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PRICE-SETTING NEWSVENDOR OPTIMAL POLICIES WITH MEAN-VARIANCE CRITERIA

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ABSTRACT OF THE DISSERTATION

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The newsvendor problem has been widely studied since it first appeared in the literature at the end of the XIX century. It is still the subject of further research that addresses more complex and realistic situations based on previous work. The amount of research work done on this model and its applications is so vast that a simple search in Google Scholar under the keyword "newsvendor" will return almost $8,000$ entries between 2010 and 2015. The problem, in its basic formulation, aims at finding an optimal replenishment policy of a perishable product in the face of uncertain, stochastic demand. Such a solution is selected in a way that maximizes the expected profit, which is calculated as the difference between the income and the purchase cost of the good in question.

This thesis elaborates on the conditions needed to guarantee the existence of a unique maximum of the objective function in the price-setting newsvendor problem with price-dependent demand. This function is presented as a mean-variance trade-off between the expected profit and the variance of the profit, weighted with a risk parameter.

The main goal of this thesis is to simplify any instance of the risk-sensitive newsvendor problem for the two most common price-dependent demand functions, namely, additive and multiplicative functions. When possible, we will provide sufficient conditions for the

unimodality of the problem. Unlike many other results previously published, we aim at expressing such conditions by using a metric that captures managerial attention. To this end, we use the lost sales rate elasticity as a measure of the level of service provided by the seller and express these sufficient conditions as a function of this metric.

Acknowledgement

When I came to the United States in 2008, little I knew that I would end up completing a Ph.D. For different reasons, I came to know myself much better in the span of three years, between 2010 and 2012. When life brought me the opportunity to do what I enjoy the most, study and learn, I could not but embrace that chance.

Being in a foreign country for 8 years can take a toll. Had I been a single person, I might have returned to Spain several years ago. I thank God that I have a wife who is more than I could have ever asked for and who stayed with me through thick and thin. Patricia made enormous efforts to stay here and make us a family. To her I owe completing this degree and having a full, meaningful life far from what was our home many years ago.

I want to extend my deepest gratitude to my parents, Baltasar and Marisa. I know complete happiness because of them and for that I will be forever in debt. In their role as parents, they always supported my American adventure over their deepest feeling of sorrow and solitude for seeing their youngest son go away. They have always been present in my thoughts and if I ever questioned my decision to come to the United States was because of them. Along this path my mother passed away, leaving me a legacy of unconditional love, happiness, and memories that will live in me forever.

When I came to Rutgers, I was determined to work with an advisor with whom I could have a feeling of comradeship and mutual respect. Because of her professional interests and her excellent attitude and aptitude, Dr. Melike-Baykal-Gürsoy has been a fantastic advisor. My gratitude goes to her for her help during almost 4 years. She shaped me into a researcher through her patience and her own skills. She gave me freedom to work and was always willing to discuss results and propose new ideas. Thanks to her, my path through my Ph.D. studies has been one to remember.

I also would like to extend my gratitude to the rest of my thesis committee members: to Dr. Adi Ben-Israel for being an accessible, friendly professor to look up to; to Dr. Fred Roberts because of the invaluable help that he has given me, academically and extra-academically, since I started contributing to some projects in CCICADA; to Dr. Ana Muriel and Dr. Michael Katehakis for their availability and comments for improving this dissertaion;

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Contents

Appendices

List of Tables

List of Figures

Chapter 1

Introduction and Literature Review

1.1 The Classic Newsvendor Problem

The newsvendor problem has been widely studied since it first appeared in the literature. Albeit an equivalent problem applied to banking theory already appeared at the end of the XIX century (Edgeworth, 1888) when Edgeworth tried to determine optimal cash reserves to satisfy random withdrawals from depositors (see Figure 1.1), its modern formulation is due to Arrow, Harris, and Marschak (1951). In their article, a policymaker wants to maximize what they called the expected net utility. In the particular case of a company, this measure is the expected monetary profit as defined by the difference between revenue and cost. The net utility is expressed as a function of a set of controllable and uncontrollable (random) variables. It is the goal of the policymaker to select the appropriate values of these controllable variables so that, taking into account the probability distributions of the uncontrollable variables, the expected net utility is maximized.

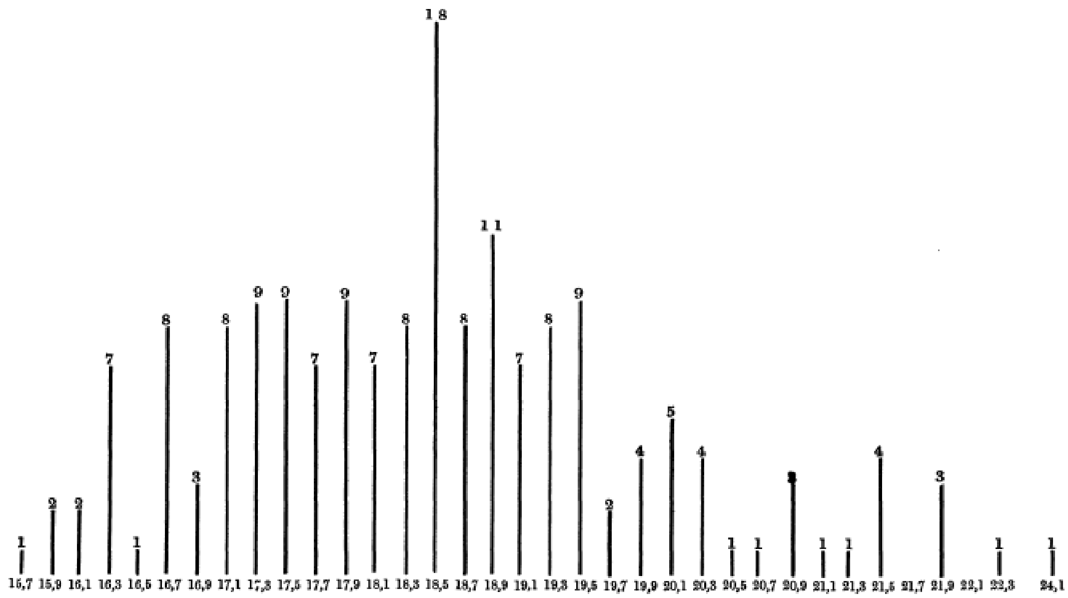


Figure 1.1: 173 returns of Bank of England notes between 1826 and 1844 and their quantities, as shown in Edgeworth (1888)

An immediate application of this model is its use in inventory theory. Thus, in its most basic formulation, the newsvendor problem utilizes only one controllable variable, namely the stock quantity, and one uncontrollable variable, the demand of the product. The newsvendor wants to know how many newspapers he should stock in order to maximize his profit, given that the demand is random. The product is assumed to be bought at a unit cost c and sold at a unit price p. The newsvendor has to make a decision on the stock quantity x before observing a realization of the demand D. If this realization is greater than the the stock quantity, the excess of demand is lost. On the contrary, if the stock quantity is greater than the realization of the demand, the excess of stock cannot be returned. In summary, the newsvendor seeks the maximization of the function below:

$$\mathbb{E}\left(\Pi(x)\right) = \underbrace{\mathbb{E}\left(p \cdot \min\{D, x\}\right)}_{\textit{Expected revenue}} - \underbrace{cx}_{\textit{Expected cost}}, \tag{1.1}$$

where $\Pi(\cdot)$ denotes the net utility function, in this case a profit function. The maximization of the expected profit is straightforward. If we denote by $f(\cdot)$ the probability distribution function of the demand, the expected minimum of the stock quantity x and the demand D is clearly

$$\mathbb{E}\left(\min\{D,x\}\right) = \int_0^x u dF(u) + \int_x^\infty x dF(u),$$

where $dF(u) = f(u)du$. Hence, equation (1.1) can now be written in term of the probability distribution function of the demand $f(\cdot)$ and the cumulative density function of the demand $F(\cdot)$:

$$\mathbb{E}\left(\Pi(x)\right) = p\left(\int_0^x uf(u)du + x\left(1 - F(x)\right)\right) - cx.$$

An application of Leibniz's Rule to the first-order derivative of $\mathbb{E}\left(\Pi(\cdot)\right)$ yields

$$\frac{d}{dx}\mathbb{E}\left(\Pi(x)\right) = p\left(1 - F(x)\right) - c,$$

whence it easily follows that

$$x^* = F^{-1}\left(1 - \frac{c}{p}\right), \tag{1.2}$$

is the only critical point of $\mathbb{E}\left(\Pi(\cdot)\right)$, and it represents a maximum, since $\frac{d^2}{dx^2}\mathbb{E}\left(\Pi(x)\right) = -pf(x) \leq 0$. Therefore, the classic newsvendor problem uses the $(1 - \frac{c}{p})^{\underline{th}}$ quantile of $F(\cdot)$ as the optimal ordering point that maximizes the expected profit (see Figure 1.2).

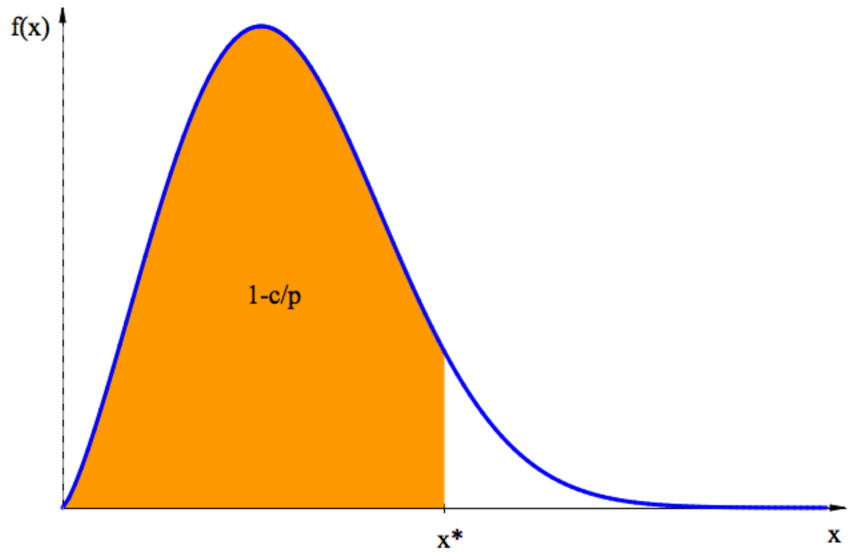


Figure 1.2: Graphical interpretation of the solution to the classic newsvendor problem

1.2 Adding Complexity to the Newsvendor Problem

The model presented in §1.1 has been subject to many modifications throughout the years. One of the reasons why the newsvendor problem has been continuously researched throughout the last decades is its applicability to real-world problems and its flexibility. Some examples of this applicability can be found in very different industry sectors: from the straightforward application to inventory of perishable products (Van Donselaar, van Woensel, Broekmeulen, and Fransoo, 2006), to scheduling the shifts of medical personnel in a hospital (Olivares, Terwiesch, and Cassorla, 2008); from energy dispatch (Densing, 2013) to game theory in economics (Cachon and Netessine, 2006), revenue management in the airline industry (Deshpande and Arikan, 2012), or other service systems (Green et al., 2006; Haugen and Hill, 1999). In the area of inventory theory, the literature offers us a very diverse range of approaches to different demand and inventory models. However, the global maxima of the objective functions presented in those problems are in many cases not straightforward to derive and the inclusion of several assumptions is often needed.

Until the 1950's there used to be a disconnection between businessmen and economists in inventory problems. For example, the classic result of the economic lot size did not consider a price-dependent demand. In other words, the demand was assumed to be given or, in the best-case scenario, to be a realization of a random variable. It was not until the 1950's (Whitin, 1955) that the effect of price in the stochastic demand was introduced. This fact, in turn added the price as a decision variable to the problem on top of the stocking quantity. Petruzzi and Dada (1999) used this price-demand relationship in several ways while presenting a single-period approach that maximized expected profit and established theorems that indicated how to select the stocking policy based on the statistical distribution that conferred the demand its stochastic nature. Moreover, they presented a closed, analytical expression to determine the optimal price as a function of such stocking policy and compared it to the so-called riskless price as defined in Mills (1959). Also, they showed how pricing decisions affect demand uncertainty under various modeling assumptions. Federgruen and Heching (1999) introduced a multi-period model for inventory control with backlogging which included a price-dependent stochastic demand and analyzed the optimal pricing and replenishment strategies to be set simultaneously in each period and how they compared when prices can be set bi-directionally or when only markdowns are allowed.

Kocabıyıkoğlu and Popescu (2011) unified and introduced the concept of lost sales rate (LSR) elasticity and explained the monotonicity of the optimal price and stock as a function of this new concept. Moreover, they set the properties that the LSR elasticity must possess for the profit to be jointly concave as a function of pricing and stocking decisions and for the problem to have a unique, optimal solution.

For a general overview of the newsvendor model applied to inventory theory, there are excellent literature reviews (Petruzzi and Dada, 1999; Khouja, 1999) that encompass results and findings of different versions of this problem: single-product, multi-product, single-stage, multi-stage, infinitely-staged.

On the other hand, and as opposed to the norm in financial analysis, a tradeoff between expected return and risk in planning problems was absent for many years. All the works mentioned above, plus some others (Xu, Cai, and Chen, 2011; Xu, Chen, and Xu, 2010; Wang, Jiang, and Shen, 2004; Federgruen and Heching, 1999; Mantrala and Raman, 1999), presented a profit-optimizer decision maker that is risk-neutral. This decision maker seeks to maximize the expected profit by finding an appropriate balance between expected income and expected costs. However, these models do not take into account the variance of the income, and therefore it is likely to select an optimal policy that maximizes the expected income but presents a variability that turns this decision into a risky bet. Some research efforts have taken place with respect to this approach. However, albeit the demand keeps its stochastic nature, in many cases it is not presented as a function of the price and therefore the optimization is sought by means of only selecting an optimal quantity of product to purchase, thus disregarding pricing decisions. To the best of our knowledge, it was Lau (1980) who first considered a mean-standard deviation payoff criterion within the newsvendor model, although he only gave the equation whose root provided the optimal quantity. Chen and Federgruen (2000) presented this mean-variance analysis for different planning problems, namely, the newsvendor problem, the base stock problem and the (R, nQ) model. When dealing with the newsvendor problem, they applied a single-stage model that optimized a utility function for risk-neutral, risk-averse and risk-seeking decision maker over a feasible region given by an efficient frontier. Later on, Choi et al. (2008) introduced stockout costs in the mean-variance analysis and presented results for various risk attitudes under demands that followed well-known statistical distributions. Wu, Li, Wang, and Cheng

(2009) focused on the impact that stockout costs have on the optimal ordering decisions when comparing classic models and mean-variance analysis models, showing via numerical results that, for a given stockout price, a mean-variance analysis yields a lower optimal order quantity and a lower optimal value of the problem. Özler, Tan, and Karaesmen (2009) offered a one-stage, multi-product approach with value-at-risk considerations that included mathematical programming results for the case of one and two products and an approximation to the N-product case. Wang and Webster (2009) introduced an alternative approach by using a piecewise linear loss-averse utility function. In Wang, Webster, and Suresh (2009); Eeckhoudt, Gollier, and Schlesinger (1995); Agrawal and Seshadri (2000); Gaur and Seshadri (2005) the authors studied the newsvendor problem within the expected utility framework and considered three different utilities, namely, constant absolute risk aversion (CARA), decreasing absolute risk aversion (DARA), and increasing absolute risk aversion (IARA), whereas Choi and Ruszczyński (2011) examined this model with an exponential utility function used to model a risk-averse decision-maker with a constant risk coefficient in the sense of the Arrow-Pratt measure. An exponential utility function was also used by Bouakiz and Sobel (1992) to show that a base-stock policy is indeed optimal in a dynamic version of the newsvendor problem.

In this thesis, our aim is to combine the two approaches introduced above. That is, we present a mean-variance analysis of the newsvendor problem that includes a stochastic, price-dependent demand. This problem was presented before in Agrawal and Seshadri (2000), but it was approached from the perspective of the expected utility framework, selecting concave utility functions to model risk-averse situations and finding the relation of optimal pricing and stocking strategies with respect to the levels of risk aversion. There have been other similar efforts; in particular, Choi and Ruszczyński (2008); Ahmed, Çakmak, and Shapiro (2007) proposed models based on a conditional value-at-risk decision criterion (CVaR) for stock optimization, whereas Chen, Xu, and Zhang (2009) analyzed how both the optimal price and stock quantity changed when varying the $\eta - quantile$ of the function they sought to maximize. Our work is focused on an alternative approach that penalizes the variability of the profit in the objective function as done by Markowitz several decades ago (Markowitz, 1952).

We propose a joint optimization approach whose goals can be summarized as follows:

- Find the conditions for the unimodality of the price-setting newsvendor problem with price-dependent demand functions, either additive or multiplicative (isoelastic).
- Consider those conditions for risk-neutral, risk-averse, and risk-seeking individuals.
- Write those conditions in managerial terms so they can be better understood. This poses an important difference with respect to previous works, where conditions are given in much more technical terms involving failure rates and generalized failure rates. Our results will be given in terms of the lost sales rate (LSR) elasticity, a concept introduced by Kocabıyıkoğlu and Popescu (2011), which directly relates to the level of service given to the customer, even though it depends implicitly on the failure rate of the random term of the demand. These authors already gave conditions for the concavity of the risk-neutral problem in terms of the LSR elasticity both price-dependent demand functions. We aim at giving a complete framework for any instance of the risk-sensitive problem.

1.3 The Mean-Variance Trade-Off Model

Mean-variance models have been widely used in the literature for incorporating risk to decision-making (Choi, Li, and Yan, 2008; Choi and Chiu, 2012; Chen and Federgruen, 2000; Wu, Li, Wang, and Cheng, 2009). This measure of risk does not fall within the category of coherent measures of risk as defined in Artzner, Delbaen, Eber, and Heath (1999): in particular the variance does not possess any of the four characteristics that characterize such risk measures, namely, subadditivity, monotonicity, translation equivariance, and positive homogeneity. Other measures like VaR, also used for modeling risk-sensitivity, are not subadditive and therefore not coherent either (Szegö, 2005). Other risk measures, denominated spectral risk measures (Acerbi, 2002) have been found to present a unique maximum in inventory and pricing problems if a) the demand error has an increasing failure rate (IFR) or is a positive random variable with increasing generalized failure rate (IGFR), b) the riskless demand has increasing price elasticity (IPE), and c) the risk-transformed distribution preserves IFR or IGFR (Fichtinger, 2010). A spectral risk measure is defined as an average of the quantiles of the distribution of the returns weighted with a non-increasing function, referred to as the spectrum. While all spectral risk measures are coherent, not all coherent risk measures are spectral. In this sense, the mean-CVaR is both spectral and coherent, but fails to preserve IFR.

Consider a decision maker who sells a perishable product, and has the ability to decide on the quantity to produce or buy and the price to set for the good he or she sells. Moreover, such a decision must be based on the expected revenue for the upcoming period, the production or procurement costs and the variability of the revenue. Assuming that the demand for a period is random and a function of the price we introduce the following risk-sensitive performance measure, which we aim at maximizing:

$$\tilde{P}(p,x) \quad = \quad pE(min\{D(p,\epsilon),x\}) - cx - \lambda Var(p \cdot min\{D(p,\epsilon),x\}), \tag{1.3}$$

where p is the price set for the good and ϵ is a continuous random variable. This random variable has support $[A, B]$, where the sign and magnitude of A and B depend on the demand function being used. It also has a probability distribution given by the continuous density function $f(\cdot)$ and the cumulative distribution function $F(\cdot)$. Furthermore, x is the inventory level set for the good, c is the production or procurement cost per unit of finished product, and $D(p, \epsilon)$ is the demand for a single period given the price p and the realization of the random variable ϵ. Finally, λ is a risk parameter greater than 0 for risk-averse cases, smaller than 0 for risk-seeking cases and equal to 0 in risk-neutral cases. Thus, the sign of this parameter reveals the attitude of the newsvendor towards the variance of the profit: a positive parameter penalizes volatility (risk-averse); a negative parameter favors volatility (risk-seeking). The former is commonly found in the literature, that has historically relied on Utility Theory. The latter is in general much scarcer and we do not know of many papers that have studied this problem in risk-seeking situations, except for some earlier attempts on simpler newsvendor models with one decision variable (Choi, Li, and Yan, 2008). However, risk-seeking behavior may arise in situations in which an individual that has lost an important amount of money wants to recoup his losses in one lucky strike. In other words, humans are usually loss-averse (not risk-averse) and make decisions in term of losses, as explained by Prospect Theory (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992; Levy, 1992). An individual will show a risk-seeking behavior if he or she is in a state of loss or if the riskier option offers the possibility of eliminating loss (Scholer, Zou, Fujita, Stroessner, and Higgins, 2010).

A common feature of the classic newsvendor model is the use of a salvage value at which the excess of stock can be sold at the end of the period, This salvage value, s, is such that $p > c > s \geq 0$ and its effect can be included without loss of generality in the model presented above by simply redefining a new cost $\bar{c} = c - s > 1$ and a new price $\bar{p} = p - s$ (Choi and Ruszczyński, 2008).

The problem that consists of the maximization of (1.3) is, of course, the unconstrained version of the problem

$$\begin{aligned} \max_{p,x} \quad & p\mathbb{E}\left[min\{D(p,\epsilon), x\}\right] - cx \\ \text{s.t.} \quad & Var\left[(p \cdot min\{D(p,\epsilon), x\}\right] \lesseqgtr k, \end{aligned}$$

after defining the corresponding Lagrangian. The constrained problem is solved for a particular value of k. The unconstrained problem, parametric programing formulation, in turn, fixes the value λ^* of the Lagrange multiplier and finds the duple (x^*, p^*) for which (x^*, p^*, λ^*) is the optimal solution of the constrained problem for some k. The value of λ for which an instance of the problem is solved is clearly related to the order of magnitude of k. For example, usually the expected profit and the standard deviation of the profit have similar orders of magnitude ($\sim 10^m$). k has therefore an order of magnitude of $\sim 10^{2m}$ and λ must be of order $\sim 10^{-m}$ so the performance measure has an overall order of magnitude of $\sim 10^m$. After an initial selection of the value of the risk parameter, λ^*, it is very likely that the duple (x^*, p^*) yields a combination of the expected profit and the variance of the profit that does not fit exactly with our risk preferences. We will need to adjust the value of the risk parameter and for this reason it is very convenient to know, qualitatively, how the expected profit and the variance of the profit will change with λ. This relationship will be studied in 3.4 and 4.4.

Although the risk-sensitive performance measure defined in equation (1.3) lacks economic meaning, optimizing equation (1.3) must be understood as maximizing profit while minimizing variance. Furthermore, the parameter λ can be considered as a scaling factor that balances the expected profit and the variance of the profit. In addition, the expression above contains several assumptions. First, it assumes the procurement costs increase linearly with the quantity bought. Implicitly, this means that batch production is not more convenient economically than item-by-item production and the economies of scale do not apply. The introduction of the variability of the revenue weighted with the parameter λ is a different approach compared to that analyzed in Chen and Federgruen (2000). In this paper, the authors presented the attitude towards risk modeled by an utility function that was concave, convex and linear for risk-averse, risk-seeking and risk-neutral settings, respectively. It is also different from CVaR-based models as studied in Chen, Xu, and Zhang (2009).

Another important feature of this problem is the demand of the goods being considered. Often, the demand is assumed to take on values according to a given statistical distribution. In other cases, the stochasticity of the demand is only given by a random perturbation of a function. This function usually depends on a very reduced number of variables. It is common, for example, to consider that the demand can be represented as a function of the price and modified according to a random variable. Thus, we introduce the demand $D(p, \epsilon)$ in the same fashion as Young (1978):

$$D(p, \epsilon) \quad = \quad g(p)\epsilon + y(p), \tag{1.4}$$

where both $g(\cdot)$ and $y(\cdot)$ model nonincreasing functions of p. If $g(p) \equiv 1$ and $y(p) \equiv a - bp$ with $a, b > 0$, the demand is said to be in additive form. On the other hand, if $y(p) \equiv 0$ and $g(p) \equiv ap^{-b}$ with $a > 0, b > 1$, the demand is said to be in multiplicative form.

For every context, it remains an open question if a given commodity exhibits a change in its demand location, consequence of an additive demand model, or scale, consequence of a multiplicative demand model. The same product may have different behavior in different contexts. For instance, retailers that have in their geographical location its main competitive advantage usually see a change in the scale of the demand of their products (Agrawal and Seshadri, 2000). The additive demand has a clear advantage in its tractability and simplicity of the estimation of its parameters via simple linear regression. In turn, the multiplicative demand curve is particularly convenient because it assumes that the price elasticity of the demand remains constant at every price, unlike the linear demand function, that presents much more negative elasticities at very low prices (Li, Sethi, and Zhang, 2014). Moreover, when converted to a logarithmic scale it also allows an easy estimation of the parameters via regression techniques (Shi and Guo, 2012; Monahan, Petruzzi, and Zhao, 2004). The isoelastic demand is widely used for measuring supply and demand in agricultural products: AGRISIM (Agricultural Simulations Model) uses demand and supply models with constant elasticities to model multi-region multi-commodity flow of agricultural goods. This model was used for assessing European Union's agricultural policies and bilateral trade liberalization between EU-member states and non-member states in the Mediterranean basin within the frame of the MEDFROL project (Kavallari, Borresch, and Schmitz, 2006; Britz and Heckelei, 2008), that includes the analysis of commodities such as apples, rice, olive oil, wheat, and tomatoes among others. However, when it comes to using this demand model in optimization, one of its main drawbacks is its lack of tractability, especially when compared to the much simpler linear demand model.

1.4 A Tale of Two Elasticities

Two recurrent concepts that will be used throughout this thesis are the price-elasticity of demand and the lost sales rate (LSR) elasticity. We find it convenient to define both before proceeding with the optimization of the two price-dependent functions.

Definition 1.1. *Price elasticity of the demand. Given a price-dependent demand $D(p)$, the price elasticity of the demand is defined as*

$$e_p = \frac{dD}{dp}\frac{p}{D}. \tag{1.5}$$

This definition returns, for a given price p, the percentage change in the demand. As mentioned before, one of the main differences between the additive and the multiplicative demand functions is that the former has a price-dependent price elasticity whereas the latter has a constant price elasticity, and hence it is also known as isoleastic demand function (see Figure 1.3). Indeed, applying (1.5) to an additive demand function ($D(p) = a - bp$) yields $e_p = -b\frac{p}{D}$. Conversely, applying this concept to a multiplicative demand function ($D(p) = ap^{-b}$) yields $e_p = -b$. In other words, when the demand is multiplicative an increase of 1% in the price of the good will produce a decrease of b% in its demand, regardless the price of the product.

Definition 1.2. *(Kocabıyıkoğlu and Popescu, 2011)The lost sales rate (LSR) elasticity for*

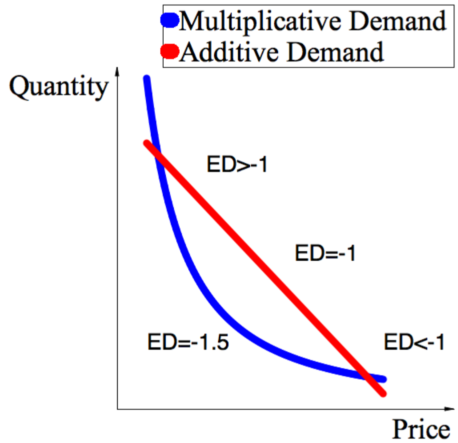


Figure 1.3: Price elasticity of the demand in additive and multiplicative demand functions

a given price p and inventory level x is defined as

$$\tilde{\kappa}(p,x) \quad = \quad \frac{p(G(p,x))^{'}_{p}}{1-G(p,x)}, \tag{1.6}$$

where $G(p,x) := Pr(D(p,\epsilon) \leq x)$ and $(G(p,x))^{'}_{p} \equiv \dfrac{\partial G(p,x)}{\partial p}$.

By definition, the level of service is to the LSR elasticity what the demand is to the price elasticity. That is, given that $Pr(D(p,\epsilon) \leq x)$ shows the probability of servicing the demand, the LSR elasticity tells us what is the change in this probability when we increase the price of our product, for a given stock quantity. This is analogous to the price elasticity of demand, which states the change in the demand when there is an increase in the price.

The LSR elasticity will be the metric in terms of which we will write sufficient conditions for the unimodality of the objective function. Even though it is very much dependent on the failure rate of ϵ (defined as $h(u) = \dfrac{f(u)}{1-F(u)}$), it provides a better managerial insight to our results, as it directly links the change in the level of service provided to the customer to the unimodality of the problem.

Chapter 2

Concavity with Additive Demand

The first chapter of this thesis focuses on an analysis to find the conditions under which the price-setting newsvendor problem with additive demand presents a concave mean-variance criterion.

Consider the performance measure introduced in §1, which we aim at maximizing:

$$\tilde{P}(p,x) \quad = \quad pE(min\{D(p,\epsilon),x\}) - cx - \lambda Var(p \cdot min\{D(p,\epsilon),x\}), \tag{2.1}$$

where p is the price set for the good and ϵ is a continuous random variable with expected value $E(\epsilon)$ and finite variance $Var(\epsilon)$. We will assume that this expected value can take on any value, although without loss of generality we could also assume that it is equal to 0 and rescale the demand accordingly. This random variable has support $[A, B]$, where $A < 0$ and $B > 0$, and has a probability distribution given by the continuous density function $f(\cdot)$

and the cumulative distribution function $F(\cdot)$. We assume that $F(\cdot)$ is twice differentiable everywhere on its domain. Furthermore, x is the inventory level set for the good, c is the production or procurement cost per unit of finished product, and $D(p, \epsilon)$ is the demand for a single period given the price p and the realization of the random variable ϵ. Finally, λ is a risk parameter greater than 0 for risk-averse cases, smaller than 0 for risk-seeking cases and equal to 0 in risk-neutral cases.

The demand function $D(\cdot)$ is in additive form. This function is attained by letting $g(p) = 1$ and $y(p) = a - bp$ in (1.4), whence we obtain

$$D(p, \epsilon) \quad = \quad a - bp + \epsilon. \tag{2.2}$$

The non-random term, $y(p) = a - bp$, is usually referred to as riskless demand (Mills, 1959). Furthermore, this additive model implies that pricing decisions do not affect the variability of the demand, as $Var[D(p, \epsilon)] = Var(\epsilon)$ (Petruzzi and Dada, 1999).

2.1 Risk-Averse Newsvendor

The decision maker seeks to maximize the performance measure (1.3). We set forth the following assumptions:

(A1) $p \in (c, p_{max}]$ where $p_{max} \leq \dfrac{a}{b}$, and $y(p) = 0,\ \forall p \notin (c, p_{max}]$,

(A2) $\dfrac{a + E[\epsilon]}{b} - p_{max} \leq p_{max} - c$,

(A3) $\lambda < \dfrac{1}{4(B - E(\epsilon))p_{max}}$,

(A4) $A + y(c) > 0$.

(A1) indicates that the price to be set has to be bigger than the unit cost of production and smaller than p_{max}. The latter will never be greater than the price at which the riskless demand for the product, $y(p)$, equals 0. When $y(\cdot)$ is a linear function, this value turns out to be $\frac{a}{b}$. (A2) imposes that p_{max} will always be at least as close to $\frac{a+E[\epsilon]}{b}$ as it is to c. The third assumption will become important in the subsequent analysis and the last assumption imposes that regardless of how small A is, the lowest price that can be set guarantees that the realization of the demand, $D(p,\epsilon)$ will still be positive.

