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THE CLASS OF THE AFFINE LINE IS A ZERO DIVISOR IN
THE GROTHENDIECK RING

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Abstract. We show that the class of the affine line is a zero divisor in the
Grothendieck ring of algebraic varieties over complex numbers. The argument
is based on the Pfaffian-Grassmannian double mirror correspondence.

1. Introduction

The Grothendieck ring $K_0(\text{Var}/\mathbb{C})$ of complex algebraic varieties is a fundamen-
tal object of algebraic geometry. It is defined as the quotient of the group of formal
integer linear combinations $\sum a_i[Z_i]$ of isomorphism classes of complex algebraic
varieties modulo the relations

$$[Z] - [U] - [Z\setminus U]$$

for all open subvarieties $U \subseteq Z$. The product structure is induced from the Carte-
sian product.

The main result of this paper is the following.

Theorem 2.13. The class $L$ of the affine line is a zero divisor in the Grothendieck
ring of varieties over $\mathbb{C}$.

The class $L = [\mathbb{C}^1]$ of the affine line plays an important role in the study
of $K_0(\text{Var}/\mathbb{C})$. For example, it has been proved in [10] that the quotient of
$K_0(\text{Var}/\mathbb{C})$ by $L$ has a natural basis indexed by the classes of projective alge-
braic varieties up to stable birational equivalence. In other instances one needs to
localize $K_0(\text{Var}/\mathbb{C})$ by $L$ (see [4, 12]) so it is important to know whether $L$ is a
nonzero divisor. While it has been shown in [14] that $K_0(\text{Var}/\mathbb{C})$ is not a domain,
there remained a hope that $L$ is nonetheless a non-zero-divisor in $K_0(\text{Var}/\mathbb{C})$.

This problem was brought to our attention by an elegant recent preprint of
Galkin and Shinder [5] in which the authors prove that if $L$ is a nonzero divisor in
$K_0(\text{Var}/\mathbb{C})$ (a weaker condition that $L^2a = 0$ implies $a \in \langle L \rangle$ in fact suffices) then
a rational smooth cubic fourfold in $\mathbb{P}^5$ must have its Fano variety of lines birational
to a symmetric square of a $K3$ surface. This paper puts a dent in this approach to
(ir)rationality of cubic fourfolds.

The consequence of our construction is another important result, which was
pointed to us by Evgeny Shinder. A cut-and-paste conjecture (or question) of
Larsen and Lunts [10, Question 1.2] asks whether any two algebraic varieties $X$
and $Y$ with $[X] = [Y]$ in the Grothendieck ring can be cut into disjoint unions
of pairwise isomorphic locally closed subvarieties.

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The negative answer to this conjecture is important in view of its potential applications to rationality of motivic zeta functions, see [4], [11].

The main idea of the proof of Theorems 2.13 and 2.14 is to compare the two sides $X_W$ and $Y_W$ of the Pfaffian-Grassmannian double mirror correspondence. These are non-birational smooth Calabi-Yau threefolds which are derived equivalent. There is a natural variety (a frame bundle over the Cayley hypersurface of $X_W$) whose class in the Grothendieck ring can be expressed both in terms of $[X_W]$ and in terms of $[Y_W]$. This provides a relation

$$([X_W] - [Y_W])(L^2 - 1)(L - 1)L^7 = 0$$

in the Grothendieck ring, which then implies that $L$ is a zero divisor.

After this preprint has appeared, the result has been improved to

$$([X_W] - [Y_W])(L + 1)L^6 = 0$$

by Kuznetsov, [9] and then later to

$$([X_W] - [Y_W])L^6 = 0$$

independently by Chambert-Loir and Martin, [3, 13].

Acknowledgements. This paper came about as a byproduct of joint work with Anatoly Libgober on higher dimensional version of Pfaffian-Grassmannian double mirror correspondence. The author is indebted to Prof. Libgober for stimulating conversations, useful references and comments on the preliminary version of the paper. I also thank Evgeny Shinder who pointed out that the construction of the paper gives a counterexample to the cut-and-paste conjecture of Larsen and Lunts, see Theorem 2.14.

