Comparison of the Discrete-Ordinates Method and the Finite-Volume Method for Steady-State and Ultrafast Radiative Transfer Analysis in Cylindrical Coordinates


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Comparison of Discrete-Ordinates Method and Finite Volume Method for Steady-State and Ultrafast Radiative Transfer Analysis in Cylindrical Coordinates

Brian Hunter and Zhixiong Guo

Department of Mechanical and Aerospace Engineering,
Rutgers, The State University of New Jersey,
98 Brett Road, Piscataway, NJ 08854

Running Head: Comparison of DOM and FVM

1 Author to whom correspondence should be addressed, zguo@rci.rutgers.edu
Abstract

The time-dependent equation of radiative transfer is solved for an axisymmetric cylindrical medium using both the discrete-ordinates method and the finite volume method. Steady and transient flux profiles are determined for absorbing and scattering media. Results for each solution method are compared and shown for various grid numbers, scattering albedos and optical thicknesses. A comparison of computational time and memory usage between the methods is presented. It is found that the finite volume method uses more memory and has a longer convergence time than the discrete-ordinates method for all cases due to the difference in angular treatment.
Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$c$</td>
<td>Speed of light in medium</td>
</tr>
<tr>
<td>$D_i^m$</td>
<td>Directional weight at face $i$ in direction $m$</td>
</tr>
<tr>
<td>$I$</td>
<td>Radiative intensity ($W/m^2sr$)</td>
</tr>
<tr>
<td>$I_b$</td>
<td>Blackbody emissive power, $= \sigma T^4/\pi$ ($W/m^2sr$)</td>
</tr>
<tr>
<td>$M$</td>
<td>Total number of directions, $= (N_\phi \times N_\theta)$</td>
</tr>
<tr>
<td>$N_r, N_z, N_\phi, N_\theta$</td>
<td>Number of divisions in each direction</td>
</tr>
<tr>
<td>$\hat{n}$</td>
<td>Surface outward unit normal vector</td>
</tr>
<tr>
<td>$Q_r, Q_z$</td>
<td>Heat flux at radial side wall, axial end wall</td>
</tr>
<tr>
<td>$r$</td>
<td>Position vector, $= r\hat{e}<em>r + \phi \hat{e}</em>\phi + z\hat{e}_z$</td>
</tr>
<tr>
<td>$r$</td>
<td>Radial location, (m)</td>
</tr>
<tr>
<td>$\hat{s}$</td>
<td>Unit direction vector, $= \mu \hat{e}<em>r + \eta \hat{e}</em>\phi + \xi \hat{e}_z$</td>
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<tr>
<td>$S$</td>
<td>Source function, Eq. (2)</td>
</tr>
<tr>
<td>$T$</td>
<td>Temperature ($K$)</td>
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<td>$t$</td>
<td>Time ($s$)</td>
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<tr>
<td>$w$</td>
<td>Discrete direction weight, DOM</td>
</tr>
<tr>
<td>$z$</td>
<td>Axial location, (m)</td>
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</tbody>
</table>

Greek Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
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<tr>
<td>$\alpha_i^m$</td>
<td>Discretization equation coefficient at node $i$ in direction $m$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Extinction coefficient, $= \sigma_a + \sigma_s (m^{-1})$</td>
</tr>
<tr>
<td>$\Delta A, \Delta V$</td>
<td>Surface area and volume of CV, ($m^2, m^3$)</td>
</tr>
<tr>
<td>$\Delta \Omega$</td>
<td>Discrete solid angle ($sr$)</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>Emissivity</td>
</tr>
</tbody>
</table>
Φ

Scattering phase function

\( \Phi \)

Average scattering phase function

\( \phi \)

Azimuthal angle measured from \( \bar{e}_r \), see Figure 1

\( \phi_0 \)

Spatial azimuthal angle measured from x-axis, see Figure 1

\( \phi_r \)

Angular azimuthal angle measured from \( x' \)-axis, see Figure 1

\( \rho \)

Reflectivity

\( \sigma \)

Stefan-Boltzmann constant, = 5.67 * 10^{-8} (W/m^2K^4)

\( \sigma_a \)

Absorption coefficient (m^{-1})

\( \sigma_s \)

Scattering coefficient (m^{-1})

\( \theta \)

Polar angle measured from \( \bar{e}_z \), see Figure 1

\( \omega \)

Scattering albedo, = \( \sigma_s / \beta \)

Subscripts

\( b \)

Blackbody

\( med \)

Medium

\( N, S, E, W, T, B \)

Neighboring control volume nodal centers

\( n, s, e, w, t, b \)

Faces of control volume

\( P \)

Control volume nodal center

\( w \)

Boundary wall

Superscripts

\( m0 \)

Previous iteration

\( m, m' \)

Discrete radiation directions

\( m^+, m^- \)

Edges of discrete solid angle
1. Introduction

For many applications, ranging from determining the thermal performance of boilers and furnaces to modeling laser ablation of cancerous cells in skin tissue, the solution of the Equation of Radiative Transfer (ERT) in an axisymmetric cylindrical enclosure is necessary. There are many methods to solve for the intensity field in such an enclosure. Two of the most common methods, due to their relative simplicity and elegance, are the Discrete-Ordinates Method (DOM) and the Finite Volume Method (FVM).

