CONVENTIONS OF THE LANGUAGE OF MATHEMATICAL PROOF WRITING

AT THE UNDERGRADUATE LEVEL

By

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ABSTRACT OF THE DISSERTATION

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This dissertation consists of three studies investigating the language of mathematical proof writing at the undergraduate level by considering a series of potential breaches of the conventions of this language. The first study describes interviews with eight mathematicians and fifteen undergraduates and their opinions on the potential breaches present in student-produced proofs. In the second study, mathematicians and undergraduates from top mathematics departments in the United States completed a survey investigating how a larger sample of mathematicians and undergraduates perceived some of the potential breaches discussed in Study 1. The third study focused on how some undergraduates described and justified their use of potential breaches of the conventions of proof writing in the proofs they wrote throughout their introduction to proof courses. These three studies present a jumping-off point for future research considering how undergraduates learn to write proofs and how mathematics instructors present the linguistic aspect of proof writing.

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INTRODUCTION

Proof has been described as central to mathematical practice by many mathematicians and mathematics educators including Thurston (1994), Wu (1996), Rav (1999), Ball & Bass (2003), and Houston (2009). For instance, Thurston (1994) described proof as an important part of how mathematicians communicate ideas and contended, "We mathematicians need to put far greater effort into communicating mathematical ideas. To accomplish this, we need to pay much more attention to communicating not just our definitions, theorems, and proofs, but also our ways of thinking" (p.8). Indeed, much of the existing research in mathematics education concerning proofs at the undergraduate level focuses on the content and the way of thing. However, I argue that we mathematics educators should also pay attention to the *ways* in which professional mathematicians communicate.

While it is important for mathematics students at all levels to engage with mathematical language, this work focuses on advanced undergraduate mathematics students writing in the genre mathematical proofs. One of the primary goals of mathematics instruction at the advanced undergraduate level is to foster students' abilities to construct valid arguments and proofs. These abilities of constructing valid arguments and proofs are most often assessed through in-class written exams in which students are expected to write mathematical proofs. However, one of the reasons Moore (1994) suggested for the difficulties students have with proofs is the students' unfamiliarity with mathematical language.

In professional mathematical practice, mathematical proofs are an essential type of communication. Rav (1999) contended that proofs "are the heart of mathematics" and they play an "intricate role [...] in generating mathematical knowledge and understanding" (p. 6). In advanced mathematics courses, undergraduate students must necessarily engage with mathematical proofs. However, existing research has shown that students have immense difficulty when reading and constructing their own proofs (Weber, 2003). As the language of mathematical proof writing is more precise and rigorous than expositional mathematical writing, it seems natural to consider the hypothesis that the language is a contributing factor to undergraduate students' difficulties with proofs.

Existing Literature on the Language of Mathematics

While the existing research considering mathematical language is lacking at the undergraduate level, studies have considered mathematical language in K-12 mathematics. A constant in the various views on mathematical language is that it differs significantly from natural language. Addressing this linguistic difference, Halliday (1978) introduced the concept of the mathematical register, leading to further research on mathematical language:

> We can refer to a 'mathematics register', in the sense of the meanings that belong to the language of mathematics (the mathematical use of natural language, that is: not mathematics itself), and that a language must express if it is being used for mathematical purposes. (p.195)

This suggests that the mathematical register contains not only technical vocabulary and symbols, but also phrases and the associated syntax structures. Pimm (1987) brought this concept of the mathematical register to the scope of mathematics education and expanded the designation of technical definitions to natural English words. Other educators (e.g., Moschkovich, 2010) have considered how mathematical language affects English language learners at the K-12 level. While these studies have provided insights into students' use of mathematical language at the K-12 level, current research has not yet provided insight into the learning of mathematical language of proof writing at the undergraduate level. Moreover, mathematical meanings and how these meanings may differ from colloquial meanings of the same words are often the focus in existing research on the mathematical register. An additional aspect of the mathematical register is understanding the linguistic conventions of the language, which includes understanding the words of the language, how they interact, and how sentences are formed with them.

While few studies consider linguistic conventions of the language of mathematical proof writing at the undergraduate level or beyond, a number of mathematicians and a mathematics educator have written texts that address how mathematicians and undergraduate students should write mathematics. In particular, AMS (1962), Halmos (1970), Gillman (1987), Krantz (1997), and Higham (1998) wrote texts describing how professional mathematicians should properly and effectively use mathematical language in published settings such as journal articles, textbooks, and dissertations. Meanwhile, Houston (2009), Alcock (2013), and Vivaldi (2014) wrote student guides that included discussions of how students should use mathematical language. The guides, for both students and mathematicians, provide numerous recommendations, including stylistic suggestions such as avoiding a passive voice (Halmos, 1970; Gillman, 1987; Krantz, 1997; Higham, 1998; Vivaldi, 2014) and focusing on brevity (Halmos, 1970; Gillman, 1987; Krantz, 1997; Higham, 1998; Houston, 2009), and other linguistic suggestions such as using correct grammar with respect to how mathematical symbols are included in sentences (Gillman, 1987; Krantz, 1997; Higham, 1998; Houston, 2009; Vivaldi (2014) and avoiding the use of unclear referents (Higham, 1998; Houston, 2009; Alcock, 2013; Vivaldi, 2014). The suggestions made in these writing guides informed the studies described in this dissertation.

The guides described above were written based on the experiences and personal opinions of the authors, rather based on findings of research investigating mathematical language. In fact, empirical and systematic research considering the language of mathematical proof writing at the level of advanced undergraduate mathematics or above is lacking. There are two studies of note addressing the mathematical language used in proofs written by mathematicians. First, Konior (1993) identified a common 'segmentation of the text' in mathematical proof signaling the organization of a proof's argument for the reader. His analysis of over 700 mathematical proofs revealed these organizational signals: "plan of procedure" which indicates the upcoming steps of the proof, "delimiting fame" which partitions the proof, and "areas of directing the reader by the plan" which designates the parts of a proof. Second, in their investigation of the roles of authority and autonomy in mathematician's language use, Burton and Morgan (2000) found that especially those mathematicians highly regarded in the field sometimes break the norms of mathematical language.

Considering the Language of Mathematical Proof Writing

This body of work extends the existing literature by considering the conventions of the language of mathematical proof writing at the undergraduate level. To focus on the form of mathematical language, this body of work considers Scarcella's (2003) framework for academic English. A "register of English used in professional books and characterized by the specific linguistic features associated with academic disciplines" (p. 9), academic English was developed to address the multi-faceted complexity of writing expected of undergraduate students. In particular, Scarcella (2003) noted that academic disciplines have their own sub-registers of academic English. As such, this body of work considers the mathematical sub-register, with a particular focus on investigating the linguistic conventions of the genre of undergraduate proof writing.

To do so, it is necessary to first consider what it means to be a convention of a language. The influential book *Convention* (Lewis, 1969) attempted to describe the regularities in the behavior of members of a group and examine conventions of the group's language. The relationship between a group of people and the language it uses, according to Lewis (1969, 1972) is dependent upon the conventions of that language. Describing Lewis' formal theory of convention, Garrod and Doherty (1994) noted the implicit and rational nature of conventions, but Jackman (1998) proposed to expand the definition of convention to be "a custom or practice counts as a convention if it admits the sort of rational reconstruction according to which it could be maintained." (p. 308). That is, conventions are a subset of rationally justifiably customs. In this body of work, I adopt Jackman's conception of what a linguistic convention is – that is, I assume that a convention of language has rational purpose. In these studies, I investigate what some of the linguistic conventions of mathematical proof writing are and what the rational purposes of these conventions are.

Meanwhile, this literature did not indicate how these conventions might be identified. Naturally, it can be difficult for a member of a community to describe conventions and normative behavior of a practice of that community. Thus, in order to identify conventions of the language of mathematical proof writing, this body of work adopts the concept of breaching experiments (Mehan & Wood, 1975) in the style of Herbst and Chazan (2003) in their studies of practical rationality. This ethnomethodological concept of breaching experiments is based on the assumption that identifying a breach of a norm or practice is more natural than trying to describe that norm or practice otherwise. Thus, Herbst and Chazan's (2003) design is built on the hypothesis that when a participant is presented with a situation in which a norm or practice is breached, then the participant will identify what practice is being breached. In discussion of the breached practice, a participant may also attempt to repair the breach and explain the role of the norm or practice. Based on the work Herbst and Chazan's (2003) design, this work investigates linguistic conventions of mathematical proof writing by way of understanding potential breaches of convention made by undergraduate students when writing mathematical proofs.

As a preliminary step towards understanding what some potential breaches of the conventions of the language of mathematical proof writing commonly made by undergraduate students are, Lew and Mejía-Ramos (2015) inspected the student-produced proofs found in 149 exams at the introduction to proof level. Lew and Mejía-Ramos (2015) identified, based on their experience with proofs at the introduction to proof level and the existing literature (writing guides discussed above; Selden & Selden, 2003) addressing the language of proof writing, what they believed to be potential breaches of mathematical proof writing. Fourteen potential breaches were identified in this preliminary study; these potential breaches are provided in Table 1 and inform the design of the three studies in this dissertation.

Study 1

The preliminary study by Lew and Mejía-Ramos (2015) created a list of what we believed to be potential breaches of the conventions of the language of mathematical proof writing at the undergraduate level. The goals of the first study of this dissertation were to verify whether mathematicians and undergraduate students indeed found these uses of mathematical language to be unconventional, to identify what participants recognized as additional potential breaches, to investigate how the participants understood the corresponding linguistic conventions and the roles they played in proof writing, and to investigate how these potential breaches might affect how mathematicians grade proofs written by their students.

Eight mathematicians and fifteen undergraduate students participated in this study. I interviewed each participant individually using a semi-structured interview protocol. I conducted the interviews with the mathematicians prior to interviewing the undergraduate students, to first verify whether mathematicians found these potential breaches to be unconventional and allow the opportunity for the mathematicians to identify additional potential breaches.

In the interviews, I presented to the participants seven partial student-generated proofs and asked the participants to identify and describe uses of mathematical language that they perceived to be unconventional or out of the ordinary. In order to understand how writing proofs with these potential breaches might affect how mathematicians grade their students' proofs, mathematicians were asked if they would deduct points or make a note to their students about the potential breach and undergraduate students were asked if they would expect their professor to deduct points or make a note based on the inclusion of the potential breach.

Moreover, the interview procedure included a second pass through the data in which I showed the participants the partial proofs in which I had marked potential breaches that the participant had not identified themselves and asked the participants to assess whether or not the potential breach was unconventional; thus, the participants discussed their understanding of the potential breaches identified in Lew and Mejía-Ramos (2015), not just what they perceived to be breaches of linguistic conventions of mathematical proof writing.

This first study provided a clearer picture of how some undergraduate students and mathematicians perceived the language of mathematical proof writing. Moreover, the exploratory nature of this first study brought forth new research questions to be investigated. However, this study also had limitations. First, the sample sizes were limited; this first study only considered eight mathematicians and fifteen undergraduate students. Second, the design of the study was somewhat unnatural; I asked mathematicians and undergraduate students to evaluate unfamiliar and truncated proofs.

Study 2

To address the sample size limitation of Study 1, the survey was designed to investigate the extent to which the findings from Study 1 generalized to a larger sample of mathematicians and undergraduate students using an online survey. That is, this study investigated whether these larger samples agreed whether or not the potential breaches were unconventional in mathematical proof writing, and how, if at all, the participants' evaluations of these potential breaches were affected by the context in which the proofs were situated.

Using the methods employed by Inglis and Mejia-Ramos (2009), this second study used an online survey to maximize the sample size of mathematicians and students. These data collection methods have been validated by Reips (2000), Gosling et al (2004) and Krantz and Dalal (2000) and used in several mathematics education studies (see Inglis, Mejía-Ramos, Weber, & Alcock, 2013; Lai, Weber, & Mejía-Ramos, 2012; Mejía-Ramos &Weber, 2014).

The survey itself included fourteen pages of marked partial proofs (similar to those presented in the second pass of Study 1). For each of the marked partial proofs presented, the potential breach was identified and presented with a reason explaining why the particular use of language might be unconventional. The reason provided with each potential breach was based on the findings from Study 1. Thus, for each of the potential breaches evaluated in the survey, participants responded if they believed the potential breach presented was unconventional for the reason provided, and to what extent they believed the quality of the exposition of the proof was affected by that particular use of language in each of three contexts: a textbook proof, a blackboard proof, and a student-produced proof. Recruited from top mathematics departments among universities the United States, 128 mathematicians and 135 undergraduate students participated in the survey.

Including the reasoning for which the participants in Study 1 found the potential breach to be unconventional, provided a clear indication of whether the larger sample of participants in this study agreed with the participants from Study 1. Moreover, by extending the design to address how each participant viewed the potential breaches in each of the three contexts of textbook proofs, blackboard proofs, and student-produced proofs, the results from this study allows for a more detailed picture of how mathematicians and undergraduate students view different aspects of the language of mathematical proof writing at the undergraduate level.

Meanwhile, the design of this second study shared a limitation of Study 1 in that it does not address the unnatural setting in which mathematicians and undergraduate students are asked to participate and evaluate the potential breaches found in unfamiliar partial proofs. Moreover, the nature of this study considered only how the participants evaluated the potential breaches at one point in the semester and any consideration of how the undergraduate students' understandings develop over time is outside the scope of the design of Study 2.

Study 3

To address the limitations of Studies 1 and 2, this third study investigated how undergraduate students enrolled in an introduction to proof course wrote mathematical proofs throughout the term. The goals of this study were to understand how undergraduate students' view of the potential breaches changed throughout the semester, if undergraduate students believed the potential breaches identified in their proofs were unconventional of the language of mathematical proof writing, and if the undergraduate students believed that their proofs should conform to the conventions of the language of mathematical prof writing.

The design of this study involved the participants comparing the proofs they wrote for homework assignments and in-class examinations to proofs approved by the professors who taught the classes. By using the participants' own mathematical proofs, the participants discussed the potential breaches of the conventions of the language of mathematical proof writing, instead of evaluating unfamiliar proofs. Moreover, since the students were familiar with the proofs being discussed, the proofs were not truncated in this study. Finally, since this study's design involved multiple meetings with each participant throughout the semester, changes in how the participants discussed and explained the presence of potential breaches could be detected.

Six undergraduate students participated in this study in the same semester that they were enrolled in an introduction to proof course. Participants submit their graded homework assignments and in-class examinations in advance of the interviews so that the professor-approved proofs could be constructed for comparison. In the interviews, participants identified and discussed differences between their proofs and the professor-approved proofs. Following this discussion, I identified potential breaches not discussed by the student and asked the participant to discuss the breach and if they believed the breach to be unconventional. Each student participated in five to six interviews.

This third study represented preliminary work towards understanding how undergraduate students learn and understand the language of mathematical proof writing. The study considered how six undergraduate students wrote mathematical proofs throughout their introduction to proof courses and how the students viewed the potential breaches of the conventions of mathematical proof writing over time. By using a semester-long study design, the third study of this dissertation provides a more comprehensive view of how undergraduate students understand and use the language of mathematical proof writing.

Discussion

As mathematical proofs are an integral part of mathematics, advanced undergraduate students of mathematics must learn to read the proofs presented by their professors, to construct the arguments that comprise a proof, and to write their proofs. Their written proofs are often a major aspect of their assessments in advanced mathematics courses; however, little is known in the field of mathematics education of how these students learn to *write* mathematical proofs or how mathematics professors expect their students to write mathematical proofs. These three studies offer important first strides toward understanding the language of mathematical proof writing at the undergraduate level. Further understanding how both mathematicians and undergraduate students perceive mathematical proof writing at the undergraduate level will inform future research investigating how undergraduate students write mathematical proofs and future endeavors to improve instruction of conventional writing practices in undergraduate classrooms.

STUDY 1

This study examined the genre of mathematical proof writing at the undergraduate level by asking mathematicians and undergraduate students to read seven partial proofs based on student-generated work and to identify and discuss uses of mathematical language that were out of the ordinary with respect to what they considered standard mathematical proof writing. In particular, three themes emerged from analyses of the data. First, mathematicians believe that the language of mathematical proof writing should obey the rules of natural language whereas some students believe the two are independent. Second, students may not fully understand the nuances involved in the careful ways that mathematicians introduce objects in proofs. Third, mathematicians focus on the context of the proof to decide how rigorous or formal a proof should be, whereas students may not be aware of this distinction.

Key words: Mathematical language, Proof, Mathematicians, Undergraduate students

Introduction

Mathematicians and mathematics educators have found undergraduate mathematics students to have difficulties when constructing (Weber, 2001), reading (Conradie & Frith, 2000), and validating (Selden & Selden, 2003) mathematical proofs. Moore (1994) suggested that one reason for these difficulties is the students' unfamiliarity with the language of mathematical proof writing. However, mathematical language at the level of advanced undergraduate proof writing has not been examined in a systematic and empirical way in mathematics education research. As a result, the field knows little about how mathematicians and students understand and use this technical language. Having a clearer picture of the language of mathematical proof writing at the undergraduate level, from the perspectives of both professional mathematicians and undergraduate mathematics students, could enable researchers and mathematics instructors to create interventions and curriculum to help undergraduate students in their transition to abstract and advanced mathematics courses.

Related Literature and Theoretical Perspective

Halliday's (1978) introduction of the notion of register (and mathematical register in particular) was groundbreaking for the study of mathematical language:

> A set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings. We can refer to a 'mathematics register', in the sense of the meanings that belong to the language of mathematics […], and that a language must express if it is being used for mathematical purposes. (p.195)

Thus, the mathematical register contains not only technical vocabulary and symbols, but also phrases and their associated syntax structures. Various mathematics educators have considered how the mathematical register plays a role in mathematics learning and classrooms. For instance, Pimm (1987) discussed how students develop the mathematical register and Schleppegrell (2007) noted students' difficulties with differentiating between the mathematically precise and colloquial uses of words like 'if', 'when', and 'then'. However, much of this existing work on mathematical language focuses on K-12 mathematics. In particular, Pimm's (1987) *Speaking Mathematically* focuses on how elementary students interact with number sentences and other bodies of work (e.g., Moschkovich, 2010) consider how English language learners interact with mathematical language at the K-12 level. While these studies have provided insights into students' use of mathematical language at the K-12 level, current research has not yet provided insight into the learning of mathematical language of proof writing at the undergraduate level.

Moreover, existing studies considering the mathematical register focuses on meanings and how mathematical and colloquial meanings may differ. Another aspect of the mathematical register is the format of the language itself; yet, the field has not explored the professional form of writing typically found in undergraduate courses. Moreover, the field has not investigated the introduction of the formality of mathematical proofs and abstract mathematical concepts that bring new levels of complexity to the study of mathematical language. The goal of the current study is to identify important aspects of the form of mathematical proof writing and to study what sense of this form that students get from what their professors present in lectures.

Focusing on the form of mathematical language.

This study uses Scarcella's (2003) framework for academic English, a "register of English used in professional books and characterized by the specific linguistic features associated with academic disciplines" (p. 9), as a conceptual framing with the intent to address the complexity of writing expected at the undergraduate level. Scarcella argued that academic disciplines have their own sub-registers of academic English and, as such, the mathematical register can be seen as a sub-register of academic English. This study considers the mathematical sub-register in this study with a focus on the linguistic conventions of the genre of undergraduate proof writing.

Understanding how a language works and what it means to be a norm, rule, or convention of a language is a central part of philosophy. Lewis's (1969) influential book *Convention* made an attempt to describe regularities in the behavior of members of a population and to examine conventions of a language spoken by a population. In this and other philosophical publications, Lewis (1969, 1972) described the relationship between a population and the language it speaks as dependent upon conventions in the language. Thus a speaker uses the language and attempts not to speak untrue utterances. Suggesting a refinement of Lewis's original definition of conventions of language, Jackman (1998) argued that Lewis's definition relies on the assumption that speakers are 'conceptually autonomous' and that conventions are arbitrarily constructed. Instead, Jackman (1998) proposed that "a custom or practice counts as a convention if it admits the sort of rational reconstruction according to which it could be maintained" (p. 308). Thus, Jackman (1998) emphasized conventions are a subset of customs, which are rationally justifiable.

Advanced undergraduate and professional mathematical language.

With Jackman's definition for conventions of language as background, this paper considers the linguistic conventions of mathematical language at the advanced undergraduate and professional levels. The existing literature on how professional mathematicians view and use the language of mathematics was critical to informing the design of the study, yet was lacking in breadth and depth. While little empirical research exists, two studies are of importance. First, Konior's (1993) analysis of over 700 mathematical proofs revealed a common style of construction, which he called segmentation of the text, of mathematical proofs that signals the organization of the proof's arguments. This segmentation included organizational signals including: "plan of procedure" that provides the reader an indication of the upcoming steps of the proof, "delimiting frame" that breaks the proof into different parts, and "areas of directing the reader by the plan" that use language to indicate the start and finish of the proof and it's different parts. Second, Burton and Morgan (2000) identified the roles that research papers' author's identity and focus played in the extent to which the authors paid attention to the study's assumed linguistic conventions in their mathematical writing. Operating under the assumption that the professional writing guides in the literature (e.g. Gillman, 1987; Krantz, 1998), Burton and Morgan found that mathematicians sometimes break norms regarding authority and autonomy, especially those mathematicians who were highly regarded in the field.

Meanwhile, a number of manuals (Alcock, 2013; AMS, 1962; Gillman, 1987; Halmos, 1970; Higham, 1998; Houston, 2009; Krantz, 1997; Vivaldi, 2014) have been written describing how mathematicians and students should effectively use mathematical language. These guides highlight various recommendations including stylistic suggestions such as avoiding a passive voice (Gillman, 1987; Halmos, 1970; Higham, 1998; Krantz, 1997; Vivaldi, 2014) and focusing on brevity (Gillman, 1987; Halmos, 1970; Higham, 1998; Krantz, 1997; Houston, 2009). They also include linguistic suggestions addressing grammar and word choice. For instance, Gillman (1987), Higham (1998), Houston (2009), Krantz (1997), and Vivaldi (2014) all discuss the use of correct grammar, especially with respect to how mathematical symbols are grammatically included in sentences, and Alcock (2013), Higham (1998), Houston (2009), and Vivaldi (2014) addressed the use of unclear referents (or pronouns for which the antecedent is not explicit) in mathematical writing. While these guides offer valuable contributions to the field's current understanding of the language of mathematics, the suggestions made by in these guides are based on personal experiences and opinions rather than a systematic, empirical investigation of mathematical language usage. The current study focuses on empirically identifying potential linguistic conventions of mathematical language at the advanced undergraduate level and the extent to which students can identify them.

Research Questions

In particular, this study aimed to investigate the following questions:

1. How do mathematicians view and describe common unconventional uses of mathematical language in undergraduate mathematical proof writing?
2. How does these unconventional uses affect how mathematicians evaluate student-constructed proofs?
3. How do students understand the conventions of mathematical proof writing at the undergraduate level?

Methods

The above philosophical discussion on conventions of language does not address how one might identify such conventions. In order to identify conventions of mathematical language, this study adopts the ethnomethodological concept of breaching experiments (Mehan & Wood, 1975) in the style of Herbst and Chazan (2003) in their body of work on practical rationality. Intending to study norms by evoking repairing reactions from their participants, the hypothesis underlying Herbst and Chazan’s design is that when a participant of a practice is presented with a situation in which a norm of such practice is breached, he or she will attempt to repair the breach highlighting not only what the norm is, but also the role that the norm has in the practice (Herbst, 2010). Adapting this methodology, I examined the linguistic dimension of undergraduate proof writing by adapting the methods of breaching experiments. In this adaptation, I present participants with student-generated proofs and ask them to identify and describe uses of mathematical language in those proofs that they perceive to be out of the ordinary with respect to undergraduate proof writing. The idea is that by identifying these non-standard uses of mathematical language, the participants discuss their understanding of the conventions of proof writing in this context and role these conventions play in mathematical practice at the undergraduate level.

Participants

This study was conducted at a large research university in the United States. Eight mathematicians and sixteen undergraduate students participated in semi-structured interviews (eight of the undergraduates were “advanced” mathematics majors who had completed the proof-based courses required for graduation and eight were “novice” undergraduates who were enrolled in an introduction to proof course at the time of the study.) The mathematicians had 1-38 years of experience teaching undergraduate proof mathematics courses, with 1-15 years of experience teaching introduction to proof courses. Of the sixteen undergraduate participants, the data presented in this paper represents fifteen – one of the novice undergraduate students focused solely on reconstructing arguments of the propositions presented in the study and thus did not discuss his views on the language of mathematical proof writing at the undergraduate level.

