STUDENT COMPREHENSION OF MATHEMATICS THROUGH ASTRONOMY

By

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Student Comprehension of Mathematics Through Astronomy

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The purpose of this study is to examine how knowledge of astronomy can enhance college-level learning situations involving mathematics. The fundamental symbiosis between mathematics and astronomy was established early in the 17th century when Johannes Kepler deduced the 3 basic laws of planetary motion. This mutually harmonious relationship between these sciences has been reinforced repeatedly in history. In the early 20th century, for example, astronomer Arthur Eddington used photographic evidence from a 1919 solar eclipse to verify Einstein’s mathematical theory of relativity. This study was conducted in 5 undergraduate mathematics classes over the course of 2 years. An introductory course in ordinary differential equations, taught in Spring Semester 2013, involved 4 students. A similar course in Spring Semester 2014 involved 6 students, a Summer Semester 2014 Calculus II course involved 2 students, and a Summer 2015 Astronomy course involved 8 students. The students were asked to use Kepler’s astronomical evidence to deduce mathematical laws normally encountered on an undergraduate level. They were also asked to examine the elementary mathematical aspects involved in a theoretical trajectory to the planet Neptune. The summer astronomy class was asked to draw mathematical conclusions about large numbers from the recent discoveries concerning the dwarf planet Pluto. The evidence consists primarily of videotaped PowerPoint presentations conducted by the students in both differential equations classes, along with interviews and tests given in all the classes. All presentations were transcribed and examined to determine the effect of astronomy as a generator of student understanding of mathematics. An analysis of the data indicated two findings: definite student interest in a subject previously unknown to most of them and a desire to make the mathematical connection to celestial phenomena.

Preface

When I was 12 years old, I dragged my parents out on our front lawn to observe the annual Perseid meteor shower. It was a warm August night, and the midnight sky was clear. We lived in suburban Long Island, and the moon was new; the conditions were ideal for viewing, but there were no meteors! A half-hour passed, and my pajama-clad parents were becoming impatient. Suddenly, a burst of 15 to 20 meteors filled the night sky. My parents were fascinated, but the event merely reaffirmed my life-long love of astronomy. Years later, when I had to choose a college major and subsequent career path, I pursued the more practical option of mathematics. This led to a lifetime of teaching mathematics on the high school and college level. However, I have never lost my love of astronomy.

In the early 17th century, Johannes Kepler discovered the elliptical nature of planetary motion. In doing so, Kepler used mathematics to construct the bridge from ancient to modern astronomy. Indeed, modern astronomy and mathematics are almost genetically intertwined. From a teaching perspective, it would seem logical and obvious to ask a simple question: Could examinations of phenomena such as a meteor shower, a solar eclipse, or even a distant dwarf planet result in a greater student appreciation of the mathematics involved in such phenomena? As a college math teacher (and, more recently, an astronomy teacher), I am in a position to explore this question. The following project is the culmination of several years of work and presentations conducted by various undergraduate classes at Centenary College, New Jersey. I believe that once the evidence of my work is revealed, the results will be illuminating.

Dedication

To Donna

Acknowledgements

I could not have completed this dissertation without the help and support of many individuals. First and foremost is my wife Donna. Without her love, patience, and motivating efforts, this project might have been forever stalled.

I would also like to give special thanks to my mentor, Dr. Keith Weber. The word patience is particularly applicable in his case. Keith has suffered through several readings of this manuscript; each reading resulted in a meticulous and profound review on his part. His long-suffering guidance has been invaluable.

I have taught mathematics at Centenary College, New Jersey, since 1993. My colleagues, who are also my friends, have offered a great deal of help and encouragement over the years. I would like to acknowledge the administrative support and blessing offered by the following individuals: Dr. Thomas Brunner, Dr. Sandra Caravella, Provost Dr. James Patterson, and President Dr. Barbara Lewthwaite. Several teaching colleagues have also been a great help in the particulars involved in writing this manuscript: Dr. Lynn Taylor, Dr. Simon Saba, and Professor Kathy Turrisi. Also, three members of the Taylor Memorial Library staff, Wendi Blewett, Rachel Ceddia, and Steve Macmillan, provided invaluable help with the various mechanics of Microsoft Word and APA formatting. I cannot thank you all enough.

I would be quite remiss if I did not recognize the efforts of the students involved in the five classes that participated in this project. Hopefully, this manuscript will prove that their work was not in vain! Finally, I would like to thank my constant writing companion Josie. Although she usually sleeps under my desk, her timely barking has broken me free of many attacks of writer's block!

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Chapter 1: Introduction

Mathematics can be a difficult concept for students who do not appreciate its wonders and intricacies. Although mathematics is universally noted as the language of the sciences, students often regard the subject as a necessary evil. Astronomy is similar to mathematics, in that it is an acquired taste. For every student who finds this subject fascinating, there are others who regard it as merely a chapter in the earth science textbook. Even the *Journal for Research in Mathematical Education* has published few articles relevant to a possible educational symbiosis between the two topics. However, mathematics and astronomy are inextricably linked in history. After publishing his theory of special relativity in 1905, Albert Einstein struggled for years to incorporate gravity into his relativity framework. In 1916, he presented a paper to the Prussian Academy of Science in which he detailed what are now known as the Einstein equations. These equations formed the core of his general theory of relativity. This theory went unheeded until it was famously verified by astronomer Arthur Eddington in 1919 (Douglas, 1957).

Eddington travelled to Africa to photograph a rare total solar eclipse. He also photographed stars the night before and contrasted them with the same stars photographed when they were visible during the eclipse. He pinpointed the stars in the photographs and used a planetarium effect in a display to several renowned scientists. The shift in the star positions dramatically illustrated the effect of gravity on light. His display made Einstein globally famous, and the general theory of relativity subsequently revolutionized 20th century science. For educators who look upon astronomy with wonder, it is clear that mathematics can only enhance their interest. Indeed, mathematics can be viewed as the key to the comprehension of traveling vast distances, describing alien environments, and understanding the complex mechanics of rocket propulsion.

Purpose

The purpose of this study was to examine students' learning and comprehension of mathematics through an exploration of astronomy. This study is a journey of learning similar to the Voyager project and pays homage to our ancestors who relied upon celestial movement to govern the course of their daily lives.

Our immediate astronomical neighborhood is, of course, the group of planets that comprise the solar system. The motion of the five planets visible to the naked eye stymied humankind for thousands of years. It was not until the early 17th century that Johannes Kepler mathematically deduced the elliptical nature of the orbits. Although it took Kepler 4 years to determine the nature of the orbit of Mars, the dynamics of elliptic motion are known to any student studying calculus or even upper-level high school algebra. Given relevant information on any planet, considerable insight can be gained by calculating an ellipse that would fit its orbital parameters. Johannes Kepler deduced the same result when he discovered the laws of planetary motion in the 17th century (Burton, 2011).

Neptune is the eighth most distant planet in our solar system. Its nearly circular elliptical orbit has a radius of approximately 30 AU (one astronomical unit [AU] is about 93 million miles). It was the first planet whose existence was predicted before its discovery. In the 19th century, Carl Freidrich Gauss speculated the existence of another body beyond the recently discovered seventh planet, Uranus. Specifically, he used Kepler's laws of planetary motion to account for abnormalities in the orbit of Uranus (Burton, 2011). Neptune was eventually discovered by various astronomers in 1846. Named after the Roman god of the sea, Neptune has a distinct bluish tint due to the heavy amounts of methane in its atmosphere. It is one of the four gas giants with a mass roughly equivalent to four Earth masses. A day is about 10 hours and a year is 165 Earth years. Neptune has one principal natural satellite, Triton, which was named, appropriately enough, after one of the mythical god Neptune’s offspring. Not much was known about either body until August 1989, when the NASA space probe Voyager II made a historic flyby before departing the solar system into deep space. Among many other revelations, Voyager measured the violence of Neptune’s atmosphere, including storms with winds approaching 2100 km/hr. It also revealed that Triton has a measurable atmosphere, a surface with active geysers spewing liquid nitrogen, and the coldest recorded temperature in the solar system: 40 degrees above absolute zero. These discoveries led to speculation that Triton is actually an object from the Kuiper Belt (a system of asteroids beyond Pluto’s orbit) captured by Neptune’s intense gravitational field. All of these unexpected discoveries have scientists and astronomers continuing to envision a more sophisticated version of the Voyager probe that will answer new questions about the nature of the Neptune system.

For nearly 10 years, the Cassini probe, a joint venture by NASA and the European Space Agency, has been orbiting the neighborhood of the sixth planet, Saturn, and its myriad moons. Known as the most successful space probe in history, Cassini has made several astonishing discoveries, including warm-water active geysers on the tiny moon Enceladus, chaotic behavior within Saturn’s rings, and lakes of liquid methane on the principal moon, Titan. In effect, the probe has become a permanent satellite in the Saturn system. With a sophisticated, Earth-guided propulsion system, the probe has made several flybys of the ring system and the nine major moons and will continue its mission through 2017. This successful tour has further fueled scientists’ hope for a similar probe to the planet Neptune, with a similar mission: to explore the planetary system with several flybys of the moon Triton. The journey would take 15 years, and the subsequent exploratory mission would take 3 years.

With respect to a Cassini-type probe to the planet Neptune, the relevant celestial mechanics involve extremely complex mathematical models. In fact, a graduate student at Delft University published a thesis report on this subject (Melman, 2007). Melman explained the probe trajectory through graduate-level mathematical modeling. This study, however, involved a teaching experiment to exploit the links between astronomy and calculus in an introductory differential equations classroom. For this undergraduate-level study, students were required to have an understanding of basic mathematic principles learned in calculus and a basic understanding of the solar system.

Research Articles of Question

Can applying fundamental astronomy knowledge enhance learning situations involving ordinary differential equations (ODE) or elementary calculus? This theory was explored by participating undergraduate students, who were challenged with analyzing the nature of the orbits of the planets Neptune and Mars. This was accomplished in two different subject matters (ODE and calculus) and considered two different orbital trajectories (planning a theoretical journey of an unmanned space probe to the planet Neptune and recreating Kepler’s discoveries of the orbit of Mars). The impetus to this learning is to demonstrate how an understanding of various concepts in astronomy, ODE, and calculus can lead to a meaningful connection that generates a learning outcome of a greater appreciation of both astronomy and mathematics. The study used actual data from the Voyager probes of the 1970s and 1980s, as model guides. Specifically, students were asked to calculate the orbital trajectories of Neptune and Mars. To calculate these trajectories, students independently learned basic astronomical and calculus principles that would apply to the exercises.

The learning outcome for this study was the students' ability to solve problems normally covered in calculus and ODE that are relevant to solar system astronomy. The following research articles of question were explored to substantiate this study:

1. What is the level of student understanding of astronomy?
2. What evidence is there that students, either individually or as teams, use astronomy as a tool for a better understanding of mathematics?
3. Are visual displays of astronomy (i.e., PowerPoint slides) conducive to a greater understanding of mathematics?
4. Is there any evidence that astronomy can be used as a device leading toward a better understanding of difficult mathematical concepts (i.e., arc length, ellipses)?
    a. What calculus and ODE algorithms were applied to the tasks of determining planetary motion and orbital length?
5. Is there any evidence that exposure to the history of astronomy, and its connection to mathematics, is conducive to a greater student appreciation of mathematics?

The types of data used to gather evidence needed to answer each question, along with an analysis to support conclusions, can be categorized with respect to each question:

1. The level of student familiarity with astronomy was examined with tests given to the Spring Semester 2014 Differential Equations class. The evidence gathered included test results, interview transcripts, and subsequent presentations. Analysis of the mean test scores indicated initial unfamiliarity with the subject matter. The later presentations were used to confirm or deny subsequent gained knowledge.
2. Student visual taped presentations in 2013 and 2014 were used as data to explore the use of astronomy as a catalyst toward a deeper understanding of mathematics. The

primary evidence consisted of written transcripts of the student visual tapes. The visual evidence displayed the extent of student recognition of mathematical foundations of astronomical phenomena (e.g., the orbit of Mars is an ellipse).

3. PowerPoint displays were used in the presentations given in 2013 and 2014. The instructor also showed PowerPoint displays to all classes to enhance student understanding and interest. These displays constituted the data used to explore the effectiveness of visual displays in astronomy. Evidence gathered included videotaped transcripts and interview transcripts from the Spring Semester 2013. Analysis shows the effectiveness of PowerPoint lectures designed to engender student interest. Subsequent student presentations were used to confirm or deny the influence of these lectures.
4. The data, and subsequent evidence, gleaned from the student presentations of 2013 and 2014 were used to explore the question of whether astronomy can be used as a device leading toward a better understanding of difficult mathematical concepts. The project results from the Summer 2014 Calculus class were also examined. Analysis explored the extent of student understanding of such subjects as arc length and ellipse construction.
5. The test results, interviews, and student presentations were used as evidence to examine the effectiveness of exposure to historical examples of mathematical modeling in astronomy. In particular, the spring and summer classes in 2014 created projects exploring the laws of planetary motion discovered by Johannes Kepler in the 17th century.

Chapter 2: Literature Review

Recent literature connecting college mathematics to astronomy is notably scarce. Specific articles in both *Educational Studies in Mathematics* and the *Journal for Research in Mathematical Education*, two of the leading journals in mathematics education, have been virtually nonexistent over the past decade. Since 2000, no articles have been written, in either publication, that link astronomy and mathematics. Related literature relevant to this study covers mathematical modeling, reform calculus, and ODE. Mathematical modeling and astronomy share many intricate relationships, such as charting the phenomena of elliptical orbits. Mathematical modeling also uses algorithms found in reform calculus and ODE; these shared principles form the equations from which we can understand the mechanics of celestial phenomena. Undergraduates' perspectives on these relationships have remained unknown.

Mathematics and Astronomy Through History

From the dawn of history, humankind has tried to explain the mysterious movements of the heavenly bodies. In an address delivered before the American Mathematical Society, Otto Neugebauer discussed the mathematical techniques employed by the ancient Greek and Babylonian cultures to explain these movements. Ptolemy's *Almagest* is regarded as the main source of knowledge of ancient astronomy (Neugebauer, 1948, p. 1013). This work relied upon the discoveries of the astronomer Hipparchus, whose findings represented a "milestone in the development of mathematical astronomy" (Neugebauer, 1948, p. 1013). The Babylonians also studied the movements of the Moon and the planets from 400 to 250 BC. To summarize these efforts, Neugebauer mentioned three seemingly observational astronomy problems that are actually "essentially dependent upon mathematical theories." (Neugebauer, 1948, p. 1014). These problems involve determining the diameter of the Moon, the constant of precession, and the parameters of geographical longitude. Solutions to these problems involved both knowledge of celestial mechanics and transformational geometry between celestial and terrestrial coordinates.

In the *Almagest*, Ptolemy assumed that the Earth was the center of the solar system and that the planets' motions were based on circular orbits around the Earth, with each planet traveling in epicycles around its orbit. This theory explained the occasional retrograde motion of a planet in the night sky. In summary, Ptolemy created a mathematical model with "a consistency and inner perfection that seemed hardly open to improvement" (Neugebauer, 1948, p. 1015). Ptolemy's work provided a justification for continued public interest in both astronomy and astrology, and his system remained unquestioned through the Middle Ages.

The dawn of modern astronomy began in 1543, when Nicholas Copernicus published *De Revolutionibus Orbium Coelestium* (*On the Revolution of Celestial Spheres*). In this work, he proposed that the Earth and the other planets revolved in circular orbits about the Sun. This revolutionary concept contradicted the accepted geocentric vision of Ptolemy's *Almagest.* The revolution continued with the efforts of Johannes Kepler and Galileo Galilei. In the early 17th century, Kepler studied the recorded observations of the astronomer Tycho Brahe. From these records, he deduced how the planets moved. His three laws of planetary motion read as follows (Burton, 2011, p. 360):

1. The orbits of the planets are ellipses, with the Sun as a focal point.
2. The velocity of a planet increases as it approaches the Sun.
3. The period (year) of a planet and its average distance from the Sun are proportionally related.

During the same time period, Galileo became the first person to use a telescope to observe the Moon and the planets. Politically dangerous at the time, Galileo's observations nonetheless set the solar system to an entirely new perspective. The revolution became complete with the work of Isaac Newton. While Kepler explained how the planets moved, Newton explained why with his law of universal gravitation. His work provided the framework for most of the astronomy explored over the next 200 years. During this time, two more planets were discovered. William Herschel discovered the planet Uranus by accident in 1781, and the perturbations of its orbit led to the discovery of the planet Neptune by J. C. Adams and U. J. J. Le Verrier in 1846. In 1801, Giuseppe Piazzi discovered Ceres, the largest asteroid in the belt between Mars and Jupiter. At one point, astronomers lost visual contact with Ceres, and mathematician Carl Freiderich Gauss used Kepler's laws and advanced gravitational theory developed by J. L. LaGrange to compute the position where the asteroid was subsequently rediscovered. LaGrange, Leonhard Euler, and Pierre LaPlace were also responsible for pioneering advances in the burgeoning science of celestial mechanics. Another contemporary, Frederich Bessel, developed the parallax algorithm to determine "the first measurement of distance" to a star (in this case, the star 61Cygni). The parallax method involves taking positional readings of a star 6 months apart. From the different perspectives, an angle is formed, from which the star's distance can be calculated by elementary trigonometry.

The advent of the 20th century heralded another revolution in the field of astronomy. The architects of this revolution were Albert Einstein and Edwin Hubble. On a cosmological scale, Einstein's theories of relativity reworked Newton's law of gravitation. On a local scale (i.e., within the realm of the solar system), the law of gravity routinely applies in any interaction between two objects. With respect to physically large, nonlocal objects, such as stars and galaxies, Einstein proposed that gravity should be regarded as a geometrical consequence between two objects within the perspective of a space–time continuum. As mentioned previously, Einstein's theory was confirmed when the astronomer Arthur Eddington presented visual proof in star photographs taken during a total solar eclipse in 1919.

In addition to Einstein's theories, the latter-day revolution was further fueled by the astronomer Edwin Hubble. On January 1, 1925, Hubble offered a presentation on stars known as Cepheid variables. The groundwork for his findings had been previously established by two other astronomers, Henrietta Leavitt and Vesto Slipher. Slipher investigated the star-like clouds known as nebulae. Centuries earlier, Galileo had confirmed that these clouds contained stars. Slipher found that these nebulae were moving at very high velocities. Leavitt established a period-luminosity relationship for Cepheid variable stars located within nebulae that could be used to determine distance. Hubble used these findings to confirm that nebulae previously thought to be confined to the limits of our Milky Way galaxy were actually galaxies themselves. Furthermore, these galaxies were at a tremendous distance from the Milky Way and traveling at tremendous velocities away from us. The expanding universe was revealed, and Edwin Hubble dramatically altered the perspective of cosmology.

In the latter half of the 20th century, further discoveries were made in the exponential growth of cosmology. In 1964, Arno Penzias and Robert Wilson, scientists at Bell Laboratory in Holmdel, New Jersey, conducted experiments to detect radio waves reflecting off artificial satellites. In doing so, they discovered traces of cosmic microwave background radiation considered to be the remnants of the spontaneous explosion popularly known as the Big Bang that gave birth to the universe 13 billion years ago. At about the same time, Maarten Schmidt, an astronomer working at the Mt. Palomar Observatory in California, discovered quasars. These radio sources, which were star-like but with the brightness of galaxies, are the most distant objects in the universe to date. In the 1970s, the scientist Stephen Hawking fused astronomy and quantum mechanics with his ground-breaking theories on black holes, which are the ultimate fates of aging giant stars collapsing under the inevitable pull of gravity. The 1990s saw the launchings of the first orbital telescopes free of the influence of Earth's atmosphere. One direct result was the discovery of several exoplanets orbiting stars outside our own solar system.

With all the breakthroughs that occurred, along with humankind's first physical journeys into outer space, "Astronomy was revolutionized in the 20th century" (Hughes, 2007, p. 1). The theoretical boundaries of astronomy will continue to expand into the 21st century. *Astronomy and Astrophysics in the New Millennium* is a series of reports published by the National Research Council. Three long-term quests are recommended for pursuit by the Astronomy and Astrophysics program:

1. To comprehend our cosmic origin.
2. To image black holes and elucidate relativistic gravity.
3. To understand the elements essential for forming Earth-like planets and life.

The foundations for research in these topics will be based in such fields as physics, quantum mechanics, string symmetry, chaos theory, and even fractal geometry. All of these fields share one thing: all theoretical advances that will be made in the course of this century will be written in the language of mathematics. Meeus (2004), in his five-volume *Mathematical Astronomy Morsels*, refers to the "classical, mathematical science of the sky" (p.1). Even though computer technology has rendered the tortured hand calculations of *The Almagest* obsolete, the need for a sound mathematical perspective remains strong.

Recent Discoveries

On July 14, 2015, an unmanned spacecraft reached the neighborhood of the dwarf planet Pluto. The New Horizons probe was launched by NASA in January 2006 and traveled over 3 billion miles to accomplish its pioneer mission, which was to record and gather information on the heretofore unexplored world.

A. Stern and Mitton (1999) chronicled the history of the discoveries in the Pluto system. The existence of Pluto was first postulated by the astronomer Percival Lowell in the late 19th century. In 1930, another astronomer, Clyde Tombaugh, discovered the planet after years of painstakingly examining photographic plates at the Arizona observatory named in Lowell's honor. In 1978, Pluto's main moon, Charon, was discovered by astronomer James Christy. By the end of the 1980s, all the major planets in the solar system, with the exception of Pluto, had been visited by unmanned probes. Shortly after New Horizons was launched, Pluto was reclassified as a dwarf planet by the International Astronomical Union (Overbye, 2006). The New Horizons probe successfully transmitted the first close-up photographs of Pluto, along with a wealth of scientific information about the planet and its five moons.

Mathematical Modeling

A central theme in mathematics education over the past 30 years has been the relationship "between mathematics and the real world" (Blum, 2002, p. 1). Mathematical modeling is the foundation of understanding in real-world astronomy. Classic historical models, such as the elliptical planetary orbits deduced by Johannes Kepler, emphasize the importance of mathematical modeling in a proper understanding of the movements of the heavens.

Educational scholars have closely studied mathematical modeling in recent years. A recent international research forum was called "Mathematical Modeling in School Education:

Mathematical, Cognitive, Curricular, Instructional, and Teacher Education Perspectives." A key concern was the apparent schism between the creative mathematical modeling process and the way this process is taught by mathematical educators. "Even those researchers who have long been conducting research on mathematical modeling have not come to an agreement on the processes of mathematical modeling and how to conceptualize mathematical modeling" (Cai et al., 2014, p. 2). According to the forum, mathematical modeling is viewed as a "bidirectional process of translating between the real-world and mathematics" (Cai et al., 2014, p. 2). Since mathematical modeling "is practiced far and wide—across the natural sciences" (Cai et al., 2014, p. 5), this process is critical in the discoveries made in modern astronomy.

One very prominent historical example was the advent of modern astronomy, the three laws of planetary motion discovered by Johannes Kepler in the early 17th century. The first law states that the orbits of the planets are ellipses with the Sun as a focal point. This conclusion was the direct result of a 4-year study by Kepler in which he constructed a mathematical model based on the observations of the planet Mars by the astronomer Tycho Brahe. Using navigational instruments on an elaborate island observatory, Brahe made painstaking positional recordings of all the visible planets, and Mars in particular. These recordings were made without the use of telescopes, which had yet to be invented. Kepler took these recordings and constructed a mathematical model. The model turned out to be an ellipse with low eccentricity. Brahe made these recordings of the planet's longitudinal placements over a several years. These positional recordings provide ideal data for students to construct the ellipse for themselves. This was done, in fact, in the Summer Semester 2014 Calculus class. This endeavor is an example of how astronomy is relevant when exploring both the mathematical modeling process and the basic properties of an ellipse. These concepts are explored in any standard Calculus II class.

Other topics explored by the mathematical modeling forum included the nature of a mathematical modeling curriculum and the existence of mathematical modeling in current mathematics textbooks. In designing most curricula, the choice of a relevant text is critical. Three texts, published over the past 30 years, have examined mathematical models in astronomy: *Concepts of Mathematical Modeling* (Meyer, 1984), *Mathematical Models and Their Analysis* (Wan, 1989), and at least three editions of *A First Course in Mathematical Modeling* (Giordano, Weir, & Fox, 2003). All three texts have either chapters or illustrated problems about planetary orbits, gravitation, acceleration, and rocket flight. Meyer's work, in particular, contains examples in astronomy illustrating various qualities of a strong mathematical model. One such quality is *generalization*, where two models can be used to illustrate each other. Meyer (1984) contrasted Kepler's second law, which explains *how* a planet moves in its orbit, with Newton's law of universal gravitation, which explains *why* a planet moves in its orbit. Meyer (1984) quoted Bertrand Russell: "I remember a sense almost of intoxication when I first read Newton's deduction of Kepler's second law from the law of gravitation. "Few joys are so useful as this" (p. 211). Meyer (1984) also illustrated the concept of *robustness* with the efforts of ancient astronomers to calculate the distance of Earth from various celestial objects. From these examples, it is obvious that astronomy is a field open to the explorations of mathematical modeling.

The mathematical modeling process is "something uniquely defined for each individual through the mathematical activity in which they take part" (Barba & Rubba, 1992, p. 12**)**. The process is fueled by motivation and interest. Meyer (1984) pointed out that, while planetary motions have little impact on modern everyday life, these motions were of great import in the 17th century. Meyer noted, "Nearly everyone" (p. 212) in the medieval world believed in astrology, where the movements of the planets were thought to influence everyday life. This was the primary impetus for Johannes Kepler in formulating his laws that revolutionized the science of astronomy. The question of motivation survives to modern times and is relevant to teaching strategies “involved in getting students engaging in school mathematics” (Coles, 2016, p. 3).

Reform Calculus and ODE

Mathematical modeling is evident in all of mathematics, but especially in the related fields of calculus and ODE. The solution of real-world problems in the sciences invariably involves the dynamics of equations of motion. Nowhere is this more evident than in current and historical discoveries in astronomy. A great deal of attention has been given, over the past two decades, to the pedagogy involved in teaching ODE. Several innovations in instructional design have been suggested over this time to improve traditional lecture approaches. Prominent among these is the inquiry-oriented ODE (IO-DE) project (Rasmussen, Kwon, Allen, Marrongelle, & Burtch, 2006). The central aim of this project was to create “a learning environment where students routinely offer explanations of and justifications for their reasoning” (Rasmussen et al., 2006, p. 42). Assessments were given that compared the IO-DE group with a group of students exposed to a traditional teaching approach. The assessments, along with a subsequent follow-up study, gave evidence that “the students in the IO-DE group scored significantly higher on the conceptually-oriented items” (Rasmussen et al., 2006, p. 69), which would tend to indicate independent thinking and creativity, as opposed to reliance on procedurally oriented algorithms. Discourse analysis in one of the IO-DE classes indicated students viewed mathematics as “a product of their own engagements” (Rasmussen et al., 2006, p. 90).

As illustrated by this study, the perception of mathematics by students has undergone a great deal of scrutiny. In the late 1980s, the National Council of Teachers of Mathematics established standards that stressed "opportunities to solve many kinds of problems, and encounters with real-world situations" (Romberg, 1989, p. 209) such as astronomy. Blum (2002) later noted "a substantial gap between the forefront of research and development in mathematics education, on the one hand, and the mainstream of mathematics instruction, on the other." (p. 150). With such revolutionary standards, educators have worked to make mathematical concepts more palatable to students. Mathematical applications for various physical and industrial phenomena inevitably come to the forefront when one speaks of "willingness within the community of mathematics teaching in higher education to explore the potential of innovative practices" (Nardi, Jaworski, & Hegedus, 2005, p. 285). The practical application relevant to this particular study was the science of astronomy. The curriculum in this case was ODE, which is a subject that is a practical tool used in all the sciences. A project involving an astronomy activity was a platform of interest to explore mathematical application techniques involving ODE.

A course in ODE is, by its nature, an extension of the theorems and techniques normally encountered in calculus. The emphasis in any ODE syllabus is typically on scientific and industrial applications. In the 1990s, the Mathematical Association of America enacted major reforms in teaching calculus. The reforms were summarized in *Calculus: The Dynamics of Change* (Roberts, 1996). It was recognized that the ideal calculus course should be "at the same time a culmination and a beginning" (Roberts, 1996, p. 1). In the same article, particular emphasis was placed on the importance of students using calculus as a springboard to pursue the other sciences. Roberts (1996) noted, "Not only must these students have a thorough grounding in calculus, but they need to be encouraged in their interests with some indications of how the subject relates to these interests" (p. 1). This reason in particular would lend credence to the belief that “the calculus course has been, and in all likelihood will continue to be, the central course in the undergraduate mathematics curriculum” (Roberts, 1996, p. 5).

Cole (1996), in defining a reform calculus course, referred to other courses, such as differential equations, that normally succeed calculus in undergraduate mathematics sequences. He examined three preliminary texts, *Exploring Differential Equations via Graphics and Data* (Lomen & Lovelock, 1996), *Differential Equations* (Blanchard, Devaney, & Hall, 1998), and *Differential Equations: A Modeling Perspective* (Borelli & Coleman, 1996). Borelli and Coleman’s text, in particular, emphasized modeling and visualization. The problems examined in their text included such topics as carbon dating and logistic population growth, where solutions involve the construction of models involving appropriate differential equations.

A mathematical modeling project of a deep space probe involving students who have a genuine interest in pursuing such a project was a logical first step as part of an ODE project. With the undergraduate students involved in this study, there was a decided emphasis on elementary mathematical modeling. There has been an actual proposal for a deep space exploration of the Neptune system, “Trajectory optimization for a mission to Neptune and Triton” (Melman, 2007) which contains complex mathematical models. The students in this study learned the foundations in calculus and ODE on which such models rest. A primary example of one such principle is the simple concept of arc length. Many advanced concepts in celestial mechanics have their bases in calculus topics such as angular momentum, gravitational acceleration, and, of course, the conic sections.

Although only one unmanned space probe (Voyager II) has briefly explored the immediate neighborhood of Neptune, the planet has been an object of continuing interest and speculation for over 200 years. Earth-based telescopes and instruments revealed the basic structure and composition of Neptune and Triton. The Voyager II probe exponentially expanded this body of knowledge. Neptune has a "uniquely active atmosphere" (Cruikshank, 1995, p.xiii) and a complex magnetic field that envelops Triton and at least six other smaller satellites. Voyager II also revealed the nature of Triton as a body possibly captured from the outer solar system by Neptune's gravitational pull. It has a uniquely eccentric orbit and an "astonishing surface geology" (Cruikshank, 1995, p.xiii), despite having the coldest recorded temperature in the solar system to date. More discoveries are made on a regular basis by the Hubble Space Telescope and other instruments in near-space Earth orbit. This continuing research generates interest in possible missions to the planet; "*in-situ* scientific investigations" (S. Stern, Lunine, Friedlander, & Chenge, 1995, p. 1151) have, in fact, been proposed. This interest provides a fertile landscape of curiosity for sufficiently motivated students. It is with this motivation that the students in this study were given the opportunity to plan a hypothetical journey of an unmanned probe to travel to Neptune and conduct a 3-year survey of the system. To get them started and adequately prepared, they were asked to describe the formulation of the arc length formula in calculus and subsequently show competence in solving related problems. These students elaborated on their work in a PowerPoint presentation describing how the mathematics would relate to the mechanics involved in making the space probe journey. These efforts stand as a concrete example of what Dowling (2013) referred to as an esoteric domain: "This domain is here conceived as a hybrid domain of, first, linguistic and extra linguistic resources that are unambiguously mathematical in terms of both expression and content and, second, pedagogic theory . . . that enables the mathematical gaze onto other practices" (Dowling, 2013, p. 1).

There are many factors to consider in the student pursuit of such a study. Particularly relevant to this project is the fact that educators expect students to absorb "in a short time, basic principles (in mathematics, but also in other scientific disciplines) that took humanity thousands of years to construct" (Sinclair, 1990, p. 19). Also relevant is the mathematical modeling process itself, and the need for "standing outside mathematics and looking into mathematics to find things that conceivably might help resolve the driving question" (Dossey, 2010, p. 88). Students in this study were asked to explore the oldest science to understand the mathematics involved in constructing the flight plan of a planetary probe. Student interest was a major consideration, as the opportunity existed for practical analytic scaffolding to "show promise for improving students' learning of mathematical skills with deeper conceptual understanding" (Speer & Wagner, 2009, p. 530).

Although the mathematics involved in Melman's (2007) study was on the upper graduate level, the basic principles (e.g., conic sections, arc length) are accessible on an undergraduate level. Mathematical modeling in astronomy can be approached from a historical viewpoint. The longitudinal position readings of the planet Mars, as originally observed and recorded by Tycho Brahe, are readily available for study (Gingerich, 1983).

Astronomy Education

The symbiosis between mathematics and astronomy in student learning should also be examined from the perspective of astronomy education. Despite the fact that astronomy is one of the oldest sciences, "research in astronomy education is a very new field" (Slater & Bailey, 2003, p. 20). The *Astronomy Education Review (AER)* began publication early in the new millennium. Prior to this, no such journals existed. Perhaps this could be attributed to the general state of astronomy education in the 20th century. In most major undergraduate liberal arts institutions, astronomy was either an elective or a topic covered in an Earth science course.

Since 2001, the *AER* has published significant results from several quantitative studies. “A Review of Astronomy Research” (Slater & Bailey, 2003, p. 20) was one of the first articles published in the journal. The article included a summary of research in astronomy education. In particular, the authors examined the Astronomy Diagnostic Test, an assessment tool conceived in 1999. The test was first intended as an introductory astronomy survey for liberal arts majors. It was later redesigned, “revised and validated by extensive student interviews, to probe student understanding in a quantitative way” (Slater & Bailey, 2003, p. 30). It became a standardized test to explore misconceived astronomy notions (Deming & Hufnagel, 2000). The device has been used as both a pretest and a post-instructional indicator. The results verified significant initial student misconceptions on a wide variety of topics, from seasonal star positions and relative planetary distances to the nature of solar energy. Although there was evidence of increased awareness after traditional instruction, test results still indicated significant student misconceptions on various astronomical concepts. Comins (2001, pp. 46, 47) outlined many misconstrued statements assumed to be true by many undergraduate students. Some commonly mistaken notions include the following:

- The Sun is solid.
- Mercury, the closest planet to the Sun, is hot everywhere on its surface.
- Saturn is the only planet with rings.
- We see all sides of the Moon from the Earth.
- Black holes are holes in space.
- Comet tails are always behind the comet.

These are among a great many mistaken facts in astronomy that a significant portion of the general student population assumes to be true. Since the beginning of the new millennium, the

AER has chronicled systematic research conducted on the learning and teaching of astronomy. As mentioned before, the Astronomy Diagnostic Test has been streamlined to cover a broad area of topics (Deming & Hufnagel, 2000). Instructors are advised “to not use this as a cumulative test of astronomy knowledge, but rather as a test to compare instructional interventions and to characterize populations” (Slater & Bailey, 2003, p. 30).

Student misunderstandings are closely associated with the notion of teacher understanding of the relevant subject material. The American Association for the Advancement of Science urges that teachers join with their students to “learn about the excitement and process of inquiry, with adequate content background and an appreciation for the philosophical, historical, and cultural importance of science” (Slater & Bailey, 2003, p. 30). Various pedagogical paradigms have been investigated by Barba and Rubba (1992), Atwood and Atwood (1996), and Trundle, Atwood, and Christopher (2003). Michael Zeilik, a pioneer of astronomical education research, has concentrated on university studies “that can verify and pragmatically inform instruction” (Slater & Bailey, 2003, p. 36). These educators and many others have contributed to a reform of astronomy education that has exploded in recent years.

Closely associated with this reform movement is a continuing debate on “the role of mathematics in introductory college courses” (Slater & Bailey, 2003, p. 36). Mathematics is emphasized as a device to lead to greater understanding in any science course, especially astronomy. Reasoning skills are emphasized over rote memorization of algorithms. The resulting experience is, hopefully, “more rewarding to students and is more appropriate when modeling real astronomy” (Slater & Bailey, 2003, p. 36).

Chapter 3: Methodology

The Testing Environment

This study was conducted at Centenary College, Hackettstown, New Jersey, during the spring semesters of 2013 and 2014 and the summer semesters of 2014 and 2015. Centenary College is an independent 4-year liberal arts institution with a combined enrollment of approximately 4,000 graduate and undergraduate students. This study focused on the undergraduate liberal arts mathematics courses.

Course Background

The courses involved in this study cover first semester Introductory ODE, second semester Calculus II, and Introductory Astronomy. The main objective of both courses was to use various algorithms to solve any equation involving derivatives. The calculus text used was one of the many versions written by Larson, Hostetler, and Edwards (*Calculus of a Single Variable: Early Transcendental Functions*, 1999). The material over two semesters covered the first 10 chapters of the book, which contained the following topics: differentiation, integration, practical applications, convergence and divergence of infinite series, and conic sections. The ODE text was written by Rice and Strange (*Ordinary Differential Equations With Applications*, 1994). The one-semester course covered six chapters, including such topics as first-order ODE, approximation methods, homogeneous and nonhomogeneous equations, and Laplace transforms.

The prerequisite for the ODE course is two semesters of college calculus. The main objectives of elementary calculus are to become proficient in the techniques of differentiation and integration. Many of the concepts normally encountered in calculus, especially arc length, are also emphasized in ODE. The astronomy course was a science elective, with no prerequisites, offered to undergraduates and local high school students.

Student Background and Data Collection

Three classes were involved in this study and the study covered the spring 2013, spring 2014, summer 2014, and summer 2015 time frame. The study population consisted of traditional and international Centenary College undergraduate students, representing ODE, Calculus II, and Astronomy. Of the total student population who participated in this study (n = 19), only one student had a background in astronomy. The background included basic celestial terminology. Specific course data are summarized in Table 1. Student data are summarized in Table 2.

Table 1

Study Course Breakdown

Semester	Course	No. students	Median age	Major
Spring 2013	ODE	4	21	Math (3) / Math education (1)
Spring 2014	ODE	6	21	Math (3) / Math education (3)
Summer 2014	Calculus II	2	21	Math (2)
Summer 2015	Astronomy	7	20	Elective

Table 2

Study Data Breakdown

Semester	Course	Lecture	Independent study	Testing	No. students	Interview	Presentation/ final project
Spring 2013	ODE	X	X	X - Oral	4	X - 4	X - 2 groups
Spring 2014	ODE	X	X	X - Written	6		X - 1
Summer 2014	Calculus II	X	X	Written	2		X
Summer 2015	Astronomy	X	X	Written	7		X

Specific subject learning in ODE was as follows:

- The subject material covered included first-order ODE, homogeneous and nonhomogeneous equations, approximation methods, and Laplace transforms. Concepts from earlier mathematics courses, including arc length, conic sections, and the ellipse were also emphasized.
- The instructor created three PowerPoint presentations:
  - PowerPoint 1: Titled "Planets in Review," the presentation covered basic solar system factoids and historical mileposts, including the number of planets; the age of each planet; and the surface features, atmosphere, size, and history of planet exploration (see Appendix G).
  - PowerPoint 2: Titled "Neptune," it concentrated on the planet Neptune with attention to the possibility of life on the planet Neptune (see Appendix G).
  - PowerPoint 3: Titled "Creativity in Mathematical Thought," it was a former graduate school project highlighting historical perspectives on mathematical modeling. In particular, the presentation highlights Johannes Kepler's laws of planetary motion (see Appendix G).
- Two accompanying diagnostic tests were given (see Appendix D) and a fact sheet about Neptune was distributed (see Appendix E).
  - The first test was designed to assess student comprehension of basic solar system knowledge and to serve as a benchmark of learning. For example, the test was designed to cover seemingly easy (e.g., general), moderate, and more challenging knowledge of the solar system. It was important for students to

grasp and apply their knowledge, as it was a key variable for determining the journey to the planet Neptune.

- The fact sheet about the essential parameters of Neptune was from NASA's website: http://nssdc.gsfc.nasa.gov/planetary/factsheet/neptunefact.html. Students were tasked with comprehending the information provided in this fact sheet and using it as a springboard to take a deeper dive into understanding the intricacies of Neptune, thereby enhancing and contributing to the project of plotting a journey to the planet Neptune. The students were given a second test to measure their level of comprehension of the fact sheet.

- The students were asked to pool their results to produce a PowerPoint presentation highlighting the elliptic properties of Neptune's orbit. They were also asked to verify Kepler's laws of planetary motion relative to the orbits of both Earth and Neptune. Three videotaped presentations constitute the main body of evidence for this study.
  - The PowerPoint presentations were videotaped:
    - PowerPoint 1 – Tuesday, May 7, 2013
    - PowerPoint 2 – Monday, May13,2013
    - PowerPoint 3 – Tuesday, May 13, 2014
- The interviews were only conducted in the Spring Semester 2013 class ($n = 4$). The purpose of the interviews was to examine the extent of student comprehension relative to the understanding of the mathematical principles. The interviews were conducted two thirds of the way through the semester and served as a gauge to refine the project for this particular group of students.

Specific subject learning in Calculus II was as follows:

- The study was continued in a Summer Semester 2014 Calculus class. The learning method primarily implemented was lecture, with an atmosphere that promulgated easy responses and questions from the students.
- The main problem studied was a laboratory exercise developed at Clark College and based on an article written by Gingerich (1983). The problem involved calculating the orbit of the planet Mars using the longitudinal observations recorded by Tycho Brahe in the latter part of the 16th century.

The students' video-recorded presentations, along with the accompanying PowerPoints, constituted the main body of evidence for analysis. Each presentation and PowerPoint is presented and examined in the following section. The work performed by the Calculus II class is also presented. The following transcriptions, along with a display of a student solution to the problem posed in the calculus course of summer 2014, constitute a systematic recording of the relevant data collected.

Subject learning in the astronomy course of summer 2015 was as follows:

- Astronomical distances and the mathematical foundation of astronomical science
- The solar system: Sun, planets, moons, asteroid belts
- The New Horizons space probe flyby of the dwarf planet Pluto (July 2015)
- The history of astronomy
- Constellations, star formation, and galaxies
- Pulsars, quasars, and black holes
- Comets and meteors; the Perseid meteor shower (August 2015)
- Text used: *ASTRO2*, Michael Seeds and Dana Blackman (2014)

- Several PowerPoint presentations were employed, including
    - Planets in review (see Appendix G)
    - Pluto rocks (see Appendix G)

Chapter 4: Student Videotaped Presentations and Projects

This section is a compilation of the data collected from all five classes. The data include three videotaped presentations, one orbital ellipse project, and a worksheet project.

Spring Semester 2013

The following is a transcript of a presentation given by three students on May 7, 2013. These students were members of a class in ODE. The transcript is followed by the accompanying PowerPoint, along with a brief summary of the students' efforts.

Trajectory Toward Neptune – A Presentation

> Instructor: This is a presentation in a class in differential equations. I am the instructor, and our hypothetical little project involves the journey of an unmanned probe to the planet Neptune, the exploration of the planet Neptune thereof, and the mathematics behind such an adventure. Here I present my class [the students designated B, C, and D introduce themselves].
>
> Student C: I am [Student C] and I'm from South Korea. Actually I am not a math major, like [my colleagues]. I am a business major.
>
> Student B: Actually I will start. Today we are going to start by talking about Neptune, which is the eighth planet of the solar system [Displays a slide titled "Facts and Figures"]. This planet was discovered by three people, Urban Le Verrier, John Couch Adams, and Jonathan Galle, in 1846. The orbit of Neptune around the Sun is almost 30 times as far as Earth's orbit.
>
> Instructor: Neptune's orbit has a radius of almost 3 billion miles.
>
> Student B: The velocity of this planet is less than our planet, because the planet is further away

Instructor: The velocities are less.

Student B: [Displays another "Facts and Figures" slide]. The volume is much heavier than Earth. Almost 58 times more than Earth.

Instructor: It's one of the gas giants.

Student B: Compared to this planet, it [Neptune] is really, really huge. Its density is much, much lighter than Earth's because it's composed mostly of gas.

Instructor: It's mostly gas.

Student B: There's no logs, no earth, no heavy things.

Instructor: It's not land as we know it.

Student B: It's just made mostly of gas. That is why it is so light. It's also very, very cold. The temperature is close to -300 degrees because this planet is the eighth planet from the Sun, very far from the Sun. That's why it's so cold compared to Earth. The atmosphere is composed of hydrogen and helium. Usually they are kind of a gas, so any kind of life cannot exist in this planet because there is no oxygen.

Instructor: Not the kind of right mix that occurs on Earth. Earth's atmosphere is mostly oxygen and nitrogen. The major part of the atmosphere on Neptune is methane, which accounts for the bluish color of the planet.

Student B: So any kind of life cannot exist on this planet.

Student C: [Displays another slide titled "Part 2: Voyager 2"]. So here is my part. It's about Voyager, the second one. There are, like, two Voyagers: Voyager 1 and Voyager 2. You can see Voyager 2 [displays another slide] made to investigate other planets.

Instructor: The first one was made to explore the inner planets (i.e., Jupiter and Saturn). This one was made to explore Jupiter, Saturn, Uranus, and Neptune.

Student C: So, like you said, it [Voyager 2] was launched in 1977. It was made to explore the outer Solar System and eventually interstellar space. As of today, it has been operating 35 years, 8 months, and 17 days so far. The spacecraft receives and transmits data via the Deep Space Network. So it is working still, along with its sister craft, Voyager 1.

Instructor: They were both launched the same year: 1977.

Student C: So now we know its extended mission. It was tasked with locating and studying the boundaries of the solar system. It is going further away, further and further. I would like to talk about the history of Voyager [displays another slide]. It's very interesting, really. Conceived in the 1970s, Voyager was an idea of NASA. They proposed a planetary grand tour to study the outer planets. It was determined that utilizing gravity assists would enable a single probe to visit the four gas giants: Jupiter, Saturn, Uranus, and Neptune.

Instructor: It's known as the gravitational slingshot effect, using a planet's gravitational force as a boost to, in effect, slingshot the probe to the next planet.

Student C: At the same time requiring only a minimum amount of propellant and a shorter transit duration between planets. Originally, Voyager 2 was planned as Mariner 12 of the Mariner program. Due to congressional budget cuts, the mission was scaled back to be a flyby of Jupiter and Saturn and renamed the Mariner Jupiter-Saturn Probes. As the program progressed, the name was later changed to Voyager, as the probe designs began to differ from previous Mariner missions [Displays another slide]. Each Voyager probe carries a gold-plated audio visual disk in the event that either spacecraft is ever found by intelligent life forms from other planetary systems. The disks carry photos of the Earth and its life forms. It actually has [audio] recordings of the United Nations and the president of the United States [Jimmy Carter] and the children of the planet Earth. It also has songs and sounds, including babies wailing, waves breaking on the shore, and various songs.

Instructor: If I'm not mistaken, it actually has a Chuck Berry record, "Johnnie Be Goode."

Student D: [Displays another slide titled Part 3: Trajectory]. I'm going to talk about the Voyager trajectories. The Voyagers [Voyager 2 before Voyager 1] were launched in 1977 [Displays a slide showing a diagram of the Voyager trajectories against the backdrop of the Gas Giant orbits]. Voyager 1 was faster than 2 [Traces the trajectories on the diagram]. Voyager 1 went to interstellar space.

Instructor: The Voyager 2 probe went by the orbits of Jupiter, Saturn, Uranus, and Neptune. In years, both Voyagers passed Jupiter in 1979, Saturn in 1981. Voyager 2 reached Uranus in 1986, and Neptune in August of 1989.

Student D: [Displays another slide titled "Gravity Assist"]. I'm going to explain how the Voyagers work. The Voyagers work on the principle of gravity assist. It uses the relative movement of the rocket and the gravity of the planet to alter, to change, the path and speed of the spacecraft. It saves propellant, time, and expenses. It accelerates, or redirects.

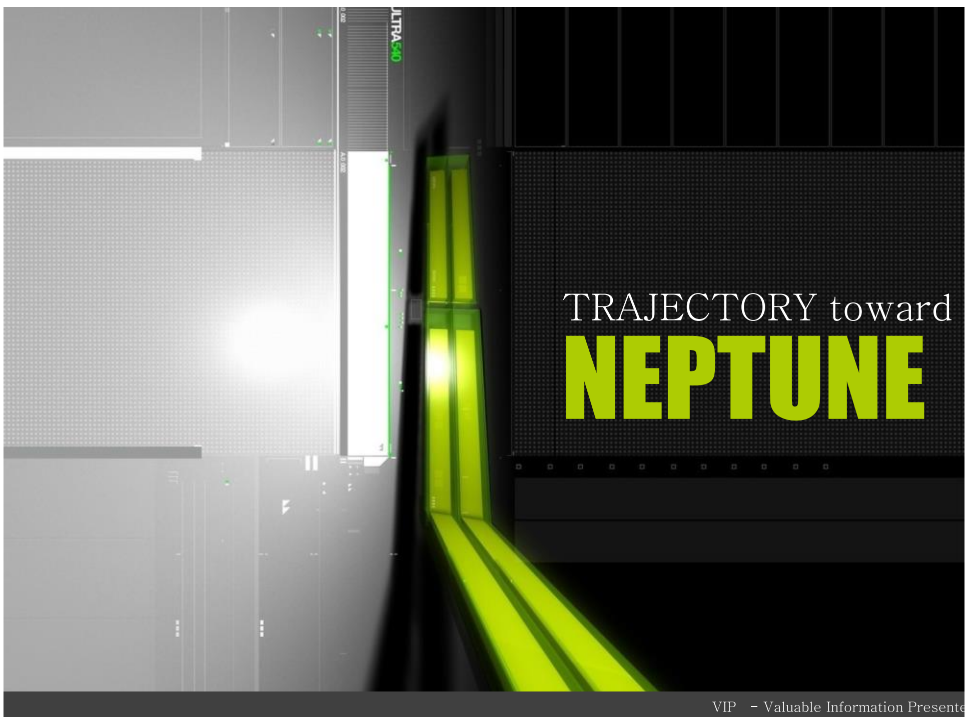
Instructor: Uses the gravitational force as a boost.

