REDUCED-ORDER KALMAN FILTER FOR A CLASS OF CONTINUOUS - TIME SYSTEMS WITH SLOW AND FAST MODES

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ABSTRACT OF THE THESIS

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In this thesis, complete decomposition of the Kalman filter into the reduced-order Kalman filter with slow and fast modes is addressed. First, we investigate the decomposition so that the slow and fast filters are completely separated with both of filters driven by the system measurements. The simulation results are presented for such a decomposition using an aircraft example. In the second part, this thesis presents the design of reduced order Kalman filters for systems with both slow and fast modes for the case of perfect measurement. The main advantage of the reduced order approach is moderating and reducing mathematical difficulties to obtain the optimal state estimation. This will facilitate the use of Kalman filter for a class of real-time physical systems. In this thesis, we explain the effectiveness of the proposed design through theoretical studies and simulation results.

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Chapter 1

Introduction

1.1 Continuous - Time Kalman Filter Technique

The Kalman filter is represented by mathematical equations that have been applied and utilized to reduce the mean and variance of the estimation error for linear dynamic stochastic systems.

The standard Kalman filter is considered and treated as an optimal estimator for linear systems with Gaussian noise disturbing the system and its measurements. It is nonlinear suboptimal equivalent, the Extended Kalman filter (EKF) is the nonlinear version of the Kalman filter [3].

In this chapter, we present how the Kalman filter works and how it can be used under the Gaussian white noise assumptions. In addition, we present its dual counterpart, the linear-quadratic regulator (LQR), and consider both the Kalman filter and LQR for singularly perturbed linear systems.

We consider the Kalman filter equation using the state-space model [1], [2]

$$\frac{d}{dt}x(t) = F(t)x(t) + B(t)u(t) + w(t) \tag{1.1}$$

$$z(t) = H(t)x(t) + v(t) \tag{1.2}$$

where:

$x(t)$: is the state vector representing the variables of interest for the system (e.g., velocity, position) in time t.

$u(t)$: is the vector having the control inputs (braking force, throttle setting, steering angle)

$F(t)$: is the system state matrix.

$B(t)$: is the control input matrix.

$w(t)$: is the vector representing the system noise process.

$z(t)$: is the vector of measurements.

$H(t)$: is the transformation matrix that maps the parameters of the state vector into the measurements.

$v(t)$: is the vector having the measurement noise terms for every observation in the measurement vector.

$\{v(t)\}$, $\{w(t)\}$ are white Gaussian independent random processes that are both assumed to be zero-mean with covariances $Q(t)$ and $R(t)$ consequently.

The white Gaussian noise statistics is given by

$v(t) \sim (0,R)$ and $w(t) \sim (0,Q)$

The statistic for state initial conditions is

$$x(0) \sim (\bar{x}_0, P(0)) \tag{1.3}$$

and $x(0)$, $v(t)$ and $w(t)$ are assumed to be uncorrelated.

The Kalman filter includes two differential equations, [5] the first one for the state estimate and the second one for the covariance of the estimation error defined by:

$$e(t) = x(t) - \hat{x}(t)$$

$$\frac{d}{dt}\hat{x}(t) = F(t)\hat{x}(t) + B(t)u(t) + K(t)(z(t) - H(t)\hat{x}(t)) \quad (1.4)$$

$$\frac{d}{dt}P(t) = F(t)P(t) + P(t)F(t)^T + Q(t) - K(t)R(t)K(t)^T \quad (1.5)$$

$$P(t_0) = P(0)$$

The Kalman gain $K(t)$ is given by

$$K(t) = P(t)H(t)^T R(t)^{-1} \quad (1.6)$$

It can be seen form (1.6) that the measurement white noise intensity matrix $R(t)$ must be nonsingular for every t.

The Kalman filter innovation process is defined by [4]:

$$\tilde{y}(t) = z(t) - H(t)\hat{x}(t) \quad (1.7)$$

1.2 Continuous Time Linear-Quadratic Regulator Controller

A dual counterpart to the Kalman filter is the linear quadratic regulator (LQR). In Chapter 2, we will use duality to find the Kalman filter gain K using the results for the corresponding linear-quadratic regulator (LQR) design [17].

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0 \tag{1.8}$$

We look for control $u(t)$ that minimizes the performance measure

$$J = \frac{1}{2} \int_{t_0}^{t_1} \left(x^T Q x + u^T R u \right) dt \tag{1.9}$$

where

$$Q = Q^T = CC^T \geqslant 0, \quad R = R^T > 0 \tag{1.10}$$

are weighted matrices. If feedback control $u(t)$ exists, it will have the form

$$u(x(t)) = -Kx(t) \tag{1.11}$$

where K is a feedback gain matrix.

Furthermore, the closed-loop system will be:

$$\dot{x}(t) = Ax(t) - BKx(t) = (A - BK)x(t) \tag{1.12}$$

The following assumption is imposed, [7].

Assumption 1:

The pair $(A;C)$ detectable and the pair $(A;B)$ stabilizable.

Under such assumption, there exists a unique positive semi definitive stabilizing solution to the algebraic Riccati equation, [7] defined by

$$0 = A^T P + PA + Q - PBR^{-1}B^T P \tag{1.13}$$

The optimal feedback gain is given by [6] :

$$K = R^{-1}B^T \bar{P} \tag{1.14}$$

The closed loop system (1.12) is asymptotically stable [7]-[8].

1.3 Decoupling Transformation for Singularly Perturbed Linear Systems

Consider a singularly perturbed linear system without control input [9]

$$\begin{aligned} \dot{x}(t) &= A_{11}x(t) + A_{12}z(t), \quad x(t_0) = x_0 \\ \epsilon\dot{z}(t) &= A_{21}x(t) + A_{22}x(t), \quad z(t_0) = z_0, \end{aligned} \tag{1.15}$$

where $x(t)$ are n_1 dimensional slow state variables ; $z(t)$ are n_2 dimensional fast state variables and ϵ is a small positive parameter.

To analyze (1.15), the common method is to use the Chang transformation to convert the equation into a block-diagonal form where the slow and fast parts of (1.15) are entirely decoupled [20]. The Chang transformation is given by

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} I_1 & \epsilon H \\ -L & I_2 - \epsilon LH \end{bmatrix} \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} = T^{-1} \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} \tag{1.16}$$

and its inverse transformation is

$$\begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} I_1 - \epsilon HL & -\epsilon H \\ L & I_2 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = T \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \tag{1.17}$$

where H and L matrices satisfy algebraic equations

$$A_{21} - A_{22}L + \epsilon L A_{11} - \epsilon L A_{21} L = 0 \tag{1.18}$$

and

$$\epsilon(A_{11} - A_{12}L)H - H(A_{22} + \epsilon L A_{12}) + A_{12} = 0 \tag{1.19}$$

Matrices *H* and *L* can be solved using several methods. For example, the Newton method is given in [22]. The decoupled form of the system is given by:

$$\begin{bmatrix} \dot{\xi}(t) \\ \dot{\eta}(t) \end{bmatrix} = \begin{bmatrix} A_{11} - A_{12}L & 0 \\ 0 & A_{22} + \epsilon L A_{12} \end{bmatrix} \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} \tag{1.20}$$

Now we present a brief summary of the classical method for feedback control of continuous time singularly perturbed linear systems [12]. Consider the controlled system

$$\begin{aligned} \dot{x}(t) &= A_{11}x(t) + A_{12}z(t) + B_1 u(t), \quad x(t_0) = x_0 \\ \epsilon \dot{z}(t) &= A_{21}x(t) + A_{22}z(t) + B_2 u(t), \quad z(t_0) = z_0. \end{aligned} \tag{1.21}$$

where $x(t) \in R^{n_1}$, $z(t) \in R^{n_2}$, and $u(t) \in R^{m}$.

The system can decomposed into two sub-systems:

n_1 - dimensional slow subsystem and n_2 - dimensional fast subsystem, by setting $\epsilon = 0$ in (1.21). The slow approximate subsystem is

$$\dot{x}_s(t) = A_s x_s(t) + B_s u_s(t), \quad x_s(t_0) = x_0 \tag{1.22}$$

$$z_s(t) = -A_{22}^{-1}(A_{21}x_s(t) + B_2 u_s(t) \tag{1.23}$$

where

$$A_s = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad B_s = B_1 - A_{12}A_{22}^{-1}B_2 \tag{1.24}$$

the vectors x_S *(t)*, u_S *(t)* and z_S *(t)*, reference to the slow subsystem parts of the variables *x(t)*, *u(t)* and *z(t)*.

The fast approximate subsystem is

$$\epsilon \dot{z}_f(t) = A_{22}z_f(t) + B_2 u_f(t), \quad z_f(t_0) = z_0 - z_s(t_0) \tag{1.25}$$

where $u_f(t) = u(t) - u_s(t)$ and $z_f(t) = z(t) - z_s(t)$ denote the fast sub-system part of the variables *z(t)* and *u(t).* A combined control law contains slow and fast parts

$$u(t) = u_s(t) + u_f(t) \tag{1.26}$$

it is known as composite control.

Chapter 2

Kalman Filter for Linear Singularly Perturbed Stochastic Systems

2.1 Introduction

In this chapter, we first present a technique which allows full decomposition of the optimal Kalman filter for a linear singularly perturbed system into pure-slow and pure-fast optimal filters both driven by the system measurement. The method depends on the decomposition of the singularly perturbed system global algebraic Riccati equation in to pure-slow and pure-fast local algebraic Riccati equations. In the second part, we perform simulation of the pure slow and pure-fast Kalman filters using MATLAB, for an aircraft example [10].

2.2 Filtering for Singularly Perturbed Linear Systems

Filtering for singularly perturbed continuous-time linear systems has been well considered in control theory [11]-[16]. In [11]-[13] the suboptimal slow and fast Kalman filters were built to generate an $O(\epsilon)$ accuracy of the approximate estimate for the state trajectories, where ϵ is a small singular perturbation positive parameter that shows the separation between the slow and fast state variables. In [14], [15] the local slow and fast Kalman filters were obtained with accuracy, that is $O(\epsilon^k)$, where k represents either the number of terms of the Taylor series expansions [14] or the number of the fixed-point iterations [15] utilized to calculate the coefficients of the reduce order filters. It is necessary to indicate that the slow and fast (local) Kalman filters in [14], [15] are driven by the innovation process so that the extra communication channels are needed to form the innovation process.

In the presented results, the Kalman filters will be driven by the system measurements. Furthermore, the optimal Kalman filter gains will be completely determined in terms of the exact pure-slow and exact pure-fast reduced-order algebraic Riccati equations [10].

2.3 Linear Continuous-Time Invariant Singularly Perturbed Stochastic Systems

Consider the linear continuous-time invariant singularly perturbed stochastic control systems:

$$\dot{x}_1(t) = A_1x_1(t) + A_2x_2(t) + G_1w_1(t) + B_1u(t) \tag{2.1}$$

$$\epsilon\dot{x}_2(t) = A_3x_1(t) + A_4x_2(t) + G_2w_1(t) + B_2u(t) \tag{2.2}$$

with the corresponding measurements

$$y(t) = C_1x_1(t) + C_2x_2(t) + w_2(t) \tag{2.3}$$

where $x_1(t) \in R^{n_1}$ and $x_2(t) \in R^{n_2}$ are state vectors. $w_1(t) \in R^{n_1}$ and $w_2(t) \in R^{n_2}$ are zero-mean, white Gaussian noise stochastic processes with intensities $W_1 > 0$, $W_2 > 0$ and $y(t) \in R^{n_2}$ are filter measurements.

In the following A_i, G_j, C_j, $i = 1, 2, 3, 4,$ $j = 1, 2$, are constant matrices of appropriate dimensions.

We assume that the filter under consideration has the standard singularly perturbed form [21] so that it satisfies the following assumption: matrix A_4 is nonsingular.