Materials

The *potential* conventions of the language of proof writing at the core of this study – such as avoiding the use of unclear reference when writing proofs – emerge from prior work (Lew & Mejía-Ramos, 2015), the suggestions of the existing literature (including the writing guides described above and Selden and Selden (1987)), and personal experience with proof writing at the undergraduate level. In prior work, Lew and Mejía-Ramos analyzed the proofs in 149 student exams from introduction to proof courses and categorized what they believed to be common unconventional uses of the language of mathematical proof writing at the undergraduate level. These exams offered a source of common naturalistic potential breaches of convention, as they were evident in actual student constructed proofs. The categories were based on the authors' experiences and the suggestions made in mathematical writing guides for mathematicians and undergraduate students (Gillman, 1987; Higham, 1998; Houston, 2009; Krantz, 1997; Vivaldi, 2014). These categories are in the Appendix and informed the design of the study.

The materials for this study included seven partial proofs that are based on student-generated work. Each of the purported proofs was truncated to help participants focus on the use of mathematical language and not the purported proof's logical validity. One of the partial proofs used in the study is provided below in Figure 1a. The partial proofs were chosen from student exams given in introduction to proof classes at the same university of the study. Moreover, each of the proofs was coded for breaches of the potential conventions identified by Lew and Mejía-Ramos (2015). For each of these breaches of potential conventions, a copy of the partial proofs was created and marked for the instance of the categories of common unconventional uses of the language of mathematical proof writing at the undergraduate level, one example is shown in Figure 1b.

Let R and S be relations on a set A. Prove: $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

Suppose $(S \circ R)^{-1}$ such that $(x, z) \in (S \circ R)^{-1}$, $x, z \in A$.
Since $(x, z) \in (S \circ R)^{-1}$, then $(z, x) \in S \circ R$.
Since $(z, x) \in S \circ R$, then $(y, x) \in S$ and $(z, y) \in R$.

*Figure 1a***.** Example of partial proof, Proof A.

Let R and S be relations on a set A. Prove: $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

Suppose $(S \circ R)^{-1}$ such that $(x, z) \in (S \circ R)^{-1}$, $x, z \in A$.
Since $(x, z) \in (S \circ R)^{-1}$, then $(z, x) \in S \circ R$.
Since $(z, x) \in S \circ R$, then $(y, x) \in S$ and $(z, y) \in R$.

*Figure 1b***.** Example of marked partial proof.

Procedures

The semi-structured interview procedures for mathematicians and for undergraduate students were nearly identical. The interviews were videotaped and lasted one to two hours. Participants were presented with one student-constructed partial proof at a time. They were asked to mark the partial proofs for anything that was out of the ordinary with respect to the use of the language in undergraduate mathematical proof writing. I made two passes through the materials.

In the first pass, participants were asked to explain why they had made each mark. Next for each mark made, the participant was asked if the breach at hand was a logical issue, if it affected the validity of the proof, if it was an issue of mathematical writing, if it was definitely unconventional or a matter of personal preference, if it lowered the quality of the proof significantly, and if they (or in the students' case, if they thought a mathematician) would deduct points based on this issue when grading the proof in an introductory proof course. These prompts were designed to elicit the participants' views on what they thought were conventional uses of mathematical language in proof writing. In particular, the prompts addressed the severity of each breach and enabled a differentiation between issues of logic and issues of mathematical writing in the analysis of the data. Moreover, the design of this study was such that the interviewed mathematicians had the opportunity to identify alternative types of unconventional uses of language of mathematical proof writing and to discuss whether they agreed that the fifteen categories identified in the Appendix were indeed unconventional uses of the language of mathematical proof writing.

In the second pass, for each of the predicted instances of unconventional use of mathematical language that had not been identified by the participant in the first pass of the data, participants were asked if they would agree that this was an issue of mathematical language. Specifically, mathematicians were asked whether or not they would agree with a colleague of theirs who had suggested these were unconventional uses of mathematical language and the undergraduate students were asked if they would agree with a classmate of theirs who believed a mathematician would think these were unconventional uses of mathematical language. If they agreed, they would be prompted to discuss the breach as in the first pass.

Analysis

I transcribed the interview videos and scanned the materials generated in the interviews. The interview protocol created clear episodes of discussion in the interview transcripts, each concerning a single breach of mathematical language. Thus the data is organized by these episodes and was then analyzed using open ended thematic analysis in the style of Braun and Clarke (2006). That is, I first familiarized myself with the data by marking for ideas and transcribing videos, generated initial codes by organizing the ideas into meaningful groups, searched for themes by focusing analysis at a broad level, and reviewed the themes to verify that the themes reflect the data set as a whole.

When developing the codes, I began by reading through the episodes from the interviews and organizing them into groups. The initial codes began as the list of fifteen categories of common unconventional uses of the language of mathematical proof writing at the undergraduate level identified in Lew and Mejía-Ramos (2015), but as the analysis proceeded, new categories emerged (such as the need to attend to the part of speech of a symbol) and existing categories were clarified and/or merged based on the participants' discussions within the interview episodes. For instance, participants described the categories of unspecified variables (those used without quantification and specification of the set to which they belong) and underspecified variables (those with some, but insufficient specification) similarly, leading to the merging of those two categories.

To illustrate the analysis protocol, I will discuss how the code *Mathematical symbols and notation have parts of speech* developed. By *Mathematical symbols and notation have parts of speech*, I mean that mathematical objects should be used correctly within a sentence or phrase according to the type of object the notation represents, or the part(s) of speech of the word(s) the notation itself represents. The instance of proof writing that lead to this code reads, "None of the sets are $\emptyset$." This line was originally coded as *Mixing mathematical notation and prose* as the use of the empty set symbol, $\emptyset$, did not match the style of the sentence, which was otherwise written completely in

English prose. As such, in the initial sorting for the episodes into meaningful groups, episodes of discussion concerning this instance of mathematical proof writing, "None of the sets are ∅.", were originally coded as *Mixing mathematical notation and prose*. However, upon closer inspection of the explanations given by the mathematician participants, it became clear that this instance of mathematical proof writing was unconventional for multiple reasons and that a new code was developed. While some of the explanations focused on the use of notation within a sentence written in English words, some explanations instead discussed the mismatch of the parts of speech used in this sentence (in particular, one interpretation of this sentence uses the symbol, ∅, as an adjective[1]) and subsequently the grammatical issues, which arose from this mismatch. As a result, the code *Mathematical symbols and notation have parts of speech* was included in the analysis.

As the analysis proceeded, the codes were organized into preliminary themes. Early themes focused on the grain-size of the potential unconventional use of mathematical language; these themes focused on introducing variables, coordinating variables in an argument, and organizing arguments within the proof. However, as the codes developed and the justifications given by mathematicians and undergraduate participants were considered and compared, the emergent themes described in the following section are more descriptive, representative, and meaningful. Finally, I reviewed each of the themes, each of the codes, and the original interview episodes to verify that the themes were representative of the data set.

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[1] The symbol, ∅, may also be read as "the empty set" which would eliminate this concern. However, the analysis described highlights the interpretation of the mathematicians in the study.

Results

Three main themes emerged from the data analyses, each theme highlighting differences between how mathematicians view the conventions of mathematical language and how students understand these conventions. First, mathematicians believe that the language of mathematical proof writing should obey the rules of natural language, whereas some students believe the two are independent. Second, students may not fully understand the nuances involved in the careful ways that mathematicians introduce new mathematical objects in proofs. Third, mathematicians focus on the context of the proof to decide how rigorous or formal a proof should be whereas students may not be aware of this distinction. Below, each of these themes is expanded.

Theme 1: Mathematical language obeys the rules of natural language

This theme highlights that mathematicians believe that the language of mathematical proof writing obeys the rules of the English language whereas some students believe the two are independent. This theme emerged from the mathematicians' attention to the need for correct grammar and complete sentences, in addition to some of the undergraduate students' responses indicating that the rules of English do not apply in mathematical settings. In particular, Theme 1 emerged from three categories of responses from participants discussing what they considered to be non-standard mathematical language use. These responses are described below using interview data three different proofs: Proof A, Proof B, and Proof C.

Let $\mathcal{P}$ be the following collection of the subsets of the integers $\mathbb{Z}$:

$$\mathcal{P} = \{A_0, A_{+e}, A_{+o}, A_{-e}, A_{-o}\}, \text{ with } A_0 = \{0\},\ A_{+e} = \{2, 4, 6, 8, \ldots\},\ A_{+o} = \{1, 3, 5, 7, \ldots\},$$
$$A_{-e} = \{-2, -4, -6, -8, \ldots\},\ A_{-o} = \{-1, -3-5, -7, \ldots\}$$

Prove that $\mathcal{P}$ is a partition of $\mathbb{Z}$.

None of the sets are $\emptyset$.

All are pairwise disjoint, since the positive sets share nothing with the negative sets and the evens share nothing with the odds, and $\{0\}$ shares nothing with the rest.

Figure 2. Proof B.

Let A, B, and C be sets. Let $f : A \to B$ and $g : B \to C$.
Prove: If f and $g \circ f$ are bijections, then g is one-to-one.

let A, B, C be sets and $f : A \to B$ and $g : B \to C$
$\forall a, b \in B$, let $g(a) = g(b)$
Need to show $a = b$.
Because $f : A \xrightarrow[\text{onto}]{1\text{-}1} B$, there $\exists x \in A$ s.t. $f(x) = a$ and $f(y) = b$

Figure 3. Proof C.

Uses mathematical symbols and notation as an incorrect part of speech.

As shown in Figure 2, the first line of Proof B reads, “None of the sets are ∅.” Participants were expected to indicate this sentence as an unconventional use of language because the mathematical symbol for the empty set, ∅, was used in a sentence that was otherwise written in English words. This was generally the case, however, two mathematicians further explained that the issue of using the symbol ∅, was an issue of grammar. For example, M8 indicated in Pass 1 through the proof that there is a problem with the part of speech of the symbol “because *empty* is an adjective and the *empty set* is a noun". So M8's comment highlights that when read, the statement says, "none of the sets are empty set" rather than the possible intended meaning, "none of the sets are empty". M5 explained similarly why the use of the symbol $\emptyset$ was inappropriate in this statement. Both M5 and M8 said that they would make a note to the student suggesting that they avoid using mathematical symbols as an incorrect part of speech in the future.

Since these mathematicians' justification was unexpected, this discussion was not included in Pass 2 for the undergraduate students' interviews. However, when prompted in Pass 2 if the sentence "None of the sets are $\emptyset$." is unconventional because of the mixing of natural language and mathematical notation, one undergraduate participant S2 made a statement arguing that words and symbols can be used interchangeably since "the symbol for the empty set is just as rigid as saying empty". From this quote, it appears that S2 is (at least implicitly) aware of the difference between the noun and the adjective forms, however, disregards the issue. With the exception of S2, no other student mentioned the grammatical issue of using the mathematical symbol.

Since this part of speech justification for why the empty set symbol is unconventional in this proof was not anticipated, the data does not allow me to speak to what extent other mathematicians agreed with the necessity to attend to the parts of speech of mathematical symbols and notation. Further investigation is necessary to assess whether mathematicians and undergraduate students recognize using mathematical symbols and notation without careful attention to the corresponding part of speech and grammar as an unconventional use of mathematical language.

Lacks punctuation and capitalization.

Proof C (as shown in Figure 3) has a lack of punctuation and capitalized letters to indicate the ending and beginning of sentences. My expectation was that mathematicians would identify these issues as unconventional of mathematical proof writing. Mathematician M7 pointed this out during Pass 1 of the interview, saying: “the expression and the punctuation are not good” and “we can’t allow writing like that”. Although the remaining mathematicians did not indicate the lack of punctuation and capitalization as unconventional in Pass 1, they all agreed after prompting that lacking punctuation and capitalization is definitely unconventional of mathematical proof writing. Although all eight mathematicians in the study agreed that proofs should be presented in complete sentences, including appropriate punctuation and capitalization of letters, not all believed they should discuss the necessity of using punctuation and capitalization in class. For example, M4 explained:

> I look for understanding of the construction of the mathematical arguments. So I’m not sure you can require that deep understanding at the same time pushing them to be correct with punctuation and so on. […] And I consider that my task is to teach them reasoning, rather than to use punctuation.

Mathematicians M3 and M5 similarly explained that they would not address punctuation and capitalization issue in their introduction to proof classes. Only M7 indicated he would deduct points from his students’ work for missing punctuation and lack of capitalized letters. Meanwhile, M4, M6, and M8 indicated that they would mark the punctuation and capitalized letters when grading, without deducting points, to illustrate that one should use complete sentences in proofs.

None of the 15 undergraduate participants discussed the lack of punctuation or capitalization in Pass 1 through the proof. In fact, during Pass 2, 12 of the 15 undergraduate participants disagreed with the suggestion that this is an issue of mathematical proof writing. When asked why they disagreed, S2 explained “well, in my experience in my classes, some of my proofs were not full sentences with punctuation and capitalization and there was never really an issue about it.” Thus, it seems that students are not made aware of issues with their proof writing until points are deducted. So students may not learn the conventions of mathematical writing by simply observing mathematicians write proofs in class or mathematicians may not write proofs in class that follow the conventions of more formal mathematical writing. Other students were even surprised that issues of English would be important in a math class. For example, S4 exclaimed “Oh my god, this is a mathematics major, not a linguistic major right? I think it’s fine!” and S8 noted this is not an issue because “it’s not an English class.” This is not to say, however, that none of the undergraduates believed that mathematical language obeys the rules of natural language; three undergraduate participants did believe that capitalization and punctuation belonged in mathematical proof writing; for instance, S3 explained, “a proof is like a math essay of sorts and it should still be like grammatically correct”.

The discussion above suggests that the mathematicians in this study agreed that full sentences should be used when writing proofs. On the other hand, some of the responses from mathematicians and students indicated that their beliefs about proper English does not play a role in proof writing at the introduction to proof level. With only one mathematician deducting points for lacking capitalization and punctuation, it is unsurprising that students do not see or attend to the necessity of proper grammar in proof writing.

Uses non-statement.

Proof A (shown in Figure 1a) included the following phrase: “Suppose $(R \circ S)^{-1}$ s.t. $(x, z) \in (R \circ S)^{-1}$”. As an imperative phrase with a transitive verb, English grammar dictates the need for both a direct object and an object complement to be a complete sentence. That is, the sentence must suppose the direct object in relation to another object or a property about the direct object. Thus, I expected mathematicians to identify this sentence as ungrammatical. While they did not give this exact explanation, the mathematicians noted the incompleteness of the sentence.

In Pass 1, seven of eight mathematicians discussed that the proof’s first line is not a complete sentence and has no meaning. M8 explained, “the way that I would parse this sentence is, suppose $(S \circ R)^{-1}$. That’s in itself a part and again it has no verb. Suppose $(S \circ R)^{-1}$?” M5 similarly noted “Students sometimes say ‘let a set’ which doesn’t mean anything. This is just a nonsense thing to say, suppose this set.” Thus, the statement does not suppose a property of the relation, is not a complete sentence, and conveys no meaning. Moreover, seven of eight mathematicians indicated they would deduct points for a nonsensical and incomplete statement. The eighth indicated they would make a note to the student to show the student that the statement was incomplete, but would not deduct points.

Meanwhile five undergraduate participants (two advanced students and three novice students) saw an issue with the statement and attempted to resolve this issue by completing the sentence in Pass 1, but they were unable to articulate what was wrong with it in the first place. For instance, N3 explained, "I would say 'Suppose $(S \circ R)^{-1}$ is a relation such that (x, z)' is in this relation". The other four students made similar corrections to the sentence, but when asked to explain their corrections, the students explained that their revisions "would be more understandable" (N3), "it would be a lot better" (N8), or "it sounds more proper to hear" (S5), but could not offer further justification of why they made these adjustments to the proof.

When prompted in Pass 2, three more undergraduate students (two advanced students and one novice student) agreed with similar comments to N3. However, seven of the undergraduates (four advanced students and three novice students) found no problem with the incomplete sentence. In particular, N4 explained that she saw no difference between saying 'Suppose $(S \circ R)^{-1}$' and 'Suppose there is a relation $(S \circ R)^{-1}$'. This suggests that some students do not believe that the language of mathematical proof writing should obey the rules of natural language and do not see the importance of using compete sentences in mathematical proof writing.

The above indicates that mathematicians believe that the incomplete sentences are unconventional of the language of mathematical proof writing, but that some undergraduate students may not be aware that the completeness of sentences should be attended to when writing mathematical proofs. Moreover, those undergraduates who did view the incomplete sentence as unconventional could not articulate why the statement was unconventional.

In summary. Some mathematicians emphasized the need to be cognizant of the parts of speech of mathematical symbols and notation and said they would make a note to their students (without deducting points) about this type of mismatch. Moreover, evidence exists that undergraduate students may be least implicitly aware of this distinction, but dismissed the issue. While all eight mathematicians agreed that the lack of capitalization and punctuation to indicate the beginning and ending of sentences is unconventional in mathematical proofs, half of these mathematicians said they would not attend to this in their introduction/transition to proof courses. Meanwhile, twelve of the fifteen undergraduate participants did not believe this is an issue of mathematical proof writing. All eight mathematicians also agreed that the use of (grammatically) incomplete sentences was definitely unconventional of mathematical language. While half of the undergraduates believed the incomplete sentences were unconventional, they were unable to express why. Moreover, we have evidence that some students may not even see the distinction.

Theme 2: Careful attention to new mathematical objects in proofs

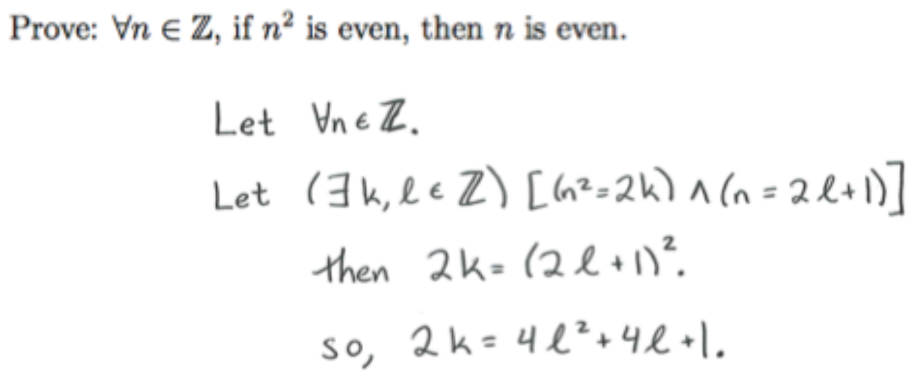
This theme highlights that mathematicians attend to new mathematical objects in mathematical proofs, while students may not fully understand the nuances involved in this process. This theme was brought forth by the mathematicians' attention to the use of unspecified variables, overquantified variables, and over using variable names.

Let A be a set. Prove: If S is a relation on A, then the relation $R = S \circ S^{-1}$ is symmetric.

let $x, z \in A$ s.t.
$(x, z) \in R$
$R = S \circ S^{-1}$
$(x, z) \in S \circ S^{-1}$

$\exists y$ s.t. $(x, y) \in S^{-1}$ and $(y, z) \in S$
$(y, z) \in S \quad \therefore \quad (z, y) \in S^{-1}$
$(x, y) \in S^{-1} \quad \therefore \quad (y, x) \in S$

Figure 4. Proof D.


Prove: $\forall n \in \mathbb{Z}$, if n^2 is even, then n is even.

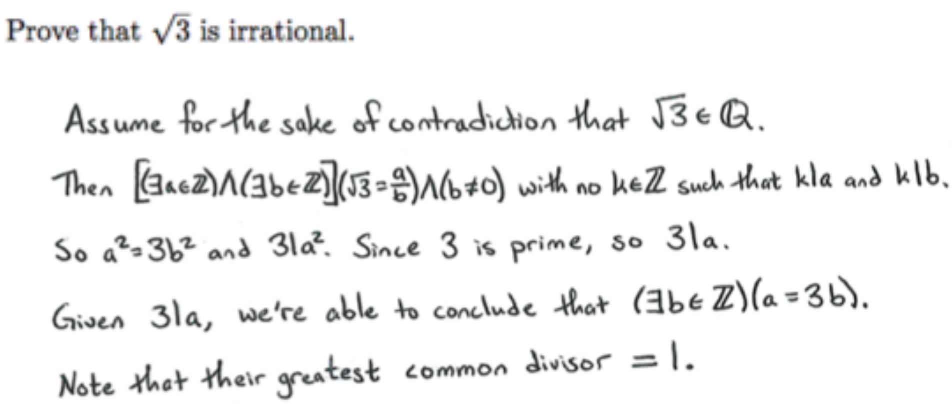
Let $\forall n \in \mathbb{Z}$.

Let $(\exists k, \ell \in \mathbb{Z})[(n^2=2k) \wedge (n = 2\ell+1)]$

then $2k = (2\ell+1)^2$.

so, $2k = 4\ell^2+4\ell+1$.

Figure 5. Proof E.


Prove that $\sqrt{3}$ is irrational.

Assume for the sake of contradiction that $\sqrt{3} \in \mathbb{Q}$.

Then $[(\exists a \in \mathbb{Z}) \wedge (\exists b \in \mathbb{Z})](\sqrt{3} = \frac{a}{b}) \wedge (b \neq 0)$ with no $k \in \mathbb{Z}$ such that $k|a$ and $k|b$.

So $a^2 = 3b^2$ and $3|a^2$. Since 3 is prime, so $3|a$.

Given $3|a$, we're able to conclude that $(\exists b \in \mathbb{Z})(a = 3b)$.

Note that their greatest common divisor $= 1$.

Figure 6. Proof F.

Uses an unspecified variable.

The necessity of variables is a common focus in mathematical proof writing. However, the last line of Proof A includes the use of the variable *y*, which is not introduced prior to its use. As such, the mathematicians were expected to attend to this unspecified variable *y*. Indeed, seven of eight mathematicians identified this in Pass 1 of the interview, with the one remaining mathematician agreeing in Pass 2. Each of the mathematicians gave similar responses; as an example, M5 explained, “This *y* showed up out of nowhere. I tell my students they have to define the variables before you start using them. So it needs to be stated—what y is in the first place." Moreover, each of the eight mathematicians indicated that they would deduct points for such an unconventional use of mathematical language.

Six of the fifteen undergraduate students did identify this unspecified variable y in Pass 1 of the interview; however, two of these undergraduates also believed this was not an issue of mathematical proof writing, explaining that they did not think it was "a big problem" (S4). In Pass 2 of the interview, seven more undergraduates agreed that use of the unspecified variable y was unconventional. The two remaining undergraduate students did not believe that this was an issue of mathematical writing as it was simply a mistake that could be easily fixed. Meanwhile, all 15 of the undergraduate participants expected that a mathematician would deduct points for using an unspecified variable.

It is clear that the undergraduate students recognize that a specifying a variable is important when writing proofs, despite not always identifying this as a serious issue of mathematical writing. Meanwhile, each of the mathematicians indicated that this is unconventional of the language of mathematical proof writing and would deduct points. These two findings suggest that mathematicians likely do deduct points for failing to specify variables and undergraduate students attend to the points deducted in their mathematical proofs. Moreover, there is a high level of agreement between both mathematicians and undergraduate students that using an unspecified variable is unconventional.

Uses an overquantified variable.

In Proof E, the first sentence is “Let $\forall n \in \mathbb{Z}$.” I expected the mathematician participants to suggest that using both the quantifier and the word ‘let’ is unconventional. Indeed, seven of eight mathematicians identified this sentence as an example of unconventional mathematical writing in Pass 1 of the interviews, with the remaining mathematician agreeing in Pass 2. Four mathematicians noted that the proof requires the writer make a choice of an arbitrary variable, for example, M4 said “there’s a confusion between the use of the quantifier and introducing a variable as an ingredient in the proof, so the student seems confused about being an arbitrary variable.” The other mathematicians gave similar explanations suggesting this use of unconventional mathematical language highlights that the proof’s author does not understand the nuances of choosing an arbitrary variable. Of these mathematicians, three indicated that they would deduct points off for this overquantified variable. Meanwhile, M5 indicated that he would not deduct points because he believed the mistake to be harmless and suggested that the rest of the proof could still be “completely correct”. The remaining four mathematicians highlighted that the statement “Let $\forall n \in \mathbb{Z}$.” is ungrammatical in the sense of the non-statement category described above; each indicating they would deduct points for the use of an incomplete sentence.