2. The construction

2.1. Pfaffian and Grassmannian double mirror Calabi-Yau varieties. Let $V$ be a 7-dimensional complex vector space. Let $W \subset \Lambda^2 V^\vee$ be a generic 7-dimensional space of skew forms on $V$. These data encode two smooth Calabi-Yau varieties $X_W$ and $Y_W$ as follows.

Definition 2.1. We define $X_W$ as a subvariety of the Grassmannian $G(2, V)$ of dimension two subspaces $T_2 \subset V$ which is the locus of all $T_2 \in G(2, V)$ with $w|_{T_2} = 0$ for all $w \in W$. We define $Y_W$ as a subvariety of the Pfaffian variety $Pf(V) \subset \mathbb{P} \Lambda^2 V^\vee$ of skew forms on $V$ whose rank is less than 6. It is defined as the intersection of $Pf(V)$ with $\mathbb{P}W \subset \mathbb{P} \Lambda^2 V^\vee$.

The following proposition summarizes the properties of $X_W$ and $Y_W$ that will be used later.

Proposition 2.2. The following statements hold for a general choice of $W$.

- The varieties $X_W$ and $Y_W$ are smooth Calabi-Yau threefolds.
- The varieties $X_W$ and $Y_W$ are not isomorphic, or even birational, to each other.
- All forms $Cw \in Y_W$ have rank 4. All forms $Cw \in PW \setminus Y_W$ have rank 6.
Proof. Smoothness of $X_W$ and $Y_W$ has been shown by Rødland [15]. They are not isomorphic to each other because the ample generators $D_X$ and $D_Y$ of their respective Picard groups have $D_X^3 = 42$ and $D_Y^3 = 14$. The statement that $X_W$ and $Y_W$ are not birational follows from the fact that they are non-isomorphic Calabi-Yau threefolds with Picard number one, see [2].

The statement about the rank of the forms follows from the fact that $W$ is generic, since the locus of rank 2 forms in $\mathbb{P}^4 V^\vee$ is of codimension 10. Alternatively, if $cw \in Y_W$ has rank 2, then $Y_W$ is automatically singular at $cw$. □

Remark 2.3. The varieties $X_W$ and $Y_W$ are double-mirror to each other, in the sense that they have the same mirror family. This is just a heuristic statement, but it does indicate that geometry of $X_W$ is intimately connected to that of $Y_W$. For example, it was shown independently in [2] and [8] that $X_W$ and $Y_W$ have equivalent derived categories.

2.2. Cayley hypersurface and its frame bundle. The main technical tool of this paper is the so-called Cayley hypersurface of $X_W$. It is the hypersurface in $G(2, V) \times \mathbb{P}W$ which consists of pairs $(T_2, cw)$ with the property $w\Big|_{T_2} = 0$. The class of $X_W$ in the Grothendieck ring of varieties over $\mathbb{C}$ is related to that of $H$ as follows.

Proposition 2.4. The following equality holds in the Grothendieck ring.

$$[H] = [G(2, 7)]\mathbb{P}^5 + [X_W]L^6$$

Proof. Consider the projection of $H$ onto $G(2, V)$. The restriction of this map to the preimage of $X_W$ is a trivial fibration with fiber $\mathbb{P}W = \mathbb{P}^6$. The restriction of it to the complement of $X_W$ is a Zariski locally trivial fibration with fiber $\mathbb{P}^5$. Indeed, the hyperplanes of $w$ that vanish on a given $T_2$ can be Zariski locally identified with a fixed $\mathbb{P}^5$ by projecting from a fixed point in $\mathbb{P}W$. This gives

$$[H] = [X_W]\mathbb{P}^6 + ([G(2, 7)] - [X_W])\mathbb{P}^5 = [G(2, 7)]\mathbb{P}^5 + [X_W](\mathbb{P}^6 - \mathbb{P}^5)$$

which proves the claim. □

Remark 2.5. In the proof of Proposition 2.4 we used the statement that for a Zariski locally trivial fibration $Z \rightarrow B$ with fiber $F$ there holds $[Z] = [B][F]$ in $K_0(Var/\mathbb{C})$. We will use this statement repeatedly in the subsequent arguments.

