the Rips machine, named relative Rips machine, to study subgroup actions.

Let X and X' be band complexes with their underlying unions of bands Y and Y'. A relative cellular map $\iota : X \to X'$ is called a *morphism* (Definition 4.1) if it restricts to a morphism of their underlying real graphs (isometry on edges up to finite subdivisions) and takes each band of Y homeomorphically to a subband of Y'. The input of the machine is a *pair* (Definition 4.3) of band complexes: two band complexes X and X' with a morphism $\iota : X \to X'$.

This relative Rips machine is made of moves on pairs of band complexes. Suppose that $H < G$ are finitely presented groups, T_G is a G-tree and $T_H < T_G$ is a minimal H-tree, and $(T_H \hookrightarrow T_G)$ is called a *pair of trees*. If $(X \xrightarrow{\iota} X')$ and $(X^* \xrightarrow{\iota} X^{*\prime})$ are related by a sequence of moves and $(X \xrightarrow{\iota} X')$ resolves $(T_H \hookrightarrow T_G)$ (see definition 4.4) then $(X^* \xrightarrow{\iota} X^{*\prime})$ also resolves $(T_H \hookrightarrow T_G)$. The goal of this machine is that using sequence of moves to put a pair of band complexes into a normal form while it still resolves the same pair of trees.

In section 4, we describe some further assumptions $(A1)$–$(A5)$ we may make on any given pair of band complexes. In particular, after a finite sequence of folding on the level of bands and real graphs (see definition 4.7), we may always assume that $\iota : X \to X'$ is locally injective on the level of union of bands.

In section 5, we will describe three processes (sequences of moves) for relative Rips machine: Process I, Process II and Process III. As with the Rips machine,

this relative machine studies one pair of components $(Y_0 \xrightarrow{\iota} Y_0')$ of $(Y \xrightarrow{\iota} Y')$ at a time. In each step of Process I and Process II, we first apply a sequence of moves, called an operation, to Y_0'. It is the same operation one would apply to Y_0' in the original Rips machine. Then we apply suitable moves to Y_0 and modify ι correspondingly so that ι remains an immersion. Successive applications of Process I and Process II will convert Y_0' into a *standard form* (Definition 2.14), whereas Y_0 may not be in its standard form yet. In particular, there may exist weight 1 bands in Y_0 map to weight 0 or weight $\frac{1}{2}$ bands. Such bands in Y_0 are called pre-weight 0 bands and pre-weight $\frac{1}{2}$ bands. Process III is then needed to deal with such bands. Roughly speaking, for a fixed pre-weight 0 or pre-weight $\frac{1}{2}$ band $\mathbf{B}_{Y_0} \subset Y_0$, we may always assume one of its bases b_{Y_0} contains all the other bases of bands intersect it. We may slide all the other bands across $\mathbf{B}_{Y_0}$ and then collapse $\mathbf{B}_{Y_0}$. However, the induced map between the resulting component of this sequence of moves, still name it Y_0, and Y_0' may fail to be an immersion. Bands in Y_0 are then folded according to their ι-images. Folding reduces the number of generalized band in Y_0 and so stops in finitely many steps. By that time, we will have a new pair of components still name it $(Y_0 \xrightarrow{\iota} Y_0')$ with the property $(*)$ that Y_0' is in standard form and bands in Y_0 have the same weights as their images in Y_0'. Now to convert Y_0 into standard form, we will go back to Process I with $(Y_0 \xrightarrow{\iota} Y_0')$. Property $(*)$ ensures that Process III will not appear again.

In section 6, as machine output, we will show that one is able to tell the type of Y_0 as the machine successively applied and the machine will eventually convert Y_0 into standard form if Y_0 is of the surface or thin type.

We show in Proposition 5.2 that Process I and Process II will convert ι into an *almost partial covering map* (A technical term. It is slightly weaker than a *partial covering map* which is defined as a locally injective morphism that maps every band onto its image band. See details in definition 5.3) when Y_0' is a surface or thin component. Based on this, in section 6, as machine output, we have:

Theorem 1.2. *Let $(Y \xrightarrow{\iota} Y')$ be a pair of unions of bands and $(Y^0 \xrightarrow{\iota^0} Y'^0) = (Y \xrightarrow{\iota} Y'), (Y^1 \xrightarrow{\iota^1} Y'^1), \ldots$ be a sequence of unions of bands formed by successively application of relative Rips machine. Then for each pair of components $(Y_0 \xrightarrow{\iota_0} Y_0') \subset (Y \xrightarrow{\iota} Y')$ where Y_0' is a minimal component, Y_0 is either a minimal component of the same type as Y_0' or Y_0 is simplicial.*
Moreover, let the pair of components corresponding to $(Y_0 \xrightarrow{\iota_0} Y_0')$ in $(Y^n \xrightarrow{\iota^n} Y'^n)$ be $(Y_n \xrightarrow{\iota_n} Y_n')$. Then Y_0 and Y_0' are both surface or thin components if and only if omitting weight 0 bands, $\iota_n : Y_n \to Y_n'$ is a finitely covering map for sufficiently large n. In addition, for sufficiently large n, Y'^n is in standard form and every surface or thin component of Y^n is also in standard form.

Corollary 1.3. *Let $H < G$ be two finitely presented groups. Further let T_G be a G-tree with trivial edge stabilizers and $T_H \subset T_G$ be a minimal H-subtree[4]. Suppose that $(X \xrightarrow{\iota} X')$ is a pair of band complexes, that X and X' resolve T_H and T_G correspondingly, that Y and Y' are single minimal components of either surface or thin type and that $\pi_1(Y)$ generates H, $\pi_1(Y')$ generates G. Then $[G : H]$ is finite.*

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[4]i.e. T_H contains no proper H-subtrees.

This whole thesis was motivated by an attempt to simplify and fill in details of the Bestvina-Feighn's note on Zlil Sela's work on the Tarski problem [Sel01]-[Sel06b], also see Kharlampovich-Myasnikov's appoarch in [KM06]. In the last section, we will describe Bestvina-Feighn's enlargement argument using machinery developed in this thesis. There is a joint paper in preparation.

2 Background

In this section we briefly review the relevant definitions and list some useful properties without proofs of the Rips machine, see details in [BF95], [Bes02].

Definition 2.1. A **real tree**, or an R**-tree**, is a metric space T such that for any x, y in T there is a unique arc from x to y and this arc is isometric to an interval of the real line.

Definition 2.2. A **real graph** Γ is a finite union of simplicial trees such that each interval of these tress is identified with an arc of the real line.

Let $I = [0, 1]$. A *band* B is a space of the form $b \times I$ where b is an arc of the real line. We identify b with $b \times \{0\}$. $b \times \{0\}$ and $b \times \{1\}$ are called the *bases* of B. Subsets of $b \times \{point\}$ are *horizontal* and subsets of $\{point\} \times I$ are *vertical*. A *vertical fiber* is a set of the form $\{point\} \times I$. A band has an involution δ_B given by reflection in $b \times \{\frac{1}{2}\}$, known as the *dual map*. Bases of B then denoted by b and $dual(b)$.

Definition 2.3. Let $B_1, \ldots, B_n$ be bands and Γ be a real graph. For each base b_i of each B_i, let f_{b_i} be a length-preserving homeomorphism from b_i to Γ. A

union of bands is the quotient space Y of the disjoint union $\Gamma \sqcup B_1 \sqcup \cdots \sqcup B_n$ modulo the union of the f_{b_i}'s.

Definition 2.4. A band B (or each of its bases) of a union of bands Y is assigned a **weight** according to the attaching map. B is a weight 0 band if it is an annulus. B is a weight $\frac{1}{2}$ band if it is a Mobius band. Otherwise B is a weight 1 band. A **block** is the closure of a connected component of the union of the interiors of the bases. The **complexity of a block** is $\max\{0, -2 + \sum \text{weight}(b) | b \subset \text{block}\}$. The **complexity** of Y is the sum of the complexities of its blocks. A *leaf* of Y is an equivalence class of the equivalence relation on Y generated by $x \sim x'$ if $\{x, x'\}$ is a vertical subset of a band in Y. Y is foliated with a natural transverse measure by these leaves [MS84].

Remark 2.5. Let Y be a union of bands constructed by gluing bands to Γ_Y where Γ_Y is a disjoint union of simplicial trees. According to [BF95, Lemma 6.1], up to a finite sequence of moves, we may assume Y has the following property.

($A1$): *Its underlying real graph Γ_Y is the disjoint union of edges. Each edge is either a block or meets no bands.*

We will make this assumption for all union of bands throughout the paper unless otherwise mentioned.

Definition 2.6. Let Y be a union of bands with its underlying real graph Γ. For $z \in \Gamma$ denote by $N_\Gamma(z, \epsilon)$ the closed ϵ-neighborhood of z in Γ. If $z \notin \Gamma$ we take $N_\Gamma(z, \epsilon)$ to be empty. For a band $B = b \times I$ of Y, $z = (z, t) \in b \times I = B$, and $\epsilon > 0$, let $N_B(z, \epsilon)$ be the closed horizontal segment $\{(x', t) \in B | |x - x'| \leq$

ϵ}. For $z \in Y$, define $N(z, \epsilon)$ to be the union of $N_\Gamma(z, \epsilon)$ and of the $N_B(z, \epsilon)$ over all B containing z. A subset of Y is **horizontal** if it is a subset of Γ or if it is contained in a single band and is horizontal there. let $s, s' \subset Y$ be horizontal and let p be a path in a leaf of Y. We say s **pushes into** s' **along** p if there is a homotopy H of s into s' through horizontal sets such that

- for every $z \in s$, $H(\{z\} \times I)$ is contained in a leaf, and

- there is $z_0 \in s$ so that $p(t) = H(z_0, t)$ for all $t \in I$.

Given a horizontal s and path p in a leaf with $p(0) \in s$, we say that s *pushes along p* if there is a horizontal set s' such that s pushes into s' along p.

As subset S of Y is **pushing saturated** if given a path p with $p(0) \in S$ and $\epsilon > 0$ such that $N(p(0), \epsilon)$ pushes along p then $p(1) \in S$.

Definition 2.7. Let Y be a union of bands. Y is **minimal** if every pushing saturated subset of Y is dense in Y and is **simplicial** if every leaf of Y is compact. A leaf is **singular** if it contains a proper pushing saturated subset. Otherwise, it is **nonsingular**.

Proposition 2.8. *[BF95, Proposition 4.8] Let Y be a union of bands. There are only finitely many isotopy classes of compact leaves in Y. Suppose that no leaf of Y has a subset that is proper, compact and pushing saturated. Then, the following holds.*

- *Each component of Y is a union of bands that is either simplicial or minimal.*

- *Each simplicial component of Y is an I-bundle over some leaf in that component.*

- *All but finitely many leaves of Y are nonsingular.*

Definition 2.9. A **band complex** X is a relative CW 2-complex based on a union of bands Y. X is obtained from Y with $0, 1,$ and 2-cells attached such that

- only finitely many (closed) cells of X meet Y,

- the 1-cells of X intersect Y in a subset of Γ,

- each component of the intersection of Γ and a 2-cell of X is a point, and

- each component of the intersection of a band of Y and a 2-cell of X is vertical.

Intersections between attaching cells and Y are call *attaching regions.*

Here is a terminology convention. If X is a band complex then Y is always its underlying union of bands and Γ is always its real graph. Similarly, the union of bands for X' is Y', etc. Further, $\overline{Y}$ denotes the union of bands obtained from Y by omitting weight 0 bands.

A *leaf* of a band complex X is a leaf of Y. Similarly, we say that X is *minimal* or *simplicial* if Y is minimal or simplicial. The *complexity* of X is the complexity of Y. The transverse measure on Y can then be integrated along a path in X. A *generalized leaf* of X is an equivalence class of points of X under the equivalence relation $x \sim x'$ if there is a path in X joining x and

x' with measure 0. $X^* \subset X$ is a *band subcomplex* if X^* is a band complex with its underlying real graph $\Gamma^* \subset \Gamma$ and its underlying union of bands $Y^* \subset Y$. Let G be a finitely presented group acting on a R-tree T by isometries (called a G-tree for short). We may obtain a band complex from T in the following way. Let X be a 2-complex whose fundamental group is G. A **resolution** for T is a G-equivariant map $r : \widetilde{X} \to T$ ($\widetilde{X}$ is the universal cover of X) such that the image of a generalized leaf of $\widetilde{X}$ is a point and r embeds the lifts of bases. We say that X a *resolving complex*.

Proposition 2.10. *[BF95, Proposition 5.3] Let G be a finitely presented group. Every G-tree T has a resolution.*

The goal of the Rips machine is to put a band complex X into a normal form. There is a list of 6 moves (M0)-(M5) that can be applied to a band complex. The complete list is in section 6 of [BF95]. These moves are elementary homotopic moves with respect to the underlying measured lamination. If a band complex X' is obtained from a band complex X by a sequence of these moves, the following holds.

- There are maps $\phi : X \to X'$ and $\psi : X' \to X$ that induces an isomorphism between fundamental groups and preserve measure.

- If $f : \widetilde{X} \to T$ is a resolution, then the composition $f\tilde{\psi} : \widetilde{X}' \to T$ is also a resolution, and if $g : \widetilde{X}' \to T$ is a resolution, then so is $g\tilde{\phi} : X \to T$.

- ϕ and ψ induce a $1-1$ correspondence between the minimal components of the laminations on X and X'

We now describe two moves which will be used a lot in the following sections.

$(M4)$ *Slide.* Let $B = b \times I$ and $C = c \times I$ be distinct bands in X. Suppose $f_b(b) \subset f_c(c)$. Then a new band complex X' can be created by replacing f_b by $f_{dual(c)} \circ \delta_C \circ f_c^{-1} \circ f_b$. This is a slide of b across C from c to $dual(c)$. We say that c is the *carrier*, and b is *carried*. See figure 3.

Figure 3: Slide B across C.

Definition 2.11. Let $B = b \times I$ be a band of Y. A subarc J of a base b is *free* if either

- b has weight 1 and the interior of J meets no other base of positive weight (see figure 4), or

- b has weight $\frac{1}{2}$, the interior of J doesn't contain the midpoint of b, and b and dual(b) are the only positive weight bases that meet the interior of J.

$(M5)$ *Collapse from a free subarc.* Let $B = b \times I$ be a band, and let $J \subset b$ be a free subarc. If b has weight 1, collapse $J \times I$ to $dual(J) \cup (\partial J \times I)$ to obtain X'. Typically, the band B will be replaced by two new bands. But if J contains one or both endpoints of b, say x, then we also collapse $x \times [0,1]$. Thus B
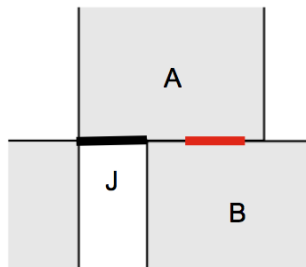
Figure 4: J is a free subarc, and the red subarc is not.

is replaced by 1 or 0 bands. Attaching maps of relative 1- and 2-cells whose image intersect interior of $J \times [0, 1)$, can be naturally homotoped upwards. See figure 5.

If b is of weight $\frac{1}{2}$, we may subdivide B over the end-point of J nearest the midpoint of b such that J is contained in a band of weight 1 ([BF95, Lemma 6.5]). Then we can collapse from J as before.
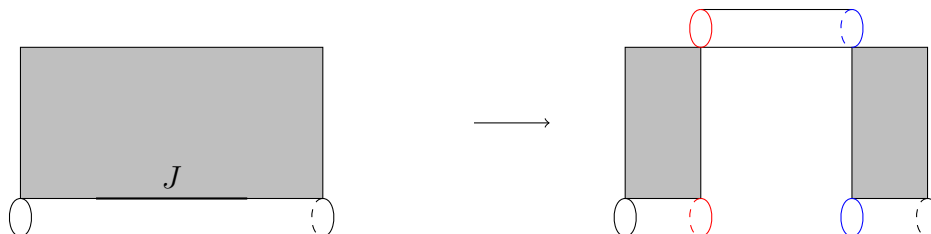
Figure 5: Collapse from J within a weight 1 band. Subdivison annuli are attached between red loops and between blue loops.

In this paper, we will use some additional moves.

($M6$) *Attach a band*: Glue $b \times I$ to X via measure-preserving $b \times \partial I \to \Gamma(X)$ transverse to the measured lamination.

($M7$) *Adding an arc to the real graph*: Add an extra segment c of the real line to Γ_X. A special case is *extend* Γ_X by gluing one end point of c to an end

point of Γ_X.

($M8$) *Attach a disk*: Glue a 2-disk D to X via $\partial D \to X^{(1)}$ representing a measure 0 loop, where $X^{(1)}$ is the union of bands Y and the relative 1-cells.

Remark 2.12. Note that unlike the moves in [BF95], $(M6) - (M8)$ may change the fundamental group of X. But we never perform such moves alone. For example, after a ($M6$)-attaching a band to X, we always perform ($M8$)-attaching a disk (multiple times if needed) to make sure the resulting band complex X' have the same fundamental group and resolves the same tree.

By [BF95], after applying a sequence of moves to X, we may arrange that no leaf of the complex has a subset that is proper, compact, and pushing saturated. Therefore, each component of X (Y) is either simplicial or minimal. The Rips machine consists of two processes (sequence of moves), Process I and Process II. Neither process increases complexity. It is designed to study one minimal component at a time. Fix a minimal component Y^0 of Y. First, Process I is successively applied until no base of Y^0 has a free subarc. It is possible that Process I continues forever. If no base has a free subarc then Process II is successively applied until the complexity of Y^0 decreases, or else it is applied forever. Thus eventually only one of the processes is applied. After putting Y^0 in the "normal form", we continue by choosing another component.

Definition 2.13. If, in the Rips machine, eventually only Process I is applied to Y^0, then Y^0 is of **thin type** (also called **Levitt or exotic type**). If eventually only Process II is applied and excess is 0 then Y^0 is of **surface type**. If eventually only Process II is applied and excess is positive then Y^0 is of **toral type** (or **axial type**).

Definition 2.14. In applying the Rips machine to Y^0, we will obtain an infinite sequence $Y^0, Y^1 \ldots$ where each component is obtained from the previous by a process. Further, there exists an integer N such that only a unique process (either Process I or II) is applied in the sequence after Y^N, and $Complexity(Y^n) = Complexity(Y^{n+1})$ for all $n > N$. Y^n is said to be in **standard form** for $n > N$. A band complex is **standard** if its every minimal component is standard.

Definition 2.15. Given a union of bands Y, let $q \in Y$ be a point and $l_q \subset Y$ be the leaf containing q. Each component d_p of $l_q - q$ is called a **direction** of Y at q. A direction is **infinite** if the corresponding component of the leaf is infinite. For $q \in \Gamma_Y$, a band in Y containing q determines a unique direction at q. Denote the **direction set** of Y at q by $T_q Y$. For a given a morphism between union of bands $\iota : Y \to Y'$, we define the **derivative** map $D_q \iota : T_q Y \to T_{\iota(q)} Y$ by defining $D_q \iota(d_q) = \iota(d_q)$ for each direction d_q at q, where $\iota(d_q)$ represents the direction at $\iota(q)$ containing the image of d_q in Y'. For each $q \in \bar{Y}$, we define its **index** by

$$i_Y(q) = \#\{\text{infinite directions at } q\} - 2.$$

The **limit set** Ω_Y of Y is the set of points in $\bar{Y}$ whose index is at least zero. Its intersection with the real graph (i.e. $\Omega_Y \cap \Gamma_Y$) is called the **limit graph**.

Each type of component in Y then can be characterized in the following way.

Proposition 2.16. *[BF95, section 8] Let Y be a union of bands in the standard form defined in definition 2.14 and Y_0 be one of its component. Then*

1. Y_0 *is a simplicial component if and only if* Ω_{Y_0} *is empty;*

2. Y_0 *is a surface component if and only if* $i_Y(q) = 0$ *for* $\forall q \in \Omega_{Y_0}$ *and the closure of* Ω_{Y_0} *is* Y_0;

3. Y_0 *is a thin component if and only if there are finitely many points in* Ω_{Y_0} *have positive index and* $\Omega_{Y_0} \cap \Gamma_{Y_0}$ *is a dense* G_δ *set in* Γ_{Y_0};

4. Y_0 *is a toral component with rank* $n > 2$ *if and only if* Ω_{Y_0} *contains infinitely may points of positive index.*

Associated to a band complex is a GD which is a generalization of the GAD's of [BF09].

