# AN ANALYSIS OF PEDAGOGICAL MOVES FOR FACILITATING THE DEVELOPMENT OF IN-SERVICE MIDDLE-SCHOOL MATHEMATICS TEACHERS' RECOGNITION OF REASONING

**By**

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ABSTRACT OF THE DISSERTATION

# An Analysis of Pedagogical Moves for Facilitating the Development of In-service Middle-School Mathematics Teachers' Recognition of Reasoning

By PHYLLIS J. CIPRIANI

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A constructivist approach for teaching and learning mathematics was the foundation for a longitudinal study at Rutgers University in 1987 (Maher, 2011). One of the objectives of the longitudinal study was to provide an environment where students solve problems in collaborative groups (Maher, 2011). Videos from the longitudinal study are stored in the Video Mosaic Collaborative Repository and are resources to use for professional development programs to gain insight in recognizing students' reasoning (Maher et al., 2010).

A qualitative case study was used to examine the effect of a semester-long course entitled Topics in Mathematics Education: A Lesson Study on Reasoning with ten in-service middle-school mathematics teachers from five districts in the southern region of New Jersey during fall 2013. Findings from this study revealed that (1) teachers' expectations of students' abilities increased, particularly with special education students; (2) teachers showed evidence of growth in their abilities to use non-leading questioning and pedagogical practices; and (3) teachers recognized that attending to students' reasoning is a gradual and continual process. Implications of the study and future research recommendations include comparing the results of the other cohorts.

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*"Children must be taught how to think, not what to think."*
*– Margaret Mead, Ph.D. in cultural anthropology*

# Chapter 1 - Introduction

## 1.1 The Significance of Attending to Students' Mathematical Reasoning

As states adopted the Common Core State Standards (CCSS) for Mathematical Content and Practice, teachers in those states were expected to implement the CCSS into their classroom lessons (National Governors Association Center for Best Practices and Council of Chief State School Officers (NGA & CCSSO), 2010). To successfully implement the mathematical content and practices of the CCSS, teachers must facilitate a technology-rich classroom environment that attends to students' mathematical reasoning (NGA & CCSSO, 2010). As a result, the development of teachers' beliefs, ideas, practices, and professional knowledge has generated much attention (Ball, Ben-Peretz, & Cohen, 2014; Battey & Franke, 2008; Borko, Mayfield, Marion, Flexer, & Cumbo, 1997; Maher, Landis, & Palius, 2010).

To meet the needs of teachers and students, effective professional development (PD) intervention programs emphasizing nontraditional pedagogical approaches such as constructivism must be provided. The data from one such intervention program with New Jersey middle school teachers will be analyzed based on a qualitative case study. The results of this case study will potentially fill a gap in research by providing knowledge about effective interventions and insight for facilitating practitioners' knowledge of recognizing students' mathematical reasoning.

Several publications of NCTM have reported that the emphasis on mathematical reasoning has increasingly grown in intensity (NCTM, 1980, 1989, 1991, 1995, 2000, 2007, 2009). One publication from the NCTM (2009) called *Focus in High School*

*Mathematics Reasoning and Sense Making* reported that mathematical reasoning "can take many forms, ranging from informal explanation and justification to formal deduction, as well as inductive observations" (p. 4). Although mathematical reasoning is represented in several forms, attending to students' mathematical reasoning shall be defined as the actions or moves incorporated into the lesson which promote and encourage students to actively engage with other students and the teacher to think and reason about a given task.

Debates on attending to students' mathematical reasoning can be traced back as early as the forties when Buell (1944) wrote a commentary in the *Mathematics Teacher* supporting the idea of teaching mathematics procedurally. Opposing this notion, Wheat (1945) responded by defining meaning in mathematics as "knowing what one does when he does it" (p. 100) and made a strong argument for teaching mathematics with meaning. The value of teaching mathematics more meaningfully was also supported by Brownell (1947) who posited that traditional mathematics programs "failed to develop arithmetical competence" (p. 265).

One example of this failure to develop competence in mathematics can be observed in a classic study done by Erlanger (1973) about a sixth grade boy named Benny. Benny was considered to be excelling in a program called the Individually Prescribed Instruction (IPI) (Erlanger, 1973). When Benny was questioned by a researcher, several misconceptions surfaced about the mathematics he was learning such as claiming "2/1+1/2 was equal to one, and 2/10 as a decimal was 1.2" (Erlanger, 1973, p. 7).

Maher & Alston (1989) also addressed mathematical misconceptions with a fifth grade student named Ling. Ling was participating in a summer enrichment program and struggled to find half of one- third using division. Ling was unable to connect the meaning to represent the relationship between these numbers symbolically by writing three incorrect division attempts using the wrong procedures (Maher & Alston, 1989). However, when Ling used manipulatives shaped as hexagons, parallelograms, trapezoids, and equilateral triangles to model the difference, she correctly solved the problem that one-half was bigger than one-third by one-sixth (Maher & Alston, 1989).

### 1.2 A CCSS Approach to Learning and Teaching Mathematics

These studies suggest how students can make major errors in learning mathematics when teachers emphasize procedural learning rather than meaningful learning (Erlwanger, 1973, Maher & Alston, 1989). Making mathematics learning more meaningful promotes reasoning and problem solving emphasized in a CCSS approach to learning and teaching mathematics (NGA & CCSSO, 2010). A CCSS approach to learning and teaching mathematics, allows for participating schools to be supplied with content standards of the mathematical knowledge necessary to attain success in mathematics for students at the postsecondary level (Achieve, 2015; Achieve the Core, 2015; Cipriani, 2015; Heck, Weiss, & Pasley, 2011; NGA & CCSSO, 2010; Rothman, 2011; School Improvement Network (SIN), 2015).

The NGA and the CCSSO collaborated with teachers, school administrators and other consultants to form the CCSS (Achieve, 2015; Achieve the Core, 2015; Cipriani, 2015; Heck et al., 2011; NGA & CCSSO, 2010). Their mission was to provide students with a framework in order to be prepared for college and work after high school

(Achieve, 2015; Achieve the Core, 2015; Cipriani, 2015; NGA & CCSSO, 2010; SIN, 2015). This mission involved three main shifts to successfully transition to a CCSS approach to learning and teaching mathematics (Achieve the Core, 2015; NGA & CCSSO, 2010).

The first shift involves decreasing the number of content standards so that more time and energy is used to form a strong conceptual foundation of understanding (Achieve the Core, 2015, NGA & CCSSO, 2010). A second shift links topics throughout the grade levels by connecting prior knowledge to newly learned knowledge (Achieve the Core, 2015, NGA & CCSSO, 2010). The third shift involves rigor where higher expectations of students allows for the application of knowledge in real-world situations to use higher-order reasoning (Achieve the Core, 2015, NGA & CCSSO, 2010). These shifts are essential in order to implement a CCSS approach to mathematics learning and teaching which emphasizes mathematical reasoning and problem solving (Achieve the Core, 2015, NGA & CCSSO, 2010).

However, the high-school standards are organized differently from the kindergarten through eighth-grade standards. At the high school level, the standards are not listed as ninth, tenth, eleventh, or twelfth-grade standards (NGA & CCSSO, 2010). Rather, the standards are divided into content topics which include number and quantity, algebra, functions, geometry, statistics and probability, and modeling (NGA & CCSSO, 2010). For the kindergarten through eighth-grade standards, each of the grade levels of standards are separately listed (NGA & CCSSO, 2010).

There are also eight CCSS for Mathematical Practice that teachers are expected to incorporate within their lessons as much as possible (NGA & CCSSO, 2010). The CCSS for Mathematical Practice explicitly describe how students should “engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years” (NGA & CCSSO, 2010, p. 8). The eight mathematical practices include sense-making and persevering to solve problems, abstract and quantitative reasoning, making and judging mathematical arguments, mathematical modeling, using tools in appropriate and strategic contexts, making mathematical arguments precisely, looking for patterns or structure, and evaluating results to see if they are reasonable (NGA & CCSSO, 2010). This focus on teachers attending to students’ reasoning will help to make the teaching and learning of mathematics more meaningful (NGA & CCSSO, 2010).

**1.3 PD Challenges**

The idea of making the teaching and learning of mathematics more meaningful has created certain challenges for school administrators. Administrators must now address how to prepare teachers to successfully implement the CCSS by restructuring lessons and classroom environments while facing challenges regarding time and budget cuts (Rothman, 2011). In fact, school districts must purchase new assessments that align to the CCSS (Leinwand, 2012; Rothman, 2011). These assessments come from the Partnership for Assessments of College and Career Readiness (PARCC) and the Smarter Balanced Assessment Consortium (SBAC) (Leinwand, 2012; Rothman, 2011). Because the testing contracts are costly to districts, funding teacher preparation in implementing the CCSS are a major challenge for school districts to overcome (Leinwand, 2012; Rothman, 2011).

Moreover, the CCSS does not provide pedagogical strategies and practitioners still have the freedom to decide the order and way the CCSS are taught (NGA & CCSSO, 2010). Even with the pedagogical freedom to decide this, the CCSS do not provide specific methods for teaching special education students, limited English-speaking learners, or students performing below their grade level (NGA & CCSSO, 2010).

Because the CCSS does not provide pedagogical strategies, a plethora of practical issues for both administrators and teachers is created. The practical issues include curriculum, teacher effectiveness, and assessments (Leinwand, 2012). To address practical issues, Rothman (2011) recommended revising curriculum, professional development, and assessments claiming that improved student learning will occur “only when teachers make the Standards part of their everyday classroom instruction, when they are prepared to teach them effectively, when the Standards are aligned with assessments that measure them faithfully, and when higher education institutions integrate the Standards into placement decisions and teacher education programs (and parents understand them)” (p. 119).

**1.4 The Longitudinal Study at Rutgers**

A longitudinal study that originally started as a PD intervention for mathematics teachers began in 1987 at Rutgers University (Maher, 2011b). The longitudinal study focused on “students building meaning of mathematical ideas and working collaboratively with each other” (Maher, 2011b, p. 5). In this study, students were videotaped participating in solving mathematical problems from first grade through high school (Maher, 2011b). In addition, these students had further discussions with researchers while in college and also after college (Maher, 2011b). Myriad research studies at Rutgers University originated from the longitudinal study (Alston & Davis, 1996; Francisco & Maher, 2011; Francisco, Maher et al., 2005; Maher, 1996, 2011a, 2011b; Maher et al., 2011a, 2011b; Martino & Maher, 1999; Palius & Maher, 2011).

One of the math strands for the longitudinal study included investigations in combinatorics tasks (Maher et al., 2010). These tasks differed from the existing school curriculum in that they contained mathematical content that was not previously studied in school and that they evoked justifications for their solutions (Maher, 2011b). The longitudinal study included the use of problems in combinatorics that were open-ended and different from the standard curriculum (Maher, 2011b). The problems used throughout the longitudinal study are listed in Appendix A of the book *Combinatorics and Reasoning: Representing, Justifying and Building Isomorphisms* (Maher, 2011b). Some of the problems included finding how many different outfits can be made from three different colored shirts and two different colored jeans; finding all possible four-tall towers that could be made selecting from two colors of unifix cubes™; finding how many different pizza combinations could be made selecting from four toppings; and finding the shortest route and the number of routes to three different points on a grid (Maher, 2011b).

**1.5 The Video Mosaic Collaborative Repository**

Many of the video-taped hours from the longitudinal study at Rutgers resulted from research funded by the National Science Foundation[1]. A subset of this videotaped data is currently stored in the Video Mosaic Collaborative repository (VMC are available resources for both pre-service and in-service teachers (see: www.videomosaic.org).

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[1] The research was supported by grants from the National Science Foundation: (DR:-0822204, REC 9814846, dIIS-1217087, directed by C.A. Maher, and MDR 9053597, directed by R. B. Davis and C.A. Maher).

Video-taped data of students' reasoning while engaging in various mathematical tasks from elementary grades through high school can be accessed, along with metadata such as transcripts, written work, and an Analytic tool, also developed from NSF funding[2].

For this case study, videos from the VMC were used by the instructor in the PD intervention to show teachers how the tasks were implemented in other classes. Video was also collected by this researcher from the PD intervention sessions along with teacher work and the work of their students to trace teachers' recognition of students' reasoning.

**1.6 The Lesson Study on Reasoning Course**

The purpose of this qualitative case study is to analyze the effect of a PD intervention used in a course for middle-school mathematics teachers at the Rutgers University Graduate School of Education called Topics in Mathematics Education: A Lesson Study on Reasoning. The objective was to help teachers learn to recognize and analyze students' mathematical reasoning in doing mathematics.

The Lesson Study on Reasoning course was an adaptation of the model from Catherine Lewis (2000). In 1993, Lewis (2000) was observing science classrooms in about fifty elementary schools in Japan and observed many similarities in the way science was taught. The Japanese teachers used an interesting activity or problem on a topic and encouraged students to explore the topic with hands-on experiments and topic and encouraged students to explore the topic with hands-on experiments and discussions of their findings. This pedagogy helped to improve the conceptual understanding of the

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[2] *EXP: Constructing Multimedia Artifacts Using a Video Repository*, Award IIS-1217087 was supported by grants from the National Science Foundation, C. A. Maher (PI), C. E. Hmelo-Silver, G. Agnew, and M.Palius.

topic which spanned over ten to twelve lessons (Lewis, 2000; Linn, Lewis, Tsuchida & Songer, 2000). This Japanese model was the foundation for Lewis's Lesson Study model which included a full cycle of teachers working collaboratively to plan, observe, analyze and discuss student responses, and revise the Lesson Study model which included a full cycle of teachers working collaboratively to plan, observe, analyze and discuss student responses, and revise instruction (Lewis, 2000; Linn, Lewis, Tsuchida & Songer, 2000).

The PD intervention studied for this research is a modification of the Lewis model. The PD intervention model was implemented with middle-school mathematics teachers from New Jersey that were participants in a special project. This project was called the New Jersey Partnership for Excellence in Middle School Mathematics (NJPEMSM).

### *1.6.1 NJPEMSM*

The NJPEMSM had the goal of preparing middle-school mathematics teachers from New Jersey to have a deeper understanding of mathematics, to involve their students in more effective engagement in learning mathematics, and to become facilitators of information by sharing their knowledge and experience with students and colleagues (NJPEMSM, 2009)[3]. The program grant was for five years beginning in 2009 adding another cohort of teachers each year. During fall 2013, the participants were members of the fourth cohort of project teachers.

The teachers selected for this program were experienced in-service teachers. Different cohorts of teachers were recruited and offered fellowships to complete all but

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[3] *NJ Partnership for Excellence in Middle School Mathematics* was supported by a grant from the National Science Foundation, Award DUE-0934079 (with A. Cohen (PI), C. A. Maher, J. W. Bennett, J. Coleman, and R. M. Beals). Any opinions, findings, conclusions, and recommendations expressed in this proposal are those of the author and do not necessarily reflect the views of the National Science Foundation.

nine credits of a program leading to a master's degree in mathematics education. Participants who completed the NJPEMSM program received a middle-grades mathematics specialization endorsement (NJPEMSM, 2009). For participating in the NJPEMSM (2009), teachers were able to have tuition and student fees waived for seven masters-level courses at Rutgers University, receive stipends after successfully completing summer institutes, and use the courses towards a Master's degree in Mathematics Education after successful admission to the Graduate School of Education.

The required program for teachers lasted twenty months and included seven graduate courses offered by the Graduate School of Education at Rutgers University (NJPEMSM, 2009). One of these seven courses is called Topics in Mathematics Education: A Lesson Study on Reasoning. This course was designed and conducted by researchers at the Robert B. Davis Institute for Learning (RBDIL) in the Graduate School of Education at Rutgers University for the NSF funded research project where a PD intervention was designed to facilitate building teachers' knowledge of recognizing students' reasoning[4]. The goal for the teachers enrolled in this lesson study intervention was to provide a series of experiences based on research where teachers participated in doing the mathematical tasks, implemented the tasks with students, watched VMC videos of the tasks being implemented, and discussed the findings (Landis, 2013).

Twenty-eight teachers registered for the Lesson Study on Reasoning course in 2013 and the teachers were divided into three separate groups. Two groups included the

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[4] *The Video Mosaic Collaborative (VMC)* is a research and development project sponsored by the National Science Foundation, award DRL-0822204 directed by Carolyn. A. Maher, Rutgers University. Any opinions, findings, conclusions, and recommendations expressed in this proposal are those of the author and do not necessarily reflect the views of the National Science Foundation.

teachers in the Northern and central New Jersey regions and were instructed by Dr. Alice Alston. The northern and central regions included eighteen teachers from Berkeley Heights, Carteret, Edison, Elizabeth, Franklin, Linden, North Brunswick, Orange, Plainfield, and Union Township. The third group was the southern region instructed by Dr. Judy Landis.

The southern region of New Jersey included ten teachers from Long Branch, Matawan-Aberdeen, Old Bridge, Sayreville, and Toms River (Landis, 2013). My research data comes from observations of the aforementioned southern region group of ten teachers and the instructor in one section of the fall 2013 course. The southern region group was chosen for my source of data research because I lived in the proximity of the southern region during the time of the intervention, which enabled me to attend all the meetings on time.

### *1.6.2 The PD Intervention Model*

Through the duration of the four month course, the southern region teachers attended were expected to participate in three cycles of tasks. The three cycles of tasks were chosen from the combinatorics field of mathematics. For each of the three task cycles, the participants worked in small groups on mathematical tasks and then discussed their task solutions, made one original online post responding to the assigned questions, VMC videos, and readings from a previous implementation of the same tasks with other students, made at least two additional posts responding to two other participants' original posts, observed and discussed the implementation from the in-district classroom visits with students, implemented the same tasks with their own students, and discussed examples of student work teachers shared after the implementation of these mathematical tasks in their own classes at the regional meetings.

**1.7 Research Purpose and Questions**

This qualitative case study will describe the impact of the PD intervention model on the fourth cohort of teachers from the NJPEMSM program. The purpose is to study the obstacles and successes experienced during this PD intervention of ten middle-school mathematics teachers in the southern region of New Jersey by examining how the teachers' identified reasoning forms from their own solutions to the mathematical tasks, their students' solutions to the same tasks, and other students' solutions to the same tasks after viewing VMC videos and reading articles about the research students' work.

The reasoning forms will also be identified and analyzed for any changes from the teachers' pretest responses to the posttest responses to the Gang of Four VMC video. Other analyses for this study will identify any evidence that the instructor's moves during the problem-solving sessions appeared to impact the teachers' knowledge construction while working on the mathematical tasks. Also, any changes in responses from the Beliefs pre and post-tests about the teaching and learning of mathematics will be analyzed.

The study will contribute significant information because relevant opportunities need to be provided that help mathematics educators attend to students' reasoning. Results of this research will provide insight for relevant PD, increased student engagement and achievement, and for making mathematics learning more meaningful. The data sources comprising this research include videos of the meetings, the online discussion course thread, the teacher portfolios, the Beliefs Pre- and Post- Assessment responses, the Gang of Four Pre- and Post- Assessment responses, instructor interviews, a group interview, and course materials. The following research questions guided the study:

1) What reasoning forms do middle school mathematics teachers identify from the following:

   (a) Their solutions to given mathematical tasks during a PD intervention;

   (b) Their current students' solutions to the same mathematical tasks implemented in their own classrooms;

   (c) The research students' solutions working on the same mathematical tasks from assigned articles to read and VMC videos;

   (d) Teachers' pre and post-assessment responses of the reasoning forms used by fourth-grade students to solve mathematical tasks in the Gang of Four VMC video?

2) What pedagogical moves are used by the instructor to facilitate the teachers' knowledge construction about mathematical reasoning as teachers:

   (a) Worked on combinatorics tasks;

   (b) Attended to research students' reasoning from VMC video and scholarly articles;

   (c) Analyzed current students' written task work?

3) In what ways, if any, do the teachers' beliefs about the teaching and learning of mathematics change?

# Chapter 2 - Theoretical Framework and Literature Review

## 2.1 Theoretical Framework

The theoretical lens that frames this case study is based on constructivism. Some views on constructivism as a theory for learning include encouraging students "to hypothesize, try things out, execute mathematical procedures, communicate and defend results, and reflect on the methods selected and the results generated" (Davis, Maher, & Noddings, 1990, p. 2). Mayer (2004) adds to this description of constructivism by claiming that "educators who wish to use constructivist methods of instruction are often encouraged to focus on discovery learning—in which students are free to work in a learning environment with little or no guidance" (p. 14). Other views include students build meaning through the application of prior knowledge and active engagement (Davis, Maher, & Noddings, 1990). From a more social constructivist perspective, Palincsar (1998) posited that "learning and understanding are regarded as inherently social; and cultural activities and tools (ranging from symbol systems to artifacts to language) are regarded as integral to conceptual development" (p. 348).

Early contributions to the constructivist perspective of learning came from Jean Piaget (Noddings, 1990, Tracey & Morrow, 2012). Noddings (1990) wrote that Piaget had a pragmatic view and "insisted that certain logical structures, developed through the coordination of actions, precede linguistic development and make the construction of linguistic structures possible" (p. 8). An early contributor to social constructivist ideas was Lev Semionovich Vygotsky (Noddings, 1990, Tracey & Morrow, 2012). Noddings (1990) reported that Vygotsky emphasized that interacting in groups helped to develop mental ideas. Noddings (1990) also reported that Vygotsky "suggested that children gradually internalize the talk that occurs in groups" (Noddings, 1990, p. 17).

Teaching implications of constructivism include the role of teachers as a facilitator of knowledge where teachers listen to students' thinking without giving specific strategies to use and facilitate discussions by questioning the ideas related to the students' construction of knowledge. Teachers must also arrange the classroom environment to encourage students to share ideas in groups and promote that all students' contributions are valued "with dignity and respect" (Maher, 1996, p. 40).

