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We prove three results.

First, an old conjecture of Zs. Tuza says that for any graph $G$, the ratio of the minimum size, $\tau_3(G)$, of a set of edges meeting all triangles to the maximum size, $\nu_3(G)$, of an edge-disjoint triangle packing is at most 2. Disproving a conjecture of R. Yuster [40], we show that for any fixed, positive $\alpha$ there are arbitrarily large graphs $G$ of positive density satisfying $\tau_3(G) > (1 - o(1))|G|/2$ and $\nu_3(G) < (1 + \alpha)|G|/4$.

Second, write $\mathcal{C}(G)$ for the cycle space of a graph $G$, $\mathcal{C}_\kappa(G)$ for the subspace of $\mathcal{C}(G)$ spanned by the copies of $C_\kappa$ in $G$, $\mathcal{T}_\kappa$ for the class of graphs satisfying $\mathcal{C}_\kappa(G) = \mathcal{C}(G)$, and $\mathcal{Q}_\kappa$ for the class of graphs each of whose edges lies in a $C_\kappa$. We prove that for every odd $\kappa \geq 3$ and $G = G_{n,p}$,

$$\max_p \Pr(G \in \mathcal{Q}_\kappa \setminus \mathcal{T}_\kappa) \to 0;$$

so the $C_\kappa$’s of a random graph span its cycle space as soon as they cover its edges. For $\kappa = 3$ this was shown in [12].

Third, we extend the seminal van den Berg–Kesten Inequality [9] on disjoint occurrence of two events to a setting with arbitrarily many events, where the quantity of interest is the maximum number that occur disjointly. This provides a handy tool for bounding upper tail probabilities for event counts in a product probability space.
Acknowledgements

• Each result of this thesis has been or will be published elsewhere in one of various papers; see [7] (a version of Chapter 3), [6] (a version of Chapter 4), [5] (a version of Section 2.3).

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Dedication

to Margie
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Chapter 1

Introduction

The main content of this thesis is two results on cycles in random graphs.\(^1\) In the first, we use a (partly) random construction to exhibit a family of graphs whose triangles are clustered in a certain desired way. In the second, we determine for which \(p = p(n)\) the standard Erdős–Rényi random graph \(G_{n,p}\) is likely to satisfy a certain property dealing with its cycles of a fixed odd length. These results are detailed in Chapters 3 and 4, respectively. Our third result, extending the van den Berg–Kesten Inequality [9] on disjoint occurrence of events, was conceived as a lemma for our second; but since it is quite general and likely to be of broader interest, we present it in Section 2.3, in our chapter on preliminary tools. Here we just give a nontechnical overview of the three results and how they relate, deferring detail to when we state them formally.

For a graph \(G\), write \(\tau_3(G)\) for the minimum size of a set of edges meeting all triangles, and \(\nu_3(G)\) for the maximum size of a set of edge-disjoint triangles. While \(\tau_3(G) \leq 3\nu_3(G)\) is trivial, a conjecture of Zsolt Tuza from 1981 [39] holds that the 3 can be improved to 2, which is tight for the complete graphs of orders 4 and 5. But is it asymptotically tight for arbitrarily large graphs with quadratically many edges in triangles?\(^2\) Raphael Yuster conjectured not (Conjecture 3.2 and (3.2); see [40]), at least for graphs with \(\tau_3(G) > (1 - o(1))|G|/2\) (which by a routine exercise [3, Theorem 2.2.1] is as large as \(\tau_3(G)\) can be). We disprove Yuster’s conjecture (Theorem 3.3), exhibiting, for any \(\alpha > 0\), an infinite family of dense graphs for which \(\tau_3(G) > (1 - o(1))|G|/2\) and \(\nu_3(G) < (1 + \alpha)|G|/4\).

\(^1\)All graphs in this thesis are finite, simple and undirected.

\(^2\)Without some requirement on the number of edges in triangles, the answer is obviously yes, because nonedges and edges not in triangles are invisible to both \(\tau_3(G)\) and \(\nu_3(G)\).
Write $\mathcal{E}(G)$ for the vector space $\mathbb{F}_2^{E(G)}$ (the edge space of $G$), $\mathcal{C}(G)$ for the subspace of $\mathcal{E}(G)$ generated by the (indicators of) cycles (the cycle space of $G$), and $\mathcal{C}_\kappa(G)$ for the subspace of $\mathcal{C}(G)$ generated by the $\kappa$-gons. It is not hard to see (Proposition 4.1) that for any $n \geq \kappa$,

$$\mathcal{C}_\kappa(K_n) = \mathcal{C}(K_n).$$  \hspace{1cm} (1.1)

Following a theme in combinatorics that has lately been prominent, we wondered to what extent (1.1) remains true in a “sparse random” setting; or, to be precise, for which $p = p(n)$ (1.1) is likely to hold with $K_n$ replaced by $G_{n,p}$. Let $\mathcal{T}_\kappa$ be the class of graphs satisfying $\mathcal{C}_\kappa(G) = \mathcal{C}(G)$, and $\mathcal{Q}_\kappa$ the class of graphs each of whose edges lies in a $\kappa$-gon. We prove (Theorem 4.4) that for every odd $\kappa \geq 3$,

$$\max_p \Pr(G_{n,p} \in \mathcal{Q}_\kappa \setminus \mathcal{T}_\kappa) \to 0;$$  \hspace{1cm} (1.2)

so the $\kappa$-gons of a random graph span its cycle space as soon as they cover its edges. Even the $\kappa = 3$ case of (1.2) (which was proved by DeMarco, Hamm and Kahn in [12]) had been open and of interest, being the first unsettled case of a conjecture of M. Kahle (see [24, 25]) on the homology of the clique complex of $G_{n,p}$. In addition to wanting to generalize the result of [12], we were motivated to study $\mathcal{C}(G_{n,p})$ by a certain aspect of our construction of the graphs disproving Yuster’s conjecture; see Section 4.1.1 for elaboration.

In proving Theorem 4.4, at several points we needed a good bound on the upper tail probability of a random variable on $G_{n,p}$ that returns, for some set $S$ of subgraphs of $K_n$, the largest size of a set of pairwise edge-disjoint elements of $S$ that appear in $G_{n,p}$. We prove a stochastic domination result (Theorem 2.7) generalizing the van den Berg–Kesten Inequality [9] that implies such a bound. Perhaps unsurprisingly, the bounds this gives, at least those relevant to the present work, are not new; but the basic result is very natural and seems of independent interest.
Chapter 2

Preliminaries

Here we establish usage conventions in force throughout the thesis (usage specific to certain sections will be introduced later as appropriate), and then give the tools we will use in Chapters 3 and 4. Most of the tools are fairly standard, the notable exception being Theorem 2.7, our extension of the van den Berg–Kesten Inequality advertised earlier. (Lemma 2.18 extending Mantel’s Theorem is also new, but seems not worth fussing over.)

Given a graph $G$, we will use $V$ and $E$ for $V(G)$ and $E(G)$ when the meaning is clear. We will often identify graphs with their edge sets.

For $v \in V$ and $F \subseteq G$ we use $N_F(v) = \{x : vx \in F\}$ and $d_F(v) = |N_F(v)|$. For disjoint $A, B \subseteq V$, $\nabla_F(A, B)$ is the set of $F$-edges joining $A$ and $B$, and we use $\nabla_F(A)$ for $\nabla_F(A, V \setminus A)$—these are the cuts of $G$—and $\nabla_F(v)$ for $\nabla_F(\{v\})$. In all cases we drop the subscript when $F = G$.

As usual $\alpha(G)$ and $\Delta(G)$ (or $\Delta_G$) denote independence number and maximum degree of $G$. We will sometimes use $v_G$ and $e_G$ for the numbers of vertices and edges of $G$. The eigenvalues of $G$ are those of its adjacency matrix; see e.g. [10, Section VIII.2].

As defined in Chapter 1, the edge space of a graph $G$, denoted $\mathcal{E}(G)$, is the vector space $\mathbb{F}_2^{E(G)}$. Its elements are naturally identified with the (spanning) subgraphs of $G$. The cycle space of $G$, denoted $\mathcal{C}(G)$, is the subspace of $\mathcal{E}(G)$ generated by the (indicators of) cycles of $G$ (see e.g. [13, Section 1.9] for an exposition).

We use $[n]$ for $\{1, \ldots, n\}$ (for a positive integer $n$), log for $\ln$ and $a = (1 \pm b)c$ for $(1-b)c \leq a \leq (1+b)c$. Asymptotic notation ($\sim, O(\cdot), \Omega(\cdot)$ and so on) is standard, with $a \ll b$ and $a \asymp b$ replacing $a = o(b)$ and $a = \Theta(b)$ when convenient. An asymptotic probabilistic statement holds with high probability (w.h.p.) if it holds with probability
tending to 1 as some specified parameter tends to infinity. We always assume parameters that tend to infinity are large enough to support our various assertions, and usually pretend large numbers are integers.

### 2.1 Deviation

Here we recall a few standard bounds on the probability that a random variable differs by some specified amount from its mean.

Set

$$\varphi(x) = (1 + x) \log(1 + x) - x$$

for $x > -1$ and (for continuity) $\varphi(-1) = 1$. We use “Chernoff’s Inequality” in the following form; see for example [23, Theorem 2.1].

**Theorem 2.1.** If $X \sim \text{Bin}(n, p)$ and $\mu = \mathbb{E}[X] = np$, then for $t \geq 0$,

$$\Pr(X \geq \mu + t) \leq \exp \left[ -\mu \varphi \left( \frac{t}{\mu} \right) \right] \leq \exp \left[ -\frac{t^2}{2(2\mu + t/3)} \right],$$

$$\Pr(X \leq \mu - t) \leq \exp \left[ -\mu \varphi \left( -\frac{t}{\mu} \right) \right] \leq \exp \left[ -\frac{t^2}{2\mu} \right].$$

(2.2)

(2.3)

For larger deviations the following consequence of the finer bound in (2.2) will be convenient.

**Theorem 2.2.** For $X \sim B(n, p)$ and any $K$, letting $\mu = \mathbb{E}[X] = np$,

$$\Pr(X > K\mu) < \exp[-K\mu \log(K/e)].$$

(Of course this is only helpful if $K > e$.)

In fact the above bounds hold with $X$ being any sum of independent Bernoullis (and $\mu = \mathbb{E}[X]$); see [23, Theorem 2.8].

We will make substantial use of the following fundamental lower tail bound of Svante Janson ([22] or [23, Theorem 2.14]), for which we need a little notation. Suppose $A_1, \ldots, A_m$ are subsets of the finite set $\Gamma$. Let $\Gamma_p$ be the random subset of $\Gamma$ gotten
by including each $x \ (\in \Gamma)$ with probability $p$, these choices made independently. For $j \in [m]$, let $I_j$ be the indicator of the event $\{\Gamma_p \supseteq A_j\}$, and set $X = \sum I_j$, $\mu = \mathbb{E}X = \sum_j \mathbb{E}I_j$ and

$$\Delta = \sum \sum \{\mathbb{E}I_i I_j : A_i \cap A_j \neq \emptyset\}. \quad (2.4)$$

(Note this includes the diagonal terms.)

**Theorem 2.3.** With notation as above, for any $t \in [0, \mu],$

$$\Pr(X \leq \mu - t) \leq \exp[-\varphi(-t/\mu)\mu^2/\Delta] \leq \exp[-t^2/\Delta].$$

This has an upper tail counterpart, but with the major restriction that the events counted must be independent. It is proved in [22, 23] for events as in Theorem 2.3, but either proof works, with a tiny modification, in the greater generality of:

**Lemma 2.4 ([22, Lemma 2] or [23, Lemma 2.4]).** For events $A_1, \ldots, A_n$ in a probability space, and $\mu = \sum \Pr(A_i),$

$$\Pr(\text{some }\mu + t \text{ independent }A_i \text{'s occur}) \leq \exp[-\varphi(t/\mu)]$$

$$\leq \exp[-t^2/(2(\mu + t/3))].$$

Note the bound here is the same as the one in (2.2), which is thus contained in this lemma. The lemma implies the weaker but sometimes convenient

**Proposition 2.5.** For events $A_1, \ldots, A_k$ in a probability space, and $\mu = \sum \Pr(A_i),$

$$\Pr(\text{some }l \text{ independent }A_i \text{'s occur}) \leq \mu^l/l!,$$

(2.5)

observed in [15] (or see [3, Lemma 8.4.1]). (An analogue of Theorem 2.2, this has no content until $l > e\lambda$, whereas Lemma 2.4 gives a usable bound even when $\mu + t \ (= l)$ $= \mu + \Omega(\sqrt{\mu}).$)
2.2 Correlation

The setting for the next theorem is a finite product probability space \( \Omega = \prod_{i=1}^{t} \Omega_i \) with each factor linearly ordered. As usual an event \( A \subseteq \Omega \) is *increasing* if its indicator is a nondecreasing function (with respect to the product order on \( \Omega \)) and *decreasing* if its complement is increasing. The seminal “correlation inequality” is essentially due to Harris [19]:

**Theorem 2.6.** If \( A, B \subseteq \Omega \) are either both increasing or both decreasing, then

\[
\Pr(A \cap B) \geq \Pr(A) \Pr(B);
\]

if one is increasing and the other decreasing then the inequality is reversed.

2.3 Disjoint Occurrence

Here we discuss our third result advertised in the abstract and introduction.

Recall that for (real-valued) random variables \( X \) and \( Y \), \( Y \) *stochastically dominates* \( X \) (written \( X \preceq Y \)) if \( \Pr(Y \geq r) \geq \Pr(X \geq r) \) \( \forall r \in \mathbb{R} \). Recall also that a probability measure \( m \) on a partially ordered \( \Gamma \) is *positively associated* (PA) if \( m(A \cap B) \geq m(A)m(B) \) whenever both \( A \) and \( B \subseteq \Gamma \) are increasing (or, equivalently, whenever both are decreasing), and note that any probability measure on a linearly ordered \( \Gamma \) is PA.

The setting for this section is a finite product probability space \( (\Omega, \mu) = \prod_{i=1}^{n} (\Omega_i, \mu_i) \) with each \( \Omega_i \) partially ordered. Events \( A_1, A_2, \ldots, A_k \subseteq \Omega \) are said to *occur disjointly at* \( \omega \in \Omega \) if there are disjoint \( S_1, \ldots, S_k \subseteq [n] \) such that for each \( i \in [k] \) and \( \omega' \in \Omega \), we have \( \omega' \in A_i \) whenever \( \omega' \) agrees with \( \omega \) on \( S_i \). We write

\[
\square_{i=1}^{k} A_i = \{ \omega \in \Omega : A_1, \ldots, A_k \text{ occur disjointly at } \omega \}.
\]

The study of disjoint occurrence was initiated by van den Berg and Kesten [9], who
showed
\[ \Pr(A \sqcap B) \leq \Pr(A) \Pr(B) \] (2.6)
for increasing \( A, B \subseteq \{0, 1\}^n \) (see also e.g. [18, Section 2.3]). The following extension of this seminal “BK Inequality” is apparently new [8].

**Theorem 2.7.** Let \((\Omega, \mu) = \prod_{i=1}^{n}(\Omega_i, \mu_i)\) be a finite product probability space with the \(\Omega_i\)’s partially ordered and the \(\mu_i\)’s PA. Given \(A_1, A_2, \ldots, A_k \subseteq \Omega\), let
\[
X = \max\{|I| : I \subseteq [k] \text{ and } \sqcap_{i \in I} A_i \text{ occurs}\}.
\]

Let \(Y_1, \ldots, Y_k\) be independent Bernoullis with \(EY_i = \Pr(A_i)\), \(Y = \sum Y_i\), and \(\lambda = \sum EY_i\). Then:

(a) If the \(A_i\)’s are all increasing, or all decreasing, then \(X \leq Y\);

(b) If the \(\Omega_i\)’s are linearly ordered, then for \(t \geq 0\),
\[
\Pr(X \geq \lambda + t) \leq \exp[-\lambda \varphi(t/\lambda)] \leq \exp[-t^2/(2(\lambda + t/3))].
\]

**Remarks.**

(i) Taking \(\Omega = \{0, 1\}^n\), \(k = 2\) and \(r = 2\) in the definition of “\(X \leq Y\)” recovers (2.6) from (a).

(ii) The most spectacular of the developments growing out of [9] is Reimer’s proof [32] of the “BK Conjecture” (of [9]) which says that (2.6) doesn’t require that \(A, B\) be increasing. In contrast, trivial examples show this requirement (or some requirement) to be necessary in (a); for instance if \(\Omega = \{0, 1\}\) with uniform measure, \(k = 2\), \(A_1 = \{0\}\) and \(A_2 = \{1\}\), then \(\Pr(X \geq 1) = 1 > 3/4 = \Pr(Y \geq 1)\).

(iii) For the same reason, (a) does not hold in the generality of Lemma 2.4 (even modified to make sense there). In other words, if \(A_1, \ldots, A_k\) are events in an
arbitrary probability space, $Y$ is as in Theorem 2.7 and $Z$ is the maximum number of independent $A_i$’s that occur, then $Z \preceq Y$ does not hold in general, as the example in (ii) also shows.

(iv) On the other hand, for increasing [or decreasing] $A_i$’s, (a) with Theorem 2.1 implies Lemma 2.4 (since independent increasing [or decreasing] events, if they occur, necessarily occur disjointly, a standard observation easily extracted from the usual proof of Theorem 2.6). In fact in this setting (a) is much stronger than Lemma 2.4, because dependent events can easily occur disjointly—so $X$ can be much larger than the $Z$ of (iii), even though the bounds given for the upper tails of $X$ and $Z$, by (a)+Chernoff and Lemma 2.4 respectively, are the same.

For example, if $x_1, \ldots, x_k, y_1, \ldots, y_k$ are distinct vertices of $G_{n,p}$ and, for $i \in [k]$, $A_i = \{\text{there is an } x_iy_i\text{-path}\}$, then $Z \leq 1$ but $X$ can be large.

**Historical Note.** As mentioned in Chapter 1, our motivation for Theorem 2.7 (a) was to obtain something like Lemma 2.4, as in Remark (iv). We learned of the lemma about a year after proving (a). Shortly thereafter, we realized the lemma’s proof (which is quite different from our proof of (a)) could be tweaked to give (b).

The proof of Theorem 2.7 (a), which is similar to the original proof of [9], is not hard but is a little awkward to write, and a few additional definitions will be helpful. We prove it for increasing $A_i$’s; the decreasing case is of course analogous.

For $\Omega = \prod_{i \in I} \Omega_i$ and $S \subseteq I$, we take $\Omega_S = \prod_{i \in S} \Omega_i$ and, for $\omega \in \Omega$, $\omega_S = (\omega_i : i \in S)$. For $A \subseteq \Omega$ and $\omega \in \Omega_J$ for some $J \subseteq I$, $S \subseteq J$ is said to witness $\omega \in A$ if $\omega' \in A$ whenever $\omega' \in \Omega$ and $\omega'_S = \omega_S$. (This is of course abusive since we can’t have $\omega \in A$ unless $J = I$.) We then (that is, for $\omega \in \Omega_J$) say $A_1, \ldots, A_k (\subseteq \Omega)$ occur disjointly at $\omega$ if there are disjoint $S_1, \ldots, S_k \subseteq J$ such that $S_j$ witnesses $\omega \in A_j \ \forall j$ and, for $A = \{A_1, \ldots, A_k\}$, set

$$X_A(\omega) = \max\{|R| : R \subseteq [k], \text{ the } A_j\text{'s indexed by } R \text{ occur disjointly at } \omega\}.$$

Thus the $X$ of Theorem 2.7 is $X_A$ evaluated at a random $\omega \in \Omega$.  

Proof of Theorem 2.7 (a). Say \(i \in [n]\) affects \(A \subseteq \Omega\) if there are \(\omega \in A\) and \(\omega' \in \Omega \setminus A\) with \(\omega[i] = \omega'[i]\), and for a collection \(\mathcal{B}\) of events in \(\Omega\), let \(\psi(\mathcal{B})\) be the number of \(i \in [n]\) that affect at least two members of \(\mathcal{B}\).

We proceed by induction on \(\psi(\mathcal{A})\). If this number is zero then the laws of \(X\) and \(Y\) agree (since the \(A_j\)'s are independent). So we may assume \(\psi(\mathcal{A}) \neq 0\), say (without loss of generality) the index 1 affects at least two of the \(A_j\)'s.

Let \((\Omega_{n+j}, \mu_{n+j}), j \in [k]\), be copies of \((\Omega_1, \mu_1)\), independent of each other and of \((\Omega_1, \mu_1), \ldots, (\Omega_n, \mu_n)\). Let \((\Omega^*, \mu^*) = \prod_{i=2}^{n+k} (\Omega_i, \mu_i)\) and (for \(j \in [k]\))

\[
B_j = \{\omega \in \Omega^*: (\omega_{n+j}, \omega_2, \ldots, \omega_n) \in A_j\}.
\]

Thus, apart from irrelevant variables, \(B_j\) is a copy of \(A_j\) gotten by replacing \((\Omega_1, \mu_1)\) by \((\Omega_{n+j}, \mu_{n+j})\). In particular \(\Pr(B_j) = \Pr(A_j)\) and, with \(\mathcal{B} = \{B_1, \ldots, B_k\}\), we have \(\psi(\mathcal{B}) = \psi(\mathcal{A}) - 1\) (since \(i \in [2, n]\) affects \(B_j\) iff it affects \(A_j\), and \(n + i\) affects \(B_j\) iff \(j = i\) and 1 affects \(A_i\)). So by inductive hypothesis it is enough to show

\[
\mu(X_A \geq r) \leq \mu^*(X_B \geq r) \tag{2.7}
\]

for each positive integer \(r\). Here it’s convenient to work with the stronger conditional version:

Claim. For each \(y \in \Omega_{[2,n]}\) (with \(\mu_i(y_i) > 0 \ \forall i \in [2, n]\)),

\[
\mu(X_A(\omega) \geq r \mid \omega_{[2,n]} = y) \leq \mu^*(X_B(\omega) \geq r \mid \omega_{[2,n]} = y). \tag{2.8}
\]

Proof. Since, for any \(y \in \Omega_{[2,n]}\) and \(\omega \in \Omega\) with \(\omega_{[2,n]} = y\),

\[
X_B(y) = X_A(y) \leq X_A(\omega) \leq X_A(y) + 1,
\]

we need only show (2.8) for \(y\) with \(X_A(y) = r - 1\) (since the left hand side of (2.8) is zero if \(X_A(y) \leq r - 2\) and both sides are 1 if \(X_A(y) \geq r\)).
Given such a \( y \), set \( \mathcal{F} = \{ x \in \Omega_1 : X_A(x, y) = r \} \) and, for \( i \in [k] \), let \( \mathcal{F}_i \subseteq \Omega_1 \) consist of those \( x \)'s for which there are \( I \in \binom{[k]}{r} \) containing \( i \) and disjoint \( S_j \)'s in \([n]\) \((j \in I)\) such that \( S_j \) witnesses \((x, y) \in A_j \) \((j \in I)\) and \(1 \in S_i\). Then, evidently,

- each \( \mathcal{F}_i \) is increasing,
- \( \mathcal{F} = \cup_{i \in [k]} \mathcal{F}_i \),
- for \( \omega \in \Omega \) with \( \omega_{[2,n]} = y \), \( X_A = r \) iff \( \omega_1 \in \mathcal{F} \), and
- for \( \omega \in \Omega^* \) with \( \omega_{[2,n]} = y \), \( X_B \geq r \) iff \( \omega_{n+j} \in \mathcal{F}_j \) for some \( j \in [k] \),

whence

\[
\mu(X_A(\omega) \geq r \mid \omega_{[2,n]} = y) = \mu_1(\mathcal{F}) = 1 - \mu_1(\cap_{j \in [k]} \mathcal{F}_j) \\
\leq 1 - \prod_{j \in [k]} \mu_1(\mathcal{F}_j) = \mu^*(X_B(\omega) \geq r \mid \omega_{[2,n]} = y),
\]

where the inequality follows from that assumption that \( \mu_1 \) is PA.

For the proof of Theorem 2.7 (b) we need just one little observation, which follows immediately from Reimer’s Theorem [32] by induction: for events \( \{A_i\}_{i \in I} \) in a product probability space with each factor linearly ordered,

\[
\Pr(\Box_{i \in I} A_i) \leq \prod_{i \in I} \Pr(A_i). \quad (2.9)
\]

**Proof of Theorem 2.7 (b).** For some to-be-determined integer \( r \leq k \) and each \( I \subseteq [k] \) of size \( r \), let \( B_I \) be the indicator of \( \Box_{i \in I} A_i \). Let \( \chi = r! \sum B_I \), so that

\[
\mathbb{E}\chi = r! \sum_{|I|=r} \Pr(\Box_{i \in I} A_i) \leq r! \sum_{|I|=r} \prod_{i \in I} \Pr(A_i) \leq \lambda^r
\]

(by (2.9)).

The rest of the proof follows [23, Lemma 2.46] verbatim, so we will be brief. If
$X \geq \lambda + t$ then $\chi \geq (\lambda + t)_r = \prod_{i=0}^{r-1}(\lambda + t - i)$, so by Markov,

$$\Pr(X \geq \lambda + t) \leq \Pr(\chi \geq (\lambda + t)_r) \leq \frac{\lambda^r}{(\lambda + t)_r} = \prod_{i=0}^{r-1} \frac{\lambda}{\lambda + t - i}.$$ 

Setting $r = t$ (to minimize the right hand side) yields

$$\log \Pr(X \geq \lambda + t) \leq \sum_{i=0}^{t-1} \log(\lambda/(\lambda + t - i)) \leq \int_0^t \log(\lambda/(\lambda + t - x)) \, dx,$$

which, with calculus, gives the stronger bound in Theorem 2.7 (b).

### 2.4 Path Counts

Here we discuss what can be said about the numbers of paths of various lengths joining pairs of vertices in a random graph. Throughout the section we use $G$ for $G_{n,p}$.

**Notation.** For $l \geq 1$ and (distinct) $x, y \in V$, we use $P_l(x, y)$ for the set of $P_l$’s ($l$-edge paths) in $G$ joining $x$ and $y$, $\tau_l^i(x, y)$ for $|P_l(x, y)|$, and $\sigma_l^i(x, y)$ for the maximum size of a collection of internally disjoint $P_l$’s of $G$ joining $x$ and $y$. (Though $l = 1$ is uninteresting, it’s convenient to allow this.) These notations will show up again in Chapter 4. In this section only, we use $V(P)$ for the set of internal vertices of a path $P$ and write $\Gamma_{x,y}^l$ for the graph on $P_l(x, y)$ with $P \sim Q$ iff $V(P) \cap V(Q) \neq \emptyset$.

Conveniently, most of what we need here has been worked out (in far greater generality) by Joel Spencer in [36] (see also [3, Section 8.5]), and we begin with two special cases of what’s proved there.

**Theorem 2.8.** For any $l \geq 2$ and $\varepsilon > 0$ there exists $K$ such that if $n^{l-1}p^l \geq K \log n$, then w.h.p.

$$\tau_l^i(x, y) = (1 \pm \varepsilon)n^{l-1}p^l \quad \forall \{x, y\} \in \binom{V}{2}.$$  

(2.10)
Proposition 2.9. For any $l \geq 1$ and $\delta > 0$, if $n^{2l-3}p^{2l-1} < n^{-\delta}$ then w.h.p.

$$\tau^i(x,y) - \sigma^i(x,y) < C \quad \forall \{x,y\} \in \binom{V}{2}, \ i \in [l], \quad (2.11)$$

where $C$ depends only on $l$ and $\delta$.