The DOM was first introduced by Carlson and Lathrop [1] in 1968 in their discussion on neutron transport. Fiveland expanded the theory for use in predicting radiative heat transfer in three-dimensional enclosures with isotropic and anisotropic scattering [2,3]. Menguc and Viskatan [4] and Jamaluddin and Smith [5] provided detailed solutions for radiative transfer in an axisymmetric, finite cylindrical enclosure to predict behavior in furnaces and combustion chambers. Jendoubi et al. [6,7] completed the framework of radiative transfer in an anisotropically scattering medium. More recently, Guo and co-authors expanded the DOM to solve the time-dependent ERT for ultrafast laser applications [8,9], such as laser tissue welding and soldering [10]. Other works by Guo and co-authors have combined the DOM for radiative transfer with Penne’s bio-heat equation for applications involving laser-skin tissue interactions for the removal of cancerous tumors via laser ablation [11,12].

First used mainly for predicting convective heat transfer, the FVM was proposed for simulation of radiative transfer by Chui and Raithby [13]. They also introduced a novel mapping scheme to ease the prediction of radiative transfer in axisymmetric cylindrical media. Chai et al. [14,15] expanded the method to consider both two- and three-dimensional enclosures, and examined the treatment of irregular geometries. Kim and Baek [16] further validated the method for use in both axisymmetric and non-axisymmetric cylindrical enclosures, and also applied the method to solutions with non-gray gas for combustion applications [17]. Murthy and Mathur [18] also expanded the method for solution of radiative transfer problems with unstructured grid meshes.

Although many studies have been completed using either of the methods, few have looked at a detailed comparison of the two. Comparisons between the two methods can be found in few steady-state cases, but comparisons for transient results in which the ERT is time-dependent have not been...
reported. Computational times have been reported in a limited fashion (for limited scattering albedos, optical thicknesses and grid numbers), and there presently are no comparisons of computational memory usage to our knowledge.

In this study, radiative transfer in an axisymmetric cylindrical enclosure is predicted by the discrete-ordinates and finite volume methods. Results for both methods are compared to exact results in a purely absorbing medium for validation. The medium is assumed isotropic. If the scattering is anisotropic, then the isotropic scaling rule [19] can be applied. Steady-state and transient heat flux profiles are analyzed and compared for various optical thicknesses and scattering albedos. Committed memory usage for each method is calculated for various angular quadratures and spatial grid numbers. The effects of scattering albedo, grid number, and optical thickness on computational convergence time for each method are determined. The ratio of computational time between the two methods is compared to previous literature results.

2. Mathematical Model

2.1. Governing Equations

2.1.1. Equation of Radiative Transfer

For a gray, absorbing-emitting and scattering medium, the time-dependent ERT has the following form [8,9,20]:

\[ \frac{1}{c} \frac{\partial I(r, s)}{\partial t} + \frac{\partial I(r, s)}{\partial s} = -\beta I(r, s) + S(r, s) \]  

(1)

where the source function \( S(r, s) \) is

\[ S(r, s) = \sigma_a I_b(r, s) + \frac{\sigma_s}{4\pi} \int_{4\pi} I(r, s') \Phi(s', s) \, d\Omega' \]  

(2)

In this work, Eq. (1) is applied to the cylindrical enclosure shown in Figure 1. The vectors \( r \) and \( s \) are the position and unit direction vectors locating the radiative intensity, with direction cosines \( \mu = \sin \theta \cos \phi, \eta = \sin \theta \sin \phi, \) and \( \xi = \cos \theta. \)

The ERT for an axisymmetric cylindrical medium can be expressed as [10]

\[ \frac{1}{c} \frac{\partial I}{\partial t} + \frac{\mu}{r} \frac{\partial}{\partial r} [rl] - \frac{1}{r} \frac{\partial}{\partial \phi} [\eta l] + \frac{\xi}{\partial z} = -\beta I + S \]  