The framework for this research is based on the PD intervention model which combines constructivist views on teaching and learning mathematics. In the PD intervention, the instructor models how to facilitate participants' recognition of reasoning with particular pedagogical moves so that participants can transform into a facilitator role to implement the same tasks with their own students. The PD intervention model for this research also allows the participants to apply prior knowledge and build on their own conceptual understanding as they provide justifications for their task solutions, view VMC videos and read articles pertaining to the tasks, and to discuss and reflect their ideas with other participants.

## 2.2 Literature Review

The research on attending to students' reasoning can be grouped in three main sections. The first section that will be discussed is the role of the instructor as a facilitator which involves the moves of the instructor or what the instructor does and says to encourage teachers' reasoning in problem solving. A second section of discussion is the classroom learning environment or how the teacher creates an atmosphere and selects problems conducive to encouraging students' reasoning in problem solving. The third section discusses several studies that involve PD models structured for attending to students' mathematical reasoning where the role of the instructor and the learning environment are emphasized and significant.

### *2.2.1 The Role of the Instructor*

Noddings (1990) claimed "the cognitive premises of constructivism can dictate only guidelines for teaching" (p. 15). The distinction between constructivism and constructivist teaching is also discussed by Maher (1996). Maher (1996) posited that "the 'constructivist teacher' encourages children to make conjectures and pursue the reasonableness of their ideas by constructing models, comparing them, developing arguments, discussing ideas, and negotiating conflicts while working on problematic situations that either have been presented to them or that they themselves have initiated and extended" (p. 39). To pursue this pedagogical approach, the role of the teacher must transform from a lecturer to a facilitator of knowledge (Maher, 1996).

Maher (1996) gave an example from a classroom session that was part of the longitudinal study where ten-year olds worked on a problem called "Guess My Tower" (p. 30). This problem was structured as a game where the winner had to choose one of four choices and then correctly match this choice with a tower picked from a box that was covered (Maher, 1996). Inside the box were all of the possible three-tall towers that can be made selecting from two colors of red and yellow unifix cubes™ (Maher, 1996). The four possible winning towers included towers where the colors were all the same, towers with one red only, towers with exactly two reds, and towers with at least two yellows

(Maher, 1996). Students were asked about the choice they made and why their choice would be best as compared to other tower choices (Maher, 1996).

Maher (1996) also discussed six episodes of video data from this problem-solving session that "illustrate the complexity of learning and teaching from a constructivist perspective" (p. 31). The first and second episode involved two students, Matt and Stephanie (Maher, 1996).[1] In these episodes, the teacher listened and questioned Matt and Stephanie about their ideas to find how many four-tall towers can be made selecting from two colors (Maher, 1996). Questions included why the students were convinced their tower ideas would work and whether there was a more convincing argument for another idea (Maher, 1996).

The third and fourth episodes involved two different students, Milin and Michelle working on the same task of finding how many four-tall towers can be made selecting from two colors (Maher, 1996). During these episodes, the teacher asked Michelle about her understanding of Milin's idea and whether Milin's explanation made sense (Maher, 1996). Milin explained an inductive argument of how he took two of the three-tall towers and removed one unifix cube™ off each tower (Maher, 1996).

In the fifth episode, the teacher extended the problem to five-tall towers and asked Milin to show that his idea still works (Maher, 1996). The teacher asked if Milin and Michelle would like to share what they learned with Matt and Stephanie and Michelle communicated Milin's idea (Maher, 1996). In the sixth episode, Stephanie had the opportunity to explain to the whole class why the pattern of doubling worked (Maher, 1996). From these episodes, teaching from a constructivist perspective encouraged

[1] Building Towers, Selecting from two colors for Guess My Tower, Clip 2 of 5: Does the Number Double? [video]. Retrieved from: http://dx.doi.org/doi:10.7282/T32V2FBZ

teachers to "facilitate discussions and probe for better understanding of student thinking through appropriate questioning that is related to the students' constructions" (Maher, 1996, p. 39).

Research by Martino and Maher (1999) also emphasized the significance of the questioning and listening by the teacher to promote students' to generalize and justify their mathematical ideas. Generalizing and justifying mathematical ideas were facilitated by the teacher by creating a classroom environment where students explored mathematical tasks and discussed possible solutions with other students (Martino & Maher, 1999). As the students explored and discussed the solutions to the mathematical tasks, teachers asked questions related to the ideas that students discovered as they participated in the tasks (Martino & Maher, 1999).

One hundred fifty one students in third, fourth, and fifth grade classrooms including urban, blue collar, and suburban New Jersey school sites were given the opportunity to explore and discuss solutions with other students (Martino & Maher, 1999). The students from these schools were given two isomorphic tasks during 1992 to 1993 (Martino & Maher, 1999). The first task included having the students find all possible four-tall towers that could be made using two colors of unifix cubes™ by working in pairs to convince their partner of their solution (Martino & Maher, 1999). For the second task, students were asked to find how many different pizza combinations could be made with four toppings (Martino & Maher, 1999).

Although the third, fourth, and fifth grade students participated in this study, only the data from the third and fourth grade students are analyzed in the paper by Martino and Maher (1999). Martino and Maher (1999) analyzed data from four different episodes of the third and fourth grade students participating in the tasks. The analyzed data from these examples showed “strong relationships between (1) a teacher’s monitoring the progress of a student’s constructions of a problem solution and (2) the teacher’s posing a timely question which invites or challenges students to revisit earlier thinking, revise it in the light of new experience, and, if appropriate, move forward to deeper, stronger understanding” (Martino & Maher, 1999, p. 74).

### *2.2.2 The Learning Environment*

According to Schorr and Amit (2005), learning environments are comprised of the classroom atmosphere and the problem-solving activities experienced by the students. The classroom atmosphere and the problem-solving activities experienced by the students result in reasoning or sense-making when students are given the opportunity to “talk about their ideas, reflect on the reasonableness of their solutions (orally and in writing), listen to the solutions of others, discuss different representations of the same problem and the relationship among representations, and share, defend and justify their solutions—orally and in written form” (Schorr & Amit, 2005, p. 138). Students’ thoughts and reflections are valued in this type of classroom environment (Schorr & Amit, 2005).

A central goal of the Schorr and Amit (2005) research was to study students’ modeling cycles. A model is “a system for describing, explaining, constructing or manipulating a complex series of experiences” (Schorr & Amit, 2005, p. 138). Although a main purpose of the Schorr and Amit (2005) research was to study students’ modeling cycles, a problem-solving activity needed to be chosen that had “the potential to elicit a thoughtful, sensible solution” (Schorr & Amit, 2005, p. 138) and be presented in a valued environment.

All eight students who participated in this study had just completed twelfth grade from local urban high schools (Schorr & Amit, 2005). Students were given a problem to solve called the "Radio Problem" (Schorr & Amit, 2005, p. 139). The problem is as follows:

> The editors of Consumer Reports want to make a new consumer guide for products that is important to teenagers. The first items that they want to rate are portable radio cassette players with headsets. They need your help to develop a rating system… The editors want a rating system that readers can use to rate any model (even if it is not listed on the attached list), and compare the models to determine which are the "best buys". The editors have also gathered the attached information for some models. They plan to use these as examples to show readers how to use the rating system. To help the editors, please: I) Develop a rating system for these players. Be sure that the system can be used to identify overall "best buys" which take into account the factors that the survey indicates are important. Also, readers should be able to use the rating system with ANY other players, including those not listed in the guide, so include any tables or charts that are part of your system. II) Write clear step-by-step instructions that make it easy for readers to use your rating system. III) Write a letter to the editors explaining why you decided on your rating system and describe its advantages and disadvantages. (Schorr & Amit, 2005, p. 139-140)

To solve this problem, students were given a chart of eleven radio brand choices and were warned that there could be several different solutions (Schorr & Amit, 2005). The students had a choice of working alone or with another person (Schorr & Amit, 2005).

The modeling cycles in solving the radio problem of one student named James was analyzed for this study (Schorr & Amit, 2005). James started with a first model by making and rating a checklist of the advantages and disadvantages of the radios (Schorr & Amit, 2005). For the second model, James used "a multi-dimensional approach in which he selected information from the data (table), intentionally ignored some of the other information (such as brand name), and then defined ranges of "good" with associated numerical values" (Schorr & Amit, 2005, p. 141). The third model involved James revising his checklist and scaling system (Schorr & Amit, 2005). In the fourth model, James accounted for the price and weight of any radio and made a table with including all of the qualitative and quantitative variables (Schorr & Amit, 2005). A list of advantages and disadvantage was also included along with “a ‘key’ so that the user could easily discern how to use the point value” (Schorr & Amit, 2005, p. 142).

Solving the problem required having James progress through a number of cycles (Schorr & Amit, 2005). By using a problem that had a real-world application and designing the problem to be open-ended, James was able to reflect on his prior work (Schorr & Amit, 2005). James was also able to make necessary revisions to solve a problem that he initially thought was unsolvable (Schorr & Amit, 2005).

Although classroom atmosphere and problem-solving activities are important factors of the learning environment, the teacher also has a responsibility to create a learning environment that builds a mathematical community (Davis et al., 1990). It is also important to note that “the role of the community-other learners and teacher-is to provide the setting, pose the challenges, and offer the support that will encourage mathematical construction” (Davis et al., 1990, p. 3). Encouraging mathematical knowledge through collaborative activities allows students to build and revise knowledge within a mathematical community (Davis et al., 1990).

A mathematical community can be created within the classroom that promotes learning from a constructivist perspective (Davis et al., 1990). This constructivist perspective of learning allows for the community of the teacher and students to support and challenge each other in constructing mathematical knowledge by forming, questioning, and revising mathematical arguments (Davis et al., 1990). However, practitioners should note that “constructivist premises imply that there are many roads to most instructional endpoints” (Noddings, 1990, p. 16).

Teachers can create these roads or opportunities by allowing students to work on open-ended problems in small group collaborations (Noddings, 1990). Research by Boaler (2006) provided evidence that social activeness in a community can have a central effect on learning mathematics. In this four year longitudinal study, Boaler (2006) created a community in a detracked mathematics classroom at an urban high school in California. Teachers created an environment where students respected and valued each individual’s contribution regardless of social class, race, gender, ethnicity or the skill levels possessed (Boaler, 2006).

The classroom activities were highly structured in multidimensional tasks where open-ended problems were used in collaborative group work (Boaler, 2006). In this community, the learning of each individual group member became the responsibility of the entire student group (Boaler, 2006). All students were provided a safe environment to share their ideas which resulted in improved assessments on group tests, an occasional rating on group conversation quality, and random calls of any person in the group (Boaler, 2006).

Small group collaborations are also encouraged and mandated in order to be certified by the National Board for Professional Teaching Standards (NBPTS, 2014). Although this certification does not replace the mandatory state certification, it is a nationally recognized ten year teaching credential (NBPTS, 2014). To be certified in early adolescence or adolescence and young adulthood for mathematics, teachers must show documented accomplishments in four portfolio entries where one entry includes a fifteen minute video of small-group collaboration. The other entries include a fifteen minute video of whole-class discourse, an analysis of a mathematical activity for two students, and documented artifacts showing that the teacher has experience in three main categories which include the teacher as a learner, the teacher as a partner with the student's family and the community, and the teacher as a collaborator or leader at the local, state, or national level (NBPTS, 2014). In addition, teachers are required to pass a six part test on mathematical content (NBPTS, 2014). However, this PD opportunity is costly and not mandatory (NBPTS, 2014).

***2.2.3 PD Models***

Several studies at Rutgers University have focused on PD with an emphasis on teachers attending to students' mathematical reasoning from a constructivist perspective of learning (Alston & Davis, 1996; Francisco & Maher, 2011; Francisco, Maher, Powell, & Weber, 2005; Maher, 1996, 2011a, 2011b; Maher et al., 2010; Martino & Maher, 1999; Palius & Maher, 2011). One PD model of attending to students' reasoning from a constructivist perspective of learning was done by Maher et al. (2010) where twenty middle-school classroom and special-education teachers from two New Jersey middle-schools participated in a one year PD workshop intervention.

Similar to the intervention previously described in this paper for my research on studying teachers' recognition of reasoning, the PD workshop intervention also used videos located in the VMC that can be publicly accessed (see www.videomosaic.org) (Maher et al., 2010). The videos show children participating in solving combinatorics problems by sharing and justifying their solutions with other students and teachers.

These videos were used within the PD intervention for the middle-school mathematics teachers and special-education teachers (Maher et al., 2010).

Maher et al., (2010) described how the videos in the PD intervention were used to study any effects that the videos of students' reasoning had on the beliefs of teachers. However, studying the effects on students' reasoning was not enough (Maher et al., 2010). Teachers were also required to participate in the various mathematical tasks as learners to "improve their mathematical reasoning skills so that they are better prepared not only to study the videos of children's reasoning, but also to promote and evaluate the mathematical reasoning of their own students" (Maher et al., 2010, p. 4).

An objective of this study was to trace how the beliefs of teachers were modified after the PD intervention (Maher et al., 2010). During the intervention, three cycles of tasks were administered that included four parts (Maher et al., 2010). These parts included "(1) teachers doing mathematics, (2) teachers studying videos of children doing mathematics, (3) teachers implementing in their classrooms, and (4) teachers analyzing their students' work" (Maher et al., 2010, p. 4). The same task cycles were used for my research for recognizing students' reasoning with the omission of the second cycle half-topping pizza combination problems.

The first cycle included the task of having the teachers work to find all possible four-tall towers that could be made selecting from two colors of unifix cubes™ (Maher et al., 2010). After working on the problem, teachers shared their solutions and arguments (Maher et al., 2010). When the discussion of the first task solutions was finished, teachers were shown the video of two students named Stephanie and Dana that worked together to build sixteen four-tall towers selecting from two colors of unifix cubes™

(Maher et al., 2010). After teachers watched this video, a discussion followed comparing the reasoning strategies used by the children and the teachers (Maher et al., 2010). Teachers were then given the task of predicting how many possible three-tall towers could be made selecting from two colors of unifix cubes™ and then watched two additional videos after making their predictions (Maher et al., 2010).

One video involved a student named Meredith who removed a cube from the top of the four-tall tower and initially predicted that there would be the same number of 3-tall towers (Maher et al., 2010). Another video included third-grade students named Stephanie and Dana and their argument that there would be more 3-tall towers because the blocks that were removed could be used to build more towers (Maher et al., 2010). In both videos, the students were asked to investigate their arguments (Maher et al., 2010). After teachers watched both of these videos, teachers were asked to predict how many five-tall towers could be made selecting from two colors and then were asked to use the unifix cubes™ to find the total number of five-tall towers (Maher et al., 2010). Teachers worked in small groups to complete the task (Maher et al., 2010).

Teachers then discussed their solutions and then watched two more videos. One video involved Stephanie and Dana a year older in the fourth grade building five-tall towers selected from two colors and finding a total of thirty-two towers (Maher et al., 2010). Discussion about the comparison of the strategies and solutions between the teachers and students occurred and then teachers watched the final video in the first cycle (Maher et al., 2010). In this video, another student named Milin shared his inductive approach to solving how many five-tall towers could be made selecting from two colors (Maher et al., 2010). However, the cycle was not completed until teachers implemented the tasks of the first cycle with their own students and shared students' work during the next workshop with the other teachers in the PD intervention (Maher et al., 2010).

Once teachers finished their discussion of students' work from the first cycle, teachers began the second cycle (Maher et al., 2010). The second cycle involved tasks that invited students to explore finding the number of pizzas that could be made with a given number of toppings available (Maher et al., 2010). Because of the complex solutions that resulted from the longitudinal study, Maher et al. (2010) suggested using the following order:

> Cycle II, task 5. A local pizza shop has asked us to help them design a form to keep track of certain pizza sales. Their standard "plain" pizza contains cheese. On this cheese pizza, one or two toppings could be added to either half of the plain pizza or the whole pie. How many choices do customers have if they could choose from two different toppings (sausage and pepperoni) that could be placed on either the whole pizza or half of a cheese pizza? List all possibilities. Show your plan for determining these choices. Convince us that you have accounted for all possibilities and there could be no more.
>
> Cycle II, task 6. The local pizza shop was so pleased with your help on the first problem that they have asked us to continue our work. Remember that they offer a cheese pizza with tomato sauce. On this cheese pizza, one or more of the following toppings could be added to either half of the plain pizza or the whole pie: peppers, sausage, mushrooms, and pepperoni. How many choices does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possible choices. (Maher et al., 2010, p. 10)
>
> Cycle II, task 7. Capri Pizza has asked you to help design a form to keep track of certain pizza choices. They offer a standard "plain" pizza with cheese and tomato sauce. A customer can then select form the following toppings: peppers, sausage, mushrooms, and pepperoni. How many choices for pizza does a customer have?
> List all possible choices. Find a way to convince each other that you have accounted for all possibilities. (Maher et al., 2010, p. 11)

After completion of these tasks, teachers discussed their solutions and then watched a video that involved fifth grade students working on the same pizza problems and a video interview with a student named Brandon in which Brandon recognizes the isomorphism between the towers and the pizza problems (Maher et al., 2010). Teachers then were given the opportunity to implement the pizza problems in their own classes

(Maher et al., 2010). At the next workshop, teachers shared and analyzed students' work (Maher et al., 2010).

The third cycle included finding all possible three-tall towers that could be made selecting from three colors of unifix cubes™ with the extended task of finding all four-tall towers selecting from three different colors of unifix cubes™ using at least one of each color called *Ankur's Challenge* since it was designed by Ankur, a participant in the study (Maher et al., 2010). After working on these tasks, teachers were given an opportunity to discuss their solutions (Maher et al., 2010). Then teachers watched one video showing five students in tenth grade working to solve *Ankur's Challenge* (Maher et al., 2010). Teachers then implemented the problems in their own classrooms and brought back students' work samples to share at the next workshop (Maher et al., 2010).

One objective of this study was "to track changes, if any, in teacher held beliefs during the course of the intervention" (Maher et al., 2010, p. 14). Before and after this PD workshop intervention, a Beliefs Inventory pre-test and post-test was given assessing how students learn, situations for teaching effectively, and the possible influences affecting students' learning of mathematics (Maher et al., 2010). Findings from earlier studies showed significant change in beliefs" (Maher et al., 2010).

Palius and Maher (2011) also described two models that helped teachers to attend to students' reasoning that used the available resources found in the VMC. Palius and Maher (2011) first described a model for pre-service teachers used in a semester course at Rutgers University before beginning their internship experience. Elementary pre-service teachers worked in pairs on tasks emphasizing place-value, counting, and fractions using Cuisenaire Rods and secondary pre-service teachers focused on combinatorics using unifix cubes™ (Palius & Maher, 2011). Pre-service teachers were shown videos of students performing the same tasks and participated in discussions about the videos and their experiences (Palius & Maher, 2011).

Palius and Maher (2011) also described a second model for in-service teachers that included all aspects of the pre-service model. However, there was an addition of implementing the tasks in the classroom and bringing samples of their students' work to analyze and discuss with the other teachers (Palius & Maher, 2011). Both models include the teachers participating in solving the problems (Palius & Maher, 2011). This research was based on a constructivist perspective where students were encouraged to explore their mathematical ideas and provide convincing arguments to each other during the tasks (Palius & Maher, 2011).

Constructivism was also the foundation for another study using a pre-service model which included "thirty-five experimental pre-service teachers and twelve comparison pre-service teachers" (Maher, 2011a, p. 87). All participating teachers were taught by the same instructor at a private New Jersey university (Maher, 2011a). Both pre-service groups worked on and discussed solutions of the exact same tasks in about a three to four week time span (Maher, 2011a). However, the pre-service comparison group did not watch the videos of the children participating in the solving of the tasks (Maher, 2011a).

Twenty-two New Jersey middle and special education teachers participated in the in-service model that lasted five months (Maher, 2011a). The assessments were comprised of a pre and post-assessment twenty minute video with open-ended questions

(Maher, 2011a). For the open-ended questions, teachers responded in writing about the students' reasoning they observed (Maher, 2011a).

Maher (2011a) reported that overall growth and a comparison each type of argument such as argument-by-cases, induction, contradiction and generalization were calculated using ANOVA tests with the in-service experimental group as well as both the comparison and experimental pre-service groups (Maher, 2011a). The results suggest that studying videos about students' reasoning helped teachers to recognize reasoning forms in students' problem solving (Maher, 2011a). The average growth for the in-service teacher group was 59.7%; the pre-service experimental teacher group had an average growth of 35.8%; and the pre-service comparison group had an average growth of 4.9% (Maher, 2011a).

Another study by Francisco and Maher (2011) supported the hypothesis that "effective teaching requires knowledge of students' mathematical reasoning" (p. 2) where two elementary teachers, four middle school mathematics teachers, and three mathematics coaches from various schools in the same district voluntarily participated for one year as interns in the Informal Mathematical Learning Project (IML).[2] The IML was an after-school project designed for urban, low-income, and minority middle school students (Francisco & Maher, 2011). The IML provided a voluntary opportunity for the twenty-four sixth-grade students to develop convincing arguments in mathematical explorations with other students (Francisco & Maher, 2011).

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The mathematical explorations included one task of building all possible three-tall towers selecting from three different colors of unifix cubes™ (Francisco & Maher, 2011). Other tasks involved using Cuisenaire Rods to answer questions about fractions (Francisco & Maher, 2011). To complete the tasks, students worked together and made convincing arguments of their mathematical ideas to other students (Francisco & Maher, 2011). The interns usually observed the students in groups of two or three but occasionally four to six students sat at a table.

