TWO PROBLEMS ON CYCLES IN RANDOM GRAPHS

BY JACOB D. BARON

A dissertation submitted to the

Graduate School—New Brunswick

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Jeff Kahn

and approved by

New Brunswick, New Jersey

October, 2016

ABSTRACT OF THE DISSERTATION

Two Problems on Cycles in Random Graphs

by Jacob D. Baron

Dissertation Director: Jeff Kahn

We prove three results.

First, an old conjecture of Zs. Tuza says that for any graph G, the ratio of the minimum size, $\tau_3(G)$, of a set of edges meeting all triangles to the maximum size, $\nu_3(G)$, of an edge-disjoint triangle packing is at most 2. Disproving a conjecture of R. Yuster [40], we show that for any fixed, positive α there are arbitrarily large graphs G of positive density satisfying $\tau_3(G) > (1 - o(1))|G|/2$ and $\nu_3(G) < (1 + \alpha)|G|/4$.

Second, write $\mathcal{C}(G)$ for the cycle space of a graph G, $\mathcal{C}_\kappa(G)$ for the subspace of $\mathcal{C}(G)$ spanned by the copies of C_κ in G, $\mathcal{T}_\kappa$ for the class of graphs satisfying $\mathcal{C}_\kappa(G) = \mathcal{C}(G)$, and $\mathcal{Q}_\kappa$ for the class of graphs each of whose edges lies in a C_κ. We prove that for every odd $\kappa \geq 3$ and $G = G_{n,p}$,

$$\max_p \Pr(G \in \mathcal{Q}_\kappa \setminus \mathcal{T}_\kappa) \to 0;$$

so the C_κ's of a random graph span its cycle space as soon as they cover its edges. For $\kappa = 3$ this was shown in [12].

Third, we extend the seminal van den Berg–Kesten Inequality [9] on disjoint occurrence of two events to a setting with arbitrarily many events, where the quantity of interest is the maximum number that occur disjointly. This provides a handy tool for bounding upper tail probabilities for event counts in a product probability space.

Acknowledgements

- Each result of this thesis has been or will be published elsewhere in one of various papers; see [7] (a version of Chapter 3), [6] (a version of Chapter 4), [5] (a version of Section 2.3).

- This material is based upon work supported by the U.S. Department of Homeland Security under Grant Award Number 2012-ST-104-000044. The views and conclusions contained in this document are those of the author and should not be interpreted as necessarily representing the official policies, either express or implied, of the U.S. Department of Homeland Security.

- I thank the professors who gave their time to serve on my committees: Doron Zeilberger, József Beck and Simon Thomas on my oral exam committee, and János Komlós, Mike Saks and Raphy Yuster on my thesis committee.

- I thank the regulars of the Rutgers Applied Game Theory Seminar and its offshoot, the bridge club, for camaraderie and for moral support as necessary.

- Most of all, thanks, Jeff.

Dedication

to Margie

Table of Contents

Chapter 1

Introduction

The main content of this thesis is two results on cycles in random graphs.[1] In the first, we use a (partly) random construction to exhibit a family of graphs whose triangles are clustered in a certain desired way. In the second, we determine for which $p = p(n)$ the standard Erdős–Rényi random graph $G_{n,p}$ is likely to satisfy a certain property dealing with its cycles of a fixed odd length. These results are detailed in Chapters 3 and 4, respectively. Our third result, extending the van den Berg–Kesten Inequality [9] on disjoint occurrence of events, was conceived as a lemma for our second; but since it is quite general and likely to be of broader interest, we present it in Section 2.3, in our chapter on preliminary tools. Here we just give a nontechnical overview of the three results and how they relate, deferring detail to when we state them formally.

For a graph G, write $\tau_3(G)$ for the minimum size of a set of edges meeting all triangles, and $\nu_3(G)$ for the maximum size of a set of edge-disjoint triangles. While $\tau_3(G) \leq 3\nu_3(G)$ is trivial, a conjecture of Zsolt Tuza from 1981 [39] holds that the 3 can be improved to 2, which is tight for the complete graphs of orders 4 and 5. But is it asymptotically tight for arbitrarily large graphs with quadratically many edges in triangles?[2] Raphael Yuster conjectured not (Conjecture 3.2 and (3.2); see [40]), at least for graphs with $\tau_3(G) > (1 - o(1))|G|/2$ (which by a routine exercise [3, Theorem 2.2.1] is as large as $\tau_3(G)$ can be). We disprove Yuster's conjecture (Theorem 3.3), exhibiting, for any $\alpha > 0$, an infinite family of dense graphs for which $\tau_3(G) > (1 - o(1))|G|/2$ and $\nu_3(G) < (1 + \alpha)|G|/4$.

[1] All graphs in this thesis are finite, simple and undirected.

[2] Without some requirement on the number of edges in triangles, the answer is obviously yes, because nonedges and edges not in triangles are invisible to both $\tau_3(G)$ and $\nu_3(G)$.

Write $\mathcal{E}(G)$ for the vector space $\mathbb{F}_2^{E(G)}$ (the *edge space* of G), $\mathcal{C}(G)$ for the subspace of $\mathcal{E}(G)$ generated by the (indicators of) cycles (the *cycle space* of G), and $\mathcal{C}_\kappa(G)$ for the subspace of $\mathcal{C}(G)$ generated by the κ-gons. It is not hard to see (Proposition 4.1) that for any $n \geq \kappa$,

$$\mathcal{C}_\kappa(K_n) = \mathcal{C}(K_n). \tag{1.1}$$

Following a theme in combinatorics that has lately been prominent, we wondered to what extent (1.1) remains true in a "sparse random" setting; or, to be precise, for which $p = p(n)$ (1.1) is likely to hold with K_n replaced by $G_{n,p}$. Let $\mathcal{T}_\kappa$ be the class of graphs satisfying $\mathcal{C}_\kappa(G) = \mathcal{C}(G)$, and $\mathcal{Q}_\kappa$ the class of graphs each of whose edges lies in a κ-gon. We prove (Theorem 4.4) that for every odd $\kappa \geq 3$,

$$\max_p \Pr(G_{n,p} \in \mathcal{Q}_\kappa \setminus \mathcal{T}_\kappa) \to 0; \tag{1.2}$$

so the κ-gons of a random graph span its cycle space as soon as they cover its edges. Even the $\kappa = 3$ case of (1.2) (which was proved by DeMarco, Hamm and Kahn in [12]) had been open and of interest, being the first unsettled case of a conjecture of M. Kahle (see [24, 25]) on the homology of the clique complex of $G_{n,p}$. In addition to wanting to generalize the result of [12], we were motivated to study $\mathcal{C}(G_{n,p})$ by a certain aspect of our construction of the graphs disproving Yuster's conjecture; see Section 4.1.1 for elaboration.

In proving Theorem 4.4, at several points we needed a good bound on the upper tail probability of a random variable on $G_{n,p}$ that returns, for some set S of subgraphs of K_n, the largest size of a set of pairwise edge-disjoint elements of S that appear in $G_{n,p}$. We prove a stochastic domination result (Theorem 2.7) generalizing the van den Berg–Kesten Inequality [9] that implies such a bound. Perhaps unsurprisingly, the bounds this gives, at least those relevant to the present work, are not new; but the basic result is very natural and seems of independent interest.

Chapter 2

Preliminaries

Here we establish usage conventions in force throughout the thesis (usage specific to certain sections will be introduced later as appropriate), and then give the tools we will use in Chapters 3 and 4. Most of the tools are fairly standard, the notable exception being Theorem 2.7, our extension of the van den Berg–Kesten Inequality advertised earlier. (Lemma 2.18 extending Mantel's Theorem is also new, but seems not worth fussing over.)

Given a graph G, we will use V and E for $V(G)$ and $E(G)$ when the meaning is clear. We will often identify graphs with their edge sets.

For $v \in V$ and $F \subseteq G$ we use $N_F(v) = \{x : vx \in F\}$ and $d_F(v) = |N_F(v)|$. For disjoint $A, B \subseteq V$, $\nabla_F(A, B)$ is the set of F-edges joining A and B, and we use $\nabla_F(A)$ for $\nabla_F(A, V \setminus A)$—these are the *cuts* of G—and $\nabla_F(v)$ for $\nabla_F(\{v\})$. In all cases we drop the subscript when $F = G$.

As usual $\alpha(G)$ and $\Delta(G)$ (or Δ_G) denote independence number and maximum degree of G. We will sometimes use v_G and e_G for the numbers of vertices and edges of G. The eigenvalues of G are those of its adjacency matrix; see e.g. [10, Section VIII.2].

As defined in Chapter 1, the *edge space* of a graph G, denoted $\mathcal{E}(G)$, is the vector space $\mathbb{F}_2^{E(G)}$. Its elements are naturally identified with the (spanning) subgraphs of G. The *cycle space* of G, denoted $\mathcal{C}(G)$, is the subspace of $\mathcal{E}(G)$ generated by the (indicators of) cycles of G (see e.g. [13, Section 1.9] for an exposition).

We use $[n]$ for $\{1, \ldots, n\}$ (for a positive integer n), log for ln and $a = (1 \pm b)c$ for $(1-b)c \le a \le (1+b)c$. Asymptotic notation ($\sim$, $O(\cdot)$, $\Omega(\cdot)$ and so on) is standard, with $a \ll b$ and $a \asymp b$ replacing $a = o(b)$ and $a = \Theta(b)$ when convenient. An asymptotic probabilistic statement holds *with high probability* (*w.h.p.*) if it holds with probability tending to 1 as some specified parameter tends to infinity. We always assume parameters that tend to infinity are large enough to support our various assertions, and usually pretend large numbers are integers.

2.1 Deviation

Here we recall a few standard bounds on the probability that a random variable differs by some specified amount from its mean.

Set

$$\varphi(x) = (1+x)\log(1+x) - x \tag{2.1}$$

for $x > -1$ and (for continuity) $\varphi(-1) = 1$. We use "Chernoff's Inequality" in the following form; see for example [23, Theorem 2.1].

Theorem 2.1. *If $X \sim \mathrm{Bin}(n,p)$ and $\mu = \mathbb{E}[X] = np$, then for $t \geq 0$,*

$$\Pr(X \geq \mu + t) \leq \exp\left[-\mu\,\varphi(t/\mu)\right] \leq \exp\left[-t^2/(2(\mu + t/3))\right], \tag{2.2}$$

$$\Pr(X \leq \mu - t) \leq \exp[-\mu\,\varphi(-t/\mu)] \leq \exp[-t^2/(2\mu)]. \tag{2.3}$$

For larger deviations the following consequence of the finer bound in (2.2) will be convenient.

Theorem 2.2. *For $X \sim B(n,p)$ and any K, letting $\mu = \mathbb{E}[X] = np$,*

$$\Pr(X > K\mu) < \exp[-K\mu\log(K/e)].$$

(Of course this is only helpful if $K > e$.)

In fact the above bounds hold with X being any sum of independent Bernoullis (and $\mu = \mathbb{E}[X]$); see [23, Theorem 2.8].

We will make substantial use of the following fundamental lower tail bound of Svante Janson ([22] or [23, Theorem 2.14]), for which we need a little notation. Suppose $A_1, \ldots, A_m$ are subsets of the finite set Γ. Let Γ_p be the random subset of Γ gotten by including each x ($\in \Gamma$) with probability p, these choices made independently. For $j \in [m]$, let I_j be the indicator of the event $\{\Gamma_p \supseteq A_j\}$, and set $X = \sum I_j$, $\mu = \mathbb{E}X = \sum_j \mathbb{E}I_j$ and

$$\overline{\Delta} = \sum\sum\{\mathbb{E}I_iI_j : A_i \cap A_j \neq \varnothing\}. \tag{2.4}$$

(Note this includes the diagonal terms.)

Theorem 2.3. *With notation as above, for any $t \in [0, \mu]$,*

$$\Pr(X \leq \mu - t) \leq \exp[-\varphi(-t/\mu)\mu^2/\overline{\Delta}] \leq \exp[-t^2/(2\overline{\Delta})].$$

This has an upper tail counterpart, but with the major restriction that the events counted must be independent. It is proved in [22, 23] for events as in Theorem 2.3, but either proof works, with a tiny modification, in the greater generality of:

Lemma 2.4 ([22, Lemma 2] or [23, Lemma 2.46])**.** *For events $A_1, \ldots, A_n$ in a probability space, and $\mu = \sum \Pr(A_i)$,*

$$\begin{aligned}\Pr(\textit{some } \mu + t \textit{ independent } A_i\textit{'s occur}) &\leq \exp\left[-\mu\, \varphi(t/\mu)\right] \\ &\leq \exp\left[-t^2/(2(\mu + t/3))\right].\end{aligned}$$

Note the bound here is the same as the one in (2.2), which is thus contained in this lemma. The lemma implies the weaker but sometimes convenient

Proposition 2.5. *For events $A_1, \ldots, A_k$ in a probability space, and $\mu = \sum \Pr(A_i)$,*

$$\Pr(\textit{some } l \textit{ independent } A_i\textit{'s occur}) \leq \mu^l/l!, \tag{2.5}$$

observed in [15] (or see [3, Lemma 8.4.1]). (An analogue of Theorem 2.2, this has no content until $l > e\lambda$, whereas Lemma 2.4 gives a usable bound even when $\mu + t$ ($= l$) $= \mu + \Omega(\sqrt{\mu})$.)

2.2 Correlation

The setting for the next theorem is a finite product probability space $\Omega = \prod_{i=1}^{t} \Omega_i$ with each factor linearly ordered. As usual an event $A \subseteq \Omega$ is *increasing* if its indicator is a nondecreasing function (with respect to the product order on Ω) and *decreasing* if its complement is increasing. The seminal "correlation inequality" is essentially due to Harris [19]:

Theorem 2.6. *If $A, B \subseteq \Omega$ are either both increasing or both decreasing, then*

$$\Pr(A \cap B) \geq \Pr(A)\Pr(B);$$

if one is increasing and the other decreasing then the inequality is reversed.

2.3 Disjoint Occurrence

Here we discuss our third result advertised in the abstract and introduction.

Recall that for (real-valued) random variables X and Y, Y *stochastically dominates* X (written $X \preccurlyeq Y$) if $\Pr(Y \geq r) \geq \Pr(X \geq r)\ \forall r \in \mathbb{R}$. Recall also that a probability measure m on a partially ordered Γ is *positively associated* (PA) if $m(A \cap B) \geq m(A)m(B)$ whenever both A and $B \subseteq \Gamma$ are increasing (or, equivalently, whenever both are decreasing), and note that any probability measure on a linearly ordered Γ is PA.

The setting for this section is a finite product probability space $(\Omega, \mu) = \prod_{i=1}^{n}(\Omega_i, \mu_i)$ with each Ω_i partially ordered. Events $A_1, A_2, \ldots, A_k$ ($\subseteq \Omega$) are said to *occur disjointly at* $\omega \in \Omega$ if there are disjoint $S_1, \ldots, S_k \subseteq [n]$ such that for each $i \in [k]$ and $\omega' \in \Omega$, we have $\omega' \in A_i$ whenever ω' agrees with ω on S_i. We write

$$\square_{i=1}^{k} A_i = \{\omega \in \Omega : A_1, \ldots, A_k \text{ occur disjointly at } \omega\}.$$

The study of disjoint occurrence was initiated by van den Berg and Kesten [9], who showed

$$\Pr(A\Box B) \le \Pr(A)\Pr(B) \tag{2.6}$$

for increasing $A, B \subseteq \{0,1\}^n$ (see also e.g. [18, Section 2.3]). The following extension of this seminal "BK Inequality" is apparently new [8].

Theorem 2.7. *Let $(\Omega,\mu) = \prod_{i=1}^n (\Omega_i,\mu_i)$ be a finite product probability space with the Ω_i's partially ordered and the μ_i's PA. Given $A_1, A_2, \ldots, A_k \subseteq \Omega$, let*

$$X = \max\{|I| : I \subseteq [k] \text{ and } \Box_{i\in I} A_i \text{ occurs}\}.$$

Let $Y_1,\ldots,Y_k$ be independent Bernoullis with $\mathbb{E}Y_i = \Pr(A_i)$, $Y = \sum Y_i$, and $\lambda = \sum \mathbb{E}Y_i$. Then:

(a) If the A_i's are all increasing, or all decreasing, then $X \preccurlyeq Y$;

(b) If the Ω_i's are linearly ordered, then for $t \ge 0$,

$$\Pr(X \ge \lambda + t) \;\le\; \exp\left[-\lambda\,\varphi(t/\lambda)\right] \le \exp\left[-t^2/(2(\lambda + t/3))\right].$$

Remarks.

(i) Taking $\Omega = \{0,1\}^n$, $k = 2$ and $r = 2$ in the definition of "$X \preccurlyeq Y$" recovers (2.6) from (a).

(ii) The most spectacular of the developments growing out of [9] is Reimer's proof [32] of the "BK Conjecture" (of [9]) which says that (2.6) doesn't require that A, B be increasing. In contrast, trivial examples show this requirement (or some requirement) to be necessary in (a); for instance if $\Omega = \{0,1\}$ with uniform measure, $k = 2$, $A_1 = \{0\}$ and $A_2 = \{1\}$, then $\Pr(X \ge 1) = 1 > 3/4 = \Pr(Y \ge 1)$.

(iii) For the same reason, (a) does not hold in the generality of Lemma 2.4 (even modified to make sense there). In other words, if $A_1, \ldots, A_k$ are events in an arbitrary probability space, Y is as in Theorem 2.7 and Z is the maximum number of independent A_i's that occur, then $Z \preccurlyeq Y$ does *not* hold in general, as the example in (ii) also shows.

(iv) On the other hand, for increasing [or decreasing] A_i's, (a) with Theorem 2.1 implies Lemma 2.4 (since independent increasing [or decreasing] events, if they occur, necessarily occur disjointly, a standard observation easily extracted from the usual proof of Theorem 2.6). In fact in this setting (a) is much stronger than Lemma 2.4, because dependent events can easily occur disjointly—so X can be much larger than the Z of (iii), even though the bounds given for the upper tails of X and Z, by (a)+Chernoff and Lemma 2.4 respectively, are the same. For example, if $x_1, \ldots, x_k, y_1, \ldots, y_k$ are distinct vertices of $G_{n,p}$ and, for $i \in [k]$, $A_i = \{\text{there is an } x_iy_i\text{-path}\}$, then $Z \leq 1$ but X can be large.

Historical Note. As mentioned in Chapter 1, our motivation for Theorem 2.7 (a) was to obtain something like Lemma 2.4, as in Remark (iv). We learned of the lemma about a year *after* proving (a). Shortly thereafter, we realized the lemma's proof (which is quite different from our proof of (a)) could be tweaked to give (b).

The proof of Theorem 2.7 (a), which is similar to the original proof of [9], is not hard but is a little awkward to write, and a few additional definitions will be helpful. We prove it for increasing A_i's; the decreasing case is of course analogous.

For $\Omega = \prod_{i\in I} \Omega_i$ and $S \subseteq I$, we take $\Omega_S = \prod_{i\in S} \Omega_i$ and, for $\omega \in \Omega$, $\omega_S = (\omega_i : i \in S)$. For $A \subseteq \Omega$ and $\omega \in \Omega_J$ for some $J \subseteq I$, $S \subseteq J$ is said to *witness* $\omega \in A$ if $\omega' \in A$ whenever $\omega' \in \Omega$ and $\omega'_S = \omega_S$. (This is of course abusive since we can't have $\omega \in A$ unless $J = I$.) We then (that is, for $\omega \in \Omega_J$) say $A_1, \ldots, A_k$ ($\subseteq \Omega$) occur disjointly at ω if there are disjoint $S_1, \ldots, S_k \subseteq J$ such that S_j witnesses $\omega \in A_j$ $\forall j$ and, for $\mathcal{A} = \{A_1, \ldots, A_k\}$, set

$$X_{\mathcal{A}}(\omega) = \max\{|R| : R \subseteq [k], \text{ the } A_j\text{'s indexed by } R \text{ occur disjointly at } \omega\}.$$

Thus the X of Theorem 2.7 is $X_{\mathcal{A}}$ evaluated at a random $\omega \in \Omega$.

Proof of Theorem 2.7 (a). Say $i \in [n]$ *affects* $A \subseteq \Omega$ if there are $\omega \in A$ and $\omega' \in \Omega \setminus A$ with $\omega_{[n]\setminus\{i\}} = \omega'_{[n]\setminus\{i\}}$, and for a collection $\mathcal{B}$ of events in Ω, let $\psi(\mathcal{B})$ be the number of $i \in [n]$ that affect at least two members of $\mathcal{B}$.

We proceed by induction on $\psi(\mathcal{A})$. If this number is zero then the laws of X and Y agree (since the A_j's are independent). So we may assume $\psi(\mathcal{A}) \neq 0$, say (without loss of generality) the index 1 affects at least two of the A_j's.

Let $(\Omega_{n+j}, \mu_{n+j})$, $j \in [k]$, be copies of (Ω_1, μ_1), independent of each other and of $(\Omega_1, \mu_1), \ldots, (\Omega_n, \mu_n)$. Let $(\Omega^*, \mu^*) = \prod_{i=2}^{n+k}(\Omega_i, \mu_i)$ and (for $j \in [k]$)

$$B_j = \{\omega \in \Omega^* : (\omega_{n+j}, \omega_2, \ldots, \omega_n) \in A_j\}.$$

Thus, apart from irrelevant variables, B_j is a copy of A_j gotten by replacing (Ω_1, μ_1) by $(\Omega_{n+j}, \mu_{n+j})$. In particular $\Pr(B_j) = \Pr(A_j)$ and, with $\mathcal{B} = \{B_1, \ldots, B_k\}$, we have $\psi(\mathcal{B}) = \psi(\mathcal{A}) - 1$ (since $i \in [2, n]$ affects B_j iff it affects A_j, and $n+i$ affects B_j iff $j = i$ and 1 affects A_i). So by inductive hypothesis it is enough to show

$$\mu(X_{\mathcal{A}} \geq r) \ \leq \ \mu^*(X_{\mathcal{B}} \geq r) \tag{2.7}$$

for each positive integer r. Here it's convenient to work with the stronger conditional version:

Claim. *For each $y \in \Omega_{[2,n]}$ (with $\mu_i(y_i) > 0 \ \forall\, i \in [2, n]$),*

$$\mu(X_{\mathcal{A}}(\omega) \geq r \mid \omega_{[2,n]} = y) \ \leq \ \mu^*(X_{\mathcal{B}}(\omega) \geq r \mid \omega_{[2,n]} = y). \tag{2.8}$$

Proof. Since, for any $y \in \Omega_{[2,n]}$ and $\omega \in \Omega$ with $\omega_{[2,n]} = y$,

$$X_{\mathcal{B}}(y) = X_{\mathcal{A}}(y) \leq X_{\mathcal{A}}(\omega) \leq X_{\mathcal{A}}(y) + 1,$$

we need only show (2.8) for y with $X_{\mathcal{A}}(y) = r - 1$ (since the left hand side of (2.8) is zero if $X_{\mathcal{A}}(y) \leq r - 2$ and both sides are 1 if $X_{\mathcal{A}}(y) \geq r$).

Given such a y, set $\mathcal{F} = \{x \in \Omega_1 : X_{\mathcal{A}}(x, y) = r\}$ and, for $i \in [k]$, let $\mathcal{F}_i \subseteq \Omega_1$ consist of those x's for which there are $I \in \binom{[k]}{r}$ containing i and disjoint S_j's in $[n]$ $(j \in I)$ such that S_j witnesses $(x, y) \in A_j$ (for $j \in I$) and $1 \in S_i$. Then, evidently,

- each $\mathcal{F}_i$ is increasing,
- $\mathcal{F} = \cup_{i\in[k]}\mathcal{F}_i$,
- for $\omega \in \Omega$ with $\omega_{[2,n]} = y$, $X_{\mathcal{A}} = r$ iff $\omega_1 \in \mathcal{F}$, and
- for $\omega \in \Omega^*$ with $\omega_{[2,n]} = y$, $X_{\mathcal{B}} \geq r$ iff $\omega_{n+j} \in \mathcal{F}_j$ for some $j \in [k]$,

whence

$$\begin{aligned}\mu(X_{\mathcal{A}}(\omega) \geq r \mid \omega_{[2,n]} = y) = \mu_1(\mathcal{F}) = 1 - \mu_1(\cap_{j\in[k]}\overline{\mathcal{F}}_j) \\ \leq 1 - \textstyle\prod_{j\in[k]} \mu_1(\overline{\mathcal{F}}_j) = \mu^*(X_{\mathcal{B}}(\omega) \geq r \mid \omega_{[2,n]} = y),\end{aligned}$$

where the inequality follows from that assumption that μ_1 is PA. $\square$

For the proof of Theorem 2.7 (b) we need just one little observation, which follows immediately from Reimer's Theorem [32] by induction: for events $\{A_i\}_{i\in I}$ in a product probability space with each factor linearly ordered,

$$\Pr(\square_{i\in I} A_i) \;\leq\; \prod_{i\in I} \Pr(A_i). \tag{2.9}$$

Proof of Theorem 2.7 (b). For some to-be-determined integer $r \leq k$ and each $I \subseteq [k]$ of size r, let B_I be the indicator of $\square_{i\in I} A_i$. Let $\chi = r! \sum B_I$, so that

$$\mathbb{E}\chi = r! \sum_{|I|=r} \Pr(\square_{i\in I} A_i) \leq r! \sum_{|I|=r} \prod_{i\in I} \Pr(A_i) \leq \lambda^r$$

(by (2.9)).

The rest of the proof follows [23, Lemma 2.46] verbatim, so we will be brief. If

$X \geq \lambda + t$ then $\chi \geq (\lambda+t)_r = \prod_{i=0}^{r-1}(\lambda+t-i)$, so by Markov,

$$\Pr(X \geq \lambda+t) \leq \Pr(\chi \geq (\lambda+t)_r) \leq \frac{\lambda^r}{(\lambda+t)_r} = \prod_{i=0}^{r-1}\frac{\lambda}{\lambda+t-i}.$$

Setting $r = t$ (to minimize the right hand side) yields

$$\log \Pr(X \geq \lambda+t) \leq \sum_{i=0}^{t-1}\log(\lambda/(\lambda+t-i)) \leq \int_0^t \log(\lambda/(\lambda+t-x))\,\mathrm{d}x,$$

which, with calculus, gives the stronger bound in Theorem 2.7 (b). □

2.4 Path Counts

Here we discuss what can be said about the numbers of paths of various lengths joining pairs of vertices in a random graph. Throughout the section we use G for $G_{n,p}$.

Notation. For $l \geq 1$ and (distinct) $x, y \in V$, we use $P^l(x,y)$ for the set of P_l's (l-*edge* paths) in G joining x and y, $\tau^l(x,y)$ for $|P^l(x,y)|$, and $\sigma^l(x,y)$ for the maximum size of a collection of internally disjoint P_l's of G joining x and y. (Though $l = 1$ is uninteresting, it's convenient to allow this.) These notations will show up again in Chapter 4. In this section only, we use $V(P)$ for the set of *internal* vertices of a path P and write $\Gamma^l_{x,y}$ for the graph on $P^l(x,y)$ with $P \sim Q$ iff $V(P) \cap V(Q) \neq \varnothing$.

Conveniently, most of what we need here has been worked out (in far greater generality) by Joel Spencer in [36] (see also [3, Section 8.5]), and we begin with two special cases of what's proved there.

Theorem 2.8. *For any $l \geq 2$ and $\varepsilon > 0$ there exists K such that if $n^{l-1}p^l \geq K \log n$, then w.h.p.*

$$\tau^l(x,y) = (1 \pm \varepsilon)n^{l-1}p^l \qquad \forall\,\{x,y\} \in \tbinom{V}{2}. \tag{2.10}$$

Proposition 2.9. *For any $l \geq 1$ and $\delta > 0$, if $n^{2l-3}p^{2l-1} < n^{-\delta}$ then w.h.p.*

$$\tau^i(x,y) - \sigma^i(x,y) < C \quad \forall \{x,y\} \in \binom{V}{2},\ i \in [l], \tag{2.11}$$

where C depends only on l and δ.