We note for use below that the assumption on $p$ in Proposition 2.9 implies

$$n^{l-2}p^{l-1} < n^{-\zeta}, \quad (2.12)$$

with $\zeta = (1 + \delta(l - 1))/(2l - 1) (= \Omega(1))$. Strictly speaking, the proposition is a little stronger than what one gets from [36], where the assumption would be $n^{l-1}p^l = O(\log n)$. (The $n^{2l-3}p^{2l-1}$ is more or less the expected number of non-edge-disjoint pairs of paths joining a given $x$ and $y$.)

Proposition 2.9, though not difficult, is a key point in Spencer’s proof of Theorem 2.8, and from our perspective is in a sense the main point, since, as indicated in the remark below, it easily gives the latter when combined with Theorem 2.3 and Lemma 2.4 (or Theorem 2.7).

Since the proof of the proposition itself is not so easy to extract from Spencer’s presentation (see his “third part” on p. 253), we next sketch an argument along lines similar to his for the present situation.

Proof of Proposition 2.9. It is enough to handle $i = l$ (since the assumption on $p$ implies a stronger assumption when we replace $l$ by $i < l$). Noting that $\tau^l(x,y) - \sigma^l(x,y) \leq |E(\Gamma^l_{x,y})|$, we find that (2.11) (with an appropriate $C$) holds at $x, y$ provided

(i) the maximum number of vertices in a component of $\Gamma^l_{x,y}$ is $O(1)$ and

(ii) the maximum size of an induced matching in $\Gamma^l_{x,y}$ is $O(1)$;

so we want to say that w.h.p. these conditions hold for all $x, y$. (Of course replacing (i) by an $O(1)$ bound on degrees would also suffice.)
For (i) we show that, for some fixed $M$, w.h.p. there do not exist $x, y$ and a collection, $Q_1, \ldots, Q_M$, of $P_l$’s joining $x$ and $y$ such that, for $i \geq 2$, $V(Q_i)$ meets, but is not contained in, $\bigcup_{j<i} V(Q_j)$. This bounds (by $(l-2)M+1$) the number of internal vertices (of $G$) in the paths belonging to a component of $\Gamma_{x,y}$, so gives (i).

Suppose $Q_1, \ldots, Q_M$ are $P_l$’s joining $x$ and $y$, with $R_i = \bigcup_{j \leq i} Q_j$ and, for $i \geq 2$, $|E(Q_i) \setminus E(R_{i-1})| = b_i$ and $|V(Q_i) \setminus V(R_{i-1})| = a_i \in [1, l-2]$. Then $b_i \geq a_i + 1$ and $a_i \leq l-2$ imply $n^{a_i} p^{b_i} \leq n^{l-2} p^{l-1}$ (for $i \geq 2$) and

$$n^{a_i} p^{b_i} \leq np(n^{l-2} p^{l-1})^M, \quad (2.13)$$

which is thus an upper bound on the probability of finding, for a given $x, y$, $(Q_1, \ldots, Q_M)$ as above of a given isomorphism type (defined in the obvious way). So the probability that there are such $Q_i$’s for some $x, y$ (and some isomorphism type) is $O(n^3 p(n^{l-2} p^{l-1})^M) = O(n^3 p^{n-M})$ (see (2.12)), so is $o(1)$ for large enough $M$.

The argument for (ii) is similar. Here we want to rule out, again for some fixed $M$, existence of $P_l$’s, say $Q_1, R_1, \ldots, Q_M, R_M$, joining some specified $x, y$, with $V(Q_i) \cap V(R_i) \neq \emptyset$ and the $V(Q_i)$’s and $V(R_i)$’s otherwise disjoint. A discussion like the one above shows that for any such sequence, with $|\bigcup_i (E(Q_i) \cup E(R_i))| = b$ and $|\bigcup_i (V(Q_i) \cup (V(R_i))| = a$, we have

$$n^a p^b < (n^{2l-3} p^{2l-1})^M < n^{-M \delta}, \quad (2.14)$$

which bounds the probability of existence by $O(n^{2-M \delta})$.

Remark. The lower bound in Theorem 2.8 is given by Theorem 2.3 (a recent development at the time). The main issue for the upper bound is handling $p$ with $n^{l-1} p^l \asymp \log n$, for which Proposition 2.9 allows replacing $\tau^l$ by $\sigma^l$. Spencer’s nice observation is that, to bound $\sigma^l(x, y)$, one need only bound the probability of having a maximal disjoint family (of $P_l$’s joining $x, y$) of a given size, and that one can use Theorem 2.3 to bound the probability that a particular (disjoint) family is maximal. His uses of this device could now be replaced by Lemma 2.4 (or Theorem 2.7), yielding (in the authors’ unbiased
Theorem 2.8 and Proposition 2.9 (with bits of Section 2.1) easily imply the following bounds on the \( \tau^l(x, y) \)'s for different ranges of \( p \).

**Corollary 2.10.** W.h.p. for all (distinct) vertices \( x, y \),

\[
\begin{align*}
\tau^l(x, y) &\sim n^{l-1} p^l & \text{if } n^{l-1} p^l = \omega(\log n), \\
\tau^l(x, y) &\sim O(\log n) & \text{if } n^{l-1} p^l = O(\log n), \\
\tau^l(x, y) &\sim O(1) & \text{if } n^{l-1} p^l < n^{-\Omega(1)}.
\end{align*}
\]

**Proof.** The first two items are easy consequences of Theorem 2.8: (2.15) is immediate and (2.16) is given by the observation that, for \( K \) as in the theorem (for some specified \( \varepsilon \)) and \( p_0 \) defined by \( n^{l-1} p_0 = K \log n \), the theorem implies that w.h.p.

\[
\tau^l(x, y) \leq (1 + \varepsilon)n^{l-1}(\max\{p, p_0\})^l \quad \forall \{x, y\} \in \binom{V}{2}
\]  
(2.18)

(since the probability of the event in (2.18) is increasing as \( p \) decreases below \( p_0 \)).

For (2.17), suppose \( n^{l-1} p^l < n^{-\alpha} \), with \( \alpha > 0 \) fixed. Since this implies \( n^{2l-3} p^{2l-1} < n^{-\delta} \) with \( \delta = \delta_\alpha > 0 \) fixed, Proposition 2.9 says it suffices to show that for given \( x, y \) and suitable fixed \( D \) (depending on \( \alpha \)),

\[
\Pr(\sigma^l(x, y) > D) = o(n^{-2}).
\]

But Proposition 2.5 bounds this probability by

\[
n^{-\alpha D}/D! < \exp[-D \log(n^\alpha D/e)],
\]

which is \( o(n^{-2}) \) for large enough \( D \).

We will also sometimes need lower bounds on path counts, as summarized in the next result, which again follows easily from what we already know.
Corollary 2.11. For any \( l \geq 2 \) there is a \( K \) such that if \( n^{l-1}p^l \geq K \log n \), then w.h.p. \( \sigma^l(x,y) = \Omega(\pi) \) for all \( x,y \), with \( \pi = \pi(n,p) \) equal to

\[
\begin{align*}
n^{l-1}p^l & \quad \text{if } n^{l-2}p^{l-1} < n^{-\Omega(1)}, \\
n^{l-1}p^l / \log n & \quad \text{if } n^{-o(1)} < n^{l-2}p^{l-1} = O(\log n), \\
np & \quad \text{if } n^{l-2}p^{l-1} = \omega(\log n).
\end{align*}
\] (2.19) (2.20) (2.21)

(Of course in view of the routine Proposition 2.12, \((1+o(1))np\) is a trivial upper bound.)

Proof. Let \( K \) be as in Theorem 2.8, for the given \( l \) and, say, \( \epsilon = 1/2 \) (since we don’t worry about constants). Since the theorem says that w.h.p. \(|V(\Gamma^l_{x,y})| > \Omega(n^{l-1}p^l)\) for all \( x,y \), the present assertion(s) will follow if we show

\[
\text{w.h.p. } \Delta(\Gamma^l_{x,y}) = O(n^{l-1}p^l/\pi) \forall x,y,
\] (2.22)

where we use the the trivial \( \alpha \geq |V|/\Delta \) (recall \( \Delta \) and \( \alpha \) are maximum degree and independence number and note \( \sigma^l(x,y) = \alpha(\Gamma^l_{x,y}) \)).

Now the degree in \( \Gamma^l_{x,y} \) of a given vertex \( Q \) (that is, a \( P_l \) joining \( x \) and \( y \)) is at most

\[
\sum_v \sum_i \tau^i(x,v)\tau^{l-i}(v,y) \leq (l-1)^2 \max\{\tau^i(x,v)\tau^{l-i}(v,y)\},
\] (2.23)

where the sums are over \( v \in V(Q) \) and \( i \in [l-1] \), and the max is over \( i \in [l-1] \) and \( v \in V \setminus \{x,y\} \) (the initial \((l-1)^2\) is of course irrelevant). On the other hand, Corollary 2.10 (with \( i \) in place of \( l \)) says that w.h.p. we have, for all \( u,v \):

\[
\tau^i(u,v) < O(1) \quad \text{if either } i \leq l-2 \text{ and } p \text{ is as in (2.19) or (2.20), or } i = l-1, \text{ and } p \text{ is as in (2.19),}
\]

and \( \tau^i(u,v) < O(\max\{n^{i-1}p^i, \log n\}) \) in general; and combining these bounds with (2.23) easily yields (2.22). \( \square \)
2.5 Density

Here we review various density properties of $G_{n,p}$. Throughout the section we use $G$ for $G_{n,p}$ and $V$ for $[n] = V(G)$. Theorems 2.1 and 2.2 easily imply the next two standardish propositions, whose proofs we omit.

**Proposition 2.12.** For $p \gg n^{-1} \log n$, w.h.p.

$$|G| \sim n^2 p/2 \quad \text{and} \quad d(v) \sim np \quad \forall \ v \in V.$$  

(Of course the second conclusion implies the first, which just needs $p \gg n^{-2}$.)

**Proposition 2.13.** (a) For any $\varepsilon > 0$ there is a $K$ such that w.h.p. for all disjoint $S, T \subseteq V$ with $|S|, |T| > K p^{-1} \log n$

$$|\nabla_G(S, T)| = (1 \pm \varepsilon) |S| |T| p$$

and

$$|G[S]| = (1 \pm \varepsilon) \binom{|S|}{2} p.$$

(b) For $K > 3$ w.h.p.

$$|G[S]| < K |S| \log n \quad \text{for all } S \subseteq V \text{ with } |S| \leq K p^{-1} \log n.$$

(c) For each $\varepsilon > 0$ there is a $K$ such that if $p > Kn^{-1} \log n$ then w.h.p.

$$|\nabla_G(S)| = (1 \pm \varepsilon) |S|(n - |S|) \quad \forall S \subseteq V.$$

**Proposition 2.14.** For fixed $\varepsilon > 0$ and $p \gg 1/n$, w.h.p.: if $H \subseteq G$ satisfies

$$d_H(v) > (1 - \varepsilon) np/2 \quad \forall \ v \in V,$$  

(2.24)

then no component of $H$ has size less than $(1 - 2\varepsilon)n/2$. 

Proof. For a given \( W \subseteq V \) of size \( w < (1 - 2\varepsilon)n/2 \), let \( \chi = |G[W]| \). Then \( \mu := E\chi = C(p) < w^2p/2 \), while if \( W \) is a component of an \( H \) satisfying (2.24) then

\[ \chi \geq |H[W]| > w(1 - \varepsilon)np/4 > \frac{(1-\varepsilon)n}{2w}\mu =: K\mu. \]

But (since \( K > (1 - \varepsilon)/(1 - 2\varepsilon) = 1 + \Omega(1) \)) Theorems 2.1 and 2.2 give

\[ \gamma_w := \Pr(\chi > K\mu) < \begin{cases} \exp[-\Omega(\mu)] & \text{if } K < e^2 \text{ (say)}, \\ \exp[-K\mu \log(K/e)] & \text{otherwise}. \end{cases} \]

Thus, with sums over \( w \in (0, (1 - 2\varepsilon)n/2) \), the probability that some \( H \) as in the lemma admits a component of size less than \( (1 - 2\varepsilon)n/2 \) is less than

\[ \sum \binom{n}{w} \gamma_w < \sum \exp[w \log(en/w)]\gamma_w, \]

which for \( p \gg 1/n \) is easily seen to be \( o(1) \).

Finally, we need to know a little about the eigenvalues of \( G \). A version of (2.25) below was proved in [16] (see also [2]) and (2.26) is shown (e.g.) in [30].

**Proposition 2.15.** Let \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \) be the eigenvalues of \( G \) and \( v_1, v_2, \ldots, v_n \) associated orthonormal eigenvectors, say with \( \max_j v_{1,j} > 0 \). If \( p \gg n^{-1}\log n \), then w.h.p.

\[ \lambda_1 \sim np \quad \text{and} \quad \max\{|\lambda_2|, |\lambda_n|\} < (2 + o(1))\sqrt{np}. \quad (2.25) \]

If \( p > n^{-1}\log^6 n \), then w.h.p.

\[ \max_j v_{1,j} < (1 + o(1)) \min_j v_{1,j}. \quad (2.26) \]

2.6 Expanders

We will want to say that good eigenvalues imply good density properties for a graph, for which our (standard) tool is
Lemma 2.16 (Expander Mixing Lemma [3, Cors. 9.2.5-6]). Let $H$ be a $d$-regular graph on $t$ vertices for which every eigenvalue except $d$ has absolute value at most $\lambda$. Let $A, B \subseteq V(H)$ be disjoint with $|A| = a, |B| = b$. Then

$$\left| \nabla(A, B) - \frac{abdt}{t} \right| \leq \lambda \sqrt{ab},$$

and

$$\left| H[A] - \frac{a^2d}{2t} \right| \leq \frac{\lambda a}{2}.$$

2.7 Short Cycles

Recall that the distance between a pair of vertices in a graph is the number of edges in a shortest path between them, and the diameter of a graph is the maximum distance between a pair of its vertices.

We will want to say that the cycle space of any graph with low diameter is spanned by short cycles.

Proposition 2.17. For any graph $G$ of diameter $D$, $\mathcal{C}(G)$ is generated by the cycles of $G$ of length up to $2D + 1$.

Proof. It suffices to show that any cycle of length at least $2D + 2$ is the sum of two shorter ones. Let $x, y$ be vertices at maximum distance along such a cycle $C$, and let $P$ be a shortest $xy$-path, say with $x = v_0, \ldots, v_k = y$ being the vertices that $P$ shares with $C$ (as ordered by $P$). For some $i \in [k]$, $v_{i-1}$ and $v_i$ are closer along $P$ than along $C$, and we get the two desired shorter cycles by replacing each of the $v_{i-1}v_i$-paths in $C$ by the one in $P$. \qed

Erratum. In [7], we gave an incorrect proof of this.
2.8 Mantel

Here we prove a new strengthening of Mantel’s Theorem [29], which may be of independent interest. Recall that Mantel’s Theorem is the first case of Turán’s Theorem ([38], or e.g. [13, Theorem 7.1.1]) and the first result in extremal graph theory, proved in 1907.

Lemma 2.18 (Mantel’s Theorem for “Crossing Triangles”). Let \( K \) be the complete graph on \( X \cup Y \), where \( X \) and \( Y \) are disjoint sets of size \( n \). Let \( F \) be a subgraph of \( K \) containing no (“crossing”) triangles meeting both \( X \) and \( Y \). Then \( |F| \leq n^2 \).

Proof. We first claim that for any largest \( F \) containing no crossing triangles, \( F[X] \) and \( F[Y] \) are complete multipartite. For convenience set \( G = F[X] \). If \( G \) is not complete multipartite, then it has vertices \( x, y, z \) satisfying \( xy \in G \) and \( xz, yz \notin G \). If \( d_F(x) > d_F(z) \), then replacing \( N_F(z) \) by \( N_F(x) \) strictly increases \( |F| \) without introducing forbidden triangles. Thus we may assume \( d_F(z) \geq d_F(x) \), and similarly \( d_F(z) \geq d_F(y) \). But then replacing both \( N_F(x) \) and \( N_F(y) \) by \( N_F(z) \) strictly increases \( |F| \) without introducing forbidden triangles. (This neighborhood-switching is a standard trick; see e.g. [13, Theorem 7.1.1]. We use it again in our proof of Theorem 3.3.)

So any largest \( F \) is complete multipartite in \( X \) with parts \( X_1, X_2, \ldots, X_r \) of sizes \( x_1 \geq x_2 \geq \cdots \geq x_r \), and in \( Y \) with parts \( Y_1, Y_2, \ldots, Y_r \) of sizes \( y_1 \geq y_2 \geq \cdots \geq y_r \) (some of the \( x_i \)'s or \( y_i \)'s being 0 if one of the partitions has more nonempty parts than the other). Since \( F \) has no triangles meeting both \( X \) and \( Y \), for any \( a \in X_i \) and \( b \in Y_j \) we have

\[
ab \in F \implies N_F(a) \cap Y \subseteq Y_j \quad \text{and} \quad N_F(b) \cap X \subseteq X_i,
\]

so by the so-called rearrangement inequality we have

\[
|F| \leq \sum_{1 \leq i < j \leq r} (x_ix_j + y_iy_j) + \sum_{i=1}^r x iy_i
= \frac{1}{2} \sum_{i=1}^r [x_i(n-x_i+y_i) + y_i(n-y_i+x_i)]
\]
\[
\begin{align*}
&= \frac{1}{2} \sum_{i=1}^{r} \left[ n(x_i + y_i) - (x_i - y_i)^2 \right] \\
&= n^2 - \frac{1}{2} \sum_{i=1}^{r} (x_i - y_i)^2. 
\end{align*}
\]

2.9 Stability

The following statement is a small instance of a recent major result of Conlon and Gowers [11]. As we will see in Section 4.4, it is the main (essentially only) ingredient in the proof of one of our lemmas in Chapter 4 (Lemma 4.10).

**Theorem 2.19.** For each odd \( \kappa \geq 3 \) and \( \varepsilon > 0 \) there is a \( C \) such that if \( p > Cn^{-(\kappa-2)/\kappa-1)} \), then w.h.p. every \( C_\kappa \)-free subgraph of \( G = G_{n,p} \) of size at least \( |G|/2 \) can be made bipartite by deleting at most \( \varepsilon n^2 p \) edges.

This (or the more general result of [11]) is a “sparse random” analogue of the Erdős-Simonovits “Stability Theorem” [14, 35] that was conjectured by Kohayakawa et al. in the seminal [26].

2.10 Regularity

Here we recall Szemerédi’s Regularity Lemma [37], or, more precisely, a generalization thereof due to Kohayakawa [27] and Rödl (unpublished). Our presentation here follows [23, Section 8.3].

**Definitions 2.20** (for the Regularity Lemma). Given a graph \( H \), a real number \( s \in (0,1) \) (called a scaling factor), and disjoint \( U,W \subseteq V(H) =: V \), the \( (s;H)\)-density \( d_{s,H}(U,W) \) between \( U \) and \( W \) is

\[
d_{s,H}(U,W) = \frac{|\nabla H(U,W)|}{s|U||W|}.
\]

For \( \epsilon > 0 \), the pair \( U,W \) is called \( (s;H,\epsilon)\)-regular if for all \( U' \subseteq U \) and \( W' \subseteq W \) with \( |U'| \geq \epsilon |U| \) and \( |W'| \geq \epsilon |W| \) we have

\[
|d_{s,H}(U,W) - d_{s,H}(U',W')| \leq \epsilon.
\]
A partition $\Pi = (V_0, V_1, \ldots, V_k)$ of $V$ is called $(\epsilon, k)$-equitable if $|V_1| = |V_2| = \cdots = |V_k|$ and $|V_0| \leq \epsilon |V|$, and it is called $(s; H, \epsilon, k)$-regular if it is $(\epsilon, k)$-equitable and all but at most $\epsilon \binom{k}{2}$ of the pairs $V_i, V_j$ ($1 \leq i < j \leq k$) are $(s; H, \epsilon)$-regular. In such a partition, $V_0$ is called the exceptional part. If $k' > k$ and $\Pi'$ is an $(\epsilon, k')$-equitable partition of $V$, then we say $\Pi'$ refines $\Pi$ if every nonexceptional part of $\Pi'$ is contained in some nonexceptional part of $\Pi$.

For $b \geq 1$ and $\beta > 0$, $H$ is called $(s; b, \beta)$-bounded if whenever $U, W \subseteq V$ are disjoint with $|U|, |W| \geq \beta |V|$ we have $d_{s,H}(U,W) \leq b$. Intuitively, when $H$ is sparse and $s$ is the (tiny) density of $H$, $(s; b, \beta)$-boundedness ensures that no substantial chunk of $H$ is much denser than it should be.

Lemma 2.21 (Szemerédi Regularity Lemma, [23, Lemma 8.18]). For all $\epsilon > 0, b \geq 1$ and natural numbers $m$ and $r$ there exist $\beta = \beta(\epsilon, b, m, r) > 0$ and $M = M(\epsilon, b, m, r) \geq m$ such that the following holds. For every choice of scaling factors $s_i$ ($i \in [r]$) and $(s_i; b, \beta)$-bounded graphs $H_i$ ($i \in [r]$) on the same vertex set $V$ with $|V| \geq m$, there exists $k \in [m, M]$ and a partition $\Pi$ of $V$ that is $(s_i; H_i, \epsilon, k)$-regular for all $i \in [r]$.

Since the proof of the Regularity Lemma starts with any partition of $V$ into $m$ nonexceptional parts of size $\lfloor |V|/m \rfloor$ and repeatedly refines this partition so that at each step each part is broken into the same number of subparts (see e.g. [27, 17] for details), we may further assume that

(i) $\Pi$ refines a specified partition of $V$ with $m$ nonexceptional parts of size $\lfloor |V|/m \rfloor$, and

(ii) For any two nonexceptional parts $S_i, S_j$ of the starting partition we have $|V_0 \cap S_i| = |V_0 \cap S_j|$, where $V_0$ is the exceptional part of $\Pi$.

Observe also that since every graph is trivially $(1; 1, \beta)$-bounded for all $\beta$, taking $b = r = s_1 = 1$ in Lemma 2.21 recovers the usual Regularity Lemma, which on one occasion (to prove Theorem 3.3) is all we will need from Lemma 2.21. Our other use of Lemma 2.21 (to prove Lemma 3.10) will require its full power.
Associated with the Regularity Lemma is the so-called Counting Lemma, which we will use in the following unusual form.

**Lemma 2.22** (Counting Lemma). Let $H$ be a graph, $\epsilon \in (0, 1/2)$, $s \in (0, 1]$, and $A, B, B'$ pairwise disjoint subsets of $V(H)$ each of size $l$. If the pairs $A, B$ and $A, B'$ are $(1; H, \epsilon)$-regular with $(1; H)$-density at least $2\epsilon$, and the pair $B, B'$ is $(s; H, \epsilon)$-regular with $(s; H)$-density at least $2\epsilon$, then $H$ contains a triangle $abb'$ with $a \in A$, $b \in B$, $b' \in B'$.

**Proof.** Since $d_{1,H}(A, B) \geq 2\epsilon$, we have $|\{a \in A \mid |\nabla(a, B)| < \epsilon l\}| < \epsilon l$, or else this subset of $A$, along with $B \subseteq B$, would violate the $(1; H, \epsilon)$-regularity of the pair $A, B$. Similarly $|\{a \in A \mid |\nabla(a, B')| < \epsilon l\}| < \epsilon l$. Thus since $\epsilon < 1/2$, there exists $a \in A$ satisfying $|N(a) \cap B|, |N(a) \cap B'| \geq \epsilon l$. Then since the pair $B, B'$ is $(s; H, \epsilon)$-regular with $(s; H)$-density at least $2\epsilon$, we have $\nabla(N(a) \cap B, N(a) \cap B') \neq \emptyset$, yielding a triangle in $H$ of the stated form. \qed

### 2.11 Containers

Here we give a specialization, adequate for present purposes, of the celebrated recent “container” theorems of [4, 34].

First we need a few definitions. Recall that a **hypergraph**, $\mathcal{H}$, is simply a collection of subsets (“edges”) of a set $\mathcal{V}$ of “vertices.” (We allow repeated edges, though we won’t actually see any.) All our hypergraphs are $r$-uniform, i.e. have all edges of size $r$, and finite, with $|\mathcal{V}| = N$. An independent set of $\mathcal{H}$ is a subset of $\mathcal{V}$ containing no edges and $\mathcal{I}(\mathcal{H})$ is the collection of such sets.

For $\sigma \subseteq \mathcal{V}$, the degree of $\sigma$ is $d(\sigma) = d_{\mathcal{H}}(\sigma) = |\{e \in \mathcal{H} : \sigma \subseteq e\}|$, which we shorten to $d(v)$ if $\sigma = \{v\}$. We use $d$ and $\Delta$ for the average and maximum values of $d(v)$ ($v \in \mathcal{V}$) and, for $l \in [r]$,

$$\Delta_l = \max\{d(\sigma) : |\sigma| = l\}$$

(so $\Delta_1 = \Delta$).

The next assertion is easily derivable from Theorem 2.2 of [4].
Theorem 2.23. For all $r$, $\delta > 0$ and $b$ there is a $B$ such that: if $\mathcal{H}$ is $r$-uniform with

$$
\Delta_l < b\theta^{l-1}d \quad \forall l \in [r],
$$

(2.27)

then there exists $C : 2^\mathcal{V} \to 2^\mathcal{V}$ such that for each $I \in \mathcal{I}(\mathcal{H})$ there is a $T \subseteq \mathcal{V}$ with:

(a) $|T| < BN\theta$,

(b) $T \subseteq I \subseteq C(T)$,

(c) $|\mathcal{H}[C(T)]| < \delta |\mathcal{H}|$

(\text{where } \mathcal{H}[X] = \{E \in \mathcal{H} : E \subseteq X\}).
Chapter 3

Tuza’s Conjecture is Asymptotically Tight for Dense Graphs

3.1 Introduction

Following [40] we write $\tau_3(G)$ for the minimum size of a triangle edge cover (set of edges meeting all triangles) in a graph $G$ and $\nu_3(G)$ for the maximum size of a triangle packing (collection of edge-disjoint triangles) in $G$. (In standard language these are the matching and vertex cover numbers of the hypergraph with vertex set $E(G)$ and edges the triangles of $G$.)

While $\tau_3(G) \leq 3\nu_3(G)$ is trivial (for any $G$), a 35-year-old conjecture of Zsolt Tuza [39] holds that this can be improved:

**Conjecture 3.1.** For any $G$, $\tau_3(G) \leq 2\nu_3(G)$.

(This is sharp for the complete graphs of orders 4 and 5.)

The best general result in this direction remains that of Haxell [20], who showed

$$\tau_3(G) \leq (66/23)\nu_3(G).$$

On the other hand, as noted in [40], a combination of results of Krivelevich [28] and Haxell and Rödl [21] implies that for any $G$,

$$\tau_3(G) < 2\nu_3(G) + o(n^2)$$

(limits as $n := |V(G)| \to \infty$). In particular, for any fixed $\beta > 0$ and $G$ ranging over
graphs satisfying $\tau_3(G) \geq \beta n^2$,\[\tau_3(G) < (2 + o(1))\nu_3(G). \quad (3.1)\]

That is, Tuza’s conjecture is asymptotically correct for such graphs.