Student D: It causes elastic collision, even though there is no actual contact.

Instructor: It uses centrifugal force at a certain speed and for a certain length. It [the probe] is acted upon by another force.

Student C: [Explains the diagram]. The velocity v of the probe is boosted by the gravitational force U as it travels around the planet, resulting in a net velocity v + 2U. The planet pulls the Voyager and gives it a 2U boost. The example is a tennis ball bouncing off a moving train. Imagine throwing a ball at 30 mph and a train approaching at 50 mph. The engineer of the train sees the ball approaching at 80 mph. After the ball bounces elastically, the bounce off the front will produce a velocity of 130 mph relative to the station. This is the end of our presentation [Displays a slide of references].

5-7-13




PRESENTATION
Contents
01 NEPTUNE
02 VOYAGER2
03 TRAJECTORY
04 REFERENCE
VIP - Valuable Information Presente


1
PART
NEPTUNE
VIP - Valuable Information Presente

01 NEPTUNE

Facts & Figure

- Discovered by Urbain Le Verrier, John Couch Adams and Johann Galle
- Date of discovery: 23 September 1846
- Orbit size around sun: 2,795,173,960 miles (*30.070 x Earth)*
- Average orbit velocity: 12,158 mph ($0.182 \times Earth$)

PRESENTATIO

VIP - Valuable Information Presente

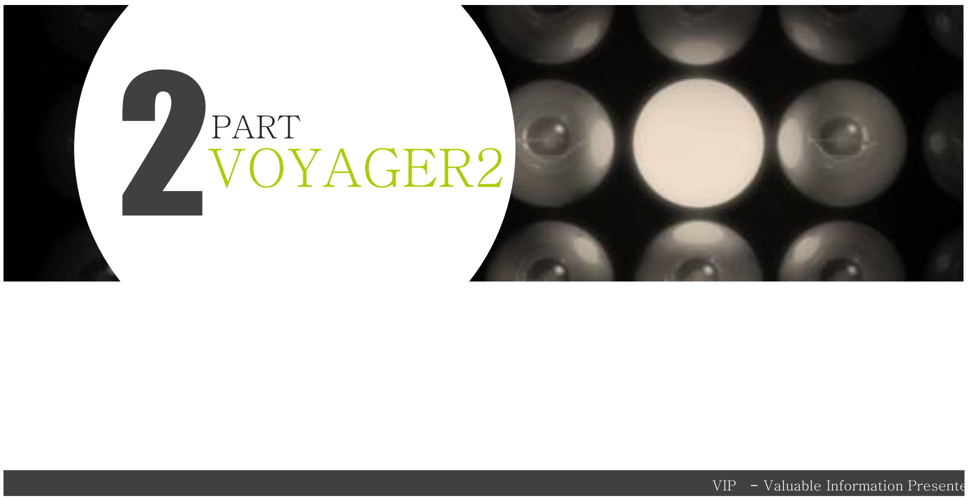
01 NEPTUNE

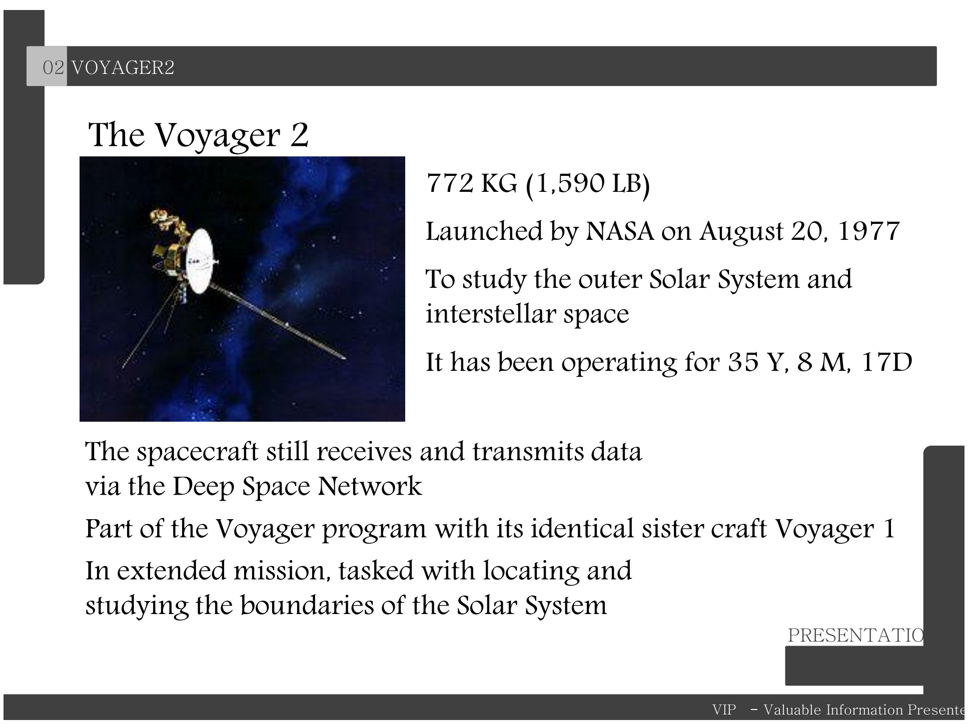
Facts & Figure

- Volume: 15,000,714,125,712 mi^3 (*57.723 x Earth)*
- Density: 1.638 g/cm^3 (*0.297 x Earth)*
- Effective temperature: -353 °F
- Atmospheric constituents: Hydrogen, Helium, Methane (Earth atmosphere consists mostly of N_2 *and* O_2

PRESENTATIO

VIP - Valuable Information Presente


2
PART
VOYAGER2
VIP - Valuable Information Presente


02 VOYAGER2
The Voyager 2
772 KG (1,590 LB)
Launched by NASA on August 20, 1977
To study the outer Solar System and interstellar space
It has been operating for 35 Y, 8 M, 17D
The spacecraft still receives and transmits data via the Deep Space Network
Part of the Voyager program with its identical sister craft Voyager 1
In extended mission, tasked with locating and studying the boundaries of the Solar System
PRESENTATIO
VIP - Valuable Information Presente

02 VOYAGER2

History

Conceived in the 1960s, a Planetary Grand Tour proposal to study the outer planets, prompted NASA to begin work on a mission in early 1970s

It was determined that utilizing gravity assists would enable a single probe to visit the four gas giants (Jupiter, Saturn, Uranus, and Neptune)

While requiring a minimal amount of propellant and a shorter transit Duration between planets

Originally, Voyager 2 was planned as Mariner 12 of the Mariner program

Due to congressional budget cuts, the mission was scaled back to be a flyby of Jupiter and Saturn, and renamed the Mariner Jupiter-Saturn probes

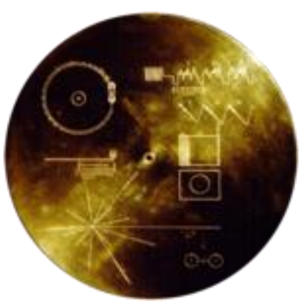
As the program progressed, the name was later changed to Voyager as the probe designs began to differ from previous Mariner missions

PRESENTATIO

VIP - Valuable Information Presente

02 VOYAGER2

Voyager Golden Record



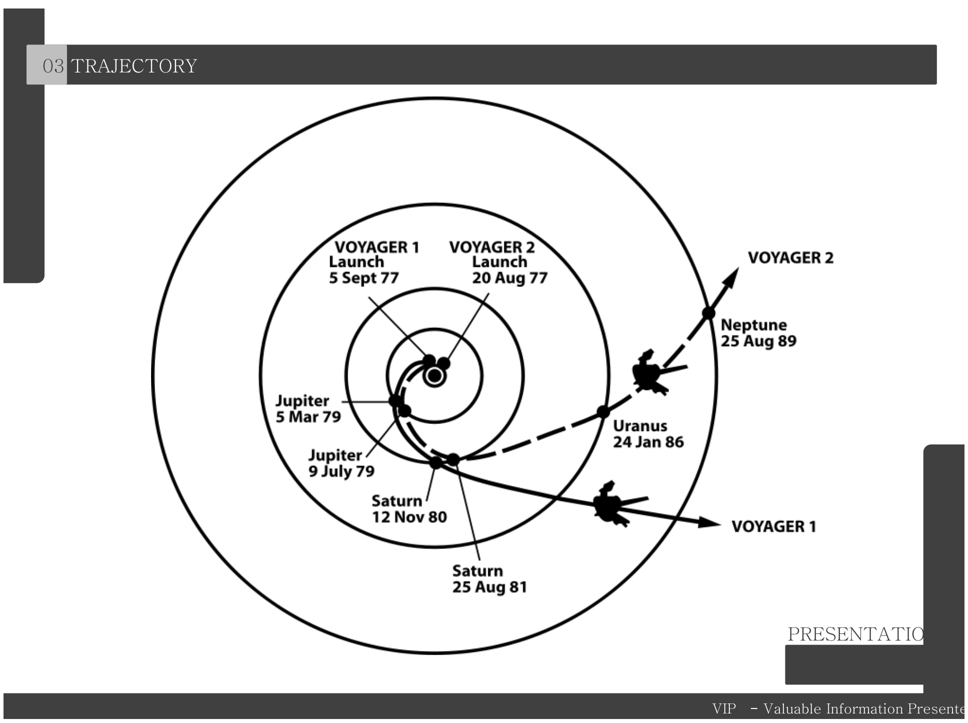
Each Voyager space probe carries a gold-plated audio-visual disc in the event that either spacecraft is ever found by intelligent life-forms from other planetary systems

The discs carry photos of the Earth and its life-forms

PRESENTATIO

VIP - Valuable Information Presente

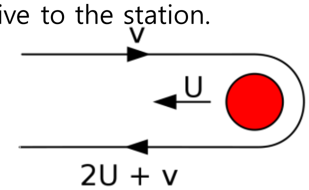

3
PART
TRAJECTORY
VIP - Valuable Information Presente


03 TRAJECTORY
VOYAGER 1
Launch
5 Sept 77
VOYAGER 2
Launch
20 Aug 77
VOYAGER 2
Neptune
25 Aug 89
Jupiter
5 Mar 79
Jupiter
9 July 79
Uranus
24 Jan 86
Saturn
12 Nov 80
VOYAGER 1
Saturn
25 Aug 81
PRESENTATIO
VIP - Valuable Information Presente

03 TRAJECTORY

Gravity Assist

- The use of the relative movement and gravity of a planet to alter the path and speed of a spacecraft (typically in order to save propellant, time, and expense).
- Accelerates and/or re-direct the path of a spacecraft.
- Elastic Collision (no actual contact…)
  - Ex) a tennis ball bounces off a moving train. Imagine throwing a ball at 30 mph toward a train approaching at 50mph. The engineer of the train sees the ball approaching at 80 mph and then departing at 80 mph after the ball bounces elastically off the front of the train. Because of the train's motion, the departure is at 130 mph relative to the station.



PRESENTATIO

VIP - Valuable Information Presente

VIP - Valuable Information Presente

04 REFERENCE

Reference

- National Aeronautics and Space Administration. (2013). *Neptune: Facts & Figures*. Retrieved from http://solarsystem.nasa.gov/planets/profile.cfm?Object=Neptune&Display=Facts
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PRESENTATIO

VIP - Valuable Information Presente

Summary

The three students were in the college's international program and were from South Korea. Student C was a business major, while Students B and D were mathematics majors. The presentation lasted approximately 15 minutes. They were asked to formulate a theoretical journey of an unmanned space probe from Earth to the planet Neptune. They were also tasked with describing the mathematics involved in planning such an adventure. The students presented their project in three sections, with each student explaining the relevant PowerPoint slides. Student B started with a section titled "Neptune." He summarized two slides containing relevant facts about the planet. These slides included a history of the planet's discovery in 1846 by the astronomers Le Verrier, Adams, and Galle. The physical characteristics of Neptune were also summarized, including orbit size and velocity, volume and density, atmospheric constituents, and average temperature. Student B also referred to the designation of Neptune as a gas giant and the unlikelihood of any life existing on the planet.

Student C then presented the second section titled "Voyager 2." The three slides contained information on the flights of the Voyager probes launched by NASA in 1977. Their joint mission was to use a rare planetary alignment to study the outer solar system by visiting the four gas giants: Jupiter, Saturn, Uranus, and Neptune. Student C noted that the probes are presently still receiving and transmitting data via the Deep Space Network. He then gave a brief history of the Voyagers' mission, from its genesis in the 1960s as a planned grand tour of the outer planets to its final realization when the Voyagers were launched in 1977. Student C finally gave a brief description of the gold-plated audio-visual disks contained in both Voyager probes. These disks contain photos and audial recordings of Earth and its life forms.

Student D then presented the final section titled “Trajectory.” The two slides he presented delved into the mathematics behind the mission flight. He displayed the flight paths for both Voyagers on the first slide, and he explained why those paths differed. He noted that the Voyager I trajectory was altered to gain more information on Saturn’s moon Titan. This alteration caused Voyager I to take a path outside the ecliptic plane of the solar system and thus bypass the outer planets Uranus and Neptune. Voyager II stayed its original course and performed a close flyby of Uranus in 1986 and Neptune in 1989.

Student D’s second slide was a diagram illustrating the mathematical physics behind the acceleration boosts known as gravity assists. Such boosts involve the use of the relative movement and gravity of a planet to alter the path and speed of a spacecraft. The ultimate result would be an acceleration and redirection of the spacecraft. Typically, this maneuver would be done to conserve propellant, save time, and defray expenses. Student D provided an example involving a tennis ball and a moving train. He concluded by noting references.

The students who presented the first two sections were business majors. The presentations, “Neptune” and “Voyager 2,” were mainly expository. The third student, a mathematics major, conducted the final section, “Trajectory,” which contained a mathematical analogy to illustrate the phenomenon of gravity assist.

The following is a transcript of a presentation given on May 13, 2013, by a student in a spring semester introductory course in ODE. The student, a senior, majored in mathematics, and her presentation was approximately 8 minutes long. A PowerPoint follows the transcript.

Neptune: A Presentation

Instructor: This is a presentation in a class in differential equations. I am the instructor, and our hypothetical little project involves the journey of an unmanned probe to the planet Neptune, and the exploration of the planet Neptune thereof, and the mathematics behind such an adventure. Here I proudly present my student [introduces Student A]:

Student A: Thank you. [Introduces herself] I would like to give my presentation on the planet Neptune. [Displays slide titled "Neptune Facts"] Neptune was, originally, in Greek mythology, the God of the Sea. It's the eighth planet from the Sun, and it was discovered in 1846. Its orbit is approximately, what is it, 4 billion km. Its diameter is about 50,000 km. Its mass is about 1.0247E26. That's scientific notation, times 10 to the 26th power.

Instructor: You could fit about four planets the size of the Earth in Neptune.

Student A: Wow! That's huge! [Displays another "Neptune Facts" slide] I think the reason why they named the planet Neptune as because of its bluish tint and that Neptune is the [Greek] God of the Sea.

Instructor: Its atmosphere is mostly methane.

Student A: Neptune has between 12 and 19 moons of varying size

Instructor: That's a whole bunch of small ones, but the one big one is Triton.

Student A: The Voyager spacecraft recorded the lowest temperature reading in history: 230 degrees below zero centigrade (40 degrees above absolute zero degrees Kelvin). I cannot even imagine such coldness. Neptune is about 30 times farther from the Sun than Earth. Its atmosphere is blue because it is composed mostly of hydrogen, helium, and methane. Neptune also has a ring system made of dark dust particles difficult to see, as opposed to Saturn's bright ring system of ice particles. [Displays another slide] Neptune is the stormiest planet, with wind speeds approaching 2100 miles per hour.

Instructor: These wind speeds were recorded by Voyager II.

Student A: Compared to hurricanes on Earth, these storms were 10, or was it 30, I forget, times more powerful. Significantly more powerful than storms on planet Earth! And the spot that's on there [points to a spot on the slide] compares to a similar spot on the planet Jupiter. It's a great storm, which, by the way, no longer exists. It was photographed in 1989, but recent photos taken by the Hubble Space Telescope reveal that it no longer exists. [Displays another slide titled "Neptune and Arc Length"] So a little bit with the math. For the trajectory to the planet Neptune, this is the arc length formula [displayed on screen]. I have an example of very simple curves, displaying how it's broken down. So we're using the distance formula, the Pythagorean Theorem. [Displays another slide] This is a picture of the orbit of Neptune. The red line is the orbit. [Displays another slide] This is a diagram of the trajectories of Voyager 1 and Voyager 2 [along with trajectories for Pioneer 10 and Pioneer 11, two similar probes that were launched in the early 1970s]. So it intersects here and here [points to Neptune's orbit and Voyager II's trajectory]. It can't just go in a straight line; it has to curve around because of the gravitational forces.

Instructor: The trajectory has to curve to take advantage of the gravitational slingshot effects. Otherwise, the trip would have taken 40 years. You can't just aim a rocket at where a planet is now. You have to figure out where it's going to be 20 years from now.

Student A: Right. [Displays another slide] Yes, this is a kind of simpler form of the previous slide. This basically shows the trajectory from the Earth, curving around the planet Jupiter, and on to Neptune. If it were launched in January 2018 and employed a gravitational boost from Jupiter, it would arrive at Neptune in January of 2033. So basically you would use the formulas of the previous slides to calculate the trajectory path of this voyage, and thank you!

5-13-13

Neptune

Neptune facts

- Neptune- God of the Sea
- 8th planet from the sun
- Discovered September of 1846
- Orbit- 4,504,000,000 km from the Sun
- Diameter- 49,532 km
- Mass – 1.0247e26

Neptune Facts

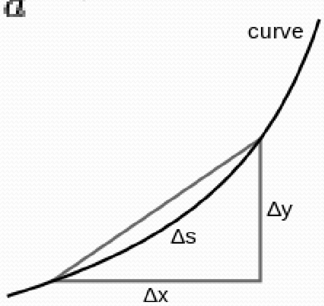
- Largest Moon is Triton
- Coldest Temperature recorded on Triton (-230 C)
- 30 times farther form the sun then Earth
- Atmosphere blue because its made of mostly gas
- Hydrogen
- Helium
- Methane
- Has rings made of dust

Neptune Facts

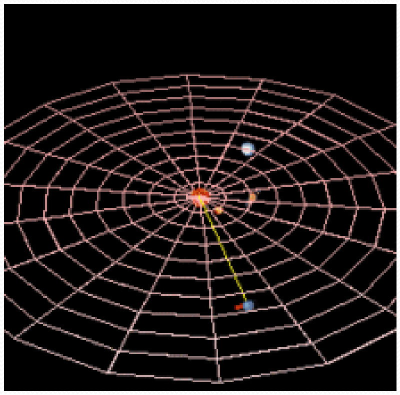
- Stormiest planet
- Winds reaching up to 1,240 MPH

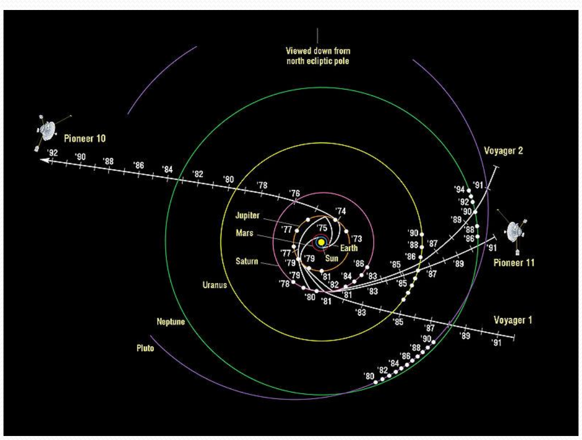
Neptune and Arc Length

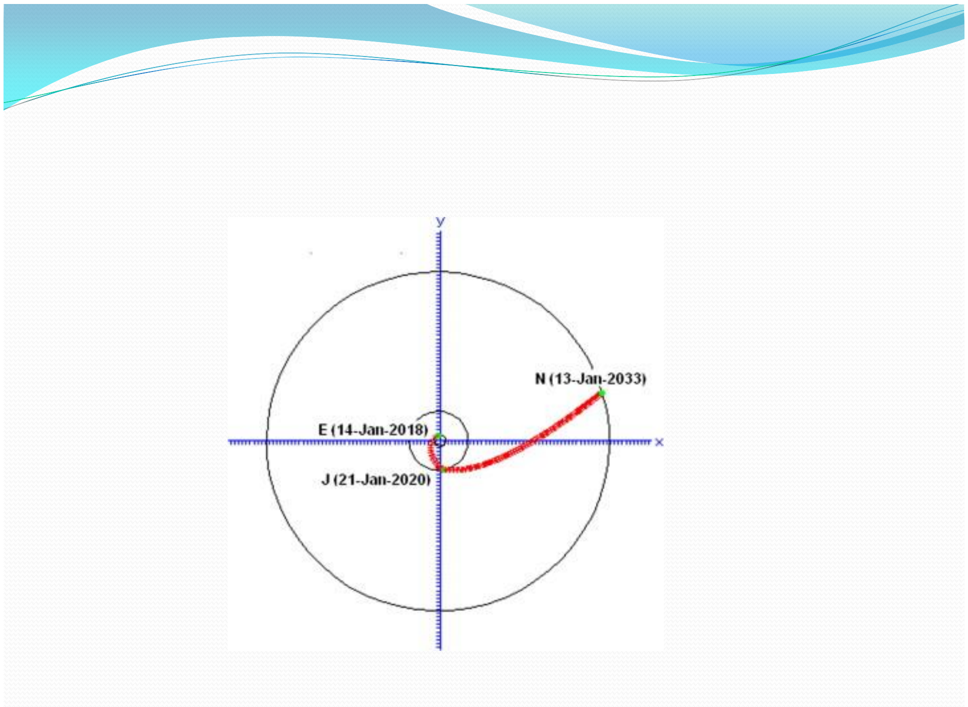
- $$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$



Neptunes Orbit







Summary

Student A chose to give an individual presentation because of an overriding interest in astronomy. The first three slides involved basic information on the planet Neptune, including the history of its discovery. Physical characteristics were also mentioned, including mass, diameter, atmosphere, and distance from the Sun. Student A also mentioned Neptune's main moon, Triton, and the findings of the Voyager 2 spacecraft when it flew by the planet in August 1989. Unique physical phenomenon were also mentioned, including frigid temperatures and ferocious storms.

In addition to expository slides, Student A displayed four slides explaining the mathematics behind the proposed mission to Neptune. She began with an illustration of the arc length formula, which is normally covered in first-semester calculus. She gave a brief explanation of the formula and its relevance to orbital trajectories. She then displayed the historic paths of both Voyager probes in their grand tour of the outer planets. She also mentioned the difficult procedure of predicting the location of a planet years after the launch of a probe. In her final slide, she explained the orbital path of a probe to Neptune, which would take advantage of a gravitational boost from the planet Jupiter. Student A concluded with references.

Spring Semester 2014

The following is a transcript of a videotaped group presentation given by six students designated TC, HD, KC, TT, CV, and RL, who took an introductory course in ODE in Spring Semester 2014. The students were traditional upperclassmen. The presentation lasted approximately 8 minutes. The accompanying PowerPoint follows the transcript.

Neptune: A Presentation

TC: [Displays slide] Neptune sidereal period: the time required for a celestial body within the solar system to complete one revolution with respect to the fixed stars. This can be calculated if its synodic period (time for it to return to the same position relative to Sun and Earth) is known.

HD: [Displays slide] Tropical period: customary to specify positions of celestial bodies with respect to the vernal equinox. Because of precession, this point moves back slowly along the ecliptic.

TT: [Displays slide] Aphelion is a point in the orbit of a planet or a comet at which it is farthest from the Sun. Perihelion is the point in the orbit of a planet or a comet at which it is nearest to the Sun.

HD: [Displays slide] Semi-major: one half of the major axis of an ellipse (as that formed by the orbit of a planet).

TT: [Displays slide] Eccentricity: an astronomical object is a parameter that determines the amount by which its orbit around another body deviates from a perfect circle. This is the equation [points to equation]. E is the total orbital energy, L is the angular momentum, m{red} is the reduced mass, and alpha is the coefficient of the inverse-square law central force.

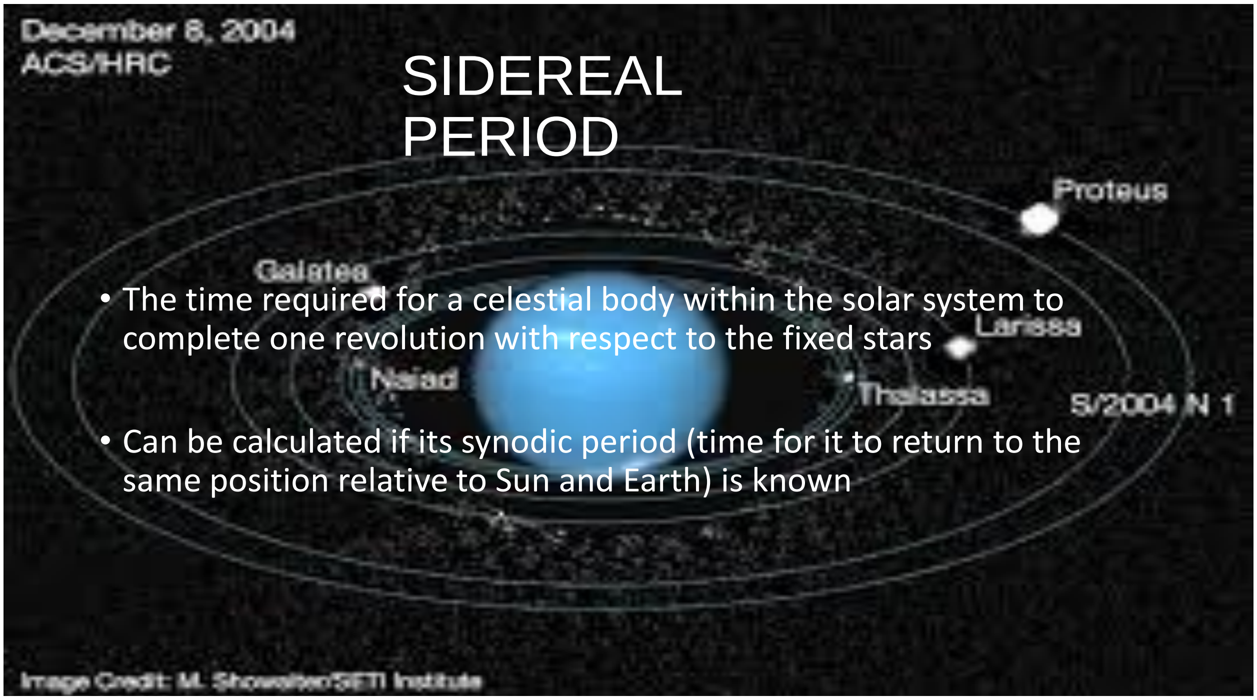
KC: [Displays another slide] Okay, Neptune is the eighth planet from the Sun. So calculating the average distance of Neptune from the Sun, you’re going to use Kepler’s third law. This states that the square of the period is proportional to the cube of the average distance. In other words, the ratio of the period squared to the distance cubed of one planet is the same as the similar ratio for another planet. The period is in years, and the distance is in terms of astronomical units (AU) [where 1 AU = 93,000,000 miles]. So the algebra is right there [points to figures on slides]. The precise numbers are 164.79 years and 30.104 AU.

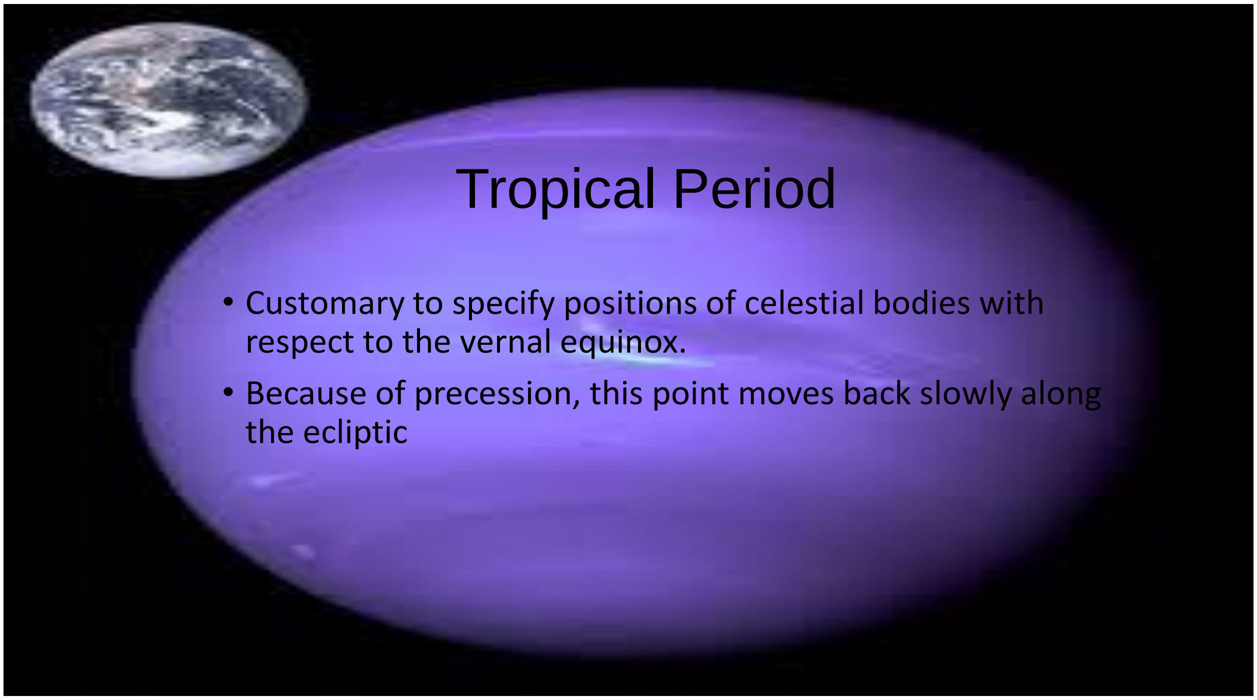
TC: [Displays another slide] So the equation for Neptune’s orbit is given by (1-e^2)/(1-e*cos(Φ). This is the equation of an ellipse. Neptune’s elliptical, which means that it’s almost an exact circle. So we were tasked with the question of distance. That is, the time it would take to travel 50,000 km at the perigee as opposed to 50,000 km at the apogee. Since it’s so closely related to a circle, it [the time] doesn’t actually change that much. So using Kepler’s second law, equal area in the arc of a circle, you would simply calculate the arc length divided by the radius, which would give you theta. Once you calculated theta, you would get a fraction of the total years in one period. It turns out to be, it’s tough to read these slides, but essentially it is 2.59 hours [at perigee].

CV: [Displays another slide] versus 2.538 hours at apogee. It’s actually moving faster, which is why [garbled audio]. The slight differences in the numbers indicates an almost perfect circle.

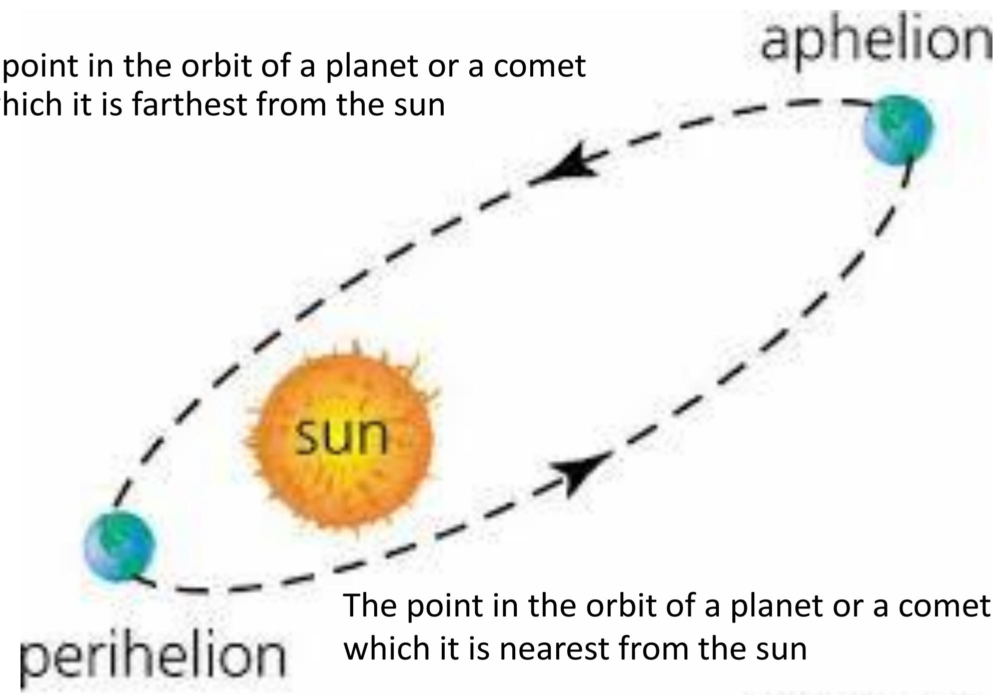
RL: [Displays another slide] So I'm going to look at Kepler's third law. This states that the square of the period of any planet is proportional to the cube of the semi-major axis of its orbit. This third law can be applied to anything. It doesn't necessarily have to be planets in our system. It can be applied to satellites as well. It's useful in finding orbits of moons and binary stars. [Displays another slide] So we're solving for Neptune's orbital period using his [Kepler's] law. I basically plugged in numbers [points to equations on board]. So I came up with 164.8 years. [Displays another slide] Strange facts about Neptune. The strongest winds in the solar system have been recorded on Neptune, at speeds of up to 2000 km per hour. Neptune sometimes orbits the Sun further away than Pluto. From 1979 to 1999, Pluto was closer to the Sun than Neptune. As Pluto was classified as a planet at the time, Neptune was then the ninth planet from the Sun. [Displays another slide] Neptune was almost named Le Verrier, [after] the French astronomer that first saw it. In certain regions of Neptune, the length of the day varies by as much as 6 hours. Because of the pressure on Neptune's surface, it may be a giant diamond or oil factory. [Displays another slide] So this is the proof of Kepler's third law. You derive it from the second law. I used this, which was the centripetal acceleration. So you make these substitutions, you get the square of the period. That's our presentation.


Neptune
(Differential Equations Class)

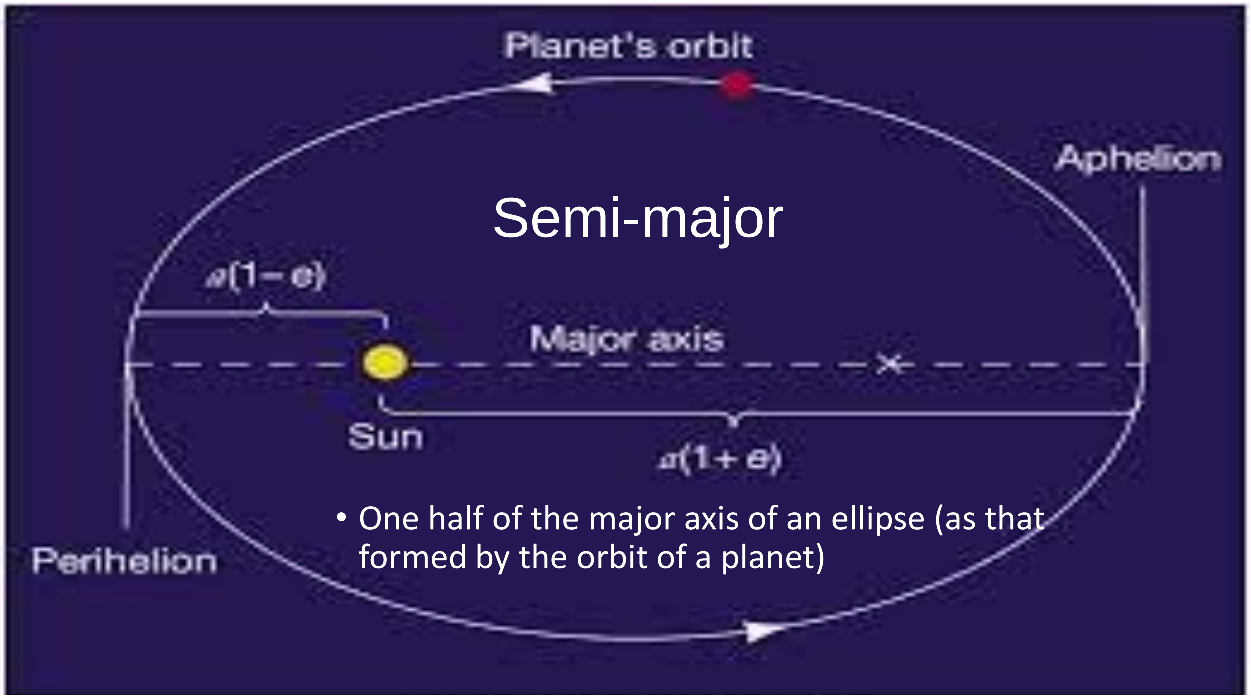

December 8, 2004
ACS/HRC
SIDEREAL PERIOD
Proteus
Galatea
• The time required for a celestial body within the solar system to complete one revolution with respect to the fixed stars
Larissa
Naiad
Thalassa
S/2004 N 1
• Can be calculated if its synodic period (time for it to return to the same position relative to Sun and Earth) is known

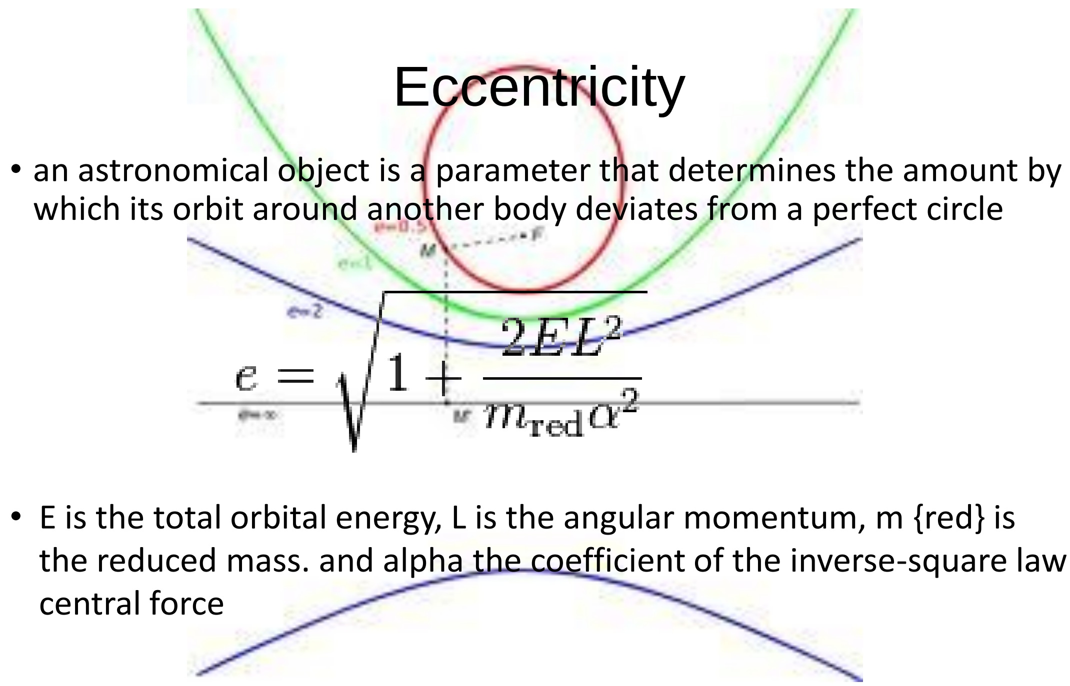


- The point in the orbit of a planet or a comet at which it is farthest from the sun



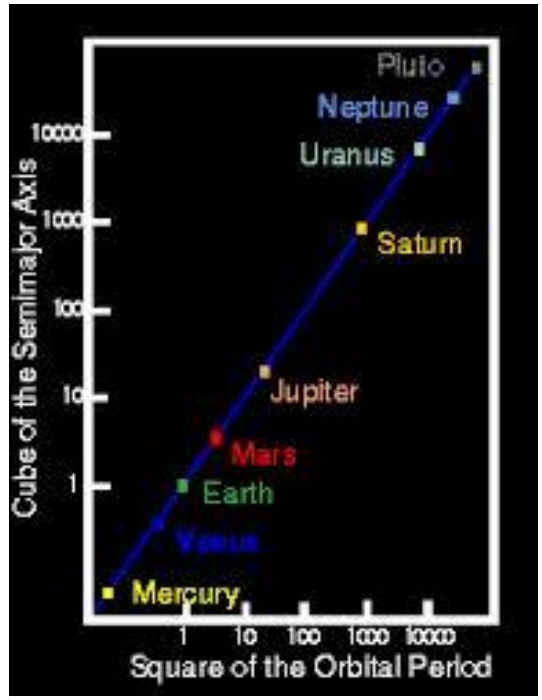
The point in the orbit of a planet or a comet at which it is nearest from the sun

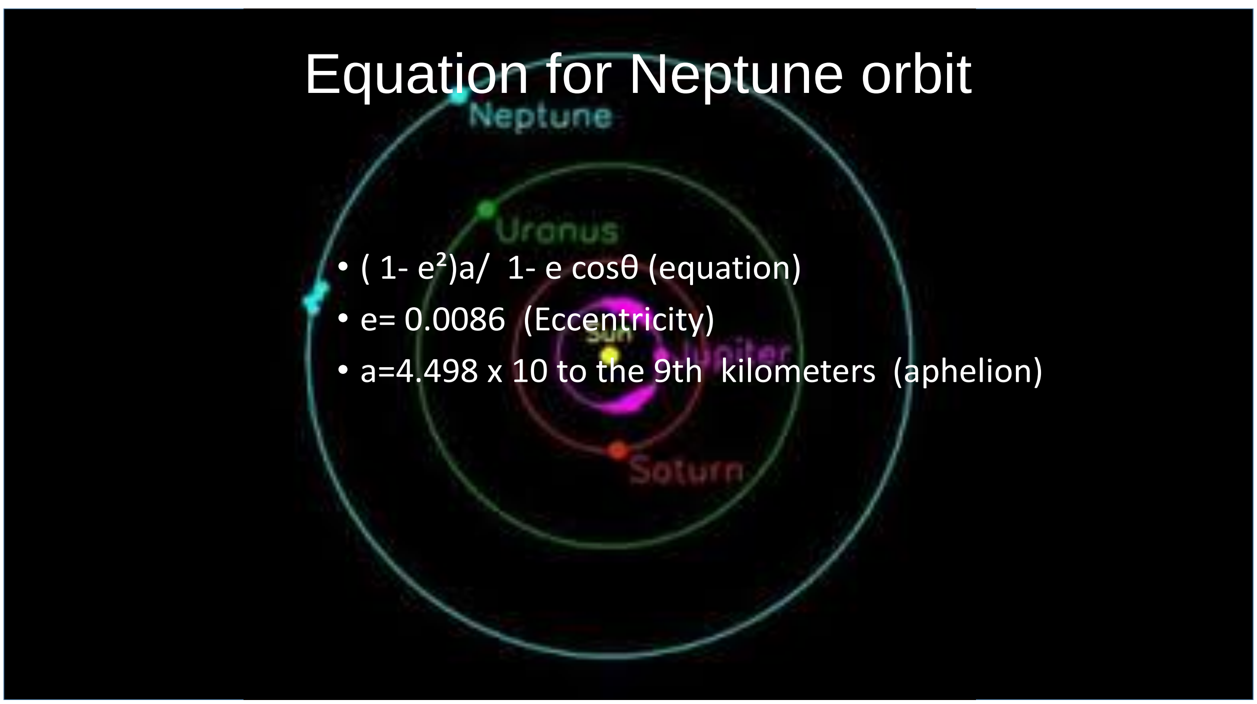

Planet's orbit
Aphelion
Semi-major
a(1− e)
Major axis
Sun
a(1+ e)
• One half of the major axis of an ellipse (as that formed by the orbit of a planet)
Perihelion

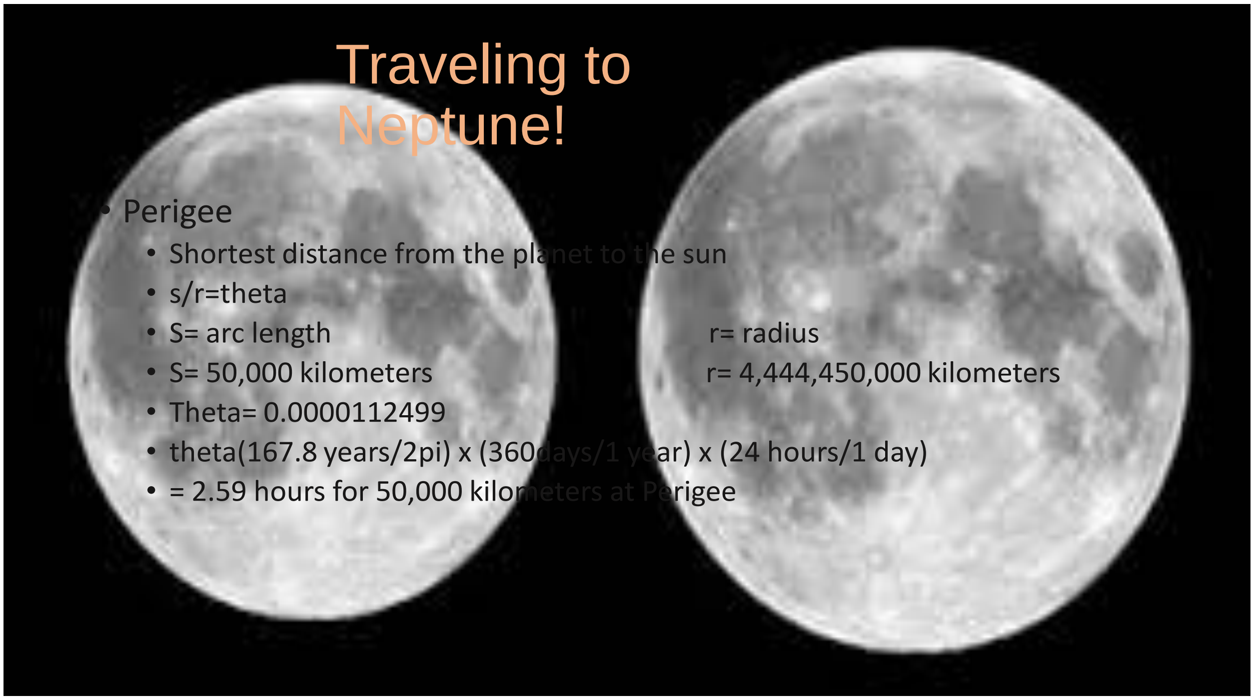

Eccentricity
• an astronomical object is a parameter that determines the amount by which its orbit around another body deviates from a perfect circle
$$e = \sqrt{1 + \frac{2EL^2}{m_{\text{red}}\alpha^2}}$$
• E is the total orbital energy, L is the angular momentum, m {red} is the reduced mass. and alpha the coefficient of the inverse-square law central force

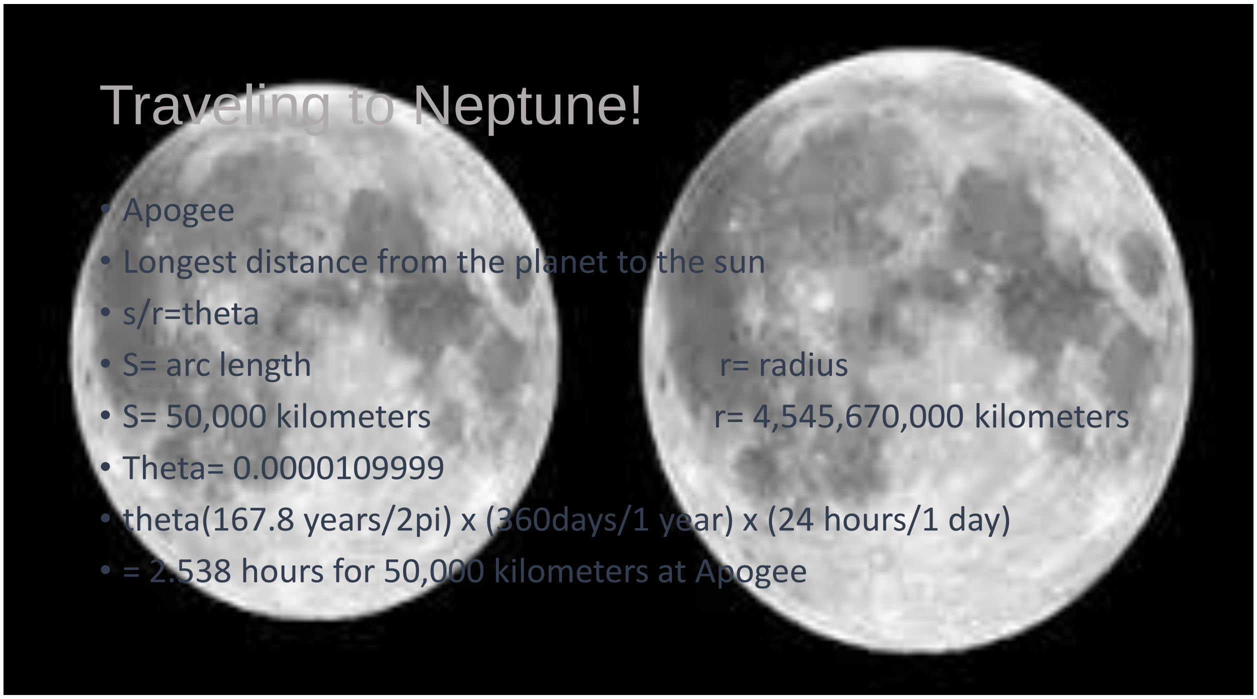
Average Distance to the Sun!