We note for use below that the assumption on p in Proposition 2.9 implies

$$n^{l-2}p^{l-1} < n^{-\zeta}, \tag{2.12}$$

with $\zeta = (1 + \delta(l-1))/(2l-1)$ $(= \Omega(1))$. Strictly speaking, the proposition is a little stronger than what one gets from [36], where the assumption would be $n^{l-1}p^l = O(\log n)$. (The $n^{2l-3}p^{2l-1}$ is more or less the expected number of non-edge-disjoint pairs of paths joining a given x and y.)

Proposition 2.9, though not difficult, is a key point in Spencer's proof of Theorem 2.8, and from our perspective is in a sense the main point, since, as indicated in the remark below, it easily gives the latter when combined with Theorem 2.3 and Lemma 2.4 (or Theorem 2.7).

Since the proof of the proposition itself is not so easy to extract from Spencer's presentation (see his "third part" on p. 253), we next sketch an argument along lines similar to his for the present situation.

Proof of Proposition 2.9. It is enough to handle $i = l$ (since the assumption on p implies a stronger assumption when we replace l by $i < l$). Noting that $\tau^l(x,y) - \sigma^l(x,y) \leq |E(\Gamma^l_{x,y})|$, we find that (2.11) (with an appropriate C) holds at x, y provided

(i) the maximum number of vertices in a component of $\Gamma^l_{x,y}$ is $O(1)$ and

(ii) the maximum size of an induced matching in $\Gamma^l_{x,y}$ is $O(1)$;

so we want to say that w.h.p. these conditions hold for all x, y. (Of course replacing (i) by an $O(1)$ bound on degrees would also suffice.)

For (i) we show that, for some fixed M, w.h.p. there do not exist x, y and a collection, $Q_1, \ldots, Q_M$, of P_l's joining x and y such that, for $i \geq 2$, $V(Q_i)$ meets, but is not contained in, $\cup_{j<i} V(Q_j)$. This bounds (by $(l-2)M+1$) the number of internal vertices (of G) in the paths belonging to a component of $\Gamma^l_{x,y}$, so gives (i).

Suppose $Q_1, \ldots, Q_M$ are P_l's joining x and y, with $R_i = \cup_{j \leq i} Q_j$ and, for $i \geq 2$, $|E(Q_i) \setminus E(R_{i-1})| = b_i$ and $|V(Q_i) \setminus V(R_{i-1})| = a_i \in [1, l-2]$. Then $b_i \geq a_i + 1$ and $a_i \leq l-2$ imply $n^{a_i} p^{b_i} \leq n^{l-2} p^{l-1}$ (for $i \geq 2$) and

$$n^{|V(R_M)|} p^{|E(R_M)|} \leq np(n^{l-2}p^{l-1})^M, \tag{2.13}$$

which is thus an upper bound on the probability of finding, for a given x, y, $(Q_1, \ldots, Q_M)$ as above of a given isomorphism type (defined in the obvious way). So the probability that there are such Q_i's for *some* x, y (and some isomorphism type) is $O(n^3 p(n^{l-2}p^{l-1})^M) = O(n^3 p n^{-\zeta M})$ (see (2.12)), so is $o(1)$ for large enough M.

The argument for (ii) is similar. Here we want to rule out, again for some fixed M, existence of P_l's, say $Q_1, R_1, \ldots, Q_M, R_M$, joining some specified x, y, with $V(Q_i) \cap V(R_i) \neq \varnothing$ and the $V(Q_i)$'s and $V(R_i)$'s otherwise disjoint. A discussion like the one above shows that for any such sequence, with $|\cup_i (E(Q_i) \cup (E(R_i))| = b$ and $|\cup_i (V(Q_i) \cup (V(R_i))| = a$, we have

$$n^a p^b < (n^{2l-3} p^{2l-1})^M < n^{-M\delta}, \tag{2.14}$$

which bounds the probability of existence by $O(n^{2-M\delta})$. □

Remark. The lower bound in Theorem 2.8 is given by Theorem 2.3 (a recent development at the time). The main issue for the upper bound is handling p with $n^{l-1}p^l \asymp \log n$, for which Proposition 2.9 allows replacing τ^l by σ^l. Spencer's nice observation is that, to bound $\sigma^l(x, y)$, one need only bound the probability of having a *maximal* disjoint family (of P_l's joining x, y) of a given size, and that one can use Theorem 2.3 to bound the probability that a particular (disjoint) family is maximal. His uses of this device could now be replaced by Lemma 2.4 (or Theorem 2.7), yielding (in the authors' unbiased view) more natural arguments.

Theorem 2.8 and Proposition 2.9 (with bits of Section 2.1) easily imply the following bounds on the $\tau^l(x,y)$'s for different ranges of p.

Corollary 2.10. *W.h.p. for all (distinct) vertices x, y,*

$$\tau^l(x,y) \sim n^{l-1}p^l \quad \text{if } n^{l-1}p^l = \omega(\log n), \tag{2.15}$$

$$\tau^l(x,y) = O(\log n) \quad \text{if } n^{l-1}p^l = O(\log n), \tag{2.16}$$

$$\tau^l(x,y) = O(1) \quad \text{if } n^{l-1}p^l < n^{-\Omega(1)}. \tag{2.17}$$

Proof. The first two items are easy consequences of Theorem 2.8: (2.15) is immediate and (2.16) is given by the observation that, for K as in the theorem (for some specified ε) and p_0 defined by $n^{l-1}p_0^l = K\log n$, the theorem implies that w.h.p.

$$\tau^l(x,y) \le (1+\varepsilon)n^{l-1}(\max\{p,p_0\})^l \quad \forall\, \{x,y\} \in \binom{V}{2} \tag{2.18}$$

(since the probability of the event in (2.18) is increasing as p decreases below p_0).

For (2.17), suppose $n^{l-1}p^l < n^{-\alpha}$, with $\alpha > 0$ fixed. Since this implies $n^{2l-3}p^{2l-1} < n^{-\delta}$ with $\delta = \delta_\alpha > 0$ fixed, Proposition 2.9 says it suffices to show that for given x, y and suitable fixed D (depending on α),

$$\Pr(\sigma^l(x,y) > D) = o(n^{-2}).$$

But Proposition 2.5 bounds this probability by

$$n^{-\alpha D}/D! < \exp[-D\log(n^\alpha D/e)],$$

which is $o(n^{-2})$ for large enough D. $\square$

We will also sometimes need *lower* bounds on path counts, as summarized in the next result, which again follows easily from what we already know.

Corollary 2.11. *For any $l \geq 2$ there is a K such that if $n^{l-1}p^l \geq K\log n$, then w.h.p. $\sigma^l(x,y) = \Omega(\pi)$ for all x, y, with $\pi = \pi(n,p)$ equal to*

$$n^{l-1}p^l \quad \text{if} \;\; n^{l-2}p^{l-1} < n^{-\Omega(1)}, \tag{2.19}$$

$$n^{l-1}p^l/\log n \quad \text{if} \;\; n^{-o(1)} < n^{l-2}p^{l-1} = O(\log n), \tag{2.20}$$

$$np \quad \text{if} \;\; n^{l-2}p^{l-1} = \omega(\log n). \tag{2.21}$$

(Of course in view of the routine Proposition 2.12, $(1+o(1))np$ is a trivial upper bound.)

Proof. Let K be as in Theorem 2.8, for the given l and, say, $\varepsilon = 1/2$ (since we don't worry about constants). Since the theorem says that w.h.p. $|V(\Gamma^l_{x,y})| > \Omega(n^{l-1}p^l)$ for all x, y, the present assertion(s) will follow if we show

$$\text{w.h.p.} \;\; \Delta(\Gamma^l_{x,y}) = O(n^{l-1}p^l/\pi) \;\; \forall\, x, y, \tag{2.22}$$

where we use the the trivial $\alpha \geq |V|/\Delta$ (recall Δ and α are maximum degree and independence number and note $\sigma^l(x,y) = \alpha(\Gamma^l_{x,y})$).

Now the degree in $\Gamma^l_{x,y}$ of a given vertex Q (that is, a P_l joining x and y) is at most

$$\textstyle\sum_v \sum_i \tau^i(x,v)\tau^{l-i}(v,y) \leq (l-1)^2 \max\{\tau^i(x,v)\tau^{l-i}(v,y)\}, \tag{2.23}$$

where the sums are over $v \in V(Q)$ and $i \in [l-1]$, and the max is over $i \in [l-1]$ and $v \in V \setminus \{x,y\}$ (the initial $(l-1)^2$ is of course irrelevant). On the other hand, Corollary 2.10 (with i in place of l) says that w.h.p. we have, for all u, v:

> $\tau^i(u,v) < O(1)$ if either $i \leq l-2$ and p is as in (2.19) or (2.20), or $i = l-1$ and p is as in (2.19),

and $\tau^i(u,v) < O(\max\{n^{i-1}p^i, \log n\})$ in general; and combining these bounds with (2.23) easily yields (2.22). $\square$

2.5 Density

Here we review various density properties of $G_{n,p}$. Throughout the section we use G for $G_{n,p}$ and V for $[n] = V(G)$. Theorems 2.1 and 2.2 easily imply the next two standardish propositions, whose proofs we omit.

Proposition 2.12. *For $p \gg n^{-1}\log n$, w.h.p.*

$$|G| \sim n^2p/2 \quad \textit{and} \quad d(v) \sim np \ \forall\ v \in V.$$

(Of course the second conclusion implies the first, which just needs $p \gg n^{-2}$.)

Proposition 2.13. (a) *For any $\varepsilon > 0$ there is a K such that w.h.p. for all disjoint $S, T \subseteq V$ with $|S|, |T| > Kp^{-1}\log n$*

$$|\nabla_G(S,T)| = (1 \pm \varepsilon)|S||T|p$$

and

$$|G[S]| = (1 \pm \varepsilon)\tbinom{|S|}{2}p.$$

(b) *For $K > 3$ w.h.p.*

$$|G[S]| < K|S|\log n \textit{ for all } S \subseteq V \textit{ with } |S| \leq Kp^{-1}\log n.$$

(c) *For each $\varepsilon > 0$ there is a K such that if $p > Kn^{-1}\log n$ then w.h.p.*

$$|\nabla_G(S)| = (1 \pm \varepsilon)|S|(n - |S|) \quad \forall S \subseteq V.$$

Proposition 2.14. *For fixed $\varepsilon > 0$ and $p \gg 1/n$, w.h.p.: if $H \subseteq G$ satisfies*

$$d_H(v) > (1 - \varepsilon)np/2 \quad \forall\ v \in V, \tag{2.24}$$

then no component of H has size less than $(1 - 2\varepsilon)n/2$.

Proof. For a given $W \subseteq V$ of size $w < (1-2\varepsilon)n/2$, let $\chi = |G[W]|$. Then $\mu := \mathbb{E}\chi = \binom{w}{2}p < w^2p/2$, while if W is a component of an H satisfying (2.24) then

$$\chi \geq |H[W]| > w(1-\varepsilon)np/4 > \tfrac{(1-\varepsilon)n}{2w}\mu =: K\mu.$$

But (since $K > (1-\varepsilon)/(1-2\varepsilon) = 1 + \Omega(1)$) Theorems 2.1 and 2.2 give

$$\gamma_w := \Pr(\chi > K\mu) < \begin{cases} \exp[-\Omega(\mu)] & \text{if } K < e^2 \text{ (say)}, \\ \exp[-K\mu\log(K/e)] & \text{otherwise.} \end{cases}$$

Thus, with sums over $w \in (0, (1-2\varepsilon)n/2)$, the probability that some H as in the lemma admits a component of size less than $(1-2\varepsilon)n/2$ is less than

$$\sum \tbinom{n}{w}\gamma_w < \sum \exp[w\log(en/w)]\gamma_w,$$

which for $p \gg 1/n$ is easily seen to be $o(1)$. $\square$

Finally, we need to know a little about the eigenvalues of G. A version of (2.25) below was proved in [16] (see also [2]) and (2.26) is shown (e.g.) in [30].

Proposition 2.15. *Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of G and $v_1, v_2, \ldots, v_n$ associated orthonormal eigenvectors, say with $\max_j v_{1,j} > 0$. If $p \gg n^{-1}\log n$, then w.h.p.*

$$\lambda_1 \sim np \quad \text{and} \quad \max\{|\lambda_2|, |\lambda_n|\} < (2+o(1))\sqrt{np}. \tag{2.25}$$

If $p > n^{-1}\log^6 n$, then w.h.p.

$$\max_j v_{1,j} < (1+o(1))\min_j v_{1,j}. \tag{2.26}$$

2.6 Expanders

We will want to say that good eigenvalues imply good density properties for a graph, for which our (standard) tool is

Lemma 2.16 (Expander Mixing Lemma [3, Cors. 9.2.5-6])**.** *Let H be a d-regular graph on t vertices for which every eigenvalue except d has absolute value at most λ. Let $A, B \subseteq V(H)$ be disjoint with $|A| = a, |B| = b$. Then*

$$\left| |\nabla(A, B)| - \frac{abd}{t} \right| \leq \lambda\sqrt{ab},$$

and

$$\left| |H[A]| - \frac{a^2 d}{2t} \right| \leq \frac{\lambda a}{2}.$$

2.7 Short Cycles

Recall that the *distance* between a pair of vertices in a graph is the number of edges in a shortest path between them, and the *diameter* of a graph is the maximum distance between a pair of its vertices.

We will want to say that the cycle space of any graph with low diameter is spanned by short cycles.

Proposition 2.17. *For any graph G of diameter D, $\mathcal{C}(G)$ is generated by the cycles of G of length up to $2D + 1$.*

Proof. It suffices to show that any cycle of length at least $2D + 2$ is the sum of two shorter ones. Let x, y be vertices at maximum distance along such a cycle C, and let P be a shortest xy-path, say with $x = v_0, \ldots, v_k = y$ being the vertices that P shares with C (as ordered by P). For some $i \in [k]$, v_{i-1} and v_i are closer along P than along C, and we get the two desired shorter cycles by replacing each of the $v_{i-1}v_i$-paths in C by the one in P. $\square$

Erratum. In [7], we gave an incorrect proof of this.

2.8 Mantel

Here we prove a new strengthening of Mantel's Theorem [29], which may be of independent interest. Recall that Mantel's Theorem is the first case of Turán's Theorem ([38], or e.g. [13, Theorem 7.1.1]) and the first result in extremal graph theory, proved in 1907.

Lemma 2.18 (Mantel's Theorem for "Crossing Triangles")**.** *Let K be the complete graph on $X \cup Y$, where X and Y are disjoint sets of size n. Let F be a subgraph of K containing no ("crossing") triangles meeting both X and Y. Then $|F| \le n^2$.*

Proof. We first claim that for any largest F containing no crossing triangles, $F[X]$ and $F[Y]$ are complete multipartite. For convenience set $G = F[X]$. If G is not complete multipartite, then it has vertices x, y, z satisfying $xy \in G$ and $xz, yz \notin G$. If $d_F(x) > d_F(z)$, then replacing $N_F(z)$ by $N_F(x)$ strictly increases $|F|$ without introducing forbidden triangles. Thus we may assume $d_F(z) \ge d_F(x)$, and similarly $d_F(z) \ge d_F(y)$. But then replacing both $N_F(x)$ and $N_F(y)$ by $N_F(z)$ strictly increases $|F|$ without introducing forbidden triangles. (This neighborhood-switching is a standard trick; see e.g. [13, Theorem 7.1.1]. We use it again in our proof of Theorem 3.3.)

So any largest F is complete multipartite in X with parts $X_1, X_2, \ldots, X_r$ of sizes $x_1 \ge x_2 \ge \cdots \ge x_r$, and in Y with parts $Y_1, Y_2, \ldots, Y_r$ of sizes $y_1 \ge y_2 \ge \cdots \ge y_r$ (some of the x_i's or y_i's being 0 if one of the partitions has more nonempty parts than the other). Since F has no triangles meeting both X and Y, for any $a \in X_i$ and $b \in Y_j$ we have

$$ab \in F \quad \Longrightarrow \quad N_F(a) \cap Y \subseteq Y_j \;\text{ and }\; N_F(b) \cap X \subseteq X_i,$$

so by the so-called rearrangement inequality we have

$$\begin{aligned} |F| &\le \sum_{1 \le i < j \le r} (x_i x_j + y_i y_j) + \sum_{i=1}^{r} x_i y_i \\ &= \frac{1}{2} \sum_{i=1}^{r} [x_i(n - x_i + y_i) + y_i(n - y_i + x_i)] \end{aligned}$$

$$= \frac{1}{2}\sum_{i=1}^{r}\left[n(x_i+y_i)-(x_i-y_i)^2\right]$$
$$= n^2 - \frac{1}{2}\sum_{i=1}^{r}(x_i-y_i)^2. \qquad \square$$

2.9 Stability

The following statement is a small instance of a recent major result of Conlon and Gowers [11]. As we will see in Section 4.4, it is the main (essentially only) ingredient in the proof of one of our lemmas in Chapter 4 (Lemma 4.10).

Theorem 2.19. *For each odd $\kappa \geq 3$ and $\varepsilon > 0$ there is a C such that if $p > Cn^{-(\kappa-2)/(\kappa-1)}$, then w.h.p. every C_κ-free subgraph of $G = G_{n,p}$ of size at least $|G|/2$ can be made bipartite by deleting at most $\varepsilon n^2 p$ edges.*

This (or the more general result of [11]) is a “sparse random” analogue of the Erdős-Simonovits “Stability Theorem” [14, 35] that was conjectured by Kohayakawa *et al.* in the seminal [26].

2.10 Regularity

Here we recall Szemerédi’s Regularity Lemma [37], or, more precisely, a generalization thereof due to Kohayakawa [27] and Rödl (unpublished). Our presentation here follows [23, Section 8.3].

Definitions 2.20 (for the Regularity Lemma)**.** Given a graph H, a real number $s \in (0,1]$ (called a *scaling factor*), and disjoint $U, W \subseteq V(H) =: V$, the *$(s;H)$-density* $d_{s,H}(U,W)$ between U and W is

$$d_{s,H}(U,W) = \frac{|\nabla_H(U,W)|}{s|U||W|}.$$

For $\epsilon > 0$, the pair U, W is called *$(s;H,\epsilon)$-regular* if for all $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \geq \epsilon|U|$ and $|W'| \geq \epsilon|W|$ we have

$$|d_{s,H}(U,W) - d_{s,H}(U',W')| \leq \epsilon.$$

A partition $\Pi = (V_0, V_1, \ldots, V_k)$ of V is called (ϵ, k)-*equitable* if $|V_1| = |V_2| = \cdots = |V_k|$ and $|V_0| \le \epsilon|V|$, and it is called $(s; H, \epsilon, k)$-*regular* if it is (ϵ, k)-equitable and all but at most $\epsilon\binom{k}{2}$ of the pairs V_i, V_j $(1 \le i < j \le k)$ are $(s; H, \epsilon)$-regular. In such a partition, V_0 is called the *exceptional part*. If $k' > k$ and Π' is an (ϵ, k')-equitable partition of V, then we say Π' *refines* Π if every nonexceptional part of Π' is contained in some nonexceptional part of Π.

For $b \ge 1$ and $\beta > 0$, H is called $(s; b, \beta)$-*bounded* if whenever $U, W \subseteq V$ are disjoint with $|U|, |W| \ge \beta|V|$ we have $d_{s,H}(U, W) \le b$. Intuitively, when H is sparse and s is the (tiny) density of H, $(s; b, \beta)$-boundedness ensures that no substantial chunk of H is much denser than it should be. $\Diamond$

Lemma 2.21 (Szemerédi Regularity Lemma, [23, Lemma 8.18])**.** *For all $\epsilon > 0, b \ge 1$ and natural numbers m and r there exist $\beta = \beta(\epsilon, b, m, r) > 0$ and $M = M(\epsilon, b, m, r) \ge m$ such that the following holds. For every choice of scaling factors s_i $(i \in [r])$ and $(s_i; b, \beta)$-bounded graphs H_i $(i \in [r])$ on the same vertex set V with $|V| \ge m$, there exists $k \in [m, M]$ and a partition Π of V that is $(s_i; H_i, \epsilon, k)$-regular for all $i \in [r]$.*

Since the proof of the Regularity Lemma starts with any partition of V into m nonexceptional parts of size $\lfloor |V|/m \rfloor$ and repeatedly refines this partition so that at each step each part is broken into the same number of subparts (see e.g. [27, 17] for details), we may further assume that

(i) Π refines a specified partition of V with m nonexceptional parts of size $\lfloor |V|/m \rfloor$, and

(ii) For any two nonexceptional parts S_i, S_j of the starting partition we have $|V_0 \cap S_i| = |V_0 \cap S_j|$, where V_0 is the exceptional part of Π.

Observe also that since every graph is trivially $(1; 1, \beta)$-bounded for all β, taking $b = r = s_1 = 1$ in Lemma 2.21 recovers the usual Regularity Lemma, which on one occasion (to prove Theorem 3.3) is all we will need from Lemma 2.21. Our other use of Lemma 2.21 (to prove Lemma 3.10) will require its full power.

Associated with the Regularity Lemma is the so-called Counting Lemma, which we will use in the following unusual form.

Lemma 2.22 (Counting Lemma)**.** *Let H be a graph, $\epsilon \in (0, 1/2)$, $s \in (0, 1]$, and A, B, B' pairwise disjoint subsets of $V(H)$ each of size l. If the pairs A, B and A, B' are $(1; H, \epsilon)$-regular with $(1; H)$-density at least 2ϵ, and the pair B, B' is $(s; H, \epsilon)$-regular with $(s; H)$-density at least 2ϵ, then H contains a triangle abb' with $a \in A$, $b \in B$, $b' \in B'$.*

Proof. Since $d_{1,H}(A, B) \geq 2\epsilon$, we have $|\{a \in A \mid |\nabla(a, B)| < \epsilon l\}| < \epsilon l$, or else this subset of A, along with $B \subseteq B$, would violate the $(1; H, \epsilon)$-regularity of the pair A, B. Similarly $|\{a \in A \mid |\nabla(a, B')| < \epsilon l\}| < \epsilon l$. Thus since $\epsilon < 1/2$, there exists $a \in A$ satisfying $|N(a) \cap B|, |N(a) \cap B'| \geq \epsilon l$. Then since the pair B, B' is $(s; H, \epsilon)$-regular with $(s; H)$-density at least 2ϵ, we have $\nabla(N(a) \cap B, N(a) \cap B') \neq \varnothing$, yielding a triangle in H of the stated form. $\square$

2.11 Containers

Here we give a specialization, adequate for present purposes, of the celebrated recent "container" theorems of [4, 34].

First we need a few definitions. Recall that a *hypergraph*, $\mathcal{H}$, is simply a collection of subsets ("edges") of a set $\mathcal{V}$ of "vertices." (We allow repeated edges, though we won't actually see any.) All our hypergraphs are *r-uniform*, i.e. have all edges of size r, and finite, with $|\mathcal{V}| = N$. An *independent* set of $\mathcal{H}$ is a subset of $\mathcal{V}$ containing no edges and $\mathcal{I}(\mathcal{H})$ is the collection of such sets.

For $\sigma \subseteq \mathcal{V}$, the *degree* of σ is $d(\sigma) = d_{\mathcal{H}}(\sigma) = |\{e \in \mathcal{H} : \sigma \subseteq e\}|$, which we shorten to $d(v)$ if $\sigma = \{v\}$. We use d and Δ for the average and maximum values of $d(v)$ ($v \in \mathcal{V}$) and, for $l \in [r]$,

$$\Delta_l = \max\{d(\sigma) : |\sigma| = l\}$$

(so $\Delta_1 = \Delta$).

The next assertion is easily derivable from Theorem 2.2 of [4].

Theorem 2.23. *For all r, $\delta > 0$ and b there is a B such that: if $\mathcal{H}$ is r-uniform with*

$$\Delta_l < b\theta^{l-1}d \quad \forall\, l \in [r], \tag{2.27}$$

then there exists $C : 2^{\mathcal{V}} \to 2^{\mathcal{V}}$ such that for each $I \in \mathcal{I}(\mathcal{H})$ there is a $T \subseteq \mathcal{V}$ with:

(a) $|T| < BN\theta$,

(b) $T \subseteq I \subseteq C(T)$,

(c) $|\mathcal{H}[C(T)]| < \delta|\mathcal{H}|$

(where $\mathcal{H}[X] = \{E \in \mathcal{H} : E \subseteq X\}$).

Chapter 3

Tuza's Conjecture is Asymptotically Tight for Dense Graphs

3.1 Introduction

Following [40] we write $\tau_3(G)$ for the minimum size of a *triangle edge cover* (set of edges meeting all triangles) in a graph G and $\nu_3(G)$ for the maximum size of a *triangle packing* (collection of edge-disjoint triangles) in G. (In standard language these are the matching and vertex cover numbers of the hypergraph with vertex set $E(G)$ and edges the triangles of G.)

While $\tau_3(G) \le 3\nu_3(G)$ is trivial (for any G), a 35-year-old conjecture of Zsolt Tuza [39] holds that this can be improved:

Conjecture 3.1. *For any G, $\tau_3(G) \le 2\nu_3(G)$.*

(This is sharp for the complete graphs of orders 4 and 5.)

The best general result in this direction remains that of Haxell [20], who showed

$$\tau_3(G) \le (66/23)\nu_3(G).$$

On the other hand, as noted in [40], a combination of results of Krivelevich [28] and Haxell and Rödl [21] implies that for any G,

$$\tau_3(G) < 2\nu_3(G) + o(n^2)$$

(limits as $n := |V(G)| \to \infty$). In particular, for any fixed $\beta > 0$ and G ranging over graphs satisfying $\tau_3(G) \geq \beta n^2$,

$$\tau_3(G) < (2 + o(1))\nu_3(G). \tag{3.1}$$

That is, Tuza's conjecture is asymptotically correct for such graphs.

The question of Raphael Yuster [40] that motivates us here is: is the constant 2 in (3.1) optimal? That is, is Tuza's conjecture still (asymptotically) tight for dense graphs with no subquadratic triangle cover? Yuster suggested not, at least in the special case where $\tau_3(G)$ is nearly as large as possible:

Conjecture 3.2 ([40]). *For fixed $\beta > 0$ and G ranging over graphs of density at least β,*

$$\tau_3(G) > (1 - o(1))|G|/2 \quad \Longrightarrow \quad \nu_3(G) > (1 - o(1))|G|/3$$

(where density is $|G|/\binom{n}{2}$, and $|G| = |E(G)|$). This would of course (for the graphs considered) be a big improvement over (3.1), which promises only $\nu_3(G) > (1 - o(1))|G|/4$.

Note that the inequalities $\tau_3(G) < |G|/2$ and $\nu_3(G) \leq |G|/3$ are easy and trivial (respectively), so Yuster's conjecture says that if G is dense and $\tau_3(G)$ is close to its trivial upper bound, then so must be $\nu_3(G)$.

Yuster also suggested weakening Conjecture 3.2 to say only that there is *some* fixed $\alpha \in (0, 1/3)$ (not depending on β) such that

$$\tau_3(G) > (1 - o(1))|G|/2 \quad \Longrightarrow \quad \nu_3(G) > (1 + \alpha)|G|/4, \tag{3.2}$$

which would still significantly improve on (3.1) (when $\tau_3(G) > (1 - o(1))|G|/2$). (Yuster did show that (3.2) is true if we allow α to depend on β.)

Surprisingly it turns out that even the weaker conjecture is wrong:

Theorem 3.3. *For all $\alpha > 0$, there exist $\beta > 0$ and arbitrarily large graphs G satisfying*

- $|G| \geq \beta\binom{n}{2}$,
- $\tau_3(G) > (1 - o(1))|G|/2$, *and*

- $\nu_3(G) < (1+\alpha)|G|/4$

(limits as $n \to \infty$). Thus even for dense graphs—and moreover for dense graphs where $\tau_3(G)$ is near $|G|/2$—Tuza's conjecture is essentially best possible.