The standard Kalman filter, corresponding to (2.1)-(2.3), driven by the innovation process is given by

$$\dot{\hat{x}}_1(t) = A_1\hat{x}_1(t) + A_2\hat{x}_2(t) + K_1 v(t) \tag{2.4}$$

$$v(t) = y(t) - C_1\hat{x}_1(t) - C_2\hat{x}_2(t) \tag{2.5}$$

where the optimal Kalman filter gains K_1 and K_2 are given by [14]

$$K_1 = (P_1 C_1^T + P_2 C_2^T) W_2^{-1} \tag{2.6}$$

$$K_2 = (\epsilon P_2 C_1^T + P_3 C_2^T) W_2^{-1} \tag{2.7}$$

with matrix P and sub-matrices P_1, P_2, and P_3 representing the positive semidefinite stabilizing solution of the Kalman filter algebraic Riccati equation

$$AP + PA^T - PSP + GW_1G^T = 0 \tag{2.8}$$

where

$$A = \begin{bmatrix} A_1 & A_2 \\ \frac{A_3}{\epsilon} & \frac{A_4}{\epsilon} \end{bmatrix} \tag{2.9}$$

$$G = \begin{bmatrix} G_1 \\ \frac{G_2}{\epsilon} \end{bmatrix} \tag{2.10}$$

$$S = C^T W_2^{-1} C \tag{2.11}$$

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & \frac{1}{\epsilon} P_3 \end{bmatrix} \tag{2.12}$$

For the decomposition and approximation of the singularly perturbed Kalman filter the Chang transformation [20] has been used in [14]-[16]

$$\begin{bmatrix} \hat{\eta}_1 \\ \hat{\eta}_2 \end{bmatrix} = \begin{bmatrix} I - \epsilon HL & -\epsilon H \\ L & I \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \tag{2.13}$$

which lead to

$$\hat{\eta}_1(t) = (I_n - \epsilon HL)\hat{x}_1(t) - \epsilon H \hat{x}_2(t) \tag{2.14}$$

$$\epsilon \hat{\eta}_2(t) = \epsilon L \hat{x}_1(t) - \hat{x}_2(t) \tag{2.15}$$

In the following we will use duality with the linear-quadratic optimal control problem to design the slow and fast reduced-order independent Kalman filters driven by the system measurement.

2.4 Linear Quadratic Optimal Control Problem for Singularly Perturbed Systems

Consider the linear-quadratic optimal control problem of (2.1), that is

$$\dot{x}_1(t) = A_1 x_1(t) + A_2 x_2(t) + B_1 u(t) \tag{2.16}$$

$$\epsilon \dot{x}_2(t) = A_3 x_1(t) + A_4 x_2(t) + B_2 u(t) \tag{2.17}$$

$$J = \int_0^{\infty} \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T Q \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + u^T R u \right] dt, \;\; Q > 0, R > 0 \tag{2.18}$$

where the control vector $u(t) \in R^m$ should be chosen such that the performance criterion, J is minimized. The very will-known solution to this problem is given by

$$u(t) = -R^{-1} B^T P_r x(t) = -F_1 x_1(t) - F_2 x_2(t) \tag{2.19}$$

where *Pr*, is the positive semidefinite solution of the regulator algebraic Riccati equation

$$A^T P_r + P_r A + Q - P_r Z P_r = 0 \tag{2.20}$$

with

$$Q = \begin{bmatrix} Q_1 & \epsilon Q_2 \\ \epsilon Q_2^T & \epsilon Q_3 \end{bmatrix} \tag{2.21}$$

$$Z = B R^{-1} B^T \tag{2.22}$$

$$B = \begin{bmatrix} B_1 \\ \frac{B_2}{\epsilon} \end{bmatrix} \tag{2.23}$$

$$P = \begin{bmatrix} P_{1r} & \epsilon P_{2r} \\ \epsilon P_{2r}^T & \epsilon P_{3r} \end{bmatrix} \tag{2.24}$$

The algebraic Riccati equation of singularly perturbed control systems is completely and perfectly decomposed into two reduced-order algebraic Riccati equations corresponding to the slow and fast time scales in [19].

The pure-slow and pure-fast algebraic Riccati equations obtained in [19] are not symmetric equations. The Newton technique is very efficient for solving the non-symmetric algebraic Riccati equations. A lot of real physical systems are singularly perturbed, for example: aircraft, robots, nuclear reactors, electrical machines, tunnel diode circuits, power electrical circuits, control system of a pendulum, chemical reactors, DC and induction motors, synchronous machines, and automobiles.

The slow subsystem variables are represented by the eigenvalues that are near to the imaginary axis, on the other hand the fast system variables are represented by the eigenvalues that are far away from the imaginary axis [24].

Consider now the optimal closed-loop Kalman filter (2.4) driven by the measurements that is

$$\dot{\hat{x}}_1(t) = (A_1 - K_1C_1)\hat{x}_1(t) + (A_2 - K_1C_2)\hat{x}_2(t) + K_1y(t) \tag{2.25}$$

$$\epsilon\dot{\hat{x}}_2(t) = (A_3 - K_2C_1)\hat{x}_1(t) + (A_4 - K_2C_2)\hat{x}_2(t) + K_2y(t) \tag{2.26}$$

with the optimal filter gains K_1 and K_2 calculated from (2.6)-(2.7). By duality between the regulator and optimal filter, the filter Riccati equation [9] can be solved by using the same decomposition method for solving (2.20) with

$$A \rightarrow A^T$$

$$Q \rightarrow GW_1G^T$$

$$F^T = K$$

$$Z = BR^{-1}B^T$$

$$Z = BR^{-1}B^T \rightarrow S = C^T W_2^{-1} C \tag{2.27}$$

By invoking results from [19], and using duality, the following matrices have to be formed (see also [10])

$$T_1 = \begin{bmatrix} A_1^T & -C_1^T W_2^{-1} C_1 \\ -G_1 W_1 G_1^T & -A_1 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} A_3^T & -C_1^T W_2^{-1} C_2 \\ -G_1 W_1 G_2^T & -A_2 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} A_2^T & -C_2^T W_2^{-1} C_1 \\ -G_2 W_1 G_1^T & -A_3 \end{bmatrix}$$

$$T_4 = \begin{bmatrix} A_4^T & -C_2^T W_2^{-1} C_2 \\ -G_2 W_1 G_2^T & -A_4 \end{bmatrix} \tag{2.28}$$

Since matrices T_1, T_2, T_3, T_4 correspond to the system matrices of a singularly perturbed system, then the slow-fast decomposition is achieved and completed by using the Chang decoupling method whose algebraic equations are given by [20]

$$T_4 L - T_3 - \epsilon L(T_1 - T_2 L) = 0 \tag{2.29}$$

$$-H(T_4 + \epsilon L T_2) + T_2 + \epsilon(T_1 - T_2 L)H = 0 \tag{2.30}$$

We solve (2.30) by using Newton method [22]. The algebraic equations in (2.30) are weakly nonlinear equations and a linear Lyapunov type equation. They play the crucial role in a method proposed for the solution.

The existing methods for solving (2.30) are recursive type algorithms with a rate of convergence $O(\epsilon)$ so that the accuracy of $O(\epsilon^{2k})$ can be achieved after k iterations [15].

In this section the method for solving (2.30) with a quadratic rate of convergence, that is $O(\epsilon^2)$, will be presented. This method is based on the Newton type recursive scheme. It is a very well-known fact that the Newton method converges quadratically in the neighborhood of the sought solution and that its main problem is in the choice of the initial guess. The initial guess is easily obtained with the accuracy of $O(\epsilon)$, by setting $\epsilon = 0$ in (2.30), that is

$$L^{(0)} = T_4^{-1}T_3 = L + O(\epsilon) \tag{2.31}$$

Thus, the Newton sequence will be $O(\epsilon^2)$,$O(\epsilon^4)$,$O(\epsilon^6)$,$O(\epsilon^8)$,.......,$O(\epsilon^{2k})$ close to the exact solution, respectively, in each iteration.

The Newton type algorithm can be constructed by setting:

$$L^{(i+1)} = L^{(i)} + \Delta L^{(i)} \tag{2.32}$$

This will produce a Lyapunov type equation of the form:

$$D_1^{(i)} L^{(i+1)} + L^{(i+1)} D_2^{(i)} = Q^{(i)} \tag{2.33}$$

with the initial condition given by

$$H^{(i)} D_1^{(i)} + D_2^{(i)} H^{(i)} = T_2 \tag{2.34}$$

where

$$D_1^{(i)} = T_4 + \epsilon L^{(i)} T_2 \tag{2.35}$$

$$D_2^{(i)} = -\epsilon(T_1 - T_2 L^{(i)}) \tag{2.36}$$

$$Q^{(i)} = T_3 + \epsilon L^{(i)} T_2 L^{(i)}, i = 0, 1, 2, ... \tag{2.37}$$

This implies

$$H^{(i)} = H + O(\epsilon^{2i}) \tag{2.38}$$

By using the permutation matrices dual to those from [15], (note E_1) is different than the corresponding one from [19].

$$E_1 = \begin{bmatrix} I_{n1} & 0 & 0 & 0 \\ 0 & 0 & I_{n1} & 0 \\ 0 & \frac{I_{n2}}{\epsilon} & 0 & 0 \\ 0 & 0 & 0 & I_{n2} \end{bmatrix} \tag{2.39}$$

$$E_2 = \begin{bmatrix} I_{n1} & 0 & 0 & 0 \\ 0 & 0 & I_{n1} & 0 \\ 0 & I_{n2} & 0 & 0 \\ 0 & 0 & 0 & I_{n2} \end{bmatrix} \tag{2.40}$$

we can define

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} \tag{2.41}$$

$$= E_2^T \begin{bmatrix} I - \epsilon NM & -\epsilon N \\ M & I \end{bmatrix} E_1 \tag{2.42}$$

Then, the desired transformation will be given by

$$T_2 = (\Pi_1 + \Pi_2 P) \tag{2.43}$$

The transformation T_2 is applied to the filter variables as:

$$\begin{bmatrix} \hat{\eta}_s \\ \hat{\eta}_f \end{bmatrix} = T_2^{-T} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \tag{2.44}$$

such that the complete closed-loop decomposition is achieved, that is:

$$\dot{\hat{\eta}}_s = (a_1 + a_2 P_s)^T \hat{\eta}_s + K_s y \tag{2.45}$$

$$\epsilon\dot{\hat{\eta}}_f = (b_1 + b_2 P_f)^T \hat{\eta}_f + K_f y \tag{2.46}$$

The matrices in (2.22) are given by:

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = (T_1 - T_2 M) \tag{2.47}$$

$$\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = (T_4 + \epsilon M T_2) \tag{2.48}$$

$$\begin{bmatrix} K_s \\ \frac{K_f}{\epsilon} \end{bmatrix} = T_2^{-T} \begin{bmatrix} K_1 \\ \frac{K_2}{\epsilon} \end{bmatrix} \tag{2.49}$$

$$P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s = 0 \tag{2.50}$$

$$P_f b_1 - b_4 P_f - b_3 + P_f b_2 P_f = 0 \tag{2.51}$$

In addition, it can be shown that $O(\epsilon)$ perturbation of the first-order approximate slow algebraic Riccati equation obtained in [23] is symmetric, that is

$$P_s A_s + A_s P_s + Q_s - P_s S_s P_s = 0 \tag{2.52}$$

Having obtained a good initial guess, the Newton-type algorithm can be used very efficiently for solving, the pure slow and pure fast nonsymmetric algebraic Riccati equations [19].