(3)
2.1.2. Boundary and Initial Conditions

For the cylindrical geometry being analyzed, the radial and end walls are assumed to be diffuse emitters and reflectors. The intensity leaving these walls, by definition, is then independent of direction [20]. For a diffusely emitting and reflecting wall, the intensity at a given point \( r_w \) on that surface is given by [8,9]:

\[
I(\mathbf{r}_w, \mathbf{s}) = \epsilon(\mathbf{r}_w)I_b(\mathbf{r}_w) + \frac{\rho(\mathbf{r}_w)}{\pi} \int_{\hat{n} \cdot \mathbf{s}' < 0} I(\mathbf{r}_w, \mathbf{s}') |\hat{n} \cdot \mathbf{s}'| d\Omega'
\]  

(4)

where \( \hat{n} \) is the outward normal vector at the surface, and the quantity \( \hat{n} \cdot \mathbf{s}' \) is the cosine of the angle between the surface normal \( \hat{n} \) and an arbitrary incoming direction \( \mathbf{s}' \). The first component of the boundary condition accounts for blackbody emission of the surface, while the second component accounts for reflection of incoming intensities. Finally, the centerline of the radial geometry \( (r = 0) \) is specified as an axisymmetric boundary condition [10, 11].

An initial condition must be specified to solve for the ultrafast radiative transfer equation. In general, this condition will be given as a known temperature or intensity field. If the temporal term in the radiative transfer equation is neglected, the initial condition will serve as an initial guess for an iterative solution procedure.

2.2. Solution Methods

2.2.1. Discrete-Ordinates Method

In the DOM, the ERT is solved for a number of discrete directions, replacing all integrals by quadrature sums [2, 3]. Eq. (3) can be written, for a discrete direction \( m \), in the following form [10-12]

\[
\frac{1}{c} \frac{\partial I^m}{\partial t} + \frac{\mu^m}{r} \frac{\partial}{\partial r} [r I^m] - \frac{1}{r} \frac{\partial}{\partial \phi} [\eta^m I^m] + \xi^m \frac{\partial I^m}{\partial z} + \beta I^m = \beta S^m, \quad m = 1, 2, ..., M
\]

(5)

The source term \( S^m \) can also be expressed as a quadrature sum:

\[
S^m = (1 - \omega)I_b + \frac{\omega}{4\pi} \sum_{m' = 1}^{M} w^{m'} \Phi^{m'm} I^{m'}, \quad m = 1, 2, ..., M
\]

(6)

The boundary condition given by Eq. (4) can be written for the various boundary walls. For example, the boundary condition at the radial side-wall is

\[
I_w^m = \epsilon_w I_{bw} + \frac{\rho_w}{\pi} \sum_{m', \mu^m > 0} w^{m'} I^{m'} |\mu^m|, \quad \mu^m < 0
\]

(7)
Conditions for the other walls can be obtained by manipulation of the direction cosines in the previous equation.

Once the intensity field is obtained by solving the governing equation, the net radiative heat fluxes at the radial side wall and axial end walls can be obtained as follows

\[ Q_r = \sum_{m=1}^{M} \mu_w^m I^m, \quad Q_z = \sum_{m=1}^{M} \xi_w^m I^m \]  

These summations are over all directions \( m \), accounting for heat flux entering and leaving the surface.

### 2.2.2 Finite Volume Method

Using a control-volume approach, Eq. (3) is integrated over an arbitrary control volume, described by volume \( \Delta V \) and solid angle \( \Delta \Omega \) [21]. Following this integration, and application of Gauss’s theorem [14], Eq. (3) can be written, for a discrete direction \( m \), as

\[
\frac{1}{c} \frac{\partial I^m}{\partial t} \int_{\Delta \Omega^m \Delta V} (I^m) dV d\Omega + \int_{\Delta \Omega^m \Delta A} I^m (\hat{s}^m \cdot \hat{n}) dA d\Omega = \int_{\Delta \Omega^m \Delta V} \beta (-I^m + S^m) dV d\Omega
\]  

where \( \hat{n} \) is the outward normal vector for a given surface. Using the important assumption that the intensity over a given control volume and solid angle remains constant [21], the integrals can be replaced by summations. Eq. (9) can therefore be simplified, giving the following form:

\[
\frac{1}{c} \frac{\partial I^m}{\partial t} \Delta V \Delta \Omega^m + \sum_{i=1}^{6} I_i^m \Delta A_i D_i^m = (-\beta I^m + \beta S^m) \Delta V \Delta \Omega^m
\]  

where the summation on the left hand side is carried out over the six faces of the control volume. The typical control volume used for this analysis is shown in Figure 2. The discrete solid angle \( \Delta \Omega^m \) is defined as