Let us define $z = x - y(p)$. We can transform $\tilde{P}$ in (1.3) into a function of (p, z) as follows:

$$\begin{aligned}
\tilde{P}(p,x) &= pE(\min\{\epsilon, z\}) + py(p) - c(z + y(p)) - \lambda Var(p\min\{\epsilon, z\}) \\
&= p\mu(z) - \lambda p^2\sigma^2(z) + py(p) - c(z + y(p)) =: P(p,z),
\end{aligned}$$

where

$$\begin{aligned}
\mu(z) \quad &= E(min\{\epsilon, z\}) = \quad E(\epsilon) + \int_z^B (z-u)f(u)\,du, \quad z \in [A,B], \\
\sigma^2(z) \quad &= Var(min\{\epsilon, z\}) = \quad Var(\epsilon) + \int_z^B (z^2-u^2)f(u)\,du - \left[\int_z^B (z-u)f(u)\,du\right]^2 \\
&\qquad -2E(\epsilon)\int_z^B (z-u)f(u)\,du, \quad z \in [A,B].
\end{aligned}$$

As indicated in Petruzzi and Dada (1999), for a selected value of z we face shortages if $\epsilon > z$ and leftovers if $\epsilon < z$. A very immediate interpretation of z is that of a safety stock, for it is defined as the difference between the actual stock level and expected demand. Understanding the behavior of the functions above is crucial for the analysis that will be shown later. On the one hand, $\mu(\cdot)$ is always an increasing function of z in $[A, B]$, for

$\mu(A) = A < 0$, $\mu(B) = E(\epsilon)$ and $\frac{d\mu(z)}{dz} = 1 - F(z)$. On the other hand, $\sigma^2(\cdot)$ is a nonnegative, increasing function of z, with $\sigma^2(A) = 0$, $\sigma^2(B) = Var(\epsilon)$ and $\frac{d\sigma^2(z)}{dz} = 2[1 - F(z)][z - \mu(z)] \geq 0$. After further simplifications, the decision-maker's problem can be written as:

$$\max_{p,z} P(p, z) = \max_{p,z} \left(-p^2[\lambda\sigma^2(z) + b] + p[\mu(z) + a + cb] - c(z + a)\right). \tag{2.3}$$

2.1.1 Sequential Optimization

Sequential optimization seeks optimization of a function of several variables by sequentially selecting the optimal values of each variable that will, at the end, produce the maximum of the function that we need to maximize. Zabel (1970) proposes a method by which it is possible to find the optimal price that maximizes the performance measure for a given z. Then, this function can be expressed in terms of only one variable, z, and consequently optimized. However, finding the optimal price requires concavity of p with respect to z. To this end, we have

$$\frac{\partial P(p, z)}{\partial p} = -2p(\lambda\sigma^2(z) + b) + (\mu(z) + a + cb), \tag{2.4}$$

$$\frac{\partial^2 P(p, z)}{\partial p^2} = -2(\lambda\sigma^2(z) + b). \tag{2.5}$$

The risk setting, $\lambda > 0$, and the nonnegativity of b and $\sigma^2(\cdot)$ defines the performance measure $P(\cdot)$ such that $P(\cdot, z)$ is concave, since $\frac{\partial^2 P(p, z)}{\partial p^2} < 0$. Forcing (2.4) to be equal to 0 yields the price p that maximizes $P(\cdot, z)$ for a given z:

$$p^*(z) = \frac{\mu(z) + a + cb}{2[\lambda\sigma^2(z) + b]}. \tag{2.6}$$

Lemma 2.1. *The optimal price $p^{*(\cdot)}$ is uniquely determined by (2.6) for any $z \in [A, B]$ and it is an increasing function. Moreover, $p^*(z) \in (c, p_{max}]$, for any $z \in [A, B]$.*

Proof. The optimal price at $z = A$ is greater than the cost c, for $p^*(A) = \dfrac{A + a + cb}{2b} > c$. On the other hand, an upper bound of $p^*(\cdot)$ is given by

$$p^*(z) \leq \quad \frac{E(\epsilon) + a + cb}{2b} \quad \leq p_{max},$$

which is guaranteed by assumption (A2). It remains to prove that $p^*(\cdot)$ is increasing. Indeed:

$$\frac{dp^*(z)}{dz} \quad = \quad \frac{1 - F(z)}{2[\lambda\sigma^2(z) + b]}[1 - 4\lambda(z - \mu(z))p^*(z)]. \tag{2.7}$$

The expression above is always positive (i.e. $p^*(\cdot)$ is increasing) provided that $1 - 4\lambda(z - \mu(z))p^*(z) > 0$. However, we know that

$$1 - 4\lambda(z - \mu(z))p^*(z) \geq 1 - 4\lambda(B - E(\epsilon))p_{max},$$

whence we obtain the condition that for the expression above to be positive we need:

$$\lambda \quad < \quad \frac{1}{4(B - E(\epsilon))p_{max}},$$

which was assumed by (A3). Therefore, $p_{max} \geq p^*(z) > c, \ \forall z \in [A, B]$. □

Remark 2.1. The optimal price for a given z, $p^*(z)$, is smaller in the risk-averse case ($\lambda > 0$) than in the risk-neutral case ($\lambda = 0$). This conclusion is correct both mathematically and intuitively, for a risk-averse individual will set lower prices to make sure that sales are as high as possible, and endorses the results obtained by Agrawal and Seshadri (2000). Under the light of an additive demand, $Var[D(p,\epsilon)] = Var(\epsilon)$ (Petruzzi and Dada, 1999) and therefore changing the price will not affect the variance of the demand but will decrease the variance of the income, as shown in (1.3). The price in the risk-neutral case is in turn smaller than or equal to the optimal riskless price as observed in Petruzzi and Dada (1999). However, the results in Agrawal and Seshadri (2000) claim that the optimal risk-neutral price equals the optimal riskless price, whereas the model proposed in this chapter suggests that the optimal risk-neutral price is in between the optimal risk-averse price and the optimal riskless price. This difference occurs because the authors use an expected utility framework where they assume that the utility function is increasing and concave.

Definition 2.1. *The risk-sensitive performance measure under the best price function of the safety stock z is defined as*

$$\begin{aligned} P^*(z) := \quad P(p^*(z), z) \quad &= \frac{1}{4}\frac{(\mu(z)+a+cb)^2}{\lambda\sigma^2(z)+b} - c(z+a) \\ &= \frac{1}{2}p^*(z)(\mu(z)+a+cb) - c(z+a). \end{aligned} \tag{2.8}$$

Its first derivative with respect to z is given by

$$\frac{dP^*(z)}{dz} = p^*(z)(1-F(z))[1-2\lambda(z-\mu(z))p^*(z)] - c. \tag{2.9}$$

In order to write our first sufficient condition for the unimodality of the problem, we refer to the LSR elasticity as defined in equation (1.6):

$$\tilde{\kappa}(p,x) = \frac{p(G(p,x))'_p}{1-G(p,x)},$$

where $G(p,x) := Pr(D(p,\epsilon) \leq x)$ and $(G(p,x))'_p \equiv \dfrac{\partial G(p,x)}{\partial p}$.

Furthermore, for the additive case we know that

$$Pr(y(p) + \epsilon \leq x) = \quad Pr(\epsilon \leq x - y(p)) \quad = F(z).$$

Hence, with a direct application of Leibniz's Rule we can further simplify the expression above and write it in terms of z as shown below:

$$\tilde{\kappa}(p,x) = \quad \frac{p(G(p,x))'_p}{1 - G(p,x)} = \frac{pbf(z)}{1 - F(z)} \quad =: \xi(p,z).$$

Moreover, by means of *Lemma* 1, we can introduce the LSR elasticity at the optimal price as $\xi^*(z) := \xi(p^*(z), z)$.

Theorem 2.1. *Assume that*

$$\xi^*(z) := \quad \frac{bp^*(z)f(z)}{1 - F(z)} \quad \geq \frac{1}{2}.$$

Then, the single-period optimal stocking and pricing policy for the case of additive demand is to stock $x^* = y(p^*) + z^*$ units to sell at the unit price p^*, where p^* is specified by Lemma 1 *and z^* is the unique root of the equation*

$$p^*(z)(1 - F(z))[1 - 2\lambda(z - \mu(z))p^*(z)] - c \quad = \quad 0.$$

Proof. See Appendix A. □

Remark 2.2. The result shown by Theorem 2.1 matches that found in Kocabıyıkoğlu and Popescu (2011) for a risk-neutral individual ($\lambda = 0$).

Albeit we have analyzed the difference between the price $p^*(z)$ that maximizes the performance measure for a given value of z in risk-neutral and risk-averse environments, it remains to see what happens to $z^*(p)$, the value of z that maximizes this measure for a given price p. The first and second partial derivatives of $P(\cdot)$ with respect to z as well as the cross partial derivative yield

$$
\begin{aligned}
\frac{\partial P(p,z)}{\partial z} &= p(1-F(z))[1-2\lambda p(z-\mu(z))]-c, && (2.10)\\
\frac{\partial^2 P(p,z)}{\partial z^2} &= pf(z)[2\lambda p(z-\mu(z))-1]-2\lambda p^2 F(z)(1-F(z)), &&\\
\frac{\partial^2 P(p,z)}{\partial p \partial z} &= [1-F(z)][1-4\lambda p(z-\mu(z))]. && (2.11)
\end{aligned}
$$

A closer look to the formulae above reveals that in a risk-neutral setting $P(p,\cdot)$ is concave for a given price p and the maximum of the performance measure is obtained at $z = F^{-1}(1-\frac{c}{p})$, which is a well-known result. When $\lambda > 0$ this objective function is still concave, for (A3) guarantees that $2\lambda p(z-\mu(z))-1$ is negative. However, there is not a closed form that yields the optimum value z^*, which solves the equation

$$
(1-F(z))[1-2\lambda p(z-E(\epsilon)-\int_z^B (z-u)f(u)\,du)]-\frac{c}{p} \quad = \quad 0. \qquad (2.12)
$$

Let us fix $p \in (c, p_{max}]$. In the following lemma we examine the dependence of z^* on λ. This dependence is expressed by $\tilde{z}^*(\lambda)$ to indicate the value of z^* at a given point p for some λ.

Lemma 2.2. *The function $\tilde{z}^*(\cdot)$ is decreasing in λ.*

Proof. See Appendix A. □

Remark 2.3. The result above endorses that obtained under CVaR considerations in Chen, Xu, and Zhang (2009) and also that yielded by the use of the expected utility framework in Agrawal and Seshadri (2000). In that sense, the model proposed here provides certainty with respect to the behavior of the optimal order quantity in additive models of the form $a - bp + \epsilon$ in the face of risk-averse environments.

2.1.2 Simultaneous Optimization

Unlike we proceeded in §2.1.1, we focus now on giving conditions to jointly optimize price and quantity decisions simultaneously, thus guaranteeing that (2.3) has a unique solution.

Theorem 2.2. *If $\xi(p, z) \geq \frac{1}{2}$, then $P(\cdot)$ is jointly concave in p and z and the problem referenced by (2.3) has a unique price-quantity solution $(p^*, z^* + y(p^*))$.*

Proof. See Appendix A. □

Remark 2.4. The result shown by Theorem 2 also matches that found in Kocabıyıkoğlu and Popescu (2011) for a risk-neutral individual ($\lambda = 0$).

Notice that this condition for joint concavity is very restrictive as it requires the LSR elasticity to be greater or equal than $\frac{1}{2}$ in the whole domain of the function under study. This condition is sufficient to guarantee that (p^*, z^*) is a maximum, but it is not necessary. If the function was not jointly concave, the state of (p^*, z^*) as a critical point would not be altered, for the only necessary and sufficient condition for criticality is that the Hessian matrix is negative semidefinite at that precise point and this is guaranteed if $\xi^*(z) \geq \frac{1}{2}$, which is in turn a sufficient condition for the existence of a unique maximum in $P(\cdot)$ (Cachon and Netessine, 2006).

Example 2.1. In order to illustrate the ideas previously exposed, we proceed with several numerical examples using different distribution functions for the random variable ϵ. In particular, we present ϵ as a random variable uniformly distributed in $[A, B]$, and a random variable normally distributed with mean $\mu = 0$ and standard deviation $\sigma = 10$, and truncated below A and above B. For both cases, we use the following parameters to define the problem: $A = -10$, $B = 10$, $a = 35$, $b = 1$, $c = 10$, and $\lambda_{max} < \frac{1}{4Bp_{max}} \leq \frac{1}{1400}$. In addition, we include the case of a uniform distribution with expectation different from 0 with $A = -3$, $B = 40$, $a = 35$, $b = 1.5$, $c = 10$, and $\lambda_{max} < \frac{1}{4(B - E(\epsilon))p_{max}}$. Per Assumption (A2), $p_{max} \geq 22.5$ in the first two cases and $p_{max} \geq 23.83$ in the case of the uniform distribution with expectation different from 0. Table 2.1 contains, for these three distributions, the optimal values of p, z and $P(p, z)$ for each value of λ that was put to the test.

	$\lambda = 0$					$\lambda = 1/11200$				
Distribution	p^*	z^*	P^*	$E[P^*]$	$SD[P^*]$	p^*	z^*	P^*	$E[P^*]$	$SD[P^*]$
Trunc. $N(0, 100)$	21.49	0.60	106.04	106.04	70.23	21.45	0.50	105.60	106.03	69.34
Uniform $[-10, 10]$	21.04	0.66	101.77	101.77	74.51	21.21	0.54	101.28	101.76	73.34
Uniform $[-3, 40]$	21.25	19.76	129.46	129.46	157.73	21.13	19.34	127.95	129.42	153.45

	$\lambda = 1/5600$					$\lambda = 1/2800$				
Distribution	p^*	z^*	P^*	$E[P^*]$	$SD[P^*]$	p^*	z^*	P^*	$E[P^*]$	$SD[P^*]$
Trunc. $N(0, 100)$	21.41	0.41	105.18	106.02	68.46	21.33	0.23	104.36	105.96	66.78
Uniform $[-10, 10]$	21.31	0.42	100.81	101.74	72.19	21.36	0.19	99.91	101.66	70.00
Uniform $[-3, 40]$	21.02	18.93	126.52	129.30	149.39	20.83	18.17	123.88	128.90	141.86

	$\lambda = 1/1400$				
Distribution	p^*	z^*	P^*	$E[P^*]$	$SD[P^*]$
Trunc. $N(0, 100)$	21.19	-0.11	102.85	105.74	63.62
Uniform $[-10, 10]$	21.41	-0.24	98.26	101.37	65.97
Uniform $[-3, 40]$	20.49	16.84	119.32	127.61	128.93

Table 2.1: Optimum values of $P(p, z)$ as a function of λ.

With respect to the uniform distribution centered at 0, Figure 2.1 shows the optimal price $p^*(\cdot)$ and its first two derivatives for different values of λ that range from λ_{max}, representing the highest risk-averse newsvendor for which $p^*(\cdot)$ is increasing, to 0, representing a risk-neutral newsvendor. The impact of risk-aversion on the optimal price $p^*(\cdot)$ is not very significant, being of 2.32% when $z = 10$ but it is interesting to verify that increasing risk-aversion leads to smaller optimal prices and how this price is concave in $[A, B]$.

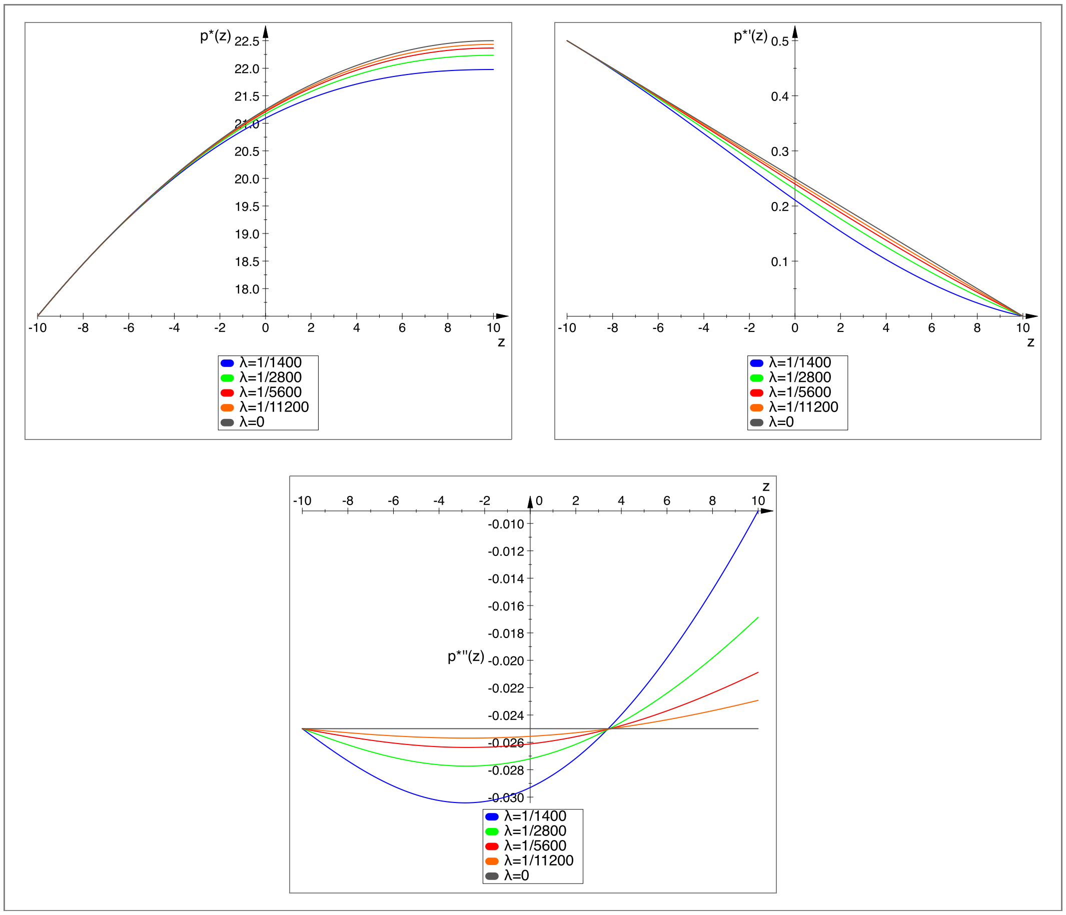


Figure 2.1: $p^*(\cdot)$ and its first two derivatives for a uniform distribution in $[-10, 10]$

Also, Figure 2.2 shows how the LSR elasticity ξ^* is always greater than $1/2$, which in turn guarantees that $P^*(\cdot)$ is concave. Note that since $p^*(\cdot)$ differs very little for different values of λ, the curves of $\xi^*(\cdot)$ almost overlap. Also, the optimal value of the objective function $P^*(\cdot)$ is inversely proportional to the value of λ and its optimum value changes only 3.45% between the most risk-averse situation and the risk neutral situation. It is worth mentioning that in this case $\xi(p, z) \geq 1/2$ and therefore the performance measure is jointly concave.

In the case of a truncated normal distribution, when comparing the most risk-averse

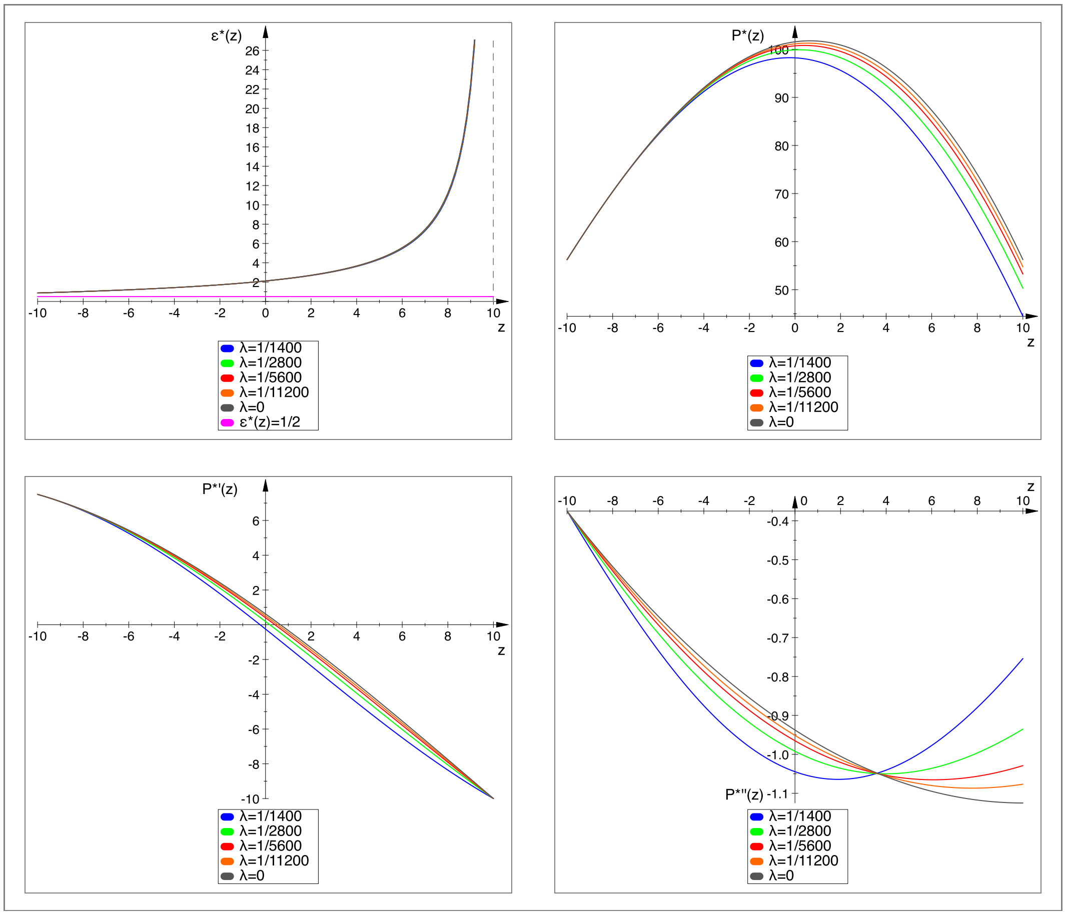


Figure 2.2: $\xi^*(\cdot)$, $P^*(\cdot)$ and its first two derivatives for a uniform distribution in $[-10, 10]$

situation with the risk-neutral case, the optimal price p^* reveals a difference of 2.44% at $z = 10$, which is very similar to that is found in the previous numerical example. Again, and endorsing the theoretical results, $p^*(\cdot)$ is concave and $\xi^*(z) \geq \frac{1}{2}$, which guarantees the existence of a unique optimum. Such optimum yields a gap of 3.02% between the most risk-averse case and the risk-neutral scenario. This difference is as well very similar to that is shown for the uniform distribution (3.02% vs. 3.45%). Unlike the uniform case, this distribution does not provide $P(\cdot)$ with joint concavity, as there are some values of $\xi(p, z)$ under $\frac{1}{2}$.

Finally, the uniform distribution with mean different from 0 yields a bigger gap of the optimal price at the right extreme of the interval, with this price at the risk-neutral case being 5.12% greater than in the most-risk averse case for $z = B = 40$. The optimal lost sales rate elasticity remains greater than 1/2 at all times, which produces a concave curve for $P^*(\cdot)$ that has a more significant variation between the risk neutral case and the most risk-averse situation (7.83%). Like in the case of the truncated normal distribution, $\xi(p, z)$ is smaller than 1/2 in some regions, which leads to a surface that is not jointly concave.

We also calculate the values of $\tilde{z}^*(\lambda)$ for different λ and a given price $p = 20$, using the same parameter values specified at the beginning of this section. The results can be found on Table 2.2 and show, as claimed by Lemma 2.2, that the risk-aversion is inversely proportional to the optimum value of $z^*(20)$.

	$\lambda = 0$				$\lambda = 1/11200$			
Distribution	z^*	P^*	$E[P^*]$	$SD[P^*]$	z^*	P^*	$E[P^*]$	$SD[P^*]$
Trunc. $N(0, 100)$	0.00	104.01	104.01	60.89	-0.07	103.69	104.01	60.36
Uniform $[-10, 10]$	0.00	100.00	100.00	64.55	-0.09	99.63	100.00	63.87
Uniform $[-3, 40]$	18.5	127.50	127.50	138.78	18.22	126.32	127.48	136.61

	$\lambda = 1/5600$				$\lambda = 1/2800$			
Distribution	z^*	P^*	$E[P^*]$	$SD[P^*]$	z^*	P^*	$E[P^*]$	$SD[P^*]$
Trunc. $N(0, 100)$	-0.14	103.36	104.00	59.83	-0.27	102.73	103.97	58.84
Uniform $[-10, 10]$	-0.18	99.27	99.98	63.15	-0.34	98.57	99.94	61.91
Uniform $[-3, 40]$	17.94	125.18	127.43	134.43	17.41	122.99	127.22	130.30

	$\lambda = 1/1400$			
Distribution	z^*	P^*	$E[P^*]$	$SD[P^*]$
Trunc. $N(0, 100)$	-0.53	101.54	103.85	56.87
Uniform $[-10, 10]$	-0.66	97.26	99.78	59.40
Uniform $[-3, 40]$	16.44	119.01	126.51	122.70

Table 2.2: Behavior of $z^*(20)$ as a function of λ.

In order to show the effect that risk aversion has on the profit, Table 2.1 and Table 2.2 also include the expected profit and the standard deviation of the profit for each case considered. These can be calculated directly from (1.3). Intuitively, the expected profit and the standard deviation of the profit should decrease as the level of risk aversion increases (we will prove this mathematically in the more general case presented in §3). Indeed, this is the case in all the different scenarios presented.

2.2 Risk-Seeking Newsvendor

Whereas the expressions of all the partial derivatives previously obtained are still valid for risk-seeking situations, the conditions under which the performance measure is concave are greatly modified by the fact that $\lambda < 0$. We turn now our attention to this case in which, again, we are seeking the maximization of

$$P(p,z) \quad = \quad p\mu(z) - \lambda p^2\sigma^2(z) + py(p) - c(z + y(p)).$$

Furthermore, we set forth the following assumptions, some of which were explained formerly (we refer the reader to §2.1 for the justification of the first three of them). The fourth assumption will become relevant in the next subsection. Note that in this case we have not included any assumption on the value of λ; this condition will be addressed as part of the analysis that will follow.

(B1) $p \in (c, p_{max}]$ where $p_{max} \leq \dfrac{a}{b}$, and $y(p) = 0 \; \forall p \notin (c, p_{max}]$,

(B2) $\dfrac{a + E(\epsilon)}{b} - p_{max} \leq p_{max} - c$,

(B3) $A + y(c) > 0$,

(B4) $E(\epsilon) < y(c)$.

2.2.1 Sequential Optimization

Lemma 2.3. *The risk-sensitive performance measure $P(\cdot, z)$ is concave for each z, if $\lambda \geq \dfrac{-b}{Var(\epsilon)}$.*

Proof. We proceed as in the risk-averse case. However, the inequality $\dfrac{\partial^2(p,z)}{\partial p^2} = -2(\lambda\sigma^2(z)+b) < 0$ does not hold for every value of λ. Therefore, setting $\lambda \geq \max_z \dfrac{-b}{\sigma^2(z)}$ solves this problem and, given that $\sigma^2(\cdot)$ is an increasing function, it is enough to set $\lambda \geq \dfrac{-b}{\sigma^2(B)} = \dfrac{-b}{Var(\epsilon)}$. □

The result shown by Lemma 2.3 may not respect (B1). In fact, as shown analytically in (2.6), for $\lambda = \dfrac{-b}{Var(\epsilon)}$ the price $p^*(z)$ goes to infinity and violates the assumption that this quantity cannot be greater than p_{max}. For that reason, we need to further restrict the possible selection of values for λ.

Lemma 2.4. *For a fixed z and a value of λ in the interval $\left[\dfrac{b(E(\epsilon)-y(c))}{2aVar(\epsilon)}, 0\right)$ the optimal price is determined uniquely as a function of z as shown by (2.6) and its value is always contained in $(c, p_{max}]$.*

Proof. See Appendix A. □

Remark 2.5. In the risk-seeking case ($\lambda < 0$), the optimal price for a given z, $p^*(z)$, is greater than that is found for the risk-neutral case ($\lambda = 0$). This conclusion is correct both mathematically and intuitively, for a risk-seeking individual will set higher prices in his pursuit of greater profit, accepting the risk of selling less as a result of such decision.

Once the conditions for $P(\cdot, z)$ to be concave is established, analogous to what was shown for the risk-averse case, we address the concavity of $P^*(\cdot)$.

Theorem 2.3. *Assume that for every $z \in [A, B]$*

$$\xi^*(z) \quad \geq \quad b\frac{dp^*(z)}{dz} - \frac{2\lambda p^*(z) b \left[F(z)p^*(z) + (z-\mu(z))\dfrac{dp^*(z)}{dz}\right]}{1 - 2\lambda(z-\mu(z))p^*(z)}. \tag{2.13}$$

Then the single-period optimal stocking and pricing policy for the case of additive demand is *to stock $x^* = y(p^*) + z^*$ units to sell at the unit price p^*, where p^* is specified by Lemma 4 and z^* is the unique root of equation (2.10).*

Proof. See Appendix A. $\square$

Obviously this condition for concavity requires evaluating the function for any $z \in [A, B]$. There exist, however, more expeditive approaches to rule out concavity: given any distribution, equation (2.13) shows that $\xi^*(A) \geq \frac{1}{2}$ and $\xi^*(B) \geq \frac{-2\lambda b {p^*}^2(B)}{1 - 2(B - E(\epsilon))p^*(B)\lambda}$. This means that if we find the optimal LSR elasticity values at the extreme of the intervals to be less than these quantities, $P^*(\cdot)$ will not be concave. Also, by the definition of the LSR elasticity in the case of additive demand, if the chosen distribution has an increasing failure rate, so does its optimal elasticity. This means that if $\xi^*(z) < \frac{1}{2}$ for any given z, then its value at $z = A$ will not be greater or equal to $\frac{1}{2}$ and therefore $P^*(\cdot)$ will not be concave.

In an effort to come up with a friendlier condition for the concavity of $P^*(\cdot)$, we can estimate an upper bound of the right hand side of (2.13) and set

$$\xi^*(z) \quad \geq \quad bK - 2\lambda a \left(\frac{a}{b} + (B - E(\epsilon))K \right), \tag{2.14}$$

with K being the maximum value that $\frac{dp^*(z)}{dz}$ attains in $[A, B]$. A priori, if the behavior of the function $z \mapsto \frac{dp^*(z)}{dz}$ is unknown, this maximum value can be large, with a rough upper bound given by $\frac{1 - 4\lambda(B - E(\epsilon))\frac{a}{b}}{2(\lambda Var(\epsilon) + b)}$, and therefore this expression might not be effective. If, like it was the case in risk-averse situations, this function is decreasing (i.e., $p^*(\cdot)$ is concave), then it attains a maximum value of $\frac{1}{2b}$ at $z = A$ and the bound above becomes

$$\xi^*(z) \quad \geq \quad \frac{1}{2} - \lambda\frac{a}{b}(2a + B - E(\epsilon)).$$

This is potentially a very useful result, but requires the concavity of $p^*(\cdot)$. In what follows, we show the conditions that are needed for such a case to take place.

Lemma 2.5. *If*

$$\lambda \quad \geq \quad \max_{z\in[A,B]} \left\{ \frac{-f(z)b}{4(a + B - E(\epsilon)) + f(z)Var(\epsilon)} \right\},$$

then the function $z \mapsto \frac{dp^*(z)}{dz}$ *is decreasing.*

Proof. See Appendix A. □

The previous result allows us to state the following theorem.

Theorem 2.4. *Assume that*

$$\lambda \quad \geq \quad \max_{z} \left\{ \frac{-f(z)b}{4(a + B - E(\epsilon)) + f(z)Var(\epsilon)} \right\}, \quad z \in [A, B],$$

and

$$\xi^*(z) \quad \geq \quad \frac{1}{2} - \lambda\frac{a}{b}[2a + B - E(\epsilon)].$$

Then the single-period optimal stocking and pricing policy for the case of additive demand is to stock $x^* = y(p^*) + z^*$ *units to sell at the unit price* p^**, where* p^* *is specified by Lemma* 4 *and* z^* *is the unique root of equation (2.10).*

Proof. Since λ is contained in a range that guarantees that the function $z \mapsto \dfrac{dp^*(z)}{dz}$ is decreasing, then we have that $K = \left.\dfrac{dp^*(z)}{dz}\right|_{z=A} = \dfrac{1}{2b}$ and therefore (2.14), which establishes a condition for the concavity of $P^*(\cdot)$, can be written as

$$\xi^*(z) \quad \geq \quad \frac{1}{2} - \lambda\frac{a}{b}[2a + B - E(\epsilon)].$$

By virtue of (A.2) and (A.3) there exists a point $z^* \in (A, B)$ at which the function $P^*(\cdot)$ attains a maximum, and such a point is uniquely determined by the root of (2.10). □

Remark 2.6. Given an appropriate range of values for λ, the LSR elasticity that is required in risk-seeking cases is greater than that is required in risk-averse situations.