In Pass 1, five undergraduate students (one advanced and four novice) indicated that there was a problem with the first statement of this proof, “Let $\forall n \in \mathbb{Z}$.” Six more undergraduate students (four advanced, and two novice) agreed in Pass 2 that the quantification of the variable n was unconventional. Of these students, five expected that a mathematician would make a note to a student for the use of a statement such as this, explaining that they did not believe that the unusual use of quantification would affect the validity of the logical argument. On the contrary, four other undergraduate students (three advanced and one novice student) disagreed that the statement “Let $\forall n \in \mathbb{Z}$.” is unconventional in Pass 2 of the interview. Despite disagreeing that this statement was unconventional of mathematical proof writing, three of these four students indicated they would expect a mathematician to make a note to a student again because they did not believe the logical validity was affected by the statement. The one remaining disagreeing undergraduate student, S4, explained that there are different ways of writing the first line of the proof and that the statement “Let $\forall n \in \mathbb{Z}$.” was conventional. As a result, S4 did not expect that a mathematician would deduct points or make a note to the student about this statement.

While both mathematicians and undergraduate students seemed to be in agreement that the statement “Let $\forall n \in \mathbb{Z}$.” is somehow unconventional, the severity of this use of unconventional mathematical language differs in these two perspectives. Seven of eight mathematicians indicated they would deduct points for such a statement, however, ten of fifteen undergraduate students would not expect a mathematician to do so because they do not believe that the inclusion of that statement affects the quality of the proof.

Overuses variable names.

In Proof F, the variable *b* is used to represent two different values. I anticipated that the mathematicians would identify this over use of the variable name *b* as unconventional mathematical writing. The mathematicians did point out the overuse of variable names in this proof—seven of eight identified the overuse of the variable *b* in Pass 1 of the interview. M3 explained, “they used the *b* again to mean something else I think. So they should have picked a different letter for what they were trying to say." Moreover, six of these mathematicians indicated that they would deduct points for this use of mathematical language, with M8 saying, "I want to communicate to the student that this is a dangerous thing to do. And one way is to add lots of remarks, but on top of that I would take away points. That's always easier to remember." Other mathematicians similarly indicated that they would deduct points because they want the students to avoid potential confusions and errors caused by using the same variable name for different values. The remaining mathematician, M3, who identified this use of the variable name *b* as unconventional indicated that he would make a note to the student, but not deduct points as he wanted the student to realize that the variable had been duplicated, but believed there was no logical issue with the proof. The mathematician, M4, who did not believe this use of the variable name *b* was definitely unconventional of mathematical language, explained that

> Because after this [second] sentence, *b* no longer plays a role. So in a sense, the second use of *b* has been redefined in a new context and students are told they're allowed to do this. It's a little bit sloppy, but it's not very serious.

So M4 agreed that a student should not reuse the variable name, but did not view the use as unconventional and indicated that this is a personal preference. M4 did not identify this as a mathematical error, citing "it's not incorrect mathematically and it doesn't impact on the line of reasoning on any of these sentences" and as a result, indicated that this use of the variable *b* was a matter of personal preference. However, she did indicate that she would make a note to highlight to the student that the variable *b* was used twice.

In Pass 1, only two of the advanced undergraduate students (S4 and S7) identified the double use of the variable b as unconventional. While both indicated that they would expect a math professor to deduct points for this, one of the two did not view this as an issue of mathematical language. Rather, S4 explained the use of the variables b caused a logical issue. S7 did indicate that you should not "use the letter *b* because that's already defined and you're already using it", which is consistent with the mathematicians' comments. In Pass 2, six undergraduate students (three advanced and three novice students) agreed, but seven disagreed. The six agreeing undergraduate students indicated that they expected a mathematician would deduct points from this proof because the double use of the variable *b* lead to a logically invalid proof. The seven who disagreed argued that the variable *b* is not being used in a different way in the second part of the proof; for instance, N5 said "I actually don't see how it's different, they define it with the same quantifier, with the same, all the integers." Moreover, these seven students did not believe any points would be deducted or notes would be made.

For this use of the variable name *b*, it is clear that the mathematicians carefully attend to the introduction of variables and the choice of the names for those variables. Meanwhile, the students do not seem to attend to this quite as carefully. Only one of the eight undergraduates who did indicate that this use of the variable name *b* was unconventional believed that the issues was related to the language of mathematical proof writing, whereas the other seven only identified the potential logical ramifications as cause for alarm. [2]

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[2] While discussing Proof F, mathematicians M1 and M8 also suggested that students may not understand the difference between a dummy variable and a variable used in the proof. In particular M1 noted "in my mind, having said that there exists an *a* and *b* , he ought to say, 'Okay, let's take one of these *a*'s and one of these *b*'s and call them *a* and *b*'" suggesting that M1 believes introducing a quantified dummy variable is

In summary. Both undergraduate students and mathematicians broadly agreed that variables should be introduced and specified before being used in a proof. Meanwhile, the undergraduates may view this as an issue of logical validity rather than an issue related to the writing of a mathematical proof. Moreover, we see that it may be the case that mathematicians' routine deduction of points on this type of error may emphasize the importance of introducing variables to students. Meanwhile, it is also clear that the introduction to variables has multiple moving parts and there are nuances involved that mathematicians attend to, which undergraduate students may not understand. For instance the use of the statement "Let $\forall n \in \mathbb{Z}$." is unconventional and mathematicians claim to often deduct points for this use of mathematical language when writing proofs, yet undergraduate students do not necessarily expect a mathematician to do so. Similarly, mathematicians consider the importance of choosing an appropriate variable name when introducing a new variable, whereas students might not see this as necessary or recognize when variable names are repeated.

Theme 3: Formality of a proof depends on context

Even though mathematicians were asked to judge conventions of formal textbook writing, occasionally the conversation naturally shifted to the conventions of other contexts. Thus, mathematicians indicated that conventions should be specific to a context, whereas students may not be aware of this distinction. This theme emerged from the mathematicians' discussion on how certain aspects of mathematical language are acceptable at different levels of experience and expertise. In particular this theme

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not the same as introducing a variable into a proof. M8 explained that local classroom conventions can side step this formality, but that "in perfect writing", the introduction of a new variable, such as a_0 or $\tilde{a}$, would be necessary.

emerged from discussing the use of formal propositional logic, the lack of using verbal connectives in a proof, the lack of using technical or mathematical language, and the use of unclear referents.

Uses formal propositional language.

In Proof F, the second line of the proof uses a long expression using formal logical operators. Based on the formality of this statement, I predicted the mathematicians would indicate that the use of this formal statement was unconventional of mathematical language. Four mathematicians indicated in their interviews that using words to describe the formal statement would be preferable, but that using the purely symbolic phrases is appropriate for students in the specific context of an introductory proof course. Of these four mathematicians, M7 was the only one who noted this in Pass 1 of the interview. He explained that students in an introductory proof course are taught to use formal propositional logic:

> The book wrote, taught at the beginning of the book to write things this way so that the students can formalize the logic. […] And then I hope they can put that formality away and instead eventually use English, more English.

So M7 believed that purely symbolic mathematical phrases are appropriate to use in an introductory proof course, but would prefer students to use more English as they progress through their mathematical education. Similarly, M1 said, “many textbooks will introduce this logical language and it’s hard for me to take points off of something that the book exhibits”.

On the other hand, three mathematicians indicated that it is indeed conventional for all undergraduate students to use this kind of formal mathematical language in their proof writing, indicating this was because the writing was in the specific context of undergraduate mathematics. For instance, M3 said, “this is something we talk about and we use this kind of shorthand, so I would say that I disagree as far as the [introduction] sort of level goes.” So this mathematician explained that since they used the formal language in class, this type of writing is acceptable and disagreed when asked if the formal language use was unconventional. The remaining mathematician indicated that using this formal type of language is mathematical and is conventional regardless of the context. [3]

Four undergraduate students (three advanced and one novice student) identified this use of formal logical conjunctions in Pass 1 of the interview but only one, N8, believed the use was definitely unconventional of the language of mathematical proof writing. N8 explained the formal expression was unconventional because “it’s just kind of in the middle of the sentence where it’s like English words … and you know it makes the proof harder to read”. The other three undergraduates who identified the formal statement in Pass 1 indicated that this is a preference but not necessarily unconventional. For example, S7 suggested he would “prefer the word ‘and’ over the and symbol, $\wedge$”. In Pass 2 of the interviews, only one undergraduate agreed that the use of the formal statement was unconventional with the remaining ten (four advanced and six novice) students indicated that this is a conventional use of mathematical language. These undergraduates argued that the notation is mathematical and communicates the same meaning. Overall only one of the fifteen undergraduate students expected a math professor to deduct points and four expected a note to be made to the student. That is, ten

[3] M2 indicated that this use of formal logic was unconventional, but his explanation discussed only on the validity of the proof. He explained the mathematical conjunction was inappropriately used; since conjunctions should be used connect propositions.

of the students would not expect a mathematician to make a note about this use of formal logical conjunctions.

While the mathematicians agreed that the use of formal logic in written proofs is likely to be unconventional at some level of mathematical practice, the mathematicians in this study also indicated that it is conventional for undergraduate students at the introduction/transition to proof level to use formal logic in their proofs. Thus, it is not surprising then that so many undergraduate participants did not believe that using formal propositional logic in a written proof was unconventional.

Lacks verbal connectives.

In Proof E, the partial proof presents as a series of mathematical expressions. As a result, I expected mathematicians to think this is unconventional mathematical language because the logic of a proof can be difficult to follow without words to guide the reader. Seven of eight mathematicians, three in Pass 1 of the interview, indicated that this is indeed unconventional mathematical writing because proofs should include verbal connectives to indicate the flow of reasoning. In Pass 1 of the interview, M7 said

> This person doesn't write with enough connectives for me. They tend to be unconnected parts of sentences, just displayed asking me to fill in the connections, which I don't think is good proof writing. I mean I can follow the logic, but it would be much clearer if there were connectives […] but I have to do it since the student doesn't do it for me.

In this quote, M7 pointed out that a proof lacking verbal connectives does not relate the steps of a proof together and leaves this work to the reader. M6 similarly believed that the lack of verbal connectives was unconventional but highlighted how context might affect the whether or not this proof is conventional, “if you’re giving a lecture, you would write this because you can say it—but in a written paper, it wouldn’t be good.” That is, outside of the formal context of a proof found in a mathematical textbook, M6 believes that a proof written on a blackboard in lecture would not need such extensive use of verbal connectives.

On the contrary, one mathematician disagreed in Pass 2 that this is unconventional because he believed the logic of the argument was clear without additional words. Four mathematicians indicated they would deduct points for this lack of words; two reported they would make a note to the student (without deducting points); and two reported they would not make a note or deduct points. The mathematicians who indicated they would leave a note explained that despite the unconventional nature of the write up, they were still able to follow the proof. Mathematicians who would neither leave a note nor deduct points similarly believed that the proof was sufficiently clear.

The discussion of Proof E in the student interviews brought forth the only category of unconventional uses of the language of mathematical proof writing in which the two groups of students differed significantly. In particular, four advanced students noted the need for connecting words in Pass 1 and the remaining four advanced students agreed that this was unconventional of mathematical language in Pass 2 of the interviews. Seven of these advanced students expected a mathematician would deduct points because “some of these [statements] could be ‘then’, some of them could be ‘suppose’” (S6) and “without connection the proof may seem like a mess” (S4). The remaining S student expected a note to be made by a professor for similar reasons.

Meanwhile, only one novice student identified the lack of verbal connectives in Proof E in Pass 1 of the interview, while in Pass 2 only two novice students agreed that this was an issue and four novice students disagreeing that this is an issue of mathematical proof writing at all. Three of these students expected that points would be deducted because as N8 explained, “you never really read a proof by a mathematician that’s just like a long list of steps with nothing in between” and one expected a note to be made because he believed that a math professor would understand what the student was trying to say, but that the proof would be more clear if more words were included.

Nearly all of the mathematicians believed that omitting the use of verbal connectives made Proof E unclear as the reader may have difficulty following the flow of the argument, but one focused on the context to discuss the possible conventionality of the partial proof. Moreover six of the eight mathematicians indicated that they would deduct points or make a note to their students about this type of unconventional use of mathematical language. On the other hand, the results show that four of seven of the novice students did not believe that the lack of verbal connectives was an issue of mathematical writing; in fact, one student said that the lack of words “doesn’t change the argument”. On the contrary, it is clear that the advanced students appeared to recognize that the lack of words in the proof’s presentation was unconventional.

Uses informal language.

The second sentence in Proof B uses informal and imprecise language to justify that one condition of the definition of a partition holds. I expected the mathematicians to identify this informal use of language as unconventional mathematical language. In Pass

1 of the interviews, three mathematicians indicated that this use of colloquial language is not mathematical and that the proof was too informal. For instance, M5 explained:

> The way that a mathematician would write it, it wouldn't be … evens share nothing with the odds. A mathematician would say this set intersect this set is empty. It's just a very informal discussion of it as opposed to actually saying mathematical content.

This quote suggests that M5 does not believe that this informal language is conventional of mathematical writing in the context of a proof written by a professional mathematician. M5 continued further explaining that "you would never see this in a journal – for undergraduates, this is conversational but it shouldn't be written this way." M4 similarly suggested that the proof lacked the precision of mathematical language and stressed that "a very important part of an [introduction to proof course] is to learn how to write things in a way that is exact". Thus, M4 saw it as particularly necessary for a student of an introduction to proof course to be able to precisely write mathematical proofs. One mathematician, M2, argued the partial proof is not even a proof, due to its extremely informal nature. Meanwhile, three mathematicians (two in Pass 1 through the interview) indicated that the use of this colloquial language is unconventional but believed that the use was acceptable because the student's reasoning is clear.

In the undergraduate interviews, all but one of the students indicated this in Pass 1 of the interview and identified the informal language as definitely unconventional. In particular, eleven students (five advanced and six novice) indicated that they would expect a mathematician to deduct points for such informal language. These eleven undergraduate students explained that the writing was "too colloquial" (S6) and "not that mathematical" (N3). Meanwhile, the remaining three (two advanced and one novice) who identified the informal language in Pass 1 indicated that they would expect a mathematician to make a note to the student author because, as S7 said, “the logic is sound, there’s no problems there, it’s just the exposition isn’t the best.” So these undergraduates see that the second sentence of this partial proof is written informally, but do not view it as essential to the proof. The one undergraduate who did not identify this in Pass 1, S1, agreed in Pass 2 that the informal language is unconventional but did not believe a mathematician would deduct points or make a note.

Some of the mathematicians believe that informal writing is appropriate based on the context of undergraduate mathematical proof writing, whereas others believe it is definitely unconventional. Thus, it may be possible that mathematicians consider undergraduate mathematical proof writing to require different levels of formality. Meanwhile, there was a high level of agreement among the undergraduate participants who found the use of the informal language to not be mathematical.

Uses unclear referent.

In Proof F, the last sentence in the partial proof says “Note that their greatest common divisor =1.” In this sentence, the referent of the word ‘their’ is unclear. As such I expected mathematicians to identify the use of ‘their’ as unconventional of the language of proof writing. Indeed, each of the eight mathematicians identified the word ‘their’ as an issue of mathematical language. M5 indicated that pronouns are used in mathematics, but that “it is a little imprecise, it’s a little bit sloppy” and “students write sloppy things a lot”, suggesting that the use of unclear referents is too imprecise for formal mathematical writing but it is acceptable for undergraduate students. The remaining seven mathematicians all explained that using pronouns make the writer’s argument unclear; for instance, M8 said, “it’s not clear whose greatest common divisor is 1. At minimum, there are three different letters here: […] *a*, the first usage of *b*, then the second and very different usage of *b*.” Despite each discussing the inappropriate use of pronouns in Pass 1 of the interview, only four mathematicians indicated they would deduct points for the unclear use of the word ‘their’. The other mathematicians explained that they would not deduct points as they believed the antecedent to the word ‘their’ was clear or because the use of the word ‘their’ did not impact the reasoning of the proof. Two of these mathematicians would make a note to the student, but two suggested they would not because as M5 said, “as long as they know the mathematical reasoning, I let those kinds of things slide.”

Only one of the undergraduate participants identified the use of the word ‘their’ in Pass 1 through Proof F. The advanced undergraduate S8 indicated that the last line of the proof was “written in a funny way”, but had difficulty explaining why aside from indicating that the word ‘their’ was odd to use in the proof. Meanwhile, nine other undergraduate participants (five advanced and four novice) agreed in Pass 2 that this use of pronouns make the proof unclear and expected points to be deducted by a math professor. For example N4 indicated that the ‘their’ “could refer to almost anything, any one of these [variables]”. Meanwhile, five undergraduate participants (two advanced and three novice) believed the proof was written conventionally because they believed it was clear to what the ‘their’ was referring, two of which expected a math professor would make a note to the student about the use of the pronoun because the proof would be more clear without the use of the pronoun.

While only four of the mathematicians indicated that they would deduct points for this use of the pronoun ‘their’, all eight did agree that using pronouns and unclear references in written proofs is unconventional. One of the mathematicians, M5, emphasized that she “let[s] those kinds of things slide” for undergraduates, despite the sloppiness of using the word ‘their’. Thus it is possible that mathematicians consider the context of the written proof when deciding what is conventional mathematical language. Meanwhile, there was less agreement among the undergraduate students on whether or not the use of the word ‘their’ was conventional, some of which believed that the word ‘their’ could refer to anything, others who believed that it was clear that ‘their’ referred to variables *a* and *b*.

In summary. Thus, the context in which a proof is written is vital information to mathematicians when evaluating whether or not the exposition of a mathematical proof is conventional. Meanwhile, the students did not discuss the importance of context when evaluating the conventionality of the proofs. The participants in this study were asked to evaluate the proofs in this study based on the formal proof writing in published contexts, such as textbooks. It is clear that despite this instruction, the mathematicians still discussed how different situations affected their evaluation. However, it may be possible that unlike the mathematicians, the undergraduate students were following the directions and focused solely on formal writing found in mathematics textbooks. Regardless, the design of the study did not investigate whether or not students are aware of this distinction and it is currently unclear if students consider the context in which they are writing or evaluating proofs.

Discussion

In summary, this paper presented three themes that emerged from analysis of the data: First, mathematicians generally believed that the language of mathematical proof writing should obey the rules of natural language, whereas some students believe the two are independent. Second, students may not fully understand the nuances involved in the careful ways that mathematicians introduce new mathematical objects in proofs. Third, mathematicians focus on the context of a proof to decide how formal it should be whereas students may not be aware of this distinction.

As this qualitative study considers only a small sample of mathematicians and undergraduate students, the findings are simply suggestive of how mathematicians and undergraduates view the language of mathematical proof writing at the undergraduate level. However these findings indicate a disconnect between what mathematicians believe is good mathematical proof writing and what their students believe is important when writing proofs.

For example, based on the above, mathematicians in the study believed that grammar and the parts of speech of mathematical words should be attended to when writing mathematical proofs. This need for complete sentences and attention to grammar is supported by the mathematical writing guides written by mathematicians (Gillman, 1987; Higham, 1998; Houston, 2009; Krantz, 1997; Vivaldi, 2014), who indicate that correct grammar and complete sentences should be used in proof writing. Moreover, many of the undergraduate students indicated that the use of punctuation and capitalization was not conventional when writing mathematical proofs because their prior experiences in their introduction to proof courses did not indicate that these were necessary. When discussing the use of non-statements in incomplete sentences, the undergraduate students had difficulty identifying why the statement was unconventional, despite recognizing that something about the statement needed to be corrected. These behaviors of the student participants suggest that these issues related to grammar may not be explicitly discussed or attended to by their mathematics instructors in introductory proof courses.

Next, when discussing how new mathematical objects are introduced in proofs, mathematicians and undergraduate students alike broadly agreed that variables must be introduced and specified prior to the use of the objects in a proof. In particular, it appears that undergraduate students are acutely aware of the importance of specifying the universe in which a new variable exists due to their mathematics instructors' routine deduction of points on this type of error. However, the introduction of new mathematical objects can be more complicated than simply indicating that a variable n is a natural number and may require attention to multiple things at once (such as the representation object itself, the type of object it is, and the sentence structure surrounding the object). In these more complicated situations, the undergraduate students may not recognize the necessity of attending to these new mathematical objects.

Finally, this study highlighted the importance of context when discussing the conventions of mathematical proof writing. Moreover, findings from this study do not indicate that undergraduate students are aware of the importance of the specific context of a mathematical proof. If students are not aware of this need for attention to the contexts of a proof, it is possible that students would be confused by the different styles of proofs that are presented informally in their classrooms and are presented formally in their textbooks.

Implications

One implication from this study is that it may be the case that students attend most to the graded aspects of their assignments. That is, it is possible if a student can recall losing points for a particular reason, it is more likely that they will avoid repeating this occurrence. While this is far from surprising, it is important to consider when deciding what one's goals are when teaching an introductory proof course and how one might effectively convey these goals to their students.

Another implication from this study is that if students are unaware of the importance of the specific context of a mathematical proof and how the context affects the conventions of mathematical proof writing, students may not recognize that a professor's sketch of a proof in class may not be written in the same way that their assignments should be written. If this is indeed the case, it would be unsurprising for students to not have the same expectations of what the proofs they construct in their homework assignments and exams should look like.

Further, two groups of student participants were used in this study to investigate if students' understandings of the conventions of the language of mathematical proof writing resolve 'naturally' throughout their undergraduate proof-writing career. Meanwhile, the difference between the responses and discussion from the novice and more advanced undergraduate students was negligible at best, save for the discussion of one of the categories of common unconventional uses of the language of mathematical proof writing. Thus, this highlights the need for further research on the exposition of mathematical proof writing at the undergraduate level and how mathematicians and mathematics educators can help students to better understand the language of mathematical proof writing.

Directions for Future Research

The findings from this study suggest two avenues for future research. First, to investigate if other mathematicians attend to the part of speech assigned to mathematical notation and how this may affect their assessment of mathematical proofs. Second, to investigate the effect of the context of a written mathematical proof on whether undergraduate students (and mathematicians) view that proof as conventional. Both of these directions of research are currently being explored using quantitative online survey sent to both mathematicians and undergraduate students and a semester-long study considering how undergraduate students' understanding of the conventions of the language of mathematical proof writing develop while enrolled in an introduction to proof course.

STUDY 2

This paper presents the findings from a survey used to investigate how mathematicians and undergraduate students perceive the genre of mathematical proof writing at the undergraduate level. Mathematician and undergraduate student participants were asked whether various proof excerpts highlighted in four partial proofs were unconventional in each one of three different contexts: undergraduate mathematics textbooks, what instructors write in the blackboard in undergraduate mathematics courses, and how students write mathematics in these courses. There are four main findings from this survey. First, there are some potential breaches of mathematical language that participants found unconventional regardless of the context in which they occur. Second, there are some differences in how mathematicians and students perceive the contexts of blackboard proofs and student-produced proofs. Third, textbook authors are expected to adhere to stricter writing norms than mathematics instructors and undergraduate students when writing proofs. Fourth, the participants agreed there were some potential breaches of mathematical language that the literature suggests were unconventional, which were not unconventional.

Key words: Mathematical language, Proof, Mathematicians, Undergraduate students, Online survey, Contexts

Introduction

Undergraduate mathematics students have difficulties when constructing (Weber, 2001), reading (Conradie & Frith, 2000), and validating (Selden & Selden, 2003) mathematical proofs. Among several different reasons for why undergraduates struggle with constructing mathematical proofs, Moore (1994) included students' unfamiliarity with the language of mathematical proof writing. However, there is a dearth of empirical and systematic research in the field of mathematics education on the language of mathematical proof writing at the level of advanced undergraduate mathematics.

In particular, how advanced undergraduate mathematics students and mathematicians understand and use the technical language of mathematical proof writing is largely unknown to the field. In Study 1, I conducted semi-structured clinical interviews with mathematicians and two groups of undergraduate students (who were at different stages of their undergraduate mathematics programs). Employing the use of breaching experiments, participants were asked to identify and discuss potential breaches of mathematical language in student-constructed proofs. The main findings of this study showed that the mathematicians and undergraduate students did not agree on the extent to which one should attend to English grammar, the introduction of new objects in proofs, and the context in which the proof was constructed. While these qualitative interviews provided a clearer picture of how some mathematicians and undergraduate students perceived the language of mathematical proof writing at the undergraduate level, the present study investigated how a larger population of mathematicians and undergraduate students evaluated particular parts of the same proofs presented in Study 1 via an online survey.

This quantitative approach lends a different perspective on how mathematicians and undergraduate mathematics students understand this technical mathematical language and further informs researchers' and mathematics instructors' understanding of both mathematicians' and undergraduate students' expectations regarding the presentation of mathematical proofs at the undergraduate level. In turn, such understandings could enable the creation of interventions and curriculum to help undergraduate students in their transition to abstract and advanced mathematics courses.