The mathematical explorations included one task of building all possible three-tall towers selecting from three different colors of unifix cubes™ (Francisco & Maher, 2011). Other tasks involved using Cuisenaire Rods to answer questions about fractions (Francisco & Maher, 2011). To complete the tasks, students worked together and made convincing arguments of their mathematical ideas to other students (Francisco & Maher, 2011). The interns usually observed the students in groups of two or three but occasionally four to six students sat at a table (Francisco & Maher, 2011). In addition, the interns minimized interaction with the students so as to not have any influence on the direction of the students' thoughts (Francisco & Maher, 2011). Although the researchers facilitated problem solving, they did not offer solutions to the problems but rather sought justifications from the students (Francisco & Maher, 2011).

In this study, the debriefing meetings were salient because it provided an opportunity for the researchers to discuss how teachers attended to students' mathematical reasoning (Francisco & Maher, 2011). The meetings were videotaped and transcribed which provided a substantial dataset (Francisco & Maher, 2011). Students' work was also an important part of the dataset (Francisco & Maher, 2011).

Five themes emerged from this study that included conceptual understanding, reasoning forms, communication of mathematical ideas, justifying mathematically, and necessary supports regarding students' growth with mathematical reasoning (Francisco & Maher, 2011). The themes contributed to providing insight to teachers in attending to students' reasoning (Francisco & Maher, 2011). From the IML experience, teachers observed that their students can be
successful in providing convincing mathematical arguments (Francisco & Maher, 2011).

Another PD model that stemmed from the IML project involved having teachers attend to students' reasoning with reflections that impacted their teaching (Francisco et al., 2005). For this PD model, there are three nonlinear and intersecting phases (Francisco et al., 2005). The first phase involved having the teacher-researchers attend to the students' thoughts by observing and documenting students' reasoning during mathematical investigations (Francisco et al., 2005). The second phase was comprised of three modalities which included reflecting after the research sessions, studying the mathematical tasks in order to plan the future direction of the lesson, and describing the videotaped data (Francisco et al., 2005). The third phase involved implementing the intervention in their classrooms the following year with a different student group (Francisco et al., 2005).

For this particular research study, the focus was on the first modality of the second phase with the purpose of using the reflection sessions to provide evidence of teachers attending to students' mathematical reasoning (Francisco et al., 2005). Although the project lasted three years, the data for this study came from the first year (Francisco et al., 2005). Thirty sixth-grade students participated in the research study in 2003 but only two episodes were discussed (Francisco et al., 2005).

In one episode, students were asked to find all the possible pizzas that could be made from four available toppings (Francisco et al., 2005). Three students named Channel, Kori, and Nia correctly found sixteen possible pizzas and then the teacher-researcher asked how many pizzas would there be if onions were added to the possible toppings (Francisco et al., 2005). Nia asked if multiplying could be used and the teacher-researcher asked if Nia could find a way to use multiplication to solve the problem (Francisco et al., 2005). Teachers discussed their observations of Nia, Channel, and Kori in the reflection session (Francisco et al., 2005).

In the second episode, students used a computer-based simulation tool to infer by sampling the number of marbles of each color in a bag containing one hundred marbles (Francisco et al., 2005). Evidence of teachers attending to students' mathematical reasoning included an observation where one group of students used an extremely big sample size and teachers questioned the purpose of the larger sample size number (Francisco et al., 2005). Other evidence of teachers attending to students' reasoning included the clarification and development of students' ideas during the mathematical activity (Francisco et al., 2005).

The development of students' ideas and reasoning during mathematical activity was also studied by researchers outside Rutgers University (Ball, Ben-Peretz, & Cohen, 2014; Grossman, Compton, Igra, Ronfeldt, Shahan, & Williamson, 2009; Kazemi, Franke, & Lampert, 2009; Lampert, Franke, Kazemi, Ghousseini, Turrou, Beasely, Cunard, & Crowe, 2013; Leong, Leong, Tay, Toh, Quek, & Dindyal, 2011; Pólya, 1945;

Sample-McMeeking, Orsi, & Cobb, 2012). George (György) Pólya (1945) in his classic book *How to Solve It,* encouraged students to solve problems using four stages including understanding the problem, making a plan, implementing the plan, and looking back or reflecting on the plan. A recent study focused on Pólya's fourth stage which included "the consideration of alternative solutions and representations, the re-examination of the solution for a more efficient strategy, and the extension of the solution to other related problems" (Leong et al., 2011, p. 182).

This particular PD program was implemented in an independent, secondary school in Singapore to analyze the teachers' thoughts of Pólya's fourth stage in problem solving and how they promoted this in their classrooms (Leong et al., 2011). The purpose of the PD program was to help teachers specifically implement the fourth stage in their pedagogy (Leong et al., 2011).

The PD program included three different components (Leong et al., 2011).

The first component included revising the curriculum (Leong et al., 2011). Within this first component, a four-page worksheet was created that included a practical problem where students had to apply Pólya's four stages. The final page of the worksheet emphasized using the fourth stage and the task was not considered to be successfully completed until the fourth page was completed (Leong et al., 2011).

The second component involved five ninety-minute PD meetings (Leong et al., 2011). For these meetings, Leong et al. trained the teachers with a guidebook of their creation which contained "an overview of Pólya's stages, a set of problems, and a recommended module plan to implement the teaching of these problems, and details for each lesson within the module" (Leong et al., 2011, p. 184). The training was given to teachers in an independent Singapore Secondary school with the purpose to "provide teachers with time to experience problem-solving themselves and to help teachers develop problem solving habits" (Leong et al., 2011, p. 185).

The third component involved implementation within the classroom from the same person instructing the teachers (Leong et al., 2011). Twenty-one students participated in a ten hourly lessons which mirrored the teachers' experience but was modified for students' needs (Leong et al., 2011). After the three components of the program were complete, three teachers were expected to implement the lessons with 164 students using the resources from the training (Leong et al., 2011). However, only two teachers named Raymond and William attended the second and third components of the training (Leong et al., 2011).

Researchers came back to the school to observe the teachers implementing the program but "the main sources of data were (1) the reflections of Raymond and William about their lessons preparation and implementation; and (2) the classroom activities of these two teachers" (Leong et al., 2011, p.185). Leong et al. (2011) reported that teachers liked how Pólya's fourth stage helped students see relationships with other problems that allowed for exploration beyond the original problem. In addition, Leong et al. (2011) posited "that unless teachers 'buy-in' to the scheme and have developed relevant skills to carry out the plans chances of success in such efforts to change practices are slim" (p. 184). Major reforms to PD including changing thought processes involved with mathematical problem-solving were proposed supporting why there is a need to study students' reasoning in today's classrooms and reexamine the essential elements that must be included for successful professional development of teachers (Leong et al., 2011).

However, one might ask whether participation in a PD program for teachers leads to higher student achievement in mathematics. Sample-McMeeking et al. (2012) studied the effect that a PD program had on student achievement in mathematics. In this particular study, one hundred twenty-eight middle- school mathematics teachers participated in a five-year PD project in Colorado called the Rocky Mountain Middle School Math and Science Partnership (RM-MSMSP) (Sample- McMeeking et al., 2012). The RM-MSMSP was funded by the NSF to bring together both teachers from seven districts and faculty from four universities and "increase the subject-matter content knowledge and pedagogical content knowledge of elementary and middle school mathematics and science teachers" (Sample-McMeeking et al., 2012, p. 160).

The project goal was to have teachers use the knowledge from this PD project to improve mathematics achievement for students in grades five through eight (Sample-McMeeking et al., 2012). For this study, a standardized test called the Colorado Student Assessment Program (CSAP) was used as the assessment to measure the effect of the project on students' mathematical achievement (Sample-McMeeking et al., 2012). The assessment scores of students from the year before the PD project were compared to the assessment scores of the students after the teachers participated in the program using a generalized linear mixed model for analysis (Sample-McMeeking et al., 2012). According to Sample-McMeeking et al. (2012), "results showed that students' odds of achieving a score of Proficient or better increased with teacher participation in the PD program" (p. 156) which provides evidence that successful PD can lead to increased achievement in mathematics.

If successful PD can lead to increased achievement in mathematics, would it be of interest to know whether a specific pedagogical knowledge base exists to prepare teachers to be effective? Ball, Ben-Peretz, and Cohen (2014) suggest the idea that records of practice have the potential to build a foundation of salient knowledge regarding learning and pedagogy. Records of practice are defined by Ball et al. (2014) as "a collection of primary materials that represent core elements of an experience, an event, or an interaction" (p. 321).

Ball et al. (2014) reported three records of practice examples. The first example discussed the experience of an educator from Norway named Hartwig Nissen (Ball et al., 2014). Nissen visited several elementary schools in Scotland in 1852 (Ball et al., 2014). From these visits, Nissen made written records that included the school's appearance, educational resources used, and observations about the educational processes (Ball et al., 2014).

A second example of a record of practice included a collection of records from a third- grade public school mathematics class. The various records were collected for one year. The records included items such as "mathematics lessons – video and audio recordings of each day, copies of the children's scribbles, drawings, notes, and work, as well as the teachers' notes (Ball et al., 2014, p. 323).

Ball et al. (2014) reported a third example of a record of practice from a teacher named Sarah who taught a regular instructed class and an English-language development class (ELD) which was instructed in English. During her first two years, Sarah reported that "when her teaching was in Spanish, her students were lively and engaged, but they were quiet during ELD lessons" (Ball et al., 2014, p. 324). In her third year, Sarah decided to use both Spanish and English for her ELD and wrote records of practice which included "narratives created explicitly for documenting and reflecting on her teaching, lesson plans, assignments, assessments and student work" (Ball et al., 2014, p. 324).

Using the three examples, Ball et al., (2014) analyzed commonalities and differences. The commonalities of the three examples included their personalized structure and specific detail and differences included the mediums of the records of practice which varied from hand-written records to video and audio data (Ball et al., 2014). Ball et al. (2014) claimed that "records make possible a special kind of study of practice, a kind of work that can lead to the generation of professional knowledge" (p. 328). According to Ball et al. (2014), this collective professional knowledge creates opportunities for the improvement of practice.

Lampert et al. (2013) posited two challenges of improving practice with new elementary mathematics teachers. The challenges included "preparing beginning teachers to actually be able to do teaching when they get into classrooms, and preparing them to do teaching that is more socially and intellectually ambitious than the current norm" (Lampert et al., 2013, p. 226). To address these challenges, Lampert et al. (2013) analyzed ninety videos that incorporated a specific pedagogical approach called "rehearsal" (227).

They define rehearsal as "a social setting for building novices' commitment to teach ambitiously" (Lampert et al., 2013, p. 227). The rehearsal videos were used to study the Master's level methods courses at the University of California, University of Michigan, and the University of Washington (Lampert et al., 2013). The methods courses were designed on the basis that "mathematics teachers need to learn to elicit, observe, and interpret student reasoning, language, and arguments and adjust their instruction to promote learning" (Lampert et al., 2013, p. 227).

The activities used during mathematics instruction included choral counting, games, computational problems involving sequencing, and having students share strategies of computation and word problems (Kazemi, Franke, & Lampert, 2009; Lampert et al., 2013). Cycles of enactment and investigation (CEI) were implemented with the new teachers (NT) (Lampert et al., 2013). Each CEI began with the NT watching a video of an instructional activity (IA) or observing a live demonstration of an IA (Lampert et al., 2013).

After this first phase, a teacher educator (TE) facilitated a discussion with the NT about their observations (Lampert et al., 2013). The next phase has the NT practice the same IA with their peers (Grossman et al., 2009; Lampert et al., 2013). Upon finishing the rehearsal with their peers, the NT implemented the IA with classroom students (Lampert et al., 2013).

Lampert et al. (2013) coded ninety rehearsal videos with Studiocode© video-analysis software to observe "*what* was worked on (the *substance* of the interaction) and *how* it was worked on (the *structure* of the interaction)" (p. 230). Four items were used to categorize interaction structure which included making suggestions, making critiques, role-playing as the teacher or student, or facilitating discussion (Lampert et al., 2013). Fifteen items categorized the substance interactions including elicit and respond, representation, engagement, attending to the IA, content goals, thinking, mathematics, student error, orienting students, process goals, IA launch, managing space, body and voice use, and closing the IA (Lampert et al., 2013).

According to Lampert et al., (2013) the mean TE/NT interactions for each rehearsal was fourteen and “22% of the TE/NT exchanges were initiated either by the rehearsing NT (e.g., to ask about how many different student ideas to elicit) or by another NT (to raise a question, for example, about how to deal with an ongoing interaction)” (p. 233). Results for substance coding revealed that 36% of the TE/NT interactions were coded for elicitation and response to students in 95% of the rehearsal videos. Although more than 70% of the rehearsal videos involved attending to students’ mathematical thinking and mathematics, comparatively a small percent of TE/NT interactions were coded, 14% and 12% respectively. From the results, Lampert et al. (2013) reported that rehearsal allowed higher-level pedagogical approximations to help novice teachers adapt their pedagogical approaches as they developed regarding their identity, skill, and knowledge (Lampert et al., 2013).

This rehearsal idea is similar to the University of Colorado Assessment Project (Borko et al., 1997; & Putnam & Borko, 2000). For the University of Colorado Assessment Project, teachers worked on tasks, implemented the tasks in their own classrooms, and discussed any experiences from participating in the project (Borko et al., 1997; & Putnam & Borko, 2000). The University of Colorado Assessment Project is similar to this research project along with many studies from Rutgers University but missing one additional aspect. This additional aspect involved having the teacher participants view, analyze, and discuss VMC videos of students working on the same mathematical tasks as the students in the video.

# Chapter 3 - Methodology

## 3.1 Research Context

Rutgers researchers at the Robert B. Davis Institute of Learning (RBDIL) designed a PD intervention to facilitate building teachers' knowledge of recognizing students' mathematical reasoning as a component of two NSF funded projects.[1] The PD intervention was implemented with teachers in the first project, the New Jersey Partnership for Excellence in Middle School Mathematics. The NJPEMSM initiative sought to prepare teachers to have a deeper understanding of mathematics, engage students more effectively in learning mathematics, and give support and encouragement for teachers to become facilitators of student mathematical learning (NJPEMSM, 2009).

Publications about student learning and video data of students working on and discussing solutions to mathematical tasks were produced through the second NSF project and used in the RBDIL PD intervention. Videos used in the intervention were obtained from the longitudinal study at Rutgers University with funding from the NSF (Awards MDR-9053597, REC-9814846, REC-0309062, and DRL-0723475) and can be accessed from the VMC (Award DRL-0822204)[2].

The RBDIL PD intervention model was implemented as a one-semester graduate course: Lesson Study on Student Reasoning. Rutgers eCollege and Sakai were the

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websites where teachers posted on-line responses to weekly questions, completed pre- and post-assessments, and the end of the course survey.

## 3.2 Professional Development Intervention Model

### *3.2.1 Definitions*

Definitions to describe the people and work in the PD intervention model follow:

**Teachers:** The teachers enrolled in the fall 2013 course, Lesson Study on Student Reasoning.
**Instructor:** The teacher of the Lesson Study Course on reasoning who assigned tasks, readings, and videos and facilitated discussion.
**Current Students:** Students of the teachers enrolled in the course.
**Research Students**: Students captured on video in research problem-solving sessions provided video data and samples of students' work to be studied as assignments.
**Intervention**: The section of the course "Lesson Study on Reasoning" being studied. The intervention used a similar set of combinatorics tasks in each cycle. (McGowan, 2016)

### *3.2.2 Assessments*

For this PD intervention model, teachers were required to take pre- and post-assessments. One of these required assessments was a Pre- and Post- Beliefs Assessment about the teaching and learning of mathematics. The Beliefs Assessment can be found in Appendix A.

Teachers also completed a Pre- and Post- assessment after watching a VMC video called *The Gang of Four of* fourth-grade research students Jeff, Michelle, Milin, and Stephanie participating in a small group interview facilitated by Researcher, Carolyn Maher. The research students had available paper and pencil and were asked to share their solutions to the three-tall towers problem, selecting from 2-colors. Participating teachers were asked to describe and give evidence from the interview of each example of reasoning that was offered by the students, whether or not the reasoning was valid, and whether the argument was convincing, indicating why or why not they were convinced. Teachers' responses were evaluated on their recognition of children's arguments, assessment or not about the children's reasoning, evidence to support claims, and whether the claims are partial or complete. The Gang of Four Assessment can be found in Appendix C along with a transcript of the video found in Appendix D.

### *3.2.3 Tasks*

In addition to the assessments, teachers were required to participate in three cycles of tasks chosen from the combinatorics strand of mathematics. For each of the three task cycles, the participants worked in small groups on mathematical tasks; made one original online post responding to the assigned questions, VMC videos, and literary articles about previous implementations of the same tasks with research students; made at least two additional posts responding to two other teachers' original posts, observed and discussed the in-district classroom visits with current students, and discussed examples of students' work after the implementation of these mathematical tasks in the teachers own classes at the regional meetings.

#### *3.2.3.1 First Cycle Tasks*

For the Cycle 1 tasks, participants were instructed to work in pairs to find a solution to the following task:

> **Building 4-tall towers, selecting from 2 colors**
>
> You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all. (Maher et al., 2010, p. 5)

A detailed framework and timeline for the Cycle 1 intervention can be found in Appendix G. In addition to this task, extension problems were provided after the completion of the four-tall tower task, selecting from two colors. The following tasks were provided:

> **Extension: Predicting 3-tall, 5-tall towers, selecting from 2 colors**
>
> Make a prediction about a solution for finding all possible towers 3 cubes high (without building them) [selecting from 2 colors]. Do you think there will be more, fewer, or the same number of possible towers as you found for towers that were 4 cubes high? [Why do you think that?]
> Make a prediction about a solution for finding all possible towers 5 cubes high (without building them) [selecting from 2 colors]. Do you think there will be more, fewer, or the same number of possible towers as you found for towers that were 4 cubes high? [Why do you think that?**]** (Maher et al., 2010, p. 7)
>
> **Extension: Building 5-tall towers, selecting from 2 colors**
>
> You have two colors of unifix cubes to build towers. Your task is to make as many different looking towers as possible, each exactly five cubes high. Find a way to convince yourself and others that you have found all possible towers five cubes high, [and that you have no duplicates. Record your towers below and provide a convincing argument why you think you have them all. (Maher et al., 2010, p. 8)

*3.2.3.2 Second Cycle Tasks*

For the Cycle 2 tasks, participants were instructed to work in pairs to find a solution to the following task:

> **The Pizza Problem**
>
> Capri Pizza has asked you to help design a form to keep track of certain pizza choices. They offer a standard "plain" pizza with cheese and tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many choices for pizza does a customer have? List all possible choices. Find a way to convince each other that you have accounted for all possibilities. (Maher et al., 2010, p. 11)

A framework and timeline for the Cycle 2 intervention can be found in Appendix G.

*3.2.3.3 Third Cycle Tasks*

For the Cycle 3 tasks, participants were instructed to work in pairs to find a solution to the following task:

> **Building Towers Three Colors**
>
> Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and provide a convincing argument that you have found them all. (Maher et al., 2010, p.12)

A framework and timeline for the Cycle 3 intervention can be found in Appendix G.

In addition to this task, an extension problem was provided after the completion of the three-tall tower task, selecting from three colors. The following task was provided:

> **Building Towers Three Colors Extension**
>
> Find all possible towers that are four cubes tall, selecting from cubes available in three different colors, so that each of the resulting towers has at least one of each color. Show your solution and provide a convincing argument that you have found them all. (Maher et al., 2010, p.13)

***3.2.4 Components***

Four components of the PD intervention model were experienced by the teachers. In the first component, teachers worked on and discussed solutions to the mathematical tasks as learners. For the second component, teachers had a discussion after reading articles and watching videos of children working on the same tasks. For the third component, teachers implemented the same mathematical tasks with their own students. In the fourth component, teachers discussed the reasoning forms of their students' work.

***3.2.5 Timeline***

The intervention activities occurred during two on-campus meetings [9/7 and 12/7/13] and regional meetings at respective school sites. The on-campus meetings were attended by the northern and southern cohort teacher groups and held at Rutgers

University Graduate School of Education (GSE). The southern region cohort group attended three regional meetings [10/2, 10/22, and 11/20]. There were also three in-district classroom visits [9/17, 10/22, and 11/20]. Detailed activities and timelines for each cycle are located in Appendix G.

**3.3 Participants**

Data for this research came from ten teacher participants from the fall 2013 southern New Jersey school regions. The five regions were Long Branch, Sayreville, Matawan-Aberdeen, Old Bridge, and Toms River. Each of the five regions had two teacher participants. The ten participants implemented the three cycles of mathematical tasks in the following grade levels:, three in sixth grade, four in seventh grade, and three in eighth grade. The tasks were implemented with regular, advanced, and special education classes as shown in table 3.1.

Table 3.1
*Classroom Demographics*

| Grade | Regular Education | Advanced Education | Special Education | | | |
|---|---|---|---|---|---|---|
| | | | Alternate School | In-Class Support | Resource Pull-out | Self-Contained |
| 6 | 1 | 1 | | 2 | | |
| 7 | | | | | 2 | |
| 8 | 1 | | 1* | | | |
| 6-8 | | | | | | 2** |
| Total | 2 | 1 | 7 | | | |

**Non-classified*
***One Mildly Cognitively Impaired (MCI) and One Language Learning Disabled (LLD)*
*Sources:* Teachers' final projects, video transcripts 9/7, 9/17, 10/2, 10/22, 11/20

One of the sixth-grade classes and two of the eighth-grade classes were described by the teachers as regular education, according to Webster (2015), as the standard education experienced by children. Another teacher described one of the sixth-grade classes as Pinnacle or gifted, consisting of students who had previously demonstrated advanced ability in all subjects on state standardized tests.