Lemma 2.6. *Define*

$$\hat{\varepsilon}^*(p) \quad = \quad \xi(p, z^*(p)),$$

and let p_A and p_B be prices in the interval $(c, p_{max}]$ which yield $z^*(p_A) = A$ and $z^*(p_B) = B$, respectively, and that may or may not exist. Then, if

$$\hat{\varepsilon}^*(p) > bp\frac{1}{B - E(\epsilon)}, \quad c < p \leq p_{max},$$

the optimal safety stock $\tilde{z}^*(\cdot)$ is a decreasing function of λ, i.e., given a price p, the optimal safety stock increases as we face more risk-seeking situations.

Proof. See Appendix A. □

Example 2.2. We proceed now to present some examples for the risk-seeking case. As the reader may have noticed, this case is more intrincate than the risk-averse case, with more conditions to be satisfied for obtaining a desirable behavior of the performance measure. Our experience has been that selecting λ according to Theorem 2.3 induces a more significant impact on the optimal value of the objective function. These examples gave way to a larger range of values for the risk parameter, as we only require that $\lambda \in \left[\frac{b(E(\epsilon)-y(c))}{2aVar(\epsilon)}, 0\right)$. As far as the optimal elasticity, $\xi^*(z)$ is concerned, Theorem 2.3 provides a more complex expression than Theorem 2.4 that needs to be evaluated at any $z \in [A, B]$. Despite this hindrance, it was not particularly difficult to find instances where this condition was satisfied and allowed us to get significant results.

We present the case that ϵ is uniformly distributed in the range $[-30, 200]$ with demand given by $D(p, \epsilon) = 600 - 60p + \epsilon$. The purchase cost of the item we sell is $c = 7$. It can be seen in Figure 2.3 how optimal price $p^*(\cdot)$ may not be concave for some values of λ but still is an increasing function of z and its value is held between $c = 7$ and $p_{max} = a/b = 10$ as expected whenever λ is greater than or equal to $\frac{b(E(\epsilon)-y(c))}{2aVar(\epsilon)}$. The risk-neutral case yields as well an optimal price which is 9.33% smaller than that of the most risk-seeking setting. Figure 2.4 shows for the most risk-seeking case that $\xi^*(\cdot)$ is always above the necessary and sufficient condition expressed by (2.13), thus conferring concavity to the performance measure for this value of λ. For clarity, we omit similar curves for other tested scenarios but, as the fourth chart in this figure shows, all cases satisfy (2.13). In this example, there is an important gap in the optimal value of the objective function between a risk-neutral scenario and the most risk seeking case, with the latter being 29.31% smaller than the former.

The last example introduces a normal distribution with mean 25 and variance 1600 truncated below by -30 and above by 100. The demand follows the expression $D(p, \epsilon) = 175 - 35p + \epsilon$ where ϵ represents the random variable with the said distribution. The purchase cost of the item that we sell is now $c = 2.7$. These parameters produce a noteworthy effect

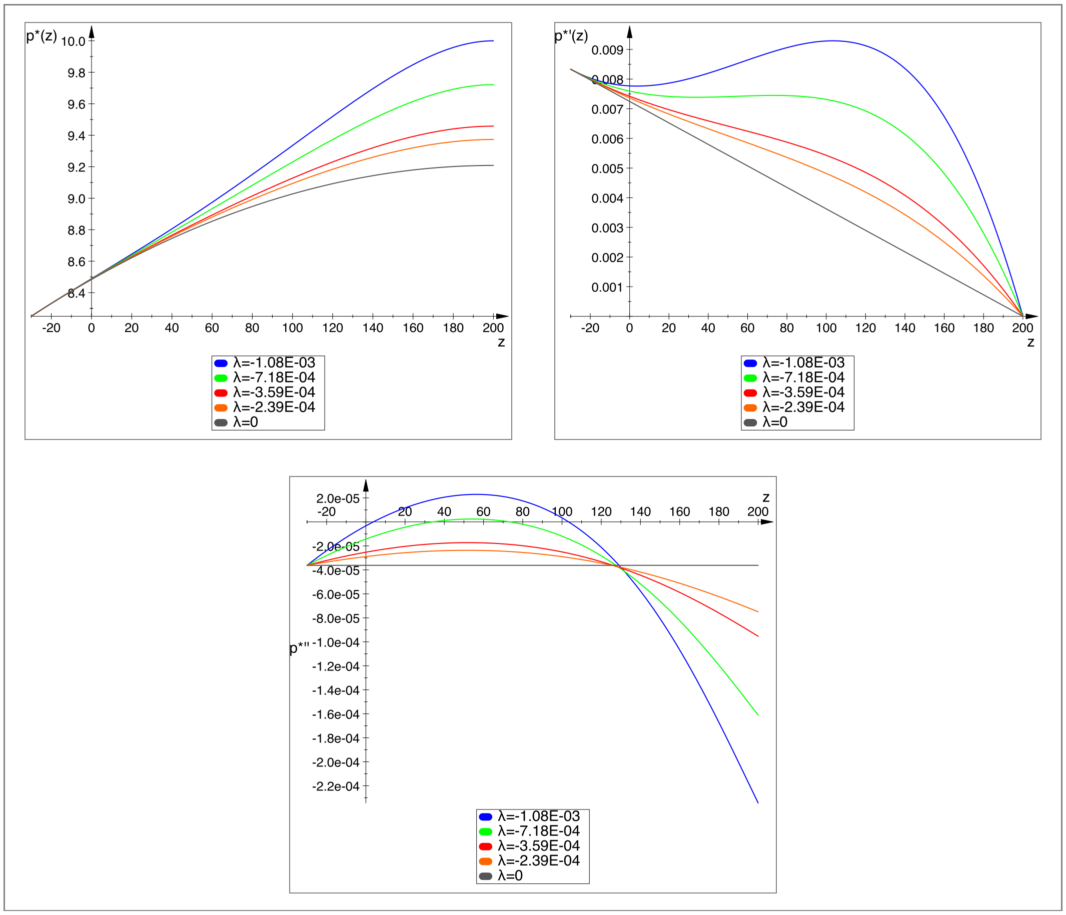


Figure 2.3: $p^*(z)$ and its first two derivatives for a uniform distribution in $[-30, 200]$

on the objective function that is even more remarkable than that in the last example: the risk-neutral case yields an optimum value which is 77.80% lower than that of the most risk-seeking scenario. Likewise, the optimal price, is 17.81% lower when $\lambda = 0$ than when it is at its minimum value. Albeit $p^*(\cdot)$ is not concave in this case either, the condition for the concavity of $P^*(\cdot)$, as established in Theorem 2.3, holds for all λ considered. Table 2.3, in turn, displays the optimal pricing and safety stock decisions along with the optimum value of the objective function, the expected profit, and the standard deviation of the profit that they produce for the different values of λ that were attempted. Note that in both cases the impact of risk-seekingness in the nature of the profit as a stochastic variable is remarkable, to the point that, when $\lambda = \lambda_{min}$, the decision-maker faces scenarios where he or she must expect a loss in exchange for a much greater variability in the profit.

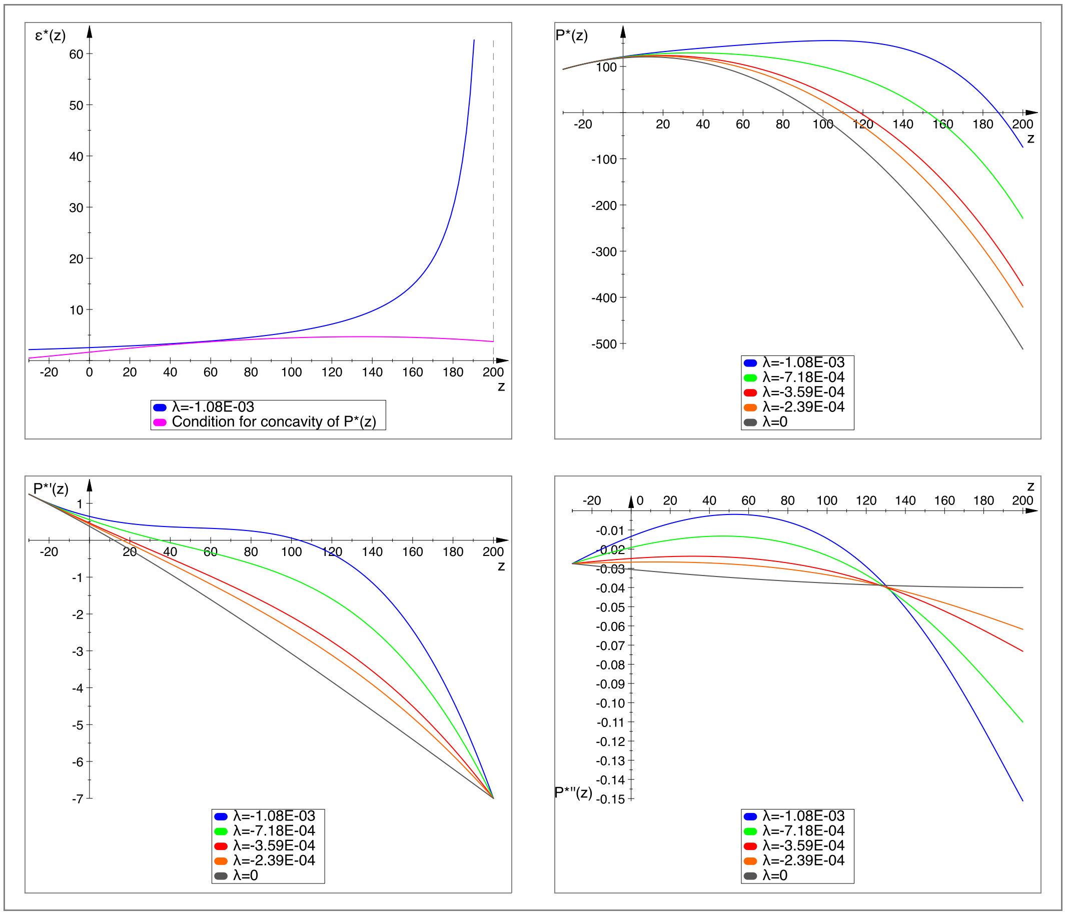


Figure 2.4: Test of $\xi^*(z)$, $P^*(z)$ and its first two derivatives for a uniform distribution in $[-30, 200]$

	$\lambda = 0$					$\lambda = \lambda_{min}/4.5$				
Distribution	p^*	z^*	P^*	$E[P^*]$	$SD[P^*]$	p^*	z^*	P^*	$E[P^*]$	$SD[P^*]$
Trunc. $N(25, 1600)$	3.93	10.75	37.80	37.80	40.45	3.99	15.84	40.13	37.32	48.66
Uniform $[-30, 200]$	8.57	12.11	120.68	120.68	82.79	8.60	16.36	122.57	120.39	95.23

	$\lambda = \lambda_{min}/3$					$\lambda = \lambda_{min}/1.5$				
Distribution	p^*	z^*	P^*	$E[P^*]$	$SD[P^*]$	p^*	z^*	P^*	$E[P^*]$	$SD[P^*]$
Trunc. $N(25, 1600)$	4.04	19.23	41.70	36.44	54.37	4.27	34.38	49.70	26.01	81.43
Uniform $[-30, 200]$	8.62	19.20	123.75	119.88	103.80	8.75	35.01	129.38	112.13	154.97

	$\lambda = \lambda_{min}$				
Distribution	p^*	z^*	P^*	$E[P^*]$	$SD[P^*]$
Trunc. $N(25, 1600)$	4.63	52.40	67.21	-3.48	115.05
Uniform $[-30, 200]$	9.37	103.74	156.05	-28.85	414.25

Table 2.3: Optimum values of $P(p, z)$ as a function of λ.

2.3 Conclusions

The approach that our model intends to give to the newsvendor problem aims at serving as a comparison with previously used models based on CVaR measures and the expected utility framework. In this chapter, as well as in the rest of this thesis, we introduce a simple yet powerful variation of the single-period, price-dependent demand newsvendor problem with two decision variables (namely, price and stock quantity) by including the variability of the demand scaled by the attitude towards risk that the seller has. Such attitude can be risk-averse or risk-seeking. The latter is much scarcer in the literature and can be taken as a starting point for future research efforts. It also presents more difficult situations under this model given the complexity of the conditions that have to hold for the performance measure to behave appropriately. However, we show that when those conditions apply, the impact of a risk-seeking newsvendor on the objective function can be remarkable.

We present results that back those found for risk-averse situations in other works with different models, plus we add conclusions for risk-seeking cases along with other findings that, despite of being intuitive, need mathematical endorsement. For instance, it was shown that the optimal price for a given safety stock z, $p^*(z)$, is smaller in risk-averse cases than in risk-neutral cases. Conversely, in risk-seeking cases, this price is greater. In both scenarios, however, it is an increasing function of z. This price is also concave in z for the risk-averse case, whereas such concavity is guaranteed in a smaller range of λ for the risk-seeking case.

Furthermore, it was found that the optimal safety stock as a function of the price, $z^*(\cdot)$, always decreases as we face an increase in risk-aversion. Intuitively, one might think that the opposite would happen in risk-seeking cases (i.e., $\tilde{z}^*(\cdot)$ always increases as we turn to be more risk-seeking). However, this result is true only provided that $\hat{\varepsilon}^*(p) \geq bp\frac{1}{B - E(\epsilon)}$.

Finally, we comment on the values of λ. The restrictions that our models set on the range of values that this parameter may take have to be considered as a scaling measure of risk if we want to preserve the concavity of the performance measure. Outside of the proposed ranges for λ, the concavity of the objective function is not guaranteed, although it might occur. In particular, if we are concerned about a risk-averse environment, it makes sense to set the most averse case to the maximum value that λ can take (i.e., $\frac{1}{4p_{max}(B - E(\epsilon))}$) and scale our risk situations according to the range $\left[0, \frac{1}{4p_{max}(B - E(\epsilon))}\right)$. Similarly, we can identify the most favorable risk-seeking case according to the lower bounds described on Theorem 2.3 or Theorem 2.4, whatever suits us best, and evaluate different risk measures according to this scale. Having such bounds on the risk parameter is reasonable. It is seldom that the results hold for the entire range of λ . Even in models with the exponential utility function (in which the decision maker is equipped with a constant risk coefficient) or other measures of risk, the degree of risk aversion is somewhat bounded to obtain certain results. On the other hand, in real life problems, λ is usually small, or close to zero.

Chapter 3

Unimodality with Additive Demand

The scope of the first chapter was to analyze the conditions for the concavity of a performance measure that uses a mean-variance criterion in the price-setting newsvendor problem with additive demand. Concavity, however, can be a restrictive feature when it comes to finding the global maximum of a function. A much ampler approach instead is that of considering the unimodality of the function under study. The next two chapters approach this problem in a similar way as done in §2, but with a focus on examining the conditions that guarantee not the concavity, but the unimodality of the problem when the demand is additive and multiplicative. It is thus a more comprehensive approach than that presented in the previous chapter. Moreover, we aim at analyzing these conditions for two of the most used price-dependent demand forms, as derived from (1.4): the additive demand and the multiplicative demand.

3.1 Problem Formulation

Consider a retailer that aims at maximizing her expected profit while keeping the variance of the profit under control. This retailer sells a good over a single period. This product may or may not be perishable. In the latter case, she may sell back the excess of stock at a salvage value. Without loss of generality, we assume that the good is perishable and does not have a salvage value. If there exists a salvage value, it can be incorporated by just a change of variables (Choi and Ruszczyński, 2008). In any case, the decision maker decides how much product to buy from the wholesaler at a given cost and sets a price that this good will sell for. Since the demand is uncertain, so will be the profit, but she is interested in setting both price and stock quantity in a way that satisfies her sensitivity to risk. This sensitivity is modeled according to the following performance measure:

$$\tilde{P}(p,x) = \underbrace{p\mathbb{E}(min\{D(p,\epsilon),x\}) - cx}_{Expected\ profit} - \lambda \underbrace{Var(p \cdot min\{D(p,\epsilon),x\})}_{Variance\ of\ the\ profit},$$

where the variables p and x are the retailer's price set for the good and the stock quantity, respectively. The replenishment cost cx is given by a constant cost of c monetary units per unit of product. We assume that the unit cost is constant and does not depend on the replenished quantity. The demand is random and price-dependent and has additive form (Petruzzi and Dada, 1999):

$$D(p,\epsilon) = a - bp + \epsilon,$$

where $a, b > 0$ (the demand is downward sloping with respect to the price) and ϵ is a continuous random variable with finite variance $Var(\epsilon)$. The term $y(p) = a - bp$ is usually referred to as the riskless demand. In what follows, we will assume that $\mathbb{E}(\epsilon) = 0$. This assumption can be made without loss of generality because if the expected value of ϵ is not 0, its value can be absorbed by a. Moreover, this random variable has convex and compact support $[A, B]$ ($A < 0$, $B > 0$), density function $f(\cdot)$ and a twice differentiable cumulative distribution function $F(\cdot)$ with a continuous second derivative. The range of prices that the retailer will consider is $[c, p_{max}]$: obviously, one will not retail a product at a lower price than its wholesale price; on the other hand, the upper bound is given by the maximum price at which the worst possible realization of the demand is nonnegative:

$$p_{max} = \max_{\{p:y(p)+A\geq 0\}} p = \frac{A+a}{b}.$$

On the other hand, for each price selected the stock quantity x will not be smaller than $y(p) + A$ (the minimum demand attainable at price p) and will not be larger than $y(p) + B$ (the maximum demand attainable at price p). In our model, λ is a risk parameter that decreases the value of the performance measure in risk-averse cases ($\lambda > 0$) and increases its value in risk-seeking cases ($\lambda < 0$).

In order to simplify the derivations we will redefine the objective function in terms of the safety stock $z = x - y(p)$, that is, the difference between the replenished quantity and the expected (or riskless) demand at a price p (Petruzzi and Dada, 1999; Thowsen, 1975). This new variable is contained in the interval $[A, B]$. After this change of variables and some algebraic calculations, we introduce our new performance measure:

$$P(p, z) = p\left(\mu(z) + y(p)\right) - c\left(z + y(p)\right) - \lambda p^2 \sigma^2(z), \tag{3.1}$$

where

$$\begin{aligned}
\mu(z) = \quad & \mathbb{E}\left(min\{\epsilon, z\}\right) && = \int_z^B (z-u)f(u)\,du, \quad z \in [A,B],\\
\sigma^2(z) = \quad & Var\left(min\{\epsilon, z\}\right) && = Var\left(\epsilon\right) + \int_z^B (z^2-u^2)f(u)\,du\\
& && - \left[\int_z^B (z-u)f(u)\,du\right]^2, \quad z \in [A,B].
\end{aligned}$$

These two functions of z and their characteristics will play a key part in the development of the conditions that will follow. The function $\mu(\cdot)$ is an increasing ($\mu'(z) = 1-F(z)$), concave ($\mu''(z) = -f(z)$) function between A and 0. Moreover, the function $\sigma^2(\cdot)$ is an increasing function ($\sigma^{2'}(z) = 2\left(z-\mu(z)\right)\left(1-F(z)\right)$) between 0 and $Var\left(\epsilon\right)$. The proceeding sections and subsections are dedicated to finding the conditions that guarantee that the problem

$$\max_{\substack{p\in[c,p_{max}]\\ z\in[A,B]}} P(p,z), \tag{3.2}$$

has a unique solution. More specifically, we will look at the conditions for the quasiconcavity (i.e. unimodality) of $P(\cdot,\cdot)$. These conditions will be found by means of sequential optimization (Zabel, 1970) and therefore we will follow the steps below:

1. Select a safety stock and find the price that maximizes $P(\cdot,\cdot)$ for that value of z, $p^*(z)$.

2. Substitute this closed-form expression of the optimal price in the objective function in order to come up with a function of only one variable, $P(p^*(z),z) =: P^*(z)$.

3. Find the safety stock z^* that maximizes $P^*(\cdot)$.

In this sequence of steps, we will proceed by finding the situations in which this maximizer

$(p^*(z^*), z^*)$ is unique. If such conditions do not hold, we also propose shortcuts to finding the optimal solution of (3.2).

3.2 Risk-averse Newsvendor

In order to analyze the risk-averse newsvendor, we assume that $y(c) + 2A \geq 0$. This is a mild assumption that forces the riskless demand at face-value c to be, in the worst-case scenario, at least as much as $-2A$. In general, perturbations or errors around the expected demand at a given price should not be excessively large and therefore we do not consider this to be a strong condition. The purpose of this assumption is to bound the optimal price from above, as explained in the proof of Lemma 3.1, which simplifies greatly the shape of the optimal price function and makes the optimization of $P(\cdot, \cdot)$ more accessible.

3.2.1 Characteristics of the Optimal Price

As introduced at the end of §3.1, the first step in our optimization process is to fix a safety stock factor and find the price that maximizes the performance measure. For any $z \in [A, B]$, solving the first-order optimality condition of (3.1) as a function of p yields a closed form for the optimal price:

$$\frac{\partial P}{\partial p} = 0 \quad \Longrightarrow p^*(z) \quad = \frac{\mu(z) + a + cb}{2(\lambda\sigma^2(z) + b)}. \tag{3.3}$$

This critical point is a maximizer because $\partial^2 P/\partial p^2 = -2(\lambda\sigma^2(z) + b) < 0$ (i.e. $P(\cdot, z)$ is concave). Also, clearly, $p^*(z) \leq p^*(z)|_{\lambda=0} = \dfrac{\mu(z) + a + cb}{2b}$ and therefore given a safety stock z the optimal price decreases with the level of risk-aversion. It is of great importance to know whether this optimal price is hedged by the interval $[c, p_{max}]$. To that end, the upcoming lemmas and results are intended to shed some light on the shape of this function $p^* : [A, B] \to \mathbb{R}$, which is found to be any of the two shown in Figure 3.1.

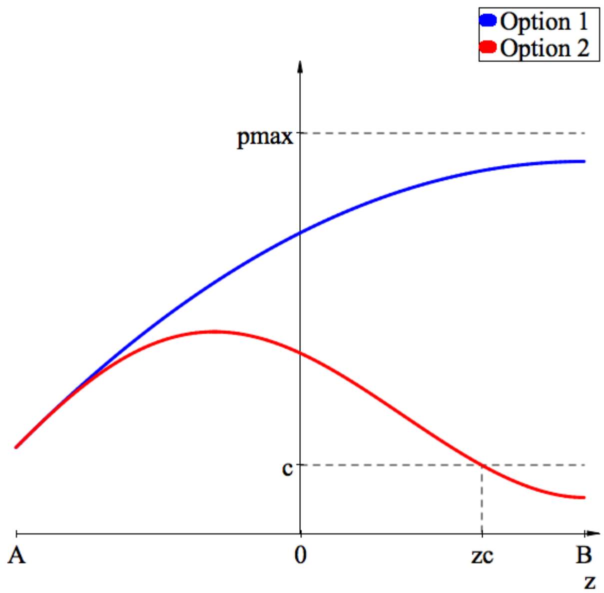


Figure 3.1: Typical optimal price functions in risk-averse cases.

Lemma 3.1. *The optimal price $p^*(\cdot)$ is a strictly positive function in $[A, B]$ and $p^*(z) \leq p_{max}$.*

Proof. See Appendix B. □

Lemma 3.2. *The optimal price $p^*(\cdot)$ is either a nondecreasing or a unimodal function of z.*

Proof. See Appendix B. □

Remark 3.1. Per (B.3), if the optimal price $p^*(\cdot)$ is increasing in a subinterval of $[A, B]$, then it is also concave in that subinterval.

In view of the lemmas above, we can guarantee that the optimal price is not greater than p_{max} but we cannot guarantee that it is not smaller than c. This hindrance resolved in §2 by assuming that λ is bounded above by $\dfrac{1}{4Bp_{max}}$, which is the minimum value of the right-hand side of (B.2). This assumption is enough to guarantee that $p^*(\cdot)$ is an increasing function of z which, along with the fact that $p^*(A) > c$, is sufficient to conclude that the optimal price is always greater than the replenishment cost. In this case, however, we do not bound the value of the risk parameter and therefore it is possible that the optimal price falls under c. Since we are only concerned about prices in $[c, p_{max}]$, we define the hedged optimal price function π^* as the following piecewise function:

$$\pi^*(z) = \begin{cases} c, & \text{if } p^*(z) < c, \\ \\ p^*(z), & \text{if } c \leq p^*(z) \leq p_{max}. \end{cases} \tag{3.4}$$

Clearly this function intends to bound the optimal price within the allowed range of prices in those cases where the risk parameter λ is such that the optimal price eventually falls under the replenishment cost. The performance measure $P(\cdot, \cdot)$ is a concave function with respect to p and this means that $\pi^*(z) = c$ maximizes $P(\cdot, \cdot)$ within the allowed interval $[c, p_{max}]$ whenever $p^*(z) < c$. In general we will use this function to further optimize the performance measure $P(\pi^*(z), z) = P^*(z)$ with respect to z. Nevertheless, there exists a range of nonnegative values for the risk parameter in which $\pi^*(z) = p^*(z),\ \forall z \in [A, B]$. This is shown in the next lemma.

Lemma 3.3. *The optimal price $p^*(z)$ is in $[c, p_{max}]$, $\forall z \in [A, B]$ if and only if $\lambda \in \left[0, \dfrac{y(c)}{2cVar(\epsilon)}\right]$.*

Proof. See Appendix B. □

3.2.2 Optimization of $P^*(\cdot)$

The next step in our optimization procedure is to redefine the objective function as a function of only the stock factor z. This is achieved by substituting p for the hedged optimal price function. Let us define the following functions of z:

$$\begin{aligned} P_1^*(z) = \quad & P(c, z) & = -c^2(\lambda\sigma^2(z) + b) + c(\mu(z) + cb - z), \\ P_2^*(z) = \quad & P(p^*(z), z) & = \frac{1}{2}p^*(z)(\mu(z) + a + cb) - c(z + a). \end{aligned}$$

The performance measure at the hedged optimal price $\pi^*(z)$ can be expressed in terms of these two functions above as a piecewise nonlinear function:

$$P^*(z) = P^*(\pi^*(z), z) = \begin{cases} P_1^*(z), & \text{if } p^*(z) < c, \\ P_2^*(z), & \text{if } c \leq p^*(z) \leq p_{max}. \end{cases}$$

The derivative of this function is:

$$P^{*'}(z) = \begin{cases} -c^2\lambda\sigma^{2'}(z) - cF(z) \leq 0, & \text{if } p^*(z) < c, \\ (1 - F(z))p^*(z)(1 - 2\lambda(z - \mu(z))p^*(z)) - c, & \text{if } c \leq p^*(z) \leq p_{max}. \end{cases} \tag{3.5}$$

By taking left and right derivatives at the point where $p^*(z) = c$, we can see that $P^*(\cdot)$ is a smooth function (i.e. its first derivative is continuous). Moreover, since $p^*(\cdot)$ is quasiconcave with $0 < p^*(z) \leq p_{max}$, and $p^*(A) > c$, it turns out that $\pi^*(\cdot)$ will have at most two pieces. Consequently, $P^*(\cdot)$ will have at most two pieces: only $P_2^*(\cdot)$ if $\lambda \in \left[0, \dfrac{y(c)}{2cVar(\epsilon)}\right]$ (as Lemma 3.3 dictates for moderately risk-averse situations) or $P_2^*(\cdot)$ and $P_1^*(\cdot)$ (in this order) if $\lambda \in \left(\dfrac{y(c)}{2cVar(\epsilon)}, \infty\right)$.

Because we will use it in the subsequent sections, we include below the second derivative of the performance measure at the hedged optimal price $\pi^*(z)$ when $\pi^*(z) = p^*(z)$:

$$\begin{aligned} P^{*''}(z)\Big|_{\pi^*(z)=p^*(z)} = P_2^{*''}(z) &= \left(p^{*'}(z)(1-F(z)) - f(z)p^*(z)\right)(1 - 2\lambda(z-\mu(z))p^*(z)) \\ &\quad -2\lambda p^*(z)(1-F(z))\left(F(z)p^*(z) + (z-\mu(z))p^{*'}(z)\right). \end{aligned} \tag{3.6}$$

As a function of λ, we find two different situations: if $\lambda \in \left[0, \dfrac{y(c)}{2cVar(\epsilon)}\right]$, we have that $c \leq p^*(z) \leq p_{max}$ for all $z \in [A, B]$ and therefore $P^*(z) = P_2^*(z)$. If $\lambda > \dfrac{y(c)}{2cVar(\epsilon)}$, there is a point z_c (and only one because of the quasiconcavity of $p^*(\cdot)$ and because $p^*(A) > c$) that solves the equation $p^*(z) = c$. In this case, $P^*(\cdot)$ will have two pieces, namely, $P_2^*(\cdot)$ and $P_1^*(\cdot)$ (in this order). However, since $P^*(\cdot)$ is a continuous, smooth function and $P_1^*(\cdot)$ is a nonincreasing function, it follows that

$$\max_{z\in[A,B]} P^*(z) = \max_{z\in[A,z_c]} P_2^*(z). \tag{3.7}$$

In other words, the optimal value of the performance measure at the hedged optimal price $\pi^*(z)$ can be found by analyzing only the subinterval in which $\pi^*(z) = p^*(z)$. In general, numerical optimization may help to find this global maximum. There are many examples in the literature where complex newsvendor models are tackled with simulation or optimization algorithms (Bisi et al., 2015; Kim, 2006; O'Neil et al., 2015; Sempolinski and Chaudhary, 2010). Analytically, we will derive a sufficient condition for the unimodality of the performance measure at the hedged price $\pi^*(\cdot)$. For this condition, and for some more that will be derived later on, we build our analysis on the lost sales rate (LSR) elasticity, $\xi(p, z)$, which we defined already in (1.6) and particularized for the additive case in §2.1.1.

The following theorem presents a sufficient condition for the unimodality of $P^*(\cdot)$:

Theorem 3.1. *Let $\lambda \geq 0$ and $z_c = \min\{z : p^*(z) = c, B\}$. If*

$$\xi^*(z) \qquad b\left(p^{*'}(z) - \frac{2\lambda(1 - F(z))p^*(z)^2}{c}\left[(z - \mu(z))p^*(z)\right]'\right), \tag{3.8}$$

$\forall z \in [A, z_c]$, then the performance measure $P^*(\cdot)$ is quasiconcave in $[A, z_c]$ and the price-setting newsvendor problem with additive demand (3.2) has a unique optimal solution $(z^*, p^*(z^*))$, where z^* solves $P_2^{*'}(z) = 0$ and $p^*(z^*)$ is given by (3.3).

Proof. See Appendix B. □

The condition above is very general but it requires the comparison of the LSR elasticity with another function at every point in $[A, B]$. Ideally, we want to come up with a constant value that we can compare $\xi^*(\cdot)$ to. Simpler conditions that guarantee the existence of a unique maximum in $P^*(\cdot)$ have been found in particular cases and have been illustrated in previous publications. For instance, Kocabıyıkoğlu and Popescu (2011) show that in the risk-neutral case, the LSR elasticity has to be at least 1/2 for the objective function to be concave. Similarly, in §2 we extend this lower bound for moderate risk-averse cases: when $\lambda < \dfrac{1}{4Bp_{max}}$, the objective function is still concave if the LSR elasticity is at least 1/2. By taking into account the shape of the optimal price function $p^*(\cdot)$ and Theorem 3.1, we can obtain these bounds as well. For example, in the risk neutral case, $\lambda = 0$ and $p^*(\cdot)$ is an increasing and concave function. Therefore $\xi^*(z) \geq bp^{*'}(z) \leq bp^{*'}(A) = 1/2$. In moderately risk-averse cases ($0 < \lambda < \dfrac{1}{4Bp^*_{max}}$) the optimal price is still increasing and concave and the second term of (3.8) is nonnegative, which allows us to write again $\xi^*(z) \geq bp^{*'}(z) \leq bp^{*'}(A) = 1/2$.