Since what follows is not entirely easy, a little orientation may be helpful. Our construction itself is not very difficult; in rough outline it does:

1. start with a triangle-free graph H with certain nice degree and eigenvalue properties (we use the well-known graphs described by Noga Alon in [1]—see Proposition 3.11);

2. join two disjoint copies of H by a complete bipartite graph to produce K;

3. replace each vertex of K by a *large* clique; and finally

4. take a suitable random subgraph of this blowup, yielding the graph G_a found in the third paragraph of Section 3.3.

So again, there is nothing very exotic here. What seems most interesting in what follows is how strange a route we needed to take to arrive at a proof that this relatively simple construction actually works.

Also interesting is whether one could simplify our argument (or give an easier example) if the goal were only to disprove the stronger Conjecture 3.2 (rather than (3.2)). We don't see how to do this, and in fact most of what follows was originally developed with the lesser goal in mind.

The rest of this chapter is organized as follows. The next section gives a long string of essential definitions, most of them nonstandard, leading up to the crucial Lemma 3.10. In Section 3.3 we prove Theorem 3.3 assuming Lemma 3.10. In Section 3.4, we prove the lemma.

3.2 Definitions

A *fractional triangle edge cover* of a graph G is an assignment of nonnegative weights to the edges of G such that the weight of each triangle (this being the sum of the weights of its edges) is at least 1. We denote by $\tau_3^*(G)$ the minimum total weight of such a cover. Dually, a *fractional triangle packing* of G is an assignment of nonnegative weights to the triangles of G such that the weight of each edge (the sum of the weights of the triangles containing it) is at most 1. We denote by $\nu_3^*(G)$ the maximum total weight of such a packing. Note we have

$$\nu_3(G) \leq \nu_3^*(G) = \tau_3^*(G) \leq \tau_3(G),$$

where the inequalities are trivial and the equality is by linear programming duality.

Given graphs G_1, G_2, the *lexicographic product* $G_1 \cdot G_2$ is the graph on vertex set $V(G_1) \times V(G_2)$ where (u_1, u_2) is adjacent to (v_1, v_2) iff either $u_1v_1 \in G_1$, or $u_1 = v_1$ and $u_2v_2 \in G_2$. Note that the lexicographic product is not commutative.

The following original definitions are critical to our arguments.

Definition 3.4 (double of a graph)**.** For a graph H, the *double of* H, denoted $K_{H,H}$, is the graph $K_2 \cdot H$. To be explicit, this is the graph whose vertex set is $X \cup Y$, where X and Y are disjoint sets of size $|V(H)|$, and whose edges satisfy $K_{H,H}[X] \simeq K_{H,H}[Y] \simeq H$ and $\{xy \mid x \in X, y \in Y\} \subseteq E(K_{H,H})$. The sets X and Y (we will always use these names) are called the *sides* of $K_{H,H}$.

Of course the notation $K_{H,H}$ is intended to suggest the notation $K_{t,t}$ for a complete bipartite graph. When the H is understood, we will frequently abbreviate $K_{H,H}$ by K.

We denote by E the copy of K_2 on vertex set $\{\mathtt{b}, \mathtt{s}\}$. Here E is for "edge," `b` is for "big," and `s` is for "small," for reasons that will now become clear.

Definition 3.5 (compound vertex)**.** Let G be a graph. Then G *on compound vertices*, denoted G^+, is the graph $G \cdot E$. This term is intended to be suggestive—we imagine G^+ as G with each of its vertices v replaced by a new compound structure with a big part $(v, \mathtt{b})$ and a small part $(v, \mathtt{s})$. We will always abbreviate, e.g., $(v, \mathtt{b})$ by $v^{\mathtt{b}}$. For a generic vertex of G^+ we write $v^{\mathtt{x}}$, $v^{\mathtt{y}}$, etc., understanding $\mathtt{x}, \mathtt{y} \in \{\mathtt{b}, \mathtt{s}\}$.

Definition 3.6 (edge types)**.** In the context of a given $K = K_{H,H}$, an edge $uw \in K$ is called *internal* if u and w belong to the same side, and *external* otherwise. Similarly, an edge $u^{\mathtt{x}}w^{\mathtt{y}} \in K^+$ with $u \neq w$ is *internal* if $uw \in K$ is internal and *external* if $uw \in K$ is external. An edge $v^{\mathtt{b}}v^{\mathtt{s}} \in K^+$ is called a *vertex edge*.

Definition 3.7 (external triangles)**.** Let H be a graph and $K = K_{H,H}$. A triangle in K or K^+ is an *external triangle* if it contains an external edge. A subgraph F of K or K^+ is *external triangle free* (*ETF*) if it contains no external triangles.

Definitions 3.8 (configurations and weight)**.** Let H be a graph with t vertices and m edges, and $K = K_{H,H}$. A *configuration on* K is a pair (F, ϕ), where $F \subseteq E(K^+)$ and $\phi : V(K^+) \to [0,1]$ satisfy the following conditions. Viewing F as a subgraph of K^+, F is ETF, contains all vertex edges of K^+, and satisfies $N_F(v^{\mathtt{b}}) \cap N_F(v^{\mathtt{s}}) = \varnothing \; \forall\, v \in V(K)$; and ϕ, which we call a *mass function*, satisfies $\phi(v^{\mathtt{b}}) \in [\frac{1}{2}, 1]$ and $\phi(v^{\mathtt{s}}) = 1 - \phi(v^{\mathtt{b}}) \; \forall\, v \in V(K)$. Given a configuration and $c \in [0,1]$, the configuration's *c-weight* is

$$\mathtt{w}_c(F,\phi) = \frac{1-c}{4m} \sum_{\substack{u^{\mathtt{x}}w^{\mathtt{y}} \in F \\ \text{internal}}} \phi(u^{\mathtt{x}})\phi(w^{\mathtt{y}}) + \frac{1-c}{2t^2} \sum_{\substack{u^{\mathtt{x}}w^{\mathtt{y}} \in F \\ \text{external}}} \phi(u^{\mathtt{x}})\phi(w^{\mathtt{y}}) + \frac{c}{t} \sum_{v \in V(K)} \phi(v^{\mathtt{b}})\phi(v^{\mathtt{s}}). \tag{3.3}$$

Here's the idea behind c-weight. Given H, we think of the vertices and edges of K as having weights attached, as follows. Each vertex weighs $\frac{c}{2t}$, each internal edge weighs $\frac{1-c}{4m}$, and each external edge weighs $\frac{1-c}{2t^2}$, for a total of unit weight on K. Passing to K^+, an adversary tries to maximize the amount of this weight he can capture in a configuration (F, ϕ). For each edge $uw \in K$, the fraction of that edge's weight that he captures is $\sum_{u^{\mathtt{x}}w^{\mathtt{y}} \in F} \phi(u^{\mathtt{x}})\phi(w^{\mathtt{y}})$, because we think of the weight of $uw \in K$ as being split among the four corresponding edges of K^+ with a $\phi(u^{\mathtt{x}})\phi(w^{\mathtt{y}})$-fraction residing in the edge $u^{\mathtt{x}}w^{\mathtt{y}}$. For each vertex $v \in V(K)$, the fraction of that vertex's weight that our adversary captures is $2\phi(v^{\mathtt{b}})\phi(v^{\mathtt{s}})$, because we think of the weight of a vertex in K as being split up in K^+ analogously to the way the weight of an edge in K is split up in K^+, with a $\phi(v^{\mathtt{b}})^2$-fraction of the weight of v residing in $v^{\mathtt{b}}$, a $\phi(v^{\mathtt{s}})^2$-fraction in $v^{\mathtt{s}}$, and the remaining $2\phi(v^{\mathtt{b}})\phi(v^{\mathtt{s}})$-fraction in the vertex edge $v^{\mathtt{b}}v^{\mathtt{s}}$. This 2 cancels the $\frac{1}{2}$ in the vertex weight $\frac{c}{2t}$ to yield the coefficient of the third sum in (3.3). To see that the 2 is natural, observe that it lets our adversary capture exactly half the weight of every vertex and edge of K by taking $F = \{u^{\mathtt{b}}w^{\mathtt{s}} \mid uw \in K\} \cup \{v^{\mathtt{b}}v^{\mathtt{s}} \mid v \in V(K)\}$ and $\phi \equiv \frac{1}{2}$. We call this the *naïve configuration.*

Definition 3.9 (fairness)**.** For $c \in [0,1]$, a graph H is called *c-fair* if

$$\max\big(\mathtt{w}_c(F,\phi)\big) = 1/2, \tag{3.4}$$

where the max is over configurations (F,ϕ) on K.

Observe that the $1/2$ in (3.4) is best possible, since the naïve configuration has c-weight $1/2$ for any c. This explains the term "fair"—our adversary can't capture more than half the weight of K, the amount to which he is naïvely entitled.

Observe also that increasing c can only make life harder for our adversary. That is, if H is c-fair, then it is c'-fair for any $c' \in [c,1]$. To see this, notice that $\mathtt{w}_c(F,\phi)$ is a convex combination of the nonnegative quantities

$$\frac{1}{2m}\sum_{\substack{u^{\mathtt{x}}w^{\mathtt{y}}\in F\\ \text{internal}}}\phi(u^{\mathtt{x}})\phi(w^{\mathtt{y}}), \qquad \frac{1}{t^2}\sum_{\substack{u^{\mathtt{x}}w^{\mathtt{y}}\in F\\ \text{external}}}\phi(u^{\mathtt{x}})\phi(w^{\mathtt{y}}) \quad \text{and} \quad \frac{1}{t}\sum_{v\in V(K)}\phi(v^{\mathtt{b}})\phi(v^{\mathtt{s}}),$$

with coefficients $\frac{1-c}{2}$, $\frac{1-c}{2}$, c. Since the first two coefficients are decreasing in c and the third quantity is at most $1/2$ (note each of the $2t$ terms in its sum is at most $1/4$), increasing c cannot raise $\mathtt{w}_c(F,\phi)$ above $1/2$. At the extremes, it is easy to see that no graph is 0-fair and every graph is 1-fair. This, finally, motivates

Lemma 3.10. *For any $c \in (0,1]$ and $N \in \mathbb{N}$, there exists a triangle-free, d-regular, c-fair graph H with $d \geq N$.*

3.3 Proof of Theorem 3.3

Fixing $\alpha > 0$ (we may assume $\alpha < 1/3$), our goal is to show there are arbitrarily large graphs G of positive density satisfying $\tau_3(G) > (1-o(1))|G|/2$ but nonetheless $\nu_3(G) < (1+\alpha)|G|/4$. To do this, we use a probabilistic construction starting with a graph promised by Lemma 3.10.

Set $c = \alpha/6$ and let H be a triangle-free, d-regular, c-fair graph on t vertices, where $d \geq (2c)^{-1}$. Let $p = \frac{1-c}{2cd}$ and $q = \frac{1-c}{2ct}$, noting that $p, q \in (0,1)$. Let $K = K_{H,H}$, and observe that $K \cdot K_a$ is the graph obtained from K when each vertex is "blown up" to a clique of size a. Call each of these K_a's in $K \cdot K_a$ a *block*, and for each $v \in V(K)$, denote by B_v the block corresponding to v. Also, consistent with Definition 3.6, call an edge $xy \in K \cdot K_a$ an *internal edge*, *external edge*, or *vertex edge* according to whether it comes from an internal edge, external edge, or vertex of K.

For each $a \in \mathbb{N}$ (think: large), let G_a be the random graph obtained from $K \cdot K_a$ by deleting each internal edge with probability $1-p$ and each external edge with probability $1-q$, these choices made independently. Then since $|\nabla_{G_a}(B_u, B_w)| \sim \mathrm{Bin}(a^2, p)$ for each internal $uw \in K$ and $|\nabla_{G_a}(B_u, B_w)| \sim \mathrm{Bin}(a^2, q)$ for each external $uw \in K$, Theorem 2.1 says that each of these numbers $|\nabla_{G_a}(B_u, B_w)|$ is typically close to its expectation. To be precise, for each $uw \in K$ (internal or external), if we set $X_{uw} = |\nabla_{G_a}(B_u, B_w)|$, $\mu_{uw} = \mathbb{E}X_{uw}$ and $x = a \log a$, then Theorem 2.1 gives $\mathbb{P}(|X_{uw} - \mu_{uw}| \geq x) = O(a^{-2}) = o(1)$ as $a \to \infty$. Since $|K| = t^2 + td$ is fixed, $\mu_{uw} = \Theta(a^2)$ and $x = o(a^2)$, it holds w.h.p. as $a \to \infty$ that $X_{uw} \sim \mu_{uw}$ for all $uw \in K$. We may thus assume G_a satisfies this property, whence

$$|\{xy \in G_a \mid xy \text{ internal}\}| \sim tda^2p = \frac{a^2t(1-c)}{2c}; \tag{3.5}$$

$$|\{xy \in G_a \mid xy \text{ external}\}| \sim t^2a^2q = \frac{a^2t(1-c)}{2c}; \tag{3.6}$$

$$|\{xy \in G_a \mid xy \text{ vertex}\}| = 2t\binom{a}{2} \sim a^2t. \tag{3.7}$$

We claim that, w.h.p. as $a \to \infty$, G_a meets the requirements of Theorem 3.3. The first and third conditions are easy to check. For density, letting $n = |V(G_a)| = 2ta$ and $m = |G_a|$, we have

$$m \sim a^2t + 2\frac{a^2t(1-c)}{2c} = \frac{a^2t}{c} = n^2(4tc)^{-1}, \tag{3.8}$$

where $(4tc)^{-1} < 1/2$ is a constant.

To see that $\nu_3(G_a) < (1+\alpha)m/4$, it suffices to find a fractional triangle edge cover of

G_a of total weight less than $(1+\alpha)m/4$, since (recall) $\nu_3(G_a) \le \nu_3^*(G_a) = \tau_3^*(G_a)$. But this is easy: simply placing weight 1 on all vertex edges and weight 1/2 on all external edges yields a fractional triangle edge cover of G_a (here the triangle-freeness of H is crucial) with total weight asymptotic to

$$a^2t + \frac{1}{2}\frac{a^2t(1-c)}{2c} = \frac{a^2t}{4c}(1+3c) = (1+\alpha/2 \pm o(1))\frac{m}{4} < (1+\alpha)\frac{m}{4}.$$

The real work is showing that $\tau_3(G_a) > (1-o(1))m/2$. To this end let $F \subseteq G_a$ be triangle-free; we need to show $|F| \le (1+o(1))m/2$. More precisely, we show that given any $\delta > 0$, we have $|F| < (1+\delta)m/2$ for large enough a. For this we apply the usual Regularity Lemma—i.e. Lemma 2.21 with $b = r = s_1 = 1$—to F. Pick (with foresight) $\epsilon < \delta/(48tc)$, and let $2t\lceil\epsilon^{-1}\rceil$ be the "m" of the lemma. Let $\Pi = (V_0, V_1, \ldots, V_k)$ be the partition given by the lemma. By comments (i) and (ii) after the lemma, we may assume Π refines the partition of $V(F) = V(G_a)$ into blocks and splits each block into exactly $k/(2t) =: \eta$ nonexceptional parts plus some vertices in V_0.

For a pair $V_i, V_j \in \Pi$ with $V_i \subseteq B_u$ and $V_j \subseteq B_w$, call the pair *internal* or *external* if uw is an internal or external edge of K (respectively), and a *vertex pair* if $u = w$. Consider the graph on $[k]$ where ij is an edge iff V_i, V_j is an internal, external or vertex pair. Notice that this graph is (isomorphic to) $K \cdot K_\eta$, with blocks $B'_v = \{i \in [k] \mid V_i \subseteq B_v\}$, $v \in V(K)$. Letting $l = |V_1|$, observe also that

$$\text{each} \left\{\begin{array}{l} \text{vertex} \\[1ex] \text{internal} \\[1ex] \text{external} \end{array}\right\} \text{edge } uw \in K \cdot K_\eta \text{ corresponds to } \left\{\begin{array}{l} \text{exactly } l^2 \\[1ex] \text{about } l^2 p \\[1ex] \text{about } l^2 q \end{array}\right\} \text{edges of } G_a, \tag{3.9}$$

where just as in (3.5)–(3.7), each "about" in (3.9) hides an $\tilde{O}(l) = \tilde{O}(n) = o(m)$ Chernoff error as $a \to \infty$.

To account for the different quantities on the right side of (3.9), we assign weights to the edges of $K \cdot K_\eta$: each vertex edge weighs $c/(t\eta^2)$, each internal edge $pc/(t\eta^2) = \frac{1-c}{2td\eta^2}$, and each external edge $qc/(t\eta^2) = \frac{1-c}{2t^2\eta^2}$, so that the weight $\mathtt{w}(uw)$ of $uw \in K \cdot K_\eta$ is $c/(t\eta^2 l^2)$ times the (approximate) number of corresponding edges in G_a. With these weights, the total weight of the edges corresponding to an internal $uw \in K$ is $\frac{1-c}{2td}$, the total weight of the edges corresponding to an external $uw \in K$ is $\frac{1-c}{2t^2}$, and the total weight of the edges in a block B'_v is $\binom{\eta}{2}\frac{c}{t\eta^2} \lesssim \frac{c}{2t}$ (where $\lesssim$ means approximate equality and $\leq$).

Leaving the topic of edge weights for a moment, we now let F' be the subgraph of F obtained after we delete the following edges from F: edges incident to V_0; edges inside some V_i, $i \in [k]$; edges that join pairs that are not $(1; F, \epsilon)$-regular; and edges that join pairs with $(1; F)$-density less than 2ϵ. (This cleanup is of course a standard concomitant of the Regularity Lemma.) Since $l \leq n/k$, this deletes at most

$$\epsilon n^2 + k\binom{l}{2} + \epsilon\binom{k}{2}l^2 + 2\epsilon l^2\binom{k}{2} \leq 3\epsilon n^2 \tag{3.10}$$

edges from F.

Let $\tilde{F}$ be the subgraph of $K \cdot K_\eta$ with $ij \in \tilde{F}$ iff there is an edge joining V_i and V_j in F'. By Lemma 2.22 (with $s = 1$) and the triangle-freeness of F, $\tilde{F}$ is also triangle-free. Let F'' be the subgraph of G_a defined by

$$\nabla_{F''}(V_i, V_j) = \begin{cases} \nabla_{G_a}(V_i, V_j) & \text{if } ij \in \tilde{F} \\ \varnothing & \text{if } ij \notin \tilde{F} \end{cases}.$$

With these definitions, (3.9), (3.10) and the calculations between them give

$$|F| \leq |F'| + 3\epsilon n^2 \leq |F''| + 3\epsilon n^2 \sim \mathtt{w}(\tilde{F})/(c/(t\eta^2 l^2)) + 3\epsilon n^2, \tag{3.11}$$

where (of course) $\mathtt{w}(\tilde{F}) = \sum_{uw \in \tilde{F}} \mathtt{w}(uw)$.

Our next goal is to massage $\tilde{F}$ until it resembles a configuration on K. For each $x \in V(\tilde{F}) = V(K \cdot K_\eta)$, let $\mathtt{w}(x)$ be the sum of the weights of its incident $\tilde{F}$-edges.[1] Fix some order π of $V(K)$, and for each $v \in V(K)$, in the chosen order, do the following, making changes to $\tilde{F}$ as necessary. We continue to write $\tilde{F}$ for the evolving graph.

[1]For the rest of the argument we use x, y, z and w, rather than i and j, for vertices of $K \cdot K_\eta$, since we want several letters from the same part of the alphabet. We use u and v for vertices of K.

1. Pick $x \in B'_v$ such that $\mathtt{w}(x) = \max_{y \in B'_v} \mathtt{w}(y)$.

2. Set $S_v = \{y \in B'_v \mid xy \in \tilde{F}\}$ and $T_v = B'_v \setminus S_v$.

3. For each $y \in T_v \setminus \{x\}$, replace $N_{\tilde{F}}(y)$ by $N_{\tilde{F}}(x)$.

4. Pick $z \in S_v$ such that $\mathtt{w}(z) = \max_{w \in S_v} \mathtt{w}(w)$.

5. For each $w \in S_v \setminus \{z\}$, replace $N_{\tilde{F}}(w)$ by $N_{\tilde{F}}(z)$.

Let $\tilde{F}' \subseteq K \cdot K_\eta$ be the graph obtained from $\tilde{F}$ after performing these steps for each $v \in V(K)$. We make the following observations about $\tilde{F}'$:

(i) $\mathtt{w}(\tilde{F}') \geq \mathtt{w}(\tilde{F})$;

(ii) $\tilde{F}'$ is triangle-free, since $\tilde{F}$ is—note in particular that $S_v \subseteq N_{\tilde{F}}(x)$ implies $\tilde{F}[S_v] = \varnothing$;

(iii) For each $v \in V(K)$, $\tilde{F}'[B'_v]$ is the complete bipartite graph between S_v and T_v; and

(iv) For each $v \in V(K)$, $z, w \in S_v$, and $x, y \in T_v$, we have $N_{\tilde{F}'}(z) = N_{\tilde{F}'}(w)$ and $N_{\tilde{F}'}(x) = N_{\tilde{F}'}(y)$.

The only tricky point here is (iv). Clearly for a given $u \in V(K)$, the condition in (iv) holds at u immediately after we perform steps 1–5 at u. But how do we know we don't violate the condition at u in the process of doing 1–5 at some other $v \in V(K)$ coming later in π? Assume we do, so that there exist $x, y \in R_u \in \{S_u, T_u\}$ and $z \in B'_v$ such that $xz \in \tilde{F}'$ and $yz \notin \tilde{F}'$. Just before we began 1–5 at v, z was $\tilde{F}$-adjacent to either both of x, y or neither, so we must have replaced $N_{\tilde{F}}(z)$ in the course of doing 1–5 at v. So there was some $w \in B'_v$ (whose $\tilde{F}$-neighborhood replaced that of z) which, just before beginning 1–5 at v, was $\tilde{F}$-adjacent to exactly one of x, y. But this is a contradiction.

For each $v \in V(K)$, let R_v be the larger of S_v, T_v, and P_v the smaller (choose arbitrarily if they are the same size). Let $\hat{F}$ be the subgraph of K^+ obtained from $\tilde{F}'$ by collapsing each R_v to a vertex v^{b} and each P_v to a vertex v^{s}, and set $\phi(v^{\mathsf{b}}) = |R_v|/\eta$ and $\phi(v^{\mathsf{s}}) = |P_v|/\eta = 1 - \phi(v^{\mathsf{b}})$ for each $v \in V(K)$. Then (ii)–(iv) imply that $(\hat{F}, \phi)$

is a configuration on K, after adding vertex edges $v^{\mathtt{b}}v^{\mathtt{s}}$ for those $v \in V(K)$ for which $P_v = \varnothing$ (if any).

Now since H is c-fair, we have $\mathtt{w}_c(\hat{F}, \phi) \leq 1/2$. By the weight calculations after (3.9), we have $\mathtt{w}_c(\hat{F}, \phi) \geq \mathtt{w}(\tilde{F}')$ (the only error here comes from the weight in a block of $K \cdot K_\eta$ being $\binom{\eta}{2}\frac{c}{t\eta^2}$ instead of exactly $\frac{c}{2t}$). Thus by (3.11) and (i), using $\eta l \leq n/(2t)$ and $\epsilon < \delta/(48tc)$, we have

$$\begin{aligned}
|F| \leq \frac{\mathtt{w}(\tilde{F})}{c/(t\eta^2 l^2)} + 3\epsilon n^2 + o(m) &\leq \frac{\mathtt{w}(\tilde{F}')}{c/(t\eta^2 l^2)} + 3\epsilon n^2 + o(m) \\
&\leq \frac{1/2}{c/(t\eta^2 l^2)} + 3\epsilon n^2 + o(m) \\
&< n^2(8tc)^{-1} + n^2\delta(16tc)^{-1} + o(m) \\
&< (1 + \delta/2 + o(1))m/2 \\
&< (1+\delta)m/2,
\end{aligned}$$

where the penultimate inequality recalls (3.8) and the last holds for large enough a. □

3.4 Proof of Lemma 3.10

We now turn to the proof of Lemma 3.10, that for any $c > 0$ there are triangle-free, d-regular, c-fair graphs H with arbitrarily large d. Luckily we need not invent anything here; rather we show—though not so easily—that for any fixed c, all sufficiently large graphs from a well-known family are c-fair. The relevant family was described by Noga Alon in [1]; since he proved therein that all graphs in this family are triangle-free and regular, with degree going to infinity, this will prove Lemma 3.10. We first list the relevant properties of these graphs.

Proposition 3.11 ([1, Theorem 2.1]). *For all $t_0 \in \mathbb{N}$, there exist $t \geq t_0$ and a triangle-free graph H_t on t vertices satisfying*

- *H_t is d-regular, with $d = \Theta(t^{2/3})$, and* (3.12)
- *all eigenvalues λ_i of H_t, other than the largest, satisfy $|\lambda_i| = O(\sqrt{d}) = O(t^{1/3})$.* (3.13)

Alon gives much more detailed information about these graphs, including a precise formula for d and bounds on the eigenvalues, but the above properties are all we will need. In fact, a weaker eigenvalue bound than (3.13) would suffice for our purposes. (We need such a bound primarily to guarantee good density properties for H, for which our (standard) tool is Lemma 2.16). It is probably not too hard—e.g. by random methods, somewhat relaxing the regularity requirement of Lemma 3.10—to produce other families of graphs, less nice than Alon's, that would be adequate here. Recognizing this, we nonetheless gladly use Alon's graphs because they are convenient and they work.

Setup for the rest of this chapter. We fix $c \in (0, 1]$ at the outset, and throughout we let (F, ϕ) be a configuration on $K = K_{H,H}$, where $H = H_t$ for some t. We denote the degree of H by d and its eigenvalues by $d = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_t$, and set $\lambda = \max_{i>1} |\lambda_i|$.

Goal: *To show that H is c-fair whenever t is sufficiently large.* (3.14)

Each proposition in what follows is an asymptotic statement, making some claim about H or (F, ϕ) as t grows to infinity; thus our asymptotic notation all refers to $t \to \infty$. Our usage here may be a little confusing, since we treat t as tending to infinity, whereas the discussion in Section 3.3 calls for a *fixed* $H = H_t$ depending on c (that is, on α). But of course what we are showing here is that *given* c, H_t is c-fair for large enough t, so that for our application in Section 3.3 we can fix such a t. We always assume (as we may) that $\mathtt{w}_c(F, \phi) \geq 1/2$; we want to show that in fact $\mathtt{w}_c(F, \phi) = 1/2$.

Though a configuration on K is defined *via* K^+, it will be more convenient in what follows to think of it in terms of K itself. We next set up some notation and terminology for this purpose.

Definitions 3.12 (edge classes, weight captured, gain/loss)**.** Given a graph $H = H_t$ and a configuration (F, ϕ) on $K = K_{H,H}$, we divide the edges of K into four classes. An edge $uw \in K$ is of

- *class 1* if $u^{\mathsf{b}}w^{\mathsf{b}}, u^{\mathsf{s}}w^{\mathsf{s}} \in F$,
- *class 2* if $u^{\mathsf{b}}w^{\mathsf{b}} \in F$, $u^{\mathsf{s}}w^{\mathsf{s}} \notin F$,
- *class 3* if $u^{\mathsf{b}}w^{\mathsf{s}}, u^{\mathsf{s}}w^{\mathsf{b}} \in F$, and
- *class 4* otherwise.

For each $uw \in K$, we will say our configuration *captures* the fraction $\sum_{u^{\mathsf{x}}w^{\mathsf{y}} \in F} \phi(u^{\mathsf{x}})\phi(w^{\mathsf{y}})$ of the weight of the edge. This weight is $\frac{1-c}{2td}$ for internal edges and $\frac{1-c}{2t^2}$ for external edges. Similarly, we say our configuration *captures* the fraction $2\phi(v^{\mathsf{b}})\phi(v^{\mathsf{s}})$ of the weight of each vertex v of K. This weight is $\frac{c}{2t}$. For $v \in V(K)$, set $\delta_v = \phi(v^{\mathsf{b}}) - 1/2$, so that δ_v measures how far from evenly the configuration splits the mass of v. Then e.g. if $uw \in K$ is of class 1, our configuration captures the fraction $(1/2 + \delta_u)(1/2 + \delta_w) + (1/2 - \delta_u)(1/2 - \delta_w) = 1/2 + 2\delta_u\delta_w$ of the weight of uw, and if uw is of class 3 then it captures the fraction $1/2 - 2\delta_u\delta_w$. Similarly, it captures the fraction $1/2 - 2\delta_v^2$ of the weight of each vertex v.