The question of Raphael Yuster [40] that motivates us here is: is the constant 2 in (3.1) optimal? That is, is Tuza’s conjecture still (asymptotically) tight for dense graphs with no subquadratic triangle cover? Yuster suggested not, at least in the special case where $\tau_3(G)$ is nearly as large as possible:

**Conjecture 3.2 ([40]).** For fixed $\beta > 0$ and $G$ ranging over graphs of density at least $\beta$,\[\tau_3(G) > (1 - o(1))|G|/2 \implies \nu_3(G) > (1 - o(1))|G|/3\]

(where density is $|G|/(\binom{n}{2})$, and $|G| = |E(G)|$). This would of course (for the graphs considered) be a big improvement over (3.1), which promises only $\nu_3(G) > (1 - o(1))|G|/4$.

Note that the inequalities $\tau_3(G) < |G|/2$ and $\nu_3(G) \leq |G|/3$ are easy and trivial (respectively), so Yuster’s conjecture says that if $G$ is dense and $\tau_3(G)$ is close to its trivial upper bound, then so must be $\nu_3(G)$.

Yuster also suggested weakening Conjecture 3.2 to say only that there is some fixed $\alpha \in (0, 1/3)$ (not depending on $\beta$) such that\[\tau_3(G) > (1 - o(1))|G|/2 \implies \nu_3(G) > (1 + \alpha)|G|/4, \quad (3.2)\]

which would still significantly improve on (3.1) (when $\tau_3(G) > (1 - o(1))|G|/2$). (Yuster did show that (3.2) is true if we allow $\alpha$ to depend on $\beta$.)

Surprisingly it turns out that even the weaker conjecture is wrong:

**Theorem 3.3.** For all $\alpha > 0$, there exist $\beta > 0$ and arbitrarily large graphs $G$ satisfying

- $|G| \geq \beta \binom{n}{2}$,
- $\tau_3(G) > (1 - o(1))|G|/2$, and
\[ \nu_3(G) < (1 + \alpha)|G|/4 \]

(limits as \( n \to \infty \)). Thus even for dense graphs—and moreover for dense graphs where \( \tau_3(G) \) is near \( |G|/2 \)—Tuza’s conjecture is essentially best possible.

Since what follows is not entirely easy, a little orientation may be helpful. Our construction itself is not very difficult; in rough outline it does:

1. start with a triangle-free graph \( H \) with certain nice degree and eigenvalue properties (we use the well-known graphs described by Noga Alon in [1]—see Proposition 3.11);
2. join two disjoint copies of \( H \) by a complete bipartite graph to produce \( K \);
3. replace each vertex of \( K \) by a large clique; and finally
4. take a suitable random subgraph of this blowup, yielding the graph \( G_a \) found in the third paragraph of Section 3.3.

So again, there is nothing very exotic here. What seems most interesting in what follows is how strange a route we needed to take to arrive at a proof that this relatively simple construction actually works.

Also interesting is whether one could simplify our argument (or give an easier example) if the goal were only to disprove the stronger Conjecture 3.2 (rather than (3.2)). We don’t see how to do this, and in fact most of what follows was originally developed with the lesser goal in mind.

The rest of this chapter is organized as follows. The next section gives a long string of essential definitions, most of them nonstandard, leading up to the crucial Lemma 3.10. In Section 3.3 we prove Theorem 3.3 assuming Lemma 3.10. In Section 3.4, we prove the lemma.

### 3.2 Definitions

A fractional triangle edge cover of a graph \( G \) is an assignment of nonnegative weights to the edges of \( G \) such that the weight of each triangle (this being the sum of the
weights of its edges) is at least 1. We denote by \( \tau_3^*(G) \) the minimum total weight of such a cover. Dually, a fractional triangle packing of \( G \) is an assignment of nonnegative weights to the triangles of \( G \) such that the weight of each edge (the sum of the weights of the triangles containing it) is at most 1. We denote by \( \nu_3^*(G) \) the maximum total weight of such a packing. Note we have

\[
\nu_3(G) \leq \nu_3^*(G) = \tau_3^*(G) \leq \tau_3(G),
\]

where the inequalities are trivial and the equality is by linear programming duality.

Given graphs \( G_1, G_2 \), the lexicographic product \( G_1 \cdot G_2 \) is the graph on vertex set \( V(G_1) \times V(G_2) \) where \( (u_1, u_2) \) is adjacent to \( (v_1, v_2) \) iff either \( u_1v_1 \in G_1 \), or \( u_1 = v_1 \) and \( u_2v_2 \in G_2 \). Note that the lexicographic product is not commutative.

The following original definitions are critical to our arguments.

**Definition 3.4** (double of a graph). For a graph \( H \), the double of \( H \), denoted \( K_{H,H} \), is the graph \( K_2 \cdot H \). To be explicit, this is the graph whose vertex set is \( X \cup Y \), where \( X \) and \( Y \) are disjoint sets of size \( |V(H)| \), and whose edges satisfy \( K_{H,H}[X] \simeq K_{H,H}[Y] \simeq H \) and \( \{xy \mid x \in X, y \in Y\} \subseteq E(K_{H,H}) \). The sets \( X \) and \( Y \) (we will always use these names) are called the sides of \( K_{H,H} \).

Of course the notation \( K_{H,H} \) is intended to suggest the notation \( K_{t,t} \) for a complete bipartite graph. When the \( H \) is understood, we will frequently abbreviate \( K_{H,H} \) by \( K \).

We denote by \( E \) the copy of \( K_2 \) on vertex set \( \{b, s\} \). Here \( E \) is for “edge,” \( b \) is for “big,” and \( s \) is for “small,” for reasons that will now become clear.

**Definition 3.5** (compound vertex). Let \( G \) be a graph. Then \( G \) on compound vertices, denoted \( G^+ \), is the graph \( G \cdot E \). This term is intended to be suggestive—we imagine \( G^+ \) as \( G \) with each of its vertices \( v \) replaced by a new compound structure with a big part \( (v, b) \) and a small part \( (v, s) \). We will always abbreviate, e.g., \( (v, b) \) by \( v^b \). For a generic vertex of \( G^+ \) we write \( v^x, v^y \), etc., understanding \( x, y \in \{b, s\} \).

**Definition 3.6** (edge types). In the context of a given \( K = K_{H,H} \), an edge \( uw \in K \) is called internal if \( u \) and \( w \) belong to the same side, and external otherwise. Similarly, an
edge \( u^xw^y \in K^+ \) with \( u \neq w \) is internal if \( uw \in K \) is internal and external if \( uw \in K \) is external. An edge \( v^bw^s \in K^+ \) is called a vertex edge.

**Definition 3.7** (external triangles). Let \( H \) be a graph and \( K = K_{H,H} \). A triangle in \( K \) or \( K^+ \) is an external triangle if it contains an external edge. A subgraph \( F \) of \( K \) or \( K^+ \) is external triangle free (ETF) if it contains no external triangles.

**Definitions 3.8** (configurations and weight). Let \( H \) be a graph with \( t \) vertices and \( m \) edges, and \( K = K_{H,H} \). A configuration on \( K \) is a pair \((F, \phi)\), where \( F \subseteq E(K^+) \) and \( \phi : V(K^+) \to [0,1] \) satisfy the following conditions. Viewing \( F \) as a subgraph of \( K^+ \), \( F \) is ETF, contains all vertex edges of \( K^+ \), and satisfies \( N_F(v^b) \cap N_F(v^s) = \emptyset \) \( \forall v \in V(K) \); and \( \phi \), which we call a mass function, satisfies \( \phi(v^b) \in \left[ \frac{1}{2}, 1 \right] \) and \( \phi(v^s) = 1 - \phi(v^b) \) \( \forall v \in V(K) \). Given a configuration and \( c \in [0,1] \), the configuration’s \( c \)-weight is

\[
w_c(F, \phi) = \frac{1-c}{4m} \sum_{u^xw^y \in F \text{ internal}} \phi(u^x)\phi(w^y) + \frac{1-c}{2t^2} \sum_{u^xw^y \in F \text{ external}} \phi(u^x)\phi(w^y) + \frac{c}{t} \sum_{v \in V(K)} \phi(v^b)\phi(v^s).
\]  

(3.3)

Here’s the idea behind \( c \)-weight. Given \( H \), we think of the vertices and edges of \( K \) as having weights attached, as follows. Each vertex weighs \( \frac{c}{2t} \), each internal edge weighs \( \frac{1-c}{4m} \), and each external edge weighs \( \frac{1-c}{2t^2} \), for a total of unit weight on \( K \). Passing to \( K^+ \), an adversary tries to maximize the amount of this weight he can capture in a configuration \((F, \phi)\). For each edge \( uw \in K \), the fraction of that edge’s weight that he captures is \( \sum_{u^xw^y \in F} \phi(u^x)\phi(w^y) \), because we think of the weight of \( uw \in K \) as being split among the four corresponding edges of \( K^+ \) with a \( \phi(u^x)\phi(w^y) \)-fraction residing in the edge \( u^xw^y \). For each vertex \( v \in V(K) \), the fraction of that vertex’s weight that our adversary captures is \( 2\phi(v^b)\phi(v^s) \), because we think of the weight of a vertex in \( K \) as being split up in \( K^+ \) analogously to the way the weight of an edge in \( K \) is split up in \( K^+ \), with a \( \phi(v^b)^2 \)-fraction of the weight of \( v \) residing in \( v^b \), a \( \phi(v^s)^2 \)-fraction in \( v^s \), and the remaining \( 2\phi(v^b)\phi(v^s) \)-fraction in the vertex edge \( v^b v^s \). This \( 2 \) cancels the \( \frac{1}{2} \) in the vertex weight \( \frac{c}{2t} \) to yield the coefficient of the third sum in (3.3). To see that the 2 is natural, observe that it lets our adversary capture exactly half the weight of every
vertex and edge of $K$ by taking $F = \{u^b w^a \mid uw \in K\} \cup \{v^b v^a \mid v \in V(K)\}$ and $\phi \equiv \frac{1}{2}$.

We call this the na"ive configuration.

**Definition 3.9 (fairness).** For $c \in [0, 1]$, a graph $H$ is called $c$-fair if

$$\max \{w_c(F, \phi)\} = \frac{1}{2},$$

where the max is over configurations $(F, \phi)$ on $K$.

Observe that the $1/2$ in (3.4) is best possible, since the na"ive configuration has $c$-weight $1/2$ for any $c$. This explains the term “fair”—our adversary can’t capture more than half the weight of $K$, the amount to which he is na"ively entitled.

Observe also that increasing $c$ can only make life harder for our adversary. That is, if $H$ is $c$-fair, then it is $c'$-fair for any $c' \in [c, 1]$. To see this, notice that $w_c(F, \phi)$ is a convex combination of the nonnegative quantities

$$\frac{1}{2m} \sum_{u^x w^y \in F \text{ internal}} \phi(u^x)\phi(w^y), \quad \frac{1}{t^2} \sum_{u^x w^y \in F \text{ external}} \phi(u^x)\phi(w^y) \quad \text{and} \quad \frac{1}{l} \sum_{v \in V(K)} \phi(v^b)\phi(v^a),$$

with coefficients $\frac{1-c}{2}$, $\frac{1-c}{t^2}$, $c$. Since the first two coefficients are decreasing in $c$ and the third quantity is at most $1/2$ (note each of the $2t$ terms in its sum is at most $1/4$), increasing $c$ cannot raise $w_c(F, \phi)$ above $1/2$. At the extremes, it is easy to see that no graph is $0$-fair and every graph is $1$-fair. This, finally, motivates

**Lemma 3.10.** For any $c \in (0, 1]$ and $N \in \mathbb{N}$, there exists a triangle-free, $d$-regular, $c$-fair graph $H$ with $d \geq N$.

### 3.3 Proof of Theorem 3.3

Fixing $\alpha > 0$ (we may assume $\alpha < 1/3$), our goal is to show there are arbitrarily large graphs $G$ of positive density satisfying $\tau_3(G) > (1 - o(1))|G|/2$ but nonetheless $\nu_3(G) < (1 + \alpha)|G|/4$. To do this, we use a probabilistic construction starting with a graph promised by Lemma 3.10.
Set $c = \alpha/6$ and let $H$ be a triangle-free, $d$-regular, $c$-fair graph on $t$ vertices, where $d \geq (2c)^{-1}$. Let $p = \frac{1-c}{2cd}$ and $q = \frac{1-c}{2ct}$, noting that $p, q \in (0, 1)$. Let $K = K_{H,H}$, and observe that $K \cdot K_a$ is the graph obtained from $K$ when each vertex is “blown up” to a clique of size $a$. Call each of these $K_a$’s in $K \cdot K_a$ a block, and for each $v \in V(K)$, denote by $B_v$ the block corresponding to $v$. Also, consistent with Definition 3.6, call an edge $xy \in K \cdot K_a$ an internal edge, external edge, or vertex edge according to whether it comes from an internal edge, external edge, or vertex of $K$.

For each $a \in \mathbb{N}$ (think: large), let $G_a$ be the random graph obtained from $K \cdot K_a$ by deleting each internal edge with probability $1-p$ and each external edge with probability $1-q$, these choices made independently. Then since $|\nabla G_a(B_u, B_w)| \sim \text{Bin}(a^2, p)$ for each internal $uw \in K$ and $|\nabla G_a(B_u, B_w)| \sim \text{Bin}(a^2, q)$ for each external $uw \in K$, Theorem 2.1 says that each of these numbers $|\nabla G_a(B_u, B_w)|$ is typically close to its expectation. To be precise, for each $uw \in K$ (internal or external), if we set $X_{uw} = |\nabla G_a(B_u, B_w)|$, $\mu_{uw} = \mathbb{E}X_{uw}$ and $x = a \log a$, then Theorem 2.1 gives $\mathbb{P}(|X_{uw} - \mu_{uw}| \geq x) = O(a^{-2}) = o(1)$ as $a \to \infty$. Since $|K| = t^2 + td$ is fixed, $\mu_{uw} = \Theta(a^2)$ and $x = o(a^2)$, it holds w.h.p. as $a \to \infty$ that $X_{uw} \sim \mu_{uw}$ for all $uw \in K$. We may thus assume $G_a$ satisfies this property, whence

\begin{align*}
|\{xy \in G_a \mid xy \text{ internal}\}| &\sim tda^2 p = \frac{a^2t(1-c)}{2c}; \quad (3.5) \\
|\{xy \in G_a \mid xy \text{ external}\}| &\sim t^2a^2 q = \frac{a^2t(1-c)}{2c}; \quad (3.6) \\
|\{xy \in G_a \mid xy \text{ vertex}\}| &\sim 2t \left(\frac{a}{2}\right) \sim a^2t. \quad (3.7)
\end{align*}

We claim that, w.h.p. as $a \to \infty$, $G_a$ meets the requirements of Theorem 3.3. The first and third conditions are easy to check. For density, letting $n = |V(G_a)| = 2ta$ and $m = |G_a|$, we have

\begin{equation}
m \sim a^2t + 2a^2t(1-c) = \frac{a^2t}{c} = n^2(4tc)^{-1}, \quad (3.8)
\end{equation}

where $(4tc)^{-1} < 1/2$ is a constant.

To see that $\nu_3(G_a) < (1+\alpha)m/4$, it suffices to find a fractional triangle edge cover of
\(G_a\) of total weight less than \((1 + \alpha)m/4\), since (recall) \(\nu_2(G_a) \leq \nu_3^*(G_a) = \tau_3^*(G_a)\). But this is easy: simply placing weight 1 on all vertex edges and weight 1/2 on all external edges yields a fractional triangle edge cover of \(G_a\) (here the triangle-freeness of \(H\) is crucial) with total weight asymptotic to

\[
a^2t + \frac{1}{2}a^2t(1-c) = \frac{a^2t}{4c}(1 + 3c) = (1 + \alpha/2 \pm o(1)) \frac{m}{4} < (1 + \alpha) \frac{m}{4}.
\]

The real work is showing that \(\tau_3(G_a) > (1 - o(1))m/2\). To this end let \(F \subseteq G_a\) be triangle-free; we need to show \(|F| \leq (1 + o(1))m/2\). More precisely, we show that given any \(\delta > 0\), we have \(|F| < (1 + \delta)m/2\) for large enough \(a\). For this we apply the usual Regularity Lemma—i.e. Lemma 2.21 with \(b = r = s_1 = 1\) to \(F\). Pick (with foresight) \(\epsilon < \delta/(48tc)\), and let \(2t \lceil \epsilon - 1 \rceil\) be the “\(m\)” of the lemma. Let \(\Pi = (V_0, V_1, \ldots, V_k)\) be the partition given by the lemma. By comments (i) and (ii) after the lemma, we may assume \(\Pi\) refines the partition of \(V(F) = V(G_a)\) into blocks and splits each block into exactly \(k/(2t) =: \eta\) nonexceptional parts plus some vertices in \(V_0\).

For a pair \(V_i, V_j \in \Pi\) with \(V_i \subseteq B_u\) and \(V_j \subseteq B_w\), call the pair internal or external if \(uw\) is an internal or external edge of \(K\) (respectively), and a vertex pair if \(u = w\). Consider the graph on \([k]\) where \(ij\) is an edge iff \(V_i, V_j\) is an internal, external or vertex pair. Notice that this graph is (isomorphic to) \(K \cdot K_\eta\), with blocks \(B'_v = \{i \in [k] \mid V_i \subseteq B_v\}, v \in V(K)\). Letting \(l = |V_1|\), observe also that

\[
\begin{align*}
\text{vertex} & \quad \text{exactly } l^2 \\
\text{internal} & \quad \text{about } l^2p \\
\text{external} & \quad \text{about } l^2q
\end{align*}
\]

where just as in (3.5)–(3.7), each “about” in (3.9) hides an \(\tilde{O}(l) = \tilde{O}(n) = o(m)\) Chernoff error as \(a \to \infty\).

To account for the different quantities on the right side of (3.9), we assign weights to the edges of \(K \cdot K_\eta\): each vertex edge weighs \(c/(tn^2)\), each internal edge \(pc/(tn^2) = \frac{1}{24n^2}\), and each external edge \(qc/(tn^2) = \frac{1-\epsilon}{24n^2}\), so that the weight \(w(uw)\) of \(uw \in K \cdot K_\eta\) is \(c/(tn^2l^2)\) times the (approximate) number of corresponding edges in \(G_a\). With these
weights, the total weight of the edges corresponding to an internal $uw \in K$ is $\frac{1-c}{2d^2}$, the total weight of the edges corresponding to an external $uw \in K$ is $\frac{1-c}{2d^2}$, and the total weight of the edges in a block $B'_v$ is $(\frac{n}{2}) \frac{c}{t \eta^2} \lesssim \frac{c}{t}$ (where $\lesssim$ means approximate equality and $\leq$).

Leaving the topic of edge weights for a moment, we now let $F'$ be the subgraph of $F$ obtained after we delete the following edges from $F$: edges incident to $V_0$; edges inside some $V_i$, $i \in [k]$; edges that join pairs that are not $(1;F,\epsilon)$-regular; and edges that join pairs with $(1;F)$-density less than $2\epsilon$. (This cleanup is of course a standard concomitant of the Regularity Lemma.) Since $l \leq n/k$, this deletes at most

$$en^2 + k \left(\frac{l}{2}\right) + \epsilon \left(\frac{k}{2}\right) l^2 + 2d^2 \left(\frac{k}{2}\right) \leq 3en^2$$

(3.10)

dges from $F$.

Let $\tilde{F}$ be the subgraph of $K \cdot K_\eta$ with $ij \in \tilde{F}$ iff there is an edge joining $V_i$ and $V_j$ in $F'$. By Lemma 2.22 (with $s = 1$) and the triangle-freeness of $F$, $\tilde{F}$ is also triangle-free. Let $F''$ be the subgraph of $G_a$ defined by

$$\nabla_{F''}(V_i, V_j) = \begin{cases} \nabla_{G_a}(V_i, V_j) & \text{if } ij \in \tilde{F} \\ \emptyset & \text{if } ij \notin \tilde{F} \end{cases}$$

With these definitions, (3.9), (3.10) and the calculations between them give

$$|F| \leq |F'| + 3en^2 \leq |F''| + 3en^2 \sim \mathcal{w}(\tilde{F})/(c/(t\eta^2l^2)) + 3en^2,$$

(3.11)

where (of course) $\mathcal{w}(\tilde{F}) = \sum_{uw \in \tilde{F}} \mathcal{w}(uw)$.

Our next goal is to massage $\tilde{F}$ until it resembles a configuration on $K$. For each $x \in V(\tilde{F}) = V(K \cdot K_\eta)$, let $\mathcal{w}(x)$ be the sum of the weights of its incident $\tilde{F}$-edges.\(^1\) Fix some order $\pi$ of $V(K)$, and for each $v \in V(K)$, in the chosen order, do the following, making changes to $\tilde{F}$ as necessary. We continue to write $\tilde{F}$ for the evolving graph.

\(^1\)For the rest of the argument we use $x, y, z$ and $w$, rather than $i$ and $j$, for vertices of $K \cdot K_\eta$, since we want several letters from the same part of the alphabet. We use $u$ and $v$ for vertices of $K$. 
1. Pick $x \in B_v'$ such that $w(x) = \max_{y \in B_v'} w(y)$.

2. Set $S_v = \{ y \in B_v' \mid xy \in \tilde{F} \}$ and $T_v = B_v' \setminus S_v$.

3. For each $y \in T_v \setminus \{ x \}$, replace $N_{\tilde{F}}(y)$ by $N_{\tilde{F}}(x)$.

4. Pick $z \in S_v$ such that $w(z) = \max_{w \in S_v} w(w)$.

5. For each $w \in S_v \setminus \{ z \}$, replace $N_{\tilde{F}}(w)$ by $N_{\tilde{F}}(z)$.

Let $\tilde{F}' \subseteq K \cdot K_\eta$ be the graph obtained from $\tilde{F}$ after performing these steps for each $v \in V(K)$. We make the following observations about $\tilde{F}'$:

(i) $w(\tilde{F}') \geq w(\tilde{F})$;

(ii) $\tilde{F}'$ is triangle-free, since $\tilde{F}$ is—note in particular that $S_v \subseteq N_{\tilde{F}}(x)$ implies $\tilde{F}[S_v] = \emptyset$;

(iii) For each $v \in V(K)$, $\tilde{F}'[B_v']$ is the complete bipartite graph between $S_v$ and $T_v$; and

(iv) For each $v \in V(K)$, $z, w \in S_v$, and $x, y \in T_v$, we have $N_{\tilde{F}'}(z) = N_{\tilde{F}'}(w)$ and $N_{\tilde{F}'}(x) = N_{\tilde{F}'}(y)$.

The only tricky point here is (iv). Clearly for a given $u \in V(K)$, the condition in (iv) holds at $u$ immediately after we perform steps 1–5 at $u$. But how do we know we don’t violate the condition at $u$ in the process of doing 1–5 at some other $v \in V(K)$ coming later in $\pi$? Assume we do, so that there exist $x, y \in R_u \subseteq \{ S_u, T_u \}$ and $z \in B_v'$ such that $xz \in \tilde{F}'$ and $yz \notin \tilde{F}'$. Just before we began 1–5 at $v$, $z$ was $\tilde{F}$-adjacent to either both of $x, y$ or neither, so we must have replaced $N_{\tilde{F}}(z)$ in the course of doing 1–5 at $v$. So there was some $w \in B_v'$ (whose $\tilde{F}$-neighborhood replaced that of $z$) which, just before beginning 1–5 at $v$, was $\tilde{F}$-adjacent to exactly one of $x, y$. But this is a contradiction.

For each $v \in V(K)$, let $R_v$ be the larger of $S_v, T_v$, and $P_v$ the smaller (choose arbitrarily if they are the same size). Let $\tilde{F}$ be the subgraph of $K^+$ obtained from $\tilde{F}'$ by collapsing each $R_v$ to a vertex $v^b$ and each $P_v$ to a vertex $v^a$, and set $\phi(v^b) = |R_v|/\eta$ and $\phi(v^a) = |P_v|/\eta = 1 - \phi(v^b)$ for each $v \in V(K)$. Then (ii)–(iv) imply that $(\tilde{F}, \phi)$
is a configuration on $K$, after adding vertex edges $v^b v^s$ for those $v \in V(K)$ for which $P_v = \emptyset$ (if any).

Now since $H$ is $c$-fair, we have $w_c(\hat{F}, \phi) \leq 1/2$. By the weight calculations after (3.9), we have $w_c(\hat{F}, \phi) \geq w(\tilde{F}')$ (the only error here comes from the weight in a block of $K \cdot K_\eta$ being \( \left(\frac{n}{\eta}\right) T \eta^2 \) instead of exactly \( \frac{\eta^2}{2} \)). Thus by (3.11) and (i), using $\eta l \leq n/(2t)$ and $\epsilon < \delta/(48tc)$, we have

\[
|F| \leq \frac{w(\tilde{F}')} {c/(t\eta^2 l^2)} + 3\epsilon n^2 + o(m) \leq \frac{w(\tilde{F}')}{c/(t\eta^2 l^2)} + 3\epsilon n^2 + o(m) \\
\leq \frac{1/2}{c/(t\eta^2 l^2)} + 3\epsilon n^2 + o(m) \\
< n^2(8tc)^{-1} + n^2 \delta (16tc)^{-1} + o(m) \\
< (1 + \delta/2 + o(1))m/2 \\
< (1 + \delta)m/2,
\]

where the penultimate inequality recalls (3.8) and the last holds for large enough $a$. □

### 3.4 Proof of Lemma 3.10

We now turn to the proof of Lemma 3.10, that for any $c > 0$ there are triangle-free, $d$-regular, $c$-fair graphs $H$ with arbitrarily large $d$. Luckily we need not invent anything here; rather we show—though not so easily—that for any fixed $c$, all sufficiently large graphs from a well-known family are $c$-fair. The relevant family was described by Noga Alon in [1]; since he proved therein that all graphs in this family are triangle-free and regular, with degree going to infinity, this will prove Lemma 3.10. We first list the relevant properties of these graphs.
Proposition 3.11 ([1, Theorem 2.1]). For all \( t_0 \in \mathbb{N} \), there exist \( t \geq t_0 \) and a triangle-
free graph \( H_t \) on \( t \) vertices satisfying

\begin{align*}
\bullet \text{ } H_t & \text{ is } d\text{-regular, with } d = \Theta(t^{2/3}), \text{ and} \\
\bullet \text{ all eigenvalues } \lambda_i \text{ of } H_t, \text{ other than the largest, satisfy } |\lambda_i| = O(\sqrt{d}) = O(t^{1/3}).
\end{align*}

(3.12) (3.13)

Alon gives much more detailed information about these graphs, including a precise
formula for \( d \) and bounds on the eigenvalues, but the above properties are all we will
need. In fact, a weaker eigenvalue bound than (3.13) would suffice for our purposes. (We
need such a bound primarily to guarantee good density properties for \( H \), for which our
(standard) tool is Lemma 2.16). It is probably not too hard—e.g. by random methods,
somewhat relaxing the regularity requirement of Lemma 3.10—to produce other families
of graphs, less nice than Alon’s, that would be adequate here. Recognizing this, we
nonetheless gladly use Alon’s graphs because they are convenient and they work.

Setup for the rest of this chapter. We fix \( c \in (0, 1] \) at the outset, and throughout
we let \((F, \phi)\) be a configuration on \( K = K_{H,H} \), where \( H = H_t \) for some \( t \). We denote the
degree of \( H \) by \( d \) and its eigenvalues by \( d = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_t \), and set \( \lambda = \max_{i>1} |\lambda_i| \).