Definition 2.17. A *generalized decomposition*, or GD for short, is a graph of groups presentation Δ where some vertices have certain extra structure. Namely, the underlying graph is bipartite with vertices in one class called *rigid* and vertices in the other called *foliated*. Further each foliated vertex has one of four types: *simplicial, toral, thin, or surface.*

A band complex X is naturally a graph of spaces where vertex spaces are components of Y and components of the closure of $X \setminus Y$ (the complements of Y). An edge corresponds to a component of the intersection of the closures of sets defining two vertices. The GD associated to X, denoted by $\Delta(X)$, is the graph of groups decomposition coming from this graph of spaces.

3   Thin Type components

3.1   Review.

As described in section 2, Y_0 is of thin type if eventually only Process I is applied. In this section, we will review Process I and restate some properties proved in [BF95] for our convenience.

Definition 3.1. Let $B_1, B_2, \ldots, B_n$ be a sequence of weight 1 bands in Y. We say that they form a **generalized band B** provided that :

- the top of B_i is identified with the bottom of B_{i+1} and meets no other positive weight bands for $i = 1, 2, \ldots, n-1$, and

- the sequence of bands is maximal with respect to above property.

We say B_i and B_{i+1} are *consecutive bands*. The bottom of B_1 and the top of B_n are *bases* of **B**, and denote them by b and $dual(b)$. Let $I_n = [0, n]$, then a generalized band **B** has the form $b \times I_n$. A *vertical fiber* of **B** is a set of the form $\{point\} \times I_n$. The union of a sub-sequence of consecutive bands $B_i, B_{i+1}, \ldots, B_{i+k}$ called a *section* of **B**, where $i \geq 1, i + k \leq n$. The *length* of **B**, denoted by $l(\mathbf{B})$, is n (the number of bands in **B**). The *width* of **B**, denoted by $w(\mathbf{B})$, is the transverse measure of its base b. Let $c \subset b$ be a sub-interval, $\mathbf{C} = c \times I_n$ is a *sub-generalized band* of **B**. Similar to the definition for bands, we may talk about the weight of a generalized band.

Definition 3.2. For a given band, the midpoint of a base divides the base into **halves**. A weight 1 base is **isolated** if its interior does not meet any other

positive weight base. A half h of a weight $\frac{1}{2}$ base b is **isolated** if the interior h meets no positive weight bases other than b and dual(b). A weight 1 base b is **semi-isolated** if a deleted neighborhood in b of one of its endpoints meets no other positive weight base. A half h of a weight $\frac{1}{2}$ base b is **semi-isolated** if a deleted neighborhood in h of one of its endpoints meets no positive weight base other than b and dual(b).

We are now ready to describe Process I.

Process I. We define X' to be the band complex obtained from X by the following operation. Find if possible, a maximal free subarc J of a base b of Y^0. If such a J does not exist, define $X' = X$ and go on to process II. Now use $(M5)$ to collapse from J. If there are several J's to choose from, abide by the following rules:

1. If there is an isolated (half-) base c, set $J = c$. This is called an I_1-collapse.

2. If there is no isolated (half-) base, but there is a semi-isolated (half-) base c then choose J so that it contains an endpoint of c. This is called an I_2-collapse.

3. If there are no isolated or semi-isolated (half-bases), we can use any free subarc as J. This is called an I_3-collapse.

4. Generalized bands are treated as units.

Let $X = X_0, X_1, \ldots$ be an infinite sequence of band complexes formed by Process I defined above (i.e. $X_{i+1} = X_i'$). Denote the underlying union of

bands for X_i by Y_i. Further denote the collapse from Y_i to Y_{i+1} by δ_i, the natural inclusion $\overline{Y}_{i+1} \hookrightarrow \overline{Y}_i$ by ι_i and the number of generalized bands in Y_i that have positive weight by a_i. We say a generalized band $\mathbf{B}' \subset Y_{i+1}$ is an **image** of a generalized band $\mathbf{B} \subset Y_i$, if a *section* of $\mathbf{B}'$ is contained in $\mathbf{B}$ under ι_i. Then $\mathbf{B}$ has no image under an I_1-collapse, exactly one image under an I_2-collapse and at most two images under an I_3-collapse. It is clear that in either case $l(\mathbf{B}) \leq l(\mathbf{B}')$. A collapse δ_i is said to be **increasing** if the length of some generalized band in Y_i is strictly less than the length of one of its images in Y_{i+1}.

Lemma 3.3. *Following the above notation, for a given collapse* $\delta_i : Y_i \to Y_{i+1}$ *we have:*

- *If δ_i is an I_1-collapse, then $a_{i+1} = a_i - 1$ and δ_i is not increasing.*

- *If δ_i is an I_2-collapse, either $a_{i+1} = a_i$ or $a_{i+1} = a_i - 1$. Moreover, δ_i is an increasing collapse in the latter case.*

- *If δ_i is an I_3-collapse, then $a_{i+1} = a_i + 1$ or $a_{i+1} = a_i$ or $a_{i+1} = a_i - 1$. Moreover, δ_i is an increasing collapse in the latter two cases.*

Proof. The first item is clear since a whole generalized band vanishes in an I_1-collapse.

For the second item, let $\mathbf{C}_i$ be a generalized band in Y_i. Suppose one of its bases c_i is a semi-isolated base and δ_i collapses from a maximal free subarc $J \subset c_i$. In Y_{i+1}, let the remaining part of $\mathbf{C}_i$ be $\mathbf{C}'_i$ and the image of $\mathbf{C}_i$(exactly one image due to I_2-collapse) be $\mathbf{C}_{i+1}$, see Figure 6. If $\mathbf{C}'_i = \mathbf{C}_{i+1}$, then $a_{i+1} = a_i$

and δ_i is not increasing. If $\mathbf{C}'_i \subsetneqq \mathbf{C}_{i+1}$, then $\mathbf{C}_{i+1}$ must be the concatenation of $\mathbf{C}'_i$ and another generalized band (can not concatenate with more than one band due to no proper compact subset of each leaf). In this case, $a_{i+1} = a_i - 1$ and δ_i is increasing. The third item can be argued in the same fashion.

$\square$
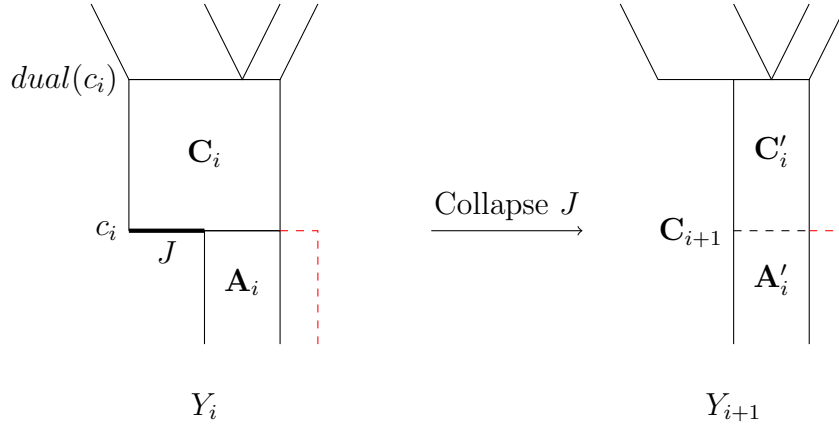
Figure 6: Without the red dashed lines, the picture showes the case that $\mathbf{C}'_i \subsetneqq \mathbf{C}_{i+1}$ and $a_{i+1} = a_i - 1$. In Y_{i+1}, $\mathbf{C}'_i$ and $\mathbf{A}_i$ form a longer generalized band $\mathbf{C}_{i+1}$. If $\mathbf{A}_i$ has the red dash as its boundary, then $\mathbf{C}'_i = \mathbf{C}_{i+1}$ and $a_{i+1} = a_i$.

Proposition 3.4. *Let X be a band complex with its underlying union of bands Y. Suppose Y consists a single component of thin type. Further let $X = X_0, X_1, \ldots$ be an infinite sequence of band complexes formed by the Rips machine. Denote the underlying union of bands for X_i by Y_i. Then the following holds.*

1. *Complexity$(Y_{i+1}) \leq$ Complexity(Y_i) for $i = 0, 1, 2 \ldots$. In particular, eventually the complexity is a fixed number C.*

2. *All bands in each $\overline{Y}_i$ can be organized into at most $N = 6C+1$ generalized bands.*

3. *Let L be the induced lamination on $\overline{Y}$ and $L_\infty = L \cap \bigcap_i^\infty \overline{Y}_i$. Then each subband of a band in $\overline{Y}$ either fully collapses within finitely many steps or meets L_∞ in infinitely many vertical fibers.*

Proof. (1) This follows from the definition of complexity. Suppose δ_i is an I_1-collapse and b is the isolated base of the collapsed generalized band $\mathbf{B}$. If the complexity of the block containing $dual(b)$ is positive, then $Complexity(Y_{i+1}) < Complexity(Y_i)$. Otherwise $Complexity(Y_{i+1}) = Complexity(Y_i)$. A similar analysis can be done for both I_2- and I_3-collapse.

(2) According to the first item, without loss, we may assume all Y_i have the fixed complexity C. Therefore in Γ_i, the number of blocks of positive complexity is bounded by C. Recall that the number of generalized bands in $\overline{Y}_i$ is denoted by a_i. Suppose $a_{i+1} > a_i$, then this occurs only if $\delta_i : Y_i \to Y_{i+1}$ is an I_3-collapse (lemma 3.3). Thus Y_i contains no isolated or semi-isolated bases. Therefore, each block of complexity 0 consists of two coinciding bases of weight 1, or of a weight $\frac{1}{2}$ pair that coincides with a weight 1 base. So for a given generalized band $\mathbf{B}$, either at least one of its bases is contained in a block of positive complexity or $\mathbf{B}$ is of weight $\frac{1}{2}$ whose bases form a block of complexity 0 along with another weight 1 base. It follows that a_i is bounded above by $6C$, and so $a_{i+1} \le 6C + 1$. If $Y_{i+1} \to Y_{i+2}$ is an I_1 or I_2-collapse, then $a_{i+2} \le a_{i+1} \le 6C + 1$. If it is an I_3-collapse, then the same analysis shows that $a_{i+1} \le 6C, a_{i+2} \le 6C + 1$. Therefore, the number of generalized bands is bounded by $N = 6C + 1$.

(3) Assume $B \subset \overline{Y}$ is a subband that does not fully collapse within finitely many steps. By [BF95, Proposition 7.2], I_3-collapses occur infinitely often. So the number of components of the intersection between B and $\overline{Y}_i$ goes to infinity with i. Each of these components then have nonempty intersection with L_∞ and we are done. $\qquad\square$

Let $X = X_0, X_1, \ldots$ be an infinite sequence formed by the Rips machine. In general, it is possible that the process of Rips machine applied to X_i bounces between Process I and Process II for a while before it eventually stabilizes with Process I. Moreover, the complexity of Y_i may actually decrease at the beginning stages. Nonetheless, there exists some integer N such none of these situations happens after Y_N. So without loss, for the rest of this section, we always make the assumption that in the sequence $X = X_0, X_1, \ldots$ only Process I occurs and the complexity is fixed for every X_i.

Let $B \subset Y_0$ be a band. We say B *vanishes* along the process if $B \cap Y_i = \emptyset$ for all sufficiently large i's ($Y_i \subset Y_0$). It is clear that bands only vanish in I_1-collapses. Since I_1-collapses lead to reductions of complexity, under our assumption of fixed complexity, we may further assume that only I_2-collapses and I_3-collapses occur along the sequence $X_0, X_1, \ldots$. In particular, no band vanishes.

3.2 Structure of Thin Type.

We will continue with the same notation as in section 3.1. Denote the union

of bases of the generalized bands in Y_i by E_i. Then E_i is the union of at most $2N$ closed intervals where N is the uniform upper bound of the number of generalized bands in Y_i as in Proposition 3.4. $Y_{i+1} \subset Y_i$ implies that the sequence $E_1 \supset E_2 \supset \dots$ is nested. According to [BF95, proposition 8.12], $\max\{$widths of generalized bands of $Y_i\} \to 0$ as $i \to \infty$. Therefore, the intersection $\cap_{i=0}^{\infty} E_i$ consists of at most $2N$ points. Denote these points by $e_1, e_2, \dots, e_n$ where $n \le 2N$. If a vertical fiber of a generalized band in Y_i contains some e_j, by definition, e_j must be an endpoint of that fiber.

A vertical fiber of a generalized band in $\overline{Y_i}$ is **short** if both its endpoints are contained in $\{e_1, e_2, \dots, e_n\}$. Since leaves of the limiting lamination L_∞ contain no loops (Proposition 8.12 [BF95]), if there are more than two vertical fibers from e_i to e_j, $i \ne j$, then one of them has to be in the boundary of Y_i. Therefore, the total number of *short* vertical fibers is bounded above by $\binom{n}{2} + 2N$. Let $s_1, s_2, \dots, s_m$ be the list of *short* vertical fibers. Furthermore, since widths of bands in Y_i converge to 0 with i, each generalized band in Y_i contains at most one of $\{s_j\}_j$ for i sufficiently large. In particular, $m \le \min\{\binom{n}{2} + 2N, N\} = N$.

Definition 3.5. In Y_i, we say a generalized band of positive weight is **short** if one of its vertical fibers is a subset of a *short* vertical fiber. Otherwise, we say the generalized band is **long**. In particular, for i sufficiently large, a generalized band is short if and only if it contains one of $\{s_j\}_j$.

Proposition 3.6. *Following the same notation as in proposition 3.4, we have that in Y_i's:*

1. *Lengths of short bands are bounded above by* $l_s = \max\limits_{j \in \{1,\ldots,m\}} \{l(s_j)\}$.

2. *There exists an integer M such that for any $i > M$, Y_i contains exactly m short bands. In particular, for all $i > M$, there is a natural bijection between short bands in Y_i and Y_{i+1}. More precisely, every short band $\mathbf{B}_{Y_{i+1}}$ in Y_{i+1} is a sub-generalized band of a short band $\mathbf{B}_{Y_i}$ in Y_i (induced by the inclusion $Y_{i+1} \hookrightarrow Y_i$).*

3. *For any fixed number k, there exists a number $N(k)$ such that the lengths of long bands are all greater than k after the stage $Y_{N(k)}$. In particular lengths of long bands are going to ∞ as $i \to \infty$ in Y_i.*

4. *Y_i contains at least one long band.*

Proof. (1), (2) follow directly from Definition 3.5.

(3) Let l_i be the length of a shortest long bands in Y_i. Then the sequence $\{l_i\}_i$ is non-decreasing as $i \to \infty$. We need to prove that the sequence is indeed increasing. Assume it is bounded above, then for all sufficiently large i, l_i is a fixed number l^*. Note that each long band in Y_{i+1} with length l^* is a sub-generalized band of some long band in Y_i, which also has length l^*. This implies that there exists a sequence of long bands $\{\mathbf{B}_{Y_i}\}_i$ such that $\mathbf{B}_{Y_{i+1}} \hookrightarrow \mathbf{B}_{Y_i}$ and $l(\mathbf{B}_{Y_i}) = l^*$ for all i's. Thus $\cap_i^{\infty} \mathbf{B}_{Y_i} \in \{s_1, s_2, \ldots, s_m\}$. But this contradicts the assumption that $\mathbf{B}_{Y_i}$'s are long bands.

(4) Since the number of generalized bands in Y_i is uniformly bounded, we only need to show that there are infinitely many increasing δ_i's in the the sequence $Y_0 \xrightarrow{\delta_0} Y_1 \xrightarrow{\delta_1} Y_2 \ldots$. We are going to construct an infinite subset of $\{Y_i\}_i$ such that for each Y_i in that subset, its corresponding δ_i is increasing. Firstly, since

a_i is bounded by N, there exists a subsequence $A = \{Y_{k_1}, Y_{k_2}, \ldots\}$ with the property that for any fixed j, $a_{k_j} \geq a_i$ for all $i \geq k_j$. Starting with Y_{k_1}, if δ_{k_j} is increasing, we continue to check $\delta_{k_{j+1}}$. Otherwise, by lemma 3.3, either δ_{k_1} is an I_2-collapse with $a_{k_1+1} = a_{k_1}$, in which case we replace Y_{k_1} by Y_{k_1+1} in the subsequence A; Or δ_{k_1} is an I_3-collapse with $a_{k_1+1} > a_{k_1}$, which contradicts to the choice of Y_{k_1}. Since I_3-collapses happen infinitely often in $Y_0, Y_1, \ldots$, A is an infinite set. $\qquad\qquad\square$

According to above proposition, for $i > M$, a generalized band in Y_i is short if and only if its length is less than l_s, otherwise it is long (thus the name).

Definition 3.7. $\{e_1, \ldots, e_n\}$ are defined as above. In Y_i, an **island** is either

- *Type 1* A connected component of the union of short bands in Y_i along with the blocks containing their bases, or

- *Type 2* A block containing an e_j which is not contained in an island of type 1.

An island is **trivial** if it is null-homotopic. Then by definition, every type 2 island is trivial.

Definition 3.8. A generalized band $\mathbf{B}$ of a band complex X is **naked** if the interior of $\mathbf{B}$ (measured lamination on $\mathbf{B}$) is disjoint from the attaching region of the 2-cells in Y. Further $\mathbf{B}$ is **very naked** if there are no subdivision annuli attached to $\mathbf{B}$.

Proposition 3.9. *Let X be a band complex with its underlying union of bands Y. Suppose Y consists a single component of thin type. Then we have the following,*

1. *There exists an integer $M' > M$ such that the number of islands is fixed for all $i > M'$. In particular, let $\mathcal{S} = \cup_{j=1}^{m} s_i$, $\mathcal{E} = \cup_{j=1}^{n} e_i$ and the fixed number for islands be N_I. Then N_I equals the number of components of $\mathcal{S} \cup \mathcal{E}$. Moreover, δ_i induces a natural bijection between islands in Y_i and Y_{i+1}. Further after passing to a subsequence, each island in Y_i and its image island in Y_{i+1} have the same homeomorphism type. We say that islands eventually stabilize.*

2. *For $i > M'$, let $\mathcal{I} \subset Y_i$ be a type 1 island and $\mathcal{S}_{\mathcal{I}}$ be the component of $\mathcal{S}$ it contains. $\mathcal{I}$ is trivial if and only if $\mathcal{S}_{\mathcal{I}}$ contains no loops. In particular, if $\mathcal{I}$ is non-trivial, $\mathcal{S}_{\mathcal{I}}$ must contain some s_j that is in the boundary of Y_i.*

3. *Every long band is naked [5]. Each base of a long band is contained in an island.*

4. *For $i > M'$, there is an isomorphism f_i between islands in Y_i and islands in Y_{i+1} which preserve measure and is the identity map on the non-trivial attaching region of 2-cells in Y_i.*

Proof. (1) To show the existence of M', we only need to show that eventually two distinct short bands either stay connected or disconnected and every e_j stays either in a type 1 island or not. Let $\mathbf{B}_{Y_i}^1$ and $\mathbf{B}_{Y_i}^2$ be two short bands in Y_i for $i > M$, $\mathbf{B}_{Y_{i+1}}^1$ and $\mathbf{B}_{Y_{i+1}}^2$ be the corresponding short bands in Y_{i+1}. Without loss, we may assume that $s_1 \subset \mathbf{B}_{Y_i}^1$, $s_2 \subset \mathbf{B}_{Y_i}^2$ are short vertical fibers determining them. If s_1 and s_2 share an endpoint, it is clear that $\mathbf{B}_{Y_i}^1$ and

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[5]This is proved in [BF95]

$\mathbf{B}^2_{Y_i}$ stay connected for all $i > M$. Otherwise, there exists an $\epsilon > 0$ such that ϵ-neighborhoods of endpoints of s_1 and s_2 are disconnected. Since widths of generalized bands in Y_i converge to 0 as $i \to \infty$, there exists $M^* > 0$ such that $\mathbf{B}^1_{Y_i}$ and $\mathbf{B}^2_{Y_i}$ are disconnected for all $i > M^*$. Exactly argument shows that there exists M^{**} such that every $e_j \notin \mathcal{S}$ is not contained in a type 1 island. Thus for all $i > M' = \max\{M^*, M^{**}\}$, every component of $\mathcal{S} \cup \mathcal{E}$ determines a unique island in Y_i. In particular, the combinatorial type of an island as a union of bands is bounded. After passing to a sub-sequence, each island in Y_i is homeomorphic to its image in Y_{i+1}.