Given $uw \in K$, we sometimes want to compare the fraction of the weight of uw captured by our configuration to the fraction of the weight of uw captured by the naïve configuration, namely $1/2$. We call this difference $\sum_{u^{\mathsf{x}}w^{\mathsf{y}} \in F} \phi(u^{\mathsf{x}})\phi(w^{\mathsf{y}}) - 1/2 \in [-1/2, 1/2]$ the *fractional gain at uw*, and its negative the *fractional loss at uw*. (Either of these can be positive or negative.) More often we want to weight the fractional gain (loss) at an edge by the appropriate edge weight ($\frac{1-c}{2td}$ or $\frac{1-c}{2t^2}$); we call this product simply the *gain* (*loss*) at the edge (no "fractional"). (Examples: if the fractional gain at internal edge uw is .16, then the gain at uw is $.16(\frac{1-c}{2td})$; if vz is an external edge of class 3, then the loss at vz is $2\delta_v\delta_z(\frac{1-c}{2t^2})$.) We use analogous terminology for vertices: the *fractional loss at v* is $2\delta_v^2$, and the *loss at v* is $c\delta_v^2/t$.

Write ζ_i (respectively ζ_e) for the average fraction of the weight of an internal (respectively external) edge captured by our configuration—that is,

$$\zeta_i = \frac{1}{td} \sum_{\substack{u^{\mathrm{x}} w^{\mathrm{y}} \in F \\ \text{internal}}} \phi(u^{\mathrm{x}})\phi(w^{\mathrm{y}}) \quad \text{and} \quad \zeta_e = \frac{1}{t^2} \sum_{\substack{u^{\mathrm{x}} w^{\mathrm{y}} \in F \\ \text{external}}} \phi(u^{\mathrm{x}})\phi(w^{\mathrm{y}})$$

—and set $\gamma_i = \zeta_i - 1/2$, $\gamma_e = \zeta_e - 1/2$. Thus γ_i and γ_e represent the average fractional gain of our configuration on internal and external edges of K, respectively. Lastly, write δ for the average of the δ_v's over $V(K)$. $\Diamond$

With these definitions, notice that $\left(\frac{1-c}{2}\right)(\gamma_i + \gamma_e)$ is the total gain over all edges of K. So, to reiterate (3.14), our goal is to show that this is always counterbalanced by an equal or larger loss in the vertices of K whenever t is sufficiently large. What follows is a long string of propositions culminating in a proof of this.

Proposition 3.13. *Let R be an ETF subgraph of K containing fractions $\xi_i(R)$ and $\xi_e(R)$ of the internal and external edges of K, respectively. Then*

$$\xi_i(R) + \xi_e(R) < 1 + o(1). \tag{3.15}$$

Proof. We apply Lemma 2.21 with $r = b = 2$, ϵ arbitrarily small but fixed, $m = 2\lceil \epsilon^{-1} \rceil$, $H_1 = R[X] \cup R[Y]$, $H_2 = \nabla_R(X, Y)$, $s_1 = d/t$, and $s_2 = 1$.

We must first check that (for large enough t) H_1 is $(d/t; 2, \beta)$-bounded and H_2 is $(1; 2, \beta)$-bounded, where $\beta = \beta(\epsilon, b, m, r) > 0$ is given by the lemma (but of course the statement is really that these hold for any fixed β and, again, sufficiently large t). The second of these is trivial. For the first, letting $U, W \subseteq V(K)$ be disjoint with

$|U|, |W| \geq 2t\beta$, we have, using Lemma 2.16,

$$\begin{aligned} d_{d/t,H_1}(U,W) &= \frac{|\nabla_{H_1}(U,W)|}{(d/t)|U||W|} = \frac{|\nabla_H(U\cap X, W\cap X)|}{(d/t)|U||W|} + \frac{|\nabla_H(U\cap Y, W\cap Y)|}{(d/t)|U||W|} \\ &\leq \frac{|U\cap X||W\cap X|d/t + \lambda\sqrt{|U\cap X||W\cap X|}}{(d/t)|U||W|} \\ &\qquad + \frac{|U\cap Y||W\cap Y|d/t + \lambda\sqrt{|U\cap Y||W\cap Y|}}{(d/t)|U||W|} \\ &\leq \frac{|U||W|d/t + \lambda\sqrt{|U||W|}}{(d/t)|U||W|} \leq 1 + o(1), \end{aligned}$$

which is at most 2 for large enough t.

Let $\Pi = (V_0, V_1, \ldots, V_k)$ be the partition given by Lemma 2.21. By comment (i) following the lemma we may assume each nonexceptional part of Π is contained in either X or Y, and by comment (ii) we may assume $|V_0 \cap X| = |V_0 \cap Y|$, implying that X and Y each contain exactly $k/2$ parts of Π. Given a pair of nonexceptional parts of Π, we say the pair is *external* if exactly one of them is contained in X, and *internal* otherwise.

We now delete the following edges from R: edges incident to V_0; edges inside some V_i, $i \in [k]$; edges that join (internal) pairs that are not $(d/t; H_1, \epsilon)$-regular; edges that join (external) pairs that are not $(1; H_2, \epsilon)$-regular; edges that join internal pairs with $(d/t; H_1)$-density less than 2ϵ; and edges that join external pairs with $(1; H_2)$-density less than 2ϵ. The following table lists upper bounds for the numbers of edges deleted from H_1 and H_2 in each of these categories. For convenience we set $l := |V_1| \leq 2t/k$.

	$H_1 = R[X] \cup R[Y]$	$H_2 = \nabla_R(X,Y)$
edges incident to V_0	$\leq d\|V_0\| \leq 2\epsilon td$	$\leq t\|V_0\| \leq 2\epsilon t^2$
edges inside some V_i	$\leq k\left(\frac{l^2 d}{2t} + \frac{\lambda l}{2}\right) \leq 2td/k + t\lambda$ $\leq \epsilon(td + t\lambda k) \leq \epsilon td(1+o(1))$	0
edges joining pairs that are not $(d/t; H_1, \epsilon)$-regular	$\leq \epsilon\binom{k}{2}\left(\frac{l^2 d}{t} + \lambda l\right) \leq \epsilon(2td + \lambda tk)$ $\leq \epsilon td(2 + o(1))$	0
edges joining pairs that are not $(1; H_2, \epsilon)$-regular	0	$\leq \epsilon\binom{k}{2}l^2 \leq 2\epsilon t^2$
edges joining internal pairs with $(d/t; H_1)$-density less than 2ϵ	$\leq 2\binom{k/2}{2}(2\epsilon l^2 d/t) \leq 2\epsilon dt$	0

edges joining external pairs with $(1; H_2)$-density less than 2ϵ	0	$\leq (k/2)^2 2\epsilon l^2$ $\leq 2\epsilon t^2$
TOTAL	$\leq (7 + o(1))\epsilon td$	$\leq 6\epsilon t^2$

Let $\tilde{X} = \{i \in [k] \mid V_i \subseteq X\}$ and $\tilde{Y} = [k] \setminus \tilde{X}$. Let $\tilde{R}$ be the graph on $\tilde{X} \cup \tilde{Y}$ where $ij \in \tilde{R}$ iff there is an undeleted edge joining V_i and V_j in R. Then since R is ETF, Lemma 2.22 gives that $\tilde{R}$ is as well (meaning, as usual, that it contains no triangles meeting both $\tilde{X}$ and $\tilde{Y}$).

Now each internal edge of $\tilde{R}$ corresponds to a pair in Π whose R-edges contribute a total of at most

$$\frac{l^2 d/t + \lambda l}{td} \leq \frac{4}{k^2} + \frac{2\lambda}{kd} = 4/k^2 + o(1)$$

to the fraction $\xi_i(R)$. Similarly each external edge of $\tilde{R}$ corresponds to a pair in Π whose R-edges contribute a total of at most $l^2/t^2 \leq 4/k^2$ to the fraction $\xi_e(R)$. By Lemma 2.18 $|\tilde{R}| \leq k^2/4$, so the contribution to $\xi_i(R) + \xi_e(R)$ from undeleted R-edges is at most $1 + k^2 o(1) = 1 + o(1)$. And as computed in the table above, the contribution to $\xi_i(R) + \xi_e(R)$ from deleted R-edges is at most $13\epsilon + o(1)$. Thus $\xi_i(R) + \xi_e(R) \leq 1 + 13\epsilon + o(1)$. Since ϵ was arbitrarily small, the proposition is proved. $\square$

We now return to our configuration (F, ϕ).

Proposition 3.14. *We have $\zeta_i + \zeta_e < 1 + o(1)$, or equivalently,*

$$\gamma_i + \gamma_e < o(1). \tag{3.16}$$

Proof. Suppose that for each $v \in V(K)$ we randomly choose one of $v^{\mathsf{b}}, v^{\mathsf{s}}$, with $\Pr(v^{\mathsf{x}}) = \phi(v^{\mathsf{x}})$ and these choices made independently. This produces a random ETF subgraph R of K in the obvious way: $uw \in R$ iff $u^{\mathsf{x}} w^{\mathsf{y}} \in F$, where we chose $u^{\mathsf{x}} \in \{u^{\mathsf{b}}, u^{\mathsf{s}}\}$ and $w^{\mathsf{y}} \in \{w^{\mathsf{b}}, w^{\mathsf{s}}\}$. Observe that $\Pr(uw \in R)$ is the fraction of the weight of uw captured by our configuration. With this observation, we calculate

$$\begin{aligned}\zeta_i + \zeta_e &= \frac{1}{td}\sum_{\substack{uw\in K\\ \text{internal}}} \Pr(uw \in R) + \frac{1}{t^2}\sum_{\substack{uw\in K\\ \text{external}}} \Pr(uw \in R)\\ &= \mathbb{E}[|R \cap (K[X] \cup K[Y])|/td] + \mathbb{E}[|R \cap \nabla(X,Y)|/t^2]\\ &= \mathbb{E}[\xi_i(R) + \xi_e(R)]\\ &< 1 + o(1),\end{aligned}$$

where the last inequality is given by Proposition 3.13. □

Proposition 3.15. *We have $\delta = o(1)$.*

Proof. We simply calculate $\mathtt{w}_c(F,\phi)$ (which, recall, we assume is at least $1/2$):

$$\begin{aligned}\mathtt{w}_c(F,\phi) &= \frac{1-c}{2}\zeta_i + \frac{1-c}{2}\zeta_e + \frac{c}{2t}\sum_{v\in V(K)} (1/2 - 2\delta_v^2)\\ &= 1/2 + \left(\frac{1-c}{2}\right)(\gamma_i + \gamma_e) - \frac{c}{t}\sum_{v\in V(K)} \delta_v^2\\ &\le 1/2 + o(1) - 2c\left(\frac{1}{2t}\sum_{v\in V(K)} \delta_v\right)^2\\ &= 1/2 - 2c\delta^2 + o(1),\end{aligned}$$

where we used Proposition 3.14 and Cauchy-Schwarz between the second and third lines. □

From now on we call a vertex v of K *balanced* if $\delta_v < \sqrt{\delta}$, and *unbalanced* otherwise; thus, in view of Proposition 3.15, all but a $o(1)$-fraction of the vertices of K are balanced. Also, we let G be the subgraph of K consisting of all edges of classes 1–3, and Γ the subgraph of G consisting of edges of classes 1 and 2. Notice that since F is ETF,

$$\Gamma \text{ has even intersection with every external triangle in } G. \tag{3.17}$$

The next three facts say that in various senses, as t grows, G accounts for nearly all of

K.

Proposition 3.16. *The total loss on $K \setminus G$ is $o(1)$.*

Proof. The total gain on G is at most what it would be if all edges of K were of class 1. Since at most $o(t)$ vertices are unbalanced, the total weight of all edges of K incident to unbalanced vertices is $o(1)$, so this gain is at most

$$(1-c)2\sqrt{\delta}^2 + o(1)(1-c)2(1/2)^2,$$

which is $o(1)$ by Proposition 3.15. Thus if the loss on $K \setminus G$ were $\Omega(1)$, we would have $\mathtt{w}_c(F,\phi) < 1/2$ for sufficiently large t (since loss on vertices is always nonnegative). $\square$

Corollary 3.17. *There are at most $o(t^2)$ class 4 edges in K.*

Proof. Assume otherwise, so that $|K \setminus G| = \Omega(t^2)$. Then since at most a $o(1)$-fraction of the edges of K are incident to unbalanced vertices, most class 4 edges join two balanced vertices. The fractional loss at any such edge is $\Omega(1)$ (at least about $1/4$, in view of Proposition 3.15), so the total loss on $K \setminus G$ is $\Omega(1)$, contradicting Proposition 3.16. $\square$

Corollary 3.18. *There are at most $o(td)$ class 4 edges in each of $K[X]$, $K[Y]$.*

Proof. Assume for a contradiction that $|(K \setminus G)[X]| = \Omega(td)$ (the proof for Y is of course the same). Then since at most $o(td)$ edges of $K[X]$ are incident to unbalanced vertices, most class 4 edges in $K[X]$ join two balanced vertices. The fractional loss at any such edge is $\Omega(1)$ (at least about $1/4$, in view of Proposition 3.15), so the total loss on $K \setminus G$ is $\Omega(1)$, contradicting Proposition 3.16. $\square$

The next result concerns only H, not K or (F,ϕ).

Proposition 3.19. *For any $H' \subseteq H$ of size $(1-o(1))|H|$, there is a $U \subseteq V(H)$ of size $o(t)$ such that $H' - U$ is connected and $\mathcal{C}(H' - U)$ is spanned by cycles of length up to 11.*

Proof. By Proposition 2.17 (and noting that finite diameter implies connectedness), it suffices to find a U of size $o(t)$ such that $H' - U$ has diameter at most 5. To this end, let $U_1 = \{v \in V(H) \mid d_{H \setminus H'}(v) \geq d/3\}$. Then $u_1 := |U_1| \leq 2|H \setminus H'|/(d/3) = o(t)$. Let $U_2 = \{v \in V(H) \setminus U_1 \mid |N(v) \cap U_1| \geq d/3\}$. We claim $u_2 := |U_2| = o(t^{1/3})$ (we just need $o(d)$). Indeed, applying Lemma 2.16 to H, we have $(1/3 - o(1))u_2 d \leq ||\nabla(U_1, U_2)| - \frac{u_1 u_2 d}{t}| \leq \lambda\sqrt{u_1 u_2}$, which (since $d = \Theta(t^{2/3})$ and $\lambda = O(t^{1/3})$) gives $u_2 \leq O(t^{-2/3} u_1) = o(t^{1/3})$, as claimed.

Set $U = U_1 \cup U_2$ and $H'' = H' - U$, and for each $v \in V(H'')$ denote by $N_2(v)$ the *second neighborhood* of v in H''; that is, the set of vertices at distance exactly 2 from v in H''. We want to show that H'' has diameter at most 5. For this it suffices to show that every v satisfies $d_2(v) := |N_2(v)| = \Omega(t)$, since for any $S, T \subseteq V(H'')$ with $|S|, |T| = \Omega(t)$ we have $\nabla_{H''}(S, T) \neq \varnothing$ (using Lemma 2.16 on H and the fact that $|H \setminus H'| = o(|H|)$).

To see that (for any v) $d_2(v) = \Omega(t)$, note first that $d_{H''}(v) \geq (1/3 - o(1))d$ $(= \Omega(d))$, since v loses at most a third of its H-neighbors to $H \setminus H'$, at most another third to U_1, and a $o(1)$-fraction to U_2. Thus, since H is triangle-free, $|\nabla_H(N_{H''}(v), N_2(v))| = \Omega(d^2) = \Omega(t^{4/3})$. On the other hand Lemma 2.16 gives $|\nabla_H(N_{H''}(v), N_2(v))| \leq d_{H''}(v) d_2(v) d/t + \lambda\sqrt{d_{H''}(v) d_2(v)} = O(t^{1/3}) d_2(v) + O(t^{2/3})\sqrt{d_2(v)}$, implying $d_2(v) = \Omega(t)$ as claimed. □

Corollary 3.20. *Any $H' \subseteq H$ of size $(1 - o(1))|H|$ has a component with $t - o(t)$ vertices.*

(This is strictly weaker than Proposition 3.19; we include it for easy reference later.)

We now return to K and our configuration (F, ϕ). The next result does most of the heavy lifting for Lemma 3.10.

Proposition 3.21. *There exist $S \subseteq V(K)$ of size $o(t)$ and a partition $A \sqcup B$ of $V(K) \setminus S$ such that $Z := \Gamma \triangle \nabla_G(A, B)$ satisfies*

$$Z \subseteq \nabla(X, Y)$$

and

$$d_Z(v) = o(t)\ \forall\, v \in V(K) \setminus S.$$

Proof. Let $\kappa = |(K \setminus G) \cap \nabla(X, Y)|/t^2$, which is $o(1)$ by Corollary 3.17. Let $S_0 = \{v \in V(K) \mid v \text{ is incident to at least } t\sqrt{\kappa} \text{ external class 4 edges}\}$. Then $|S_0|t\sqrt{\kappa} \leq 2\kappa t^2$, implying $|S_0| = O(t\sqrt{\kappa}) = o(t)$. Now apply Proposition 3.19 to each of $G[X \setminus S_0]$ and $G[Y \setminus S_0]$, which we may do by Corollary 3.18. Let S_1 be the union of S_0 and the two deleted sets from Proposition 3.19, and set $\bar{G} = G - S_1$, $\bar{X} = X \setminus S_1$ and $\bar{Y} = Y \setminus S_1$.

Let $\mathcal{T}(\bar{G})$ be the subspace of $\mathcal{C}(\bar{G})$ generated by the external triangles of $\bar{G}$. Then we observe, crucially:

$$\text{all cycles of } G[\bar{X}] \text{ and } G[\bar{Y}] \text{ of length up to 11 belong to } \mathcal{T}(\bar{G}). \tag{3.18}$$

To see this, let $C = x_1, \ldots, x_k x_1$ be a cycle, say in $G[\bar{X}]$, with $k \leq 11$. If there exists $y \in \bar{Y}$ with $x_i y \in G\ \forall\, i \in [k]$, then $C \in \mathcal{T}(\bar{G})$, because C is the sum of the triangles $x_i x_{i+1} y x_i$, where of course we take subscripts mod k. But if there is no such y then for some x_i we have

$$|\nabla_{K \setminus G}(x_i, \bar{Y})| \geq |\bar{Y}|/11,$$

implying $x_i \in S_0$, which it isn't.

Now by (3.18) and our choice of S_1, we have

$$\Gamma[\bar{X}] = \nabla_G(\bar{X}_1, \bar{X}_2) \quad \text{ and } \quad \Gamma[\bar{Y}] = \nabla_G(\bar{Y}_1, \bar{Y}_2)$$

for some partitions $\bar{X}_1 \sqcup \bar{X}_2$ of $\bar{X}$ and $\bar{Y}_1 \sqcup \bar{Y}_2$ of $\bar{Y}$, since Γ is orthogonal (over $\mathbb{F}_2$, recall) to all external triangles in $\bar{G}$ (see (3.17)), and thus to all cycles in $G[\bar{X}]$ and $G[\bar{Y}]$ of length up to 11 (by (3.18)), and thus to all cycles in $G[\bar{X}]$ and $G[\bar{Y}]$ (see Proposition 3.19).

By Corollary 3.20 we can find a $U \subseteq \bar{X} \cup \bar{Y}$ of size $o(t)$ such that $G[\bar{X} \setminus U]$ and $G[\bar{Y} \setminus U]$ are connected. Set $S = S_1 \cup U$, producing the S of the proposition. Finally, set $X'_1 = \bar{X}_1 \setminus U$, $X'_2 = \bar{X}_2 \setminus U$ and $X' = X'_1 \cup X'_2$ $(= \bar{X} \setminus U)$, and define Y'_1, Y'_2 and Y'

similarly.

Now suppose $x \in X'$. Since all but a $o(1)$-fraction of the external edges at x belong to $\nabla_G(x, Y')$, the subgraph of G induced by the corresponding vertices (that is, $G[N_G(x) \cap Y']$) has a component of size $t - o(t)$ (Corollary 3.20 again), say with vertex set $Y_1^x \cup Y_2^x$, where $Y_1^x \subseteq Y_1'$ and $Y_2^x \subseteq Y_2'$. Since $\Gamma[Y_1^x \cup Y_2^x] = \nabla_G(Y_1^x, Y_2^x)$, (3.17) gives

$$\begin{aligned} yz \in \nabla_G(Y_1^x, Y_2^x) &\implies |\Gamma \cap \{xy, xz\}| = 1, \\ yz \in G[Y_1^x] \cup G[Y_2^x] &\implies |\Gamma \cap \{xy, xz\}| \in \{0, 2\}. \end{aligned}$$

Thus the connectivity of $G[Y_1^x \cup Y_2^x]$ implies that

$$\nabla_\Gamma(x, Y_1^x \cup Y_2^x) \in \{\nabla_G(x, Y_1^x), \nabla_G(x, Y_2^x)\}. \tag{3.19}$$

Moreover, the connectivity of $G[X']$ and the fact that any $u, w \in X'$ have common G-neighbors in $(Y_1^u \cup Y_2^u) \cap (Y_1^w \cup Y_2^w)$ (in fact many, since $u, w \notin S_0$) imply "coherence" of the choices in (3.19), meaning that u and w choose the same option iff they are on the same side of $X_1' \cup X_2'$. Of course a similar analysis applies with the roles of X and Y reversed. Assuming without loss of generality that each $x \in X_1$ chooses $\nabla_\Gamma(x, Y_1^x \cup Y_2^x) = \nabla_G(x, Y_2^x)$ in (3.19), the proposition is proved, with $A = X_1' \cup Y_1'$ and $B = X_2' \cup Y_2'$. $\square$

At long last we can accomplish the goal set forth in (3.14).

Proof of Lemma 3.10. Let $S, A, B \subseteq V(K)$ and $Z \subseteq \nabla(X, Y)$ be as in Proposition 3.21, and set $W = V(K) \setminus S$ $(= A \cup B)$. We analyze $K[W]$ first, and edges meeting S later.

Set $p = \frac{1-c}{2td}$ and $q = \frac{1-c}{2t^2}$. Let φ be the vector indexed by $X \cup Y$ with

$$\varphi_v = \begin{cases} \delta_v & \text{if } v \in A \\ -\delta_v & \text{if } v \in B \\ 0 & \text{if } v \in S \end{cases} \cdot$$

Let C be the adjacency matrix of H, J the $t \times t$ matrix of 1's, and I the $2t \times 2t$ identity matrix. Lastly, let N be the weighted adjacency matrix of K, and T the adjacency matrix of Z. These matrices look like this:

$$N = \begin{array}{c|c|c|} & X & Y \\ \hline X & pC & qJ \\ \hline Y & qJ & pC \\ \hline \end{array} \qquad T = \begin{array}{c|c|c|} & X & Y \\ \hline X & 0 & o(t) \text{ 1's per row} \\ \hline Y & o(t) \text{ 1's per row} & 0 \\ \hline \end{array} .$$

On $K[W]$, the weight our configuration captures is at most what it would be if all class 2 edges, as well as all class 4 edges in $\nabla(A, B)$, were instead class 1, and all class 4 edges in $K[A] \cup K[B]$ were instead class 3. In this case, our configuration's overall loss on $K[W]$ (edges and vertices) would be exactly

$$\varphi^{\mathsf{T}}(N - 2qT + (c/t)I)\varphi. \tag{3.20}$$

To show that our configuration captures at most half the weight of $K[W]$ it would suffice to show (3.20) to be nonnegative, but let's instead show the stronger

$$\varphi^{\mathsf{T}} M \varphi \geq 0, \tag{3.21}$$

where $M = N - 2qT + (.66c/t)I$. Thus we're showing that the gain on edges of $K[W]$ is at most $(.66c/t)\sum_{v \in W} \delta_v^2$, reserving the remaining vertex loss in W, $(.34c/t)\sum_{v \in W} \delta_v^2$, for use below in handling edges meeting S. For (3.21), we simply show M is positive definite. We first treat the N term and then the T term, helping ourselves to a little bit of the I term in each of these steps. As will be clear below, and as is perhaps hinted by the constants .66 and .34, nothing in this argument is very delicate.

Let P and Q be the "pC" and "qJ" portions of N, respectively. Since P and Q are symmetric and commute, they admit a common orthonormal basis of eigenvectors. We seek to describe these eigenvectors and their corresponding eigenvalues in terms of the eigenvectors and eigenvalues of C, so let $w_1 = t^{-1/2}\mathbb{1}, w_2, \ldots, w_t$ be an orthonormal eigenbasis for C with corresponding eigenvalues $d = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_t$. Then a common orthonormal eigenbasis for P and Q is

$$\begin{aligned} v_1 &= 2^{-1/2}(w_1, w_1), & v_2 &= 2^{-1/2}(w_1, -w_1), \quad \ldots, \\ v_{2t-1} &= 2^{-1/2}(w_t, w_t), & v_{2t} &= 2^{-1/2}(w_t, -w_t), \end{aligned}$$

where (x, y) is the concatenation of x and y. These eigenvectors have corresponding eigenvalues $pd, pd, p\lambda_2, p\lambda_2, \ldots, p\lambda_t, p\lambda_t$ for P and $qt, -qt, 0, 0, \ldots, 0$ for Q, and therefore $pd + qt = \frac{1-c}{t}$, $pd - qt = 0$, $p\lambda_2, p\lambda_2, \ldots, p\lambda_t, p\lambda_t$ for N. Call these N-eigenvalues $\mu_1, \ldots, \mu_{2t}$ (for use below). Now since $|\lambda_t| \leq O(t^{1/3})$ (see (3.13)), all eigenvalues of N are at least $-O(t^{-4/3}) = -o(t^{-1})$. Thus (e.g.) $N + (.33c/t)I$ is (eventually) positive definite.

We now turn to the T term in M, which is easier. As every absolute row sum of T is $o(t)$, so is every eigenvalue of T. Thus every eigenvalue of $-2qT$ is at least $-o(t^{-1})$, so (e.g.) $-2qT + (.33c/t)I$ is (eventually) positive definite. Therefore M is positive definite, as claimed.

Finally we deal with contributions involving S. For this let $\bar{\delta} = \langle \delta_v \mid v \in V(K) \rangle$, $\bar{\delta}' = \mathbb{1}_W \circ \bar{\delta}$ (where $\circ$ denotes componentwise product), $\alpha_i = \bar{\delta} \cdot v_i$ and $\alpha_i' = \bar{\delta}' \cdot v_i$, $i \in [2t]$ (where $\cdot$ denotes the usual inner product). The total gain from edges meeting S is at most what it would be if all these edges were class 1, which is exactly

$$\begin{aligned} \bar{\delta}^t N \bar{\delta} - (\bar{\delta}')^t N \bar{\delta}' &= \sum_{i=1}^{2t} \mu_i(\alpha_i^2 - (\alpha_i')^2) \\ &= \mu_1(\alpha_1^2 - (\alpha_1')^2) + \sum_{i=2}^{2t} \mu_i(\alpha_i^2 - (\alpha_i')^2). \end{aligned} \tag{3.22}$$

In view of what we know about the μ_i's, the sum in (3.22) is at most

$$p\lambda_2 \sum_{v \in V(K)} \delta_v^2 - (\min \mu_i) \sum_{v \in W} \delta_v^2 \;\leq\; O(t^{-4/3}) \left[\sum_{v \in V(K)} \delta_v^2 + \sum_{v \in W} \delta_v^2 \right], \tag{3.23}$$

while, with ε defined by $\alpha_1' = (1-\varepsilon)\alpha_1$, the first term in (3.22) is

$$\begin{aligned}\mu_1(2\varepsilon-\varepsilon^2)\alpha_1^2 &= \frac{1-c}{t}(2\varepsilon-\varepsilon^2)\frac{1}{2t}\bigg(\sum_{v\in V(K)}\delta_v\bigg)^2 \\ &< \varepsilon t^{-2}\bigg(\sum_{v\in V(K)}\delta_v\bigg)^2 \qquad (3.24)\\ &\le \min\left\{\varepsilon^{-1}t^{-2}\bigg(\sum_{v\in S}\delta_v\bigg)^2, \quad 2\varepsilon t^{-1}\sum_{v\in V(K)}\delta_v^2\right\} \qquad (3.25)\end{aligned}$$

(actually (3.24) is equal to the first expression in (3.25)).