Goal: To show that \( H \) is \( c \)-fair whenever \( t \) is sufficiently large. \hfill (3.14)

Each proposition in what follows is an asymptotic statement, making some claim about
\( H \) or \((F, \phi)\) as \( t \) grows to infinity; thus our asymptotic notation all refers to \( t \to \infty \).
Our usage here may be a little confusing, since we treat \( t \) as tending to infinity, whereas
the discussion in Section 3.3 calls for a fixed \( H = H_t \) depending on \( c \) (that is, on \( \alpha \)).
But of course what we are showing here is that given \( c \), \( H_t \) is \( c \)-fair for large enough \( t \),
so that for our application in Section 3.3 we can fix such a \( t \). We always assume (as we
may) that \( w_c(F, \phi) \geq 1/2 \); we want to show that in fact \( w_c(F, \phi) = 1/2 \).

Though a configuration on \( K \) is defined via \( K^+ \), it will be more convenient in what
follows to think of it in terms of \( K \) itself. We next set up some notation and terminology
for this purpose.

**Definitions 3.12** (edge classes, weight captured, gain/loss). Given a graph \( H = H_t \) and a configuration \((F, \phi)\) on \( K = K_{H,H} \), we divide the edges of \( K \) into four classes.

An edge \( uw \in K \) is of

- **class 1** if \( u^bw^b, u^aw^a \in F \),
- **class 2** if \( u^bw^b \in F, u^aw^a \notin F \),
- **class 3** if \( u^bw^a, u^aw^b \in F \), and
- **class 4** otherwise.

For each \( uw \in K \), we will say our configuration *captures* the fraction \( \sum_{uxwy \in F} \phi(u^x)\phi(w^y) \) of the weight of the edge. This weight is \( \frac{1-c}{2t} \) for internal edges and \( \frac{1-c}{2t^2} \) for external edges. Similarly, we say our configuration *captures* the fraction \( 2\phi(v^b)\phi(v^a) \) of the weight of each vertex \( v \) of \( K \). This weight is \( \frac{c}{2t} \). For \( v \in V(K) \), set \( \delta_v = \phi(v^b) - 1/2 \), so that \( \delta_v \) measures how far from evenly the configuration splits the mass of \( v \). Then e.g. if \( uw \in K \) is of class 1, our configuration captures the fraction \( (1/2 + \delta_u)(1/2 + \delta_w) + (1/2 - \delta_u)(1/2 - \delta_w) = 1/2 + 2\delta_u\delta_w \) of the weight of \( uw \), and if \( uw \) is of class 3 then it captures the fraction \( 1/2 - 2\delta_u\delta_w \). Similarly, it captures the fraction \( 1/2 - 2\delta_v^2 \) of the weight of each vertex \( v \).

Given \( uw \in K \), we sometimes want to compare the fraction of the weight of \( uw \) captured by our configuration to the fraction of the weight of \( uw \) captured by the naïve configuration, namely 1/2. We call this difference \( \sum_{uxwy \in F} \phi(u^x)\phi(w^y) - 1/2 \in [-1/2,1/2] \) the *fractional gain at \( uw \)*, and its negative the *fractional loss at \( uw \).* (Either of these can be positive or negative.) More often we want to weight the fractional gain (loss) at an edge by the appropriate edge weight \( (\frac{1-c}{2t}) \) or \( \frac{1-c}{2t^2} \); we call this product simply the *gain (loss) at the edge* (no “fractional”). (Examples: if the fractional gain at internal edge \( uw \) is .16, then the gain at \( uw \) is \( .16(\frac{1-c}{2t}) \); if \( vz \) is an external edge of class 3, then the loss at \( vz \) is \( 2\delta_v\delta_z(\frac{1-c}{2t^2}) \).) We use analogous terminology for vertices: the *fractional loss at \( v \) is \( 2\delta_v^2 \), and the loss at \( v \) is \( c\delta_v^2/t \).
Write $ζ_i$ (respectively $ζ_e$) for the average fraction of the weight of an internal (respectively external) edge captured by our configuration—that is,

$$ζ_i = \frac{1}{td} \sum_{u^xw^y ∈ F_{\text{internal}}} φ(u^x)φ(w^y) \quad \text{and} \quad ζ_e = \frac{1}{t^2} \sum_{u^xw^y ∈ F_{\text{external}}} φ(u^x)φ(w^y)$$

—and set $γ_i = ζ_i - 1/2$, $γ_e = ζ_e - 1/2$. Thus $γ_i$ and $γ_e$ represent the average fractional gain of our configuration on internal and external edges of $K$, respectively. Lastly, write $δ$ for the average of the $δ_v$’s over $V(K)$.

With these definitions, notice that $(1 - c^2) (γ_i + γ_e)$ is the total gain over all edges of $K$. So, to reiterate (3.14), our goal is to show that this is always counterbalanced by an equal or larger loss in the vertices of $K$ whenever $t$ is sufficiently large. What follows is a long string of propositions culminating in a proof of this.

**Proposition 3.13.** Let $R$ be an ETF subgraph of $K$ containing fractions $ξ_i(R)$ and $ξ_e(R)$ of the internal and external edges of $K$, respectively. Then

$$ξ_i(R) + ξ_e(R) < 1 + o(1).$$

**Proof.** We apply Lemma 2.21 with $r = b = 2$, $c$ arbitrarily small but fixed, $m = 2[ε^{-1}]$, $H_1 = R[X] ∪ R[Y]$, $H_2 = ∇_R(X,Y)$, $s_1 = d/t$, and $s_2 = 1$.

We must first check that (for large enough $t$) $H_1$ is $(d/t; 2, β)$-bounded and $H_2$ is $(1; 2, β)$-bounded, where $β = β(ε, b, m, r) > 0$ is given by the lemma (but of course the statement is really that these hold for any fixed $β$ and, again, sufficiently large $t$). The second of these is trivial. For the first, letting $U, W ⊆ V(K)$ be disjoint with
If $|U|, |W| \geq 2t\beta$, we have, using Lemma 2.16,

$$d_{d/t,H_1}(U, W) = \frac{\nabla_{H_1}(U, W)}{(d/t)|U||W|} = \frac{\nabla_H(U \cap X, W \cap X)}{(d/t)|U||W|} + \frac{\nabla_H(U \cap Y, W \cap Y)}{(d/t)|U||W|}$$

$$\leq \frac{|U \cap X||W \cap X|d/t + \lambda \sqrt{|U \cap X||W \cap X|}}{(d/t)|U||W|} + \frac{|U \cap Y||W \cap Y|d/t + \lambda \sqrt{|U \cap Y||W \cap Y|}}{(d/t)|U||W|}$$

$$\leq \frac{|U||W|d/t + \lambda \sqrt{|U||W|}}{(d/t)|U||W|} \leq 1 + o(1),$$

which is at most 2 for large enough $t$.

Let $\Pi = (V_0, V_1, \ldots, V_k)$ be the partition given by Lemma 2.21. By comment (i) following the lemma we may assume each nonexceptional part of $\Pi$ is contained in either $X$ or $Y$, and by comment (ii) we may assume $|V_0 \cap X| = |V_0 \cap Y|$, implying that $X$ and $Y$ each contain exactly $k/2$ parts of $\Pi$. Given a pair of nonexceptional parts of $\Pi$, we say the pair is external if exactly one of them is contained in $X$, and internal otherwise.

We now delete the following edges from $R$: edges incident to $V_0$; edges inside some $V_i$, $i \in [k]$; edges that join (internal) pairs that are not $(d/t; H_1, \epsilon)$-regular; edges that join (external) pairs that are not $(1; H_2, \epsilon)$-regular; edges that join internal pairs with $(d/t; H_1)$-density less than $2\epsilon$; and edges that join external pairs with $(1; H_2)$-density less than $2\epsilon$. The following table lists upper bounds for the numbers of edges deleted from $H_1$ and $H_2$ in each of these categories. For convenience we set $l := |V_1| \leq 2t/k$.

<table>
<thead>
<tr>
<th>Category</th>
<th>$H_1 = R(X) \cup R(Y)$</th>
<th>$H_2 = \nabla_R(X, Y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Edges incident to $V_0$</td>
<td>$\leq d</td>
<td>V_0</td>
</tr>
<tr>
<td>Edges inside some $V_i$</td>
<td>$\leq k \left(\frac{\epsilon^2 d}{2\epsilon} + \frac{k}{2}\right) \leq \epsilon d + \epsilon t k \leq \epsilon t d (1 + o(1))$</td>
<td>$0$</td>
</tr>
<tr>
<td>Edges joining pairs that are not $(d/t; H_1, \epsilon)$-regular</td>
<td>$\leq \epsilon \left(\frac{k}{2}\right) \left(\frac{\epsilon^2 d}{t} + k\lambda l\right) \leq \epsilon \left(2 \epsilon d + \lambda t k\right) \leq \epsilon t d (2 + o(1))$</td>
<td>$0$</td>
</tr>
<tr>
<td>Edges joining pairs that are not $(1; H_2, \epsilon)$-regular</td>
<td>$0$</td>
<td>$\leq \epsilon \left(\frac{k}{2}\right) l^2 \leq 2\epsilon t^2$</td>
</tr>
<tr>
<td>Edges joining internal pairs with $(d/t; H_1)$-density less than $2\epsilon$</td>
<td>$\leq 2 \left(\frac{\epsilon^2 d^2}{2\epsilon} \right)(2\epsilon t d/t) \leq 2\epsilon t d$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
edges joining external pairs with \((1; H_2)\)-density less than \(2\epsilon\) & 0 & \(\leq (k/2)^22\epsilon t^2\) & \(\leq 2\epsilon t^2\) \\
TOTAL & \(\leq (7 + o(1))\epsilon td\) & \(\leq 6\epsilon t^2\) \\

Let \(\tilde{X} = \{i \in [k] \mid V_i \subseteq X\}\) and \(\tilde{Y} = [k] \setminus \tilde{X}\). Let \(\tilde{R}\) be the graph on \(\tilde{X} \cup \tilde{Y}\) where \(ij \in \tilde{R}\) iff there is an undeleted edge joining \(V_i\) and \(V_j\) in \(R\). Then since \(R\) is ETF, Lemma 2.22 gives that \(\tilde{R}\) is as well (meaning, as usual, that it contains no triangles meeting both \(\tilde{X}\) and \(\tilde{Y}\)).

Now each internal edge of \(\tilde{R}\) corresponds to a pair in \(\Pi\) whose \(R\)-edges contribute a total of at most
\[
\frac{l^2d/t + \lambda l}{td} \leq \frac{4k^2}{k^2} + \frac{2\lambda}{kd} = \frac{4k^2}{k^2} + o(1)
\]
to the fraction \(\xi_i(R)\). Similarly each external edge of \(\tilde{R}\) corresponds to a pair in \(\Pi\) whose \(R\)-edges contribute a total of at most \(l^2/t^2 \leq 4/k^2\) to the fraction \(\xi_e(R)\). By Lemma 2.18 \(|\tilde{R}| \leq k^2/4\), so the contribution to \(\xi_i(R) + \xi_e(R)\) from undeleted \(R\)-edges is at most \(1 + k^2o(1) = 1 + o(1)\). And as computed in the table above, the contribution to \(\xi_i(R) + \xi_e(R)\) from deleted \(R\)-edges is at most \(13\epsilon + o(1)\). Thus \(\xi_i(R) + \xi_e(R) \leq 1 + 13\epsilon + o(1)\). Since \(\epsilon\) was arbitrarily small, the proposition is proved.

We now return to our configuration \((F, \phi)\).

**Proposition 3.14.** We have \(\zeta_i + \zeta_e < 1 + o(1)\), or equivalently,
\[
\gamma_i + \gamma_e < o(1). 
\] (3.16)

**Proof.** Suppose that for each \(v \in V(K)\) we randomly choose one of \(v^b, v^a\), with \(\Pr(v^x) = \phi(v^x)\) and these choices made independently. This produces a random ETF subgraph \(R\) of \(K\) in the obvious way: \(uw \in R\) iff \(u^xw^y \in F\), where we chose \(u^x \in \{u^b, u^a\}\) and \(w^y \in \{w^b, w^a\}\). Observe that \(\Pr(uw \in R)\) is the fraction of the weight of \(uw\) captured
by our configuration. With this observation, we calculate
\[
\zeta_i + \zeta_e = \frac{1}{td} \sum_{uw \in K_{\text{internal}}} \Pr(uw \in R) + \frac{1}{t^2} \sum_{uw \in K_{\text{external}}} \Pr(uw \in R)
\]
\[
= \mathbb{E}[|R \cap (K[X] \cup K[Y])|/td] + \mathbb{E}[|R \cap \nabla(X,Y)|/t^2]
\]
\[
= \mathbb{E}[(\xi_i(R) + \xi_e(R)]
\]
\[
< 1 + o(1),
\]
where the last inequality is given by Proposition 3.13.

**Proposition 3.15.** We have \( \delta = o(1) \).

**Proof.** We simply calculate \( w_e(F,\phi) \) (which, recall, we assume is at least \( 1/2 \)):
\[
w_e(F,\phi) = \frac{1-c}{2} \zeta_i + \frac{1-c}{2} \zeta_e + \frac{c}{2t} \sum_{v \in V(K)} (1/2 - 2\delta_v^2)
\]
\[
= 1/2 + \left( \frac{1-c}{2t} \right) (\gamma_i + \gamma_e) - \frac{c}{t} \sum_{v \in V(K)} \delta_v^2
\]
\[
\leq 1/2 + o(1) - 2c \left( \frac{1}{2t} \sum_{v \in V(K)} \delta_v^2 \right)^2
\]
\[
= 1/2 - 2c\delta^2 + o(1),
\]
where we used Proposition 3.14 and Cauchy-Schwarz between the second and third lines.

From now on we call a vertex \( v \) of \( K \) *balanced* if \( \delta_v < \sqrt{\delta} \), and *unbalanced* otherwise; thus, in view of Proposition 3.15, all but a \( o(1) \)-fraction of the vertices of \( K \) are balanced. Also, we let \( G \) be the subgraph of \( K \) consisting of all edges of classes 1–3, and \( \Gamma \) the subgraph of \( G \) consisting of edges of classes 1 and 2. Notice that since \( F \) is ETF,

\[ \Gamma \text{ has even intersection with every external triangle in } G. \] (3.17)

The next three facts say that in various senses, as \( t \) grows, \( G \) accounts for nearly all of
Proposition 3.16. The total loss on $K \setminus G$ is $o(1)$.

Proof. The total gain on $G$ is at most what it would be if all edges of $K$ were of class 1. Since at most $o(t)$ vertices are unbalanced, the total weight of all edges of $K$ incident to unbalanced vertices is $o(1)$, so this gain is at most

$$(1 - c)2\sqrt{\delta} + o(1)(1 - c)2(1/2)^2,$$

which is $o(1)$ by Proposition 3.15. Thus if the loss on $K \setminus G$ were $\Omega(1)$, we would have $\nu_c(F, \phi) < 1/2$ for sufficiently large $t$ (since loss on vertices is always nonnegative).

Corollary 3.17. There are at most $o(t^2)$ class 4 edges in $K$.

Proof. Assume otherwise, so that $|K \setminus G| = \Omega(t^2)$. Then since at most a $o(1)$-fraction of the edges of $K$ are incident to unbalanced vertices, most class 4 edges join two balanced vertices. The fractional loss at any such edge is $\Omega(1)$ (at least about $1/4$, in view of Proposition 3.15), so the total loss on $K \setminus G$ is $\Omega(1)$, contradicting Proposition 3.16.

Corollary 3.18. There are at most $o(td)$ class 4 edges in each of $K[X], K[Y]$.

Proof. Assume for a contradiction that $|(K \setminus G)[X]| = \Omega(td)$ (the proof for $Y$ is of course the same). Then since at most $o(td)$ edges of $K[X]$ are incident to unbalanced vertices, most class 4 edges in $K[X]$ join two balanced vertices. The fractional loss at any such edge is $\Omega(1)$ (at least about $1/4$, in view of Proposition 3.15), so the total loss on $K \setminus G$ is $\Omega(1)$, contradicting Proposition 3.16.

The next result concerns only $H$, not $K$ or $(F, \phi)$.

Proposition 3.19. For any $H' \subseteq H$ of size $(1 - o(1))|H|$, there is a $U \subseteq V(H)$ of size $o(t)$ such that $H' - U$ is connected and $\mathcal{C}(H' - U)$ is spanned by cycles of length up to 11.
Proof. By Proposition 2.17 (and noting that finite diameter implies connectedness), it suffices to find a $U$ of size $o(t)$ such that $H' - U$ has diameter at most 5. To this end, let $U_1 = \{v \in V(H) : d_{H \setminus H'}(v) \geq d/3\}$. Then $u_1 := |U_1| \leq 2|H \setminus H'|/(d/3) = o(t)$. Let $U_2 = \{v \in V(H) \setminus U_1 : |N(v) \cap U_1| \geq d/3\}$. We claim $u_2 := |U_2| = o(t^{1/3})$ (we just need $o(d)$). Indeed, applying Lemma 2.16 to $H$, we have $(1/3 - o(1))u_2d \leq |\nabla(U_1, U_2)| - \frac{u_1u_2d}{t} \leq \lambda \sqrt{u_1u_2}$, which (since $d = \Theta(t^{2/3})$ and $\lambda = O(t^{1/3})$) gives $u_2 \leq O(t^{-2/3}u_1) = o(t^{1/3})$, as claimed.

Set $U = U_1 \cup U_2$ and $H'' = H' - U$, and for each $v \in V(H'')$ denote by $N_2(v)$ the second neighborhood of $v$ in $H''$; that is, the set of vertices at distance exactly 2 from $v$ in $H''$. We want to show that $H''$ has diameter at most 5. For this it suffices to show that every $v$ satisfies $d_2(v) := |N_2(v)| = \Omega(t)$, since for any $S, T \subseteq V(H'')$ with $|S|, |T| = \Omega(t)$ we have $\nabla_{H''}(S, T) \neq \emptyset$ (using Lemma 2.16 on $H$ and the fact that $|H \setminus H'| = o(|H|)$).

To see that (for any $v$) $d_2(v) = \Omega(t)$, note first that $d_{H''}(v) \geq (1/3 - o(1))d = \Omega(d)$, since $v$ loses at most a third of its $H$-neighbors to $H \setminus H'$, at most another third to $U_1$, and a $o(1)$-fraction to $U_2$. Thus, since $H$ is triangle-free, $|\nabla_H(N_{H''}(v), N_2(v))| = \Omega(d^2) = \Omega(t^{4/3})$. On the other hand Lemma 2.16 gives $|\nabla_H(N_{H''}(v), N_2(v))| \leq d_{H''}(v)d_2(v)d/t + \lambda \sqrt{d_{H''}(v)d_2(v)} = O(t^{1/3})d_2(v) + O(t^{2/3})\sqrt{d_2(v)}$, implying $d_2(v) = \Omega(t)$ as claimed. \qed

Corollary 3.20. Any $H' \subseteq H$ of size $(1 - o(1))|H|$ has a component with $t - o(t)$ vertices.

(This is strictly weaker than Proposition 3.19; we include it for easy reference later.)

We now return to $K$ and our configuration $(F, \phi)$. The next result does most of the heavy lifting for Lemma 3.10.

Proposition 3.21. There exist $S \subseteq V(K)$ of size $o(t)$ and a partition $A \sqcup B$ of $V(K) \setminus S$ such that $Z := \Gamma \Delta \nabla_G(A, B)$ satisfies

$$Z \subseteq \nabla(X, Y)$$
and
\[ d_Z(v) = o(t) \forall v \in V(K) \setminus S. \]

Proof. Let \( \kappa = \frac{|(K \setminus G) \cap \nabla(X,Y)|}{t^2} \), which is \( o(1) \) by Corollary 3.17. Let \( S_0 = \{ v \in V(K) \mid v \text{ is incident to at least } t\sqrt{\kappa} \text{ external class 4 edges} \} \). Then \(|S_0|t\sqrt{\kappa} \leq 2\kappa t^2\), implying \(|S_0| = O(t\sqrt{\kappa}) = o(t)\). Now apply Proposition 3.19 to each of \( G[X \setminus S_0] \) and \( G[Y \setminus S_0] \), which we may do by Corollary 3.18. Let \( S_1 \) be the union of \( S_0 \) and the two deleted sets from Proposition 3.19, and set \( \bar{G} = G - S_1 \), \( \bar{X} = X \setminus S_1 \) and \( \bar{Y} = Y \setminus S_1 \).

Let \( \mathcal{T}(\bar{G}) \) be the subspace of \( \mathcal{C}(\bar{G}) \) generated by the external triangles of \( \bar{G} \). Then we observe, crucially:

all cycles of \( G[\bar{X}] \) and \( G[\bar{Y}] \) of length up to 11 belong to \( \mathcal{T}(\bar{G}) \). \hspace{1cm} (3.18)

To see this, let \( C = x_1, \ldots, x_k x_1 \) be a cycle, say in \( G[\bar{X}] \), with \( k \leq 11 \). If there exists \( y \in \bar{Y} \) with \( x_i y \in G \ \forall i \in [k] \), then \( C \in \mathcal{T}(\bar{G}) \), because \( C \) is the sum of the triangles \( x_i x_{i+1} y x_i \), where of course we take subscripts mod \( k \). But if there is no such \( y \) then for some \( x_i \) we have

\[ |\nabla_{K \setminus G}(x_i, \bar{Y})| \geq |\bar{Y}|/11, \]

implying \( x_i \in S_0 \), which it isn’t.

Now by (3.18) and our choice of \( S_1 \), we have

\[ \Gamma[\bar{X}] = \nabla_G(\bar{X}_1, \bar{X}_2) \quad \text{and} \quad \Gamma[\bar{Y}] = \nabla_G(\bar{Y}_1, \bar{Y}_2) \]

for some partitions \( \bar{X}_1 \sqcup \bar{X}_2 \) of \( \bar{X} \) and \( \bar{Y}_1 \sqcup \bar{Y}_2 \) of \( \bar{Y} \), since \( \Gamma \) is orthogonal (over \( \mathbb{F}_2 \), recall) to all external triangles in \( \bar{G} \) (see (3.17)), and thus to all cycles in \( G[\bar{X}] \) and \( G[\bar{Y}] \) of length up to 11 (by (3.18)), and thus to all cycles in \( G[\bar{X}] \) and \( G[\bar{Y}] \) (see Proposition 3.19).

By Corollary 3.20 we can find a \( U \subseteq \bar{X} \cup \bar{Y} \) of size \( o(t) \) such that \( G[\bar{X} \setminus U] \) and \( G[\bar{Y} \setminus U] \) are connected. Set \( S = S_1 \cup U \), producing the \( S \) of the proposition. Finally, set \( X'_1 = \bar{X}_1 \setminus U \), \( X'_2 = \bar{X}_2 \setminus U \) and \( X' = X'_1 \cup X'_2 \) (= \( \bar{X} \setminus U \)), and define \( Y'_1 \), \( Y'_2 \) and \( Y' \).
similarly.

Now suppose \( x \in X' \). Since all but a \( o(1) \)-fraction of the external edges at \( x \) belong to \( \nabla_G(x, Y') \), the subgraph of \( G \) induced by the corresponding vertices (that is, \( G[N_G(x) \cap Y'] \)) has a component of size \( t - o(t) \) (Corollary 3.20 again), say with vertex set \( Y_1^x \cup Y_2^x \), where \( Y_1^x \subseteq Y_1' \) and \( Y_2^x \subseteq Y_2' \). Since \( \Gamma[Y_1^x \cup Y_2^x] = \nabla_G(Y_1^x, Y_2^x) \), (3.17) gives

\[
yz \in \nabla_G(Y_1^x, Y_2^x) \implies |\Gamma \cap \{xy, xz\}| = 1,
\]
\[
yz \in G[Y_1^x] \cup G[Y_2^x] \implies |\Gamma \cap \{xy, xz\}| \in \{0, 2\}.
\]

Thus the connectivity of \( G[Y_1^x \cup Y_2^x] \) implies that

\[
\nabla_G(x, Y_1^x \cup Y_2^x) \in \{\nabla_G(x, Y_1^x), \nabla_G(x, Y_2^x)\}.
\]

Moreover, the connectivity of \( G[X'] \) and the fact that any \( u, w \in X' \) have common \( G \)-neighbors in \( (Y_1^u \cup Y_2^u) \cap (Y_1^w \cup Y_2^w) \) (in fact many, since \( u, w \notin S_0 \)) imply “coherence” of the choices in (3.19), meaning that \( u \) and \( w \) choose the same option iff they are on the same side of \( X_1' \cup X_2' \). Of course a similar analysis applies with the roles of \( X \) and \( Y \) reversed. Assuming without loss of generality that each \( x \in X_1 \) chooses \( \nabla_G(x, Y_1^x \cup Y_2^x) = \nabla_G(x, Y_2^x) \) in (3.19), the proposition is proved, with \( A = X_1' \cup Y_1' \) and \( B = X_2' \cup Y_2' \).

At long last we can accomplish the goal set forth in (3.14).

**Proof of Lemma 3.10.** Let \( S, A, B \subseteq V(K) \) and \( Z \subseteq \nabla(X, Y) \) be as in Proposition 3.21, and set \( W = V(K) \setminus S = A \cup B \). We analyze \( K[W] \) first, and edges meeting \( S \) later.

Set \( p = \frac{1-c}{2td} \) and \( q = \frac{1-c}{2tz} \). Let \( \varphi \) be the vector indexed by \( X \cup Y \) with

\[
\varphi_v = \begin{cases} 
\delta_v & \text{if } v \in A \\
-\delta_v & \text{if } v \in B \\
0 & \text{if } v \in S
\end{cases}
\]
Let \( C \) be the adjacency matrix of \( H \), \( J \) the \( t \times t \) matrix of 1’s, and \( I \) the \( 2t \times 2t \) identity matrix. Lastly, let \( N \) be the weighted adjacency matrix of \( K \), and \( T \) the adjacency matrix of \( Z \). These matrices look like this:

\[
N = \begin{pmatrix}
X & Y \\
\text{pC} & \text{qJ} \\
\text{qJ} & \text{pC}
\end{pmatrix},
\quad
T = \begin{pmatrix}
X & Y \\
0 & \text{o}(t) \text{ 1’s per row} \\
\text{o}(t) \text{ 1’s per row} & 0
\end{pmatrix}.
\]

On \( K[W] \), the weight our configuration captures is at most what it would be if all class 2 edges, as well as all class 4 edges in \( \nabla(A, B) \), were instead class 1, and all class 4 edges in \( K[A] \cup K[B] \) were instead class 3. In this case, our configuration’s overall loss on \( K[W] \) (edges and vertices) would be exactly

\[
\varphi^\top (N - 2qT + (c/t)I)\varphi.
\]  

(3.20)

To show that our configuration captures at most half the weight of \( K[W] \) it would suffice to show (3.20) to be nonnegative, but let’s instead show the stronger

\[
\varphi^\top M \varphi \geq 0,
\]  

(3.21)

where \( M = N - 2qT + (.66c/t)I \). Thus we’re showing that the gain on edges of \( K[W] \) is at most \((.66c/t) \sum_{v \in W} \delta_v^2 \), reserving the remaining vertex loss in \( W \), \((.34c/t) \sum_{v \in W} \delta_v^2 \), for use below in handling edges meeting \( S \). For (3.21), we simply show \( M \) is positive definite. We first treat the \( N \) term and then the \( T \) term, helping ourselves to a little bit of the \( I \) term in each of these steps. As will be clear below, and as is perhaps hinted by the constants .66 and .34, nothing in this argument is very delicate.