Students from the other six classes from this study were described as special-education students. In one of the eighth-grade self-contained special education classes, students were classified as mildly-cognitively-impaired. Four of the seventh grade classes from this study consisted of three resource classes and one self-contained class. Teachers of the three resource classes described their classes as pull-out where students are removed from regular education to receive instruction designed specifically for special education students, as described by Mastropieri & Scruggs (2010). One class was classified as an in-class support (ICS) special education class but consisted of both regular and special education students and was taught with an instructor and a paraprofessional as the in-class support. This class is also referred to as an inclusion class.

### 3.4 Data Sources

Data from the participants were collected from nine sources. The data sources for this study are the beliefs pre- and post-assessments about the teaching and learning of mathematics, the pre- and post-video assessments from the Gang of Four video on identifying students' reasoning, instructor interviews, video data with transcriptions of all on-campus and regional meetings of the teachers discussion and participation in solving the mathematical tasks, and the weekly responses to questions provided by the instructor and posted on the eCollege online discussion threads about required readings and videos relating to the mathematical tasks (Landis, 2013). Other data sources also included final teacher projects, final reflections at the last meeting, and samples of task work from students and teacher participants. Table 3.2 below shows the data sources and dates of data collection.

*Table 3.2*
*Data Sources and Collection Dates*

| Data Sources | Collection Dates |
|---|---|
| Beliefs Pre-&Post Assessments | Before 9/7/13 & After 12/7/13 |
| Pre-&Post Gang of Four Video Assessments | Before 9/7/13 & After 12/7/13 |
| Students' Work Samples | 9/17; 10/2; 10/22; 11/20/13 |
| Discussion Threads Online | 9/13 to 12/13 Weekly |
| Final Projects | 12/7/13 |
| Videos: | |
| Transcripts of Regional Meetings | 9/7; 9/17; 10/2; 10/22; 11/20 |
| Reflection Discussion | 12/7/13 |
| Instructor Interviews | 10/2/13 & 11/3/13 (Audio only) |
| Teachers' Task Work Discussion | 9/7; 9/17; 10/2; 10/22/13 |

### *3.4.1 Beliefs Pre- and Post-Assessment*

The pre- and post-assessments on teachers' beliefs about the learning and teaching of mathematics were given at the beginning and the ending of the PD intervention respectively and are located in Appendix A. Although the test is comprised of thirty-four items, a twenty-two statement subset of this assessment that is pertinent to this study has been analyzed. The twenty-two items that correspond to students' reasoning regarding the mathematics, the learning of mathematics, and the teaching of mathematics are located in Appendix B. The response categories use a 5-point Likert scale for each item, ranging from strong agreement (e.g. 1) to strong disagreement (e.g. 5).

Both the beliefs pretest and posttest have been analyzed using the methods for the PD project [DRL-0822204, directed by Carolyn A. Maher]. Specifically, the tests from the ten participants from the southern regions are descriptively analyzed. The purpose is to analyze the data for any changes from the Pre- to the Post- assessments in the teachers' beliefs regarding the teaching and learning of mathematics.

### *3.4.2 Pre- and Post-Assessment Responses to Gang of Four Video*

Teachers were required to complete a pre-video assessment to identify students' reasoning after watching a VMC video called The Gang of Four (www.videomosaic.org) before participating in the PD intervention. The Gang of Four video showed a group of four, fourth-grade students discussing their solutions of constructing all the possible 3-tall towers that can be made selecting from two colors of unifix cubes™. The video had the students in a small group sharing and justifying their solutions to the aforementioned task as well as discussion from one student's approach to the towers task.

After watching the video, the in-service teachers were asked to provide responses identifying students' reasoning and determine whether the reasoning was a valid and a convincing argument by providing evidence from the video. The responses were evaluated according to the recognition of children's justifications, the validity of the arguments, the supportive evidence, and whether the cited evidence was partial or complete. The forms of reasoning shown in the video were reasoning by cases, contradiction, and induction. After the PD intervention, the teachers completed a post-video assessment using the same Gang of Four video.

From the responses, any changes from pre- to post-test are noted and comparisons will be made between the teachers' solutions and what the teachers recognized from watching the fourth graders in the videos. The pretest/posttest and the video transcripts are located in Appendix C and D respectively.

### *3.4.3 Videotaped Meetings*

Video was used to document the on-campus and regional meetings of the participating middle school mathematics teachers using both a still camera and a roving camera. The transcribed recordings were coded to analyze teachers' own reasoning forms as they worked on the mathematical tasks, teachers' discussions of students' written work samples after the in-district classroom visits and students' samples of work brought by each teacher, and the instructor's moves throughout the activities and discussions of the intervention. The videos were transcribed by this researcher and verified by a graduate student. Transcripts are located in Appendix K.

### *3.4.4 Reflection Discussion*

Video was used to document the reflections about the intervention at the final meeting on December 7, 2013. Participants were given four questions provided and asked by the instructors of the intervention. These questions are located in Appendix F. Teachers were asked to reflect about their activities to discuss in small groups first and then share their thoughts with the larger group when the instructors called for the group attention.

### *3.4.5 Interviews*

Two interviews were conducted with the facilitator, Dr. Judith Landis, who instructed the middle school teachers of the southern region. The first interview occurred in person at Carl Sandburg Middle School in Old Bridge, New Jersey on October 2, 2013 and was videotaped. In this brief interview, this researcher asked general questions about any concerns with the teachers' progress in the NJPEMSM project. The second interview was conducted over the telephone on November 3, 2013 where more specific questions were asked (see Appendix E) about how the NJPEMSM project promoted teachers attending to students' reasoning. This interview was audiotaped.

A teacher interview was videotaped on December 7, 2013 with all twenty-eight teachers of the fourth cohort group. Teachers shared their reflections about the PD activities and discussed how their participation in the NJPEMSM project had affected how they attended to students' reasoning. The interview-protocols that were used can be found in Appendix F.

### *3.4.6 Teachers' Discussion of Tasks*

Teachers were asked to work on the same mathematical tasks as their students prior to giving the tasks to their students. On 9/7/13, Teachers worked on the first cycle four-tall towers problem selecting from two colors and the first extension problem to predict three-tall and five-tall towers selecting from two colors. Teachers worked on the second cycle pizza problem, finding all possible pizza combinations selecting from four different toppings on 10/2/13. On 10/22/13, teachers worked on the third cycle three-tall towers problem selecting from three colors and the Ankur's Challenge extension problem.

### *3.4.7 Students' Work Samples*

At particular meetings, teachers were asked to bring samples of students' work from the implementation of the tasks in their own classrooms to share with the other teacher participants. Samples of students work were used to discuss the reasoning forms that teachers identified from student work. Before discussing student samples, teachers discussed and identified the reasoning forms used in teacher samples when the teachers participated in solving the same mathematical tasks. These discussions were videotaped with two cameras, one still and one roving. The discussions based on students' work samples are coded using the frameworks already described for students' reasoning and instructor moves. Coding framework examples are in Appendix H and Appendix I.

### *3.4.8 Online Discussion*

Teachers were asked to respond weekly on line to questions posted by the instructor beginning with the second week. These questions referred to required readings, assigned VMC videos, and experiences with implementing the tasks in their respective classrooms, or the teachers' work after completing the tasks. Teachers were also asked to respond to a minimum of two other posts.

### *3.4.9 Final Teacher Projects*

The teachers were assigned a portfolio project to be completed by the final meeting. Teachers were asked to choose three samples of student work that impressed, surprised, and concerned them for each of the three task cycles as well as make reflections for each cycle.

## 3.5 Reasoning Strategies Framework for Analysis

The reasoning strategies framework for analysis was developed collaboratively by a research team, each studying one cohort of teachers from the three-year period of the project and is in Appendix L. Video transcripts, online discussion threads, and final projects were coded using the reasoning strategies framework for analysis. The framework was used to code the observed reasoning strategies of teachers after working on the three cycles of mathematical problems on 9/7/13, 10/2/13, and 11/20/13. The reasoning strategies framework for analysis was also used to code students' work samples, on-line discussions, and the teachers' final projects. The following definitions describe the framework.

1. **Heuristic/ Strategy**: This characteristic describes the method by which the work was organized in building a solution. Codes for identifying types of strategies and heuristics based on this body of research were developed in collaboration with other researchers analyzing similar tasks making use of common heuristics and strategies used in solving combinatorics problems that have been identified from the research literature (Maher and Martino 1996, Maher, Sran, and Yankelewitz, 2011). Names for heuristics and strategies arose from students' work on the towers problems, but in some cases the strategies can be applied to pizzas as well. The heuristic or strategy used was recorded as fitting one of the following types:

a. **Guess and Check**-The strategy of guess and check involves first guessing a solution then testing that the solution is correct. Students can be observed using the guess and check method when building towers or listing pizzas in a random order and then double-checking for duplicate towers or pizza toppings (Maher & Martino, 1996).

b. **Opposites-** The opposite of a tower in two colors is a tower of the same height where each position holds the opposite color of the first tower. For example, a 4-tall tower with yellow, blue, blue, blue and one with blue, yellow, yellow, yellow are opposites. (Maher, Sran &Yankelewitz, 2011) This strategy can be applied to pizzas as well. For example two pizzas, one with peppers and pepperoni, and the other with sausage and mushrooms could be considered opposites because there is no topping shared by both pizzas, and all of the toppings that appear on one pizza do not appear in the other.

*Figure 3.1: Opposite four-tall towers.*

c. **Cousins-** Two towers are said to be cousins if one tower can be flipped to form the second tower. For example, a 4-tall tower with yellow, blue, blue, blue and a tower with blue, blue, blue, yellow are cousins (Maher & Martino, 1996) flipped to form a three-tall tower with the top and middle cube blue and the bottom cube yellow (Maher & Martino, 1996). Figure 3.2 shows an example of cousin towers.

*Figure 3.2: Cousin three-tall towers.*

**d. Elevator-** The elevator pattern is used when finding all possible towers containing one cube of one color and the remaining cubes of the other color. The single colored cube is placed in the first position of the first tower. To create a second tower, the cube is then moved to the second position. The cube is continuously lowered one position to create new towers until it is placed in the final position (Maher, Sran &Yankelewitz, 2011). This strategy can appear in the pizza problem as well.

*Figure 3.3: Elevator pattern of three-tall towers.*

**e. Staircase-** The staircase pattern is named as such due to its resemblance to a staircase. In towers of two colors, the first tower begins with the first three positions as the same color followed by the 2nd color in the last position. In each new tower, the number of cubes of the 2nd color increases from the bottom by one cube until the final tower is a solid tower of that color (Maher, Sran &Yankelewitz, 2011). This strategy can appear in the pizza problem, for example when a student starts with a one topping pizza, and successively adds toppings to identify new pizzas. An example of the staircase pattern is shown in Figure 3.4.

*Figure 3.4: Staircase pattern of five-tall towers.*

f. **Controlling for Variables-** Controlling for variables is a method in which one variable is held constant while adjusting another variable. When building towers, one color of the tower is held constant in one position while the color arrangements in all other positions are varied (Maher & Martino, 1996). This is also referred to as holding a constant.

g. **Other-** Any strategy or heuristic other than those previously defined.

2. **Representation** - This characteristic describes the format used to monitor progress or describe a solution. Maher (2011) lists some common representations (physical objects, words, and symbols) and describes how existing representations are elaborated upon or related to new representations. To analyze the development of representations in this intervention, representations used by students or teachers were recorded as fitting one of the following types:

a. **Manipulatives** - Tangible objects used by students or teachers to help them solve the mathematical tasks. While the objects mostly used in the study included unifix cubesTM, other tangible items may be used.

b. **Drawings-** Pictures or diagrams used by students or teachers to help them solve the mathematical tasks. These may include tree diagrams.

c. **Charts-** Any graphic form or table used to represent a student's or teacher's work.

d. **Symbols-** Numbers, letters, or any other symbols (including written words) that are used to help students or teachers represent their work.

3. **Form of Argument**: This characteristic describes the structure of the argument used to justify that a solution set is complete accounting for all possible elements fitting the task criteria. Initial definitions of argument type were developed by Wright (2015, personal correspondence). The definitions were then discussed and evaluated by a team of researchers (Maher, Wright, Cipriani, Krupnik, and

McGowan). The form of argument was recorded as fitting one of the following types:

a. **Case Argument-** In a proof by cases, a statement is proved by proving all of the smaller subsets of statements the union of which make up the whole set. For example, the solution to the Four-tall Tower Task when selecting from two colors (i.e. blue and yellow) can be justified by separating the towers into cases using a characteristic of the tower. One such characteristic is the number of cubes of a specific color that the towers contain. In this situation, the towers can be broken down into 5 cases; (1) towers containing 0 yellow, (2) towers containing 1 yellow, (3) towers containing 2 yellow, (4) towers containing 3 yellow and (5) towers containing 4 yellow. A complete argument by cases would include justifications that (1) the cases describe the entire set of four-tall towers when selecting from two colors (2) all towers fitting each case have been identified and (3) no towers can be described by more than one of the cases.

**b. Inductive Argument-** In an inductive argument, the particular solution is considered to be an extension of an initial problem. To make an inductive argument, (1) an initial case is identified and a solution is presented. (2) The relationship between one case's solution and the subsequent case's solution is shown to hold up to some arbitrary point. (3) It is demonstrated in a general way that the solution can be extended beyond the arbitrary point identified in step 2. The general solution to the

Towers Task, 2 to the nth power ($2^n$) where 2 represents the number of colors selected from and *n* represents the height of the tower, can be proved through an inductive argument.

The first step is to prove the result is true for a basis case (often $n = 0$ *or* $n = 1$). In the case of towers, we prove the basis case n=1 or towers of one cube in height. Since there are only two cubes from which to select, i.e. yellow or blue, there are only two towers that can be built. $2^1 = 2$. Thus, the justification is established for the case, n=1.

In the second step, an inductive hypothesis is made. The inductive hypothesis assumes the result of step 1 is true for $n=k$. Therefore, it is assumed that the total number of different towers of height $k$ is $2^k$. In the third step, this assumption is used to prove the next case ($n = k+1$). The total number of towers that are $k + 1$ tall can be found by placing another cube on the top of each of the $2^k$ towers that are $k$ tall.

That additional cube can take on one of the two colors, e.g., yellow or blue. Therefore, for each of the existing $2^k$ towers, two new towers of height k+1 can be created; one with a yellow cube added to the top and one with a blue cube added to the top. Therefore, the total number of towers that can be created of height $k + 1$ is $2^k \cdot 2 = 2^k \cdot 2^1 = 2^{(k+1)}$. Thus, the argument is made for the case of $n= k+1$.

The provision of an induction argument coded in this research of the general solution 2n includes the basis step (n=1) in which a teacher (or student) describes that the total number of 1-tall towers created when selecting from two colors is 2, i.e. one of only blue and one of only yellow. The second step is less formal but describes that the total number of towers of a given height can be found by placing either a yellow or blue cube on the top of all of the towers of the previous height, therefore doubling the total number of towers created in the previous height.

c. **Recursion-** Recursion is defined as an operation on one or more preceding elements according to a rule or formula involving a finite number of steps (Merriam-Webster, 2015). An example of recursive reasoning can be seen in one possible solution of the 4-topping Pizza with Halves problem. The total number of 4-topping combinations is 24 or 16, thus there are 16 different whole 4-topping pizzas (same topping(s) on each side). When determining the total number of 4-topping pizzas in which the two sides of the pizza are not the same, a recursive calculation can be used. First choose one topping on one side, i.e. plain, leaving 15 remaining toppings for the other side. Next choose a different topping for one side, i.e. pepperoni. Again there are 15 toppings for the remaining side but one would create a duplicate from the previous set, thus only 14 remaining toppings can be used. Choose a third topping for one side, i.e. peppers. Again there are 15 toppings for the remaining side but two would create a duplicate from the two previous sets, thus only 13 remaining toppings can be used and so on. Each new set of pizzas can be found by subtracting one from the previous set. The total number of different 4-topping pizzas that can be created is the sum of 1 through 16.

d. **Contradiction-** When a situation arises that is inconsistent or contrary to known or inherent facts, a contradiction has been reached. In the 4-tall Tower Problem, when selecting from two colors, (e.g., yellow and blue), a proof by contradiction can be used to prove the total number of towers that can be built in the case of exactly one yellow cube. The yellow cube can be placed in either first, second, third or fourth position. If other towers can be built with one yellow cube, the yellow cube would have to be in a different position, the fifth position. Placing a cube in the fifth position would require the tower to be a height of at least five. This is a contradiction of the requirement that the tower has a height four.

**e. Rule-** Features of a given task may be used to identify numbers and perform calculations leading to a solution. In that case, the work is justified with a procedure or "rule", which is a statement that relates the mathematical operations to features of the problem. For example, in the 4-tall towers problem, selecting from two colors, a student may incorrectly claim that $4^2 = 16$ makes sense as a solution because there are four blocks in each tower, and two colors to choose from.

**4. Teacher Evaluation:** In addition to recording the forms of reasoning identified by teachers as they progressed through the intervention, this study aims to identify which arguments (if any) were found convincing.

**a. Convincing -** When a teacher made a claim that a particular argument was convincing, the argument was recorded as “convincing” for that teacher.

**b. Not convincing -**When a teacher made a claim that a particular argument was not convincing, that argument was recorded as “not convincing” for that teacher. In some instances, the teacher provided a reason as to why the argument was not convincing. Instances in which the teacher claimed the argument was not convincing because it was incomplete will be coded as “incomplete.” Instances in which the teacher claimed the argument was not convincing because it was not a valid argument will be coded as “invalid.”

**5. Researcher Evaluation:** In order to gain a truer picture of each teacher’s recognition of forms of reasoning, it was necessary to identify missed opportunities, or situations in which teachers may have failed to recognize a particular form of reasoning. In order to identify these situations, the researcher evaluated each form of reasoning presented or discussed by the teachers. This evaluation was done using codes identical to those used in the “Teacher Evaluation” section- with one exception. The Researcher Evaluation includes an additional code “Undetailed Description” This code is applied to indicate situations in which there is not enough information about the particular argument to allow a code of “Convincing” or “Not Convincing” to be applied. (Wright, 2015)

### 3.6 Instructor Moves Framework for Analysis

The instructor moves framework for analysis was a second framework for analysis used to code the strategies used by the instructor to facilitate teachers’ recognition of reasoning. This framework is used to code the video data of the observed instructor moves after teachers worked on mathematical tasks, discussed the in-district classroom implementation of the tasks, and discussed students' work samples.

The framework draws from the Smith and Stein (2011) framework for practices that encourage mathematical discussions. Smith and Stein (2011) posited five practices describing how teachers can productively facilitate mathematical discussions in an NCTM publication called *5 Practices for Orchestrating Mathematics Discussions.* The reported practices included *anticipating* students' actions and possible strategies used when problem solving, *monitoring* students' work during the task of problem solving, *selecting* salient work from the students, *sequencin*g students' regarding the order in which the work is shared, and *connecting* the strategies and ideas for conceptual understanding (Smith & Stein, 2011). Although not a part of research by Smith and Stein (2011), an additional practice involved *motivating* was included to account for celebrating participants' work (Marzano, 2007).

The Herbal-Eisenmann, Steele, and Cirillo (2013) framework for teacher discourse moves served as a foundation to the framework for analysis for coding this study. These pedagogical practices include *waiting* or pausing to allow time for participants to process and then respond to questions posed by the instructor or another participant; *inviting* which asks participants to contribute solutions to share different strategies and forms of argument; and *re-voicing* which is defined as restating, repeating, reporting, or paraphrasing the ideas of the participants out loud (Herbel-Eisenmann et al., 2013). The instructor moves framework for analysis is located in Appendix H and is organized in two different categories: observed representations and forms of pedagogical practice. The first category, used representations, is defined as in the reasoning strategies framework.

The second category describes the pedagogical practices that the instructor used to facilitate the teachers' recognition of students' reasoning. The forms of pedagogical practice are founded on the aforementioned research from Herbel-Eisenmann et al. (2013) and Smith and Stein (2011) as well as the research from Maher and Martino (1999). Martino and Maher (1999) emphasized the significance of the questioning and listening by the teacher to promote students to generalize and justify their mathematical ideas. The following definitions describe the types of questioning:

1. Types of Questioning:

    a. **Explanation**: Questions that invite a teacher or group of teachers to describe what they are doing or did. Explanation questions might be used while teachers are working on a task, in contrast to describing a completed task. (Maher and Martino, 1999)

    b. **Justification**: Questions that elicit how the teachers are convinced that the solution is correct. (Maher and Martino, 1999)

    c. **Generalization**: Questions that invite teachers to consider a similar problem with the goal of encouraging them to consider patterns that suggest a solution to the original problem. For example, by considering building towers of different heights, with different color choices, students can begin to consider how the height of a tower might be related to the number of color choices in finding the total number of towers that can be made. (Maher and Martino, 1999, p. 65)

d. **Connection**: Questions that invite teachers to consider whether they can identify similar problems, and if so, to describe similarities and/or differences. (Maher and Martino, 1999)

e. **Probing**: Questions that invite teachers "to elaborate on particular ideas." (Herbel-Eisenmann et al., 2013, p. 183) For the purposes of this study, "probing" will be distinguished from "inviting." "Probing" refers to situations in which one particular teacher is invited to elaborate on his or her particular idea, whereas "inviting" refers to situations in which the question is asked in a way to encourage many teachers to respond.

f. **Other Solution**: Questions that make public to other teachers various solutions. (Maher and Martino, 1999) For the purposes of this study, "Other Solutions" are used to describe the first time a particular solution is presented, but not for each time the solution is mentioned by the instructor.