When the scenario becomes more risk-averse, the optimal price function turns unimodal. For values of λ greater than $\dfrac{1}{4Bp^*_{max}}$, there exists now a point z_ψ such that $z_\psi = \min\{z : p^{*'}(z) = 0\}$. This point identifies the maximum of $p^*(\cdot)$ and divides the optimal price function in two subintervals: in $[A, z_\psi)$, $p^*(\cdot)$ is increasing and concave; in $[z_\psi, z_c]$, $p^*(\cdot)$ is nonincreasing and has exactly two critical points, located at the extremes of this subinterval. This particular and predefined shape of this function allows us to propose a constant lower bound for the LSR elasticity when $\lambda \geq \dfrac{1}{4Bp^*_{max}}$.

Theorem 3.2. *Let $\lambda \geq 0$, $z_c = \min\{z : p^*(z) = c, B\}$ and $z_\psi = \min\{z : p^{*'}(z) = 0\}$. If the LSR elasticity ξ^* is bounded below by*

$$\xi^*(z) > \frac{1}{2}, \ \forall z \in [A, z_\psi], \tag{3.9}$$

$$\xi^*(z) > -\frac{2(1 - F(z_\psi))\lambda b p^*(z_\psi)^2}{c} \cdot \left(F(z_\psi)p^*(z_c) - \frac{z_c - \mu(z_c)}{2\left(\lambda\sigma^2(z_\psi) + b\right)} \right), \ \forall z \in (z_\psi, z_c], \tag{3.10}$$

then the performance measure $P^*(\cdot)$ is quasiconcave in $[A, z_c]$ and the price-setting newsvendor problem with additive demand (3.2) has a unique optimal solution $(z^*, p^*(z^*))$, where z^* solves $P_2^{*'}(z) = 0$ and $p^*(z^*)$ is given by (3.3).

Proof. See Appendix B. □

This new theorem tackles any risk-averse scenario given that if $\lambda < \dfrac{1}{4Bp_{max}}$ only the first bound applies because $z_\psi = z_c = B$. The second bound can be useful for many more risk-averse scenarios, as we will see in the following example. It requires to obtain the safety stock z_ψ at which the apex of the function p^* occurs by solving the equation $p^{*'}(z) = 0$. While we can think of $-\dfrac{2\lambda B p_{max}^2}{c}$ as a simpler lower bound for ξ^* in the second subinterval, our proposed bound is much sharper.

Example 3.1. In order to illustrate how Theorem 3.2 improves the results obtained in §2, we will work on the basis of one of the examples in the previous chapter. We consider again the demand function $D(p, \epsilon) = 35 - p + \epsilon$, where $\epsilon \sim U[-10, 10]$. The cost of the commodity is $c = 10$. Under their assumptions made in §2, the most risk-averse case under which the concavity of $P^*(\cdot)$ can be guaranteed corresponds to a value of the risk parameter $\lambda = \dfrac{1}{4Bp_{max}} = \dfrac{1}{1400}$. With our focus on unimodality, we are able to prove that there is a unique maximum for values of λ beyond $\dfrac{1}{1400}$. In Table 3.1 we show this for several instances by applying our constant bounds. However, these bounds do not work for the cases $\lambda = 0.02$ and $\lambda = 0.05$ and therefore we cannot claim the unimodality of $P^*(\cdot)$ by using them. Still, it is possible to use the non-constant bound (3.8) to prove the quasiconcavity of $P^*(\cdot)$. Figure 3.2 displays how bound (3.8) from Theorem 3.1 can be applied to those cases where our constant bounds (3.9) and (3.10) do not hold.

λ	z_ψ	z_c	**Cond. (3.9)**	**Cond. (3.10)**	**Cond. met?**
0	10	10	1/2	1/2	✓
0.01	-2.85	10	1/2	-0.61	✓
0.02	-4.88	10	1/2	1.47	×
0.05	-6.71	4.91	1/2	1.24	×
0.06	-6.99	3.39	1/2	-1.32	✓

Table 3.1: Example of the conditions for the unimodality of the objective function for risk-averse scenarios

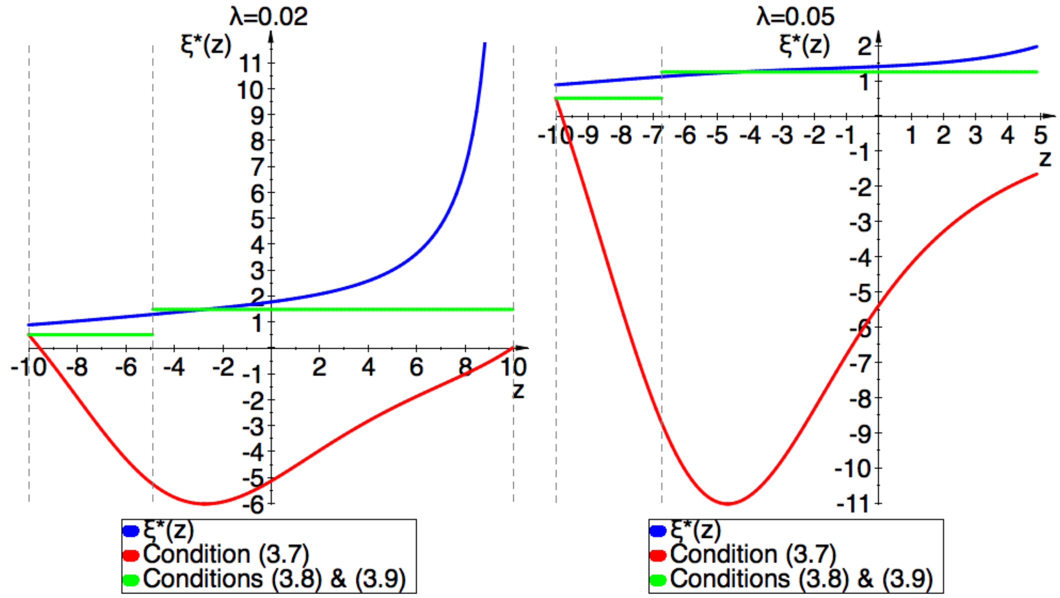


Figure 3.2: Condition (3.8) applied to cases where conditions (3.9) and (3.10) did not hold

3.3 Risk-seeking Newsvendor

3.3.1 Characteristics of the Optimal Price

When the retailer is risk-seeking, the optimal price $p^*(\cdot)$ presents very different characteristics. From the first-order condition (3.3), it is easy to see that $p^*(z) < 0$ when $\lambda < -b/\sigma^2(z)$ and presents a discontinuity when $\lambda = -b/\sigma^2(z)$. Since $\sigma^2(\cdot)$ is an increasing function, as z increases from A the optimal price presents three possible, well differentiated pieces, that may appear in the following order: first the optimal price is positive and nondecreasing with respect to z in the region where $\lambda > -b/\sigma^2(z)$; then this price tends to $+\infty$ when $\lambda = -b/\sigma^2(z)$; finally the optimal price surges from $-\infty$ and attains finite negative values when $\lambda < -b/\sigma^2(z)$.

While the first-order optimality condition is the same that was obtained for the risk-averse case, the critical point $p^*(z)$ is not always a maximizer. Indeed, the second partial derivative of $P(\cdot,\cdot)$ with respect to p

$$\partial^2 P/\partial p^2 \quad = \quad -2(\lambda\sigma^2(z) + b),$$

indicates that the performance measure is concave with respect to p if $\lambda > -b/\sigma^2(z)$, convex with respect to p if $\lambda < -b/\sigma^2(z)$ and linear in p if $\lambda = -b/\sigma^2(z)$. In other words, positive values of $p^*(\cdot)$ correspond to a maximizer of $P(\cdot,\cdot)$, whereas negative values of $p^*(\cdot)$ correspond to a minimizer of the performance measure. In the former case, the concavity of $P(\cdot,\cdot)$ with respect to p when $\lambda > -b/\sigma^2(z)$ implies that the maximizer of $P(\cdot,\cdot)$ in the interval $[c, p_{max}]$ when $p^*(z) > p_{max}$ is obtained at p_{max}. In the latter case, the convexity of $P(\cdot,\cdot)$ with respect to p when $\lambda < -b/\sigma^2(z)$ implies that the positive maximizer of $P(\cdot,\cdot)$ in the interval $[c, p_{max}]$ is also obtained at p_{max} when $p^*(z) < c$. This idea is illustrated in Figure 3.3, where we chose two different risk scenarios for the same problem and plotted the objective function at $z = 0$. In one scenario, the objective function is concave in p for $z = 0$ and the optimal price is outside of the interval $[c, p_{max}]$ and it is greater than p_{max}. In the other scenario, the objective function is convex in p for $z = 0$ and the optimal price is outside of the interval $[c, p_{max}]$ and it is smaller than c.

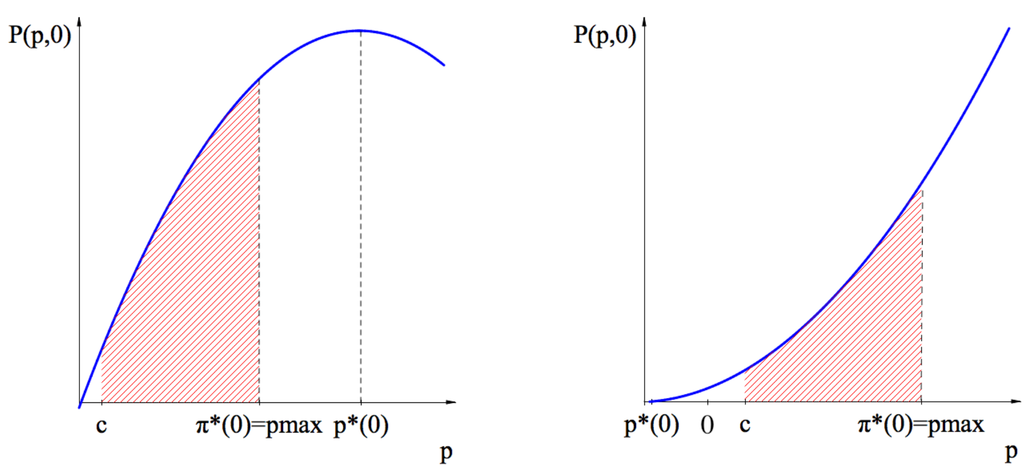


Figure 3.3: Obtention of the optimal hedged prices in risk-seeking cases.

Let $\tilde{z} = \{z : \lambda = -b/\sigma^2(z)\}$ if $\lambda \leq -b/Var(\epsilon)$ and $\tilde{z} = B$ if $\lambda > -b/Var(\epsilon)$. Note that when $\lambda \leq -b/Var(\epsilon)$ the function $p^(\cdot)*$ is not defined at $z = \tilde{z}$.

Lemma 3.4. *Let $\lambda < 0$. The optimal price $p^*(\cdot)$ is strictly increasing at all points in $[A, B)$ where it is defined and has a critical point at $z = B$.*

Proof. See Appendix B. □

The importance of Lemma 3.4 is that it gives us a good idea of what $p^*(\cdot)$ looks like. In particular, we know that in many risk-seeking scenarios, the optimal price will go over p_{max}. When that happens, the function will never return to the interval $[c, p_{max}]$. As a matter of fact, only two options will occur at that point: either the function increases to a point $p^*(B) \geq p_{max}$ or the function presents an asymptote at $z = \tilde{z}$ and $p^*(z) < 0$ in $(\tilde{z}, B]$. Hence, if we let $z_{p_{max}} = \min\{\{z : p^*(z) = p_{max}\}, B\}$ be the safety stock that produces an optimal price equal to p_{max}, or B (whichever is smaller), we can define the hedged optimal price function $\pi(\cdot)$ in the same spirit as in the previous section:

$$\pi^*(z) = \begin{cases} p^*(z), & \text{if } z \leq z_{p_{max}}, \\ p_{max}, & \text{if } z > z_{p_{max}}. \end{cases}$$

Obtaining $z_{p_{max}}$ will play a central role in characterizing the properties of the objective function $P^*(\cdot)$. An illustration of a typical optimal price function and its corresponding hedged optimal price function is presented in Figure 3.4.

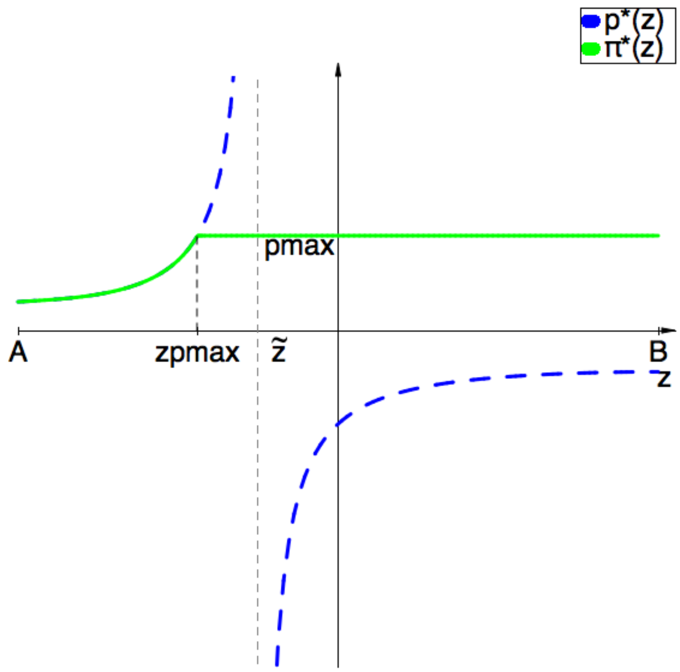


Figure 3.4: Typical optimal price and hedged optimal price functions in risk-seeking cases.

3.3.2 Optimization of $P^*(\cdot)$

Let us define the function $z \mapsto P_3^*(z) = P(p_{max}, z) = -p_{max}^2(\lambda\sigma^2(z) + b) + p_{max}(\mu(z) + a + cb) - c(z + a)$. The performance measure at the hedged optimal price $\pi^*(z)$ in the risk-seeking case can be expressed in terms of $P_2^*(\cdot)$ and $P_3^*(\cdot)$ as:

$$P^*(z) = P^*(\pi^*(z), z) = \begin{cases} P_2^*(z), & \text{if } z \leq z_{p_{max}}, \\ P_3^*(z), & \text{if } z > z_{p_{max}}. \end{cases}$$

The derivative of this piecewise, nonlinear function is shown below. Like in the risk-averse case, the left and right derivatives of this function at $z = z_{p_{max}}$ are equal and the function is smooth.

$$P^{*'}(z) = \begin{cases} (1-F(z))p^*(z)(1-2\lambda(z-\mu(z))p^*(z)) - c, & \text{if } z \le z_{p_{max}}, \\ -p_{max}^2 \lambda \sigma^{2'}(z) + p_{max}(1-F(z)) - c, & \text{if } z > z_{p_{max}}. \end{cases}$$

Its critical points are attained where $P^{*'}(z) = 0$:

$$P^{*'}(z) = 0 \quad \Longrightarrow \quad \begin{cases} (1-F(z))(1-2\lambda(z-\mu(z))p^*(z)) = \dfrac{c}{p^*(z)}, & \text{if } z \le z_{p_{max}}, \\ F(z) = 1 - \dfrac{c}{p_{max}} - \lambda p_{max} \sigma^{2'}(z), & \text{if } z > z_{p_{max}}. \end{cases} \tag{3.11}$$

Like in the risk-averse case, where $P_1^*(\cdot)$ was always monotonic, the second piece of $P^*(\cdot)$, $P_3^*(\cdot)$, has a well predefined shape, as shown in the next lemma.

Lemma 3.5. *The function $P_3^*(\cdot)$ is unimodal in $[A, B]$.*

Proof. See Appendix B. □

In analyzing the optimization of $P^*(\cdot)$ we will break down the value of λ in two thresholds. For this, we will use the following definitions:

- Let $\lambda_{z_{p_{max}}}$ be the risk parameter that gives way to a scenario where $p^*(B) = p_{max}$. In other words, $\lambda_{z_{p_{max}}}$ represents the scenario with the lowest value of λ such that $z_{p_{max}} = B$. By using (3.3) we conclude that

$$\lambda_{z_{p_{max}}} = \frac{a + b(c - 2p_{max})}{2p_{max} Var(\epsilon)}.$$

- Let λ_t be the value of the risk parameter that would make $P_2^{*'}(z_{p_{max}}) = 0$. By using

(3.11) we conclude that

$$\lambda_t(\lambda) = \begin{cases} -\infty, & \text{if } \lambda \geq \lambda_{z_{p_{max}}}, \\ \dfrac{1 - \dfrac{c}{(1 - F(z_{pmax}(\lambda)))p_{max}}}{2(z_{p_{max}}(\lambda) - \mu(z_{p_{max}}(\lambda))p_{max}}, & \text{if } \lambda < \lambda_{z_{p_{max}}}. \end{cases}$$

where we have made it clear that λ_t changes with the level of risk-seekingness λ through the value of $z_{p_{max}}$.

Clearly $\lambda > \lambda_t(\lambda)$ implies $P_2^{*'}(z_{p_{max}}) = P_3^{*'}(z_{p_{max}}) < 0$. Conversely, $\lambda \leq \lambda_t(\lambda)$ implies $P_2^{*'}(z_{p_{max}}) = P_3^{*'}(z_{p_{max}}) \geq 0$. The sign of the derivative of $P^*(\cdot)$ at $z = z_{p_{max}}$ is important because it determines how $P_2^*(\cdot)$ and $P_3^*(\cdot)$ are joined at this breakpoint. The following result helps to predict how this occurs for every instance of the problem.

Lemma 3.6. *There is only one solution to the equation* $\lambda = \lambda_t(\lambda)$ *in* $(-\infty, 0)$.

Proof. See Appendix B. □

This result is important because it means that $P^{*'}(z_{p_{max}}) = 0$ only once in $[A, B]$. Given that $\lambda_t(0) = -\infty$ (because $z_{p_{max}}(0) = B$), this means that $P^{*'}(z_{p_{max}}) < 0$ when $\lambda \in (\lambda_t(\lambda), 0)$, $P^{*'}(z_{p_{max}}) = 0$ when $\lambda = \lambda_t(\lambda)$, and $P^{*'}(z_{p_{max}}) > 0$ when $\lambda \in (-\infty, \lambda_t(\lambda))$.

We can also have some insight about how the critical points of $P_2^*(\cdot)$ and $P_3^*(\cdot)$ change with λ. To this end we will define, for a given risk parameter λ, the values

$$\begin{aligned} \zeta_2(\lambda) &= \min\left\{z \in [A, B] : P_2^{*'}(z) = 0\right\}, \\ \zeta_3(\lambda) &= \min\left\{z \in [A, B] : P_3^{*'}(z) = 0)\right\}. \end{aligned}$$

In other words, $\zeta_2(\lambda)$ and $\zeta_3(\lambda)$ denote the minimum safety stock that produces a critical point in $P_2^*(\cdot)$ and $P_3^*(\cdot)$ respectively, within the interval $[A, B]$ and for a given risk parameter λ. Since $P_2^{*'}(A) > 0$, $\zeta_2(\lambda)$ always represents a maximum. Since, per Lemma 3.5 $P_3^*(\cdot)$ is unimodal, $\zeta_3(\lambda)$ always represents the unique maximum of this function.

Lemma 3.7. *Let $\lambda_A < \lambda_B \leq 0$. Then $\zeta_2(\lambda_A) > \zeta_2(\lambda_B)$ and $\zeta_3(\lambda_A) > \zeta_3(\lambda_B)$. In other words, the safety stock at which the first maximum in $P_2^*(\cdot)$ and the only maximum in $P_3^*(\cdot)$ occur over the interval $[A, B]$ increases as λ decreases.*

Proof. See Appendix B. □

We introduce now two sufficient conditions for P_2^* to be unimodal.

Lemma 3.8. *Let $\lambda \leq 0$ and $\tilde{\lambda} \in (\lambda, 0]$. If the LSR elasticity ξ^* is bounded below by*

$$\xi^*(z) \quad > \quad \frac{2bc}{\mu(\zeta_2(\tilde{\lambda})) + a + cb}, \ \forall z \in [\zeta_2(\tilde{\lambda}), B], \tag{3.12}$$

then the performance measure $P_2^*(\cdot)$ is quasiconcave in $[A, B]$.

Proof. See Appendix B. □

Corollary 3.1. If condition (3.12) holds for λ, it also holds for any instance with risk parameter in the interval $(-\infty, \lambda)$. This follows because if, after selecting a value of $\tilde{\lambda}$, condition (3.12) holds for an instance of this problem, it will also hold for a more risk-seeking instance due to $\xi^*(\cdot)$ increasing when we select a lower λ ($p^*(\cdot)$ increases as λ decreases).

Corollary 3.2. A less sharp but easier lower bound for $P_2^*(\cdot)$ to be unimodal in $[A, B]$ is

$$\xi^*(z) > \quad \frac{2bc}{A + a + cb} \quad = \frac{c}{p^*(A)}. \tag{3.13}$$

Note that this bound is always less than 1. Again, if this condition holds for an instance with risk parameter λ, it will hold for all more risk-seeking instances.

Condition (3.12) provides an auto-adaptative bound that decreases as $\tilde{\lambda}$ gets closer to λ. This bound also reduces the interval over which it has to hold as we decrease $\tilde{\lambda}$ (i.e. as we increase $\zeta_2(\tilde{\lambda})$). An advantage of this auto-adaptative bound is that we are able to reformulate it should a bound is not satisfied when we use the solution of the problem with risk parameter $\tilde{\lambda}$ in order to determine the unimodality of the problem with risk parameter λ. A good strategy to check if $P_2^*(\cdot)$ is unimodal for all risk-seeking instances is to test (3.13) for the risk-neutral case. If it holds, per Corollary 3.2 this function will be unimodal for any $\lambda < 0$. If it does not hold, we can use condition (3.12) to see if an instance with parameter λ, and per Corollary 3.1, if all instances with risk parameter less than λ, are unimodal. It is a good idea to use $\tilde{\lambda} = 0$ so $\zeta_2(\tilde{\lambda})$ is the first maximum of the risk-neutral problem. If $\xi^*(z) > 1/2$, $\zeta_2(0)$ is the only solution to first-order optimality condition of the risk-neutral problem and it is easy to find.

We are now prepared to establish the theorems that tackle the unimodality of the risk-seeking newsvendor problem.

Theorem 3.3. *Let $\lambda \in [\lambda_t(\lambda), 0)$ and $\tilde{\lambda} \in (\lambda, 0]$. Then,*

$$\max_{z\in[A,B]} P^*(z) = \max_{z\in[A,z_{pmax}]} P_2^*(z) = \max_{z\in[\zeta_2(\tilde{\lambda}),z_{pmax}]} P_2^*(z). \tag{3.14}$$

Moreover, if $P_2^*(\cdot)$ is unimodal in $[A, B]$, then the performance measure $P^*(\cdot)$ is quasiconcave and the price-setting newsvendor problem with additive demand (3.2) has a unique optimal solution $(\zeta_2(\lambda), p^*(\zeta_2(\lambda)))$, where $\zeta_2(\lambda)$ solves $P_2^{*'}(z) = 0$ and $p^*(\zeta_2(\lambda))$ is given by (3.3).

Proof. See Appendix B. $\square$

Theorem 3.4. *Let* $\lambda \in (-\infty, \lambda_t(\lambda))$ *and* $\tilde{\lambda} \in (\lambda, 0]$. *Then,*

$$\max_{z\in[A,B]} P^*(z) \quad = \quad \max\{P^*(\zeta_3(\lambda)), \max_{z\in[\zeta_2(\tilde{\lambda}), z_{p_{max}}]} P_2^*(z)\}. \tag{3.15}$$

Moreover, if $P_2^*(\cdot)$ *is unimodal in* $[A, B]$, *then the performance measure* $P^*(\cdot)$ *is quasiconcave and the price-setting newsvendor problem with additive demand (3.2) has a unique optimal solution* $(\zeta_3(\lambda), p^*(\zeta_3(\lambda)))$, *where* $\zeta_3(\lambda)$ *solves* $P_3^{*'}(z) = 0$ *and* $\pi^*(\zeta_3(\lambda))$ *is given by (3.4).*

Proof. See Appendix B. $\square$

Example 3.2. Consider the same demand function as in Example 3.1, $D(p, \epsilon) = 35 - p + \epsilon$, where $\epsilon \sim U[-10, 10]$. The cost of the commodity is $c = 10$ and $p^*(A) = \dfrac{A + a + cb}{2b} = 17.5$. Let us consider two risk-seeking instances: $\lambda_1 = -0.001$ and $\lambda_2 = -0.01$. A simple application of equation (3.13) for the case of $\lambda = 0$ yields the condition $\xi^*(z) > 0.57$, which holds in $[A, B]$ because $\xi^*(A) = 0.875$ and the failure rate is increasing for a uniform distribution. Per Corollary 3.2, $P_2^*(\cdot)$ is unimodal in this interval for any risk-seeking instance. For these two scenarios, we obtain that $z_{p_{max}}(-0.001) = 10$ and $z_{p_{max}}(-0.01) = 1.28$. These two values yield $\lambda_t(-0.001) = -\infty$ and $\lambda_t(-0.01) = -5.2 \cdot 10^{-4}$. Since $\lambda_t(-0.001) < -0.001$, by virtue of Theorem 3.3, the only solution to $P_2^{*'}(z) = 0$ provides the triple that solves the problem $(\zeta_2(-0.001) = 2.24, p^*(\zeta_2(-0.001)) = 22.11, P^*(\zeta_2(-0.001)) = 108.5)$. Since $\lambda_t(-0.01) > -0.01$, by virtue of Theorem 3.4, the only solution to $P_3^{*'}(z) = 0$ provides the triple that solves the problem $(\zeta_3(-0.01) = 8.48, \pi^*(\zeta_3(-0.01)) = 33.19, P^*(\zeta_3(-0.01)) = 265.58$ (see Figure 3.5).

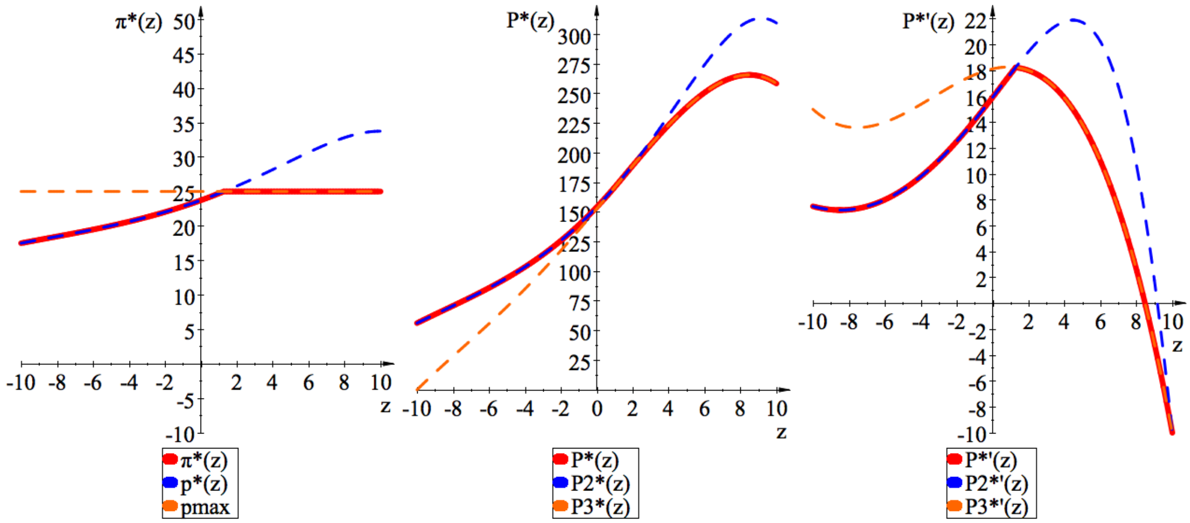


Figure 3.5: Example of a risk-seeking newsvendor problem ($D(p, \epsilon) = 35 - p + \epsilon$, $\epsilon \sim U[-10, 10]$, $c = 10$, $\lambda = -0.01$).

3.4 Sensitivity Analysis of the Expected Profit and the Variance of the Profit

Managerially speaking, the ultimate goal of this analysis is to know the mean and the second central moment (i.e. the variance) of the random variable profit. The selection of an appropriate risk parameter λ is done according to these values and how acceptable they are for the decision maker. It is important to know beforehand how these two measures will change as a function of risk sensitivity λ. Desirably, increasing the value of λ (i.e. becoming more risk-averse) will reduce the expected profit in exchange for a lower variance of the profit. Likewise, decreasing the value of λ (i.e. becoming more risk-seeking) will reduce the expected profit in exchange for a higher variance of the profit. Our results for the additive, price-dependent demand confirm this behavior.

Lemma 3.9. *In risk-averse cases, the expected profit and the variance of the profit at the hedged optimal price $\pi^*(z)$ decrease as λ increases. In risk-seeking cases, as λ decreases, the expected profit decreases and the variance of the profit increases.*

Proof. See Appendix B. □

Remark 3.2. Let $\lambda_1 > \lambda_2 > 0$. Then the optimal pair $(z^*_{\lambda_1}, \pi^*(z^*_{\lambda_1}))$ produces lower expected profit and a higher variance of the profit than the optimal pair $(z^*_{\lambda_2}, \pi^*(z^*_{\lambda_2}))$.

Remark 3.3. Let $\lambda_1 < \lambda_2 < 0$. Then the optimal pair $(z^*_{\lambda_1}, \pi^*(z^*_{\lambda_1}))$ produces lower expected profit and higher variance of the profit than the optimal pair $(z^*_{\lambda_2}, \pi^*(z^*_{\lambda_2}))$.

3.5 Conclusions

The present chapter seeks to find a general solution framework and a full characterization for the mean-variance newsvendor with price-dependent, additive demand. The performance measure must be seen as a weighted combination of the expected profit and the variance of the profit. The relative importance of the variance of the profit as well as the sign of its contribution to such measure is given by a risk parameter λ. The decision maker should see this maximization problem as a method to attain optimal stocking and pricing policies in view of his or her risk profile. For each value of λ the maximization problem (3.2) produces a pair of policies that will in turn yield an expected profit and variance of the profit. These two quantities are ultimately the basis of the decision maker's actions. It is then when he or she will have to resolve whether these levels of expectation and variance of the profit are acceptable and fine-tune the value of λ accordingly. We showed that the expected profit and the variance of the profit decrease with the level of risk-aversion, whereas they decrease and increase respectively with the level of risk-seekingness. This fact makes tuning the value of λ an easier task because we know beforehand what to expect when changing it.

We believe that the analysis of this problem must be done in a more managerial fashion, and to this end we base our study on a recently introduced metric, the lost sales rate elasticity, and express our conditions for the unimodality of the objective function in terms of this measure. One of the major difficulties that we encountered was that the optimal price for a given safety stock is not necessarily contained in the allowed range of prices. This produces a performance measure $P^*(\cdot)$ that in general is piecewise and nonlinear. However, the additive demand model allows a very precise description of the optimal price function. Having knowledge about the characteristics of this function was crucial to develop constant lower bounds for the LSR elasticity in order for the objective function to be unimodal.

The optimal price function $p^*(\cdot)$ was found to be either nondecreasing or unimodal in the risk-averse case, and nondecreasing in the risk-seeking case. This allowed us to define a piecewise optimal hedged price function $\pi^*(\cdot)$ that capped the maximum value of the optimal price to p_{max}.

The most important achievement of this chapter is that we are able to simplify the solution of every single instance of the risk-sensitive newsvendor problem with mean-variance trade-off and additive demand. Unlike other authors, we do not assume that the random variable ϵ has any particular property like increasing failure rate or increasing generalized failure rate, which makes our approach as general as possible. For the risk-averse case, we use equations (3.9) and (3.10) to extend the bounds provided by Kocabıyıkoğlu and Popescu (2011) (risk-neutral case) and those found in §2 (moderately risk-averse cases) and create a unified framework for this type of problems. For risk-seeking scenarios, we follow the thread of §2 and find two alternative simple conditions for any possible scenario of this kind. Equation (3.12) is an auto-adaptative bound that can be used should the easier bound (3.13) does not hold. Moreover, by taking advantage of the structure of the solutions in risk-seeking cases, we can show that if any of these two conditions hold for a given value of λ, then the property of unimodality is extended to all more risk-seeking instances. Finally, regardless of the sensitivity to risk, we can simplify the original optimization problem even if the conditions proposed do not hold (equation (3.7) in risk-neutral and risk-averse situations and equations (3.14) and (3.15) in risk-seeking environments).