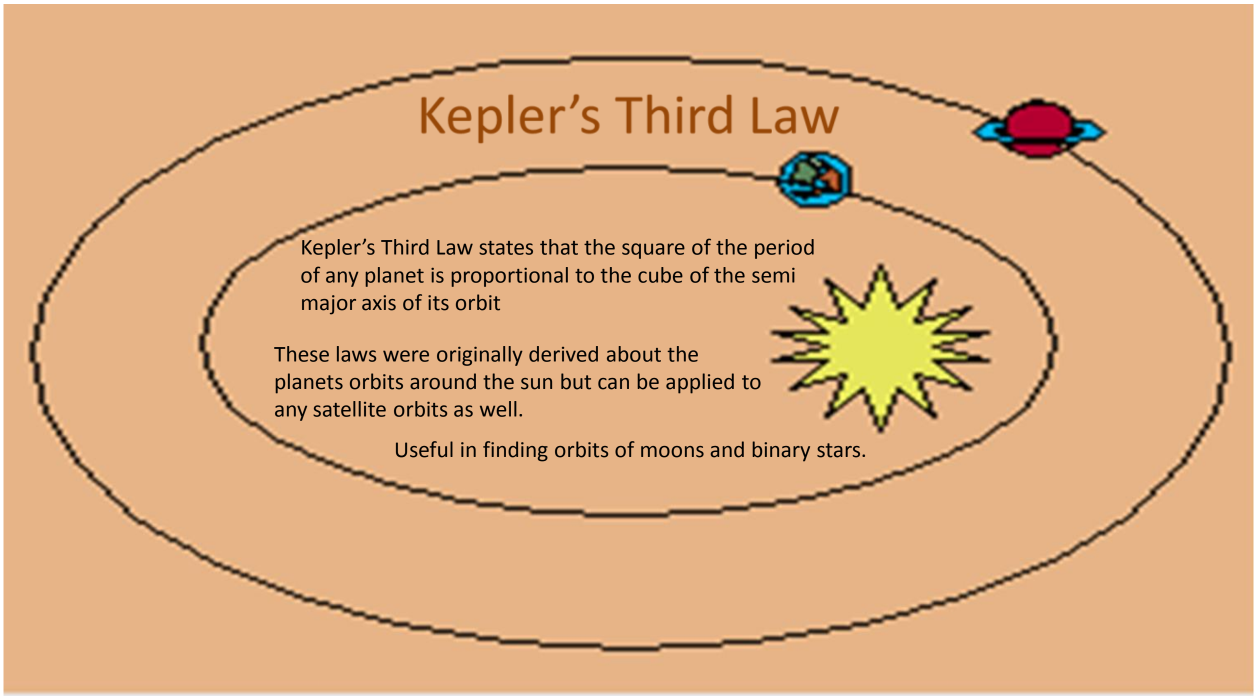
- Kepler's Third Law
  - Period^2 = Distance^3 (with periods in years and distance in AU)
    - You are given period = 164.8 years
    - 164.8^2 = 27,159.04
    - cube root of 27,159.04 = 30.059
    - Distance = 30.059 AU
    - Miles= 3 billion miles
  - (The precise numbers are 164.79 years and 30.104 AU.)

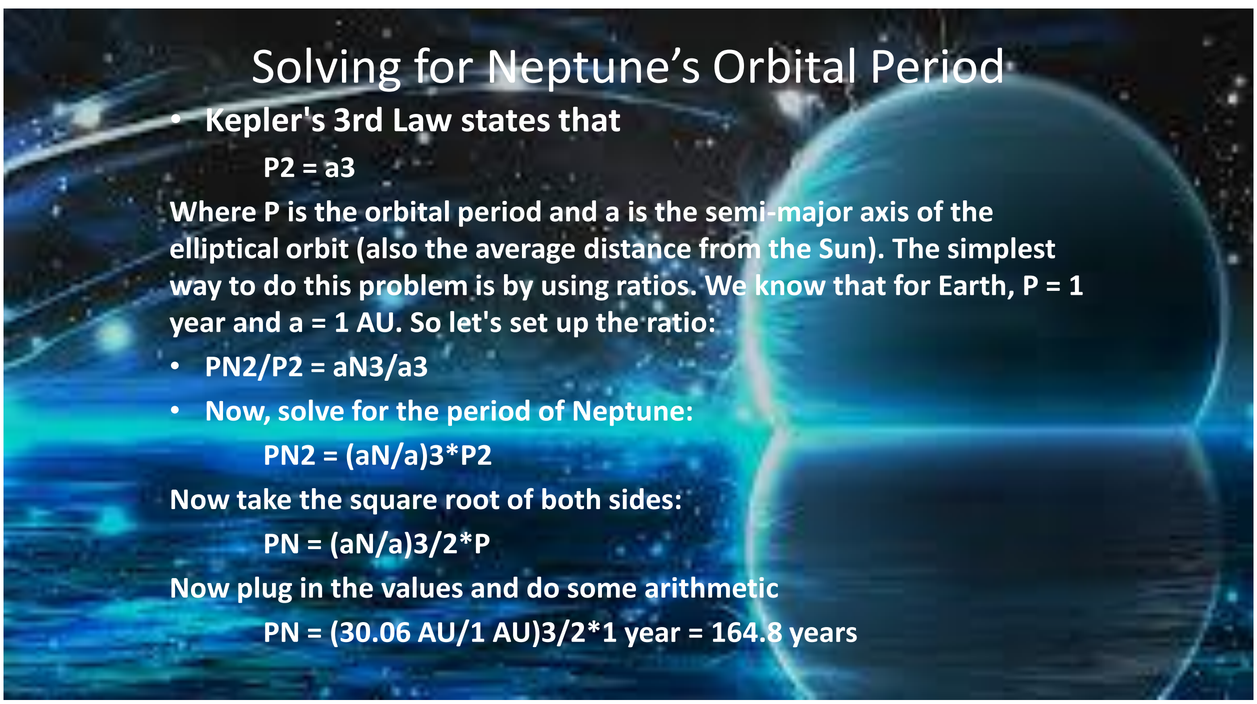


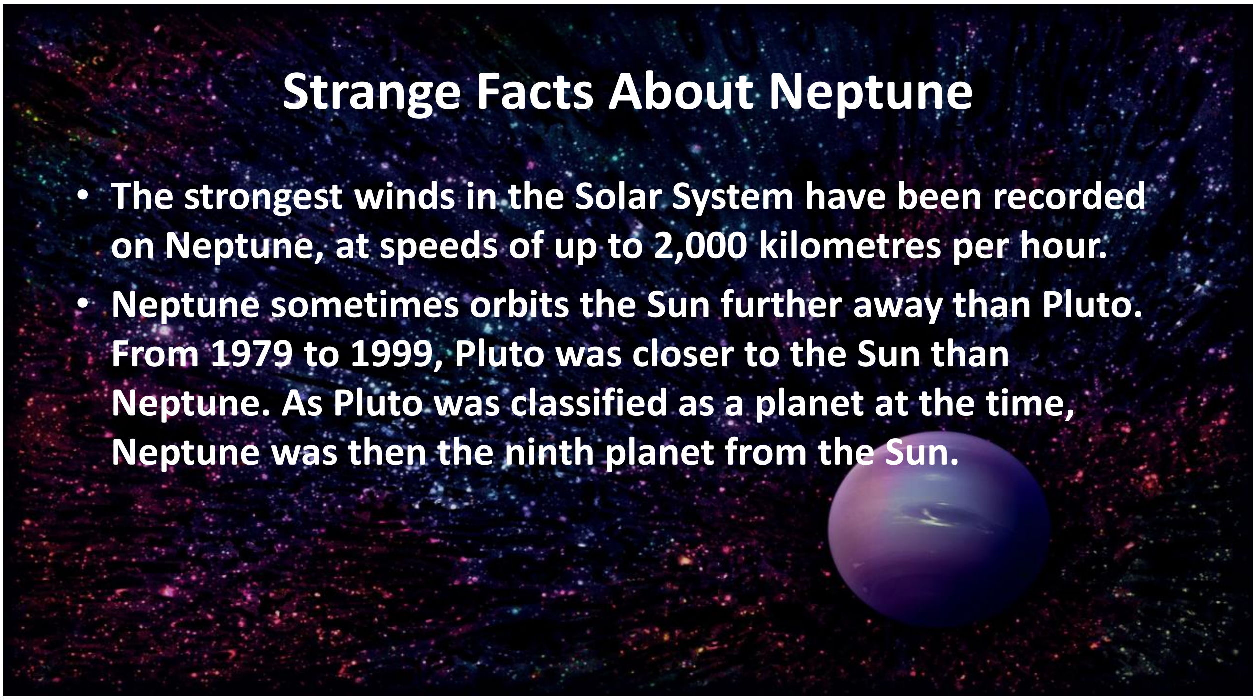


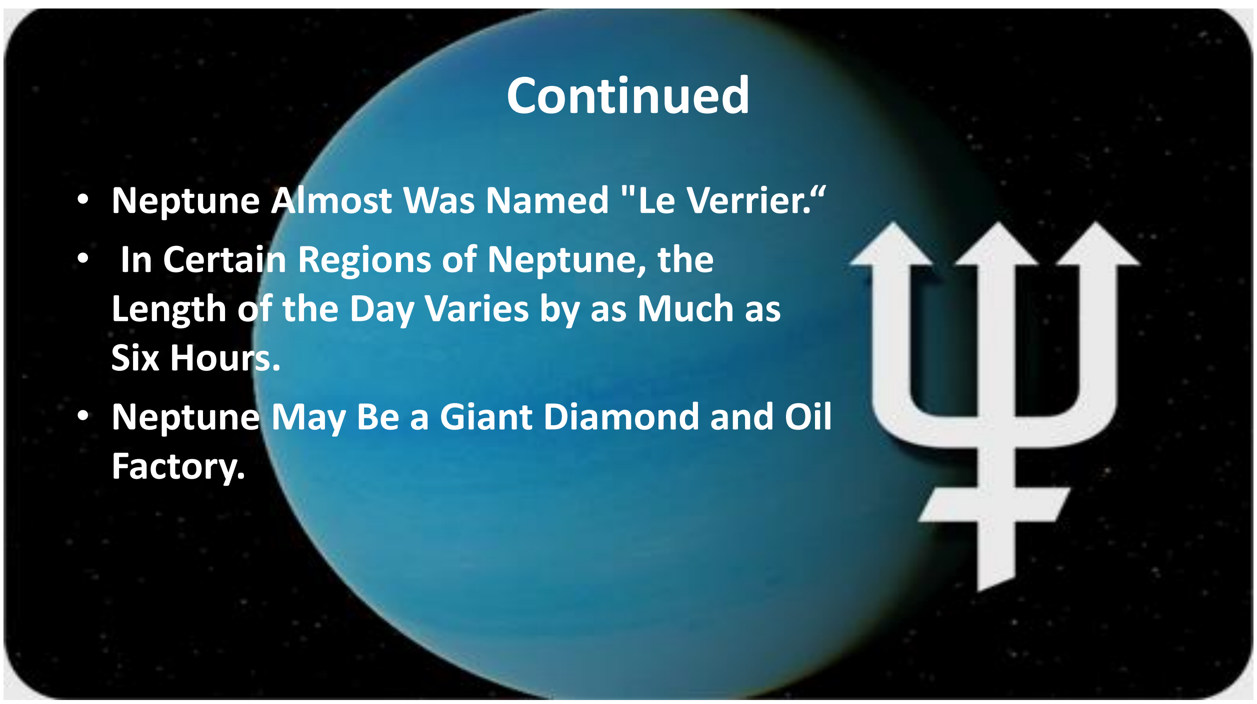

Traveling to Neptune!
• Perigee
• Shortest distance from the planet to the sun
• s/r=theta
• S= arc length
r= radius
• S= 50,000 kilometers
r= 4,444,450,000 kilometers
• Theta= 0.0000112499
• theta(167.8 years/2pi) x (360days/1 year) x (24 hours/1 day)
• = 2.59 hours for 50,000 kilometers at Perigee


Traveling to Neptune!
• Apogee
• Longest distance from the planet to the sun
• s/r=theta
• S= arc length
r= radius
• S= 50,000 kilometers
r= 4,545,670,000 kilometers
• Theta= 0.0000109999
• theta(167.8 years/2pi) x (360days/1 year) x (24 hours/1 day)
• = 2.538 hours for 50,000 kilometers at Apogee


Kepler’s Third Law
Kepler’s Third Law states that the square of the period of any planet is proportional to the cube of the semi major axis of its orbit
These laws were originally derived about the planets orbits around the sun but can be applied to any satellite orbits as well.
Useful in finding orbits of moons and binary stars.


Solving for Neptune’s Orbital Period
• Kepler's 3rd Law states that
P2 = a3
Where P is the orbital period and a is the semi-major axis of the elliptical orbit (also the average distance from the Sun). The simplest way to do this problem is by using ratios. We know that for Earth, P = 1 year and a = 1 AU. So let's set up the ratio:
• PN2/P2 = aN3/a3
• Now, solve for the period of Neptune:
PN2 = (aN/a)3*P2
Now take the square root of both sides:
PN = (aN/a)3/2*P
Now plug in the values and do some arithmetic
PN = (30.06 AU/1 AU)3/2*1 year = 164.8 years


Strange Facts About Neptune
The strongest winds in the Solar System have been recorded on Neptune, at speeds of up to 2,000 kilometres per hour.
Neptune sometimes orbits the Sun further away than Pluto. From 1979 to 1999, Pluto was closer to the Sun than Neptune. As Pluto was classified as a planet at the time, Neptune was then the ninth planet from the Sun.


Continued
Neptune Almost Was Named "Le Verrier.“
In Certain Regions of Neptune, the Length of the Day Varies by as Much as Six Hours.
Neptune May Be a Giant Diamond and Oil Factory.

Proof to Kepler's Law

Summary

The first two students presented five expository slides. The information on these slides included basic astronomical terms (sidereal and tropical period, aphelion and perihelion). They also displayed a diagram of an orbital ellipse, along with the equation for eccentricity of an ellipse. The third student attempted to use Kepler's third law to calculate Neptune's distance from the sun using astronomical units. In her equation, she used the average distance of the Earth from the Sun (1 AU) and the orbital periods of the Earth (1 year) and Neptune (164.8 years). The student calculated an average distance of 30 AU for the planet Neptune.

The fourth student presented an expository slide displaying the equation of Neptune's orbit. The fifth student gave an example illustrating Kepler's second law, which basically states that a planet moves faster in its orbit when it is closer to the sun. He noted the fact that the orbit of the planet Neptune is an ellipse with low eccentricity. This makes the orbit close to being a perfect circle. As a result, his comparison speeds were very close. The final student attempted to derive Kepler's third law by creating a mathematical proportion model using actual figures. He also mentioned several strange facts about the planet Neptune. These included high winds and varying day lengths.

Calculus II, Summer 2014

Two students in this class were asked to create a mathematical model of the orbit of the planet Mars using the longitudinal readings first recorded by the astronomer Tycho Brahe in the 16th century. Both students successfully calculated the elliptical orbit of the planet using the following data set:

Date	Heliocentric longitude of Earth	Geocentric longitude of Mars
February 17, 1595	159'	135'
January 5, 1597	115'	182'
September 19, 1591	6'	284'
August 6, 1593	323'	347'
December 7, 1593	86'	3'
October 25, 1595	42'	50'
March 28, 1587	197'	168'
February 12, 1589	154'	219'
March 10, 1585	180'	132'
January 26, 1587	136'	185'

Both students used instructions developed by a Clark College astronomy course (see Appendix C). One student successfully completed the task and constructed a model of the orbit of Mars. She followed the instructions to triangulate five positions of the planet in its orbit (see Figure 1).

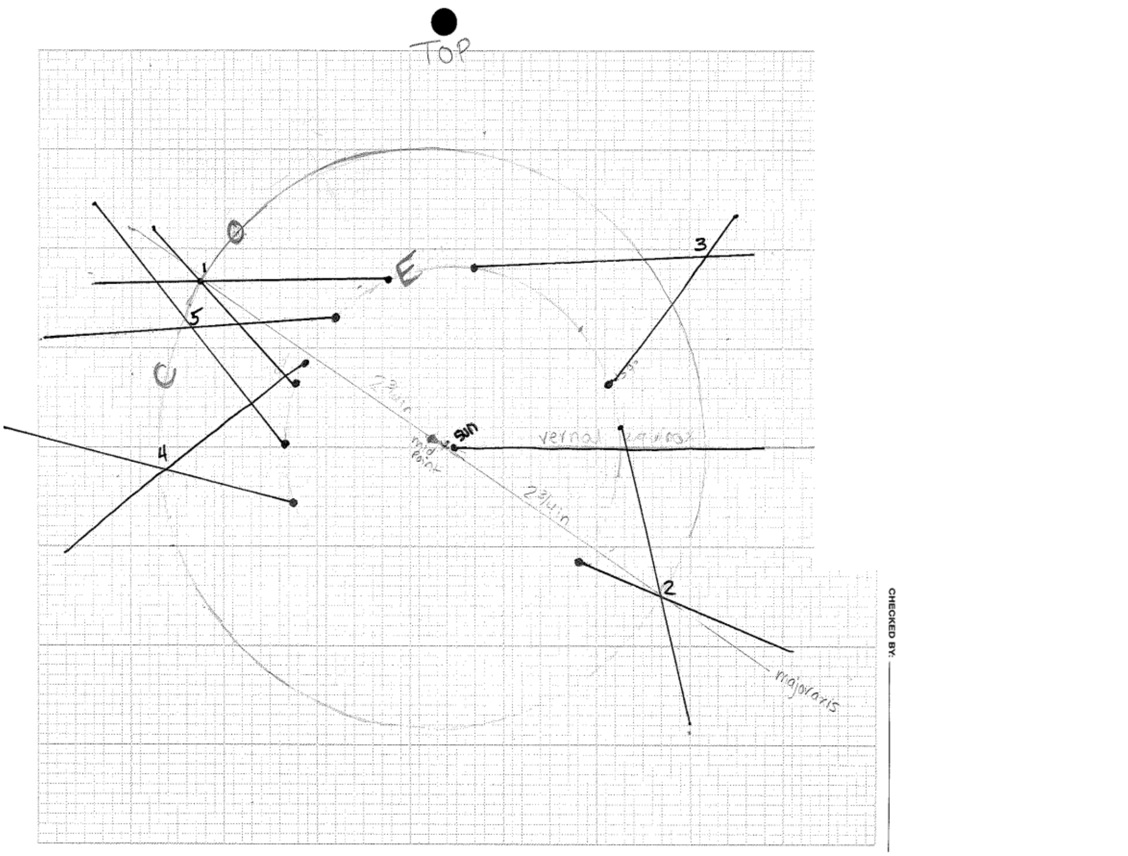


*Figure 1.*The orbit of Mars.

The student was asked to calculate the length of the planet’s semimajor axis based on the scale she had used in constructing her diagram. The results of her calculations were as follows:

1. Semimajor axis (scale) = 6.985 cm
2. Semimajor axis = 1.397 AU
3. Percent error = 8.8%
4. Semimajor axis = 209,550,000 km
5. Distance of closest approach = 57,150,000 km
6. Distance of greatest separation = 133,350,000 km
7. (Your) value of eccentricity = 0.27
8. Percent error = 65.9%

The actual semimajor axis of Mars is 1.52 AU, which corresponds to approximately 227,920,000 km.

Astronomy, Summer 2015

This elective course was taken by six undergraduate science majors and two local high school students. The text used was *Astro 2: Instructor Edition* (Seeds & Backman, 2014). The teaching method was primarily lecture driven, with the instructor giving daily PowerPoint presentations. The course content was informational; although mathematics was not emphasized overall, the introduction involved explanations of the terms light year, astronomical unit, and parsec. Since these terms involve exponentially large numbers and distances that are difficult to comprehend, the introduction covered mathematical notions such as scientific notation and elementary distance equations. The dwarf planet Pluto was also emphasized, as the New Horizons space probe encountered the world in July 2015. A worksheet was given with five questions:

- It took nine years for the New Horizons probe to reach Pluto. How fast was it travelling in miles per hour?
- If there was a straight road from Earth to Pluto, and your car was travelling at a constant speed of 65 mph, how long would it take to reach Pluto?
- How long does it take for any electronic transmission to travel from Pluto to Earth? (You should come up with about 4.5 hours)
- How big is a billion? (What does that figure mean to you?)

Chapter 5: Analysis and Results of Student Data Collection

The goal of this study was to examine student learning techniques and problem-solving abilities when tasked with formulating a mathematical model of the orbits of our planetary neighbors, specifically Neptune and Mars. Topics of learning included the study research, the conclusions, the laws of planetary motion as discovered by Johannes Kepler in the beginning of the 17th century, the analysis of basic calculus and ODE laws and algorithms, and basic celestial mechanics. The specific mathematical principles needed for this study included arc length, accelerated forces, angular momentum, conic figures, and a fundamental understanding of cosmological distances. Students were also encouraged to read elementary astronomy texts to become familiar with relevant celestial terminology.

The subject of this interdisciplinary relationship between astronomy and mathematics was explored from Spring Semester 2013 through Summer Semester 2014.

- The Spring Semester 2013 included four students in two different ODE classes. The students participated in lecture, independent study, oral testing, and interviews. They were tasked with preparing a PowerPoint presentation that showcased their learning. (Their presentations and accompanying interviews are indexed in Appendix A.)
- This investigation continued in a Spring Semester 2014 ODE class with six students. The students participated in lecture, independent study, and testing. They were also tasked with preparing a PowerPoint presentation showcasing their learning. (Their presentations and accompany interviews are indexed in Appendix B.)
- The investigation continued with the Summer Semester 2014 Calculus class consisting of two students. The questions raised by the participating study students in Spring Semester 2014 were incorporated into this class. The students participated in

lecture and independent study by way of an experiment in orbital ellipses. Their final project did not require a PowerPoint presentation. They focused instead on documentation of an orbital ellipse of the planet Mars using Johannes Kepler's laws (see Appendix C).

- The investigation concluded with the Summer 2015 Astronomy class composed of eight students. The class was an elective the students chose to fulfill various degree requirements. The class also included two local high school students. This class was included in the investigation to examine student familiarity (or, perhaps, lack thereof) of the mathematics behind various astronomical concepts and to take advantage of several astronomical events occurring at that time.

Astronomy has often been called the oldest science. What is seemingly old assumes new relevance when students discover the value of a forgotten science. In this study, students were asked to make the connection between the skies above and modern-day undergraduate mathematics. The specific data examined included the following:

1. The presentation and accompanying PowerPoint "Trajectory Toward Neptune" conducted by Students B, C, and D on May 7, 2013.
2. The presentation and accompanying PowerPoint "Neptune" conducted by Student A on May 13, 2013.
3. The presentation and accompanying PowerPoint "Neptune" conducted by six students in the Spring Semester 2014 Introductory Differential Equations class.
4. The calculation of the orbit of Mars provided by a student in the Summer Semester 2014 Calculus II class.
5. The survey given to the Summer Semester 2015 Astronomy class.

The data were examined with the specific purpose of answering the research articles of question posed in the Introduction. With respect to these data, several claims are made, followed by the reasons why these claims are made:

1. There are varying levels of student familiarity with solar system knowledge: In the 5-7-13 presentation, Student B gave a detailed profile of Neptune, and Student D gave an analysis of the Voyager II journey. In the 5-13-13 presentation, Student A also described slides of Neptune and the Voyager II probe. In the May 2014 group presentation, some of the students merely presented definitions with accompanying pictures culled from the Internet. One student presented a slide with strange facts about Neptune. The Summer 2015 Astronomy class had little background in either mathematics or astronomy. Their appreciation of astronomy can only be represented by positive student evaluations.

2. Astronomy can be used as a tool for better understanding of mathematics: In the 5-7-13 presentation, Student C gave a mathematical explanation of the phenomenon of gravity assist, along with an illustrative slide. In the 5-13-13 presentation, Student A explored the concept of arc length by illustrating the formula with a slide juxtaposed with a slide illustrating the orbit of Neptune. Three students in the Spring Semester 2014 class examined mathematical implications of astronomical phenomena; Student KC used Kepler's third law to calculate the average distance of Neptune from the Sun, Student TC used Kepler's second law to approximate the rate differences between Neptune's orbit at apogee and perigee, and Student RL used and derived Kepler's third law to calculate Neptune's orbital period successfully. The student in the 2014 Calculus class successfully used longitudinal readings to trace the orbit of Mars.

3. Visual displays were used to illustrate aspects of mathematical and astronomical synchronicity: The three PowerPoint presentations offered in Spring Semesters 2013 and 2014 constituted visual evidence of student efforts and interest.

4. Historical applications of mathematical modeling were used to support the student learning process: In the Spring Semester 2014 class, Student KC used Kepler's third law to calculate planetary distance, Student TC used Kepler's second law to illustrate varying orbital speeds, and Student RL used the third law to calculate an orbital period. In the summer 2014 class, the student used Kepler's first law, along with Tycho Brahe's actual longitudinal readings, to calculate to calculate the orbit of Mars. As noted before, Kepler's laws are mathematical models that forged a bridge from ancient to modern astronomy.

5. Mathematical algorithms were used to construct astronomical models: Although no actual algorithms were displayed, the 5-7-13 presentation given by Student D used a diagram illustrating the mathematical physics behind acceleration boosts. In the 5-13-13 presentation, Student A displayed the arc length formula to illustrate orbital trajectories. The Spring Semester 2014 students displayed equations for ellipses and used proportionality models to demonstrate Kepler's laws. The summer 2014 student used longitudinal readings to formulate the ellipse demonstrating Kepler's first law.

The students' work during the semester and post semester feedback indicated evidence supporting the benefits of independent and collective mathematical application coupled with a genuine interest in discovering the symbiotic relationship between astronomy and mathematics. An examination of the student data collection, as compiled from various teaching methods, supported the observation that students are open to different learning techniques. Lecture tends to be the most common teaching technique, but students are equally receptive when lectures are supplemented with visual displays, preliminary testing, or interviews.

Diagnostic Test Results

The results from the written test (see Appendix D) given to the six students at the beginning of the Spring Semester 2014 ODE class revealed a lack of basic knowledge of the solar system. When the instructor reviewed the results with the students, he learned that astronomy was not part of their education process. Therefore, students had difficulty connecting celestial knowledge to basic mathematical principles such as the ellipse, arc length, and basic acceleration mechanics. All three of these mathematical principles are covered in either ODE or Calculus II. More important, command of these mathematical principles is essential to plotting the journey to the planet Neptune. The average scores of the six students who participated in the diagnostic testing reflected less than 40% comprehension of basic solar system knowledge.

As mentioned previously, one student had a background in astronomy. This student was in the Spring Semester 2013 ODE class. She was not tested, as the instructor was aware of her previous astronomical knowledge. Upon post presentation discussion with the instructor, she acknowledged that her astronomical ability aided her in making an immediate connection between the two sciences.

Lecture

Based on the student test performance (spring 2014), the instructor created three PowerPoints designed to provide basic celestial knowledge, supplement the fundamental differential equation course requirements, and motivate independent learning to fulfill the assignment (all three PowerPoints are displayed in Appendix F). The first, “Planets in Review,” gave a brief summary of the nine major planetary bodies in the solar system. Brief anecdotes were displayed for each planet. These included the following:

1. If Mercury replaced the Moon in orbit around Earth, tidal waves would be over 400 feet high, and coastal cities would no longer exist.
2. The Venera probes of the 1970s revealed the surface of Venus to be utterly inhospitable to any life-forms.
3. Earth is the only place in the solar system where life of any kind is actually known to exist. Life-forms range from intelligent (Albert Einstein) to not so intelligent (Stan Laurel and Oliver Hardy).
4. Deimos, a moon of Mars, has a gravitational force so weak that one could literally jump off its surface into space.
5. The Shoemaker-Levy comet impacted the surface of Jupiter in 1994. If the same comet had struck Earth, global extinctions would have resulted.
6. The Cassini probe, a joint project of NASA and the European Space Agency, has been exploring the Saturn system since 2004. In 2005, the Huygens sub probe made a soft landing on the surface of Saturn's moon Titan, the only satellite in the solar system with a significant atmosphere. In the same year, Cassini also viewed active warm water geysers on the tiny moon Enceladus.
7. Uranus was visited by Voyager II in 1984. Its rotational axis is tilted nearly 90 degrees. Its five major moons are named after characters created by William Shakespeare and Alexander Pope.

8. Voyager II flew by Neptune in 1989. It recorded the fiercest winds in the solar system (nearly 2100 mph). It also recorded the coldest temperature (40 degrees above absolute zero) on Neptune's cantaloupe moon, Triton.
9. The picture of Pluto and its moon Charon was taken by the Hubble Space Telescope. The Discovery probe, launched in 2005, arrived at the planet in July 2015.

Each planet description was accompanied with a list of movies made about the planet. This was done not only for entertainment purposes, but also to contrast actual exploratory evidence with popular fanciful depictions. The ultimate objective of this particular presentation was twofold: to provide information in an entertaining form and to display phenomena associated with each planet of the solar system visually.

The second PowerPoint, "Neptune," was created to provide necessary background information for the implementation of student projects. The planet Neptune was chosen as the basis for this project for a number of reasons. The inner planets (Mercury, Venus, and Mars) have been extensively studied and explored. The gas giants Jupiter and Saturn have also been examined and visited by unmanned space probes. In contrast, Neptune has been visited only once, by the Voyager II space probe in August 1989. The brief flyby revealed a planet in meteorological turmoil, along with a geologically active main satellite, Triton. The discoveries in this encounter included the following:

1. The highest recorded wind velocities (2100 mph) and a great storm (designated the Great Blue Spot) on Neptune itself.
2. The lowest recorded temperature (40 degrees above absolute zero) on the moon Triton.
3. Active geysers spewing liquid nitrogen on Triton.

Melman (2007) wrote a thesis report titled *Trajectory Optimization for a Mission to Neptune and Triton*. The mathematics displayed in the report were extremely complex and outside the range of the undergraduates involved in this study. The concepts of orbital ellipses and arc lengths, however, form the bases of trajectory analysis. These elemental notions are commonly covered in undergraduate calculus and differential equations courses. The ultimate objective for this PowerPoint was to provide background and familiarity with the planet, which was also the objective of this mathematical exercise.

The third PowerPoint presented was titled “Creative Thought in Mathematical History.” The purpose of this PowerPoint was to present the creative processes evident in several historical breakthroughs in mathematical history. From the tile proof of the Pythagorean Theorem to the creation of the Mandelbroit set, most historical mathematical discoveries have been characterized by intuitive innovation and plenty of hard work. This presentation of the mathematical thought processes used by great mathematicians in history was displayed to the students in this study to inspire them and to aid them in their own creative approaches to the problem at hand.

One of the scientists examined in this presentation was Johannes Kepler, who used mathematical reasoning to arrive at the three basic laws of planetary motion. In particular, he examined the positions of the planet Mars, as recorded by the astronomer Tycho Brahe. Over a period of 4 years of research, Kepler concluded that the orbit of Mars is an ellipse with the sun as one of the focal points. This discovery marked the dawn of modern astronomy, and it also served as evidence of the importance of mathematics in the scientific method. The students in the Summer 2014 Calculus class used Tycho Brahe’s readings to calculate the elliptical orbit of Mars. The ultimate purpose of this PowerPoint was to provide students with insights into the nature of mathematical reasoning. The presentation was also meant to highlight the efforts of Johannes Kepler in creating a mathematical model.

Students reacted positively to the PowerPoint lecture style and expressed a desire to proceed with the assignment. Specifically, students were intrigued by the intricacies of planetary motion and fascinated by the various factoids about the nature of our cosmic neighborhood. An analysis of each PowerPoint presentation follows, in which each of the research articles of question is addressed:

1. What is the level of student understanding of astronomy?
2. What evidence is there that students, either individually or as teams, use astronomy as a tool for a better understanding of mathematics?
3. Are visual displays of astronomy (i.e., PowerPoints) conducive to a greater understanding of mathematics?
4. Is there any evidence that astronomy can be used as a device leading toward a better understanding of difficult mathematical concepts (i.e., arc length, ellipses)?
    a. What calculus and ODE algorithms were applied to the tasks of determining planetary motion and orbital length?
5. Is there any evidence that exposure to the history of astronomy, and its connection to mathematics, is conducive to greater student appreciation of mathematics?

The first presentation, “Trajectory to Neptune,” was performed on May 7, 2013. The participants were three students in the college’s international program. Each one presented a section. The first two sections were expository in nature, while the third section delved into the mathematics involved. Students B and C, who presented the first two sections, were business majors. Their sections reflected a nominal understanding of astronomy, to answer Question 1. A visual display of astronomy was evident in both sections. However, there was no evidence of relevance to a greater understanding of mathematics (Questions 2, 3, and 4). A partial answer to Question 5 was given by Student C, who presented a history of the Voyager II probe. Student D, who presented the final section, was a mathematics major. In his section, he addressed the relevant mathematical questions. He used a solid understanding of astronomy (Question 1), along with an effective visual display (Question 3), to use astronomy as a tool for deeper mathematical insight (Question 2). In particular, he used a model involving a tennis ball and a moving train to illustrate the phenomenon of gravity assist (Question 4). It should be noted that this example is sometimes examined in first-semester calculus and physics. Student D provided sufficient evidence in his presentation to answer the first four articles of question.

The second PowerPoint presentation was held on May 13, 2013. It was given by Student A, who was an upper-class mathematics major. Her solo presentation was titled “Neptune.” She had an inherent interest and nominal understanding of astronomy (Question 1). She used her knowledge of orbital paths to come to a greater understanding of arc length (Question 2). She used visual displays in her presentation to make the connection between orbits and arc lengths (Question 3). She also investigated the large numbers involved in calculating the distance and mass parameters of distant planets. Citing orbital trajectories as examples, she displayed the arc length formula and explained this formula in terms of the Pythagorean Theorem (Question 4). She successfully addressed four articles of question in a presentation marked by enthusiasm and interest.

The third PowerPoint presentation was held on May 13, 2014. It was a group presentation given by six members of a class in differential equations and also titled “Neptune.” The students displayed nominal interest in astronomy, as evidenced by the quality of the slides they prepared

(Questions 1 and 3). Each student was responsible for explaining a slide. Two of the students presented slides displaying astronomical terms. One student displayed the equation of the eccentricity of an ellipse. Three students addressed the mathematics involved in Johannes Kepler's laws of planetary motion. In doing so, they gave evidence of using astronomy as a means to come to a better understanding of mathematics. Specifically, each student explored mathematical modeling (Question 2). One student used Kepler's third law to calculate the mean orbital radius of Neptune, which is basically an example of exponential proportionality. She successfully calculated a mean radius of 30.1 AU. Another student used Kepler's second law to compare the planet's rates of speed at different locations in orbit. This law states that a given planet's velocity increases as it approaches the sun. He successfully calculated two mean rates that were very close in value. However, he noted how close Neptune's elliptical orbit came to being a perfect circle. The third student used relevant figures, comparing Earth and Neptune, to derive Kepler's third law as a mathematical model. He displayed his figures and noted several anomalies about the planet. All three students successfully used astronomical models to verify underlying mathematical theories. They also used Kepler's laws, which represent a milestone in the history of astronomy. In doing so, they successfully addressed all five articles of question.

With regard to the Summer 2014 Calculus class, lecture was the primary motivational tool because the course curriculum demanded a focus on the fundamentals of differentiation and integration algorithms. Opportunities to create mathematical models were therefore limited compared to the ODE classes. Nonetheless, when these students were presented with the opportunity to optimize their learning by including the development of a mathematical model (specifically, the orbit of Mars as opposed to Neptune), they welcomed the assignment.

It should be noted that the instructor intentionally chose the planet Mars, as opposed to Neptune for the end point of the orbital project for the following reasons:

- Johannes Kepler’s laws are relevant to the orbits of all planets.
- Johannes Kepler’s laws are essential and fundamental to both calculus and ODE mathematical models.
- Johannes Kepler relied upon the actual longitudinal recordings of Tycho Brahe to calculate the orbit of Mars. These data are commonly used in a standard calculus course to calculate the equation of an ellipse. Mastering calculus is the precursor to mastering ODE.
- At the time of Johannes Kepler’s breakthroughs, Neptune had yet to be discovered, which made it a more relevant and ultimately a more challenging project for a class in ODE as opposed to calculus.

PowerPoint Analysis

The PowerPoint presentations were given by 10 students, including four students from Spring Semester 2013 and six students from Spring Semester 2014. All 10 students involved in the PowerPoint presentations represented an unusual amalgam of interest, intelligence, and enthusiasm. These qualities are not present in every student in every class. It has been noted that the actual mathematics involved in the calculation of the celestial mechanics of such a deep space voyage (the trajectory to the planet Neptune) is extremely complex. The students’ assignment involved the basic mathematical principles upon which the more complex equations are based.

While the PowerPoint presentations reflected students’ appreciation and ability to connect mathematics and astronomy, there was a noticeable difference in the expression of the content. For example, the student from Spring Semester 2013 who developed and presented the presentation on her own displayed a solid command of mathematics and a distinct passion for astronomy. It was noted that this student already possessed a strong command of ODE and Calculus II, as well as a basic knowledge of astronomy. This student successfully intertwined her knowledge of astronomy with the mathematics involved. This was evident in the illustrations she displayed in her comprehensive presentation.

The group presentations (Spring Semester 2013 and Spring Semester 2014) took a strategically methodical path of first laying out the foundations of astronomy and then progressively incorporating the relevant mathematics to support the astronomical phenomena. The groups worked synergistically, keeping the end goal of the project in mind. Each participating student focused on one particular component of the project, thereby building the presentation story. The project objective, along with a proof of Kepler’s third law of planetary motion, was eloquently stated at the conclusion of the presentation. The students thus demonstrated their ability to integrate the mathematical principles with the astronomical phenomena.

In summary, the mathematics explored by these students reflected a solid grasp of the mathematical subject material (differential equations and Calculus II), and an appreciation for the shared connection with astronomy. Specific topics examined included the following:

1. The equation of an ellipse: Kepler’s first law of planetary motion states that the orbit of any planet is an ellipse with the sun as a focal point. Ellipses were examined in all three PowerPoints.

2. The equation for arc length: This formula from calculus is used to calculate lengths involving orbital trajectories. This equation was specifically examined by Student A in the May 13, 2013, presentation.
3. Mathematical modeling involving exponential proportionality: Kepler's third law of planetary motion states the proportionality between the square of its yearly period and the cube of its mean distance from the sun. This law was specifically examined by two students in the May 13, 2014, presentation.
4. Mathematical modeling involving orbital velocity: Kepler's second law of planetary motion states that a planet's radial speed varies with its distance from the sun. This law was specifically examined by one student involved in the May 13, 2014, presentation.
5. Comprehension of the mathematical nature of the phenomenon of gravitational assist: An illustration of this effect was specifically examined by a student in the May 7, 2013, presentation.
6. Comprehension of the large numbers involved in the data of astronomical phenomena.

Videotape Analyses

May 2013 Presentation 1. This was a videotape of a PowerPoint presentation conducted by three students in a differential equations class during the Spring Semester 2013. The three students (JS, JB, CP) were in the college's international Program; all hailed from South Korea. JS was a math major, while JB and CP majored in business. The presentation lasted approximately 15 minutes, and an analysis follows.

Each student presented a subdivision of the PowerPoint. The first two sections, "Neptune" and "Voyager 2," were presented by JB and CP, who were business majors. These sections were expository rather than mathematical. JS, a mathematics major, conducted the third section, "Trajectory." The student presented a mathematical analogy to illustrate the concept of gravity assist.

The videotaped evidence shows a definite familiarity, on the part of all three students, with knowledge of the solar system. One student, JS, used a mathematical model to illustrate the Voyager trajectory; in doing so, he displayed a better understanding of the mathematical undertones of this space journey. The PowerPoint itself, constructed by the students, is a visual display in astronomy where mathematical modeling was evident. An example of this is the description of the eccentricity of Mars' orbit.

> Mars' orbit has an eccentricity of e = 0.0086. The aphelion for the planet is 4.498 * 10^9 km.
>
> CV: [Displays another slide] The eccentricity is .00865.

May 2013 Presentation 2. This presentation was conducted by a single student who displayed a marked interest in astronomy. Her enthusiasm for the subject matter was reflected in her performance. She recognized the mathematical foundations of astronomy in two instances. First, she examined the formula for arc length and its importance in calculating the orbital trajectory of the Voyager II probe. She also discussed the nature of Neptune's orbit without going into the specifics of ellipse calculations.

May 2014 presentation. The presentation was created by all six members of a differential equations class conducted at Centenary College in the spring semester of 2014. Three of the students (TC, HD, TT) read off the PowerPoint. One student (KC) described the

PowerPoint page on Kepler's third law relative to the planet Neptune. She explained the numbers on the page and tried to derive the law using Neptune's characteristics (semimajor axis = 30.059 AU; period = 164.8 years). Another student (CV) explained two PowerPoint pages on Kepler's second law using a given distance of 50,000 km. The final student (TM) derived Kepler's third law using the second law as a basis. He also explained the last few pages of Neptune facts on the PowerPoint.

By their own admission, the students who participated in the demonstration became familiar with previously unfamiliar astronomical expressions. The evidence for three of the students (TC, HD, TT) involved simply reading off the PowerPoint slides. Evidence for the other three (KC, CV, TM) involved mathematical interpretations of the slides they presented.

KC presented a mathematical example of Kepler's third law. Specifically, the student calculated the mean distance of Neptune's orbit from the sun using the planet's period (year) of 164.8 Earth years. KC correctly calculated a mean distance of 30.059 AU, which she then correctly converted to approximately 3 billion miles.

CV gave a demonstration of Kepler's second law using an arbitrary distance of 50,000 km. He was attempting to show that a planet travels faster at its closest approach to the sun (perigee) than it does at its most distant point in the orbit from the sun (apogee). While his calculations actually produced a greater time for the perigee (2.59 hours) as opposed to the apogee (2.538 hours), this could be attributed to approximation errors. More important, CV correctly realized that the numbers indicated an orbit of very low eccentricity; Neptune's orbit, in fact, is very close to being a perfect circle.

Like KC, TM delved into the mechanics of Kepler's third law. He actually proved this law, demonstrating how it could be derived from Kepler's second law. It should be noted that this student, in Spring Semester 2014, also engaged in a celestial mechanics research project involving the famous three-body problem.

The evidence provided in this demonstration provides an answer to the first research question: What is the level of familiarity of student knowledge of the solar system? Although test scores clearly indicated initial unfamiliarity with the subject matter, the demonstration confirmed subsequent understanding.

The results showed that half of the students involved demonstrated a mathematical curiosity with the astronomical orbital figures. Although the evidence is not overwhelmingly positive, this demonstration shows that astronomy can be used as a tool for a better understanding of mathematical modeling.

Summer 2014 Analysis

One student successfully completed the assignment, which involved drawing a diagram of the orbit of the planet Mars based on the recorded longitudinal observations by Tycho Brahe in the latter half of the 16th century. Based on her scaled diagram (see Figure 1), she then calculated the length of the semimajor axis of the orbit, along with its eccentricity. In the diagram, the semimajor axis would have corresponded to half the length between Position 1 and Position 2. Her semimajor axis length, 6.985 cm, was then converted to astronomical units using the scaled radius of Earth’s orbit (the diagrammed inner circle with the Sun at center). She used a radius of 5 cm to produce a length of 1,397 AU. The actual mean distance of Mars from the Sun is 1.52 AU. This resulted in a relative error of about 8.8%. In the diagram, the actual radius of the inner circle (corresponding to Earth’s orbit) is 4.2 cm, which would produce a semimajor axis of 1.66 AU and a corresponding percentage error of 9.4%. Given the approximate values of the longitudes in the chart, either value would have come within 10% of the actual mean distance. Furthermore, her value of 1.397 AU would correspond to her converted value of 209,550,000 km (using a scale of 1 AU = 1.5 * 10^8 km). The actual value of the mean distance, which corresponds to the semimajor axis, is 227,920,000 km (see Appendix G). This would give a relative error of approximately 8%.

To calculate the eccentricity, the distance from the center of the major axis to one of the focal points is divided by the length of the semimajor axis. The eccentricity of Mars is 0.0935 (see Appendix G) based on the fact that Kepler's first law states the Sun is a focal point of every planet's orbit. In the student's diagram, the distance of the Sun from the midpoint of the major axis is approximately 0.5 cm. Using the scaled semimajor axis length of 6.985 cm, the value of the eccentricity is approximately 0.0716. This would result in a percentage error of about 23.44%. The student's calculated value of 0.27 results in a much greater error percentage. Two other calculations involved the distance of closest approach and the distance of greatest separation of Mars from Earth. Both calculations involve the alignment of Earth, Mars, and the Sun along the Martian major axis. From the student's diagram, the closest approach, or perigee, would be about 75,000,000 km. The greatest separation, or apogee, would be about 425,000,000 km. The student's values (57,150,000 km and 133,350,000 km) differed significantly.

An analysis of the data led to two conclusions. First, the student successfully interpreted the longitudinal locations used to locate the five critical points in her diagram of the orbit of Mars. This led to the determination of the Martian semimajor axis within a 9% degree of error, which would indicate the successful use of astronomical data to determine the basic properties of an ellipse. The calculations for determining the eccentricity, along with the apogee and perigee, were less successful. This would seem to lead to either of two conclusions: a mistaken definition was used in the calculation or, more likely, the scale measurements in the diagram resulted in widely varying degrees of accuracy. Ultimately, the student's efforts resulted in positive answers to two of the research questions:

1. Astronomy can be used as a tool for a better understanding of mathematics. The student used the longitudinal readings of the planet Mars to draw an ellipse successfully.
2. A historical data set in astronomy can be used to construct a mathematical model. In this case, the student used the longitudinal readings of the astronomer Tycho Brahe to construct a mathematical model of an ellipse.

Summer 2015 Analysis

The astronomy class in summer 2015 was an elective course with no mathematics majors. The students had the opportunity to solve the worksheet problems (p. 74) independently. The problems were then examined in class, guided by the instructor. The first question was as follows: "It took 9 years for the Discovery probe to reach Pluto. How fast was it travelling?" A distance of 3 billion miles was assumed. The standard equation was also assumed:

Distance = Rate * Time.

With a time of 9 years converted to 78,840 hours, the rate was calculated to be approximately 38,052 mph. The second question once again involved the basic distance equation: "If there was a straight road . . . and a constant rate of 65 mph, how long would it take to reach Pluto?" The answer, 46,153,846 hours, converts to 5,269 years.

The third question involved the speed of light: 186,000 miles per second: "How long does it take an electronic transmission to travel from Pluto to Earth?" Using the distance equation and various conversions, the answer, 4.5 hours, merely confirmed a figure prominently mentioned in news broadcasts. The answers to all these questions were arrived at through mutual discussion between instructor and students. The fourth question involved speculation and investigation for the students: “How big is a billion?” Some of the results included the following:

- If a billion pennies were stacked one on top of the other, the height would be 870 miles.
- One billion flies grouped together would be the equivalent of the mass of an elephant.

The investigation of these questions produced various answers to the research questions. For the first question, the students involved definitely gained familiarity with basic solar system knowledge. The second question was actually answered in reverse; elementary mathematical notions were employed to gain a deeper understanding of the vast distances involved in astronomy. To answer the third question, visual PowerPoint displays were used throughout the course to illustrate the variety of topics explored. The fourth question involved the history of astronomy; the mathematical model of an ellipse was examined to illustrate Kepler’s laws of orbital motion. The fifth question was only answered indirectly; the only mathematical algorithms used were the distance equation and elementary conversion processes.

Interviews

Interviews were only conducted during the middle of the Spring Semester 2013 ODE class. The class was split into two groups (Group A and Group B). Group A was represented by one student. This student had previous astronomical knowledge. Group B was comprised of three international students, none of whom had previous astronomical knowledge. The interviews were informal and recorded. They focused on the course syllabus and additional lecture data about astronomy as provided by the instructor.

The outcome of these interviews resulted in clarification of student perspective about the connection between astronomy and mathematics. Specifically, the instructor modified the course lecture to enhance learning about astronomy and made its connection to mathematics (e.g., arc length, ellipse, and acceleration mechanics). In turn, students were motivated to embark on independent learning to enhance the connection between both sciences.

Interviews were not incorporated in the spring 2014 or summer 2014 courses for two reasons: (a) a strict course syllabus and (b) the syllabus content and compressed course timelines negated an effective interview process. Two interview sessions were conducted with the three international students (A, B, C) in Spring Semester 2013. The first session covered a specific topic in ODE: homogeneous equations with constant coefficients (see Appendix H). The majority of the session consisted of lecture demonstrating various examples of homogeneous equations and the accompanying algorithms used to solve these equations. This lecture was conducted with considerable verbal interaction between teacher and students. Specifically, four examples were covered. Toward the end of the session, the instructor suggested Internet research on the planet Neptune. A short discussion on astronomy, NASA, and the Apollo missions of the 1960s followed.

The second session began with a short discussion of the teacher's family and career background (see Appendix H). This was followed by a discussion of the differential equations algorithm known as reduction of order. Specifically, quadratic differential equations involving two unknowns were examined. With the first example presented, one of the students asked for a clarification of the phrase *reduction of order*. Once this was explained, two more examples were presented. Again, there was considerable interaction between students and teacher. After these examples were analyzed, time ran out on the session.