Research Questions

One of the main findings from Study 1 was that the participating mathematicians focused on the context in which the proof was written when considering the exposition of a mathematical proof. As such, this consideration of context frames Study 2, which investigates the extent to which the findings from Study 1 generalize to a larger sample of mathematicians and undergraduate students and whether these samples agree on the linguistic conventions of mathematical proof writing at the undergraduate level. In particular, I investigate the following questions:

1. To what extent do mathematicians and undergraduate students differentiate between the contexts of textbook proofs, blackboard proofs, and student-produced proofs when evaluating the exposition of a mathematical proof?
2. To what extent do mathematicians and undergraduate students agree among themselves on what the linguistic conventions of mathematical proof writing are in each one of these three contexts?
3. In what ways do mathematicians and undergraduate students disagree in the way they evaluate and assess the exposition of mathematical proofs in these three different contexts?
4. In what ways do mathematicians and undergraduate students disagree in their expectations of how these breaches affect the grade of proofs written in homework assignments and exams?

Related Literature and Theoretical Perspective

Prior Work on the Language of Mathematical Proof Writing

There is little systematic, empirical work on the language of mathematical proof writing. Konior (1993) studied over 700 mathematical proofs written in academic textbooks and mathematical monographs investigating the construction of mathematical proofs, finding a common structure that framed the arguments of a proof by highlighting the plan of procedure and using cues to direct the reader through the proof. Burton and Morgan (2000) identified the different roles that mathematicians' demonstrations of authority, community membership, and knowledge of the field played in mathematical writing. Operating under the assumption that the professional writing guides in the literature (e.g. Gillman, 1987; Krantz, 1998) describe the norms of mathematical writing, Burton and Morgan found that these norms are sometimes broken by mathematicians, especially by those who were highly regarded in the field. While these studies do begin to further the understanding of mathematical proof writing at the professional level, research on the language of mathematical proof writing at the undergraduate level is lacking.

As stated above, a number of mathematicians (AMS, 1962; Halmos, 1970; Gillman, 1987; Krantz, 1997; Higham, 1998) have written texts describing how to properly and effectively use the language of mathematics for professional purposes such as published journal articles, dissertations, and books. Common suggestions for proof writing included: (1) making the logical structure evident in the exposition of the proof, (2) avoiding statements using too many symbols and mathematical notation, (3) being consistent with respect to both notation and word choice, (4) aiming to be concise while still using full and correct sentences, 5) using correct grammar when considering symbols as words and mathematical expressions as phrases, and 6) avoiding using a passive voice. A number of mathematicians and a mathematics educator have also written guides for undergraduate students, three of which (Vivaldi, 2014; Alcock, 2013; Houston, 2009) have sections addressing the language of mathematical proof writing. While these guides showed some overlap with the suggestions for mathematicians, the focus of suggestions for undergraduates was on more specific issues regarding mathematical grammar and clarity including: 1) using correct words to describe mathematical objects, 2) using connecting words and phrases, 3) avoiding using unclear referents, 4) defining symbols and notation used in a proof, 5) using the symbols and notation commonly used in mathematical practice, and 6) revising their mathematical writing.

Since the suggestions provided in both guides for mathematicians and students were written based on the authors' own assumptions and personal experiences, further work is necessary to investigate the extent to which these expectations of advanced mathematical proof writing are shared by mathematicians and undergraduate students. This study serves as a first step toward this goal.

Linguistic Conventions of Proof Writing in Different Contexts

Researchers in higher education (Becher, 1987), linguistics (Hyland, 2004), and composition (Bizzell, 1982; Batholomae, 1985) have highlighted that different disciplines have characteristic discourse practices. Hyland (2004) further emphasized that "students entering university must acquire the specialized literacy of their community" (p. 105). Similarly, Berkenkotter, Huckin, and Ackerman (1988) summarized the work of composition scholars Bizzell and Batholomae stating, "students entering academic disciplines must learn the genres and conventions that members of the disciplinary community employ. Without this knowledge, they contend, students remain locked outside of the community's discourse" (p. 10). These quotes highlight how important it is for students of a discipline to understand and use the genres and discursive conventions of that discipline. As such, it is important for undergraduate students studying advanced mathematics to understand the genres and conventions of mathematical discourse. To investigate these genres and conventions of mathematical discourse, it is thus necessary to discuss the relevant theories of genres and conventions and how these theories may apply in mathematical discourse.

Conventions of language

Philosopher David Lewis's (1969) influential book *Convention* presented an early articulation of what a convention of language is. In particular, Lewis described conventions of language as a manifestation of the relationship between a population and the language it speaks. Garrod and Doherty (1994) expounded on the mechanism of Lewis' formal theory of convention:

> For a behavior to become conventional in Lewis's sense it must be common knowledge in the community that all will conform to it on the grounds that they expect all others also to conform. In this way conventions represent the rational solutions to the recurrent co-ordination problems faced by the community based on, at least an implicit, recognition that all should conform to them.[4] (p. 185)

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[4] Garrod and Dougherty (1994) further emphasize that despite the description that 'all will conform', "encountering one person who violates the convention is not sufficient to undermine it; there are always the others who can be expected to conform" (p. 186). That is, while it is expected for all to conform to a convention, the reality may not come to fruition.

In this quote, Garrod and Dougherty (1994) noted the rational and potentially implicit nature of conventions of language. Meanwhile, Jackman (1998) argued that the rationality of convention is not clear in Lewis's formal theory and proposed an explicit amendment such that "a custom or practice counts as a convention if it admits the sort of rational reconstruction according to which it could be maintained." (p. 308). Thus, Jackman (1998) emphasized that conventions are a subset of customs, which are rationally justifiable. In this paper, I follow Jackman (1998) and assume that linguistic conventions have rational purposes and use findings from Study 1 to explicitly provide justifications for potential breaches of mathematical language, and investigate levels of agreement on these conventions among mathematicians and undergraduate students.

Genre Theory

In Ken Hyland's chapter on genre in the 2002 Annual Review of Applied Linguistics, he provided an overview of genre theory and described studies that investigated different aspects of linguistic genres. Adopting Kress's (1989) definition of genre, Hyland explained that there are two main assumptions of genre analysis:

> [T]hat the features of a similar group of texts depend on the social context of their creation and use, and that those features can be described in a way that relates a text to others like it and to the choices and constraints acting on text producers. (p. 114)

Thus, genres are ways of using language defined by their particular contexts and the ways the texts of that type are produced and used.

In this body of work, I subscribe to the English for Specific Purposes (ESP) approach, which considers genres as defined by both formal properties and structures of language as well as the communicative purposes of texts in particular contexts. ESP researchers have focused on genre structures and grammatical features of language with the goal of helping their students understand and follow the linguistic conventions of the various genres of a discourse (Hyon, 1996). Moreover, much of the work of ESP scholars has focused on the pedagogical applications of genre theory in tertiary and professional contexts (Hyland, 2002).

In particular within the body of ESP research, there have been studies investigating language variation across genres; for example, Bondi (1999) studied the variety of key genres used in economics, including both the professional discourse within academic research papers and discourse in textbooks and newspaper articles. A central conclusion of Bondi's study is that the producer of a text creates the audience of a dialogue using different argumentative structures when writing in differing genres. Hyland (2002) emphasizes the importance of Bondi's study as a good example "of what detailed analysis of genre variation can tell us about similarities and differences in the contexts created by discourse" (p. 118).

In the present study, I investigate these differences in the contexts created by mathematical discourse in an undergraduate introduction to proof course. This investigation was motivated by the mathematicians' focus on the context in which the proof was constructed in Study 1. The mathematicians who participated in that study discussed the importance of knowing what course the student was enrolled in, where the proof was written (in a textbook, on the board, in an assignment to be turned in for grading), and what type of assessment the proof was a part of (in a homework assignment, or a timed exam setting). As a result of these discussions in the interviews with mathematicians in Study 1, the current study examines how three different proof contexts (textbook, blackboard, and student-produced) affect mathematicians' and undergraduate students' evaluations of potential breaches of mathematical language. Further, in the case of student-produced proofs, participants were asked to differentiate between homework assignments and exams.

Methods

Online Studies in Mathematics Education

Following the methods of data collection employed by Inglis and Mejía-Ramos (2009), this study uses an online survey in order to maximize the sample size of mathematicians and undergraduate students. Using an online survey to conduct research, naturally, presents some practical difficulties including the possibility of individual participants submitting multiple responses. Meanwhile, using the steps described by Reips (2000), Gosling et al (2004) and Krantz and Dalal (2000) showed that online studies can produce results that are consistent with more traditional methods of research. Moreover, a number of mathematics education publications have used these methods of web-based research (see Inglis, Mejía-Ramos, Weber, & Alcock, 2013; Lai, Weber, & Mejía-Ramos, 2012; Mejía-Ramos &Weber, 2014).

Design of the Study

The survey website was created using Qualtrics and included seventeen pages: one page asking participants to give their informed consent to participation in the study, one page with demographic questions, one page giving participants instructions on how to complete the survey (See Appendix A), and fourteen pages asking participants to make evaluations regarding the language used in several partial proofs (for one example page, see Appendix B). By partial proofs I mean proofs based on student work that were truncated to discourage participants from focusing on the logical validity of the purported proof being evaluated, and to instead focus the evaluation on the use of mathematical language.

The tasks of the survey were designed based on the design and analysis of Study 1. I chose four of the seven partial proofs from Study 1 to include in the survey as shown in Table 1. Partial proofs were chosen to maximize the number of types of potential breaches of mathematical language included in the survey and to include proof excerpts for which mathematicians and undergraduate students disagreed in Study 1. Each of the four proofs included in the survey included three or four types of potential breaches of mathematical language.

Potential breaches of the language of mathematical proof writing

Each of the potential breaches of mathematical language included in the survey is briefly described in Table 1 below. The table includes the highlighted portions of the partial proofs for each of the potential breaches along with the exact explanations provided in the survey. The explanations are based on the mathematicians' discussion of the same potential breaches and proofs in Study 1. These potential breaches of the language of mathematical proof writing are at the core of both this study and Study 1. Based on prior work, Lew and Mejía-Ramos (2015) identified what they believed were common unconventional uses of language in mathematical proof writing at the undergraduate level. These common unconventional uses of mathematical language were identified in student-produced proofs found in 149 exams at the introduction to proof level and categorized using their personal experiences with proof writing at the undergraduate level, suggestions from the mathematical writing guides described above, and existing literature discussing the genre of mathematical proof writing (Selden & Selden, 2003),

The fourteen potential breaches of the language of mathematical proof writing considered in this study are provided in Table 1. More specifically, Table 1 shows each of the four partial proofs marked for all of the potential breaches of the language of mathematical proof writing included in the survey. Below each of the partial proofs, each proof excerpt is given with the marked potential breach and the explanation of why one might believe the potential breach is unconventional that was provided in the survey. In each page of the survey presenting a potential breach of the language of mathematical proof writing, the entire partial proof appeared marked with only one of the potential breaches (for an example, see Appendix B).

	Marked Partial Proof	
	Potential Breach and Corresponding Proof Excerpt	**Explanation for Potential Breach** *(A mathematician suggested / A classmate suggested that a mathematician would think this is unconventional mathematical writing because…)*
Partial Proof 1	Suppose $f : A \to B$, $g : B \to C$, $h : B \to C$, for sets A, B, and C. Prove: If f is onto B and $g \circ f = h \circ f$, then $g = h$. Suppose f is onto B and $g \circ f$ NTS: $g=h$ If f is onto B, $Rng(f)=B$ $\boxed{\forall y \in B,\ \exists x \in A \text{ such that } f(x)=y}$ let $(x, z) \in g \circ f$ such that $\exists y \in B$ $g(y)=z$	
	Uses non-statement Suppose f is onto B and $g \circ f$ NTS: $g=h$	… the statement "suppose g∘f" is incomplete/meaningless.
	Uses an unspecified variable let $(x, z) \in g \circ f$ such that $\exists y \in B$ $g(y)=z$	… the variable z should be introduced prior to its use in the proof.
	Includes statements of definitions $\boxed{\forall y \in B,\ \exists x \in A \text{ such that } f(x)=y}$	… complete statements of definitions should not be stated in proofs, rather they should be applied.
	Lacks punctuation and capitalization let $(x,z) \in g \circ f$ such that $\exists y \in B$ $g(y)=z$.	… proofs should be written in full sentences, which includes correct capitalization and punctuation.
Partial Proof 2	Prove that $\sqrt{3}$ is irrational. Assume for the sake of contradiction that $\sqrt{3} \in \mathbb{Q}$. Then $[(\exists a \in \mathbb{Z})(\exists b \in \mathbb{Z})](\sqrt{3} = \frac{a}{b}) \wedge (b \neq 0)$ with no $k \in \mathbb{Z}$ such that $k \mid a$ and $k \mid b$. So $a^2 = 3b^2$ and $3 \mid a^2$. Since 3 is prime, so $3 \mid a$. Given $3 \mid a$, we're able to conclude that $(\exists b \in \mathbb{Z})(a = 3b)$. Note that their greatest common divisor $= 1$.	
	Uses formal propositional language Then $[(\exists a \in \mathbb{Z})(\exists b \in \mathbb{Z})](\sqrt{3} = \frac{a}{b}) \wedge (b \neq 0)$ with no $k \in \mathbb{Z}$	… entire statements of formal propositional logic are difficult to read.
	Uses unclear referent Note that their greatest common divisor $= 1$.	… this use of pronouns is imprecise and makes what the writer is discussing unclear.
	Overuses variable names Then $[(\exists a \in \mathbb{Z})(\exists b \in \mathbb{Z})](\sqrt{3} = \frac{a}{b}) \wedge (b \neq 0)$ with no $k \in \mathbb{Z}$ such that $k \mid a$ and $k \mid b$. So $a^2 = 3b^2$ and $3 \mid a^2$. Since 3 is prime, so $3 \mid a$. Given $3 \mid a$, we're able to conclude that $(\exists b \in \mathbb{Z})(a = 3b)$.	… the variable b is used in two different ways, representing different values.
	Mixes mathematical notation and text Note that their greatest common divisor $= 1$.	… the equal symbol should not be used to connect a verbal statement and a number.

<table>
<tr><td rowspan="4">Partial Proof 3</td><td colspan="2">Prove that $\mathcal{P}$ is a partition of $\mathbb{Z}$.
None of the sets are $\emptyset$.
All are pairwise disjoint, since the positive sets share nothing with the negative sets and the evens share nothing with the odds, and $\{0\}$ shares nothing with the rest.</td></tr>
<tr><td>Fails to make the proof structure explicit
Prove that $\mathcal{P}$ is a partition of $\mathbb{Z}$.
None of the sets are $\emptyset$.</td><td>… the writer should include a statement about the structure of the proof to indicate the flow of the argument to the reader.</td></tr>
<tr><td>Uses mathematical symbols or notation as an incorrect part of speech
None of the sets are $\emptyset$.</td><td>… the symbol ø represents the phrase "empty set". So the first line reads "none of the sets are empty set", which is not a grammatically correct sentence.</td></tr>
<tr><td>Uses informal language
All are pairwise disjoint, since the positive sets share nothing with the negative sets and the evens share nothing with the odds, and $\{0\}$ shares nothing with the rest.</td><td>… the second sentence of the proof is insufficiently rigorous and formal.</td></tr>
<tr><td rowspan="4">Partial Proof 4</td><td colspan="2">Let A be a set. Prove: If S is a relation on A, then the relation $R = S \circ S^{-1}$ is symmetric.
let $x, z \in A$ s.t.
$(x,z) \in R$
$R = S \circ S^{-1}$
$(x,z) \in S \circ S^{-1}$
$\exists y$ s.t. $(x,y) \in S^{-1}$ and $(y,z) \in S$
$(y,z) \in S \therefore (z,y) \in S^{-1}$
$(x,y) \in S^{-1} \therefore (y,x) \in S$</td></tr>
<tr><td>Fails to state assumptions of hypotheses
Let A be a set. Prove: If S is a relation on A, then the relation $R = S \circ S^{-1}$ is symmetric.
let $x, z \in A$ s.t.
$(x,z) \in R$</td><td>… the writer should include a statement about what is being assumed in the proof (in this case that S is a relation on A).</td></tr>
<tr><td>Uses an unspecified variable with an existential quantifier
$\exists y$ s.t. $(x,y) \in S^{-1}$ and $(y,z) \in S$
$(y,z) \in S \therefore (z,y) \in S^{-1}$</td><td>… the introduction of the variable y should specify the set to which y belongs.</td></tr>
<tr><td>Lacks verbal connectives
let $x, z \in A$ s.t.
$(x,z) \in R$
$R = S \circ S^{-1}$
$(x,z) \in S \circ S^{-1}$
$\exists y$ s.t. $(x,y) \in S^{-1}$ and $(y,z) \in S$
$(y,z) \in S \therefore (z,y) \in S^{-1}$
$(x,y) \in S^{-1} \therefore (y,x) \in S$</td><td>… the writer should use verbal connectives (e.g. therefore, then, since) to indicate the flow of the argument to the reader.</td></tr>
</table>

Table 1. Potential breaches included in the survey by partial proof.

Survey Tasks

For each of the potential breach presented, participants were provided an explanation of why a colleague (as a mathematician) or classmate (as an undergraduate student) might believe the corresponding proof excerpt had been written in an unconventional manner. Participants were then asked if they agreed that this proof excerpt was indeed unconventional for the stated reason, and to what extent it affected the quality of the proof. These questions were asked for each of the contexts of a textbook proof, a blackboard proof, and a student-produced proof. Finally, mathematicians were asked if they would make a note to a student author or deduct points for this use of language and undergraduate students were asked if they would expect a mathematics instructor to make a note or deduct points. An example page from the survey is provided in Appendix B. Mathematician participants were presented the four proofs in a randomized order and undergraduate student participants were randomly assigned to three of the four proofs, also in a randomized order. Since, I expected that undergraduate students would take more time to read proofs than the mathematicians, student participants viewed and evaluated three of the four proofs to lower the likelihood that students would not complete the survey due to the time it took to complete. Indeed, the mean times spent on the survey (excluding time spent responding to demographics questions and reading instructions) were 14.8 minutes for the mathematician participants and 13.8 minutes for the student participants, where the mathematicians made evaluations on an additional proof.

Participants

Participants were recruited from 25 of the top mathematics departments in the United States through email solicitation through their department secretaries. Mathematicians and undergraduates were sent separate links, and the link directed those who chose to participate in the study to the survey website. In total, 128 mathematicians (75 PhD students, 16 Postdoctoral fellows, and 37 faculty members) and 135 undergraduate students (11 first-year, 29 second-year, 42 third-year, 40 fourth-year, and 13 in year five or above) participated in the survey. Of the 135 undergraduate student participants, 106 evaluated Proof 1, 102 evaluated Proof 2, 101 evaluated Proof 3, and 96 evaluated Proof 4. When asked how many proof-based courses they had taken, 8 of the undergraduates reported having taken none, 22 reported having taken one course, 16 reported having taken two courses, 21 reported having taken three courses, 19 reported having taken four proof courses, and 49 reported having taken five or more courses. In addition, 87 of the 135 students reported having taken an introduction to proof course.[5]

[5] Note that not all universities offer an introduction to proof course and those that do may not require the course, thus some of the 48 students who had not taken an introduction to proof course reported having taken other proof-based courses.

	Potential Breach of Mathematical Language	Do you agree that this is an unconventional use of mathematical language for the reason provided? *(% Agree)*					
		Mathematicians			Students		
		Textbook Context	Blackboard Context	Student Context	Textbook Context	Blackboard Context	Student Context
Partial Proof 1	Uses non-statement	100% ++	100% ++	98% ++	96% ++	89%	85%
	Uses an unspecified variable	59%	34%	37%	83%	52%	61%
	Includes statements of definitions	41%	18%	12% --	41%	24%	19%
	Lacks punctuation and capitalization	95% ++	34%	50%	82%	17%	30%
	N(mathematicians)=128, N(students)=108						
Partial Proof 2	Uses formal propositional language	88% ++	74%	66%	68%	60%	59%
	Uses unclear referent	93% ++	67%	70%	80%	41%	49%
	Overuses variable names	98% ++	95% ++	93% ++	94% ++	90% ++	86%
	Mixes mathematical notation and text	88% ++	28%	45%	79%	23%	34%
	N(mathematicians)=128, N(students)=102						
Partial Proof 3	Fails to make the proof structure explicit	70%	29%	28%	70%	46%	40%
	Uses mathematical symbols or notation as an incorrect part of speech	72%	19%	24%	58%	18%	24%
	Uses informal language	77%	45%	48%	82%	64%	58%
	N(mathematicians)=128, N(students)=101						
Partial Proof 4	Fails to state assumptions of hypotheses	64%	34%	40%	77%	51%	56%
	Uses an unspecified variable with an existential quantifier	85% ++	55%	54%	93% ++	73%	78%
	Lacks verbal connectives	97% ++	52%	72%	84%	47%	61%
	N(mathematicians)=128, N(students)=96						

++ Significantly different from and greater than 75% of the sample.
-- Significantly different from and less than 25% of the sample.
(These tests were all evaluated with a level of significance $\alpha=0.05/42$.)

Table 2. Mathematicians' and students' responses indicating if they agree that the proof excerpt was unconventional for the reason provided in each context.

	Potential Breach of Mathematical Language	**Are there differences in how mathematicians view different contexts of mathematical proof writing at the undergraduate level?** ***(% Agree in Context 1 / % Agree in Context 2)***					
		Mathematicians			**Students**		
		Textbook Context vs Blackboard Context	**Textbook Context vs Student Context**	**Blackboard Context vs Student Context**	**Textbook Context vs Blackboard Context**	**Textbook Context vs Student Context**	**Blackboard Context vs Student Context**
Partial Proof 1	**Uses non-statement**	100% / 100%	100% / 98%	100% / 98%	96% / 89%	96% / 85%	89% / 85%
	Uses an unspecified variable	59% / 34% *	59% / 37% *	34% / 37%	83% / 52% *	83% / 61% *	52% / 61%
	Includes statements of definitions	41% / 18% *	41% / 12% *	18% / 12%	41% / 24%	41% / 19% *	24% / 19%
	Lacks punctuation and capitalization *N(mathematicians)=128, N(students)=108*	95% / 34% *	95% / 50% *	34% / 50% *	82% / 17% *	82% / 30% *	17% / 30%
Partial Proof 2	**Uses formal propositional language**	88% / 74% *	88% / 66% *	74% / 66%	68% / 60%	68% / 59%	60% / 59%
	Uses unclear referent	93% / 67% *	93% / 70% *	67% / 70%	80% / 41% *	80% / 49% *	41% / 49%
	Overuses variable names	98% / 95%	98% / 93%	95% / 93%	94% / 90%	94% / 86%	90% / 86%
	Mixes mathematical notation and text *N(mathematicians)=128, N(students)=102*	88% / 28% *	88% / 45% *	28% / 45% *	79% / 23% *	79% / 34% *	23% / 34%
Partial Proof 3	**Fails to make the proof structure explicit**	70% / 29% *	70% / 28% *	29% / 28%	70% / 46% *	70% / 40% *	46% / 40%
	Uses mathematical symbols or notation as an incorrect part of speech	72% / 19% *	72% / 24% *	19% / 24%	58% / 18% *	58% / 24% *	18% / 24%
	Uses informal language *N(mathematicians)=128, N(students)=101*	77% / 45% *	77% / 48% *	45% / 48%	82% / 64% *	82% / 58% *	64% / 58%
Partial Proof 4	**Fails to state assumptions of hypotheses**	64% / 34% *	64% / 40% *	34% / 40%	77% / 51% *	77% / 56% *	51% / 56%
	Uses an unspecified variable with an existential quantifier	85% / 55% *	85% / 54% *	55% / 54%	93% / 73% *	93% / 78% *	73% / 78%
	Lacks verbal connectives *N(mathematicians)=128, N(students)=96*	97% / 52% *	97% / 72% *	52% / 72% *	84% / 47% *	84% / 61% *	47% / 61%

* The sample's responses for the two contexts were significantly different.. (Evaluated with a level of significance α=0.05/42.)

Table 3. Pairwise comparisons across contexts of the mathematicians' and students' responses indicating if they agree that the proof excerpt was unconventional for the reason provided.