Chapter 4

Unimodality with Multiplicative Demand

4.1 Introduction and Problem Formulation

In this chapter we study the condition for the problem to be unimodal when the demand is multiplicative. The price-setting newsvendor model with isoelastic, price-dependent demand was studied by Choi and Chiu (2012); nevertheless, in this work the authors assume that the pricing and stocking decisions can be taken in different stages, namely, first by deciding on the stock quantity and, once the demand is observed, by setting the price. Although we will follow the same sequential optimization approach that was presented in prevous chapters, it is important to remark that such an approach is intended to make pricing and stocking decisions simultaneously.

Consider the single-stage, single-product, newsvendor problem with two decision variables, namely stock quantity and price. A risk-neutral retailer would thus pursue the maximization of his expected profit, $\mathbb{E}(\Pi(p,x)) = p\mathbb{E}(min\{D(p,\epsilon),x\}) - cx$, where $c \in \mathbb{R}^+$ is the cost of the product and $D(p,\epsilon)$ is its demand. Note that $c \geq 1$ can be assumed without loss of generality, since any currency can be easily reconverted to a new scale. The first term represents the income collected in a single-period, whereas the second term represents the cost incurred when manufacturing or procuring x units of product. We assume that such a cost increases linearly with the number of units procured or manufactured. The demand is considered to be multiplicative (isoelastic), price-dependent, and stochastic, and can be obtained by setting $y(p) \equiv 0$ and $g(p) \equiv ap^{-b}$ in (1.4). The demand function takes on the form:

$$D(p,\epsilon) \quad = \quad ap^{-b}\epsilon, \tag{4.1}$$

where ϵ is a random variable with finite variance $Var(\epsilon)$ and compact and convex support $[A,B]$, $0 < A < 1 < B$. We assume that ϵ has a probability distribution characterized by a probability density function f and a cumulative density function $F \in \mathcal{C}^2$ (i.e. F is twice differentiable with continuous second derivative). We also consider, without loss of generality, that $E(\epsilon) = 1$.

The term $g(p)$ denoted as riskless price (Petruzzi and Dada, 1999) represents in this case an isoelastic demand, i.e., $g(p) = ap^{-b}$ with $a \in \mathbb{R}^+$ and $b \in \mathbb{Q}^+$. We define b as the quotient of two natural numbers, $b = b_1/b_2$, b_1, $b_2 \in \mathbb{N}^+$, b_2 odd, such that $b > 1$. The assumptions on the rationality of b and the odd parity of b_2 will be discussed later on. Since in general we use the term elasticity for referring to the price elasticity of demand, $b > 1$ represents products that are commonly known to present a constant, elastic demand, meaning that an increase of 1% in the price of the commodity in question will always produce a drop greater than 1% in its demand and equal to b. Examples of elastic goods are usually products that are not critically needed by consumers or for which they can readily find a substitute. The archetypal example is the elasticity of soft drinks like Coca-Cola or Mountain Dew, that have elasticities of 3.8 and 4.4, respectively (Ayers and Collinge, 2003). On the contrary, goods that see a reduction of their demand by less than 1% after an increase of their price are called inelastic (i.e. $b < 1$). Examples of such goods are alcoholic beverages or cigarettes. These types of goods will not be covered in this chapter.

Again, the objective function is presented as a combination of the expected profit and the variance of the profit weighted with a risk parameter. The sign of this parameter determines whether the newsvendor is risk-averse or risk-seeking: $\lambda > 0$ for the former (i.e. the variance of the profit decreases the objective function), $\lambda < 0$ for the latter (i.e. the variance of the profit increases the objective function).

$$\begin{aligned}\tilde{P}(p,x) &= \mathbb{E}(\Pi(p,x)) - \lambda Var(\Pi(p,x)) \\ &= p\mathbb{E}(min\{D(p,\epsilon),x\}) - cx - \lambda Var(p \cdot min\{D(p,\epsilon),x\}).\end{aligned}$$

Another characteristic that is usually added is the stockout cost. Although we do not take this cost into account in our analysis, it would be interesting to see how it affects the optimal decision as a function of the retailer's risk attitude. It is known that when we optimize only the stock quantity this optimal quantity need not be inversely proportional to the risk aversion when this cost is present and a mean-variance tradeoff is implemented (Wu, Li, Wang, and Cheng, 2009).

Following the procedure presented in Petruzzi and Dada (1999), we define the price-sensitive stock factor $z = x/g(p)$. Since $x \in [g(p)A, g(p)B]$, it follows that $z \in [A, B]$. We can rewrite the objective function as a function of (p, z):

$$\begin{aligned}
\tilde{P}(p,x) &= p\mathbb{E}(min\{D(p,\epsilon),x\}) - cx - \lambda Var(p \cdot min\{D(p,\epsilon),x\}) \\
&= g(p)p\mathbb{E}(min\{\epsilon,z\}) - czg(p) - \lambda Var(pg(p) \cdot min\{\epsilon,z\}) \\
&= pg(p)\mu(z) - czg(p) - \lambda(g(p)p)^2\sigma^2(z) \\
&= \mathbb{E}(\Pi(p,z)) - \lambda Var(\Pi(p,z)) =: P(p,z), \qquad (4.2)
\end{aligned}$$

where $\mu(z) = \mathbb{E}(min\{\epsilon, z\}) = \int_z^B (z-u)f(u)du+1$ and, $\sigma^2(z) = Var(min\{\epsilon, z\}) = Var(\epsilon)+ \int_z^B (z^2-u^2)f(u)du - \mu^2(z)+1,\ z \in [A,B]$. Note that $\mu(\cdot)$ is always an increasing, concave function, for $\mu'(z) = 1-F(z)$ and $\mu''(z) = -f(z),\ z \in [A,B]$. On the other hand, $\sigma^2(\cdot)$ is an increasing function with $\sigma^{2'}(z) = 2\left(1-F(z)\right)\left(z-\mu(z)\right)$, although not much can be said about its second derivative $\sigma^{2''}(z) = 2(1-F(z))F(z) - 2f(z)(z-\mu(z))$. Examples of the analytical expressions of the integrals needed to calculate the functions $\mu(\cdot)$ and $\sigma(\cdot)$ are shown in tables (4.1) and (4.2), where it was always assumed that $\mathbb{E}(\epsilon) = 1$.

Distribution	**Support**	$\int_z^B (z-u)f(u)du$
Uniform	$[A,B]$	$\frac{(B-z)^2}{2(A-B)}$
Shifted expo.(θ)[a]	$[A,\infty)$	$-\frac{1}{\theta}e^{-\theta(z-A)}$
Gamma (α,β)[b]	$(0,\infty)$	$\frac{z\tilde{\Gamma}(\alpha,z/\beta)-\beta\tilde{\Gamma}(\alpha+1,z/\beta)}{\Gamma(\alpha)}$[c]
Normal $(1,v)$[d]	$(0,\infty)$	$-\frac{z-1}{2}\left(erf\left(\frac{z-1}{\sqrt{2v}}\right)-1\right) - vf(z)$[e]
LogNormal (m,v)	$(0,\infty)$	$\frac{z}{2}\left(1-erf\left(\frac{ln(z)-m}{\sqrt{2v}}\right)\right) - \frac{1}{2}\left(1+erf\left(\frac{v+m-ln(z)}{\sqrt{2v}}\right)\right)$
Triangular (A,B,m)	$[A,B]$	$\begin{cases} -\frac{(B-m)(B+2m-3z)}{3(B-A)} - \frac{(m-z)^2(2m-3A+z)}{3(B-A)(m-a)}, & \text{if } A \le z \le m \\ -\frac{(B-z)^3}{3(B-A)(B-m)}, & \text{if } m < z \le B \end{cases}$

[a] The shifted exponential has pdf $f(u) = \theta e^{-\theta(u-A)}$. Its mean is given by $A + 1/\theta$ and by assumption it equals 1, whence $\theta = \frac{1}{1-A}$.
[b] The gamma distribution has pdf $f(u) = \beta^\alpha/\Gamma(\alpha)x^{\alpha-1}e^{-\beta x}$ where α and β are the shape and rate parameters respectively
[c] $\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt$ is the Gamma function and $\tilde{\Gamma}(a,x) = \int_x^\infty t^{a-1}e^{-t}dt$ is the upper incomplete Gamma function.
[d] Assume that the variance v of this distribution is such that $F(A) \approx 0$ and thus we can consider that all the mass of the distribution is in $[A,\infty)$.
[e] $erf(x) = \frac{2}{\pi}\int_0^x e^{-t^2}dt$ is the error function.

Table 4.1: Analytical results for some commonly used statistical distributions

Distribution	**Support**	$\int_z^B (z^2 - u^2) f(u) du$
Uniform	$[A, B]$	$\frac{(B-z)^2(B+2z)}{3(A-B)}$
Shifted expo.(θ)	$[A, \infty)$	$-\frac{2}{\theta} e^{-\theta(z-A)} \left(z + \frac{1}{\theta}\right)$
Gamma (α, β)	$(0, \infty)$	$\frac{z^2 \tilde{\Gamma}(\alpha, z/\beta) - \beta^2 \tilde{\Gamma}(k+2, z/\beta)}{\Gamma(\alpha)}$
Normal $(1, v)$	$(0, \infty)$	$\frac{1}{2}(1 + v - z^2)\left(erf\left(\frac{z-1}{\sqrt{2v}}\right) - 1\right) - \frac{\sqrt{v}(z+1)e^{-\frac{(z-1)^2}{2v}}}{\sqrt{2\pi}}$
LogNormal (m, v)	$(0, \infty)$	$\frac{z^2}{2}\left(1 - erf\left(\frac{ln(z)-m}{\sqrt{2v}}\right)\right) - \frac{1}{2}e^{(m+v)}\left(1 + erf\left(\frac{2v+m-ln(z)}{\sqrt{2v}}\right)\right)$
Triangular (A, B, m)	$[A, B]$	$\begin{cases} -\frac{(B-m)(B^2+2Bm+3m^2-6z^2)}{6(B-A)} & \\ -\frac{(m-z)^2(3m^2+6mz+3z^2-4Am-8Az)}{6(B-A)(m-a)}, & \text{if } A \leq z \leq m \\ -\frac{(B-z)^3(B+3z)}{6(B-A)(B-m)}, & \text{if } m < z \leq B \end{cases}$

Table 4.2: Analytical results for some commonly used statistical distributions (ctd.)

Since they will be used frequently throughout this chapter, we next present some partial derivatives of the objective function $P(p, z) = ap^{-b}\left(\mu(z)p - cz - \lambda ap^{2-b}\sigma^2(z)\right)$ with respect to (p, z):

$$\begin{aligned}
\frac{\partial P(p,z)}{\partial p} &= ap^{-(b+1)}\left(2\lambda\sigma^2(z)a(b-1)p^{-(b-2)} - (b-1)\mu(z)p + bcz\right), && (4.3)\\
\frac{\partial^2 P(p,z)}{\partial p^2} &= ap^{-(b+2)}(-2(b-1)\lambda\sigma^2(z)a(2b-1)p^{-(b-2)} + b(b-1)\mu(z)p && (4.4)\\
&\quad -(b+1)bcz), \\
\frac{\partial^2 P(p,z)}{\partial z^2} &= -\lambda a^2 p^{2(1-b)}\sigma^{2''}(z) + ap^{1-b}\mu''(z), && (4.5)\\
\frac{\partial^2 P(p,z)}{\partial p \partial z} &= ap^{-(2b-1)}\left(bcp^{b-2} - (b-1)p^{b-1}\mu'(z) + 2(b-1)a\lambda\sigma^{2'}(z)\right). && (4.6)
\end{aligned}$$

Following the same sequential optimization procedure that was introduced in §2, we define the performance measure at the optimal price as $P^*(z) := P(p^*(z), z) = ap^*(z)^{-b}(\mu(z)p^*(z) - cz - \lambda ap^*(z)^{2-b}\sigma^2(z))$. This function is subsequently maximized and hence the pair $(z^*, p^*(z^*))$ is optimal.

4.2 Optimization with Respect to p

Given a stock factor z finding the function $p^*(z)$ reduces to solving the condition $\partial P/\partial p = 0$, i.e.

$$2\lambda\sigma^2(z)a(b-1)p^*(z)^{-(b-2)} - (b-1)\mu(z)p^*(z) + bcz \quad = \quad 0. \tag{4.7}$$

Clearly, for any given λ, finding a closed-form for $p^*(z)$ requires knowledge of b as well. An exception to this rule is the risk-neutral case ($\lambda = 0$), for which the optimal price is always $p^*(z) = \dfrac{bcz}{(b-1)\mu(z)}$. We refer the reader to Petruzzi and Dada (1999) for a thorough analysis of this case. However, when the newsvendor is risk-sensitive, we may find multiple solutions to equation (4.7). Since the price elasticity of the demand b is a rational number, this equation can be transformed into a polynomial after a suitable change of variables. The roots of this resulting polynomial can be real or imaginary and this gives way to questioning whether we can have better information about the solution of (4.7). Our goal is to be able to identify a positive real solution, greater than c, that maximizes the performance measure $P(\cdot, z)$. In the subsections that follow we study how to identify such a solution, but in the meantime we include in Table 4.3 below some closed-form solutions of the optimal price for specific values of the demand elasticity. Note that for $b = 2$ and $b = 3$ there are some values of λ for which $p^*(z)$ is a negative real number or an imaginary number. We will see later on under which circumstances and for which values of the risk parameter this occurs. The proofs of all the results that follow in this chapter are provided in Appendix C.

b	$p^*(z)$
1.5	$\left(\dfrac{a\lambda\sigma^2(z)+\sqrt{(a\lambda\sigma^2(z))^2+3\mu(z)cz}}{\mu(z)}\right)^2$
2	$\dfrac{2cz+2\lambda a\sigma^2(z)}{\mu(z)}$
3	$\dfrac{3cz+\sqrt{(3cz)^2+32\lambda a\sigma^2(z)\mu(z)}}{4\mu(z)}.$

Table 4.3: Closed-form solutions of some optimal prices

4.2.1 Risk-Averse Retailer

When the retailer is risk-averse ($\lambda > 0$) equation (4.7) will only have one real positive solution, and such a solution will always be greater than the cost c and will also represent a maximizer of $P(\cdot, z)$. This is a remarkable result because it allows us to have certainty about the behavior of the performance measure $P^*(\cdot)$ that we will use later on and shows that there is no need to evaluate $P(\cdot, z)$ at each of those different solutions in order to find the price that maximizes this function for a given z.

Lemma 4.1. *Let $\lambda > 0$. Given a stock factor there is exactly one positive real solution to equation (4.7), $p^*(z)$, for each $z \in [A, B]$.*

Theorem 4.1. *Let $\lambda > 0$. The objective function $P(\cdot, z)$ is unimodal with respect to p in $(0, \infty)$ and the optimal price $p^*(z)$ is a maximizer.*

Proof. See Appendix C. □

Note that, since $\sigma^2(A) = 0$, the optimal price at $z = A$ is independent of the level of risk aversion and equals $\dfrac{cb}{b-1}$. This price depends only on the elasticity of the demand of the product and its cost, thus matching the results by Wang, Jiang, and Shen (2004); Petruzzi and Dada (1999).

4.2.2 Risk-Seeking Retailer

When the newsvendor is risk-seeking ($\lambda < 0$) the number of positive real roots depends on the value of b. The results yielded for this case are more complicated, but we can still predict the number of positive real roots, whether they maximize the performance measure $P(\cdot, z)$, and whether they are contained in the range $[c, \infty)$. We introduce the threshold value λ_{min} as

$$\lambda_{min} = \max_{z \in [A,B]} \frac{-bcz}{(2a(b-1)\sigma^2(z))},$$

and propose the following statements:

Lemma 4.2. *Let $\lambda < 0$. Then, for each stock factor z:*

- *If $1 < b < 2$, there is exactly one positive real solution to equation (4.7), $p^*(z)$.*
- *If $b = 2$, there is a unique solution to equation (4.7).*
- *If $b > 2$, there are either two positive real solutions to equation (4.7), or there are none.*

Proof. See Appendix C. □

Theorem 4.2. *Let $\lambda < 0$. The objective function $P(\cdot, z)$ is unimodal with respect to p in $(0, \infty)$ $\forall z \in [A, B]$ if and only if $1 < b < 2$, or $b = 2$ and $\lambda \geq \lambda_{min}$, and $p^*(z)$ is a maximizer. If $b > 2$ and (4.7) has two positive real solutions $\forall z \in [A, B]$, then $P(\cdot, z)$ is bimodal with respect to p in $(0, \infty)$.*

Proof. See Appendix C. □

Corollary 4.1. Consider the case $b > 2$ in which (4.3) has two positive real roots. Let those roots be $p_1(z)$ and $p_2(z)$ such that $0 < p_1(z) < p_2(z)$. Given the limits of $\partial P/\partial p$ at 0^+ and ∞, $p^*(z)$ refers to $p_2(z)$, since it is clear that $p_1(z)$ is a minimum.

Because we want a function of maximizers, p^*, that is continuous in z, we will not consider the cases in which, for $b > 2$, positive real roots for some stock factors and imaginary roots for others may arise. If this happens, p^* will have discontinuities. Hence, we assume in what follows that, when $b > 2$, $P(\cdot, z)$ has two positive real roots $\forall z \in [A, B]$.

Although we know in any case that $p^*(A) = \dfrac{bc}{b-1} > c$, neither Lemma 4.2 nor Theorem 4.2 guarantee that the roots of (4.3) are contained in $[c, \infty)$. Let $\pi^* : \mathbb{R}^+ \to \mathbb{R}^+$ be the hedged optimal price function. We define this function in the range of prices where the economic activity of the retailer is meaningful, $[c, \infty)$. Our goal is to define this function such that it is continuous in its domain for a given b and λ. The newsvendor is interested in selling a product at a price no less than its cost and therefore it is mandatory that $\pi^*(z) \geq c$, $\forall z \in [A, B]$. The hedged optimal price function $\pi^*(\cdot)$ is thus constructed as

$$\pi^*(z) = \begin{cases} c, & \text{if } p^*(z) < c, \\ p^*(z), & \text{if } p^*(z) \geq c. \end{cases}$$

It is important to remark that this construction method yields a function $\pi^*(\cdot)$ that is indeed piecewise continuous, although not differentiable at those points where $p^*(z) = c$.

In order to continue with our analysis, we define the function $t : [A, B] \to \mathbb{R}^-$ as

$$t(z) = \frac{(b-1)\mu(z) - bz}{2a(b-1)\sigma^2(z)} c^{b-1},$$

and another threshold value of the risk parameter, λ_{lim}, as the value of λ needed for c to be a solution to (4.7) for some stock factor z, that is:

$$\lambda \geq \max_{z \in [A,B]} t(z) \quad = \lambda_{lim}. \tag{4.8}$$

Lemma 4.3. *Let $\lambda < 0$ and let (4.7) have one or two real positive roots $\forall z \in [A, B]$ as stated in Lemma 4.2. These roots are <u>always</u> greater than or equal to c if and only if $\lambda \geq \lambda_{lim}$.*

Proof. See Appendix C. □

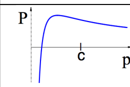
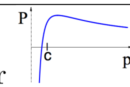
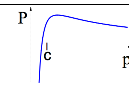
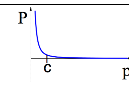
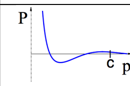
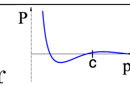
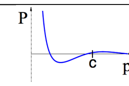
Corollary 4.2. Let $b = 2$. Since

$$\frac{((b-1)\mu(z) - bz)\, c^{b-1}}{2a(b-1)\sigma^2(z)} > \frac{-bcz}{2a(b-1)\sigma^2(z)}, \; \forall z \in [A, B],$$

it follows that $\lambda_{lim} > \lambda_{min}$. Hence, from Theorem 4.2 and Lemma 4.3 we conclude that given a stock factor, $P(\cdot, z)$ is unimodal with respect to p in $(0, \infty)$ and $p^*(z) \geq c \; \forall z \in [A, B]$ if and only if $\lambda \geq \lambda_{lim}$.

Note that when $\lambda = \lambda_{lim}$ there will always be a root that is equal to c. However, this root might not represent the optimal price that maximizes the profit for a given safety stock. For example, when $P(\cdot, z)$ has two positive real roots $p_1(z)$ and $p_2(z)$, $0 < p_1(z) < p_2(z)$, it is possible that it is $p_1(z)$ that is equal to c and not $p_2(z)$. In view of the results from Theorem 4.2 and Lemma 4.3, we can summarize all the different possibilities in the optimization of $P(\cdot, z)$ in $[c, \infty)$ for a given stock factor. This is shown in Table 4.4.

Lemma 4.3 shows that given for a stock factor z, $p^*(z) > c$ provided that $\lambda > t(z)$ (see Figure 4.1 below, where we have denoted by z_i the i^{th} point where $\lambda = t(z)$). It is then clear that $\pi^*(z) = p^*(z), \; \forall z \in [A, B]$ whenever $\lambda \geq \lambda_{lim}$. In Figure 4.1 we represent a

Case	$\boldsymbol{b}$	$\boldsymbol{\lambda}$	**Roots in $\mathbb{R}^+$**	**Max. in $[c,\infty)$**	**Shape of $P(\cdot, z)$**
1	$1 < b < 2$	$(-\infty, \lambda_{lim})$	1	c or $p^*(z)$	 or 
2	$1 < b < 2$	$[\lambda_{lim}, 0)$	1	$p^*(z)$	
3	$b = 2$	$(-\infty, \lambda_{lim})$	0	c	
4	$b = 2$	$(-\infty, \lambda_{lim})$	1	c or $p^*(z)$	Same as in Case 1
5	$b = 2$	$[\lambda_{min} \leq \lambda < \lambda_{lim}]$	1	c or $p^*(z)$	Same as in Case 1
6	$b = 2$	$[\lambda_{lim}, 0)$	1	$p^*(z)$	Same as in Case 2
7	$b > 2$[a]	$(-\infty, \lambda_{lim})$	2	c or $p^*(z)$	 or 
8	$b > 2$	$[\lambda_{lim}, 0)$	2	$p^*(z)$	

[a] In all cases where $b > 2$ it is assumed that there are two roots in $\mathbb{R}^+$, $\forall z \in [A, B]$

Table 4.4: Analysis of the optimality in $[c, \infty)$ of the objective function with respect to the price for a given stock factor (risk-seeking cases)

function t that is unimodal; in such a case, it follows that the equation $\lambda = t(z)$ will have at most two solutions or, in other words, the function $\pi^*(\cdot)$ will have at most three pieces. However, an important difficulty of this model is that t does not seem to have a predefined shape, and therefore one cannot know a priori the number of pieces that $\pi^*(\cdot)$ will have. For this reason, and even though we will continue to use the function $\pi^*(\cdot)$ for the sake of generality, we will only tackle the optimization of cases for $\lambda \geq \lambda_{lim}$. We restrict our study to risk-averse and moderately risk-seeking cases and refer the reader to numerical optimization for solving instances where $\lambda < \lambda_{lim}$.

Example 4.1. Let $D(p, \epsilon) = 10^6 p^{-b} \epsilon$ with $\epsilon \sim U[0.001, 1.999]$. Let $c = 100$. The first-order condition (4.7) as a function of b and z is $2 \cdot 10^6 \lambda (b-1) \sigma^2(z) p^{-(b-2)} - (b-1)\mu(z) p + 100bz = 0$, with $\mu(z) = -0.2503z^2 + 1.0005z - 2.503 \cdot 10^{-7}$ and $\sigma^2(z) = -0.06263z^4 + 0.1671z^3 - 5.009 \cdot 10^{-4} + 5.009 \cdot 10^{-7} z - 2.001 \cdot 10^{-10}$.

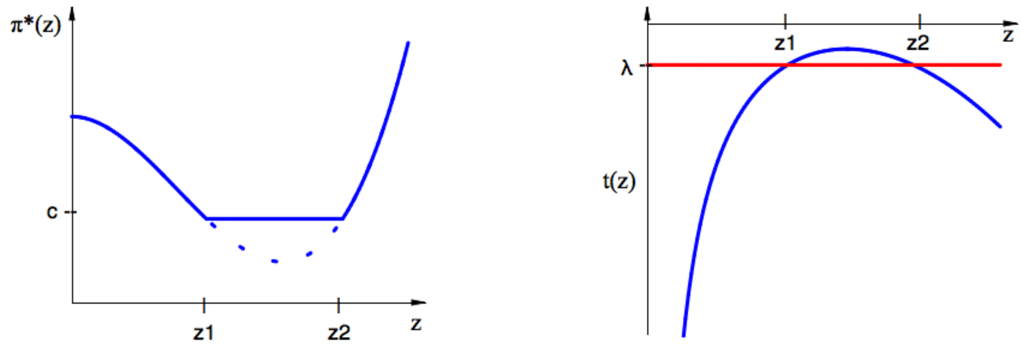


Figure 4.1: Piecewise continuous optimal price function

- $\underline{b = 1.5:}$ the first-order condition becomes $10^6\lambda\sigma^2(z)\sqrt{p} - \frac{1}{2}\mu(z)p + 150z = 0$. The lower bound for λ, as shown in Lemma 4.3, can be obtained numerically. It turns out that (4.8) attains its maximum at $z = 1.7215$ with a value of $\lambda_{lim} = \max(\mu(z) - 3z)5 \cdot 10^{-6} = -6.952 \cdot 10^{-5}$. The only positive real root in this case is given by

$$p^*(z) = \left(\frac{10^6\lambda\sigma^2(z) + \sqrt{(10^6\lambda\sigma^2(z))^2 + 300\mu(z)z}}{\mu(z)} \right)^2.$$

  Observe that $b = 1.5$ implies that b_2 is even, which contradicts one of the assumptions set earlier in Section 4.1. However, this example is fairly simple and it can be easily seen that the first-order condition only yields one real positive root.

- $\underline{b = 2:}$ the first-order condition becomes $2 \cdot 10^6\lambda\sigma^2(z) - \mu(z)p + 200z = 0$. Since $b = 2$, per the corollary from Lemma 4.3 we use (4.8) to set a lower bound for λ and thus set $\lambda_{lim} = \max(\mu(z) - 2z)5 \cdot 10^{-5} = -4.085 \cdot 10^{-4}$. The only positive real root is

$$p^*(z) = 2\frac{10^6\lambda\sigma^2(z) + 100}{\mu(z)}.$$

- $\underline{b = 3:}$ the first-order condition becomes $\dfrac{4 \cdot 10^6\lambda\sigma^2(z)}{p} - 2\mu(z)p + 300z = 0$. λ_{lim} is set to $\max_{z\in[A,B]} \dfrac{2\mu(z)-3z}{400} = -0.02639$ and the equation above has two roots:

$$p_1^*(z) = \frac{300z - \sqrt{9\cdot 10^4 z^2 + 32\cdot 10^6 \mu(z)\sigma^2(z)\lambda}}{4\mu(z)},$$

$$p_2^*(z) = \frac{300z + \sqrt{9\cdot 10^4 z^2 + 32\cdot 10^6 \mu(z)\sigma^2(z)\lambda}}{4\mu(z)}.$$

These roots are real and positive if $\lambda \geq \dfrac{-(300z)^2}{32\cdot 10^6 \mu(z)\sigma^2(z)}$, the first being a minimum and the second being the maximum we are interested in. In this case, $\lambda_{lim} \geq -(300z)^2 32\cdot 10^6 \mu(z)\sigma^2(z)$, $\forall z \in [A,B]$ and therefore $p_2^*(z)$ is always real, positive, and not smaller than c. Hence, the assumption $p^*(z) \geq c$ holds. As mentioned before, when $\lambda = \lambda_{lim}$, c is a root of (4.7). Further analysis shows that this root occurs at $z = 1.6214$. However, this point corresponds to a minimizer and $\pi(\cdot)$ is composed of maximizers only. For this reason, the right-most graph in Figure 4.2 does not show a curve that reaches $c = 100$ when $\lambda = \lambda_{lim}$. This can be seen in further detail in Figure 4.3, where it is clear that $c = 100$ is only a root of (4.7) when $\lambda = \lambda_{lim}$ (in this case at $z = 1.6214$), but this root does not correspond to a maximizer.

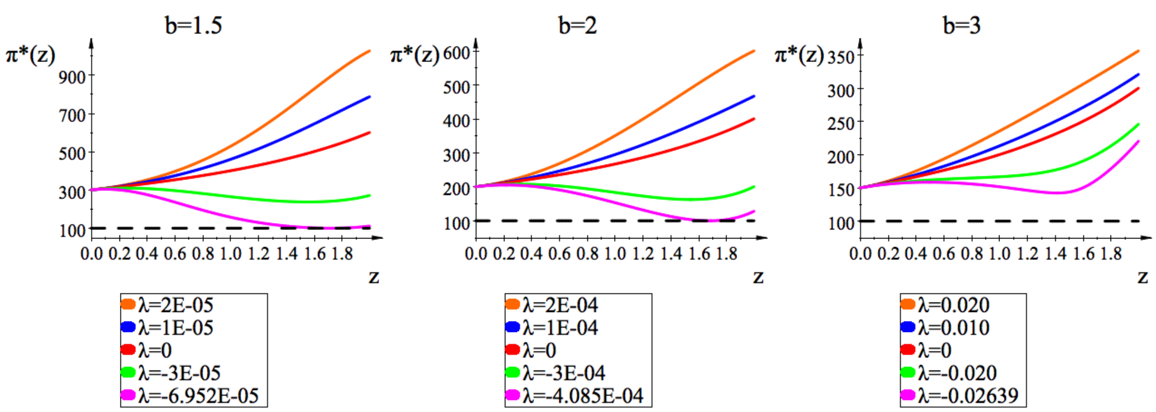


Figure 4.2: Optimal price function under different risk scenarios

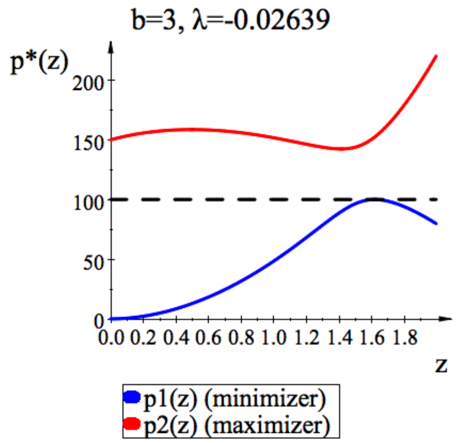


Figure 4.3: Maximizing and minimizing prices for $\lambda = \lambda_{lim}$

4.3 Optimization with Respect to z

As commented in the introduction of this thesis, it is usual in the literature to find examples based on different risk measures that guarantee the unimodality of the objective function under more restrictions, usually related to the generalized failure rate of ϵ. For instance, Xu, Cai, and Chen (2011); Wang, Jiang, and Shen (2004) show unimodality for the risk-neutral case and multiplicative demand models if the random variable has an increasing generalized failure rate. For risk-averse cases with CVaR considerations, Chen, Xu, and Zhang (2009) show that a strictly increasing generalized failure rate in the risk distribution is required to attain unimodality. In what follows, we proceed to attain different conditions for unimodality depending on the risk-parameter in a mean-variace setting. We want to give a managerial meaning to our results and, to that end, we use again the lost sales rate (LSR) elasticity, as defined by (1.6):

$$\tilde{\kappa}(p, x) \quad = \quad \frac{p(G(p, x))'_p}{1 - G(p, x)},$$

where $G(p,x) := Pr(D(p,\epsilon) \leq x)$ and $(G(p,x))_p^{'} \equiv \dfrac{\partial G(p,x)}{\partial p}$. In particular, when the demand is multiplicative, $Pr(y(p)\epsilon \leq x) = Pr\left(\epsilon \leq \dfrac{x}{y(p)}\right) = F(z)$, and therefore we obtain that

$$\tilde{\kappa}(p,x) = \frac{p(G(p,x))_p^{'}}{1-G(p,x)} = \frac{bzf(z)}{1-F(z)} =: \xi(z). \tag{4.9}$$

Just like the price elasticity of demand in isoelastic demand curves, the LSR elasticity is not a function of the price when the demand is multiplicative. In other words: the price-isoelastic demand is also LSR-isoelastic because, given a stock factor, the change in the level of service will be the same regardless of the price from which that increase takes place.