The effectiveness of these two interview sessions should be judged against the PowerPoint presentation given by these students at the end of Spring Semester 2013. The presentation was mostly expository. The mathematics covered in the gravity assist model is normally encountered in first-semester calculus or physics. While the interest in astronomy was evident in the presentation, the level of mathematics did not reflect the difficulty normally encountered in an introductory course in differential equations.

In addition to these sessions, two interviews were conducted with the single student who presented the second Neptune PowerPoint presentation in Spring Semester 2013. Both interviews were conducted prior to the presentation (see Appendix H for the transcript). The first interview session began with a discussion of the student's mathematical background, along with her long-standing interest in astronomy. A pretest taken earlier was then discussed and analyzed. Ultimately, the student appreciated the opportunity to brush up on her calculus background.

The second interview developed into a productive learning experience. Two examples were discussed. The first one involved an advanced calculus integral, presented in another class, that the student was having difficulty solving. The instructor suggested simple substitution or integration by parts, as both algorithms are normally encountered in first-year calculus. Through productive interaction between student and instructor, the problem was solved by successive applications of the integration-by-parts algorithm. With some guidance, the student succeeded in applying the algorithm to solve a spontaneously presented problem.

The second example involved deriving the formula for arc length. This formula was chosen because, on a simple level, the formula can be used to calculate the length of orbital segments and trajectories. The topic itself is normally encountered in Calculus I. With some guidance, the student successfully derived the formula using simple derivatives and the Pythagorean Theorem. The instructor then presented two problems involving arc length. The first problem involved the length of a line segment. This problem was chosen because the answer arrived at by use of the arc length formula can easily be verified by simple geometry. The student had no difficulty applying the formula to obtain the correct result. A second, more complex example involved using trigonometric formulae. The student solved the problem, but needed assistance with the various trigonometric identities involved. Throughout the course of this session, the student showed considerable input and enthusiasm.

The effectiveness of these two sessions must be judged against the presentations the students conducted at the end of Spring Semester 2013. As in the previous student presentation, there was considerable exposition, specifically on the planet Neptune. In this case, considerable attention was paid to the arc formula and its relevance in calculating orbital trajectories. The student demonstrated an enthusiasm in astronomy that translated to mathematical models involving arc length and ellipses.

Chapter 6: Conclusions

The analysis and results of this research reveal positive student motivation and comprehension of new learnings when exposed to an unfamiliar scientific milieu. Despite having a limited or complete lack of basic solar system knowledge, the students expressed a desire to learn the connection between astronomy and mathematics. Their final presentations, whether a PowerPoint (ODE) or written analysis (Calculus II), demonstrated their comprehension of mathematics through the application of astronomy. In each presentation, all students exhibited a command of astronomy, thereby expanding their knowledge base and connecting the symbiotic relationship between the two sciences. Additional post class discussion revealed student appreciation and satisfaction in exploring astronomy, specifically its fundamental relevance to a real-life scenario where mathematical modeling is applied. Participating students in this study successfully applied and expanded their problem-solving abilities. The astronomy course conducted in summer 2015 provided an opportunity to deal with the symbiosis between mathematics and astronomy from a different perspective, that of displaying the importance of elementary mathematics in understanding the dimensional aspects of the world’s oldest science.

Testing oral or written methodology at the beginning of each class gauged students’ current understanding of astronomical knowledge and provided ample guidance for enhancing the lecture content. Only one student (Student A) had previous astronomical familiarity. This student made her familiarity known to the instructor prior to formally enrolling in the class. Sheer interest was the primary impetus in raising her level of knowledge. Therefore, the instructor selected oral testing, as this method provided flexibility to delve deeper into the specifics of her knowledge based on her initial verbal responses.

The other students did not have astronomy as part of a formal class curriculum. Therefore, the oral or written testing methodology provided a benchmark by which the instructor tailored the content and delivery of the lecture.

Although the interview technique was helpful, the instructor used it only for the Spring Semester 2013 class. The decision to discontinue the interview process was based on the class size of the spring 2014 course and the compressed time frame of the summer 2014 class. Strict syllabus adherence was a priority.

In all three courses (Spring Semester 2013, Spring Semester 2014, and Summer Semester 2014), the lecture method provided the students with ample exposure to the celestial background needed to engage in these projects, as evidenced by the content expressed by all students in their final projects. The instructor did supplement the lecture method with several PowerPoint presentations for both spring 2013 and spring 2014 classes. This decision was made for two reasons: the complexity of the final project (developing the trajectory of an unmanned space probe to the planet Neptune) required a deeper understanding of astronomy and the course length of 16 weeks provided ample time to introduce astronomy into the ODE course curriculum. The summer 2014 course curriculum for Calculus II, however, required strict attention to the course syllabus and was scheduled to run in a compressed learning timeline of 6 weeks. The nature of the Calculus II course did not require an extensive understanding or learning of astronomy; a PowerPoint presentation was not paramount to aid the students with their final project on recreating Kepler's discovery of the orbit of Mars. Additionally, the course material covered in a Calculus II class sets the framework for the material covered in ODE. A future goal would be to challenge these students, who now have a fundamental understanding of astronomy, to also present an unmanned space probe to the planet Neptune. It would be interesting to see the results and the methodology for completing this project.

The lecture technique was a sufficient motivator for independent, supplemental learning. This statement is reflected in the successful completion of the student projects, including a PowerPoint or visual example. A notable observation was that classes introduced to astronomy through PowerPoint presentations displayed more curiosity. They asked many questions and engaged in a collective conversation at the conclusion of each presentation. The class response to learning astronomy with the aid of PowerPoint supports a conclusion that visual representation positively supports lecture. Similarly, with regard to the benefits of using visual displays of astronomy to demonstrate the solution to a mathematical problem (specifically the ODE class project of planning the trajectory of an unmanned space probe to the planet Neptune), students methodically built each slide to support the relationship between both sciences.

Student A, from the Spring Semester 2013 ODE course, supplemented her astronomical knowledge by laying down the foundation for the inherent mathematical principle of an arc length. While her presentation did not overtly visualize the connection, she eloquently explained the formula for arc length and its connection with orbital trajectories.

The Spring Semester 2013 international students also explored arc length. They explained how a planetary gravity boost was necessary to complete the journey. The incorporation of gravity boost into the explanation of the journey to Neptune demonstrated a profound understanding of the mechanics of propulsion necessary to complete such a planetary journey. The Spring Semester 2014 students explored multiple aspects of the mathematics behind a proposed Neptune probe, including arc length, gravity boost, conic sections, and Kepler’s third law. They successfully implemented the planetary motion laws established by Johannes Kepler.

For instance, one student summarized and proved Kepler's third law and integrated this with the actual mechanics of the proposed journey to the planet Neptune.

The Summer 2014 Calculus II students explored Kepler's first law of motion by using longitudinal recordings to construct the orbit of the planet Mars. This illustrated a basic understanding of the essential characteristics of an ellipse.

The efforts of all five classes demonstrated that astronomy can be used as a device leading to a more profound understanding of mathematics. Further, the combined efforts of the two spring semester ODE classes demonstrated noticeable differences in team and individual approaches. The summer 2014 class demonstrated that astronomical history is conducive to a greater understanding of the relevant mathematics. The students in all five classes recognized the learning connection between the two sciences. Furthermore, student feedback revealed that interest in astronomy was a positive impetus to explore the relevant mathematical principles. This feedback also established the fundamental importance of mathematical modeling in making the connection.

The students who attended the astronomy course offered in summer 2015 had only a general background knowledge of mathematics. By taking the course as an elective, the students displayed genuine interest in the subject. Their interest was further enhanced by fortuitous timing; in this case, the summer of 2015 marked the successful arrival of the New Horizons space probe to the distant dwarf planet Pluto. The 9-year journey of the NASA spacecraft covered a distance of over 3 billion miles. This real-life event was covered by virtually all major news media and offered the unique opportunity to explore the mathematical perspective behind the exponentially large distances involved in space exploration. The mathematical principles involved were essentially simple, as they involved the conversion process and the distance formula. The students nonetheless gained an appreciation of the vastness of outer space dimensions, without being overwhelmed by higher level mathematics. Later, the same students worked with the simple model of an ellipse to illustrate Kepler’s laws of planetary motion.

The answers to the five research articles of question can be summarized as follows:

1. What is the level of familiarity of student knowledge of the solar system? Initial pretests indicated a low level in all four classes, with one student in the Spring 2013 ODE class having a working knowledge of the solar system. Subsequent PowerPoint presentations in the three ODE classes offered proof of an increased student awareness of basic astronomy facts and terminology. The Summer 2014 Calculus II class demonstrated a working knowledge of the nature of planetary orbits and accompanying historical background. The Summer 2015 Astronomy class offered a survey of general information on stars and planets to students with little previous background knowledge.
2. How can astronomy be used as a tool for better understanding of mathematics? There were several instances in the five classes where basic principles in astronomy were illustrated by mathematical models. For example, Kepler’s laws of planetary motion were illustrated using mathematical models involving ellipses. Also, space probe trajectories were examined using the standard calculus formula for arc length. In yet another instance, the large numbers involved in planetary and galactic distances were illustrated with proportional perspective models.
3. Are visual displays of astronomy conducive to this understanding? Visual PowerPoint displays were used extensively to support the lectures given in all four classes. The astronomy course, in particular, was characterized by daily displays of current topics,

such as the New Horizons encounter with the dwarf planet Pluto. In the upper-level mathematics classes, the response to this stimulus was evident in the quality of the individual PowerPoint projects.

4. Can exposure to historical examples of mathematical modeling in astronomy support this learning process? A specific historical example used in all five classes was Johannes Kepler's discovery and development of the laws of planetary motion in the early 17th century. His achievement marked the dawn of modern astronomy. The primary focus of his work was his examination of the orbit of the planet Mars. The Calculus 2014 class, in particular, used Kepler's figures, painstakingly recorded by the astronomer Tycho Brahe, to construct the mathematical model of an ellipse. All four mathematics classes specifically referenced Kepler's laws in their individual projects.
5. How were calculus and ODE algorithms used to construct astronomical mathematical models? The algorithms used to construct an ellipse are covered in Calculus I. As mentioned before, these algorithms were used to illustrate the elliptical nature of planetary orbits. Arc length is also a concept covered in Calculus I. This concept was used to calculate spacecraft trajectories. Physical models involving distance, velocity, and acceleration are routinely used in both calculus and ODE to develop the equations governing spacecraft journeys.

Relevant articles in prominent mathematical educational journals have been scarce. Nonetheless, history has shown that there is a vital link between astronomy and theoretical mathematics. The students at Centenary College who participated in this study recently demonstrated this through their projects. The students' eagerness to express their revelation about the significance of astronomy was revealing.

A few conclusions can be drawn from this study. From the presentations given, student interest in astronomy as it relates to mathematics is evident, and the immediate future of astronomy can only fuel that interest. From continuing revelations on Pluto as electronic feedback from the probe is translated and disseminated, to continued explorations of the planet Mars by mobile rover vehicles, and to the summer 2017 solar eclipse that will be visible across North America, opportunities to exploit student curiosity exist. Also, the lack of relevant literature emphasizes the need for further research articles on educational ramifications. The present astronomical revelations require mathematical modeling techniques involving differential equations. Opportunities will exist for examining student success in applying these techniques. In short, this study showed the promise for astronomy to connect mathematical techniques and mathematical modeling, as well as to motivate student engagement.

A subtext to this study relates to the role of the instructor, who relied upon the sage guidance offered by one of his first trusted colleagues: "Remember, the students who are signing up for your class aren't just taking math. They're taking you" (T. Griesbach, personal communication, Jan. 12, 1975).

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Appendix A: Spring Semester 2013

There were two student groups; student A by herself formed one group, and three international students (B, C, and D) formed the second group.

Table 3

Student Course Breakdown: Differential Equations (ODE), Spring 2013

COURSE	# of CLASSES	# of STUDENTS PER CLASS
ODE	2	1 – Student A 3 – Student B, C, D

Student A was a junior at the time with an equine major and mathematics minor. An achiever with a high GPA, her work ethic is matched by her enthusiasm to explore new topics. When presented with the opportunity to participate in a mathematics related astronomy project, she immediately agreed.

Students B, C, D were three Korean students in their junior year participating in the College's International Program. As with Student A, they were enthusiastic about participation in the project.

Typical to students in this program, all had completed a calculus program in their pre-collegiate school experience overseas. One particularly notable quality was their mutual curiosity, as they took many opportunities to ask questions about the instructor's educational background and life experiences, along with American life experiences in general. All three were Business majors with Mathematics minors.

Both student groups were tasked with developing and presenting a PowerPoint presentation. Both presentations were videotaped.

Student A PowerPoint Presentation

Student A's presentation, entitled NEPTUNE, began with a physical description of the planet and the history of its discovery. Then an updated version of relevant information on both Neptune and Triton was given, based on the August 1989 flyby of the Voyager II probe. Student A then discussed the relative distances involved, including lengths of the orbits and the time length such a journey would involve. (Anywhere from 12 to 20 years.). Related calculus principles were then discussed, including the concept of arc length with its associated formula. Finally, the concept of planetary based gravity "slingshot" effects were discussed. This principle was successfully utilized by both Voyager probes to boost their velocities and shorten the mission times considerably. An estimated 40-year journey to Neptune was reduced to 12 years, as Voyager II, launched in 1977 and reached Neptune in 1989.

In addition to the presentation, student A submitted to three audio interviews. The first two were of an introductory nature. Student A's background included four semesters of calculus and several upper level courses, including the present course. The material being presently covered compared to her second semester of calculus. With this as a background, the third in-depth interview concentrated on the subject of arc length. Specifically, the derivation of the arc length formula was covered (See fig. 1). Student A's extensive background made it easy for her to follow the derivation, but certain identities (specifically, involving trigonometry) were not immediately forthcoming. However, throughout the interview, Student A followed the formula derivation without difficulty.

$$L = \int_a^b \sqrt{1 + \left[f'(x) \right]^2} \, dx$$

Figure 2. Arc length formula

The interview in question began with an integral problem. The interview was held in a classroom with normal whiteboards. Although technology was available, it was not used. The problem in question was solved, and discussion proceeded to the subject of arc length. In setting up the question and referring to the following diagram (See Fig. 3)

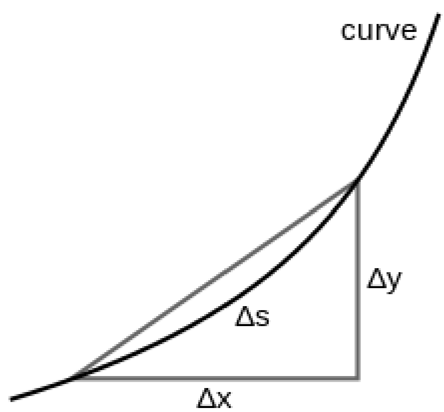

*Figure 3.*Arc length diagram

Student A made the following comment:

”Doesn’t this have to do with …the Pythagorean Theorem?”

From this point, the transition from geometry to differentials was smooth. At one point the instructor made a reference to the “magic trick” of multiplying an expression by one (in disguise as a ratio of differentials) to which Student A remarked:

“Well…I’ve never seen this (trick) before.”

The next stage of the derivation involved the use of algebra to bring the formula to its recognizably familiar final form (above, Fig. 2). Then an example was used, involving a linear formula where the length could be calculated without using calculus, to verify the formula. This reinforced the validity of the formula in the student's mind. The discussion then proceeded to a more complicated problem where both calculus and trigonometry were used. At this point Student A made a reference indicative of the present state of educational reality;

"I was actually trying to relate it to, like, real world problems. And I was GOOGLING…."

In the seemingly never-ending attempts at educational reform, technological fingertip information is an undeniable, indelible fact of life. Closely related to this fact is the educational de-emphasis on using rote memory to retrieve facts, like trigonometric identities. Prior to the technological revolution of the early 1980s, retention of certain facts was vital to success in Mathematics. Now, with such information at the tap of a button, the emphasis has shifted. In the subsequent discussion, Student A, an extremely motivated mathematics major, did not have the command of certain trigonometric identities that she certainly would have had in earlier decades. At one point she remembered the fundamental trigonometric identity:

"Sine squared plus cosine squared equals one."

She used this to derive the less familiar identity involving tangent and secant.

"Tangent squared plus one equals secant squared."

More trigonometric identities and mathematical "tricks" were subsequently employed to decipher a particularly difficult integral; namely, the definite integral, between two points, of secant cubed. At one point Student A had to convert from degree to radian measure, which she did readily.

From this moment on, Student A, with little instructor help, deciphered the integral and calculated the arc length in question.

Shortly thereafter, the interview concluded. Ultimately, Student A displayed an aptitude for analytical reasoning along with an avid enthusiasm for the subject material. Once again it should be pointed out that the actual celestial and propulsion dynamics involved in a theoretical journey to a distant planet would be extremely difficult. However, an appreciation of the trajectories involved was evidenced in Student A's presentation (See Fig. 4 below). In fact, such a journey has already been outlined; Trajectory Optimization for a Mission to Neptune and Triton (Melman, 2007).

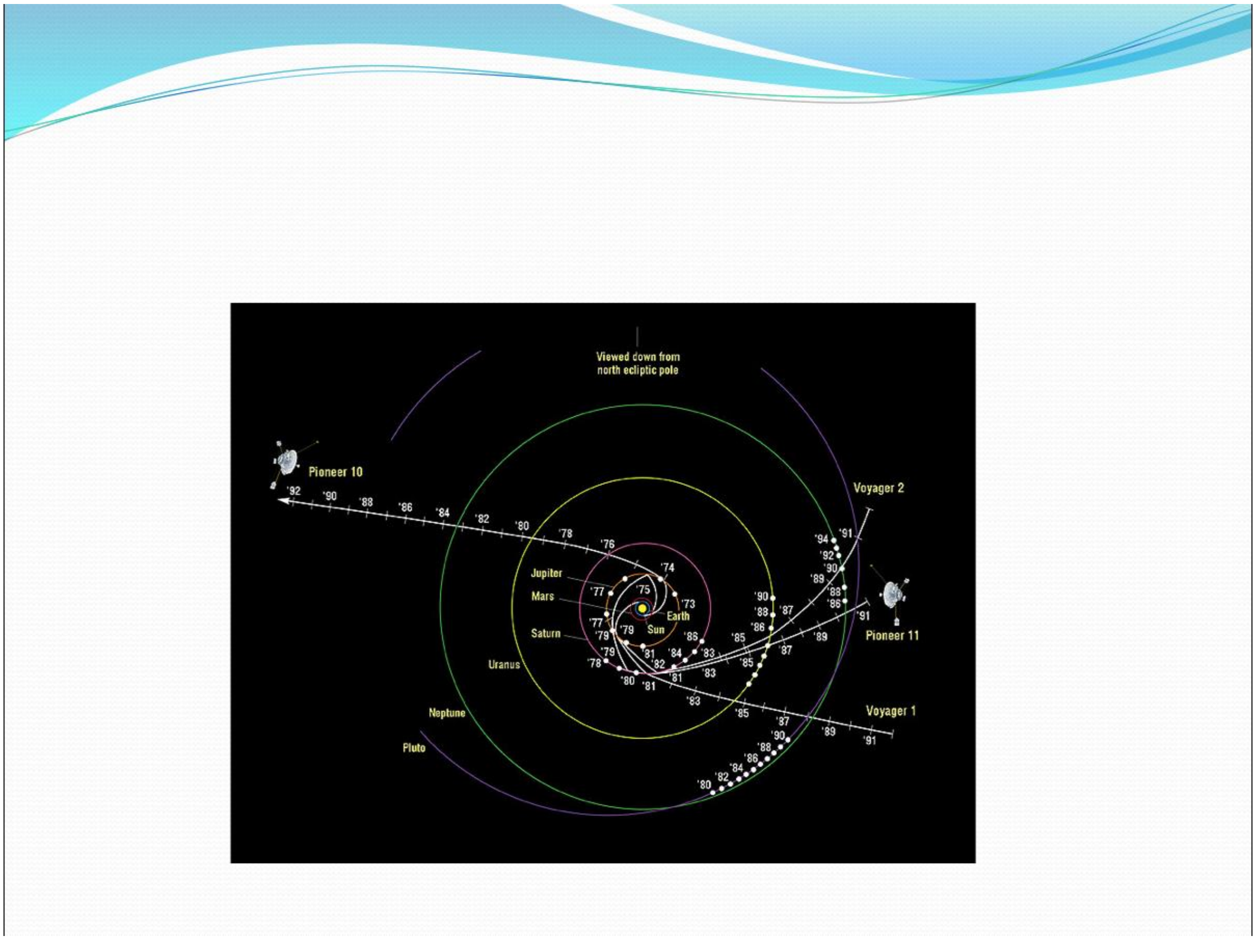

Figure 4. Pioneer and Voyager Orbital Trajectories

The relevance of arc length (in a simplified analogy) is obviously present in Student A’s presentation.

“Why would this be called reduction of order?”

In the instructor’s previous experience, such a question was never actually asked. Most students would seem to accept the expression as they would any other mathematical term! The rest of the class (interview) proceeded in the normal fashion, with problems assigned and work continuing with the project.

Student B, C, D PowerPoint Presentations

In addition to Student A's presentation and interviews, students B, C, and D also gave a presentation and consented to interviews. Their PowerPoint presentation, "Trajectory toward Neptune", was videotaped. Each student (B, C, D) explained a program segment: Student B gave Neptune planetary facts, Student C explained the historic Voyager II flyby of the outer planets in the late 20th century, and Student D presented the mathematics of gravity assisted "boosts". As in the other presentation, trajectory models (specifically, the Voyager I and II probes were displayed. (See Fig. 5)

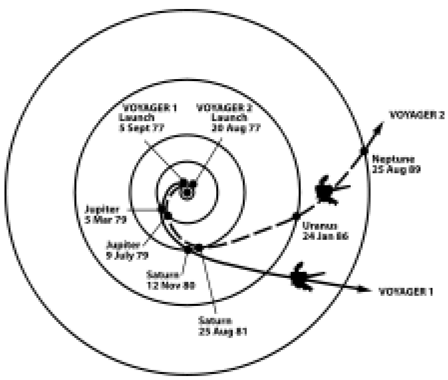

Figure 5. Voyager Trajectories

Additionally, Student D presented a specific illustration of the mathematics of a planetary assisted gravity "boost" (See fig. 6)

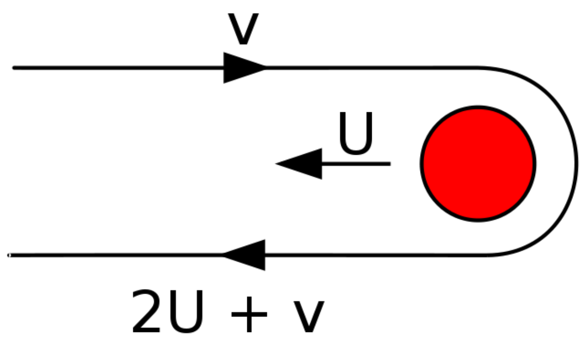

Figure 6. Gravity Assist Diagram

The students displayed enthusiasm and knowledge on a difficult subject, particularly because only one specialized in Mathematics (The other two were business majors). In addition to the presentation, two videotaped interview sessions were held, in which pertinent topics in ODE were covered. The first interview covered the topic of homogeneous equations with constant coefficients. Specific problems involved the use of solutions of the form e^mx to break equations down to quadratic or cubic polynomials solvable by factoring or the use of formulas. Related topics included synthetic division. The methods were quite familiar to the students. The problems were solved quite smoothly, with little help from the instructor.

After this, the class proceeded to deal with a cubic polynomial and the use of synthetic division to render it factorable. The students then dealt with the converse: given a solution, how does one derive the original differential equation? A typical problem was examined, and homework problems were then assigned.

The class concluded with an update on student progress with the proposed planetary projection project. The students had done some research on the internet. They asked the instructor about his interest in Astronomy, at one point asking him if he had ever worked for NASA. (For the instructor, the subject has merely been a lifelong hobby.) The informal discussion then ranged from the Apollo missions of the 1960s and 1970s to the more recent

Space Shuttle journeys. The students were asked to GOOGLE the Neptune trajectory by the following class.

A second interview session was held with the same students a few weeks later. This time the main topic was reduction of order.

A differential equation was presented which involved reduction of order. The solution of this equation was interspersed with further references to the instructor's past. The present difficulty of finding any meaningful teaching job was also mentioned.

Eventually several problems were presented and solved. As in the previous interview, the students' analytical skills were evident. Also evident were attempts at a deeper understanding of the material itself.

Appendix B: Spring Semester 2014

Table 4

Study Course Breakdown: Differential Equations (ODE), Spring 2014

COURSE	# of CLASSES	# of STUDENTS PER CLASS
ODE	1	6

May 12, 2014 Neptune Presentation – ODE

Prior to the presentation the students were given survey pre-test concerning general astronomy. (Appendix D, p.38) The results, though inconclusive, showed a not-unexpected unfamiliarity in the subject.

The six students involved gave a videotaped presentation revolving around mathematical knowledge as it related to astronomy; the planet Neptune, its nature, and its orbital characteristics. An analysis of the presentation follows.

The students presented the topic as a team, with each one speaking on a specific aspect. The first three discussed terminology; they showed that comprehension of arcane terms can be achieved and readily presented. The final three dealt with the mathematical terms at hand. The first student explained Kepler’s Third Law and independently used Neptune’s “year” to correctly establish its mean distance from the sun. The second student used Kepler’s second law to establish Neptune’s orbital length and various speeds along its orbit. The third student independently derived Kepler’s third law and discussed various unusual aspects of the eighth planet.

Overall, the students independently;

1. Achieved familiarity with difficult astronomical terms
2. Discovered Kepler’s laws and used effective examples from their investigation of Neptune to solve mathematical problems
3. In one case, actually explored and derived Kepler’s third law

Appendix C: Calculus II, Summer Semester 2014

Table 5

Study Course Breakdown: Calculus II, Summer 2014

COURSE	# of CLASSES	# of STUDENTS PER CLASS
Calculus II	1	2

The following longitudinal positions of the planet Mars were originally recorded by the astronomer Tycho Brahe from his observatory in Denmark in the late 16th Century.

Date	Heliocentric Longitude of Earth	Geocentic Longitude of Mars
Feb 17, 1595	159’	135’
Jan 5, 1587	115’	182’
Sept.19,1591	6’	284’
Aug. 6, 1593	323’	347’
Dec 7, 1593	86’	3’
Oct. 25, 1595	42’	50’
Mar. 28, 1587	197’	168’
Feb. 12, 1589	154’	219’
Mar. 10, 1585	180’	132’
Jan. 26, 1587	136’	185’

The degree pairings are based on the Martian "year" of 687 days, and the Spring Equinox corresponds to a Solar reading of 0'.

(Note: the following instructions were provided in an Astronomy course conducted at Clark College)

Once every 687 days, Mars returns to the same point in its orbit around the sun. If we plot the lines of sight from the earth to Mars at this interval we can triangulate a point on the orbit of Mars. Materials needed include graph paper, protractor, compass, and centimeter ruler.

1. Turn the graph paper so that the long edge is horizontal. Place a small dot near the center labeled "Sun."
2. Using the ruler, draw a straight line to the right, starting at the Sun and ending roughly two centimeters from the right side of the page. This line is the "Vernal Equinox. "This is the direction an observer on earth would look to see the Sun on March 21. All angles will be measured counter-clockwise from this line.
3. Using the compass, draw a 5.0 cm. radius circle centered on the Sun. This is "Earth's Orbit." We know that the earth's orbit is actually an ellipse. You will see that the effect that this difference has on our model when you finish constructing the orbit of Mars.
4. This sets the scale of our drawing at 5.0 cm. = 1 A.U. (Astronomical Unit) where 1 A.U. is the average distance between the Earth and the Sun. (93 million miles or 1.5 *10^8 km.)
5. Note that the data table is divided into pairs of dates. Each pair represents an interval of one Martian year. Starting with February 17, 1585, use a protractor to plot the heliocentric longitude of the Earth (159') given in the table as a point on the earth's orbit.

6. Next center the protractor on the point you just marked and plot the geocentric position of Mars (135’) with a line.
7. Now repeat this procedure for the January 5, 1587 date. The point of intersection of the lines is the position Mars had occupied on these two dates. Label this as position “1.”
8. Repeat this procedure for the next four pairs of data, numbering each point in sequence. You should have five positions for Mars.
9. Kepler chose the first two sets of data to represent the aphelion and perihelion for Mars, respectively. Draw a line from the first position for Mars to the second position for Mars. This line is the major axis of the orbit.

Appendix D: Quizzes

Quiz 1

March 17, 2004

SURVEY QUIZ

(Easy)

1. How many planets are in our Solar System?
2. What is the planet closest to the sun?
3. What is the largest planet?
4. What is the planet whose orbit is closest to Earth's?
5. Which planet is dubbed "the red planet"?

(Harder)

1. The Galilean moons orbit which planet?
2. Which moon orbiting which planet is the only moon with an appreciable atmosphere?
3. Which planet was recently (2006) declassified to "planetoid" status?
4. For the past ten years the CASSINI probe has been in orbit about which planetary system?
5. In the late 1970s NASA sent unmanned space probes to fly by the outer planets of the Solar System. What were they named?

(Tough)

1. Name the Galilean moons.
2. What is the only moon in the solar system in retrograde orbital motion? (Hint; it also has the coldest recorded temperature in history.)
3. On what planet or moons is there evidence of man-made artifacts?

4. What is the only planet in retrograde rotation? (e.g. from the surface, the sun would be seen rising in the west and setting in the east.)
5. What is the largest body in the asteroid belt? (Hint; it made Gauss famous.)

Quiz 2

April 28, 2014

ODE further questions in Astronomy

Given the Neptune Orbital Fact Sheet and the opportunity to find information on the internet

1. Define the following terms; sidereal period, tropical period, aphelion, perihelion, semi-major, eccentricity
2. Determine the average distance of Neptune from the sun
3. What is this distance in AU (Astronomical units; 1 AU = 93,000,000 mi.)
4. Given that the major axis lies on the x-axis and the center is the origin, determine the equation of the Neptune (elliptic) orbit. (Use the Neptune Mean Orbital Elements in the table and a unit distance of 1 AU)
5. Using the comparative facts in the sheet on Earth and Neptune, verify Kepler's third law of planetary motion.
6. Travelling in its orbit calculate the time difference it takes Neptune to travel 50000 km at perigee to the same distance at apogee.
7. Given the polar spiral r = (1/2)∗Φ^2, calculate the arc length from theta =0 το Φ =6∗Π.

Appendix E: Neptune/Earth Comparison

Bulk parameters

	Neptune	Earth	Ratio (Neptune/Earth)
Mass (10^{24} kg)	102.42	5.9726	17.147
Volume (10^{10} km^3)	6,254	108.321	57.74
Radius (1 bar level) (km)			
Equatorial	24,764	6,378.1	3.883
Polar	24,341	6,356.8	3.829
Volumetric mean radius (km)	24,622	6,371.0	3.865
Ellipticity (Flattening)	0.01708	0.00335	5.10
Mean density (kg/m^3)	1,638	5,514	0.297
Gravity (eq., 1 bar) (m/s^2)	11.15	9.80	1.14
Acceleration (eq., 1 bar) (m/s^2)	11.00	9.78	1.12
Escape velocity (km/s)	23.5	11.19	2.10
GM (x 10^6 km^3/s^2)	6.8351	0.3986	17.15
Bond albedo	0.290	0.306	0.95
Visual geometric albedo	0.41	0.367	1.12
Visual magnitude V(1,0)	-6.87	-3.86	-
Solar irradiance (W/m^2)	1.51	1,367.6	0.0011
Black-body temperature (K)	46.6	254.3	0.183
J_2 (x 10^{-6})	3411.	1082.63	3.151
Number of natural satellites	14	1	

Planetary ring system	Yes	No

Orbital parameters

	Neptune	Earth	Ratio (Neptune/Earth)
Semimajor axis (10^6 km)	4,495.06	149.60	30.047
Sidereal orbit period (days)	60,189.	365.256	164.79
Tropical orbit period (days)	59,799.9	365.242	163.73
Perihelion (10^6 km)	4,444.45	147.09	30.216
Aphelion (10^6 km)	4,545.67	152.10	29.886
Synodic period (days)	367.49	-	-
Mean orbital velocity (km/s)	5.43	29.78	0.182
Max. orbital velocity (km/s)	5.50	30.29	0.182
Min. orbital velocity (km/s)	5.37	29.29	0.183
Orbit inclination (deg)	1.769	0.000	-
Orbit eccentricity	0.0113	0.0167	0.677
Sidereal rotation period (hours)	16.11*	23.9345	0.673
Length of day (hrs)	16.11	24.0000	0.671
Obliquity to orbit (deg)	28.32	23.44	1.208

* Magnetic coordinates (as determined by the Voyager 2 Radio Science experiment)

Neptune Observational Parameters

Discoverer: Johann Gottfried Galle (based on predictions by

John Couch Adams and Urbain Leverrier)

Discovery Date: 23 September 1846

Distance from Earth

Minimum (10^6 km) 4305.9

Maximum (10^6 km) 4687.3

Apparent diameter from Earth

Maximum (seconds of arc) 2.4

Minimum (seconds of arc) 2.2

Mean values at opposition from Earth

Distance from Earth (10^6 km) 4347.31

Apparent diameter (seconds of arc) 2.3

Apparent visual magnitude 7.8

Maximum apparent visual magnitude 7.78

Neptune Mean Orbital Elements (J2000)

Semi-major axis (AU) 30.06896348

Orbital eccentricity 0.00858587

Orbital inclination (deg) 1.76917

Longitude of ascending node (deg) 131.72169

Longitude of perihelion (deg) 44.97135

Mean Longitude (deg) 304.88003

North Pole of Rotation

Right Ascension: 299.36 + 0.70 sin N

Declination: 43.46 - 0.51 cos N

Reference Date: 12:00 UT 1 Jan 2000 (JD 2451545.0)

N = 359.28 + 549.308T degrees

T = Julian centuries from reference date

Neptunian Magnetosphere

Goddard Space Flight Center OTD (O8) Model

Dipole field strength: 0.142 gauss-Rn^3

Dipole tilt to rotational axis: 46.9 degrees

Longitude of tilt: 288 degrees (IAU convention)

Dipole offset (planet center to dipole center) distance: 0.55 Rn

Note: Rn denotes Neptunian radii, here defined to be 24,765 km

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Neptunian Atmosphere

Surface Pressure: >>1000 bars

Temperature at 1 bar: 72 K (-201 C)

Temperature at 0.1 bar: 55 K (-218 C)

Density at 1 bar: 0.45 kg/m^3

Wind speeds: 0-580 m/s

Scale height: 19.1 - 20.3 km

Mean molecular weight: 2.53 - 2.69 g/mole

Atmospheric composition (by volume, uncertainty in parentheses)

Major: Molecular hydrogen (H_2) - 80.0% (3.2%); Helium (He) - 19.0% (3.2%);

Methane (CH_4) 1.5% (0.5%)

Minor (ppm): Hydrogen Deuteride (HD) - 192; Ethane (C_2H_6) - 1.5

Aerosols: Ammonia ice, water ice, ammonia hydrosulfide, methane ice (?)

Appendix F: Earth/Mars Comparison

Bulk parameters

	Mars	Earth	Ratio (Mars/Earth)
Mass (10^{24} kg)	0.64174	5.9726	0.107
Volume (10^{10} km^3)	16.318	108.321	0.151
Equatorial radius (km)	3396.2	6378.1	0.532
Polar radius (km)	3376.2	6356.8	0.531
Volumetric radius (km)	3389.5	6371.0	0.532
Core radius (km)	1700	3485	0.488
Ellipticity (Flattening)	0.00589	0.00335	1.76
Mean density (kg/m^3)	3933	5514	0.713
Surface gravity (m/s^2)	3.71	9.80	0.379
Surface acceleration (m/s^2)	3.69	9.78	0.377
Escape velocity (km/s)	5.03	11.19	0.450
GM (x 10^6 km^3/s^2)	0.04283	0.3986	0.107
Bond albedo	0.250	0.306	0.817
Visual geometric albedo	0.170	0.367	0.463
Visual magnitude V(1,0)	-1.52	-3.86	-
Solar irradiance (W/m^2)	589.2	1367.6	0.431
Black-body temperature (K)	210.1	254.3	0.826
Topographic range (km)	30	20	1.500
Moment of inertia (I/MR^2)	0.366	0.3308	1.106
J_2 (x 10^{-6})	1960.45	1082.63	1.811

Number of natural satellites	2	1
Planetary ring system	No	No

Orbital parameters

	Mars	Earth	Ratio (Mars/Earth)
Semimajor axis (10^6 km)	227.92	149.60	1.524
Sidereal orbit period (days)	686.980	365.256	1.881
Tropical orbit period (days)	686.973	365.242	1.881
Perihelion (10^6 km)	206.62	147.09	1.405
Aphelion (10^6 km)	249.23	152.10	1.639
Synodic period (days)	779.94	-	-
Mean orbital velocity (km/s)	24.07	29.78	0.808
Max. Orbital velocity (km/s)	26.50	30.29	0.875
Min. orbital velocity (km/s)	21.97	29.29	0.750
Orbit inclination (deg)	1.850	0.000	-
Orbit eccentricity	0.0935	0.0167	5.599
Sidereal rotation period (hrs)	24.6229	23.9345	1.029
Length of day (hrs)	24.6597	24.0000	1.027
Obliquity to orbit (deg)	25.19	23.44	1.075

Mars Observational Parameters

Discoverer: Unknown

Discovery Date: Prehistoric

Distance from Earth

Minimum (10^6 km) 55.7

Maximum (10^6 km) 401.3

Apparent diameter from Earth

Maximum (seconds of arc) 25.1

Minimum (seconds of arc) 3.5

Mean values at opposition from Earth

Distance from Earth (10^6 km) 78.39

Apparent diameter (seconds of arc) 17.9

Apparent visual magnitude -2.0

Maximum apparent visual magnitude -2.91

Mars Mean Orbital Elements (J2000)

Semi-major axis (AU) 1.52366231

Orbital eccentricity 0.09341233

Orbital inclination (deg) 1.85061

Longitude of ascending node (deg) 49.57854

Longitude of perihelion (deg) 336.04084

Mean Longitude (deg) 355.45332

North Pole of Rotation

Right Ascension: 317.681 - 0.106T

Declination : 52.887 - 0.061T

Reference Date: 12:00 UT 1 Jan 2000 (JD 2451545.0)

T = Julian centuries from reference date

Martian Atmosphere

Surface pressure: 6.36 mb at mean radius (variable from 4.0 to 8.7 mb depending on season)

[6.9 mb to 9 mb (Viking 1 Lander site)]

Surface density: ~0.020 kg/m^3

Scale height: 11.1 km

Total mass of atmosphere: ~2.5 x 10^{16} kg

Average temperature: ~210 K (-63 C)

Diurnal temperature range: 184 K to 242 K (-89 to -31 C) (Viking 1 Lander site)

Wind speeds: 2-7 m/s (summer), 5-10 m/s (fall), 17-30 m/s (dust storm) (Viking Lander sites)

Mean molecular weight: 43.34 g/mole

Atmospheric composition (by volume):

Major: Carbon Dioxide (CO_2) - 95.32%: Nitrogen (N_2) - 2.7%

Argon (Ar) - 1.6%; Oxygen (O_2) - 0.13%; Carbon Monoxide (CO) - 0.08%

Minor (ppm): Water (H_2O) - 210; Nitrogen Oxide (NO) - 100; Neon (Ne) - 2.5; Hydrogen-Deuterium-Oxygen (HDO) - 0.85; Krypton (Kr) - 0.3

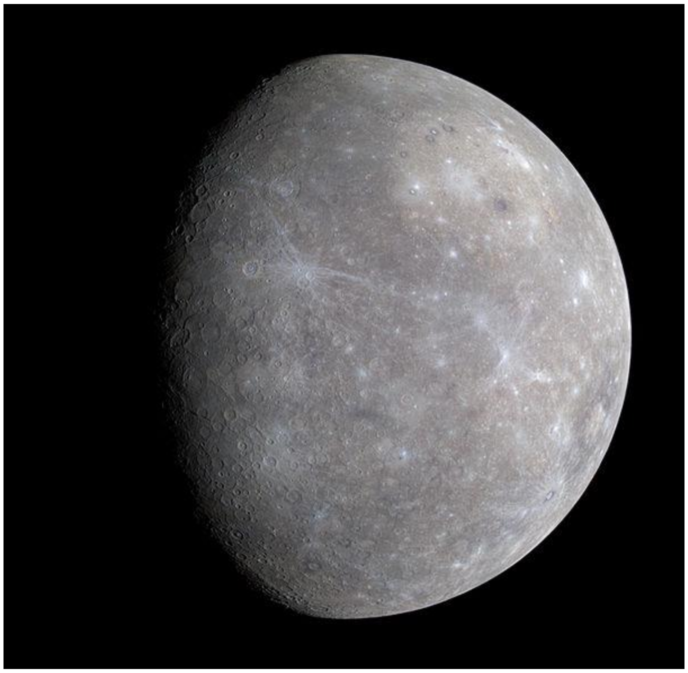
Appendix G: PowerPoint Presentations

The PowerPoints displayed in Appendix G were presented to all classes involved in this study.

PLANETS IN REVIEW !

STARRING

MERCURY

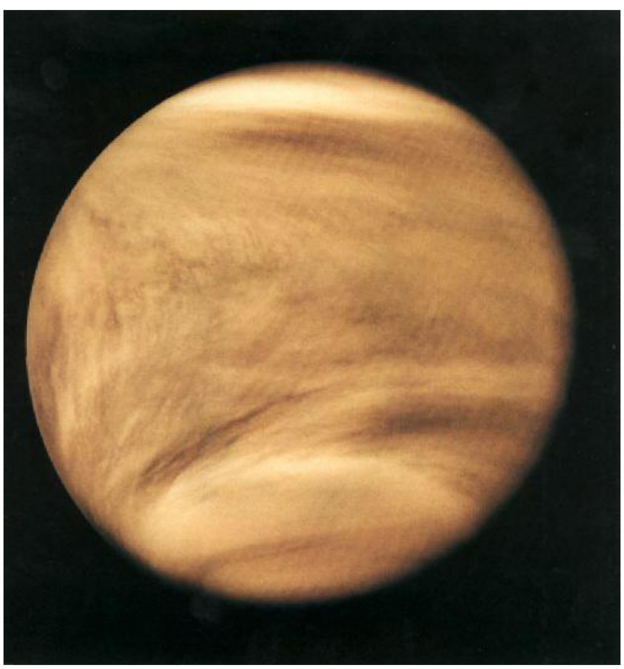


MERCURY

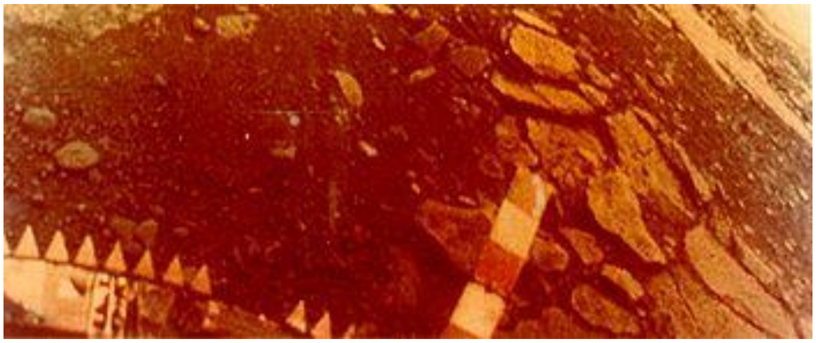
- Smallest (Pluto)
- Fastest …1 year = 88 (earth) days
- 1 “day” = 56 (earth) days….”sunstop”
- Greek God of speed
- Tides if Mercury “replaced” Moon (by 4x)
- Einstein and the 1919 eclipse

VENUS

]

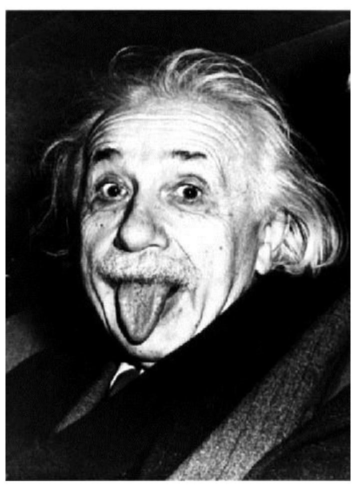


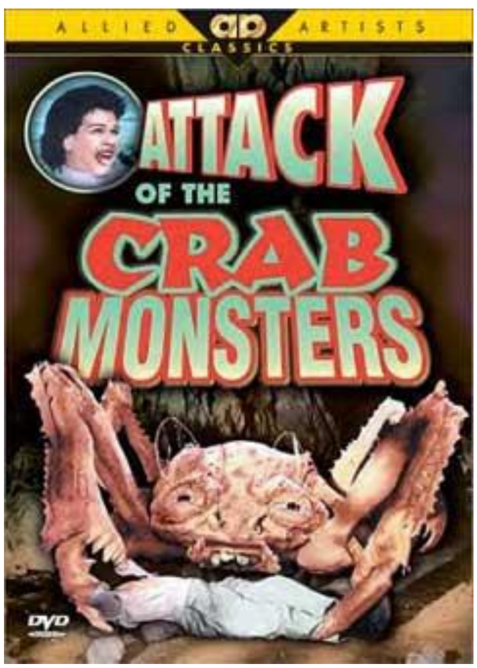
THE VENERA PROBES



VENUS

- Brightest (Galileo phases)..Alaska…(cast shadows)
- Greek godess of love
- Phosphorus & Hesperus
- “Sister” planet…closer to Sun.. Cloud enshrouded …life possibilities
- Year = 225 days…”Day” = 243 days
- Retrograde motion (Sun rises in West, sets in East)
- The Venera probes
  - Surface temp 800 deg F (melts lead)
  - Air pressure 90x Earth (Pacific Ocean bottom)
  - Carbon Dioxide atmosphere..sulfur clouds..sulfuric acid rain



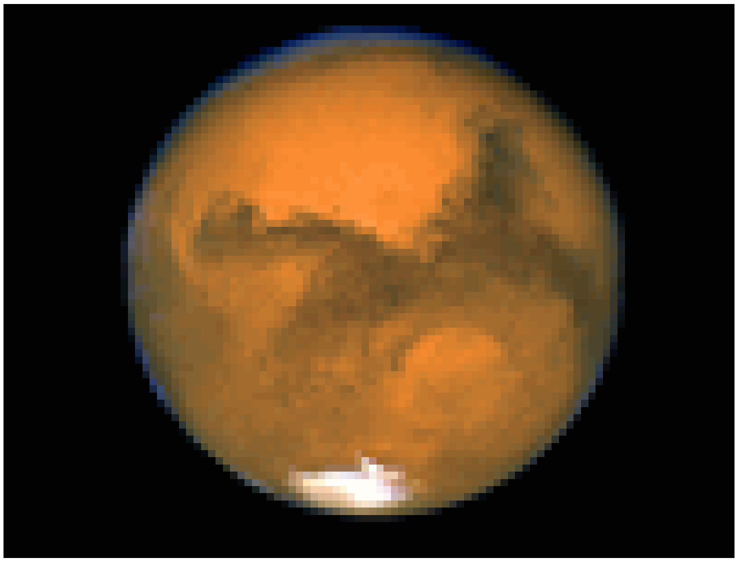

ALLIED ARTISTS
CLASSICS
ATTACK
OF THE
CRAB
MONSTERS
DVD

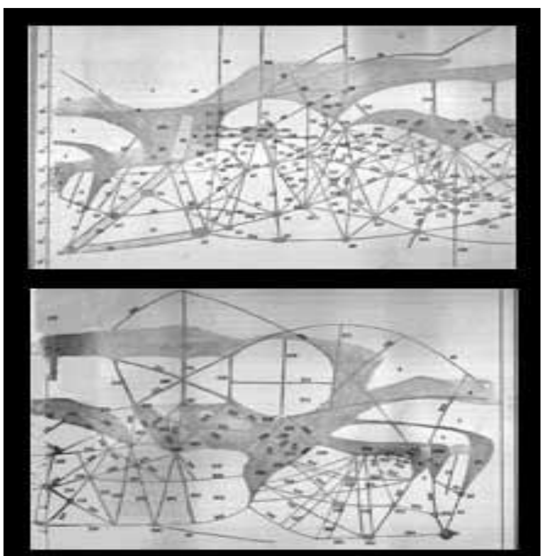
EARTHRISE (CHRISTMAS, 1968)

Le Voyage dans le lune (1902)

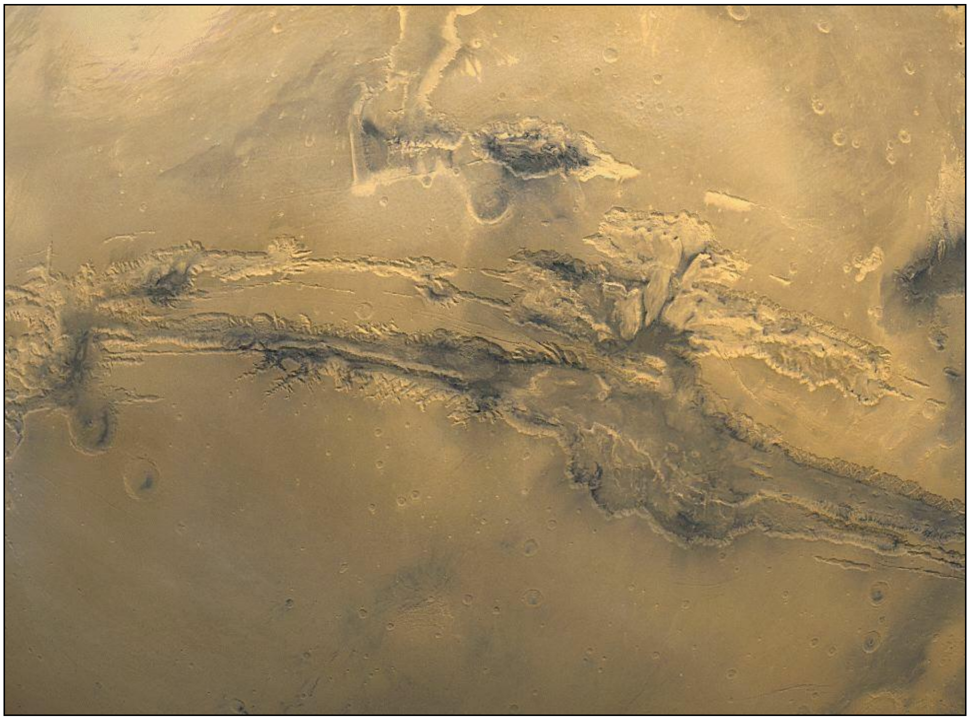

ARTHUR HILTON'S
CAT-WOMEN
OF THE MOON
LOVE-STARVED MOON MAIDENS
ON THE PROWL!
STARRING
MARIE WINDSOR
SONNY TUFTS · VICTOR JORY
MUSIC BY ELMER BERNSTEIN

MARS

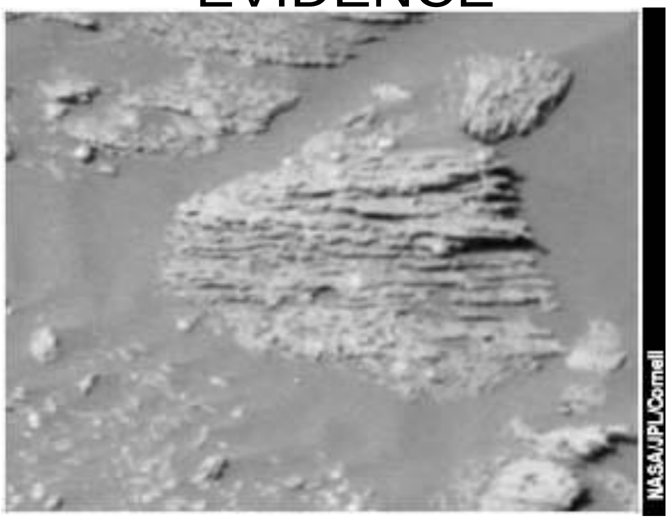




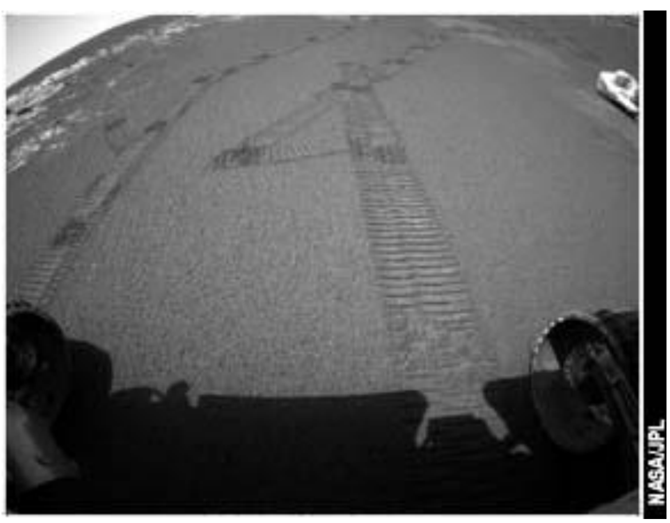
Martian "canals" as mapped by
Percival Lowell in the late 1800s.