(2) If $\mathcal{S}_{\mathcal{I}}$ contains no loops, $\mathcal{I}$ can be viewed as an I-bundle over $\mathcal{S}_{\mathcal{I}}$ and so is a trivial island. Otherwise $\mathcal{S}_{\mathcal{I}}$ contains loops. Note that every I-bundle neighborhood of $\mathcal{S}_{\mathcal{I}}$ intersects the limiting lamination L_∞ infinitely often. Leaves in L_∞ have no loops[6] implies that $\mathcal{S}_{\mathcal{I}}$ must contain some s_j that is in the boundary of Y_i.

(3) Since long bands are getting infinitely long, up to homotopy, we may assume that each long band meets the attaching region only at points. Further, up to homotopy, we may assume that these points are contained in its bases. Therefore, it is a naked band. By the definition of $\{e_j\}_j$, every block of Γ_i contains at least a point in $\{e_j\}_j$ for i sufficiently large. Thus the block containing a base of a long band must be part of an island.

(4) The bijection f_i induced by δ_i between islands in Y_i and islands in Y_{i+1} in fact is an measure preserving isomorphism since each island in Y_i and its image in Y_{i+1} have the same homeomorphism type. There are in total finitely many

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[6]See for example [BF95, Proposition 8.9] or [Gui00].

components of the attaching regions in Y_i. By composing with homotopies within bands, we may assume that f_i is identity on attaching regions that are homotopy equivalence to a point. Now for each attaching region (contained in a leaf) that is not null-homotopic, if it is not contained in a boundary leaf of a weight 0, this attaching region must contain a segment in a boundary of an non-trivial island By item (2), such an attaching region are contained in a component $\mathcal{S}_\mathcal{I}$ of $\mathcal{S}$ that is not null-homotopic. f_i is identity on such $\mathcal{S}_\mathcal{I}$'s and so f_i is identity on attaching regions.

$\square$

It follows from proposition 3.9 that every long band travels from one island to another island (maybe the same one). After a finite sequence of sliding (M4) weight 0 bands, we may assume that every long band is very naked. Thus every subdivision annulus is attached to an island. Each island along with weight 0 bands attached to it is called a **generalized island**. Considering each generalized island as a vertex and each long band as an edge, a thin component then has a "graph of spaces" like structure (see figure 2).

More generally, for band complex with more than one component, we have the following corollary .

Corollary 3.10. *Let X be a band complex with its underlying union of bands Y, $\Delta(X)$ be the associated GD of X and Y_0 be a thin component. Further let $X^0 = X, X^1, \ldots$ be a sequence of band complexes and $Y_i \subset X^i$ be the component corresponding to Y_0. Then for n sufficiently large, intersections between Y_n and other vertex spaces of $\Delta(X^n)$ (attaching regions) are contained in generalized islands of Y^n. Moreover, $f_n : Y_n \to Y_{n+1}$ constructed in Proposition*

3.9 can be extended to $f^n : X^n \to X^{n+1}$ with the property that f^n is the identity on the closure of the complement of Y_n in X^n.

Proof. By construction, every long band is very naked and so every attaching region of Y_n is contained in a generalized island of Y_n. According to Item (2) of Proposition 3.9, $f_n : Y_n \to Y_{n+1}$ is an identity on the attaching regions. Therefore f_n can uniquely extend to $f^n : X^n \to X^{n+1}$ by defining f^n to be the identity on the closure of the complement of Y_n in X^n. $\square$

3.3 Shortening Thin Type

Now let X be a band complex with its underlying union of bands Y, m be the transverse measure on X and $\mathcal{B}$ be a set of loops in Y generating $\pi_1(X)$. Set $|X|_{\mathcal{B}} := \sum_{\mu \in \mathcal{B}} m(\mu)$ be the *length* of X with respect to $\mathcal{B}$. Further let $Y_0 \subset Y$ be a thin component, $X = X^0, X^1, \ldots$ be an infinite sequence of band complexes formed by the Rips machine and $Y_n \subset X^n$ be the component corresponding to Y^0. In this section, we will use $f^n : X^n \to X^{n+1}$ constructed in Corollary 3.10 to show that in analogy to Sela's shortening argument for surfaces, lengths of X can also be shortened using thin components.

Definition 3.11. Let u be a subarc of an edge of the underlying real graph Γ of X with basepoint z_0 in the interior of u. A **short** (with respect to u) loop is a loop $p_1 * \lambda * p_2$ based at z_0 where p_1 and p_2 are paths in the interior of u and λ is a path within a leaf.

By [BF95, Proposition 5.8], given any non-degenerated segment u in Γ, there

exists a generating set of the image of $\pi_1(Y)$ in $\pi_1(X)$ consisting of short loops with respect to u. In particular, we may choose a sequence $\{u_i\}_i$ such that each u_i is a spanning arc of a long band in Y_i (transverse to the lamination). To be more precise, if we cut a long band of Y_i along a spanning arc u_i, Y_i is then decomposed into a finite union of simplicial components. Each of these simplicial components has the form of finite tree times a foliated interval. Further Y_i is homotopy equivalent to the space obtained by gluing the ends of these simplicial components. This simplicial structure allow us to find a collection of short loops generating the fundamental group of Y_i.

Proposition 3.12. *There is an infinite subsequence of $Y_1, Y_2, \ldots$ such that for any $l > k$, there exists a homeomorphism $h_{k,l} : Y_k \to Y_l$ that is the identity on the attaching regions and with the following additional property. There exists transverse subarcs $u_k \subset Y_k$ and $u_l \subset Y_l$ such that $h_{k,l}$ induces a isomorphism between the set of simplicial components of Y_k formed by cutting open along u_k to the set of simplicial components of Y_l formed by cutting open along u_l.*

Proof. By proposition 3.9, the homeomorphism types of islands stabilize for i sufficiently large. Further there are bounded number of long bands, so the combinatorics of $\{Y_i\}$'s is bounded. Thus there is an infinite subsequence of $Y_1, Y_2, \ldots$ has the same homeomorphism type. Name the subsequence also by $Y_1, Y_2, \ldots$. Let the homeomorphism from Y_k to Y_l induced by the isomorphism $f_l \circ f_k$ be $h_{k,l}$. So $h_{k,l}$ is an identity on the attaching regions. Let u_k be a spanning arc of a long band in Y_k which induces a simplicial decomposition. $u_l = h_{k,l}(u_k)$ must also induce the same decomposition. Thus the corresponding sequence of union of bands $Y_1, Y_2, \ldots$ satisfies the requirement and we are

done. □

We may now shorten by thin components as in the surface case. Let $\mathcal{B}_n$ be a generating set of $\pi_1(Y_n)$ consists of short loops. Further let $H = \pi_1(X)$, m_n be the transverse measure on X^n and $\mathcal{B}^n$ (containing $\mathcal{B}_n$) be a generating set of $\pi_1(X^n) = H$. Then the length of X^n with respect to $\mathcal{B}^n$ is $|X^n|_{\mathcal{B}^n} := \sum_{\mu \in \mathcal{B}^n} m_n(\mu)$. For each $\mu \in \mathcal{B}_n$ represented by a short loop of the form $p_1 * \lambda * p_2$ in Y_n, we have $m_n(\mu) = m_n(p_1) + m_n(p_2)$.

Let $\{Y_i\}_i$ be a sequence of the property described in Proposition 3.12, $\mathcal{B}_i$ be the generating set induced by cutting Y_i along u_i. Every loop in Y_l ($\hookrightarrow Y_k$) can be viewed as a loop in Y_k, where $l > k$. Thus every short loop in $\mathcal{B}_l$ can be written in terms of short loops in $\mathcal{B}_k$. This induces an automorphism of H, denoted by $\alpha_{k,l}$. More precisely, $\alpha_{k,l}$ is the identity map on $\mathcal{B}^k - \mathcal{B}_k$ and is defined in the following way on $\mathcal{B}_k$. For every μ [7] in $\mathcal{B}_k$, $\alpha_{k,l}$ maps μ to $h_{kl}(\mu)$ which is a short loop in $\mathcal{B}_l$ and so can be realized as an element generated by $\mathcal{B}_k$.

Now fix $k = 1$. For $\forall \mu \in \mathcal{B}_1$, $m_i(\alpha_{1j}(\mu)) \to 0$ as $i \to \infty$ since the weights of bands in Y_i goes to 0 as $i \to \infty$. In particular, for any ϵ satisfying the following,

$$\min\{m_1(\mu) | \mu \in \mathcal{B}_1\} > \epsilon > 0,$$

we may pick l sufficiently large such that $m_l(\alpha_{1,l}(\mu)) < \epsilon$ for $\forall \mu \in \mathcal{B}_1$. Then

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[7]We abuse the notation μ here for both the element and the short loop representing the element.

we have

$$|X^1|_{\mathcal{B}^1} > \epsilon * |\mathcal{B}_1| + \sum_{\mu \in \mathcal{B}^1 - \mathcal{B}_1} m_1(\mu) > \sum_{\mu \in \mathcal{B}^1} m_l(\alpha_{1,l}(\mu)) = |X^1|_{\alpha_{1,l}(\mathcal{B}^1)}.$$

Thus X^1 is shortened by $\alpha_{1,l}$.

In fact, $|Y_1|_{\mathcal{B}_1}$ can be shortened as much as one wants to by taking l sufficiently large. Every generating set of Y_1 can be written in terms of $\mathcal{B}_1$. Therefore Y_1 can be shortened with any given fixed generating set, and so is Y^1.

4   Overview for the Relative Rips Machine

Given a group acting on a real tree, we may ask how its subgroups act on their minimal subtrees. In section 4 and section 5, we will construct a version of Rips machine, called relative Rips machine, to study subgroup actions. Relative Rips machine takes as input a *pair* of band complexes (see definition 4.3): two band complexes X and X' with a *morphism* $\iota : X \to X'$, denoted by $(X \xrightarrow{\iota} X')$. The goal of this machine is to put both X and X' into some normal form simultaneously as well as improve the map between them. As with the Rips machine, this relative machine studies one pair of components $(Y_0 \xrightarrow{\iota} Y_0')$ of $(Y \xrightarrow{\iota} Y')$ at a time. To be more precise, fix a minimal component Y_0' of Y', and let Y_0 (possibly simplicial) be a component of Y such that $\iota(Y_0)$ is contained in Y_0'. In general, for a fixed Y_0', there are finitely many choices for Y_0. To simplify the notation, we will first work on the special case where there is only one such pair $(Y_0 \xrightarrow{\iota} Y_0')$ (the choice for Y_0 is unique), then justify the machine described in the special case for the general case in section 6.2.

Definition 4.1. Let X and X' be band complexes. A map $\Gamma \to \Gamma'$ is a **morphism** if every edge of Γ has a finite subdivision such the restriction on each segment of the subdivision is an isometry. A **morphism** $Y \to Y'$ restricts to a morphism of real graphs and takes each band of Y homeomorphically to a sub-band of Y. Finally, a **morphism** $X \to X'$ is a relative cellular map restricting to a morphism $Y \to Y'$. Inclusion of band subcomplex into a band complex is an example of morphism.

Definition 4.2. Let Y, Y' be two unions of bands and $\iota : Y \to Y'$ be a morphism. Let q be a point in Y, ι is **locally injective** near q, if there exists a neighborhood of q in Y such that the restriction of ι on that neighborhood is injective. ι is an **immersion** if ι is locally injective near every point in Y. Further let $p = \iota(q) \in \iota(Y) \subset Y'$. ι is **locally surjective** near q, if there exists $\epsilon > 0$ such that for $\forall \epsilon' < \epsilon$, ϵ'-neighborhood U_p of p has a corresponding neighborhood U_q of q such that $U_p = \iota(U_q)$. ι is a **submersion** if ι is locally surjective near every point in Y. We say ι is a **local isometry** if ι is an immersion and also a submersion.

Definition 4.3. Given two band complexes X and X', we say they form a *pair* if there is a morphism $\iota : X \to X'$, denoted by $(X \xrightarrow{\iota} X')$. Correspondingly, $(Y \xrightarrow{\iota} Y')$ is a *pair* of union of bands. $(Y_0 \xrightarrow{\iota} Y'_0)$ is a pair of components where $Y_0 \subset Y$ is a component and Y'_0 is a component in Y' with the property that $\iota(Y_0) \subset Y'_0$.

Definition 4.4. Let $H < G$ be two finitely presented groups, T_G be a G-tree, $T_H \subset T_G$ be a minimal H-subtree. $(T_H \hookrightarrow T_G)$ is called a *pair of trees*. Further

let $(X_H \xrightarrow{\iota} X_G)$ be a pair of band complexes. We say $(X_H \xrightarrow{\iota} X_G)$ resolves $(T_H \hookrightarrow T_G)$ if there exists resolutions $r_G : \widetilde{X}_G \to T_G$ and $r_H : \widetilde{X}_H \to T_H$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\widetilde{X}_G & \xrightarrow{\ r_G\ } & T_G \\
\tilde{\iota} \uparrow & & \uparrow \\
\widetilde{X}_H & \xrightarrow{\ r_H\ } & T_H
\end{array}
$$

Remark 4.5. In this section and section 5, we will apply moves on a pair of band complexes $(X \xrightarrow{\iota} X')$. Suppose that $H < G$ are finitely presented groups, T_G is a G-tree and $T_H < T_G$ is a minimal H-tree. If $(X \xrightarrow{\iota} X')$ and $(X^* \xrightarrow{\iota} X^{*\prime})$ are related by an operation (a sequence of moves) and $(X \xrightarrow{\iota} X')$ resolves $(T_H \hookrightarrow T_G)$ then $(X^* \xrightarrow{\iota} X^{*\prime})$ also resolves $(T_H \hookrightarrow T_G)$. The goal is to use the moves to put a pair of band complexes into a normal form while it still resolves the same pair of trees.

We now describe some further assumptions we will make throughout the rest sections of this paper for a pair $(X \xrightarrow{\iota} X')$.

For a given band complex X, according to [BF95, Lemma 6.1], we may always arrange that no leaf of the complex has a subset that is proper, compact, and pushing saturated by a sequence of moves. This further allows us to make the assumption that the underlying union of bands Y of this complex is a disjoint union of components. We can arrange this for a pair of band complexes $(X \xrightarrow{\iota} X')$ as well. There are only finitely many proper, compact, pushing saturated subsets of leaves in X and X' (since each singular leaf contains at least one vertical fiber in the boundary of Y). Firstly, such "bad" subsets in X

can be removed by splitting as in [BF95, Lemma 6.1] (composition of cutting its underlying real graph and subdividing bands). Denote the resulting band complex by X^*. Define morphism $\iota^* : X^* \to X'$ as the following. On the level of union of bands, ι^* is the composition of the inclusion $Y^* \hookrightarrow Y$ and $\iota : Y \to Y'$ which remains an morphism. ι^* maps subdivision annuli and cones in $\Gamma(X^*)$ created by above splits to the ι-images of the corresponding split vertical fibers and split vertices in Y, and remains the same as ι on all the other 2-cells. Secondly, we may remove "bad" subsets in X' again by splitting to obtain X'^* and these splittings are disjoint from $\iota^*(X^*)$ since leaves of X^* are made containing no "bad" subsets. So $(X^* \xrightarrow{\iota} X'^*)$ is a pair with the desired property. Hence, from now on we will assume that for a pair of band complexes $(X \xrightarrow{\iota} X')$ we have the following,

- $(A2)$: No leaf of X and X' has a subset that is proper, compact, and pushing saturated. In particular, their underlying union of bands Y and Y' are disjoint unions of components.

In definition 4.1, a morphism $\iota : X \to X'$ takes each band of Y homeomorphically to a subband of Y'. In general, it may not hold on the level of generalized bands. The following lemma allows us to work only with generalized bands.

Proposition 4.6. *Let $(Y \xrightarrow{\iota} Y')$ be a pair of union of bands. We may convert ι into a morphism that takes each generalized band of Y homeomorphically to a sub-generalized band of Y' by applying finitely many moves to Y.*

Proof. Let $\mathbf{B}_Y$ be a generalized band of Y and its image in Y' be $\widehat{\mathbf{B}}_Y = \iota(\mathbf{B}_Y)$.

1. First, we may assume that $\widehat{\mathbf{B}}_Y$ is contained in a single generalized band $\mathbf{B}_{Y'}$ in Y'. Otherwise $\widehat{\mathbf{B}}_Y$ is contained in the union of two or more consecutive generalized bands in X. In this case, we can horizontally (transverse to the foliation) subdivide $\mathbf{B}_Y$ into several generalized bands such that each new generalized band of the induced subdivision on $\widehat{\mathbf{B}}_Y$ is contained in a single generalized band, see figure 7;
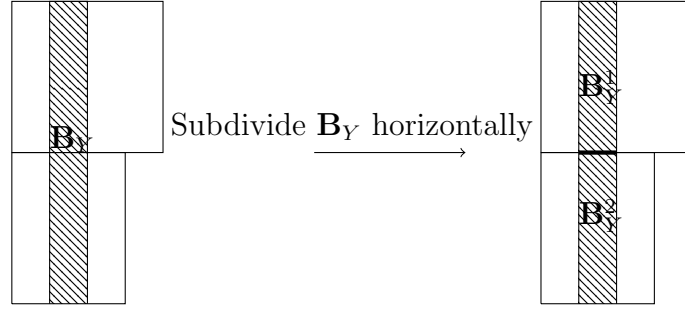

Figure 7: Blank rectangles represent generalized bands in Y'. The figure shows the case where $Y \hookrightarrow Y'$ (locally this is always true), i.e. $Y = \iota(Y)$. Shaded parts are generalized bands in Y.

2. Then we may assume that there is no two or more consecutive generalized bands $\mathbf{B}_Y^1, \mathbf{B}_Y^2 \ldots$ of Y map into the same generalized band of Y'. Otherwise, let b_Y^i and $dual(b_Y^i)$ be the bottom and top bases of $\mathbf{B}_Y^i$. $dual(b_Y^{i-1}) \cup b_Y^i - dual(b_Y^{i-1}) \cap b_Y^i$ are union of free subarcs. After collapsing these free subarcs, $\mathbf{B}_Y^1, \mathbf{B}_Y^2, \ldots$ form a new longer generalized band, whose image in Y' is a single sub-generalized band. See figure 8;

3. Finally, we may assume $l(\widehat{\mathbf{B}}_Y) = l(\mathbf{B}_{Y'})$ i.e. $\widehat{\mathbf{B}}_Y$ is a sub-generalized band. Otherwise $l(\widehat{\mathbf{B}}_Y) < l(\mathbf{B}_{Y'})$. One of the bases of $\widehat{\mathbf{B}}_Y$ is a type I free subarc. This implies that one of the bases of $\mathbf{B}_Y$ is a free subarc of type I and so the whole $\mathbf{B}_Y$ can be collapsed. See figure 9.
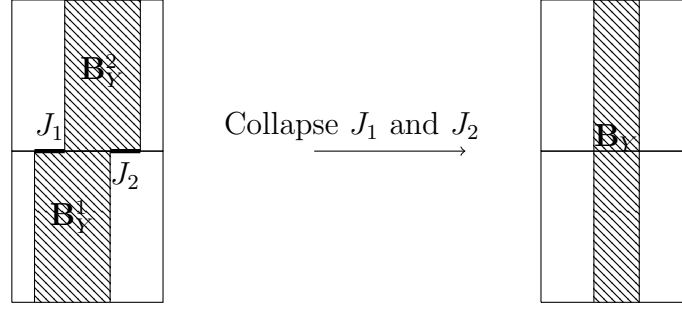
Figure 8: This is the case where $Y \hookrightarrow Y'$ ($Y = \iota(Y)$). $dual(b_Y^1) \cup b_Y^2 - dual(b_Y^1) \cap b_Y^2 = J_1 \cup J_2$. Shaded parts are generalized bands in Y and blank rectangles are generalized bands in Y'.
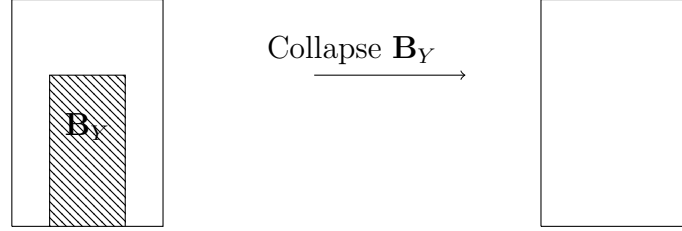
Figure 9: This is the case where $Y \hookrightarrow Y'$ ($Y = \iota(Y)$). Shaded parts are generalized bands in Y and blank rectangles are generalized bands in Y'.