	Potential Breach of Mathematical Language and Context of the Assignment	Points deducted for this unconventional use of mathematical language in a student-produced proof? *(% Points off / % Note made / % No mark)*	
		Mathematicians	**Undergraduate Students**
Partial Proof 1	**Uses non-statement**		
	Homework context **	58% / 41% / 2%	40% / 50% / 10%
	Exam context	55% / 45% / 1%	54% / 37% / 9%
	Uses an unspecified variable		
	Homework context **	20% / 27% / 54%	20% / 52% / 28% *
	Exam context **	20% / 27% / 54%	43% / 31% / 25%
	Includes statements of definitions		
	Homework context	3% / 9% / 88%	2% / 23% / 75%
	Exam context	2% / 13% / 86%	3% / 20% / 77%
	Lacks punctuation and capitalization		
	Homework context	6% / 38% / 56% *	3% / 25% / 73%
	Exam context	2% / 32% / 66%	6% / 20% / 75%
	N(mathematicians)=128, N(students)=108		
Partial Proof 2	**Uses formal propositional language**		
	Homework context	14% / 63% / 23% *	11% / 53% / 36%
	Exam context	8% / 60% / 32%	20% / 42% / 38%
	Uses unclear referent		
	Homework context **	25% / 50% / 25% *	8% / 55% / 37%
	Exam context	17% / 48% / 34%	20% / 43% / 37%
	Overuses variable names		
	Homework context	57% / 37% / 6%	49% / 45% / 6%
	Exam context	49% / 43% / 8%	56% / 37% / 8%
	Mixes mathematical notation and text		
	Homework context	3% / 39% / 58% *	2% / 44% / 54%
	Exam context	1% / 31% / 68%	8% / 34% / 58%
	N(mathematicians)=128, N(students)=102		
Partial Proof 3	**Fails to make the proof structure explicit**		
	Homework context	9% / 34% / 57%	15% / 44% / 42%
	Exam context	11% / 21% / 68%	20% / 35% / 46%
	Uses mathematical symbols or notation as an incorrect part of speech		
	Homework context	2% / 27% / 71%	2% / 32% / 66%
	Exam context	2% / 21% / 77%	3% / 32% / 65%
	Uses informal language		
	Homework context	34% / 27% / 39%	42% / 34% / 25%
	Exam context	30% / 29% / 41%	50% / 27% / 24%
	N(mathematicians)=128, N(students)=101		
Partial Proof 4	**Fails to state assumptions of hypotheses**		
	Homework context	16% / 30% / 54%	23% / 42% / 35% *
	Exam context **	14% / 25% / 61%	38% / 30% / 32%
	Uses an unspecified variable with an existential quantifier		
	Homework context **	19% / 42% / 39%	34% / 49% / 17%
	Exam context **	20% / 38% / 42%	49% / 38% / 14%
	Lacks verbal connectives		
	Homework context	27% / 48% / 24% *	19% / 46% / 35%
	Exam context	21% / 45% / 34%	25% / 36% / 39%
	N(mathematicians)=128, N(students)=96		

* The samples expected the proof to be graded differently because of the potential breach by a level of significance $\alpha=0.05/14$.
** Mathematicians and students responses were significantly different by a level of significance $\alpha=0.05/28$.

Table 4. Reponses by mathematicians and students of their expectations of how a student-constructed proof would be assessed in classroom assignments.

Analysis

The plan for analysis included investigating if the mathematicians and students answered the various aspects of the survey differently – in particular, whether they agreed or disagreed that the potential breaches were unconventional in each of the three contexts (textbook proofs, blackboard profs, and student-produced proofs), whether they agreed or disagreed on the extent to which these potential breaches affected the quality of the proof in each of the three contexts, whether they believed points would be deducted or a note would be made in a student-produced proof, and the extent to which the samples viewed these contexts differently. Moreover, I planned to consider how the different demographics of undergraduate students may have responded to the survey items differently. However, the sizes of the subgroups were not appropriate for statistical analyses; for example, testing if students in different years responded differently to the items was inappropriate with sample sizes ranging from seven to thirty-eight.

The findings from the study are summarized in Tables 2, 3, and 4. Descriptive statistics were first considered to provide a holistic view of the data sets. I then conducted a number a statistical tests. For each of the tests conducted, I used an appropriate Bonferroni correction of the level of significance to account for the problem of multiple comparisons.

Table 2 focuses on the mathematicians and students responses indicating if they agreed that the proof excerpt was unconventional for the reason provided in each context. In order to evaluate if the proportions of agreement that a potential breach was unconventional indicated a high level of agreement within the samples, I considered 75% of the sample to be the threshold. Similarly, I considered 25% of the sample to be the threshold of a high level of agreement that a potential breach was not unconventional within the samples. As such, I conducted Chi-squared tests for equality of proportions checking for proportions $p=0.25$ and 0.75 with a level of significance of $\alpha=0.05/42$. The results of these Chi-squared tests are indicated with ++ and - - as described below Table 2. When considering this binomial data throughout this paper, the proportions of agreement were categorized in the following ways: high agreement that the use is unconventional (significantly different from and greater than 75%), high agreement that the use is not unconventional (significantly different from and less than 25%), or inconclusive (not shown to have high agreement within the sample).

Table 3 provides pairwise comparisons of the responses across the three contexts. To investigate if a single sample (mathematicians or undergraduate students) evaluated the three different contexts differently when evaluating if the potential breaches were or were not unconventional of mathematical language, I conducted Cochran Q tests. Then, to understand more clearly which contexts differed, I conducted further pairwise Cochran Q tests comparing how the samples responded in the following contexts: textbook proofs versus blackboard proofs, textbook proofs versus student-produced proofs, and blackboard proofs versus student-produced proofs. These tests were also evaluated with a level of significance of $\alpha=0.05/42$ and are indicated in Table 3 with * as described below Table 3.

Table 4 presents the response by mathematicians and students of their expectations of how a student-constructed proof would be assessed in classroom assignments. I conducted Fisher exact tests with a level of significance $\alpha=0.05/14$ to investigate if the mathematicians' and students' expectations differed on how a student-produced proof might be graded based on these potential breaches of mathematical language. Significant items are indicated with * as described below Table 4. I conducted Stuart-Maxwell Tests with a level of significance $\alpha=0.05/28$ to evaluate if the single samples evaluated the contexts of proofs written in homework assignments and proofs written in an in-class exam differently. Significant items are indicated with ** as described below Table 4.

Results

In this section, I will present the findings from this study by addressing each of the research questions introduced above.

To what extent does the context affect evaluations of proof writing?

As shown in Table 3, the pairwise Cochran Q tests indicate that neither the mathematicians nor the undergraduate students responded independently of the three different contexts when responding to the survey items. Overall, these potential breaches are more commonly judged to be unconventional in the context of textbook proofs than in the classroom contexts of proofs (proofs written on the blackboard and student-produced proofs). Moreover, the tests failed to find a difference in most of the judgments between the contexts of blackboard proofs and student-produced proofs. This may suggest that mathematicians and students often do not see a difference between these two contexts. Finally, there are some potential breaches in which the context did not elicit differing judgments both by mathematicians and students. Thus, there is evidence that there are some conventions of mathematical language that cut across contexts.

The context of textbook proofs

For twelve of the fourteen breaches presented in the survey, the mathematicians' responses varied in different contexts when indicating if they agreed that a potential breach was unconventional for the reason provided. In particular, in each of these twelve potential breaches, the table illustrates that the context of textbook proofs elicited a significantly higher percentage of mathematicians agreeing that the potential breach was unconventional for the reason presented than that of both the contexts of blackboard proofs and student-produced proofs.

Similarly, the undergraduate students' responses in different contexts were different in eleven of the fourteen potential breaches with a significantly higher percentage of students agreeing that the proof excerpt was unconventional in the context of textbook proofs than in the contexts of blackboard proofs and student-produced proofs[6].

Contexts of blackboard proofs and student-produced proofs

When comparing the classroom contexts of blackboard proofs and student-produced proofs, the results of the study indicated that the mathematicians responded significantly differently in three of the fourteen potential breaches. In particular, the Cochran-Q tests indicated that the mathematicians judged the proof excerpts that lacked punctuation and capitalization, mixed mathematical notation and text, and lacked verbal connectives differently in the contexts of blackboard proofs and student-produced proofs. Moreover, for each of these types of potential breaches, a larger percent of the mathematicians found the potential breach to be unconventional for the reason provided in the context of student-produced proofs than in the context of blackboard proofs.

This highlights that it is possible that instructors of mathematics may be able write proofs on the blackboard in certain ways that they recognize should not be used in formal

[6] The proportion of students agreeing that including statements of definitions was unconventional in textbook proofs was significantly different than the proportion of students who agreed this was also a breach in student-produced proofs, but it was not significantly different from the proportion of students who thought it was unconventional in blackboard proofs.

published proofs (such as a proof presented in a textbook), but that undergraduate students should not write in these ways when producing proofs themselves for classroom assessments. Thus, it may be the case that mathematicians believe that a proof written on a blackboard in an undergraduate mathematics course is an informal sketch of a proof and that instructors of mathematics do not have the time to be careful with complete sentences and notation.

When the context of the proof may not matter

The responses from mathematicians (and students) were not significantly different across contexts for two of the potential breaches: using non-statements and overusing variable names. Furthermore, both mathematicians and students widely agreed that these potential breaches were unconventional across contexts (with agreement percentages ranging from 85% to 96% across contexts for students and from 93% to 100% for mathematicians). Participants' evaluations of these two types of potential breaches in the contexts of textbook proofs, blackboard proofs, and student-produced proofs suggest that there are cases in which the extent to which a particular use of mathematical language is seen as unconventional may not depend on context.

The undergraduate students' responses also did not differ significantly according to context for the proof excerpt including the use of formal propositional language (with agreement percentages ranging from 59% to 68%).

Summary

This analysis revealed that the mathematicians and the students treated the context differently in a number of types of potential breaches. Moreover in addition to the two potential breaches (overuses variable names and uses non-statements) for which the mathematicians did not differentiate between the three contexts; students also did not treat any of the pairs of contexts differently when evaluating the proof excerpt using formal propositional logic. These differences between students and mathematician may not be surprising based on the findings from Study 1. In particular, the mathematicians in Study 1 identified using formal propositional language as unconventional depending on the author of the proof – in fact, all but one found the use of formal propositional language to be conventional at the level of an introduction to proof course due to the inclusion of formal logic in most course syllabi.

Moreover, this analysis shows not only that the mathematicians treated these contexts differently when discussing proof excerpts exhibiting a lack of punctuation and capitalization, mixing mathematical notation and text, and lacking verbal connectives, but also that fewer mathematicians identified the same potential breaches to be unconventional in the context of blackboard proofs than in student-produced proofs. This suggests that the formality of a blackboard proof is less important to mathematicians than the formality of a student-produced proof. This is particularly interesting mathematicians most often present proofs to their students on the blackboard. As such, these results would suggest the most common manner in which mathematicians present proofs to their students is less formal than the way in which they might expect their students to produce proofs. I note that while the students similarly had lower levels of agreement for each of potential breaches, the difference was not significant in this sample.

To what extent do mathematicians and students agree among themselves in these contexts?

The results from the mathematicians' and students' survey confirmed that some of these potential breaches are indeed unconventional of mathematical language in at least one context. In particular, both samples' responses showed more internal agreement in the context of textbook proofs than the other two contexts. For more than half (eight of fourteen) of the types of potential breaches, the mathematicians' agreement percentage was significantly different from and greater than 75%. The students' results confirmed fewer of these breaches, but the students' responses did have agreement percentages significantly different from and greater than 75% for three types of potential breaches. Mathematicians' and students' results in the other two contexts suggest a lack of agreement amongst themselves in whether the potential breaches are indeed unconventional of mathematical language, with less than half of all judgments made shown to be significantly different from and above 75% or significantly different from and below 25%. Table 5 shows the percentage of mathematicians who agreed that the potential breaches were unconventional for the reason provided in each of the three contexts. Table 6 presents the level of agreement among students. Lines connect the agreement percentages for each type of potential breach according to context, and the shaded sections indicate the percentages which indicate that the mathematicians' agreement is significantly different and greater than 75% or significantly different and less than 25%. The un-shaded, center section of the graph presents the results of the participants' surveys, which did not show high levels of agreement according to the Chi-squared tests for equality of proportions.

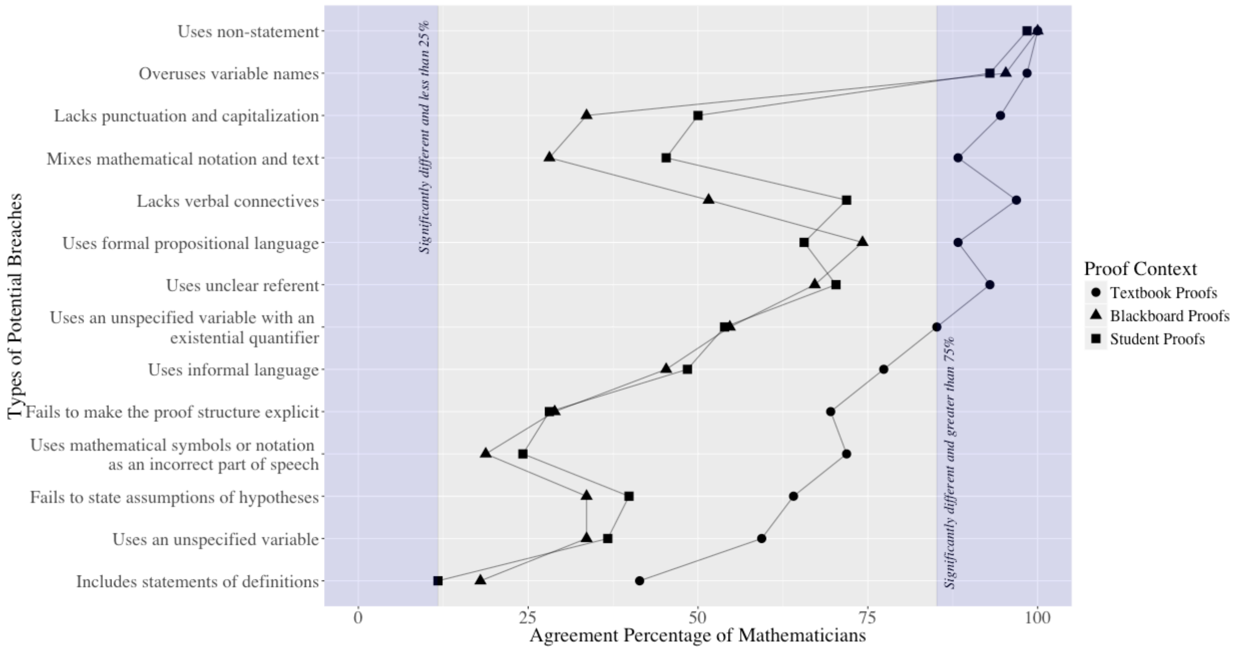

Table 5. The mathematicians' agreement percentage for each potential breach in each of the three contexts.

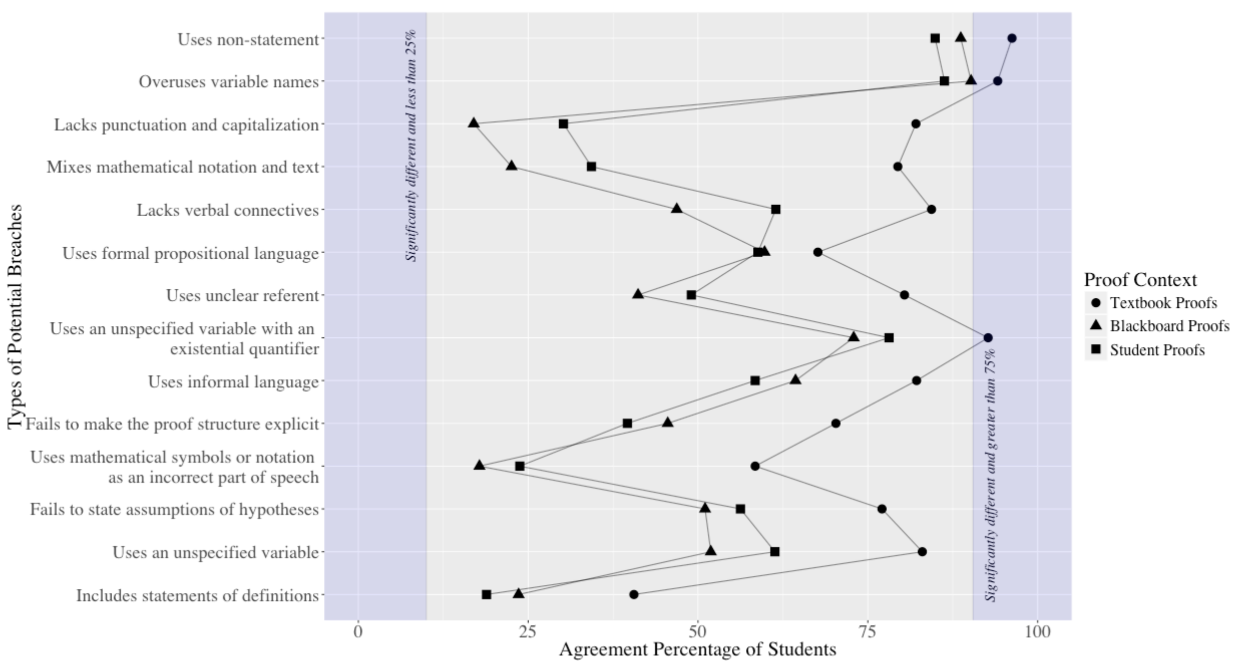

Table 6. The students' agreement percentage for each potential breach in each of the three contexts.

In this section, I will first discuss the types of potential breaches for which the samples' responses showed high agreement. Then, I will provide a post hoc analysis of the types of potential breaches for which the samples' responses did not show this high agreement.

Types of potential breaches for which the samples' responses showed high agreement

In the context of textbook proofs, the mathematicians' responses showed that they found eight of the fourteen types of potential breaches to be unconventional for the reasons presented in the survey with a high level of agreement (as shown in in Table 2). Based on Table 5, it can be seen that the percent of mathematicians who agreed that proof excerpts that used non-statements, overused variable names, lacked punctuation and capitalization, mixed mathematical notation and text, lacked verbal connectives, used formal propositional language, used unclear referents, and used an unspecified variable with an existential quantifier were unconventional of mathematical proof writing in the context of textbooks was significantly different and greater than 75%. Moreover, we see again that there is a high level of agreement among mathematicians that the proof excerpts exhibiting the use of non-statements or overuse of variable names are unconventional in each of the three contexts.

These findings indicate that these eight potential breaches of the conventions of mathematical languages are indeed unconventional in the context of textbook proofs for the reasons provided in the survey and that overusing variable names and lacking punctuation and capitalization are also unconventional in classroom contexts. Finally, Table 5 also shows that the percent of mathematicians that agreed that the inclusion of statements of definitions was unconventional is significantly different and less than

25%in the context of a student-produced proof. That is, there is a high level of agreement among the mathematicians that a proof excerpt including the statement of definitions is not unconventional in that context.

As can be seen in Table 2, the students' responses in the context of textbook proofs showed that they found four of the fourteen types of potential breaches to be unconventional for the reasons presented in the survey with a high level of agreement. Table 6 shows that the percent of students who agreed that the potential breach was unconventional was significantly different and greater than 75% for the proof excerpts that overused variable names, used non-statements, used formal propositional language, and used an unspecified variable in an existential quantifier. I note that these types of potential breaches identified by the students as unconventional is a subset of those identified by the mathematicians.

When the samples' responses did not show high agreement

For 29 of the 42 judgments made by mathematicians (fourteen potential breaches in each of three contexts), percentages of agreement among mathematicians did not cross the threshold for high level of agreement, i.e. these percentages were not significantly different from and higher that 75% or significantly different from and lower than 25%.

Table 5 shows that for five of the types of potential breaches of the conventions of the language of mathematical proof writing, the level of agreement among mathematicians was not shown to be significantly different and greater than 75% or significantly different and less than 25% in any of the three contexts. Meanwhile, when considering only the two classroom contexts (blackboard proofs and student-produced proofs), Table 5 shows that for eleven of the fourteen types of potential breaches the level of agreement was not shown to meet the threshold of high agreement. Finally, Table 5 further highlights that a number of these agreement percentages are close to 50%. In particular, eight of the 42 judgments had percentage agreements between 40% and 60%, including two judgments in the context of textbook proofs. Thus, these findings may indicate that the mathematicians may not see these types of potential breaches as unconventional, as a whole. Particularly in the classroom contexts of proofs, the results suggest that there is not a clear, shared understanding of what is expected of the exposition of proofs at the level of an undergraduate introduction to proof course.

Moreover, it is clear that a larger percent of the mathematicians agreed that a potential breach was unconventional in the context of a textbook proof than when the same potential breach was assessed in either of the other two contexts. In fact, Table 5 suggests that that the fewer mathematicians agreed a proof excerpt is unconventional mathematical proof writing in the context of textbook proofs, the fewer perceive that the same excerpt is unconventional in the classroom contexts.

Much like the mathematicians, the students' responses indicated that proofs written in the context of a textbook should be held to stricter standards of writing than those written in a classroom context. As can be seen in Table 6, the students' evaluations made in the context of textbook proofs always had the highest agreement percentage. However, the students had less agreement amongst themselves than the mathematicians when evaluating if a proof excerpt was unconventional for the reason provided in the survey. In fact, for 39 of the 42 judgments made by students, the agreement percentage did not cross the threshold for high level of agreement.

Table 6 shows that for twelve of the fourteen types of potential breaches, the students' agreement percentages were not shown to be significantly different and greater than 75% or significantly different and less than 25% in any of the three contexts. When considering only the two classroom contexts of blackboard proofs and student-produced proofs, Table 6 shows that the students' agreement percentages for all fourteen types of potential breaches were not shown to cross the threshold for high level of agreement. Finally, Table 6 also shows that a sizable number of these judgments fall between 40% and 60%. For the students, twelve of the 42 judgments made yielded an agreement percentage between 40% and 60%, with two of these judgments made in the context of a textbook proof. These findings suggest that like the mathematicians who took the survey, these students might not see these types of potential breaches as unconventional. Meanwhile, with nearly one-third of the judgments made showing particularly extreme disagreement amongst the students and with none of the agreement percentages meeting the threshold in the classroom contexts of proofs, these findings may indicate that there may not be agreement among the undergraduate students on what proofs should look like within classroom contexts. Therefore, this would in turn suggest that students may not have a clear understanding of what is expected of the proofs they produce in the classroom and what is conventional of their proof writing in these classroom contexts.

The findings discussed in this section were the result of a post-hoc analysis on data that failed to reject the null hypothesis and thus are not absolute. Meanwhile, this analysis offered an interpretation of the data, which may be used to direct future research concerning the linguistic conventions of mathematical proof writing.

Summary

What this survey does show is that mathematicians believe that the authors of textbook proofs should avoid the following breaches of conventions of mathematical proof writing: using non-statements, overusing variable names, lacking punctuation and capitalization, mixing mathematical notation and text, lacking verbal connectives, using formal propositional logic, using unclear referents, and using an unspecified variable with an existential quantifier. Moreover, within classroom contexts, mathematicians also agree both instructors and students should not write proofs that lack punctuation and capitalization, mix mathematical notation and text, and lack verbal connectives. Meanwhile, the survey also shows that undergraduate students believe that the authors of textbook proofs should avoid overusing variable names, using non-statements, using formal propositional logic, and using an unspecified variable in an existential quantifier.

However, these findings suggest a possible lack of clarity from the perspectives of both mathematicians and undergraduate students of what a student-produced proof should look like. Both samples' responses did not show many judgments with high agreement regarding types of potential breaches of unconventional mathematical language in different contexts of proofs, specifically when considering classroom proof contexts. These findings suggest the possible lack of agreement of how students in an introduction to proof course should be using mathematical language. If it is the case that mathematicians do not have a shared understanding of how mathematical language should be used in student-produced proofs, it would then be unsurprising that the sample of undergraduate students would similarly lack a shared understanding of mathematical proof writing at the undergraduate level. The lack of a shared understanding within or between mathematicians and students would indicate a need for a discussion among mathematicians and students on the expositional expectations of their students' proofs in introduction to proof courses.

In what ways do mathematicians and undergraduates disagree in their assessments?

Based on the previous section, it is evident that for both samples, the fewer participants that agreed a proof excerpt was unconventional in the context of a textbook proof, the fewer that perceived in the classroom contexts that the same proof excerpt as unconventional as can be seen in Tables 5 and 6. However, the mathematicians and students both had high levels of agreement that the potential breaches were unconventional of mathematical language in only two of the 42 judgments made: overusing variable names and using non-statements, both in the context of textbook proofs. Thus, to further investigate differences between how the mathematicians and students found the potential breaches to be unconventional to the same extent for each of the three contexts, I conducted Chi-squared tests of independence. Similarly, I conducted Fisher Exact tests to compare how the mathematicians and undergraduate students expected the quality of the exposition of the proof to be affected by the potential breaches in each of the three contexts. For both of these sets of tests, the level of significance were adjusted by a Bonferroni correction of 42 (fourteen items were evaluated in three contexts each) to give $\alpha=0.05/42$.