On the other hand, we get to subtract from these gains

$$\begin{aligned}\frac{c}{t}\sum_{v\in S}\delta_v^2 + \frac{.34c}{t}\sum_{v\in W}\delta_v^2 &= \frac{.66c}{t}\sum_{v\in S}\delta_v^2 + \frac{.34c}{t}\sum_{v\in V(K)}\delta_v^2 \\ &\ge \frac{.66c}{t|S|}\bigg(\sum_{v\in S}\delta_v\bigg)^2 + \frac{.34c}{t}\sum_{v\in V(K)}\delta_v^2. \qquad (3.26)\end{aligned}$$

We need to say this is larger than the sum of the right hand sides of (3.23) and (3.25), which is easy. For example, half the second term of (3.26) dominates the right hand side of (3.23), while the right hand side of (3.25) is at most half the second term of (3.26) if $\varepsilon \le .17c/2$ (to be unnecessarily precise), and otherwise, since $|S| = o(t)$, is dominated by the first term of (3.26). □

Chapter 4

The Cycle Space of $G_{n,p}$

4.1 Introduction

An issue of considerable interest in combinatorics over the last few decades has been the extent to which various standard facts, for instance the classic theorems of Turán, Ramsey and Szemerédi, remain true in a "sparse random" setting. Thus, for example, one may ask for which $p = p(n)$ a given (deterministic) assertion regarding the complete graph K_n is likely to hold in the ("Bernoulli") random graph $G_{n,p}$. The main result of this chapter follows this theme.

Our underlying deterministic statement is Proposition 4.1 below, for which we need a definition: for a fixed graph H, the *H-space* of G is the subspace of $\mathcal{E}(G)$ generated by the copies of H in G; this will be denoted $\mathcal{C}_H(G)$, or simply $\mathcal{C}_\kappa(G)$ if $H = C_\kappa$.

Proposition 4.1. *If $\kappa \geq 3$ is odd, then for any $n \geq \kappa$, $\mathcal{C}_\kappa(K_n) = \mathcal{C}(K_n)$.*

(Below, in Theorem 4.5, we will characterize $\mathcal{C}_H(K_n)$ for any fixed H and large enough n.)

When, in terms of p $(= p(n))$, are the κ-gons of $G_{n,p}$ likely to span its cycle space? Let $\mathcal{T}_\kappa$ be the class of graphs G satisfying $\mathcal{C}_\kappa(G) = \mathcal{C}(G)$ and let $\mathcal{Q}_\kappa$ be the class of nonempty graphs each of whose edges lies in a copy of C_κ. For any G, it's easy to see that $G \notin \mathcal{T}_\kappa$ unless every edge of G that lies in a cycle in fact lies in a κ-gon. On the other hand, if $p > (1 + \Omega(1)) \log n/n$ then w.h.p. every edge of $G_{n,p}$ does lie in a cycle (see [23, p. 105]). So for such p, $G_{n,p} \in \mathcal{T}_\kappa$ w.h.p. at least requires $G_{n,p} \in \mathcal{Q}_\kappa$ w.h.p., and we should first understand when this is true. Let

$$p_\kappa^* = p_\kappa^*(n) = [(\kappa/(\kappa - 1))n^{-(\kappa-2)} \log n]^{1/(\kappa-1)} \tag{4.1}$$

(where we always use log for ln). Note $\mathcal{Q}_\kappa$ is not an *increasing* property—that is, it is not preserved by adding edges. Nonetheless, p_κ^* is a *sharp threshold* for $\mathcal{Q}_\kappa$, in the sense that:

Lemma 4.2. *For any fixed $\kappa \geq 3$ and $\varepsilon > 0$,*

$$\Pr(G_{n,p} \in \mathcal{Q}_\kappa) \to \begin{cases} 0 & \text{if } p < (1-\varepsilon)p_\kappa^*, \\ 1 & \text{if } p > (1+\varepsilon)p_\kappa^*. \end{cases} \tag{4.2}$$

(Throughout this chapter limits are taken as $n \to \infty$.) We prove this routine observation in Section 4.3. The cases in (4.2) are called the *0-statement* and the *1-statement* (respectively).

Given Lemma 4.2, one might hope that p_κ^* is also a sharp threshold for $\mathcal{T}_\kappa$, and it essentially is, but for a small glitch in the 0-statement: for $p < (1-\Omega(1))/n$, we have $\lim \Pr(G_{n,p} \in \mathcal{T}_\kappa) > 0$ for the silly reason that the probability of having no cycles at all is (asymptotically) positive (see e.g. [31, Theorem 1]). Thus we will show:

Theorem 4.3. *For any fixed odd $\kappa \geq 3$ and $\varepsilon > 0$,*

$$\Pr(G_{n,p} \in \mathcal{T}_\kappa) \to \begin{cases} 0 & \text{if } (1-o(1))/n < p < (1-\varepsilon)p_\kappa^*, \\ 1 & \text{if } p > (1+\varepsilon)p_\kappa^*. \end{cases}$$

We actually prove the following stronger statement (see Section 4.3 for "stronger"), which says that edges not in κ-gons are the obstruction to $\mathcal{T}_\kappa$ in a precise sense. This is the main result of this chapter.

Theorem 4.4. *For any fixed odd $\kappa \geq 3$,*

$$\max_p \Pr(G_{n,p} \in \mathcal{Q}_\kappa \setminus \mathcal{T}_\kappa) \to 0; \tag{4.3}$$

equivalently,

$$\forall\, p = p(n), \quad \Pr(G_{n,p} \in \mathcal{Q}_\kappa \setminus \mathcal{T}_\kappa) \to 0. \tag{4.4}$$

(The (trivial) equivalence is given by the observation that (4.4) holds iff it holds when, for each n, $p = p(n)$ is a value achieving the maximum in (4.3) (and in this case the two statements are the same).)

Theorems 4.3 and 4.4 for $\kappa = 3$ were proved in [12]; even the former had been open and of interest, being the first unsettled case of a conjecture of M. Kahle (see [24, 25]) on the homology of the clique complex of $G_{n,p}$. Perhaps surprisingly, the argument of [12] does not extend to $\kappa \geq 5$, though, as discussed below, it does share a starting point with what we do here.

What happens if we replace the C_κ of Proposition 4.1 by some other graph? With $\mathcal{D}(G) = \{D \in \mathcal{E}(G) : |D| \equiv 0 \pmod 2\}$, the proposition generalizes neatly:

Theorem 4.5. *For any graph H with at least one edge and n large enough with respect to H,*

$$\mathcal{C}_H(K_n) = \begin{cases} \mathcal{C}(K_n) & \text{if } H \text{ is Eulerian and } |H| \text{ is odd,} \\ \mathcal{C}(K_n) \cap \mathcal{D}(K_n) & \text{if } H \text{ is Eulerian and } |H| \text{ is even,} \\ \mathcal{E}(K_n) & \text{if } H \text{ is not Eulerian and } |H| \text{ is odd,} \\ \mathcal{D}(K_n) & \text{if } H \text{ is not Eulerian and } |H| \text{ is even.} \end{cases} \tag{4.5}$$

Here $|H| = |E(H)|$ and "Eulerian" means degrees are even, but not that the graph is necessarily connected. Of course the left-to-right containments ($\mathcal{C}_H(K_n) \subseteq \mathcal{C}(K_n)$ and so on) are obvious.

The natural value of $\mathcal{C}_H(G)$, which we will denote $\mathcal{W}_H(G)$, is then what one gets by replacing K_n by G in the appropriate expression on the right hand side of (4.5); e.g. for $H = C_\kappa$,

$$\mathcal{W}_H(G) = \begin{cases} \mathcal{C}(G) & \text{if } \kappa \text{ is odd,} \\ \mathcal{C}(G) \cap \mathcal{D}(G) & \text{if } \kappa \text{ is even.} \end{cases} \tag{4.6}$$

(We could instead set $\mathcal{W}_H(G) = \mathcal{E}(G) \cap \mathcal{C}_H(K_n)$, which by Theorem 4.5 is the same for all but a few values of n.) So we are interested in understanding when $G_{n,p}$ is likely to lie in

$$\mathcal{T}_H := \{G : \mathcal{C}_H(G) = \mathcal{W}_H(G)\}.$$

(Again, $\mathcal{C}_H(G) \subseteq \mathcal{W}_H(G)$ is trivial for any H and G.)

As before, membership in $\mathcal{T}_H$ will (in non-silly cases) at least require that the copies of H cover the edges of $G := G_{n,p}$, but when H is non-Eulerian there is a second requirement: each vertex (of G) should have odd degree in some copy of H in G (since for any $v \in V(G)$, $\mathcal{W}_H(G)$ will contain graphs in which v has odd degree). For example if H is a pair of triangles joined by a slightly long path and $n^{-1+\varepsilon} < p \ll n^{-2/3}$ for a suitable small ε depending on the length of the path, then (w.h.p.) all edges of G are in copies of H, but most vertices fail to lie in triangles, so have even degree in every copy.

Generalizing $\mathcal{Q}_\kappa$, let $\mathcal{Q}_H$ be the class of nonempty graphs G satisfying (i) each edge of G is in a copy of H, and (ii) if H is not Eulerian, then each vertex of G has odd degree in some copy of H; so we have just said that we "essentially" have $\mathcal{T}_H \subseteq \mathcal{Q}_H$. Though we hesitate to make it a conjecture, we don't know that the following generalization of Theorem 4.4 is wrong.

Question 4.6. *Could it be that for each (fixed) H,*

$$\max_p \Pr(G_{n,p} \in \mathcal{Q}_H \setminus \mathcal{T}_H) \to 0? \tag{4.7}$$

Understanding when $G_{n,p} \in \mathcal{Q}_H$ w.h.p. is easier, so this would also tell us when $\mathcal{T}_H$ is likely to hold. (Note that in general we don't expect a statement like Theorem 4.3, since the "threshold" for $\mathcal{Q}_H$ itself may not be sharp.) Even if (4.7) is not true in general, it seems likely to hold for reasonably nice H (even, say, edge-transitive to start, though this should be much more than is needed). One could also relax (4.7) to an Erdős-Rényi-like threshold statement; e.g. with $p_{\mathcal{Q}_H} = \min\{p_0 : \Pr(G_{n,p} \in \mathcal{Q}_H) \geq 1/2\ \forall p \geq p_0\}$,

$$\text{if } p \gg p_{\mathcal{Q}_H} \text{ then } G_{n,p} \in \mathcal{T}_H \text{ w.h.p.}$$

Outline. The rest of this chapter is organized as follows. Section 4.1.1 digresses breifly to discuss how Theorem 4.4 was inspired in part by the work of Chapter 3. Section 4.2

recalls edge space preliminaries, outlines the main points (Lemmas 4.8–4.10) for the proof of Theorem 4.4, and introduces the (standard) coupling critical to two of their proofs. Section 4.3 proves Lemma 4.2 and gives the easy derivation of Theorem 4.3 from Theorem 4.4. The heart of the chapter is Sections 4.4–4.6, particularly the last of these. They supply the proofs of the lemmas of Section 4.2, in ascending order of difficulty. Section 4.7 gives the easy proof of Theorem 4.5, which we postpone as it is unrelated to the rest of the chapter. Finally (and a bit tangentially), Section 4.8 gives the proof of a theorem (Theorem 4.12) that takes a step towards settling Question 4.6 (see the remark after Lemma 4.10).

4.1.1 Digression: A Connection to Chapter 3

In Chapter 1, we mentioned that we were motivated to study $\mathcal{C}(G_{n,p})$ in part by an aspect of our construction of the graphs disproving Yuster's conjecture. We are now in a position to elaborate.

Recall that our construction started with a fixed, random-looking but triangle-free graph H (see the paragraph after Theorem 3.3). We had this idea early on in the course of developing the construction. Much later we realized that one of the properties we would need from this H is (more or less) that its cycle space be spanned by short cycles (see Proposition 3.19). This discovery, combined with the fact that we had earlier considered taking as our H a de-triangled instance of $G_{n,p}$,[1] led us to wonder what we could say generally about $\mathcal{C}(G_{n,p})$.

Despite our historical link between Theorem 4.4 and the construction for Theorem 3.3, the former theorem is not directly relevant to the construction, for several reasons. For one, as already mentioned, a starting H for the construction must be triangle-free. For two, the property of H asserted in Proposition 3.19 is somewhat peculiar, dealing with the cycle space of subgraphs of H in addition to that of H itself. For three, and most importantly, Theorem 4.4 is overkill in the sense that Proposition 3.19 doesn't require any graph (H or any of its subgraphs) to be spanned by cycles of a *fixed* length,

[1] Recall we eventually settled on a better H—see Proposition 3.11 and the succeeding paragraph.

but only by cycles *up to* a fixed length. This, as the first two lines of the proposition's proof indicate, is *much* easier.

4.2 Main Points for the Proof of Theorem 4.4

Before outlining the proof of Theorem 4.4, we need to review just a little more background.

4.2.1 Edge Space Basics

The edge space $\mathcal{E}(G)$ of a graph G (defined early in Chapter 2), being an $\mathbb{F}_2$-vector space, comes equipped with a standard inner product: $\langle J, K\rangle = \sum_{e\in E(G)} J(e)K(e) = |J\cap K|$, where the sum and cardinality are interpreted mod 2. (The first expression thinks of J and K as vectors, the second as subgraphs of G.) With this, the orthogonal complement, $\mathcal{S}^\perp$, of a subspace $\mathcal{S}$ of $\mathcal{E}(G)$ is defined as usual. Then $\mathcal{C}^\perp(G)$, called the *cut space* of G, consists of the (indicators of) cuts of G (which, note, includes $\varnothing$); $(\mathcal{C}(G)\cap\mathcal{D}(G))^\perp$ consists of cuts and their complements; and $\mathcal{C}_H^\perp(G)$ is the set of subgraphs of G having even intersection with every copy of H (in G).

As mentioned earlier, $\mathcal{C}_H(G) \subseteq \mathcal{W}_H(G)$ always; dually, $\mathcal{W}_H^\perp(G) \subseteq \mathcal{C}_H^\perp(G)$. In particular, for odd $\kappa \geq 3$,

$$\mathcal{C}^\perp(G) \subseteq \mathcal{C}_\kappa^\perp(G), \text{ and equality here is the same as } G \in \mathcal{T}_\kappa. \tag{4.8}$$

The next (trivial) observation will be useful at a few points.

Proposition 4.7. *Let G be a graph and $L \subseteq G$, and suppose L', L'' are (respectively) smallest and largest members of the coset $L + \mathcal{C}^\perp(G)$. Then*

$$\forall\, v \in V \quad d_{L'}(v) \leq d_G(v)/2 \leq d_{L''}(v).$$

(For example if $d_{L'}(v) > d_G(v)/2$, then $L' + \nabla(v)$ ($\in L + \mathcal{C}^\perp(G)$) is smaller than L'.)

In particular, if $G \notin \mathcal{T}_\kappa$, then since $\mathcal{C}_\kappa^\perp(G) \setminus \mathcal{C}^\perp(G) \supseteq L + \mathcal{C}^\perp(G)$ for any $L \in$

$\mathcal{C}_\kappa^\perp(G) \setminus \mathcal{C}^\perp(G)$, a smallest element F of $\mathcal{C}_\kappa^\perp(G) \setminus \mathcal{C}^\perp(G)$ satisfies

$$d_F(v) \le d_G(v)/2 \quad \forall\, v \in V. \tag{4.9}$$

4.2.2 Structure of the Proof

From now through the end of Section 4.7 we fix an odd $\kappa \ge 5$ (as mentioned earlier, the case $\kappa = 3$ of Theorem 4.4 was proved in [12]), *and set* $p^* = p_\kappa^*$, $\mathcal{Q} = \mathcal{Q}_\kappa$, $\mathcal{T} = \mathcal{T}_\kappa$ *and* $G = G_{n,p}$; so our objective, (4.3), becomes

$$\max_p \Pr(G \in \mathcal{Q} \setminus \mathcal{T}) \to 0. \tag{4.10}$$

As sometimes happens, though (4.10) should become "more true" as p ($> p^*$) grows, some points in the proof run into difficulties for larger p, and it seems easiest to deal first with smaller p and then derive the full statement from this restricted version. The next two lemmas, the first of which is our main point, implement this plan.

Lemma 4.8. *For any fixed* K *and* $p \le Kp^*$,

$$\Pr(G \in \mathcal{Q} \setminus \mathcal{T}) \to 0. \tag{4.11}$$

(The interest here is really in p at least about p^*, smaller values being handled by Lemma 4.2; see (4.28).)

Lemma 4.9. *There exists* $K > 1$ *such that if* $p > q := Kp^*$, *then*

$$\Pr(G \notin \mathcal{T}) < \Pr(G_{n,q} \notin \mathcal{T}) + o(1).$$

Applying Lemmas 4.9 and 4.8, together with (the 1-statement of) Lemma 4.2 to $p'(n) := \min\{p(n), Kp^*(n)\}$ then easily gives Theorem 4.4. (For n's with $p(n) > Kp^*$, we have, using Lemma 4.9 for the first inequality and Lemmas 4.8 and 4.2 for the final $o(1)$,

$$\begin{aligned}\Pr(G \in \mathcal{Q} \setminus \mathcal{T}) &< \Pr(G_{n,p'} \notin \mathcal{T}) + o(1)\\ &< \Pr(G_{n,p'} \in \mathcal{Q} \setminus \mathcal{T}) + \Pr(G_{n,p'} \notin \mathcal{Q}) + o(1) = o(1),\end{aligned}$$

and for the remaining n's we have $p = p'$ and Lemma 4.8 applies directly.)

The following device will play a central role in the proofs of both of these lemmas (so in most of this chapter). For the rest of the chapter we fix some rule that associates with each finite graph G a subgraph $F(G)$ satisfying

$$F(G) = \begin{cases} \varnothing & \text{if } G \in \mathcal{T}, \\ \text{some smallest element of } \mathcal{C}_\kappa^\perp(G) \setminus \mathcal{C}^\perp(G) & \text{if } G \notin \mathcal{T}. \end{cases} \tag{4.12}$$

(By (4.8), this makes sense.)

We will use this only with G, so set $F(G) = F$ throughout. A crucial point is that G determines F (see the paragraph preceding Proposition 4.11). That F is a minimizer will be used only to say that it is small and has small degrees, as promised by (4.9). Another useful observation (recalling the notation of Section 2.4):

$$xy \in F \implies |F| \geq \sigma^{\kappa-1}(x,y) + 1. \tag{4.13}$$

(Proof: Since F lies in $\mathcal{C}_\kappa^\perp(G)$, it must contain a second edge of each κ-gon of G containing xy, and there is a set of $\sigma^{\kappa-1}(x,y)$ such κ-gons that share no edges except xy.)

With F thus defined we may replace the event $\{G \notin \mathcal{T}\}$ by the more convenient $\{F \neq \varnothing\}$, which in particular allows us to tailor our treatment to the size of a hypothetical F. As we will see, ruling out fairly large F's is easy—not from scratch, but with the help of a powerful result from [11] (Theorem 2.19), which more or less immediately yields:

Lemma 4.10. *For fixed $c > 0$ and $p \gg n^{-(\kappa-2)/(\kappa-1)}$,*

$$\Pr(|F| > cn^2 p) \to 0. \tag{4.14}$$

Thus the real problem in proving Lemma 4.8, and the most interesting part of the whole business, is dealing with F's that are small relative to G (but nonempty). Thus far—and a little further; see the preview following the statement of Lemma 4.14—our structure mirrors that of [12]; but the (two-page) argument handling this main point there offers no help here.

Remark. In connection with Question 4.6, it seems worth observing that Lemma 4.10, at least, can be considerably extended. In fact we can prove a statement of this type with the odd cycle C_κ replaced by a general H, though not always with the (conjecturally correct) lower bound on p that would correspond to a positive answer to Question 4.6. See Section 4.4 for a statement and Section 4.8 for a (sketchy) proof.

4.2.3 Coupling

A critical role in the proofs of Lemmas 4.8 and 4.9 is played by the usual coupling of G $(= G_{n,p})$ and $G_{n,q}$, where p will always be the value we're really interested in and $q < p$ will depend on what we're trying to do.

So, *from now on we set* $G_0 = G_{n,q}$.

A standard description: let λ_e, $e \in E(K_n)$, be chosen uniformly and independently from $[0, 1]$ and set

$$G = \{e : \lambda_e < p\}, \quad G_0 = \{e : \lambda_e < q\}.$$

In particular $G_0 \subseteq G$. Probabilities in the proofs of Lemmas 4.8 and 4.9 will refer to the joint distribution of G and G_0.

We will get most of our leverage from two alternate ways of viewing the choice of the pair (G, G_0):

(A) Choose G first; thus we choose G in the usual way and let G_0 be the ("(q/p)-random") subset of G gotten by retaining edges of G with probability q/p, these choices made independently (a.k.a. *percolation on* G).

(B) Choose G_0 first; that is, we choose G_0 in the usual way, define p' by $(1-q)(1-p') = 1-p$, and let G be the random superset of G_0 gotten by adding each edge of $K_n \setminus G_0$ to G_0 with probability p', these choices again made independently.

We will often refer to these as "coupling down" and "coupling up" (respectively).

The proof of Lemma 4.9 is based naturally (or inevitably) on the viewpoint in (A); namely, we show that (with p, q as in the lemma) if G is "bad" (meaning $G \notin \mathcal{T}$) then the coupled G_0 is likely to be bad as well. For the proof of Lemma 4.8, viewpoint (B) is the primary mover, though the smaller role of (A) is also crucial.

With reference to the setup introduced at (4.12), when working with G and G_0 as above, we set $F_0 = G_0 \cap F$ (a (q/p)-random subset of F; note this has nothing to do with $F(G_0)$, which will play no role here). Then automatically

$$F_0 \in \mathcal{C}_\kappa^\perp(G_0), \tag{4.15}$$

since $F_0 \cap C = F \cap C$ for any κ-gon C of G_0.

We will want to say that certain features of (G, F) are reflected in (G_0, F_0). A simple but crucial point here is that there is no summing (of probabilities) over possible F's, since there is just one F for each G. The following proposition will be sufficient for our purposes.

Proposition 4.11. *With the above setup, for any p, q and $g = g(n) = \omega(1)$, w.h.p.*

$$|F_0| \sim |F|q/p \quad \textit{if} \quad |F| > gp/q$$

and

$$d_{F_0}(v) \begin{cases} \sim d_F(v)q/p & \forall\, v \textit{ with } d_F(v) > (g\log n)p/q, \\ < 3g\log n & \forall\, v \textit{ with } d_F(v) \le (g\log n)p/q. \end{cases}$$

(This is true for any rule that specifies a particular subgraph (in place of F) for each graph; but we will only use it with F $(= F(G))$, so just give the statement for this case.)

Proof. These are straightforward applications of Theorems 2.1 and 2.2, so we will be brief. For the first assertion we want to say that for any fixed $\varepsilon > 0$,

$$\Pr\left(\{|F| > gp/q\} \wedge \{|F_0| \neq (1 \pm \varepsilon)|F|q/p\}\right) \to 0.$$

But the probability here is less than

$$\Pr\left(|F_0| \neq (1 \pm \varepsilon)|F|q/p \;\mid\; |F| > gp/q\right),$$

which by Theorem 2.1 is less than $\exp[-\Omega(\varepsilon^2 g)]$.

The second assertion (pair of assertions) is similar, following from

$$\sum_v \Pr\left(d_{F_0}(v) \neq (1 \pm \varepsilon)d_F(v) \;\middle|\; d_F(v) > (g \log n)p/q\right) < n \exp[-\Omega(\varepsilon^2 g \log n)]$$
$$= o(1)$$

for any fixed $\varepsilon > 0$, and (now switching to Theorem 2.2)

$$\sum_v \Pr\left(d_{F_0}(v) > 3g \log n \;\middle|\; d_F(v) \leq (g \log n)p/q\right) < n \exp[-(3g \log n)\log(3/e)]$$
$$= o(1). \qquad \square$$

4.3 Two Simple Points

Here we dispose of Lemma 4.2 and the derivation of Theorem 4.3 from Theorem 4.4. (Recall we are using G for $G_{n,p}$ and V for $V(G)$.)

Proof of Lemma 4.2. We begin with the 1-statement, a typical application of Theorem 2.3. We assume $p > (1 + \varepsilon)p^*$ and $p = O(p^*)$ (as we may, since for larger p, the

1-statement is contained in Theorem 2.8). Given $x, y \in V$, let the A_i's (in the paragraph preceding Theorem 2.3) be the (edge sets of) the $(\kappa-1)$-paths joining x and y in K_n; so $X = \tau^{\kappa-1}(x,y)$, $\mu \sim n^{\kappa-2}p^{\kappa-1}$ and $\overline{\Delta} = \mu + O(\mu n^{\kappa-3}p^{\kappa-2}) \sim \mu$. Thus (note $\varphi(-1) = 1$) Theorem 2.3 gives

$$\Pr(\tau^{\kappa-1}(x,y) = 0) \leq \exp[-(1-o(1))\mu]. \tag{4.16}$$

So the probability that $\mathcal{Q}$ ($= \mathcal{Q}_\kappa$) fails—that is, that there is some xy in G with $\tau^{\kappa-1}(x,y) = 0$—is less than

$$\tbinom{n}{2}pe^{-(1-o(1))\mu} < \exp[\log(n^2p) - (1-o(1))\mu] = o(1)$$

(since $\mu > (1-o(1))(1+\varepsilon)^{\kappa-1}(\kappa/(\kappa-1))\log n \sim (1+\varepsilon)^{\kappa-1}\log(n^2p)$).