The number of generalized bands in Y' is bounded. So after finitely many moves, ι maps every generalized band in Y to a sub-generalized band in Y'. □

Proposition 4.6 allows us to make the following assumption and only work with generalized bands unless otherwise stated.

- $(A3)$: A morphism between two band complexes always represents the morphism it induced on the level of generalized band.

Definition 4.7. Let Y be an union of bands and B_1, B_2 be two distinct bands (both have length 1, i.e. the original bands, not the generalized bands) in Y. If a base of B_1 is identified with a base of B_2, we say that these two bands

have a **common base**, denote the base by b. Let $\phi : B_1 \to B_2$ be the linear homeomorphism fixing b. Define " $\sim$ " be the equivalence relation on Y such that $x \sim \phi(x)$, for all $x \in B_1$. The quotient space $Y/\sim$ is still an union of bands. Call the quotient map $f : Y \twoheadrightarrow Y/\sim$ a **fold** between union of bands. If a base of B_1 overlaps a base of B_2 (B_1 *overlaps* B_2 for short), i.e. $b_1 \cap b_2 = o \neq \emptyset$, see Figure 10 , we may fold the subbands [8] determined by the common segment o in B_1 and B_2 (shaded parts in figure 10). Moreover, we may also fold two generalized bands with a common base (or an overlap) if they have the same length.
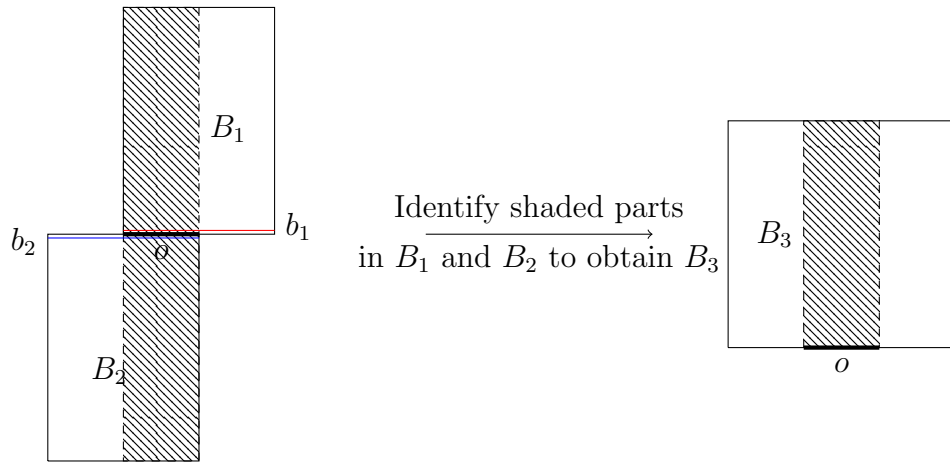
Figure 10: B_1 and B_2 are two bands in Y. Their bases overlap at o. Fold subbands in B_1 and B_2 determined by o to obtain a new band B_3.

Remark 4.8. In previous sections, we always assumed that a given union of bands Y has the structure constructed in definition 2.9, in particular $(A1)$ is satisfied. Roughly speaking, in this section, for a given pair of band complexes $(X \xrightarrow{\iota} X')$ (with its underlying pair of union of bands $(Y \xrightarrow{\iota} Y')$) we will fold Y

---

[8]Here "subband" is abused for the case where o is a single point

(according to $\iota : Y \to Y'$) to get a new union of bands Y^1 ($= Y/\sim$) such that $\iota^1 : Y^1 \to Y'$ is "closer" to an immersion. In general, the Γ_{Y^1} obtained after folding is a simplicial forest. To fix this, we could do subdivision. However, the problem is that we may have to subdivide infinitely many times before we reach immersion and the underlying real graph of the limiting union of bands may not be simplicial anymore. To bypass this, whenever Γ_{Y^1} fails $(A1)$, we will replace $(Y^1 \xrightarrow{\iota^1} Y')$ by a new pair $(Y^{1*} \xrightarrow{\iota^{1*}} Y')$ resolving the same pair of trees where $\Gamma_{Y^{1*}}$ satisfies $(A1)$. This replacement procedure is discussed in proposition 4.10.

Definition 4.9. Let $\iota : Y \to Y'$ be a morphism between two union of bands where Γ_Y is a simplicial forest ($\Gamma_{Y'}$ is a union of edges), $\mathbf{B}_Y^1, \mathbf{B}_Y^2$ be two generalized bands in Y that have some overlap. Let the overlap be $o = b_Y^1 \cap b_Y^2$ where $b_Y^1 \subset \mathbf{B}_Y^1$ and $b_Y^2 \subset \mathbf{B}_Y^2$ are bases. Denote $\iota(b_Y^1) \cap \iota(b_Y^2)$ by $\widehat{o}$. o and $\widehat{o}$ are segments. We say the overlap between $\mathbf{B}_Y^1$ and $\mathbf{B}_Y^2$ is **wide** if o and $\widehat{o}$ have the same length.

Proposition 4.10. *Let $(X \xrightarrow{\iota} X')$ be a pair of band complexes where Γ_X is a simplicial forest. We may replace $(X \xrightarrow{\iota} X')$ by a new pair $(X^* \xrightarrow{\iota^*} X')$ such that it resolves same pair of trees as $(X \xrightarrow{\iota} X')$ and the restriction of ι^* on the underlying real graphs $\iota^* : \Gamma_{X^*} \to \Gamma_{X'}$ is an immersion. In particular, all overlaps between bands are wide.*

Proof. The restriction of ι on the underlying real graphs $\iota : \Gamma_X \to \Gamma_{X'}$ is a morphism between graphs, which can be realized as composition of finitely many folds (for graphs) and an immersion [BF91]. Let X^* be the resulting

band complex obtained from X by folding Γ_X according to ι. Let $\iota^* : X^* \to X'$ be the induced morphism. $(X^* \xrightarrow{\iota} X')$ is then a new pair of union of bands with the desire property. $\qquad\square$

Proposition 4.11. *Let $(X \xrightarrow{\iota} X')$ be a pair of band complexes where Γ_X is a simplicial forest. Up to a finite sequence of folds between union of bands, we may replace $(X \xrightarrow{\iota} X')$ by a new pair $(X^* \xrightarrow{\iota^*} X')$ such that it resolves same pair of trees as $(X \xrightarrow{\iota} X')$ and $\iota^* : Y^* \to Y'$ is an immersion.*

Proof. We may assume that $\iota : \Gamma_X \to \Gamma_{X'}$ is an immersion by Proposition 4.10. We are done if $\iota : Y \to Y'$ is an immersion. Otherwise there exists some band $\mathbf{B}_{Y'} \subset Y'$ has two preimages $\mathbf{B}_Y^1, \mathbf{B}_Y^2 \subset Y$ with the property that $\mathbf{B}_Y^1$ overlaps $\mathbf{B}_Y^2$ and the overlap is wide. Then fold $\mathbf{B}_Y^1$ and $\mathbf{B}_Y^2$ according to the overlap. Let the resulting band complex be X^* and $\iota^* : X^* \to X'$ be the induced morphism. $\mathbf{B}_{Y^*} = \mathbf{B}_Y^1 \cup \mathbf{B}_Y^2 / \sim_{fold}$ is then a new band in Y^* and ι^* maps $\mathbf{B}_{Y^*}$ into $\mathbf{B}_{Y'}$. In particular, the number of generalized bands in Y^* is less than the number in Y due to the fold. Therefore, after finitely many steps of folding, ι^* will be an immersion. $\qquad\square$

Proposition 4.11 allows us to make the following assumption to any given pair of band complexes $(X \xrightarrow{\iota} X')$.

- (A4) The restriction of ι on the level of union of bands $\iota : Y \to Y'$ is an immersion

Let Y_0' be a minimal component of Y'. Its ι-preimage may consists several components in Y. In each step of relative Rips machine, we will do moves

on Y_0' first, then apply induced moves to its preimages in Y and modify ι properly. To simplify our notation, we will focus on the special case $(A5)$ first, then come back to the most general case in section 6.2.

- $(A5)$ For a fixed minimal component Y_0' in Y', its ι-preimage, if not empty, is a unique component Y_0 in Y.

Remark 4.12. Here we note that to describe a morphism $\iota : Y \to Y'$ between two unions of bands, one only need specify a well-defined map between some neighborhoods of their underlying real graphs. This is because morphisms map every generalized band $\mathbf{B}_Y$ in Y homeomorphically to a sub-generalized band in Y'. If a map is *well defined* between neighborhoods of their underlying real graphs, i.e. for $\forall \mathbf{B}_Y \in Y$, neighborhoods of its bases b_Y and $dual(b_Y)$ map to dual positions near $b_{Y'}$ and $dual(b_{Y'})$ which uniquely determine a band $\mathbf{B}_{Y'}$ in Y', it induces a unique morphism from Y to Y' (up to homotopy within a band) by sending $\mathbf{B}_Y$ to $\mathbf{B}_{Y'}$. Moreover, for a given morphism $\iota : Y \to Y'$, to show ι is an immersion/submersion, one only need to check local injectivity /surjectivity near their real graphs. We will use this fact throughout this section.

Definition 4.13. Let $(X \xrightarrow{\iota} X')$ be a pair of band complexes with their underlying unions of bands $(Y \xrightarrow{\iota} Y')$. In relative Rips machine, described in section 5, some sequences of moves will be applied to Y and Y' as units. We list these combinations of moves here for better reference.

1. *Subdivide X at a point.* Let $q \in \Gamma_X$ be a point on the real graph of X. We obtain a new band complex X^* by subdividing bands in Y, whose

bases contain q as an interior point. X^* is well-defined since there are only finitely many such bands in Y (See figure 11). $(X^* \xrightarrow{\iota^*} X')$ is a new pair of band complexes with immersion ι^* defined as follows. On the level of union of bands, $\iota^* : Y^* \to Y'$ is the composition of inclusion $Y^* \hookrightarrow Y$ and $\iota : Y \to Y'$. On the level of attaching cells, ι^* maps subdivision cells newly created from subdivisions to their corresponding vertical fibers in X' and remains the same on all the other attaching cells.
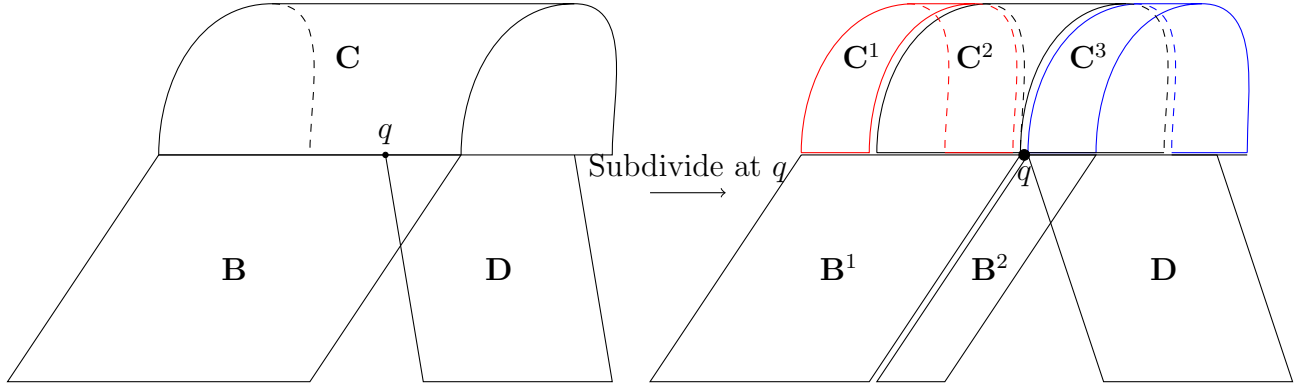
Figure 11: The left figure is a part of Y. There are three generalized bands $\mathbf{B}, \mathbf{C}, \mathbf{D}$ containing point q. In particular $\mathbf{B}$ and $\mathbf{C}$ contains q as an interior point of their bases. So they are subdivided during this move.

2. *Subdivide X' at a point.* Let $p \in \Gamma_{X'}$ be a point on the real graph of X' and $q_1, \ldots, q_n \in \Gamma_X$ be its ι-preimages in Γ_X. As above move, we may subdivide X' at p to obtain X'^*. To get a new pair $(X^* \xrightarrow{\iota^*} X'^*)$, we may then obtain X^* by subdividing X at $q_1, \ldots, q_n$. Let $\iota^* : X^* \to X'^*$ be the induced map from ι. In particular newly created subdivision cells in X^* map to corresponding newly created subdivision cells in X'^*.

3. *Duplicate a segment of the underlying real graph of a union of bands.*

Let c be a segment of Γ_Y. Add an extra segment c' of length $l(c)$ to Γ_Y. Then attach to Y a new band $\mathbf{C} = c \times I$ via a measure-preserving map $c \times \{0\} \to c$ and $c \times \{1\} \to c'$. It is clear that there are two copies of c in the new Y and the new Y resolves the same tree as the original one. In particular, a band $\mathbf{B} \subset Y$ with one of its bases b contained in c can now be attached to the new Y either along c or c' (slide $\mathbf{B}$ across $\mathbf{C}$). We will use this move to solve the problem of keeping ι as an immersion when there are overlapping bases in Y that map to disjoint bases after applying moves to Y'. The ι-image of $\mathbf{C}$ is uniquely determine by ι-images of other bands with their bases contained in c and c', see section 5.2 for more details.

Notation 4.14. For the rest of this paper unless otherwise stated, we will use the following notations:

- $(X \xrightarrow{\iota} X')$ is a pair of band complexes satisfying assumptions $(A1)-(A5)$;

- $(Y \xrightarrow{\iota} Y')$ are their underlying unions of bands (a pair of unions of bands) and $\widehat{Y} = \iota(Y) \subset Y'$;

- $(Y_0 \xrightarrow{\iota_0} Y_0')$ is a pair of components, where $Y_0 \subset Y$, $Y_0' \subset Y$, and $\widehat{Y}_0 = \iota(Y_0) \subset Y_0'$;

- $\mathbf{B}_Y \subset Y$ is a generalized band with bases b_Y and $dual(b_Y)$ (Similarly for other generalized bands, e.g. c_Y and $dual(c_Y)$ for $\mathbf{C}_Y \subset Y$, $b_{Y'}$ and $dual(b_{Y'})$ for $\mathbf{B}_{Y'} \subset Y'$).

5 The relative Rips machine

We will describe relative Rips machine in this section. The machine consists three processes (sequences of moves): Process I, Process II and Process III. Let $(X \xrightarrow{\iota} X')$ be a pair of band complexes, $(Y_0 \xrightarrow{\iota} Y_0')$ be a pair of components of it. In each step of Process I and Process II, we first apply a move to Y_0'. It is the same move one would apply to Y_0' in the original Rips machine. Then we apply moves to Y_0 and modify ι correspondingly so that ι remains a morphism. Successive applications of Process I and Process II will convert Y_0' into a *standard form*, whereas Y_0 may not be in its standard form yet. In particular, there may exist weight 1 bands in Y_0 that map to weight 0 or weight $\frac{1}{2}$ bands. Such bands in Y_0 are called pre-weight 0 bands and pre-weight $\frac{1}{2}$ bands. Process III is then needed to deal with such bands (see section 5.3). After finitely many steps in Process III, we will again have a new pair of components still call it $(Y_0 \xrightarrow{\iota} Y_0')$ with the property $(*)$ that Y_0' is in standard form and bands in Y_0 have the same weights as their images in Y_0'. We will go back to Process I with $(Y_0 \xrightarrow{\iota} Y_0')$. Property $(*)$ ensures that Process III will not appear again. In section 6, as machine output, we will show that one is able to tell the type of Y_0 as the machine successively applied and the machine will eventually convert Y_0 into standard form if Y_0 is of the surface or thin type. Once we finish analyzing $(Y_0 \xrightarrow{\iota} Y_0')$, we continue by choosing another pair. From section 5.1 to section 5.3, we will describe these three processes.

5.1 Process I

Let $(Y_0 \xrightarrow{\iota_0} Y_0')$ be a pair of components and $\iota_0 = \iota|_{Y_0}$. We define $(X^* \xrightarrow{\iota^*} X'^*)$ to be a pair of band complexes obtained from $(X \xrightarrow{\iota} X')$ by the following operation. Find, if possible, a maximal free subarc $J_{Y'}$ of a base $b_{Y'}$ in Y_0'. If such a $J_{Y'}$ does not exist, define $(X^* \xrightarrow{\iota^*} X'^*) = (X \xrightarrow{\iota} X')$ and go on to Process II. Else in Y_0', use $(M5)$ to collapse from $J_{Y'}$ to get Y_1'. If there are several $J_{Y'}$'s to choose from, abide the rule described in section 3.1.

Now we need to make corresponding changes to Y_0 and ι_0 to obtain Y_1 and ι_1 . It depends on if the collapsed region in Y_0', union of $J_{Y'}$ and the interior of $\mathbf{J}_{Y'} = J_{Y'} \times I_n$ ($J_{Y'}$ is identified with $J_{Y'} \times \{0\}$), intersects the image of Y_0 (denoted by $\widehat{Y}_0$). If the collapsed region in Y_0' does not intersect $\iota_0(Y_0) = \widehat{Y}_0$, let $Y_1 = Y_0$ and $\iota_1 = \iota_0$.
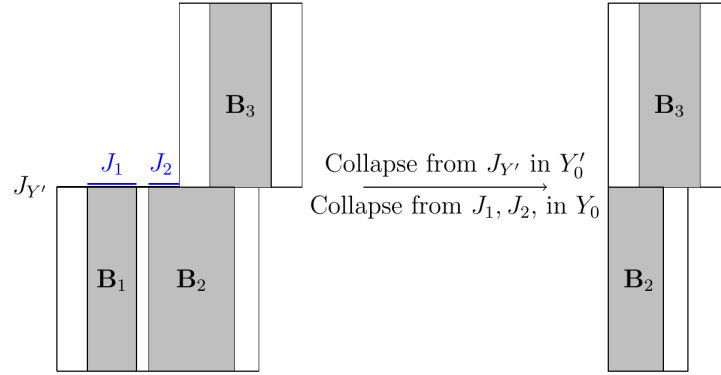
Figure 12: The figure is a special case where we assume that Y_0 is identified with $\widehat{Y}_0$ near this portion of the union of bands. Blank rectangles are bands in Y_0' and gray parts are bands in $\widehat{Y}_0$.

Otherwise let $\{\mathbf{J}_i\}_{i=1,\ldots,n}$ be the set of preimages of $\mathbf{J}_{Y'} \cap \widehat{Y}_0$ in Y_0, and let J_i be the base of $\mathbf{J}_i$ whose ι_0-image is contained in $J_{Y'}$. Since ι_0 is an immersion,

J_i's are also free subarcs in Y_0. Y_1 is produced by collapsing $\{J_i\}$'s in Y_0. Then restricting ι_0 on Y_1, we will get $\iota_1 : Y_1 \to Y_1'$ (See Figure 12). In particular, ι_1 maps subdivision annuli created by collapsing J_i's to the subdivision annuli created by collapsing $J_{Y'}$.

For the case where $\mathbf{J}_{Y'}$ is contained in a weight $\frac{1}{2}$ band, each $\mathbf{J}_i$ is either contained in a weight $\frac{1}{2}$ or weight 1 band. The above process is well defined for both cases. In more details, suppose the band containing $\mathbf{J}_{Y'}$ is $\mathbf{B}_{Y'}$ of weight $\frac{1}{2}$ and the band containing $\mathbf{J}_i$ is $\mathbf{B}_Y^i$. Let w be the midpoint of the base $b_{Y'}$ of $\mathbf{B}_{Y'}$. Recall that to collapse $J_{Y'}$ from $\mathbf{B}_{Y'}$, we will firstly replace $\mathbf{B}_{Y'}$ by bands of weight $0, \frac{1}{2}$, and 1 (In the degenerate case where w is one end point of $J_{Y'}$, there is no weight $\frac{1}{2}$). $J_{Y'}$ is now a free arc in a weight 1 base ($\mathbf{J}_X$ is contained in the weight 1 band). In each $\mathbf{B}_Y^i$, we will do the corresponding replacement (induced subdivision). Each $\mathbf{J}_i$ is then contained in a weight 1 band that maps to the weight 1 band of $\mathbf{B}_{Y'}$. We may now collapse from $J_{Y'}$ in Y_0' and J_i's in Y_0 as described above .