We can project the Cayley hypersurface $H$ onto the second factor $\pi : H \rightarrow \mathbb{P}W$. We will have different fibers depending on whether the image lies in $Y_W$ or not. While we would like to say that the restriction of $\pi$ to the preimages of $Y_W$ and its complement are Zariski locally trivial, we do not know if this is true or not. So instead of using $H$ itself we will pass to the frame bundle $\tilde{H}$ over $H$.

Definition 2.6. We denote by $\tilde{H}$ the frame bundle of $H$, i.e. the space of triples $(v_1, v_2, w)$ where $v_1$ and $v_2$ are linearly independent vectors in $V$ and $w$ is an element of $\mathbb{P}W$ such that $w(v_1, v_2) = 0$.

Remark 2.7. Since $\tilde{H}$ is the frame bundle of the Zariski locally trivial vector bundle (pullback of the tautological subbundle on $G(2, V)$) on $H$, the fibration $\tilde{H} \rightarrow H$ is Zariski locally trivial. An easy calculation shows that

$$[\tilde{H}] = [H](L^2 - 1)(L^2 - L)$$

in the Grothendieck ring.
We now consider the projection $\tilde{H} \to \mathbb{P}W$. Notice that we have

\[(2)\] 
\[\tilde{H} = \tilde{H}_1 \sqcup \tilde{H}_2\]

where $\tilde{H}_1$ is the preimage of $Y_W$ and $\tilde{H}_2$ is the preimage of its complement in $\mathbb{P}W$.

**Proposition 2.8.** The following equality holds in the Grothendieck ring.

\[[\tilde{H}_1] = [Y_W](L^3 - 1)(L^7 - L) + (L^7 - L^3)(L^6 - L)\]

**Proof.** There is a subvariety $\tilde{H}_{1,1}$ in $\tilde{H}_1$ given by the condition $v_1 \in \text{Ker}(w)$. Forgetting $v_2$ realizes $\tilde{H}_{1,1}$ as a Zariski locally trivial fibration with fiber $\mathbb{C}^7 - \mathbb{C}$ over the space of pairs $(v_1, w)$ with $v_1 \in \text{Ker}(w)$, $v_1 \neq 0$. This in turn is a Zariski locally trivial fibration over $Y_W$ with fiber $(\mathbb{C}^3 - \text{pt})$, since all $Cw \in Y_W$ have rank 4. Putting all this together, we have

\[[\tilde{H}_{1,1}] = [Y_W](L^3 - 1)(L^7 - L)\]

in the Grothendieck ring. Similarly, the complement $\tilde{H}_{1,2}$ of $\tilde{H}_{1,1}$ in $\tilde{H}_1$ satisfies

\[[\tilde{H}_{1,2}] = [Y_W](L^7 - L^3)(L^6 - L)\]

Indeed, $\tilde{H}_{1,2}$ forms a vector bundle of rank 6 over the space of pairs $(v_1, w)$, since the condition $w(v_1, v_2) = 0$ is now nontrivial. The result of the proposition now follows from $[\tilde{H}_1] = [\tilde{H}_{1,1}] + [\tilde{H}_{1,2}]$. \qed

**Proposition 2.9.** The following equality holds in the Grothendieck ring.

\[[\tilde{H}_2] = \left( [\mathbb{P}^6] - [Y_W] \right) \left( (L - 1)(L^7 - L) + (L^7 - L^3)(L^6 - L) \right)\]

**Proof.** The argument is completely analogous to that of Proposition 2.8. The only difference is that a form $Cw \notin Y_W$ has rank 6 and thus a one-dimensional kernel. \qed

As a corollary of Propositions 2.8 and 2.9 we get the formula for $[\tilde{H}]$.

**Proposition 2.10.** The following equality holds in the Grothendieck ring.

\[[\tilde{H}] = [\mathbb{P}^6](L^7 - L)(L^6 - 1) + [Y_W](L^2 - 1)(L - 1)L^7\]

**Proof.** This follows immediately from (2) and Propositions 2.8 and 2.9. \qed

2.3. **Main theorem.** We are now ready to prove our main result. We start with the following formula derived from the calculations of the previous subsection.