For the 0-statement we use the second moment method (see e.g. [3, Chapter 4]) and, again, Theorem 2.3. Let Z_{xy} be the indicator of the event $\{xy \in G\} \wedge \{\tau^{\kappa-1}(x,y) = 0\}$ $(x, y \in V)$ and $Z = \sum Z_{xy}$. Theorem 2.6 gives $\Pr(\tau^{\kappa-1}(x,y) = 0) > (1-p^{\kappa-1})^{n^{\kappa-2}} > \exp[-\mu - o(1)]$ (μ as above), whence

$$\mathbb{E}[Z_{xy}] > p\exp[-\mu - o(1)]. \tag{4.17}$$

In particular $\mathbb{E}[Z] = \omega(1)$ (using $p < (1-\varepsilon)p^*$ and ignoring the rather trivial case $p = O(n^{-2})$), so for $\mathbb{E}Z^2 \sim \mathbb{E}[Z]^2$ (which gives the 0-statement *via* Chebyshev's Inequality), it's enough to show

$$\mathbb{E}[Z_{xy}Z_{uv}] < (1+o(1))\mathbb{E}[Z_{xy}]^2$$

for distinct $\{x,y\}$,$\{u,v\} \in \binom{V}{2}$, which in view of (4.17) follows from

$$\begin{aligned}\mathbb{E}[Z_{xy}Z_{uv}] &\leq p^2\Pr(\tau^{\kappa-1}(x,y) = \tau^{\kappa-1}(u,v) = 0)\\ &\leq p^2\exp[-(1-O(n^{\kappa-3}p^{\kappa-2}))2\mu] = p^2\exp[-2\mu + o(1)].\end{aligned}$$

Here the first inequality is given by Theorem 2.6 (since the events $\{xy, uv \in G\}$ and

$\{\tau^{\kappa-1}(x,y) = \tau^{\kappa-1}(u,v) = 0\}$ are increasing and decreasing respectively), and the second by Theorem 2.3, where the A_i's are the $(\kappa-1)$-edge paths joining either x and y or u and v, for which $\mathbb{E}X \sim 2\mu$ (recall X is the number of A_i's that occur) and it's easy to see that $\overline{\Delta} - \mu = O(n^{2\kappa-5}p^{2\kappa-3}) = O(n^{\kappa-3}p^{\kappa-2})\mu$ $(= o(\mu))$. □

Proof that Theorem 4.4 implies Theorem 4.3. This is routine and we aim to be brief. Lemma 4.2 gives the 1-statement (which is the interesting part). For the 0-statement, it is enough to say that for p in the stated range, G w.h.p. contains an edge lying in a cycle but not in a C_κ. This is again given by Lemma 4.2 if p is large enough that *all* edges are in cycles (w.h.p), which is true if $p > (1+\Omega(1))\log n/n$ (see [23, p. 105]). For smaller p, w.h.p. G contains cycles of length $\omega(1)$ if $p > (1-o(1))/n$ and of length $\Omega(n^{3/10})$ (say) if $p \geq 1/n$ (see e.g. [23, Theorem 5.18(i)]). On the other hand, since the expected number of C_κ's in G is less than $(np)^\kappa$, the number of edges in C_κ's is w.h.p. less than $\omega(np)^\kappa$ for any $\omega = \omega(1)$; so in the range under discussion, the C_κ's w.h.p. don't cover even one longest cycle in G. □

4.4 Proof of Lemma 4.10

Here we give the easy proof of Lemma 4.10 and then state the extension to general H mentioned in the remark after the lemma.

For the lemma it's enough to show that the conclusions of Proposition 2.12, Theorem 2.19 and Proposition 2.13 (c), the latter two with $\varepsilon = c/3$, imply $|F| < cnp^2$ (deterministically).

Let F' be a largest element of $F + \mathcal{C}^\perp(G)$. Then $|F'| \geq |G|/2$ (by Proposition 4.7), so, since F' is C_κ-free, the conclusion of Theorem 2.19 gives an $A \subseteq V$ with

$$|F' \setminus \nabla_G(A)| < \varepsilon n^2 p. \tag{4.18}$$

It then remains to observe that (under our assumptions), (4.18) implies

$$(|F| \leq) \quad |F' \triangle \nabla_G(A)| < 3\varepsilon n^2 p.$$

But the conclusion of Proposition 2.13 (c) gives $|\nabla_G(A)| < (1+\varepsilon)n^2p/4$, whence

$$|\nabla_G(A) \setminus F'| \leq (1+\varepsilon)n^2p/4 - (|G|/2 - \varepsilon n^2 p) < 2\varepsilon c n^2 p$$

(where we again used Proposition 2.12 to say $|G| \sim n^2p/2$). $\square$

Generalization. For this discussion we restrict to H with $e_H \geq 2$. For such an H, set

$$m_2(H) = \max\left\{\frac{e_K - 1}{v_K - 2} : K \subseteq H, v_K \geq 3\right\}. \tag{4.19}$$

This parameter plays a central role in various contexts, in particular in results more or less related to (the general version of) Theorem 2.19; see e.g. [33] for an overview.

Theorem 4.12. *For any fixed H with $e_H \geq 2$, the following is true. For any $\varepsilon > 0$ there is an M such that if $p > Mn^{-1/m_2(H)}$ then w.h.p.: for each $F \in \mathcal{C}_H^\perp(G)$ there is an $X \in \mathcal{W}_H^\perp(G)$ with $|F \Delta X| < \varepsilon n^2 p$; in particular, if $\mathcal{C}_H(G) \neq \mathcal{W}_H(G)$, then*

$$\min\{|F| : F \in \mathcal{C}_H^\perp(G) \setminus \mathcal{W}_H^\perp(G)\} < \varepsilon n^2 p.$$

This is proved in Section 4.8.

Remarks. Notice that Theorem 4.12 *contains* an extension of Lemma 4.10, whereas in the preceding discussion we did need a few lines to get from Theorem 2.19 to the lemma. But the two theorems live in somewhat different worlds, since Theorem 2.19 assumes only that F is C_κ-*free*, which is much weaker than requiring that it have odd intersection with every C_κ.

As mentioned in Section 2.9, the value $n^{-1/m_2(H)}$ is not necessarily what's needed for Question 4.6. For instance, if H is two triangles joined by a P_l, then $m_2(H) = 2$ (take K to be one of the triangles), but the range where the question is most interesting (the point at which $\mathcal{Q}_H$ becomes likely) is at $p \asymp n^{-2/3}\log^{1/3} n$, corresponding to all *vertices* being in triangles. On the other hand, in many (or most) natural cases—e.g. the ("balanced") H's for which $K = H$ achieves the max in (4.19)—Theorem 4.12 does give what should be the correct extension of Lemma 4.10. (It would be interesting to see if one could push the theorem to give the correct extension in general; with our current approach this would mainly require a fairly significant extension of what we are getting from “containers,” and we haven’t yet thought about plausibility.)

4.5 Proof of Lemma 4.9

By Corollary 2.11 with $l=\kappa-1$, there is a $K>1$ such that if $p>Kp^*$, then w.h.p.

$$\text{every } \{x,y\}\in\binom{V}{2} \text{ satisfies } \sigma^{\kappa-1}(x,y)=\Omega(\pi) \tag{4.20}$$

(where $\pi=\pi(n,p)$ is as in the corollary). We work in the coupling framework of Section 4.2.3, taking $q=Kp^*$.

For Lemma 4.9 it is of course enough to show

$$\Pr(\{G\notin\mathcal{T}\}\ \wedge\ \{G_0\in\mathcal{T}\})\to 0. \tag{4.21}$$

Note that $G_0\in\mathcal{T}$ implies $F_0\in\mathcal{C}^{\perp}(G_0)$, since we always have $F_0\in\mathcal{C}^{\perp}_{\kappa}(G_0)$ (see (4.15)); thus (4.21) will follow from

$$\Pr(\{F\neq\varnothing\}\ \wedge\ \{F_0\in\mathcal{C}^{\perp}(G_0)\})\to 0. \tag{4.22}$$

So it will be enough to show that

$$F_0\notin\mathcal{C}^{\perp}(G_0) \tag{4.23}$$

follows (deterministically) from

$$F\neq\varnothing \tag{4.24}$$

combined with various statements that we already know to hold w.h.p. This is not hard, but is more circuitous than one might wish. Roughly we show that, barring occurrence of some low probability event, (i) presence of even one edge in F forces F to be large enough (not very large) that $F_0\neq\varnothing$, and (ii) F_0 is not substantial enough to meet all

xy-paths in $G_0 - xy$ for an $xy \in F_0$, so any such xy is contained in a cycle witnessing (4.23).

A convention. To slightly streamline the presentation we agree that in this argument, appeals to a probabilistic statement X—e.g. "X implies" or "by X"—actually refer to *the conclusion of* X, which conclusion will always be something that X asserts to hold w.h.p. See the references to (4.20), Lemma 4.10 and Proposition 4.11 in the next paragraph for first instances of this.

If (4.24) holds, then (4.20) and (4.13) (for the lower bound) together with Lemma 4.10 (for the upper) imply that

$$\Omega(\pi) < |F| < n^2p/10. \tag{4.25}$$

Since $\pi q/p \gg 1$, the lower bound in (4.25) and the first part of Proposition 4.11 give $|F_0| \sim |F|q/p$, so

$$0 \neq |F_0| < (1 + o(1))n^2q/10. \tag{4.26}$$

In addition, Proposition 2.12, (4.9) and the second part of Proposition 4.11 give

$$d_{F_0}(v) < (1 + o(1))nq/2 \ \ \forall\, v \in V.$$

Thus, setting $H_0 = G_0 \setminus F_0$ and recalling the approximate (nq)-regularity of G_0 given by Proposition 2.12, we have

$$d_{H_0}(v) > (1 - o(1))nq/2 \ \ \forall\, v \in V. \tag{4.27}$$

Now choose an $xy \in F_0$ (recall (4.26) says $F_0 \neq \varnothing$) and let X, Y be the H_0-components of x and y. By (4.27) and Proposition 2.14 (applied to G_0), we have $|X|, |Y| > n/3$, which implies $X = Y$: otherwise X and Y are disjoint and we have the contradiction

$$(1 - o(1))n^2q/9 < |\nabla_{G_0}(X, Y)| \leq |F_0| < (1 + o(1))n^2q/10,$$

where the first inequality is given by Proposition 2.13 (a) (applied to G_0), the second holds because $\nabla_{G_0}(X,Y) \subseteq F_0$, and the third is given by (4.26).

But this (i.e. $X = Y$) gives an xy-path in H_0, and adding xy to this path produces a cycle meeting F_0 only in xy; so we have (4.23).

4.6 Proof of Lemma 4.8

Here we introduce the two main assertions, Lemmas 4.13 and 4.14, underlying Lemma 4.8, and prove the latter assuming them. The supporting lemmas are proved in Sections 4.6.1 and 4.6.2.

Note that for the proof of Lemma 4.8, Lemma 4.2 allows us to restrict attention to the range

$$(1-\varepsilon)p^* < p < Kp^* \tag{4.28}$$

(for any fixed $\varepsilon > 0$), and recall that, as observed following (4.14), it's enough to show that for a given $\lambda = \lambda(n) \to 0$,

$$\Pr(\{G \in \mathcal{Q}\} \ \wedge \ \{0 < |F| < \lambda n^2 p\}) \to 0. \tag{4.29}$$

We again work with the coupling of Section 4.2.3, now taking $q = \vartheta p$ with a *fixed* $\vartheta \in (0,1)$ small enough to support the discussion below (the rather mild constraints on ϑ are at (4.40) and (4.47)). Define the random variables α and α_0 by

$$|F| = \alpha n^2 p/2 \quad \text{and} \quad |F_0| = \alpha_0 n^2 q/2. \tag{4.30}$$

Definitions. Henceforth a *path* (with length unspecified) is a $P_{\kappa-1}$ (and an xy-path is a path whose endpoints are x and y). Our paths will always lie in G and often in G_0. We now write $\sigma(x,y)$ for $\sigma^{\kappa-1}(x,y)$ (recall from Section 2.4 that this is the maximum size of a set of internally disjoint xy-paths in G), and $\sigma_0(x,y)$ for the analogous quantity in G_0. For $S \subseteq G$, a path P is *S-central* if it contains an odd number of edges of S, at least one of which is internal. Let $\sigma(x, y; S)$ be the maximum size of a collection of internally disjoint S-central xy-paths, and $\sigma_0(x, y; S)$ the corresponding quantity in G_0. An (S, t)-*rope* is a P_t whose terminal edges lie in S. Set

$$R(S) = \{\{x, y\} \in \tbinom{V}{2} : \sigma_0(x, y; S) > .25n^{\kappa-2}q^{\kappa-1}\} \tag{4.31}$$

and define events

$$\mathcal{R} = \{|F \cap R(F_0)| \geq .12\alpha n^2 p\}$$

and

$$\mathcal{P} = \{0 < |F| < \lambda n^2 p\}$$

(the second conjunct in (4.29)).

Lemma 4.13. *There is a fixed $\varepsilon > 0$ such that for p as in (4.28), w.h.p.*

$$G \in \mathcal{Q} \wedge \mathcal{P} \implies G \in \mathcal{R}. \tag{4.32}$$

(In other words, $\Pr(G \in \mathcal{Q} \wedge \mathcal{P} \wedge \overline{\mathcal{R}}) \to 0$. Of course $\mathcal{R}$ holds trivially if $F = \varnothing$, so it's only the upper bound in $\mathcal{P}$ that's of interest here.)

Remarks. For $\{x, y\} \in \binom{V}{2}$, $\sigma_0(x, y)$ should be around $n^{\kappa-2}q^{\kappa-1}$. Lemma 4.13 says that, provided $G \in \mathcal{Q} \wedge \mathcal{P}$, it's likely that for a decent fraction of the edges xy of F, even $\sigma_0(x, y; F_0)$ is of this order of magnitude—which is *un*natural if F_0 is small relative to G_0 (since then paths should typically avoid F_0). Viewed from Lemma 4.13 the parity requirement in the definition of "central" may look superfluous, since a path of G_0 joining ends of an edge of F necessarily has odd intersection with F_0; but this extra condition will later play a brief but important role in justifying (4.36).

For the next lemma we temporarily expand the range of q and G_0, assuming only what's needed for the proof (though we will use the lemma only with q and G_0 as above).

Lemma 4.14. *For fixed $t \geq 3$, $q = q(n) > n^{-1}\log^6 n$ and $G_0 = G_{n,q}$, w.h.p.: for*

$S \subseteq G_0$, say with $|S| = \beta n^2 q/2$, the number of (S,t)-ropes in G_0 is

$$O(\max\{\beta^2 n^{t+1} q^t,\ \beta n^{t/2+2} q^{t/2+1}\}). \tag{4.33}$$

Remarks. Note this is of interest only when $\beta \ll 1$, since Proposition 2.12 bounds (w.h.p.) the number in question by $(1 + o(1))n^{t+1}q^t$; see Section 4.6.2 for a little more on the bounds in (4.33). The bound is also correct, but more trivial, when $t = 2$. The lemma doesn't actually require $S \subseteq G_0$: the proof shows that, for any $S \subseteq E(K_n)$ (of the stated size) with $\Delta_S = O(nq)$ (where Δ is maximum degree), we have the same bound for the number of P_t's with terminal edges in S and internal edges in G_0.

Preview. The proof of Lemma 4.8, which we are about to give, is based mainly on "coupling up": using information about (G_0, F_0) to constrain what happens when we choose $G\backslash G_0$. (To this extent our strategy is similar to that of [12], but the resemblance ends there.) On the other hand, the proof of the crucial Lemma 4.13 in Section 4.6.1 is based on "coupling down": most of the work there is devoted to the proof of a similar statement (Lemma 4.15) involving only G (not G_0), from which the desired hybrid statement follows easily *via* coupling. In sum, we couple down to show that $\mathcal{R}$ is likely (precisely, the conjunction of its failure with $\mathcal{Q} \wedge \mathcal{P}$ is unlikely), and couple up to show it is *un*likely. A little more on the latter:

We would like to say that if G_0 is sufficiently nice—as it will be w.h.p.—then $\mathcal{P} \wedge \mathcal{R}$ is unlikely; this gives (4.29) *via* Lemma 4.13. The main point we need to add to Lemmas 4.13 and 4.14 is a deterministic one: if G_0 enjoys relevant genericity properties, together with the conclusion of Lemma 4.14, then, for each $S \subseteq G_0$, $R(S)$ is fairly small (depending on $|S|$; see (4.37)). Combined with $F \neq \varnothing$ (from $\mathcal{P}$), this will allow us to say that the lower bound on $|G \cap R(F_0)|$ ($= |F \cap R(F_0)|$) in $\mathcal{R}$ is larger by a crucial factor $\alpha^{-\Omega(1)}$ than $|R(F_0)|p$—its natural value when we "couple up"—which *ought* to make $\mathcal{R}$ unlikely. But of course F_0 depends on G, so, given G_0, we are forced to sum the probability of this supposedly unlikely event over possible values S of F_0. This turns out to mean that the whole argument would collapse if we were to replace the above

$\alpha^{-\Omega(1)}$ by $\alpha^{-o(1)}$. (Here we again use $\mathcal{P}$, in this case to say α is small.)

A word on presentation. We prove the desired

$$\Pr(\mathcal{Q} \wedge \mathcal{P}) = o(1) \tag{4.34}$$

(= (4.29)) by producing a list of unlikely events and showing that at least one of these must hold if $\mathcal{Q} \wedge \mathcal{P}$ does. A more intuitive formulation might, for example, begin: "By Lemma 4.13 (since we assume $\mathcal{Q} \wedge \mathcal{P}$), *we may assume* $\mathcal{R}$." But note this would really mean, not that we *condition* on $\mathcal{R}$ (which is not something we could hope to understand), but that we need only bound probabilities $\Pr(\mathcal{S} \wedge \mathcal{R})$ for $\mathcal{S}$'s of interest, and for a formal discussion this seems most clearly handled by something like the present approach.

We need two additional events (supplementing $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ above). The first of these is simply

$$\mathcal{S} = \{\alpha_0 \sim \alpha\}$$

(i.e. for any $\eta > 0$, $\alpha_0 = (1 \pm \eta)\alpha$ for large enough n; recall α, α_0 were defined in (4.30)). The second, which we call $\mathcal{T}$, is the conjunction of a few properties of G_0 that we already know hold w.h.p., namely: $|G_0| \sim n^2 q/2$ (see Proposition 2.12); (2.15) and (2.16) for $l \in [\kappa - 1, 2\kappa - 6]$ (meaning, in view of (4.28), (2.16) if $l = \kappa - 1$ and (2.15) otherwise); and the conclusion of Lemma 4.14 for $t \leq \kappa - 1$ (actually we only need this for even t). We first outline and then fill in details.

We have $\Pr(\mathcal{Q} \wedge \overline{\mathcal{R}}) = o(1)$ (by Lemma 4.13; this is the only role $\mathcal{Q}$ plays in the present argument), and will show

$$\Pr(\mathcal{R} \wedge \{F \neq \varnothing\} \wedge \overline{\mathcal{S}}) = o(1). \tag{4.35}$$

(This is easy and a secondary use of $\mathcal{R}$. Note $\{F \neq \varnothing\}$ is implied by $\mathcal{P}$.)

We will also show that, *deterministically,*

$$\mathcal{R} \wedge \{F \neq \varnothing\} \wedge \mathcal{S} \implies |(G \setminus G_0) \cap R(F_0)| > .1\alpha n^2 p \tag{4.36}$$

provided ϑ is sufficiently small (this is again easy), and, as mentioned in the preview,

$$\mathcal{T} \implies |R(S)| = O(\alpha_S^{1+\delta} n^2) \tag{4.37}$$

for some fixed $\delta > 0$ and all $S \subseteq G_0$, where we set $\alpha_S = 2|S|/(n^2 q)$. Thus the conjunction of $\mathcal{P}, \mathcal{R}, \mathcal{S}$ and $\mathcal{T}$ implies (again, deterministically), the event—call it $\mathcal{U}$—that $|G_0| < n^2 q$ (say) and there is an $S \subseteq G_0$ (namely the one that will become F_0) satisfying (say):

$$\alpha_S < 2.1\lambda, \quad |R(S)| = O(\alpha_S^{1+\delta} n^2), \quad \text{and} \quad |(G \setminus G_0) \cap R(S)| > .09\alpha_S n^2 p. \tag{4.38}$$

Thus, finally, for (4.29) it is enough to show (by a routine calculation)

$$\Pr(\mathcal{U}) = o(1). \tag{4.39}$$

(Because: since $\overline{\mathcal{U}}$ implies $\overline{\mathcal{P}} \vee \overline{\mathcal{R}} \vee \overline{\mathcal{S}} \vee \overline{\mathcal{T}}$, (4.39) implies

$$\Pr(\mathcal{Q} \wedge (\overline{\mathcal{P}} \vee \overline{\mathcal{R}} \vee \overline{\mathcal{S}} \vee \overline{\mathcal{T}})) = \Pr(\mathcal{Q}) - o(1);$$

but the left hand side here is at most

$$\Pr(\mathcal{Q} \wedge \overline{\mathcal{P}}) + \Pr(\mathcal{Q} \wedge \mathcal{P} \wedge \overline{\mathcal{R}}) + \Pr(\mathcal{P} \wedge \mathcal{R} \wedge \overline{\mathcal{S}}) + \Pr(\overline{\mathcal{T}}) = \Pr(\mathcal{Q} \wedge \overline{\mathcal{P}}) + o(1)$$

(the second and third terms on the left being bounded by Lemma 4.13 and (4.35) respectively), so we have $\Pr(\mathcal{Q} \wedge \mathcal{P}) = \Pr(\mathcal{Q}) - \Pr(\mathcal{Q} \wedge \overline{\mathcal{P}}) = o(1)$.)

Proof of (4.35). If $F \neq \varnothing$ (i.e. $\alpha > 0$) and $\mathcal{R}$ holds, then $F \cap R(F_0) \neq \varnothing$, while by

(4.13), for any $xy \in F \cap R(F_0)$,

$$|F| > \sigma(x,y) \geq \sigma_0(x,y) > .25n^{\kappa-2}q^{\kappa-1} = \Omega(\log n).$$

But then (since $\log n \gg p/q$) Proposition 4.11 says that w.h.p. $|F_0| \sim \vartheta|F|$, which is the same as $\mathcal{S}$. $\square$

Proof of (4.36). Note it is always true that $G_0 \cap R(F_0) \subseteq F_0$, since the endpoints of an $xy \in (G_0 \cap R(F_0)) \setminus F_0$ would be joined by a path (many paths) having odd intersection with F_0, and adding xy to such a path would produce a C_κ having odd intersection with F_0. (As mentioned earlier, this is the reaon for "odd" in the definition of central.) So if $\mathcal{R}$, $\mathcal{S}$ and $\{F_0 \neq \varnothing\}$ hold (and ϑ is slightly small) then

$$|(G \setminus G_0) \cap R(F_0)| > .12\alpha n^2 p - (1+o(1))\alpha n^2 q/2 > .1\alpha n^2 p. \tag{4.40}$$

$\square$

Proof of (4.37). Set $c = (\kappa-3)/2$. For $l \in [c]$ and $\varnothing \neq S \subseteq G_0$ (for $S = \varnothing$ there is nothing to show), call an xy-path (S,l)-*central* if it is S-central and at least one of its S-edges is at distance l (along the path) from one of x, y. (So a path may be (S,l)-central for several l's.) Let $\sigma_0(x,y;S,l)$ be the maximum size of a collection of internally disjoint (S,l)-central xy-paths in G_0 and

$$R_l(S) = \{\{x,y\} \in \tbinom{V}{2} : \sigma_0(x,y;S,l) > (.25/c)n^{\kappa-2}q^{\kappa-1}\}, \tag{4.41}$$

and notice that

$$R(S) \subseteq \cup_{l\in[c]} R_l(S). \tag{4.42}$$

Supposing temporarily (through (4.46)) that S and l have been specified, we abbreviate $\sigma_0(x,y;S,l) = \varsigma(x,y)$, $R_l(S) = R_l$ and use simply *"rope"* for "$(S,2l+2)$-rope" (defined before Lemma 4.13). Set $|R_l| = \rho_l n^2$ and

$$r = 2(\kappa-1) - 2(l+1) = 2(\kappa-l) - 4 \in [\kappa-1, 2\kappa-6]. \tag{4.43}$$

We next show that if G_0 satisfies

$$T := \max_{u,v} \tau^r(u,v) = O(n^{r-1}q^r) \tag{4.44}$$

(as implied by (2.15) and (2.16), so by $\mathcal{T}$), then

$$\text{the number of ropes is } \Omega(\rho_l n^{2l+3}q^{2l+2}). \tag{4.45}$$

Proof. Say a rope $P = (u_{l+1}, \dots, u_1, z, v_1, \dots, v_{l+1})$ is *generated by* $\{x,y\}$ if there are internally disjoint paths $(z, u_1, \dots, u_{\kappa-2}, w)$ and $(z, v_1, \dots, v_{\kappa-2}, w)$ with $\{z,w\} = \{x,y\}$. Each $\{x,y\} \in \binom{V}{2}$ generates at least $2\binom{\lfloor \varsigma(x,y)/2 \rfloor}{2}$ such ropes (since a set of a internally disjoint (S,l)-central xy-paths, each with an S-edge at distance l from x, produces $\binom{a}{2}$ of them), while the number of pairs generating a given rope is at most T (since in the scenario above, the complement of P in the cycle $(z, u_1, \dots, u_{\kappa-2}, w, v_{\kappa-2}, \dots, v_1, z)$ is a path of length r (see (4.43)) centered at w, so with P determines $\{x,y\}$). Thus the number of ropes is at least

$$T^{-1} \sum_{\{x,y\} \in R_l} 2\binom{\lfloor \varsigma(x,y)/2 \rfloor}{2} = \Omega(|R_l|(n^{\kappa-2}q^{\kappa-1})^2/T) = \Omega(\rho_l n^{2l+3}q^{2l+2}). \qquad \square$$

If we now also assume the conclusion of Lemma 4.14 for $t = 2l+2$ (again, this is contained in $\mathcal{T}$), then combining that upper bound with the lower bound in (4.45) gives

$$\rho_l = O(\max\{\alpha_S^2, \alpha_S(nq)^{-l}\}) = O(\alpha_S^{1+\delta}), \tag{4.46}$$

with $\delta > 0$ depending only on κ. (Here we use $\alpha_S \geq n^{-2}$, valid since $S \neq \varnothing$.)

So, now letting l vary, it follows that if G_0 satisfies $\mathcal{T}$ (and so all relevant instances of (4.44) and (4.33)), then (4.46) holds for all $l \in [c]$, which in view of (4.42) bounds

$|R(S)|$ as in (4.37). □

(It may be worth noting that for $l = 0$ the above argument gives only $\rho_l = O(\alpha_S)$, which loses the crucial δ in (4.46); thus the insistence on *central* paths in $\mathcal{R}$ and Lemma 4.13.)

Proof of (4.39). Given G_0, S, we have $|(G \setminus G_0) \cap R(S)| \sim \text{Bin}(m, p')$, with $m \le |R(S)|$ and $p' < p$ defined by $(1-q)(1-p') = 1-p$ (as in (B) of Section 4.2.3). So for $|R(S)|$ as in (4.38), Theorem 2.2 gives

$$\Pr(|(G \setminus G_0) \cap R(S)| > .09\alpha_S n^2 p) < \exp[-\Omega(\alpha_S n^2 p \log(1/\alpha_S))],$$

where the implied constant depends on δ but not on ϑ. Thus, assuming $|G_0| < n^2 q$ (as given by $\mathcal{U}$), setting $\alpha_s = 2s/(n^2 q)$ (where s will be $|S|$, so $\alpha_s = \alpha_S$), and summing over $s < 2.1\lambda n^2 q$, we have

$$\begin{aligned}\Pr(\mathcal{U} \mid G_0) &< \textstyle\sum_s \binom{n^2 q}{s} \exp[-\Omega(\alpha_s n^2 p \log(1/\alpha_s))] \\ &< \textstyle\sum_s \exp[\alpha_s n^2 p\{(\vartheta/2)\log(2e/\alpha_s) - \Omega(\log(1/\alpha_s)\}],\end{aligned} \tag{4.47}$$

which is $o(1)$ for small enough ϑ (implying (4.39) since

$$\Pr(\mathcal{U}) = \sum\{\Pr(G_0)\Pr(\mathcal{U} \mid G_0) : |G_0| < n^2 q\}).$$ □

4.6.1 Proof of Lemma 4.13

Fix $\varepsilon > 0$ (as in (4.28)) small enough to support the proofs of Propositions 4.17 and 4.20 below; these are our only constraints on ε, and it will be clear they are satisfiable. We continue to assume that p is as in (4.28).

Most of our effort here is devoted to proving the following variant of Proposition 4.13 in which we replace $\sigma_0(x, y, F_0)$ by $\sigma(x, y, F)$ and q by p.

Lemma 4.15. *W.h.p.*

$$G \in \mathcal{Q} \wedge \mathcal{P} \implies |\{xy \in F : \sigma(x, y; F) > .26n^{\kappa-2}p^{\kappa-1}\}| \geq .13\alpha n^2 p. \tag{4.48}$$

"Coupling down" will then easily get us to Lemma 4.13 itself. (The extra .01's, relative to the pretty arbitrary .25 and .12 in (4.31) and (4.32), leave a little room for this.)

Preview. The proof of Lemma 4.15 breaks into two parts, roughly (w.h.p.): (a) if $G \in \mathcal{Q}$ (here we don't need to assume $G \in \mathcal{P}$), then $\sigma(x, y)$ is close to its natural value for most $xy \in F$ (see the paragraph following the proof of Proposition 4.19); (b) a decent fraction of the paths produced in (a) are F-central (shown by limiting the number that are *not*; this is based on Proposition 4.20 and does assume $G \in \mathcal{P}$).