EVIDENCE

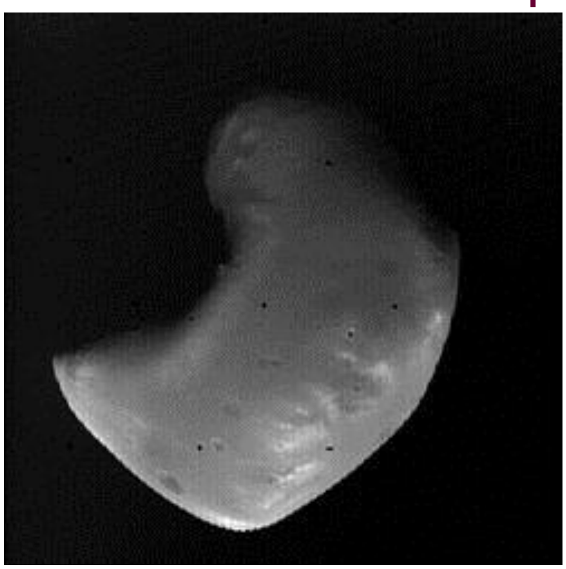


U-TURN



MARTIAN LANDSCAPE

DEIMOS – “Able to leap…”



WAR OF THE WORLDS


2-D
ROBOT
MONSTER
SCIENCE FICTION GOLD

MARS – PLANETARY BOX-OFFICE KING !

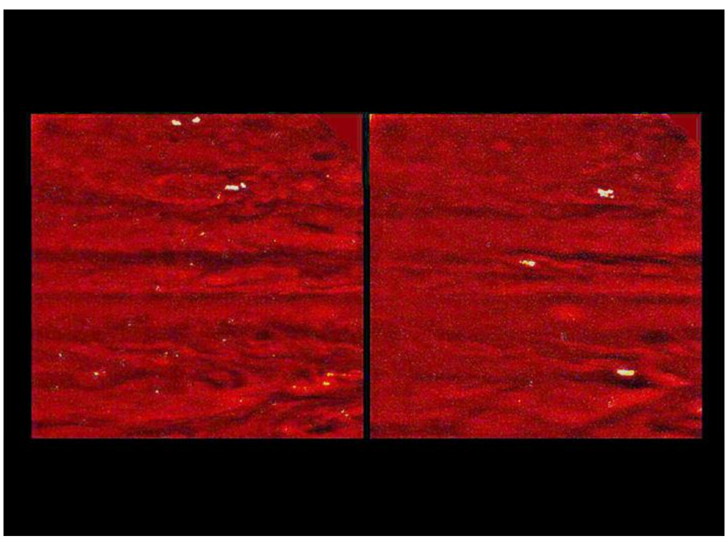
- Flash Gordon's Trip to Mars (1938)
- Flying Disc Man from Mars (1951)
- Abbott & Costello Go to Mars (1953)
- Robinson Crusoe on Mars (1964)
- Santa Claus Conquers the Martians (1964)
- The Maid and the Martians (Pajama Party) (1964)
- The 12-handed Men of Mars (Mexican) (1964)
- Mars Needs Women! (1966)
- Lobster Men from Mars (1989)
- Bad Girls from Mars (1990)
- Martians Go Home! (1990)
- Mars Attacks! (1996)
- Brave Little Toaster Goes to Mars (1998)

MARS

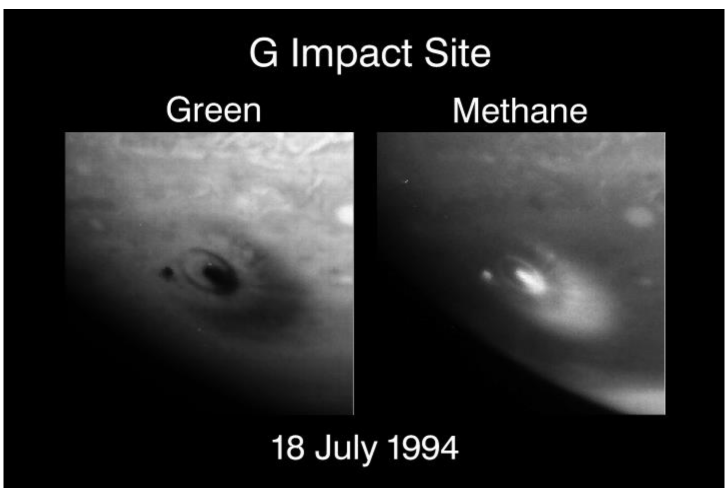
- Roman god of war
- Day = 23.5 hrs….Year = 670 days
- Orbit tilt = seasons like Earth
- Percival Lowell's "canals"…deepest canyons
- H. G. Wells' "War of the Worlds"
- Closest (35 mil miles) to feasibly explore
- Mass = land mass of Earth
- Olympus Mons (highest mt. in solar system
- Landings – Viking (1970s) & Rovers
- Water and micro-organic fossils

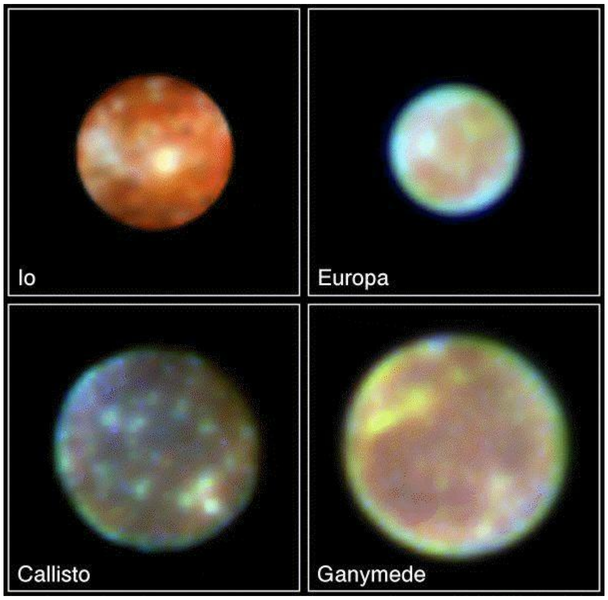
JUPITER

JUPITER LIGHTNING



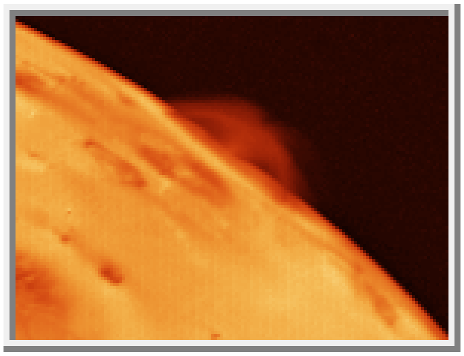
SCHOMAKER COMET 1994



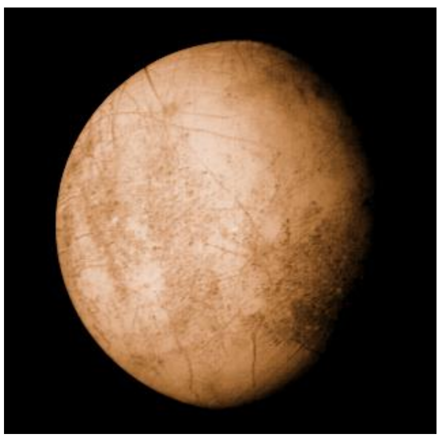


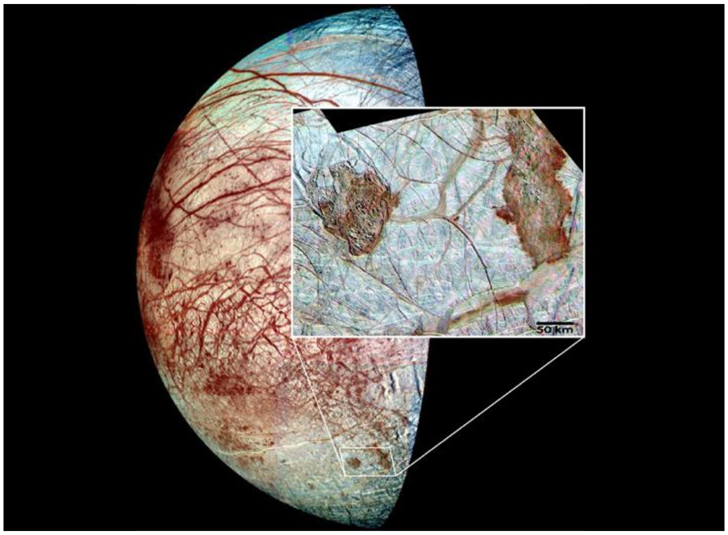
IO

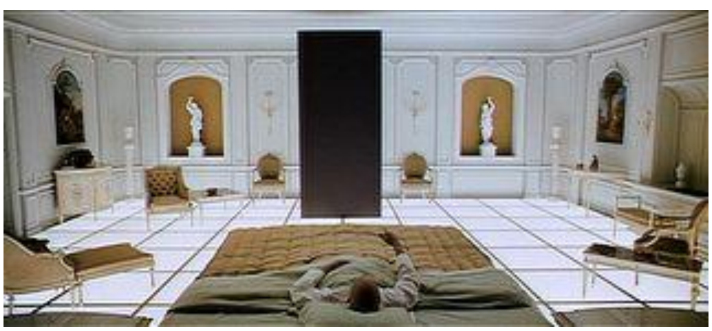
PROMETHEUS

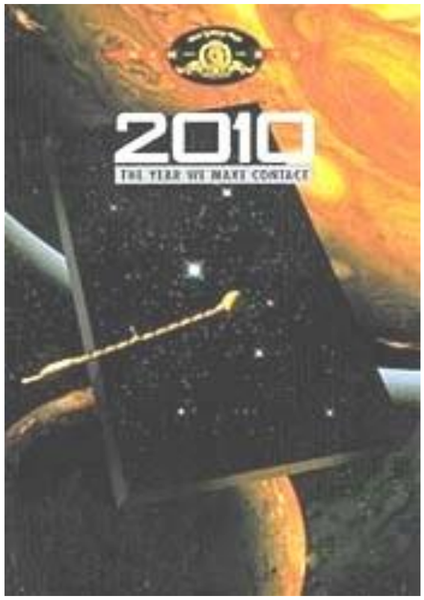


EUROPA




50 km

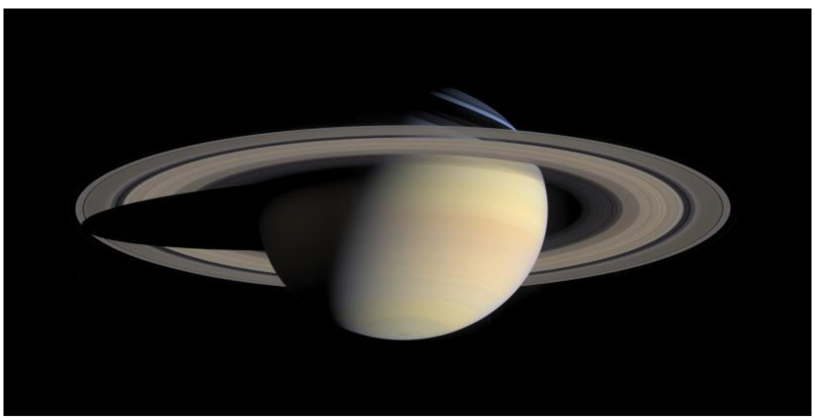


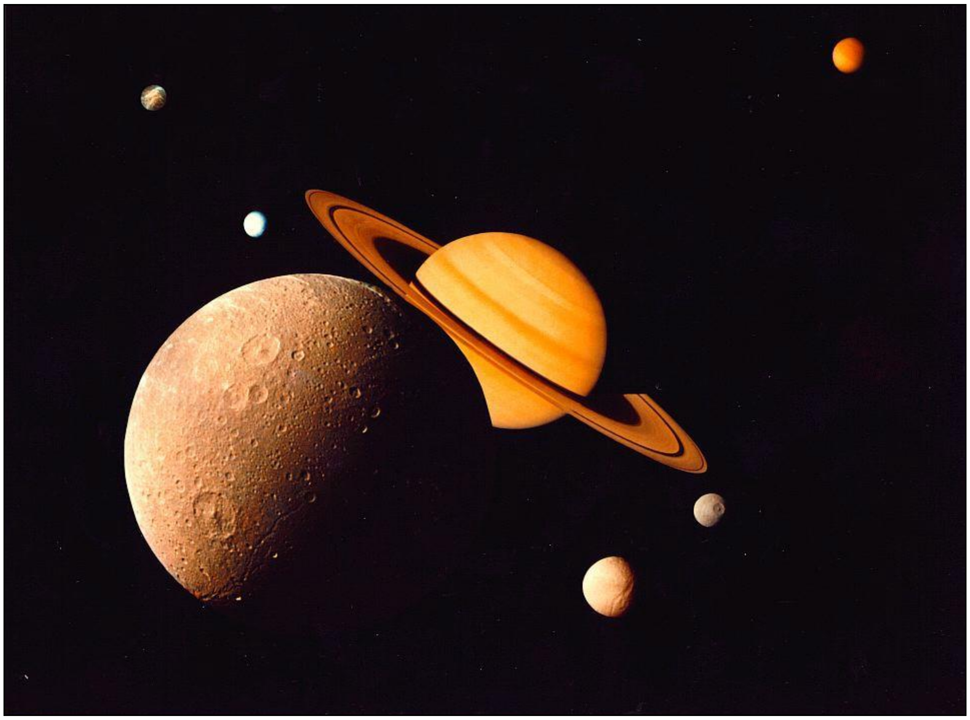


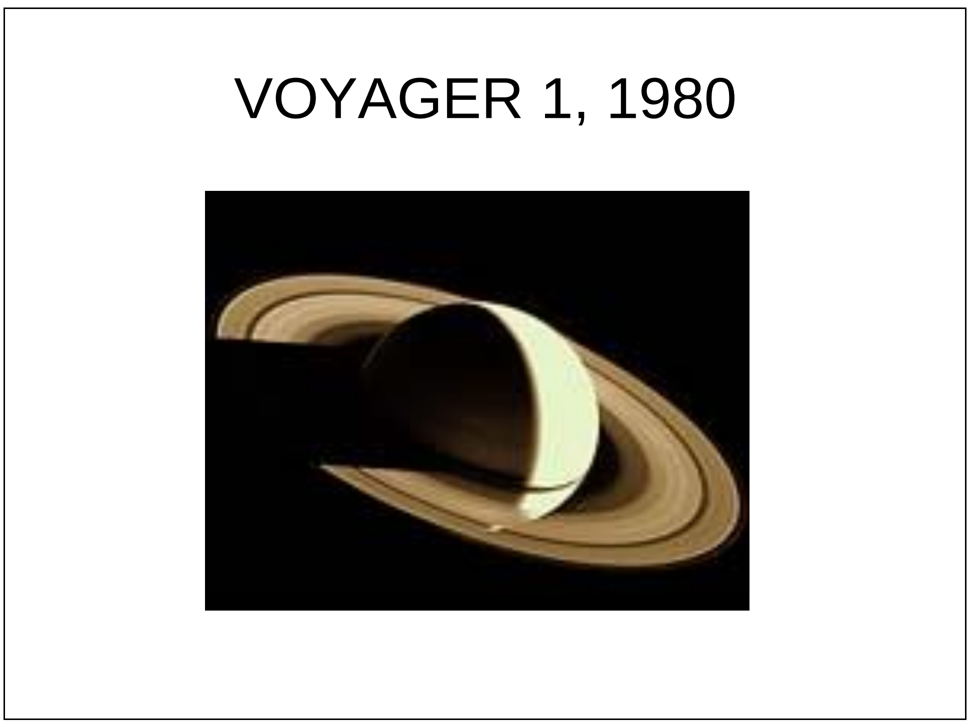
JUPITER

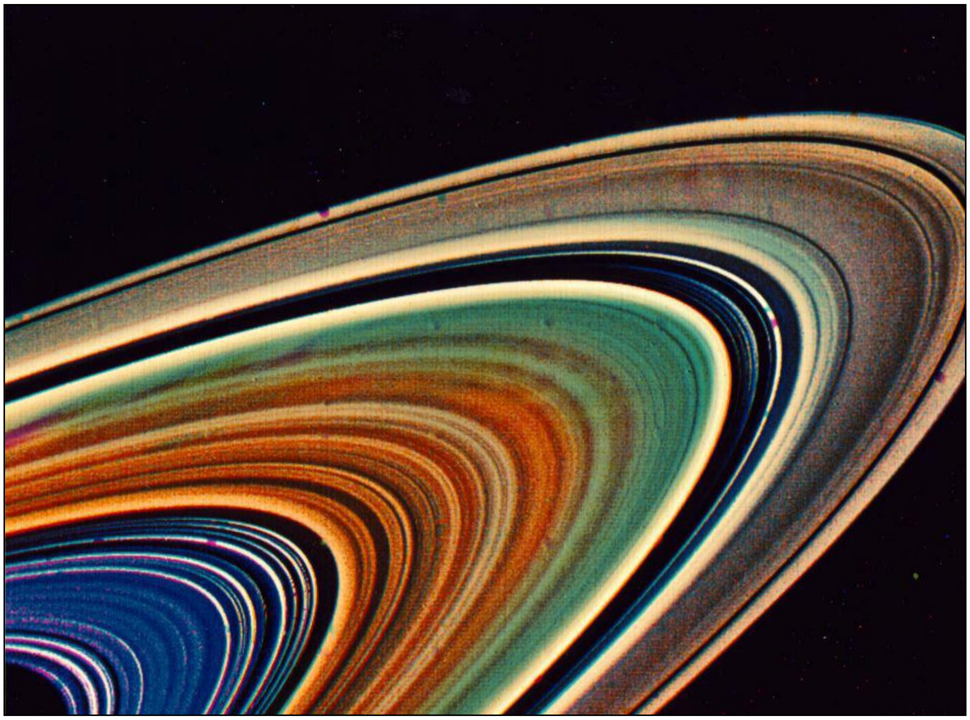
- Roman king of the Gods
- Largest “Gas Giant” (almost a small sun)…dime to a dinner plate
- Great Red Spot (storm hundreds yrs old)
- Voyager probes 1979
- Galileo
- Io, Europa, Ganymede, Callisto

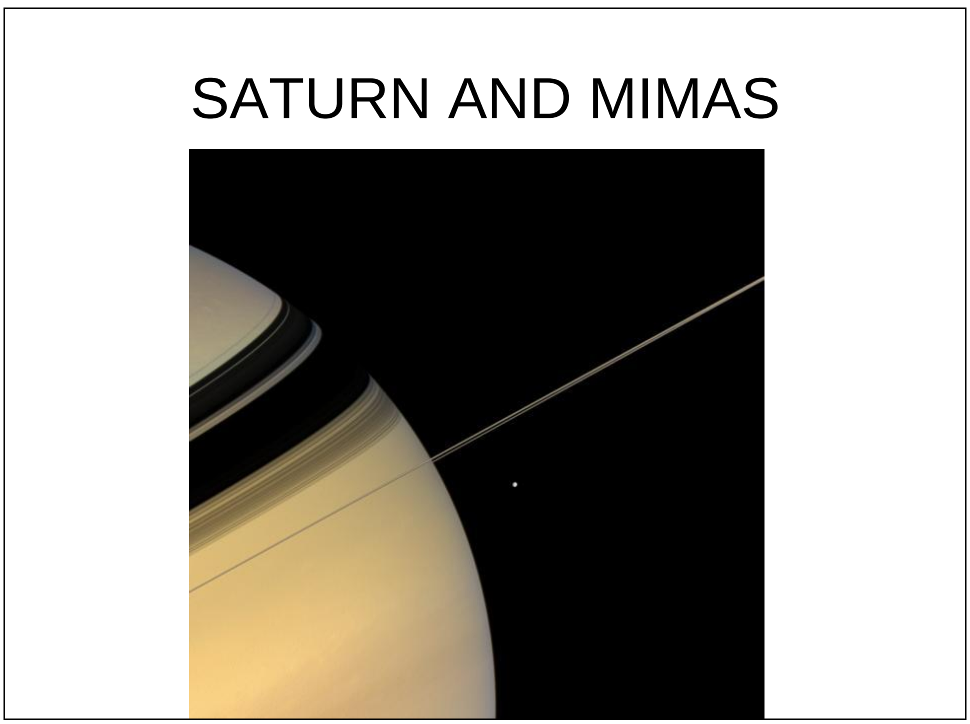
SATURN

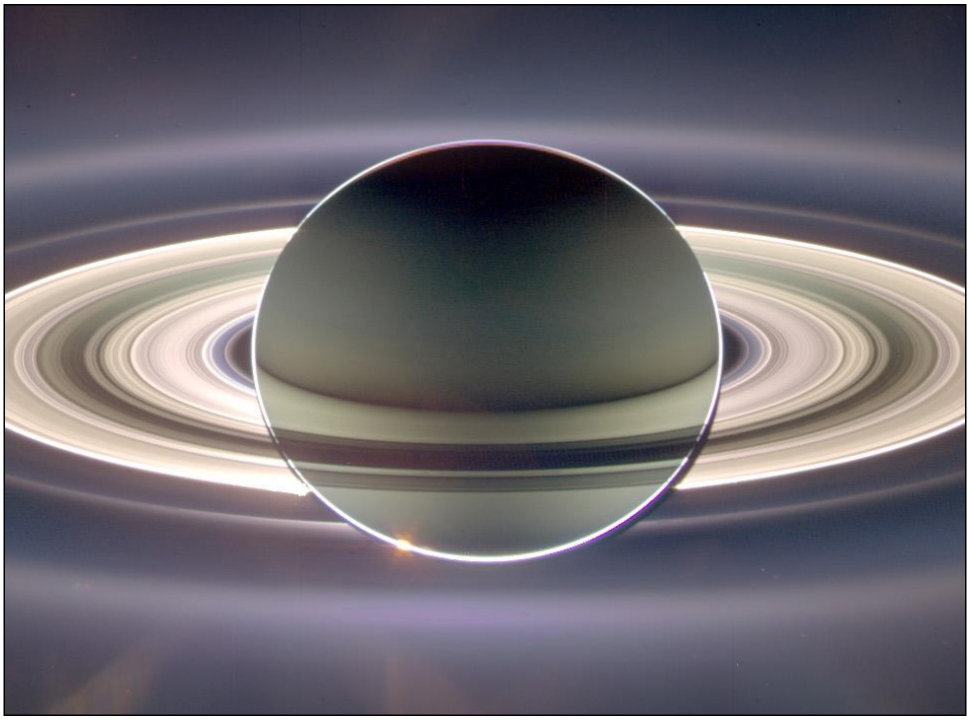




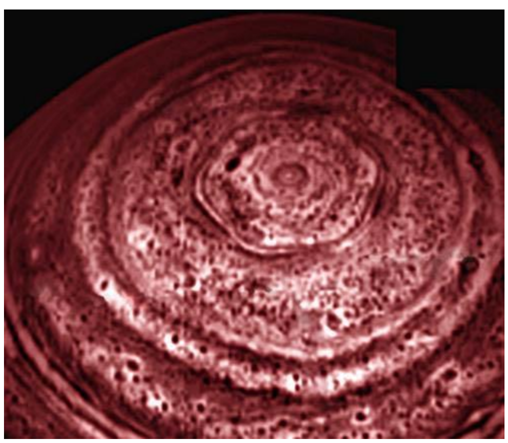

VOYAGER 1, 1980

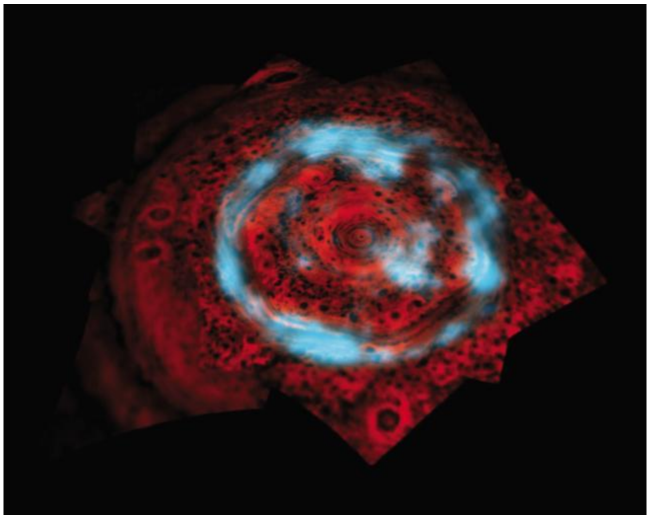



SATURN AND MIMAS



THE HEXAGON

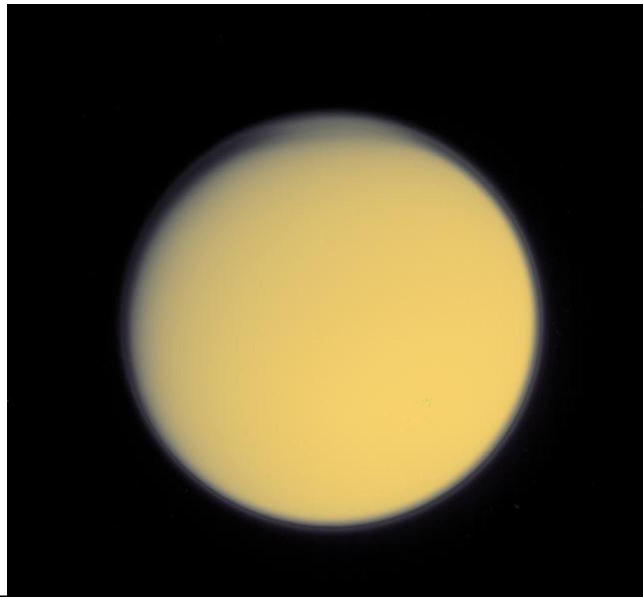




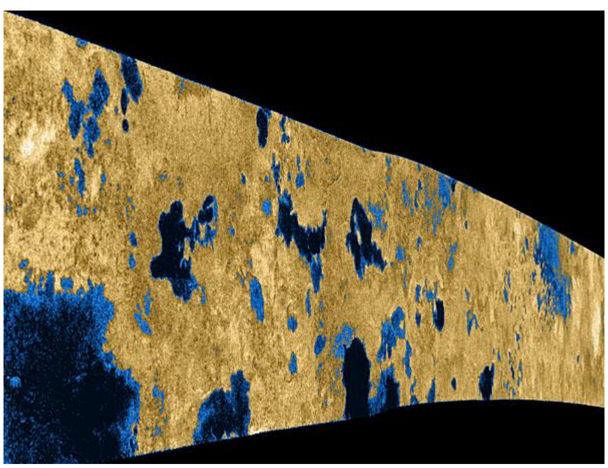
MET. DR. THIP

- Mimas
- Enceladus
- Tethys
- Dionne
- Rhea
- Titan
- Hyperion
- Iapetus
- Phoebe

TITAN

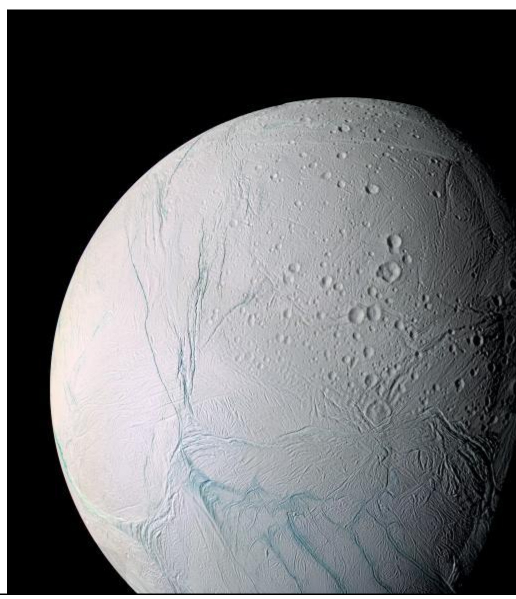


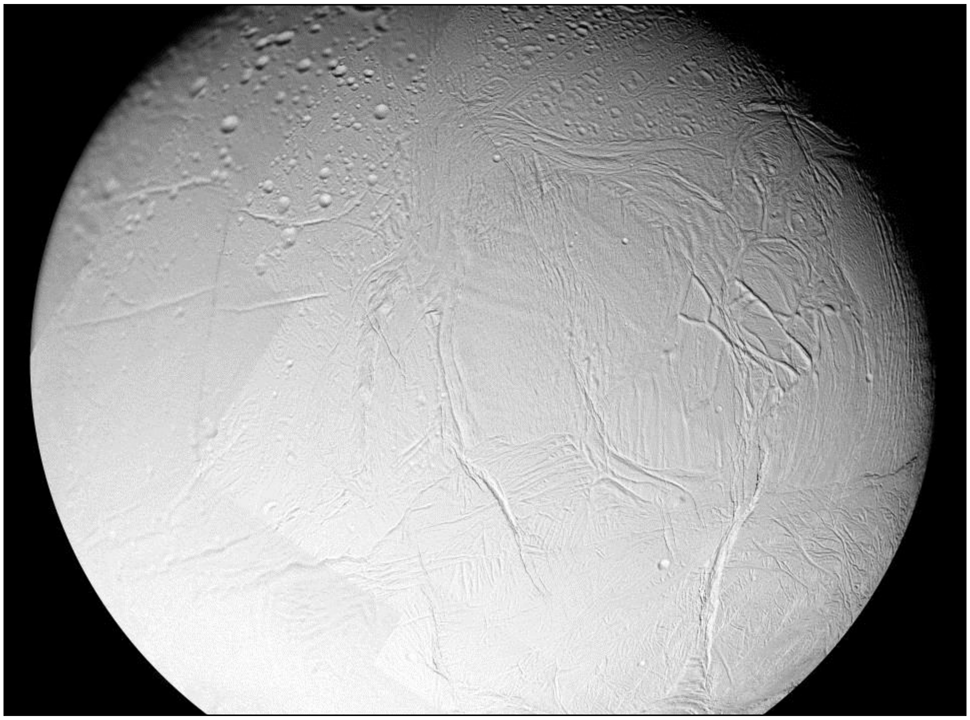
LAKES OF METHANE

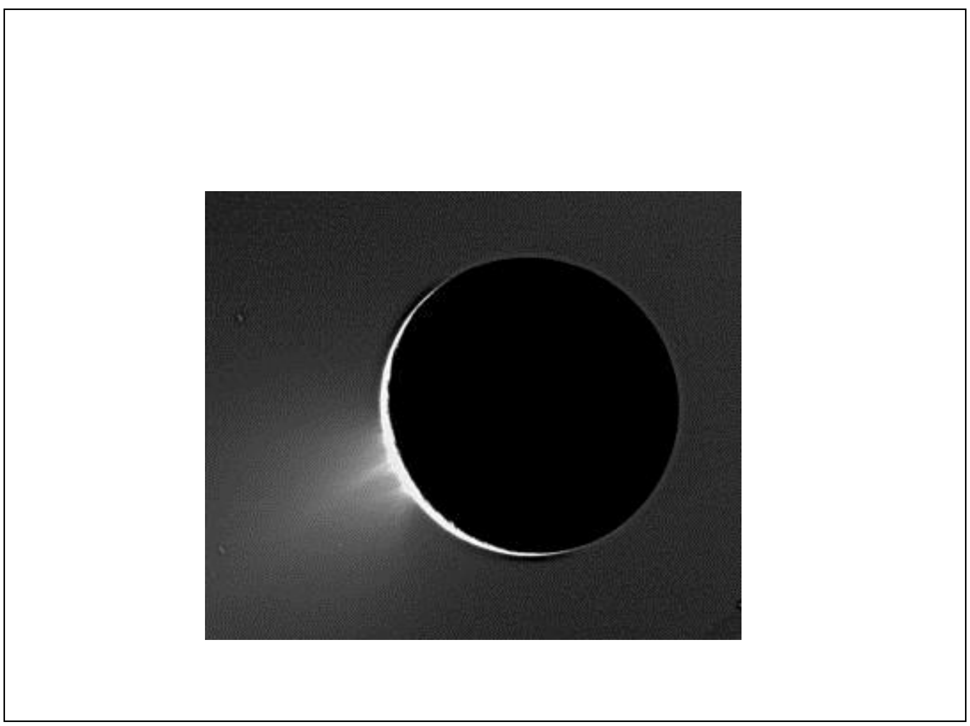


HUYGENS PROBE

ENCELADUS







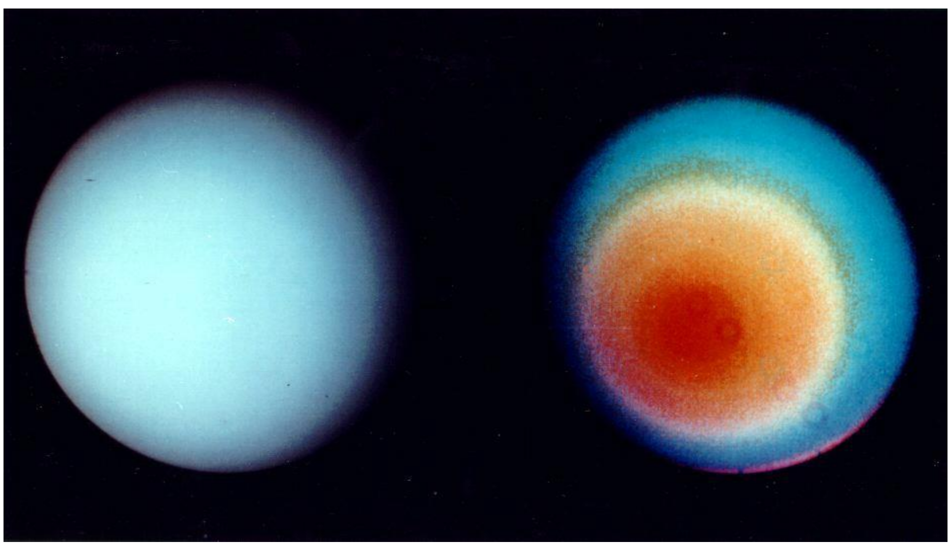

SILENT RUNNING

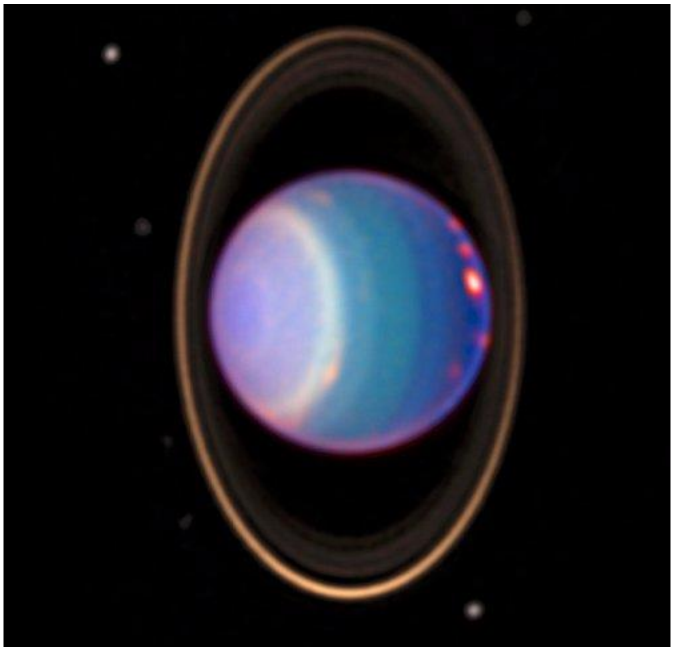

SATURN 3

SATURN

- Rings (less than 1 mile thick!)
- Roman God of Agriculture
- Voyager probes 1980, 1981
- Cassini Probe 1997…in orbit since 2004
- Huygens probe Jan. 2005 …TITAN
- ENCELADUS… Jan. 2006
- “Met Dr. Thip”

URANUS



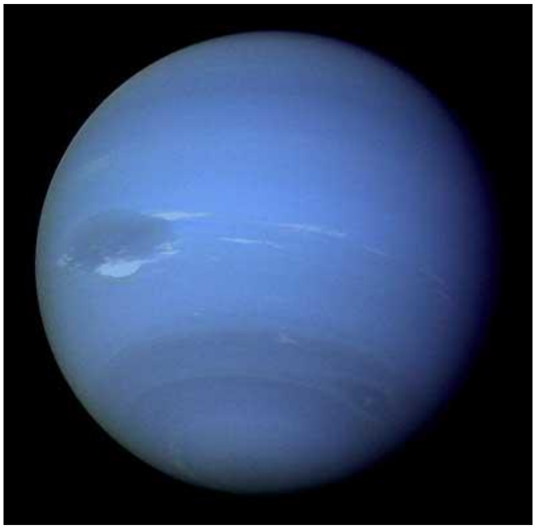


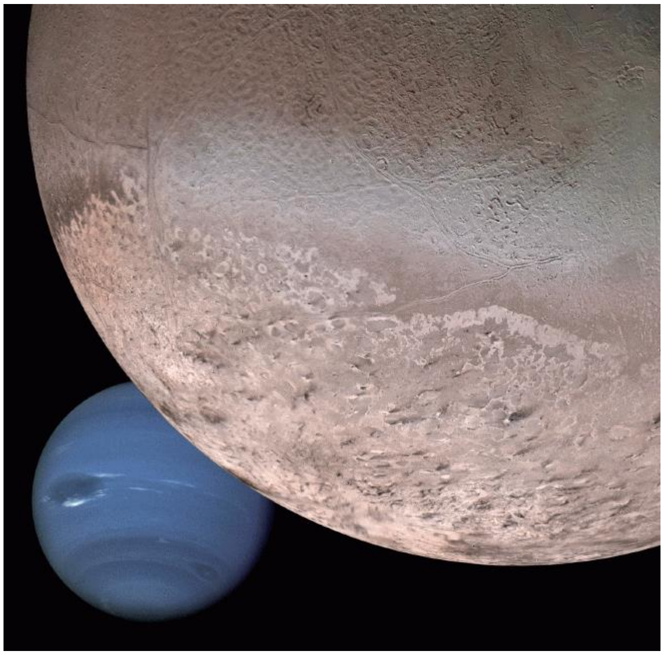

MIRANDA
ARIEL
UMBRIEL
TITANIA
OBERON

URANUS

- Greek deity of Heavens
- Voyager 2 – Jan. 24, 1986
- Retrograde orbit and 89 deg tilt
- MAUTO – moons named after Shakespeare & Pope
- 2 billion miles away

NEPTUNE



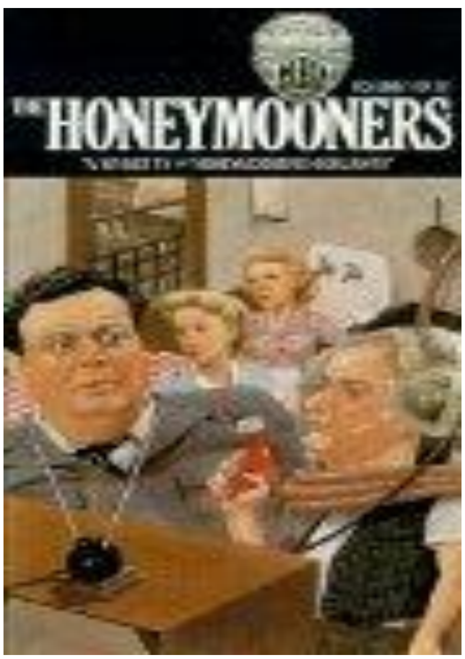



INVASION of the
NEPTUNE MEN

NEPTUNE

- Discovered by Gauss and LeVerrier
- Voyager 2 – Aug. 25, 1989
- Roman god of the Sea
- Fastest winds – 2100 km/hr
- Triton – coldest recorded temp (-400 deg F. – 40 deg above absolute zero!)

PLUTO


HONEYMOONERS

PLUTO THE PUP

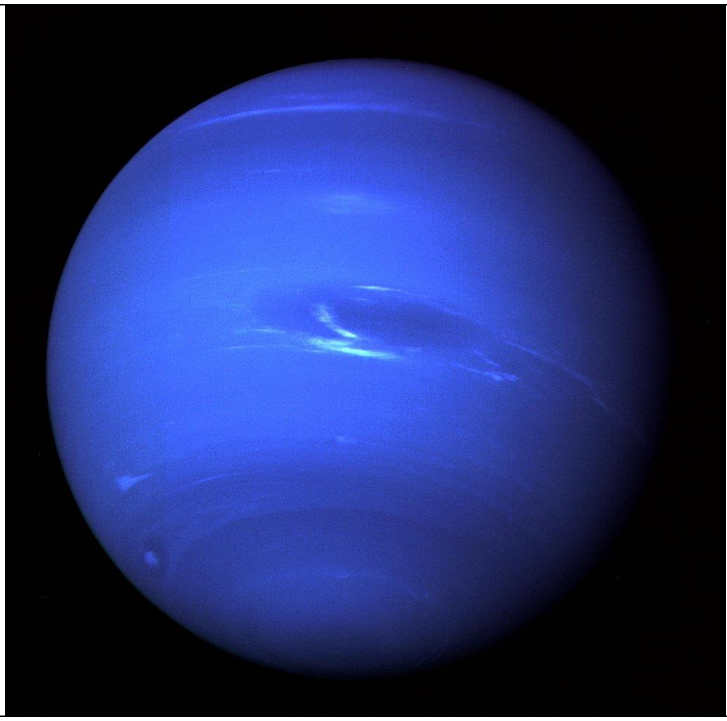
PLUTO

- Clyde Tombaugh – 1930
- Roman God of War
- 1 year = 243 (Earth) years
- Now a “Dwarf” planet
- “New Horizons” Probe to reach Pluto in 2015
- Charon (1978) – ferried the dead across Styx

NEPTUNE

.

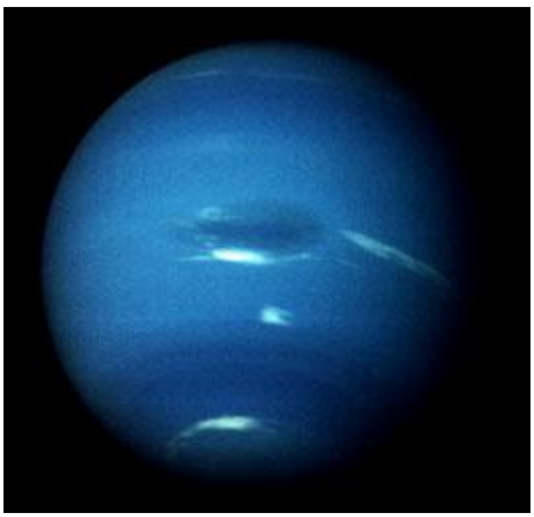
NEPTUNE



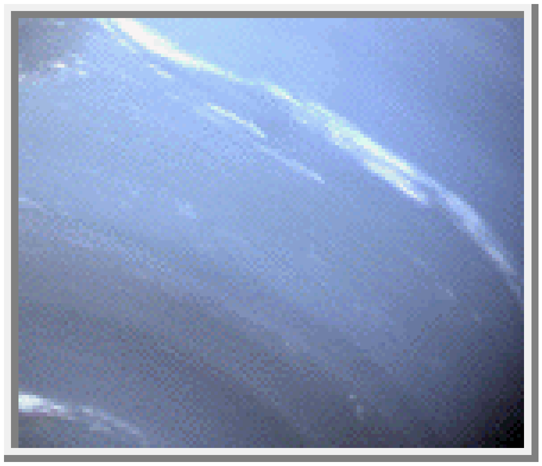
NEPTUNE – A THUMBNAIL SKETCH

- Eighth planet from the sun, with a mean distance of 3 billion miles
- A gas giant of about 4 Earth-masses, it is the fourth largest planet in the solar system
- Its “day” is 16.4 hours, its “year” is 165 Earth-years
- Discovered in 1846, it recently celebrated its first Earth “birthday”
- Named after the Roman god of the sea.