In either case, let $(X^* \xrightarrow{\iota} X'^*)$ be the resulting pair of band complexes and we say it is produced from $(X \xrightarrow{\iota} X')$ by Process I. It is clear ι_1 is obtained by restricting ι_0 on Y_1, thus remains an immersion. We then continue by applying Process I to $(X^* \xrightarrow{\iota} X'^*)$.

Note that by the first item of proposition 3.4,

$$Complexity(X'^*) \leq Complexity(X').$$

5.2   Process II

Follow the same notation as above, we now describe Process II which again will produce from $(X \xrightarrow{\iota} X')$ another pair of band complexes $(X^* \xrightarrow{\iota} X'^*)$. Process II will only be applied after Process I. So, in this section, we also assume that Y_0' has no free subarc (Y_0 may contain some free subarcs), i.e. for each point $z \in \Gamma_{Y_0'}$, we have

$(*_2)$: *the sum of the weights of the bases containing z is at least 2.*

Therefore, if the Rips machine constructed in [BF95] takes Y_0' as an input, its Process II will be applied. Moves described below for Y_0' (to obtain Y_1') is in fact the same moves one would apply to Y_0' in the Rips machine.

We may orient $\Gamma_{Y_0'}$ and order the components of $\Gamma_{Y_0'}$. This induces a linear order on $\Gamma_{Y_0'}$. Let K be the first component and z be the initial point of K. Let $b_{Y'}$ be the longest base of positive weight containing z, chosen to have weight 1 if possible. Further let $\mathbf{B}_{Y'}$ be the corresponding generalized band. The union of bands Y_1' is the result of the composition of the following two operations.

Operation 1 (Slide) $Y_0' \to Y_0'^*$: First slide from $b_{Y'}$ all these positive weight bases contained in $b_{Y'}$ (except $b_{Y'}$ and $dual(b_{Y'})$) whose midpoint is moved away from z as a result of the slide.

Operation 2 (Collapse) $Y_0'^* \to Y_1'$: Then, Collapse from the maximal free initial segment of $b_{Y'}$.

Then we only need to define Y_1 and $\iota_1 : Y_1 \to Y_1'$. Once Y_1 and ι_1 have been

properly defined, since

$$Complexity(Y_1') \leq Complexity(Y_0')$$

and if they are equal, $(*_2)$ holds for Y_1' ([BF95, Proposition 7.5]), we are in position to apply Process II again. Thus we successively apply Process II unless the complexity of Y_i' decreases at some stage. In this case, we say Process II sequence ends, and we go back to Process I.

Now we will describe how to obtain Y_1 and ι_1 by cases depends whether $\widehat{Y}_0$ intersects $\mathbf{B}_{Y'}$.
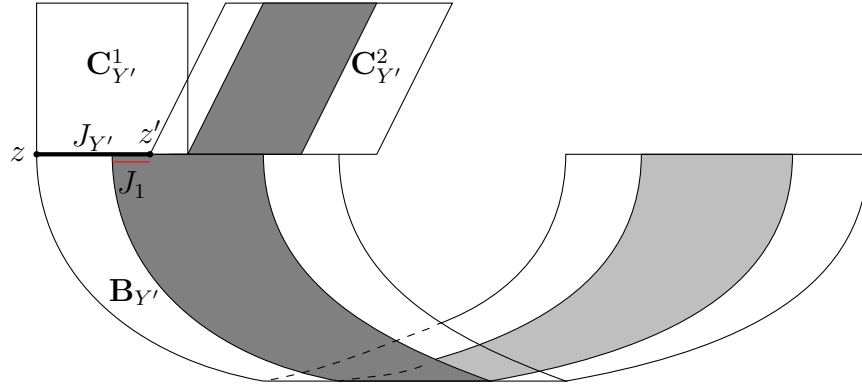
Figure 13: The figure is a special case where we assume that Y_0 is identified with $\widehat{Y}_0$ near this portion of the union of bands. Blank rectangles are bands in Y'_0 and gray parts are bands in $\widehat{Y}_0$. In Y'_0, $\mathbf{B}_{Y'}$ is the carrier. $\mathbf{C}_{Y'}^1$ will be carried and $\mathbf{C}_{Y'}^2$ will not. J_1 is defined as shown.

Let $J_{Y'}$ be the maximal free initial segment of $b_{Y'}$ produced in operation 1 and let the sub-generalized band containing the collapsed region in operation 2 be $\mathbf{J}_{Y'}$. We will assume for the rest of this section that:

$(*_3)$: *In Y'_0, $\widehat{Y}_0$ intersects some generalized band that is carried in operation 1.*

Suppose not. If further $\widehat{Y}_0$ also has no intersection with the interior of $\mathbf{J}_{Y'}$, let

$Y_1 = Y_0$, $\iota_1 = \iota_0$ and we are done; If $\widehat{Y}_0$ only intersects $\mathbf{J}_{Y'}$ (See Figure 13), let the set of preimages in Y_0 of these intersections be $\mathbf{J}_1, \ldots, \mathbf{J}_k$. Let J_i's be bases of $\mathbf{J}_i$'s as in Process I. Y_1 is obtained by collapsing from all J_i's in Y_0 and ι_1 is obtained by restricting ι_0 on Y_1.

Under assumption $(*_3)$, there are now two cases.

Case 1 $\widehat{Y}_0$ only intersects generalized bands that are carried from Y_0' to $Y_0'^*$ (No intersection with the carrier $\mathbf{B}_{Y'}$).

Let c_Y be a base of a generalized band $\mathbf{C}_Y$ in Y_0. If $\iota_0(c_Y)$ is contained in a base that is carried by $\mathbf{B}_Y$, we say c_Y is also **carried**, otherwise, we say c_Y is not carried. According to remark 4.12, to define Y_1 and ι_1 we only need to work on neighborhoods of Γ_{Y_0}. The definition of Y_1 and ι_1 will be made block by block depending on whether bases contained in that block is carried.

Let I be a block of Γ_{Y_0}. If every base contained in I is not carried, Y_1 and ι_1 are defined to be the same as Y_0 and ι_0 near this block. If every base contained in I is carried, then let Y_1 be the same as Y_0 near this block, while ι_1 is defined by mapping every base $c_Y \subset I$ to the dual position of $\iota_0(c_Y)$ in $dual(b_{Y'})$. To be more precise, let $\mathbf{C}_{Y'}$ be the band in Y_0' containing the ι_0-image of $\mathbf{C}_Y$, where $\mathbf{C}_Y \subset Y_0$ is the band with base c_Y. During the operation $Y_0' \to Y_1'$, $\mathbf{C}_{Y'}$ is slid across $\mathbf{B}_{Y'}$ to its new position in Y_1'. ι_1 is defined to map $\mathbf{C}_Y$ into $\mathbf{C}_{Y'}$'s new position in Y_1'. Otherwise

$(*_4)$: *Within a block I, some bases are carried and some bases are not.*

Near blocks satisfying $(*_4)$, Y_1 is defined as the result of the composition of the following two operations.

Operation $1'$ (Slide) $Y_0 \rightarrow Y_0^*$: Let the union of carried bases within I be $\omega = \omega_1 \cup \cdots \cup \omega_k$, where each ω_i is a component of ω. Near I, we define Y_0^* by modifying Y_0 as below. Duplicate the real graph of Y_0 on each ω_i (Definition 4.13) by adding a new band $\Omega_i = \omega_i \times [0,1]$ along it. Denote the other base of Ω_i by $dual(\omega_i)$ (See figure 14). In I, every carried base c_Y is contained in some ω_i. Slide each band $\mathbf{C}_Y$ corresponding to a c_Y across the Ω_i.

Do this modification on every block satisfying $(*_4)$ and call the resulting union of bands Y_0^*.

By above construction, Y_0^* is the same as Y_0 away from Ω_i's. To define ι_0^* : $Y_0^* \rightarrow Y_0'^*$, we only need to describe the map near Ω_i's. In $\Gamma_{Y_0^*}$, firstly, consider the blocks containing ω_i's. All bases other than ω_i's in such blocks are bases in Y_0 that are not carried. Define $\iota_0^* = \iota_0$ near these bases and $\iota_0^*(\omega_i) = \iota_0(\omega_i) \subset b_{Y'}^* = b_{Y'}$. Secondly, consider the blocks of $\Gamma_{Y_0^*}$ containing $dual(\omega_i)$'s. Every base, say c_Y, other than $dual(\omega_i)$'s in such blocks is a base come from Y_0 that is carried. So define ι_0^* as mapping c_Y to the new position of $\iota_0(c_Y)$ in Y'^* and $\iota_0^*(dual(\omega_i)) = dual(\iota_0(\omega_i))$ in $dual(b_{Y'^*})$. Finally, let the unique subband of $\mathbf{B}_{Y'^*}$ determined by the image of ω_i and $dual(\omega_i)$ be $\widehat{\Omega}_i$, define $\iota_0^*(\Omega_i) = \widehat{\Omega}_i$.

Now let's check that ι_0^* is an immersion. Near the blocks of $\Gamma_{Y_0^*}$ containing ω_i's, ι_0^* is the restriction of ι_0 on all the bands other than Ω_i's. ι_0 is an immersion and $\widehat{Y}_0$ does not intersect $\mathbf{B}_{Y'}$ implies ι_0^* is injective near these blocks. Near the blocks of $\Gamma_{Y_0^*}$ containing $dual(\omega_i)$'s, ι_0^* is injective for the same reason.

Operation $2'$ (Collapse) $Y_0^* \rightarrow Y_1$: Let $\widehat{Y}_0^* \subset Y_0'^*$ be the image of Y_0^* under ι_0^*. According to operation $1'$, $\widehat{Y}_0^*$ may intersect $\mathbf{J}_{Y'}$ (The intersection part will

collapse in this operation). Exactly the same as in Process I, let the set of preimages of these intersections be $\mathbf{J}_i$'s. Y_1 is obtained by collapsing from J_i's in Y_0^*. ι_1 is then the restriction of ι_0^* on Y_1. ι_1 is an immersion since ι_0^* is.
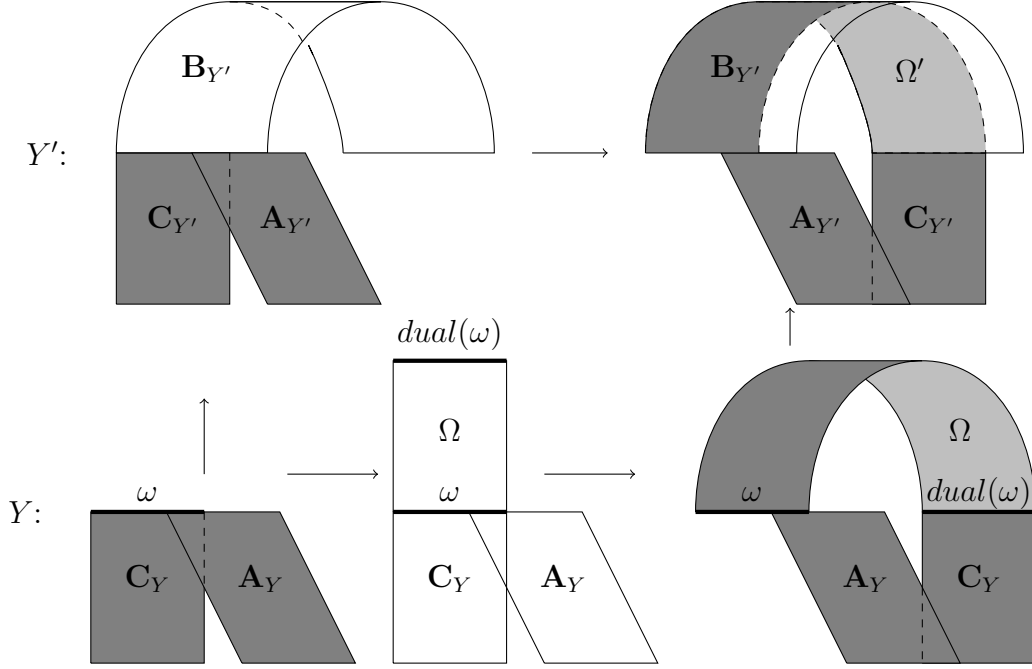
Figure 14: The shaded part is $\widehat{Y}$ (within Y').

Case 2 If $\widehat{Y}_0$ also intersects $\mathbf{B}_{Y'}$, Y_1 and ι_1 is produced as follows.

Operation $0'$ (Subdivide) Let $\mathbf{B}_1, \ldots, \mathbf{B}_k$ be generalized bands of Y_0, whose images under ι_0 are contained in $\mathbf{B}_{Y'}$. Further for each $\mathbf{B}_i$, let its base that maps into $b_{Y'}$ be b_i with end points z_i and z_i' and the block of Γ_{Y_0} containing b_i be I_i. Subdivide Y_0 at every z_i or z_i', if z_i or z_i' is an interior point of I_i. Denote the resulting union of bands still by Y_0. Each band in the new Y_0 is a subband of the original Y_0. So the original ι_0 will induce a map from the new Y_0 to Y_0'. The induced map, denoted still by ι_0, remains an immersion

since every block in the new Y_0 is contained in a block of the original Y_0. On the level of band complexes, annuli produced from subdivision are mapped to corresponding subdivided leaves in Y_0'.

As a result of operation $0'$, we may assume the following property for each block I of Y_0.

$(*_5)$ *For generalized bands with one of their bases contained in I, either none of them has its ι_0-image intersects the interior of $\mathbf{B}_{Y'}$ or exactly one of them has its ι_0-image that intersects the interior of $\mathbf{B}_{Y'}$. Further in the later type, that base coincides with I.*

Operation $1'$ (Slide) $Y_0 \to Y_0^*$: For blocks of Y_0 in the former type of $(*_5)$, proceed exactly as in case 1 to get Y_0^* and ι_0^*. Otherwise, let $\mathbf{B}_Y$ be the generalized band that map into $\mathbf{B}_{Y'}$ with one of its bases $b_Y = I$ mapping to $b_{Y'}$. Further let $\mathbf{C}_Y \subset Y_0$ be a generalized band with one of its bases c_Y contained in b_Y, and denote the band in Y' containing its ι_0-image by $\mathbf{C}_{Y'}$. Then Y_0^* is obtained by sliding every carried $\mathbf{C}_Y$ across $\mathbf{B}_Y$. ι_0^* is defined by mapping carried $\mathbf{C}_Y$ in its new position in Y_0^* into $\mathbf{C}_{Y'}$'s new position in $Y_0'^*$. It is well-defined since $\mathbf{C}_Y$ is carried if and only if $\mathbf{C}_{Y'}$ is carried. Thus ι_0^* is also an immersion.

Operation $2'$ (Collapse) $Y_0^* \to Y_1$: We can define J_i's as in case 1 and collapse from J_i's to get Y_1. In particular, for a given block I in the later type of $(*_2)$, if its image under ι_0 is contained in $J_{Y'}$, all the other bases in this block must be carried. Therefore such I's become free arcs in Y_0^*, and hence completely collapse. Restricting ι_0^* on Y_1, we will get ι_1 and ι_1 is an immersion since ι_0^* is.

Example 5.1. The 2-sheeted covering surface example. Y_0 is a surface com-

ponent in standard form, and $\iota : Y_0 \to Y_0'$ is an inclusion. Y_0' is not in standard form. Run relative Rips machine, we get a sequence of pairs $(Y_0 \overset{\iota}{\to} Y_0'), (Y_1 \overset{\iota}{\to} Y_1') \dots$. Note that Y_3 is a 2-sheeted cover of Y_3' and both Y_3' and Y_3 are in standard form.

Proposition 5.2. *Let $(Y_0 \overset{\iota_0}{\to} Y_0')$ be a pair of components where Y_0' is of either thin or surface type and $(Y_0 \overset{\iota_0}{\to} Y_0'), (Y_1 \overset{\iota_1}{\to} Y_1'), \dots$ be a sequence of pairs of components formed by Process I and Process II. There exists an integer N such that for any generalized band $\mathbf{B}_{Y_n'}$ in Y_n' with $n \geq N$, $\iota_n(Y_n) \cap \mathbf{B}_{Y_n'}$ is a unique sub-generalized band. Further let $\mathbf{B}_{Y_n}^1, \dots, \mathbf{B}_{Y_n}^{k_n}$ be generalized bands in Y_n that map into $\mathbf{B}_{Y_n'}$ ($\mathbf{B}_{Y_n'}$'s ι_n-preimage set), then $\cap_{j=1}^{k_n} \iota_n(\mathbf{B}_{Y_n}^j)$ contains a vertical fiber that is in the limiting lamination L_∞ of Y_i''s.*

Proof. By construction, there exists an integer $N_0 \geq 0$ such that for any pair $(Y_i \overset{\iota_i}{\to} Y_i')$ with $i \geq N_0$, Y_i' is in a standard form.

Firstly, assume that there exists some $(Y_n \overset{\iota_n}{\to} Y_n')$ with $n \geq N_0$ with the following desire property

$(*_6)$: *For any generalized band $\mathbf{B}_{Y_n'}$ in Y_n', $\iota_n(Y_n) \cap \mathbf{B}_{Y_n'}$ is a unique sub-generalized band and $\cap_{j=1}^{k_n} \iota_n(\mathbf{B}_{Y_n}^j)$ contains a vertical fiber in the limit set.*

We will check that this property $(*_6)$ is also true for $(Y_{n+1} \overset{\iota_{n+1}}{\to} Y_{n+1}')$.

Suppose $(Y_{n+1} \overset{\iota_{n+1}}{\to} Y_{n+1}')$ is obtained from $(Y_n \overset{\iota_n}{\to} Y_n')$ by Process I and $\mathbf{J}_{Y_n'}$ is the sub-generalized band in Y_n' containing the collapsed region. If $\widehat{Y}_n$ does not intersect the interior of $\mathbf{J}_{Y_n'}$, then $Y_{n+1} = Y_n$ and property $(*_6)$ is automatically preserved. Otherwise, we need check property $(*_6)$ for the band $\mathbf{B}_{Y_n'}$ that contains $\mathbf{J}_{Y_n'}$. Let $\mathbf{B}_{Y_{n+1}'}$ be an image of $\mathbf{B}_{Y_n}$ in Y_{n+1}' ($\mathbf{B}_{Y_i'}$ may have two images if the collapse is an I_3 collapse, see section 3 for more details.) and $\{\mathbf{B}_{Y_{n+1}}^j\}_j$'s

be the set of $\mathbf{B}_{Y'_{n+1}}$'s ι_{n+1}-preimage in Y_{n+1}. Further, let the vertical boundary of $\mathbf{J}_{Y'_n}$ contained in the interior of $\mathbf{B}_{Y'_{n+1}}$ be l. Each $\mathbf{B}^j_{Y_{n+1}}$ is an image of some $\mathbf{B}^{j'}_{Y_n}$. So $\cap^{k_n}_1 \iota_n(\mathbf{B}^j_{Y_n})$ is not empty implies every $\widehat{\mathbf{B}}^j_{Y_{n+1}}$ contains a neighborhood of l in $\mathbf{B}_{Y'_{n+1}}$. Moreover the limit set is a prefect set in Y'_{n+1} ([BF95]). Thus the statement is also true for $(Y_{n+1} \overset{\iota_{n+1}}{\to} Y'_{n+1})$.

Suppose $(Y_{n+1} \overset{\iota_{n+1}}{\to} Y'_{n+1})$ is obtained from $(Y_n \overset{\iota_n}{\to} Y'_n)$ by Process II (This implies that Y'_n is a surface component). If $\mathbf{B}_{Y'_n}$ is not the carrier for $Y'_n \to Y'_{n+1}$, the property is true for $\mathbf{B}_{Y'_{n+1}}$ since $\widehat{\mathbf{B}}^j_{Y_{n+1}} = \widehat{\mathbf{B}}^j_{Y_n}$ for $j = 1, \ldots, k_n$. Assume $\mathbf{B}_{Y'_n}$ is the carrier. Then Y'_n is a surface component implies that for any $z \in \mathring{b}_{Y'_n}$ (a base of $\mathbf{B}_{Y'_n}$), other than $\mathbf{B}_{Y'_n}$ there is exactly one positive weight band containing z. Therefore no block of Γ_{Y_n} has property $(*_4)$ and no move of duplicating real graph is applied from $Y_n \to Y_{n+1}$. In particular, preimages of $\mathbf{B}_{Y'_{n+1}}$ in Y_{n+1} are only produced from $\mathbf{B}^j_{Y_n}$ by collapsing. Follow the same argument as above for Process I, property $(*_6)$ is also true for carrier.

Now we will show there exists some $n > N_0$ such that the property $(*_6)$ holds for $(Y_n \overset{\iota_n}{\to} Y'_n)$.