In addition to questioning, the following pedagogical practices are also defined:

2. **Anticipating**: Predicting teachers' or students' behaviors or strategies while working on a mathematical task. (Smith & Stein, 2011)

3. **Monitoring**: Checking for teachers' understanding as they are working on the task. The instructor monitors to make decisions about which solutions or strategies to make public without direct interaction. (Smith & Stein, 2011)

4. **Selecting**: Choosing to share a particular teacher's work. (Smith & Stein, 2011)

5. **Sequencing**: Asking for teachers' work to be presented in a certain order as opposed to allowing teachers to choose the order of work shared. (Smith & Stein,

2011)

6. **Motivating**: Celebrating students' or teachers' work through praise or encouragement. Marzano (2011)

7. **Waiting**: Pausing to allow time for teachers to process and then respond to questions posed by the instructor or another teacher. (Herbel-Eisenmann, Steele, & Cirillo, 2013)

8. **Inviting**: Soliciting multiple solution strategies, often with the goal of "making diverse solutions available for public consideration" or "including multiple students in the discussion. (Herbel- Eisenmann et al., 2013, p. 183)

9. **Re-voicing**: "Restating or rephrasing a teacher's contribution." (Herbel-Eisenmann et al., 2013, p. 183)

The frameworks for analysis were formed in collaboration with Will McGowan, Erica Wright, and me, with support from our advisor, Dr. Carolyn Maher. The foundation of the frameworks was a combination of both the previously aforementioned research and our own data collection. We began our discussion to form the instructor moves and reasoning frameworks for analysis beginning in the summer of 2014. Will, Erica, and I had weekly conversations on a conference line to discuss additions and revisions resulting in numerous editions of the frameworks.

Each member of the team selected video clips from his or her data set and the group coded examples together using the frameworks and made revisions as necessary. The frameworks for the instructor moves and reasoning strategies were finalized in October of 2015 and were used to code the video data and portfolio work. Weekly phone conversations were held and another graduate student, Victoria Krupnik, joined the team in 2016 to help with the verification of codes for reasoning strategies and instructor moves.

## 3.7 Beliefs Framework for Analysis

Teachers' beliefs about the learning and teaching mathematics were also analyzed. The pre- and post-assessments on participant beliefs about the learning and teaching of mathematics were given at the beginning and the ending of the PD intervention respectively, using the same instrument for each assessment. That instrument is located in Appendix A. The assessment is comprised of thirty-four statements where some of the statements are consistent with NCTM standards; and some statements are inconsistent with NCTM standards. Relevant to this study, a subset of twenty-two statements of this assessment were analyzed. The twenty-two items, located in Appendix B, correspond to students' reasoning regarding the mathematics, the learning of mathematics, and the teaching of mathematics. The response categories use a 5-point Likert scale for each item, ranging from strong agreement (e.g. 1) to strong disagreement (e.g. 5).

### *3.7.1 Categories of Beliefs*

The assessment was used to analyze teacher beliefs over the duration of the intervention. Some assessment items were statements considered to be consistent with recent NCTM standards. Other assessment items were considered to be inconsistent with the standards and are marked below with an asterisk. The videos of regional meetings, online discussions, and final projects were analyzed for any changes in knowledge about teacher beliefs.

For analyzing the beliefs, the questions were organized into the following categories:

**Expectations and Student Abilities**: Q1, Q7, *Q13, *Q29

**Mathematical Discourse**: Q4, *Q23

**Concepts and Procedures**: Q2, *Q5, Q9, *Q11, Q18, Q19, Q21,

**Manipulatives**: *Q10, *Q17

**Student and Teacher Roles**: Q24, *Q30, *Q32

**Differentiated Instruction**: *Q6, Q15, Q28, Q31 (McGowan, 2016)

The teachers' responses from the pre- and post-assessments are coded as consistent, inconsistent or undecided concerning the standard described in each category. Undecided is used as the code for a "3" rating (neutral). Consistent is the code used for ratings showing agreement with statements consistent with the standards or disagreement with statements inconsistent with standards. Inconsistent is the code used for ratings showing disagreement with statements consistent with standards, as well as ratings expressing agreement with statements inconsistent with standards.

### *3.7.2 Intervention Data*

Based on the above-described groups, codes were formed to relate teacher responses made during the intervention. In addition, codes were formed that identified beliefs as referring to the learning and teaching of mathematics. Each belief response is coded relating to the NCTM Standards from the beliefs inventory assessments and were coded as inconsistent, consistent, or undecided. The following descriptions determine whether the beliefs from the question categories are consistent or inconsistent with standards from the beliefs assessments. Unclear claims are coded as undecided.

**Expectations and Student Abilities:**
Statements indicating lower expectations for some learners, of that only some students are capable of mathematical success are marked as inconsistent with standards.
Statements indicating beliefs that all students are capable of mathematical success are marked as consistent with standards.

**Mathematical Discourse:**
Statements claiming that student mathematical discourse is not valuable or that mathematical discourse is only valuable to students actively discussing the mathematics are marked as inconsistent with standards.
Statements claiming that mathematical discourse is valuable for all students are marked as consistent with standards.

**Concepts and Procedures:**
Statements claiming that mathematics is more about procedures than concepts are marked as inconsistent with standards.
Statements claiming that concepts and procedures are both important in mathematics are marked as consistent with standards.

**Manipulatives:**
Statements claiming that manipulatives have a limited value or are only useful for certain learners are marked as inconsistent with standards.
Statements claiming that manipulatives are valuable for all learners, particularly as reasoning and communication tools, are marked as consistent with standards.

**Student and Teacher Roles:**
Statements claiming that the teacher is the sole authority in the classroom are marked as inconsistent with standards.
Statements claiming that students can have mathematical authority, particularly be making and supporting claims are marked as consistent with standards.

**Differentiated Instruction:**
Statements claiming that all students learn the same way and that teachers do not need to accommodate a range of student abilities are marked as inconsistent with standards.
Statements claiming that teachers do need to accommodate a range of student abilities are marked as consistent with standards. (McGowan, 2016)

## 3.8 Summary

The data are coded by groups to compare the beliefs pre- and post-assessment responses. Each response is identified as consistent, inconsistent, or undecided in regard to the standards from the Beliefs Inventory. Data on teacher beliefs are analyzed by the teacher for any noted patterns, trends, or changes in categories of questions and beliefs on the learning and teaching of mathematics and then compared to pre- and post-assessment Beliefs Inventory data.

# Chapter 4 – Cycle 1 Session Summary and Analysis

This research focuses on one section of three cohorts participating in the intervention sessions. Prior to the initial session, teachers were required to complete pre-assessments on students' reasoning from the Gang of Four video and on beliefs of the learning and teaching of mathematics. For the beginning session, all three teacher cohort groups met as a large group at the Rutgers Graduate School of Education (GSE) to start the intervention. For the final session, all three cohort groups met again to discuss and reflect on the intervention. The intervention sessions were comprised of on-campus meetings, regional meetings, on-line discussion threads, and in-district classroom visits. During the regional meetings, teachers discussed students' work samples at district schools. For the in-district classroom visits, teachers observed an implementation with current students for each of the three task cycles.

During the intervention, teachers met on-line weekly to discuss the videos and articles using guided questions provided by the instructor. The instructor partitioned the on-line discussion threads into fifteen separate units. For each unit, a meeting was scheduled or a weekly on-line discussion was scheduled. This chapter is a summary and analysis of five session units for the first cycle of mathematical tasks.

## 4.1 Unit 1: Initial On-Campus Meeting 9/7/13

At this meeting, the instructors introduced themselves and briefly discussed the course requirements. Dr. Palius then spoke with the teachers about completing a pre-course assessment for the NJPEMSM program evaluation. Then, the instructors began the Cycle 1 intervention by giving the teachers a mathematical task, tools, and encouragement to work collaboratively.

### *4.1.1 Teachers Work on First Cycle Task*

Teachers were asked to work in pairs to find a solution for building four-tall towers selecting from two colors and convince each other of their solution. Once a solution was found, teachers were asked also to convince one of the circulating researchers of their solution. If successful, they were then asked to produce a written solution. If they did not successfully convince one of the researchers, teachers were invited to rethink their solution.

The first teacher pair that the instructor of the southern region cohort briefly monitored was solving the four-tall towers task by starting with a tower and then creating another tower with opposite colors. After a few minutes, the instructor encouraged the pair to by saying "That's good, you are checking for duplicates" (9/7 meeting transcript, line 19).

Then the instructor of the southern region cohort stopped at a second pair of teachers to ask "how many did you find?" and "how do you know you have them all?" (9/7 meeting transcript, line 20) This pair of teachers explained to the instructor how they "started with two reds and two yellows one of a color" (9/7 meeting transcript, line 22) and "took this red and moved it to the second position" (9/7 meeting transcript, line 26). The strategy the second teacher pair used is defined as the elevator strategy.

The instructor then stopped at a third teacher pair to ask what they were doing (9/7 meeting transcript, line 34). One teacher in the pair described their work by telling the instructor that "I used two reds and so to approach that I kept the first red always on the bottom" (9/7 meeting transcript, line 41). The instructor informed the teachers that they were "holding a constant" as their strategy (9/7 meeting transcript, line 52).

The instructor then moved on to a fourth teacher pair where they had arranged the four-tall towers in pairs by building one tower and then making another tower with opposite colors. The instructor asked the fourth teacher pair "how do I know that you found all the towers?" (9/7 meeting transcript, line 69) The fourth teacher pair was unable to provide the instructor with a convincing argument, so the instructor asked the teacher pair to "think about rearranging the towers so you can convince me" (9/7 meeting transcript, line 82).

The instructor then returned to the first teacher pair to ask "What is that?" regarding their task solution (9/7 meeting transcript, line 26). One of the teachers (T2) in the pair began to explain how they found all the towers. The explanation is as follows:

> So we are adding two additional ones Every time you are adding an extra position. So for the ones that are one high, when you add a second block, you have a yellow, you could add either a yellow or red again; and to that red you add another yellow again. (9/7 meeting transcript, line 89)

The instructor responded by saying to the pair that they were using an inductive argument (9/7 meeting transcript, line 90).

The other teacher in the pair (T3) added to the discussion by saying "You are doubling it. Two times two is four." (9/7 meeting transcript, line 91) The instructor then facilitated a discussion by asking them what they meant by doubling (9/7 meeting transcript, line 92). T2 responded by saying they solved the problem with exponents. The instructor asked "So which is it? Two to the fourth to get 16 or doubled" (9/7 meeting transcript, line 91) and T3 replied "both" (9/7 meeting transcript, line 97).

The instructor asked this pair to predict how many 5-tall towers could be made (9/7 meeting transcript, line 98). T2 explained there were "2 different colors and 2 to the fifth power is 32" (9/7 meeting transcript, line 103). The instructor explained that they were using exponents and that they needed to write their argument down (9/7 meeting transcript, line 104). Then the instructor moved to a fifth teacher pair.

The instructor asked the fifth teacher pair to explain what they did to solve the task (9/7 meeting transcript, line 110). For one of the arguments, one of the teachers in the fifth pair said "we move red down and started with one yellow and started to take the two red down to this position so it is kind of like a rebuilding by moving the reds down" (9/7 meeting transcript, line 123). The instructor informed the fifth teacher pair that they were describing a recursive argument (9/7 meeting transcript, line 124).

The instructor returned to each teacher pair to tell them to write their convincing arguments on paper and that each teacher had to write their own argument. After forty minutes, the instructors asked the teachers to stop working in their pairs to share and discuss their reasoning strategies used to solve the first cycle task with all three cohort groups in the larger group.

### *4.1.2 Teachers' Discussion of First Cycle Task Solutions*

The large group discussion began when the northern and central cohort instructor asked "what was the first strategy in here?" (9/7 meeting transcript, line 210) One teacher from the southern region cohort group volunteered the first response with the following:

> So I had four of one color and I had it in one of the other color. And it could be in the first, the second, the third, or the fourth position. And then I knew if I did it that way, I could reverse it and do it with the other colors as well. (9/7 meeting transcript, line 217)

After this teacher shared her reasoning, the northern and central cohort instructor mentioned how "it's okay to not understand what they are doing" (9/7 meeting transcript, line 234) because then you can ask questions to provide an opportunity for the student to explain his or her reasoning. This was also the situation with another teacher from the southern region cohort. This teacher began solving the four-tall tower problem, selecting from two colors using opposites; but then rearranged her towers using the staircase strategy and explained her reasoning to both instructors (9/7/13 meeting transcript, line 202).

The northern and central cohort instructor could not understand the teacher's reasoning and kept asking questions until an understanding was gained. The instructor said "it took me the longest time to understand. What she was saying is: Look it! There are four spots that I can change a red for a yellow. And yet this one, then there are four spots and I can change this red for the yellow then there are four spots and I can change this red for a yellow" (9/7 meeting transcript, line 238). To help with the conceptual understanding of this argument, the instructor demonstrated what she was saying by moving the Unifix cubes to the four different spots as the instructor described the argument.

Another teacher pair was asked to share the strategy they used to solve the problem. The argument is as follows:

> First we started with all of one color and then we decided to just keep the bottom ones consistently red; not to change that. So then we went and said, well this one has no yellows so let's just use one yellow at a time and there is three positions that one yellow can occupy, keeping the bottom one red. (9/7 meeting transcript, line 238)

The southern region cohort instructor asked the larger group to compare what the previous teacher pair did to this teacher pair by asking "does that look at all similar to what they did?" (9/7 meeting transcript, line 271) and several teachers recognized that "they held a constant" (9/7 meeting transcript, line 272).

Another teacher pair in the northern and central cohort group shared their strategy by responding “we built one tower high, two tower high, three tower high” (9/7 meeting transcript, line 298). After asking the teachers to predict the possible 4-, 5-, and 6-tall towers that can be made, the instructor asked “how do you get those numbers?” (9/7 meeting transcript, line 306) and one teacher responded “you are doubling the outcome” (9/7 meeting transcript, line 307). When the instructor asked for the teacher to explain their argument, the teacher responded “if I have 5-tall towers, 5 times 2 are ten but you are not doubling the position you are doubling the outcome. So you would have 2, then 4 times 2 is 8” (9/7 meeting transcript, line 309). The instructor then asked if anyone in the large group could explain the argument in a different way.

One of the southern region teacher pairs said “We started off by doubling the outcome and then we looked at it in the form of exponents. We saw that if you kept the base, exponents change depending on how high and then we saw that four cubes high you would get…you would get 16 results” (9/7 meeting transcript, line 309). The instructor then asked the teachers “when they are told to build towers 5 tall, what do you think that they could possibly predict as their solution?” (9/7 meeting transcript, line 314) and several teachers answered the instructor by saying 25 (9/7 meeting transcript, line 315). The instructor ended the discussion by telling teachers to try to arrange to have an hour to do the cycle one task or implement the task over two days, give the extension problems to students that have completed the 4-tall tower task, and make an on-line post for next week. Then the instructors met with their respective groups separately to plan for other meetings and to introduce expectations and requirements.

## 4.2 Unit 2: On-line Discussion and In-District Classroom Visit

For the on-line discussion, the instructor wrote on-line to sign in at the front office of one middle school for the first in-district classroom visit on 9/17/13. Teachers met briefly after the in-district classroom visit to debrief and discuss students' work from the classroom visit. The instructor also wrote that the teachers should be implementing the first cycle towers problems in their classrooms during the week of 9/18/13 and 9/27/13.

### *4.2.1 On-line Discussion 9/11/13 to 9/16/13*

After the initial on-campus meeting, teachers were asked to participate in a weekly on-line discussion by making one original post and responding to two other teacher posts. For this discussion, teachers were asked to respond on-line to the following questions:

> 1. Before doing the classroom implementation, what do you think your children will predict (without building them) for 3-tall and 5-tall towers? Do you think they will say that there will be more, fewer, or the same number of towers as there were for towers 4 cubes high? What reasons will they give?
>
> 2. Before doing the towers problems with your children, predict how they might arrange their towers and what kind of convincing arguments they might give for their solutions. (Unit 2 on-line discussion thread, questions 1and 2)

#### *4.2.1.1First Question Responses*

The first question asked the teachers to predict how many 3-tall and 5-tall towers could be made without building the towers. Six out of ten participants responded that students would get 6 towers for 3-tall, 8 towers for 4-tall, and 10 towers for 5-tall using doubling as the reason. The response that one of the teachers predicted follows:

> Before doing any towers, I think students will predict that 3 high will produce 6 towers, 4 will give 8, and 5 will give 10. I teach in the resource classroom and students struggle with their facts. Many times, coming into the new school year, the students are not aware or familiar with exponents. I think it will be difficult for them to predict to use exponents. Instead, they may just assume they need to multiply the 2 from the colors with the number of cubes high it will be. I think

> after completing the tower activity, it would be a good opportunity to introduce and discuss exponents. (Unit 2 on-line discussion thread, line 1)

Two teachers speculated that students would predict 9 towers for 3-tall and 25 for 5-tall by squaring the numbers. The response of one of the teachers follows:

> I think that my students will predict that there will be 9 towers for the 3-tall and 25 towers for the 5-tall. I think some of them will be able to visualize an increase in the number of towers as the tower height increases and some of them will think there are an equal number of towers no matter what the height is. (Unit 2 on-line discussion thread, line 9)

Another teacher predicted that his students would say 3 for 3-tall towers and 5 for 5-tall based on the tower height. This teacher had mildly cognitively impaired (MCI) students. His response follows:

> I believe my students will say that the possibilities will be the same number as the amount high the towers can be. So for 3 towers high I think they will say there are 3 possible outcomes, for 4 towers high there will be 4 possible outcomes, and for 5 towers high there will be 5 possible outcomes. I do believe that they will associate a lower number in height with a lower amount of outcomes. I think they will say that 3 will be less because it is a lower number. Also, 5 will be more, because it is a higher number. (Unit 2 on-line discussion thread, line 31)

Another teacher didn't give a specific number but responded that her students would guess a smaller amount than would be possible to make because her students would just try to build them using trial and error. Her response follows:

> I think my students will predict a smaller number of towers than what is actually possible to make. I predict that students will think of different color combinations, but fail to realize that a different position constitutes as a different tower. I think they will think two towers of one color, one tower with three and one, another tower of three and one and one of two and two. I do think that students will think that five would make more towers and three would make fewer towers. I think the reasons will be that the more blocks you have the more ways you can pair the two different colors. (Unit 2 on-line discussion thread, line 32)

*4.2.1.2 Second Question Responses*

The second questions asked the teachers to predict how their students might arrange the towers and what students might give as convincing arguments. Three of the teachers responded by mentioning the opposite strategy to create pairs of towers with opposite colors. The responses of the three teachers follow:

> I definitely think the students will try to create patterns with the colored cubes. I would think the students might create patterns then do the opposite patterns, which is how I did them. (Unit 2 on-line discussion thread, line 1)
>
> After some exploration, some groups may begin to see that they have pairs of towers that appear to be opposites and may look for opposites that they are missing. (Unit 2 on-line discussion thread, line 12)
>
> Another way students could approach the problem would be to build a tower then to build the tower with the opposite colors in each position. (Unit 2 on-line discussion thread, line 37)

Six teachers responded by mentioning that students would begin to just randomly begin to build towers with no organization. The responses of the six teachers follow:

> I think that they will approach the problem slowly and look to me for help (which I will hold back on giving) and might beginning by making random towers until they compare their creations and see a pattern. (Unit 2 on-line discussion thread, line 9)
>
> I believe that many of my students will start building without a strategy, rather just start looking for possibilities. (Unit 2 on-line discussion thread, line 12)
>
> For my students, I could see this problem being about trial and error. They will attempt to make as many possible towers that they can without any type of organization or thought process before jumping right in. (Unit 2 on-line discussion thread, line 17)
>
> I believe that they will just start by building all types of towers. (Unit 2 on-line discussion thread, line 19)
>
> I also have many students who don't seem to have a good grasp on organization. I think they would just jump right in and create the towers randomly. (Unit 2 on-line discussion thread, line 22)

> There are a few students that I think will try this activity with no plan in mind and then as they are working realize that having a plan may work best and decide to start over. (Unit 2 on-line discussion thread, line 36)

One teacher mentioned the elevator strategy by saying “Once students are building the towers, students will realize that changing the position of a block will create a different tower” (Unit 2 on-line discussion thread, line 32). Another teacher mentioned a proof by cases strategy with the following response “Students may start by arranging their towers according to how many of a certain color the tower contains starting with towers with no red, 1 red, 2 red, 3 red, and then 4 red (Unit 2 on-line discussion thread, line 37). All teachers were asked to complete their on-line responses by 9/16/13, the day before the first in-district classroom visit.

#### *4.2.2 In-District Classroom Visit 9/17/13*

The first in-district classroom visit was scheduled for a different day than the regional meeting due to the large cohort size of ten teachers. For the first in-district classroom visit, the Cycle 1 task was implemented with twenty eighth-grade current students of a teacher. The eighth-grade current students were sitting in individual student desks that were pushed together in pairs. Similar to the way their teachers were organized the week before, the students worked in pairs. They were asked to convince their partners of their solutions and write down the solutions after convincing one of the researchers.

After the current students left, the instructor held a debriefing meeting with the ten teachers from the southern region group to discuss the students’ solutions from the in-district classroom implementation using an overhead and transferred the students’’ work to transparencies for sharing. The teachers first discussed two students' work where a long written explanation for each group was written on their paper.