The pure-slow equation can be solved by using the Newton-algorithm with an initial guess obtained from [19]

$$P_1^{(i+1)}(a_1 - a_2 P_1^{(i+1)}) - (a_4 - P_1^{(i)} a_2) P_1^{(i+1)} = a_3 + P_1^{(i)} a_2 P_1^{(i)} \tag{2.53}$$

$i = 0, 1, 2, 3, \ldots.$

The pure-fast equation will be

$$P_2^{(i+1)}(b_1 - b_2P_2^{(i)}) - (b_4 - P_2^{(i)}b_2)P_2^{(i+1)} = b_3 + P_2^{(i)}b_2P_2^{(i)} \tag{2.54}$$

with

$$i = 0, 1, 2, 3, ...$$

We can now define the corresponding approximate (in the spirit of the theory of singular perturbations, [13]-[16]) pure-slow and pure-fast decoupled filters as

$$\dot{\hat{\eta}}_s^k(t) = (a_1^k + a_2^kP_s^k)^T\hat{\eta}_s^k(t) + K_s^ky(t) \tag{2.55}$$

$$\epsilon\dot{\hat{\eta}}_f^k(t) = (b_1^k + b_2^kP_f^k)^T\hat{\eta}_f^k(t) + K_f^ky(t) \tag{2.56}$$

$$\begin{bmatrix} a_1^k & a_2^k \\ a_3^k & a_4^k \end{bmatrix} = (T_1^k - T_2^kM^k) \tag{2.57}$$

$$\begin{bmatrix} b_1^k & b_2^k \\ b_3^k & b_4^k \end{bmatrix} = (T_4^k + \epsilon M^{k-1}T_2^{k-1}) \tag{2.58}$$

$$P_s^k = P_s + O(\epsilon)^k \tag{2.59}$$

$$P_f^k = P_f + O(\epsilon)^k \tag{2.60}$$

$$M^k = M + O(\epsilon)^k \tag{2.61}$$

$$\begin{bmatrix} K_s^k \\ \frac{K_f^k}{\epsilon} \end{bmatrix} = T_2^{k^{-T}} \begin{bmatrix} K_1^k \\ \frac{K_2^k}{\epsilon} \end{bmatrix} \tag{2.62}$$

2.5 Slow - Fast for Kalman Filter Decoupling Result

The proposed decomposition is such that the slow and fast filters are completely separated and both of filters are driven by the system measurement, see Figure 2.1.

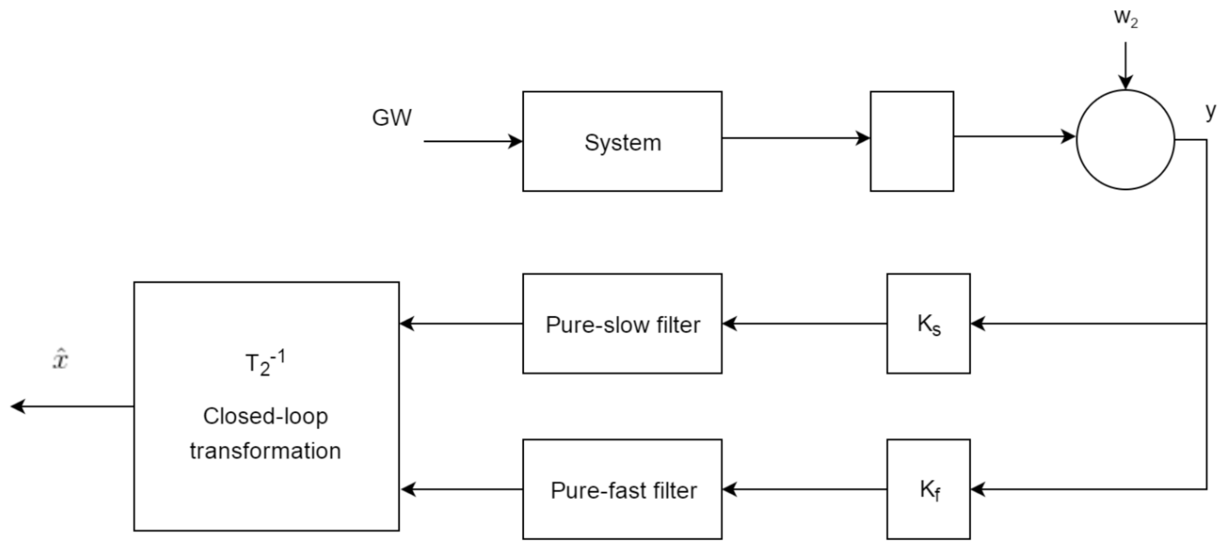

Figure 2.1: Filtering method from [19]

2.6 Aircraft Example

To demonstration this method we consider the example F-8 aircraft, the same example as the one that done in [13] - [15]. The matrices of the problem are given in the paper [10]

$$A_1 = \begin{bmatrix} 0.278386 & -0.965256 \\ 0.089833 & -0.290700 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.074210 & 0.016017 \\ 0.012815 & -0.001398 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -0.001815 & 0.005873 \\ 0.002850 & -0.009223 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} -0.030344 & 0.075024 \\ -0.075092 & -0.016777 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 0 \\ 1 & -3.236 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 0.00500 \\ -0.003152 & 0.01302 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} -46.62696 \\ 7.858776 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} -18.210002 \\ -45.049998 \end{bmatrix}$$

with

$$W_1 = 0.000315\,, W_2 = diag\begin{bmatrix} 0.000686 & 40 \end{bmatrix}, \epsilon = 0.025.$$

The slow and fast reduced-order Kalman filters driven by the system measurements are given by:

$$\dot{\hat{\eta}}_s(t) = \begin{bmatrix} 0.2755 & -0.9558 \\ 0.0903 & -0.2923 \end{bmatrix} \hat{\eta}_s(t) + \begin{bmatrix} -0.2561 & 0.0018 \\ -0.0958 & 0 \end{bmatrix} y(t)$$

$$\epsilon\dot{\hat{\eta}}_f(t) = \begin{bmatrix} -1.21151 & 1.1831 \\ -2.9733 & -5.1789 \end{bmatrix} \hat{\eta}_f(t) + \begin{bmatrix} 9.1085 & 0.0028 \\ 22.5077 & 0.0039 \end{bmatrix} y(t)$$

2.6.1 Simulation Results

The simulation result are obtained using the block diagram built in Simulink/ MATLAB

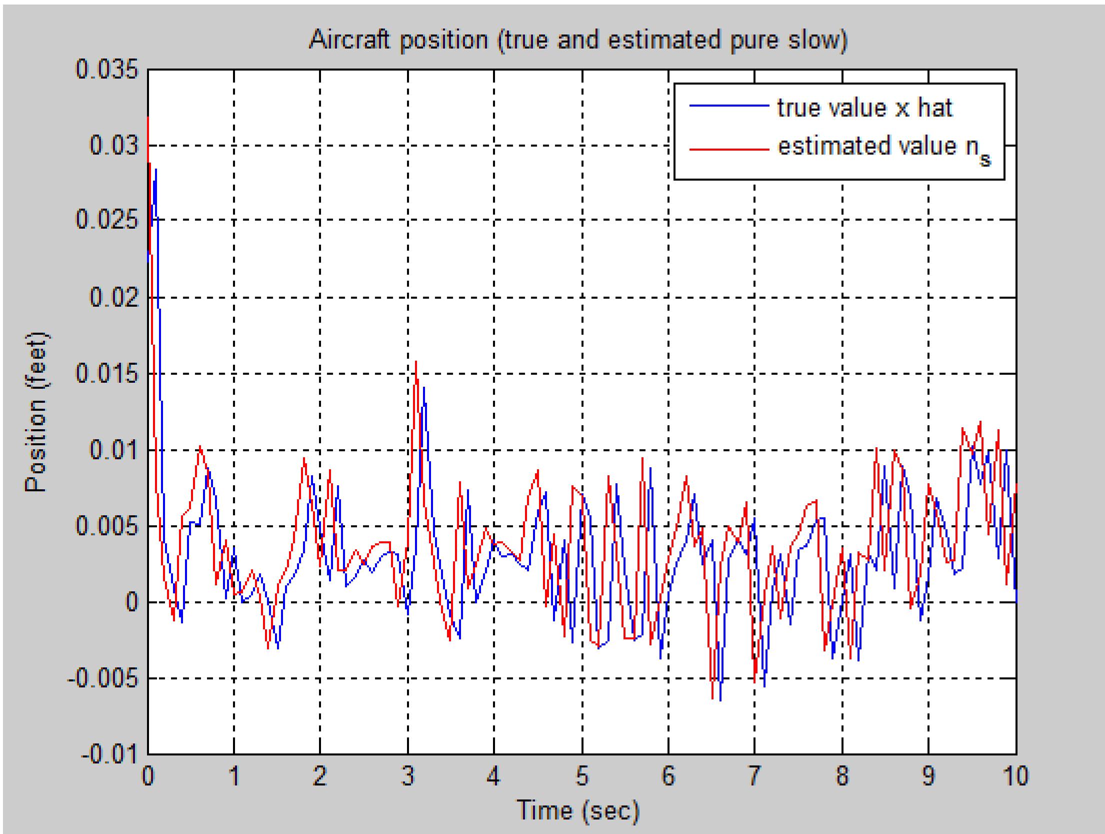

Figure 2.2 - Aircraft position (true and estimated pure slow)

The simulation result in Figure 2.2 shows the true and estimated aircraft position response using pure slow reduced-order Kalman filter. We can see from Figure 2.2 that estimated results are very closely following the actual results.

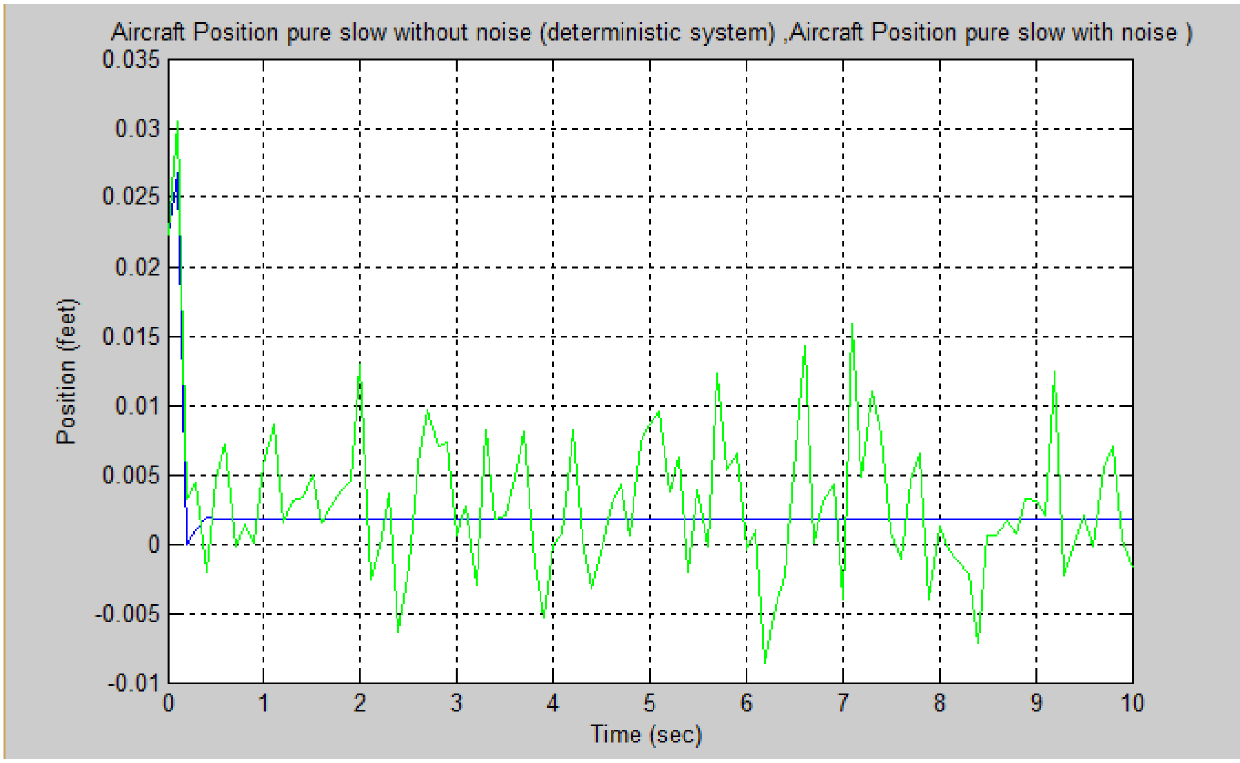

Figure 2.3 -Aircraft position pure slow without noise (deterministic system), aircraft position pure slow with noise

We can see from Figure 2.3 that the noise is really small for the pure slow reduced-order Kalman filter.