\[
\Delta \Omega^m = \int_{\Delta \Omega^m} d\Omega^m = \int_{\phi^m+}^{\phi^m-} \int_{\theta^m-}^{\theta^m+} \sin \theta \ d\theta \ d\phi = (\cos \theta^m- - \cos \theta^m+)(\phi^m+ - \phi^m-)
\]  

where \( \phi^m+, \phi^m-, \theta^m+, \) and \( \theta^m- \) are the azimuthal and polar angles defining the edges of the discrete solid angle, as shown in Figure 3. The source term is defined as

\[
S^m = (1 - \omega)I_b + \frac{\omega}{4\pi} \sum_{m'=1}^{M} \Phi^{m'm} I^{m'} \Delta \Omega^{m'}, \quad m = 1, 2, \ldots, M
\]  

where \( \Phi^{m'm} \) is the average scattering phase function from direction \( m' \) to direction \( m \). The directional weight at control volume face \( i \) in direction \( m \), represented by the symbol \( D_i^m \) in Eq. (10), can be calculated for each face using the following integral expansion:
\[ D_i^m = \int_{\phi_{m-}}^{\phi_{m+}} \int_{\theta_{m-}}^{\theta_{m+}} (\mathbf{s}^m \cdot \mathbf{n}) \sin \theta \, d\theta \, d\phi \]  

(13)

As an example, for the “north” face in the control volume shown in Figure 2 \((i = n)\), the normal vector is \(\mathbf{n}_n = \mathbf{e}_r\), and thus the dot product \(\mathbf{s}^m \cdot \mathbf{n}_n = \sin \theta \cos \phi\). Thus, the north directional weight is

\[ D_n^m = (\sin \phi_{m+} - \sin \phi_{m-}) \cdot \left[ \frac{1}{2} (\theta_{m+} - \theta_{m-}) - \frac{1}{4} (\sin 2\theta_{m+} - \sin 2\theta_{m-}) \right] \]  

(14)

Calculations for all six faces are presented in detail by Kim [17].

A special consideration is needed when describing the scattering phase function for this method [21]. In order to satisfy all moment constraints, the discrete solid angles must be subdivided into smaller sub-angles. The average scattering phase function from a solid angle \(\Delta \Omega^m\) into a second solid angle \(\Delta \Omega^m\) can be accurately calculated as

\[ \Phi^{m'm} = \frac{1}{\Delta \Omega^m \Delta \Omega^m} \sum_{m_d=1}^{M_d} \sum_{m_d'=1}^{M_d'} \Phi^{m_d'm_d} \Delta \Omega^{m_d} \Delta \Omega^{m_d} \]  

(15)

where \(M_d\) and \(M_d'\) are the number of subdivisions in each discrete solid angle.

Similarly to the DOM, once the intensity field is found by solution of the governing equation, the total radiant heat fluxes at the radial side wall and axial end wall can be calculated using [13]

\[ Q_r = \sum_{m=1}^{M} I^m D^m_n, \quad Q_z = \sum_{m=1}^{M} I^m D^m_t \]  

(16)

Again, these summations are calculated over all directions \(m\), accounting for both incoming and outgoing fluxes.

### 2.3. Numerical Scheme

#### 2.3.1. Discrete Ordinates Method

The ERT is solved using the transient discrete-ordinates method (TDOM). In this treatise, attention was focused on the \(S_{16}\) (288 discrete ordinates) quadrature, although multiple quadrature schemes \((S_4, S_6, S_{10}, S_{12}, S_{14})\) were used to determine the effect of the scheme on CPU time and committed memory. The step scheme [20] was used to relate the control volume facial intensities to nodal intensities. For more details on the discretization of the governing equations and the numerical scheme used, please refer to previous publications by Guo and co-authors [8-11].
2.3.2. Finite Volume Method

To obtain the discretized equation for the FVM, Eq. (3) is expanded over the control volume in Figure 2. Analyzing the summations over the faces \( i = n, s, e, w, t, b \) shown in Figure 2, the discretized form of Eq. (3) becomes

\[
\frac{I_p^m - I_p^{m0}}{c\Delta t} \Delta V \Delta \Omega^m + \sum_{i=n,s,e,w,t,b} I_i^m \Delta A_i D_i^m = -\beta I_p^m \Delta V \Delta \Omega^m + \beta S_p^m \Delta V \Delta \Omega^m
\]  

(17)

where the partial derivative in time has been evaluated using a forward-differencing technique with \( I_p^{m0} \) as the nodal intensity from the previous time step [10], the subscript \( P \) represents the control volume center, \( \Delta A_i \) and \( \Delta V \) are the facial area and control volume, respectively.