Thin Case. In general, for a fixed generalized band $\mathbf{B}_{Y'_0}$ in Y'_0, $\widehat{Y}_0 \cap \mathbf{B}_{Y'_0}$ consists of finitely many sub-generalized bands. Given two fixed generalized bands $\mathbf{B}^1_{Y_0}, \mathbf{B}^2_{Y_0} \subset Y_0$, let $\mathbf{B}^j_{Y_i}$ be an image of $\mathbf{B}^j_{Y_0}$ in $\subset Y_i$ for $j = 1, 2$.

$(*_7)$ *Claim that for i sufficiently large, $\mathbf{B}^1_{Y_i}$ and $\mathbf{B}^2_{Y_i}$ map to the same band $\mathbf{B}_{Y'_i} \subset Y'_i$ if and only if $\widehat{\mathbf{B}}^1_{Y_i} \cap \widehat{\mathbf{B}}^2_{Y_i}$ contains a vertical fiber in the limit set.*

The if direction of the claim is clear. To prove the only if direction, suppose $\mathbf{B}^1_{Y_i}$ and $\mathbf{B}^2_{Y_i}$ map to the same band and $\widehat{\mathbf{B}}^1_{Y_i} \cap \widehat{\mathbf{B}}^2_{Y_i}$ does not contain any vertical

fiber in the limit set. Then there exists a vertical fiber $l' \subset \mathbf{B}_{Y_i'}$ between $\widehat{\mathbf{B}}_{Y_i}^1$ and $\widehat{\mathbf{B}}_{Y_i}^2$ that is not in the limit set. l' will collapse along the process. So there exists some $m > 0$ such that l' collapse from $X_{i+m} \to X_{i+m+1}$. Then each image of $\mathbf{B}_{Y_i}^1$ and each image of $\mathbf{B}_{Y_i}^2$ in Y_{i+m+1} will either be exactly the same generalized band X_{i+m+1} (contradiction!) or two different bands in X_{i+m+1}. So the claim $(*_7)$ holds.

We may pick n sufficiently large such that the claim is true over all choices of $\mathbf{B}_{Y_0'}$'s and $\mathbf{B}_{Y_0}$'s, then property $(*_6)$ follows.

Surface Case. If Y_0' is of surface type, then eventually only Process II is applied. For $m > N_0$, Y_m' is of standard form and there is an infinite sequence $m_1 = m < m_2 < \ldots$ such that Y_{m_i}' is a scaling down version of $Y_{m_{i-1}}'$. Thus all generalized bands in Y_m' are getting thinner and their limits are the boundary leaves. Therefore, over all components of $\iota_m(Y_m) \cap \mathbf{B}_{Y_m'}$, only the one containing the limiting boundary (if there is any) survives till the end. Thus $\cap_1^{k_m} \mathbf{B}_m$ contains a neighborhood of the limiting boundary and we are done. $\qquad\square$

Definition 5.3. Let $\iota : Y \to Y'$ be a morphism between two unions of bands. ι is **graph like** if each generalized band of Y maps onto a generalized band in Y'. If ι is a graph like immersion, we say ι is a **partial covering map**. Further an immersion ι is **an almost partial covering map** if every pair of components $(Y_0 \xrightarrow{\iota_0} Y_0') \subset (Y \xrightarrow{\iota} Y')$ has property $(*_6)$ described in Proposition 5.2.

Thus Process I and Process II will eventually convert Y' into standard form and turn $\iota_0 : Y_0 \to Y_0'$ into an almost partial covering map when Y_0' is a surface

or thin component. We say $(Y \xrightarrow{\iota} Y')$ is **stabilized** in this case.

5.3 Process III

Let $(Y_0 \xrightarrow{\iota_0} Y_0')$ be a pair of components and $\mathbf{B}_{Y_0'} \subset Y_0'$ be a generalized band. If $weight(\mathbf{B}_{Y_0'}) = 1$, then all of its preimages in Y_0 are of weight 1. On the other hand, if $weight(\mathbf{B}_{Y_0'})$ is 0 or $\frac{1}{2}$, it is possible that some of its preimages in Y_0 are of weight 1 instead of having the same weight as $weight(\mathbf{B}_{Y_0'})$ (Note that preimage of a weight 0 (or $\frac{1}{2}$) band can not be a weight $\frac{1}{2}$ (or 0) band). We call such weight 1 preimages **pre-weight** 0 **bands** or **pre-weight** $\frac{1}{2}$ **bands** depending on the weight of $\mathbf{B}_{Y'}$. Moreover, we say that $\iota_0 : Y_0 \to Y_0'$ has a certain property **up to weight** 0 **bands** if the restriction of ι_0 to the preimage of $\overline{Y}_0'$ in Y_0 (i.e. $\overline{Y}_0$ omitting pre-weight 0 bands, denote it by $\check{Y}_0$) has that property.

One goal of relative Rips machine is to convert both Y_0' and Y_0 into some normal form simultaneously. Let $(Y_0 \xrightarrow{\iota_0} Y_0'), (Y_1 \xrightarrow{\iota_1} Y_1'), \ldots$ be a sequence of pairs of components formed by Process I and Process II. As from section 5.2, $(Y_n \xrightarrow{\iota_n} Y_n')$ stabilizes for sufficiently large n. We would like to conclude that Y_n is also in standard form by then. However, Y_n may fail to be standard due to the existence of pre-weight 0 or pre-weight $\frac{1}{2}$ bands (see example 5.4). In Process III, we will "get rid" of pre-weight 0 and pre-weight $\frac{1}{2}$ bands. Two bases of a pre-weight 0 (or $\frac{1}{2}$) band have the same image in Y_0'. Intuitively, we may collapse pre-weight 0 and pre-weight $\frac{1}{2}$ bands by viewing them as ultra thick real graphs. More details are discussed below.

Example 5.4. Let Y_n' be a minimal component in standard form and let Y_n

be a double cover of Y_n'. Suppose $\mathbf{B}_{Y_n'} \subset Y_n'$ is a weight 0 band and its two preimages in Y_n are $\mathbf{B}_{Y_n}^1, \mathbf{B}_{Y_n}^2$. Assume that $\mathbf{B}_{Y_n}^1$ and $\mathbf{B}_{Y_n}^2$ are both pre-weight 0 bands and form an annulus in Y_n as shown in figure 15. Then Y_n is not in standard form since the complexity of Y_n goes down if we slide $\mathbf{B}_{Y_n}^1$ across $\mathbf{B}_{Y_n}^2$.

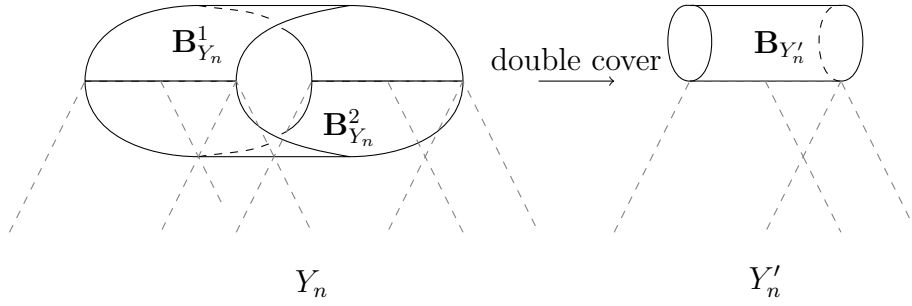Figure 15: Y_n is a double cover of X_n. Two preimages of weight 0 band $\mathbf{B}_{X_n}$ are two pre-weight 0 bands $\mathbf{B}_{Y_n}^1$ and $\mathbf{B}_{Y_n}^1$. Dashed bands are possible other bands attached to the same block.
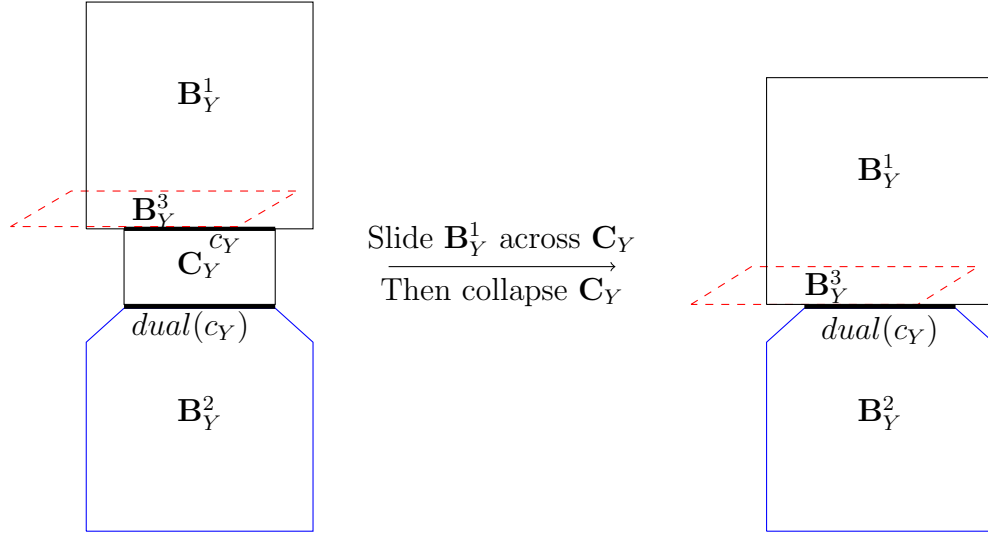

Figure 16: We may slide $\mathbf{B}_Y^1$ across $\mathbf{C}_Y$ even if c_Y does not contain b_Y. If the red dashed $\mathbf{B}_Y^3$ was there, $\mathbf{B}_Y^3$ also can also be slided as shown

Definition 5.5. Let Y be a union of bands and $\mathbf{B}_Y \subset Y$ be a generalized band with bases b_Y and $dual(b_Y)$. We say a base $b \subset \Gamma_Y$ is **wide** if b coincides with the block containing it, i.e. any other base intersecting b is contained in b . $\mathbf{B}_Y$ is **wide** if either b_Y or $dual(b_Y)$ is wide. Suppose b_Y is wide, we may collapse $\mathbf{B}_Y$ by first sliding all bases contained in b_Y across $\mathbf{B}_Y$ and then collapsing $\mathbf{B}_Y$ from b_Y. This sequence of move is called **collapse a wide band**.

Let $\mathbf{B}_Y, \mathbf{C}_Y \subset Y$ be two distinct generalized bands with the property that $b_Y \cap c_Y$ is non-degenerated (not a point). Using move $(M4)$ defined in section 2 (which will preserve $(A1)$ for Y), we can only slide $\mathbf{B}_Y$ across $\mathbf{C}_Y$ if $b_Y \subset c_Y$. If we further allow Y's underlying real graph to be a simplicial forest, $\mathbf{B}_Y$ can be slid across $\mathbf{C}_Y$ as in figure 16 even without the assumption that $b_Y \subset c_Y$. Let the block of Γ_Y (a component of the union of all open bases) containing b_Y and c_Y be K_Y. In fact, we may slide every band with a base contained in K_Y across $\mathbf{C}_Y$. After all sliding, c_Y is then a free arc in the resulting union of bands and so can be collapsed. The new union of bands produced above is the same as viewing $\mathbf{C}_Y$ as an ultra thick segment of the underlying real graph of Y. This sequence of move is called **collapse a general band**.

We are now ready to describe Process III which again will produce from $(X \xrightarrow{\iota} X')$ another pair of band complexes $(X^* \xrightarrow{\iota^*} X')$ (X' remains the same). Process III will only be applied when Y_0' is in standard from and Y_0 contains some pre-weight 0 or pre-weight $\dfrac{1}{2}$ bands.

Suppose Y_0 contains in total N_w $(N_w > 0)$ pre-weight 0 and pre-weight $\frac{1}{2}$ bands. Let $\mathbf{B}_Y \subset Y_0$ be one of them and $\mathbf{B}_{Y'} \subset Y_0'$ be the weight 0 (or $\frac{1}{2}$) band containing $\iota_0(\mathbf{B}_Y)$. Further let the block of Γ_{Y_0} containing b_Y be K_{b_Y} and

the block of $\Gamma_{Y'_0}$ containing $b_{Y'}(= dual(b_{Y'}))$ be $K_{b_{Y'}}$. Apply either *collapse a wide band* move or *collapse a general band* move to collapse $\mathbf{B}_Y$ (i.e. slide all bands, one of whose bases is contained in K_{b_Y}, across $\mathbf{B}_Y$ and then collapse $\mathbf{B}_Y$). Denote the resulting component by Y_0^*. Let the block (may not be an edge) in Y_0^* formed in this move be $K_{b_Y}^*$. Since $\iota_0(b_Y) = \iota_0(dual(b_Y))$, $\iota_0 : Y_0 \to Y'_0$ induces a well-defined morphism $\iota_0^* : Y_0^* \to Y'_0$. In particular, $\iota_0^* : K_{b_Y}^* \to K_{b_{Y'}}$ is a morphism between graphs. Let the resulting band complex corresponding to Y_0^* be X^*, $\iota^* : X^* \to X'$ be the morphism that equals to ι_0^* when restricting to Y_0^* and remains the same as ι on all the other components. Further, according to Proposition 4.11, up to a finite folding sequence, we may assume ι^* is an immersion. $(X^* \xrightarrow{\iota^*} X')$ is then the new pair of band complexes produced from $(X \xrightarrow{\iota} X')$ by Process III. Let the total number of pre-weight 0 and pre-weight $\frac{1}{2}$ bands in X^* be N_w^*. It is clear that $N_w^* \leq N_w - 1$ as at least $\mathbf{B}_Y$ is collapsed.(If there are more than one pre-weight 0 or $\frac{1}{2}$ bases in one component, pick one to be the carrier and the other one will become weight 0 or $\frac{1}{2}$.) If $N_w^* = 0$, we say Process III sequence ends, and then go back to Process I with $(X^* \xrightarrow{\iota^*} X')$ (whichever appropriate). Otherwise, we are in the position to apply Process III again.

Proposition 5.6. *Let $(Y_0 \xrightarrow{\iota_0} Y'_0)$ be a pair of components and $(Y_0 \xrightarrow{\iota_0} Y'_0), (Y_1 \xrightarrow{\iota_1} Y'_1), \ldots$ be a sequence formed by Process I, II and III as described above. Then Process III appears only finitely many times. In particular, there exists $N > 0$ such that for any $n > N$, Y'_n is in standard form, Y_n contains no pre-weight 0 or $\frac{1}{2}$ band and $\iota_n : Y_n \to Y'_n$ is an almost partial covering map when Y'_0 is a surface or thin component.*

Proof. Suppose $(Y_{i+1} \overset{\iota_{i+1}}{\to} Y'_{i+1})$ is obtained from $(Y_i \overset{\iota_i}{\to} Y'_i)$ by Process III. Then the total number of pre-weight 0 and pre-weight $\frac{1}{2}$ bands in Y_{i+1} is less than the number of Y_i due to collapsing pre-weight 0 or pre-weight $\frac{1}{2}$ band (and possibly pre-weight 0 or $\frac{1}{2}$ bands become weight 0 or $\frac{1}{2}$ bands and some folding). Now let $(Y_k \overset{\iota_k}{\to} Y'_k) \to (Y_{k+1} \overset{\iota_{k+1}}{\to} Y'_{k+1})$ (note $Y'_{k+1} = Y'_k$) be the first time that Process III is applied. Then after finitely many steps, say m steps, the machine will return to Process I. Furthermore, Y'_k is in standard form (otherwise Process III would not be applied) implies that any weight 0 (resp. $\frac{1}{2}$) band in Y'_{k+i} is a sub-band of a weight 0 (resp. $\frac{1}{2}$) band in Y'_k for all integers $i > 0$. As consequence Y_{k+m} contains no pre-weight 0 nor pre-weight $\frac{1}{2}$ band implies Y_{k+m+i} contains no pre-weight 0 nor pre-weight $\frac{1}{2}$ band for all $i > 0$. Thus, Process III won't be applied again. Let $N = k + m$, and we are done. □

6 Machine Output

6.1 Special case

For a pair of components $(Y_0 \overset{\iota_0}{\to} Y'_0)$, in relative Rips machine, eventually either only Process I is applied (Y'_0 is of thin type) or only Process II is applied (Y'_0 is of surface or toral type). In this section, for a fixed Y'_0, we examine possible outputs for the structure of Y_0, focusing on cases where Y'_0 is either thin or surface.

We will discuss some properties of pair of components in the next few lemmas.

Lemma 6.1. *Let $(Y_0 \overset{\iota_0}{\to} Y_0')$ be a pair of components where ι_0 is an immersion, then*

1. *If Y_0 is a toral component with rank $n > 2$, then so is Y_0'.*

2. *If Y_0' is a toral component and Y_0 is not a simplicial component, then Y_0 is also a toral component .*

Proof. (1) Y_0 is a toral component implies that there are infinitely many points in the limit graph[9] of Y_0 are of positive index which must also be true for Y_0', so Y_0' is toral.

(2) Y_0' is toral component implies its dual tree is a line, and therefore the minimal subtree corresponding to Y_0 is also a line. Thus Y_0 is a toral component. $\square$

Lemma 6.2. *Let $(Y_0 \overset{\iota_0}{\to} Y_0')$ be a pair of components where ι_0 is an immersion. If further $\iota_0 : Y_0 \to Y_0'$ is also a submersion, i.e. ι_0 is a local isometry, then $\iota_0 : Y_0 \to Y_0'$ is a covering map of finite degree. In particular, Y_0 and Y_0' are of the same type.*

Proof. It is clear that ι_0 is a finite covering map as ι_0 is a local isometry and Y_0 is a finite complex, see for example in [Hat02].

By Proposition 2.16 and Lemma 6.1, we have the following. If either one of Y_0 and Y_0' is a simplicial component, then every leaf in both Y_0 and Y_0' is compact and so both of them are simplicial. If either one of Y_0 and Y_0' is a toral component, then so is the other one. Moreover if any one of them is a surface component, then all but finitely many points in both $\Gamma_{Y_0'}$ and Γ_{Y_0} are

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[9]See definition 2.15.

of zero index, thus both of them are of surface type. If any one of Y_0 and Y_0' is a thin component, then limit graphs of both Y_0 and Y_0' are dense G_δ set, and so both are of thin type. □

Lemma 6.3. *Let $(Y_0 \overset{\iota_0}{\rightarrow} Y_0')$ be a pair of components. Suppose that Y_0' is a minimal component in standard form and Y_0 is a finite-sheeted cover of Y_0'. Further assume that Y_0 contains no pre-weight 0 nor pre-weight $\frac{1}{2}$ band. Then Y_0 is also in standard form.*

Proof. By lemma 6.2, Y_0 is also a minimal component and is of the same type as Y_0'. The assumption that Y_0 contains no pre-weight 0 band nor pre-weight $\frac{1}{2}$ band implies that every generalized band $\mathbf{B}_{Y_0}$ in Y_0 and its image $\mathbf{B}_{Y_0'}$ in Y_0' have the same weight. In particular, for any $q \in \Gamma_{Y_0}$ and its image $\iota_0(q) = p \in \Gamma_{Y_0'}$, $i_{Y_0}(q) = i_{Y_0'}(p)$, i.e. p and q have the same index. Since Y_0' is in standard form, by proposition 2.16, Y_0 is also in standard form. □

Corollary 6.4. *Let $(Y_0 \overset{\iota_0}{\rightarrow} Y_0')$ be a pair of components. Suppose that ι_0 is a local isometry up to weight 0 bands. Then there exists a finite-sheeted covering map $\iota_0' : Y_0^* \rightarrow Y_0'$ (up to subdividing some weight 0 bands) extending $\iota_0 : Y_0 \rightarrow Y_0'$. Moreover, we may create Y_0^* by attaching finitely many bands to Y_0 such that all of these attached bands are either weight 0 band or will become a weight 0 band if we apply Rips' machine to Y_0^*. In particular, Y_0 is a minimal component of the same type as Y_0'.*

Proof. We may assume that ι_0 is a partial covering map (i.e. images of weight 0 bands and pre-weight 0 bands of Y_0 in Y_0' are generalized bands, not proper sub-generalized bands). Otherwise, we may archive this by subdividing weight

0 bands in Y_0' and their preimages in Y_0 correspondingly. For a given weight 0 band $\mathbf{B}_{Y_0'}$ with base $b_{Y_0'}$ in Y_0', let $b_{Y_0}^1, \ldots, b_{Y_0}^n \subset \Gamma_{Y_0}$ be preimages of $b_{Y_0'}$ in the real graph of Y_0. Every component of the union of preimages of $\mathbf{B}_{Y_0'}$ in Y_0 is either a weight 0 band or a consecutive sequence of pre-weight 0 bands.