Neither the Chi-squared tests of independence nor the Fisher Exact tests found many differences in how the mathematicians and undergraduate students evaluated the potential breaches of mathematical language. Of the 42 evaluations of Chi-squared tests of independence, eight differences were found (presented in Appendix C) regarding the following types of potential breaches: Using a non-statement (in each of the three contexts), using an unspecified variable (in the textbook and student contexts), using formal propositional logic (in the textbook context), using unclear referent (in the blackboard context), and using an unspecified variable with an existential quantifier (in the student context). I note that three of these eight differences, the differences found in using a non-statement, while statistically significant, are not practically significant since both samples found non-statements to be highly unconventional. Of the 42 evaluations of Fisher Exact tests, five significant differences were found (presented in Appendix D): a greater percentage of mathematicians deemed that using formal propositional logic and using unclear referents in the blackboard more seriously affected the quality of the proof in that context, while a greater percentage of students thought that using an unspecified variable lowered the quality of a proof (moderately or significantly) in the textbook context and that doing so with an existential quantifier lowered the quality of the proof in both classroom contexts.

One significant difference between the mathematicians and students was how the context of blackboard proofs and student-constructed proofs differed. As discussed above, the mathematicians' responses did indicate a difference in how they judged some of the potential breaches in the two classroom contexts. Meanwhile, the students' responses did not indicate any significant differences in how they assessed the potential breaches in the blackboard proof and student-produced proof contexts. The mathematicians judged the potential breaches lacking punctuation and capitalization, mixing mathematical notation and text, and lacking verbal connectives to be more unconventional in student-produced proofs than in blackboard proofs. This suggests that, in certain respects, these mathematicians expect to hold their student's proofs to higher linguistic standards than the proofs they themselves write on the blackboard in class, but also that the undergraduate students do not recognize this difference in their professors' expectations.

Summary

These findings suggest that in some ways the students and mathematicians did respond similarly when evaluating whether the potential breaches were unconventional for the reasons given. Despite the findings in Study 1 suggesting otherwise, the students' responses often mimicked the responses of the mathematicians. However, these findings also showed that there are some important differences in how the students and mathematicians answered the survey. In particular, in three of the types of potential breaches, only the mathematicians' responses identified differences in how they viewed the contexts of blackboard proofs and student proofs.

In what ways do mathematicians and undergraduates disagree in their expectations of how classroom assessments are graded?

One of the questions asked in Study 1 concerned how these potential breaches of mathematical language affected how mathematicians assessed the proofs of their students. As a follow up in this survey, I asked participants to identify if they would (or, in the case of the undergraduate participants, would expect their instructor to) deduct points, make a note, or leave no mark on a student's proof in both the context of a homework assignment and an in-class exam. Using Fisher Exact tests with a level of significance adjusted by a Bonferroni correction of 28, because the data was unequally distributed among the possible answers, I found five potential breaches of mathematical language in which the mathematicians and students expected a student-produced proof to be graded statistically differently in the homework context, in the exam context, or both contexts as marked with ** in Table 4. To investigate if the samples distinguished between the contexts of homework assignments and in-class examinations, I conducted Stuart-Maxwell Tests using a level of significance using a Bonferroni correction of 14 as marked in Table 4 with *.

An interesting finding is for two of the potential breaches in which the mathematicians and students responses were statistically different, uses non-statements and uses unclear referent, the mathematicians responded indicating they would deduct more points on a homework assignment than the undergraduate students expected a professor would. Moreover, Table 4 shows that the mathematicians differentiated between the contexts of homework assignments and in-class examinations for using unclear referents. Thus, mathematicians both assess this potential breach differently in homework assignments and exams and differently from the undergraduate students.

Meanwhile, the remaining three potential breaches of mathematical language in which the mathematicians and students responses were statistically different (fails to state assumption of hypotheses, uses an unspecified variable, and uses an unspecified variable with an existential quantifier) are those in which students have emphasized as important when considering the language of mathematical proof writing in both in this survey and in Study 1. Table 4 further shows that the undergraduate students also evaluated two of these potential breaches of mathematical language (uses an unspecified variable and fails to state assumptions of hypotheses) differently in the two contexts of homework assignments and exam.

Summary

These findings show that mathematicians and students do both expect student-produced proofs to be assessed differently in the contexts of homework assignments and examinations with regard to certain types of potential breaches of mathematical language. However, in each of the potential breaches in which mathematicians treated the two contexts differently, mathematicians assess the homework context more harshly than the exam context. In fact, in all of the potential breaches, fewer mathematicians would "do nothing" in the homework context than the exam context. On the contrary, in the two potential breaches in which students expect the issues of mathematical language to be assessed in statistically different way, more of the students expect a mathematician to deduct points in the context of an exam than in a the context of a homework assignment.

Conclusion

Limitations of This Study

Following the results from Study 1, it is somewhat surprising that the mathematicians and students had such similar proportions of agreement for so many of the proof excerpts. There are some possible reasons for why these results did not show evidence of a significant difference between mathematicians and students. First, the populations of the participants may affect the generalizability of the findings. In particular we recruited mathematicians and undergraduate students at top programs in the United States, so it is possible that the high caliber of the undergraduate students lead to this high level agreement between the mathematicians and students. Further, the majority of the students who participated were advanced undergraduate students who had taken a number of proof courses. It would be interesting to replicate this survey with a sample of students from the same universities who were concurrently enrolled in an introduction to proof course and also with a sample of students from a more diverse selection of universities and colleges who were concurrently enrolled in an introduction to proof course.

Second, the design of the survey itself may have affected the survey results. The survey was modeled to mirror Pass 2 of Study 1, which involved identifying possible potential breaches of the language of mathematical proof writing and asking the participants if they agreed or disagreed that the potential breach was unconventional. However, in Study 1 participants made a first pass through the proofs in which they first identified any potential breaches of unconventional mathematical language, an opportunity that was not afforded in the survey reported in this paper. This missing aspect of an initial pass through the proofs may have lead the students to agree that the presented potential breaches were in fact unconventional. In future iterations of this survey, it would be advantageous to allow the participants to choose from a selection of explanations of why a presented potential breach may or may not be unconventional.

Discussion

This paper reported on the results of a survey asking mathematicians and undergraduate students to evaluate potential breaches of the language of mathematical proof writing in different contexts. There are four main findings from this survey: First, the mathematicians and students who participated in this survey agreed there are some potential breaches of mathematical language that are unconventional regardless of the context in which they occur. Second, while the mathematicians and students who participated in this survey generally responded similarly when assessing if the proof excerpts are conventional in the different contexts, there are some differences in how they perceive the contexts of blackboard proofs and student-produced proofs. Third, the mathematicians and students agreed that textbook authors are expected to adhere to stricter writing norms than mathematics instructors and undergraduate students when writing proofs. Fourth, the participants' responses indicated there are some potential breaches of mathematical language that the literature (Selden & Selden, 2003) might suggest are unconventional, which the participants agreed were not unconventional.

The findings of this report highlight some potential breaches of mathematical language that mathematicians and students agree are unconventional in the context of published proofs and gives insight on how these two populations consider the language of mathematical proof writing in the classroom context at the introduction to proof level. In particular, the results regarding the linguistic conventions of mathematical proof writing in classroom contexts suggest that for both groups of participants it is unclear what a student-produced proof is expected to look like. The mathematicians' responses did not indicate significantly high levels of agreement for twelve of the fourteen types of potential breaches in the student context, which may indicate the possibility that there is no standard universal understanding or expectation among mathematicians of how students should write proofs. Discussions amongst mathematicians, especially those who teach introduction to proof courses, concerning their expectations for the writing of mathematical proofs by their students would be a useful step towards a shared understanding of linguistic conventions of proof writing in the context of student-produced proofs.

Meanwhile, if there is not currently a consensus among mathematicians of how their students in introduction to proof courses should be writing their proofs, then a natural question is, how are the instructors of these courses presenting mathematical proof writing to their students? Based on the results from the student survey concerning the context of student-produced proofs, the students' responses did not indicate high levels of agreement for any of the fourteen types of potential breaches. That is, the proportion of agreement amongst the students for each of the types of potential breaches evaluated in the context of a student-produced proof was not shown to be significantly different from and greater than 75% for any of the fourteen categories. These results indicate that as a group, students may not have a shared understanding of what is expected of them when writing proofs.

Moreover, when investigating how mathematicians report to assess student-constructed proofs and how students expect their student-constructed proofs to be assessed, there is clear mismatch in the expectations of these two populations. For each of the types of potential breaches in which the mathematicians responded differently in the two types of assessments, fewer mathematicians indicated that they would not make a note or deduct points in a homework setting than on an in-class exam. However, more of the students expected a mathematician to deduct points in the context of an examination than on a homework assignment for each of the categories. So not only are the expectations of mathematicians and students with regards to the assessment of student-constructed proofs different with regards to the contexts of homework and exams, the expectations are quite opposite with the students emphasizing the need to follow linguistic convention on examinations and the mathematicians focusing more on the these conventions in homework settings. Based on these findings, it seems important for instructors to lead explicit classroom discussions regarding the instructor's and students' expectations for mathematical proof writing.

Further, responses from students may indicate that they do not expect differences between the way their professors write proofs on the blackboard in class and the way that they should be writing their own proofs. While this may not be unanticipated if students are expected to learn to write proofs based on the proofs they see presented in class, this discrepancy with mathematicians' expectations certainly causes some alarm. One possible avenue for future research entails considering mathematical proofs, which are written by mathematicians for their students to read outside of class time such as homework solutions, and how students perceive the differences between the exposition of the proofs they produced in their own homework and the exposition of their professor's solution.

STUDY 3

This paper presents the findings from study investigating how undergraduate students perceive the language of mathematical proof writing and if their perceptions of the language develop over the course of their introduction to proof course. Six undergraduate students enrolled in an introduction to proof course participated in a series of semi-structured clinical interviews. The paper reports four different profiles of how students might perceive and enact the language of mathematical proof writing and offers possible reasons why the students may fit these profiles.

Key words: Mathematical language, Proof, Undergraduate students, Clinical interviews

Introduction

The importance of academic disciplinary discourse in the learning of that discipline has been touted by researchers in higher education (Becher, 1987), linguistics (Hyland, 2004), and composition (Bizzell, 1982; Batholomae, 1985). Specifically, these researchers emphasize that different disciplines have particular discourse practices and to truly succeed in a field, one must "acquire the specialized literacy of their community" (Hyland 2004, p. 105). In the mathematical community, one of the main ways in which mathematical results are communicated is through mathematical proofs. However, studies in mathematics education show that undergraduate mathematics students have difficulties when constructing (Weber, 2001), validating (Selden & Selden, 2003), and reading proofs (Conradie & Frith, 2000). One suggested reason that undergraduate struggle with mathematical proofs is that students are unfamiliar with the mathematical language of proof writing (Moore, 1994). In the literature, this mathematical language has been investigated by way of the mathematical register (Halliday, 1978), which includes consideration of both the technical vocabulary and symbols and the associated syntax structures of mathematical phrases. Thus far, much of the research considering the mathematical register has focused on K-12 mathematics (for example, Pimm, 1987) and little is known of the mathematical language that mathematicians use when presenting proofs to their undergraduate students and how undergraduate students understand and use it.

The current study builds on my previous work investigating the language of mathematical proof writing at the introduction to proof level of undergraduate mathematics. First, Lew and Mejía-Ramos (2014) identified what we believed were categories of common unconventional uses of mathematical proof writing in 149 written exams from introduction to proof courses. In Study 1, mathematicians and undergraduate students were interviewed to investigate how mathematicians and undergraduate students viewed and described these potential breaches of mathematical language in undergraduate mathematical proof writing. In Study 2, a survey investigated how a larger, more diverse population of mathematicians and undergraduate mathematics students evaluated particular potential breaches of the mathematical language of proof writing in the contexts of textbook proofs, blackboard proofs, and student produced proofs.

These studies focused on understanding how mathematicians and undergraduate students evaluated potential breaches in the language mathematical proof writing and identifying differences in how mathematicians and undergraduates evaluated the potential breaches. Meanwhile, these studies offered little in the way of understanding how undergraduate students come to understand such language. Moreover in these studies, the participants made these evaluations based on the student-produced partial proofs that I provided. As a result, the undergraduate student participants' evaluations were based on incomplete proofs of claims that they may not have seen or constructed previously. Furthermore, these evaluations were based on partial proofs in order to encourage the participants to focus on the language use in the proof, as opposed to focusing on the logical validity of the argument. The task of reading an incomplete proof of a claim with which the participant may not be familiar is certainly an unnatural setting.

To address these limitations of Studies 1 and 2, the current study's design involved interviews with undergraduate students throughout the term of their introduction to proof course, using proofs that the participants themselves constructed for the class. This design allowed the participants to discuss proofs with which they were familiar and that they had constructed themselves. Thus, the students discussed their own uses of mathematical language as opposed to evaluating the potential breaches of mathematical language made by an unknown student. Moreover the proofs that the participants discussed in the current study were not truncated – that is the students considered their own complete proof attempt, as it was handed in for a homework assignment or written exam.

The current study offers a first step in the direction of understanding how undergraduate students may come to understand the mathematical language of proof writing, by investigating how six undergraduate students enrolled in an introduction to proof course perceived potential breaches of the mathematical language in their own proof writing throughout the semester. Since the students discussed their uses of mathematical language in their proof writing, developments of how the students used the language of proof writing and as well as developments of how they discussed the potential breaches were documented and discussed with the students. By interviewing the students multiple times throughout the semester, this study provides a more comprehensive picture of how undergraduate students perceive the importance of using mathematical language in their proof writing. Having this clearer understanding of how undergraduate students perceive the language mathematical proof writing and how this affects their proof writing will inform future research on how undergraduates learn to use mathematical language in their proof writing and the creation of interventions and curriculum to help undergraduate students adopt conventional writing practices of the mathematical community.

Literature Review and Theoretical Perspective

Mathematical Language

Little empirical and systematic research exists on the language of mathematical proof writing. However, a number of texts exist written for and by mathematicians that attempt to describe how one should properly and effectively use the language of mathematics. These guides on mathematical writing (AMS, 1962; Halmos, 1970; Gillman, 1987; Krantz, 1997; Higham, 1998) are for professional purposes such as journal articles, dissertations, and books. Guides written for undergraduate students have also been constructed by mathematicians and a mathematics educator (Vivaldi, 2014; Alcock, 2013; Houston, 2009). The guides, both for mathematicians and undergraduate students of mathematics, are mostly comprised of suggestions of how to best use mathematical language. The guides for mathematicians focus on higher level issues such as attending to the logical structure of the proof and being concise while providing full and complete sentences. Meanwhile the guides for undergraduate students offer more specific suggestions about mathematical grammar and clarity. While these guides were written based on the authors' personal experiences and observations, they provide an illustration of what desirable mathematical language use might look like.

Prior Work Investigating the Language of Mathematical Proof Writing

As noted above, this study builds on my prior work investigating the language of mathematical proof writing. This prior work is grounded in the English for Specific Purposes (ESP) approach to language learning (Hyland, 2002; Scarcella, 2003) which is centered on the use of the English language in specific discourses and disciplines. As such, these studies view mathematical proof writing at the undergraduate level as a specific register of English.

Preliminary categories of common unconventional uses of mathematical proof writing were identified from the analysis of 149 written exams from introduction to proof courses (Lew & Mejía-Ramos, 2015) based on their personal experiences with proof writing at the undergraduate level, existing literature discussing the genre of proof writing, and mathematical writing guides.

Next, in Study 1, I conducted semi-structured clinical interviews with mathematicians and undergraduate students. This study employed the use of breaching experiments, which involves participants identifying and discussing potential breaches of conventional mathematical language use. Originally developed by Mehan and Wood (1975), the ethnomethodological concept of breaching experiments is applied the style of Herbst and Chazan (2003). Since describing customs and norms of a practice is more difficult than identifying breaches of said customs and norms, breaching experiments are used to elicit participants' understandings of the conventions of mathematical proof writing. In Study 1, the materials included seven partial proofs (based on student-produced proofs) used to elicit conversation about the use of mathematical language in the proofs. Proofs were truncated to avoid discussion centered on the logical validity of the proof. Each participant read each of the partial proofs and identified potential breaches of the mathematical language of proof writing. Participants then discussed why they believed these potential breaches were unconventional of mathematical proof writing. Finally, I pointed out potential breaches in the proofs that the participants had not identified but that I had expected that a mathematician would believe is unconventional and asked the participants if they agreed that this potential breach was unconventional of mathematical language.

The interview data from eight mathematicians and fifteen undergraduate students was presented in Study 1. Interview transcripts were coded using an open ended thematic analysis in the style of Braun and Clarke (2006). There were three main themes found in the analysis identifying disagreement between mathematicians' and undergraduate students' understanding of how mathematical language should be used in proof writing. In particular, there was disagreement in 1) how English grammar plays a role in using mathematical language, 2) how new mathematical objects are introduced in mathematical proofs, and 3) how the context in which the proof was constructed affected how conventional different aspects of mathematical language were.

Specifically, the mathematicians who participated in Study 1 indicated that they believed that the language of mathematical proof writing obeys the rules of the English language. These rules of English language include attending to the capitalization and punctuation used in sentences, writing grammatically complete sentences, and using mathematical notation correctly within a sentence or phrase according to the type of the object being represented, or the part(s) of speech of the word(s) that notation represents. Meanwhile, some of the student participants believed that English language and mathematical language are independent.

Next, the analysis showed that while both mathematicians and students identified using an unspecified variable (ie, using the variable *n* to represent a natural number without indicating that explicitly) in a proof as unconventional of mathematical proof writing, undergraduates did not always identify or recognize potential breaches with regards to introducing new mathematical objects as unconventional when the mathematical objects became more complicated. Moreover, when the undergraduate students did identify these potential breaches as unconventional of mathematical language, they did not necessarily believe that their professors would deduct points for the breach. For example, when discussing the statement "Let $\forall n \in \mathbb{Z}$.", some of the undergraduate participants did not believe that using this statement would merit a deduction of points on a written assignment, while seven of eight mathematicians indicated that they would deduct points.

Finally, while the participants were asked to judge the partial proofs based on the conventions of formal textbook writing, the mathematicians occasionally focused on the conversation to the conventions of other contexts of mathematical proof writing. In this way, the mathematicians' responses suggest that the conventions of the mathematical language of proof writing are specific to context. Meanwhile, the undergraduate participants may not have been aware of the importance of considering the context in which a proof was written when evaluating if a potential breach is conventional.

To extend the findings from Study 1, Study 2 used an online survey to investigate how a larger sample of mathematicians and undergraduate students perceived the genre of the mathematical language of proof writing at the undergraduate level. The study included four of the seven partial proofs in Study 1 and participants indicated if they agreed that the indicated proof excerpts were indeed unconventional of the mathematical language of proof writing in each of three contexts: the context of a textbook proof, a blackboard proof, and a student-produced proof. Moreover, participants indicated to what extent they believed each potential breach of mathematical proof writing would affect the quality of the exposition of the proof. A total of 128 mathematicians and 135 undergraduate students from universities with top ranked mathematics departments around the United States completed the survey.

There were four main findings from the survey used in Study 2. First, there are potential breaches of mathematical language that participants found to be unconventional independent of the context of the proof. In particular, using non-statements (or writing grammatically incomplete phrases) and overusing variable names (by using the same variable to represent multiple values) were two potential breaches which both mathematicians and undergraduate students participating in the survey found to be unconventional independent of the three contexts included in the survey prompts.

Second, there are some potential breaches of mathematical language in which mathematicians and students perceive the contexts of blackboard proofs and student-produced proofs differently. Specifically, the survey responses indicated that the mathematicians answered the prompts differently in the contexts of blackboard proofs and student-produced proofs with regard to the lacking of punctuation and capitalization in sentences, mixing mathematical notation and text by using mathematical notation as an abbreviations for word(s) rather than a representation of a mathematical object, and lacking verbal connectives such as ‘then’, ‘therefore’, and ‘hence’ to guide the flow of the argument. Meanwhile, the undergraduate students’ responses did not indicate a significant difference in how the students answered the same questions in the two contexts of blackboard proofs and student-produced proofs for any of the potential breaches.

Third, the authors of textbook proofs are expected to adhere to stricter conventions of mathematical proof writing than both undergraduate students and their mathematics instructors. For each of the potential breaches included in the survey, the context of a textbook proof elicited the greatest percentage of the participants (both undergraduate students and mathematicians) agreeing that the potential breach was unconventional. Fourth, there were some potential breaches of mathematical language that the participants did not agree were unconventional despite the literature suggesting otherwise. For example, the results showed that neither the mathematicians nor students indicated that including entire statements of definitions within a proof was unconventional of mathematical proof writing.

Each of these studies and their findings added to the current understanding of mathematicians’ and students’ expectations of mathematical proof writing. The current study narrows the focus of the previous studies to how undergraduate students enrolled in an introduction to proof course views and uses the language of mathematical proof writing in the context of student-produced proofs. By having the students identify and discuss potential breaches of the language of mathematical proof writing in their own writing, this study will investigate both how undergraduate students understand the function of the language of mathematical proof writing at the introduction to proof level and how they actualize these understanding in their own proof writing.

Research Questions

In particular this study address the following research questions:

1. How do undergraduate students' discussions of the potential breaches of the language of mathematical proof writing in the proofs they write in an introduction to proof course change throughout the course of a semester?
2. In what ways, if any, do undergraduate students believe the potential breaches identified in their proofs are unconventional of mathematical proof writing?
3. In what ways, if any, do undergraduate students believe that the proofs they write in an introduction to proof course should conform to the conventions of mathematical proof writing?

Methods

Participants

Six participants were recruited from two sections of the introduction to proof course offered at a large research institution in the United States. Students were informed that their participation involved meeting with me for a number of individual interviews throughout the semester to discuss the proofs they wrote for their introduction to proof course. Students were modestly reimbursed for their time. Participants had not taken an introduction to proof course prior to their participation in this study. The two sections in which I recruited students had different professors and three students were recruited from each.

Materials

Throughout the course of their introduction to proof courses, participants sent me scanned or photographed copies of their graded homework assignments and written exams. Each of the assignments was read and coded for any of the potential breaches of mathematical language addressed in Studies 1 and 2 (for examples and descriptions of the potential breaches, see Study 2, Table 1). Additional types of potential breaches present in the students' proofs were also considered. For each interview with each student, I chose one to four proofs to be included in the interview materials based on the number of proofs available from each participant and to maximize the potential breaches discussed in each interview.

For each of the proofs included in the interview materials, an alternative proof (which was approved by the students' professor) was provided for comparison. The alternative professor-approved proof, henceforth called the professor proof, was included in the interview materials. The instructor of the class, the teaching assistant of the class, or I constructed the professor proof. When possible, students in the same section of the class discussed their proofs of the same theorems to minimize the number of professor proofs constructed.

The professor proof was incorporated into the design of this study to assist students in identifying potential breaches of mathematical language in their own proofs, by means of comparison. Since this study involved undergraduate students discussing the use of the language of mathematical proof writing in their own proofs, the juxtaposition of their own proof with the corresponding professor proof created conditions in which the students would identify potential breaches in their own work and allow potential breaches that the student had not identified to be pointed out without abruptly accusing the students of poor writing practices.

Procedure

The interview protocol was identical for each interview with each of the six students. Two passes were made through the materials. All interviews were videotaped and lasted between 19 and 60 minutes. In the first pass through the proofs, students were asked to identify any differences they saw between their proof and the corresponding professor proof with regard to the way the proofs were written. For each of these differences, the student explained what the differences was and was asked if they would say the potential breach was definitely unconventional of mathematical language. If they said the potential breach was unconventional, they were asked to what extent the breach was unconventional, if it significantly lowered the quality of the proof or if it was harmless. If the student indicated that the ways both proofs were written were conventional, they were asked if the context in which the proof was written affected whether or not they thought it was conventional. Students were also asked why they wrote their proof in the way they did, why they thought the professor wrote the proof differently, and if they expected a mathematician to either deduct points or make a note to them about the potential breach.

After discussing each of the differences that the student identified, participants were presented with each of the potential breaches that they had not identified and were asked if they agreed that this was unconventional of mathematical writing. If they agreed, the participant explained why they did so and were prompted to discuss the potential breach as they were in the first pass through the materials.

During the interview, if the participant mentioned a change in how they wrote mathematical proofs or if the interviewer noted a change in how the participant discussed a potential breach, the student was asked what had changed their mind about that particular aspect of mathematical writing.

Analysis

At the conclusion of data collection, I scanned all interview materials for analysis and transcribed the interview videos. The protocol used in this study made clear episodes of discussion concerning each of the potential breaches of mathematical language. These individual episodes were coded according to codes developed in Study 1. When an episode discussed a breach that did not fit any of the codes developed in Study 1, a new code was developed using open-ended thematic analysis in the style of Braun and Clarke (2006), using the same analysis protocol described and illustrated in Study 1.