Definitions. It will be convenient to set

$$\Lambda = n^{\kappa-2}p^{\kappa-1},$$

since this quantity—essentially the typical number of paths in G joining a given pair of vertices—will appear repeatedly below. We write $Q \sim Q'$ when Q, Q' are distinct C_κ's sharing at least one edge. For edges e, f of G, we take

$$e \sim f \quad \Leftrightarrow \quad [\text{some } C_\kappa \text{ of } G \text{ contains both } e \text{ and } f], \tag{4.49}$$

$$e \approx f \quad \Leftrightarrow \quad [\text{there are } C_\kappa\text{'s } Q \sim Q' \text{ of } G \text{ with } e \in Q \text{ and } f \in Q'], \tag{4.50}$$

$S(e) = \{f \in G : e \sim f\}$, and $T(e) = \{f \in G : e \approx f\}$. For $\gamma \in (0, 1)$, let

$$L(\gamma) = \{\{x, y\} \in \tbinom{V}{2} : \sigma(x, y) < \gamma\Lambda\}$$

and $F(\gamma) = F \cap L(\gamma)$. Finally, with C as in Proposition 2.9 for $l = \kappa - 1$ (and, say, $\delta = 1/\kappa$), let $\mathcal{S}$ be the event that G satisfies (2.11) (so *not* the $\mathcal{S}$ used above).

Fix $\zeta = .01$. Our goal in the next four propositions is to show that $F(1 - \zeta)$ is small, accomplishing (a) of our outline above. We do this by showing separately (in

Propositions 4.18 and 4.19, using the tools provided by Propositions 4.16 and 4.17) that $F(\zeta)$ and $F(1-\zeta)\setminus F(\zeta)$ are small.

Proposition 4.16. *For $\gamma \in (0,1)$ and distinct $\{x_1,y_1\},\ldots,\{x_c,y_c\} \in \binom{V}{2}$,*

$$\Pr(\mathcal{S} \wedge \{\{x_i,y_i\} \in L(\gamma)\ \forall i \in [c]\}) \leq n^{-(c-o(1))(\kappa/(\kappa-1))(1-\varepsilon)^{\kappa-1}\varphi(\gamma-1)}. \tag{4.51}$$

(Recall $\varphi(x)$ was defined in (2.1).) Note the bound here is natural, being, for p at the lower bound in (4.28) (and up to the $o(1)$), what Theorem 2.1 would give for the probability that c independent binomials, each of mean Λ, are all at most $\gamma\Lambda$.

Proof. Since $\mathcal{S}$ gives $\tau(x,y) \leq \sigma(x,y) + C < (1+o(1))\gamma\Lambda$ for $\{x,y\} \in L(\gamma)$, the event in (4.51) implies that $X := \sum_{i\in[c]} \tau(x_i,y_i) < (1+o(1))c\gamma\Lambda$; so we just need to bound the probability of this.

In the notation of Theorem 2.3, with $A_1,\ldots,A_m$ the edge sets of the various x_iy_i-paths (in K_n), we have $\mu \sim c\Lambda$ and $\overline{\Delta} = \mu + O(\Lambda^2/(np)) \sim \mu$. (If two of our paths, say P and Q, share $l \in [1,\kappa-2]$ edges, then at least l internal vertices of P are vertices of Q; so the contribution of such pairs to $\overline{\Delta}$ is less than

$$c^2 n^{2(\kappa-2)-l} p^{2(\kappa-1)-l} = O(\Lambda^2/(np)) = o(1)$$

(using the upper bound in (4.28) for the $o(1)$)). Thus Theorem 2.3 gives

$$\Pr(X < (1+o(1))c\gamma\Lambda) < \exp\left[-(1-o(1))\varphi(\gamma-1)c\Lambda\right],$$

which, since $\Lambda > (1-\varepsilon)^{\kappa-1}(\kappa/(\kappa-1))\log n$, is less than the right hand side of (4.51). $\square$

Proposition 4.17. *W.h.p.*

$$\textit{if } Q_1 \sim Q_2 \sim Q_3 \sim Q_4 \textit{ are } C_\kappa\textit{'s of } G \textit{ then } |(\cup Q_i) \cap L(\zeta)| \leq 1. \tag{4.52}$$

Also, there is a fixed M such that w.h.p.

$$|S(e) \cap L(1-\zeta)| < M \quad \forall\, e \in G. \tag{4.53}$$

(Note the Q_i's in (4.52) need not be distinct.)

Proof. Write η_γ for the quantity $n^{-(1-o(1))(\kappa/(\kappa-1))(1-\varepsilon)^{\kappa-1}\varphi(\gamma-1)}$ appearing in (4.51) (here without the c).

Since $\mathcal{S}$ occurs w.h.p., it suffices to show that the probability that it holds while either (4.52) or (4.53) fails is $o(1)$. Thus in the case of (4.52) we want to bound the probability that $\mathcal{S} \wedge \{J \subseteq G\} \wedge \{|J \cap L(\zeta)| \geq 2\}$ holds for some $J \subseteq K_n$ of the form $\cup_{i\in[4]} Q_i$, where the Q_i's are C_κ's sharing edges as appropriate. With $\mathcal{T}(J) = \mathcal{S} \wedge \{|J \cap L(\zeta)| \geq 2\}$, this probability is at most

$$\begin{aligned} \sum \Pr(\{J \subseteq G\} \wedge \mathcal{T}(J)) \ &\leq \ \sum \Pr(J \subseteq G) \Pr(\mathcal{T}(J)) \\ &\leq \ O(n^{4\kappa-6} p^{4\kappa-3} \eta_\zeta^2) = o(1). \end{aligned}$$

Here the first inequality is an instance of Theorem 2.6 (since $\{J \subseteq G\}$ and $\mathcal{T}(J)$ are increasing and decreasing respectively), Proposition 4.16 gives $\Pr(\mathcal{T}(J)) = O(\eta_\zeta^2)$ (for any J), and the $o(1)$ holds (for small enough ε) since $n^{4\kappa-6}p^{4\kappa-3} = \tilde{\Theta}(n^{\kappa/(\kappa-1)})$. The argument for

$$\sum \Pr(J \subseteq G) = O(n^{4\kappa-6} p^{4\kappa-3}) \tag{4.54}$$

is similar to the proof of Proposition 2.9; briefly: if $Q_1, \ldots, Q_4$ are C_κ's, with $R_i = \cup_{j\leq i} Q_j$ and, for $i \geq 2$, $|E(Q_i) \setminus E(R_{i-1})| = b_i \leq \kappa - 1$ and $|V(Q_i) \setminus V(R_{i-1})| = a_i$, then $n^{a_i} p^{b_i} \leq \Lambda$ for $i \geq 2$ (since $b_i = a_i = 0$ or $b_i \geq a_i + 1$), yielding $n^{|V(R_4)|} p^{|E(R_4)|} \leq n^2 p \Lambda^4$ and (4.54).

Treatment of (4.53) is similar. Here J runs over subsets of K_n of the form $\cup_{i\in[M]} Q_i$, where the Q_i's are C_κ's with a common edge, and, with $\mathcal{T}(J) = \mathcal{S} \wedge \{|J \cap L(1-\zeta)| \geq M\}$, the probability that $\mathcal{S}$ holds while (4.53) fails is at most

$$\sum \Pr(\{J \subseteq G\} \wedge \mathcal{T}(J)) \;\leq\; O(n^2 p \Lambda^M \eta_{1-\zeta}^M) \;=\; o(1).$$

This is shown as above, with $n^{|V(J)|}p^{|E(J)|} \leq n^2 p \Lambda^M$ given by the passage following (4.54) (with M in place of 4) and the $o(1)$ valid for large enough M because $n^2 p \Lambda^M < n^{\kappa/(\kappa-1)} O(\log^{M/(\kappa-1)} n)$. □

The next assertion is the only place where we use the condition $\{G \in \mathcal{Q}\}$ of (4.32) (and (4.29)).

Proposition 4.18. *W.h.p.*

$$G \in \mathcal{Q} \implies |F(\zeta)| = o(|F|). \tag{4.55}$$

Proof. By the first part of Proposition 4.17 it is enough to show that the right hand side of (4.55) follows (deterministically) from the conjunction of $\{G \in \mathcal{Q}\}$ and (4.52). But these imply that $|T(e) \cap F| \geq \zeta\Lambda$ for each $e \in F(\zeta)$: $\{G \in \mathcal{Q}\}$ gives at least one C_κ containing e; this C_κ contains a second edge, xy, of F (since $F \in \mathcal{C}_\kappa^\perp$), which by (4.52) is not in $L(\zeta)$; and $T(e)$ contains at least $\zeta\Lambda$ (distinct) F-edges lying on xy-paths. Moreover, again by (4.52), $T(e) \cap F(\zeta) = \{e\}$ $\forall e \in F(\zeta)$ and $T(e) \cap T(f) = \varnothing$ for distinct $e, f \in F(\zeta)$. Thus $|F(\zeta)| < |F|/(\zeta\Lambda)$ $(= o(|F|))$, as desired. □

Proposition 4.19. *W.h.p.*

$$|F(1-\zeta) \setminus F(\zeta)| = o(|F|). \tag{4.56}$$

Proof. It's enough to show that (4.53) implies (4.56) (since Proposition 4.17 says (4.53) holds w.h.p.). This is again easy: Set $B = F(1-\zeta) \setminus F(\zeta)$ and consider the graph with vertex set F and adjacency as in (4.49). Each $e \in B$ has degree at least $\zeta\Lambda$ in this graph, while (4.53) says no vertex has more than M neighbors in B. Thus $|B|(\zeta\Lambda - M) \leq |F \setminus B| M$, which (since $\Lambda \gg 1$) gives (4.56). □

Combining Propositions 4.18 and 4.19 completes part (a) of the preview following the statement of Lemma 4.15:

$$\text{w.h.p.} \quad G \in \mathcal{Q} \quad \Longrightarrow \quad |F(1-\zeta)| = o(|F|). \tag{4.57}$$

The next assertion, an echo of Section 2.4, provides technical support for part (b) (getting from (4.57) to Lemma 4.15 by controlling non-F-central paths).

For $v \in V$ and $S \subseteq \nabla_G(v)$, let $T_S(v)$ be the set of C_κ's using two edges of S and $\tau_S(v) = |T_S(v)|$. (We could write simply T_S, τ_S, but keep the v as a reminder).

Proposition 4.20. *For each fixed $\theta > 0$ there exists C_θ such that w.h.p.: for all $v \in V$ and $S \subseteq \nabla_G(v)$, with $|S| = \gamma np$ and $\mu = \gamma^2 n^{\kappa-1}p^\kappa/2$,*

$$\tau_S(v) < \begin{cases} (1+\theta)\mu & \text{if } \gamma > \gamma_\theta = C_\theta \log\log n/\log n, \\ o(\mu/\gamma) & \text{in general.} \end{cases} \tag{4.58}$$

Proof. We first observe that there is a fixed B such that w.h.p. no v lies in more than B C_κ's that meet $N(v)$ more than twice (basically because—here we omit the routine details—the expected number of such C_κ's at a given v is $O(n^{\kappa-1}p^{\kappa+1}) = n^{-\Omega(1)}$). It is thus enough to prove Proposition 4.20 with T and τ replaced by T' and τ', where $T'_S(v) = \{Q \in T_S(v) : |Q \cap N(v)| = 2\}$ and $\tau'_S(v) = |T'_S(v)|$.

Here we use a reduction similar to the one given by Proposition 2.9 (though, as will appear below, we can't expect to do quite as well as in (2.11)). Let $\sigma_S(v)$ be the maximum size of a collection of C_κ's from $T'_S(v)$ that are disjoint outside $\overline{N}(v) := \{v\} \cup N(v)$. Set $\psi(S) = \min\{|S|, \log^2 n\}$.

Proposition 4.21. *There exists D such that w.h.p. for all v and $S \subseteq \nabla_G(v)$,*

$$\tau'_S(v) - \sigma_S(v) < D\psi(S). \tag{4.59}$$

Proof. For fixed v and $S \subseteq \nabla_G(v)$, let $\Gamma = \Gamma_S$ be the graph on $T'_S(v)$ with $Q \sim R$ if Q and R share a vertex not in $\overline{N}(v)$. Since $\tau'_S(v) - \sigma_S(v) \leq |E(\Gamma)|$, (4.59) holds (for a suitable D) provided

(i) the sizes of the components of Γ are $O(1)$ and

(ii) the sizes of the induced matchings of Γ are $O(\psi(S))$;

so we would like to say that w.h.p. (i) and (ii) hold for all v and S. Here (and only here) we use $V(Q)$ for the set of vertices of Q not in $\overline{N}(v)$.

Of course (i) holds for all S (at v) iff it holds for $S = \nabla_G(v)$, so we just consider this case. Here we again (as in Proposition 2.9) want, for large enough M, (probable) nonexistence of $Q_1, \ldots, Q_M \in T'_S(v)$ such that, for $i \geq 2$, $V(Q_i)$ meets, but is not contained in, $\cup_{j<i} V(Q_j)$. Arguing as for (2.13) we find that the total numbers, say a and b, of vertices (other than v) and edges used by such $Q_1, \ldots, Q_M$ satisfy

$$n^a p^b \leq n^{\kappa-1} p^{\kappa} (n^{\kappa-3} p^{\kappa-2})^{M-1}. \tag{4.60}$$

(Note here we do count neighbors of and edges at v. The bound says $n^a p^b$ is largest when each new Q_i meets what precedes it in a P_2 starting at v.) Since $n^{\kappa-1}p^{\kappa} = \Theta(np \log n)$ and $n^{\kappa-3}p^{\kappa-2} = \tilde{\Theta}(n^{-1/(\kappa-1)})$, the bound in (4.60) is $o(1/n)$ for slightly large M, as is the probability of seeing such Q_i's at v.

For (ii), it will help to condition on $\nabla_G(v)$. Using ν' for the maximum size of an induced matching and invoking Proposition 2.12, we find that it's enough to show that, for a given v, $R \subseteq \nabla(v)$ of size less than $2np$ (say) and large enough D,

$$\Pr(\exists S \subseteq R, \nu'(\Gamma_S) > D\psi(S) \mid \nabla_G(v) = R) = o(1/n). \tag{4.61}$$

So assume we have conditioned on $\{\nabla_G(v) = R\}$, with R as above. An easy verification (again similar to those in the proof of Proposition 2.9) gives, for any $S \subseteq R$ (and, again, $\gamma_S = \gamma$ and $\Gamma_S = \Gamma$),

$$\begin{aligned} \tilde{\mu} = \tilde{\mu}_S := \mathbb{E}|E(\Gamma)| &= O(\tbinom{|S|}{2} n^{\kappa-3} p^{\kappa-2} |S| n^{\kappa-4} p^{\kappa-3}) \\ &= O(\gamma^3 n^{2(\kappa-2)} p^{2(\kappa-1)}) = O(\gamma^3 \log^2 n); \end{aligned} \tag{4.62}$$

say $\tilde{\mu} < C\gamma^3 \log^2 n$ (with C fixed). On the other hand, with $\{Q_i, R_i\}$ the possible edges of Γ and $A_i = \{Q_i \cup R_i \subseteq G\}$, $\nu'(\Gamma) \geq l$ implies occurrence of some l independent A_i's, an event whose probability Proposition 2.5 bounds by $\tilde{\mu}^l/l! < (e\tilde{\mu}/l)^l$. (Here we could replace Proposition 2.5 by Lemma 2.4 (or Theorem 2.7), actually getting a slightly better bound, but it seems preferable to make clear that the more elementary result is all that's needed.)

This leaves us with the union bound arithmetic. Here we first note that for $\nu'(\Gamma_S) < D\log^2 n\ \forall S$ we just need to check $S = R$, for which, in view of (4.62), we have $(e\tilde{\mu}/l)^l = o(1/n)$ for $l = D\log^2 n$ with a suitable D ($D > Ce$ is enough). We then need to say (again, for suitable D) that with probability $1 - o(1/n)$,

$$\nu'(\Gamma_S) < D|S| \text{ for all } S \text{ with } |S| < \log^2 n. \tag{4.63}$$

But with $s = \gamma np$, $\tilde{\mu} = \tilde{\mu_s} < C\gamma^3 \log^2 n$ and sums over $s \in [1, \log^2 n]$, the probability that (4.63) fails is at most

$$\sum \binom{|R|}{s} \left(\frac{e\tilde{\mu}}{Ds}\right)^{Ds} < \sum \exp\left[\gamma np \left\{\log(2e/\gamma) + D\log\left(\frac{Ce\gamma^3\log^2 n}{D\gamma np}\right)\right\}\right],$$

which, since we are in the range $\gamma np \in [1, \log^2 n]$, is easily $o(1/n)$. □

We continue with the proof of Proposition 4.20, which, by Proposition 4.21, we now need only prove with $\tau_S(v)$ replaced by $\sigma_S(v)$. Here it will help to have a concrete $o(\cdot)$ in (4.58). Set $h = h(n) = (\log\log n)^{1/2}$ (we need $1 \ll h \ll \log\log n$) and, with C_θ (and thus γ_θ) TBA, set

$$K_\gamma = \begin{cases} 1+\theta & \text{if } \gamma > \gamma_\theta, \\ (h\gamma)^{-1} & \text{otherwise.} \end{cases}$$

Given v and $S \subseteq \nabla_G(v)$ of size γnp (so we condition on $\{S \subseteq G\}$), Lemma 2.4 (or Theorem 2.7) gives

$$\sigma_S(v) \preccurlyeq Y \sim \mathrm{Bin}(m, p^{\kappa-2}),$$

with $m = \gamma^2 n^{\kappa-1} p^2/2$ (so $\mathbb{E}Y$ is the μ of Proposition 4.20). On the other hand, Theorems 2.1 and 2.2 give, writing K for K_γ,

$$\Pr(Y > K\mu) < \begin{cases} \exp[-\theta^2\mu/3] & \text{if } \gamma > \gamma_\theta, \\ \exp[-K\mu\log(K/e)] & \text{otherwise.} \end{cases} \tag{4.64}$$

Thus, with ξ_γ denoting the appropriate bound in (4.64), the probability of violating the σ_S-version of (4.58) with an S of size γnp is less than

$$n\tbinom{n}{\gamma np}p^{\gamma np}\xi_\gamma < \exp[\log n + \gamma np\log(e/\gamma)]\cdot\xi_\gamma \tag{4.65}$$

(where the terms preceding ξ_γ correspond to summing $\Pr(S \subseteq G)$ over $v \in V$ and $S \subseteq \nabla(v)$ of size γnp).

Finally, we should make sure the bound in (4.65) is small. Recalling (4.28), we have (for slightly small ε) $\Lambda > (1-\varepsilon)^{\kappa-1}\kappa/(\kappa-1)\log n > \log n$ and

$$\mu \ \ (= (\gamma^2 np/2)\Lambda) \ \ > (\gamma^2 np/2)\log n. \tag{4.66}$$

Thus for $\gamma > \gamma_\theta$ the bound in (4.65) is less than

$$\exp[\gamma np\cdot\{\log(e/\gamma) - \theta^2\gamma\log n/6\} + \log n],$$

which is tiny ($\exp[-n^{\Omega(1)}]$) for fixed $C_\theta > 6\theta^{-2}$.

For $\gamma \le \gamma_\theta$, noting that $(\gamma K_\gamma/2)\log(K_\gamma/e) \sim \log(1/\gamma)/(2h) = \omega(1)$ (and $\gamma np \ge 1$), and again using (4.66), we find that the right hand side of (4.65) is less than

$$\exp[\gamma np\cdot\{\log(e/\gamma) - (\gamma K_\gamma/2)\log(K_\gamma/e)\log n\} + \log n] \ = \ n^{-\omega(1)}.$$

And of course summing these bounds over γ gives what we want. □

Proof of Lemma 4.15. Fix $\theta = .005$ and let $C = C_\theta$ and γ_θ be as in Proposition 4.20. Set $\gamma_v = d_F(v)/(np)$, and let φ_v be the number of C_κ's of G using two F-edges at v. Let $\sigma^*(x,y)$ be the number of xy-paths having F-edges at one or both of x, y. Write

$\sum'$ and $\sum''$ for sums over v with $\gamma_v > \gamma_\theta$ and $\gamma_v \le \gamma_\theta$ respectively. We have, w.h.p.,

$$\begin{aligned}\textstyle\sum_{xy\in F}\sigma^*(x,y) &\le 2\textstyle\sum_{v\in V}\varphi_v\\ &\le n^{\kappa-1}p^\kappa\cdot[(1+\theta)\textstyle\sum'\gamma_v^2+\sum'' o(\gamma_v)],\end{aligned}\tag{4.67}$$

where the first inequality comes from considering how many times each side counts the various C_κ's of G, and the second is given by Proposition 4.20.

Since $\sum\gamma_v = \alpha n$, the second sum in (4.67) is $o(\alpha n)$. For the first, let $B = \{v \in V : \gamma_v > \theta\}$. If we now assume $\alpha = o(1)$ (as given by $\mathcal{P}$), then we have $|B| = o(n)$; so Proposition 2.13 (parts (a) and (b)) gives (w.h.p.)

$$|G[B]| \ll |B|\theta np < \textstyle\sum_{v\in B} d_F(v) \le \alpha n^2 p,$$

whence $\sum_{v\in B}\gamma_v np \le 2|G[B]| + |\nabla_F(B)| < (1+o(1))\alpha n^2p/2$,

$$\textstyle\sum_{v\in B}\gamma_v < (1+o(1))\alpha n/2$$

and (recalling $d_F(v) \le d_G(v)/2\ \forall v$; see (4.9))

$$\textstyle\sum_{v\in B}\gamma_v^2 \le \max_v\gamma_v\sum_{v\in B}\gamma_v < (1+o(1))\alpha n/4. \tag{4.68}$$

Thus (since also $\sum_{v\in V\setminus B}\gamma_v^2 \le \theta\sum_v\gamma_v = \theta\alpha n$) we find that the expression in square brackets in (4.67) is less than $(1/4+2\theta)\alpha n$, whence

$$\textstyle\sum_{xy\in F}\sigma^*(x,y) \le (1/4+2\theta)\alpha n^\kappa p^\kappa = .26\alpha n^\kappa p^\kappa. \tag{4.69}$$

(To avoid confusion we note that the .26 here, which is more or less forced by the essentially tight bound in (4.68), has nothing to do with the .26 in (4.48).)

Now let $F^* = \{xy \in F : \sigma(x,y) \ge (1-\zeta)\Lambda\}$ $(= F\setminus F(1-\zeta))$. By (4.57), $|F^*| \sim \alpha n^2p/2$, w.h.p. provided $\mathcal{Q}$ holds. Note that (recall $\zeta = .01$) $xy \in F^*$ has $\sigma(x,y;F) > .26\Lambda$ (as in (4.48)) unless $\sigma^*(x,y) > .73\Lambda$. (As noted earlier, xy-paths necessarily have odd intersection with F, so the only real requirement for such a path to be central is that it have an internal edge in F.) It thus follows from (4.69) that for $\tilde{F} := \{xy \in F^* : \sigma(x, y; F) \le .26\Lambda\}$, we have

$$|\tilde{F}| \le \frac{.26\alpha n^\kappa p^\kappa}{.73\Lambda} \le .36\alpha n^2 p,$$

whence $|F^* \setminus \tilde{F}| \ge .13\alpha n^2 p$, implying (4.48). □

Proof of Lemma 4.13. As mentioned earlier, Lemma 4.13 follows easily from Lemma 4.15 *via* "coupling down" (viewpoint (A) of Section 4.2.3): it is enough to show that if G satisfies the right hand side of (4.48) then w.h.p. it also satisfies $\mathcal{R}$; that is, $|F \cap R(F_0)| \ge .12\alpha n^2 p$.

For $xy \in F' := \{xy \in F : \sigma(x, y; F) > .26\Lambda\}$ (see (4.48)), Theorem 2.1 gives

$$\Pr(\sigma_0(x, y; F_0) \le .25 n^{\kappa-2} q^{\kappa-1}) < \exp[-\Omega(n^{\kappa-2} q^{\kappa-1})] = n^{-\Omega(1)},$$

since members of a set of $\sigma(x, y; F)$ internally disjoint, F-central xy-paths survive in G_0 (and become F_0-central) independently, each with probability $\vartheta^{\kappa-1}$. So by Markov's Inequality, w.h.p.

$$|\{xy \in F' : \sigma_0(x, y; F_0) \le .25 n^{\kappa-2} q^{\kappa-1}\}| = o(|F'|).$$

The lemma follows. □

4.6.2 Proof of Lemma 4.14

This is a simple consequence of Proposition 2.15, but for perspective a brief comment on the bounds may helpful. The first bound—corresponding to a β^2-fraction of all P_t's having their ends in S—is the generic value, and will be the truth if q is large enough that (w.h.p.) all $\tau^{t-2}(x, y)$'s are about the same. For smaller q one can sometimes do better by, e.g. (for even t), taking S to consist of all edges at distance $t/2-1$ from some small set of "centers," producing something like the second bound.

Proof. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of the adjacency matrix, A, of G_0, with associated orthonormal eigenvectors $v_1, v_2, \ldots, v_n$, say with $\max_j v_{1,j} > 0$. Let $M = A^{t-2}$ (so M has eigenvalues λ_i^{t-2} ($i \in [n]$), with eigenvectors v_i), and $f = (d_S(x) : x \in V) = \sum \beta_i v_i$.

The number of (S,t)-ropes is w.h.p. less than

$$\begin{aligned} fMf^T &= \sum \lambda_i^{t-2}\beta_i^2 \\ &\leq \lambda_1^{t-2}\beta_1^2 + \max\{|\lambda_2|, |\lambda_n|\}^{t-2}\, \|f\|_2^2 \\ &< (1+o(1))\left[(nq)^{t-2}\beta_1^2 + (4nq)^{(t-2)/2}\|f\|_2^2\right], \end{aligned} \tag{4.70}$$

where we used $\sum \beta_i^2 = \|f\|_2^2$ and the second inequality is given by (2.25). We then need bounds on β_1^2 and $\|f\|_2^2$, both of which are easy: w.h.p.

$$\beta_1 = \langle f, v_1 \rangle \sim n^{-1/2} \sum d_S(v) = 2n^{-1/2}|S| = \beta n^{3/2} q$$

(using (2.26)) and

$$\|f\|_2^2 = \sum d_S^2(x) \leq \Delta_S \sum_x d_S(x) < (1+o(1))nq \cdot 2|S| \sim \beta n^3 q^2.$$

The lemma follows. □

4.7 Proof of Theorem 4.5

In what follows we set $\mathcal{E}(K_n) = \mathcal{E}$, $\mathcal{C}_H(K_n) = \mathcal{C}_H$ and so on. We prove (sketchily) Theorem 4.5 for $n \geq v_H + 2$—which is best possible e.g. if $H = K_\kappa$ with $\kappa \geq 4$ (e.g. since for $n \leq \kappa + 1$, $\mathcal{C}_H^\perp \supseteq \mathcal{C} \cap \mathcal{D}$)—and add a note at the end to cover $H = C_\kappa$ and $n \geq \kappa$.

We first note that $\mathcal{C}_H = \mathcal{E}$ if $|H| = 1$ (trivially) and $\mathcal{C}_H = \mathcal{D}$ if $|H| = 2$. (Since each of P_2, $2K_2$ (a 2-edge matching) is the sum of two copies of the other, the copies of an H of size 2 span all 2-edge subgraphs, and so all even subgraphs, of K_n.) Moreover, if H is a matching then $\mathcal{C}_H$ is easily seen to contain (all copies of) K_2 if $|H|$ is odd or

$2K_2$ if $|H|$ is even, so is equal to $\mathcal{E}$ or $\mathcal{D}$ as appropriate.

We may thus restrict attention to H containing a vertex x of degree at least 2, and observe that in this case $\mathcal{C}_H \supseteq \mathcal{C}_4$. (The sum of two copies of H that differ only in the copy of x is a $K := K_{2,d(x)}$, and repeating this with K and one of its divalent vertices produces a C_4.)