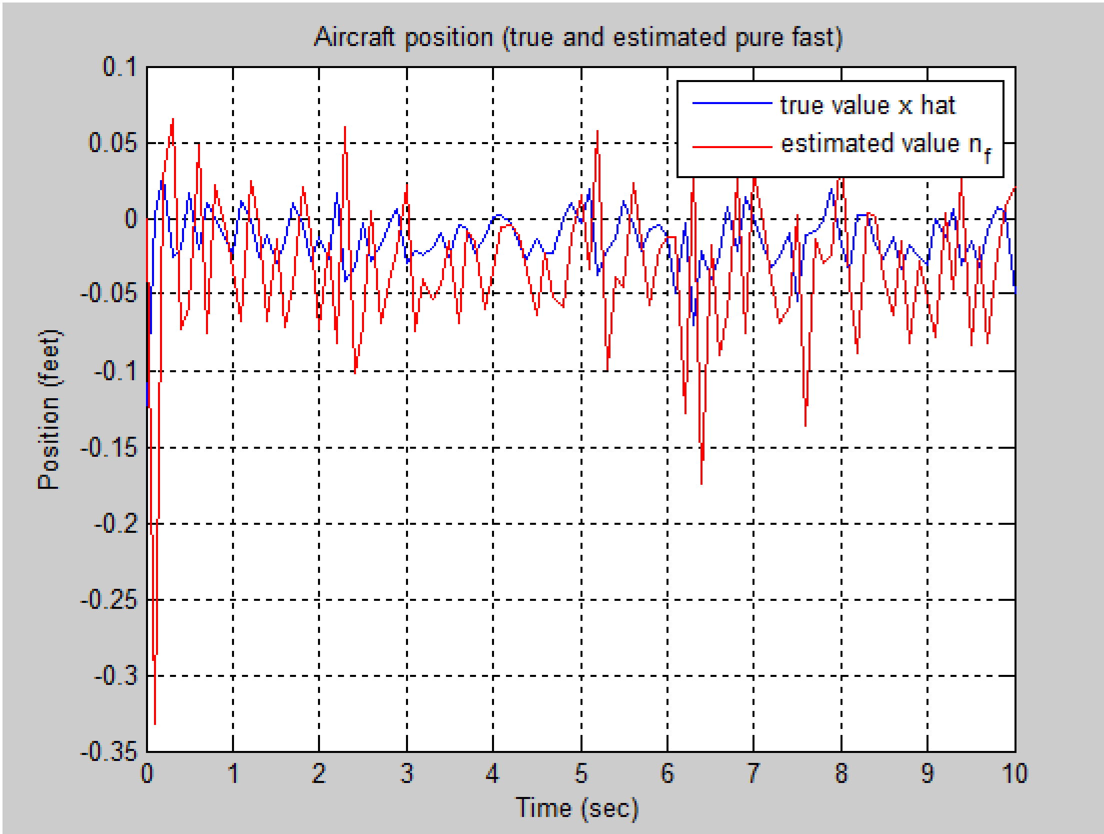

Figure 2.4 - Aircraft position (true and estimated pure fast)

The simulation result in Figure 2.4 shows the true and estimated aircraft position response using pure fast reduced-order Kalman filter.

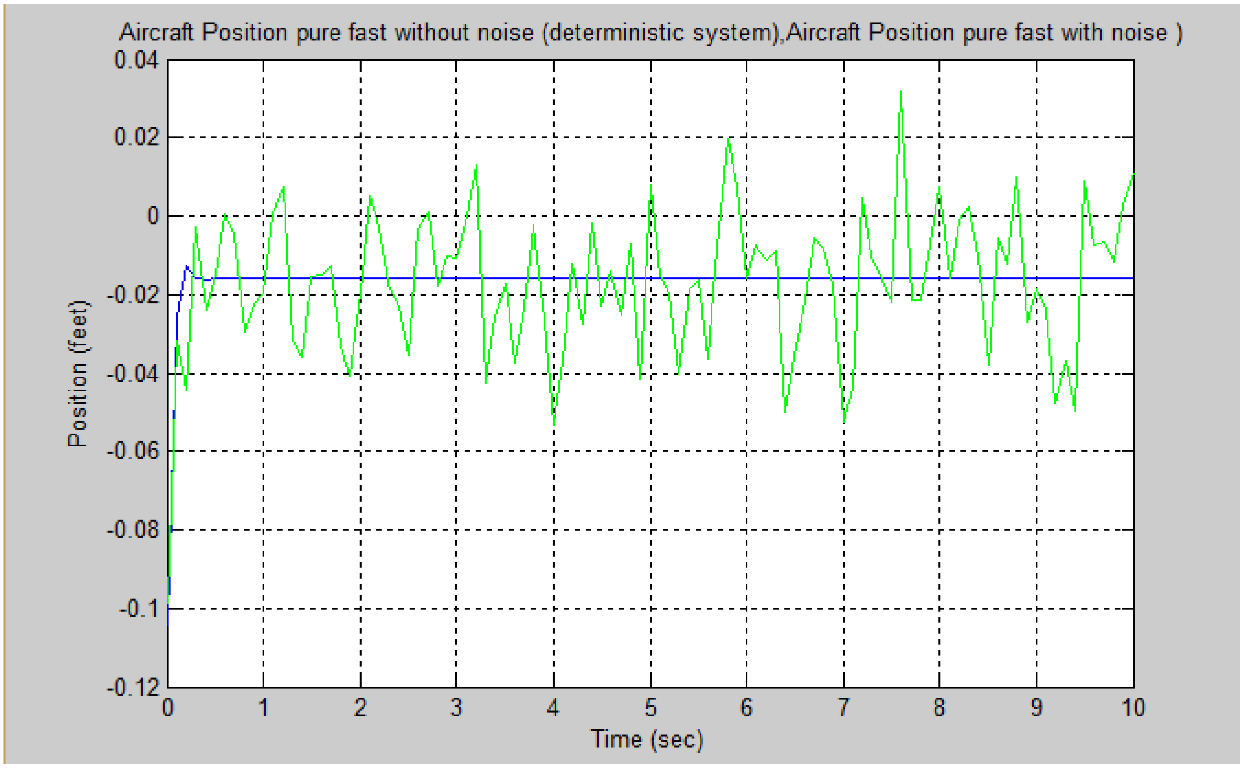

Figure 2.5 -Aircraft position pure fast without noise (deterministic system), aircraft position pure fast with noise)

We can see from Figure 2.5 that the noise is small for the pure fast reduced-order Kalman filter.

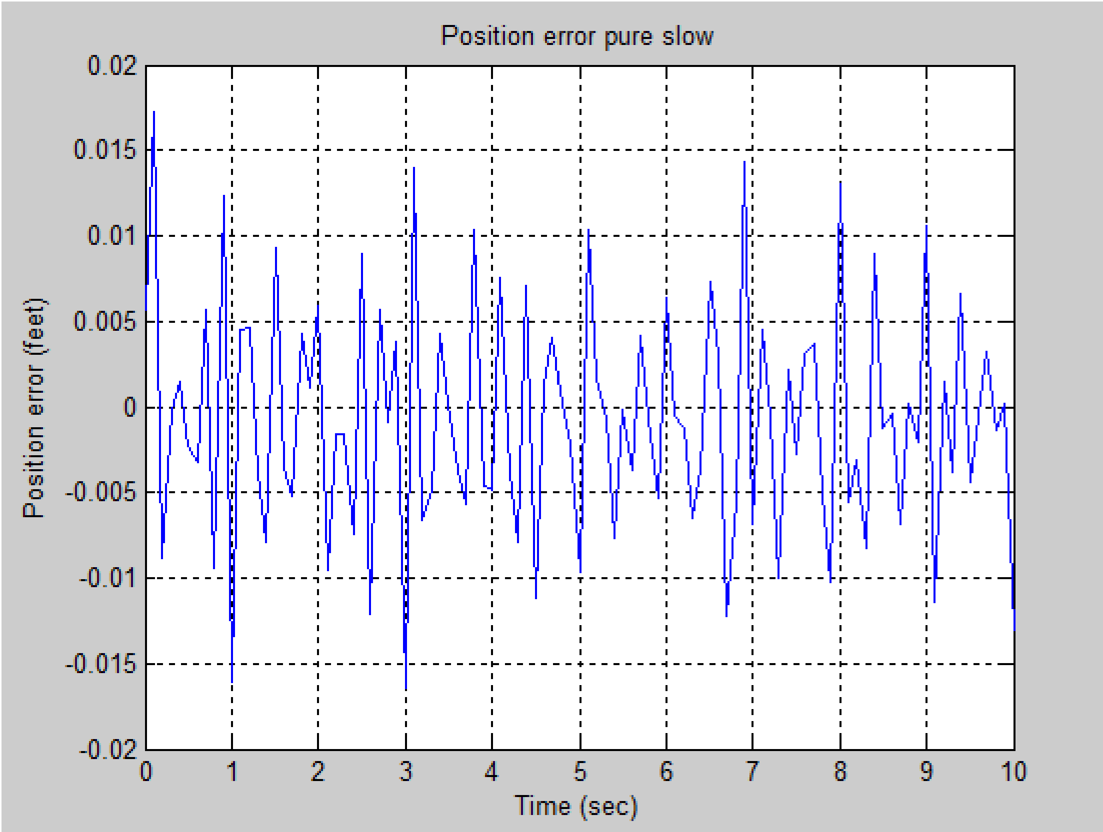

Figure 2.6 - Position error pure slow

We can see from Figure 2.6 that the position estimation error is really small for pure slow reduced-order Kalman filter and it is value between 0.015 and - 0.015.