Let \( P \) and \( Q \) be the “\( pC \)” and “\( qJ \)” portions of \( N \), respectively. Since \( P \) and \( Q \) are symmetric and commute, they admit a common orthonormal basis of eigenvectors.

We seek to describe these eigenvectors and their corresponding eigenvalues in terms of
the eigenvectors and eigenvalues of $C$, so let $w_1 = t^{-1/2}1, w_2, \ldots, w_t$ be an orthonormal eigenbasis for $C$ with corresponding eigenvalues $d = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_t$. Then a common orthonormal eigenbasis for $P$ and $Q$ is

$$v_1 = 2^{-1/2}(w_1, w_1), \quad v_2 = 2^{-1/2}(w_1, -w_1), \quad \ldots,$$

$$v_{2t-1} = 2^{-1/2}(w_t, w_t), \quad v_{2t} = 2^{-1/2}(w_t, -w_t),$$

where $(x, y)$ is the concatenation of $x$ and $y$. These eigenvectors have corresponding eigenvalues $pd, pd, p\lambda_2, p\lambda_2, \ldots, p\lambda_t, p\lambda_t$ for $P$ and $qt, -qt, 0, 0, \ldots, 0$ for $Q$, and therefore $pd + qt = \frac{1-c}{t}, \quad pd - qt = 0, \quad p\lambda_2, p\lambda_2, \ldots, p\lambda_t, p\lambda_t$ for $N$. Call these $N$-eigenvalues $\mu_1, \ldots, \mu_{2t}$ (for use below). Now since $|\lambda_t| \leq O(t^{1/3})$ (see (3.13)), all eigenvalues of $N$ are at least $-O(t^{-4/3}) = -o(t^{-1})$. Thus (e.g.) $N + (.33c/t)I$ is (eventually) positive definite.

We now turn to the $T$ term in $M$, which is easier. As every absolute row sum of $T$ is $o(t)$, so is every eigenvalue of $T$. Thus every eigenvalue of $-2qT$ is at least $-o(t^{-1})$, so (e.g.) $-2qT + (.33c/t)I$ is (eventually) positive definite. Therefore $M$ is positive definite, as claimed.

Finally we deal with contributions involving $S$. For this let $\delta = \langle \delta_v | v \in V(K) \rangle$, $\delta' = 1_W \circ \delta$ (where $\circ$ denotes componentwise product), $\alpha_i = \delta \cdot v_i$ and $\alpha'_i = \delta' \cdot v_i$, $i \in [2t]$ (where $\cdot$ denotes the usual inner product). The total gain from edges meeting $S$ is at most what it would be if all these edges were class 1, which is exactly

$$\delta^t N \delta - (\delta')^t N \delta' = \sum_{i=1}^{2t} \mu_i (\alpha_i^2 - (\alpha'_i)^2) \quad \begin{array}{l}
= \mu_1 (\alpha_1^2 - (\alpha'_1)^2) + \\
+ \sum_{i=2}^{2t} \mu_i (\alpha_i^2 - (\alpha'_i)^2).
\end{array} \quad (3.22)$$

In view of what we know about the $\mu_i$'s, the sum in (3.22) is at most

$$p\lambda_2 \sum_{v \in V(K)} \delta_v^2 - (\min_i \mu_i) \sum_{v \in W} \delta_v^2 \leq O(t^{-4/3}) \left[ \sum_{v \in V(K)} \delta_v^2 + \sum_{v \in W} \delta_v^2 \right], \quad (3.23)$$
while, with \( \varepsilon \) defined by \( \alpha'_1 = (1 - \varepsilon)\alpha_1 \), the first term in (3.22) is

\[
\mu_1(2\varepsilon - \varepsilon^2)\alpha_1^2 = \frac{1 - c}{t} (2\varepsilon - \varepsilon^2) \frac{1}{2t} \left( \sum_{v \in V(K)} \delta_v \right)^2 < \varepsilon t^{-2} \left( \sum_{v \in V(K)} \delta_v \right)^2 \tag{3.24}
\]

\[
\leq \min \left\{ \varepsilon^{-1} t^{-2} \left( \sum_{v \in S} \delta_v \right)^2, \ 2\varepsilon t^{-1} \sum_{v \in V(K)} \delta_v^2 \right\} \tag{3.25}
\]

(actually (3.24) is equal to the first expression in (3.25)).

On the other hand, we get to subtract from these gains

\[
\frac{c}{t} \sum_{v \in S} \delta_v^2 + \frac{.34c}{t} \sum_{v \in W} \delta_v^2 = \frac{.66c}{t} \sum_{v \in S} \delta_v^2 + \frac{.34c}{t} \sum_{v \in V(K)} \delta_v^2 \geq \frac{.66c}{t|S|} \left( \sum_{v \in S} \delta_v \right)^2 + \frac{.34c}{t} \sum_{v \in V(K)} \delta_v^2. \tag{3.26}
\]

We need to say this is larger than the sum of the right hand sides of (3.23) and (3.25), which is easy. For example, half the second term of (3.26) dominates the right hand side of (3.23), while the right hand side of (3.25) is at most half the second term of (3.26) if \( \varepsilon \leq .17c/2 \) (to be unnecessarily precise), and otherwise, since \( |S| = o(t) \), is dominated by the first term of (3.26).
Chapter 4
The Cycle Space of $G_{n,p}$

4.1 Introduction

An issue of considerable interest in combinatorics over the last few decades has been the extent to which various standard facts, for instance the classic theorems of Turán, Ramsey and Szemerédi, remain true in a “sparse random” setting. Thus, for example, one may ask for which $p = p(n)$ a given (deterministic) assertion regarding the complete graph $K_n$ is likely to hold in the (“Bernoulli”) random graph $G_{n,p}$. The main result of this chapter follows this theme.

Our underlying deterministic statement is Proposition 4.1 below, for which we need a definition: for a fixed graph $H$, the $H$-space of $G$ is the subspace of $E(G)$ generated by the copies of $H$ in $G$; this will be denoted $C_H(G)$, or simply $C_κ(G)$ if $H = C_κ$.

Proposition 4.1. If $κ ≥ 3$ is odd, then for any $n ≥ κ$, $C_κ(K_n) = C(K_n)$.

(Below, in Theorem 4.5, we will characterize $C_H(K_n)$ for any fixed $H$ and large enough $n$.)

When, in terms of $p (= p(n))$, are the $κ$-gons of $G_{n,p}$ likely to span its cycle space? Let $T_κ$ be the class of graphs $G$ satisfying $C_κ(G) = C(G)$ and let $Q_κ$ be the class of nonempty graphs each of whose edges lies in a copy of $C_κ$. For any $G$, it’s easy to see that $G ∉ T_κ$ unless every edge of $G$ that lies in a cycle in fact lies in a $κ$-gon. On the other hand, if $p > (1 + Ω(1)) log n/n$ then w.h.p. every edge of $G_{n,p}$ does lie in a cycle (see [23, p. 105]). So for such $p$, $G_{n,p} ∈ T_κ$ w.h.p. at least requires $G_{n,p} ∈ Q_κ$ w.h.p., and we should first understand when this is true. Let

$$p^*_κ = p^*_κ(n) = \left(\frac{(κ - 1)n^{κ/2}}{κ} \right) \frac{1}{κ-1}$$

(4.1)
(where we always use log for ln). Note $Q_κ$ is not an increasing property—that is, it is not preserved by adding edges. Nonetheless, $p_κ^*$ is a sharp threshold for $Q_κ$, in the sense that:

**Lemma 4.2.** For any fixed $κ ≥ 3$ and $ε > 0$,

$$
\Pr(G_{n,p} ∈ Q_κ) \to \begin{cases} 
0 & \text{if } p < (1 - ε)p_κ^*, \\
1 & \text{if } p > (1 + ε)p_κ^*. 
\end{cases}
$$

(4.2)

(Throughout this chapter limits are taken as $n → ∞$.) We prove this routine observation in Section 4.3. The cases in (4.2) are called the 0-statement and the 1-statement (respectively).

Given Lemma 4.2, one might hope that $p_κ^*$ is also a sharp threshold for $T_κ$, and it essentially is, but for a small glitch in the 0-statement: for $p < (1 - Ω(1))/n$, we have $\lim \Pr(G_{n,p} ∈ T_κ) > 0$ for the silly reason that the probability of having no cycles at all is (asymptotically) positive (see e.g. [31, Theorem 1]). Thus we will show:

**Theorem 4.3.** For any fixed odd $κ ≥ 3$ and $ε > 0$,

$$
\Pr(G_{n,p} ∈ T_κ) \to \begin{cases} 
0 & \text{if } (1 - o(1))/n < p < (1 - ε)p_κ^*, \\
1 & \text{if } p > (1 + ε)p_κ^*. 
\end{cases}
$$

We actually prove the following stronger statement (see Section 4.3 for “stronger”), which says that edges not in $κ$-gons are the obstruction to $T_κ$ in a precise sense. This is the main result of this chapter.

**Theorem 4.4.** For any fixed odd $κ ≥ 3$,

$$
\max_p \Pr(G_{n,p} ∈ Q_κ \setminus T_κ) \to 0; \quad (4.3)
$$

equivalently,

$$
∀ p = p(n), \quad \Pr(G_{n,p} ∈ Q_κ \setminus T_κ) → 0. \quad (4.4)
$$

(The (trivial) equivalence is given by the observation that (4.4) holds iff it holds when,
for each $n$, $p = p(n)$ is a value achieving the maximum in (4.3) (and in this case the two statements are the same).

Theorems 4.3 and 4.4 for $\kappa = 3$ were proved in [12]; even the former had been open and of interest, being the first unsettled case of a conjecture of M. Kahle (see [24, 25]) on the homology of the clique complex of $G_{n,p}$. Perhaps surprisingly, the argument of [12] does not extend to $\kappa \geq 5$, though, as discussed below, it does share a starting point with what we do here.

What happens if we replace the $C_\kappa$ of Proposition 4.1 by some other graph? With $\mathcal{D}(G) = \{ D \in \mathcal{E}(G) : |D| \equiv 0 \pmod{2} \}$, the proposition generalizes neatly:

**Theorem 4.5.** For any graph $H$ with at least one edge and $n$ large enough with respect to $H$,

$$C_H(K_n) = \begin{cases} 
C(K_n) & \text{if } H \text{ is Eulerian and } |H| \text{ is odd}, \\
C(K_n) \cap \mathcal{D}(K_n) & \text{if } H \text{ is Eulerian and } |H| \text{ is even}, \\
\mathcal{E}(K_n) & \text{if } H \text{ is not Eulerian and } |H| \text{ is odd}, \\
\mathcal{D}(K_n) & \text{if } H \text{ is not Eulerian and } |H| \text{ is even}. 
\end{cases} \tag{4.5}$$

Here $|H| = |E(H)|$ and “Eulerian” means degrees are even, but not that the graph is necessarily connected. Of course the left-to-right containments $C_H(K_n) \subseteq C(K_n)$ and so on) are obvious.

The natural value of $C_H(G)$, which we will denote $W_H(G)$, is then what one gets by replacing $K_n$ by $G$ in the appropriate expression on the right hand side of (4.5); e.g. for $H = C_\kappa$,

$$W_H(G) = \begin{cases} 
C(G) & \text{if } \kappa \text{ is odd}, \\
C(G) \cap \mathcal{D}(G) & \text{if } \kappa \text{ is even}. 
\end{cases} \tag{4.6}$$

(We could instead set $W_H(G) = \mathcal{E}(G) \cap C_H(K_n)$, which by Theorem 4.5 is the same for all but a few values of $n$.) So we are interested in understanding when $G_{n,p}$ is likely to lie in

$$\mathcal{T}_H := \{ G : C_H(G) = W_H(G) \}.$$
(Again, $C_H(G) \subseteq W_H(G)$ is trivial for any $H$ and $G$.)

As before, membership in $T_H$ will (in non-silly cases) at least require that the copies of $H$ cover the edges of $G := G_{n,p}$, but when $H$ is non-Eulerian there is a second requirement: each vertex (of $G$) should have odd degree in some copy of $H$ in $G$ (since for any $v \in V(G)$, $W_H(G)$ will contain graphs in which $v$ has odd degree). For example if $H$ is a pair of triangles joined by a slightly long path and $n^{-1+\varepsilon} < p < n^{-2/3}$ for a suitable small $\varepsilon$ depending on the length of the path, then (w.h.p.) all edges of $G$ are in copies of $H$, but most vertices fail to lie in triangles, so have even degree in every copy.

Generalizing $Q_\kappa$, let $Q_H$ be the class of nonempty graphs $G$ satisfying (i) each edge of $G$ is in a copy of $H$, and (ii) if $H$ is not Eulerian, then each vertex of $G$ has odd degree in some copy of $H$; so we have just said that we “essentially” have $T_H \subseteq Q_H$. Though we hesitate to make it a conjecture, we don’t know that the following generalization of Theorem 4.4 is wrong.

**Question 4.6.** Could it be that for each (fixed) $H$,

$$\max_p \Pr(G_{n,p} \in Q_H \setminus T_H) \rightarrow 0? \quad (4.7)$$

Understanding when $G_{n,p} \in Q_H$ w.h.p. is easier, so this would also tell us when $T_H$ is likely to hold. (Note that in general we don’t expect a statement like Theorem 4.3, since the “threshold” for $Q_H$ itself may not be sharp.) Even if (4.7) is not true in general, it seems likely to hold for reasonably nice $H$ (even, say, edge-transitive to start, though this should be much more than is needed). One could also relax (4.7) to an Erdős-Rényi-like threshold statement; e.g. with $p_{Q_H} = \min\{p_0 : \Pr(G_{n,p} \in Q_H) \geq 1/2 \forall p \geq p_0\}$,

$$\text{if } p \gg p_{Q_H} \text{ then } G_{n,p} \in T_H \text{ w.h.p.}$$

**Outline.** The rest of this chapter is organized as follows. Section 4.1.1 digresses briefly to discuss how Theorem 4.4 was inspired in part by the work of Chapter 3. Section 4.2
recalls edge space preliminaries, outlines the main points (Lemmas 4.8–4.10) for the proof of Theorem 4.4, and introduces the (standard) coupling critical to two of their proofs. Section 4.3 proves Lemma 4.2 and gives the easy derivation of Theorem 4.3 from Theorem 4.4. The heart of the chapter is Sections 4.4–4.6, particularly the last of these. They supply the proofs of the lemmas of Section 4.2, in ascending order of difficulty. Section 4.7 gives the easy proof of Theorem 4.5, which we postpone as it is unrelated to the rest of the chapter. Finally (and a bit tangentially), Section 4.8 gives the proof of a theorem (Theorem 4.12) that takes a step towards settling Question 4.6 (see the remark after Lemma 4.10).

4.1.1 Digression: A Connection to Chapter 3

In Chapter 1, we mentioned that we were motivated to study $C(G_{n,p})$ in part by an aspect of our construction of the graphs disproving Yuster’s conjecture. We are now in a position to elaborate.

Recall that our construction started with a fixed, random-looking but triangle-free graph $H$ (see the paragraph after Theorem 3.3). We had this idea early on in the course of developing the construction. Much later we realized that one of the properties we would need from this $H$ is (more or less) that its cycle space be spanned by short cycles (see Proposition 3.19). This discovery, combined with the fact that we had earlier considered taking as our $H$ a de-triangled instance of $G_{n,p}$, ¹ led us to wonder what we could say generally about $C(G_{n,p})$.

Despite our historical link between Theorem 4.4 and the construction for Theorem 3.3, the former theorem is not directly relevant to the construction, for several reasons. For one, as already mentioned, a starting $H$ for the construction must be triangle-free. For two, the property of $H$ asserted in Proposition 3.19 is somewhat peculiar, dealing with the cycle space of subgraphs of $H$ in addition to that of $H$ itself. For three, and most importantly, Theorem 4.4 is overkill in the sense that Proposition 3.19 doesn’t require any graph ($H$ or any of its subgraphs) to be spanned by cycles of a fixed length,

¹Recall we eventually settled on a better $H$—see Proposition 3.11 and the succeeding paragraph.
but only by cycles up to a fixed length. This, as the first two lines of the proposition’s proof indicate, is much easier.

4.2 Main Points for the Proof of Theorem 4.4

Before outlining the proof of Theorem 4.4, we need to review just a little more background.

4.2.1 Edge Space Basics

The edge space $\mathcal{E}(G)$ of a graph $G$ (defined early in Chapter 2), being an $\mathbb{F}_2$-vector space, comes equipped with a standard inner product: $(J, K) = \sum_{e \in E(G)} J(e)K(e) = |J \cap K|$, where the sum and cardinality are interpreted mod 2. (The first expression thinks of $J$ and $K$ as vectors, the second as subgraphs of $G$.) With this, the orthogonal complement, $S^\perp$, of a subspace $S$ of $\mathcal{E}(G)$ is defined as usual. Then $\mathcal{C}(G)$, called the cut space of $G$, consists of the (indicators of) cuts of $G$ (which, note, includes $\emptyset$); $(\mathcal{C}(G) \cap \mathcal{D}(G))^\perp$ consists of cuts and their complements; and $\mathcal{C}_H(G)$ is the set of subgraphs of $G$ having even intersection with every copy of $H$ (in $G$).

As mentioned earlier, $\mathcal{C}_H(G) \subseteq \mathcal{W}_H(G)$ always; dually, $\mathcal{W}_H^\perp(G) \subseteq \mathcal{C}_H^\perp(G)$. In particular, for odd $\kappa \geq 3$,

$$\mathcal{C}^\perp(G) \subseteq \mathcal{C}_\kappa^\perp(G),$$

and equality here is the same as $G \in \mathcal{T}_\kappa$. (4.8)

The next (trivial) observation will be useful at a few points.

**Proposition 4.7.** Let $G$ be a graph and $L \subseteq G$, and suppose $L', L''$ are (respectively) smallest and largest members of the coset $L + \mathcal{C}^\perp(G)$. Then

$$\forall v \in V \quad d_{L'}(v) \leq d_G(v)/2 \leq d_{L''}(v).$$

(For example if $d_{L'}(v) > d_G(v)/2$, then $L' + \nabla(v) (\in L + \mathcal{C}^\perp(G))$ is smaller than $L'$.)

In particular, if $G \notin \mathcal{T}_\kappa$, then since $\mathcal{C}_\kappa^\perp(G) \setminus \mathcal{C}^\perp(G) \supseteq L + \mathcal{C}^\perp(G)$ for any $L \in$
$C^\perp_\kappa(G) \setminus C^\perp(G)$, a smallest element $F$ of $C^\perp_\kappa(G) \setminus C^\perp(G)$ satisfies

$$d_F(v) \leq d_G(v)/2 \quad \forall v \in V. \quad (4.9)$$

### 4.2.2 Structure of the Proof

*From now through the end of Section 4.7 we fix an odd $\kappa \geq 5$ (as mentioned earlier, the case $\kappa = 3$ of Theorem 4.4 was proved in [12]), and set $p^* = p^*_\kappa$, $Q = Q_\kappa$, $T = T_\kappa$ and $G = G_{n,p}$; so our objective, (4.3), becomes

$$\max_p \Pr(G \not\in Q \setminus T) \to 0. \quad (4.10)$$

As sometimes happens, though (4.10) should become “more true” as $p (> p^*)$ grows, some points in the proof run into difficulties for larger $p$, and it seems easiest to deal first with smaller $p$ and then derive the full statement from this restricted version. The next two lemmas, the first of which is our main point, implement this plan.

**Lemma 4.8.** For any fixed $K$ and $p \leq Kp^*$,

$$\Pr(G \in Q \setminus T) \to 0. \quad (4.11)$$

(The interest here is really in $p$ at least about $p^*$, smaller values being handled by Lemma 4.2; see (4.28).)

**Lemma 4.9.** There exists $K > 1$ such that if $p > q := Kp^*$, then

$$\Pr(G \notin T) < \Pr(G_{n,q} \notin T) + o(1).$$

Applying Lemmas 4.9 and 4.8, together with (the 1-statement of) Lemma 4.2 to $p'(n) := \min\{p(n), Kp^*(n)\}$ then easily gives Theorem 4.4. (For $n$'s with $p(n) > Kp^*$, we have,
using Lemma 4.9 for the first inequality and Lemmas 4.8 and 4.2 for the final $o(1)$,

$$
\Pr(G \in Q \setminus T) < \Pr(G_{n,p'} \not\in T) + o(1)
< \Pr(G_{n,p'} \in Q \setminus T) + \Pr(G_{n,p'} \not\in Q) + o(1) = o(1),
$$

and for the remaining $n$’s we have $p = p'$ and Lemma 4.8 applies directly.)

The following device will play a central role in the proofs of both of these lemmas (so in most of this chapter). For the rest of the chapter we fix some rule that associates with each finite graph $G$ a subgraph $F(G)$ satisfying

$$
F(G) = \begin{cases} 
\emptyset & \text{if } G \in T, \\
\text{some smallest element of } C_\kappa(G) \setminus C_\perp(G) & \text{if } G \not\in T.
\end{cases}
(4.12)
$$

(By (4.8), this makes sense.)

We will use this only with $G$, so set $F(G) = F$ throughout. A crucial point is that $G$ determines $F$ (see the paragraph preceding Proposition 4.11). That $F$ is a minimizer will be used only to say that it is small and has small degrees, as promised by (4.9). Another useful observation (recalling the notation of Section 2.4):

$$
xy \in F \implies |F| \geq \sigma^{\kappa-1}(x,y) + 1.
(4.13)
$$

(Proof: Since $F$ lies in $C_\kappa(G)$, it must contain a second edge of each $\kappa$-gon of $G$ containing $xy$, and there is a set of $\sigma^{\kappa-1}(x,y)$ such $\kappa$-gons that share no edges except $xy$.)

With $F$ thus defined we may replace the event $\{G \not\in T\}$ by the more convenient $\{F \neq \emptyset\}$, which in particular allows us to tailor our treatment to the size of a hypothetical $F$. As we will see, ruling out fairly large $F$’s is easy—not from scratch, but with the help of a powerful result from [11] (Theorem 2.19), which more or less immediately yields:
**Lemma 4.10.** For fixed $c > 0$ and $p \gg n^{-(\kappa-2)/(\kappa-1)}$,

$$\Pr(|F| > cn^2p) \to 0.$$  \hspace{1cm} (4.14)

Thus the real problem in proving Lemma 4.8, and the most interesting part of the whole business, is dealing with $F$’s that are small relative to $G$ (but nonempty). Thus far—and a little further; see the preview following the statement of Lemma 4.14—our structure mirrors that of [12]; but the (two-page) argument handling this main point there offers no help here.

**Remark.** In connection with Question 4.6, it seems worth observing that Lemma 4.10, at least, can be considerably extended. In fact we can prove a statement of this type with the odd cycle $C_{\kappa}$ replaced by a general $H$, though not always with the (conjecturally correct) lower bound on $p$ that would correspond to a positive answer to Question 4.6. See Section 4.4 for a statement and Section 4.8 for a (sketchy) proof.

### 4.2.3 Coupling

A critical role in the proofs of Lemmas 4.8 and 4.9 is played by the usual coupling of $G (= G_{n,p})$ and $G_{n,q}$, where $p$ will always be the value we’re really interested in and $q < p$ will depend on what we’re trying to do.

So, from now on we set $G_0 = G_{n,q}$.

A standard description: let $\lambda_e, e \in E(K_n)$, be chosen uniformly and independently from $[0, 1]$ and set

$$G = \{e : \lambda_e < p\}, \hspace{0.5cm} G_0 = \{e : \lambda_e < q\}.$$  

In particular $G_0 \subseteq G$. Probabilities in the proofs of Lemmas 4.8 and 4.9 will refer to the joint distribution of $G$ and $G_0$.

We will get most of our leverage from two alternate ways of viewing the choice of the pair $(G, G_0)$:
(A) Choose \( G \) first; thus we choose \( G \) in the usual way and let \( G_0 \) be the \("(q/p)\)-random") subset of \( G \) gotten by retaining edges of \( G \) with probability \( q/p \), these choices made independently (a.k.a. percolation on \( G \)).

(B) Choose \( G_0 \) first; that is, we choose \( G_0 \) in the usual way, define \( p' \) by \((1-q)(1-p') = 1-p\), and let \( G \) be the random superset of \( G_0 \) gotten by adding each edge of \( K_n \setminus G_0 \) to \( G_0 \) with probability \( p' \), these choices again made independently.

We will often refer to these as “coupling down” and “coupling up” (respectively).

The proof of Lemma 4.9 is based naturally (or inevitably) on the viewpoint in (A); namely, we show that (with \( p, q \) as in the lemma) if \( G \) is “bad” (meaning \( G \not\in \mathcal{T} \)) then the coupled \( G_0 \) is likely to be bad as well. For the proof of Lemma 4.8, viewpoint (B) is the primary mover, though the smaller role of (A) is also crucial.

With reference to the setup introduced at (4.12), when working with \( G \) and \( G_0 \) as above, we set \( F_0 = G_0 \cap F \) (a \((q/p)\)-random subset of \( F \); note this has nothing to do with \( F(G_0) \), which will play no role here). Then automatically

\[
F_0 \in \mathcal{C}_\kappa^+(G_0),
\]

(4.15)
since \( F_0 \cap C = F \cap C \) for any \( \kappa \)-gon \( C \) of \( G_0 \).

We will want to say that certain features of \((G, F)\) are reflected in \((G_0, F_0)\). A simple but crucial point here is that there is no summing (of probabilities) over possible \( F \)’s, since there is just one \( F \) for each \( G \). The following proposition will be sufficient for our purposes.

**Proposition 4.11.** With the above setup, for any \( p, q \) and \( g = g(n) = \omega(1) \), w.h.p.

\[
|F_0| \sim |F|q/p \text{ if } |F| > gp/q
\]

and

\[
d_{F_0}(v) \begin{cases} 
\sim d_F(v)q/p & \forall v \text{ with } d_F(v) > (g \log n)p/q, \\
< 3g \log n & \forall v \text{ with } d_F(v) \leq (g \log n)p/q.
\end{cases}
\]
(This is true for any rule that specifies a particular subgraph (in place of \( F \)) for each graph; but we will only use it with \( F = F(G) \), so just give the statement for this case.)

**Proof.** These are straightforward applications of Theorems 2.1 and 2.2, so we will be brief. For the first assertion we want to say that for any fixed \( \varepsilon > 0 \),

\[
\Pr \left( \{|F| > \frac{gp}{q}\} \land \{|F_0| \neq (1 \pm \varepsilon)|F|q/p\} \right) \to 0.
\]

But the probability here is less than

\[
\Pr \left( |F_0| \neq (1 \pm \varepsilon)|F|q/p \mid |F| > \frac{gp}{q} \right),
\]

which by Theorem 2.1 is less than \( \exp[-\Omega(\varepsilon^2 g)] \).

The second assertion (pair of assertions) is similar, following from

\[
\sum_v \Pr \left( \left| \frac{d_{F_0}(v)}{d_F(v)} - (1 \pm \varepsilon) \right| d_F(v) > (g \log n)p/q \right) < n \exp[-\Omega(\varepsilon^2 g \log n)]
\]

\[
= o(1)
\]

for any fixed \( \varepsilon > 0 \), and (now switching to Theorem 2.2)

\[
\sum_v \Pr \left( \frac{d_{F_0}(v)}{d_F(v)} > 3g \log n \mid d_F(v) \leq (g \log n)p/q \right) < n \exp[-(3g \log n) \log(3/e)]
\]

\[
= o(1).
\]

### 4.3 Two Simple Points

Here we dispose of Lemma 4.2 and the derivation of Theorem 4.3 from Theorem 4.4. (Recall we are using \( G \) for \( G_{n,p} \) and \( V \) for \( V(G) \).)