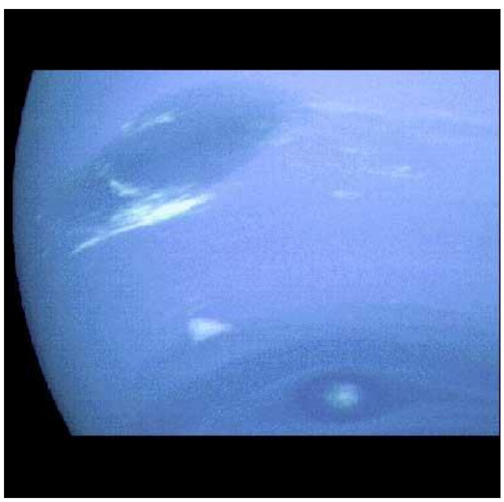
THE GREAT DARK SPOT



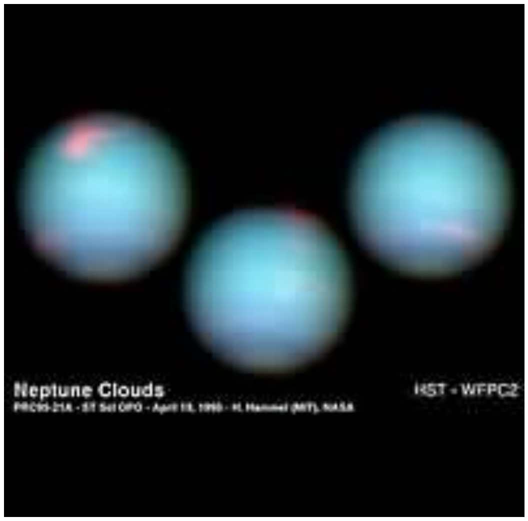
WAVE CLOUD



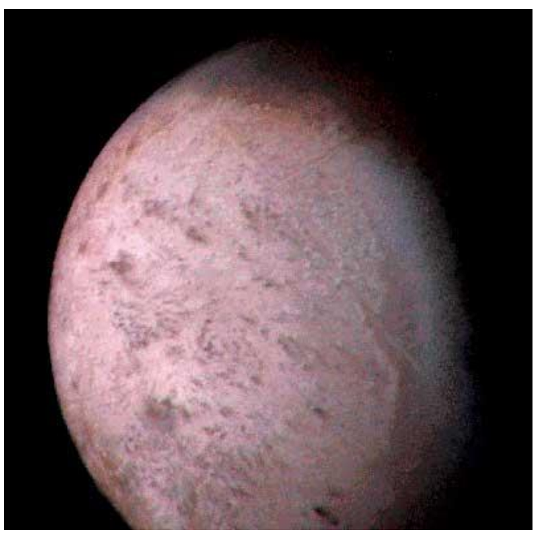
SCOOTER



THE VIEW FROM HUBBLE



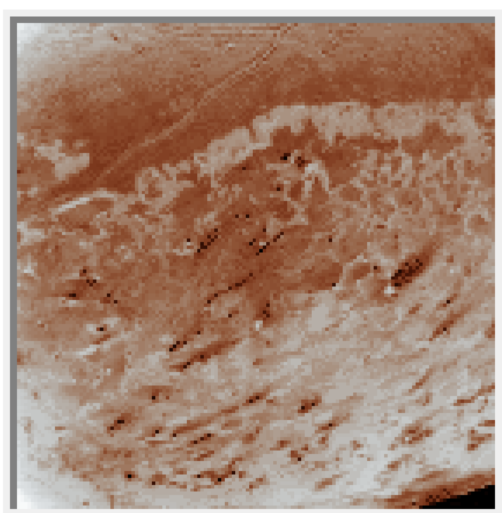
TRITON



TRITON – THUMBNAIL SKETCH

- Named after the son of Neptune
- Discovered in 1846 by William Lassell
- Last major solar system body viewed by Voyager II (August 1989)
- Retrograde orbital motion
- One of three places in the solar system where active geysers have been detected

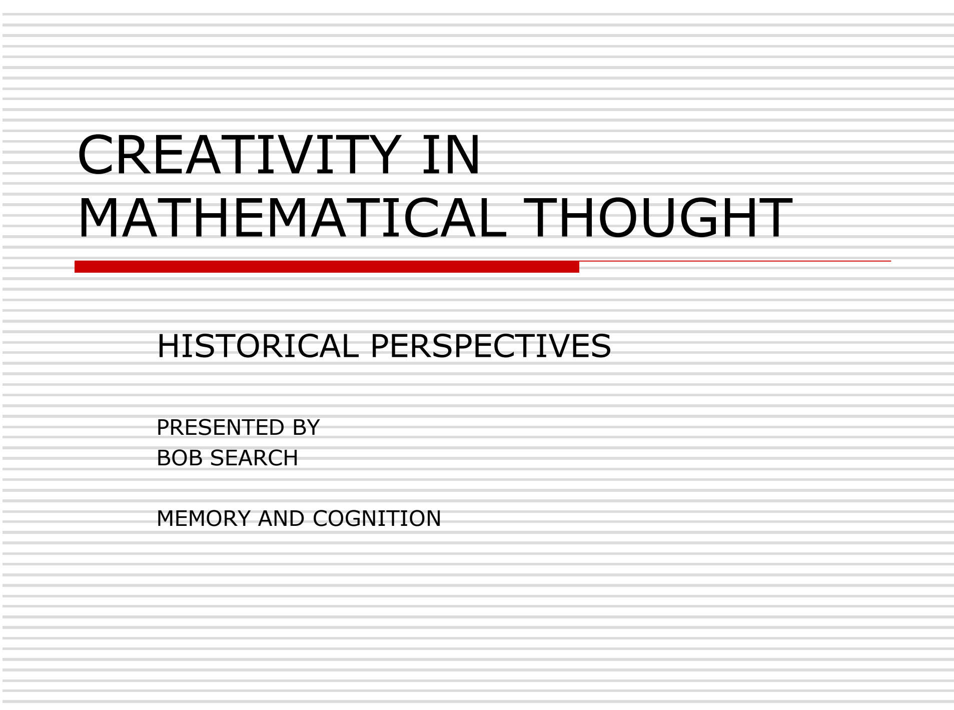
THE CANTALOUPE MOON



GEYSERS OF LIQUID NITROGEN

VOYAGER'S LAST VIEW

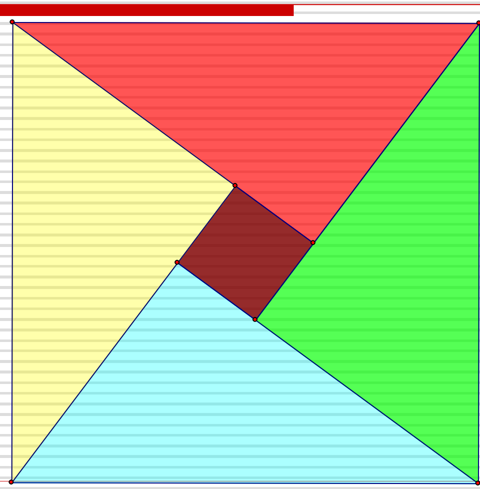
CREATIVITY IN MATHEMATICAL THOUGHT



PYTHAGORAS OF SAMOS

- (585 – 500 B.C.)
- THE RIGHT TRIANGLE THEOREM
- THE “TILE’ INSPIRATION

THE PYTHAGOREAN “TILE”



EUCLID OF ALEXANDRIA

- (323 – 285 B.C.)
- *ELEMENTS OF GEOMETRY*
- EUCLIDEAN GEOMETRY
- NUMBER THEORY

EUCLID'S *ELEMENTS…*ON PAPYRUS

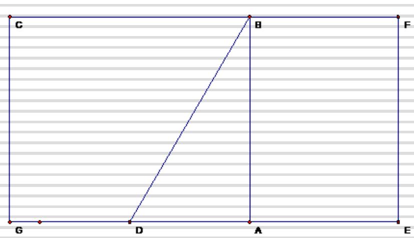
LEONARDO OF PISA (FIBONACCI)

- (1175 – 1250 A.D.)
- THE FIBONACCI SEQUENCE
  - 1,1,2,3,5,8,13,21,34,…
- THE GOLDEN RATIO

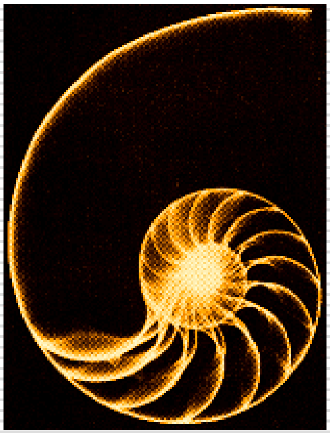
THIS IS A "GOLDEN RECTANGLE"

$m\overline{AB} = 8.00$ cm
$m\overline{CB} = 8.00$ cm
$m\overline{DB} = 8.94$ cm
$m\overline{ED} = 8.94$ cm
$m\overline{FE} = 8.00$ cm
$m\overline{GE} = 12.94$ cm

$$\frac{m\overline{GE}}{m\overline{FE}} = 1.62$$



THE GOLDEN RATIO IN NATURE

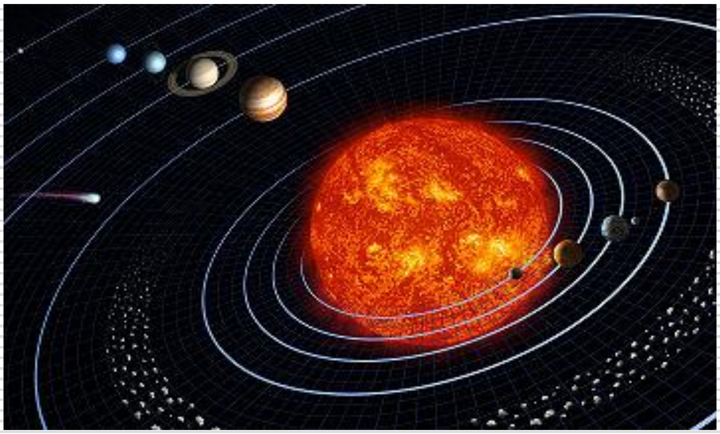


THE GOLDEN RATIO IN ART

JOHANNES KEPLER

- (1571 – 1630)
- THE THREE LAWS OF PLANETARY MOTION
  - ORBITS ARE ELLIPSES
  - EQUAL ARCS IN EQUAL TIME
  - DISTANCE AND YEAR LENGTH PROPORTIONAL
- TYCHO BRAHE
  - ASTRONOMICAL OBSERVATIONS

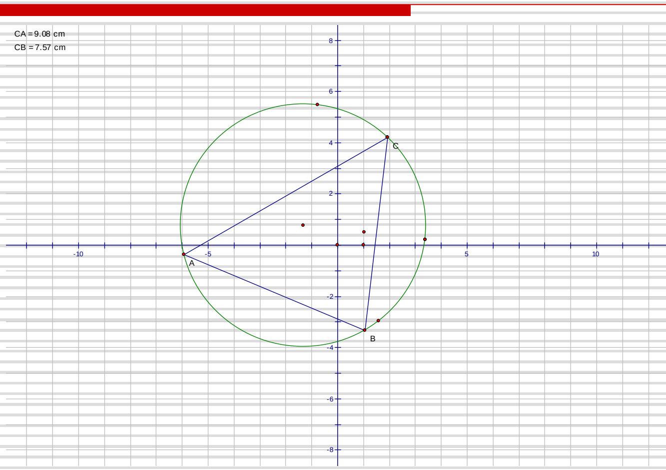
THE SOLAR SYSTEM



RENE DESCARTES

- (1596 – 1650)
- THE NUMBER LINE
  - EQUATIONS WITH ONE VARIABLE
- EQUATIONS WITH TWO VARIABLES
  - VISUAL REPRESENTATION
- INSPIRATION
  - THE CARTESIAN COORDINATE SYSTEM

THE RECTANGULAR COORDINATE SYSTEM



PIERRE de FERMAT

- (1601 – 1665)

- "FERMAT'S LAST THEOREM"
  - PROVED BY ANDREW WILES…IN 1994
  - ELLIPTIC MODULARITY
  - EXTENSIVE USE OF TECHNOLOGY

FERMAT'S LAST THEOREM

- $X^n + Y^n = Z^n$

- This equation has no non-zero integer solutions for x, y and z when $n > 2$.

- *I have discovered a truly remarkable proof which this margin is too small to contain.*

BLAISE PASCAL

- (1623 – 1662)
- PROBABILITY
- COMBINATORICS
- “PASCAL’S TREE”

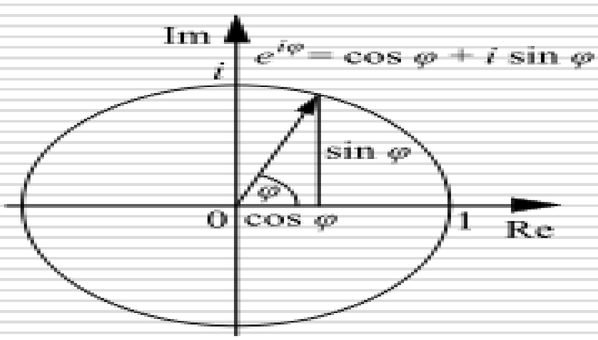
PASCAL’S TRIANGLE

EXPAND					1		1				
(A + B)^N				1		2		1			
			1		3		3		1		
		1		4		6		4		1	
	1		5		10		10		5		1

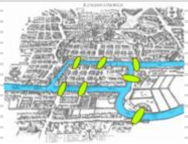
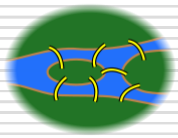
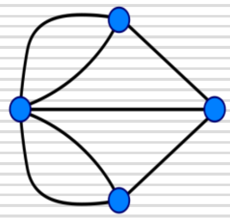
LEONHARD EULER

- (1707 – 1783)
- CALCULUS
  - SEQUENCES
- COMPLEX NUMBERS
  - THE EULER IDENTITY
- GRAPH THEORY
  - THE KONIGSBERG BRIDGE PROBLEM

EULER'S IDENTITY



THE KONIGSBERG BRIDGE PROBLEM

CARL FRIEDRICH GAUSS

- (1777 – 1855)
- NUMBER THEORY
- MATHEMATICAL INDUCTION
- *DISQUISITIONES ARITHMETICAE*
- THEORETICAL ASTRONOMY
    - THE DISCOVERY OF CERES

GAUSS...IN FOURTH GRADE

ADD THE NUMBERS FROM 1 TO 100

1 + 2 + 3 +.........+ 99 + 100 = N

100 + 99 + 98 +........+ 2 + 1 = N

101+101+101+.......+101 +101 +101 = 2N

100 * 101 = 2N

5050 = N

ARTHUR CAYLEY

- (1821 – 1895)
- *COLLECTED MATHEMATICAL PAPERS*
  - *(1889 – 1898)*
  - 13 VOLUMES
- LINEAR ALGEBRA
- EQUATIONS AS ARRAYS
  - MATRICES

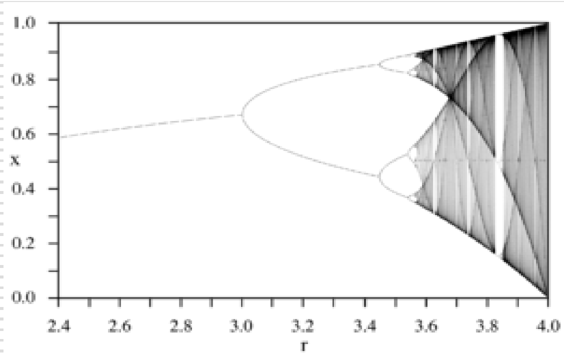
MATRIX REPRESENTATION

$$\begin{aligned} x + 3y - 2z &= 5 \\ 3x + 5y + 6z &= 7 \\ 2x + 4y + 3z &= 8 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array}\right].$$

MITCHELL FEIGENBAUM

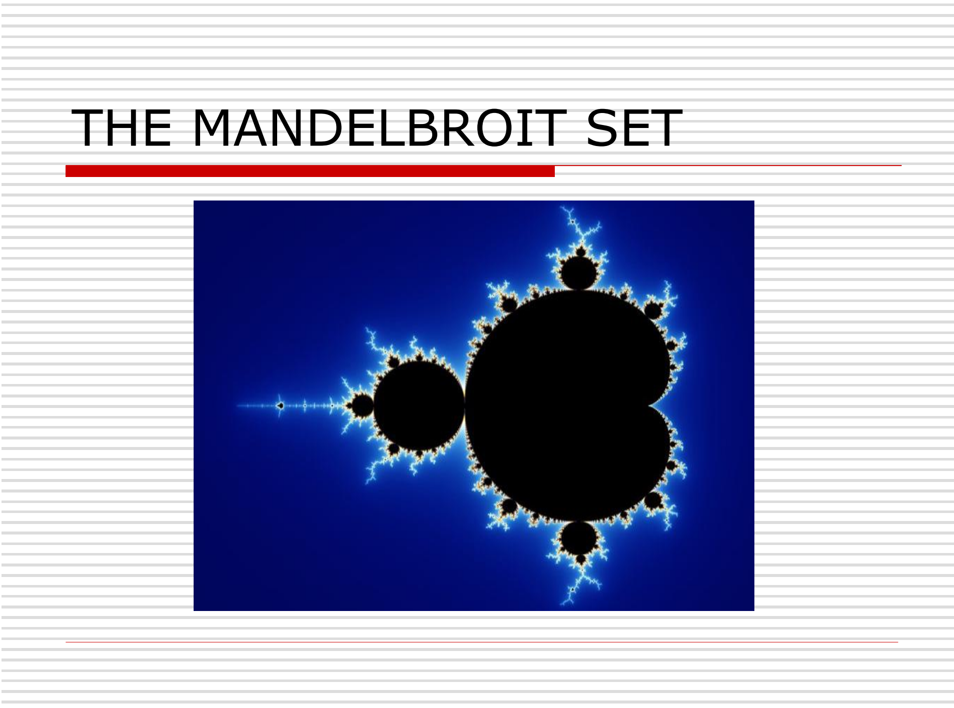
- BORN IN 1944
- CHAOS THEORY
- TURBULENCE IN FLUID DYNAMICS
- LOGISTIC POPULATION GROWTH
  - BIFURCATIONS
  - PREDICTION OF CHAOTIC EVENTS
  - THE FEIGENBAUM CONSTANT
- 37 HOUR DAILY CYCLE

BIFURCATION DIAGRAM

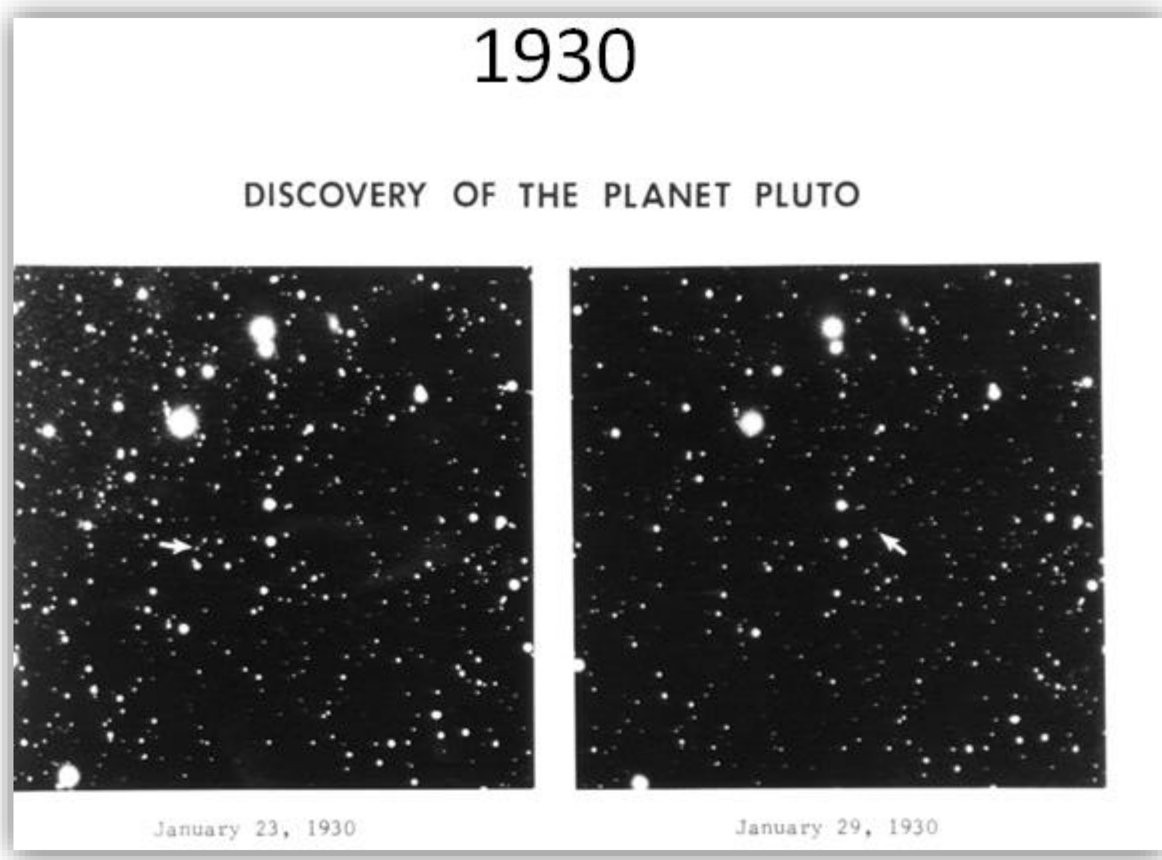


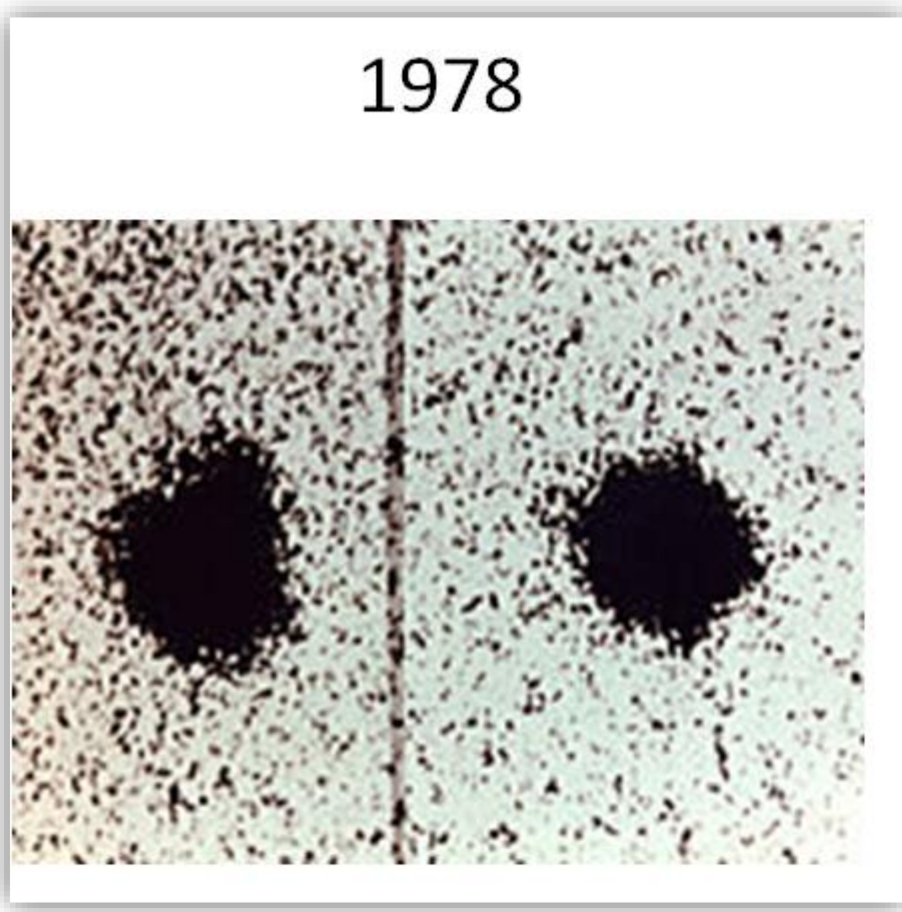
BENOIT MANDELBROIT

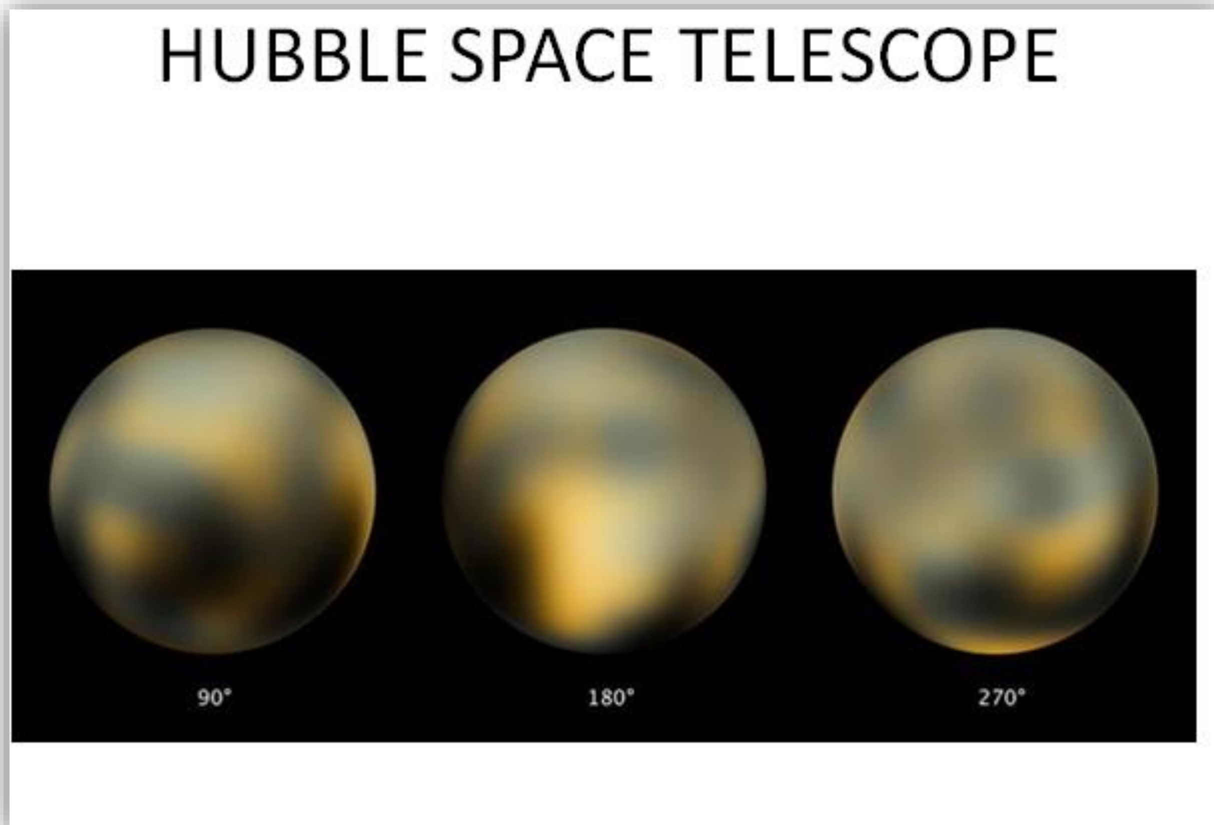
- BORN IN 1924
- COINED THE TERM "FRACTAL"
- ISOLATED CURIOSITIES
  - BEFORE THE 1980S
- MICROCHIP TECHNOLOGY
  - THE REVOLUTION OF THE 1980S
- THE MANDELBROIT SET
- FRACTAL GEOMETRY

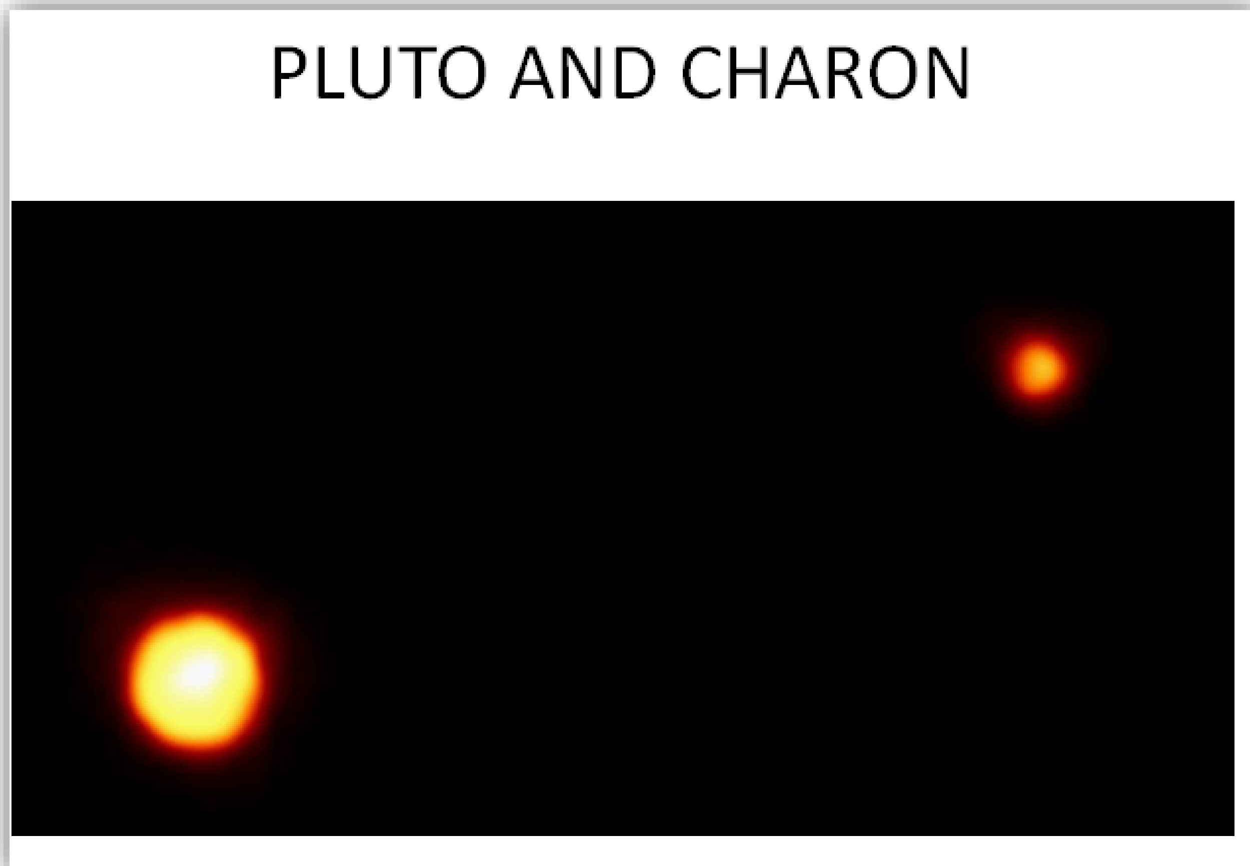


PLUTO ROCKS


1930
DISCOVERY OF THE PLANET PLUTO
January 23, 1930
January 29, 1930

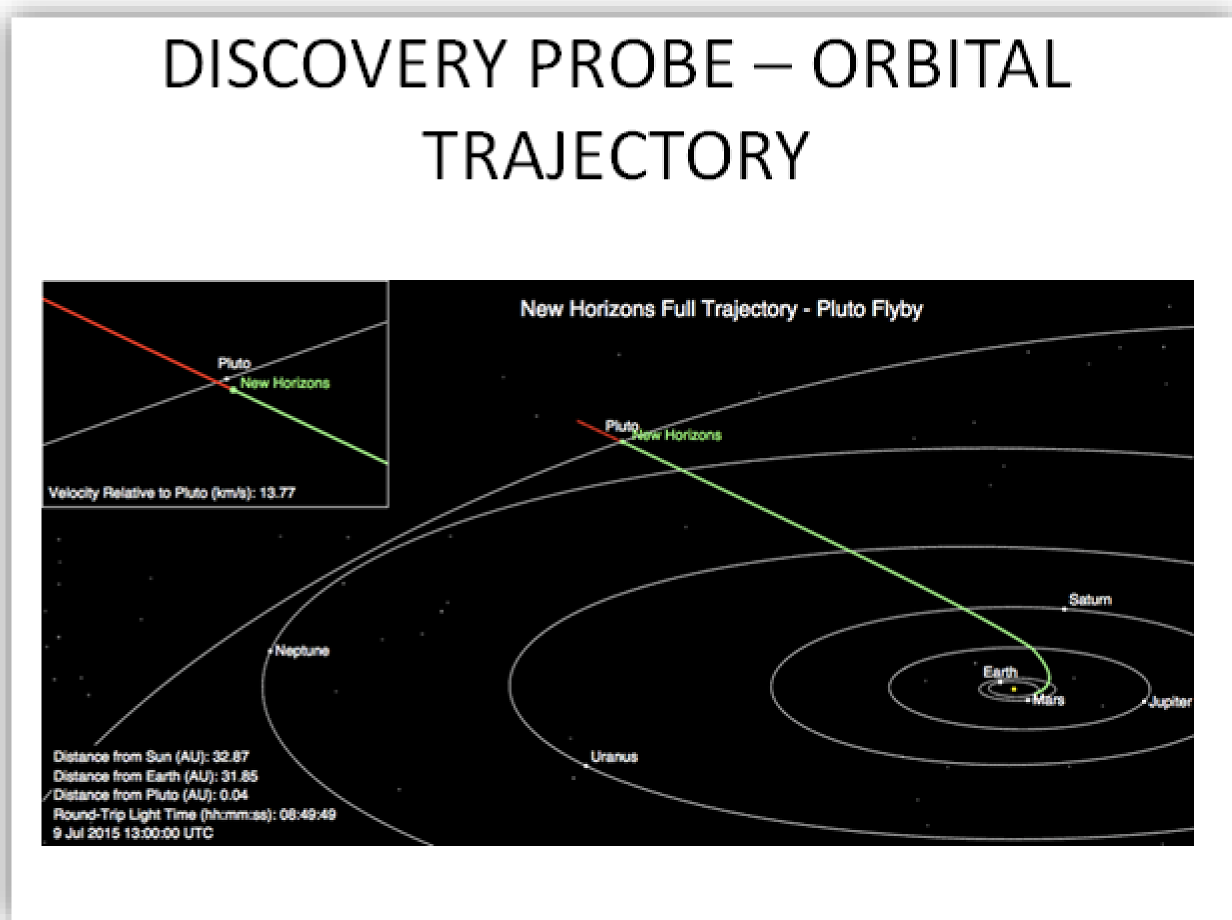

1978

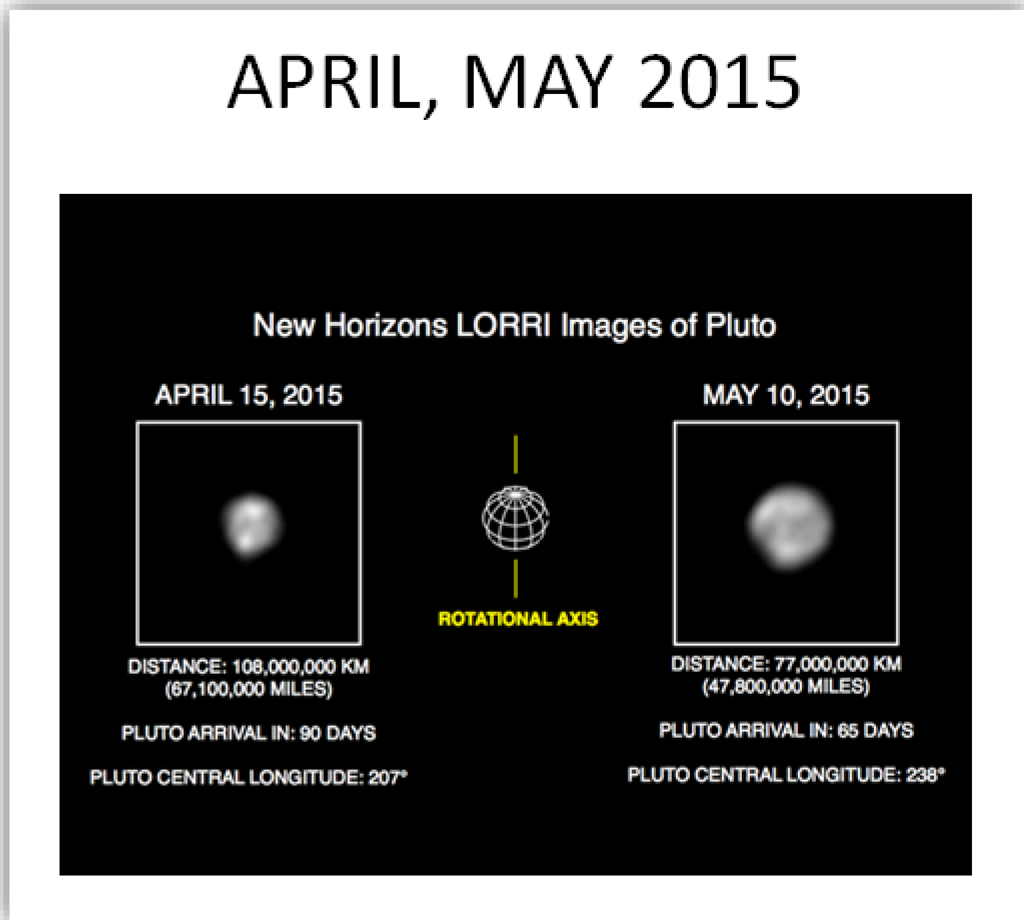

HUBBLE SPACE TELESCOPE
90°
180°
270°

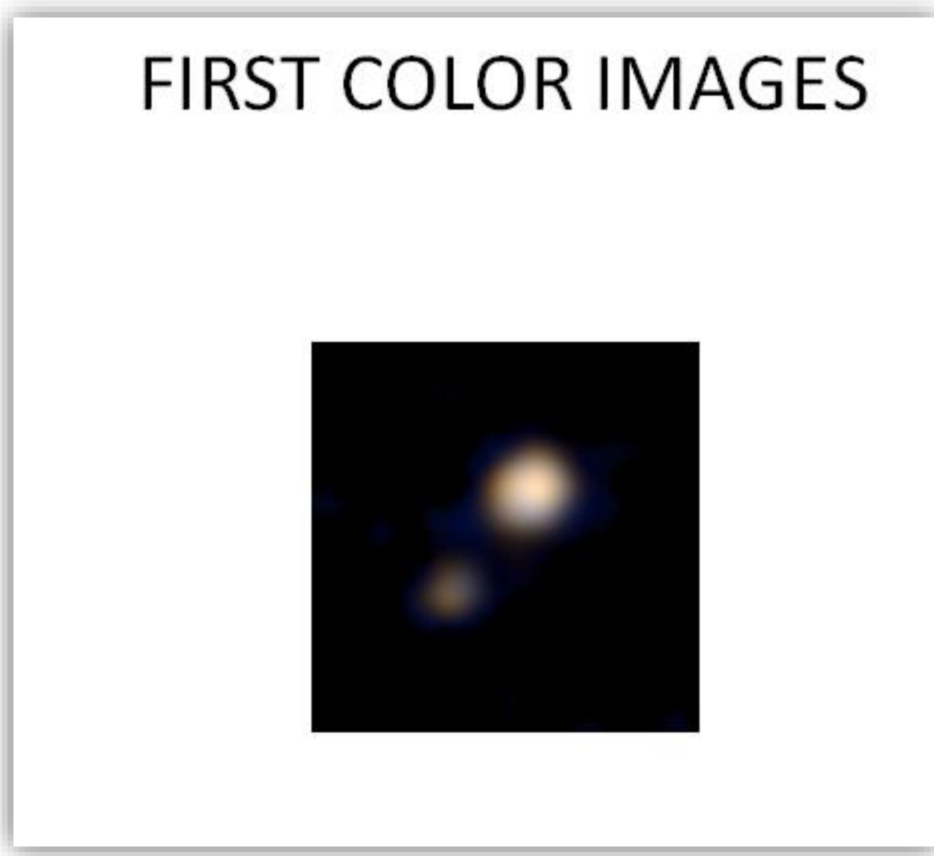

PLUTO AND CHARON


NEW HORIZONS – JANUARY, 2006


DISCOVERY PROBE

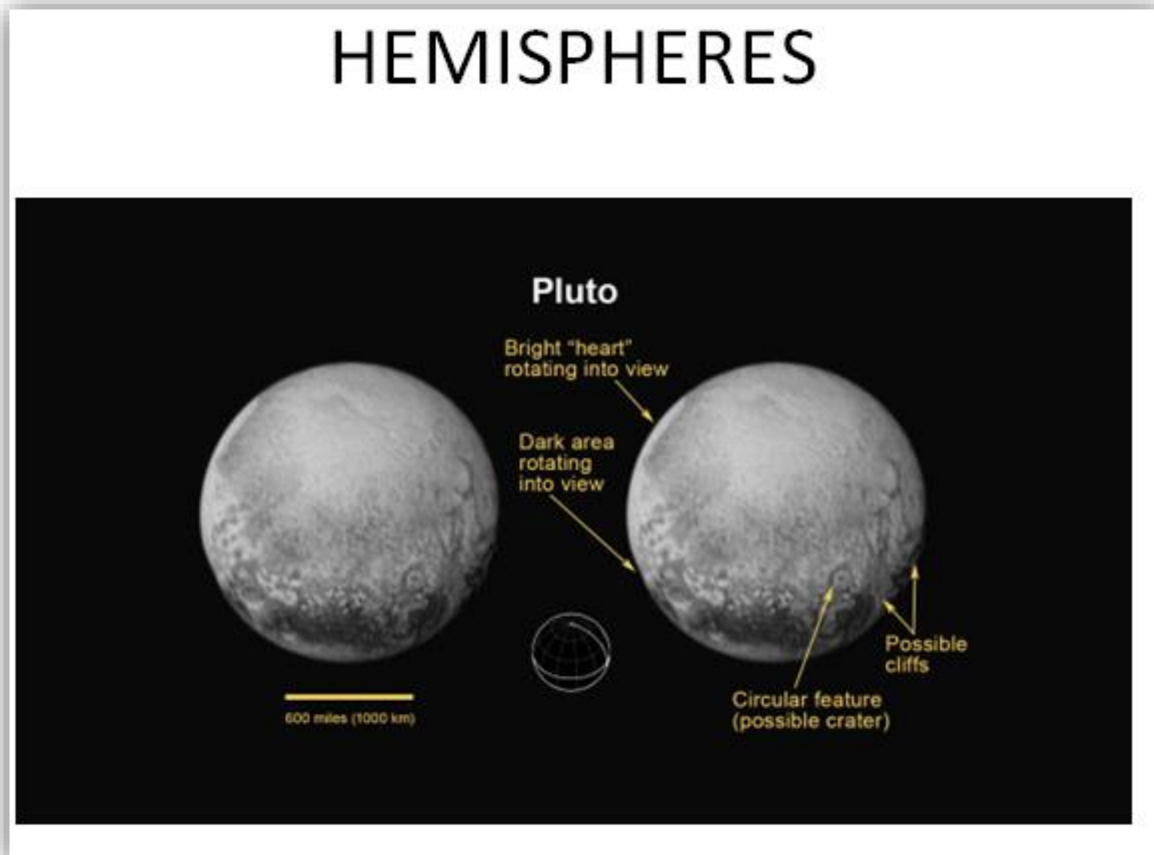

DISCOVERY PROBE – ORBITAL TRAJECTORY
New Horizons Full Trajectory - Pluto Flyby
Pluto
New Horizons
Velocity Relative to Pluto (km/s): 13.77
Pluto
New Horizons
Saturn
Neptune
Earth
Mars
Jupiter
Uranus
Distance from Sun (AU): 32.87
Distance from Earth (AU): 31.85
Distance from Pluto (AU): 0.04
Round-Trip Light Time (hh:mm:ss): 08:49:49
9 Jul 2015 13:00:00 UTC

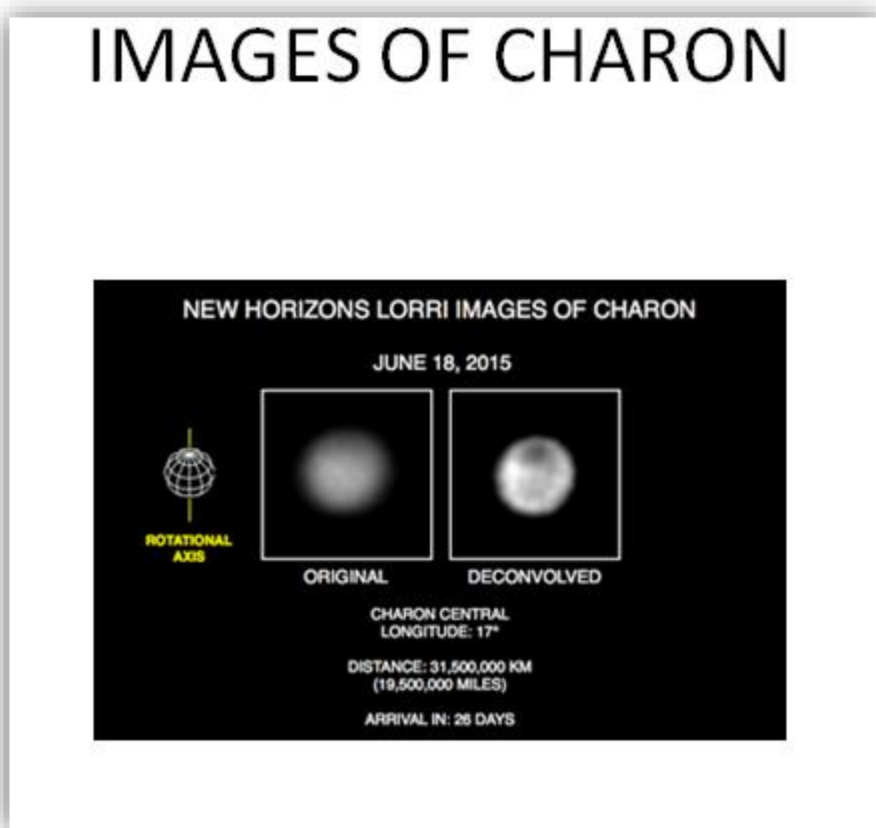

APRIL, MAY 2015
New Horizons LORRI Images of Pluto
APRIL 15, 2015
MAY 10, 2015
ROTATIONAL AXIS
DISTANCE: 108,000,000 KM
(67,100,000 MILES)
PLUTO ARRIVAL IN: 90 DAYS
PLUTO CENTRAL LONGITUDE: 207°
DISTANCE: 77,000,000 KM
(47,800,000 MILES)
PLUTO ARRIVAL IN: 65 DAYS
PLUTO CENTRAL LONGITUDE: 238°


FIRST COLOR IMAGES

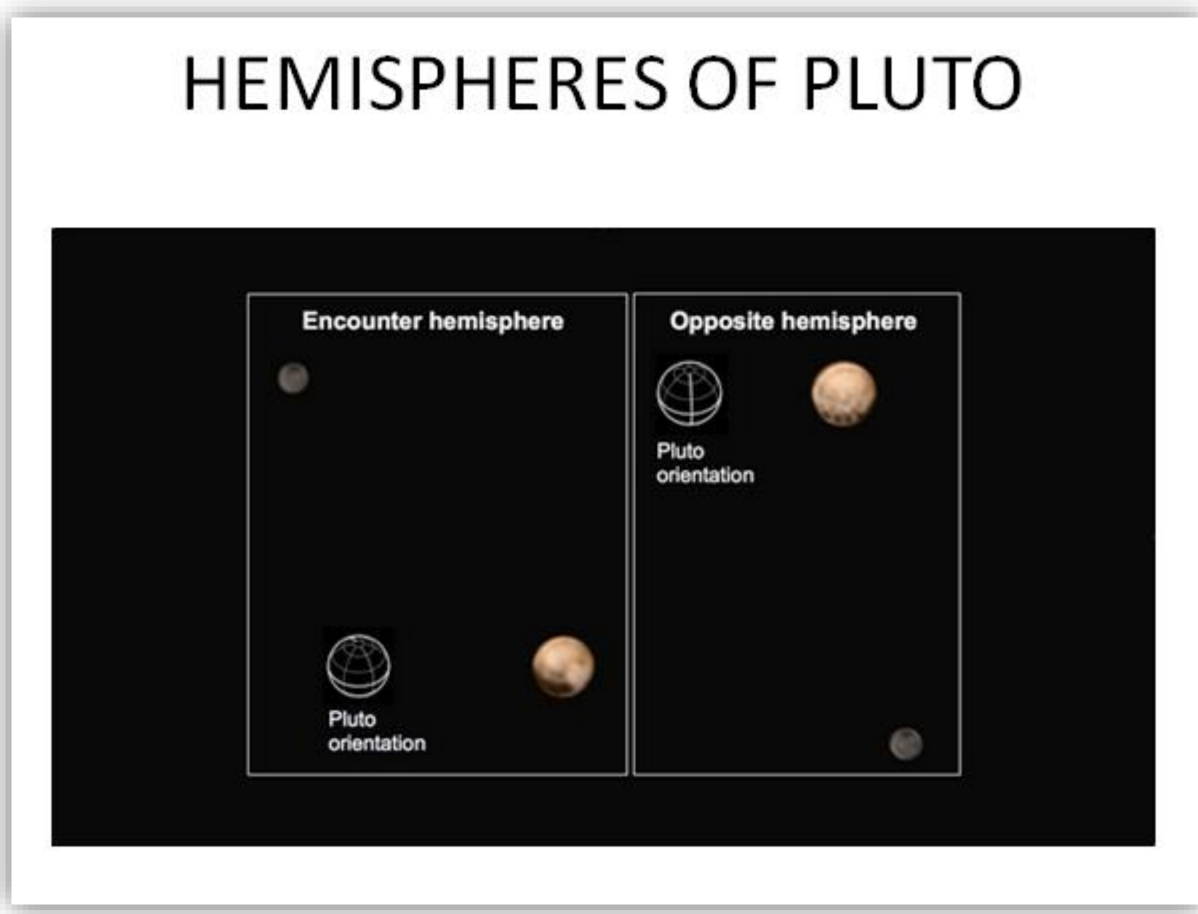

JUNE 29, 2015

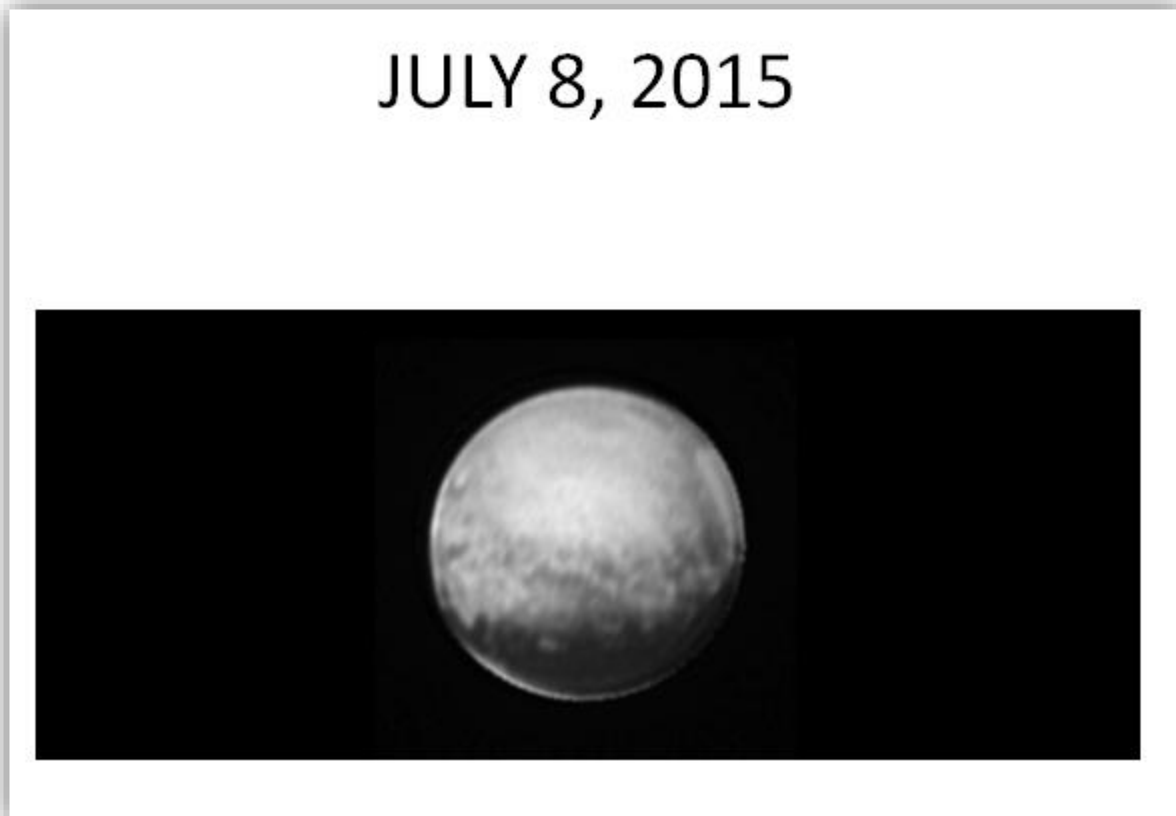

JULY 6, 2015


HEMISPHERES
Pluto
Bright "heart" rotating into view
Dark area rotating into view
Possible cliffs
Circular feature (possible crater)
600 miles (1000 km)