Since clearly $\mathcal{C}_4 = \mathcal{C} \cap \mathcal{D}$, we're done if H is in the second case of (4.5). Otherwise let $\tilde{H}$ be a copy of H in K_n and let F be a smallest element of $\tilde{H} + \mathcal{C}_4$. Evidently F is in the same case of (4.5) as H. Moreover, we claim F is either a triangle or the disjoint union of a matching and star (so possibly just a matching or just a star). Note this is enough, as the copies of F are then easily seen to generate the desired subspace of $\mathcal{E}$: if H is Eulerian then $F = K_3$; otherwise we may add two copies of F to produce a P_2, so the generated space contains $\mathcal{D}$. (Minor note: $|V(F)| \leq |V(H)| + 1$ since all odd vertices of F must also be odd in $\tilde{H}$.)

For the claim we observe that F cannot contain a P_3 (since adding a C_4 containing such a P_3 reduces $|F|$); disjoint P_2's (reduce by adding a C_6); or $K_3 + K_2$ (convert to P_4 and then reduce to P_2).

Finally, for $H = C_\kappa$ and $n \geq \kappa \geq 4$ (for $\kappa = 3$ there is nothing to show), it is enough to observe that the sum of two copies of H on the same vertex set and sharing a $P_{\kappa-3}$ is a C_4; so $\mathcal{C}_H = \mathcal{C} \cap \mathcal{D}$ if κ is even, while for odd κ, $\mathcal{C} \cap \mathcal{D} \subsetneq \mathcal{C}_H \subseteq \mathcal{C}$ implies $\mathcal{C}_H = \mathcal{C}$.

4.8 Proof of Theorem 4.12

Here, finally, we prove Theorem 4.12. Since we make no use of this material we strive for brevity (albeit with little success), and will sometimes allow considerably less formality than elsewhere.

The proof of the theorem is based on Theorem 2.23, a "container" theorem, along with Lemma 4.22, an anologue of the "Erdős-Simonovits Stability Theorem" [14, 35]. (Theorems 2.19 and 4.12 may be thought of as "sparse random" analogues of Erdős-Simonovits. Our use of Lemma 4.22 below is analogous to the use of Erdős-Simonovits in the proofs of Theorem 2.19 in [4, 34].)

We may assume—as we do throughout—that H has no isolated vertices, because, clearly, letting H' be a copy of H with any isolated vertices removed, we have $\mathcal{C}_H(L) = \mathcal{C}_{H'}(L)$ for any graph L with $v_L \geq v_H$. Recall we are also assuming $e_H \geq 2$.

For any H and $F \subseteq J \subseteq E(K_n)$, let $\tau_H(F, J)$ be the number of copies of H in J (say unlabelled, though it doesn't matter) having odd intersection with F, and abbreviate this to $\tau_H(F)$ if $J = E(K_n)$.

Lemma 4.22. *For any fixed graph H and $\varepsilon > 0$, there is a $\delta > 0$ such that if $F \subseteq E(K_n)$ satisfies $\tau_H(F) < \delta n^{v_H}$, then there is an $X \in \mathcal{W}_H^{\perp}(K_n)$ with $|F \Delta X| < \varepsilon n^2$.*

We will actually apply the following simple extension of Lemma 4.22, which is proved, together with the lemma itself, at the end of this section.

Corollary 4.23. *For any fixed H and $\varepsilon > 0$, there is a $\delta > 0$ such that if $F \subseteq J \subseteq E(K_n)$ satisfy $|J| > (1-\delta)n^2/2$ and $\tau_H(F, J) < \delta n^{v_H}$, then there is an $X \in \mathcal{W}_H^{\perp}(K_n)$ with $|F \Delta X| < \varepsilon n^2$.*

Proof of Theorem 4.12. For the rest of this discussion we take $v_H = \kappa$. The hypergraph $\mathcal{H}$ $(= \mathcal{H}_n)$ to which we will apply Theorem 2.23 is as follows. Let K be a copy of K_n and $\mathcal{V} = E(K) \times \{0, 1\}$; thus $N := |\mathcal{V}| = n(n-1)$. Let λ_i be the natural bijection $(e \mapsto (e, i))$ from $E(K)$ to $\mathcal{V}_i = \{(e, i) : e \in E(K)\}$ $(i \in \{0, 1\})$. Finally, let $\mathcal{H}$ be the e_H-uniform hypergraph on $\mathcal{V}$ whose edges are the E's satisfying

- ∘ $|E| = e_H$,
- ∘ $\bigcup_{i=0}^{1} \lambda_i^{-1}(E \cap \mathcal{V}_i)$ is (the edge set of) a copy of H in K, and
- ∘ $|E \cap \mathcal{V}_0| \equiv 1 \mod 2$.

In what follows, for $X \subseteq \mathcal{V}$, we set $X_i = X \cap \mathcal{V}_i$, $X^i = \lambda_i^{-1}(X_i)$ and $\tilde{X} = X^0 \cup X^1$. (So we may think of $\tilde{X}$ as a subgraph of K underlying X.) For orientation we note immediately that for any $F \subseteq G \subseteq K_n$,

$$F \in \mathcal{C}_H^{\perp}(G) \iff \lambda_0(F) \cup \lambda_1(G \setminus F) \in \mathcal{I}(\mathcal{H}). \tag{4.71}$$

(Recall from Section 2.11 that $\mathcal{I}(\mathcal{H})$ is the set of independent sets of $\mathcal{H}$.)

For the remainder of our discussion we set $\theta = n^{-1/m_2(H)}$ (recall the definition of $m_2(H)$ at (4.19)). We first need to check that $\mathcal{H}$ and θ satisfy the hypotheses of Theorem 2.23. Clearly $d = \Delta$ (by symmetry), and $d \asymp n^{\kappa-2}$ follows from

$$|\mathcal{H}| = (n)_\kappa 2^{e_H-1}/|\mathrm{Aut}(H)|, \tag{4.72}$$

so we want to show that for any $\sigma \subseteq \mathcal{V}$ of size l,

$$d(\sigma) = O(n^{\kappa-2-(l-1)/m_2(H)}). \tag{4.73}$$

For $\sigma \subseteq \mathcal{V}$ let K_σ be the graph with edge set $\tilde{\sigma}$ and vertices those vertices of K incident with edges of $\tilde{\sigma}$, and set $v_\sigma = |V(K_\sigma)|$, $e_\sigma = |E(K_\sigma)|$. Notice that $d(\sigma) = 0$ unless $|\tilde{\sigma}| = |\sigma|$ (i.e. $\sigma^0 \cap \sigma^1 = \varnothing$) and K_σ is (isomorphic to) a subgraph of H; so we may assume these are true. But in this case we have $d(\sigma) \asymp n^{\kappa - v_\sigma}$, so (4.73) follows from

$$v_\sigma \geq 2 + (e_\sigma - 1)/m_2(H),$$

which is the same as $m_2(H) \geq (e_\sigma - 1)/(v_\sigma - 2)$ and is true by the definition of m_2 (since $K_\sigma \subseteq H$). $\Diamond$

Let δ' be the δ given by Corollary 4.23 with $\varepsilon/4$ in place of ε, and

$$\delta = \min\{\delta', \varepsilon/4\}/\left(e_H 2^{e_H}\right). \tag{4.74}$$

Choose b so that (2.27) holds (with $r = e_H$ and the present $\mathcal{H}$ and θ) and let B and C be as in Theorem 2.23. Noting that the assumption of Theorem 4.12 is now $p > M\theta$, we will prove the theorem with M significantly larger than $\varepsilon^{-2}\delta^{-1}B\log(eM/B)$, say

$$M > 3\beta^{-1}\varepsilon^{-2}\delta^{-1}B\log(eM/B), \tag{4.75}$$

with β as in (4.81).

In view of (4.71), $F \in \mathcal{C}_H^{\perp}(G)$ implies existence of some $T \subseteq \mathcal{V}$ with

$$|T| < BN\theta, \tag{4.76}$$

$$T \ \subseteq \ \lambda_0(F) \cup \lambda_1(G \setminus F) \ \subseteq \ C(T) \tag{4.77}$$

and

$$|\mathcal{H}[C(T)]| < \delta|\mathcal{H}|. \tag{4.78}$$

For T satisfying (4.76) write $\mathcal{Q}_T$ for the event that there is an $F \in \mathcal{C}_H^{\perp}(G)$ satisfying both (4.77) and

$$\min\{|F \Delta X| : X \in \mathcal{W}_H^{\perp}(G)\} \geq \varepsilon n^2 p. \tag{4.79}$$

We will show (for any T)

$$\Pr(\mathcal{Q}_T) < p^{|T|} \exp[-\Omega(\varepsilon^2 \delta n^2 p)] = p^{|T|} \exp[-\Omega(\varepsilon^2 \delta N p)], \tag{4.80}$$

where the implied constants depend on neither T nor ε (so nor δ). This easily gives Theorem 4.12, as follows. By the above discussion (sentence containing (4.76)–(4.78)), failure of the theorem's conclusion is contained in the event $\cup \mathcal{Q}_T$ (union over T as in (4.76)), so the probability of this failure is less than

$$\sum \Pr(\mathcal{Q}_T) < \sum_t \tbinom{N}{t} p^t \exp[-\beta \varepsilon^2 \delta n^2 p] \tag{4.81}$$

for a suitable fixed $\beta > 0$ (where the second sum ranges over $t \leq BN\theta$). On the other hand,

$$\tbinom{N}{t} p^t < \exp[t \log(eNp/t)] \leq \exp[BN\theta \log(eNp/(BN\theta))];$$

so the bound in (4.81) is less than

$$BN\theta \ \exp[BN\theta \ \log(ep/(B\theta)) - \beta\varepsilon^2 \delta n^2 p], \tag{4.82}$$

which is small by our choice of M (recall $p > M\theta$). $\Diamond$

Proof of (4.80). Set $R = (C(T))^0$, $S = (C(T))^1$, $\mathsf{t} = |\tilde{T}|$ $(= |T|$—see the first inclusion in (4.77)) and $\mathsf{c} = \binom{n}{2} - |R \cup S|$. Notice first that existence of an F as in (4.77) implies

$$\tilde{T} \subseteq G \subseteq R \cup S,$$

which has probability $p^{\mathsf{t}}(1-p)^{\mathsf{c}}$. This already gives (4.80) unless

$$\mathsf{c} < \delta n^2, \tag{4.83}$$

which we may therefore assume.

We next show that

$$\tau_H(R, R \cup S) \leq |\mathcal{H}[C(T)]| \tag{4.84}$$

and

$$\begin{align} |R \cap S| &\leq |\mathcal{H}[C(T)]||\mathrm{Aut}(H)|n^{-(\kappa-2)} + 2e_H\delta n^2 \tag{4.85} \\ &< \delta n^2(2^{e_H-1} + 2e_H) < \varepsilon n^2/4. \tag{4.86} \end{align}$$

(The content in (4.85)–(4.86) is the first inequality; the second is given by (4.78) and (4.72), and the third by (4.74).)

Proof of (4.84). For any copy $X \cup Y$ of H with $X \subseteq R$, $Y \subseteq S \setminus R$ and $|X|$ odd (these are the copies counted by $\tau_H(R, R \cup S)$), we have $\lambda_0(X) \cup \lambda_1(Y) \in \mathcal{H}[C(T)]$. $\Diamond$

Proof of (4.85). For each $xy \in K$, by double counting,

$$\text{there are } \left(\frac{(n)_\kappa}{|\mathrm{Aut}(H)|}\right)\frac{e_H}{\binom{n}{2}} \text{ copies of } H \text{ in } K \text{ containing } xy. \tag{4.87}$$

Thus there are at least $|R \cap S|n^{\kappa-2}/|\mathrm{Aut}(H)|$ copies of H meeting $R \cap S$, at most $2e_H\mathsf{c}n^{\kappa-2}/|\mathrm{Aut}(H)| < 2e_H\delta n^{\kappa}/|\mathrm{Aut}(H)|$ of which are not contained in $R \cup S$. But each of the at least $(|R \cap S| - 2e_H\delta n^2)n^{\kappa-2}/|\mathrm{Aut}(H)|$ copies of H that meet $R \cap S$ and *are* contained in $R \cup S$ underlies at least one member of $\mathcal{H}[C(T)]$: for such a copy, say $\tilde{H}$, containing $xy \in R \cap S$, we may partition $\tilde{H} \setminus \{xy\} = L \cup M$ with $L \subseteq R$ and $M \subseteq S$, and then $\mathcal{H}[C(T)]$ contains $\lambda_0(L\cup\{xy\})\cup\lambda_1(M)$ if $|L|$ is even and $\lambda_0(L)\cup\lambda_1(M\cup\{xy\})$ if $|L|$ is odd. Thus $|\mathcal{H}[C(T)]|\geq(|R\cap S|-2e_H\delta n^2)n^{\kappa-2}/|\mathrm{Aut}(H)|$, implying (4.85). $\Diamond$

We may now apply Corollary 4.23 to $R\subseteq R\cup S$ with $\varepsilon/4$ in place of ε (recall the line preceeding (4.74); the hypotheses of the corollary are verified in (4.83), (4.84), (4.78), (4.74) and (4.72).) This yields some $Y\in\mathcal{W}_H^{\perp}(K_n)$ with

$$|R\Delta Y|<\varepsilon n^2/4. \tag{4.88}$$

We will show that, barring occurrence of some event(s) with probability as in (4.80), each F as in (4.77) is close to $X:=G\cap Y$ $(\in\mathcal{W}_H^{\perp}(G))$. Given F as in (4.77), we have

$$\begin{aligned} F\Delta X=F\Delta(G\cap Y)&=(F\Delta(G\cap R))\Delta((G\cap R)\Delta(G\cap Y))\\ &=(F\Delta(G\cap R))\Delta(G\cap(R\Delta Y)\\ &\subseteq(F\Delta(G\cap R))\cup(G\cap(R\Delta Y),\end{aligned}$$

which, since $F\subseteq G\cap R\subseteq F\cup(G\cap(R\cap S))$, implies

$$|F\Delta X|\leq|G\cap R\cap S|+|G\cap(R\Delta Y)|.$$

So we have

$$|F\Delta X|<\varepsilon n^2 p$$

unless

$$\max\{|G\cap R\cap S|,|G\cap(R\Delta Y)|\}>\varepsilon n^2p/2. \tag{4.89}$$

Thus, finally, for (4.80) we just need to show

$$\Pr((4.89)\text{ holds}\mid\tilde{T}\subseteq G)<\exp[-\Omega(\varepsilon^2\delta n^2p)]. \tag{4.90}$$

(Because: $\mathcal{Q}_T\Longrightarrow(4.89)\wedge\{\tilde{T}\subseteq G\}$, so $\Pr(\mathcal{Q}_T)\leq\Pr(\tilde{T}\subseteq G)\Pr((4.89)\mid\tilde{T}\subseteq G)$.)

But (4.90) is easy: by (4.86) and (4.88), the conditional expectation of each of $|G\cap R\cap S|,|G\cap(R\Delta Y)|$ given $\{G\supseteq\tilde{T}\}$ is less than $\varepsilon n^2p/4+\mathsf{t}$, which in view of

(4.76) and our choice of M (see (4.75)) is less than $\varepsilon n^2 p/3$. So (4.90) follows from Theorem 2.1. $\square$

Proof of Lemma 4.22. This will be rather sketchy and thoroughly informal, foregoing epsilons and deltas in favor of qualitative language. Thus, to begin, we use "most" to mean for all but a small fraction of relevant possibilities, where "small" can be made less than any desired (positive) constant *via* an appropriate choice of δ. For example, "most x, y" means the number of exceptions is less than $\delta_\varepsilon n^2$ for a suitable δ_ε. Similarly we say sets A and B are "close," and write $A \approx B$, if $|A \triangle B|$ is at most small fraction of what it might have been (e.g. n^2 if A, B are sets of edges, or n if A, B are sets of edges at a given vertex). And so on.

Let $V(H) = \{u_1, \ldots, u_\kappa\}$, say with $u_{\kappa-1}u_\kappa \in H$ and $N_H(u_{\kappa-1}) \backslash \{u_\kappa\} = \{u_i : i \in I\}$. Say a κ-tuple $(x_1, \ldots, x_\kappa)$ of vertices of K_n is *even* if, for $\varphi : V(H) \to V(K_n)$ given by $\varphi(u_i) = x_i$ (for $i \in [\kappa]$), $|\varphi(E(H)) \cap F|$ is even (where, of course, $\varphi(u_i u_j) = \varphi(u_i)\varphi(u_j)$). We now use x, y, z, possibly subscripted, for vertices of K_n and $N(x)$ for $N_F(x)$.

Claim 1. There is an x such that for most y, $N(y)$ is close to either $N(x)$ or $\overline{N(x)}$ $(:= V(K_n) \setminus N(x))$. (In fact this is true of most choices of x.)

Proof. Simple averaging, using the fact that $\tau_H(F)$ is small, shows that most choices of x, y satisfy

$(*)$ for most choices of $x_1, \ldots, x_{\kappa-2}$ and z,

$$\text{both } (x_1, \ldots, x_{\kappa-2}, x, z) \text{ and } (x_1, \ldots, x_{\kappa-2}, y, z) \text{ are even.} \tag{4.91}$$

So we may fix an x for which $(*)$ holds for most y. If $(*)$ holds for x, y then there is a fixed $(x_1, \ldots, x_{\kappa-2})$ such that (4.91) holds for most z, and for each such z we have

$$|N(z) \cap \{x, y\}| \equiv |N(x) \cap \{x_i : i \in I\}| + |N(y) \cap \{x_i : i \in I\}| \pmod 2; \tag{4.92}$$

thus, since there is no z on the right hand side of (4.92), $N(y)$ is close to one of $N(x)$, $\overline{N(x)}$ whenever y satisfies $(*)$ (with our fixed x). $\lozenge$

Claim 2. For x as in Claim 1, F is close to either $\nabla(N(x))$ or its complement.

Proof. Set $A = N(x)$, $B = \overline{A}$, $S = \{y : N(y) \approx A\}$ and $T = \{y : N(y) \approx B\}$. We first observe that Claim 2 will follow if we show that

$$\text{one of} \quad I := (A \cap S) \cup (B \cap T), \quad J := (A \cap T) \cup (B \cap S) \quad \text{is small.} \tag{4.93}$$

Suppose for example that I is small. It is easy to see that if $e \in F \triangle \nabla(A)$ then e either meets $I \cup \overline{S \cup T}$ or lies in $\nabla_F(y) \triangle \nabla(y, B)$ for some $y \in A \cap T$ or in $\nabla_F(z) \triangle \nabla(z, A)$ for some $z \in B \cap S$. But the number of such e's is small, since we assume I and $\overline{S \cup T}$ are small (the latter by our choice of x), while $y \in T$ implies $\nabla_F(y) \approx \nabla(y, B)$, and similarly $z \in S$ implies $\nabla_F(z) \approx \nabla(z, A)$. Thus in this case $F \approx \nabla(A)$. (Showing that J small implies $F \approx \overline{\nabla(A)}$ is of course similar.) $\lozenge$

For (4.93) it is enough to show that $A \cap S$ and $A \cap T$ cannot both be large, and similarly for the pairs $(A \cap S, B \cap S)$, $(A \cap T, B \cap T)$, $(B \cap S, B \cap T)$; there is little difference between these and we just show the first. The set $\nabla_F(A \cap S, A \cap T)$ is small since any $z \in A \cap T$ (or just T) has few neighbors in A. But we also have $|\nabla_F(A \cap S, A \cap T)| \approx |A \cap S||A \cap T|$, since, for each $y \in A \cap S$, the set $(A \cap T) \setminus N(y)$ (or even $A \setminus N(y)$) is small. So it must be that one of $A \cap S$, $A \cap T$ is small. $\lozenge$

The four flavors of Lemma 4.22 (corresponding to the possibilities for $\mathcal{W}_H(K_n)$ in (4.5)) now follow easily. If H is even Eulerian then Claim 2 is what we want (since $\mathcal{W}_H^{\perp}(K_n)$ consists precisely of cuts and their complements). If H is odd (not necessarily Eulerian), then F cannot be close to the complement of a cut, since the edges of K_n contained in the larger side of the cut would contain $\Omega(n^{\kappa})$ copies of H that are *contained in* F. So F is close to a cut, which for Eulerian (odd) H is again what we want.

For non-Eulerian H (briefly): If F is close to a cut $\nabla(X, Y)$ with both X and Y large, then there are many odd copies of H with one odd vertex in (say) X and all other vertices in Y, so F cannot be close to such a cut. In particular this says that for odd (non-Eulerian) H, F must be close to $\varnothing$ (since being close to a cut with a small side is being close to $\varnothing$, and we have already said that F is not close to the complement of a cut). Finally, if H is even (non-Eulerian) then F is either close to $\varnothing$ (if close to a cut with a small side) or to $E(K_n) \in \mathcal{W}_H^{\perp}(K_n)$ (if close to the complement of such a cut). $\square$

Proof of Corollary 4.23. Given ε, let δ be as in Lemma 4.22, and $\delta' = \delta/(2e_H)$. Then, recalling (4.87), $|J| > (1-\delta')n^2/2$ implies that the number of copies of H not contained in J is at most $(\delta' n^2/2)\left(\frac{(n)_\kappa}{|\mathrm{Aut}(H)|}\right)\frac{e_H}{\binom{n}{2}} < \delta n^\kappa/2$, which, with $\tau_H(F,J) < \delta' n^\kappa < \delta n^\kappa/2$, implies $\tau_H(F) < \delta n^\kappa$, the hypothesis of Lemma 4.22. $\square$

Bibliography

[1] Noga Alon. Explicit ramsey graphs and orthonormal labelings. *Electron. J. Combin.*, 1, Oct 1994.

[2] Noga Alon, Michael Krivelevich, and Van H. Vu. On the concentration of eigenvalues of random symmetric matrices. *Israel J. Math.*, 131:259–267, 2002.

[3] Noga Alon and Joel H. Spencer. *The Probabilistic Method*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley, Hoboken, NJ, third edition, 2008.

[4] József Balogh, Robert Morris, and Wojciech Samotij. Independent sets in hypergraphs. *J. Amer. Math. Soc.*, 28:669–7097, 2015.

[5] Jacob D. Baron and Jeff Kahn. An extension of the BK Inequality. In preparation.

[6] Jacob D. Baron and Jeff Kahn. On the cycle space of a random graph. To be submitted.

[7] Jacob D. Baron and Jeff Kahn. Tuza's conjecture is asymptotically tight for dense graphs. *Combin. Probab. Comput.*, 25(5):645–667, Sept 2016.

[8] J. van den Berg. Personal communication, Oct 2015.

[9] J. van den Berg and H. Kesten. Inequalities with applications to percolation and reliability. *J. Appl. Probab.*, 22(3):556–569, Sept 1985.

[10] Béla Bollobás. *Modern Graph Theory*, volume 184 of *Graduate Texts in Mathematics*. Springer, New York, 1998.

[11] D. Conlon and W. T. Gowers. Combinatorial theorems in sparse random sets. arXiv math.CO, 2 Feb 2015, http://arxiv.org/abs/1011.4310v2.

[12] B. DeMarco, A. Hamm, and J. Kahn. On the triangle space of a random graph. *J. Comb.*, 4(2):229–249, 2013.

[13] Reinhard Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Heidelberg, 4th edition, 2010.

[14] P. Erdős. Some recent results on extremal problems in graph theory (results). In *Theory of Graphs (Internat. Sympos., Rome, 1966)*, pages 117–123 (English), 124–130 (French). Gordon and Breach, New York, 1967.

[15] Paul Erdős and Prasad Tetali. Representations of integers as the sum of k terms. *Random Structures Algorithms*, 1(3):245–261, 1990.

[16] Z. Füredi and J. Komlós. The eigenvalues of random symmetric matrices. *Combinatorica*, 1(3):233–241, 1981.

[17] Stefanie Gerke and Angelika Steger. The sparse regularity lemma and its applications. In *Surveys in Combinatorics 2005*, volume 327 of *London Math. Soc. Lecture Note Ser.*, pages 227–258. Cambridge Univ. Press, Cambridge, 2005.

[18] Geoffrey R. Grimmett. *Percolation*, volume 321 of *Grundlehren der mathematischen Wissenschaften.* Springer-Verlag Berlin Heidelberg, Berlin, 2nd edition, 1999.

[19] T. E. Harris. A lower bound on the critical probability in a certain percolation process. *Math. Proc. Cambridge Phil. Soc.*, 56(1):13–20, Jan 1960.

[20] Penny Haxell. Packing and covering triangles in graphs. *Discrete Math.*, 195:251–254, Jan 1999.

[21] Penny Haxell and Vojtěch Rödl. Integer and fractional packings in dense graphs. *Combinatorica*, 21(1):13–38, Jan 2001.

[22] Svante Janson. Poisson approximation for large deviations. *Random Structures Algorithms*, 1(2):221–229, June 1990.

[23] Svante Janson, Tomasz Łuczak, and Andrzej Ruciński. *Random Graphs.* Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley, New York, 2000.

[24] Matthew Kahle. Topology of random clique complexes. *Discrete Math.*, 309(6):1658–1671, Apr 2009.

[25] Matthew Kahle. Sharp vanishing thresholds for cohomology of random flag complexes. arXiv math.AT, 2 Oct 2013, https://arxiv.org/abs/1207.0149v3.

[26] Y. Kohayakawa, T. Łuczak, and V. Rödl. On K^4-free subgraphs of random graphs. *Combinatorica*, 17(2):173–213, June 1997.

[27] Yoshiharu Kohayakawa. Szemerédi's regularity lemma for sparse graphs. In Felipe Cucker and Michael Shub, editors, *Foundations of Computational Mathematics*, pages 216–230. Springer Berlin Heidelberg, 1997.

[28] Michael Krivelevich. On a conjecture of tuza about packing and covering of triangles. *Discrete Math.*, 142:281–286, July 1995.

[29] W. Mantel. Problem 28. In *Wiskundige Opgaven*, volume 10, pages 60–61. 1907.

[30] Pradipta Mitra. Entrywise bounds for eigenvectors of random graphs. *Electron. J. Combin.*, 16(1), Oct 2009.

[31] B. Pittel. A random graph with a subcritical number of edges. *Trans. Amer. Math. Soc.*, 309(1):51–75, Sept 1988.

[32] David Reimer. Proof of the Van den Berg–Kesten conjecture. *Combin. Probab. Comput.*, 9(1):27–32, Jan 2000.

[33] Vojtěch Rödl and Mathias Schacht. Extremal results in random graphs. In László Lovász, Imre Z. Ruzsa, and Vera T. Sós, editors, *Erdős Centennial*, volume 25 of *Bolyai Society Mathematical Studies*, pages 535–583. Springer Berlin Heidelberg, New York, 2013.

[34] David Saxton and Andrew Thomason. Hypergraph containers. arXiv math.CO, 28 Nov 2014, http://arxiv.org/abs/1204.6595v3.

[35] M. Simonovits. A method for solving extremal problems in graph theory, stability problems. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 279–319. Academic Press, New York, 1968.

[36] Joel Spencer. Counting extensions. *J. Combin. Theory Ser. A*, 55(2):247–255, Nov 1990.

[37] Endre Szemerédi. Regular partitions of graphs. In *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, volume 260 of *Colloques Internationaux du Centre National de la Recherche Scientifique*, pages 399–401, Paris, 1978. CNRS.

[38] Paul Turán. On an extremal problem in graph theory. *Matematikai és Fizikai Lapok*, 48:436–452, 1941.

[39] Zsolt Tuza. Conjecture. In A. Hajnal, L. Lovász, and V. T. Sós, editors, *Finite and Infinite Sets*, Proc. Colloq. Math. Soc. Janos Bolyai, page 888. North-Holland, Amsterdam, 1981.

[40] Raphael Yuster. Dense graphs with a large triangle cover have a large triangle packing. *Combin. Probab. Comput.*, 21:952–962, Nov 2012.