**Proof of Lemma 4.2.** We begin with the 1-statement, a typical application of Theorem 2.3. We assume \( p > (1 + \varepsilon)p^* \) and \( p = O(p^*) \) (as we may, since for larger \( p \), the
1-statement is contained in Theorem 2.8). Given \(x, y \in V\), let the \(A_i\)'s (in the paragraph preceding Theorem 2.3) be the (edge sets of) the \((\kappa - 1)\)-paths joining \(x\) and \(y\) in \(K_n\); so \(X = \tau_{\kappa - 1}(x, y), \mu \sim n^{\kappa - 2}p^{\kappa - 1}\) and \(\Delta = \mu + O(\mu n^{\kappa - 3}p^{\kappa - 2}) \sim \mu\). Thus (note \(\varphi(-1) = 1\)) Theorem 2.3 gives

\[
\Pr(\tau_{\kappa - 1}(x, y) = 0) \leq \exp[-(1 - o(1))\mu]. \tag{4.16}
\]

So the probability that \(Q (= Q_\kappa)\) fails—that is, that there is some \(xy\) in \(G\) with \(\tau_{\kappa - 1}(x, y) = 0\)—is less than

\[
\binom{n}{2}pe^{-(1-o(1))}\mu < \exp[\log(n^{2}p) - (1 - o(1))\mu] = o(1)
\]

(since \(\mu > (1 - o(1))(1 + \varepsilon)^{\kappa - 1}(\kappa/(\kappa - 1))\log n \sim (1 + \varepsilon)^{\kappa - 1}\log(n^{2}p))\).

For the 0-statement we use the second moment method (see e.g. [3, Chapter 4]) and, again, Theorem 2.3. Let \(Z_{xy}\) be the indicator of the event \(\{xy \in G\} \land \{\tau_{\kappa - 1}(x, y) = 0\}\) \((x, y \in V)\) and \(Z = \sum Z_{xy}\). Theorem 2.6 gives \(\Pr(\tau_{\kappa - 1}(x, y) = 0) > (1 - p^{\kappa - 1})^{n^{\kappa - 2}} > \exp[-\mu - o(1)]\) (\(\mu\) as above), whence

\[
\mathbb{E}[Z_{xy}] > p \exp[-\mu - o(1)]. \tag{4.17}
\]

In particular \(\mathbb{E}[Z] = \omega(1)\) (using \(p < (1 - \varepsilon)p^*\) and ignoring the rather trivial case \(p = O(n^{-2})\)), so for \(\mathbb{E}Z^2 \sim \mathbb{E}[Z]^2\) (which gives the 0-statement \textit{via} Chebyshev’s Inequality), it’s enough to show

\[
\mathbb{E}[Z_{xy}Z_{uv}] < (1 + o(1))\mathbb{E}[Z_{xy}]^2
\]

for distinct \(\{x, y\}, \{u, v\} \in \binom{V}{2}\), which in view of (4.17) follows from

\[
\mathbb{E}[Z_{xy}Z_{uv}] \leq p^2 \Pr(\tau_{\kappa - 1}(x, y) = \tau_{\kappa - 1}(u, v) = 0)
\]

\[
\leq p^2 \exp[-(1 - O(n^{\kappa - 3}p^{\kappa - 2}))2\mu] = p^2 \exp[-2\mu + o(1)].
\]

Here the first inequality is given by Theorem 2.6 (since the events \(\{xy, uv \in G\}\) and
\( \{ \tau^{\kappa-1}(x,y) = \tau^{\kappa-1}(u,v) = 0 \} \) are increasing and decreasing respectively, and the second by Theorem 2.3, where the \( A_i \)'s are the \((\kappa - 1)\)-edge paths joining either \( x \) and \( y \) or \( u \) and \( v \), for which \( \mathbb{E}X \sim 2\mu \) (recall \( X \) is the number of \( A_i \)'s that occur) and it's easy to see that \( \Delta - \mu = O(n^{2\kappa-5}p^{2\kappa-3}) = O(n^{\kappa-3}p^{\kappa-2})\mu = o(\mu) \).

\[\square\]

**Proof that Theorem 4.4 implies Theorem 4.3.** This is routine and we aim to be brief. Lemma 4.2 gives the 1-statement (which is the interesting part). For the 0-statement, it is enough to say that for \( p \) in the stated range, \( G \) w.h.p. contains an edge lying in a cycle but not in a \( C_\kappa \). This is again given by Lemma 4.2 if \( p \) is large enough that all edges are in cycles (w.h.p.), which is true if \( p > (1 + \Omega(1))\log n/n \) (see [23, p. 105]). For smaller \( p \), w.h.p. \( G \) contains cycles of length \( \omega(1) \) if \( p > (1 - o(1))/n \) and of length \( \Omega(n^{3/10}) \) (say) if \( p \geq 1/n \) (see e.g. [23, Theorem 5.18(i)]). On the other hand, since the expected number of \( C_\kappa \)'s in \( G \) is less than \((np)^\kappa \), the number of edges in \( C_\kappa \)'s is w.h.p. less than \( \omega(np)^\kappa \) for any \( \omega = \omega(1) \); so in the range under discussion, the \( C_\kappa \)'s w.h.p. don't cover even one longest cycle in \( G \).

\[\square\]

### 4.4 Proof of Lemma 4.10

Here we give the easy proof of Lemma 4.10 and then state the extension to general \( H \) mentioned in the remark after the lemma.

For the lemma it's enough to show that the conclusions of Proposition 2.12, Theorem 2.19 and Proposition 2.13 (c), the latter two with \( \varepsilon = c/3 \), imply \( |F| < cnp^2 \) (deterministically).

Let \( F' \) be a largest element of \( F + C_\perp(G) \). Then \( |F'| \geq |G|/2 \) (by Proposition 4.7), so, since \( F' \) is \( C_\kappa \)-free, the conclusion of Theorem 2.19 gives an \( A \subseteq V \) with

\[ |F' \setminus \nabla_G(A)| < \varepsilon n^2 p. \tag{4.18} \]

It then remains to observe that (under our assumptions), (4.18) implies

\[ (|F| \leq) \quad |F' \triangle \nabla_G(A)| < 3\varepsilon n^2 p. \]
But the conclusion of Proposition 2.13 (c) gives $|\nabla G(A)| < (1 + \varepsilon)n^2p/4$, whence

$$|\nabla G(A) \setminus F'| \leq (1 + \varepsilon)n^2p/4 - (|G|/2 - \varepsilon n^2p) < 2\varepsilon cn^2p$$

(where we again used Proposition 2.12 to say $|G| \sim n^2p/2$).

**Generalization.** For this discussion we restrict to $H$ with $e_H \geq 2$. For such an $H$, set

$$m_2(H) = \max \left\{ \frac{e_K - 1}{v_K - 2} : K \subseteq H, v_K \geq 3 \right\}. \quad (4.19)$$

This parameter plays a central role in various contexts, in particular in results more or less related to (the general version of) Theorem 2.19; see e.g. [33] for an overview.

**Theorem 4.12.** For any fixed $H$ with $e_H \geq 2$, the following is true. For any $\varepsilon > 0$ there is an $M$ such that if $p > Mn^{-1/m_2(H)}$ then w.h.p.: for each $F \in \mathcal{C}_H(G)$ there is an $X \in \mathcal{W}_H^\perp(G)$ with $|F \Delta X| < \varepsilon n^2p$; in particular, if $\mathcal{C}_H(G) \neq \mathcal{W}_H(G)$, then

$$\min\{|F| : F \in \mathcal{C}_H^\perp(G) \setminus \mathcal{W}_H^\perp(G)\} < \varepsilon n^2p.$$ 

This is proved in Section 4.8.

**Remarks.** Notice that Theorem 4.12 contains an extension of Lemma 4.10, whereas in the preceding discussion we did need a few lines to get from Theorem 2.19 to the lemma. But the two theorems live in somewhat different worlds, since Theorem 2.19 assumes only that $F$ is $C_k$-free, which is much weaker than requiring that it have odd intersection with every $C_k$.

As mentioned in Section 2.9, the value $n^{-1/m_2(H)}$ is not necessarily what’s needed for Question 4.6. For instance, if $H$ is two triangles joined by a $P_t$, then $m_2(H) = 2$ (take $K$ to be one of the triangles), but the range where the question is most interesting (the point at which $Q_H$ becomes likely) is at $p \asymp n^{-2/3}\log^{1/3} n$, corresponding to all vertices being in triangles. On the other hand, in many (or most) natural cases—e.g. the (“balanced”) $H$’s for which $K = H$ achieves the max in (4.19)—Theorem 4.12 does give what should be the correct extension of Lemma 4.10. (It would be interesting to
see if one could push the theorem to give the correct extension in general; with our current approach this would mainly require a fairly significant extension of what we are getting from “containers,” and we haven’t yet thought about plausibility.)

### 4.5 Proof of Lemma 4.9

By Corollary 2.11 with \( l = \kappa - 1 \), there is a \( K > 1 \) such that if \( p > Kp^* \), then w.h.p.

\[
\text{every } \{x, y\} \in \binom{V}{2} \text{ satisfies } \sigma^{\kappa-1}(x, y) = \Omega(\pi) \tag{4.20}
\]

(where \( \pi = \pi(n, p) \) is as in the corollary). We work in the coupling framework of Section 4.2.3, taking \( q = Kp^* \).

For Lemma 4.9 it is of course enough to show

\[
\Pr(\{G / \in T\} \land \{G_0 \in T\}) \to 0. \tag{4.21}
\]

Note that \( G_0 \in T \) implies \( F_0 \in \mathcal{C}(G_0) \), since we always have \( F_0 \in \mathcal{C}_\kappa(G_0) \) (see (4.15)); thus (4.21) will follow from

\[
\Pr(\{F \neq \emptyset\} \land \{F_0 \in \mathcal{C}(G_0)\}) \to 0. \tag{4.22}
\]

So it will be enough to show that

\[
F_0 \notin \mathcal{C}(G_0) \tag{4.23}
\]

follows (deterministically) from

\[
F \neq \emptyset \tag{4.24}
\]

combined with various statements that we already know to hold w.h.p. This is not hard, but is more circuitous than one might wish. Roughly we show that, barring occurrence of some low probability event, (i) presence of even one edge in \( F \) forces \( F \) to be large enough (not very large) that \( F_0 \neq \emptyset \), and (ii) \( F_0 \) is not substantial enough to meet all
xy-paths in $G_0 - xy$ for an $xy \in F_0$, so any such $xy$ is contained in a cycle witnessing (4.23).

A convention. To slightly streamline the presentation we agree that in this argument, appeals to a probabilistic statement $X$—e.g. “$X$ implies” or “by $X$”—actually refer to the conclusion of $X$, which conclusion will always be something that $X$ asserts to hold w.h.p. See the references to (4.20), Lemma 4.10 and Proposition 4.11 in the next paragraph for first instances of this.

If (4.24) holds, then (4.20) and (4.13) (for the lower bound) together with Lemma 4.10 (for the upper) imply that

\[ \Omega(\pi) < |F| < n^2 p / 10. \]  

(4.25)

Since $\pi q / p \gg 1$, the lower bound in (4.25) and the first part of Proposition 4.11 give $|F_0| \sim |F| q / p$, so

\[ 0 \neq |F_0| < (1 + o(1)) n^2 q / 10. \]  

(4.26)

In addition, Proposition 2.12, (4.9) and the second part of Proposition 4.11 give

\[ d_{F_0}(v) < (1 + o(1)) n q / 2 \quad \forall v \in V. \]

Thus, setting $H_0 = G_0 \setminus F_0$ and recalling the approximate $(n q)$-regularity of $G_0$ given by Proposition 2.12, we have

\[ d_{H_0}(v) > (1 - o(1)) n q / 2 \quad \forall v \in V. \]  

(4.27)

Now choose an $xy \in F_0$ (recall (4.26) says $F_0 \neq \emptyset$) and let $X, Y$ be the $H_0$-components of $x$ and $y$. By (4.27) and Proposition 2.14 (applied to $G_0$), we have $|X|, |Y| > n / 3$, which implies $X = Y$: otherwise $X$ and $Y$ are disjoint and we have the contradiction

\[ (1 - o(1)) n^2 q / 9 < |\nabla_{G_0}(X, Y)| \leq |F_0| < (1 + o(1)) n^2 q / 10, \]
where the first inequality is given by Proposition 2.13 (a) (applied to $G_0$), the second holds because $\nabla_{G_0}(X,Y) \subseteq F_0$, and the third is given by (4.26).

But this (i.e. $X = Y$) gives an $xy$-path in $H_0$, and adding $xy$ to this path produces a cycle meeting $F_0$ only in $xy$; so we have (4.23).

### 4.6 Proof of Lemma 4.8

Here we introduce the two main assertions, Lemmas 4.13 and 4.14, underlying Lemma 4.8, and prove the latter assuming them. The supporting lemmas are proved in Sections 4.6.1 and 4.6.2.

Note that for the proof of Lemma 4.8, Lemma 4.2 allows us to restrict attention to the range

$$
(1 - \varepsilon)p^* < p < Kp^*
$$

(for any fixed $\varepsilon > 0$), and recall that, as observed following (4.14), it’s enough to show that for a given $\lambda = \lambda(n) \to 0$,

$$
\Pr(\{G \in Q\} \land \{0 < |F| < \lambda n^2 p\}) \to 0.
$$

We again work with the coupling of Section 4.2.3, now taking $q = \vartheta p$ with a fixed $\vartheta \in (0,1)$ small enough to support the discussion below (the rather mild constraints on $\vartheta$ are at (4.40) and (4.47)). Define the random variables $\alpha$ and $\alpha_0$ by

$$
|F| = \alpha n^2 p/2 \quad \text{and} \quad |F_0| = \alpha_0 n^2 q/2.
$$

**Definitions.** Henceforth a path (with length unspecified) is a $P_{\kappa-1}$ (and an $xy$-path is a path whose endpoints are $x$ and $y$). Our paths will always lie in $G$ and often in $G_0$. We now write $\sigma(x,y)$ for $\sigma^{\kappa-1}(x,y)$ (recall from Section 2.4 that this is the maximum size of a set of internally disjoint $xy$-paths in $G$), and $\sigma_0(x,y)$ for the analogous quantity in $G_0$. For $S \subseteq G$, a path $P$ is $S$-central if it contains an odd number of edges of $S$, ...
at least one of which is internal. Let $\sigma(x, y; S)$ be the maximum size of a collection of internally disjoint $S$-central $xy$-paths, and $\sigma_0(x, y; S)$ the corresponding quantity in $G_0$. An $(S, t)$-rope is a $P_t$ whose terminal edges lie in $S$. Set

$$R(S) = \{ \{x, y\} \in \binom{V}{2} : \sigma_0(x, y; S) > .25n^{\kappa-2}q^{\kappa-1} \} \tag{4.31}$$

and define events

$$\mathcal{R} = \{|F \cap R(F_0)| \geq .12\alpha n^2 p\}$$

and

$$\mathcal{P} = \{0 < |F| < \lambda n^2 p\}$$

(the second conjunct in (4.29)).

**Lemma 4.13.** There is a fixed $\varepsilon > 0$ such that for $p$ as in (4.28), w.h.p.

$$G \in Q \land P \implies G \in \mathcal{R}. \tag{4.32}$$

(In other words, $\Pr(G \in Q \land P \land \overline{R}) \to 0$. Of course $\mathcal{R}$ holds trivially if $F = \emptyset$, so it’s only the upper bound in $\mathcal{P}$ that’s of interest here.)

**Remarks.** For $\{x, y\} \in \binom{V}{2}$, $\sigma_0(x, y)$ should be around $n^{\kappa-2}q^{\kappa-1}$. Lemma 4.13 says that, provided $G \in Q \land \mathcal{P}$, it’s likely that for a decent fraction of the edges $xy$ of $F$, even $\sigma_0(x, y; F_0)$ is of this order of magnitude—which is unnatural if $F_0$ is small relative to $G_0$ (since then paths should typically avoid $F_0$). Viewed from Lemma 4.13 the parity requirement in the definition of “central” may look superfluous, since a path of $G_0$ joining ends of an edge of $F$ necessarily has odd intersection with $F_0$; but this extra condition will later play a brief but important role in justifying (4.36).

For the next lemma we temporarily expand the range of $q$ and $G_0$, assuming only what’s needed for the proof (though we will use the lemma only with $q$ and $G_0$ as above).

**Lemma 4.14.** For fixed $t \geq 3$, $q = q(n) > n^{-1}\log^6 n$ and $G_0 = G_{n, q}$, w.h.p.: for
$S \subseteq G_0$, say with $|S| = \beta n^2 q/2$, the number of $(S, t)$-ropes in $G_0$ is

$$O(\max\{\beta^2 n^{t+1} q^t, \beta n^{t/2 + 2} q^{t/2 + 1}\}).$$

(4.33)

**Remarks.** Note this is of interest only when $\beta \ll 1$, since Proposition 2.12 bounds (w.h.p.) the number in question by $(1 + o(1)) n^{t+1} q^t$; see Section 4.6.2 for a little more on the bounds in (4.33). The bound is also correct, but more trivial, when $t = 2$. The lemma doesn’t actually require $S \subseteq G_0$: the proof shows that, for any $S \subseteq E(K_n)$ (of the stated size) with $\Delta_S = O(nq)$ (where $\Delta$ is maximum degree), we have the same bound for the number of $P_t$’s with terminal edges in $S$ and internal edges in $G_0$.

**Preview.** The proof of Lemma 4.8, which we are about to give, is based mainly on “coupling up”: using information about $(G_0, F_0)$ to constrain what happens when we choose $G \setminus G_0$. (To this extent our strategy is similar to that of [12], but the resemblance ends there.) On the other hand, the proof of the crucial Lemma 4.13 in Section 4.6.1 is based on “coupling down”: most of the work there is devoted to the proof of a similar statement (Lemma 4.15) involving only $G$ (not $G_0$), from which the desired hybrid statement follows easily via coupling. In sum, we couple down to show that $\mathcal{R}$ is likely (precisely, the conjunction of its failure with $Q \land \mathcal{P}$ is unlikely), and couple up to show it is unlikely. A little more on the latter:

We would like to say that if $G_0$ is sufficiently nice—as it will be w.h.p.—then $\mathcal{P} \land \mathcal{R}$ is unlikely; this gives (4.29) via Lemma 4.13. The main point we need to add to Lemmas 4.13 and 4.14 is a deterministic one: if $G_0$ enjoys relevant genericity properties, together with the conclusion of Lemma 4.14, then, for each $S \subseteq G_0$, $R(S)$ is fairly small (depending on $|S|$; see (4.37)). Combined with $F \neq \emptyset$ (from $\mathcal{P}$), this will allow us to say that the lower bound on $|G \cap R(F_0)|$ ($= |F \cap R(F_0)|$) in $\mathcal{R}$ is larger by a crucial factor $\alpha^{-\Omega(1)}$ than $|R(F_0)|p$—its natural value when we “couple up”—which ought to make $\mathcal{R}$ unlikely. But of course $F_0$ depends on $G$, so, given $G_0$, we are forced to sum the probability of this supposedly unlikely event over possible values $S$ of $F_0$. This turns out to mean that the whole argument would collapse if we were to replace the above
\(\alpha^{-\Omega(1)} \leq \alpha^{-o(1)}\). (Here we again use \(P\), in this case to say \(\alpha\) is small.)

A word on presentation. We prove the desired

\[
\Pr(Q \land \mathcal{P}) = o(1)
\]  

(4.34)

(\(= (4.29)\)) by producing a list of unlikely events and showing that at least one of these must hold if \(Q \land \mathcal{P}\) does. A more intuitive formulation might, for example, begin: “By Lemma 4.13 (since we assume \(Q \land \mathcal{P}\), we may assume \(R\).” But note this would really mean, not that we condition on \(R\) (which is not something we could hope to understand), but that we need only bound probabilities \(\Pr(S \land \mathcal{R})\) for \(S\)’s of interest, and for a formal discussion this seems most clearly handled by something like the present approach.

We need two additional events (supplementing \(P, Q, R\) above). The first of these is simply

\[
\mathcal{S} = \{\alpha_0 \sim \alpha\}
\]

(i.e. for any \(\eta > 0\), \(\alpha_0 = (1 \pm \eta)\alpha\) for large enough \(n\); recall \(\alpha, \alpha_0\) were defined in (4.30)). The second, which we call \(\mathcal{T}\), is the conjunction of a few properties of \(G_0\) that we already know hold w.h.p., namely: \(|G_0| \sim n^2q/2\) (see Proposition 2.12); (2.15) and (2.16) for \(l \in [\kappa - 1, 2\kappa - 6]\) (meaning, in view of (4.28), (2.16) if \(l = \kappa - 1\) and (2.15) otherwise); and the conclusion of Lemma 4.14 for \(t \leq \kappa - 1\) (actually we only need this for even \(t\)). We first outline and then fill in details.

We have \(\Pr(Q \land \mathcal{R}) = o(1)\) (by Lemma 4.13; this is the only role \(Q\) plays in the present argument), and will show

\[
\Pr(\mathcal{R} \land \{\mathcal{F} \neq \emptyset\} \land \overline{\mathcal{S}}) = o(1). \tag{4.35}
\]

(This is easy and a secondary use of \(\mathcal{R}\). Note \(\{\mathcal{F} \neq \emptyset\}\) is implied by \(\mathcal{P}\).)
We will also show that, deterministically,

\[ \mathcal{R} \land \{F \neq \emptyset\} \land S \implies |(G \setminus G_0) \cap R(F_0)| > 0.1\alpha n^2 p \]  \hspace{1cm} (4.36)

provided \( \vartheta \) is sufficiently small (this is again easy), and, as mentioned in the preview,

\[ \mathcal{T} \implies |R(S)| = O(\alpha_S^{1+\delta} n^2) \]  \hspace{1cm} (4.37)

for some fixed \( \delta > 0 \) and all \( S \subseteq G_0 \), where we set \( \alpha_S = 2|S|/(n^2 q) \). Thus the conjunction of \( \mathcal{P}, \mathcal{R}, \mathcal{S} \) and \( \mathcal{T} \) implies (again, deterministically), the event—call it \( \mathcal{U} \)—that \( |G_0| < n^2 q \) (say) and there is an \( S \subseteq G_0 \) (namely the one that will become \( F_0 \)) satisfying (say):

\[ \alpha_S < 2.1\lambda, \quad |R(S)| = O(\alpha_S^{1+\delta} n^2), \quad \text{and} \quad |(G \setminus G_0) \cap R(S)| > 0.09\alpha_S n^2 p. \]  \hspace{1cm} (4.38)

Thus, finally, for (4.29) it is enough to show (by a routine calculation)

\[ \Pr(\mathcal{U}) = o(1). \]  \hspace{1cm} (4.39)

(Because: since \( \mathcal{U} \) implies \( \mathcal{P} \lor \overline{\mathcal{R}} \lor \overline{\mathcal{S}} \lor \mathcal{T} \), (4.39) implies

\[ \Pr(\mathcal{Q} \land (\mathcal{P} \lor \overline{\mathcal{R}} \lor \overline{\mathcal{S}} \lor \mathcal{T})) = \Pr(\mathcal{Q}) - o(1); \]

but the left hand side here is at most

\[ \Pr(\mathcal{Q} \lor \overline{\mathcal{P}}) + \Pr(\mathcal{Q} \lor \mathcal{P} \lor \overline{\mathcal{R}}) + \Pr(\mathcal{P} \lor \mathcal{R} \lor \overline{\mathcal{S}}) + \Pr(\mathcal{T}) = \Pr(\mathcal{Q} \lor \overline{\mathcal{P}}) + o(1) \]

(the second and third terms on the left being bounded by Lemma 4.13 and (4.35) respectively), so we have \( \Pr(\mathcal{Q} \land \mathcal{P}) = \Pr(\mathcal{Q}) - \Pr(\mathcal{Q} \land \overline{\mathcal{P}}) = o(1). \))

**Proof of** (4.35). If \( F \neq \emptyset \) (i.e. \( \alpha > 0 \)) and \( \mathcal{R} \) holds, then \( F \cap R(F_0) \neq \emptyset \), while by
(4.13), for any \( xy \in F \cap R(F_0) \),

\[
|F| > \sigma(x, y) \geq \sigma_0(x, y) > .25n^{\kappa-2}q^{\kappa-1} = \Omega(\log n).
\]

But then (since \( \log n \gg p/q \)) Proposition 4.11 says that w.h.p. \( |F_0| \sim \vartheta|F| \), which is the same as \( S \).

**Proof of (4.36).** Note it is always true that \( G_0 \cap R(F_0) \subseteq F_0 \), since the endpoints of an \( xy \in (G_0 \cap R(F_0)) \setminus F_0 \) would be joined by a path (many paths) having odd intersection with \( F_0 \), and adding \( xy \) to such a path would produce a \( C_\kappa \) having odd intersection with \( F_0 \). (As mentioned earlier, this is the reason for “odd” in the definition of central.) So if \( \mathcal{R}, S \) and \( \{F_0 \neq \emptyset\} \) hold (and \( \vartheta \) is slightly small) then

\[
|(G \setminus G_0) \cap R(F_0)| > .12\alpha n^2p - (1 + o(1))\alpha n^2q/2 > .1\alpha n^2p.
\]

**Proof of (4.37).** Set \( c = (\kappa - 3)/2 \). For \( l \in [c] \) and \( \emptyset \neq S \subseteq G_0 \) (for \( S = \emptyset \) there is nothing to show), call an \( xy \)-path \( (S,l) \)-central if it is \( S \)-central and at least one of its \( S \)-edges is at distance \( l \) (along the path) from one of \( x, y \). (So a path may be \( (S,l) \)-central for several \( l \)'s.) Let \( \sigma_0(x, y; S, l) \) be the maximum size of a collection of internally disjoint \( (S,l) \)-central \( xy \)-paths in \( G_0 \) and

\[
R_l(S) = \left\{ \{x, y\} \in \binom{V}{2} : \sigma_0(x, y; S, l) > (25/c)n^{\kappa-2}q^{\kappa-1} \right\},
\]

and notice that

\[
R(S) \subseteq \cup_{l \in [c]} R_l(S).
\]

Supposing temporarily (through (4.46)) that \( S \) and \( l \) have been specified, we abbreviate \( \sigma_0(x, y; S, l) = \varsigma(x, y) \), \( R_l(S) = R_l \) and use simply “rope” for “\((S,2l + 2)\)-rope”
(defined before Lemma 4.13). Set $|R_l| = \rho_l n^2$ and

$$r = 2(\kappa - 1) - 2(l + 1) = 2(\kappa - l) - 4 \in [\kappa - 1, 2\kappa - 6].$$  

We next show that if $G_0$ satisfies

$$T := \max_{u,v} \tau^{\ast}(u,v) = O(n^{r-1}q^r)$$

(as implied by (2.15) and (2.16), so by $T$), then

the number of ropes is $\Omega(\rho_l n^{2l+3}q^{2l+2})$.  