IMAGES OF CHARON
NEW HORIZONS LORRI IMAGES OF CHARON
JUNE 18, 2015
ROTATIONAL AXIS
ORIGINAL
DECONVOLVED
CHARON CENTRAL LONGITUDE: 17°
DISTANCE: 31,500,000 KM (19,500,000 MILES)
ARRIVAL IN: 26 DAYS

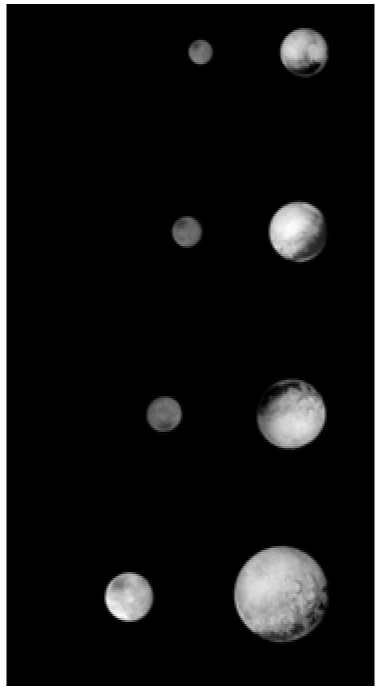

CHARON


HEMISPHERES OF PLUTO
Encounter hemisphere
Pluto
orientation
Opposite hemisphere
Pluto
orientation

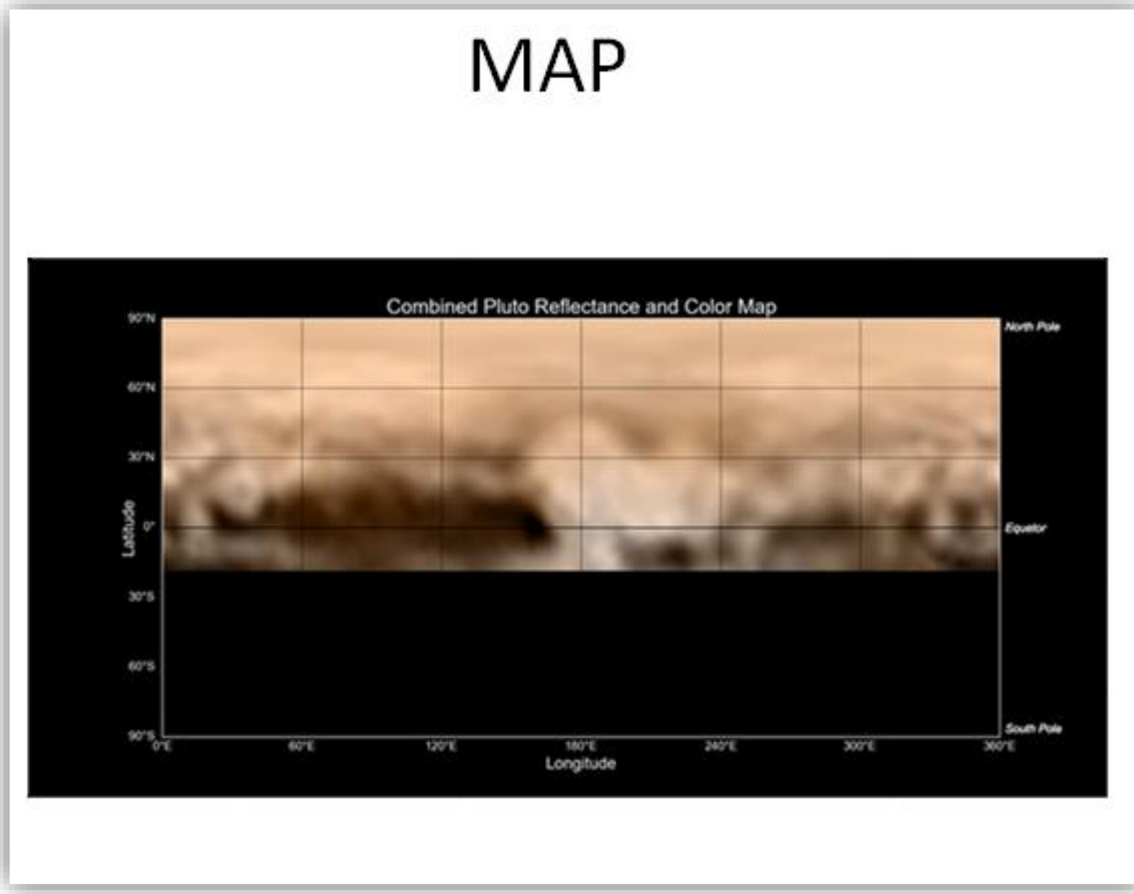

JULY 8, 2015

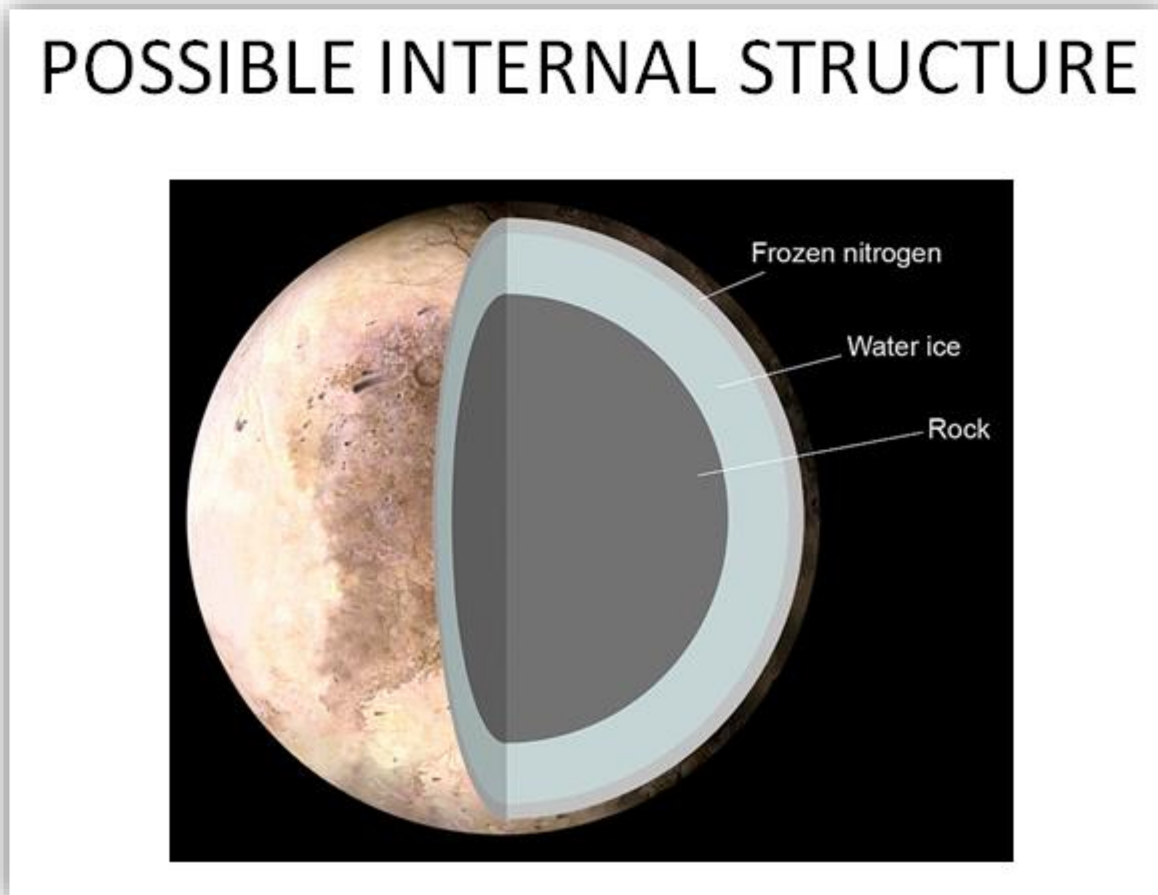

PLUTO AND CHARON

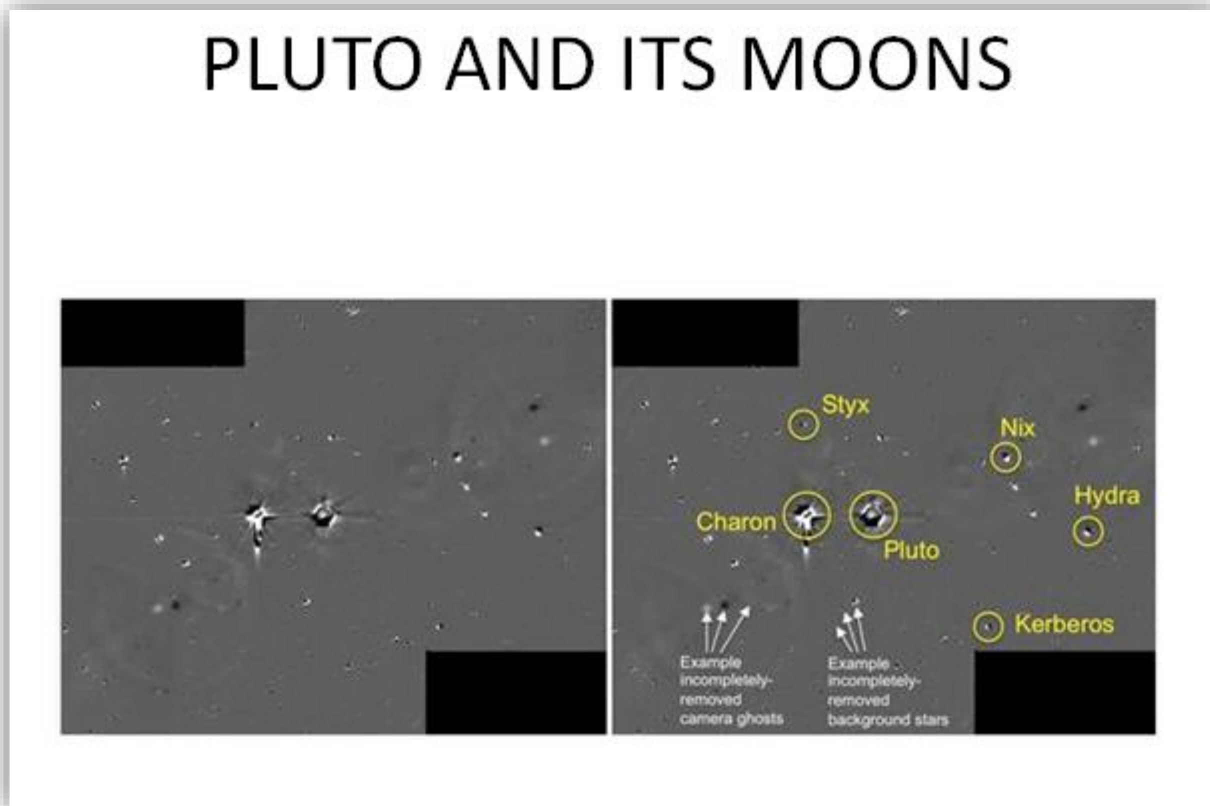
APPROACHING THE PLUTO SYSTEM

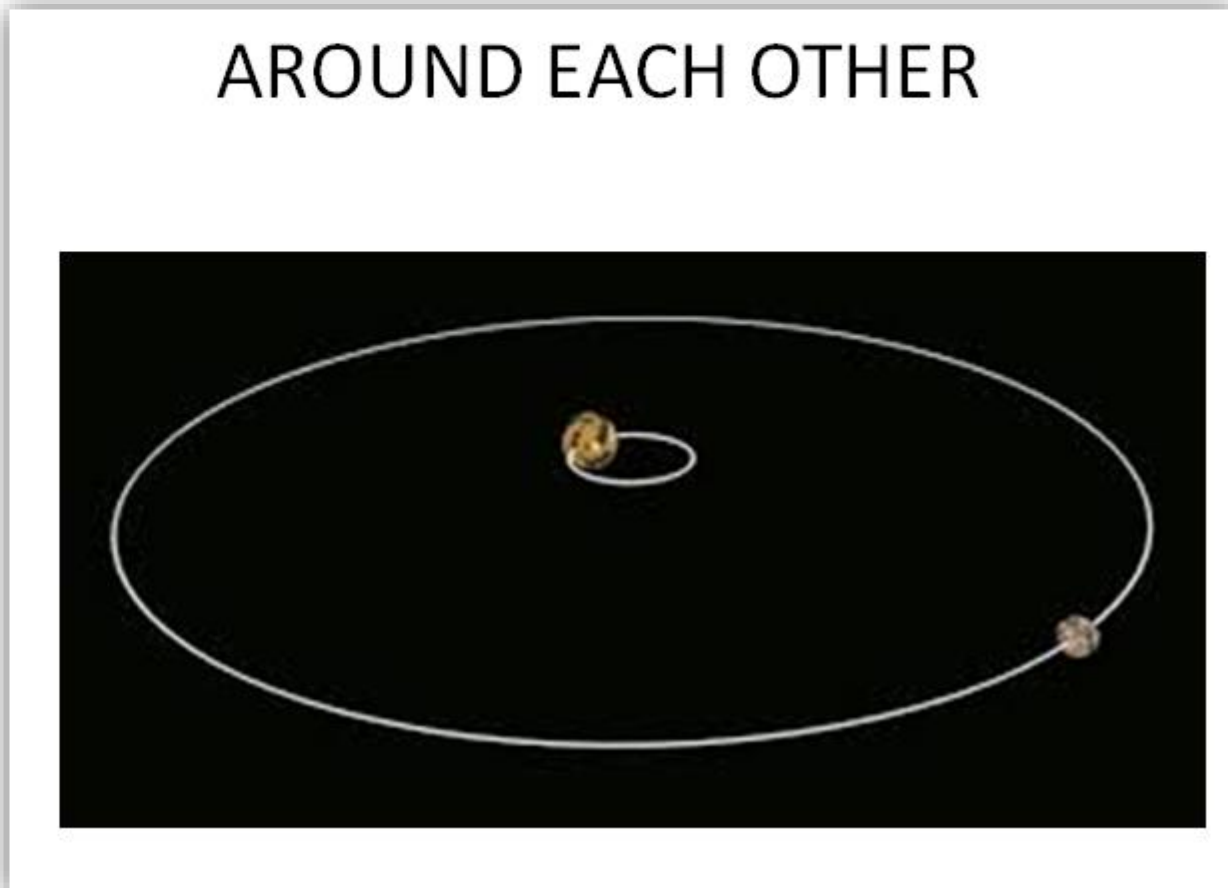


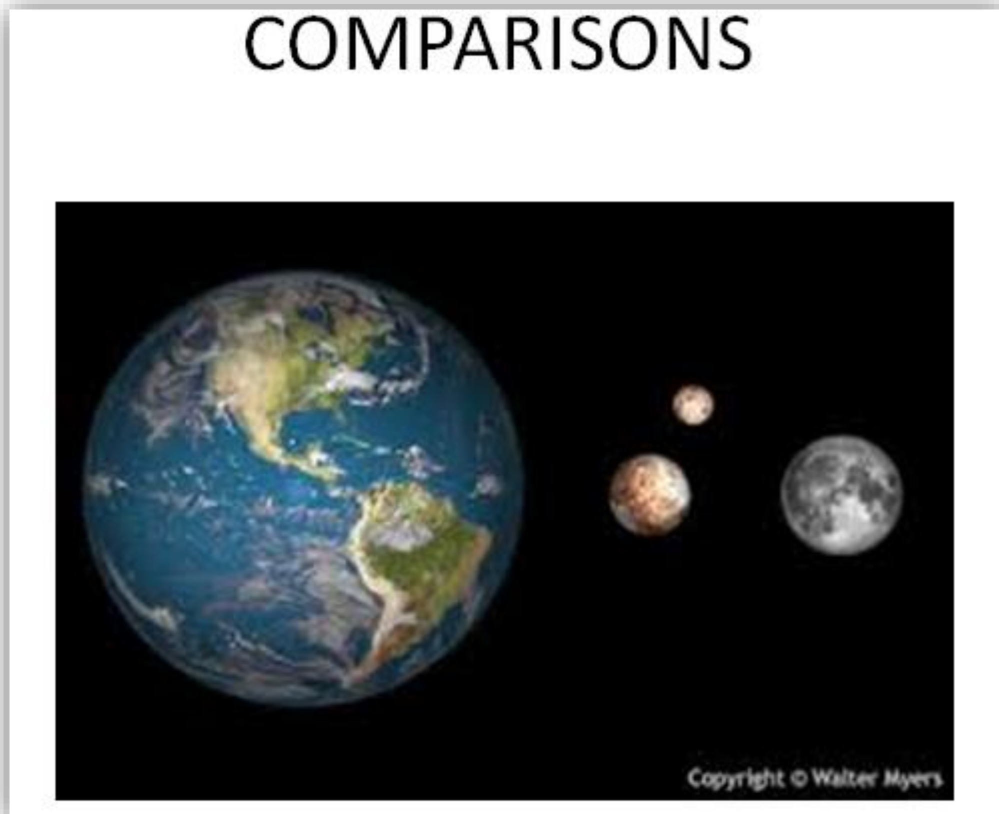

LATEST COLOR IMAGE OF PLUTO
NASA/JHUAPL/SWRI


MAP
Combined Pluto Reflectance and Color Map
90°N
60°N
30°N
0°
30°S
60°S
90°S
Latitude
North Pole
Equator
South Pole
0°E
60°E
120°E
180°E
240°E
300°E
360°E
Longitude

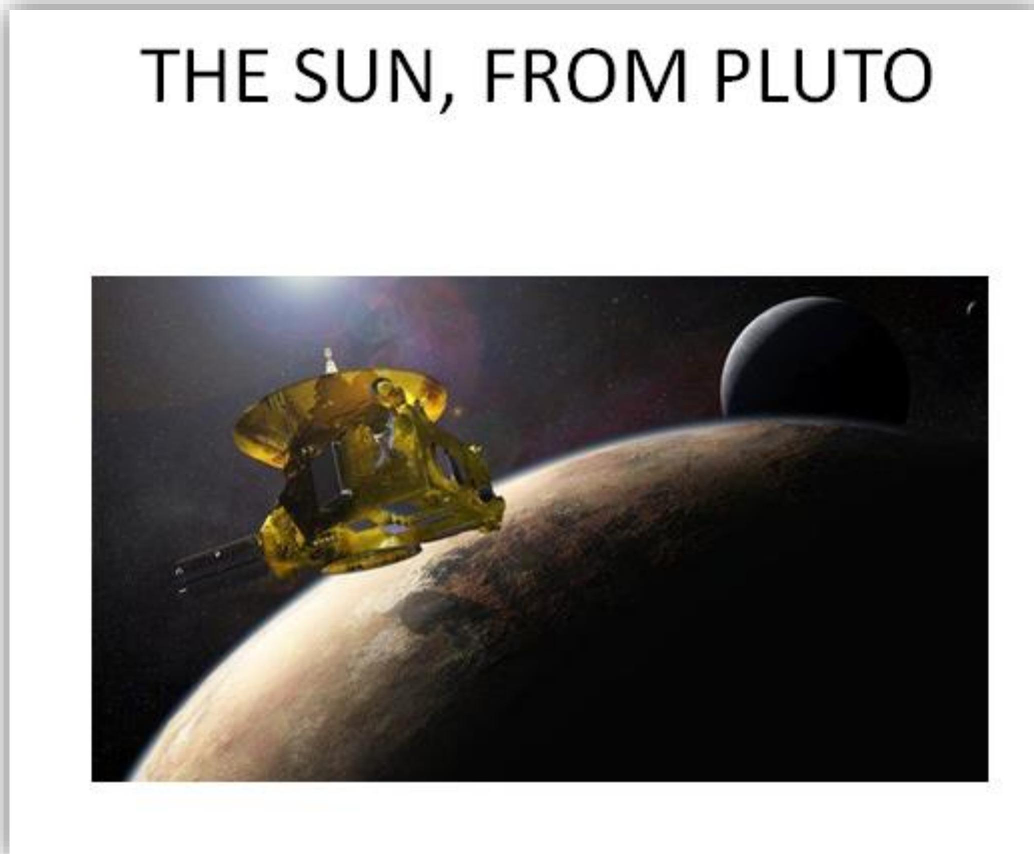

POSSIBLE INTERNAL STRUCTURE
Frozen nitrogen
Water ice
Rock


PLUTO AND ITS MOONS
Styx
Nix
Hydra
Charon
Pluto
Kerberos
Example incompletely-removed camera ghosts
Example incompletely-removed background stars

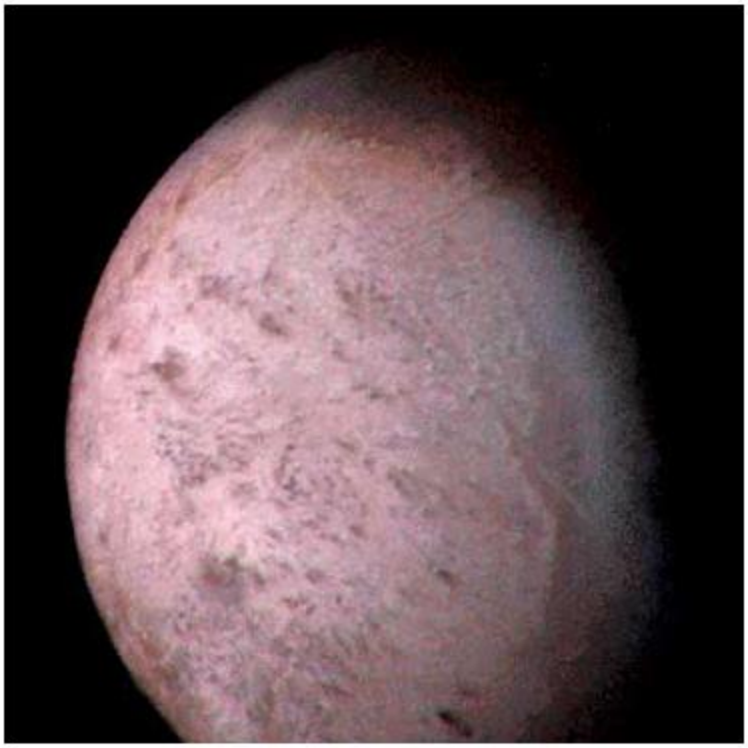

AROUND EACH OTHER

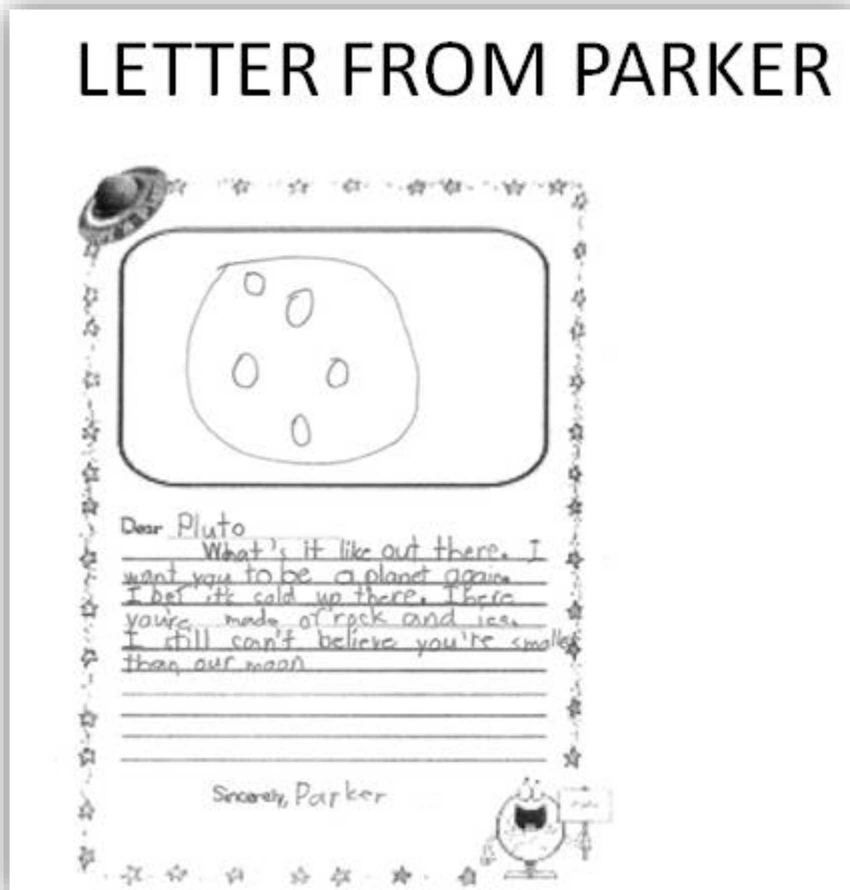

COMPARISONS
Copyright © Walter Myers


THE U.S.A.
PLUTO and CHARON


THE SUN, FROM PLUTO

COUSIN TRITON





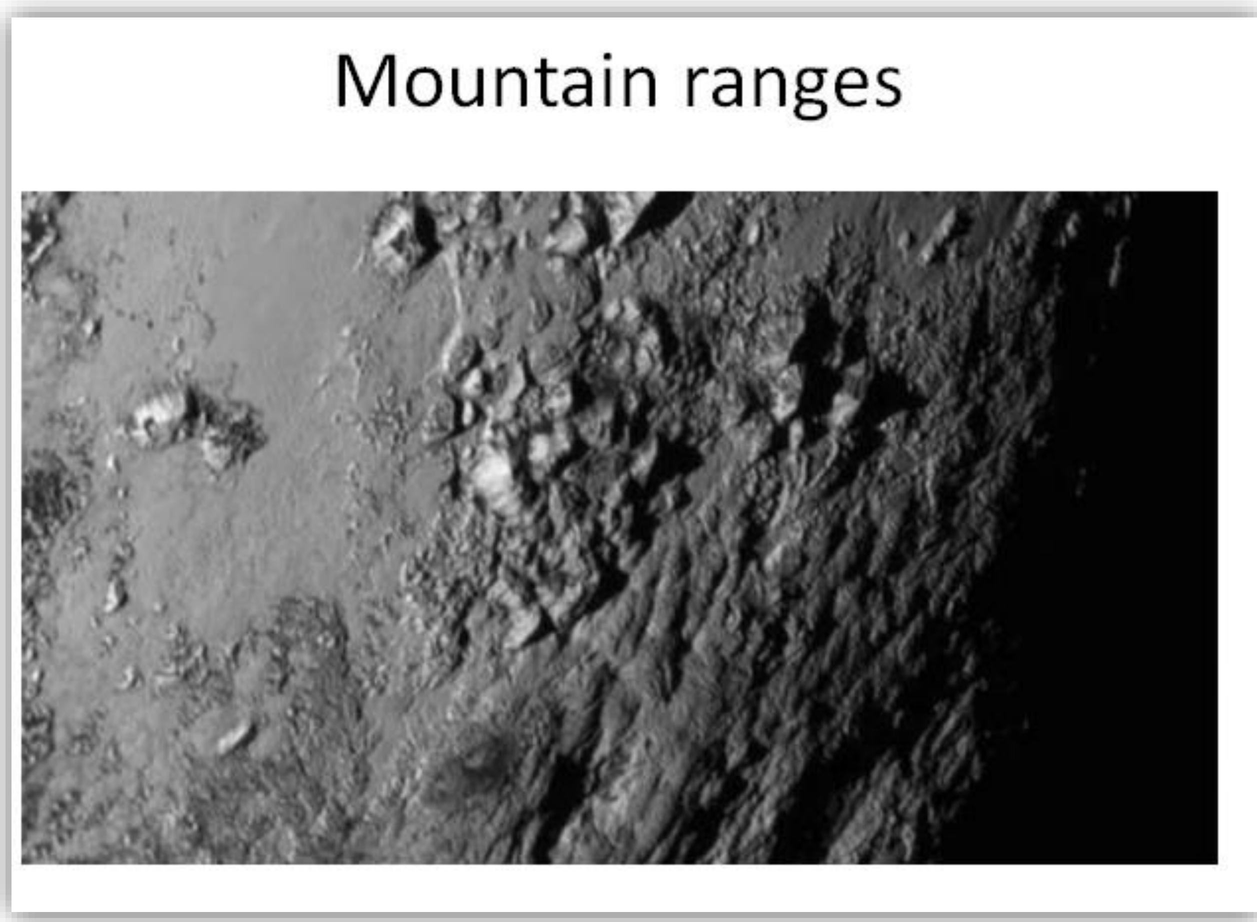
LETTER FROM PARKER

Dear Pluto

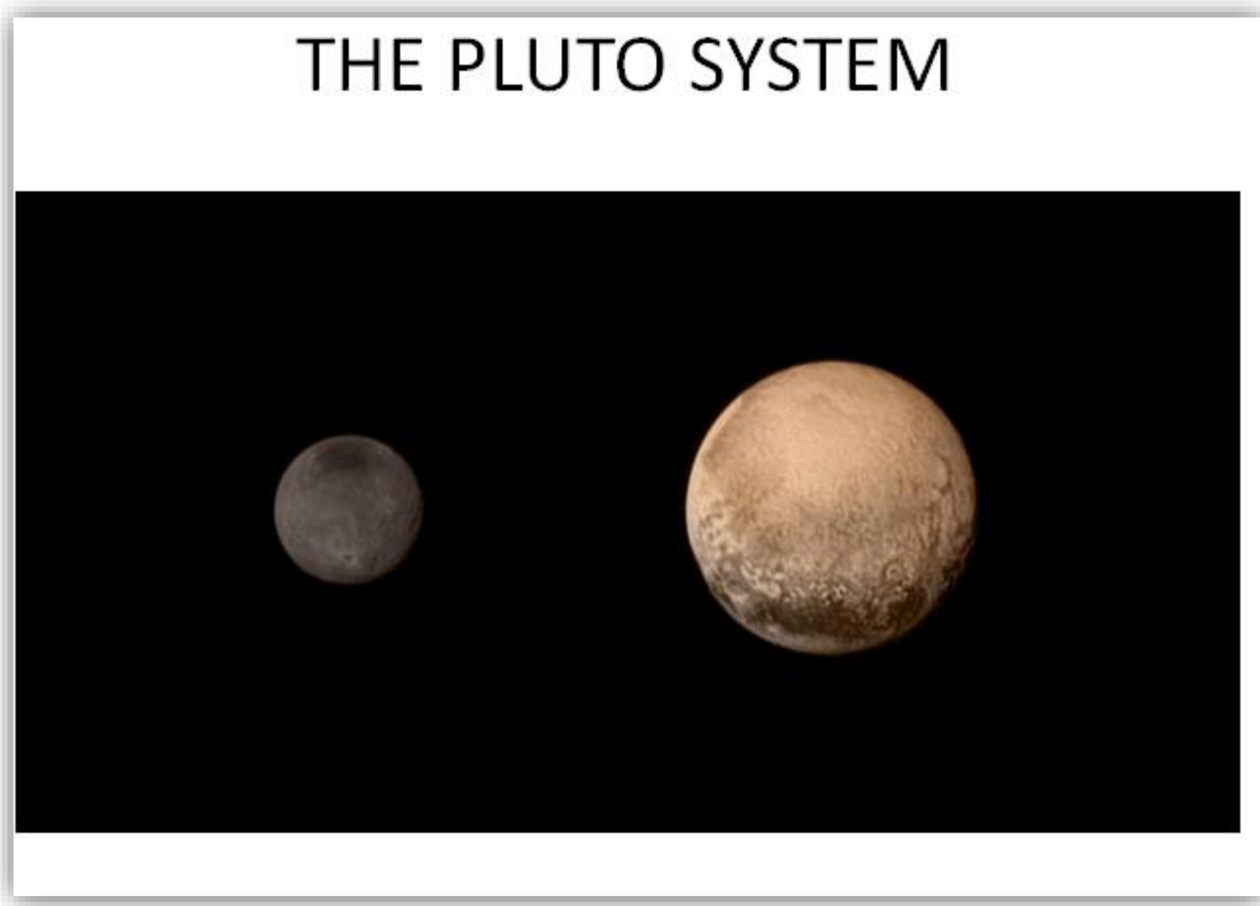
What's it like out there. I want you to be a planet again. I bet it's cold up there. I here you're made of rock and ice. I still can't believe you're smaller than our moon

Sincerely, Parker

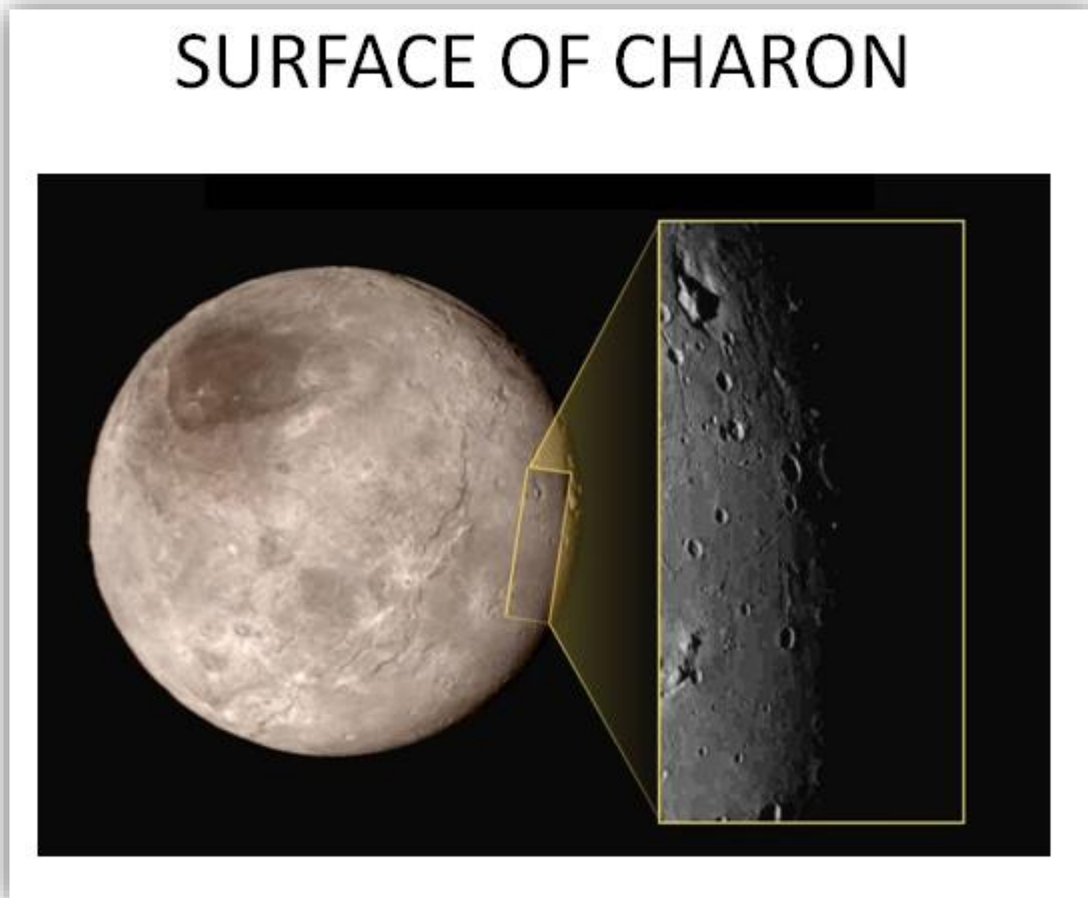

SMILE!!!

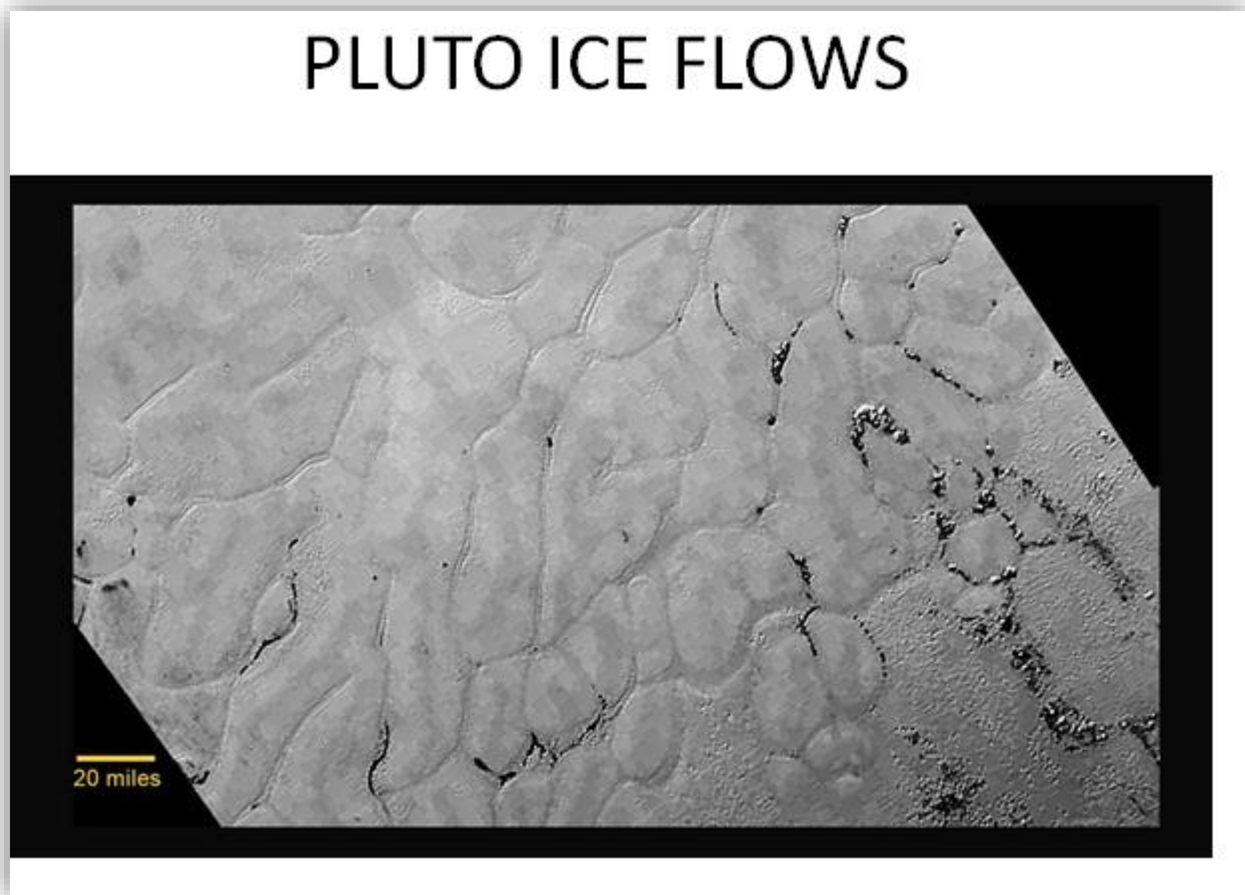

Mountain ranges

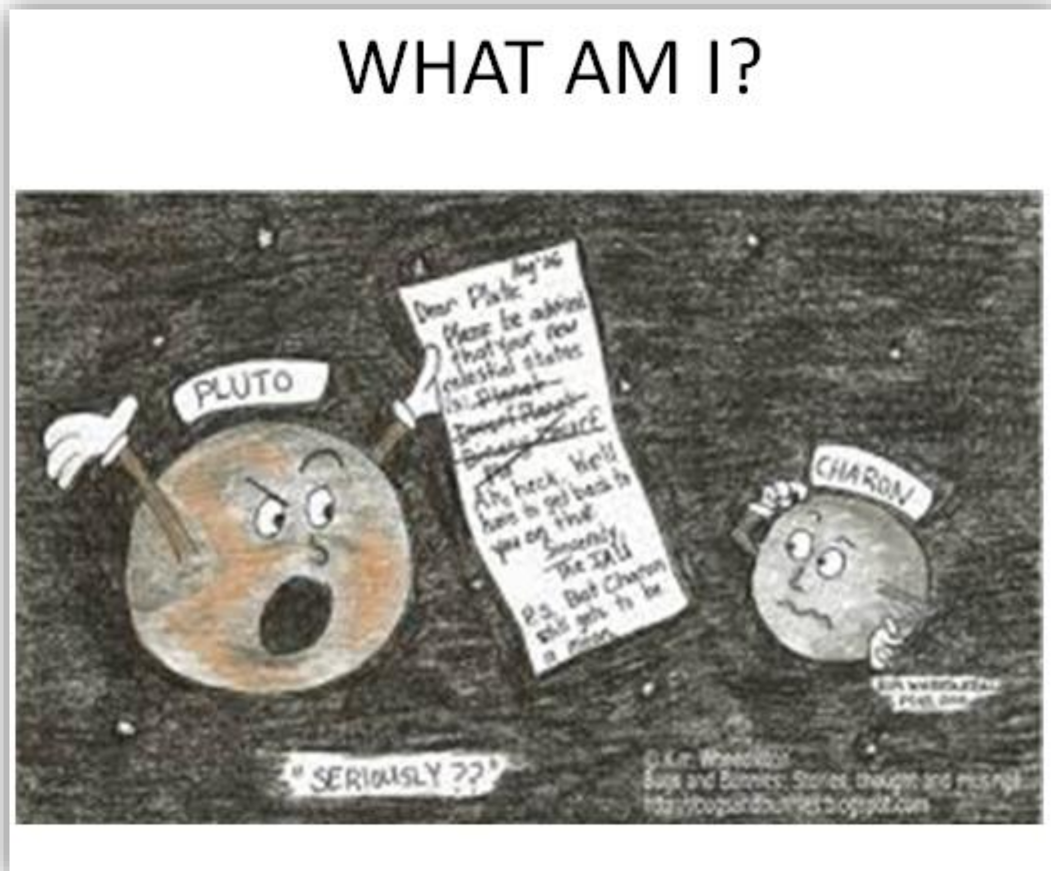

LATEST PICTURE

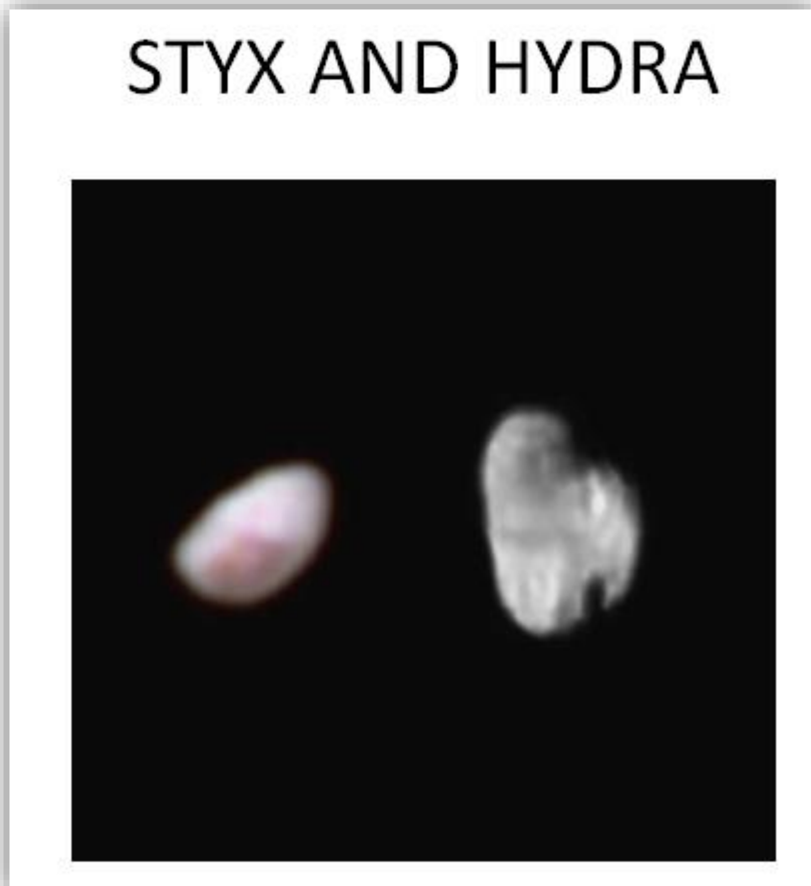

THE PLUTO SYSTEM

CHARON

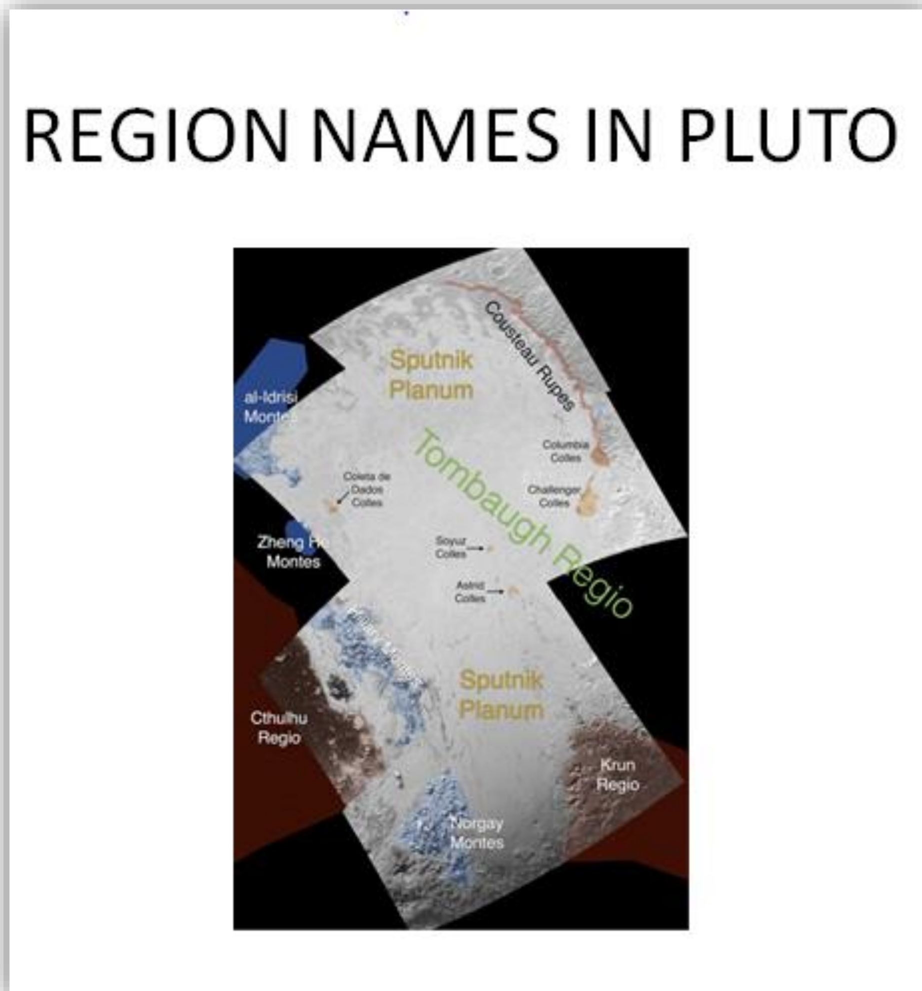

SURFACE OF CHARON

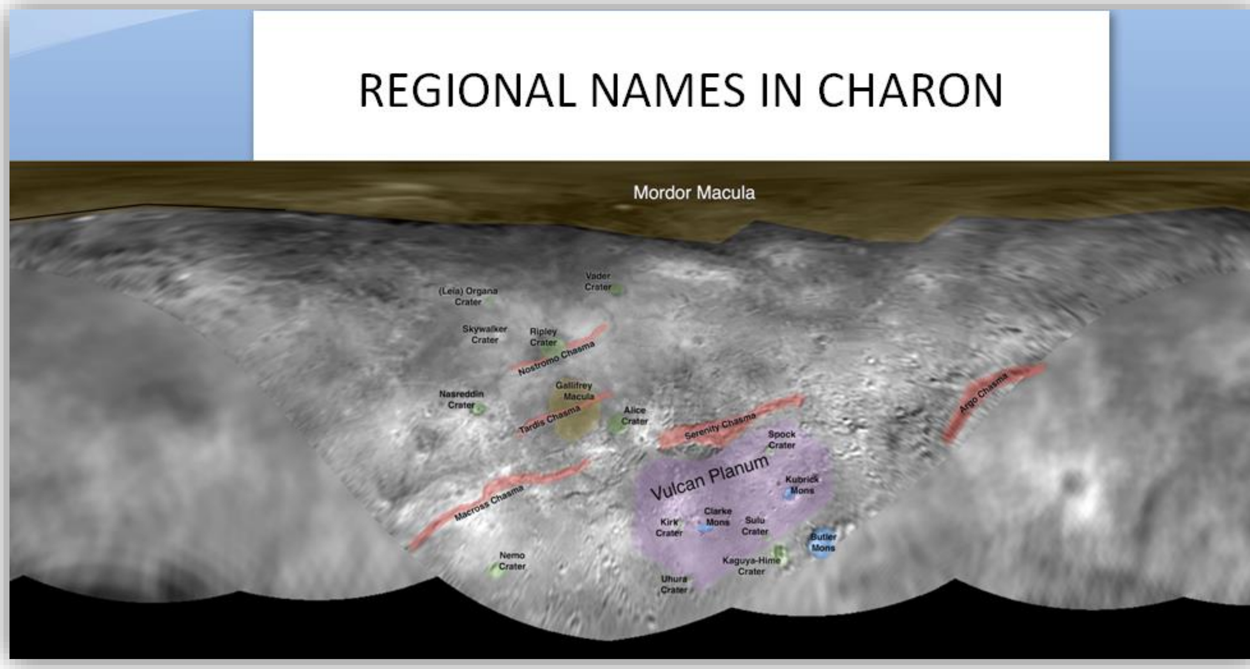

PLUTO ICE FLOWS
20 miles


WHAT AM I?
PLUTO
Dear Pluto
Please be advised that your new celestial status is:
Ah heck. We'll have to get back to you on that.
Sincerely
The IAU
P.S. But Charon still gets to be
CHARON
"SERIOUSLY ??"