Eq. (17) is more readily solved if it is written in terms of neighboring nodal intensities \( i = N, S, E, W, T, B \) instead of facial intensities \( i = n, s, e, w, t, b \). As outlined in Modest [20], there are many schemes that can be used to transform this equation into the desired form. In this treatment, the simple step scheme was used (as in the DOM, for consistency). This scheme will ensure positive intensities [17,20]. The following replacements are made for this scheme in Eq. (17)

\[
I_i^m D_i^m = I_p^m \max(D_i^m, 0) - I_i^m \max(D_i^m, 0)
\]  

(18)

The faces \( i = e, w \) get a special treatment. In the step scheme, the west facial intensity is a known quantity. This is due to the sweeping scheme, where the west face on the current control volume is equal to the east face on the previous control volume. The east facial intensity is unknown, and is approximated in the step scheme as the intensity at the center of the current control volume. Using these approximations, Eq. (17) can be rewritten in the desired form

\[
\alpha_p^m I_p^m = \sum_{i=N,S,T,B} \alpha_i^m I_i^m - \Delta A_w D_w^m I_p^{m-1} + \beta S_p^m \Delta V \Delta \Omega^m + \frac{\Delta V \Delta \Omega^m}{c\Delta t} I_p^{m0}
\]  

(19)

where the superscript \( m - 1 \) designates that the nodal intensity be taken from the previous control volume in the sweep; and the alpha factors and source term are calculated as
\[\alpha_P^m = \frac{\Delta V \Delta \Omega^m}{c \Delta t} + \sum_{i=n,s,t,b} \max(\Delta A_i D_i^m, 0) + \beta \Delta V \Delta \Omega^m + \Delta A_e D_e^m \]
\[a_i^m = \max(-\Delta A_i D_i^m, 0) \]
\[S_P^m = (1 - \omega) I_b + \frac{\omega}{4\pi} \sum_{m'=1}^M \Phi_{m'm} f_{m'}^m 0 \Delta \Omega^{m'} \]

The above equations, along with the necessary boundary and initial conditions, provide the framework for the solution of the ERT with the FVM.

As previously mentioned, for an axisymmetric geometry, the intensities are invariant with the spatial azimuthal angle \(\phi_0\). Thus, by fixing a value of \(\phi_0\), the intensities corresponding to the vector \(\hat{S}\) can be obtained by simply solving the discretized equations at different values of \(\phi_r\). However, as discovered by Chui and Raithby [13], this method introduces a lack of conservation and unphysical directional coupling. To avoid this, they introduced a simple mapping solution. Instead of fixing \(\phi_0\), the value of \(\phi_r\) for all intensity vectors was set to zero. The directional intensities were then calculated by solving the discretized equations at various values of \(\phi_0\).

The cylindrical geometry was divided into \((N_r \times N_z)\) equally spaced control volumes. The total solid angle of \(4\pi\) was divided into \((N_\phi \times N_\theta)\) equally spaced directions, with \(\Delta \theta = \pi/N_\theta\) and \(\Delta \phi = 2\pi/N_\phi\). The intensities for all points were solved by sweeping in the \(\theta\) and \(\phi\) directions. There are four possible sweep directions, corresponding to the values of the directional weights in the radial and axial directions. For example, if \(D_n^m\) and \(D_i^m\) are both positive, the solution procedure will sweep from the radial centerline out to the radial side wall, and from the bottom end wall to the top end wall \((r = 0 \rightarrow R, z = 0 \rightarrow 2H)\). Finally, the intensity distribution in the medium is symmetric about the axis \(y = 0\).

This means that only the control volumes where \(y \geq 0\) need be solved for [13].

### 2.3.3. Computational Time and Committed Memory

The computer used for the calculations is a Dell Optiplex 780, with an Intel 2 Dual Core 3.16 GHz processor, and 4.0 GB of RAM. The computation time for each case was analyzed using a built-in time monitor in FORTRAN, and the committed memory was verified through both the Performance Manager in Windows, and by checking the image size through the FORTRAN interface. For all cases, the methods were run in ideal conditions, with only other essential processes running. Due to the power of the processors, a fine computational grid of at least \((N_r \times N_z) = (150 \times 150)\) was used when comparing CPU times and memory.
In order to have accurate comparisons, an equal number of directions were used to solve the ERT in both methods. For example, if the DOM was solved with the $S_{16}$ quadrature, the FVM was solved with 288 directions (16 divisions in the azimuthal direction, and 18 divisions in the polar direction).