If ι_0 is in fact a local isometry, let $Y_0' = Y_0$ and we are done by lemma 6.2. Otherwise there exists some weight 0 band $\mathbf{B}_{Y_0'}$ in Y_0' such that ι_0 is not surjective near some $b_{Y_0}^k$. Since ι_0 is a partial covering map, there are only two possibilities. One is no band in Y_0 with base $b_{Y_0}^k$ maps to $\mathbf{B}_{Y_0'}$. We may fix this by adding a weight 0 band $\mathbf{B}_{Y_0}^k$ to Y_0 along $b_{Y_0}^k$ and defining ι_0^* to map $\mathbf{B}_{Y_0}^k$ onto $\mathbf{B}_{Y_0'}$. The other case is that there exits exactly one pre-weight 0 band $\mathbf{B}$ with base $b_{Y_0}^k$ mapping to $\mathbf{B}_{Y_0'}$. Then $\mathbf{B}$ must be one end of a consecutive sequence of pre-weight 0 bands. One end base of this sequence is $b_{Y_0}^k$ and the other end base must be one of $\{b_{Y_0}^1, \ldots, b_{Y_0}^n\}$, say it is $b_{Y_0}^{k'}$. We may fix this situation by attaching a band $\mathbf{B}'$ to Y_0 with one of its bases along $b_{Y_0}^k$ and the other one along $b_{Y_0}^{k'}$ (orient in the same direction) and defining ι_0^* to map $\mathbf{B}'$ also onto $\mathbf{B}_{Y_0'}$. Following above attaching rule, we may do this to each such $b_{Y_0}^k$ (for a fixed $\mathbf{B}_{Y_0'}$) and for every weight 0 band in Y_0'. Let the resulting union of bands be Y_0^*. Then $\iota_0^* : Y_0^* \to Y_0'$ is a local isometry by construction and so Y_0^* is a finite-sheeted covering space of Y_0'. Moreover in Y_0^*, we may alter each $\mathbf{B}'$ to a weight 0 band by sliding one of its bases across the consecutive sequence of pre-weight 0 bands containing $\mathbf{B}$ and call the resulting union of bands Y_0^{**}. Since Y_0 only differs from Y_0^{**} by weight 0 bands, Y_0 is the same type of component as Y_0^{**} is, and further the same as Y_0^* and Y_0'. □

In the proof of corollary 6.4, we start with a partial covering map $\iota_0 : Y_0 \to Y_0'$ (up to finitely many subdivisions), and extend it to a finite covering map. Mimicking the similar argument for graphs in [Sta83], the following lemma shows that this completing process can be done for any partial covering map between union of bands.

Lemma 6.5. *Let* $(Y_0 \overset{\iota_0}{\to} Y_0')$ *be a pair of components and* $\iota_0 : Y_0 \to Y_0'$ *be a partial covering map. Then by adding finitely many arcs to the real graph* Γ_{Y_0} *(using move* $(M7)$*) and attaching finitely many new bands to* Y_0 *(using move* $(M8)$*), we may extend* ι_0 *to a finite covering map. Moreover, assume* Y_0 *contains no pre-weight* $\frac{1}{2}$ *band (resp. pre-weight* 0*), then the new constructed* Y_0 *also contains no pre-weight* $\frac{1}{2}$ *(resp. pre-weight* 0*) band. In particular, on the level of fundamental group,* Y_0 *either is a finite index subgroup of* Y_0' *or is a free factor of a finite index subgroup of* Y_0'*.*

Proof. Γ_{Y_0} and $\Gamma_{Y_0'}$ contain only finitely many blocks. ι_0 is a finite covering map equivalent to ι_0 is a local isometry near every block. Y_0' and Y_0 can be viewed as graphs by considering each block as a vertex and each generalized band as an edge. ι_0 then can be viewed as an immersion between finite graphs with the property that maps each edge to exactly one edge as $\iota_0 : Y_0 \to Y_0'$ is a partial covering map.

The preimage of a block $K \subset \Gamma_{Y_0'}$ are finitely many blocks $K_1, \ldots, K_{n(K)}$ in Γ_{Y_0}. Different blocks may have different numbers of preimages ($n(K)$ depends on K). Let $n = \max\{n(K)\}_K$. For blocks in $\Gamma_{Y_0'}$, the number of whose preimages is less than n, add extra arcs (move $M7$) to $\Gamma_{Y_0'}$ to complete ι_0 as a covering map on the level of real graphs. Then we need to complete ι_0

for bands. Let $\mathbf{B}_{Y_0'} \subset Y_0'$ be a generalized band. If $b_{Y_0'}$ and $dual(b_{Y_0'})$ are contained in the same block, processed exactly as in Corollary 6.4. Otherwise, let the block containing $b_{Y_0'}$ be K, the block containing $dual(b_{Y_0'})$ be K' and $K \neq K'$. At each preimage K_i of K, there is either no band or exactly one band $\mathbf{B}_{Y_0}^i$ maps to $\mathbf{B}_{Y_0'}$ since ι_0 is an immersion. The band $\mathbf{B}_{Y_0}^i$ uniquely determines one preimage of K'. Thus each preimage of $\mathbf{B}_{Y_0'}$ in Y_0 groups one block in $\{K_1, \ldots, K_n\}$ with one block in $\{K_1', \ldots, K_n'\}$. After paring up like this, there are a same number of $\{K_i\}$ and $\{K_i'\}$ left unpaired. Pair these leftover up randomly. For each these new pair (K_i, K_j'), attaching a new band $\mathbf{B}_{Y_0}^* = b_{Y_0}^* \times I_N$ with the property that $m(b_{Y_0'}) = m(b_{Y_0}^*)$ and $l(\mathbf{B}_{Y_0'}) = N$ to Y_0 in the following way: glue $b_{Y_0}^*$ to $b_{Y_0'}$'s preimage in K_i and glue $dual(b_{Y_0}^*)$ to $dual(b_{Y_0'})$'s preimage in K_j'. Further extend ι_0 to $\mathbf{B}_{Y_0}^*$ by mapping it to $\mathbf{B}_{Y_0'}$. Do the above process for all generalized bands in Y_0'. It is easy to check that the resulting union of band is then a finite cover of Y_0'.

In the case that Y_0 contains no pre-weight $\frac{1}{2}$ (resp. pre-weight 0) band, the set of preimages of weight $\frac{1}{2}$ (resp. weight 0) bands in Y_0' contains only weight $\frac{1}{2}$ (resp. weight 0) bands in Y_0. So restricting on weight $\frac{1}{2}$ (resp. weight 0) bands of Y_0', Y_0 can be convert to a finite covering by adding only weight $\frac{1}{2}$ (resp. weight 0) bands. Thus the new constructed Y_0 also contains no pre-weight $\frac{1}{2}$ (resp. pre-weight 0) band. $\qquad\square$

In the proof of proposition 6.9, we will need to complete an almost partial cover to a cover. The following lemma shows that we may first complete an almost partial cover to a partial cover and then, using lemma 6.5, complete the partial cover to a finite cover.

Lemma 6.6. *Let $(Y_0 \overset{\iota_0}{\to} Y_0')$ be a pair of components and the immersion $\iota_0 : Y_0 \to Y_0'$ be an almost partial covering map. Then there exists a pair of components $(Y_0^* \overset{\iota_0^*}{\to} Y_0')$ such that ι_0^* is a partial covering map, $Y_0 \subset Y_0^*$ and $\iota_0^*|_{Y_0} = \iota_0$. In particular, Y_0^* can be constructed by adding finitely many arcs to the real graphs Γ_{Y_0} and attaching finitely many bands to Y_0. We call this extending ι_0 to a partial covering map.*

Proof. Let $\mathbf{B}_{Y_0}$ be a generalized band in Y_0, $\mathbf{B}_{Y_0'}$ be a generalized band in Y_0' and $\mathbf{B}_{Y_0}$ properly map into (not onto) $\mathbf{B}_{Y_0'}$. Further, let b_{Y_0} and $dual(b_{Y_0})$ be bases of $\mathbf{B}_{Y_0}$, $b_{Y_0'}$ and $dual(b_{Y_0'})$ be bases of $\mathbf{B}_{Y_0'}$. Then at least one of the endpoints of b_{Y_0}, denote it by q, maps to an interior point $p \in b_{Y_0'}$. ι_0 is an almost partial covering map implies that there is no other band with the property that its base(s) contained in the same block as b_{Y_0} or $dual(b_{Y_0})$ and it maps into $\mathbf{B}_{Y_0'}$. Otherwise that band must have some overlap with $\mathbf{B}_{Y_0}$ which contradicts to ι_0 is an immersion. Further we may assume that within the block containing b_{Y_0} or $dual(b_{Y_0})$, there is a segment $b_{Y_0}^*$ or $dual(b_{Y_0}^*)$ maps onto $b_{Y_0'}$ or $dual(b_{Y_0'})$. Otherwise, we may obtain $b_{Y_0}^*$ or $dual(b_{Y_0}^*)$ by adding new arcs to Γ_{Y_0} (γ' in figure 17). Then by attaching a new band to Y_0, $\mathbf{B}_{Y_0}$ can be extended to a new band whose bases are $b_{Y_0}^*$ and $dual(b_{Y_0}^*)$. Let the resulting band be $\mathbf{B}_{Y_0}^*$. We can then modify ι_0 to get ι_0^* such that it maps $\mathbf{B}_{Y_0}^*$ onto $\mathbf{B}_{Y_0'}$. Finally, since there are finitely many of generalized bands in Y_0, we may extend $(Y_0 \overset{\iota_0}{\to} Y_0')$ to $(Y_0^* \overset{\iota_0^*}{\to} Y_0')$ with the property that ι_0^* is a partial covering map within finite steps. $\square$

Now for a pair of unions of bands $(Y_0 \overset{\iota_0}{\to} Y_0')$, we will discuss the type of Y_0 when ι_0 is not a finite covering map. In lemma 6.7, we will show that if Y_0' is
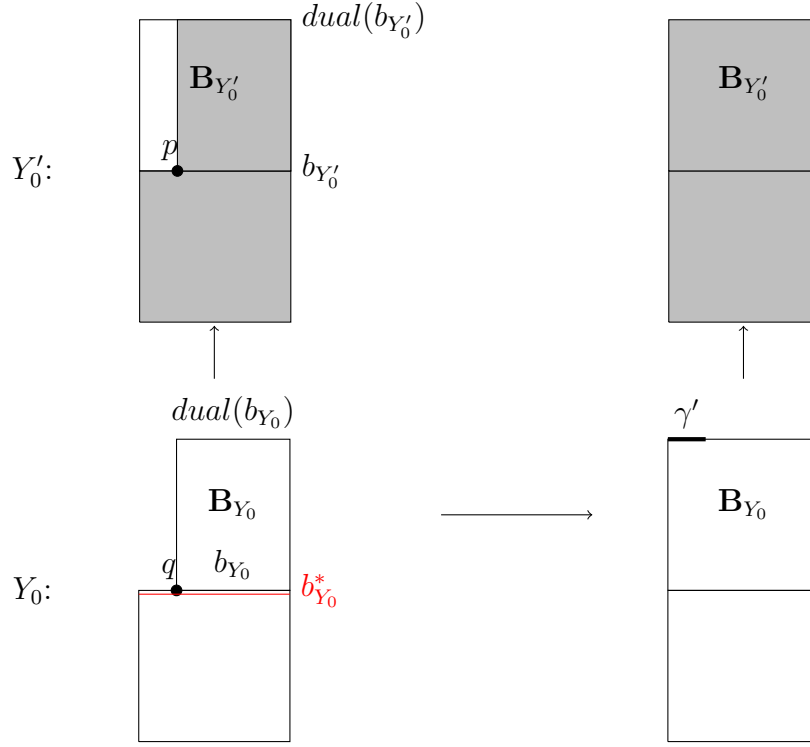
Figure 17: The gray part is $\widehat{Y}_0$ within Y_0'.

simplicial, then Y_0 must also be simplicial. In lemma 6.8, we will show that if Y_0' is a surface or thin component and relative Rips machine does not convert ι_0 into a map which is surjective (i.e. $\widehat{Y}_0 \subsetneq Y_0'$), then Y_0' must be simplicial. In proposition 6.9, we will discuss the general case where ι_0 is surjective but not locally surjective.

Lemma 6.7. *Let $(Y_0 \xrightarrow{\iota_0} Y_0')$ be a pair of components. Suppose Y_0' is a simplicial component, then Y_0 is also a simplicial component. In particular, for any pair of components $(Y_0 \xrightarrow{\iota_0} Y_0')$, if $\widehat{Y}_0 = \iota_0(Y_0)$ is simplicial, then so is Y_0.*

Proof. Y_0' is simplicial implies $\widehat{Y}_0$ is also simplicial. Suppose Y_0 is minimal, then every leaf in Y_0 is dense which implies its image in $\widehat{Y}_0$ is also dense. This

contradicts to $\widehat{Y}_0$ is simplicial. $\qquad\qquad\qquad\qquad\qquad\square$

Lemma 6.8. *Let $(Y_0 \overset{\iota_0}{\to} Y_0')$ be a pair of components and $\iota_0 : Y_0 \to Y_0'$ be an immersion. Further assume that Y_0' is in standard form and it is either of thin type or surface type. If there exists a point p in Y_0' with $i_{Y_0'}(p) \geq 0$ such that it has no pre-image in Y_0, then Y_0 is simplicial. In particular, the fundamental group of Y_0 is an infinite index subgroup of the fundamental group of Y_0'.*

Proof. Let $\widehat{Y}_0 = \iota_0(Y_0)$. By lemma 6.7, we only need to show that $\widehat{Y}_0$ is simplicial.

$\widehat{Y}_0$ is a band sub-complex (a closed set) in Y_0'. By the assumption, p is contained in $Y_0' - \widehat{Y}_0$. So there exists a neighborhood of p that is contained in $Y_0' - \widehat{Y}_0$. Without loss, we may assume that p is an interior point of some band in Y_0'. Hence, there exists a sub-generalized band $\mathbf{B}_{Y_0'}$ in Y_0' containing p which is disjoint from $\widehat{Y}_0$.

Assume that $\widehat{Y}_0$ is not simplicial, then by Lemma 6.1, $\widehat{Y}_0$ is either a surface component or a thin component. According to proposition 2.16, since Y_0' is in standard form, there is an uncountably many collection of leaves in $\overline{\widehat{Y}}_0$ ($\widehat{Y}_0$ omitting weight 0 bands) that are 2-ended trees which is quasi-isometry to lines [10]. Further in $\overline{Y'}_0$, leaves containing these leaves are also 2-ended trees expect for finitely many. Therefore we may pick a leaf l in $\overline{\widehat{Y}}_0$ which is a 2-ended tree such that the leaf l' in $\overline{Y'}_0$ containing it is also a 2-ended tree. So l' is also quasi isometry to a line and let the line be $\overline{l}'(= l' \cap \Omega_{Y_0'})$. $i_{Y_0'}(p) \geq 0$ implies that $\overline{l}'$ intersects $\mathbf{B}_{Y_0'}$ infinitely often. Hence, each component of $l' \cap \widehat{Y}_0$

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[10]In the case of surface, all but finitely many of leaves are q.i. to lines. See [BF95, section 8]

is either a finite tree or an 1-ended tree. However, this contradicts to our choice that l is a 2-ended tree. Thus $\widehat{Y}_0$ must be simplicial, and so is Y_0.

The immersion $\iota_0 : Y_0 \to Y_0'$ induces an monomorphism on the level of their fundamental groups. Since Y_0 is simplicial, the minimal translation length of elements in the fundamental group of Y_0 is bounded below by some positive number ϵ. On the other hand, Y_0' is a minimal component, so there exists sequence of elements in the fundamental group of Y_0' whose translation lengths converge to 0. Therefore, the fundamental group of Y_0 is an infinite index subgroup of Y_0'. $\square$

Proposition 6.9. *Let* $(Y_0 \overset{\iota_0}{\looparrowright} Y_0')$ *be a pair of components. Suppose that* Y_0' *is either a surface or thin component. Then either* Y_0 *is simplicial or for any sequence* $(Y_0 \overset{\iota_0}{\looparrowright} Y_0'), (Y_1 \overset{\iota_1}{\looparrowright} Y_1'), \ldots$ *formed by relative Rips machine,* $\iota_n : Y_n \to X_n$ *is a finite covering map up to weight* 0 *bands for all sufficiently large* n.

Proof. By Proposition 5.6, we may assume that Y_0' is in standard form, ι_0 is an almost partial covering map and Y_0 contains no pre-weight 0 nor pre-weight $\frac{1}{2}$ band. In particular, $\iota_0(\overline{Y}_0) \subset \overline{Y}'_0$. Therefore we may work with $(\overline{Y}_0, \overline{Y}'_0)$ instead. To simplify the notation, without loss, we will assume that both Y_0 and Y_0' do not contain any weight 0 band.

Firstly, $(Y_0 \overset{\iota_0}{\looparrowright} Y_0')$ can be extended to $(Y_0^* \overset{\iota_0^*}{\looparrowright} Y_0')$ such that ι_0^* is a partial covering map as in lemma 6.6. Then by lemma 6.5, ι_0^* can further be extended to a finite covering map $\widetilde{\iota}_0^* : \widetilde{Y}_0^* \to Y_0'$. Lemma 6.3 implies that $\widetilde{Y}_0^*$ is the same type of minimal component as Y_0' is and $\widetilde{Y}_0^*$ is also in standard form. By construction we have $Y_0 \hookrightarrow Y_0^* \hookrightarrow \widetilde{Y}_0^*$. We may assume that $Y_0 \subsetneq \widetilde{Y}_0^*$, or we

are done. If the image of Y_0 in $\widetilde{Y}_0^*$ misses any point in the limit set of $\widetilde{Y}_0^*$, by lemma 6.8, Y_0 is simplicial. Otherwise, each sub-generalized band $\mathbf{C} \in \widetilde{Y}_0^* - Y_0$ contains no point in the limit set of $\widetilde{Y}^*$ and so image of $\mathbf{C}$ in Y_0' must also contain no point in the limit set of Y_0'. Moreover, $\widetilde{Y}_0^*$ must be a thin type component since all but finitely many leaves of a surface component are in the limit set.

By construction, the following diagram commutes.

$$
\begin{array}{ccc}
Y_0' & & \\
\uparrow^{\iota_0} & \overset{\widetilde{\iota}_0^*}{\nwarrow} & \\
Y_0 \hookrightarrow & Y_0^* \hookrightarrow & \widetilde{Y}_0^*
\end{array}
$$

Consider three pairs of components $(Y_0 \overset{\iota_0}{\to} Y_0')$, $(Y_0 \hookrightarrow \widetilde{Y}_0^*)$ and $(\widetilde{Y}_0^* \overset{\widetilde{\iota}_0^*}{\to} Y_0')$. The procedure of obtaining $(Y_1 \overset{\iota_1}{\to} Y_1')$ from $(Y_0 \overset{\iota_0}{\to} Y_0')$ can be viewed as combination of obtaining $(\widetilde{Y}_1^* \overset{\widetilde{\iota}_1^*}{\to} Y_1')$ from $(\widetilde{Y}_0^* \overset{\widetilde{\iota}_0^*}{\to} Y_0')$ first, then obtaining $(Y_1 \hookrightarrow \widetilde{Y}_1^*)$ from $(Y_0 \hookrightarrow \widetilde{Y}_0^*)$.