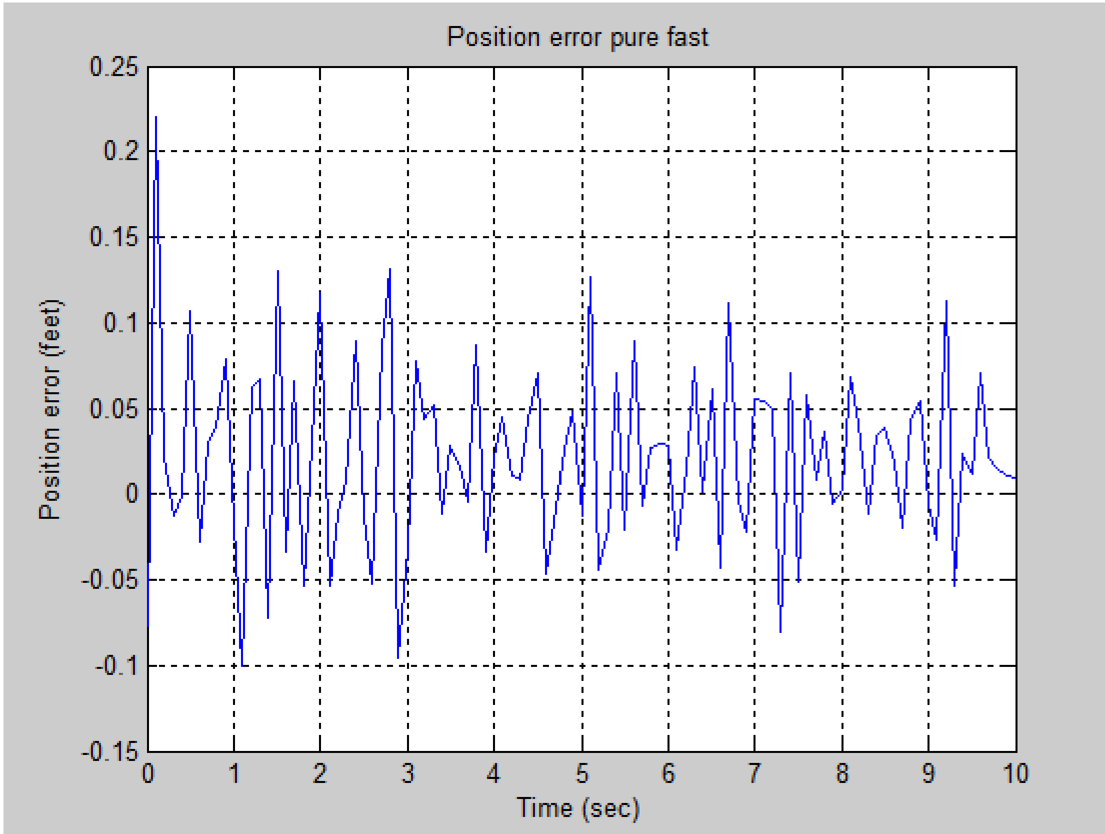

Figure 2.7 - Position error pure fast

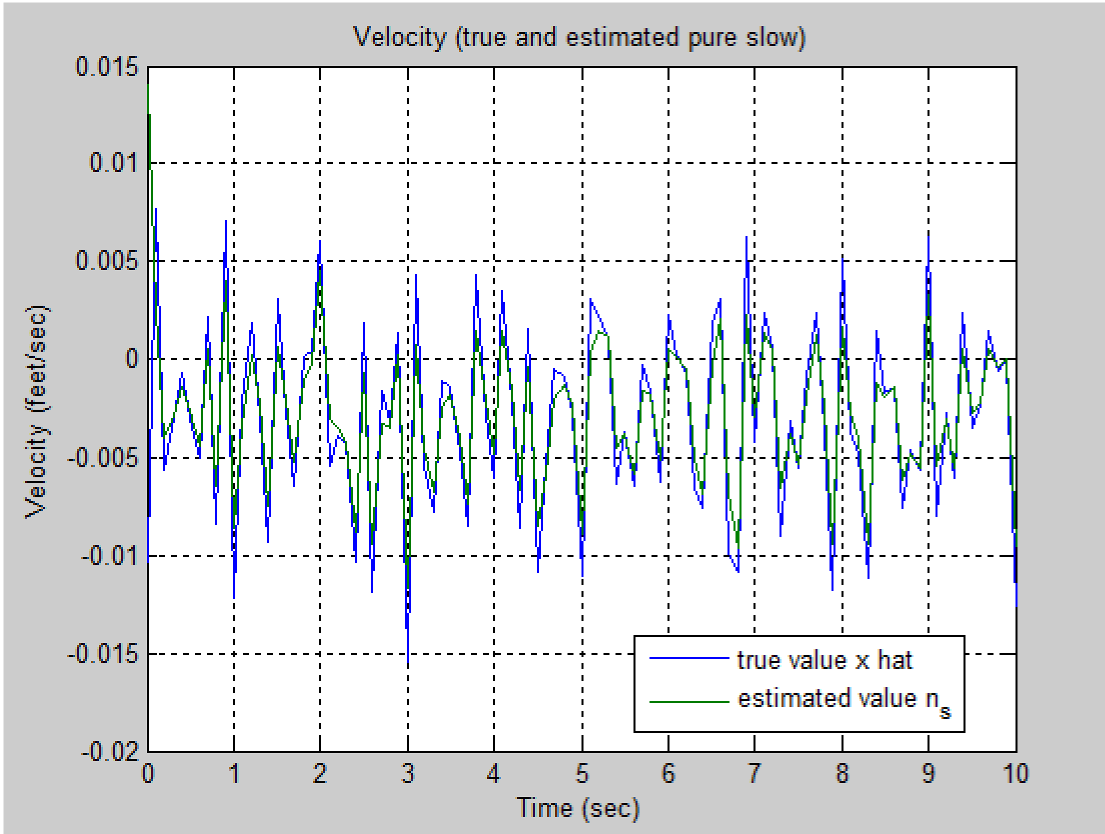

Figure 2.8 - Velocity (true and estimated pure slow)

Figure 2.8 shows the true and estimated velocity response using the pure slow reduced-order Kalman filter. We can see from Figure 2.8 and Figure 2.10 later that the velocity result is very good and very close to actual results.

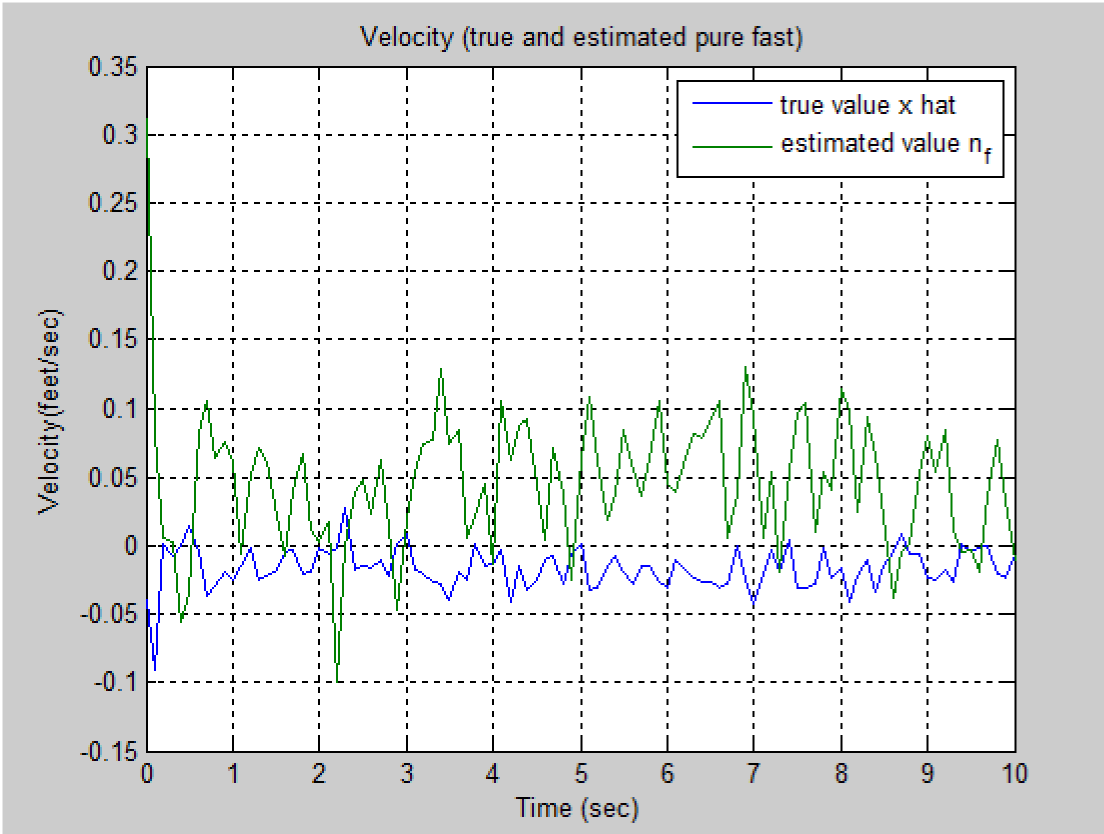

Figure 2.9 - Velocity (true and estimated pure fast)

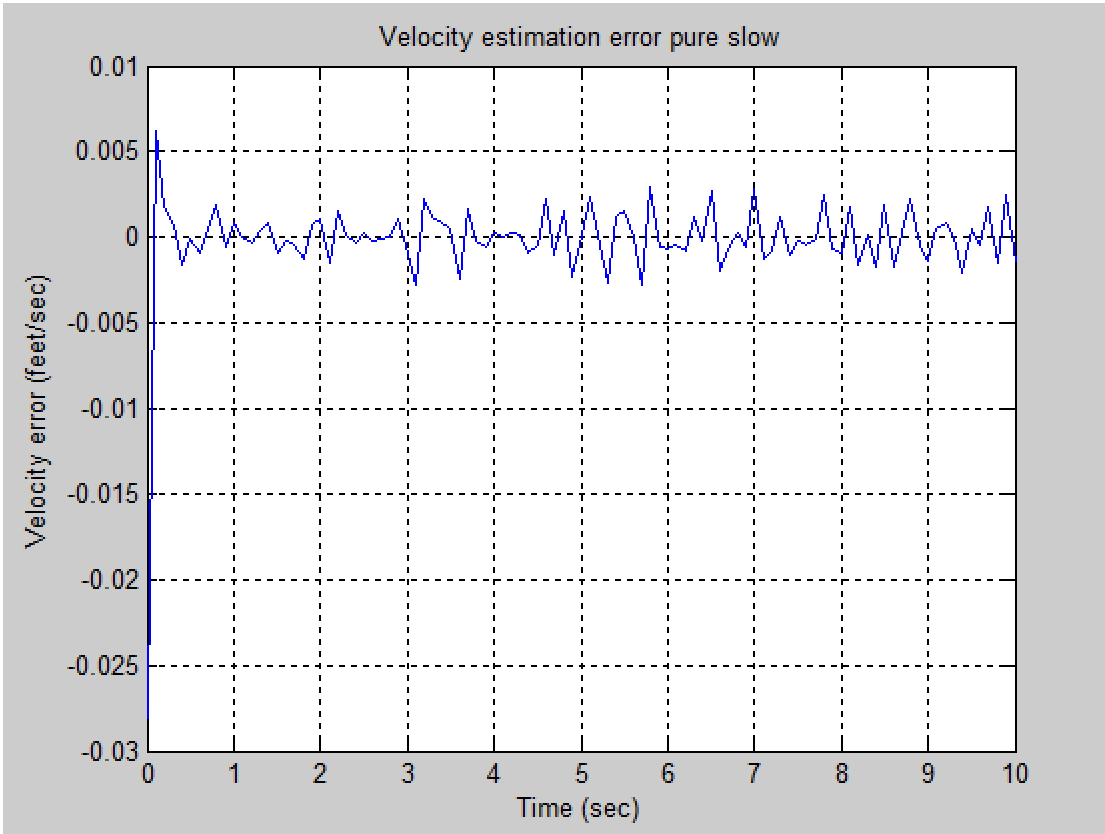

Figure 2.10 - Velocity estimation error pure slow

We can see from that the error value (noise) in pure slow reduced-order Kalman filter is really small and it is value between 0.005 and - 0.005.