Proof. Say a rope $P = (u_{l+1}, \ldots, u_1, z, v_1, \ldots, v_{l+1})$ is generated by $\{x, y\}$ if there are internally disjoint paths $(z, u_1, \ldots, u_{\kappa-2}, w)$ and $(z, v_1, \ldots, v_{\kappa-2}, w)$ with $\{z, w\} = \{x, y\}$. Each $\{x, y\} \in \binom{V}{2}$ generates at least $2(\lfloor \varsigma(x,y)/2 \rfloor)$ such ropes (since a set of $a$ internally disjoint $(S,l)$-central $xy$-paths, each with an $S$-edge at distance $l$ from $x$, produces $\binom{a}{2}$ of them), while the number of pairs generating a given rope is at most $T$ (since in the scenario above, the complement of $P$ in the cycle $(z, u_1, \ldots, u_{\kappa-2}, w, v_{\kappa-2}, \ldots, v_1, z)$ is a path of length $r$ (see (4.43)) centered at $w$, so with $P$ determines $\{x, y\}$). Thus the number of ropes is at least

$$T^{-1} \sum_{\{x,y\} \in R_l} 2(\lfloor \varsigma(x,y)/2 \rfloor) = \Omega(|R_l|(n^{\kappa-2}q^{\kappa-1})^2/T) = \Omega(\rho_l n^{2l+3}q^{2l+2}).$$

If we now also assume the conclusion of Lemma 4.14 for $t = 2l + 2$ (again, this is contained in $T$), then combining that upper bound with the lower bound in (4.45) gives

$$\rho_l = O(\max\{\alpha_S^2, \alpha_S(nq)^{-l}\}) = O(\alpha_S^{1+\delta}),$$

with $\delta > 0$ depending only on $\kappa$. (Here we use $\alpha_S \geq n^{-2}$, valid since $S \neq \emptyset$.)

So, now letting $l$ vary, it follows that if $G_0$ satisfies $T$ (and so all relevant instances of (4.44) and (4.33)), then (4.46) holds for all $l \in [c]$, which in view of (4.42) bounds
\[ |R(S)| \text{ as in (4.37)}. \]

(It may be worth noting that for \( l = 0 \) the above argument gives only \( \rho_l = O(\alpha_S) \), which loses the crucial \( \delta \) in (4.46); thus the insistence on central paths in \( R \) and Lemma 4.13.)

**Proof of (4.39).** Given \( G_0, S \), we have \( |(G \setminus G_0) \cap R(S)| \sim \text{Bin}(m, p') \), with \( m \leq |R(S)| \) and \( p' < p \) defined by \( (1 - q)(1 - p') = 1 - p \) (as in (B) of Section 4.2.3). So for \( |R(S)| \) as in (4.38), Theorem 2.2 gives

\[
\Pr(|(G \setminus G_0) \cap R(S)| > 0.9\alpha_S n^2 p < \exp[-\Omega(\alpha_S n^2 p \log(1/\alpha_S))],
\]

where the implied constant depends on \( \delta \) but not on \( \vartheta \). Thus, assuming \( |G_0| < n^2 q \) (as given by \( U \)), setting \( \alpha_s = 2s/(n^2 q) \) (where \( s \) will be \( |S| \), so \( \alpha_s = \alpha_S \)), and summing over \( s < 2.1\lambda n^2 q \), we have

\[
\Pr(U \mid G_0) < \sum_s \binom{n^2 q}{s} \exp[-\Omega(\alpha_s n^2 p \log(1/\alpha_s))] \leq \sum_s \exp[\alpha_s n^2 p \{ (\vartheta/2) \log(2e/\alpha_s) - \Omega(\log(1/\alpha_s)) \}], \tag{4.47}
\]

which is \( o(1) \) for small enough \( \vartheta \) (implying (4.39) since

\[
\Pr(U) = \sum \{ \Pr(G_0) \Pr(U \mid G_0) : |G_0| < n^2 q \}. \]

**4.6.1 Proof of Lemma 4.13**

Fix \( \varepsilon > 0 \) (as in (4.28)) small enough to support the proofs of Propositions 4.17 and 4.20 below; these are our only constraints on \( \varepsilon \), and it will be clear they are satisfiable. We continue to assume that \( p \) is as in (4.28).

Most of our effort here is devoted to proving the following variant of Proposition 4.13 in which we replace \( \sigma_0(x, y, F_0) \) by \( \sigma(x, y, F) \) and \( q \) by \( p \).
Lemma 4.15. W.h.p.

\[ G \in Q \land P \implies |\{xy \in F : \sigma(x, y; F) > 0.26n^{\kappa-2}p^{\kappa-1}\}| \geq 0.13an^{2}p. \quad (4.48) \]

“Coupling down” will then easily get us to Lemma 4.13 itself. (The extra .01’s, relative to the pretty arbitrary .25 and .12 in (4.31) and (4.32), leave a little room for this.)

Preview. The proof of Lemma 4.15 breaks into two parts, roughly (w.h.p.): (a) if \( G \in Q \) (here we don’t need to assume \( G \in P \)), then \( \sigma(x, y) \) is close to its natural value for most \( xy \in F \) (see the paragraph following the proof of Proposition 4.19); (b) a decent fraction of the paths produced in (a) are \( F \)-central (shown by limiting the number that are not; this is based on Proposition 4.20 and does assume \( G \in P \)).

Definitions. It will be convenient to set

\[ \Lambda = n^{\kappa-2}p^{\kappa-1}, \]

since this quantity—essentially the typical number of paths in \( G \) joining a given pair of vertices—will appear repeatedly below. We write \( Q \sim Q' \) when \( Q, Q' \) are distinct \( C_\kappa \)'s sharing at least one edge. For edges \( e, f \) of \( G \), we take

\[ e \sim f \iff [\text{some } C_\kappa \text{ of } G \text{ contains both } e \text{ and } f], \quad (4.49) \]

\[ e \approx f \iff [\text{there are } C_\kappa \text{'s } Q \sim Q' \text{ of } G \text{ with } e \in Q \text{ and } f \in Q'], \quad (4.50) \]

\[ S(e) = \{f \in G : e \sim f\}, \text{ and } T(e) = \{f \in G : e \approx f\}. \]

For \( \gamma \in (0, 1) \), let

\[ L(\gamma) = \{\{x, y\} \in \binom{V}{2} : \sigma(x, y) < \gamma \Lambda\} \]

and \( F(\gamma) = F \cap L(\gamma) \). Finally, with \( C \) as in Proposition 2.9 for \( l = \kappa - 1 \) (and, say, \( \delta = 1/\kappa \)), let \( S \) be the event that \( G \) satisfies (2.11) (so not the \( S \) used above).

Fix \( \zeta = .01 \). Our goal in the next four propositions is to show that \( F(1 - \zeta) \) is small, accomplishing (a) of our outline above. We do this by showing separately (in
Propositions 4.18 and 4.19, using the tools provided by Propositions 4.16 and 4.17) that $F(\zeta)$ and $F(1 - \zeta) \setminus F(\zeta)$ are small.

**Proposition 4.16.** For $\gamma \in (0, 1)$ and distinct $\{x_1, y_1\}, \ldots, \{x_c, y_c\} \in \binom{V}{2}$,

$$\Pr(S \land \{(x_i, y_i) \in L(\gamma) \forall i \in [c]\}) \leq n^{-(c-o(1))(\kappa/(\kappa-1))(1-\varepsilon)^{\kappa-1}\varphi(\gamma-1)}.$$  \hfill (4.51)

(Recall $\varphi(x)$ was defined in (2.1).) Note the bound here is natural, being, for $p$ at the lower bound in (4.28) (and up to the $o(1)$), what Theorem 2.1 would give for the probability that $c$ independent binomials, each of mean $\Lambda$, are all at most $\gamma\Lambda$.

**Proof.** Since $S$ gives $\tau(x, y) \leq \sigma(x, y) + C < (1 + o(1))\gamma\Lambda$ for $\{x, y\} \in L(\gamma)$, the event in (4.51) implies that $X := \sum_{i \in [c]} \tau(x_i, y_i) < (1 + o(1))c\gamma\Lambda$; so we just need to bound the probability of this.

In the notation of Theorem 2.3, with $A_1, \ldots, A_m$ the edge sets of the various $x_iy_i$-paths in $K_n$, we have $\mu \sim c\Lambda$ and $\Delta = \mu + O(\Lambda^2/(np)) \sim \mu$. (If two of our paths, say $P$ and $Q$, share $l \in [1, \kappa - 2]$ edges, then at least $l$ internal vertices of $P$ are vertices of $Q$; so the contribution of such pairs to $\Delta$ is less than

$$c^2n^{2(\kappa - 2)^{-1}}p^{2(\kappa - 1)^{-1}} = O(\Lambda^2/(np)) = o(1)$$

(using the upper bound in (4.28) for the $o(1)$)). Thus Theorem 2.3 gives

$$\Pr(X < (1 + o(1))c\gamma\Lambda) < \exp \left[ -(1 - o(1))\varphi(\gamma - 1)c\Lambda \right],$$

which, since $\Lambda > (1-\varepsilon)^{\kappa-1}(\kappa/(\kappa-1))\log n$, is less than the right hand side of (4.51). \hfill \Box

**Proposition 4.17.** W.h.p.

if $Q_1 \sim Q_2 \sim Q_3 \sim Q_4$ are $C_\kappa$’s of $G$ then $|\cup Q_i \cap L(\zeta)| \leq 1$. \hfill (4.52)
Also, there is a fixed $M$ such that w.h.p.

$$|S(e) \cap L(1-\zeta)| < M \quad \forall e \in G.$$  \hfill (4.53)

(Note the $Q_i$’s in (4.52) need not be distinct.)

**Proof.** Write $\eta_\gamma$ for the quantity $n^{-o(1)}/(\kappa/(\kappa-1))^{\kappa-1} \varphi(\gamma-1)$ appearing in (4.51) (here without the $c$).

Since $S$ occurs w.h.p., it suffices to show that the probability that it holds while either (4.52) or (4.53) fails is $o(1)$. Thus in the case of (4.52) we want to bound the probability that $S \land \{J \subseteq G\} \land \{|J \cap L(\zeta)| \geq 2\}$ holds for some $J \subseteq K_n$ of the form $\bigcup_{i \in [4]} Q_i$, where the $Q_i$’s are $C_\kappa$’s sharing edges as appropriate. With $T(J) = S \land \{|J \cap L(\zeta)| \geq 2\}$, this probability is at most

$$\sum \Pr(J \subseteq G) \Pr(T(J)) \leq \sum \Pr(J \subseteq G) \Pr(T(J)) \leq O(n^{4\kappa-6}p^{4\kappa-3} \eta_\gamma^2) = o(1).$$

Here the first inequality is an instance of Theorem 2.6 (since $\{J \subseteq G\}$ and $T(J)$ are increasing and decreasing respectively), Proposition 4.16 gives $\Pr(T(J)) = O(\eta_\gamma^2)$ (for any $J$), and the $o(1)$ holds (for small enough $\varepsilon$) since $n^{4\kappa-6}p^{4\kappa-3} = \tilde{\Theta}(n^{\kappa/(\kappa-1)})$. The argument for

$$\sum \Pr(J \subseteq G) = O(n^{4\kappa-6}p^{4\kappa-3})$$ \hfill (4.54)

is similar to the proof of Proposition 2.9; briefly: if $Q_1, \ldots, Q_4$ are $C_\kappa$’s, with $R_i = \bigcup_{j \leq i} Q_j$ and, for $i \geq 2$, $|E(Q_i) \setminus E(R_{i-1})| = b_i \leq \kappa-1$ and $|V(Q_i) \setminus V(R_{i-1})| = a_i$, then $n^{a_i}p^{b_i} \leq \Lambda$ for $i \geq 2$ (since $b_i = a_i = 0$ or $b_i \geq a_i + 1$), yielding $n^{|V(R_i)|}p^{|E(R_i)|} \leq n^2p\Lambda^4$ and (4.54).

Treatment of (4.53) is similar. Here $J$ runs over subsets of $K_n$ of the form $\bigcup_{i \in [M]} Q_i$, where the $Q_i$’s are $C_\kappa$’s with a common edge, and, with $T(J) = S \land \{|J \cap L(1-\zeta)| \geq M\}$,
the probability that $\mathcal{S}$ holds while (4.53) fails is at most

$$\sum \Pr(\{J \subseteq G\} \land T(J)) \leq O(n^2p\Lambda^M\eta_1^{-\zeta}) = o(1).$$

This is shown as above, with $n^{\|V(J)\|p^{\|E(J)\|}} \leq n^2p\Lambda^M$ given by the passage following (4.54) (with $M$ in place of 4) and the $o(1)$ valid for large enough $M$ because $n^2p\Lambda^M < n^{\kappa/(\kappa-1)}O(\log^{M/(\kappa-1)} n)$.

The next assertion is the only place where we use the condition $\{G \in \mathcal{Q}\}$ of (4.32) (and (4.29)).

**Proposition 4.18.** W.h.p.

$$G \in \mathcal{Q} \implies |F(\zeta)| = o(|F|). \quad (4.55)$$

**Proof.** By the first part of Proposition 4.17 it is enough to show that the right hand side of (4.55) follows (deterministically) from the conjunction of $\{G \in \mathcal{Q}\}$ and (4.52). But these imply that $|T(e) \cap F| \geq \zeta\Lambda$ for each $e \in F(\zeta)$: $\{G \in \mathcal{Q}\}$ gives at least one $C_\kappa$ containing $e$; this $C_\kappa$ contains a second edge, $xy$, of $F$ (since $F \in \mathcal{C}_\kappa^\perp$), which by (4.52) is not in $L(\zeta)$; and $T(e)$ contains at least $\zeta\Lambda$ (distinct) $F$-edges lying on $xy$-paths. Moreover, again by (4.52), $T(e) \cap F(\zeta) = \{e\} \forall e \in F(\zeta)$ and $T(e) \cap T(f) = \emptyset$ for distinct $e, f \in F(\zeta)$. Thus $|F(\zeta)| < |F|/(\zeta\Lambda) (= o(|F|))$, as desired.

**Proposition 4.19.** W.h.p.

$$|F(1 - \zeta) \setminus F(\zeta)| = o(|F|). \quad (4.56)$$

**Proof.** It's enough to show that (4.53) implies (4.56) (since Proposition 4.17 says (4.53) holds w.h.p.). This is again easy: Set $B = F(1 - \zeta) \setminus F(\zeta)$ and consider the graph with vertex set $F$ and adjacency as in (4.49). Each $e \in B$ has degree at least $\zeta\Lambda$ in this graph, while (4.53) says no vertex has more than $M$ neighbors in $B$. Thus $|B|(\zeta\Lambda - M) \leq |F \setminus B|M$, which (since $\Lambda \gg 1$) gives (4.56).
Combining Propositions 4.18 and 4.19 completes part (a) of the preview following the statement of Lemma 4.15:

\[ w.h.p. \ G \in \mathcal{Q} \implies |F(1 - \zeta)| = o(|F|). \quad (4.57) \]

The next assertion, an echo of Section 2.4, provides technical support for part (b) (getting from (4.57) to Lemma 4.15 by controlling non-\(F\)-central paths).

For \(v \in V\) and \(S \subseteq \nabla_G(v)\), let \(T_S(v)\) be the set of \(C_\kappa\)’s using two edges of \(S\) and \(\tau_S(v) = |T_S(v)|\). (We could write simply \(T_S, \tau_S\), but keep the \(v\) as a reminder).

**Proposition 4.20.** For each fixed \(\theta > 0\) there exists \(C_\theta\) such that w.h.p.: for all \(v \in V\) and \(S \subseteq \nabla_G(v)\), with \(|S| = \gamma np\) and \(\mu = \gamma^2 n^{\kappa-1} p^\kappa / 2\),

\[ \tau_S(v) < \begin{cases} (1 + \theta) \mu & \text{if } \gamma > \gamma_\theta = C_\theta \log \log n / \log n, \\ o(\mu / \gamma) & \text{in general}. \end{cases} \quad (4.58) \]

**Proof.** We first observe that there is a fixed \(B\) such that w.h.p. no \(v\) lies in more than \(B C_\kappa\)’s that meet \(N(v)\) more than twice (basically because—here we omit the routine details—the expected number of such \(C_\kappa\)’s at a given \(v\) is \(O(n^{\kappa-1} p^{\kappa+1}) = n^{-O(1)}\)). It is thus enough to prove Proposition 4.20 with \(T\) and \(\tau\) replaced by \(T'\) and \(\tau'\), where \(T'_S(v) = \{Q \in T_S(v) : |Q \cap N(v)| = 2\}\) and \(\tau'_S(v) = |T'_S(v)|\).

Here we use a reduction similar to the one given by Proposition 2.9 (though, as will appear below, we can’t expect to do quite as well as in (2.11)). Let \(\sigma_S(v)\) be the maximum size of a collection of \(C_\kappa\)’s from \(T'_S(v)\) that are disjoint outside \(\overline{N}(v) := \{v\} \cup N(v)\). Set \(\psi(S) = \min\{|S|, \log^2 n\}\).

**Proposition 4.21.** There exists \(D\) such that w.h.p. for all \(v\) and \(S \subseteq \nabla_G(v)\),

\[ \tau'_S(v) - \sigma_S(v) < D\psi(S). \quad (4.59) \]

**Proof.** For fixed \(v\) and \(S \subseteq \nabla_G(v)\), let \(\Gamma = \Gamma_S\) be the graph on \(T'_S(v)\) with \(Q \sim R\) if \(Q\) and \(R\) share a vertex not in \(\overline{N}(v)\). Since \(\tau'_S(v) - \sigma_S(v) \leq |E(\Gamma)|\), (4.59) holds (for a
suitable $D$) provided

(i) the sizes of the components of $\Gamma$ are $O(1)$ and

(ii) the sizes of the induced matchings of $\Gamma$ are $O(\psi(S))$;

so we would like to say that w.h.p. (i) and (ii) hold for all $v$ and $S$. Here (and only here) we use $V(Q)$ for the set of vertices of $Q$ not in $\overline{N}(v)$.

Of course (i) holds for all $S$ (at $v$) iff it holds for $S = \nabla_G(v)$, so we just consider this case. Here we again (as in Proposition 2.9) want, for large enough $M$, (probable) nonexistence of $Q_1, \ldots, Q_M \in T'_S(v)$ such that, for $i \geq 2$, $V(Q_i)$ meets, but is not contained in, $\cup_{j<i} V(Q_j)$. Arguing as for (2.13) we find that the total numbers, say $a$ and $b$, of vertices (other than $v$) and edges used by such $Q_1, \ldots, Q_M$ satisfy

$$n^a p^b \leq n^{\kappa-1} p^\kappa (n^{\kappa-3} p^{\kappa-2})^{M-1}.$$  \hfill (4.60)

(Note here we do count neighbors of and edges at $v$. The bound says $n^a p^b$ is largest when each new $Q_i$ meets what precedes it in a $P_2$ starting at $v$.) Since $n^{\kappa-1} p^\kappa = \Theta(np \log n)$ and $n^{\kappa-3} p^{\kappa-2} = \tilde{\Theta}(n^{-1/(\kappa-1)})$, the bound in (4.60) is $o(1/n)$ for slightly large $M$, as is the probability of seeing such $Q_i$’s at $v$.

For (ii), it will help to condition on $\nabla_G(v)$. Using $\nu'$ for the maximum size of an induced matching and invoking Proposition 2.12, we find that it’s enough to show that, for a given $v$, $R \subseteq \nabla(v)$ of size less than $2np$ (say) and large enough $D$,

$$\Pr(\exists S \subseteq R, \nu'(\Gamma_S) > D\psi(S) \mid \nabla_G(v) = R) = o(1/n).$$  \hfill (4.61)

So assume we have conditioned on $\{\nabla_G(v) = R\}$, with $R$ as above. An easy verification (again similar to those in the proof of Proposition 2.9) gives, for any $S \subseteq R$ (and, again, $\gamma_S = \gamma$ and $\Gamma_S = \Gamma$),

$$\tilde{\mu} = \tilde{\mu}_S := \mathbb{E}[E(\Gamma)] = O\left(\binom{|S|}{2} n^{\kappa-3} p^{\kappa-2} |S| n^{\kappa-4} p^{\kappa-3}\right) = O(\gamma^3 n^{2(\kappa-2)} p^{2(\kappa-1)}) = O(\gamma^3 \log^2 n);$$  \hfill (4.62)
say $\tilde{\mu} < C\gamma^3 \log^2 n$ (with $C$ fixed). On the other hand, with $\{Q_i, R_i\}$ the possible edges of $\Gamma$ and $A_i = \{Q_i \cup R_i \subseteq G\}$, $\nu'(\Gamma) \geq l$ implies occurrence of some $l$ independent $A_i$’s, an event whose probability Proposition 2.5 bounds by $\tilde{\mu}^l/l! < (e\tilde{\mu}/l)^l$. (Here we could replace Proposition 2.5 by Lemma 2.4 (or Theorem 2.7), actually getting a slightly better bound, but it seems preferable to make clear that the more elementary result is all that’s needed.)

This leaves us with the union bound arithmetic. Here we first note that for $\nu'(\Gamma_S) < D \log^2 n \forall S$ we just need to check $S = R$, for which, in view of (4.62), we have $(e\tilde{\mu}/l)^l = o(1/n)$ for $l = D \log^2 n$ with a suitable $D$ ($D > Ce$ is enough). We then need to say (again, for suitable $D$) that with probability $1 - o(1/n)$,

$$\nu'(\Gamma_S) < D|S| \text{ for all } S \text{ with } |S| < \log^2 n. \quad (4.63)$$

But with $s = \gamma np$, $\tilde{\mu} = \mu_s < C\gamma^3 \log^2 n$ and sums over $s \in [1, \log^2 n]$, the probability that (4.63) fails is at most

$$\sum (|R|/s)^D_s < \sum \exp \left[ \gamma np \left\{ \log(2e/\gamma) + D \log \left( \frac{C\gamma^3 \log^2 n}{D\gamma np} \right) \right\} \right],$$

which, since we are in the range $\gamma np \in [1, \log^2 n]$, is easily $o(1/n)$. \qed

We continue with the proof of Proposition 4.20, which, by Proposition 4.21, we now need only prove with $\tau_S(v)$ replaced by $\sigma_S(v)$. Here it will help to have a concrete $o(\cdot)$ in (4.58). Set $h = h(n) = (\log \log n)^{1/2}$ (we need $1 \ll h \ll \log \log n$) and, with $C_\theta$ (and thus $\gamma_\theta$) TBA, set

$$K_\gamma = \begin{cases} 1 + \theta & \text{if } \gamma > \gamma_\theta, \\ (h\gamma)^{-1} & \text{otherwise}. \end{cases}$$

Given $v$ and $S \subseteq \nabla_G(v)$ of size $\gamma np$ (so we condition on $\{S \subseteq G\}$), Lemma 2.4 (or Theorem 2.7) gives

$$\sigma_S(v) \asymp Y \sim \text{Bin}(m, p^{\kappa-2}),$$

with $m = \gamma^2 n^{\kappa-1}p^2/2$ (so $\mathbb{E}Y$ is the $\mu$ of Proposition 4.20). On the other hand,
Theorems 2.1 and 2.2 give, writing $K$ for $K_\gamma$,

\[
\Pr(Y > K\mu) < \begin{cases} 
\exp[-\theta^2\mu/3] & \text{if } \gamma > \gamma_\theta, \\
\exp[-K\mu \log(K/e)] & \text{otherwise.}
\end{cases}
\] (4.64)

Thus, with $\xi_\gamma$ denoting the appropriate bound in (4.64), the probability of violating the $\sigma_S$-version of (4.58) with an $S$ of size $\gamma np$ is less than

\[
n(\gamma np)^p \gamma \xi < \exp[\log n + \gamma np \log(e/\gamma)] \cdot \xi_\gamma
\] (4.65)

(where the terms preceding $\xi_\gamma$ correspond to summing $\Pr(S \subseteq G)$ over $v \in V$ and $S \subseteq \nabla(v)$ of size $\gamma np$).

Finally, we should make sure the bound in (4.65) is small. Recalling (4.28), we have (for slightly small $\varepsilon$) $\Lambda > (1 - \varepsilon)\kappa^{-1}\kappa/\kappa - 1) \log n > \log n$ and

\[
\mu \ ( = (\gamma^2 np/2)\Lambda ) \ > (\gamma^2 np/2) \log n.
\] (4.66)

Thus for $\gamma > \gamma_\theta$ the bound in (4.65) is less than

\[
\exp[\gamma np \cdot \{\log(e/\gamma) - \theta^2\gamma \log n/6\} + \log n],
\]

which is tiny ($\exp[-n^{\Omega(1)}]$) for fixed $C_\theta > 6\theta^{-2}$.

For $\gamma \leq \gamma_\theta$, noting that $(\gamma K_{\gamma}/2) \log(K_{\gamma}/e) \sim \log(1/\gamma)/(2h) = \omega(1)$ (and $\gamma np \geq 1$), and again using (4.66), we find that the right hand side of (4.65) is less than

\[
\exp[\gamma np \cdot \{\log(e/\gamma) - (\gamma K_{\gamma}/2) \log(K_{\gamma}/e) \log n\} + \log n] = n^{-\omega(1)}.
\]

And of course summing these bounds over $\gamma$ gives what we want. \qed

Proof of Lemma 4.15. Fix $\theta = .005$ and let $C = C_\theta$ and $\gamma_\theta$ be as in Proposition 4.20. Set $\gamma_v = d_F(v)/(np)$, and let $\varphi_v$ be the number of $C_\kappa$'s of $G$ using two $F$-edges at $v$. Let $\sigma^*(x,y)$ be the number of $xy$-paths having $F$-edges at one or both of $x,y$. Write
\[ \sum' \text{ and } \sum'' \text{ for sums over } v \text{ with } \gamma_v > \gamma_\theta \text{ and } \gamma_v \leq \gamma_\theta \text{ respectively.} \]

We have, w.h.p.,

\[ \sum_{xy \in F} \sigma^*(x, y) \leq 2 \sum_{v \in V} \varphi_v \]
\[ \leq n^{\kappa - 1} \varphi \cdot \left[ (1 + \theta) \sum' \gamma_v^2 + \sum'' o(\gamma_v) \right], \]

(4.67)

where the first inequality comes from considering how many times each side counts the various \( C_\kappa \)'s of \( G \), and the second is given by Proposition 4.20.

Since \( \sum \gamma_v = \alpha n \), the second sum in (4.67) is \( o(\alpha n) \). For the first, let \( B = \{ v \in V : \gamma_v > \theta \} \). If we now assume \( \alpha = o(1) \) (as given by \( P \)), then we have \( |B| = o(n) \); so Proposition 2.13 (parts (a) and (b)) gives (w.h.p.)

\[ |G[B]| \ll |B| \theta np < \sum_{v \in B} d_F(v) \leq \alpha n^2 p, \]

whence \( \sum_{v \in B} \gamma_v np \leq 2|G[B]| + |\nabla F(B)| < (1 + o(1))\alpha n^2 p/2, \)

\[ \sum_{v \in B} \gamma_v < (1 + o(1))\alpha n/2 \]

and (recalling \( d_F(v) \leq d_G(v)/2 \forall v; \) see (4.9))

\[ \sum_{v \in B} \gamma_v^2 \leq \max_v \gamma_v \sum_{v \in B} \gamma_v < (1 + o(1))\alpha n/4. \]

(4.68)

Thus (since also \( \sum_{v \in V \setminus B} \gamma_v^2 \leq \theta \sum_v \gamma_v = \theta \alpha n \) we find that the expression in square brackets in (4.67) is less than \( (1/4 + 2\theta)\alpha n \), whence

\[ \sum_{xy \in F} \sigma^*(x, y) \leq (1/4 + 2\theta)\alpha n^2 p^\kappa = .26\alpha n^2 p^\kappa. \]

(4.69)

(To avoid confusion we note that the .26 here, which is more or less forced by the essentially tight bound in (4.68), has nothing to do with the .26 in (4.48).)