For any given sequence $(Y_0 \overset{\iota_0}{\to} Y_0'), (Y_1 \overset{\iota_1}{\to} Y_1'), \dots$ formed by relative Rips machine, let $\widetilde{Y}_i^*$ be the induced intermediate finite cover of Y_i'. By above analysis, images of $\mathbf{C}$'s in Y_0' contains no point in the limit set and so fully collapse within finitely many steps (Proposition 3.4). As a consequence, $\mathbf{C}$'s in $\widetilde{Y}_0^*$ fully collapse within finitely many steps. Therefore $Y_n = \widetilde{Y}_n^*$ for sufficiently large n. Thus eventually Y_n is a finite-sheeted cover of Y_n'. $\qquad \square$

Following immediately from Proposition 5.6, Lemma 6.1, Lemma 6.2 and Proposition 6.9, we have:

Theorem 6.10. *Let $(Y \xrightarrow{\iota} Y')$ be a pair of unions of bands and $(Y^0 \xrightarrow{\iota^0} Y'^0) = (Y \xrightarrow{\iota} Y'), (Y^1 \xrightarrow{\iota^1} Y'^1), \ldots$ be a sequence of unions of bands formed by relative Rips machine. Then for each pair of components $(Y_0 \xrightarrow{\iota_0} Y'_0) \subset (Y \xrightarrow{\iota} Y')$ where Y'_0 is a minimal component, Y_0 is either a minimal component of the same type as Y'_0 or Y_0 is simplicial. Moreover, let the pair of components corresponding to $(Y_0 \xrightarrow{\iota_0} Y'_0)$ in $(Y^n \xrightarrow{\iota^n} Y'^n)$ be $(Y_n \xrightarrow{\iota_n} Y'_n)$. Then Y_0 and Y'_0 are both surface or thin components if and only if $\iota_n : \overline{Y}_n \to \overline{Y}'_n$ is a finitely covering map for sufficiently large n. In addition, for sufficiently large n, Y'^n is in standard form and every surface or thin component of Y^n is also in standard form.*

Remark 6.11. It is possible that both Y_0 and Y'_0 are of toral type but $\overline{Y}_n$ is not a finite cover of $\overline{Y}'_n$. For example, let Y_0 be a toral component that is dual to the action of $\mathbb{Z}^3 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ on a real line where $a \to 1, b \to e, c \to \pi$. Its infinite index subgroup $\langle a \rangle \times \langle b \rangle \simeq \mathbb{Z}^2$ also acts on the real line and is dual to a toral component.

Corollary 6.12. *Let $H < G$ be two finitely presented groups. Further let T_G be a G-tree with trivial edge stabilizers and $T_H \subset T_G$ be a minimal H-subtree[11]. Suppose that $(X \xrightarrow{\iota} X')$ is a pair of band complexes, that X and X' resolve T_H and T_G correspondingly, that Y and Y' are single minimal components of either surface or thin type and that $\pi_1(Y)$ generates H, $\pi_1(Y')$ generates G. Then $[G : H]$ is finite.*

Proof. Apply relative Rips machine to $(X \xrightarrow{\iota} X')$. Since Y is a surface or thin component, according to Theorem 6.10, the machine will eventually convert

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[11]i.e. T_H contains no proper H-subtrees.

$(X \xrightarrow{\iota} X')$ into a new pair $(X^* \xrightarrow{\iota} X'^*)$ such that $\overline{Y}^*$ is a finite cover of $\overline{Y'}^*$. By [BF95], T_G has trivial edge stabilizers implies that that Y^* and Y'^* contain no attaching annulus. Thus Y^* is a finite cover of Y'^*.

Claim that X^* is also a finite cover of X'^*. X^* and X'^* are obtained from Y^* and Y'^* by attaching 2-cells. If a 2-cell is attached along a loop l in Y^*, then there must be a corresponding 2-cell is attached along the image of l in Y'^*. Otherwise l is contained in the torsion of G. Moreover every loop in Y'^* has finitely many preimages in Y^*. If there is a 2-cell attached along some loop in Y'^*, there must be finitely many 2-cells attached along all its preimages in Y^* as H is a subgroup of G. Therefore, X^* is also a finite cover of X'^*, $G = \pi_1(X'^*)$, $H = \pi_1(X^*)$, and so $[G : H]$ is finite. $\qquad\square$

6.2   General case

In previous subsections, for a given pair of band complexes $(X \xrightarrow{\iota} X')$, we always assume that $(A6)$ ι is injective between components of Y and Y' to simplify our description. In fact, the assumption $(A6)$ is not necessary for relative Rips machine. Suppose there are several components $Y_0^1, \ldots, Y_0^k \subset Y$ map to the same component $Y_0' \subset Y'$. Relative Rips machine described above can then be applied to a $k + 1$-tuple $(Y_0^1, \ldots, Y_0^k, Y_0')$. Roughly speaking, in each step, we may apply proper move to Y_0' first, then correspondingly modify each Y_0^i.

In more details, if Y_0' contains some free subarc, "Process I" will be applied. Similarly as in subsection 5.1, let $\mathbf{J}_1^j, \ldots, \mathbf{J}_{n_j}^j$ be pre-images of $\mathbf{J}_{Y'} \cap \widehat{Y}_0^j$ in Y_0^j. Then collapse $\mathbf{J}_{Y'}$ in Y_0' and collapse $\mathbf{J}_1^j, \ldots, \mathbf{J}_{n_j}^j$ in Y_0^j for $j = 1, \ldots, k$

to obtain $(Y_1^1, \ldots, Y_1^k, Y_1')$. If Y_0' contains no free subarc, "Process II" will be applied. Y_1' is obtained by applying Process II of the original Rips machine to Y_0'. Each Y_1^j is obtained by modifying Y_0^j as described in subsection 5.2 (considering each $(Y_0^j \overset{\iota_0^j}{\to} Y_0')_j$ as a pair). Therefore, the sequence $(Y_0^1, \ldots, Y_0^k, Y_0'), (Y_1^1, \ldots, Y_1^k, Y_1'), \ldots$ produced by Process I and Process II will eventually be *stabilized* by proposition 5.2. Now for a stabilized $k + 1$-tuple $(Y_i^1, \ldots, Y_i^k, Y_i')$, if none of Y_i^j contains pre-weight 0 band, continue with either Process I or Process II (whichever appropriate). Otherwise, apply Process III to $(Y_i^1, \ldots, Y_i^k, Y_i')$ (considering each $(Y_i^j \overset{\iota_i^j}{\to} Y_i')_j$ as a pair) to obtain $(Y_{i+1}^1, \ldots, Y_{i+1}^k, Y_{i+1}')$. Then by Proposition 5.6, Process III appears only finitely many times.

Then by exactly the same arguments, all results proved in subsection 6.1 hold for pair of band complexes $(X \overset{\iota}{\to} X')$ without assuming $(A5)$ that the restriction of ι on the level of union of bands $\iota : Y \to Y'$ is an immersion.

7 Application

In this section, we will need a partial order on the band complexes.

Definition 7.1. A morphism $\iota : X \to X'$ induces a cellar map $\phi : \Delta(X) \to \Delta(X')$ between their GD's. We say that a vertex of $\Delta(X)$ **changes types under** the map ϕ, if the vertex and its image have different types. For example, a simplicial component maps into a minimal component.

For a vertex v of $\Delta(X)$, write $\pi(v)$ for the conjugacy class of the subgroup of $\pi(X)$ generated by the corresponding union of bands. We say v **gets bigger**

if the stabilizer of the image of v strictly contains the image of the stabilizer of v, i.e. $\phi(\pi(v)) \subset \pi(\phi(v))$. For example, a surface component maps onto another surface component as a 2-sheeted cover.

Definition 7.2. For band complexes X and X', we write $X' > X$ if there is a morphism $\iota : X \to X'$ such that:

1. $\Delta(X)$ has more vertices than $\Delta(X')$; or

2. $\Delta(X)$ and $\Delta(X')$ have the same number of vertices and under the morphism either a vertex changes type or a surface or thin vertex gets bigger.

Lemma 7.3. *Let X_i's be band complexes. Then all sequences $X_1 < X_2 < \ldots$ is finite.*

Proof. It is enough to consider the case of a sequence $Y_0 < Y_1 < Y_2 < \ldots$ where each Y_i is a union of bands of a single surface or thin component and each $\iota_i : Y_i \to Y_{i+1}$ is a morphism. View each $(Y_i \overset{\iota_i}{\to} Y_{i+1})$ as a pair of components. According to theorem 6.10, up omitting weight 0 bands, Y_i is a finitely covering space of Y_{i+1}. In the case of surface component, the length of a sequence of finitely sheeted covering spaces is bounded by the Euler characteristic of the corresponding surface of Y_1, also see in for example [Rey11]. In the case of thin type, each Y_i can be viewed as a graph in the following fashion. View each island in Y_i as a vertex and each long band as an edge. Then the length if the sequence $Y_0 < Y_1 < Y_2 < \ldots$ is bounded by the Euler characteristic of the graph corresponding to Y_1. $\qquad\square$

Throughout the rest of this section, let $H < G$ be two finitely generated groups, $\mathbb{F}$ be a fixed non-abelian free group, $h_i : H \to \mathbb{F}$ be a sequence of

homomorphisms and $g_i : G \to \mathbb{F}$ be a sequence of extensions of h_i's. Moreover, let $\mathcal{B}$ be a fixed basis for $\mathbb{F}$, $\mathcal{H} \subset \mathcal{G}$ be fixed finite generating sets for $H < G$. With some further assumptions (described below), we may construct a pair of band complexes in the following way.

Each h_i can be viewed as an action of H on the Cayley graph of $\mathbb{F}$ via h_i, denote the corresponding H-tree by T_{h_i}. Up to passing to a subsequence, there exists $T_H = \lim T_{h_i}$ in the projective space of H-trees ([Sel97], also in [Gui08]). Similarly each g_i can be realized as a G-tree , denoted by T_{g_i}, with a limiting G-tree $T_G = \lim T_{g_i}$ (up to sub-sequence). Up to passing to proper quotient, we may assume that T_H is a faithful H-tree and T_G is a faithful G-tree. Further assume that H is not elliptic in T_G, then up to scaling, T_H is the minimal H-subtree in T_G.

Let X_H be a finite complex with fundamental group H and X_G be a finite complex with fundamental group G containing X_H. Each h_i is represented by a resolution $r_i : \widetilde{X}_H \to T_{h_i}$ which induces a band complex structure on X_H, denote it by X_H^i. Each leaf of X_H^i is transversely labeled by an element of $\mathcal{B}$. For each $\alpha \in \mathcal{H}$, fix a curve $l_\alpha \subset X_H$ with $\pi_1(l_\alpha) = \alpha \in H$. We may arrange r_i such that it is tight on every l_α (map lifts of l_α to axises of $[\alpha]$ in T_{h_i}), and so the value of $h_i(\alpha)$ can be read off by following l_α and keeping track of curves it intersects. In this case we say that r_i is **exact** on $\mathcal{H}$. Let $r : \widetilde{X}_H \to T_H$ be the limiting resolution of r_i's, denote the induced band complex structure on X_H by X_H^∞.

We may then extend r and r_i to resolutions $\hat{r} : \widetilde{X}_G \to T_G$ and $\hat{r}_i : \widetilde{X}_G \to T_{g_i}$ correspondingly. Similarly, denote the band complex structure on X_G induced

by $\hat{r}_i$ by X_G^i and the band complex structure on X_G induced by $\hat{r}$ by X_G^∞. By attaching extra bands and extra 2-cells, we may arrange $\hat{r}_i$ to be exact on $\mathcal{G}$. $(X_H^\infty \hookrightarrow X_G^\infty)$ is then a pair of band complexes.

Definition 7.4. Suppose $\{h_i : H \to \mathbb{F}\}$ converges to T_H equipped with a resolution $r : \widetilde{X}_H \to T_H$ as above. A sequence $h_i' : H' \to \mathbb{F}$ converging to $T_{H'}$ together with a resolution $\widetilde{X}_{H'} \to T_{H'}$ is an *enlargement* if:

- $H < H'$

- $h_i'|_H = h_i$;

- $T_{H'} := \lim T_{h_i'}$ is a faithful H'-tree;

- H is not elliptic in $T_{H'}$;

- $X_{H'} > X_H$.

Proposition 7.5. *Let $(X_H^\infty \hookrightarrow X_G^\infty)$ be a pair of band complexes constructed as above and $\iota : X_H^\infty \to X_G^\infty$ be the induced morphism. Suppose that X_G^∞ contains either a surface or thin component, denoted by Y_G. Then either*

1. *$\iota^{-1}(Y_G)$ is empty; or*

2. *$\iota^{-1}(Y_G)$ is a single component in X_H^∞, denote it by Y_H with the property that relative Rips machine will convert $\iota : Y_H \to Y_G$ into a homeomorphism; or*

3. *we may find an enlargement of X_H^∞.*

Proof. Suppose that $\iota^{-1}(Y_G)$ is not empty. By theorem 6.10, every component of $\iota^{-1}(Y_G)$ is either a minimal component of the same type as Y_G or is simplicial. Moreover, we may assume Y_G and its preimages do not contain weight 0 bands as T_H and T_G are limiting trees admitted *super stable* and *very small* actions [BF09]. So relative Rips machine will convert every minimal component of $\iota^{-1}(Y_G)$ into a finitely covering space for Y_G.

Assume that the second bullet in the proposition fails. Then either there are more than one choices for Y_H. Or for a fixed choice of Y_H, Y_H is a simplicial component or $\iota : Y_H \to Y_G$ is a n-sheeted covering map ($n > 1$). Now we will show that in either case, we may enlarge X_H^∞.

Firstly, suppose that $\iota^{-1}(Y_G)$ contains more than one component. Then besides Y_H, there is another component $Y_H^* \in X_H^\infty$ also maps into Y_G. Intuitively, we will enlarge X_H^∞ by gluing these two components together. It will be done in the following way. Let J be an arc in the real graph of Y_G with positive transverse measure. By picking J small enough, we may assume that J has finitely many pre-images $J_1 \ldots J_n$ in Y_H and finitely many pre-images $J_1^* \ldots J_{n^*}^*$ in Y_H^* with the property that ι maps each J_i and $J_{i^*}^*$ homeomorphically onto J. Attach a band $B = J \times I$ to X_H^∞ via a measure-preserving map $J \times \{0\} \to J_m$ and $J \times \{1\} \to J_{m^*}^*$ for some $m \in \{1, \ldots, n\}, m^* \in \{1, \ldots, n^*\}$[12]. Let the resulting complex be $X_{H'}$ where H' is its fundamental group and $X_{H'}^\infty$ be the induced band complex with the new component $Y_{H'}$ formed by the union of Y_H, Y_H^* and B. In particular, $X_{H'}^\infty > X_H^\infty$ since $\Delta(X_{H'}^\infty)$ has less vertices than $\Delta(X_H^\infty)$. By construction, for large i, X_H^i have similar band complex structure

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[12]We may add one band for each pair of segments in $\{J_1 \ldots J_n, J_1^* \ldots J_{n^*}^*\}$ to get a bigger enlargement

as X_H^∞. So by adding an extra band B as for X_H^∞, we may build $X_{H'}^i$. We may extend $\mathcal{H}$ to a generating set $\mathcal{H}' \subset \mathcal{G}$ for H'. For $\forall i$, let the map obtained by reading off interactions between l_α and $X_{H'}^i$, for all $\alpha \in \mathcal{H}'$, be $h_i' : H' \to \mathbb{F}$. Then h_i' is an extension of h_i. Finally, replace H' by $\lim h_i'$ and add on extra 2-cells to $X_{H'}$ if necessary. We now have an enlargement of X_H^∞.

Secondly, suppose there is a pair of components (Y_H, Y_G) where Y_H is either a simplicial component or a finite covering space of Y_G. Intuitively, we will enlarge X_H^∞ by replacing its component Y_H by Y_G. It proceeds as follows. Let $J \subset \Gamma_G$ be an small arc as in the previous case and $J' \subset \Gamma_H$ be one of its preimages in Y_H. Since Y_G is either a surface or thin component, there is a band complex structure for Y_G whose union of bands is the result of glueing finitely many bands to J via their bases. First let $X_{H'}$ be the result of glueing these new bands to J' in Y_H where H' is the fundamental group of this new complex and $X_{H'}^\infty$ be the induced band complex with the new component $Y_{H'}$ formed by Y_H and new bands glued to J'. By construction, for large i, X_G^i have similar band complex structure as X_G^∞. Thus cutting the component of X_G^i containing J open along J, we will obtain a band complex structure similarly as in Y_G (may be a little off towards boundaries of bands). By glueing these bands to X_H^i, we may build $X_{H'}^i$. Proceed as in the previous case, we will obtain h_i'. Again replace H' by $\lim h_i'$ and add on extra 2-cells to $X_{H'}$ if necessary. Now to show $X_{H'}^\infty$ is an enlargement, we only need to check that $X_{H'}^\infty > X_H^\infty$. Indeed, either the vertex corresponding to Y_H changes type or gets bigger (up to moves, Y_H is a proper cover of Y_G whereas $Y_{H'}$ is homeomorphic to Y_G). $\qquad\square$

According to lemma 7.3 and proposition 7.5, for a given pair of $(X_H^\infty \overset{\iota}{\to} X_G^\infty)$, up to finitely many enlargement, ι induces an isomorphism between surface and thin components in X_H^∞ and $\iota(X_H^\infty)$.

This enlarging technique can be used inductively in understanding questions arising from model theory such as solving equations over groups, extension problems and decision problems through a geometric point of view.

References

[Bes02]   Mladen Bestvina, $\mathbb{R}$-*trees in topology, geometry, and group theory*, Handbook of geometric topology, North-Holland, Amsterdam, 2002, pp. 55–91. MR 1886668

[BF91]   Mladen Bestvina and Mark Feighn, *Bounding the complexity of simplicial group actions on trees*, Invent. Math. **103** (1991), no. 3, 449–469. MR 1091614 (92c:20044)

[BF95]   M. Bestvina and M. Feighn, *Stable actions of groups on real trees*, Invent. Math. **121** (1995), no. 2, 287–321. MR 1346208 (96h:20056)

[BF09]   Mladen Bestvina and Mark Feighn, *Notes on Sela's work: limit groups and Makanin-Razborov diagrams*, Geometric and cohomological methods in group theory, London Math. Soc. Lecture Note Ser., vol. 358, Cambridge Univ. Press, Cambridge, 2009, pp. 1–29. MR 2605174 (2011h:20087)

[GLP94] D. Gaboriau, G. Levitt, and F. Paulin, *Pseudogroups of isometries of **R** and Rips' theorem on free actions on **R**-trees*, Israel J. Math. **87** (1994), no. 1-3, 403–428. MR 1286836

[Gui98] Vincent Guirardel, *Approximations of stable actions on **R**-trees*, Comment. Math. Helv. **73** (1998), no. 1, 89–121. MR 1610591

[Gui00] ———, *Reading small actions of a one-ended hyperbolic group on **R**-trees from its JSJ splitting*, Amer. J. Math. **122** (2000), no. 4, 667–688. MR 1771568

[Gui08] ———, *Actions of finitely generated groups on ℝ-trees*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 1, 159–211. MR 2401220

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 1867354

[KM06] Olga Kharlampovich and Alexei Myasnikov, *Elementary theory of free non-abelian groups*, J. Algebra **302** (2006), no. 2, 451–552. MR 2293770 (2008e:20033)

[MS84] John W. Morgan and Peter B. Shalen, *Valuations, trees, and degenerations of hyperbolic structures. I*, Ann. of Math. (2) **120** (1984), no. 3, 401–476. MR 769158

[Rey11] Patrick Reynolds, *On indecomposable trees in the boundary of outer space*, Geom. Dedicata **153** (2011), 59–71. MR 2819663

[Sel97] Z. Sela, *Acylindrical accessibility for groups*, Invent. Math. **129** (1997), no. 3, 527–565. MR 1465334

[Sel01] Zlil Sela, *Diophantine geometry over groups. I. Makanin-Razborov diagrams*, Publ. Math. Inst. Hautes Études Sci. (2001), no. 93, 31–105. MR 1863735

[Sel03] Z. Sela, *Diophantine geometry over groups. II. Completions, closures and formal solutions*, Israel J. Math. **134** (2003), 173–254. MR 1972179 (2004g:20061)

[Sel04] ———, *Diophantine geometry over groups. IV. An iterative procedure for validation of a sentence*, Israel J. Math. **143** (2004), 1–130. MR 2106978 (2006j:20059)

[Sel05a] ———, *Diophantine geometry over groups. III. Rigid and solid solutions*, Israel J. Math. **147** (2005), 1–73. MR 2166355 (2006j:20060)

[Sel05b] ———, *Diophantine geometry over groups. V_1. Quantifier elimination. I*, Israel J. Math. **150** (2005), 1–197. MR 2249582 (2007k:20088)

[Sel06a] ———, *Diophantine geometry over groups. V_2. Quantifier elimination. II*, Geom. Funct. Anal. **16** (2006), no. 3, 537–706. MR 2238944 (2007k:20089)

[Sel06b] ———, *Diophantine geometry over groups. VI. The elementary theory of a free group*, Geom. Funct. Anal. **16** (2006), no. 3, 707–730. MR 2238945 (2007j:20063)

[Sta83]   John R. Stallings, *Topology of finite graphs*, Invent. Math. **71** (1983), no. 3, 551–565. MR 695906 (85m:05037a)