3. Results and Discussion

3.1. Purely Absorbing Medium

The first problem considered is that of a purely absorbing cylindrical enclosure. The cylinder has radius $R = 1$ m and height $2H = 2$ m. All boundary walls are cold ($T = 0$ K) and black ($\epsilon_W = 1$). The medium is prescribed a fixed temperature of $T_{med} = 100$ K. The non-dimensional radial heat flux at the radial side wall $q^* = Q_R/\sigma T^4_{med}$ is calculated for three different extinction coefficients. The steady-state results for both the FVM and DOM are compared with the readily-available exact solution [17] for validation. The spatial grid used for this computation was $(N_r \times N_z) = (150 \times 150)$. Several sets of grid were considered and satisfactory convergent results were obtained. To minimize computational time for the steady-state case, the time derivative was neglected in the solution of the ERT.

Figure 4 shows the steady-state axial variation of non-dimensional radial heat flux for three extinction coefficients ($\beta = 5.0, 1.0, 0.1$ m$^{-1}$). The present solutions for both methods are found to be very accurate when compared with the exact solution. The maximum percentage difference between the exact solution and our schemes occurs at the midplane for the case where $\beta = 1.0$ m$^{-1}$. The maximum error for the DOM $S_{16}$ scheme is 2.14%, whereas the FVM with 288 directions has a maximum error of 2.89%. For all three extinction coefficients, the DOM solution is slightly more accurate than the FVM solution with an equivalent number of directions. The average percentage difference between the DOM and FVM is 0.44%, 0.81%, and 0.28% for $\beta = 5.0, 1.0$, and 0.1 m$^{-1}$, respectively.

High-order quadrature (> $S_{16}$) is not available in the literature yet because the choice of a scheme must satisfy three moment equations in the discrete-ordinates method. However, the increase of angular directions in the FVM is not limited. As a test, the above problem was also solved using the FVM with 440 directions (equivalent to the DOM $S_{20}$ quadrature). The inlay in Figure 4 shows that, for $\beta = 1.0$ m$^{-1}$, the FVM with 440 directions more closely predicts to the exact solution than the FVM with 288 directions, but still underpredicts the previous DOM $S_{16}$ solution by 0.22%. The difference in scheme accuracy is very small for this case, but may arise from the differences in direction choices between the two schemes. The DOM uses weighting factors for each direction, meaning that certain
directions have more influence than others, while the present FVM has an equal weight for all directions.

Transient results for the pure absorption medium case are shown in Figure 5. Guo and Kumar [8] showed that the time step used has to satisfy the inequality:

$$c\Delta t \leq \min (\Delta x, \Delta y)$$  \hspace{1cm} (21)

This is to ensure the traveling distance between two time steps does not exceed the spatial control volume size. For the given case, the refractive index of the medium was taken as 1.4, meaning that the speed of light in the medium can be calculated as 0.214 m/ns.

The spatial grid for the transient results was taken to be \((N_r \times N_z) = (40 \times 40)\). The time step was taken to be \(\Delta t = 6.6\) ps, in order to fully capture the transient results. The axial distribution of non-dimensional heat flux at the radial side wall for \(\beta = 5.0\) m\(^{-1}\) is plotted for different values of non-dimensional time \(t^* = ct/L\). As \(t^*\) increases, the effects of the cold walls can be visualized as sharp flux decreases at the end wall boundaries. At \(t^* = 5.0\), the steady-state solution presented in Figure 4 is attained. For all non-dimensional times, the solution found via the FVM underpredicts the solution found with the DOM, as in the steady-state case.

3.2. Scattering Medium

The second test problem is a scattering cylindrical medium of the same size as in the absorbing case \((R = 1\) m, \(2H = 2\) m). All walls are considered as black, and the axial end walls are cold. The radial side wall is hot, with emissive power \(E_{bw} = 1\). The medium temperature is kept cold throughout. The scattering is scaled to be isotropic for all cases [19]. The non-dimensional radial heat flux at the side wall is defined as \(-q^* = -Q_R/E_{bw}\), where the negative sign is taken to normalize the flux range from 0 to 1.