Now let \( F^* = \{ xy \in F : \sigma(x, y) \geq (1 - \zeta)\Lambda \} (= F \setminus F(1 - \zeta)). \) By (4.57), \( |F^*| \sim \alpha n^2 p/2, \) w.h.p. provided \( Q \) holds. Note that (recall \( \zeta = .01 \)) \( xy \in F^* \) has \( \sigma(x, y; F) > .26\Lambda \) (as in (4.48)) unless \( \sigma^*(x, y) > .73\Lambda. \) (As noted earlier, \( xy \)-paths
necessarily have odd intersection with $F$, so the only real requirement for such a path to be central is that it have an internal edge in $F$.) It thus follows from (4.69) that for $\tilde{F} := \{xy \in F^* : \sigma(x, y; F) \leq .26\Lambda\}$, we have

$$|\tilde{F}| \leq \frac{.26\alpha n^\kappa p^\kappa}{.73\Lambda} \leq .36\alpha n^2 p,$$

whence $|F^* \setminus \tilde{F}| \geq .13\alpha n^2 p$, implying (4.48). \hfill \Box

Proof of Lemma 4.13. As mentioned earlier, Lemma 4.13 follows easily from Lemma 4.15 via “coupling down” (viewpoint (A) of Section 4.2.3): it is enough to show that if $G$ satisfies the right hand side of (4.48) then w.h.p. it also satisfies $\mathcal{R}$; that is, $|F \cap R(F_0)| \geq .12\alpha n^2 p$.

For $xy \in F' := \{xy \in F : \sigma(x, y; F) > .26\Lambda\}$ (see (4.48)), Theorem 2.1 gives

$$\Pr(\sigma_0(x, y; F_0) \leq .25n^{\kappa-2}q^{\kappa-1}) < \exp[-\Omega(n^{\kappa-2}q^{\kappa-1})] = n^{-\Omega(1)},$$

since members of a set of $\sigma(x, y; F)$ internally disjoint, $F$-central $xy$-paths survive in $G_0$ (and become $F_0$-central) independently, each with probability $\psi^{\kappa-1}$. So by Markov’s Inequality, w.h.p.

$$|\{xy \in F' : \sigma_0(x, y; F_0) \leq .25n^{\kappa-2}q^{\kappa-1}\}| = o(|F'|).$$

The lemma follows. \hfill \Box

4.6.2 Proof of Lemma 4.14

This is a simple consequence of Proposition 2.15, but for perspective a brief comment on the bounds may helpful. The first bound—corresponding to a $\beta^2$-fraction of all $P_t$’s having their ends in $S$—is the generic value, and will be the truth if $q$ is large enough that (w.h.p.) all $\tau^{t-2}(x, y)$’s are about the same. For smaller $q$ one can sometimes do better by, e.g. (for even $t$), taking $S$ to consist of all edges at distance $t/2-1$ from some small set of “centers,” producing something like the second bound.
Proof. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of the adjacency matrix, $A$, of $G_0$, with associated orthonormal eigenvectors $v_1, v_2, \ldots, v_n$, say with $\max_j v_{1j} > 0$. Let $M = A^{t-2}$ (so $M$ has eigenvalues $\lambda_i^{t-2}$ ($i \in [n]$), with eigenvectors $v_i$), and $f = (d_S(x) : x \in V) = \sum \beta_i v_i$.

The number of $(S, t)$-ropes is w.h.p. less than

$$f M f^T = \sum \lambda_i^{t-2} \beta_i^2$$

$$\leq \lambda_1^{t-2} \beta_1^2 + \max \{ |\lambda_2|, |\lambda_n| \}^{t-2} \| f \|_2^2$$

$$< (1 + o(1)) \left[ (nq)^{t-2} \beta_1^2 + (4nq)^{(t-2)/2} \| f \|_2^2 \right], \quad (4.70)$$

where we used $\sum \beta_i^2 = \| f \|_2^2$ and the second inequality is given by (2.25). We then need bounds on $\beta_1^2$ and $\| f \|_2^2$, both of which are easy: w.h.p.

$$\beta_1 = \langle f, v_1 \rangle \sim n^{-1/2} \sum d_S(v) = 2n^{-1/2} |S| = \beta n^{3/2} q$$

(using (2.26)) and

$$\| f \|_2^2 = \sum d_S^2(x) \leq \Delta_S \sum d_S(x) < (1 + o(1))nq \cdot 2|S| \sim \beta n^3 q^2.$$

The lemma follows. \qed

4.7 Proof of Theorem 4.5

In what follows we set $E(K_n) = E$, $C_H(K_n) = C_H$ and so on. We prove (sketchily) Theorem 4.5 for $n \geq v_H + 2$—which is best possible e.g. if $H = K_\kappa$ with $\kappa \geq 4$ (e.g. since for $n \leq \kappa + 1$, $\mathcal{C}_H \supseteq \mathcal{C} \cap \mathcal{D}$)—and add a note at the end to cover $H = C_\kappa$ and $n \geq \kappa$.

We first note that $C_H = E$ if $|H| = 1$ (trivially) and $C_H = D$ if $|H| = 2$. (Since each of $P_2$, $2K_2$ (a 2-edge matching) is the sum of two copies of the other, the copies of an $H$ of size 2 span all 2-edge subgraphs, and so all even subgraphs, of $K_n$.) Moreover, if $H$ is a matching then $C_H$ is easily seen to contain (all copies of) $K_2$ if $|H|$ is odd or
2K₂ if |H| is even, so is equal to 𝒜 or 𝒫 as appropriate.

We may thus restrict attention to H containing a vertex x of degree at least 2, and observe that in this case 𝒞₉ ⊇ 𝒞₄. (The sum of two copies of H that differ only in the copy of x is a K := K₂,d(x), and repeating this with K and one of its divalent vertices produces a 𝒞₄.)

Since clearly 𝒞₄ = 𝒜 ∩ 𝒫, we’re done if H is in the second case of (4.5). Otherwise let ˜H be a copy of H in Kn and let F be a smallest element of ˜H + 𝒞₄. Evidently F is in the same case of (4.5) as H. Moreover, we claim F is either a triangle or the disjoint union of a matching and star (so possibly just a matching or just a star). Note this is enough, as the copies of F are then easily seen to generate the desired subspace of 𝒜: if H is Eulerian then F = K₃; otherwise we may add two copies of F to produce a P₂, so the generated space contains 𝒫. (Minor note: |V(F)| ≤ |V(H)| + 1 since all odd vertices of F must also be odd in ˜H.)

For the claim we observe that F cannot contain a P₃ (since adding a 𝒞₄ containing such a P₃ reduces |F|); disjoint P₂’s (reduce by adding a 𝒞₆); or K₃ + K₂ (convert to P₄ and then reduce to P₂).

Finally, for H = Cᵦ and n ≥ κ ≥ 4 (for κ = 3 there is nothing to show), it is enough to observe that the sum of two copies of H on the same vertex set and sharing a Pκ−₃ is a 𝒞₄; so 𝒞₉ = 𝒜 ∩ 𝒫 if κ is even, while for odd κ, 𝒜 ∩ 𝒫 ⊆ 𝒞₉ ⊆ 𝒜 implies 𝒞₉ = 𝒜.

### 4.8 Proof of Theorem 4.12

Here, finally, we prove Theorem 4.12. Since we make no use of this material we strive for brevity (albeit with little success), and will sometimes allow considerably less formality than elsewhere.

The proof of the theorem is based on Theorem 2.23, a “container” theorem, along with Lemma 4.22, an analogue of the “Erdős-Simonovits Stability Theorem” [14, 35]. (Theorems 2.19 and 4.12 may be thought of as “sparse random” analogues of Erdős-Simonovits. Our use of Lemma 4.22 below is analogous to the use of Erdős-Simonovits in the proofs of Theorem 2.19 in [4, 34].)
We may assume—as we do throughout—that $H$ has no isolated vertices, because, clearly, letting $H'$ be a copy of $H$ with any isolated vertices removed, we have $C_H(L) = C_{H'}(L)$ for any graph $L$ with $v_L \geq v_H$. Recall we are also assuming $e_H \geq 2$.

For any $H$ and $F \subseteq J \subseteq E(K_n)$, let $\tau_H(F, J)$ be the number of copies of $H$ in $J$ (say unlabelled, though it doesn’t matter) having odd intersection with $F$, and abbreviate this to $\tau_H(F)$ if $J = E(K_n)$.

**Lemma 4.22.** For any fixed graph $H$ and $\varepsilon > 0$, there is a $\delta > 0$ such that if $F \subseteq E(K_n)$ satisfies $\tau_H(F) < \delta n^{v_H}$, then there is an $X \in W_{H}^+(K_n)$ with $|F \Delta X| < \varepsilon n^2$.

We will actually apply the following simple extension of Lemma 4.22, which is proved, together with the lemma itself, at the end of this section.

**Corollary 4.23.** For any fixed $H$ and $\varepsilon > 0$, there is a $\delta > 0$ such that if $F \subseteq J \subseteq E(K_n)$ satisfy $|J| > (1 - \delta)n^2/2$ and $\tau_H(F, J) < \delta n^{v_H}$, then there is an $X \in W_{H}^+(K_n)$ with $|F \Delta X| < \varepsilon n^2$.

**Proof of Theorem 4.12.** For the rest of this discussion we take $v_H = \kappa$. The hypergraph $\mathcal{H} (= \mathcal{H}_n)$ to which we will apply Theorem 2.23 is as follows. Let $K$ be a copy of $K_n$ and $V = E(K) \times \{0, 1\}$; thus $N := |V| = n(n - 1)$. Let $\lambda_i$ be the natural bijection $(e \mapsto (e, i))$ from $E(K)$ to $V_i = \{(e, i) : e \in E(K)\} \ (i \in \{0, 1\})$. Finally, let $\mathcal{H}$ be the $e_H$-uniform hypergraph on $V$ whose edges are the $E$’s satisfying

- $|E| = e_H$,
- $\bigcup_{i=0}^{1} \lambda_i^{-1}(E \cap V_i)$ is (the edge set of) a copy of $H$ in $K$, and
- $|E \cap V_0| \equiv 1 \text{ mod } 2$.

In what follows, for $X \subseteq V$, we set $X_i = X \cap V_i$, $X^i = \lambda_i^{-1}(X_i)$ and $\tilde{X} = X^0 \cup X^1$. (So we may think of $\tilde{X}$ as a subgraph of $K$ underlying $X$.) For orientation we note immediately that for any $F \subseteq G \subseteq K_n$,

$$F \in C_{H}^+(G) \iff \lambda_0(F) \cup \lambda_1(G \setminus F) \in \mathcal{I}(\mathcal{H}).$$  \hfill (4.71)
(Recall from Section 2.11 that \( \mathcal{I}(\mathcal{H}) \) is the set of independent sets of \( \mathcal{H} \).)

For the remainder of our discussion we set \( \theta = n^{-1/m_2(H)} \) (recall the definition of \( m_2(H) \) at (4.19)). We first need to check that \( \mathcal{H} \) and \( \theta \) satisfy the hypotheses of Theorem 2.23. Clearly \( d = \Delta \) (by symmetry), and \( d \asymp n^{\kappa - 2} \) follows from

\[
|\mathcal{H}| = (n)^{\kappa} 2^{e_H-1} / |\text{Aut}(H)|, 
\]

so we want to show that for any \( \sigma \subseteq V \) of size \( l \),

\[
d(\sigma) = O(n^{\kappa - 2 - (l-1)/m_2(H)}).
\]  

(4.73)

For \( \sigma \subseteq V \) let \( K_\sigma \) be the graph with edge set \( \tilde{\sigma} \) and vertices those vertices of \( K \) incident with edges of \( \tilde{\sigma} \), and set \( v_\sigma = |V(K_\sigma)|, \ e_\sigma = |E(K_\sigma)| \). Notice that \( d(\sigma) = 0 \) unless \( |\tilde{\sigma}| = |\sigma| \) (i.e. \( \sigma^0 \cap \sigma^1 = \emptyset \)) and \( K_\sigma \) is (isomorphic to) a subgraph of \( H \); so we may assume these are true. But in this case we have \( d(\sigma) \asymp n^{\kappa - v_\sigma} \), so (4.73) follows from

\[
v_\sigma \geq 2 + (e_\sigma - 1)/m_2(H),
\]

which is the same as \( m_2(H) \geq (e_\sigma - 1)/(v_\sigma - 2) \) and is true by the definition of \( m_2 \) (since \( K_\sigma \subseteq H \)).

\[\diamondsuit\]

Let \( \delta' \) be the \( \delta \) given by Corollary 4.23 with \( \varepsilon/4 \) in place of \( \varepsilon \), and

\[
\delta = \min\{\delta', \varepsilon/4\} / (e_H 2^{e_H}).
\]  

(4.74)

Choose \( b \) so that (2.27) holds (with \( r = e_H \) and the present \( \mathcal{H} \) and \( \theta \)) and let \( B \) and \( C \) be as in Theorem 2.23. Noting that the assumption of Theorem 4.12 is now \( p > M\theta \), we will prove the theorem with \( M \) significantly larger than \( \varepsilon^{-2} \delta^{-1} B \log(eM/B) \), say

\[
M > 3\beta^{-1} \varepsilon^{-2} \delta^{-1} B \log(eM/B),
\]  

(4.75)

with \( \beta \) as in (4.81).
In view of (4.71), $F \in \mathcal{C}_H^\perp(G)$ implies existence of some $T \subseteq V$ with

$$|T| < BN\theta, \quad (4.76)$$

$$T \subseteq \lambda_0(F) \cup \lambda_1(G \setminus F) \subseteq C(T) \quad (4.77)$$

and

$$|\mathcal{H}[C(T)]| < \delta|\mathcal{H}|. \quad (4.78)$$

For $T$ satisfying (4.76) write $\mathcal{Q}_T$ for the event that there is an $F \in \mathcal{C}_H^\perp(G)$ satisfying both (4.77) and

$$\min\{|F \Delta X| : X \in \mathcal{W}_H^\perp(G)\} \geq \varepsilon n^2p. \quad (4.79)$$

We will show (for any $T$)

$$\Pr(\mathcal{Q}_T) < p^{|T|}\exp[-\Omega(\varepsilon^2 \delta n^2p)] = p^{|T|}\exp[-\Omega(\varepsilon^2 \delta Np)], \quad (4.80)$$

where the implied constants depend on neither $T$ nor $\varepsilon$ (so nor $\delta$). This easily gives Theorem 4.12, as follows. By the above discussion (sentence containing (4.76)–(4.78)), failure of the theorem’s conclusion is contained in the event $\cup\mathcal{Q}_T$ (union over $T$ as in (4.76)), so the probability of this failure is less than

$$\sum \Pr(\mathcal{Q}_T) < \sum t \binom{N}{t} p^t \exp[-\beta \varepsilon^2 \delta n^2p] \quad (4.81)$$

for a suitable fixed $\beta > 0$ (where the second sum ranges over $t \leq BN\theta$). On the other hand,

$$\binom{N}{t} p^t < \exp[t \log(eNp/t)] \leq \exp[BN\theta \log(eNp/(BN\theta))];$$

so the bound in (4.81) is less than

$$BN\theta \exp[BN\theta \log(ep/(B\theta)) - \beta \varepsilon^2 \delta n^2p], \quad (4.82)$$

which is small by our choice of $M$ (recall $p > M\theta$). \hfill \diamond
Proof of (4.80). Set \( R = (C(T))^0, S = (C(T))^1, t = |\tilde{T}| (= |T|) \)—see the first inclusion in (4.77)) and \( c = \binom{n}{2} - |R \cup S| \). Notice first that existence of an \( F \) as in (4.77) implies \( \tilde{T} \subseteq G \subseteq R \cup S \), which has probability \( p^t(1 - p)^c \). This already gives (4.80) unless

\[
c < \delta n^2,
\]

which we may therefore assume.

We next show that

\[
\tau_H(R, R \cup S) \leq |\mathcal{H}[C(T)]| 
\]

and

\[
|R \cap S| \leq |\mathcal{H}[C(T)]||\text{Aut}(H)|n^{-(\kappa - 2)} + 2e_H\delta n^2
\]

\[
< \delta n^2(2e_H^{-1} + 2e_H) < \varepsilon n^2/4.
\]

(The content in (4.85)–(4.86) is the first inequality; the second is given by (4.78) and (4.72), and the third by (4.74).)

Proof of (4.84). For any copy \( X \cup Y \) of \( H \) with \( X \subseteq R, Y \subseteq S \setminus R \) and \( |X| \) odd (these are the copies counted by \( \tau_H(R, R \cup S) \)), we have \( \lambda_0(X) \cup \lambda_1(Y) \in \mathcal{H}[C(T)] \). ☐

Proof of (4.85). For each \( xy \in K \), by double counting,

\[
\text{there are } \left( \frac{(n)_{\kappa}}{\left| \text{Aut}(H) \right|} \right) \frac{e_H}{\binom{n}{2}} \text{ copies of } H \text{ in } K \text{ containing } xy.
\]

Thus there are at least \( |R \cap S|n^{\kappa - 2}/|\text{Aut}(H)| \) copies of \( H \) meeting \( R \cap S \), at most \( 2e_H\delta n^{\kappa - 2}/|\text{Aut}(H)| \) copies of \( H \) meeting \( R \cup S \), of which are not contained in \( R \cup S \). But each of the \( 2e_H\delta n^{\kappa - 2}/|\text{Aut}(H)| \) copies of \( H \) that meet \( R \cap S \) and are contained in \( R \cup S \) underlies at least one member of \( \mathcal{H}[C(T)] \): for such a copy, say \( \tilde{H} \), containing \( xy \in R \cap S \), we may partition \( \tilde{H} \setminus \{xy\} = L \cup M \) with \( L \subseteq R \) and \( M \subseteq S \),
and then $\mathcal{H}[C(T)]$ contains $\lambda_0(L \cup \{xy\}) \cup \lambda_1(M)$ if $|L|$ is even and $\lambda_0(L \cup \lambda_1(M \cup \{xy\})$ if $|L|$ is odd. Thus $|\mathcal{H}[C(T)]| \geq (|R \cap S| - 2e_H \delta n^2)n^{\alpha - 2}/|\text{Aut}(H)|$, implying (4.85). ♦

We may now apply Corollary 4.23 to $R \subseteq R \cup S$ with $\varepsilon/4$ in place of $\varepsilon$ (recall the line preceeding (4.74); the hypotheses of the corollary are verified in (4.83), (4.84), (4.78), (4.74) and (4.72).) This yields some $Y \in W_{H}^+(K_n)$ with

$$|R \Delta Y| < \varepsilon n^2/4. \quad (4.88)$$

We will show that, barring occurrence of some event(s) with probability as in (4.80), each $F$ as in (4.77) is close to $X := G \cap Y \in W_{H}^+(G))$. Given $F$ as in (4.77), we have

$$F \Delta X = F \Delta (G \cap Y) = (F \Delta (G \cap R)) \Delta \Delta ((G \cap R) \Delta (G \cap Y))$$
$$= (F \Delta (G \cap R)) \Delta (G \cap (R \Delta Y))$$
$$\subseteq (F \Delta (G \cap R)) \cup (G \cap (R \Delta Y),$$

which, since $F \subseteq G \cap R \subseteq F \cup (G \cap (R \cap S))$, implies

$$|F \Delta X| \leq |G \cap R \cap S| + |G \cap (R \Delta Y)|.$$

So we have

$$|F \Delta X| < \varepsilon n^2p$$

unless

$$\max\{|G \cap R \cap S|, |G \cap (R \Delta Y)|\} > \varepsilon n^2p/2. \quad (4.89)$$

Thus, finally, for (4.80) we just need to show

$$\Pr((4.89) \text{ holds } | \tilde{T} \subseteq G) < \exp[-\Omega(\varepsilon^2 \delta n^2 p)]. \quad (4.90)$$

(Because: $Q_T \implies (4.89) \land \{\tilde{T} \subseteq G\}$, so $\Pr(Q_T) \leq \Pr(\tilde{T} \subseteq G) \Pr((4.89) \mid \tilde{T} \subseteq G).$)

But (4.90) is easy: by (4.86) and (4.88), the conditional expectation of each of $|G \cap R \cap S|,|G \cap (R \Delta Y)|$ given $\{G \supseteq \tilde{T}\}$ is less than $\varepsilon n^2 p/4 + t$, which in view of
(4.76) and our choice of $M$ (see (4.75)) is less than $\varepsilon n^2 p/3$. So (4.90) follows from Theorem 2.1.

Proof of Lemma 4.22. This will be rather sketchy and thoroughly informal, foregoing epsilons and deltas in favor of qualitative language. Thus, to begin, we use “most” to mean for all but a small fraction of relevant possibilities, where “small” can be made less than any desired (positive) constant via an appropriate choice of $\delta$. For example, “most $x, y$” means the number of exceptions is less than $\delta \varepsilon n^2$ for a suitable $\delta \varepsilon$. Similarly we say sets $A$ and $B$ are “close,” and write $A \approx B$, if $|A \triangle B|$ is at most small fraction of what it might have been (e.g. $n^2$ if $A, B$ are sets of edges, or $n$ if $A, B$ are sets of edges at a given vertex). And so on.

Let $V(H) = \{u_1, \ldots, u_{\kappa}\}$, say with $u_{\kappa-1} u_{\kappa} \in H$ and $N_H(u_{\kappa-1}) \setminus \{u_{\kappa}\} = \{u_i : i \in I\}$. Say a $\kappa$-tuple $(x_1, \ldots, x_{\kappa})$ of vertices of $K_n$ is even if, for $\varphi : V(H) \rightarrow V(K_n)$ given by $\varphi(u_i) = x_i$ (for $i \in [\kappa]$), $|\varphi(E(H)) \cap F|$ is even (where, of course, $\varphi(u_i u_j) = \varphi(u_i) \varphi(u_j)$). We now use $x, y, z$, possibly subscripted, for vertices of $K_n$ and $N(x)$ for $N_F(x)$.

Claim 1. There is an $x$ such that for most $y$, $N(y)$ is close to either $N(x)$ or $\overline{N(x)}$ ($:= V(K_n) \setminus N(x)$). (In fact this is true of most choices of $x$.)

Proof. Simple averaging, using the fact that $\tau_H(F)$ is small, shows that most choices of $x, y$ satisfy

\[(*) \text{ for most choices of } x_1, \ldots, x_{\kappa-2} \text{ and } z,
\]

both $(x_1, \ldots, x_{\kappa-2}, x, z)$ and $(x_1, \ldots, x_{\kappa-2}, y, z)$ are even. \hspace{1cm} (4.91)

So we may fix an $x$ for which $(*)$ holds for most $y$. If $(*)$ holds for $x, y$ then there is a fixed $(x_1, \ldots, x_{\kappa-2})$ such that (4.91) holds for most $z$, and for each such $z$ we have

\[|N(z) \cap \{x, y\}| = |N(x) \cap \{x_i : i \in I\}| + |N(y) \cap \{x_i : i \in I\}| \pmod{2}; \hspace{1cm} (4.92)\]

thus, since there is no $z$ on the right hand side of (4.92), $N(y)$ is close to one of $N(x), \overline{N(x)}$ whenever $y$ satisfies $(*)$ (with our fixed $x$). \hfill \diamond
Claim 2. For $x$ as in Claim 1, $F$ is close to either $\nabla(N(x))$ or its complement.

Proof. Set $A = N(x)$, $B = \overline{A}$, $S = \{y : N(y) \approx A\}$ and $T = \{y : N(y) \approx B\}$. We first observe that Claim 2 will follow if we show that

$$\text{one of } I := (A \cap S) \cup (B \cap T), \quad J := (A \cap T) \cup (B \cap S) \text{ is small.} \quad (4.93)$$

Suppose for example that $I$ is small. It is easy to see that if $e \in F \triangle \nabla(A)$ then $e$ either meets $I \cup S \cup T$ or lies in $\nabla_F(y) \triangle \nabla(y,B)$ for some $y \in A \cap T$ or in $\nabla_F(z) \triangle \nabla(z,A)$ for some $z \in B \cap S$. But the number of such $e$’s is small, since we assume $I$ and $S \cup T$ are small (the latter by our choice of $x$), while $y \in T$ implies $\nabla_F(y) \approx \nabla(y,B)$, and similarly $z \in S$ implies $\nabla_F(z) \approx \nabla(z,A)$. Thus in this case $F \approx \nabla(A)$. (Showing that $J$ small implies $F \approx \nabla(A)$ is of course similar.)

For $(4.93)$ it is enough to show that $A \cap S$ and $A \cap T$ cannot both be large, and similarly for the pairs $(A \cap S, B \cap S)$, $(A \cap T, B \cap T)$, $(B \cap S, B \cap T)$; there is little difference between these and we just show the first. The set $\nabla_F(A \cap S, A \cap T)$ is small since any $z \in A \cap T$ (or just $T$) has few neighbors in $A$. But we also have $|\nabla_F(A \cap S, A \cap T)| \approx |A \cap S||A \cap T|$, since, for each $y \in A \cap S$, the set $(A \cap T) \setminus N(y)$ (or even $A \setminus N(y)$) is small. So it must be that one of $A \cap S$, $A \cap T$ is small.

The four flavors of Lemma 4.22 (corresponding to the possibilities for $W_H(K_n)$ in $(4.5)$) now follow easily. If $H$ is even Eulerian then Claim 2 is what we want (since $W_H(K_n)$ consists precisely of cuts and their complements). If $H$ is odd (not necessarily Eulerian), then $F$ cannot be close to the complement of a cut, since the edges of $K_n$ contained in the larger side of the cut would contain $\Omega(n^\kappa)$ copies of $H$ that are contained in $F$. So $F$ is close to a cut, which for Eulerian (odd) $H$ is again what we want.

For non-Eulerian $H$ (briefly): If $F$ is close to a cut $\nabla(X,Y)$ with both $X$ and $Y$ large, then there are many odd copies of $H$ with one odd vertex in (say) $X$ and all other vertices in $Y$, so $F$ cannot be close to such a cut. In particular this says that for odd (non-Eulerian) $H$, $F$ must be close to $\emptyset$ (since being close to a cut with a small side is being close to $\emptyset$, and we have already said that $F$ is not close to the complement
of a cut). Finally, if $H$ is even (non-Eulerian) then $F$ is either close to $\emptyset$ (if close to a cut with a small side) or to $E(K_n) \in \mathcal{W}_{\mathcal{H}}(K_n)$ (if close to the complement of such a cut).

Proof of Corollary 4.23. Given $\varepsilon$, let $\delta$ be as in Lemma 4.22, and $\delta' = \delta/(2e_H)$. Then, recalling (4.87), $|J| > (1 - \delta')n^2/2$ implies that the number of copies of $H$ not contained in $J$ is at most $(\delta'n^2/2) \left( \frac{\binom{n}{2}}{\text{Aut}(H)} \right) < \delta n^\kappa/2$, which, with $\tau_H(F, J) < \delta'n^\kappa < \delta n^\kappa/2$, implies $\tau_H(F) < \delta n^\kappa$, the hypothesis of Lemma 4.22. □
Bibliography