**Proposition 2.11.** The following equality holds in the Grothendieck ring.

\[([X_W] - [Y_W])(L^2 - 1)(L - 1)L^7 = 0\]

**Proof.** We use Proposition 2.10 and Proposition 2.4 with equation (1) to get expressions for $[\tilde{H}]$, in terms of $[Y_W]$ and $[X_W]$ respectively. By subtracting one from the other we get

\[([X_W] - [Y_W])(L^2 - 1)(L - 1)L^7 = [\mathbb{P}^6](L^7 - L)(L^6 - 1) - [G(2,7)][\mathbb{P}^5](L^2 - 1)(L^2 - L)\]

which then equals zero in view of $[G(2,7)](L^2 - 1)(L^2 - L) = (L^7 - 1)(L^7 - L)$ and $[\mathbb{P}^6](L^6 - 1) = [\mathbb{P}^5](L^7 - 1)$. \qed
Remark 2.12. It was communicated to us by Kuznetsov [9] that the factor \((L^2 - 1)(L - 1)L^7\) in the statement of Proposition 2.11 can be replaced by \((L + 1)L^6\) by considering the projectivization of the tautological subbundle instead of the frame bundle. Later, Chambert-Loir and Martin [3, 13] have independently shown that

\[ ([X_W] - [Y_W])L^6 = 0. \]

Their argument relies on the fact that a skew-symmetric form over any field has a standard basis.

Theorem 2.13. The class \(L\) of the affine line is a zero divisor in the Grothendieck ring of varieties over \(\mathbb{C}\).

Proof. In view of Proposition 2.11, it suffices to show that

\[ ([X_W] - [Y_W])(L^2 - 1)(L - 1) \]

is a nonzero element of the Grothendieck ring. In fact, we can argue that it is a nonzero element modulo \(L\). Indeed, if it were zero modulo \(L\), this would mean that \([X_W] \equiv [Y_W] \mod L\). This implies that \(X_W\) is stably birational to \(Y_W\), by [10]. This means that for some \(k \geq 0\) the varieties \(X_W \times \mathbb{P}^k\) and \(Y_W \times \mathbb{P}^k\) are birational to each other. We now consider the MRC fibration [7], which is a birational invariant of an algebraic variety. Importantly, if \(X\) is not uniruled (for example a Calabi-Yau variety) then the base of the MRC fibration of \(X \times \mathbb{P}^k\) is \(X\). Thus, birationality of \(X_W \times \mathbb{P}^k\) and \(Y_W \times \mathbb{P}^k\) implies birationality of \(X_W\) and \(Y_W\), which is known to be false, see Proposition 2.2. \(\square\)

It was observed by Evgeny Shinder that the construction of this paper provides a negative answer to the cut-and-paste question of Larsen and Lunts [10, Question 1.2] which asks whether any two varieties with equal classes in the Grothendieck ring can be cut up into isomorphic pieces.\(^1\)


Proof. The equality

\[ [X_W](L^2 - 1)(L - 1)L^7 = [Y_W](L^2 - 1)(L - 1)L^7 \]

implies that trivial \(GL(2, \mathbb{C}) \times \mathbb{C}^6\) bundles over \(X_W\) and \(Y_W\) have the same class in the Grothendieck ring. If it were possible to cut them into unions of isomorphic varieties, then \(X_W \times GL(2, \mathbb{C}) \times \mathbb{C}^6\) would be birational to \(Y_W \times GL(2, \mathbb{C}) \times \mathbb{C}^6\). This implies that \(X_W\) and \(Y_W\) are stably birational, and thus birational, in contradiction with Proposition 2.2. \(\square\)

Remark 2.15. Our method works over any field of characteristic zero. It does not appear to work in positive characteristics, since results of [10] are based on [1] which in turn relies on the resolution of singularities.

\(^1\)Another counterexample to the question was recently announced by Ilya Karzhemanov in [6].
References


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