There were 6 groups of towers and the capital letters R and Y were used to symbolize the red and yellow unifix cubes. One tower and the opposite colors of that original-made tower were in each of the first four groups. One teacher recognized that the students paired the first four groups as "one way and then the opposite. So that's why there is two in each group" (9/17 meeting transcript, line 53). The student's written argument for the first two towers was "You can only have four blocks and then you have one yellow in the first one then the second one is the alternate color" (9/17 meeting transcript, line 65). The instructor then asked the teachers "Is that convincing? What might they have said to really let you know why they have all the towers in that group?" (9/17 meeting transcript, line 66) and one teacher responded "there are only two colors" (9/17 meeting transcript, line 67). Figure 4.1 is a replication of the tower chart of one student's work from the classroom visit on 9/17/13.

| G1 | G2 | G3 | G4 | G5 | G6 |
|---|---|---|---|---|---|
| Y R | YR | R Y | YR | RRRY | YYYR |
| Y R | YR | Y R | RY | RRYR | YYRY |
| Y R | RY | Y R | YR | RYRR | YRYY |
| Y R | RY | R Y | RY | YRRR | RYYY |

*Figure 4.1 First Discussed Student's Work from 9/17/13 Class Visit*

The instructor then had a teacher read the student's argument for group 2. The student's written argument was "You can't have any other combination in this group because of the two yellow on the top, two red on bottom, and then we did the opposite; two red on top, two yellow on bottom" (9/17 meeting transcript, line 69). The instructor said that the students just reported what they did but did not provide a convincing argument as to whether the students found all possible towers (9/17 meeting transcript, line 72). This was also the case with the student's written argument for groups 3 and 4.

The instructor then asked "what about group 5" (9/17 meeting transcript, line 58). The fifth group had four towers where the fifth group had yellow unifix cubes only on the diagonal, where the first position of the yellow unifix cube started at the bottom and then was moved up one cube at a time to form the next tower. The student's written argument is as follows:

> For this group we started with 3 red on top and one yellow on the bottom. Then we moved the yellow block up one, which gave us two red, one yellow and then another red on the bottom. For the next one, we did one red, and then a yellow, and two more red. For the last one, we did one yellow on the top and three red on the bottom. (9/17 meeting transcript, line 93)

One teacher recognized that "Group 5 is that diagonal like...I just meant diagonal was all one color" (9/17 meeting transcript, line 59). Another teacher recognized that the sixth group was "the opposite of the fifth group" (9/17 meeting transcript, line 61). The instructor said to the teachers that the argument was a written argument of what the students did and then distinguished the difference between a convincing argument and a not convincing argument with the following:

> And if they exhausted all four positions there would be no other place to put a single red cube in one red and three yellow. That's a convincing argument, okay. Um, there are only four positions in tower four tall and they the single red cube in each of the four positions. That's a convincing argument; more than telling you what they did. (9/17 meeting transcript, line 96)

The instructor continued this discussion by asking the teachers which groups they thought were convincing (9/17 meeting transcript, line 104). One teacher said "I think four is pretty convincing. I think it's just alternating colors. So there's no other way…If you moved one to the top, you'd have the same tower as the second one" (9/17 meeting transcript, line 109). The instructor responded as follows: "If they said that, then you're right, that would be convincing" (9/17 meeting transcript, line 110). The instructor asked the teachers if anyone saw the students moving the cube up from the bottom. One teacher admitted to seeing the students move the cube up from the bottom. The instructor responded by saying "when you take a [Unifix] cube from the bottom and move it up to the top to build another tower, take a [Unifix] cube from the bottom and move it up to the top; that's called a recursive argument" (9/17 meeting transcript, line 112).

Two other teachers admitted to seeing other groups make a similar argument. One teacher mentioned that two girls made an argument where "they had the two alternating ones, and they were like, oh these don't fit" (9/17 meeting transcript, line 112). The argument from the two girls is as follows:

> We started with yellow and one red. Then we moved the red down on space every time and move the yellow to the top every time. Then we did the opposite with three red and one yellow. Then we did two of each color; two red, two yellow. We moved the two red down one cube and took the one yellow on the bottom and move it to the top. We put the two yellows on top, on top of each other, and had two reds…on the bottom. Two yellows on top of two reds. Oh, two reds…on the bottom. Then we moved one of the reds on top of the two yellows. (9/17 meeting transcript, line 147)

The instructor then asked "how could they have convinced you with the alternating ones that there aren't any more for that either" (9/17 meeting transcript, line 122). Another teacher responded "Take the top; put it on the bottom, and now they have a different one. But if they took the top and put it on the bottom again, they would go back to the other one so there's no more" (9/17 meeting transcript, line 125). The instructor then said "So they really could've used the recursive argument for the alternating ones too" (9/17 meeting transcript, line 126). Figure 4.2 is a replication of the drawing the student provided.

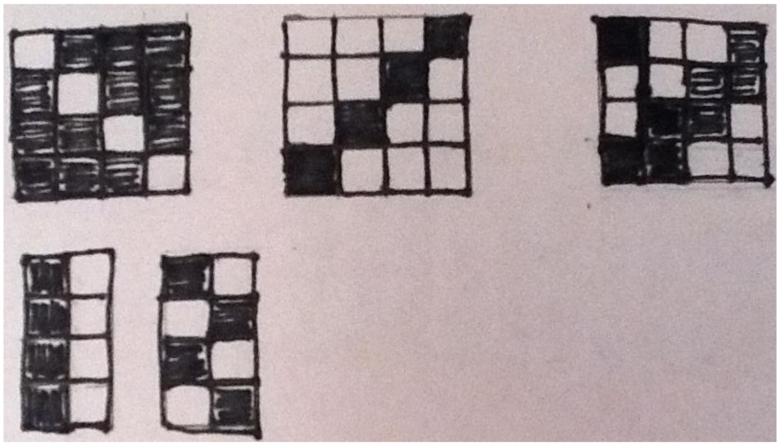


*Figure 4.2 Second Discussed Student's Work from 9/17/13 Class Visit*

Teachers also discussed the work from students where the drawing showed the same color on the top. The following student's argument was shown with the projector and read by teacher volunteers:

> Group 1 says, you can only have four blocks and then you have one yellow in the first one then the second one is the alternate color. You can't have any other combination in this group because of the two yellow on the top, two red on bottom, and then we did the opposite; two red on top, two yellow on bottom. We did two of each color, red on top, & two yellow in the middle, and one red on the bottom. For the other one, we did one yellow on top, two red in the middle, and one yellow on the bottom. We did red-yellow-red-yellow and for the other one we did yellow-red-yellow-red. For this group we started with 3 red on top and one yellow on the bottom. Then we moved the yellow block up one, which gave us two red, one yellow and then another red on the bottom. For the next one, we did one red, and then a yellow, and two more red. For the last one, we did one yellow on the top and three red on the bottom. (9/17/13 meeting transcript, lines 65-93).

One teacher recognized this reasoning as holding a constant (9/17 meeting transcript, line 176) and the drawing of this student's work is in Figure 4.3. After the teachers discussed the students' work from the in-district classroom visit, the instructor ended the meeting by asking the teachers to implement the same task in their own classrooms between 9/18 and 9/27 and complete the on-line assignments.

**Figure 4.3:** Tower Drawing, Third Discussed Student's Work from 9/17/13 Class Visit

The teachers discussed a fourth student's work sample. The student had 5 groups labeled A through E. Figure 4.4 is a replication of the fourth student's work sample. The following student's written argument was placed on the screen for the teachers to read:

> Group A: can't have any more towers because all of the red blocks that are in a different position.
> Group B: it can't have any more because of the yellow blocks are in different positions.
> Group C: was just the standard four blocks of each color and then we switched the colors.
> Group D: we just alternated the colors.
> Group E: made a tower of the full color and there are only two colors.

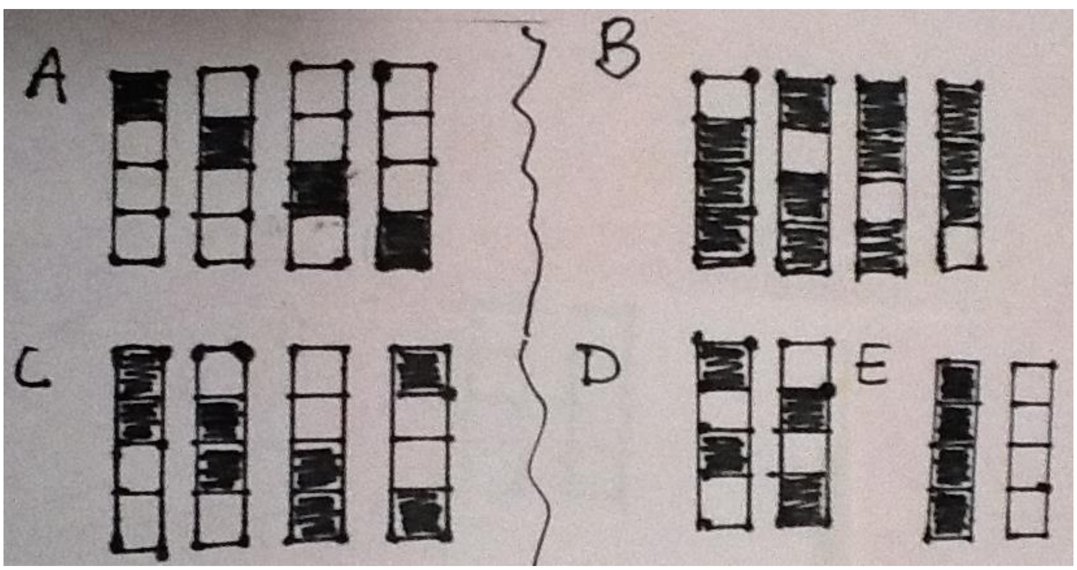



*Figure 4.4 Fourth Discussed Student's Work from 9/17/13 Class Visit*

The instructor said that Group A was "the start of a real nice convincing argument" and that B was "the opposite argument of A" (9/17/13 meeting transcript, line 215). When the instructor asked about Group C, one teacher replied "I think that the way she grouped it, like was more convincing than the way she wrote it." (9/17/13 meeting transcript, line 216). One of the teachers recognized the student may have used a recursive (9/17/13 meeting transcript, line 220). However, the instructor replied that ". And that probably is a really good guess as to what happened for these towers. She might've been using a recursive argument. She might've been taking the two together, and moved them down the way she did the one." (9/17/13 meeting transcript, line 221).

Another student's argument was discussed with the teachers but not shown on the screen. The instructor asked one of the teachers to read the following student's argument:

> Being that the tower is one cube shorter than the four cube tall tower; I would say there are less than 16 towers; we have less tower patterns to choose from. If we do 2 the amount of colors times four the amount in the towers you are going to get 8. Two which is the amount of colors times 3 which is the amount in the tower, you are going to get 6 towers. (9/17/13 meeting transcript, line 242).

The instructor said to the teachers "What she is saying is this: if we build the towers four tall which we did; okay we had four cubes tall, right and there were two colors. So she said when I build towers four tall I had the four cubes times the 2 colors; that's 8." (9/17/13 meeting transcript, line 243). This student incorrectly used the rule strategy. The instructor asked the teachers "What would you ask her to do?" (9/17/13 meeting transcript, line 251) and one of the teachers replied "Build them!" (9/17/13 meeting transcript, line 252).

### 4.3 Unit 3: On-line Discussion 9/18/13 to 9/24/13

For this discussion, teachers were instructed to watch four videos located on the VMC website: www.videomosaic.org. One video was of students Stephanie and Dana working on a shirts and pants problem. For the shirts and pants problem, students had to find the total number of outfits that could be made from three shirts and two pants of different colors. The other assigned videos were of third- grade students Stephanie and Dana working on the 4-tall towers problem selecting from two colors, Stephanie's prediction for 3-tall towers, and Meredith's prediction for 3-tall towers by removing the top cube.

Teachers were also assigned to read chapter 3 of Combinatorics and Reasoning (Maher, Powell, & Uptegrove, 2010) about the second- and third-grade students' representations to solve the shirts and pants problem. After watching the videos and reading the chapter, participants were asked to respond to the following guiding questions on-line:

> 1. At Rutgers, after building 5-tall towers, selecting from 2 colors, and listening to the arguments shared by your colleagues, what, if anything more did you notice in the video of the children that you watched building 4-tall towers, selecting from 2 colors?
>
> 2. You have watched the video of Stephanie and Dana in grade 3 building 4-tall towers, selecting from 2 colors. Are their arguments convincing? Why?
>
> 3. For the other two videos you have watched, both Stephanie and Meredith make predictions for the number of towers 3-tall. Their predictions are not the same. Does this give you any insight to the way children think or reason?
>
> 4. In Chapter 3, compare and contrast the solutions the children found as second graders to the Shirts and Pants problem to the solutions they found as third graders. (Unit 3 on-line discussion thread, questions 1-4)

#### *4.3.1 First Question Responses*

The first question asked teachers to discuss anything that was noticed in the video of the children building 4-tall towers, selecting from 2 colors. Three of the teachers mentioned opposites as a strategy. The responses of the three teachers follow:

> I did notice that the "opposite" method was very popular. (Unit 3 on –line discussion thread, line 1)

> We immediately started building opposites. This was very common amongst the students this past week. (Unit 3 on –line discussion thread, line 14)
>
> No matter what group of students I listened to, I saw them all talk about opposite pairs, patterns, diagonal movements, and recursive patterns but all were unclear as to how to explain why they did what they did. (Unit 3 on –line discussion thread, line 33)

Eight of the ten teachers mentioned the exposure to different strategies. The responses of the teachers follow:

> I learned from my colleagues and from watching the children that there are many different ways of approaching and explaining the problem. (Unit 3 on –line discussion thread, line 1)
>
> Whether it was with my partner in our own class, watching the video or seeing the students this past week, I was able to listen and see how different people approached the problem in so many different ways. (Unit 3 on –line discussion thread, line 14).
>
> What I noticed while I was watching the video and also watching the students in C's class are the different ways that people organize towers. (Unit 3 on –line discussion thread, line 26)
>
> After to speaking to my colleagues and conducting the activity in our class together it opened my eyes to the many different approaches that can be used to solve the problem. (Unit 3 on –line discussion thread, line 30)
>
> I noticed that there are many different ways to approach this problem. (Unit 3 on –line discussion thread, line 39)
>
> I have learned that there are multiple approaches toward a problem like this. (Unit 3 on –line discussion thread, line 41)
>
> I did learn many different ways from hearing others in class. (Unit 3 on –line discussion thread, line 46)
>
> I found it very interesting how many methods were used in getting to the solution of 16 towers. (Unit 3 on –line discussion thread, line 48)

### *4.3.2 Second Question Responses*

For this question, teachers were asked if Stephanie's and Dana's arguments were convincing. The teachers were also asked to justify their answer. Half of the teachers thought Stephanie and Dana provided a convincing argument. The responses of the five teachers that thought Stephanie and Dana provided a convincing argument follow:

> I thought Stephanie and Dana's argument was very convincing in that 3 tall will give 8 towers. Dana explained that when you take one cube from each 4-tall tower, that will leave you with duplicates. After getting rid of those duplicates, you will be left with 8 towers. (Unit 3 on –line discussion thread, line 1)

> I think Stephanie's and Dana's argument is convincing. They were easily able to identify the 16 total towers that can be created using 4-tall high unifix cubes. When asked how many towers they would have for 3-tall, they simply just took off one cube from the 16 towers that they already had created. They were able to then see that they had duplicates amongst the 16 towers and by eliminating the duplicates; they were left with 8 different towers. (Unit 3 on –line discussion thread, line 14)

> Stephanie and Dana did have a convincing argument. The girls came to 16 combinations, they explained that they had kept trying more and more but they kept making duplicates so that made them realize there were no more combinations. Though that usually is not enough once they were explaining how they knew to build the 3-high towers they showed that they understood how they had no more combinations from the four. Taking one off just created more duplicates that they had to get rid of and their resulting answer was 8 towers 3-high. (Unit 3 on –line discussion thread, line 41)

> Stephanie and Dana have convincing arguments. They are able to create patterns and show that they have exhausted options without repeating. (Unit 3 on –line discussion thread, line 46)

> I thought the girls' arguments were convincing. Stephanie determined right away that with less blocks there would need to be less towers because you would have duplicates. She had good reasoning to convince herself of that too. (Unit 3 on–line discussion thread, line 48)

Half of the teachers responded that Stephanie and Dana did not provide a convincing argument. The responses of the five teachers that thought Stephanie and Dana did not provide a convincing argument follow:

> Stephanie's argument for having 16 towers four tall was not convincing. She only mentioned that she had them all since she was checking and could not find more. This is not convincing since there is the possibility of missing some. She did not describe any method used in creating them. Dana's argument was not convincing either since she stated that you should always assume there is more

> until you find out the answer. She did not give any reasoning or method to how to determine the answer or know when you have them all. (Unit 3 on–line discussion thread, line 6)
>
> I do not think that Stephanie and Dana have a convincing argument. Both Stephanie and Dana believe that they have tried many different ways and are convinced that they have made all the possibilities but do not supply a convincing argument as to why they are sure. (Unit 3 on–line discussion thread, line 26)
>
> I really don't feel that their argument is very convincing. They really don't explain why they came to the conclusion and how they determined that all possibilities are done. At the end Stephanie explains that if you take one red and one blue away from the tower of four they would be the same. However, she is not thinking that there would have to be more color combinations because they are working with two colors. (Unit 3 on–line discussion thread, line 30)
>
> I do not think Stephanie and Dana's argument is completely convincing. I think they are on the right track but they still are not sure how to explain if there are truly only 16 towers. (Unit 3 on–line discussion thread, line 34)
>
> I am not completely convinced by their argument. It appears that since she was "checking and checking" she used a guess and check method, but did not have a systematic way to see if she had all of the possible outcomes. (Unit 3 on–line discussion thread, line 39)

#### *4.3.3 Third Question Responses*

The third question asked teachers for their insight to the way children think or reason based on the 3-tall tower predictions of Stephanie and Meredith. Eight teachers specifically mentioned Meredith using the unifix cubes to solidify her conceptual understanding of the task. The responses of the eight teachers follow:

> Meredith initially thought it would be 16 still because by just taking one cube off the top, there would be the same amount of towers. It wasn't until she actually tested out that she noticed the duplicates and knew to get rid of them. (Unit 3 on – line discussion thread, line 1)
>
> Meredith did not initially think that the number of towers would change since you could just remove a block and be left with towers three tall. It was not until she actually removed the blocks that she saw the duplicates. (Unit 3 on –line discussion thread, line 6)
>
> Meredith on the other hand, predicted that the number of towers would be the same. She immediately thought you could simply just remove one cube from each

> tower and 16 different towers would still remain. It wasn't until she physically removed a cube from each tower for her to visually see that duplicates would be present. (Unit 3 on –line discussion thread, line 14)
>
> After Meredith completes the actual process of removing the top block, she is able to articulate this very clearly. It was necessary for Meredith to complete the activity before being able to do so. (Unit 3 on –line discussion thread, line 26)
>
> When Meredith pulled the top cubes off it took her awhile to realize that some might become doubles. Once she realized that she started to omit the doubles. (Unit 3 on –line discussion thread, line 30)
>
> Stephanie's answer is correct but Meredith still arrived at the correct answer quickly after removing the top cube from all of her towers. (Unit 3 on –line discussion thread, line 34)
>
> Meredith does not predict the duplicates, but is able to see them after she begins removing cubes. (Unit 3 on –line discussion thread, line 39)
>
> Meredith predicting that there would be the same amount until her teacher asked her to try it (Unit 3 on –line discussion thread, line 41)

### *4.3.4 Fourth Question Responses*

For the fourth question, teachers were asked to compare and contrast the solutions of the children from second to third grade on the shirts and pants problem. Five of the teachers noted an improvement in reasoning from the second grade to the third grade. The responses of the five teachers that mentioned improvement in reasoning from the second to the third grade follow:

> Stephanie, Dana, and Michael all showed an improvement in their reasoning between second and third grade. While Dana seemed to understand there could be six outfits both years, in second grade she was adding the assumption that the outfits should match and taking out a combination she did not think matched as a result. Stephanie did not provide an explanation for why she changed the white shirt to a yellow shirt instead of making the last two outfits with those colors but in third grade, her diagram showed she could determine the number of outfits in a more organized way and ensure she did not miss any outfits as a result. Michael's understanding of the problem improved between the two years. He no longer assumed the shirt and pants had to be the same color, instead he knew to match each shirt with each pant color. He also completed the activity in an organized way by matching each with lines then making a list from the lines he drew. (Unit 4 on–line discussion thread, line 6)

> All three students' answers and organization of the work improved from second to third grade and showed that the three students had grown in the past year when given the same problem. (Unit 4 on–line discussion thread, line 15)

> The major improvement that students made from 2nd grade to 3rd grade was their ability to rationalize their problem solving strategy. The students were able to explain why they used lines to connect possibilities and the purpose of doing so. (Unit 4 on–line discussion thread, line 26)

> I also noticed in the video that the 3rd graders' reasoning was quicker and more methodical than when they were 2nd graders. (Unit 4 on–line discussion thread, line 34)

> There was obvious growth in that one year in order for these students to get to the correct solution. (Unit 4 on–line discussion thread, line 48)

All teachers were asked to complete their on-line responses by 9/24/13, the day before the next on-line discussion, and were instructed on-line to implement the Cycle 1 tasks in their classrooms sometime between September 18th and 27th**.**

**4.4 Unit 4: Online Discussion 9/25/13 to 10/1/13**

For this discussion, teachers were instructed to watch six VMC videos about students solving the Cycle 1 task. The first video showed Stephanie and Dana working on the 4-tall towers problem as third-graders and working on the 5-tall towers problem as fourth-graders. The other videos are five clips showing fifth-grade students working on building towers selecting from two colors using the Guess My Towers Problem.