In general, we can define the objective function $P^*(\cdot)$ as a function of z as follows:

$$P^*(z) := P(\pi^*(z), z) = \begin{cases} ap^*(z)^{-b}\left(\mu(z)p^*(z) - cz - \lambda a p^*(z)^{2-b}\sigma^2(z)\right), & \text{if } p^*(z) \geq c, \\ ac^{-b+1}\left(\mu(z) - z - \lambda a c^{-b+1}\sigma^2(z)\right), & \text{if } p^*(z) < c. \end{cases} \tag{4.10}$$

Its first-order derivative, after using the relation between λ and $p^*(z)$ as derived from (4.7), is

$$P^{*'}(z) = \begin{cases} ap^*(z)^{-b}R(z), & \text{if } p^*(z) \geq c, \\ -ac^{-b+1}\left(F(z) + \lambda a c^{-b+1}\sigma^{2'}(z)\right), & \text{if } p^*(z) < c, \end{cases} \tag{4.11}$$

where $R(z) = (1-F(z))p^*(z) - c - \lambda a p^*(z)^{-(b-2)}\sigma^{2'}(z)$. (4.12)

These piecewise expressions are only needed if the retailer is risk-seeking with $\lambda < \lambda_{min}$, for only in those cases it may happen that $p^*(z) < c$ for some values of z. For any other value of λ only the first piece, corresponding to the case where $p^*(z) > c$, will be needed.

4.3.1 Risk-Neutral Retailer

When $\lambda = 0$, $P^*(\cdot)$ and its first-order derivative can be greatly simplified to

$$P^*(z) = ap^*(z)^{-b}\left(\mu(z)p^*(z) - cz\right),$$

and

$$P^{*'}(z) = ap^*(z)^{-b}R(z),$$

where $R(z) = (1 - F(z))p^*(z) - c$. This is the same result obtained by Wang, Jiang, and Shen (2004); Petruzzi and Dada (1999). It is thus clear that the optimal stock factors z^* of the risk-neutral, single-stage newsvendor problem with isoelastic demand satisfy the equation $F(z^*) = 1 - c/p^*(z^*)$. When the stock factor is the only decision variable, this result particularizes for the classic, well-known result of the single-stage newsvendor problem where the stock factor that maximizes the profit is unique and equal to the $(1 - c/p)$-quantile of z (sometimes called the newsvendor quantile). However, when the price is also a decision variable it is not clear anymore whether this equation has one or multiple solutions. The following theorems intend to shed some light on some conditions that guarantee local and global optimality of the solutions to $R(z) = 0$:

Theorem 4.3. *The following local and global optimality conditions hold for the risk-neutral case:*

a) *(Local optimality) Let z^* be a solution to the equation $F(z) = 1 - c/p^*(z)$. Then the pair $(z^*, p^*(z^*))$ is a strict local maximum of $P(\cdot,\cdot)$ in $[A, B] \times [c, \infty)$ if and only if $\xi(z^*) > 1$. If $\xi(z^*) < 1$, this pair is a saddle point of $P(\cdot,\cdot)$ in $[A, B] \times [c, \infty)$.*

b) *(Global optimality) If $\xi(z) > 1,\ \forall z \in [A, B]$, then $P(\cdot,\cdot)$ is unimodal in $[A, B] \times [c, \infty)$. In other words, there is only one stock factor z^* that satisfies the equation $F(z^*) = 1 - c/p^*(z^*)$ and therefore the pair $(z^*, p^*(z^*))$ solves the risk-neutral, single-stage newsvendor problem with isoelastic demand.*

Proof. See Appendix C. □

Example 4.2. Consider the demand function $D(p, \epsilon) = 10^6 p^{-3}\epsilon$. Let $c = 50$. The random variable ϵ has a probability density function denoted by $f(z) = 0.5f_1(z) + 0.5f_2(z)$, where f_1 and f_2 are in turn the pdf's of two normal random variables with means $0.4, 1.6$ and standard deviations $0.1, 0.2$, respectively. We assume $A = 0.001$ and $B = 3$ because the density of ϵ beyond those points is negligible. The optimal price for each value of z can be calculated by using the third entry of Table 4.3. Solving the equation $F(z) = 1 - c/p^*(z)$ numerically yields the following solutions: $z_1^* = 0.4831, z_2^* = 0.8, z_3^* = 1.392$. Evaluating these points in the expression $\xi(z) = 3zf(z)/(1-F(z))$ yields the following LSR elasticities: $\xi(z_1^*) = 3.4026$, $\xi(z_2^*) = 0.0048$, $\xi(z_3^*) = 5.6998$.

These results show that the points $(z_1^*, p^*(z_1^*)) = (0.4831, 83.1294)$ and $(z_3^*, p^*(z_3^*)) = (1.392, 117.5295)$ are strict local maxima of $P(\cdot,\cdot)$, whereas $(z_2^*, p^*(z_2^*)) = (0.8, 100)$ is a saddle point of $P(\cdot,\cdot)$. The pair $(1.392, 117.5295)$ is also the global maximum of $P(\cdot,\cdot)$ in $[0.001, 3] \times [50, \infty)$ with a value of the objective function $P(z_3^*, p^*(z_3^*)) = 21.4355$. Figure 4.4 shows these three points as the solutions to $P^{*'}(z) = 0$ (i.e. as the solutions to $F(z) = 1 - c/p^*(z)$) plotted on the curve $P^*(z) = P(z, p^*(z))$ and then those three points plotted on the surface defined by $P(z,p)$. Note that $(0.8, 100)$ is a local minimum of $P^*(\cdot)$ but it is a saddle point of $P(\cdot,\cdot)$.

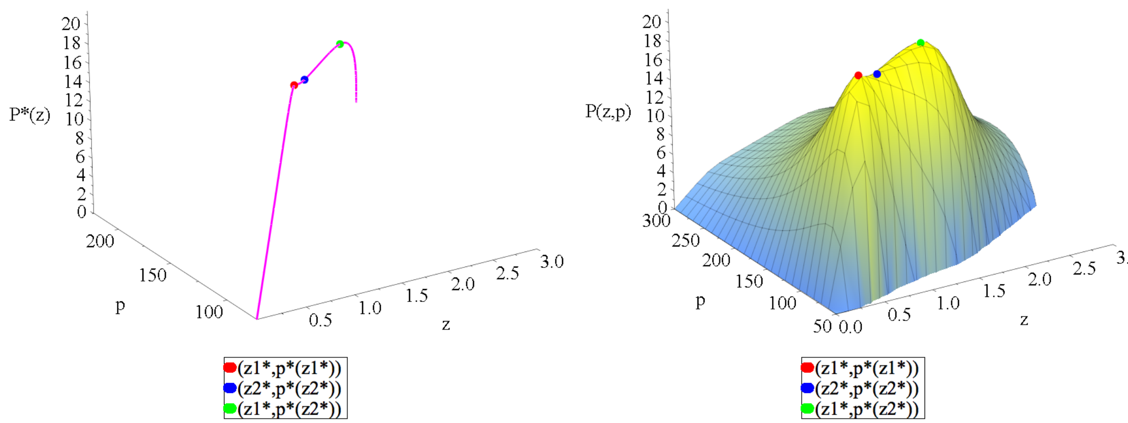


Figure 4.4: Illustration of local optimality conditions for the risk-neutral case

The theorem above gives conditions for a point to be either a local maximum or a unique maximum of $P(\cdot,\cdot)$ in the risk-neutral, single-stage newsvendor problem with multiplicative demand. Some similar results that guaranteed the unimodality of this problem were obtained in the past as a function of the failure rate of ϵ, $h(z) = f(z)/(1 - F(z))$, and the generalized failure rate of ϵ, $g(z) = zh(z)$. For instance, Petruzzi and Dada (1999) show that if $b \geq 2$ and $2h(z)^2 + h'(z) > 0$ this problem has a unique solution. Wang, Jiang, and Shen (2004) claim that ϵ having an increasing generalized failure rate is sufficient, thus uncoupling the economic parameters of the model from the uniqueness of the optimal solution. Both conditions are the consequence of imposing the unimodality of equivalent formulations of $R(z)$ (see both papers for further details). In turn, we make the Hessian of $P(\cdot,\cdot)$ negative definite in all the pairs $(z^*, p^*(z))$ for proving our local optimality condition and transform the results to give them the more economic and managerial interpretation that the LSR elasticity provides. This result also complements Theorem 2 from Kocabıyıkoğlu and Popescu

(2011) that claims the concavity of the objective function in risk-neutral cases if $\xi(x) > 1/2$. However, they assume that $2y'(p)+py''(p) < 0$, which implies that the good has an inelastic demand ($b < 1$). In this chapter, we extend the concept of concavity to that of unimodality and we consider products that have an elastic demand ($b > 1$).

The following lemma characterizes the changes in the optimal stock factor and the optimal price in the risk-neutral case.

Lemma 4.4. *Let $\lambda = 0$ and z^* be a solution to the equation $F(z) = 1 - c/p^*(z)$. If the LSR elasticity at z^* is greater than the price elasticity of the demand (i.e. $\xi(z^*) > b$) then z^* decreases in b and p^* increases in c and decreases in b.*

Proof. See Appendix C. □

Corollary 4.3. This result matches what Wang, Jiang, and Shen (2004) propose under the IGFR condition.

4.3.2 Risk-Sensitive Retailer

When $\lambda \geq \lambda_{lim}$, $P^*(\cdot)$ and its first-order derivative can be written as shown in equations (4.10) - (4.12) for the case $p^*(z) \geq c$. There exist some conditions under which the unimodality of the risk-sensitive problem, either risk-averse or moderately risk-seeking, is guaranteed.

Theorem 4.4. *(Global optimality) Let*

$$
\begin{aligned}
\eta(z) &= (1 - F(z))p^*(z) - c, \\
\Psi(z) &= (c + (b-1)\eta(z))^2(1 - F(z))(z - \mu(z)), \\
\Phi(z) &= (b-1)^2\sigma^2(z)\eta(z) + bcz(1 - F(z))(z - \mu(z)).
\end{aligned}
$$

The following sufficient conditions guarantee that the unimodality of the single-stage newsvendor problem with isoelastic demand (i.e. there is only one stock factor z^* that satisfies the equation $R(z) = 0$):

a) If $\lambda \geq \lambda_{lim}$:

$$\xi(z) \quad > \quad \left(\frac{\Psi(z)}{\Phi(z)} - \frac{F(z)\eta(z)}{z - \mu(z)}\right)\frac{bz}{c}, \ \forall z \in [A, B],$$

b) If $\lambda \geq 0$:

$$\xi(z) \quad > \quad \left(1 + \frac{(b-1)\eta(z)}{c}\right)^2, \ \forall z \in [A, B], \tag{4.13}$$

c) If $\lambda \geq 0$ and $b \geq 2$:

$$\xi(z) \quad > \quad \left(1 + \frac{2\lambda a(b-1)(B-1)}{c^{b-1}}\right)^2, \ \forall z \in [A, B], \tag{4.14}$$

Proof. See Appendix C. □

A very interesting remark to make here is that the global optimality condition from Theorem 4.3 is a particularization of the global optimality condition from Theorem 4.4. As a matter of fact, when $\lambda = 0$, it turns out that $\eta(z)|_{R(z)=0} = 0$ (because in this case $\eta(z) = R(z)$). Applying this to (4.13) yields directly the expression $\xi(z) > 1$. On the other hand, the bound provided for the risk-averse cases with $b \geq 2$ is in general very close to 1. This is because the order of magnitude of λ is generally very small: let m, k, and r be constants; if the order of magnitude of the variance of the profit is $\sim 10^{2m}$, then the objective function $P(\cdot, \cdot)$ dictates that $\lambda \sim 10^{-m}$. The values of the price elasticity b

and the upper bound B are usually of order $\sim 10^0$. The parameter a has usually a larger order of magnitude ($\sim 10^r$); the denominator $c^{b-1} \sim 10^{k(b-1)}$ (where 10^k is the order of magnitude of the cost). After all, the second term in the squared expression from (4.14) is $\sim 10^{r-k(b-1)-m} << \sim 10^0$, as long as the order of magnitude of a is not comparatively very high.

All the results in Theorem 4.4 require the evaluation of the LSR elasticity in all the points in the compact interval $[A, B]$. Under some circumstances, we can reduce our optimization problem to a smaller interval, which increases the applicability of our results.

Definition 4.1. *Let z^*_{RSE} be a solution to the equation $(1 - F(z))p^*(z) - c = 0$ where $p^*(z)$ is the optimal price as derived from solving (4.7) for the risk-sensitive problem. Such a solution is called risk-sensitive equivalent (RSE) solution.*

An RSE solution is therefore a stock factor that satisfies the optimality condition for the risk-neutral problem (i.e. $R(z) = (1 - F(z))p^*(z) - c = 0$) but uses a risk-sensitive optimal price. It turns out that if the function $(1 - F(\cdot))p^*(\cdot) - c$ is decreasing, then we can reduce our interval of optimization.

Lemma 4.5. *If $\xi(z) > bz\dfrac{p^{*'}(z)}{p^*(z)}$, $\forall z \in [A, B]$, then the risk-sensitive problem has a unique RSE solution (RSE-optimal solution) and we can reduce our optimization problem as follows:*

$$\max_{z\in[A,B]} P^*(z) = \begin{cases} \displaystyle\max_{z\in[A,z^*_{RSE}]} P^*(z), & \textit{if } \lambda > 0, \\ \displaystyle\max_{z\in[z^*_{RSE},B]} P^*(z), & \textit{if } \lambda < 0, \end{cases}$$

Proof. See Appendix C. □

Corollary 4.4. $P^*(z^*_{RSE})$ is a lower bound of the optimal solution of the problem, $P^*(z^*)$.

4.4 Sensitivity Analysis of the Optimal Price, the Expected Profit, and the Variance of the Profit

4.4.1 Relationship between the Optimal Price and the Risk Parameter

We can analyze how, for a given stock factor, the optimal price changes as a function of the risk parameter λ. For a given stock factor $\hat{z}$, let $\lambda \mapsto \tilde{p}^*(\lambda)$ denote the optimal price as a function of the risk parameter.

Lemma 4.6. *Given stock factor $\hat{z}$, the optimal price is a nondecreasing, concave function with respect to λ.*

Proof. See Appendix C. □

Corollary 4.5. In the risk-averse case, the optimal price $p^*(\lambda, z)$ is always greater than or equal to the cost c since it follows from Lemma 4.6 that $p^*(\lambda, z) \geq p^*(0, z) \geq \tilde{p}^*(0)\Big|_{z=A} = \tilde{p}^*(\lambda)\Big|_{z=A} = \dfrac{bc}{b-1} > c$.

The consequence of this lemma is that the optimal price for a given stock factor $\hat{z}$ increases with the level of risk-aversion, whereas it decreases with the level of risk-seekingness. Although this result may seem counterintuitive at first sight, it is convenient to recall that one important characteristic of the multiplicative demand is that the price affects the demand uncertainty. More concisely, the variance of the demand is in this case decreasing with respect to the price, for $Var(D(p, \epsilon)) = Var(\epsilon)y(p)^2$ (Petruzzi and Dada, 1999). Therefore, when increasing λ in the risk-averse case, a price increase will reduce the riskless demand $y(p)$, and this in turn will reduce the variance of the stochastic demand. Similarly, reducing λ in the risk-seeking case will increase the riskless demand and induce an increment in the variance of the stochastic demand.

4.4.2 Relationship between Profit and the Risk Parameter

Let $\tilde{\Pi}^*(\lambda)$ be a random variable denoting the profit for a given stock factor $\hat{z}$ and price $\tilde{p}^*(\lambda)$ as a function of the risk parameter λ.

Lemma 4.7. *The variance of the profit for a given stock factor $\hat{z}$ and price $\tilde{p}^*(\lambda)$ decreases as λ increases.*

Proof. See Appendix C. □

Corollary 4.6. As the newsvendor gets more risk-averse (risk-seeking), his optimal policy induces a smaller (greater) variance of the profit.

Lemma 4.8. *The expected profit for a given stock factor $\hat{z}$ and price $\tilde{p}^*(\lambda)$ decreases as λ increases in the risk-averse case and decreases as λ decreases in the risk-seeking case.*

Proof. See Appendix C. □

For illustration purposes, we analyze Example 4.1 with $b = 3$ after embedding $\pi^*(z)$ in $P(\cdot, \cdot)$. Figure 4.5 shows the objective function $P^*(z)$, as well as $E(\Pi^*(z))$ and $Std(\Pi^*(z)) = \sqrt{Var(\Pi^*(z))}$, for different values of λ ranging from risk-seeking to risk-averse situations. All the curves represent values of λ above λ_{lim} and therefore we should expect $\pi^*(z) = p^*(z)$. The behavior predicted by lemmas 4.7 and 4.8 can be observed in this figure: for a given stock factor $\hat{z}$ the variance of the profit decreases with the risk-aversion and increases with the risk seekingnees; in turn, the expected profit decreases with both risk-aversion and risk-seekingness. It is under the light of an example like this one where the power of a mean-variance analysis25as a tool for decision-making can be seen: first we are able to come up with an array of optimal decisions as a function of our stance towards risk. The optimal value of the objective function itself is not significant; instead, it reveals an optimal stock factor and price that can be used for determining the best combination of expected profit and standard deviation of the profit for a particular risk tolerance. These are the true metrics when it comes to making a decision.

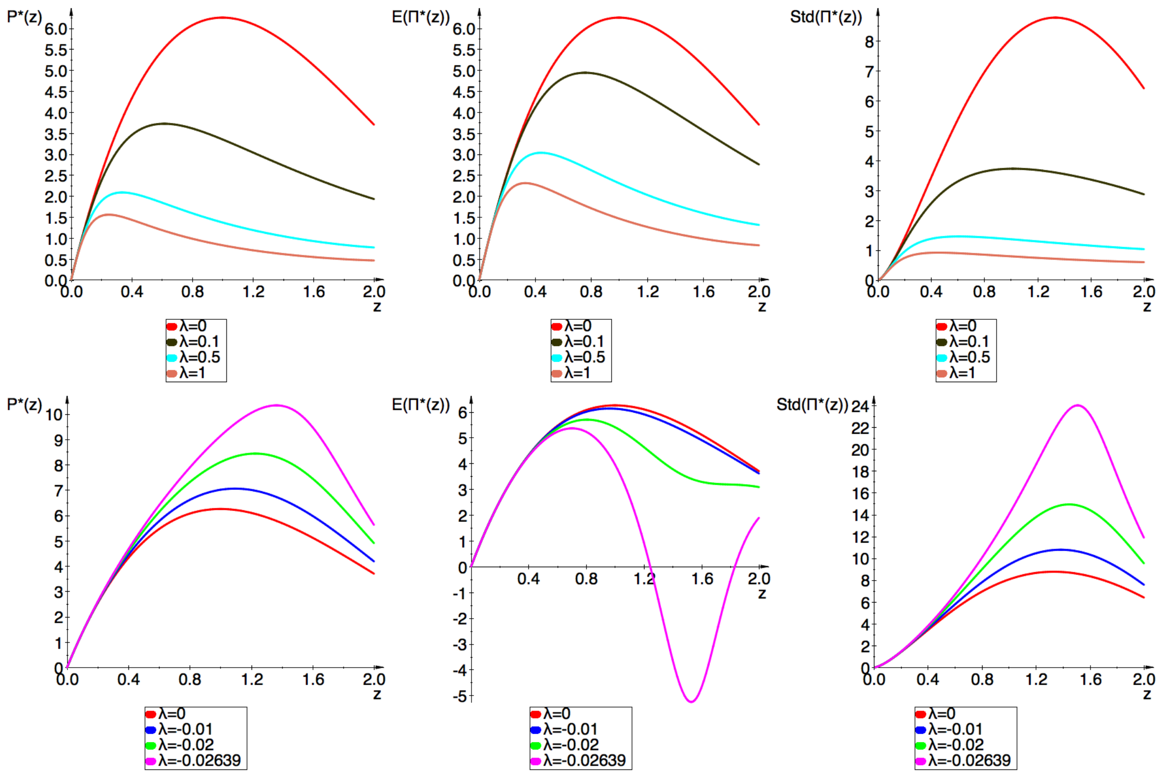


Figure 4.5: Objective function, expected profit, and standard deviation of the profit under different risk scenarios with $D(p, \epsilon) = 10^6 p^{-3} \epsilon$, $\epsilon \sim U[0.001, 1.999]$, $c = 100$

Secondly, the range of values for λ that are acceptable for every situation is given by the results that these values yield and the results derived from the sensitivity analysis previously shown: a risk-averse decision-maker does not know at first what his tolerance to risk is in terms of λ but he knows that there is a maximum standard deviation that is acceptable for him. Fine-tuning λ is thus a matter of finding the scenario that results in that maximum standard deviation. It follows from the sensitivity analysis that any λ greater than the value found will generate optimal pairs that guarantee lower standard deviations and this appreciation gives a range of values of λ. An analogous interpretation for risk-seeking individuals can be made using similar arguments in view of the results that stem from the sensitivity analysis. Finally, Table 4.5 shows several numerical results for different values of λ ranging from risk-seeking cases to risk-averse cases. For these experiments we used a demand function $D(p, \epsilon) = 10^6 p^{-1.5}\epsilon$ with ϵ being distributed as three different distributions, namely, uniform, normal, and triangular. The cost of the commodity is assumed to be $c = 100$. For the range of values of λ used, $\lambda > \lambda_{lim}$ and therefore $p^*(z) > c$. Since $b = 1.5$, this optimal price can be calculated via the closed-form result shown in Table 4.3. Every scenario met condition a) from Theorem 4.4 and therefore the solution to the optimization problem is given by a unique pair $(z^*, p^*(z^*))$. Note that some risk-seeking scenarios even incur in expected loss profit in exchange for a higher standard deviation of the profit.

	$\lambda = -2.1E-04$				$\lambda = -1.2E-04$			
Distribution	p^*	z^*	$E[\Pi^*]$	$SD[\Pi^*]$	p^*	z^*	$E[\Pi^*]$	$SD[\Pi^*]$
Uniform [0.6, 1.4]	143.49	1.36	4,321.12	16,220.60	219.26	1.34	25,982.03	15,392.70
Normal $(1{,}0.25^2)$	140.70	1.43	-1,607.21	16,962.52	218.21	1.38	24,309	16,014.08
Triangular $(0.3, 1.6, 1.1)$	115.53	1.42	-21.677.25	19,882.49	193.26	1.39	19,749.96	18,704.23
	$\lambda = -3E-05$				$\lambda = 0$			
Distribution	p^*	z^*	$E[\Pi^*]$	$SD[\Pi^*]$	p^*	z^*	$E[\Pi^*]$	$SD[\Pi^*]$
Uniform [0.6, 1.4]	334,38	1.28	33,287.76	11,180.27	365.24	1.18	33,837.41	10,092.55
Normal $(1{,}0.25^2)$	330.45	1.25	33,067.93	11,902.05	359.59	1.15	33,646.23	10,137.24
Triangular $(0.3, 1.6, 1.1)$	325.86	1.28	32,672.02	13,530.66	367.91	1.18	33,432.89	11,484.84
	$\lambda = 3E-05$				$\lambda = 3E-04$			
Distribution	p^*	z^*	$E[\Pi^*]$	$SD[\Pi^*]$	p^*	z^*	$E[\Pi^*]$	$SD[\Pi^*]$
Uniform [0.6, 1.4]	366.83	1.04	33,220.21	7,626.22	333.95	0.77	28,622.65	3,525.68
Normal $(1{,}0.25^2)$	372.07	1.04	33,168.79	8,298.95	382.56	0.76	27,532.56	4,716.51
Triangular $(0.3, 1.6, 1.1)$	388.66	1.06	32,778.30	9,274.07	378.65	0.71	25,737.30	4,738.12

Table 4.5: Summary of results of the optimization problem for several random variables $(c = 100, y(p) = 10^6 p^{-1.5})$.

4.5 Conclusions

The results presented in this chapter are oriented not only towards presenting conditions for the unimodality, but also towards giving those conditions a managerial appeal by including a metric typically used in industry: the level of service. We show that in risk-averse instances the optimal price is a unimodal function in z that is always greater than the cost. Neither the unimodality nor the monotonicity of this function is guaranteed in risk-seeking instances, but its shape and sign of its derivative can be known based on the price elasticity of the demand and the risk-parameter. In turn, the objective function $P^*(\cdot)$ is not necessarily unimodal, although we attain optimality conditions that guarantee this unimodality under some premises characterized by the LSR elasticity $\xi(\cdot)$, a measure of the change in the level of service when increasing the price of the product. We also prove that the condition found for the risk-neutral problem ($\xi(z) > 1$) is a particularization of one of our conditions for the risk-sensitive problem and extends the results obtained by Kocabıyıkoğlu and Popescu (2011) to the case of price-elastic goods. We also investigate the much less explored risk-seeking case, and find interesting research questions that stem from the complexity that arises in instances with very high risk-seeking behavior. In particular, the cases where $\lambda < \lambda_{lim}$ may result in a piecewise, nonlinear objective function which a priori complicates the search for the optimal solution of the problem.

Finally, a sensitivity analysis performed on the main variables of the model reveals some insights useful for decision-making: when compared to the risk-neutral case, the risk-averse newsvendor sets higher optimal prices for a given stock factor and the risk-seeking newsvendor prefers lower optimal prices. A risk-averse retailer should also anticipate smaller expectation and variance in his profit in comparison to a risk-neutral individual, while a risk-seeking newsvendor should predict smaller expectation and higher variance in his profit. These gaps between the results in risk-neutral and risk-sensitive cases are proportional to the degree of risk-aversion or risk-seekingness.

Chapter 5

Final Conclusions and Future Research

The present thesis was completed aiming at achieving a well-defined goal: the unification of all risk-sensitive instances of the price-setting newsvendor problem with price-dependent demand and two decision variables (namely, price and stock quantity), as well as the characterization of the conditions for the unimodality of each instance under a metric that captures managerial attention.

We achieve the first goal by introducing a mean-variance trade-off. Such a performance measure must be seen as a weighted combination of the expected profit and the variance of the profit. The relative importance of the variance of the profit as well as the sign of its contribution to such measure is given by a risk parameter λ. The sign of this risk parameter denotes whether the decision maker is risk-averse or risk-seeking. The latter still remains much less studied than the former, and for this reason we believe that our work bridges efficiently a gap in the literature. One major characteristic of our model is that we do not make any assumption on the random component of the demand. While many authors work with random variables that have an increasing failure rate or an increasing generalized failure rate, our results also hold for random perturbations that do not have those properties. We tackle the second goal by including the LSR elasticity, and hence the level of service, as the main metric for assessing the unimodality of a given instance.

While §2 analyzes the concavity of the objective function with additive demand, §3 and §4 broadens the scope of our study and finds conditions for the unimodality of the performance measure with additive demand and multiplicative demand, respectively. This scope, besides being more general, does not use the assumptions that are needed in the study of the concavity of the objective function.

There exist important similarities in the behavior of the performance measure when the demand is additive and multiplicative. In particular, we showed that the expected profit and the variance of the profit decrease with the level of risk-aversion, whereas they decrease and increase respectively with the level of risk-seekingness. The importance of this result lies in the need to calibrate the value of λ to adapt the instance of the model to the risk sensitivity of the decision maker. It is much easier to calibrate this parameter if we know beforehand what the impact on the performance measure will be after changing its value.

However, finding the conditions for the unimodality of the objective function turned out to be much simpler when the demand is additive. Albeit in both cases the optimal price is not necessarily contained in the allowed range of prices, a major advantage of the additive demand model is that it allows a very precise, closed-form description of this function. Having knowledge about the characteristics of the optimal price is crucial to develop constant lower bounds in terms LSR elasticity because it allows to know exactly how many pieces the nonlinear and piecewise performance measure will have. Conversely, when the demand is isoelastic, and except for the risk-neutral case, the functional form of the optional price remains unknown unless the value of the price-elasticity of the demand is specified. Consequently, we do not know how many pieces the performance measure will have. Its analysis is much more complex and so is finding constant bounds that guarantee the unimodality of the objective function.

Future research directions point to how to deal with the opportunities provided by the presence of massive amount of data. New trends in the research community are geared towards big data analysis and the newsvendor problem can benefit greatly from the development of new techniques. The concept of big data is spreading vastly among researchers and is considered a hot topic nowadays. Although there does not seem to be consensus on what big data really means, it deals with the use of advanced analysis techniques to extract useful information from massive amounts of data. In many occasions in the past, lack of data used to be a problem. Nevertheless, technological changes as well as the capacity to acquire and storage an unprecedented amount of data is posing a problem that did not exist before: how to extract the information we need from so much data.

There has been a very interesting attempt to incorporate the application of machine learning techniques to the newsvendor problem (Rudin and Vahn, 2013). This work does not imply any specific relationship between price and demand. On the contrary, the amount of variables that determine the demand is the result of collecting exogenous and endogenous information. All in all, the problem is to find an optimal stock function $q(\cdot)$ of those variables (or features) such that the empirical risk with respect to the dataset S_n (Alpaydin, 2014) is minimized over the period in which the data were collected:

$$\min_{q \in Q:\{f:X\to\mathbb{R}\}} \hat{R}(q(\cdot);S_n) \quad = \quad \frac{1}{n}\sum_{i=1}^{n}\left[b(d_i - q(\mathbf{x_i}))^+ + h(q(\mathbf{x_i}) - d_i)^+\right],$$

where b and h are, respectively, the unit backordering and holding costs, d_i is the demand observed in period i, and $\mathbf{x_i}$ is a vector containing the features in period i. Once this is done, we can observe the features for the period $n+1$ and use $q(\cdot)$ to make an educated decision on the ordering quantity that is more convenient. The function $q(\cdot)$ is selected among those in a class Q which can be, for instance, the class of linear functions.

We believe this approach is an excellent starting point for more advanced models. For example, we may want to include the price as a decision variable and maximize the average profit over a set of n periods. The price can also be a function of the features selected among a predefined class of functions. Moreover, we can add a mean-variance tradeoff to the analysis and study this enhanced big data newsvendor problem in order to see how the availability of large datasets improve the decision-making process and what is the impact that different features have on price and stock policies for different risk profiles.

Appendix

Proofs of Selected Theorems and Lemmas

Appendix A

Concavity with Additive Demand

Theorem 2.1

Proof. First, we show that $p^*(\cdot)$ is concave. Indeed, the second-order derivative of this function yields

$$\begin{aligned} \frac{d^2p^*(z)}{dz^2} &= \left(-f(z) - \frac{\lambda\sigma^{2'}(z)(1-F(z))}{\lambda\sigma^2(z)+b}\right)\frac{1-4\lambda(z-\mu(z))p^*(z)}{2(\lambda\sigma^2(z)+b)} \\ &\quad -4\lambda\frac{1-F(z)}{2(\lambda\sigma^2(z)+b)}\left[F(z)p^*(z)+(z-\mu(z))\frac{dp^*(z)}{dz}\right], \end{aligned} \tag{A.1}$$

which is clearly nonpositive, since $z-\mu(z) \geq 0$ for $z \in [A, B]$ and the function $p^*(\cdot)$ is increasing.