The coded interview data for each student was then organized by chronological order to investigate if the students' perceptions of the different potential breaches of mathematical proof writing had developed over time. Each student was viewed as a unit of analysis. The analysis of how each student perceived and used the language of mathematical proof writing was influenced by the quasi-judicial method developed by Bromley (1986) for case study research. This method entails careful consideration of potential explanations of data collected about a case, where both evidence supporting and refuting a claim is studied to evaluate the explanations. In my analysis of each of the student's interview data, I made multiple episode-by-episode passes through the data with the goal of describing the way in which each student perceived and used the language of mathematical proof writing. After considering various possible interpretations of the evidence, I constructed a case for each of the profiles (described below) of how a student might perceive and enact this perception of the language of mathematical proof writing.

Results

Throughout the course of the study and upon analysis of the interview data, it became clear that there were few changes in how the participants discussed the potential breaches of the language of mathematical proof writing in the proofs they wrote throughout the semester. However, the analysis of the interview data revealed that the students differed in two manners.

First, the analysis highlighted a distinction in whether the undergraduate student believed that using conventional mathematical language was necessary in the writing of mathematical proofs.

That is, during their interviews, some participants identified and recognized potential breaches in their proofs as unconventional of the language of mathematical proof writing and as harmful to the expositional quality of a proof. For instance, Student 2 described unconventional proof writing as:

> [l]ike texting, kind of. Like, it's just a quicker way of writing the proof without actually writing out the whole thing. So it's unconventional, […] but it is something that's done.

Meanwhile some participants did not believe that potential breaches in their proofs were unconventional of mathematical proof writing or did not believe that the presence of potential breaches affected the expositional quality of a mathematical proof.

The analysis presented a second distinction whether the participants believed the proofs that they write for their introduction to proof course should avoid these potential breaches of the language of mathematical proof writing. This distinction was exhibited in the actual mathematical proof writing of the participant and their discussion of the potential breaches present in their proof in one of two ways: how (if at all) a participant's proof writing changed throughout the semester or if the participant persisted in making the same type of potential breaches in their proof writing throughout the semester.

The intersection of these two characteristics/classifications/attributes revealed four different profiles in which students perceive the relevance of language use on mathematical proof writing and how the proofs they wrote for their introduction to proof courses may or may not have actualized these perceptions. Below, each profile is presented by discussing the interview data from student participants in the study, whether they identify certain potential breaches as unconventional of mathematical proof writing, whether their proof writing actually reflects their beliefs on the necessity of the use of specific mathematical language use in mathematical proof writing. I also offer possible explanations of why the students fit these profiles based on their reported rationale for writing proofs in the way that they did.

Profile 1

This first profile describes a student that both believes that using conventional mathematical language was necessary in the writing of mathematical proofs and believes their proofs should avoid these potential breaches of the language of mathematical proof writing. To discuss this profile, I used the interview data from Student 2. Student 2 was a participant who identified potential breaches in her proof writing as unconventional in mathematical proof writing, believed that the quality of her proofs would be affected by these potential breaches, and whose proof writing quickly changed to reflect this belief.

Early on in the semester, Student 2 indicated that she had previously not considered the role of language in mathematical proof writing, did not think it was necessary to attend to her language use, and hence, had not considered language use when she checked over her proofs. For instance, when explaining why she wrote a proof without using correct grammar in conjunction with mathematical symbols in the first interview Student 2 said:

> I didn't really look back at my proof after I wrote it and […] *I didn't really like look really into how it was being written*. And now that I look back at it, I see that it seems, like it made sense to me so I just kind of assumed that it would make sense to someone else. […] But, I guess I wasn't, *I wasn't really looking at the formal way of writing it.* (Emphasis made by author.)

This quote highlights the fact that at the time of the first interview Student 2 had not yet focused on the linguistic aspect of mathematical proof writing when writing proofs for her introduction to proof course. In this first interview, Student 2 also indicated that using unclear referents, using proper capitalization and punctuation, and indicating the structure of an argument were all unconventional uses of mathematical language in proof writing and believed that these breaches lowered the quality of the exposition of her proofs. After this first meeting and discussion on these breaches, Student 2 did not continue to write proofs in her homework assignments that included these breaches.

As the semester progressed, Student 2 had fewer beaches (as identified both by myself and the student herself) in the proofs written in her homework assignments. When Student 2 did write proofs with breaches (using informal language, mixing mathematical notation and text, and lacking verbal connectives) in her homework assignments, she explained that she had written these proofs in these ways because she had difficulties with the logical aspects of the proofs or was unsure of how to express the concepts clearly, not because she did not find the use of language to be unnecessary. For instance, in her second interview, Student 2 agreed that her proof failed to properly assume the hypotheses of the claim, where the professor proof did so and explained that she wrote her proof without correctly assuming the hypotheses because “like this one was where [she] was kind of scrambling so I don’t think I noticed.” Moreover, in her final interview Student 2 pointed out that the professor proof used verbal connectives like “then we see”, “hence”, and “thus” whereas her proof did not. When asked why she wrote her proof in this way, she said:

> I don’t know if I like knew how to do it and the proof itself is incorrect. So I think I was trying to write the proof rather than make it proper.

So while she acknowledged the necessity of avoiding potential breaches in her proof writing, Student 2’s priority when writing this particular proof was to construct a valid argument for the proof. These findings suggest that Student 2’s perception of mathematical proof writing came to recognize the importance of proper language use during the time of the study.

Based on the interview data, I account for this student's profile based on her perception of why professors write their proofs, her intrinsic motivation, and her personal ambition in learning to write a mathematical proof in a conventional way. When discussing why she believed that her professor wrote the proof differently than she did, Student 2 gave a variety of answers. S1B posited that the professor wrote the proof in the way he did to improve clarity, to aid the reader, to be correct, and to serve as a pedagogical tool for his students. While most of these reasons were provided by many of the participants in this study, Student 2's belief that the professor's proof was written to be a guide, "to help us get used to the way that it's actually supposed to be written—so it's something we can mimic", was not shared by any other student. This quote emphasizes that she saw the professor proof as a model for proof writing for Student 2 and her classmates to follow.

Further, there are multiple indications that Student 2 is an intrinsically motivated student. When discussing what brought about the changes in her perception of mathematical proof writing, Student 2 believed the changes were due to time spent individually studying the proofs written in the textbook, questions she had personally asked the professor, and comparing her proofs to the professor proof in the interviews for this study. Moreover, at multiple times during the interviews, Student 2 mentioned having spent a good deal of time working on the proofs and writing multiple drafts of the proofs. Spending additional time studying the textbook, asking the professor questions, and writing multiple drafts of the proofs for her homework assignments all suggest that S1B was a motivated student that put forth more than a perfunctory level of effort in her introduction to proof course.

Finally, Student 2's discussion indicated a desire to have clear exposition in proofs. When discussing a professor proof that clearly specified all the variables used in the proof, she described it as "[tying] in everything, it's like a nice ribbon, it's easy to follow" – suggesting a penchant for clarity in mathematical proofs. In multiple instances, Student 2 believed a mathematics professor should have made a note to her about the breaches in her proofs even when she had not received any notes on her graded assignments. Discussing a proof in which she failed to indicate the structure of the proof, Student 2 explained that she "would want them to point it out because it seems like a silly/careless mistake [… that] takes away from the properness of the proof". Similarly when discussing a proof in which she failed to assume the hypotheses of the proof, she expected a note to be made "because that's like the formal language of how to prove and it's something that you *should* learn". Thus, she wanted to have feedback specific to the exposition of her proofs, in order to learn from her mistakes.

Profile 2

The second profile focuses on a student who believes that using conventional mathematical language is necessary when writing proofs, but also wrote proofs making the same types of potential breaches in his/her writing throughout the semester. To discuss this profile, I will use the interview data from Student 1. This student was a participant that continually breached conventions of mathematical proof writing throughout the semester, despite having identified them as unconventional a number of times.

For instance, Student 1 both described using informal language as unconventional and continued to write proofs using informal language in his first, second, fourth, and fifth interviews. In his first interview, Student 1 explained that the corresponding professor proof,

> [S]eems to be a *real proof* in comparison to how I just state adding numbers will result in a certain type of number – *whereas the actual proof* has implementation to show a complete generality, whereas I'm far too ambiguous because once again I just used words.

So Student 1 described the professor proof as 'a real proof' and 'the actual proof' in juxtaposition with his own proof and highlighted that his proof is unconventional for being written in 'just words'. He also failed to use proper capitalization and punctuation, and used long expressions of formal mathematical language throughout the semester, despite indicating that these were all unconventional to include in mathematical proofs. Overall, Student 1 identified many breaches of conventions of mathematical proof writing over his six interviews, however, he found most of these breaches to be harmless and believed that a mathematics professor would not make a note or deduct points from a student's homework assignment for using these breaches. Student 1 only indicated that breaches might be harmful to the quality of the exposition of the proof when the breach led to issues in the proof's logical validity.

Moreover, when discussing why his professor wrote the proofs in the ways that they had (by attending to issues of language to a greater extent than he had), Student 1 indicated that the professor proof was clearer, more formal, reduced ambiguity, and as noted above was an 'actual proof' as opposed to the proofs he wrote, suggesting this student did recognize the necessity of using proper language in proof writing and believed proofs should be written this way.

If Student 1 believed that conventional language use was necessary should be used in proper mathematical proof writing, then it is interesting that he did not aim to use this conventional language when writing proofs in his homework assignments and exams in his introduction to proof course. Based on the interview data, I suggest two reasons why a student in this profile may not feel compelled to write proofs following conventions of language, despite reporting that proofs should be written with correct language use.

First, it is natural for students to rely on less formal modes of proof writing when they are not sure how articulate their arguments in mathematical language. In his first interview, Student 1 indicated that he wrote his proof with informal words rather than using mathematical notation because he “just couldn’t formulate it like [the professor proof]”. Similarly when discussing his use of informal language in the subsequent interview, Student 1 further explained that “if [he] can’t think of a way to be formal, then [he]’ll just continue to be informal anyway.”

Moreover, if there are no indications to the student that these potential breaches of mathematical language are undesirable, it is unlikely that the student will change their proof writing behavior. The person who graded Student 1’s proofs did not make a note about or deduct points for Student 1’s use of informal language at any point throughout the semester. Moreover, Student 1’s expectation for how a professor’s grade of his proof would be affected by the use of informal language became progressively less strict. In the first two interviews, Student 1 indicated that he expected a mathematics professor to deduct points from his proof due to his use of informal language; in the fourth interview, he expected a mathematics professor to make a note on his proof, but not deduct points; and expected neither a note would be made nor points be deducted in his fifth interview.

Second, students may not believe that the purpose of proofs written in homework assignments is to exhibit correct language use; rather they may view homework assignments as a vehicle to demonstrate that they understand the mathematical concepts. For instance, Student 1 explained that when writing homework he is often “lazy” and believed that proofs written in homework assignments were less formal than proofs in other contexts, such as exams, textbooks, or journal articles. When asked what he believed the purpose of writing homework was, he responded:

> To show you understand how to do said proofs and to show that understanding to somebody who already has that understanding. It doesn’t require the small little things. […] When you include the small things like the words and stuff it helps to bridge gaps, whereas for the purpose of class, it’s not necessary *because you’re giving it to a person who knows what they’re doing already. The understanding is there without the small things.*

In this quote, Student 1 explains that including the ‘small things’, meaning using words and proper language, is unnecessary in the context of proofs written for homework assignments. In particular, he identified the particular audience of homework assignments as part of the reason of why he believes that this is the case. Since the expected readers of classroom assignments are mathematics instructors and teaching assistants, these readers are familiar with the content at hand and understand the mathematical notation used in the proof. Moreover, since Student 1 believes the propose of writing proofs in homework assignments is to show mathematical understanding, as opposed to exhibiting expositional prowess, students such as Student 1 may find it superfluous to write the proofs in their homework using proper mathematical language.

Profile 3

The third profile focuses on a student who does not believe that using conventional mathematical language is necessary in the writing of mathematical proof writing, yet elected to write proofs that avoided potential breaches of the language of mathematical proof writing. The interview data from Student 4 will be used to illustrate the existence of this profile. Student 4 explained that he did not believe that it was necessary to avoid some of the potential breaches of mathematical language, such as using an unclear reference, failing to indicate the structure of a proof, and using informal language, yet also explained multiple times that he aimed to improve the expositional quality of his proofs.

For example in his first interview, Student 4 discussed of a proof in which he used more words than mathematical notation and explained that he believed it is conventional to write mathematical proofs both by “writing it all out and using, you know, all the mathematical symbols.” Moreover, he noted that he “switch[es] back and forth, but [he does] have a preference for writing it all out”. So Student 4 indicated that there may not be set conventions in mathematical proof writing with respect to using mathematical notation or text, but had a preference for writing out his proof in sentences, rather than in lines of mathematical symbols.

Moreover, in his fourth interview, Student 4 pointed out that the professor did not indicate the structure of a proof by separating the different conditions of proving the claimed statement. Student 4 believed that both his proof and the professor proof were written conventionally and when asked why he wrote the proof in the way he did, noted that he wanted to "make it easier for [him] to keep track of it" and "make it easier for the reader to keep track of it". Moreover, when asked why the professor would have written a proof without making this proof structure clear, Student 4 explained, "that's just his personal preference". Thus, Student 4 believed that a reason to use proper English language in mathematical proof writing is a matter of choice.

Since this student does not seem to believe that using proper language and clear exposition is necessary of mathematical proof writing, it is interesting that he does then aim to include full sentences and additional explanations in his own mathematical proofs. Based on the interview data, I suggest that in addition to this belief that preference dictates the use of proper mathematical language, Student 4's behavior may be related to the student's intrinsic motivation.

Throughout the semester, Student 4 included explicit descriptions and indications of the structure of his arguments even going so far as to identify when the professor's proof lacked this discussion. For instance when discussing a paragraph providing an additional explanation of his proof in his fifth interview, Student 4 explained

> I always like to end my proof with what I'm trying to say. So for example here, I'm trying to show two properties. […] I do the paragraph, a little paragraph, I always have a sentence saying this, this is that. […] Meanwhile the professor did not, he just wrote the proof style.

This quote shows that Student 4 preferred to include explanatory sentences making his argument clear to the reader and noted that the professor did not include such an explanation. Thus, this student chose to attend to certain aspects of the language of mathematical proof writing because he desired to write clear and expository proofs, despite not believing that it was necessary of mathematical proof writing.

When discussing a proof from his first homework assignment, Student 4 discussed a change in his preference in mathematical proof writing that had occurred since the due date of the assignment. In this proof, Student 4 had not indicated the structure of the proof but explained that he had "forgot[ten] to mention it was contraposition or sometimes a contradiction or something like that and [he] like[d] to do that in the beginning". When asked what caused this change, he explained that he had looked over his graded assignment and said, "even myself I would be like, wait what did I prove this? What was I using when I was proving this? […] So it even confused me, the writer." Thus, Student 4 took the time to review his graded assignment, was unable to follow the argument of his own proof, and thereafter preferred to include statements that explicitly indicated the structure of his argument. In addition to this self-reflection on the clarity of his arguments, Student 4 also attributed other developments in his perspective of mathematical proof writing to asking the professor questions about writing proofs and his participation in the study—further suggesting that the student's intrinsic motivation may have been a contributing factor to his proof writing behavior.

Profile 4

This final profile focuses on students who do not believe that using conventional mathematical language is necessary in the writing of mathematical proofs and persisted in writing proofs with these potential breaches throughout the semester. To discuss this profile, I will use interview data from two students, Student 5 and Student 3. Both students found the use of proper language to be unnecessary of mathematical proof writing and the proofs written by these students certainly embodied this perception of proof writing. [7]

Student 5 explained that he found writing to be annoying and used notation as a short cut. In particular, he explained in his fifth interview that he did not use full sentences in his proofs in order "to save space, also time." When discussing why he prefers the use of notation to the use of words, Student 5 pointed out that "[he could] write out [his] thoughts without having to do it in complete sentences". Moreover, when explaining in his first interview why he felt that proofs were conventional with or without the use of proper capitalization and punctuation, Student 5 stated "all the math is there". Thus, this student did not view the use of complete sentences as a necessary aspect of mathematical proof writing, so long as the necessary mathematical content and argument were present in the proof. Similarly Student 3 indicated throughout the semester that he found mathematical writing conventional whether it used informal language or formal language, included verbal connectives, or indicated the structure of the proof.

When asked why they believe their professors wrote the proofs differently with regard to the use of mathematical language, like Student 4 (the student described in Profile 3), these two students noted that their professors preferred to write their proofs this way. Student 3 explained in his fifth interview that the professor "wants to be more formal because that's his personal preference" and Student 5 said in his second interview "[he] guess[ed] [the professor] felt like writing it out" when explaining why their professor used mathematical notation while they wrote those proofs informally. So these

[7] I note that as these students did not believe that using conventional mathematical language was necessary, they have some similarities with Student 1 (the student described in Profile 1). Similarly, as these students persisted in writing proofs with potential breaches throughout the semester, they have some similarities with Student 4 (the student described in Profile 3). The interesting part, then, is how Students 3 and 5 differ from these other students.

two students view the use of complete sentences and proper language to be a matter of choice in mathematical proof writing.

It is worth noting however, that students that fit this profile may have this perception of mathematical proof writing according to the context in which the proof appears. For example, while Student 3 found mathematical writing conventional without the use of verbal connectives and formal language, he did indicate that some issues of language are important in professional contexts such as textbooks and journal articles, although he does not believe use of proper capitalization and punctuation is necessary in undergraduate contexts.

Since these two participants indicated that they did not believe it necessary of mathematical proof writing to use proper language, such as using full sentences, capitalization, punctuation, or verbal connectives, it is unsurprising that Student 5 and Student 3 did not do so in their own proofs. It is interesting however, to consider why they have these perspectives of mathematical proof writing. Based on the interview data, I suggest two reasons why a student with this profile may have this belief and exhibit this kind of behavior when writing proofs.

First, students in this profile may believe that the main focus of proof writing is the logical validity of a proof and that the expositional form of the proof is not of importance based on their mathematics professor's actions in class. For instance, in his third interview Student 5 reported that that he did not attend to the use of proper English grammar in his proof writing based on his professor's blackboard proofs:

> I think that's how the professor write it and I looked at it in my notes and kind of looked at the formatting and did a bunch of practice problems with that formatting.

He similarly reported in his fifth interview that that he did not use sentences in his proofs because “a lot of times [his] professor would write proofs not in complete sentences”. Thus, this student’s proof writing is modeled after the proof writing that his professor exhibited on the blackboard in class. Since a professor’s blackboard proofs are one of the main sources of a students’ examples of written mathematical proof, it is natural for students to adopt their professor’s in class proof writing habits—especially in an introduction to proof course where most students do not have prior experiences with mathematical proofs. Thus, if a student’s interpretation of the professor’s blackboard proofs does not convey the importance of the exposition of a mathematical proof, it is be unsurprising that a student would not believe that using complete sentences and proper language is necessary of mathematical proof writing.

Similarly, Student 3 explained in his third interview that he did not believe that using complete sentences was necessary when writing proofs because “it’s like not stressed in class” so he “just focus[es] on getting it out, not like the sentences and the form of it.” That is, based on the actions of the professor and the lack of focus on the use of mathematical language within the classroom, Student 3 did not believe that using proper language is an important aspect of the course. As such, when writing proofs, Student 3’s focus was on “getting the proof out there”, or the logical validity of the proof.

Second, there is evidence that both of these students are extrinsically motivated. When discussing whether capitalization and punctuation were necessary in proof writing in the first interview, Student 5 claimed that he would use full sentences in an exam setting. When asked why, his response was “I don’t want to get points off.” Thus, Student 5 claimed to be motivated to use conventional mathematical proof writing based on avoiding a deduction of points. Meanwhile, in his third interview while discussing his lack of an explicit indication of the structure of the proof, Student 3 explained that the focus of the grader was not the same as the focus of the discussions in this study:

> It seems like what you're asking for is like, the writing of it and what [the grader] is asking for is like the correctness of it. So if you look at her notes […], she'll just like correct me when I'm wrong. […] So again we haven't really been stressed on the writing of it.

Thus, Student 3 suggested that since the person who assigned his grades did not make notes or deduct points based on the 'writing' aspect of the proof, he is unconcerned with the exposition of his proof; rather Student 3 is concerned with the logical validity of his proofs. Furthermore, in the same interview when asked about how his lack of complete sentences might affect the grading of his proof, he responded "I'm not used to it, so I would be very upset if they did." In this quote, Student 3 indicated that he would be frustrated with a deduction of points because his experience did not inform him that he should use complete sentences to avoid losing points. Hence, both Students 3 and 5 indicated that their proof writing is motivated by avoiding deduction of points.

Theoretical Implications

Due to the small sample size of this study, I make no claims of sample-to-population generalizations. Meanwhile, the nature of this study does allow me to interpret how these different profiles may be related. Based on the above discussion, we see that Profile 4 are is characterized by students who gave their professor's actions in their introduction to proof courses as reasons of why they wrote their proofs in the way that they did and why they believed that the use of complete sentences and proper language was not necessary when writing mathematical proofs. More specifically, since Student 5's professor wrote proofs on the blackboard without complete sentences and careful attention to the language and exposition of the proof and because Student 3's professor did not emphasize the importance of language in mathematical proof writing, these students did not value the expositional aspect of proof writing.

On the other hand, we also see that the interview data suggested that the students in Profiles 1 and 3 were both intrinsically motivated to spend individual time considering mathematical proof writing and how the exposition and use of mathematical language affected their proof. These students cited writing multiple drafts of proofs, reviewing their graded homework assignments, and asking the professor questions about language use in proof writing. These students attended to the language use in their proofs significantly more so than the students in Profiles 2 and 4.

This being said, it is not surprising that students who may be intrinsically motivated will take extra steps to write their proofs in a more clear way attending to their language use than their classmates who believe careful language use is unnecessary. What is somewhat unexpected is the existence of Profiles 2 and 3. If a student believes that language use is necessary in mathematical proof writing, one might expect that student to write his/her proofs according to the conventions of the language of mathematical proof writing. Similarly, if a student does not think that mathematical proof writing necessarily includes careful attention to the language of the proof, one might not expect that student to attend carefully to the clear exposition of his proof. Meanwhile, the findings from this study shows that these students do exist. One common feature of these students with Profiles 2 and 3 are that they each believed that the careful use of mathematical language in proof writing was a matter of the personal preference of the author.

Another point of note is that in each pair of profiles discussed in this section, here are students from both sections of the instruction to proof courses from which I recruited participants for this study. That is, for each of these characteristics (students who were intrinsically motivated, students whose professors did not convey the importance of a proof's exposition, and students who believed that language use in a proof was a matter of personal choice), there are students of both professors. Thus, the differences in these students' perceptions of the language of mathematical proof writing may be based more on their interpretation of what their professor deems important about mathematical proof writing, rather than a direct outcome of how a professor writes proofs on the blackboard.

Discussion

This paper presented four different profiles in which students do or do not believe that conventional mathematical language is necessary in proof writing and do or do not write their proofs in compliance with conventional mathematical language. The analysis of the interview data highlighted the existence of both students who found the conventional use of mathematical language to be necessary in proof writing and students who believed this conventional use of mathematical language use was not necessary in mathematical proof writing. Moreover, regardless of whether the students believed that language use was or was not a necessary component of mathematical proof writing, some students wrote the proofs for their introduction to proof course by using conventional mathematical language and some did not. The results section proposed reasons of why these students fit these particular profiles.

Some of the findings from this study also relate to findings in previous work regarding the mismatch of mathematicians' and students' expectations of mathematical proof writing at the undergraduate level. The results of Study 2 presented evidence that when evaluating some of the linguistic conventions of mathematical proof writing, the mathematicians distinguished between proofs written on the blackboard by instructors and the proofs written by students. In particular, the mathematicians found the potential breaches to be more serious in student-produced proofs than in blackboard proofs. Meanwhile, the students' responses in Study 2 did not indicate that the students recognize this distinction. In this study, Students 3 and 5 indicated that the way their professors wrote proofs in class affected their perceptions of mathematical proof writing. In particular, Student 5 indicated that his proof writing was based on mimicking the proofs written by their professor. These findings highlight that students do not perceive that the proofs they write should be held to higher standards than the proofs their professors write in class. Moreover, this finding emphasizes the need for explicit discussions between mathematicians and their students on their expectations of mathematical proof writing.