The first clip of the Guess My Towers Problem showed Milin and Michelle working on finding all the possible 3-tall towers selecting from two colors for Question 1 of the Guess My Towers problem. Question 1 of the Guess My Towers Problem follows:

> PROBLEM STATEMENT "You have been invited to participate in a TV Quiz Show and have the opportunity to win a vacation to Disneyworld. The game is played by choosing one of the four possibilities for winning and then picking a tower out of a covered box. If the tower matches your choice, you win. You are told that the box contains all possible towers three tall that can be built when you

> select from cubes of two colors, red and yellow. You are given the following possibilities for a winning tower: a. All cubes are exactly the same color; b. There is only one red cube; c .Exactly two cubes are red; d. At least two cubes are yellow. Question 1.Which choice would you make and why would this choice be any better than any of the others? (Private Universe Project in Mathematics Workshops (PUP), Building Towers, Selecting from Two Colors for Guess My Tower, Clip 1 of 5: The Meaning of "At Least", 1993)

The second video clip showed fifth-graders Stephanie and Matt finding all possible 4-tall towers selecting from two colors for question 2 of the Guess My Towers problem. Question 2 of the Guess My Towers problem follows:

> PROBLEM STATEMENT Question 2.Assuming you won, you can play again for the Grand Prize which means you can take a friend to Disneyworld. But now your box has all possible towers that are four tall (built by selecting from the two colors, yellow and red). You are to select from the same four possibilities for a winning tower. Which choice would you make this time and why would this choice be better than any of the others?" (PUP, Building Towers, Selecting from two colors for Guess My Tower, Clip 2 of 5: Does the Number Double?, 1993

The third video clip showed Milin, a fifth-grader Milin who shared his inductive argument for building 3-tall towers with researcher Carolyn Maher and another student named Michelle. The fourth video clip, Matt explained Milin's inductive argument to Robert and Michelle. Stephanie also adds to Matt's explanation with 4-tall towers. The fifth video clip, Stephanie explained the "doubling rule" to Matt, Michelle I., Michelle R., Milin, and Robert.

In addition to the videos, teachers were also assigned to read chapters 4 and 5 of Combinatorics and Reasoning (Maher, Powell, & Uptegrove, 2010). Chapter 4 examines the strategies and representations used for solving towers problems. Chapter 5 examines how Stephanie and their classmates built their conceptual understanding of Milin's inductive argument. After watching the videos and reading chapters 4 and 5, participants were asked to respond to the following guiding questions on-line:

1. How do the children's strategies used to solve the towers problem in third grade look different than the strategies used when they were fourth graders? Which of their arguments did you find convincing?

2. During your classroom implementation of 4-tall towers, did any solutions surprise, delight, or puzzle you? Talk in detail about the solution of one of your students, so we can understand what the student did and whether you were surprised, delighted, or puzzled about his or her work.

3. In Chapter 5, we see that when children are given the opportunity to share mathematical ideas, they can contribute to the growth of understanding of their classmates. Talk about what one child in the video did that helped another child grow in their understanding.

4. In the video that you watched, did you find Milan's inductive argument convincing? Did his classmates follow his inductive argument? Give support for your answer. (Unit 3 on-line discussion thread, questions1-4)

### *4.4.1 First Question Responses*

Three teachers noted that Stephanie and Dana used a guess and check strategy in third grade and then used the opposite strategy in fourth grade. The responses of the three teachers follow:

> In 3rd grade Stephanie and Dana didn't really have a strategy. They built the towers and then held up towers to one another to see if there were duplicates. In 4th grade, they immediately started by "doing the opposite." This was convincing to me because that is how I found the towers when I did this activity. (Unit 4 on –line discussion thread, line 1)
>
> In third grade, Stephanie and Dana made towers and then put them in a line. They continued to make towers and check it with what they had already built to see if it was a duplicate. The used a similar strategy in fourth grade but this time they noticed the pairs that could be created. (Unit 4 on –line discussion thread, line 7)
>
> In third grade, Stephanie and Dana used the process of trial and error to create the different towers. They were easily able to create several towers, but used a guess and check strategy to create any additional towers. They would then compare each new tower with the towers that already existed. If the tower was in fact a duplicate, they broke down the tower and tried again. If the tower was different then, the ones that they had already created, they would add it to the group. In fourth grade, Stephanie and Dana went immediately to the idea of opposites. (Unit 4 on –line discussion thread, line 21)

### *4.4.2 Second Question Responses*

For the second question, teachers were asked to respond about solutions that surprised, delighted, or puzzled the teachers when they implemented the 4-tall towers task in their own classrooms. Six teachers noted they were surprised at their students' work. The responses of the six teachers follow:

> I was surprised by some of the work….There was one group that started with a recursive argument. They went through each grouping and were able to clearly explain why there were no more options. As the class continued, I noticed more and more groups using a recursive argument. (Unit 4 on–line discussion thread, line 7)
>
> Another group that surprised me was a pair that I had high expectations from; however, they struggled making all 16 towers. They made two towers of one color, two towers of three and one, two towers half and half and two towers with each cube color alternating. (Unit 4 on–line discussion thread, line 11)
>
> I did have one group of students who were closest to creating a convincing argument for the 16 towers that they created. They first organized the towers showing 1 blue cube and 3 yellow cubes, with the blue cube in the 1$^{st}$ position, 2$^{nd}$ position, 3$^{rd}$ position, and 4$^{th}$ position. They described how they knew there could not be any more towers in this set because if they tried to move the blue cube to another spot, they would need a tower that was 5 tall. They created the same argument for 1 yellow and 3 blue cubes. They then found all the towers that had 2 blue and 2 yellow cubes. They started with 2 blues in the 1$^{st}$ and 2$^{nd}$ positions, and then moved them to the 2$^{nd}$ and 3$^{rd}$, 3$^{rd}$ and 4$^{th}$, and 4$^{th}$ and 1$^{st}$ positions. Because there would not be any other place to put 2 blues next to each other, they had all towers in this set. Next they created the alternating towers, and the solid towers. What most surprised me about their work was that when I returned to their group to give them a recording sheet, I noticed they had decided to reorganize their towers into pairs of opposites. (Unit 4 on–line discussion thread, line 29)
>
> One of the things that surprised me the most came from two boys in my 8$^{th}$ grade class. They are both extremely quiet and withdrawn from most school work (except for gym) but during this activity I saw a lot of cooperation from them. (Unit 4 on–line discussion thread, line 36)
>
> I was pleasantly surprised with almost all of my students who worked on this problem. I think that in class about 99% of them got to 16. (Unit 4 on–line discussion thread, line 41)
>
> I was surprised because they approached it mathematically before building. They were using the reasoning that since there is two colors and four high that they

> probably will have an even amount of towers. They were trying to solve the problem, but then decided to start building. As they were building they did what most of my students did and built them by opposites. (Unit 4 on–line discussion thread, line 45)

Two of the teachers noted that they were puzzled by students' work. The two teachers' responses follow:

> One group that really puzzled me had a lot of towers built. The towers filled the desks. When I asked why they had so many, they said that they had towers standing up and towers lying down. I said that if the knob was at the top it did not matter if they were upright or lying down. It did not matter. The students insisted that they were different. They also had several duplicates within their layout but for some reason could not see them. (Unit 4 on–line discussion thread, line 43)

> I had one student that actually used the same method that I had used to group them. She created to sets, towers with red cubes on the top and towers with yellow cubes on the top. She kept a constant to organize the towers. However, once she got here she had a really hard time explaining or organizing the towers in the set to show that she had exhausted all options. This student is my highest achieving student and it was puzzling to me that she could not develop an argument. (Unit 4 on–line discussion thread, line 47)

#### *4.4.3 Third Question Responses*

For the third question, teachers were asked to discuss one child that helped another child grow in their understanding from the videos assigned. Five of the teachers noted that Milin helped Michele. The responses of the five teachers follow:

> Milin helped Michelle understand that 2 colors, 3 cubes tall will give 8 towers. He showed her that by starting with two blocks red and yellow, you can add on a red and then a yellow, giving you a total of 4. Then, you double the 4 because you have to do the opposite of those 4. (Unit 4 on–line discussion thread, line 1)

> Milin was able to display the solution clearly to Michelle by showing a red or yellow added to each of the previous towers. (Unit 4 on–line discussion thread, line 7)

> They started with 1-tall and were both able to see that there were two different towers possible. Milan then tried to build upon the towers he had already created, but the teacher prompted him to create new towers so that Michelle could see the different towers from 1-tall versus 2-tall. He explained to Michelle that the number of towers doubled from 1-tall to 2-tall because he was adding one red and

> one yellow on top of the towers he had already created. Using the different towers they created, they built two towers that were 1-tall, four towers that were 2-tall and eight towers that were 3-tall. The concept of doubling was then seen because they are adding one red and one yellow to each previous created tower. Michelle then understood where Milin got the idea of "doubling" and was able to believe that 4-tall would create 16 different towers. (Unit 4 on–line discussion thread, line 21)

> I feel that Milin was the one to first understand the inductive reasoning behind this problem. When he was explaining it to his partner Michelle at first he struggled but then when he was showing her how one base color can become two because you can put a yellow and then a red on top of each base Michelle was able to see the idea. (Unit 4 on–line discussion thread, line 41)

> Milin really helped Michelle reason through this problem. As he was building his towers bigger and bigger, he clearly showed how you can get two towers from each smaller tower because there were two color options to add. A she was explaining this, the light bulb really turned on for Michelle and she was then able to finish the explanation herself. (Unit 4 on–line discussion thread, line 43)

Two of the teachers thought Milin helped all of his classmates. The responses of the two teachers follow:

> I think Milin's use of inductive reasoning and breaking the towers down helped his classmates understand. He was able to clearly communicate with his classmates what he was doing when he added a block each time and doubled the number of towers. (Unit 4 on–line discussion, line 3)

> It seems as only Milin saw this reasoning originally, but once the other students began to understand how it worked, it provides a very convincing argument for why there are only a certain number of possible towers of any height. (Unit 4 on–line discussion thread, line 29)

Six teachers noted that Stephanie helped other students. The responses of the six teachers follow:

> Milin was able to display the solution clearly to Michelle by showing a red or yellow added to each of the previous towers. Stephanie used a similar argument describing it as a family tree. (Unit 4 on–line discussion thread, line 7)

> I also think that Stephanie's description of families aided to the students' understanding. (Unit 4 on–line discussion thread, line 11)

> Stephanie is struggling to explain to the other students at the table how she is creating the 4 towers that are 2-tall. Once Matt steps in and clarifies the explanation, it seems as though all of the students at the table are convinced. It is

> clear that Stephanie now understands this argument as she jumps back in and starts explaining the 4-tall towers that can be made, starts building, and is even confident enough to share the reasoning with the class as shown in the next segment. (Unit 4 on–line discussion thread, line 29)

> Throughout the videos Stephanie seems to be the most demonstrable child. She was extremely willing to use the cubes to demonstrate her reasoning to others and talk out her work as she progressed through the steps. (Unit 4 on–line discussion thread, line 36)

> Stephanie came back in to further show the explanation. In the final video when they were talking to the whole class and they had their examples lined up I think it was a great visual and verbal explanation to refer to a family tree; that each "parent" cube had kids and then they had kids and each time you are multiplying their kids by two. (Unit 4 on–line discussion thread, line 41)

> In the last video when Stephanie was explaining her reasoning I think she was now more effective because she had all the blocks set up, which is a good visual aid, and went through the process step by step. (Unit 4 on–line discussion thread, line 45)

One teacher mentioned Michelle helped Matt. The teacher responded that "Michelle was explaining the inductive pattern and during this explanation Matt (who did not understand the reasoning behind why there were 16 towers) was listening to her and was able to jump in and continue her explanation. He learned from watching and listening to her reasoning. He actually even said that it was a family tree" (Unit 4 on-line discussion thread, line 47).

#### *4.4.4 Fourth Question Responses*

For the fourth question, teachers were asked if Milin's arguments were convincing. The teachers were also asked to justify their answer and provide support for whether Milin's classmates followed his inductive argument. All ten teachers responded that Milin's inductive argument was convincing. Seven teachers noted that Milin's argument was convincing when Milin used the cubes to physically show Michelle his inductive argument.

> At first, she was pretty confused, and then when Milan showed her how to start with 2 cubes and add on each color, it was like it clicked for her. (Unit 4 on–line discussion thread, line 1)
>
> Starting with the basic one block tower, he build from each tower showing there were two additional outcomes from each tower when another block was added because of the two color options. (Unit 4 on–line discussion thread, line 7)
>
> He clearly showed not only that an increased number in a block led to the doubling of a new "family" but he stated why this was occurring as well. (Unit 4 on–line discussion thread, line 11)
>
> It wasn't until he physically built the towers though that Michelle was convinced of Milan's "doubling" concept, which was evident in a later video when Michelle was then able to explain the problem. (Unit 4 on–line discussion thread, line 21)
>
> When he describes how the only cubes to add on to each base he previously built are either yellow or red, he proves that there are no other possibilities than the set he created. (Unit 4 on–line discussion thread, line 29)
>
> Using only two colors you only have two options to build on the each previous tower thus multiplying the result by two works. (Unit 4 on–line discussion thread, line 41)
>
> Once he started to show Michelle the process of building the towers one set at a time she seemed to understand it. (Unit 4 on–line discussion thread, line 45)

All teachers were asked to implement the Cycle 1 tasks sometime before September 27th, complete their on-line responses by 10/2/13, and bring two or three samples of students' work to share at the regional meeting on October 2, 2013.

### 4.5 Unit 5: Regional Meeting and On-line Discussion

For the week of 10/2/13 to 10/8/13, teachers were asked to make an original post by 10/5/13 and respond to at least two other posts by 10/8/13 using the on-line questions provided by the instructor. For this week, teachers also attended the first regional meeting. At the first regional meeting, teachers discussed samples of work from the first cycle four-tall towers problem.

***4.5.1 Regional Meeting 10/2/13 Discussion of Students' Work***

Ten teachers met at a middle school in New Jersey and brought students' work samples of the 4-tall towers task. During this meeting, teachers shared and discussed students' solutions in desks pushed together to form a U-shape.

Teacher 1 (T1) was the first to share her students' work. The first pair of students concerned her because they were trying to make one big tower out of the 4-tall towers (10/2/13 meeting transcript, line 10). Although no teachers recognized a strategy the students used, one teacher suggested trying to "help them organize it differently" (10/2/13 meeting transcript, line 45). Figure 4.5 shows the written work and a picture of the towers.

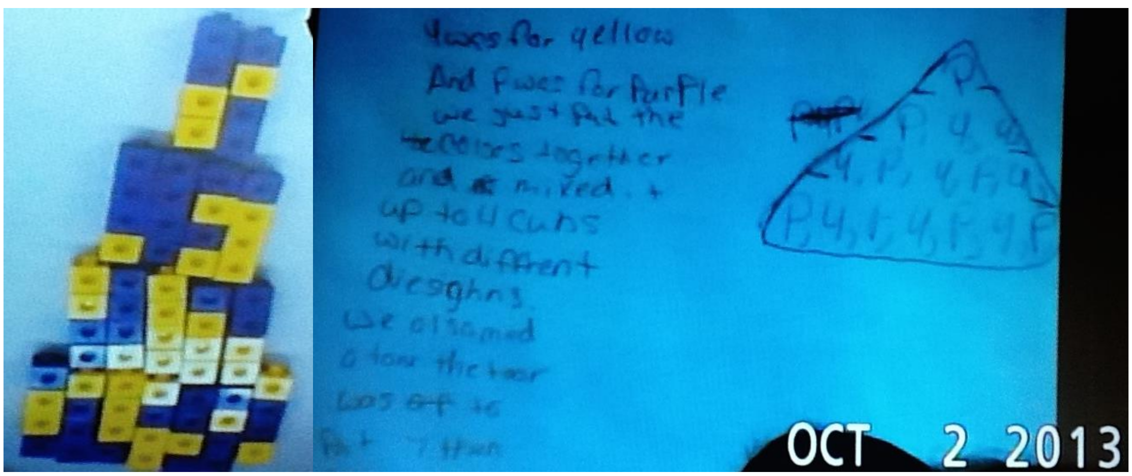



*Figure 4.5 T1's Cycle 1 Student Work Sample 1*

T1 put the student's work on the screen for the teachers to read. The following written argument was provided by the student:

> Y was for yellow and P was for purple. We just put the colors together and mixed it up to 4 cubes with different designs. We also made a tower the tower was to put 7, then 5, then 3 then 1 we did. (10/2/13 meeting transcript, line 41)

The next example shared by T1 was from a girl. T1 reported that this student correctly answered sixteen total towers but only recorded ten towers using symbols b to represent brown cubes and g to represent green cubes (10/2/13 meeting transcript, line 61). T1 read the following student's written argument to the class "We kept on making 6 of 3 and we made [a] 4 set and we found our answer. We can't make any more because there would be duplicates." (10/2/13 meeting transcript, line 78). The instructor asked the teachers "did they have any way of organizing those ten" (10/2/13 meeting transcript, line 88) and one teacher replied "First they were opposites, so you get 2. Then there they have their diagonal…with the green going up" (10/2/13 meeting transcript, lines 89, 91). Figure 4.6 shows the tower chart created by T1's student.

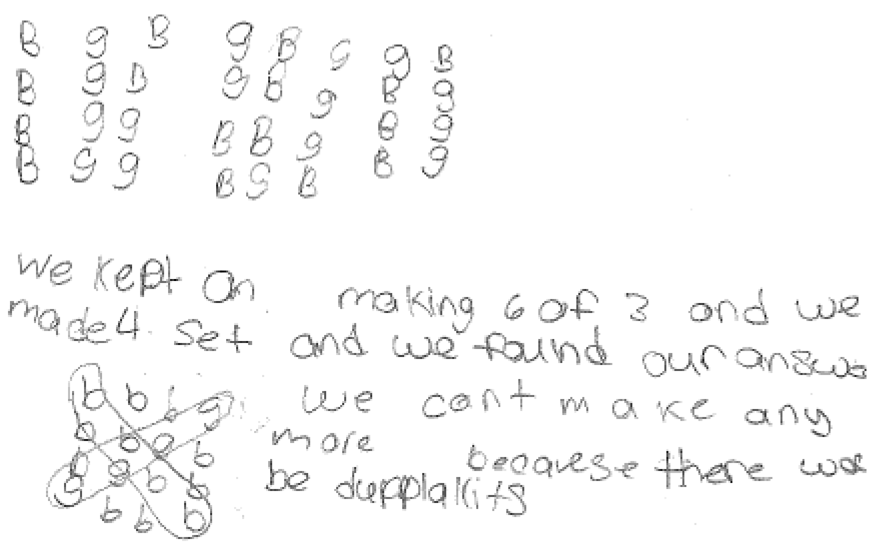



*Figure 4.6 T1's Cycle 1 Student Work Sample 2*

T1 also shared another student group's work that came up with an answer of sixteen. The student drew 16 towers with individual cubes with either r or b in the blocks to symbolize red or blue blocks respectively. T1 read the following argument out loud to the teachers: "We couldn't make anymore because we think we made all the patterns plus we found all the blocks and we all worked together to create these patterns" (10/2/13 meeting transcript, line 103). When the instructor asked if the teachers were convinced, one of the teachers replied "for this group because of the level" (10/2/13 meeting transcript, lines 108,109). The instructor informed the teachers "You can't say the kids are young so we are going to expect less. If you want a convincing argument, you want a convincing argument" (10/2/13 meeting transcript, lines 110). Figure 4.7 shows the drawing of the towers provided by T1's student.

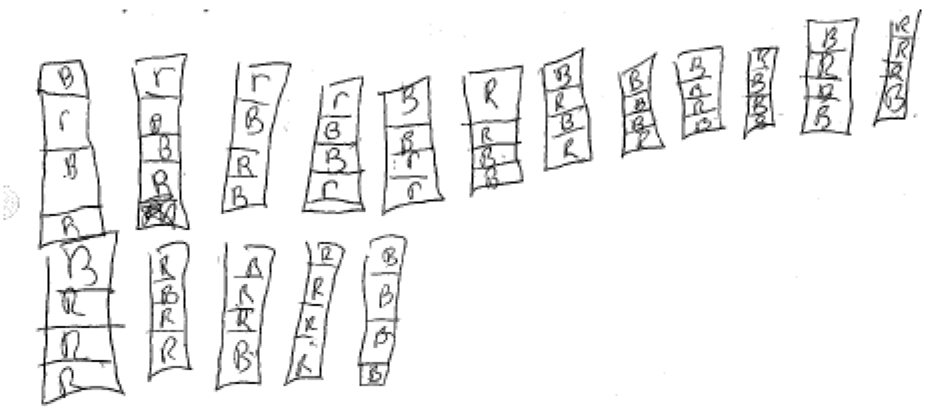


*Figure 4.7 T1's Cycle 1 Tower Drawing Student Work Sample 3*

Teacher 2 (T2) was the second teacher to share her students' work. The towers in the first-shared work sample were drawn as pairs of opposite towers. T2 read the following student's written argument to the teachers: "I did the same block twice but not the same color like a pattern sort of. The reason I did it like that is because it is easier for me" (10/2/13 meeting transcript, lines 150). Figure 4.8 shows the student's pairs of opposite-colored towers.

*Figure 4.8 T2's Cycle 1 Tower Drawing Student Work Sample 1*

T2 shared a second example of students' work where the students made a drawing of sixteen 4-tall towers using symbols of b and y to represent blue and yellow cubes. T2 recognized the opposite strategy by reporting that "when he did them, he started doing them opposite but then I guess between him and his partner they were getting confused about what they were drawing and that is probably why he didn't stick with it. But um they did originally do opposites" (10/2/13 meeting transcript, lines 183). T2 told the teachers that this student struggled to explain his solution and then read the following student's written argument to the teachers: "The tower we were building is four inches high and it is y and b" (10/2/13 meeting transcript, lines 193).