Analyzing (2.9) at the extreme points of the interval $[A, B]$ yields:

$$\left.\frac{dP^*(z)}{dz}\right|_{z=A} = \frac{A+a+cb}{2b} - \frac{2bc}{2b} > 0, \quad \text{by assumption (A5)}, \tag{A.2}$$
$$\left.\frac{dP^*(z)}{dz}\right|_{z=B} = -c < 0. \tag{A.3}$$

Therefore, there exists a point $z^* \in (A, B)$ at which the function $P^*(\cdot)$ attains its maximum. We claim that such a point is unique by showing that $P^*(\cdot)$ is a concave function. Indeed,

$$\begin{aligned}\frac{d^2P^*(z)}{dz^2} &= \left(\frac{dp^*(z)}{dz}(1-F(z)) - f(z)p^*(z)\right)[1-2\lambda(z-\mu(z))p^*(z)] \\ &\quad -2\lambda p^*(z)(1-F(z))\left[F(z)p^*(z) + (z-\mu(z))\frac{dp^*(z)}{dz}\right].\end{aligned} \tag{A.4}$$

Note the similarity of the last term on the right-hand side to the right hand side of equation (A.1); hence this term is also nonpositive. The first term of the right-hand side,however, might take on a different sign. While (A3) guarantees that $[1-2\lambda(z-\mu(z))p^*(z)] \geq 0$, it is unclear what happens with the first part of the term. If we force it to be nonpositive, we have that

$$\frac{dp^*(z)}{dz}(1-F(z)) - f(z)p^*(z) \leq 0.$$

However, since the function $z \mapsto \dfrac{dp^*(z)}{dz}$ is decreasing, it attains its maximum at $z = A$:

$$\left.\frac{dp^*(z)}{dz}\right|_{z=A} = \frac{1}{2b} \implies \frac{dp^*(z)}{dz}(1-F(z)) - f(z)p^*(z) \leq \frac{1}{2b}(1-F(z)) - f(z)p^*(z) \leq 0.$$

The last inequality is equivalent to

$$\frac{bp^*(z)f(z)}{1-F(z)} \geq \frac{1}{2},$$

from which we conclude that $P^*(\cdot)$ is concave if $\xi^*(z) \geq \frac{1}{2}, \forall z \in [A, B]$. □

Lemma 2.2

Proof. Given the complexity of equation (2.12), we proceed to see how $\tilde{z}^*$ varies with changing λ. Thus, if we rename the left-hand side of (2.12) as $g(\lambda, z)$, the following holds

$$\begin{aligned}
\frac{\partial g(\lambda, z)}{\partial \lambda} &= -2p(1-F(z)[z-\mu(z)], \\
\frac{\partial g(\lambda, z)}{\partial z} &= f(z)[2\lambda p(z-\mu(z))-1] - 2\lambda pF(z)(1-F(z)).
\end{aligned}$$

By means of the Implicit Function Theorem (Stewart, 2011) it turns out that

$$\frac{d\tilde{z}^*(\lambda)}{d\lambda} = \frac{2p(1-F(\tilde{z}^*(\lambda))[\tilde{z}^*(\lambda)-\mu(\tilde{z}^*(\lambda))]}{f(\tilde{z}^*(\lambda))[2\lambda p(\tilde{z}^*(\lambda)-\mu(\tilde{z}^*(\lambda)))-1] - 2\lambda pF(\tilde{z}^*(\lambda))(1-F(\tilde{z}^*(\lambda)))}. \quad (A.5)$$

The numerator in the formula above is always nonnegative. The second term of the denominator is always nonnegative as well but it is subtracted. Then, if the first term in the denominator is negative, the entire expression will become negative. This occurs, if:

$$2\lambda p(\tilde{z}^*(\lambda)-\mu(\tilde{z}^*(\lambda)))-1 < 0,$$

whence

$$\lambda < \frac{1}{2p(\tilde{z}^*(\lambda) - \mu(\tilde{z}^*(\lambda)))} \leq \frac{1}{2p_{max}(B - E(\epsilon))},$$

but this is guaranteed by (A3). Therefore, $\tilde{z}^*(\cdot)$ is decreasing. $\square$

Theorem 2.2

Proof. We must show that the Hessian matrix of $P(\cdot)$ is negative semidefinite. This implies that

$$\frac{\partial^2 P(p,z)}{\partial z^2} \leq 0 \quad \text{and} \quad \Delta(p,z) = \frac{\partial^2 P(p,z)}{\partial p^2}\frac{\partial^2 P(p,z)}{\partial z^2} - \left(\frac{\partial^2 P(p,z)}{\partial p \partial z}\right)^2 \geq 0.$$

Note that the validity of the conditions above also implies that $\frac{\partial^2 P(p,z)}{\partial p^2} \leq 0$. Equations (2.5) and (2.11) are negative and nonnegative respectively. Likewise, the second-order partial derivative of $z \mapsto P(p,z)$ with respect to z is nonpositive as a consequence of a straightforward application of (A3):

$$\frac{\partial^2 P(p,z)}{\partial z^2} = -2\lambda p^2 F(z)[1 - F(z)] - pf(z)[1 - 2\lambda p(z - \mu(z))] \leq 0.$$

Finally, it remains to check that the determinant of the Hessian matrix is nonnegative:

$$\begin{aligned}\Delta(p,z) &= 2[\lambda\sigma^2(z)+b][2\lambda p^2 F(z)(1-F(z))+pf(z)(1-2\lambda p(z-\mu(z)))] \\ &\quad -[1-F(z)]^2[1-4\lambda p(z-\mu(z))]^2]] \\ &\geq (1-F(z))\Big\{\big(4\lambda bp^2F(z)+(1-2\lambda p(z-\mu(z)))\big) \\ &\quad +F(z)[1-4\lambda p(z-\mu(z))]^2-[1-4\lambda p(z-\mu(z))]^2\Big\} \geq 0,\end{aligned}$$

where the inequality follows from the fact that $2(\lambda\sigma^2(z)+b) \geq 2b$ and from assuming that $\xi(p,z) \geq \dfrac{1}{2}$. □

Lemma 2.4

Proof. Given (2.6), it only remains to prove that $c < p^*(z) \leq p_{max}$. Indeed, if $\lambda \in \left(\dfrac{-b}{Var(\epsilon)}, 0\right)$, the right-hand side of (2.7) is always positive and thus $p^*(\cdot)$ is increasing. Besides, $p^*(A) = \dfrac{A+a+cb}{2b} > c$. Therefore, $p^*(z) > c,\ \forall z \in [A,B]$.

Nonetheless, as shown before, $p^*(z)$ may take values greater than p_{max}. Hence, if we require $p^*(B) \leq p_{max} \leq \dfrac{a}{b}$, then $p^*(z) \leq p_{max}$ for any $z \in [A,B]$ (because $p^*(\cdot)$ is increasing). Consequently,

$$p^*(B) = \frac{E(\epsilon)+a+cb}{2(\lambda Var(\epsilon)+b)} \leq \frac{a}{b},$$

which holds whenever

$$\lambda \geq \frac{b(E(\epsilon)-y(c))}{2aVar(\epsilon)},$$

with the right-hand side being negative, as guaranteed by (B4). Now, it remains to require the number $\frac{b(E(\epsilon)-y(c))}{2aVar(\epsilon)}$ being contained in the interval $\left(\frac{-b}{Var(\epsilon)},0\right)$. This occurs, whenever

$$\frac{b(E(\epsilon)-y(c))}{2aVar(\epsilon)} \quad \geq \quad \frac{-b}{Var(\epsilon)},$$

whence we obtain the necessary condition $E(\epsilon) \geq -y(-c)$. This condition, however, is always met by virtue of assumption (B3) and the fact that $E(\epsilon) > A$:

$$-y(c) < A < E(\epsilon) \quad \Longrightarrow \quad -y(-c) < E(\epsilon) - 2bc < E(\epsilon).$$

Therefore, for $\lambda \in \left[\frac{b(E(\epsilon)-y(c)}{2aVar(\epsilon)},0\right)$ the function $P(\cdot,z)$ is concave for all $z \in [A,B]$ and $p^*(z) \leq p_{max}$. □

Theorem 2.3

Proof. The first-order derivative of $P^*(\cdot)$ at the points A and B is given by (A.2) and (A.3), respectively. Therefore, there exists a point $z^* \in (A,B)$ at which the function $P^*(\cdot)$ attains its maximum. Such maximum is unique if $P^*(\cdot)$ is a concave function and we refer to equation (A.4) in order to prove it. Imposing concavity requires that

$$\begin{aligned}[1-2\lambda(z-\mu(z))p^*(z)]\left(\tfrac{dp^*(z)}{dz}[1-F(z)]-f(z)p^*(z)\right) \quad &\leq \\ 2\lambda p^*(z)[1-F(z)]\left(F(z)p^*(z)+(z-\mu(z))\frac{dp^*(z)}{dz}\right)&.\end{aligned}$$

This condition holds for $z = B$. For all other values of z, dividing both sides by $1 - F(z)$ and multiplying by b gives an expression in terms of $\xi^*(z)$:

$$\xi^*(z) \quad \geq \quad b\frac{dp^*(z)}{dz} - \frac{2\lambda p^*(z)b\left[F(z)p^*(z) + (z - \mu(z))\frac{dp^*(z)}{dz}\right]}{1 - 2\lambda(z - \mu(z))p^*(z)}, \quad z \in [A, B],$$

which is a necessary and sufficient condition for $P^*(\cdot)$ to be concave. $\square$

Lemma 2.5

Proof. A straightforward application of (A.1) gives a necessary condition for $\frac{dp^*(\cdot)}{dz}$ to be decreasing:

$$\begin{aligned}\frac{-f(z)(\lambda\sigma^2(z) + b) - 2\lambda(z - \mu(z))(1 - F(z))^2}{\lambda\sigma^2(z) + b}[1 - 4\lambda(z - \mu(z))p^*(z)] \quad &\leq \\ 4\lambda(1 - F(z))\left[F(z)p^*(z) + (z - \mu(z))\frac{dp^*(z)}{dz}\right], \quad z \in [A, B].&\end{aligned}$$

Bounding both sides of this equation yields a sufficient condition for the concavity of $p^*(\cdot)$. The condition above always holds for $z = A$ and $z = B$ (the expression above becomes $-f(A) \leq 0$ and $-f(B)[1 - 4\lambda(B - E(\epsilon))p^*(B)] \leq 0$ respectively). For $A < z < B$, we establish that the largest value of the left-hand side has to be at most equal to the smallest value of the right-hand side. The largest value of the left-hand side is represented by the value that is closest to 0, which is $\frac{1}{b}(-2\lambda(B - E(\epsilon)) - f(z)(\lambda Var(\epsilon) + b))$. Conversely, the smallest value of the right-hand side is $2\lambda\frac{2a + B - E(\epsilon)}{b}$. Therefore, we obtain that

$$\frac{1}{b}[-2\lambda(B - E(\epsilon)) - f(z)(\lambda Var(\epsilon) + b)] \quad \leq \quad 2\lambda\frac{2a + B - E(\epsilon)}{b},$$

whence

$$\lambda \geq \frac{-f(z)b}{4(a+B-E(\epsilon))+f(z)Var(\epsilon)}.$$

Since we want this range to be valid for all z, we can rewrite this as

$$\lambda \geq \max_{z\in[A,B]}\left\{\frac{-f(z)b}{4(a+B-E(\epsilon))+f(z)Var(\epsilon)}\right\}.$$

□

Lemma 2.6

Proof. We analyze again (A.5) under the light of the implicit function theorem, but this time with the condition $\lambda < 0$. The numerator in this equation is still nonnegative, but the second term in the denominator is now nonpositive and is being subtracted. The first term of the denominator is always nonpositive. Hence, if $\tilde{z}^*(\lambda)$ is to decrease with λ we must have that

$$f(\tilde{z}^*(\lambda))[2\lambda p(\tilde{z}^*(\lambda)-\mu(\tilde{z}^*(\lambda)))-1]-2\lambda pF(\tilde{z}^*(\lambda))(1-F(\tilde{z}^*(\lambda))) < 0,$$

but we know that this surely occurs in $p = p_A$ and $p = p_B$, for if there exists $p = p_A$ such that $\tilde{z}^*(\lambda) = A$ this condition simplifies to $-f(A) \leq 0$, which always holds. Moreover, if there exists $p = p_B$ such that $\tilde{z}^*(\lambda) = B$ this condition requires $2\lambda p(B - E(\epsilon)) - 1 \leq 0$, or equivalently $\lambda \leq \dfrac{1}{2p(B-E(\epsilon)}$, which also holds. For all other values of p we can assert that $\dfrac{d\tilde{z}^*(\lambda)}{d\lambda}$ is nonpositive if

$$\lambda \quad \leq \quad \frac{f(\tilde{z}^*(\lambda))}{2p\left[f(\tilde{z}^*(\lambda))(\tilde{z}^*(\lambda) - \mu(\tilde{z}^*(\lambda))) - F(\tilde{z}^*(\lambda))(1 - F(\tilde{z}^*(\lambda)))\right]},$$

where $c < p \leq p_{max}$ and $p \neq p_A, p_B$. If the right-hand side of this equation is positive, then the condition above always holds and $\dfrac{d\tilde{z}^*(\cdot)}{d\lambda} \leq 0$. This right-hand side is positive as long as

$$f(\tilde{z}^*(\lambda))(\tilde{z}^*(\lambda) - \mu(\tilde{z}^*(\lambda))) - F(\tilde{z}^*(\lambda))(1 - F(\tilde{z}^*(\lambda))) \quad > \quad 0,$$

whence, after dividing by $1 - F(\tilde{z}^*(\lambda))$ and multiplying by bp we obtain that

$$\hat{\varepsilon}^*(p) \quad > \quad bp\frac{F(\tilde{z}^*(\lambda))}{\tilde{z}^*(\lambda) - \mu(\tilde{z}^*(\lambda))}, \quad c < p \leq p_{max},\ p \neq p_A, p_B.$$

An upper bound of the right-hand side is given by $bp\dfrac{1}{B - E(\epsilon)}$Therefore, we can conclude that if

$$\hat{\varepsilon}^*(p) > bp\frac{1}{B - E(\epsilon)}, \quad c < p \leq p_{max},$$

then, given a price p, $\tilde{z}^*(\cdot)$ decreases. In other words, when λ decreases, i.e., as we focus on more risk-seeking situations, $\tilde{z}^*(\cdot)$ increases. $\square$

Appendix B

Unimodality with Additive Demand

Lemma 3.1

Proof. The first claim is supported by the numerator of (3.3) being strictly positive since $\mu(z) + a + cb \geq A + a + cb = A + y(c) + 2cb > 0$.

To prove that $p^*(z) \leq p_{max}$, we focus first on the risk-neutral case. When $\lambda = 0$, the optimal price $p^*(\cdot)$ is an increasing function in z. Indeed, in this case $p^*(z)|_{\lambda=0} = \dfrac{\mu(z) + a + cb}{2b}$ and $p^{*\prime}(z) = \dfrac{1 - F(z)}{2b} > 0$. Therefore, when $\lambda = 0$ the optimal price has a maximum value $p^*(B) = \dfrac{a + cb}{2b}$. This value is smaller than p_{max} because $y(c) + 2A \geq 0$. Hence, our only assumption serves the purpose of bounding the optimal price from above.

Given that, per (3.3), $p^*(z) \leq p^*(z)|_{\lambda=0}$, we conclude that in the risk-averse case $p^*(z) \leq p_{max}$. $\square$

Lemma 3.2

Proof. The derivative of the optimal price with respect to the safety stock is

$$p^{*'}(z) \quad = \quad \frac{1-F(z)}{2(\lambda\sigma^2(z)+b)}(1-4\lambda(z-\mu(z))p^*(z)). \tag{B.1}$$

It is not guaranteed that this is positive. As a matter of fact, we have that

$$p^{*'}(z) \geq 0 \quad \Longleftrightarrow \quad \lambda < \frac{1}{4(z-\mu(z))p^*(z)}. \tag{B.2}$$

It is easy to see that $p^{*'}(A) = 1/(2b)$ and $p^{*'}(B) = 0$. Also $p^{*'}(z) = 0$ in (A, B) if and only if there are solutions to the equation $p^*(z) = \dfrac{1}{4\lambda(z-\mu(z))}$. On the other hand, the second derivative of the optimal price $p^*(\cdot)$ is

$$\begin{aligned} p^{*''}(z) \quad &= \quad \left(-f(z) - \frac{\lambda\sigma^{2'}(z)(1-F(z))}{\lambda\sigma^2(z)+b}\right)\frac{1-4\lambda(z-\mu(z))p^*(z)}{2(\lambda\sigma^2(z)+b)} \\ &\quad -4\lambda\frac{1-F(z)}{2(\lambda\sigma^2(z)+b)}(F(z)p^*(z)+(z-\mu(z))p^{*'}(z)), \end{aligned} \tag{B.3}$$

which, particularized for the points where $p^{*'}(z) = 0$ is

$$p^{*''}(z)\Big|_{p^{*'}(z)=0} \quad = \quad -4\lambda\frac{1-F(z)}{2(\lambda\sigma^2(z)+b)}F(z)p^*(z) \quad < 0. \tag{B.4}$$

Therefore, any critical point that exists in (A, B) is a maximum. Since $p^{*'}(A) > 0$ and $p^{*'}(B) = 0$, the equation $p^*(z) = \dfrac{1}{4\lambda(z-\mu(z))}$ has at most one solution in (A, B) and one of the following outcomes occur: if such a solution does not exist, the function $p^*(\cdot)$ is increasing in $[A, B)$ with a maximum at $z = B$; if such a solution exists at a point z_ψ, the function $p^*(\cdot)$ increases in $[A, z_\psi)$, has a maximum at $z = z_\psi$, decreases in $[z_\psi, B)$, and has an inflection point at $z = B$. It is consequently quasiconcave (unimodal). $\square$

Lemma 3.3

Proof. By Lemma 3.1, when $\lambda \geq 0$, $p^*(z) \leq p_{max}$, $\forall z \in [A, B]$. It remains to validate the conditions for the optimal price to be greater than the replenishment cost.

If we impose in (3.3) that the optimal price is at least as large as the replenishment cost, it follows that $p^*(z) \geq c$ when $\lambda \leq \dfrac{\mu(z) + y(c)}{2c\sigma^2(z)}$. Therefore this holds $\forall z \in [A, B]$ as long as

$$\lambda \quad \leq \quad \min_{z \in [A,B]} \frac{\mu(z) + y(c)}{2c\sigma^2(z)}.$$

Let $t(z) = \dfrac{\mu(z) + y(c)}{2c\sigma^2(z)}$. We will prove that this function is decreasing. Its first derivative is $t'(z) = \dfrac{(1 - F(z))\sigma^2(z) - \sigma^{2'}(z)(\mu(z) + y(c))}{2c\left(\sigma^2(z)\right)^2}$. While the denominator is always nonnegative, we can also prove that the numerator is nonpositive. Using the equality $\sigma^{2'}(z) = 2(1 - F(z))(z - \mu(z))$ the numerator is nonpositive if

$$\sigma^2(z) \quad \leq \quad 2(z - \mu(z))(\mu(z) + y(c)).$$

Both sides of this equation are nonnegative in $[A, B]$ and equal to 0 at $z = A$. Moreover, $[2(z - \mu(z))(\mu(z) + y(c))]' = 2F(z)(\mu(z) + y(c)) + \sigma^{2'}(z) \geq \sigma^{2'}(z)$. Therefore, it is clear that $\sigma^2(z) - 2(z - \mu(z))(\mu(z) + y(c)) \leq 0$ and t is a decreasing function of z. This implies that

$p^*(z) \geq c, \ \forall z \in [A, B]$ if and only if

$$\lambda \leq \min_{z \in [A,B]} t(z) = t(B) \quad = \frac{y(c)}{2cVar(\epsilon)}.$$

□

Theorem 3.1

Proof. We will analyze $P_2^*(\cdot)$ in $[A, z_c]$. $P_2^*(\cdot)$ is a continuous function with $P_2^{*'}(A) = p^*(A) - c > 0$ and $P_2^{*'}(z_c) < 0$. The last inequality follows because in case that $P^*(\cdot)$ is a piecewise, nonlinear function, then it is also smooth (i.e. $P_2^*(z_c)$=$P_1^*(z_c)$) and $P_1^*(\cdot)$ is a decreasing function in $[z_c, B]$. Therefore, there must be at least one point in $[A, z_c]$ where $P_2^{*'}(z) = 0$. This point is unique and confers quasi-concavity to $P_2^*(\cdot)$ if $P_2^{*''}(z)|_{P_2^{*'}(z)=0} < 0$. Per (3.5), at the critical points $1 - 2\lambda(z - \mu(z))p^*(z) = \dfrac{c}{p^*(z)(1 - F(z))}$ and we can write (3.6) in terms of the failure rate $h(z)$ as

$$\frac{cp^{*'}(z)}{p^*(z)} - h(z)c - 2(1 - F(z))\lambda p^*(z)\left(F(z)p^*(z) + (z - \mu(z))p^{*'}(z)\right) \quad < \quad 0.$$

By using the expression of the LSR elasticity at the optimal price $p^*(z)$ in additive models, $\xi^*(z) = bp^*(z)h(z)$, and observing that $F(z)p^*(z) + (z - \mu(z))p^{*'}(z) = [(z - \mu(z))p^*(z)]'$ we can rewrite the formula above as

$$\xi^*(z) \quad > \quad b\left(p^{*'}(z) - \frac{2(1 - F(z))\lambda p^*(z)^2}{c}[(z - \mu(z))p^*(z)]'\right).$$

□

Theorem 3.2

Proof. Assume that $p^*(\cdot)$ is a unimodal function and consider the two subintervals $[A, z_\psi]$ and $(z_\psi, z_c]$. We will apply condition (3.8) to both subintervals. In $[A, z_\psi]$ the optimal price is nondecreasing and concave with only one critical point at $z = z_\psi$. It follows that the unimodality is guaranteed as long as $\xi^*(z) > 1/2, \forall z \in [A, z_\psi]$. In $(z_\psi, z_c]$ the optimal price is nonincreasing with only one critical point at $z = z_c$ if $z = B$ (otherwise the function is strictly decreasing in $(z_\psi, z_c]$). Therefore the first term in condition (3.8) is negative. The second term in (3.8) takes the opposite sign of $[(z - \mu(z))p^*(z)]' = F(z)p^*(z) + (z - \mu(z))p^{*'}(z)$. If $(z-\mu(z))p^*(z)$ is a nondecreasing function at the points that satisfy $P_2^{*'}(z) = 0$, then the second term in (3.8) is negative as well and a valid lower bound is $\xi^*(z) \geq 0$ which, by the definition of LSR elasticity, always holds. However, in general we do not know the sign of the slope of $(z - \mu(z))p^*(z)$ at those points and we have to bound ξ^* by using the most restrictive condition, which is given by the case when $(z - \mu(z))p^*(z)$ is a decreasing function and the second term of (3.8) is positive:

$$\begin{aligned}
\xi^*(z) &> b\left(p^{*'}(z) - \frac{2(1-F(z))\lambda p^*(z)^2}{c}\left[(z-\mu(z))p^*(z)\right]'\right) \\
&\leq -\frac{2(1-F(z))\lambda b p^*(z)^2}{c}\left[(z-\mu(z))p^*(z)\right]'. \qquad \text{(B.5)}
\end{aligned}$$

A lower bound for $\xi^*(\cdot)$ is given by the product of the factors of (B.5). In turn, a lower bound of $[(z - \mu(z))p^*(z)]'$ in $(z_\psi, z_c]$ is

$$F(z)p^*(z) + (z - \mu(z))p^{*'}(z) \geq F(z_\psi)p^*(z_c) + (z_c - \mu(z_c))p^{*'}(z),$$

where we used the fact that $[z-\mu(z)]' \geq 0$. In finding a good lower bound of $[(z - \mu(z))p^*(z)]'$

we need to find the lowest value that $p^{*'}(\cdot)$ may take in $(z_\psi, z_c]$, where this function is negative. Per (B.1) and (3.5), we have that

$$\begin{aligned}
p^{*'}(z)\Big|_{P^{*'}(z)=0} &= \frac{1-F(z)}{2\left(\lambda\sigma^2(z)+b\right)}\left(1-4\lambda(z-\mu(z))p^*(z)\right)\\
&= \frac{1-F(z)}{2\left(\lambda\sigma^2(z)+b\right)}\left(1-2\lambda(z-\mu(z))p^*(z)\right)-\frac{1-F(z)}{2\left(\lambda\sigma^2(z)+b\right)}2\lambda(z-\mu(z))p^*(z)\\
&= \frac{c}{2p\left(\lambda\sigma^2(z)+b\right)}-\frac{1-F(z)}{2\left(\lambda\sigma^2(z)+b\right)}2\lambda(z-\mu(z))p^*(z)\\
&= \frac{1}{2\left(\lambda\sigma^2(z)+b\right)}\left(\frac{c}{p}-(1-F(z))2\lambda(z-\mu(z))p^*(z)\right)\\
&= \frac{1}{2\left(\lambda\sigma^2(z)+b\right)}\left(\frac{2c}{p}-(1-F(z))\right)\\
&= -\frac{1}{2\left(\lambda\sigma^2(z)+b\right)}\left(1-\frac{2c}{p}-F(z)\right)\\
&\geq -\frac{1}{2\left(\lambda\sigma^2(z_\psi)+b\right)}.
\end{aligned}$$

Finally the lower bound of $[(z-\mu(z))p^*(z)]'$ is

$$F(z)p^*(z)+(z-\mu(z))p^{*'}(z) \quad \geq \quad \underbrace{F(z_\psi)p^*(z_c)}_{\substack{\text{smallest positive value of}\\ F(z)p^*(z)}} \quad \underbrace{-\frac{z_c-\mu(z_c)}{2\left(\lambda\sigma^2(z_\psi)+b\right)}}_{\substack{\text{largest negative value of}\\ (z-\mu(z)p^{*'}(z)}} \; .$$

Inserting this expression in (B.5) yields our lower bound on the LSR elasticity:

$$\xi^*(z) \quad > \quad -\frac{2(1-F(z_\psi))\lambda b p^*(z_\psi)^2}{c}\left(F(z_\psi)p^*(z_c)-\frac{z_c-\mu(z_c)}{2\left(\lambda\sigma^2(z_\psi)+b\right)}\right).$$

□

Lemma 3.4

Proof. We analyze this function in two subintervals, $[A, \tilde{z})$ and $(\tilde{z}, B]$. Consider equation (B.1). Clearly, there is always a critical point at $z = B$. Also, per this equation, the function $p^*(\cdot)$ is strictly increasing $\forall z : p^*(z) > 0$. When $p^*(z) < 0$ (i.e. $\tilde{z} < z \leq B$) the function $p^*(\cdot)$ tends to $-\infty$ as we approach $\tilde{z}$ from the right and therefore it is concave and increasing at its right limit towards B, reaching a value at $z = B$ of $p^*(B) = \dfrac{a + cb}{2(\lambda Var(\epsilon) + b)}$. Assume it is decreasing in some region in $(\tilde{z}, B)$. Then the function must present a local maximum in such interval. However, this is not possible since, per (B.4), if there is a critical point in $(\tilde{z}, B)$ the function is convex at such point and should be a minimum. Therefore the optimal price $p^*(\cdot)$ has only one critical point at $z = B$, which is also an inflection point. We conclude that $p^*(\cdot)$ is strictly increasing in $(\tilde{z}, B)$. $\square$

Lemma 3.5

Proof. The function $P_3^*(\cdot)$ has at least one critical point because $P_3^{*'}(A) > 0$ and $P_3^{*'}(B) < 0$. The first-order optimality condition solves the equation

$$\lambda = \frac{p_{max}(1 - F(z)) - c}{p_{max}^2 \sigma^{2'}(z)} = \frac{1 - \dfrac{c}{p_{max}(1 - F(z))}}{2p_{max}^2(z - \mu(z))}.$$

The function in the right-hand side of this equation is decreasing because its denominator is increases in z and its numerator decreases in z. Therefore, it attains the constant value λ only once. Given the signs of $P_3^{*'}(A)$ and $P_3^{*'}(B)$, this unique critical point is a maximum. $\square$

Lemma 3.6

Proof. This result follows easily because $\lim_{\lambda \to \lambda^-_{z_{p_{max}}}} \lambda_t(\lambda) = -\infty$ (remember that $z_{p_{max}}(\lambda) = B$ if $\lambda \in [\lambda_{z_{p_{max}}}, 0))$ and because $\lambda_t(\lambda)$ increases as λ decreases. The latter is easy to see because $z_{p_{max}}$ decreases when λ decreases, which makes the numerator of $\lambda_t(\lambda)$ increase and the denominator decrease as λ decreases. □

Lemma 3.7

Proof. For clarity of exposition, we will let $\lambda_B = 0$ and $\lambda_A < \lambda_B$, although this result is straightforward to show for any relation $\lambda_A < \lambda_B \leq 0$.

Let $\hat{z} = \min\left\{z : F(z) = 1 - \frac{c}{p^*(z)|_{\lambda=0}}\right\}$ be the first maximum of the risk-neutral problem. To see that this point is indeed a maximum, consider the risk-neutral problem: in this case $z_{p_{max}} = B$. There is at least a critical point because $P_2^{*'}(A) > 0$, and $P_2^{*'}(B) < 0$. Given the sign of $P_2^{*'}(A)$ this first critical point, $\hat{z}$, will be a maximum.

Compare the first-order optimality conditions of the risk-neutral problem and the risk-seeking problems of the functions $P_2^*(\cdot)$ and $P_3^*(\cdot)$ under the light of the optimal price. Taking into account that $p^*(z) > 0$ whenever $P_2^*(\cdot)$ applies, in any risk-seeking instance and for any safety stock we have that $p^*(z)|_{\lambda=\lambda_A} \geq p^*(z)|_{\lambda=\lambda_B}$. Therefore

$$1 - \frac{c}{p^*(z)|_{\lambda=\lambda_A}(1 - 2\lambda_A(z - \mu(z))p^*(z)|_{\lambda=\lambda_A})} \geq 1 - \frac{c}{p^*(z)|_{\lambda=\lambda_B}},$$

and

$$1 - \frac{c}{p_{max}} - \lambda_A p_{max} \sigma^{2'}(z) \geq 1 - \frac{c}{p_{max}}.$$

Consequently both first-order conditions will have their first solution at a safety stock higher than $\hat{z}$. This is illustrated in Figure B.1, where a risk-neutral condition and a risk-seeking condition are shown. □

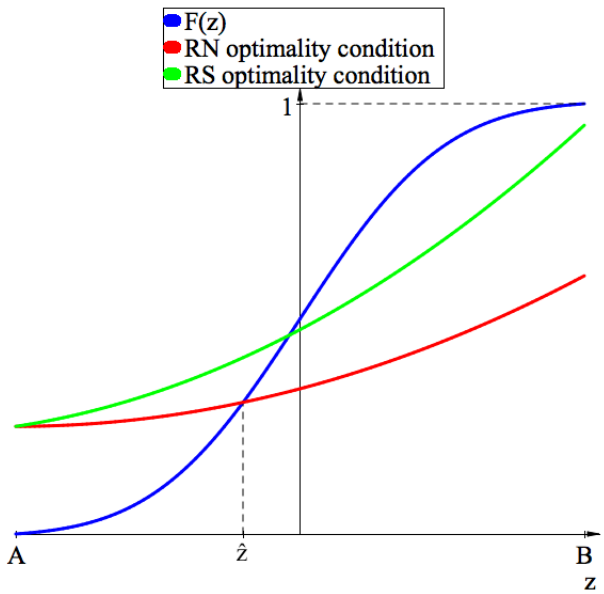


Figure B.1: Illustration of Lemma 3.7

Lemma 3.8

Proof. The function $P_2^*(\cdot)$ has at least one solution in $[A, B]$ because $P_2^{*'}(A) > 0$ and $P_2^{*'}(B) < 0$. Using (3.11), consider the equation $P_2^{*'}(z) = 0$. This can be written as

$$\lambda = \frac{1 - \dfrac{c}{p^*(z)(1 - F(z))}}{2(z - \mu(z))p^*(z)}.$$

Compare both sides of this equation. The number of times that the function of the right-hand side crosses the constant λ is the number of critical points of $P_2^*(\cdot)$. Since

$$\lim_{z\to A} \frac{1-\dfrac{c}{p^*(z)(1-F(z))}}{2(z-\mu(z))p^*(z)} \quad = \quad \infty,$$

if this function is always decreasing, it will cross λ exactly once. Taking into account that the denominator $2(z-\mu(z))p^*(z)$ is nondecreasing, it is enough that the numerator is decreasing:

$$\left[1-\frac{c}{p^*(z)(1-F(z))}\right]' \quad = \quad \frac{{p^*}'(z)(1-F(z))-f(z)p^*(z)}{p^*(z)^2(1-F(z))^2},$$

which follows if ${p^*}'(z)(1-F(z))-f(z)p^*(z)<0$ or, in terms of the LSR elasticity, if $\xi^*(z)>b{p^*}'(z)$.