Next, Study 2 also indicated a mismatch between how mathematicians and students viewed the necessity of formality in the contexts of proofs written for homework assignments and exams. In particular, the students' responses in Study 2 indicated that the students expect proofs to be graded more strictly in exam contexts than in homework contexts, while the mathematicians' responses showed that mathematicians had the opposing expectation. Some of the participants in this study provide further support of this mismatch. As discussed in the results section, Student 1 explained he expected that proofs written in homework assignments would be graded more leniently than proofs written in exams because the purpose of homework assignments are to show understanding and that careful language use is not necessary.[8]

Directions for Future Research

The findings from this study provides insight to how undergraduate students perceive the necessity of complying with the conventions of the language of mathematical proof writing and why they may choose to actualize this perception when writing proofs. This study is an important step towards understanding how undergraduate students might learn to use the language of mathematical proof writing and how instructors of mathematics might aid their students' proof writing.

In particular, the implications from this study suggest that undergraduate students' proof writing behavior is not solely dependent upon how their professors write proofs in class. Moreover, the students' proof writing is affected by their beliefs about what their professors value with regards to their proof writing. Thus, it follows that mathematics professors should explicitly discuss the importance of the exposition of mathematical proofs with their students and that future research should investigate effective ways to convey the importance of the exposition of mathematical proofs.

[8] Although not discussed in the results section, Students 4 and 5 also noted this expectation of proofs written for homework assignments and proofs written in exams.

CONCLUSION

Summary

This dissertation described three subsequent studies in which I examined the language of mathematical proof writing at the undergraduate level. At the outset of this endeavor, I proposed to investigate how mathematicians and students view and describe the linguistic norms of mathematical proof writing at the undergraduate level and how breached norms affected how mathematicians evaluated student-constructed proofs by interviewing eight mathematicians and sixteen undergraduate students (eight at the beginning of their undergraduate proof writing career and eight at the end). Second, I proposed to create an online survey to investigate to what extent mathematicians and students agreed on the linguistic norms of mathematical proof writing both within groups and between groups. Finally, I proposed to conduct a semester-long study in which I interviewed six undergraduate students, in up to eight meetings each, to reveal how students understood the linguistic norms of mathematical proof writing and how the students' understandings of these norms changed throughout the semester of an introduction to proof course.

In Study 1, I interviewed eight mathematicians and sixteen undergraduate students to better understand how these individuals viewed and described the linguistic norms of mathematical proof writing at the undergraduate level. After data collection and during analysis, it became increasingly clear that the mathematicians' descriptions did not appear to indicate precise, prescribed norms. Rather, the mathematicians discussed the conventions of the language of mathematical proof writing, sometimes remarking on the contexts in which the proofs were written. As a result, the focus in this body of work turned from aiming to identify precise, prescriptive linguistic norms. Rather, the focus turned to understanding the reasoning behind these conventions of the language.

The results from Study 1 provided rich descriptions of how these mathematicians and undergraduate students perceived the various potential breaches of convention identified in the seven partial proofs. In particular, the findings highlighted three main ways in which the mathematicians' and undergraduate students' discussions of the breaches differed. First, some of the students' responses suggested that they do not believe that mathematical proof writing should obey the rules of natural language, whereas the mathematicians in the study did believe that mathematical proofs are subject to the rules of natural language. Next, the mathematicians attended to multiple aspects of how new mathematical objects were introduced into proofs. While the undergraduate students did attend to the introduction of new variables, the students may not understand the many nuances involved when the introduction of new mathematical objects is more complicated. Finally, despite being asked to comment on whether they believed the potential breaches were unconventional, some mathematicians focused their discussion on the context of the proof to determine how formal a proof should be written. The student participants may not have been aware of this distinction; however, the students in this first study were not explicitly asked if the context would affect their evaluations of the potential breaches.

Using the findings from Study 2, I designed an online survey to investigate if a larger sample of mathematicians and undergraduate students agreed that potential breaches were unconventional of the language of mathematical proof writing for the reasons reported in Study 1. One of the planned analyses involved comparing responses of subgroups of the populations (for instance comparing pure versus applied mathematicians or comparing the undergraduate students in different years of study), however, the sample distribution for these subgroups were not conducive to this analysis. Moreover, the between-groups analysis did not reveal many significant differences between the mathematician and student participants.

Meanwhile the survey results of 128 mathematicians and 135 undergraduate students did allow analyses regarding how the entire samples responded to the prompts. There were four main findings from Study 2. First, the participants found some of the potential breaches of the conventions of the language of mathematical proof writing to be unconventional regardless of the context in which the breaches occurred. Second, mathematicians' responses suggested that they perceive the contexts of blackboard proofs and student-produced proofs to be different when evaluating certain potential breaches; meanwhile, the students' responses did not indicate any differences between how they perceived the conventions of the proof writing these two contexts. Third, both mathematicians and undergraduate students expected textbook authors to follow stricter norms when writing mathematical proofs than they expected of mathematics instructors and undergraduate students. Finally, some of the potential breaches of the conventions of the language of mathematical proof writing were not judged to be unconventional by either the mathematicians or the undergraduate students, despite the literature suggesting that the potential breaches would be viewed as unconventional.

Study 3 departed from the clinical nature of Studies 1 and 2, bringing the focus to undergraduate students' proof writing while enrolled in an introduction to proof course. I successfully recruited six students to participate in the study and each of the participants met with me for five or six interviews. Since the students needed to complete the assignment, submit the assignment, receive the graded assignment, and then submit the graded assignment to me, there was a delay between the date the professor gave an assignment and the date the interview was scheduled. Moreover, once I had the graded assignments, the professor proofs needed to be constructed and approved (if not written by the professor). These delays between the assignment of homework problems (or the date of the examination) and the interviews in which those same homework problems (or exam problems) were discussed limited the number of interviews possible.

Furthermore, few changes in how the participants discussed the potential breaches appeared as I conducted the interviews and analyzed the data. Meanwhile, the findings revealed that the students differed along two different dimensions. The first dimension in which the students differed was with regards to whether the participant believed that complying with the conventions of the language of mathematical proof writing was necessary in the writing of mathematical proofs. The second dimension in which the students differed was with regards to whether the students believed that the proofs that they wrote should comply with the conventions of the language of mathematical proof writing.

The variation among these two dimensions create a four different profiles of how undergraduate students perceive the necessity of following the conventions of the language of mathematical proof writing and whether their discussions and justifications of their own proof writing indicated that they believed that their proofs should be held to these same conventions of mathematical proof writing. The data from Study 3 presents existential evidence of each type of profile.

The first profile highlighted the student who both believed that following the conventions of mathematical proof writing was necessary and believed that their proof writing should avoid breaches of the conventions. This intrinsically motivated student identified that the professor's proofs should act as a guide for students to mimic and that she wanted her proof writing to have clear exposition.

The second profile described the student who believed it was necessary to follow the conventions of mathematical proof writing, yet did not avoid breaching those conventions in his own proof writing. Throughout the semester, no points were deducted nor were notes made with regard to the student's informal language use. Unsurprisingly, this student found his own proof writing to be decreasingly unconventional throughout the semester. Moreover, this student did not believe it was necessary to follow the conventions of mathematical proof writing in proofs written for a homework assignment, as the intended reader would understand the proof without careful exposition.

The third profile focused on a student that did not believe that the conventions of mathematical proof writing necessarily needed to be followed; however, elected to write proofs avoiding breaches of these conventions. Like the student in the first profile, this student was intrinsically motivated and indicated that he wanted his proofs to be as clear as possible. Meanwhile, he also believed that using conventional mathematical language was a matter of personal preference and thus not necessary.

The final profile described students that did not believe that one needed to conform to the conventions of mathematical proof writing and did not do so in their proofs. They both indicated that they believed that using careful exposition in proof writing was a matter of personal preference and hence, unnecessary. Moreover, both of these students appeared to be motivated extrinsically and their discussions focused on whether they would receive full grades for their proofs.

Implications

Taken as a whole, this body of work has multiple implications. First, the findings from Study 1 and Study 2 suggest that mathematicians are generally in agreement when deciding whether a potential breach is unconventional in the context of a proof writing in a textbook. These findings, however, also suggest that mathematicians are not necessarily in agreement when discussing their expectations of how their undergraduate students should write proofs. In Study 1, when discussing the use of formal propositional language within proofs, the eight mathematicians had differing opinions for which level of undergraduate student using the formal language was unconventional. Moreover, the results in Study 2 specifically highlight that in the sample of mathematicians were split in the context of student-produced proofs when deciding whether or not six of the potential breaches were unconventional.

If the mathematicians who are teaching these advanced, proof-based, undergraduate mathematics courses are unclear on their expectations of the proof writing of their students, it would be expected that the undergraduate students in these professor's classes similarly may not have clear understandings of the expectations to which they are being held. Clearly, the results from this body of work show that this is the case. With the exception of a few types of potential breaches, the student participants' results in Studies 1 and 2 did not show high levels of agreement when deciding whether the potential breach was unconventional. In particular, the student participants' responses were split in seven types of potential breaches in the context of student-produced proofs. Thus, there may be a trickle-down effect here; if the mathematicians are, as a group, unsure of the expectations they should hold their students to, their students surely are similarly unsure of these expectations.

Next, findings from Studies 1 and 2 suggest that the mathematicians in these studies view homework assignments as a setting for a stricter following of the conventions of the language of mathematical proof writing than in-class examinations. In Study 1, the mathematicians explained that the time pressures of an in-class examination would deter them from deducting points for some types of breaches of conventions of the language of mathematical proof writing. Meanwhile, these mathematicians suggested that in a homework assignment, they would be more inclined to deduct points for some types of breaches because in the setting of a homework assignment the student has a sufficient amount of time to attend to their language use. The results from Study 2 support these findings. For each of the types of potential breaches in which mathematicians' results suggested they viewed the homework and exam settings as differently, fewer of the mathematicians indicated that they would neither make a note to the student nor take points off.

Meanwhile, findings from Studies 2 and 3 clearly suggest that the undergraduate participants in these studies have the opposite expectations of their graded proofs. In Study 2, for the potential breaches in which the undergraduate participants viewed the homework and exam settings as differently, more of the students expected a mathematician to deduct points in the context of an exam than in the context of a homework assignment. Moreover, the results from Study 3 support these findings, where

Students 1, 4, 5, and 6 all described their expectations that proofs written for homework assignments would be graded less harshly than proofs written in exam settings. More specifically, Student 1 explained that he had this expectation because he believed that the purpose of homework assignments was to demonstrate his understanding, rather than using careful mathematical language.

Thus, not only is there evidence that both mathematicians and undergraduate students may be unsure of what professors' expectations of the proofs their students write are with regards to some types of breaches, but also that they have opposing expectations with respect to homework and exam settings. Based on these results, it is clear that professors' expectations of their students' proofs should be made explicit to their students. Moreover, the results also suggest that it may be advantageous for mathematicians to discuss what their expectations are for how their students use the language of mathematical proof writing in their introduction to proof courses.

Discussion

The findings from this body of work provide a research-based view of the language of mathematical proof writing, a perspective that was previously missing from the mathematics education literature. More specifically, the findings present potential breaches of conventions of the language of mathematical proof writing, justifications of why those potential breaches are unconventional of mathematical proof writing, to what extent these breaches are viewed as unconventional in the mathematical community, and how mathematicians' assessment of student-produced proofs is affected by these breaches. Moreover, the findings from Study 3 present four different ways that undergraduate students might perceive the significance of the conventions of the language of mathematical proof writing and enact their perceptions in their own proof writing.

While the field of mathematics education previously lacked a focus on empirical studies considering the language of mathematical proof writing, the literature reviews included in this dissertation highlight existing guides describing how mathematicians and undergraduates should write mathematics. These studies present some empirical evidence supporting the suggestions in those guides, and also contradicting other suggestions in the literature. Moreover, these studies also inform the now growing body of work investigating how mathematicians grade their students' proofs (Miller, Engelke-Infante, & Weber, 2016; Moore, 2014) and how students interpret their professors' feedback on graded proofs (Moore, Byrne, Fukawa-Connelly, & Hanusch; 2016). Beyond mathematics education, the methods used in these studies can be applied in investigations in other fields in which technical languages are used.

Furthermore, the results from this dissertation will inform subsequent studies of professional mathematical language and undergraduate students' acquisition of the language of mathematical proof writing. In particular, I hope to continue my work in this area during my upcoming year (2016-2017) at Arizona State University by conducting a teaching experiment to better experience and understand how students learn to use the language of mathematical proof writing and to write mathematical proofs. Based on the findings from this study, I plan to apply for an internal grant from the Research Enhancement Program at Texas State University during my first year as a tenure-track assistant professor (2017-2018) to fund a project in which I develop and test a teaching intervention to help students learn and engage with the language of mathematical proof writing. Using this teaching intervention, I plan to investigate how an undergraduate student's understanding of the language of mathematical proof writing might positively affect that student's proof comprehension and construction. Ideally, this investigation would lead to a larger scale study of the connection between the linguistic and cognitive aspects of understanding advanced mathematics (and proof in particular), for which I would apply for an NSF IUSE (Improving Undergraduate Stem Education) grant.

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APPENDIX A: SURVEY INSTRUCTIONS

Instructions for the Survey

In this survey, you will be presented with examples of student proof writing and asked a series of questions about a particular aspect of the writing. The examples of student proof writing are partial proofs, meaning that the example begins at the beginning of a student's proof attempt, but some length of the attempt has been removed from the end. Further, these partial proofs have been marked to bring your attention to a particular part of the proof. For each of the marked partial proofs you see, you will be provided with a brief explanation of why the particular part of the proof might be considered to be unconventional and asked to decide if you agree the language use is unconventional in each of the following three contexts:

1. A ***textbook proof*** is a proof that is written in the way that proofs in undergraduate textbooks are normally written.
2. A ***blackboard proof*** is a proof that is written in the way that proofs normally appear on the blackboard in an undergraduate proof-based course taught by a mathematician.
3. A ***student produced proof*** is a proof that is written in the way that you would prefer an undergraduate student to write mathematical proofs.

Below is an example of how a participant might respond to a prompt in this survey:

Consider the following task and subsequent partial proof:

Suppose $f : X \to Y$ and $g : Y \to Z$ for sets X, Y, and Z.
Prove: If f is onto Y and $g \circ f$ is one-to-one, then g is one-to-one.

Suppose f is onto Y and g∘f is one-to-one.
Let ☺, ☹ ∈ Y such that g(☺) = g(☹).

	A mathematician suggested that this is unconventional mathematical writing because **the assignment of variables should follow normal naming conventions where this uses inappropriate symbols.** Would you agree or disagree in the following contexts?		To what extent does this use of mathematical language lower the quality of the exposition of the partial proof in the following contexts? *(Please select "N/A" if you do not find this writing to be unconventional.)*			
	Agree	Disagree	Significantly	Moderately	Not at all	N/A
Textbook Proofs	◉	○	◉	○	○	○
Blackboard Proofs	○	◉	○	○	○	◉
Student Produced Proofs	◉	○	○	◉	○	○

In the context of textbook proofs, this participant agreed that using smiley faces as variables is unconventional for the reason provided in the prompt and indicated that using smiley faces significantly lowers the quality of the exposition of the partial proof.

In the context of blackboard proofs, the participant disagreed that using smiley faces is unconventional for the reason provided and marked "N/A" in the right most column. Please note that if the participant agreed that using smiley faces was unconventional but did not agree with the explanation provided in the prompt, the participant would still indicate "disagree" and "N/A" as shown.

Finally, in the context of the student produced proofs, the participant agreed that using smiley faces as variables is unconventional for the reason provided in the prompt and indicated that this use of smiley faces moderately lowers the quality of the exposition of the partial proof.

Next, the participant will be asked if she/he would deduct points for this unconventional use of mathematical language on a homework and on an exam in an introduction to proof course.

As you proceed through the study, please keep in mind this study is particularly focused on the mathematical writing, or exposition, of the proof. While it is natural to focus on the validity of the logical argument of the proof, this is not the goal of this study. You may recognize that there are other, or more serious, issues with the partial proof, but please focus on the highlighted issue and the corresponding questions at hand.

Finally, I ask you to respond to the questions based on how you believe mathematicians, as a community, view the language of mathematical proof writing in various undergraduate contexts. Please respond to the questions accordingly; even if this may not be the way you write proofs in other contexts (for example, proofs for professional journal articles or a 'back of a napkin' proof).

I have read these instructions.
(And acknowledge that I cannot return to this page later.)

○

APPENDIX B: SURVEY PAGE EXAMPLE

Please keep in mind this study is particularly focused on the mathematical writing of the proof. While it is natural to focus on the validity of the logical argument of the proof, this is not the goal of this study.

Also, keep in mind the three different contexts of undergraduate proof writing:

1. A ***textbook proof*** is a proof that is written in the way that proofs in undergraduate textbooks are normally written.
2. A ***blackboard proof*** is a proof that is written in the way that proofs normally appear on the blackboard in an undergraduate proof-based course taught by a mathematician.
3. A ***student produced proof*** is a proof that is written in the way that you expect that a mathematician would prefer an ideal undergraduate student to write mathematical proofs.

Consider the following task and subsequent partial proof:

Let A be a set. Prove: If S is a relation on A, then the relation $R = S \circ S^{-1}$ is symmetric.

let $x, z \in A$ s.t.
$(x,z) \in R$
$R = S \circ S^{-1}$
$(x,z) \in S \circ S^{-1}$
$\exists y$ s.t. $(x,y) \in S^{-1}$ and $(y,z) \in S$
$(y,z) \in S$ $\therefore$ $(z,y) \in S^{-1}$
$(x,y) \in S^{-1}$ $\therefore$ $(y,x) \in S$

	A classmate suggested that a mathematician would think this is unconventional mathematical writing because ***the writer should use verbal connectives (e.g. therefore, then, since) to indicate the flow of the argument to the reader.*** Would you agree or disagree in the following contexts?		To what extent does this use of mathematical language lower the quality of the exposition of the partial proof in the following contexts? *(Please select "N/A" if you do not find this writing to be unconventional.)*			
	Agree	Disagree	Significantly	Moderately	Not at all	N/A
Textbook Proofs	○	○	○	○	○	○
Blackboard Proofs	○	○	○	○	○	○
Student Produced Proofs	○	○	○	○	○	○

In an introduction to proof course, would you expect a mathematician to take points off for this type of unconventional use of mathematical language in the following contexts?

	I would expect a mathematician to take points off (and perhaps make a note).	I would expect a mathematician to make a note, but not take points off.	I would not expect a mathematician to take points off or make any to the student.
Homework	○	○	○
Exam	○	○	○

APPENDIX C: PAIRWISE COMPARISONS OF MATHEMATICIANS' AND STUDENTS' EVALUATIONS OF EACH POTENTIAL BREACH IN EACH CONTEXT

	Potential Breach of Mathematical Language	Do you agree that this is an unconventional use of mathematical language for the reason provided? *(% Mathematicians Agree/% Students Agree)*		
		Textbook Context	Blackboard Context	Student Context
Partial Proof 1	Uses non-statement	**100% / 96%** *	**100% / 89%** *	**98% / 85%** *
	Uses an unspecified variable	**59% / 83%** *	34% / 52%	37% / 61% *
	Includes statements of definitions	41% / 41%	18% / 24%	12% / 19%
	Lacks punctuation and capitalization *N(mathematicians)=128, N(students)=108*	95% / 82%	34% / 17%	50% / 30%
Partial Proof 2	Uses formal propositional language	**88% / 68%** *	74% / 60%	66% / 59%
	Uses unclear referent	93% / 80%	**67% / 41%** *	70% / 49%
	Overuses variable names	98% / 94%	95% / 90%	93% / 86%
	Mixes mathematical notation and text *N(mathematicians)=128, N(students)=102*	88% / 79%	28% / 23%	45% / 34%
Partial Proof 3	Fails to make the proof structure explicit	70% / 70%	29% / 46%	28% / 40%
	Uses mathematical symbols or notation as an incorrect part of speech	72% / 58%	19% / 18%	24% / 24%
	Uses informal language *N(mathematicians)=128, N(students)=101*	77% / 82%	45% / 64%	48% / 58%
Partial Proof 4	Fails to state assumptions of hypotheses	64% / 77%	34% / 51%	40% / 56%
	Uses an unspecified variable with an existential quantifier	85% / 93%	55% / 73%	**54% / 78%** *
	Lacks verbal connectives *N(mathematicians)=128, N(students)=96*	97% / 84%	52% / 47%	72% / 61%

* Mathematicians and students responses were significantly different (Chi-squared test of independence) by a level of significance $\alpha=0.05/42$.

APPENDIX D: MATHEMATICIANS' AND STUDENTS' EVALUATIONS OF HOW EACH POTENTIAL BREACH AFFECTS THE QUALITY OF THE EXPOSITION OF A PROOF

	Potential Breach of Mathematical Language	To what extent does this use of mathematical language lower the quality of the exposition of the partial proof in the following contexts? *(% Significantly / % Moderately / % Not at all / % N/A)*								
		Textbook Proof			Blackboard Proof			Student-produced proof		
		Mathematicians	Students		Mathematicians	Students		Mathematicians	Students	
Partial Proof 1	Uses non-statement	86% / 13% / 1% / 0%	76% / 16% / 6% / 2%		67% / 30% / 3% / 0%	54% / 30% / 14% / 2%		60% / 34% / 6% / 0%	45% / 32% / 20% / 3%	
	Uses an unspecified variable	**28% / 26% / 20% / 26%**	**38% / 42% / 9% / 10%**	*	16% / 20% / 23% / 41%	23% / 32% / 18% / 27%		13% / 25% / 26% / 36%	18% / 43% / 15% / 24%	
	Includes statements of definitions	8% / 25% / 30% / 38%	11% / 14% / 36% / 39%		2% / 13% / 37% / 48%	4% / 8% / 42% / 47%		1% / 10% / 34% / 55%	4% / 12% / 31% / 53%	
	Lacks punctuation and capitalization	30% / 35% / 34% / 2%	16% / 34% / 41% / 9%		4% / 23% / 34% / 39%	5% / 9% / 42% / 44%		4% / 27% / 40% / 30%	5% / 19% / 39% / 38%	
	N(mathematicians)=128, N(students)=108									
Partial Proof 2	Uses formal propositional language	50% / 34% / 9% / 6%	39% / 25% / 16% / 21%		**38% / 34% / 15% / 13%**	**16% / 38% / 20% / 26%**	*	19% / 46% / 19% / 16%	17% / 37% / 24% / 23%	
	Uses unclear referent	45% / 41% / 11% / 3%	29% / 47% / 12% / 12%		**28% / 38% / 16% / 17%**	**10% / 35% / 30% / 25%**	*	20% / 46% / 19% / 15%	10% / 38% / 31% / 21%	
	Overuses variable names	73% / 23% / 3% / 1%	70% / 26% / 2% / 2%		60% / 34% / 5% / 2%	53% / 38% / 6% / 3%		53% / 39% / 6% / 2%	46% / 41% / 8% / 5%	
	Mixes mathematical notation and text	27% / 42% / 27% / 4%	15% / 40% / 33% / 12%		4% / 17% / 45% / 34%	1% / 20% / 43% / 36%		3% / 27% / 43% / 27%	1% / 25% / 46% / 28%	
	N(mathematicians)=128, N(students)=102									
Partial Proof 3	Fails to make the proof structure explicit	24% / 42% / 15% / 19%	35% / 26% / 19% / 21%		9% / 20% / 28% / 42%	16% / 27% / 28% / 30%		4% / 27% / 29% / 41%	9% / 31% / 30% / 31%	
	Uses mathematical symbols or notation as an incorrect part of speech	21% / 33% / 33% / 13%	12% / 35% / 32% / 22%		5% / 14% / 38% / 44%	4% / 16% / 36% / 45%		1% / 17% / 40% / 42%	2% / 22% / 34% / 43%	
	Uses informal language	36% / 39% / 13% / 13%	49% / 31% / 14% / 7%		15% / 34% / 25% / 26%	22% / 42% / 20% / 17%		15% / 38% / 24% / 23%	25% / 32% / 23% / 21%	
	N(mathematicians)=128, N(students)=101									
Partial Proof 4	Fails to state assumptions of hypotheses	23% / 32% / 23% / 22%	36% / 35% / 9% / 19%		10% / 22% / 26% / 42%	19% / 31% / 21% / 29%		6% / 28% / 30% / 36%	17% / 31% / 26% / 26%	
	Uses an unspecified variable with an existential quantifier	32% / 42% / 18% / 8%	49% / 39% / 7% / 5%		**12% / 35% / 30% / 23%**	**23% / 51% / 11% / 15%**	*	**10% / 33% / 31% / 26%**	**21% / 49% / 22% / 8%**	*
	Lacks verbal connectives	62% / 29% / 9% / 1%	44% / 39% / 7% / 10%		17% / 38% / 13% / 31%	17% / 34% / 24% / 25%		23% / 46% / 13% / 17%	23% / 40% / 21% / 17%	
	N(mathematicians)=128, N(students)=96									

* Mathematicians and students responses were significantly different (Fisher Exact tests) by a level of significance α=0.05/42.