STYX AND HYDRA


ECLIPSE, FROM DISCOVERY


REGION NAMES IN PLUTO
Cousteau Rupes
Sputnik Planum
al-Idrisi Montes
Tombaugh Regio
Columbia Colles
Coleta de Dados Colles
Challenger Colles
Soyuz Colles
Astrid Colles
Montes
Sputnik Planum
Cthulhu Regio
Krun Regio
Norgay Montes


REGIONAL NAMES IN CHARON
Mordor Macula
Vader Crater
(Leia) Organa Crater
Skywalker Crater
Ripley Crater
Nostromo Chasma
Gallifrey Macula
Nasreddin Crater
Tardis Chasma
Alice Crater
Serenity Chasma
Argo Chasma
Spock Crater
Vulcan Planum
Kubrick Mons
Macross Chasma
Clarke Mons
Kirk Crater
Sulu Crater
Butler Mons
Nemo Crater
Kaguya-Hime Crater
Uhura Crater


A NEW WORLD

Appendix H: Interviews

Differential Equations Interview session one

RJS – teacher

B, C, D – students

RJS – OK, I just wanted to see if this thing worked. All right today we are going to look at homogeneous equations with constant coefficients.

For the next few moments we discussed the equation

$$Y'' - 7y' + 10y = 0$$

We discussed how all linearly independent solutions would be of the form e^mx. In the case of our example, there would be a solution of the form

$$Y = c1e^{5x} + c2e^{2x} \quad \text{(c1, c2 two arbitrary constants)}$$

Other similar equations were discussed. Cubic equations were looked at, and synthetic division was explored as a means of breaking down the cubic. The three students, Korean internationals in their mid-twenties, had no problem with the material (they had seen synthetic division in the past.) It was obvious that they understood the logic of the session.

The discussion culminated with an overall view of how to deal with homogeneous differential equations. assuming solutions of the form e^mx and how the derivatives, y' = me^mx and y" =m^2e^mx, lead to the characteristic equation, to be solved for m.

We concluded with a simple, factorable cubic where we used synthetic division (which they learned, by the way, in middle school.) Problems were assigned, and the students were dismissed. The session lasted about an hour and fifteen minutes. The rest of the transcription follows:

RJS – OK, we're dealing with homogeneous equations and I want to consider homogeneous equations with constant coefficients. We're going to look at one type today and we're going to look at more types in the coming days. Supposing let's say we've got second degree equations' Let's say:

$$Y'' - 7y' + 10y = 0$$

All the coefficients, if you noticed, are constant terms. OK, 1, -7, and 10. OK, the claim is this, whenever you have a situation like this. This is called a homogeneous equation with constant coefficients (writes on board). OK, I don't feel like writing "this" and this stands for coefficients. Once again, homogeneous means you've got a zero here, and constant coefficients means the derivatives of all the functions are all constant numbers. Now when you have a situation the claim is that the solutions are going to be something of the form e to the mx power. And then what you want to do is take the derivatives and put them in here. OK, this would be

B – m times e to the mx

RJS – Exactly. And y" would be..

B – m squared e to the mx

RJS – Exactly. And then what they claim is …what you can do is when you put that claim into that equation, into the original diff. eq., this becomes:

$$m^2e^{mx} - 7me^{mx} + 10e^{mx} = 0$$

Now as you can see every one of these term has a factor of e^{mx}. You take that out. And what's left inside, you've got:

$$m^2 - 7m + 10 = 0$$

When you do this, this equation inside is given a name. It's known as the characteristic equation. (Writes on board) and it's characteristic of degree 2. This characteristic equation will always be a quadratic equation, which is solvable. Now I deliberately picked my numbers so that this characteristic equation is something easily factorable. Now if I were to ask you to solve this;

$$m^2 - 7m + 10 = 0$$

could you do it? The thing of course with this guy is that it's pretty easy to factor. So the equation would factor into..

B - $(m - 5)(m - 2)$

RJS - Exactly

$$(m - 5)(m - 2) = 0$$

Which means, of course, $m = 5$ or $m = 2$. This means that this diff. eq. is going to have 2 linearly independent solutions. Your one solution is going to be $e^{(5x)}$ and your other will be $e^{(2x)}$. When you come to think of it, it's fairly simple. And you'll find that each one of them checks out. Because for instance if you put $e^{(5x)}$ the second derivative will become $25e^{(5x)}$ - 7 times $5e(5x)$ $+10e^{(5x)}$ and is it equal to zero? You can see pretty easily that it is. Because this is going to be $25 - 35 + 10$. They easily cancel each other out. Which means the complete solution is going to be this. To make a long story short, the total solution is going to be something times e^(5x) plus something else times e^(2x), and that would be your final solution to this problem. And again you can have situations with initial conditions or boundary conditions and that will place a value on each of those. And there are going to be certain types…we'll stay for the time being with quadratics. Suppose we have something like what we had before:

$$y'' - 2y' + y = 0$$

That's another quadratic situation. So we presume, just like before, that a solution should exist, something of the form e^(mx). So like before, y' is me^(mx) and y" is m^2 e^(mx). So when you put this guy in this becomes:

$$m^2e^{(mx)} - 2me^{(mx)} + e^{(mx)} = 0$$

And just like the other situation you'll be able to factor e^(mx) out and what you have left inside is :

$$m^2 - 2m + 1 = 0$$

This is your so-called characteristic equation. Solve it..

B – negative one..

RJS – This of course factors into (m – 1)^2 = 0. As opposed to the other equation, this has just one solution. We say it is a solution of multiplicity 2. So..(shuffles papers)..OK, so m = 1. So one solution would be e^(1x). Now y1 would be e^(1x) and y2 would also be e^(1x). These two guys are linearly dependent. So you have a situation where you have just one solution. What you basically have to do is ..quadratic…you simply do this…your other solution will always be this; x times e^(mx). It will always be that because as it turns out this guy will always solve this and in addition these two will always be linearly independent where the other two weren't . Again to make a long story short, for a quadratic of this form one solution will always be e^(mx) and the other will always be xe^(mx). And your final solution would become:

$$Y\ (\text{total}) = c1e^{\wedge}(x) + c2xe^{\wedge}(x)$$

And this isn't too bad…I'll show that it fits..

$$Y' = c1e^{\wedge}(x) + c2(\ 1e^{\wedge}(x) + xe^{\wedge}(x))$$

What we put in the brackets there we had to use the product rule. So it's the derivative of x times this + x times the derivative of this. And you put them together, you've got.

$$Y' = (c1 + c2 + xc2)e^{\wedge}(x)$$

And then we do your double derivative which would be

$$Y'' = (c1 + c2 + xc2)e^{\wedge}(x) + c2e^{\wedge}(x)$$

Again you have to apply the product rule. And you can combine like terms to get

$$(c1 + 2c2 + xc2)e^{\wedge}(x)$$

Now you don't have to do this all the time but let's see if this does, in fact, work in the original equation. Y" becomes this…(writes on board)…minus 2 times this guy + this guy. And we hope all this cancels out. Let's see ..we get..(c1 +2c2)..(more writing on board)..(everything does, eventually, cancel)…(the cancellations done collaboratively)…So let's just say..I'll throw another one up there. We'll just stick to quadratics. OK, let's do…We'll look at this guy (writes on board). It's quadratic, homogeneous, with constant coefficients. We make the assumption that a solution will always be of the form $e^{(mx)}$ This becomes, you get used to this after a while. Y' is $me^{(mx)}$ and y" is $m^2e^{(mx)}$.These get substituted in here:

$$m^2e^{(mx)} + 6me^{(mx)} + 9e^{(mx)} = 0$$

So then factor out $e^{(mx)}$ and what's left:

$$m^2 + 6m + 9 = 0$$

This thing is called the characteristic equation. Now by itself...you realize $e^{(mx)}$ can never be 0. So in order to solve this- it has to be set = 0. And that's the so-called characteristic equation. And again I deliberately picked these numbers 6 and 9. So I believe that's easily factorable. That would factor into

B – (m + 3)

RJS – exactly. Which = 0 which means of course m would have to be negative 3. And we know that's the only solution. So we say that since every quadratic of degree 2 has to have 2 solutions. So that means either 2 distinct or 1 with multiplicity 2. Or it's also possible, sometimes you could have quadratics with no solutions…you know, with imaginary numbers and that's another can of worms entirely

B – so you would multiply..

RJS – I'm sorry..this stands for..when I say mult it stands for multiplicity. All right, so if m is -3 when that happens my 2 solutions would be $y1 = e^{(-3x)}$ and $y2 = xe^{(-3x)}$ Which means my total overall solution would be:

$$Y = c1e^{(-3x)} + c2xe^{(-3x)}$$

So that in general is how to solve a….and you can have…

B – Did you get your hair cut?

RJS – Yes I did. My wife fooled me. My wife was getting her hair cut Sunday and I went with her and after she finished her hair cut she said "Come on. Your turn now." I have a very hard time saying no to my wife.

B – Aah, you..

RJS – Originally I was going to say no. I have a girl who has been cutting my hair since 1993. Her name's Tammy and I know her quite well. This girl was different from Tammy. I didn't want that initially but when my wife says "get on the chair" I get on the chair. So that's how I got my hair cut. Let me do this…(pause)..OK, again let's deal with…the principle's the same..again, remembering the degree of the equation..OK, I see what it's doing…(another pause) …Ok, let's say we have this diff. eq.:

Y triple prime…

RJS – OK, we have a third degree equation. Now for the third derivative we're going to have at least (most) 3 solutions. Two of them may be identical, one different…or all 3 might be identical. So we got to figure that out. Again just like the previous problem we assume a solution will be of the form e^(mx). I can always cancel out the e^(mx) and get the characteristic equation to solve. What's the third derivative of this going to be?

B – m cubed e^(mx)

RJS – OK, and this guy..

B – m^2 e^(mx)…(more writing on board)..

RJS – and just like the previous situation e^(mx) will cancel out and what's left will be

m^3 – 6m^2 + 11m – 6 = 0

Now the thing we have to solve is this thing. Now, are you guys familiar with synthetic substitution?

B – yeh..

RJS – Synthetic substitution is a very easy thing. I deliberately chose these numbers so that it's easily factorable. For what you want to do with this is find an easy number that works. And a very easy way to check is to use this thing called synthetic substitution. OK, synthetic division goes like this. This of course is an implied one. You write the coefficients 1, -6, 11, -6. Let's say you want to check one, see if one works. Synthetic division works like this ; if you end up with a zero here then one works. Now how do you do this? Step 1…(describes synthetic division in this case for m=1)…that means that one works. And what that means is this; if one works, (m – 1) is a factor. What's left after you take the factor out? Well, that will be determined by these guys. You bring it down one..bring it down to squares. It will be:

m^2 – 5m + 6 = 0

And I think you can see that m^2 – 5m + 6 is easily factorable. It would factor into what?

B – (m – 3)(m – 2)

RJS – Which means that m can be either +3 or +2. Which means over-all your solution is going to be this:

$$Y = c1e^{(x)} + c2e^{(2x)} + c3e^{(3x)}$$

So you're good with synthetic substitution?

B – We're good. We learned that in middle school.

(A little interactive mumbling follows)

RJS – If you wind up with a cubic..it's an easy way.. you guess…you hope ..that an easy number works like one or two or zero. You write down this number..and you multiply and add..and keep doing it. It's a pretty easy way of working on..if you have a quadratic equation all you gotta do is factor…or use the quadratic formula. With cubics you might have…well you might find this one very easy..

B – Some are very complicated..

RJS – Some are complicated, so when you wind up with something complicated and it's of degree 3 or higher. One easy way to help is, number one, you write out the coefficients and you hope an easy number works

B – (indistinguishable)..I don't understand why is that…..y = z……I don't understand..

RJS – Oh it's when you got…

B – Linear combination

RJS – If it's 1,2 or 3 that means that e^(1x) works and so does e^(2x) and e^(3x) and as it turns out when there are separate numbers they will be linearly independent. And when they are linearly independent and it's a cubic equation that means that all solutions are going to be some kind of combination of those three. That's what they say when they have linear combination. So the complete solution will be a linear combination of these guys. So overall:

Y = c1e^(x) + c2e^(2x) +c3e^(3x)

RJS – so we got a whole bunch…(pause)..on your books, if you look at page 181…I also have to show you…if you have a solution and you don't know what the diff eq. is you can figure out the diff. eq. from that

B – diff. eq.?

RJS – Differential equation. Yeah, let me show you. Suppose you have a diff. eq….and the solution is..looks like this: (writes on board) The questions is what is the diff eq. that this function is supposed to solve? And it's actually quite easy. What it means is you look at these 2 numbers and your characteristic equation is m – this number times m – this number…

B – You work it the other way around?

RJS – Before we were given the equation we had to use this. Also, if you're given this it's possible to come up with a diff. eq. The diff. eq. is going to be…simply write down this m - (-2) is going to become m + 2 times m – 2 which becomes…m^2 – 4. That is your characteristic equation which means your diff. eq. is going to be:

Y" – 4y = 0

I'll give you another example. Supposing we had:

$$Y = c1e^{(-4x)} + c2e^{(5x)}$$

What is the original diff. eq. It's going to start (m – (-4))(m – 5) and your diff.eq.will become:

$$y'' – y' -20y = 0$$

There's your diff. eq. There's a bunch of problems on p. 181. Let me just show you that some of them look rather strange. The first 20, based on what we did here, should be no problem. Let's say you have something that looks like this…(writes on board)…what you have to realize is that when you see something that looks like this….this is just the same as y" ..this is just the same as 5y'……and you assume a solution of the form y = e^(mx) and the second derivative is m^2e^(mx) + 5me^(mx) + 4e^(mx) = 0. Then it becomes standard you factor out e^(mx). This becomes

$$m^2 + 5m + 4 = 0$$

Solve this. It becomes..something easily factorable..(writes on board)…This becomes..either -4 or -1 which means your final solution is going to become:

$$y = c1e^{(-4x)} + c2e^{(-x)}$$

You should have enough now to do p.181 without too many problems. So next time we'll do some of these problems. So if you have questions, just go over what we need to do. So did you get the chance to look at anything on the internet?...On Neptune?....Like I said just look up

B – Did you work on….?

RJS –No I never did work at NASA

B-So this is a hobby?

RJS - Yes this is just a hobby for me . My hobby is astronomy

B – Astronomy…and physics..

RJS – They're closely related..

B - …I see…

RJS – exactly…I just did it…I never took a class or anything..I just did it because I liked it.

The rest of the transmission is just conversation about the Apollo missions (I watched Apollo 11 on television) Just conversation about Apollo 18 (which never happened) Some discussion about Apollo 13. ..space shuttle discussion…Challenger disaster. I actually applied to be the teacher in space…All seven astronauts died..The O rings…another disaster in 2002…2010 they completely stopped the program..which is a shame…some discussion on unmanned flights…The distances are incredibly immense…..if you could look up..just do Neptune trajectory on Google..by Thursday…then we'll do some more problems….(class ends)

Differential Equations Interview 2 April 25, 2013

RJS – teacher

Students B, C, D

RJS – Were there any problems with the homework?

B – No problem.

RJS – (garbled) Well then We'll show you something called reduction of order

B – reduction of order?

RJS – yes, go to it here

B – How old are you?

RJS – I'm 66 years old

B – 66?

RJS – 66

B – Ohh!

RJS – I was born in January 1947. Two years after the end of the war. As an actual fact, my father was a captain in the army He was in communications and he was stationed in the Pacific during World War II. He was fighting the Japanese during World War II

B – Your father was a general?

RJS – He was a captain. In World War II, originally, the plan was to invade Japan itself. There was going to be a massive invasion of Japan. If that had happened, there would have been massive destruction. My father would have been in communications. If you invade a land, one of the first things to do would be to set up proper communications. So if they actually had invaded Japan, my father would have been among the first into Japan. But as it turned out they dropped the bombs on Hiroshima and Nagasaki and they forced…I think the reason why they dropped two was because if they only dropped one, they might have thought they only had one. So what? But if they drop two they might have the capability of dropping more. So this forced the Japanese to surrender. If there had been an invasion, my father could very easily have been killed and I might never have been born. As it was I was born two years later. So I guess you might say I am the child of the atom bomb!

B – Ha! Ha!

RJS – So that's my World War II story! So I and my generation were the original baby boomers! All of the Americans who came home from WWII, they all moved into the suburbs and gave birth to people like me!! I was born in Brooklyn, and when I was 5 years old we moved out to Long Island where I grew up with a lot of other baby boomers.

B – Do you have any brothers or sisters?

RJS – Yes I have two brothers and one sister. If I tell you their ages, they will kill me! So you are sworn to secrecy. My sister is two years younger than me, my brother Steve is just turning 60, and my younger brother John is 51.

B – So you are the oldest?

RJS – Yes I am. So that's my family history. We were fortunate. We grew up in relative prosperity. We're in relatively tough economic times now, but like I said, we were fortunate back then. I eventually graduated college in 1969 and I got a job teaching high school math. I was lucky, teaching jobs are a bit tougher to get these days……

THE REST of the transcription involved a short discussion of some of the incidents (specifically, 0ne where I was mistaken for a student!) of my teaching career. This, in turn, was followed by a discussion involving reduction. Specifically, it involved the differential equation:

$X^2y'' - 6xy' + 4y = 0$

Knowing one solution, $y = x^2$, use reduction of order to find the other solution ($y = x^2 \ln x$) This was followed by a discussion of various kinds of solutions (involving real, complex, repeated, etc.) This was followed by a discussion on how the DE could be derived from the solution. A few homework problems were assigned, followed by dismissal. The entire session lasted roughly one hour fifteen minutes. The students (Korean internationals in their mid-twenties) had absolutely no difficulty following the logic of the session. The rest of the transcription is now forthcoming:

RJS – I taught at this one high school for 15 years. That's a long time. And it's really weird. All these people I taught..recently..I taught from 1969 to 1985. I got an e-mail a couple of days ago from some people from the class of 1978…is having a reunion. The class of 1978..eighty-eight…ninety eight..two thousand eight….The class of 1978. It's going to be their thirty-five year reunion. Their average age when they graduated was 18, and now they're going to be about 53. 35 plus 18. They're going to be an average age of 53. They're going to have their reunion next Thanksgiving.

B – Did you teach (mentions former student's name.)?

RJS – Oh yeah! I taught her. Yep. She's teaching here now.

B – She was your..greatest student here? Like now, greatest student mathematics here?...

RJS – Yeh, Yeh, She was a very good mathematics student. She's a very good teacher. At least that's what I've found.

B – She is much younger than us, actually.

RJS – No kidding

B - We're 24.

RJS – I had no idea you were 24.

B – About the same age as American students. It's kind of like the army.

RJS – That's cool. When I started teaching high school I was 22 years old

B – 22?..Oh..

RJS – And I was only five years older than most of my students. When you're in high school it's not like you're in college. You know in high school, if you're caught in the hall, you know you have to be in a classroom, you have to be sitting down. If you're caught in the halls, you have detention. You know, you have a detention system. One day I was walking with a bunch of my students and I lost track of time. I wasn't aware.. the bell had rung and classes had started and I was with these three or four students and the vice-principal looks at us. The high school students…they all froze. He looks at all of us, the vice-principal, and says "Don't you all have to be somewhere?" The students go "Yeah, we …uh." They all go to their separate rooms, and I just stood there. And he looks at me and goes "Well?!!" And I looked at him and I said "Um…I'm a teacher here" And he goes "OH my god! " He thought I was a student"

B – Um..

RJS – Which got me into more trouble. I shouldn't have been out in the hall. I could tell you stories for the next 5 hours of stuff I did when I was teaching high school. Excuse me! I shall do reduction of order! It kind of like goes like this …(writes on board) It's when you have…and we'll concentrate on..quadratics…and I'll best do it with an example. OK The general form of a quadratic equation..a differential quadratic equation of the form we've been looking at…looks like this…some function of x times the second derivative plus another function of x times the first derivative plus some other function times y equals zero. I say x but these could be constants. A sub zero could be 4. And what you want to do is this. You come up with…you find some kind of solution..one by one…that works here, and you're supposed to come up with another one…OK if you have a quadratic equation there's gonna be two solutions…so-called linearly independent solutions. So what you want to do is if you're given one function what you want to do is find another function you let that be equal to whatever this function is times some other function v and then you of figure out what v is. We'll find an example…what I'll do is this (shuffles paper..writes something on board) We'll look at the book. It's

X^2y" – 3xy' + 4y = 0

That's the differential equation. OK and you're given that one such solution is x squared. OK, so the first thing you want to do is ask does that solution fit? That part of it, is easy. OK, you simply get the derivative y one prime would be..

B – 2x

RJS – and y double prime would be just two. So the question is if you put these 3 things into here will it balance? And you can see without too much trouble, that it will. X squared time y" which is 2 minus 3x times y' which is 2x + 4x squared and the question is does that equal to 0? OK? And you can easily see that it is. You got 2x squared minus 6x squared + 4x squared = 0. So you have to develop a second formula from that. What you're gonna do is this. You're gonna set your second solution, whatever it is, = to something, whatever it is, times x squared. So if your y 2 prime …Now you realize this v is some function of x we don't know it..it's gonna be v' x squared + the derivative of x squared which is 2x

B – Also, this could have that in it..

RJS - Yes, and we're presuming this function v could have something in x. It could be a constant too don't forget. OK, but presuming it's a function of x, then you do this and y 2 double prime would be v"x Squared + 2v'x. And we gotta do this guy. Plus 2 x' is one plus xv'. OK, so we have to use the product rule on this.. the derivative of this times v + this times the derivative of v and the derivative of x is just 1. So 1 times v is just here and then x v' is there. So we combine the like terms and we get v"x^2 + 2xv' +2xv' is 4xv' + 2…OK, now we've got to plug all this in to this guy. OK, so we're gonna do x squared times this thing…all right then, minus 3x times ..uh..this….,+ uh 4 times (mumble)..OK, this was put in place of y", this was put in place of y' and this was put in place of y. OK, so there should be some cancelling out going on here. We get v"x to the 4rth + 4 x cubed v' + 2x squared v minus 3 x cubed v"..minus 6x squared + 4x^2 v (writes on board) = 0. And as you can see we got 4 x^2 v minus 6x^2 v + 2x^2v..they cancel and we've got 2 like terms here and we wind up with…v"x^4 + x^3v" + v"x^3 = 0 and assuming that x does not = 0 we can divide by x^3 and we find v"x + v' = 0 and we get another substitution. We're gonna let v' = some w. and we'll replace it with v afterward. Then v" just becomes w' and + w just = 0. A simple linear diff. eq. which we can easily solve by moving to minus w..we get.. I think I got this right..so this eventually means natural log w = natural log of x ^-1. W = x^-1 and w was originally our v'. So we get v' = x^-1. So v is the antiderivative of dx/x. and the (anti)derivative of dx/x is natural log of x

B – natural log of w =…..

RJS - OH,plus if two natural logs are = to each other. If natural log of f(x) = natural log of g(x) then f(x) and g(x) are themselves =.

B – (mumbles)

RJS – Natural log is a so-called monotone increasing function. And if 2 logs are =. Natural log of x2 = natural log of x1 then x2 will equal x1. So if the natural log of w, which is this is equal to the natural log of this which is 1/x. I can write this as x^-1 as natural logs go. Minus natural log of x is natural log of x^-1 and then you get this times the antiderivative of this dx/x which is natural log of x. Now originally we had our second y which is this now becomes our second y.. This guy turned out to be natural log x. So I can write my y2 as x^2 natural log of x. That process is what's called reduction of order. Does it work? Let's see if it does. Can I erase this?

B – Sure.

RJS – In order to see if it fits here we have to take the first and second derivatives of this thing. So we have to use the product rule on this thing. The derivative of this times this + the derivative of this times this. So we have 2x natural log of x + x^2 times 1/x. Which is 2x natural log of x + x. The second derivative I'll do this derivative first, which is just 1 + 2 times the derivative of this thing which is natural log of x + x times the derivative of natural log x which is just 1/x. So we simplify and combine like terms, this is just obviously 1 + 2 times 3 y". Now the question is..x^2 times 3 + 2 natural log of x minus 3x times y' = 2x natural log of x + 1…+4 times x^2 natural log of x…question is, will it be 0? OK, let's see, we've got

3x^2 +x^2 lnx - 6x^2lnx – 3x^2 + 4 x^2 lnx

Now I think we can pretty easily see that…we've got….eventually…cancellation. We can see that this solution and this solution are linearly independent. So our final solution would be c1y1 + c2y2. In other words, c1x^2 + c2x^2lnx. So then I'll show you one more thing…

B – Why would this be reduction of order?

RJS – Oh, the reason why would be that at one point remember we had natural log of ….we had v" of x + v' = 0 The reason why we did that was that if we let v" …or w = v'. This is a second degree differential and the reason they call it reduction is if v' = w then v" = w'. Then w' + w = 0. This order of this equation is 2 while the order of this is 1.

B – ooh…

RJS – that's why they call it reduction of order. Let me show you one more thing. I neglected to mention complex numbers. (shuffles papers)..uh…just show this..Oh, I'm not going to do complex #s yet. There may be equations like this guy. Y triple prime – y' =0 Again, uh, the normal way of doing this with constant coefficients…you've got an implied one here minus 5 minus 2. When you have that situation you always assume that a solution exists which will have this form…and we have to go to the third degree. That's easy enough. Y double prime..y triple prime..we've got m e to the mx. We've got m^2 e to the m x and we've got m cubed. We put those into this equation. We're going to get e to the mx times m cubed – 5m e to the mx – 2. As you can see the e to the mx factors (it is never zero) and we've got:

$$m^3 – 5m -2 = 0$$

And again, as I've told you in the past this is never 0. So in order for this thing to have a solution we have to solve this (pointing to above) and again this is a cubic equation. The standard way of doing is if you can't find a solution..hopefully there's an easy one…you resort to synthetic division. Remember there is no second degree and you have to account for every power of m. So we have 1 for m cubed no coefficient for m^2 so we have to account for that with a zero minus 5 m – 2. I believe what works ..oh, let's cheat..it says that negative 2 works…We'll do negative 2. If you do something like this (with possibilities) +1, -1 or 0 I think you'd find…the idea is to hopefully get a 0 there. You bring down this guy (1) -2 times 1 is -2 -2 times -2 is 4. -5 +4 is -1 and the negative 2 times positive 1 is negative 2..and that gets you zero. So that says that if:

P(m) = m^3 – 5m – 2

And -2 is a zero of the function. I should point out that if you put -2 in for m in that polynomial, then (-2)^3 minus 5(-2) – 2…this is easy to figure out (-2)^3 is -8 . (-5) times (-2) is 10. You can see that -8 +10 _ 2 is zero. So, if -2 is a root, and it is, then minus a minus 2

B – m + 2

RJS – right, m + 2 is a factor. And when you divide it, it's determined by these guys (referring to the bottom line of the synthetic division).Then this becomes

B – m^2 – 2m – 1

RJS – Right. Now with this guy, as it turns out, this is not factorable. So, over here, you've got to use your quadratic formula. So we get – (-2) plus or minus the square root of (-2)^2 – 4(1)(-1) over 2(1). So that 's what it comes out to (points to solution). So this would become

B – 2 plus or minus root 8 over 2

RJS – Which would reduce to..

B – 1 plus or minus root 2

RJS – so your solution would eventually be c1e^(-2x) + c2e(1 – radical 2)x + c3e^(1 +radical 2)x. I should show you one more thing. If you have the solution and you want to produce the equation. Suppose you have:

$$Y = c1e^{3x} + c2e^{(-5x)}$$

Your solution is, um,…your characteristic equation is going to be:

$$(m - 3)(m + 5)$$

Which would become:

$$m^2 + 2m - 15$$

which means your diff. eq. is going to be

$$y'' + 2y' - 15y = 0$$

So I just have to show you one more thing relative to that. What happens when you throw in a third, let's say, a constant. Let's say you have c1e^(-2x) + cl2e^(4x) You know that that's a solution, and you have to deal with that. What you do is..

B – Make it a zero x

RJS – Exactly. And that means your characteristic equation is going to be this:

$$(m - 4)(m + 2)(m - 0)$$

Which is just m so you have

m^3 – 2m^2 + 2m

which makes your differential equation;

y triple prime minus 2 y Double prime + 2y Prime = 0

RJS – So you can look at..(gives a 2 page problem set)…You do have a copy of the book

B – Yeh, Yeh

RJS – So if you look at the problems on those 2 pages you'll see pretty much what we covered today. We'll discuss more of the Neptune Project next week.

B – Thank you

RJS – Thank You!

Student A Interview 1 – Mar 12, 2013

Q. How is the course proceeding so far?

A. So far I believe it has gone well. I like the pace, and it has challenged me with calculus.

Q. Describe your mathematical background (i.e. courses taken, etc.) up to this semester.

A. Business stats, stats, Calc 1 thru 4, Advanced Geometry, Discrete Math, Quantitative Literacy, Number Theory, History of Math.

Q. How difficult was the Pre-test?

A. I found the Pre-test to be of medium difficulty. I had trouble with some of the problems, but others I found easy.

Q. How comfortable are you with your calculus background?

A. Not as comfortable as I should be with calculus, but I'm working on it.

Q. What calculus algorithms do you feel you need review on?

A. Homogeneous equations, solving different differential equations that I've seen previously. Also, doing more difficult integrals (like trig substitutions, partial fractions, etc.)

Q. From what you've seen so far, describe the relevance of this course to so-called "real life" applications.

A. I can see how math is relevant to all matters of real life science, including such subjects as astronomy.

Q. What do you hope to get out of this course?

A. A greater appreciation for the relevance of differential equations, and where they are relevant in fields such as astronomy, etc.

Q. Do you feel any changes should be made to improve the course as we move on?

A. No, I think the pace is comfortable for me.

Q. Are you enjoying yourself?

A. Yes, this class is enjoyable because it has challenged my calculus background, which needed a brush up.

Q. Let’s discuss the specifics of your Pre-test

(Actually, we discussed it in detail when she took it a few weeks ago)

(Student A) Interview 2 May 2, 2013

RJS – interviewer

A – interviewee

RJS – OK, give me the problem.

A – yeah, um, OK

RJS – Oh, yeah, this is a very interesting little thing…the integral of natural log of x quantity cubed dx..and we would figure this one out…let’s see…this is a function within a function…I would say…

A – (mumble)

RJS – yes, let’s see..

A – Do we have to figure out the problem first? (note; the problem is an identity proof.)

RJS – Let’s see if we can figure this one out. If we did a simple substitution..u would be natural log of x, v would be x cubed…and we would get nowhere…

A – That's what I did! That's what I did…and it goofed up…

RJS - Yes

A – I know the derivative of natural log of x is 1 over x

RJS – What's this in context of?

A – It's just a problem. My brother had it and he couldn't figure it out, and he asked me to figure it out…

RJS – OK, was it a textbook problem?

A – Yes it was

RJS – OK, then we should be able to figure it out..I think maybe we could do it with…integration by parts….integral u dv = uv – integral v du…that would make it look like ..uh..

A – Let me just have … uh…

RJS – pause…

A – it says ..uhh.. prove that…this is..prove this..antiderivative this…

RJS – OK, let's try this. Let dv = dx and u = natural log of x cubed. …pause..This becomes du = 3 natural log of x squared dx…and v = what?

A – uh…x.

RJS – that's right. It's going to be this times this…x times natural log of x cubed minus v du…which would be integral 3 times natural log of x squared….OK now these x's conveniently cancel..so we get ..uh.. x nl x (nl = natural log) cubed – 3integral ln (x squared) dx..Now I believe that for this we have to use integration by parts again..

A - OK

RJS – So this becomes.. let u

A – Wait, is that negative….

RJS – So we have to go through the same process with this that we went through with this (points). So u = ln(x squared) and dv = dx. So du = 2ln(x) times 1/x dx and v = x. OK, so we've got x ln(x) cubed minus 3 times now this process becomes x times ln(x) squared…pause …minus v dx 2 times ln (x) times x…times dx, I'm sorry..OK 2 ln(x) (all of this is being done on a whiteboard)…yeah, and the x's once again cancel..OK, so I get…x ln(x) cubed minus3x ln(x) squared and then we've got minus a minus which is plus 3 times 2 which is 6 and the x's cancel out and the derivative of ln x we have plus six ln(x) and we have to do it one more time which is dv = dx,.. OK, du = 1/x dx and v =

A – x

RJS – yes OK, so this finally becomes ..running out of space here..I'll put it up here..equals..we've got x cubed…pause..minus 3x ln(x) squared plus 6 now we have to do x ln(x) minus x times 1/x

A – wait it would be

RJS – v du…OK so that becomes x ln(x cubed) minus 3 x ln(x squared) plus 6x ln(x) minus this is just 1 dx which is x…OK, is that what it came out to be?

A – Yeah, it did…that was awesome..

RJS – Let me see where we were at ..uh..doing our…I believe we were doing our..where is my interview…(pause)…just a second here..let's see..give me your questions, relative to the Neptune thing..

A – This one?

RJS – Oh, yeah..now I guess, OK, when I say, arc length, what, specifically, does that mean to you?

A – Uh, the, um..arc length is the ..it's so frustrating..I just couldn't figure it out…

RJS – excessive application of integration by parts..this formula ..so you have to do it…you actually had to do it three times …OK

A – OK, so arc length…you have ..um..you have a function and you want the arc length from this point to this point

RJS – exactly…say from point a to point b

A – arc length is the actual length if you ..uh..straightened it out…

RJS – the actual physical length of this arc ….you can do it according to a formula which goes something like this

A – Yeah, its in here..

RJS – yeah, here to here (shuffles papers) and we're using the fact that we can consider this a part of the total arc length

A – doesn't this have to do with ..um..the Pythagorean theorem?

RJS – exactly dx and dy… you're breaking it up into little triangles

B – yes

RJS – and what it becomes is by Pythagoras is going to become dx squared plus dy squared

A – Yeah!

RJS – OK, and that comes out to…

A – the change in x squared and the change in y squared.

RJS – and then your arc length call it s will be whatever it is from a to b

A – ds

RJS – and the whole thing would be dx squared plus dy squared…a little mental arithmetic…multiplied by dx / dx….which is basically multiplying by one..you don't change anything..you make a further change..you go from a to b and you take the square root of dx squared and dy squared and with this you can do a little bit of mathematical trickery…and you can make this guy dx squared

A- Well, I've never seen this before..

RJS – This dx becomes .. you express it like that…you haven't really changed it

A – Right

RJS – and because what it does…you can stick this thing into that part of it….and so you get a to b of the square root of this thing in here ..the square root of dx squared plus dy squared OK times ..this becomes over dx squared and so this becomes the square root times dx on the outside. What that does is this ..you can now distribute dx over dx squared here and her ..and so this becomes the integral from a to b of radical dx squared over dx squared , which becomes one plus dy/dx quantity squared dx..and there you have your formula and if you do it in terms of functions with dy/dx, the derivative..if you write it as y it would become 1 plus y prime quantity squared dx. Now, what would such a problem look like in real life?

A – Ummm.

RJS – Well, you'd get…well, you could do something simple

A – Well they talk about axis of rotation

RJS – yes, well that stuff..when you're talking about arc length…and we're going to be using arc length to calculate the orbit to Neptune. So let's just use a very simple function for y, y = x, which would look like this (demonstrates on board) Something easily verifiable, let's see..the thing I gave you…let's go from x = 0 to x = 5. This is , of course, all I'm asking is just the physical length of this line ..something you can do without calculus..

A – Yeh

RJS – Well you can see from here to here, from zero to 5, this is just a right triangle. That's nice and simple...a 45- 45 -90…I mean you can do 5 squared plus five squared = x squared. Or, if you recognize the fact that this is a 45 - 45 - 90 triangle, all you have to do is multiply 5 by radical 2.

Now the question is if you were to do the same problem using this little formula, what would it look like? To give you a little bit of understanding of how it works…

A – Right

RJS – OK, we would calculating the arc length from 0 to 5

A – OK, this is what we're going to have to do

RJS – Yeh, yeh, so this is going to become 1 plus whatever f prime of x is squared. Now in this case f(x) is just x. What's f prime of x? What's the derivative of x with respect to x?

A – One

RJS – Just one. OK, so this formula now becomes your little s becomes the integral from 0 to 5 of radical 1 plus now f prime of x is just one..so we have 1 plus 1 squared dx 1 squared is just 1 . 1 plus 1 is (formula) radical 2. This is just a constant, radical 2, so I put it on the outside time 0 to 5 of dx and that integral is just what?

A – One

RJS – The integral of dx is just what?

A – x

RJS – So it's going to be radical 2 times x from 0 to 5. So it's radical 2 times x=5 minus x = 0 which is 5radical2. Which is what you would expect.

A – OK

RJS – OK, so what we want to do is look at it with a little more…sometimes these problems involve trigonometric substitution. The problem I'm looking for is ..let me just take a look at this section (shuffles papers)

A – I was actually trying to relate it to, like, real world problems. And I was googling….

RJS – Ah yes. And this is exactly what I was looking for. Let's say we have this problem…yeh, right,..This one's going to be, unfortunately, a little more complicated..

A – Yeh

RJS – Let's see what we have here…f(x) is now one-half x squared. (writes on board). And we're going to try to find…and this is going to be a parabola..that looks kind of like this…and we want it from x = 0 to x = 1 that's simple enough. It's going to look something this..and y = one-half x squared. So we're basically looking for that thing. And we're going to try to use…we'll call it s..we'll use the trick formula …s is..we are looking from 0 to 1 of the integral of radical one plus f prime, whatever that is, squared and dx. So, if f(x) = one-half x squared, what is f prime of x?

A Um, it's just 1x?

RJS – Exactly.

A – So it's just x

RJS – Yeh. Derivative of x-squared is 2x and multiplied by ½ is just x. So this becomes just radical 1 plus x-squared. Here's where the fun comes in. We now have to deal with the fact that radical 1 plus x-squared does not go by your normal rules

A – Right

RJS – You let ..basically you're going to use a trig formula. You're going to start by drawing a reference triangle. You're looking for an x and you're looking for a Φ. OK, you're starting with a right triangle with an angle whose opposite side is x and adjacent side is one. What formula connects Φ with x with 1?

A – Sine?

RJS – Close…

A – No,no,no,no……tangent!

RJS – yes, tangent Φ is x over one in this reference triangle. Opposite over adjacent. If the opposite is x and the adjacent is one in this right triangle, what is the hypotenuse?

A – x squared plus 1..uh, the square root of x squared plus 1!

RJS – Yep, now what you have to do next is express x in terms of another variable, namely Φ. You know that tangent Φ = x. You now have to take the derivative of both sides. What's the derivative of tangent?

A – Secant?...secant squared!

RJS – Right…secant squared Φ deΦ. So, this equation becomes transformed. We keep it from 0 to 1. So this becomes the square root of 1 plus tangent squared. Dx becomes secant squared Φ deΦ. OK, so how about this? Is there a trig identity that you can replace 1 plus tan squared Φ with?

A – Sine over cosine?

RJS – You’re on the right track. What is the fundamental trig identity?

A – Sine squared plus cosine squared = 1

RJS – and that’s used all the time. And there’s something that follows from it that is used almost as often. If I divide through by cosine squared x, what’s going to happen?

A – Sine squared over cosine squared plus 1 = 1 over cosine squared

RJS – Yeh. 1 plus this squared = 1 over cosine squared. Now, what’s sine over cosine? It’s the same as opposite over adjacent..

A – Um..x..

RJS – The same as opposite over adjacent

A – Oh! Tangent

RJS – OK, so this is 1 plus tangent squared. And what’s 1 over cosine?

A – Hesitates..

RJS – 1 over cosine….

A – Secant!

RJS – Yeh, that’s it, exactly…OK, so radical 1 plus tan squared x..is..secant x. so this becomes secant theta. So we get 0 to 1 of integral of secant cubed Φ deΦ. Now, problem is..what in God’s name do we do with that!? It’s doable but…

A – Do we have to separate it? Like, uh, secant squared and secant?

RJS – Yeh, the best way of doing it..I do this all the time…I would say the best way of dealing with this is ah..secant cubed Φ…9 times out of 10 with this we usually rely on integration by parts. You should always keep this formula (points to board) u dv = uv – v du. Let's say we got secant squared..ah, I would say

A – You mean, we'll have to do it more than once?

RJS – I don't believe we'll have to use it more than once. I would say, for secant cubed, we would use u =secant Φ, dv = secant squared Φ

A – Yeh, that's what I was going to do..

RJS – Because the derivative of tangent is..

A – Secant squared?

RJS – yeh, and the anti-derivative of secant squared would be

A – Tangent

RJS – Yeh, and how about the derivative of secant Φ?

A – Is it secant tangent?

RJS – That's it, exactly! The derivative of secant Φ is secant Φ tangent Φ.

A – I remembered something! Yay!

RJS – OK, so what does this mess become?! We have got..secant cubed Φ becomes …u times v becomes secant Φ tangent Φ minus..uh..(writes on board)…this becomes minus the antiderivative ..v du becomes secant Φ times tangent squared Φ..(pause)..OH, and tangent squared is secant squared plus 1…OK, so what do we have here? We have this…now remember, we're going to have to repeat this s = 0 to 1 secant cubed Φ deΦ. We have to do a little bit of a trick here..This becomes the integral of secant cubed Φ again plus one times secant Φ. Now, as you can see, the derivative repeats itself here

A – Yeah

RJS – OK, so what we have to do is the following. We're going to keep this ..

A – Yeh, yeh…the same thing we had before…

RJS – Yeh, exactly! A little tricky math..and then we get secant cubed minus the anti-derivative of secant Φ deΦ. OK, so now what happens is this is going to move over here. So we ultimately wind up with twice 0 to 1 of secant cubed Φ deΦ equals what we had here secant Φ tangent Φ minus antiderivative of secant Φ deΦ. OK a thousand dollars for antiderivative of secant Φ deΦ.

A – Of secant Φ?

RJS – It's cool when you figure it out…but it's not …uh…

A – Wait,…the integral?...secant Φ deΦ..Is it not secant…

RJS – Actually, it's a really neat trick. It's an illustration of a mathematical trick. It's like, OK, what I'm going to do is..you do..just concentrate on secant Φ

A – OK..Don’t you just do tangent Φ plus 1.

RJS – You’re on the right track…you are definitely on the right track…and it’s the old math trick of multiplying by 1. OK, so what does multiplying by one mean? It means secant Φ…

A – Oh, by itself!

RJS – Only I’m going to express one in the following way..

A – Theta over secant Φ

RJS – You’re on the right track. We would better express it as

A – Tan over tan

RJS – Tangent Φ plus secant Φ OVER tangent Φ plus secant Φ. Have I changed anything at all?

A – No

RJS – What I’ve done is …watch what I’ve done. I recognize the bottom as tangent Φ plus secant Φ but on the top I’m going to multiply out secant Φ times secant Φ is secant squared Φ plus secant Φ times tangent Φ. Now, what is the derivative of the denominator? What is the derivative of secant Φ plus tangent Φ?

A – Um

RJS – What is the derivative of tangent?

A – Uh…

RJS – You just said it before…

A – Secant squared

RJS – Yeah, and what is the derivative of secant Φ?

A – Secant Φ tangent Φ. I knew it was something like this

RJS – Notice, the derivative of the denominator is the numerator

A – Yeah

RJS – What you have is the situation. Keep in mind, what you now have is this dv over v. And what is the antiderivative of dv over v?

A – Uh, v, no..oh, would it be natural log

RJS – Natural log v . And the means that the anti-derivative of this is simply the natural log of the denominator.

A – Oh, yeah!

RJS – OK, so what does this mess turn out to be over here? We finally get..this turns out to be s. What we were looking for turns out to be the integral for 0 to 1 of secant cubed Φ deΦ = one-half of secant Φ tangent Φ minus one-half natural log of secant Φ plus tangent Φ. Now this would turn out to be..in calculus..keep in mind that under these circumstances we now have to convert back to x . Once we're dealing with that. Sometimes with calculus you can do either one of two things at this point. 0 to 1 is in terms of x's. We would have to convert all our secants and tangents to functions of x's.

A – (agrees)

RJS – We do have an alternative. We first said this is x's and we originally substituted for the x's…x becomes tangent Φ. When the tangent of Φ = 0 what is Φ?

A – Φ is 1?

RJS – No, zero. Now when x is 1. Give me a simple..uh…what is..when you have right triangles, when do you have tangent Φ = 1.

A – When that is equal to one…opposite over adjacent

RJS - =1. That would make Φ how many degrees?

A – Uh, forty five?

RJS – Exactly! And forty five in radians would be?

A – In radians would be ….π over 4?

RJS – Exactly. So, to make this a little easier we can covert to Φ = 0 to Φ = π/4

A - Oh, yeah, I remember this

RJS – That would become 0 to π/4. The integral becomes one-half secant π/4 tangent π/4 minus one-half natural log secant π/4 plus tangent π/4 minus this whole thing at Φ = 0… (pauses)… minus one-half natural log secant 0 plus tangent 0. Which if you look at the bottom of zeroes, what is the tangent of zero?

A – Zero!

RJS – This drops out. And what's secant zero? If cosine zero is 1, cosine is also…

A –One

RJS – And what's natural log of 1?

A – 1 over X?

RJS – Natural log of 1 ..e to the zero power is 1. So natural log of 1 is just…

A – Zero

RJS – So this drops out. So what we need to do up her is…We do one-half of secant π/4 tangent π/4 minus one-half….now we're doing π/4…so my triangle would be 1 – 1 – radical 2….π/4, forty five degrees. We know that the tangent is 1..uh..what's secant π/4? Not adjacent over hypotenuse but hypotenuse over adjacent…

A – So it's radical 2?

RJS – And the minus natural log of radical 2 plus 1. There is your arc length from 0 to 1

A – Wow. That's pretty cool, actually

RJS – Yeah, it is. You've got a whole lot of mathematical tricks in there

..the rest is checking.

RJS – Looks like we've got what we want..So I am going to stop!

Appendix I: Student Study Consent Form

Consent Form

DIFFERENTIAL EQUATIONS STUDY

Rutgers University Graduate School of Education

Description: As a student scheduled to take MTH304 Differential Equations in the Spring Semester 2013, you are invited to participate in a study based on the teaching methods that will be used in this course. This study will involve the researcher conducting observations during the first month of class sessions, and you will be asked to participate in 6 additional hours of interviews and other observed sessions. The interviews and sessions may be audio [or video] – taped for the purpose of maintaining accurate records. These tapes will only be used by personnel involved in this study.

Risks and Benefits: There are no foreseeable risks or benefits associated with your participation in this research study.

Time Involvement: Your participation in this study will take no more than normal class time and 6 additional hours.

Payment and Costs: You will receive no payment for your participation in this study, and there will be no foreseeable costs for you associated with your participation.

Subject's Rights: If you have read this form and decided to participate in this project, please understand your participation is voluntary. **You have the right to withdraw your consent or discontinue participation at any time without penalty. You have the right to refuse to answer particular questions.**

Research Products: Your name and organization will not be identified in any reports of the findings from this study. You will be given a copy of the report describing the study's findings.

The principal investigator for this study is:

Robert Search, Centenary College, Hackettstown, NJ 07840 tel. (908)852-1400 (ext. 2112)

Rutgers Graduate School of Education

10 Seminary Place

New Brunswick, NJ 08901

If you have any questions about your rights as a research subject, you may contact the Sponsored Programs Administrator at Rutgers University at (732)932- 0150 (ext. 2104)

I give consent to participate in this study.

Signature__________________ Date_____________

Name_____________________

Signature of Investigator______________

I give consent to be audio-recorded.

Signature__________________ Date_____________

Name_____________________

Signature of Investigator______________