An upper bound for ${p^*}'(\cdot)$ at the critical points can be obtained in the same fashion as in Theorem 3.2, thus obtaining:

$$\begin{aligned}
{p^*}'(z)\Big|_{{P^*}'(z)=0} &= -\frac{1}{2\left(\lambda\sigma^2(z)+b\right)}\left(1-\frac{2c}{p}-F(z)\right) \\
&\le \frac{c}{p^*(z)\left(\lambda\sigma^2(z)+b\right)} = \frac{2c}{\mu(z)+a+cb} \\
&\le \frac{2c}{A+a+cb},
\end{aligned}$$

whence the condition $\xi^*(z)>\dfrac{c}{p^*(A)}$ can be derived. A sharper bound can be obtained if we consider the smallest maximum of a less risk-seeking instance, $\zeta_2(\tilde{\lambda})$, given that, per

Lemma 3.7, the first maximum will occur in the interval $[\zeta_2(\tilde{\lambda}), B]$:

$$\xi^*(z) \quad > \quad \frac{2bc}{\mu(\zeta_2(\tilde{\lambda})) + a + cb},$$

which only needs to hold in $[\zeta_2(\tilde{\lambda}), B]$. □

Theorem 3.3

Proof. When $\lambda > \lambda_t(\lambda)$, $P^{*'}(z_{p_{max}}) < 0$. Because of Lemma 3.5, the function $P_3^*(\cdot)$ is decreasing in $[z_{p_{max}}, B]$. Therefore, $\max\limits_{z\in[A,B]} P^*(z) = \max\limits_{z\in[A,z_{p_{max}}]} P_2^*(z) = \max\limits_{z\in[\zeta_2(\tilde{\lambda}),z_{p_{max}}]} P_2^*(z)$. The last equality is a consequence of Lemma 3.7.

If $P_2^*(\cdot)$ is unimodal in $[A, B]$, then its only maximum occurs in the interval $[\zeta_2(\tilde{\lambda}), z_{p_{max}}]$ and can be easily attained by solving $P_2^{*'}(z) = 0$.

For the case $\lambda = \lambda_t(\lambda)$, $z_{p_{max}}$ is the critical point of $P_3^*(\cdot)$ and a critical point of $P_2^*(\cdot)$. As a result, equation (3.14) still holds, as well as the rest of the theorem. □

Theorem 3.4

Proof. When $\lambda < \lambda_t(\lambda)$, $P^{*'}(z_{p_{max}}) > 0$. Because of Lemma 3.5, the function $P_3^*(\cdot)$ has its only maximum in $[z_{p_{max}}, B]$. In general, the function $P_2^*(\cdot)$ may have several critical points in $[A, z_{p_{max}}]$. Therefore, $\max\limits_{z\in[A,B]} P^*(z) = \max\{P^*(\zeta_3(\lambda)), \max\limits_{z\in[\zeta_2(\tilde{\lambda}),z_{p_{max}}]} P_2^*(z)\}$.

If $P_2^*(\cdot)$ is unimodal in $[A, B]$, then its only maximum occurs at in the interval $[z_{p_{max}}, B]$, where $P^*(z) = P_3^*(z)$. Hence, the maximum of $P^*(\cdot)$ is attained at the only point that solves $P_3^{*'}(z) = 0$, which is $\zeta_3(\lambda)$. □

Lemma 3.9

Proof. Let $\Pi^*(z,\lambda)$ be the profit at the hedged price $\pi^*(z,\lambda)$. Recall that the profit is a random variable. From (3.1) we can we can redefine our performance measure at the hedged price $\pi^*(z,\lambda)$:

$$P(z,\lambda) \quad = \quad \underbrace{\pi^*(z,\lambda)\left(\mu(z) + y(\pi^*(z,\lambda))\right) - c\left(z + y(\pi^*(z,\lambda))\right)}_{\mathbb{E}(\Pi^*(z,\lambda))} - \lambda \underbrace{\pi^*(z,\lambda)^2\sigma^2(z)}_{Var(\Pi^*(z,\lambda))}.$$

Consider first the risk-averse case. When $\lambda > 0$:

$$\mathbb{E}\left(\Pi^*(z,\lambda)\right) = \begin{cases} c(\mu(z) - z) & \text{if } z > z_c, \\ p^*(z,\lambda)(\mu(z) + a - bp^*(z,\lambda)) - c(z + a - bp^*(z,\lambda)) & \text{if } z \le z_c. \end{cases}$$

$$Var\left(\Pi^*(z,\lambda)\right) = \begin{cases} c^2\sigma^2(z) & \text{if } z > z_c, \\ p^*(z,\lambda)^2\sigma^2(z) & \text{if } z \le z_c. \end{cases}$$

For any given stock factor z the derivative of these two functions are:

$$\frac{\partial}{\partial\lambda}\mathbb{E}\left(\Pi^*(z,\lambda)\right) = \begin{cases} 0 & \text{if } z > z_c, \\ \dfrac{\partial p^*(z,\lambda)}{\partial\lambda}\left(\mu(z) + a + b\left(c - 2p^*(z,\lambda)\right)\right) & \text{if } z \le z_c. \end{cases}$$

$$\frac{\partial}{\partial\lambda}Var\left(\Pi^*(z,\lambda)\right) = \begin{cases} 0 & \text{if } z > z_c, \\ -\dfrac{2\left(\sigma^2(z)\right)^2}{\lambda\sigma^2(z) + b}p^*(z,\lambda)^2 & \text{if } z \le z_c. \end{cases}$$

Given that $\dfrac{\partial p^*(z,\lambda)}{\partial\lambda} = -\dfrac{\sigma^2(z)}{\lambda\sigma^2(z) + b}p^*(z,\lambda) \le 0$ and that, per (3.3), $\mu(z) + a +$

$b\left(c-2p^*(z,\lambda)\right)>0$ in risk-averse cases, we conclude that the expected profit for a given stock factor at the hedged optimal price does not increase with λ. Also, $\frac{\partial}{\partial\lambda}Var\left(\Pi^*(z,\lambda)\right)\leq 0$ (i.e. as λ increases, the variance of the profit does not increase).

Now consider the risk-seeking case. When $\lambda<0$:

$$\mathbb{E}\left(\Pi^*(z,\lambda)\right)=\begin{cases} p^*(z,\lambda)(\mu(z)+a-bp^*(z,\lambda)-c(z+a-bp^*(z,\lambda)) & \text{if } z\leq z_{p_{max}},\\ p_{max}(\mu(z)+a-bp_{max})-c(z+a-bp_{max}) & \text{if } z>z_{p_{max}}.\end{cases}$$

$$Var\left(\Pi^*(z,\lambda)\right)=\begin{cases} p^*(z,\lambda)^2\sigma^2(z) & \text{if } z\leq z_{p_{max}},\\ p_{max}^2\sigma^2(z) & \text{if } z>z_{p_{max}}.\end{cases}$$

For any given stock factor z the derivative of these two functions are:

$$\frac{\partial}{\partial\lambda}\mathbb{E}\left(\Pi^*(z,\lambda)\right)=\begin{cases} \frac{\partial p^*(z,\lambda)}{\partial\lambda}\left(\mu(z)+a+b\left(c-2p^*(z,\lambda)\right)\right) & \text{if } z\leq z_{p_{max}},\\ 0 & \text{if } z>z_{p_{max}}.\end{cases}$$

$$\frac{\partial}{\partial\lambda}Var\left(\Pi^*(z,\lambda)\right)=\begin{cases} -\frac{2\left(\sigma^2(z)\right)^2}{\lambda\sigma^2(z)+b}p^*(z,\lambda)^2 & \text{if } z\leq z_{p_{max}},\\ 0 & \text{if } z>z_{p_{max}}.\end{cases}$$

When $z\leq z_{p_{max}}$, $\lambda\sigma^2(z)+b\geq 0$ and $p^*(z,\lambda)>c$. Therefore

$$\frac{\partial p^*(z,\lambda)}{\partial\lambda}=-\frac{\sigma^2(z)}{\lambda\sigma^2(z)+b}p^*(z,\lambda)\leq 0.$$

Per (3.3), $\mu(z)+a+b\left(c-2p^*(z,\lambda)\right)<0$ in risk-seeking cases, and we conclude that the expected profit for a given stock factor at the hedged optimal price does not decrease with λ (i.e. as λ decreases, the expected profit does not increase). Also, $\dfrac{\partial}{\partial \lambda} Var\left(\Pi^*(z,\lambda)\right) \leq 0$ (i.e. as λ decreases, the variance of the profit increases). $\square$

Appendix C

Unimodality with Multiplicative Demand

Lemma 4.1

Proof. Let us define $\alpha = 2\lambda\sigma^2(z)a(b-1) \geq 0, \beta = (b-1)\mu(z) > 0$, and $\gamma = bcz > 0$. We can rewrite (4.7) as follows:

$$\alpha p^*(z)^{\frac{2b_2-b_1}{b_2}} - \beta p^*(z) + \gamma \quad = \quad 0. \tag{C.1}$$

Let us consider the following cases:

1. $\underline{1 < b < 2}$: let $q^*(z) = p^*(z)^{1/b_2}$. Since $b_1 > b_2$ and $2b_2 - b_1 < b_2$, (C.1) can be written as the following polynomial in descending order of powers from left to right:

$$-\beta q^*(z)^{b_2} + \alpha q^*(z)^{2b_2-b_1} + \gamma \quad = \quad 0. \tag{C.2}$$

2. <u>$b = 2$:</u> in this case (C.1) has the following unique solution: $p^*(z) = \dfrac{\alpha + \gamma}{\beta}$.

3. <u>$b > 2$:</u> let $q^*(z) = p^*(z)^{1/b_2}$. Since $b_1 > 2b_2$ and $b_1 - 2b_2 < b_1 - b_2$, we can rewrite (C.1) as the following polynomial in descending order of powers from left to right:

$$\beta q^*(z)^{b_1 - b_2} - \gamma q^*(z)^{b_1 - 2b_2} - \alpha \quad = \quad 0. \tag{C.3}$$

Per Descartes' Rule of Signs, polynomials (C.2) and (C.3) only have one positive real root. Undoing the change of variables to recover $p^*(z)$ will yield a single positive real root in (C.1). It is at this point where the assumptions made on the elasticity of the demand make sense. Accepting b as a rational number allows a change of variables that helps us write (C.1) as a polynomial. It is easier to handle these equations because there are well-known theorems that state their number and sign of the roots, as it is the case with Descartes' Rule of Signs. A positive real root of any of the transformed polynomials will correspond to a positive real root of (C.1). Nevertheless, this one-to-one correspondence between the positive real roots of the polynomials and the positive real roots of (C.1) does not exist in the other direction (i.e. a positive real root in (C.1) may not come from a positive real root of any of the polynomials). A straightforward example can be found when $b_2 = 4$ and the polynomial (C.3) has the pair of conjugate imaginary roots $\pm i$ or any negative real root. Undoing the change of variables would yield a positive real root of (C.1). It is for this reason that we presume b_2 with odd parity, for in this case conjugated imaginary roots and negative real roots will never be transformed into positive real roots. Hence, the one-to-one correspondence between the positive real roots of the polynomials and the positive real roots of (C.1) is attained and we can assert that (C.1) has only one positive real root. This assumption greatly simplifies the problem and has a negligible impact on its accuracy: any irrational number can be well approximated by a rational number and any rational number expressed as a quotient can also be slightly modified if needed so the parity of the denominator is odd. □

Theorem 4.1

Proof. For a given stock factor z we have that

$$\begin{aligned}
\lim_{p\to 0^+}\frac{\partial P(p,z)}{\partial p} &= \lim_{p\to 0^+}\left(\frac{2a^2(b-1)\sigma^2(z)\lambda}{p^{2b-1}}-\frac{a(b-1)\mu(z)}{p^b}+\frac{abcz}{p^{b+1}}\right) &= \infty,\\
\lim_{p\to \infty}\frac{\partial P(p,z)}{\partial p} &= \lim_{p\to \infty}\left(\frac{2a^2(b-1)\sigma^2(z)\lambda}{p^{2b-1}}-\frac{a(b-1)\mu(z)}{p^b}+\frac{abcz}{p^{b+1}}\right) &= 0^-.
\end{aligned}$$

The sign of the partial derivatives above and Lemma 4.1, guarantee that $P(\cdot,z)$ is unimodal with respect to p in $(0,\infty)$ and that the optimal price $p^*(z)$ is indeed a maximizer. □ □

Lemma 4.2

Proof. Let us define $\alpha = 2\lambda\sigma^2(z)a(b-1) \leq 0, \beta = (b-1)\mu(z) > 0$ and $\gamma = bcz > 0$. As shown in Lemma 4.1, we can analyze the different cases as a function of the price elasticity of the demand b.

- $\underline{1 < b < 2}$: with a suitable change of variable, equation (C.1) can be rewritten again as (C.2) and, per Descartes' Rule of Signs, this equation has only one positive real root.

- $\underline{b{=}2}$: in this case (C.1) has the following unique solution: $p^*(z) = \dfrac{\alpha+\gamma}{\beta}$, which is nonnegative for $\forall z \in [A,\ B]$ if and only if $\lambda \geq \lambda_{min}$. If $\lambda < \lambda_{min}$ there will be some values of z for which $p^*(z) < 0$ and others for which $p^*(z) \geq 0$.

- $\underline{b > 2}$: as done in Lemma 4.1, equation (C.1) can be transformed into the polynomial (C.3) and, per Descartes' Rule of Signs, it has either two positive real roots or no positive real roots at all.

$\square$

Theorem 4.2

Proof. Let $1 < b < 2$ or $b = 2$ and $\lambda \geq \lambda_{min}$ so that (4.3) has only one positive real root. For a given stock factor z we have that

$$\begin{aligned}
\lim_{p\to 0^+} \frac{\partial P(p,z)}{\partial p} &= \lim_{p\to 0^+} \left(\frac{2a^2(b-1)\sigma^2(z)\lambda}{p^{2b-1}} - \frac{a(b-1)\mu(z)}{p^b} + \frac{abcz}{p^{b+1}} \right) &= \infty, \\
\lim_{p\to \infty} \frac{\partial P(p,z)}{\partial p} &= \lim_{p\to \infty} \left(\frac{2a^2(b-1)\sigma^2(z)\lambda}{p^{2b-1}} - \frac{a(b-1)\mu(z)}{p^b} + \frac{abcz}{p^{b+1}} \right) &= 0^-.
\end{aligned}$$

It follows that $P(\cdot, z)$ is unimodal with respect to p in $(0, \infty)$ in these circumstances, and that $p^*(z)$ is a maximizer. We can prove that, when $b > 2$, $P(\cdot, z)$ is bimodal with respect to p in $(0, \infty)$ by calculating again these limits. In this case, $\lim_{p\to 0^+} \frac{\partial P(p,z)}{\partial p} = \infty$ and $\lim_{p\to\infty} \frac{\partial P(p,z)}{\partial p} = 0^+$. The result follows as, per Lemma 4.2, equation (4.3) has two real positive roots. $\square$

Lemma 4.3

Proof. We prove first the condition for the optimal price to be strictly greater than c. Face value is never an optimal price if c is never a root of (4.7), i.e. $2a(b-1)\lambda\sigma^2(z)c^{-(b-2)} - (b-1)\mu(z)c + bcz \neq 0$. Since the function above is continuous in z, this condition holds as long as the left-hand side is always below 0 or above 0. If it is below 0, then it follows that $\lambda < \dfrac{((b-1)\mu(z) - bz)\, c^{b-1}}{2a(b-1)\sigma^2(z)}$. However, this is not possible, for this implies $\lambda < -\infty$ for $z = A$. If it is above 0, then $\lambda > \dfrac{((b-1)\mu(z) - bz)\, c^{b-1}}{2a(b-1)\sigma^2(z)}$. Taking into account that the numerator of this expression is always negative, a lower bound for λ is thus given by the maximum of the right-hand side of this inequality, whence the strict inequality of our result follows. The possibility of the optimal price being equal to c is allowed by introducing the equality in this lower bound. □

Theorem 4.3

Proof. We prove first the local optimality condition. If the Hessian matrix of P is negative definite at $(z^*, p^*(z^*))$, then this point is a strict local maximum of $P(\cdot,\cdot)$ in $[A, B] \times [c, \infty)$. Given that $\dfrac{\partial^2 P}{\partial z^2} = -ap^{-(b-1)}f(z) < 0$, per the second derivative test such a Hessian is negative definite as long as $\Delta(z^*, p^*(z^*)) > 0$, where $\Delta(z,p) = \dfrac{\partial^2 P}{\partial p^2}\dfrac{\partial^2 P}{\partial z^2} - \left(\dfrac{\partial^2 P}{\partial p \partial z}\right)^2$. Using equations (4.5)-(4.6) with $\lambda = 0$, we can rewrite the equation $\Delta(z,p) > 0$ as $-pf(z)\,(b(b-1)\mu(z)p - (b+1)bcz) - (bc - (b-1)p\,(1 - F(z)))^2 > 0$. If we particularize for the set of prices that are optimal, $p^*(z)$, this condition can be written now as $\Delta(z, p^*(z)) = p^*(z)bczf(z) - (bc - (b-1)p^*(z)\,(1 - F(z)))^2 > 0$, where we used the closed-form solution of the optimal price in the risk-neutral case, $p^*(z) = bcz/\left((b-1)\mu(z)\right)$, to simplify the first term of the previous equation. Moreover, z^* satisfies the equation

$F(z) = 1 - c/p^*(z)$ and thus we obtain $\Delta(z^*, p^*(z^*)) = bz^* f(z^*) > 1 - F(z^*)$, whence the condition $\xi(z^*) > 1$ follows from the definition of LSR elasticity for isolastic demand as shown in equation (4.9). Proving that $(z^*, p^*(z^*))$ is a saddle point of P in $[A, B] \times [c, \infty)$ can be done analogously by imposing that $\Delta(z^*, p^*(z^*)) < 0$.

Next, we prove the global optimality condition by reductio ad absurdum. The problem has at least a local maximum because $P^{*'}(A) > 0$ and $P^{*'}(B) < 0$. This maximum will occur at a point z^* such that, per our local optimality condition, $\xi(z^*) > 1$. Assume that $\xi(z) > 1$ for any stock factor z. If there is a second critical point, such a point will be a minimum and it will occur at z^{**}. Per our local optimality condition, $\xi(z^{**}) < 1$; however, we assumed that $\xi(z) > 1$ and therefore such a point cannot exist. We conclude then that if $\xi(z) > 1$, then the equation $R(z) = 0$ has only one solution, and this solution is a global maximum of $P^*(\cdot)$. □

Lemma 4.4

Proof. Using the formula for $p^*(z)$ in the risk-neutral case, let us redefine the optimality condition for z^* as $\tilde{R}(z^*, b) = (1 - F(z^*))bcz^* / ((b-1)\mu(z^*)) - c = 0$. By the Implicit Function Theorem, $\frac{dz^*}{db} = -\frac{\frac{\partial \tilde{R}}{\partial b}}{\frac{\partial \tilde{R}}{\partial z^*}} = \frac{z^*\mu(z^*)}{b(b-1)\left(\mu(z^*)\left(1 - z^*h(z^*)\right) - z^*(1 - F(z^*))\right)}$, where $h(z^*)$ is the failure rate of ϵ evaluated at $z = z^*$. The denominator (and the expression above) is negative if $1 - z^*h(z^*) < 0$ or, equivalently, if $\xi(z^*) > b$. As for the optimal price, simple applications of the chain rule yield $\frac{dp^*}{db} = \frac{\partial p^*}{\partial z^*}\frac{dz^*}{db} + \frac{\partial p^*}{\partial b}$ and $\frac{dp^*}{dc} = \frac{\partial p^*}{\partial c}$. In the first case we obtain $\frac{dp^*}{db} = \frac{bc}{(b-1)\,\mu(z^*)^2}\left(\int_A^{z^*} uf(u)du\right)\frac{dz^*}{db} - \frac{cz^*}{\mu(z^*)(b-1)^2}$, which is negative if $\xi(z^*) > b$. In the second case we have that $\frac{dp^*}{dc} = \frac{bz^*}{(b-1)\mu(z^*)} > 0$ (i.e. p^* is linear in

c). ☐

Theorem 4.4

Proof. The behavior of $P^{*'}$ is clearly given by that of R. At the limits of $[A, B]$ we have $R(A) = c/(b-1) > 0$, $R(B) = -c < 0$. This fact, along with the continuity of R, implies that there is at least one solution to the equation $P^{*'}(z) = 0$ (i.e. $R(z) = 0$ at least once). In fact, $P^{*'}$ has only one root if and only if $R'(z)|_{R(z)=0} < 0$. If this occurs, this root represents also a maximum of P^*, since $P^{*'}(A) > 0$ and $P^{*'}(B) < 0$. Note that,

$$\begin{aligned} R'(z) &= -f(z)p^*(z) + (1-F(z))p^{*'}(z) \\ &\quad -\lambda a p^*(z)^{-(b-1)}\left(p^*(z)\sigma^{2''}(z) - (b-2)p^{*'}(z)\sigma^{2'}(z)\right). \end{aligned} \tag{C.4}$$

In general, at the critical points of P we have that

$$R(z) = 0 \quad \Longrightarrow \quad \lambda a p^*(z)^{-(b-2)}\sigma^{2'}(z) = (1-F(z))p^*(z) - c. \tag{C.5}$$

Substituting (C.5) in (C.5) and reordering terms, we obtain $R'(z)|_{R(z)=0} = -f(z)p^*(z) + (b-1)(1-F(z))p^{*'}(z) - (b-2)cp^{*'}(z)/p^*(z) - \lambda a p^*(z)^{-(b-2)}\sigma^{2''}(z)$. Dividing by $1 - F(z)$ and using the equality $\sigma^{2''}(z) = 2(1-F(z))F(z) - 2f(z)(z-\mu(z))$ gives

$$\begin{aligned} \left.\frac{R'(z)}{1-F(z)}\right|_{R(z)=0} &= \frac{p^{*'}(z)}{p^*(z)} \underbrace{\left((b-1)p^*(z) - \frac{(b-2)c}{1-F(z)}\right)}_{\text{B}} \\ &\quad + h(z)p^*(z) \underbrace{\left((z-\mu(z))2\lambda a p^*(z)^{-(b-1)} - 1\right)}_{\text{A}} -2\lambda a F(z)p^*(z)^{-(b-2)}. \end{aligned}$$

Ⓐ can be further particularized for $R(z) = 0$ using (C.5) to get $(z - \mu(z))2\lambda a p^*(z)^{-(b-1)} - 1 = \dfrac{-c}{p^*(z)(1 - F(z))} < 0$. Furthermore, Ⓑ can be rewritten as

$$\frac{c + (b-1)((1-F(z))p^*(z) - c)}{(1 - F(z))}.$$

Now, let $\eta(z) = (1-F(z))p^*(z) - c$. All in all, our condition for the negativity of $R'(z)|_{R(z)=0}$ results in $\dfrac{{p^*}'(z)}{p^*(z)}((b-1)\eta(z) + c) - \dfrac{F(z)}{z - \mu(z)}\eta(z) - h(z)c < 0$, whence we obtain:

$$h(z) \quad > \quad \left(\frac{{p^*}'(z)}{p^*(z)} + \left((b-1)\frac{{p^*}'(z)}{p^*(z)} - \frac{F(z)}{z - \mu(z)} \right) \frac{\eta(z)}{c} \right). \tag{C.6}$$

It follows from (4.7) that

$${p^*}'(z) \quad = \quad \frac{2\lambda a(b-1){\sigma^2}'(z)p^*(z)^{-(b-2)} + bc - (b-1)\mu'(z)p^*(z)}{2\lambda a(b-1)(b-2)\sigma^2(z)p^*(z)^{-(b-1)} + (b-1)\mu(z)},$$

and

$${p^*}'(z) \quad = \quad p^*(z)\frac{bc + \dfrac{{\sigma^2}'(z)}{\sigma^2(z)}((b-1)\mu(z)p^*(z) - bcz) - (b-1)\mu'(z)p^*(z)}{(b-1)^2\mu(z)p^*(z) - (b-2)bcz}, \tag{C.7}$$

after removing the explicit dependence on λ. Note that, because of (4.7), $(b-1)\mu(z)p^*(z) - bcz$ is positive in risk-averse cases ($\lambda > 0$), negative in risk-seeking cases ($\lambda < 0$), and 0 in risk-neutral cases ($\lambda = 0$). We can find ${p^*}'(z)/p^*(z)$ at those points where $R(z) = 0$. To see this, use ${p^*}'(z)$ as shown in (C.7) and particularize for those points by means of (C.5). The result can be manipulated to get $\left.\dfrac{{p^*}'(z)}{p^*(z)}\right|_{R(z)=0} = \dfrac{c + (b-1)((1-F(z))p^*(z) - c)}{(b-1)^2\mu(z)p^*(z) - (b-2)bcz} = \dfrac{c + (b-1)\eta(z)}{(b-1)^2\mu(z)p^*(z) - (b-2)bcz}$. This result, in conjunction with (C.6), yields our first condition:

$$h(z) \quad > \quad \left(\frac{(c+(b-1)\eta(z))^2(1-F(z))(z-\mu(z))}{(b-1)^2\sigma^2(z)\eta(z)+bcz(1-F(z))(z-\mu(z))} - \frac{F(z)}{z-\mu(z)}\eta(z)\right)\frac{1}{c}. \quad \text{(C.8)}$$

Per (C.5), $\eta(z)|_{R(z)=0} \geq 0$ when $\lambda \geq 0$, and therefore we can bound (C.8) to get our second condition:

$$\begin{aligned} h(z) \quad &> \quad \left(\frac{(c+(b-1)\eta(z))^2(1-F(z))(z-\mu(z))}{(b-1)^2\sigma^2(z)\eta(z)+bcz(1-F(z))(z-\mu(z))} - \frac{F(z)}{z-\mu(z)}\eta(z)\right)\frac{1}{c} \\ &\leq \quad \frac{(c+(b-1)\eta(z))^2(1-F(z))(z-\mu(z))}{(b-1)^2\sigma^2(z)\eta(z)+bcz(1-F(z))(z-\mu(z))}\frac{1}{c} \\ &\leq \quad \frac{(c+(b-1)\eta(z))^2(1-F(z))(z-\mu(z))}{bcz(1-F(z))(z-\mu(z))}\frac{1}{c} = \frac{(c+(b-1)\eta(z))^2}{bc^2z} \\ &= \quad \frac{((b-1)\eta(z)+c)^2}{bc^2z} \implies g(z) > \frac{1}{b}\left(\frac{(b-1)\eta(z)+c}{c}\right)^2. \end{aligned} \quad \text{(C.9)}$$

Using the equality $\xi = bzh(z) = bg(z)$ we can write equations (C.8) and (C.9) as a function of the LSR elasticity, as shown in this theorem. For the last condition, assume that $b \geq 2$ and remember that the equation $R(z) = 0$ is equivalent to $\eta(z) = \dfrac{2\lambda a(1-F(z))(z-\mu(z))}{p^*(z)^{b-2}}$. Our second condition can thus be written as

$$\xi(z) > \quad \left(1+\frac{(b-1)\eta(z)}{c}\right)^2 \quad \leq \left(1+\frac{(b-1)\dfrac{2\lambda a(B-1)}{c^{b-2}}}{c}\right)^2,$$

and therefore we arrive to the our last lower bound:

$$\xi(z) > \left(1+\frac{2\lambda a(b-1)(B-1)}{c^{b-1}}\right)^2.$$

$\square$

Lemma 4.5

Proof. Per its definition, the risk sensitive problem is solved by finding the solution to the equation $(1 - F(z))p^*(z) - c = 0$. The left-hand side of this equation is positive at $z = A$ and negative at $z = B$. It is also decreasing if $\xi(z) > bz\dfrac{p^{*'}(z)}{p^*(z)}$. If this condition holds, then the equation in question has only one solution.

Let us focus know on the first-order optimality condition of the risk-sensitive problem:

$$R(z) = \underbrace{(1 - F(z))p^*(z) - c}_{Ⓐ} \underbrace{-\lambda a p^*(z)^{-(b-2)}\sigma^{2'}(z)}_{Ⓑ} = 0.$$

First, this equation will never be solved at $z = A$ $(z = B)$, since at those points Ⓐ > 0 (Ⓐ < 0), whereas Ⓑ $= 0$. For all other points in the interval $[A, B]$, part Ⓑ of the equation bears the sign of the risk parameter λ. In risk-averse cases, this part is negative and therefore $R(z^*_{RSE}) < 0$. Since part Ⓐ is decreasing, it follows that the solutions to $R(z) = 0$ take place for stock factors that are smaller than z^*_{RSE}, i.e., $z^* \in (A, z^*_{RSE})$.

Likewise, in risk-seeking cases, part Ⓑ is positive and therefore $R(z^*_{RSE}) > 0$. Since part Ⓐ is decreasing, it follows that the solutions to $R(z) = 0$ take place for stock factors that are greater than z^*_{RSE}, i.e., $z^* \in (z^*_{RSE}, B)$. □

Lemma 4.6

Proof. Let $g(\lambda, p) = \partial P(\lambda, p, \hat{z})/\partial p$. Per the Implicit Function Theorem we have that

$$\frac{d\tilde{p}^*(\lambda)}{d\lambda} = -\frac{\dfrac{\partial g(\lambda,p)}{\partial \lambda}}{\dfrac{\partial g(\lambda,p)}{\partial p^*(\hat{z})}} = -\frac{2a^2(b-1)\sigma^2(\hat{z})p^*(\hat{z})^{-2b+1}}{\left.\dfrac{\partial^2 P(\lambda,p,z)}{\partial p^2}\right|_{p=p^*(\hat{z})}} \geq 0,$$

since P is concave when $p = p^*(\hat{z})$. Moreover, differentiating again with respect to λ yields

$$\tilde{p}^{*''}(\lambda) = -\frac{(2a(b-1)\sigma^2(\hat{z}))^2(2b-1)p^*(\hat{z})^{-4b+1}}{\left(\left.\frac{\partial^2 P(\lambda,p,z)}{\partial p^2}\right|_{p=p^*(\hat{z})}\right)^2} \leq 0.$$

□

Lemma 4.7

Proof. From (4.2) and per Lemma 4.6 it follows that

$$\frac{d}{d\lambda}Var(\tilde{\Pi}^*(\lambda)) = -a^2\sigma^2(\hat{z})\tilde{p}^{*'}(\lambda)\frac{2b-2}{\tilde{p}^*(\lambda)^{2b-1}} \leq 0.$$

□

Lemma 4.8

Proof. From (4.2) it follows that $E(\tilde{\Pi}^*(\lambda)) = a\mu(\hat{z})\tilde{p}^*(\lambda)^{-b+1} - c\hat{z}a\tilde{p}^*(\lambda)^{-b}$. Therefore $\frac{d}{d\lambda}E(\tilde{\Pi}^*(\lambda)) = a\tilde{p}^{*'}(\lambda)\tilde{p}^*(\lambda)^{-b-1}\left(c\hat{z}b - \mu(\hat{z})(b-1)\tilde{p}^*(\lambda)\right)$.

Since the first factor is nonnegative, the sign of this derivative is given by that of the second factor shown above, which is nonpositive if and only if $\tilde{p}^*(\lambda) \geq \frac{cb\hat{z}}{(b-1)\mu(\hat{z})}$. Since, per (4.7), $\tilde{p}^*(0) = \frac{cb\hat{z}}{(b-1)\mu(\hat{z})}$ and given Lemma 4.6, we conclude that this factor is indeed nonpositive for $\lambda > 0$ and nonnegative for $\lambda < 0$. □

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