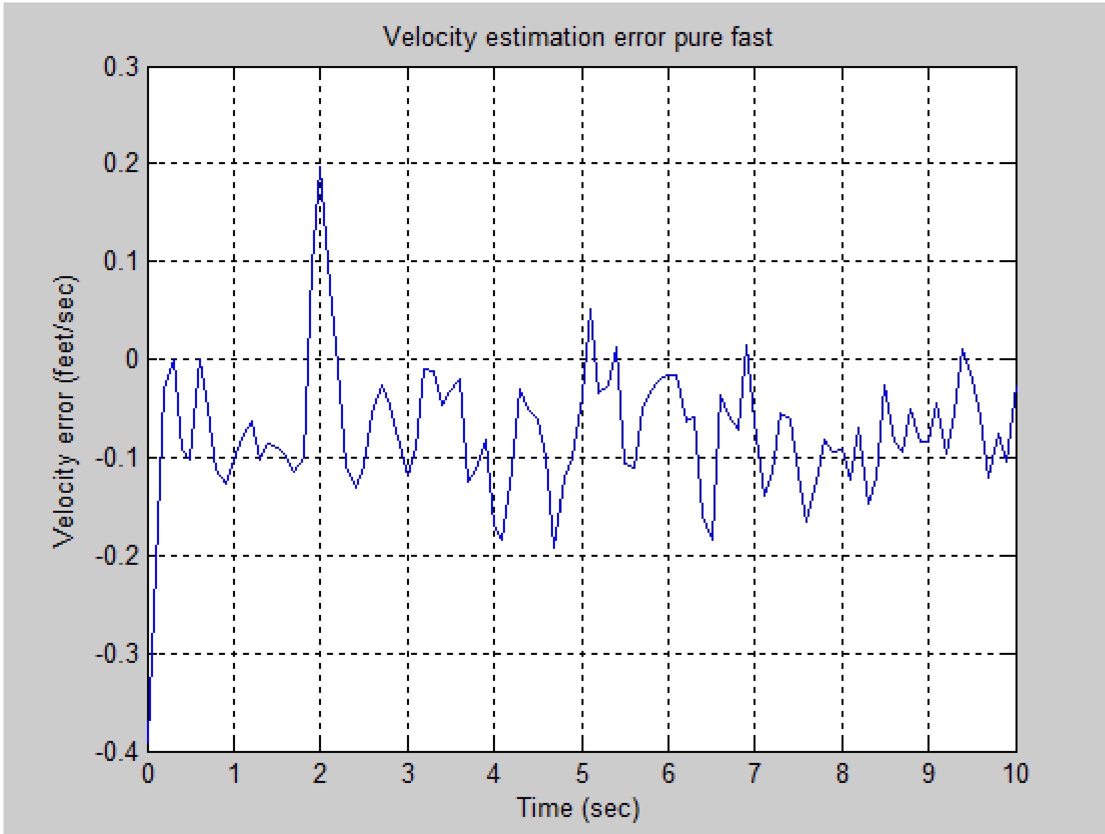

Figure 2.11 - Velocity estimation error pure fast

Figure 2.9 shows the true and estimated velocity response using pure fast reduced-order Kalman filter. We can see from Figures 2.9 and 2.11 that result for velocity are very well and very close to true results with small noise.

2.7 Conclusion

In this chapter we simulate the slow and fast reduced-order Kalman filter for singularly perturbed time-invariant systems developed in [10]. The simulation is done for an aircraft example. From the simulation results we can see that the aircraft position error in pure slow reduced-order Kalman filter is much smaller compared to the pure fast reduced-order Kalman filter. That is meaning the estimated result for aircraft position is very close to true results for pure slow reduced-order Kalman filter.

The same with the velocity error the pure slow reduced-order Kalman filter is smaller than pure fast reduced-order Kalman filter. The pure slow reduced-order Kalman filters perform better (on average) than the pure fast reduced-order Kalman filter.

Chapter 3

Reduced Order Kalman Filter for a Class of Systems with Slow and Fast Modes

3.1 Introduction

The Kalman filter is a dynamic system of the same order of the system whose state space variables are estimated. In some special cases the order of the continuous-time Kalman filter can be reduced [27]. The reduced-order Kalman filter has the obvious advantages over the full-order Kalman filter:

a- Simplicity of implementation.

b- Reduced processing.

c- Increased accuracy.

In this chapter we consider the continuous-time reduced-order Kalman filter of [27] for a special class linear system with slow and fast state variables (singularly perturbed systems).

3.2 Reduced Order Kalman Filter for System with Slow – Fast State Variables

The singular perturbation method has been used to study the reduced-order optimal Kalman filter by decoupling or breaking it down into two separate reduced-order filters operating in two different time scales. The particular mathematics for the reduced-order Kalman filter utilized the results of Friedland [27].

Consider a linear continuous-time stochastic system disturbed by white noise

$$\dot{x}(t) = Ax(t) + Bu(t) + Fw(t) \tag{3.1}$$

with perfect state measurements

$$y = Cx(t) \tag{3.2}$$

Assume that equation (3.1) has a singularly perturbed structure, that is

$$\begin{bmatrix} \dot{x_1}(t) \\ \epsilon \dot{x_2}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) + \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} w(t) \tag{3.3}$$

$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{3.4}$$

where

$x_1(t) \in R^{n_1}, x_2(t) \in R^{n_2}, n_1 + n_2 = n,$ are the state vectors.

$B_1 \in R^{n_1 \times m}, B_2 \in R^{n_2 \times m},$ are constant matrices.

$u(t) \in R^m$ is the control vector.

$y(t) \in R^r$ is the observation vector.

$w(t)$ is a zero mean white Gaussian noise with intensity (spectral density) $W \geq 0$.

ϵ is a small positive singular perturbation parameter.

We will use the results of [26]-[27] to derive the reduced-order Kalman filter for singularly perturbed linear systems. We consider two cases: (a) only slow state variables are perfectly measured, (b) only fast state variables are perfectly measured.

3.3 Main Results

3.3.1 Case 1: Reduced Order Kalman Filter for Fast State Variables

Assume that in (3.3) – (3.4) only the slow state variables are perfectly measured, that is:

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) + F_1w(t) \tag{3.5}$$

$$\epsilon\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) + F_2w(t) \tag{3.6}$$

$$y(t) = x_1(t) \tag{3.7}$$

We will assume that the observation noise here is vacant, as is the basic presumption with the reduced-order Kalman filter [25]. It is also presumed that the state variables in the fast reduced-order Kalman filter have to be estimated. We assume that the first n_1 here are measured $C = \begin{bmatrix} I_{n_1} & 0 \end{bmatrix}$.

This will match dimensions to the partitioning of the state matrices and vectors. Hence, for this case we assume:

$$\begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 \end{bmatrix} \tag{3.8}$$

where I_{n_1} is identity matrix of dimension n_1.

Since $y(t) = x_1(t)$ is known at all times, the estimate of $x_1(t)$ is given by:

$$\hat{x}_1(t) = y(t). \tag{3.9}$$

We need to design the reduced-order Kalman filter to estimate $x_2(t)$. Information about $x_2(t)$ is carried in the derivation of $y(t)$, that is

$$\dot{y}(t) = \dot{x}_1 = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) + F_1w(t) \tag{3.10}$$

The reduced-order Kalman filter for the fast state variables is given by:

$$\epsilon\dot{\hat{x}}_2(t) = A_{21}\hat{x}_1(t) + A_{22}\hat{x}_2(t) + B_2u(t) + K_2(\dot{y}(t) - \dot{\hat{y}}(t)) \tag{3.11}$$

with the corresponding measurements (3.10) that carry information about $x_2(t)$.

The signal $\dot{\hat{y}}(t)$ is defined, by

$$\dot{\hat{y}}(t) = A_{11}x_1(t) + A_{12}\hat{x}_2(t) + B_1u(t) \tag{3.12}$$

note that $x_1(t) = y(t) = \hat{x}_1(t)$.

The general observation equation for $x_2(t)$ in (3.11) can be converted into another form by eliminating $\dot{\hat{y}}(t)$ from (3.11), which leads to

$$\epsilon\dot{\hat{x}}_2(t) = A_{21}\hat{x}_1(t) + A_{22}\hat{x}_2(t) + B_2u(t) + K_2(\dot{y}(t) - A_{11}\hat{x}_1(t)$$
$$-A_{12}\hat{x}_2(t) - B_1u(t)) - K_2F_1w(t) \tag{3.13}$$

To eliminate the need for the derivative of $y(t)$, we will introduce a charge of variables as explained below. Bring the term $K_2\dot{y}(t)$ to the left-hand side of (3.13), and arrange the right-hand side of (3.13) as follows

$$\underbrace{\epsilon\dot{\hat{x}}_2(t) - K_2\dot{y}(t)}_{z(t)} = (A_{21} - K_2A_{11})y(t) + (A_{22} - K_2A_{12})\hat{x}_2(t) + (B_2 - K_2B_1)u(t) \tag{3.14}$$

where

$$z(t) = \epsilon\hat{x}_2(t) - K_2y(t) = \epsilon\hat{x}_2(t) - K_2\hat{x}_1(t) = \epsilon\hat{x}_2(t) - K_2x_1(t) \tag{3.15}$$

This leads to

$$\dot{z}(t) = (A_{22} - K_2A_{12})\,\frac{1}{\epsilon}\,z(t) + (A_{22} - K_2A_{12})\frac{K_2}{\epsilon}y(t)$$
$$+(A_{21} - K_2A_{11})y(t) + (B_2 - K_2B_1)u(t) \tag{3.16}$$

Note that

$$\hat{x}_2(t) = \frac{1}{\epsilon}z(t) + \frac{1}{\epsilon}K_2\hat{x}_1(t) = \frac{1}{\epsilon}z(t) + \frac{1}{\epsilon}K_2 y(t) \;\; \Rightarrow$$

and the reduced-order Kalman filter for fast stat variables is given by

$$\epsilon\dot{z}(t) = (A_{22} - K_2A_{12})z(t) + \epsilon(B_2 - K_2B_1)u(t) +$$

$$[(A_{22} - K_2A_{12})K_2 + \epsilon(A_{21} - K_2A_{11})]y(t) \qquad (3.17)$$

Having obtained $z(t)$ from (3.16), we have from (3.15):

$$\hat{x}_2(t) = \frac{1}{\epsilon}(z(t) + K_2y(t)) \qquad (3.18)$$

When $\epsilon = 0$, the approximate fast estimation can be obtained from (3.17) as

$$z^{(0)}(t) = (A_{22} - K_2A_{12})^{-1}(A_{22} - K_2A_{12})y(t) \quad = y(t) = x_1(t) \qquad (3.19)$$

in which case

$$\hat{x}_2^{(0)} = \frac{1}{\epsilon}(Z^{(0)}(t) + K_2)y(t) = \frac{1}{\epsilon}[(A_{22} - K_2A_{12})^{-1}(A_{22} - K_2A_{12}) + K_2]y(t) \qquad (3.20)$$

To find an expression for the estimation error we use the previous equation to form

$$\epsilon(\dot{x}_2(t) - \dot{\hat{x}}_2(t)) = \epsilon\dot{e_2}(t) \qquad (3.21)$$

$$= A_{21}(x_1(t) - \hat{x}_1(t)) + A_{22}(x_2(t) - \hat{x}_2(t)) + F_2w(t) - K_2(A_{11}x_1(t) +$$

$$A_{12}x_2(t) + B_1u(t) + F_1w(t) - A_{11}x_1(t) - A_{12}\hat{x}_2(t) - B_1u(t) - F_1w(t)) \qquad (3.22)$$

which lead to

$$\epsilon\dot{e}_2(t) = A_{22}e_2(t) - K_2A_{12}e_2(t) + F_2w(t) - K_2F_1w(t) \qquad (3.23)$$

or

$$\epsilon\dot{e}_2(t) = (A_{22} - K_2A_{12})e_2(t) + (F_2 - K_2F_1)w(t) \qquad (3.24)$$

Note that the mean value of $e_2(t)$ is zero at all times since the gain K_2 to be obtain from the corresponding algebraic Riccati equation stabilizes the reduced-order Kalman filter derived in (3.17).