Figure 6 shows the steady-state non-dimensional flux for various scattering albedos and various extinction coefficients. The spatial grid for the steady-state comparison is \((N_r \times N_z) = (150 \times 150)\). As the value of albedo increases, the wall heat flux decreases due to more radiative energy being scattered away from the side wall. The normalized heat flux is a minimum at the midplane \((z = 1)\), where the effects of the cold end walls are the least significant. The FVM and DOM produce consistent results. There does not seem to be a significant effect of scattering albedo on the accuracy/consistency of the results between the two schemes.
For a purely scattering medium, the extinction coefficient is varied to determine the effect of optical thickness on the two schemes; and the heat flux leaving the radial side wall decreases as extinction coefficient (and thus optical thickness) increases, as expected. For an optically thick medium ($\beta = 5.0 \text{ m}^{-1}$), there is an almost exact correspondence between the solutions predicted by the FVM and DOM. The percentage difference between the two schemes at the midplane ($z = 1$) is 0.06%. As the extinction coefficient decreases, deviations in the flux profiles start to appear. For the optically thin medium ($\beta = 0.1 \text{ m}^{-1}$), physically unrealistic bumps in the flux profile appear for both schemes due to ray effect [22] that occurs whenever a discrete number of directions are used to approximate a continuous angular variation. Chai et al. [22] found that ray effect is more pronounced for optically thin media, as seen in Figure 6.

Ray effect does not occur consistently between the two methods. In some regions, the FVM overpredicts the DOM solution by as much as 7.81%, but in other regions, it underpredicts by as much as 4.20%. The difference in the manifestation of the ray effect comes about from the choice of directions. In the current FVM scheme, a uniformly spaced angular grid is used, meaning that there is a constant azimuthal and polar spacing between given directions. This is not the case in the DOM, however, as the quadrature is chosen to satisfy given moment equations and will not necessarily be uniform. This difference in angular discretization leads to the discrepancies in the manifestation of the ray effect.

Transient profiles for all three extinction coefficients are shown in Figure 7. The spatial grid is $(N_x \times N_y) = (40 \times 40)$, and the time step is $\Delta t = 0.083 \text{ ns}$. The transient results, at long times, once again match the steady-state results given in the previous figures. For an optically thick medium, the FVM and DOM solutions correspond more closely at all non-dimensional times. As the optical thickness decreases, ray effect again begins to manifest in all transient profiles. We notice the ray effect is still prevalent, even though the grid size has been changed. This is consistent with literature, as ray effect is independent of the spatial discretization [22].

Figure 8 shows transient flux profiles for an absorbing-scattering medium, with $\omega = 0.5$. The heat fluxes are larger than in the case for pure scattering due to the medium absorbing more energy from the hot side wall. As the dimensionless time increases, the two methods both converge to steady-state solutions. Again, a slight difference is seen in the profiles for the two different methods. However
the difference still seems to result more from the manifestation of ray effect than the difference in scattering albedo, as discussed previously for the steady-state solutions.

3.3. Computational Time and Memory Usage

The efficiency of the FVM and DOM are compared by analyzing the computational time and memory usage of each method. In order to have accurate committed memory comparisons, only variables and arrays that were necessary for each method were included. Double-precision was used for all real numbers. The committed memory of the program (or image size) is calculated for various spatial and angular quadratures.

Figure 9 shows the variation of committed memory usage with angular discretization. As the number of directions increases, the memory usage of each method also increases. This is due to needing an increased array size to store all of the intensities and heat fluxes. For all angular quadratures, the FVM requires more memory than the DOM. The memory ratio between FVM and DOM is plotted in Figure 9. As the angular quadrature increases, the ratio between the FVM and DOM memory usage increases in a logarithmic fashion, with the FVM using nearly double the memory when compared to the DOM for the $S_{16}$ case. This memory difference arises mainly due to the angular discretization. The angular derivative, in the DOM, is approximated using angular differencing coefficients [1], as described by Jendoubi et al.[6,7]. This effectively allows for the problem to be solved as a two-dimensional problem. However, the angular derivative in the FVM is calculated using neighboring control volumes in the azimuthal direction. Thus, intensities need to be stored for three dimensions (corresponding to the six faces of the control volume).

Figure 10 compares the committed memory usage and convergence time to a steady-state solution vs. spatial discretization. Memory and time values are plotted for 288 angular directions ($S_{16}$), and for purely scattering media with extinction coefficient of $\beta = 1.0 \text{ m}^{-1}$. The spatial grid was varied from $(N_r \times N_z) = (150 \times 150)$ to $(N_r \times N_z) = (400 \times 400)$ in order to tax the computational resources of the current workstation. Convergence times are measured by solving the governing equations until a steady-state solution is reached. The committed memory values, for different angular and spatial grids, are listed in Table 1.

For both methods, the committed memory increases as the spatial grid number increases. However, we do not see the logarithmic behavior in the ratio between the FVM and DOM memory usage. As shown in Table 1, for the $S_{16}$ scheme, the ratio between the FVM and DOM stays at a
constant 1.90 regardless of the spatial discretization. This result is similar for any grid size, while keeping the angular discretization scheme similar. The constant ratio between the two methods with varying spatial grid number validates the hypothesis presented earlier that the main influence on memory difference is the angular discretization technique.