The optimal gain matrix K_2 of the reduced-order Kalman filter for the fast state variables can be obtained by using the result of [27]. The gain matrix K_2 is given by:

$$K_2 = (PA_{12}^T + F_2 W F_1^T) V^{-1} \tag{3.25}$$

where

$$V = F_1 W F_1^T \tag{3.26}$$

P matrix in (3.25) is the covariance of the error in estimating vector $x_2(t)$ and is given by the positive semi-definite stabilizing solution of the following algebraic Riccati equation:

$$P\tilde{A}^T + \tilde{A}P - PA_{12}V^{-1}A_{12}^T P + \tilde{W} = 0 \tag{3.27}$$

where

$$\tilde{A} = A_{22} - F_2 W F_1^T V^{-1} A_{12} \tag{3.28}$$

$$\tilde{W} = F_2 W F_2^T - F_2 W F_1^T V^{-1} F_1 W F_2^T \tag{3.29}$$

This Reccati equation corresponds to the case when the state and measurement white noise processes are correlated which is expected since the system (3.5)-(3.6) and measurements (3.10) have the same noise. The estimation error covariance is given by:

$$E_2 = Var\ (e_2) \tag{3.30}$$

The variance of the estimation error satisfies the following algebraic Lyapunov equation obtained from (3.24)

$$\frac{1}{\epsilon}(A_{22} - K_2 A_{12})E_2 + \frac{1}{\epsilon}(A_{22} - K_2 A_{12})^T E_2 +$$

$$\frac{1}{\epsilon}(F_2 - K_2 F_1)^T W \frac{1}{\epsilon}(F_2 - K_2 F_1) = 0 \tag{3.31}$$

It can be seen from this equation that $E_2 = O(\frac{1}{\epsilon})$, where $O(\epsilon^i)$ stands for $O(\epsilon^i) < \alpha\epsilon^i$ with α being a bounded constant and i a real number. In the case when $W=O(\epsilon)$ then $Var(e_2(t)) = O(1)$.

3.3.2 Case 2: Reduced Order Kalman Filter for Slow State Variables

Consider the same singularly perturbed linear stochastic system as is section 3.3.1

$$\begin{bmatrix} \dot{x}_1(t) \\ \epsilon\dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) + \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} w(t)$$

$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{3.32}$$

We will presume that the fast states variables are directly measured that is, $C = [0\ I]$ and the residual $n - n_2$ have to be estimated using the reduce-order Kalman filter.

Hence, we can write:

$$\begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix} \tag{3.33}$$

The system and the measurement are now defined by

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) + F_1w(t)$$

$$\epsilon\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) + F_2w(t)$$

$$y(t) = x_2(t) \tag{3.34}$$

The fast variable estimate is given by:

$$\hat{x}_2(t) = y(t) \tag{3.35}$$

In the following we will construct the reduced-order filter for estimation of the slow state variables. We start with

$$\dot{y}(t) = \dot{x}_2(t) = \frac{1}{\epsilon}(A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) + F_2w(t)) \tag{3.36}$$

Note that the measurement defined in (3.36) carries information about $x_1(t)$. The corresponding reduced-order slow Kalman filter can be designed as follows

$$\dot{\hat{x}}_1(t) = A_{11}\hat{x}_1(t) + A_{12}\underbrace{\hat{x}_2(t)}_{=y(t)} + B_1u(t) + K_1(\dot{y}(t) - \dot{\hat{y}}(t)) \tag{3.37}$$

with its measurement given by

$$\dot{\hat{y}}(t) = \frac{1}{\epsilon}(A_{12}\hat{x}_1(t) + A_{22}y(t) + B_2u(t)) \tag{3.38}$$

From equation (3.37) and (3.38) we have:

$$\dot{\hat{x}}_1(t) = A_{11}\hat{x}_1(t) + A_{12}\hat{x}_2(t) + B_1u(t) +$$

$$K_1(\dot{y}(t) - \frac{1}{\epsilon}(A_{21}\hat{x}_1(t) + A_{22}\hat{x}_2(t) + B_2u(t))) \tag{3.39}$$

Let

$$z_1(t) = x_1(t) - K_1y(t) \Rightarrow \; x_1(t) = z_1(t) + K_1y(t) \tag{3.40}$$

then

$$\dot{\hat{x}}_1(t) - K_1\dot{y}(t) = \dot{z}_1(t) \tag{3.41}$$

so that

$$\dot{z}_1(t) = A_{11}(z_1(t) + K_1 y(t)) + A_{12} y(t) + B_1 u(t)$$

$$-\frac{1}{\epsilon} K_1 A_{21}\ (\hat{z}_1(t) + K_1 y(t)) - \frac{1}{\epsilon} K_1 A_{22} y(t) - \frac{1}{\epsilon} K_1 B_2 u(t) \tag{3.42}$$

and

$$\dot{z}_1(t) = A_{11} z_1(t) + A_{11} K_1 y(t) + A_{12} y(t) + (B_1 - \frac{1}{\epsilon} K_1 B_2) u(t)$$

$$-\frac{1}{\epsilon} K_1 A_{21}\, z_1(t)\ - \frac{1}{\epsilon} K_1 A_{12} K_1 y(t) - \frac{1}{\epsilon} K_1 A_{22} y(t) \tag{3.43}$$

This filter is also fast since it has $\frac{1}{\epsilon}$ on the right hand side

$$\dot{z}_1(t) = (A_{11} - \frac{1}{\epsilon} K_1 A_{12}) z_1(t) + (B_1 - \frac{1}{\epsilon} K_1 B_2) u(t) +$$

$$(A_{11} K_1 + A_{12} - \frac{1}{\epsilon} K_1 A_{12} K_1 - \frac{1}{\epsilon} K_1 A_{22}) y(t) \tag{3.44}$$

Let

$$\alpha_1 = \epsilon A_{11} - K_1 A_{21} \tag{3.45}$$

$$\beta_1 = \epsilon B_1 - K_1 B_2 \tag{3.46}$$

$$\gamma_1 = \epsilon A_{11} K_1 + \epsilon A_{12} - K_1 A_{12} K_1 - K_1 A_{22} \tag{3.47}$$

$$\epsilon \dot{z}_1(t) = \underbrace{(\epsilon A_{11} - K_1 A_{21})}_{=\alpha_1} z_1(t) + \underbrace{(\epsilon B_1 - K_1 B_2)}_{=\beta_1} u(t) +$$

$$\underbrace{(\epsilon A_{11} K_1 + \epsilon A_{12} - K_1 A_{21} K_1 - K_1 A_{22})}_{= \gamma_1} y(t)$$

$$\epsilon \dot{z}_1(t) = \alpha z_1(t) + \beta_1 u(t) + \gamma_1 y(t) \tag{3.48}$$

From the previous equation we have:

$$\epsilon(\dot{\hat{x}}_1(t) - K_1 \dot{y}(t)) = \alpha_1 x_1(t) - \alpha_1 K_1 y(t) + \beta_1 u(t) + \gamma_1 y(t) \tag{3.49}$$

then

$$\dot{\hat{x}}_1(t) = K_1 \dot{y}(t) + \frac{1}{\epsilon}\alpha_1 x_1(t) - \frac{1}{\epsilon}(\alpha_1 K_1 - \gamma_1) y(t) + \frac{1}{\epsilon}\beta_1 u(t) \tag{3.50}$$

To find an expression for the estimation error we use the previous equation to form

$$\dot{x}_1(t) - \dot{\hat{x}}_1(t) = \dot{e}_1 \tag{3.51}$$

$$\dot{x}_1(t) - \dot{\hat{x}}_1(t) = A_{11} x_1(t) + A_{12} x_2(t)$$

$$+ B_1 u(t) + F_1 w(t) - (A_{11} \hat{x}_1(t) + A_{12} x_2(t) + B_1 u(t) + K_1(\dot{y}(t) - \dot{\hat{y}}(t))) \tag{3.52}$$

$$= A_{11} x_1(t) + A_{12} x_2(t) + B_1 u(t) + F_1 w(t) - (A_{11} \hat{x}_1(t) + A_{12} x_2(t) + B_1 u(t) +$$

$$K_1(\frac{1}{\epsilon}(A_{21} x_1(t) + A_{22} x_2(t) + B_2 u(t) + F_2 w(t)) -$$

$$\frac{1}{\epsilon}(A_{21} \hat{x}_1 + A_{22} x_2(t) + B_2 u(t)))) \tag{3.53}$$

which lead to

$$e_1(t) = A_{11} e_1(t) + F_1 w(t) - \frac{1}{\epsilon} K_1 A_{21} e_1(t) - \frac{1}{\epsilon} K_1 F_2 w(t) \tag{3.54}$$

or

$$e_1(t) = (A_{11} - \frac{1}{\epsilon}K_1 A_{21})e_1(t) + (F_1 - \frac{1}{\epsilon}K_1 F_2)w(t) \quad (3.55)$$

The optimal gain matrix K_1 of the reduced-order Kalman filter for the slow state variables can be obtained by using the result of [27].

The gain matrix K_1 is given by:

$$K_1 = (PA_{21}^T + F_2 W F_1(t))V^{-1} \quad (3.56)$$

where

$$V = F_1 W F_1^T \quad (3.57)$$

Note that the gain matrix K_1 exists under the assumption that the matrix V is invertible. P matrix in (3.56) is the covariance of the error in estimating vector $x_1(t)$ and is given by the positive semi-definite stabilizing solution of the following algebraic Riccati equation:

$$P\tilde{A}^T + \tilde{A}P - PA_{21}V^{-1}A_{21}^T P + \tilde{W} = 0 \quad (3.58)$$

where

$$\tilde{A} = A_{11} - F_2 W F_1^T V^{-1} A_{21} \quad (3.59)$$

$$\tilde{W} = F_2 W F_2^T - F_2 W F_1^T V^{-1} F_1 W F_2^T \quad (3.60)$$

This Reccati equation corresponds to the case when the state and measurement white noise processes are correlated. The estimation error covariance is given by:

$$E_1 = Var\,(e_1) \quad (3.61)$$

The variance of the estimation error satisfies the following algebraic Lyapunov equation obtained from (3.55)

$$(A_{11} - \frac{1}{\epsilon}K_1A_{21})E_1 + (A_{11} - \frac{1}{\epsilon}K_1A_{21})^T E_1 +$$
$$(F_1 - \frac{1}{\epsilon}K_1F_2)^T W \ (F_1 - \frac{1}{\epsilon}K_1F_2) = 0 \tag{3.62}$$

3.4 Numerical Example

3.4.1 Example

In this section, we use a simple example to show the performance of the reduced Kalman order filters. Consider the system given by:

$$\begin{bmatrix} \dot{x}_1(t) \\ \epsilon\dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) + \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} w_1(t)$$

We will set B_1, B_2 equal to zero and study the following equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \epsilon\dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_1(t) \tag{3.63}$$

for two cases:

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{3.64}$$

and

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{3.65}$$

We assume, $W = 0.1$.

3.4.2 Simulation Results

The simulation result are obtained using Simulink/ MATLAB

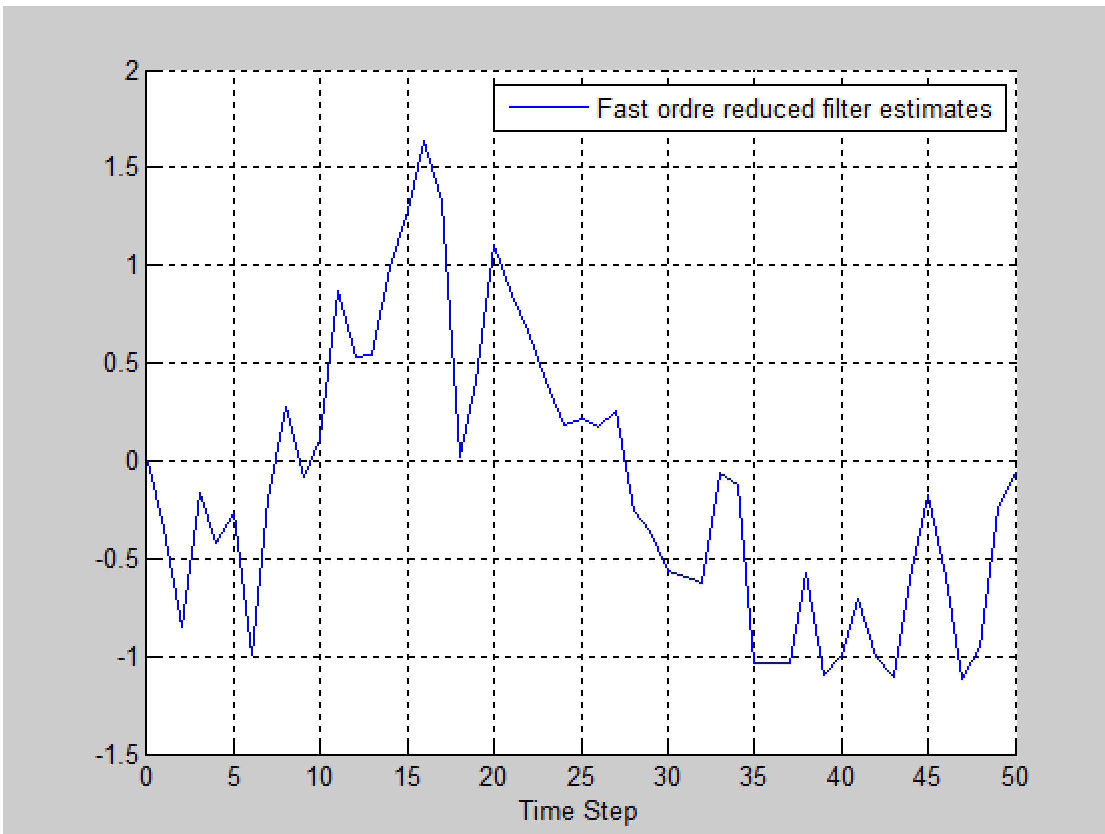

Figure 3.1: The reduced-order Kalman filter for fast state variable estimates

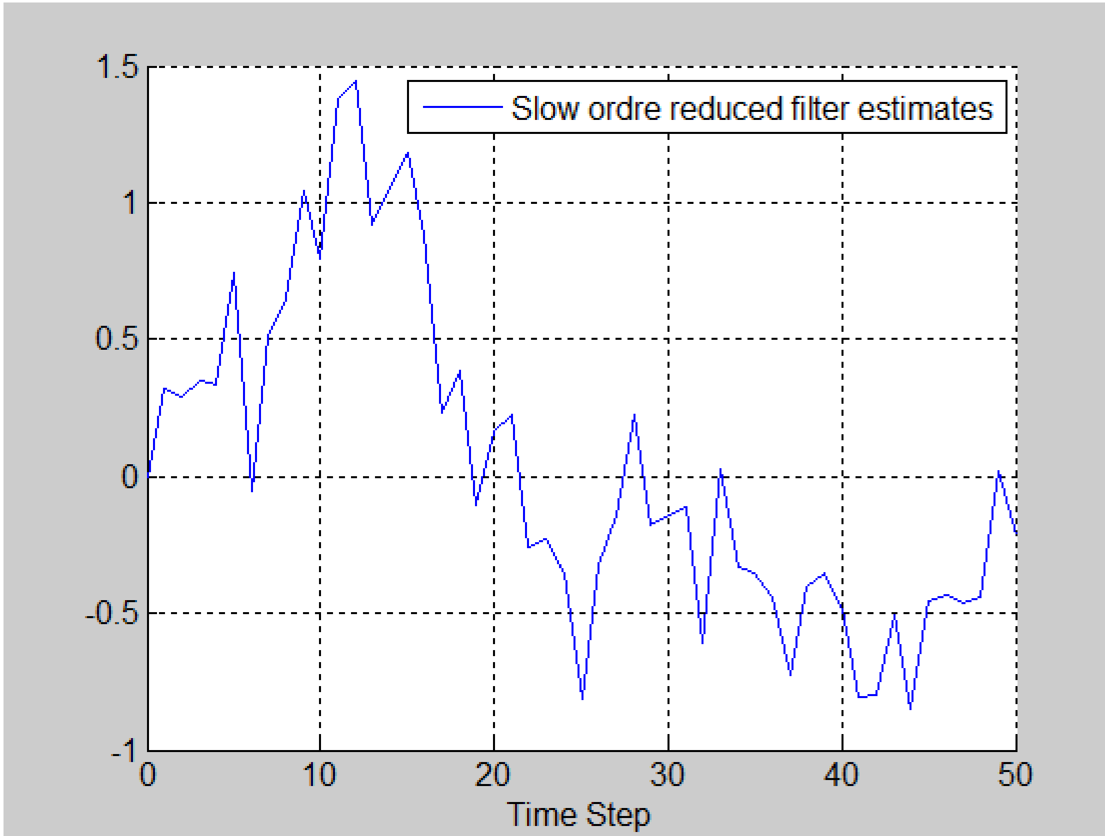

Figure 3.2: The of reduced-order Kalman filter for slow state variable estimates

3.5 Conclusion

We have shown how to design the reduced order Kalman filters composed of slow and fast Kalman filters for the case of perfect respectively slow and fast measurements. The first subsystem is the reduced-order Kalman filter for slow state variables; the second subsystem is the reduced-order Kalman filter for fast state variables. Using these two will reduce calculations in real-time systems and it will decrease mathematical and computational requirements, which encourages using the reduced-order Kalman filter method.

Chapter 5

Conclusion and Future Work

5.1 Conclusion

In the section, we summarize the contributions of the thesis. In the first part of the study, we demonstrate the process that depends on the pure-slow pure-fast decomposition method for finding a solution for the filter singularly perturbed algebraic Riccati equation. We use here the Chang transformation to convert the equation into a block-diagonal design where the slow and fast pieces of equation are entirely decoupled like compound control an approach. We perform simulation for such obtained pure-slow and pure-fast Kalman filters.

The next task of the thesis was to use the reduced-order method for slow and fast sub-systems, and find the solution of the corresponding algebraic Riccati equation. This will facilitate the use of Kalman filter for real-time physical systems.

Our results show improvement when compared to the other results available in the literature used for the same problem. All these methods provide a theoretical approach on how to design the reduced-order modeling Kalman filter for continuous- time system with slow and fast modes and perform corresponding simulation.

5.2 Future Work

In the future work, we plan to use the results of this thesis to design an extended Kalman filter for singularly perturbed systems, and derive pure-slow pure-fast decomposition model and the corresponding reduced-order extended Kalman filter. We plan also to study the same problem for discrete-time systems. Based on the results obtained, this method can be applied to practical problems in mechanical systems industry, wireless communications, chemical, biology systems, and power systems, where reduced-order method need to be addressed.

Bibliography

[1] R.S. Bucy, and P.D. Joseph, *Filtering for Stochastic Processes with Applications to Guidance*, John Wiley & Sons, 1968; 2nd Edition, ISBN 0-8218-3782-6, AMS Chelsea Publ.2005.

[2] R. Faragher, *IEEE Signal Processing Magazine* 29(5):128-132, August 2012.

[3] Kalman and Extended Kalman Filters: Concept, Derivation and Properties Maria Isabel Ribeiro Institute for Systems and Robotics Instituto Superior T´ecnico, 2004.

[4] M. Grewal and A. Andrews, "*Kalman Filtering: Theory and Practice*," Prentice-Hall, Inc., Englewood Clins, New Jersey 1993.

[5] T. Kailath, A. H. Sayed, B. Hassibi, *"Linear Estimation,"* Prentice Hall, 2000.

[6] Z. Gajic, M. Lim, D. Skataric, W-C. Su, and V. Kecman, *"Optimal Control of Weakly Coupled Systems and Applications,"* CRC Press-Taylor and Francis Group, Boca Raton, Florida, 2009.

[7] P. Dorato, V. Cerone, and C. Abdallah, *"Linear Quadratic Control: An Introduction."* Krieger Publishing Co., Inc., Melbourne, FL, USA, 2000.

[8] C.K. Hwang, K.S. Huang, K.B. Lin, and B.K. Lee, *"Observer base linear quadratic regulation with estimated state feedback control,"* In Machine Learning and Cybernetics (ICMLC), International Conference on, vol. 6, 2802-2805, 2010.

[9] Z. Gajic, and M. Lim, *"Optimal Control of Singularly Perturbed Linear Systems with Applications: High Accuracy Techniques,"* Marcel Dekker, New York, 2001.

[10] Z. Gajic and M. Lim, "A new filtering method for linear singularly perturbed systems," *IEEE Trans. Automatic Control*, vol. AC-39, 1952–1955, Sept. 1994.

[11] A. Haddad, "Linear filtering of singularly perturbed systems," *IEEE Trans. Automatic Control* vol. AC-21, pp. 515-519, 1976.

[12] A. Haddad and P. Kokotovic, "Stochastic control of linear singularly perturbed systems," *IEEE Trans. Automatic Control,* vol. AC-22, pp. 815-821, 1977.

[13] D. Teneketzis and N. Sandell, "Linear regulator design for stochastic systems by multiple time-scale method," *IEEE Trans. Automatic Control,* vol. AC-22, pp. 615421, 1977.

[14] H. Khalil and Z. Gajic, "Near-optimum regulators for stochastic linear singularly perturbed systems," *IEEE Trans. Automatic Control*, vol. AC-29, pp. 531-541, 1984.

[15] Z. Gajic, “Numerical fixed point solution for near-optimum regulators of linear quadratic Gaussian control problems for singularly perturbed systems,” *Int. J. Contr.*, vol. 43, pp. 373-387, 1986.

[16] P. Kokotovic, H. Khalil, and J. O’Reilly, *“Singular Perturbation Methods in Control: Analysis and Design,”* Orlando, FL: Academic, 1986.

[17] Brown, Robert Grover; Hwang, Patrick Y.C. *“Introduction to Random Signals and Applied Kalman Filtering”* (3 ed.). New York: John Wiley & Sons. pp. 289–293, 1997.

[18] Z. Gajic and X. Shen, *“Parallel Algorithm for Optimal Control of Large Scale Linear Systems,”* London: Springer-Verlag, 1993.

[19] W. Su, Z. Gajic, and X. Shen, “The exact slow-fast decomposition of the algebraic Riccati equation of singularly perturbed systems,” *IEEE Trans. Automatic Control*, vol. AC-37, pp. 1456-1459, 1992.

[20] K. Chang, “Singular perturbations of a general boundary value problem,” *SIAM J. Math. Anal.*, vol. 3, pp. 502-526, 1972.

[21] H. Khalil, “Feedback control of nonstandard singularly perturbed systems,” *IEEE Trans. Automatic Control*, vol. AC-34, pp. 1052-1060, 1989.

[22] T. Grodt and Z. Gajic, “The recursive reduced-order numerical solution of the singularly perturbed matrix differential Riccati equation,” *IEEE Trans. Automatic Control*, Vol. AC-33, 751–754, Aug. 1988.

[23] J. Chow and P. Kokotovic, “A decomposition of near-optimum regulators for systems with slow and fast modes,” *IEEE Trans. Automatic Control*, vol. AC-21, pp. 701-705, 1976.

[24] H. Mukaidani, “Newton method for solving cross-coupled sign-indefinite algebraic Riccati equations of weakly coupled large scale systems”. *Applied Mathematics and Computation*, vol. 188, pp. 103-115, 2007.

[25] B. F. Farrell, and P. J. Ioannou, “State estimation using a reduced order Kalman filter,” *J. Atmos. Sci., J. Atmos. Sci.*, 58, 3666-3680. J. Atmos. Sci., 2001.

[26] D. Simon, “Reduced Order Kalman Filtering without Model Reduction,” *Control and Intelligent Systems,* vol. 35, no. 2, pp. 169-174, April 2007.

[27] B. Friedland, “On the properties of reduced-order Kalman filters,” *IEEE Trans. Automatic Control*, 34(3), 321-324, 1989.