The convergence times to steady-state are also plotted in Figure 10 versus spatial grid number. A parabolic increase is seen in both methods as the number of grid points is increased. When comparing the two methods, it is seen that the ratio of convergence times between the two methods is roughly constant. The FVM/DOM time ratio is 1.09 for \((N_r \times N_z) = (150 \times 150)\), and 1.07 for \((N_r \times N_z) = (400 \times 400)\). This shows that the effects of spatial grid size are similar on both methods.

Figure 11 shows the effect of scattering albedo and extinction coefficient on computational time. Values are tabulated for various albedos and extinction coefficients in Table 2. The time to reach steady-state increases as scattering albedo increases (from pure absorption to pure scattering). This is due to the source term in the governing equation. As the extinction coefficient increases, a similar result is seen. Solution time for an optically thick medium is much larger than for an optically thin medium, as it takes longer for the radiant energy to propagate.

The ratio between the computational times for both methods is also displayed in Figure 11. Regardless of scattering albedo and extinction coefficient, the ratio was nearly constant in all cases, with a value between 1.10 and 1.15. These values are not consistent with previous literature results. Kim and Baek [16] predicted that as scattering albedo increased, the ratio of computation time between FVM and DOM increased to about 1.5. Chai et al. [14] calculated the ratio between FVM and DOM to be a maximum of 2.0 for isotropic scattering using the \(S_6\) quadrature. The discrepancy in these results can be attributed to dramatic increases in computational processing power and speed. The FVM does require almost double the memory in certain cases, but the invention of high speed Dual Core processors allows for more computationally intensive processes to be solved relatively quickly.

4. Conclusions

The equation of radiative transfer has been solved for an axisymmetric cylindrical medium, using both the discrete-ordinates method and the finite volume method. Steady-state and transient flux profiles are examined for a pure absorbing medium and an isotropic scattering medium. The variations of non-dimensional radial heat flux with scattering albedo and extinction coefficient are calculated. The
committed memory usage and computational time for each method are compared. The following conclusions are drawn from this study:

(1) For a purely absorbing medium, both methods predict closely to the exact solution. The FVM tends to underpredict the DOM for the same number of directions. Increasing the number of directions in the FVM reduces the error between the two methods. The results are consistent for both steady-state and transient cases.

(2) Variations in scattering albedo affect the magnitude of the heat flux, but do not produce noticeable variations between the two solution methods. The two methods produce nearly identical heat flux profiles for optically thick media. Ray effect is evident in both methods for optically thin media. Ray effect has a different impact on each method, due to the choice of discrete directions, which produces different flux profiles. Ray effect exists in both the steady-state and transient flux profiles.

(3) The memory usage for both methods increases greatly as the angular quadrature is increased. The memory ratio between the FVM and DOM increases in a logarithmic fashion with angular quadrature. Although an increase in memory is seen with the increase of spatial grid number, the ratio between the two methods is nearly constant with refinement of the spatial grid. Therefore, the difference in memory between the two methods is mainly impacted by the treatment of the angular derivative in the ERT. For all grid numbers, the FVM uses more memory than the DOM.

(4) The computational time for both methods increases with increasing extinction coefficient (optical thickness) and increasing scattering albedo. The computational time for the FVM is greater than the DOM. The ratio between the two methods, regardless of albedo, extinction coefficient, or spatial and angular grid, is between 1.10 and 1.15, much lower than the literature predictions of 1.5 – 2. The increase in computational power and processor speed accounts for this discrepancy.

(5) For an axisymmetric cylindrical geometry, both methods produce similar intensity and heat flux solutions. The DOM is more computationally efficient, solving in a faster time and with a lower total memory usage for these simple cases.
References


### Table 1: CPU Committed Memory (MB) vs. Spatial Grid Number and Quadrature Scheme

<table>
<thead>
<tr>
<th>Quadrature Scheme</th>
<th>FVM (Nφ x Nθ)</th>
<th>Domain (Sφ)</th>
<th>Grid = 150 x 150</th>
<th>Grid = 200 x 200</th>
<th>Grid = 300 x 300</th>
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<td></td>
<td></td>
<td>FVM</td>
<td>DOM</td>
<td>Ratio</td>
<td>FVM</td>
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### Table 2: CPU Times (seconds) for Various Extinction Coefficients and Scattering Albedos

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<th>β (m⁻¹)</th>
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<table>
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<td>7.53</td>
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