T2 also shared a third sample of students' work. This student did not provide a written argument but did make a drawing of the towers using yellow and blue crayons (10/2/13 meeting transcript, lines 209). The instructor asked the teachers "What's good about what she did?" (10/2/13 meeting transcript, lines 213) and two teachers replied "the blue diagonal" (10/2/13 meeting transcript, lines 214,215). Figure 4.9 shows the drawing of the towers provided by the student using blue and yellow crayons.

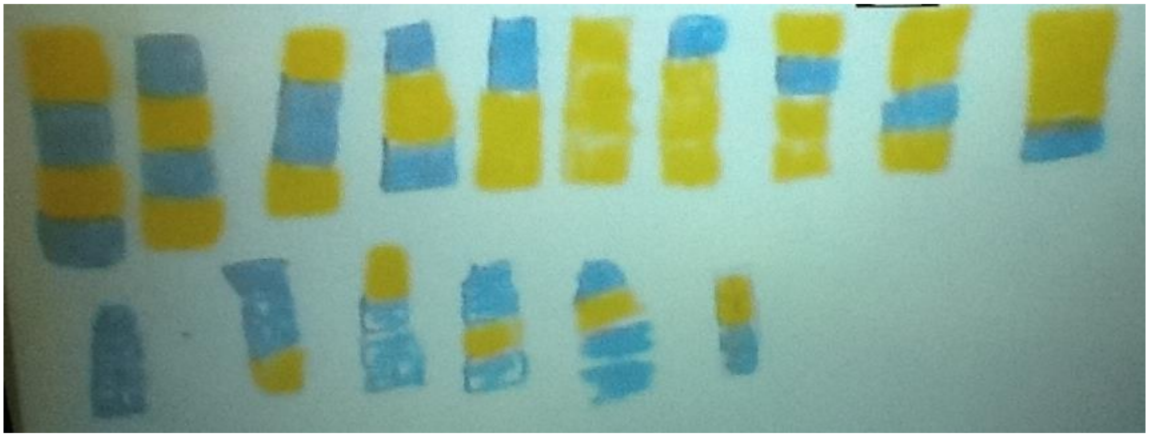


*Figure 4.9: T2's Cycle 1 Tower Drawing, Student Work Sample 3*

T2 also shared a fourth sample of work from a student that she described as "the most interesting one" (10/2 meeting transcript, line 235). The drawing shows the towers connected to form one big cube and T2 read the following student's written argument to the teachers "16 combinations total I think I have all the combinations for this worksheet" (10/2 meeting transcript, line 237).

Teacher 3 (T3) was the third teacher to share students' work. This time, teachers were asked to read the student's argument on the screen. The following student's argument was silently read by the teachers:

> There is one solid color, and 4 blocks high. There is no other way of doing this. There is only 1 yellow and 3 blues in each tower. There is a pattern, the yellow keeps moving up one. There is only 1 blue and 3 yellow in each tower. There is a pattern, the blue keeps moving up one. Two of the same color is touching, and one color isn't touching. Not one of the same is touching. Both colors are next to their twin. (10/2/13 meeting transcript, line discussion thread, line 326)

Figure 4.10 shows the student's work. In the first sample T3 shared, T3 recognized that the second and third groups were opposites of each other and that the students "were referring to that as a staircase" (10/2 meeting transcript, line 315).

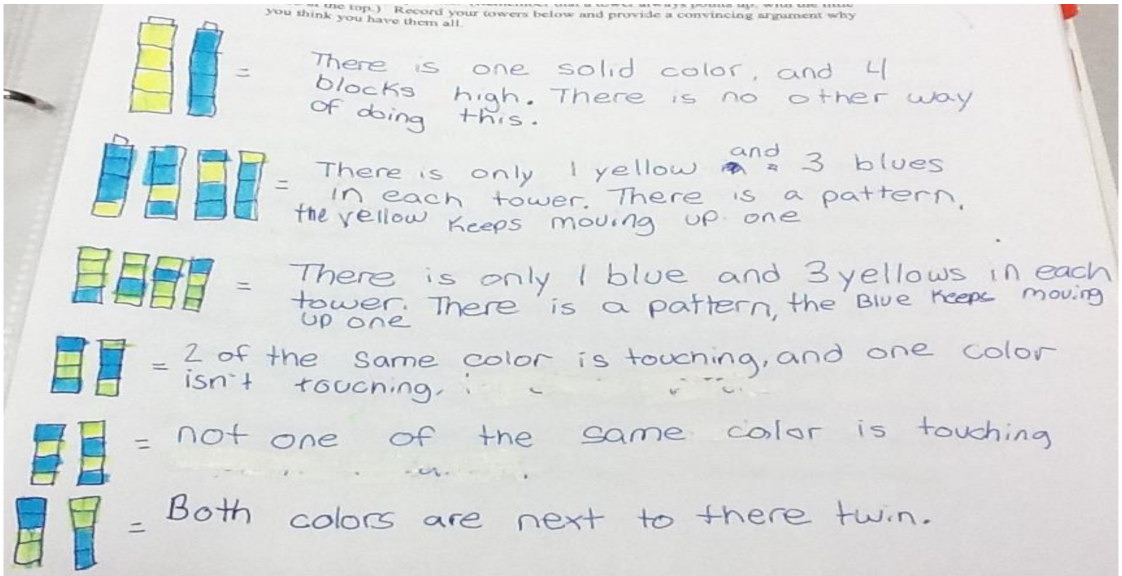



*Figure 4.10 T3's Cycle 1 Student Work Sample 1*

T3 also shared a second example with the teachers by showing the following argument on the screen:

> We have 16 in all. We think we have all the towers because each tower has 4 blocks in it and there are 2 colors to make the tower so 2x4=8. Then we realized that we can invert the colors to double the towers so 2x8=16. Then we couldn't make any more so we think that we made all the towers. We put the towers this way because it's a pattern. Every time the pattern moves it always has one small difference. (10/2/13 meeting transcript, line discussion thread, line 326)

The instructor invited the teachers to share their thoughts about the strategy the boys used (10/2 meeting transcript, line 329). T3 replied to the instructor by saying "I thought it was pretty good that they immediately jumped to the math of it" (10/2 meeting transcript, line 330). A discussion emerged on how this type of strategy was considered to be an invalid argument. The instructor reinforced that the mathematics the students use must make mathematical sense by saying "just the way 4 times 4 gives you the correct answer of 16 for towers that are four tall, that is not how you get the towers four tall" (10/2 meeting transcript, line 349). Figure 4.11 shows the tower drawing of the student's work.

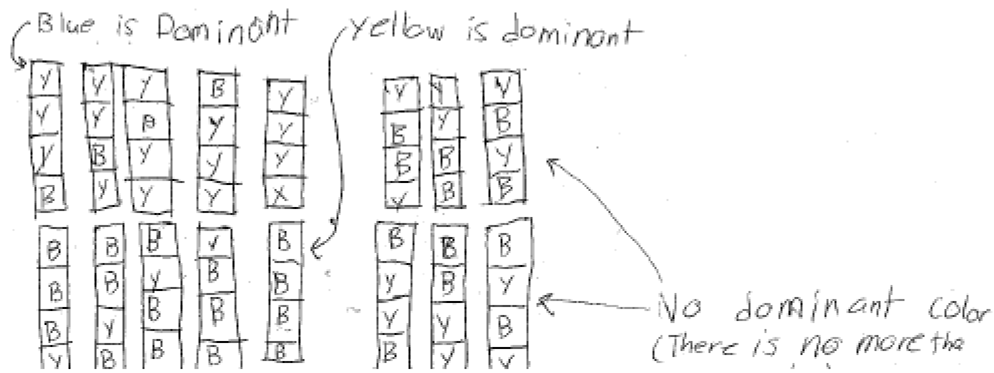



*Figure 4.11 T3's Cycle 1Tower Drawing, Student Work Sample 2*

A third example that T3 shared was of students that randomly drew towers in groups of two. This group of students concerned their teacher because they were trying to spell the word MATH with the towers (10/2 meeting transcript, line 389). However, when the instructor asked the teachers about how the students arranged the towers, multiple teachers replied "opposites" (10/2 meeting transcript, line 395). Figure 4.12 shows the student's work.


*Figure 4.12 T3's Cycle 1Tower Drawing, Student Work Sample 3*

The fourth teacher (T4) shared two samples of students' work. In the first example that T4 shared, the student provided a drawing of four groups of four towers. Under each drawn group of four towers, the student provided a written argument that described what the student did. Figure 4.13 shows the student's work.

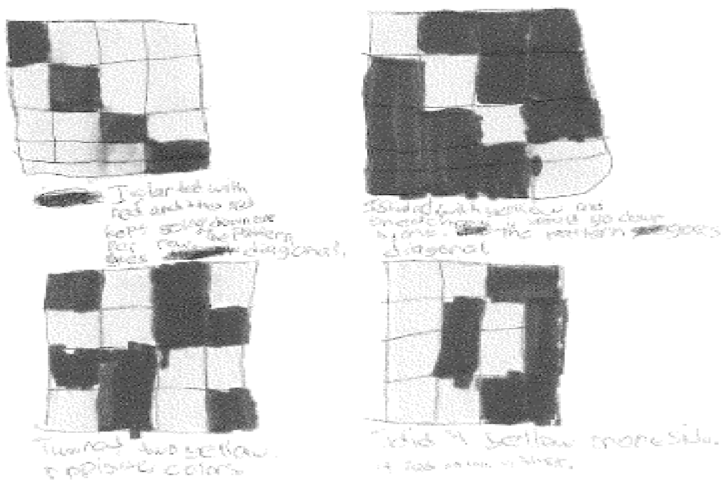


*Figure 4.13 T4's Cycle 1 Student Work Sample 1*

The following is the student's written argument:

> [Top left] I started with red and the red kept going down one per row. The pattern goes diagonal. [Top right] I started with yellow and on each one it would go down by one. The pattern goes diagonal. [Bottom left] Two red and two yellow are opposite colors. [Bottom right] I did 4 yellow on one side and 4 red on the other. (10/2/13 meeting transcript, line discussion thread, line 415)

According to T4, the student was able to verbally say "for this one it was going diagonally" (10/2 meeting transcript, line 429).

The second example that T4 shared was of a fourteen-year old girl who had been retained in the eighth grade twice (10/2 meeting transcript, line 461). T4 mentioned that this student made a drawing showing a yellow diagonal going down and that the student verbally noted an opposite pattern which would be "the same thing but red going down" (10/2 meeting transcript, line 463). The student wrote the following on her paper:

> We started with yellow and then we moved them down and moved them down until yellow gets on the bottom or the same thing but the red going down. (10/2/13 meeting transcript, line discussion thread, line 461)

Figure 4.14 shows the picture of the towers drawn by T4's student.

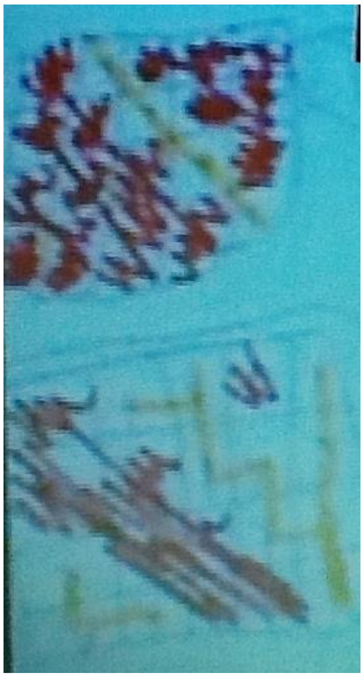


*Figure 4.14 T4's Cycle 1Tower Drawing Student Work Sample 2*

A fifth teacher (T5) was the next to share. In the first sample, T5 was impressed that the students came up with the right amount of towers but could not provide a convincing argument in writing (10/2 meeting transcript, line 491). T5 read the following student's argument to the teachers as she projected the student's work on the screen:

> The reason why we arranged the blocks this way is because we think it was easier the way we did it. But it helped us a lot better. (10/2/13 meeting transcript, line discussion thread, line 491)

T5 shared a second student's work by reading the following written argument to the teachers:

> I believe that I am done because we made the opposites from all the towers I have made. I built the towers from the way I can make them as many ways as I can. I

> have 18 towers and at the end I put the pairs in 2s two each. So I was done. I thought it would be the fast way to organize the towers, so it would be easier. I thought the opposites were easier to handle than to just wing it. (10/2/13 meeting transcript, line discussion thread, line 507)

T5 said to the teachers “when I thought they had a duplicate, I said why don’t you try and pair them up because they did have had it all scrunched together but it was laying in one and so I said separate into twos but they still put it the same” (10/2/13 meeting transcript, line discussion thread, line 517). The instructor responded by reminding the teachers to not lead the students in a certain direction (10/2/13 meeting transcript, line discussion thread, line 518).

The third sample T5 shared was from students who correctly answered 16 total towers by using the rule strategy incorrectly. Their argument follows:

> We did it this way because we had two colors and they had to be four high, so you multiply them. Then they can be the opposite what you would have to add two more so that would be 16. So there could be only 16 different combos. We organized by the opposite of the combinations (10/2 meeting transcript, line 523)

This student’s work sparked a big discussion about celebrating mathematical arguments that make sense. The mathematics the students used did not allow for generalization for solving the towers problem with varying numbers of tower heights. The instructor shared the following:

> I can remember um, I started as a middle school math teacher. I can remember very clearly when students would be able to force the numbers to get them the solution. And the solution happened to be the right answer. But the mathematics made no sense. And I had to try and let them know that I am not impressed when they get you know some Gobbledygook that turned into an answer that happens to be correct when the process is wrong. (10/2 meeting transcript, line 530)

The instructor was stressing the importance of celebrating students’ mathematical work that makes logical sense.

A sixth teacher (T6) then shared her students' work. T6 read the following argument provided by one of her students:

> First we put one yellow on the top with three red under it. Then we moved the yellow down one and moved the three reds like this from the original. Then we moved the yellow down again and again. And then we were at the bottom so that we knew we were done with those. Then we did the opposite. Then we did 4 and 4 like this. And those are all red and all yellow. And since there are only four blocks and two colors we knew that we were done with that. Then we decided to alternate red yellow red yellow like this: Then we did this: and moved it like this: then this than this: and once we reached the top we knew we were done. (10/2 meeting transcript, lines 539-549)

The instructor asked the teachers "Okay what is that called?" (10/2 meeting transcript, line 558) and one teacher replied "recursive" (10/2 meeting transcript, line 559). This was confirmed by the instructor when she said "someone said it, a recursive argument" (10/2 meeting transcript, line 560). Figure 4.15 shows the work and drawings of the student work that T6 first shared.

T6 also shared the work of this student's partner. T6 thought that the partner had "a better explanation for when they get to the twos and the twos" (10/2 meeting transcript, line 565). T6 read the following argument provided by the partner:

> Then we took two reds and two yellows. We put the two reds on top of the two yellows. After that we took the two reds and put them in between the two yellows. If you make the opposite of the last two so here are the opposite of the last two and put them next to each other in a certain way then it will be a pattern of two. Then you can put two of each color in a pattern of red yellow red yellow and yellow red yellow red. Lastly you can only have four blocks in a tower and there are only two colors so you can have a tower of only red and only yellow. (10/2 meeting transcript, lines 578-588)

The third sample of work shared byT6 was a pair of students that answered there were 20 possible towers (10/2 meeting transcript, line 592). T6 read the following argument provided by one of the students in the third-shared work sample:

> We made the basics, all red, all yellow. Then we did all the combos of one. Turns out all the combos of one were also 3. It was three because there are four

blocks in a tower. That means that there are going to be 3 different color blocks. Next we did the twos. We had two of each color places back to the explanation of the four color towers since there are only four blocks in a tower so there are no other combos except for four. (10/2 meeting transcript, lines 592, 594, 596)

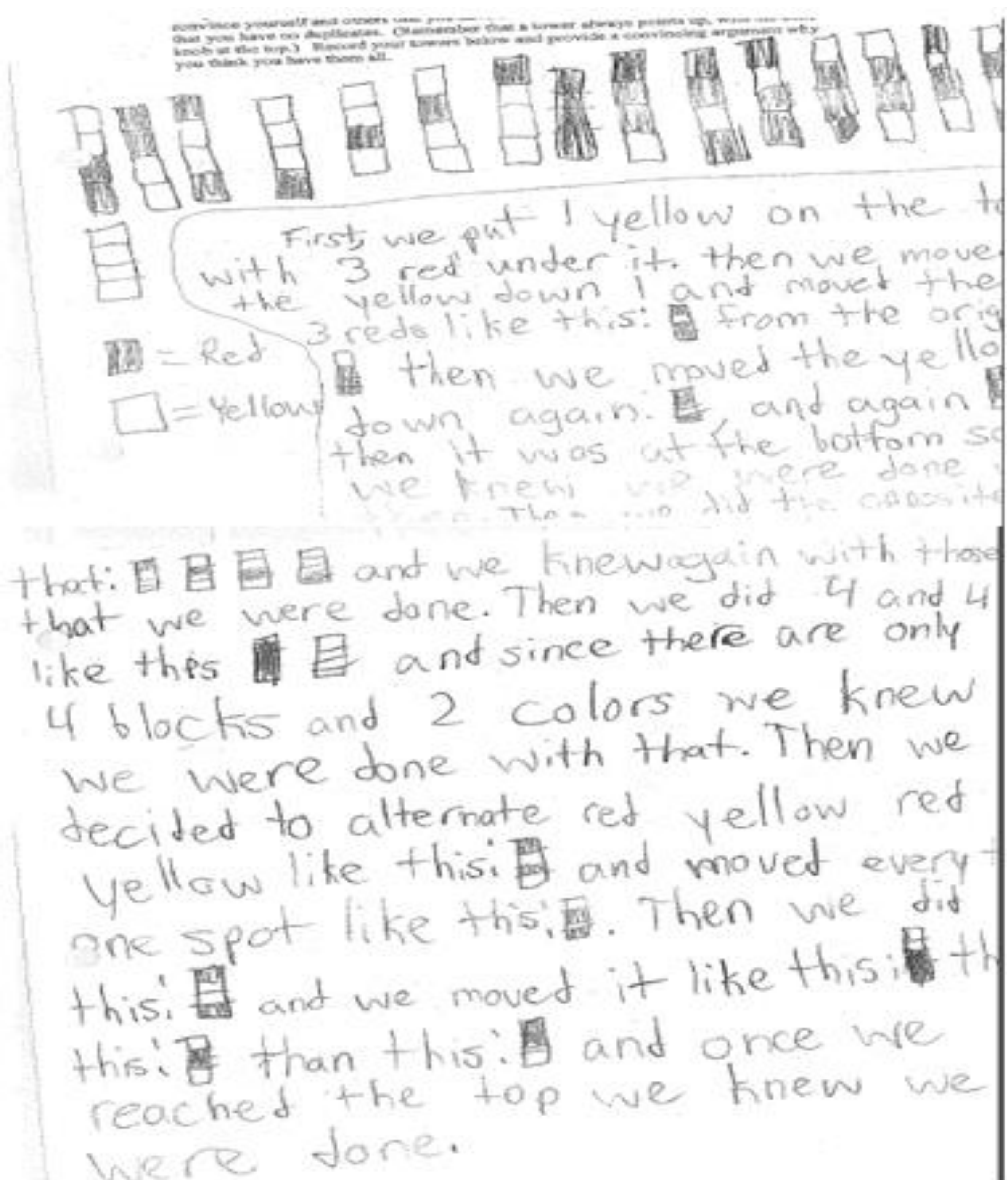



*Figure 4.15 T6's Cycle 1 Student Work Sample 1*

T6 claimed that this student was demonstrating a combination of the elevator and opposite strategy but referred to the strategy as a staircase (10/2 meeting transcript, line 596). T6 also shared the partner's work which was drawn bigger and more accurately such that the first two groups of eight towers were the elevator strategy and the last ten towers were grouped showing a recursive argument but with two cubes stuck together starting at the bottom and then moving up one position each time.

T6 shared a fifth student's work by reading the following argument to the teachers:

> I used two of each color. The first one in the set has the colors together. So there's two of one color on the bottom and two of the other color on the top. The second one of the group one only broke one color apart. With one color on top, two colors in the middle and one color on the bottom. (10/2 meeting transcript, lines 610, 612, 614)

One teacher recognized that the student controlled for a variable (10/2 meeting transcript, line 608).

The seventh teacher (T7) then shared her students' work. T7 said that this student verbally described a recursive argument but provided a different written argument (10/2 meeting transcript, line 635). T7 read the following argument to the teachers:

> The first pair of two of all the same color is there because there are four blocks and all are the same [color] but opposite from its partner. The second group of two pairs makes four different groups but they link together because if you take the bottom or top and put it completely opposite of the top or bottom one, it would make a different tower. You can only do this process 8 times before it starts to repeat itself. The third group of two pairs makes 3 different groups but the only link together twice. If you switch the top and bottom one with the opposite, it would be completely different towers and you can only do this with four towers. The last 2 towers are a set of two different colors mixed twice and switching them would just duplicate (it) further. That is why there can only be 16 different towers without any duplicates. (10/2 meeting transcript, lines 610-661)

T7 also recognized that the students used the opposite strategy (10/2 meeting transcript, line 641).

T7 then shared a second example by reading the following arguments to the teachers:

> There are four blocks high, 2 colors, 16 combos; and 64 total blocks which are all divisible by each other. (10/2 meeting transcript, line 673)
>
> There can only be two completely one cube tower (only two colors). There can be four, 3 red one yellow towers because there are only four high towers and that is the same for red. For two red and two yellow towers there can only be 2 because there are only two sides to switch to make two different towers. For the towers in the center there can only be two because there are only two colors to put in center of the tower. For the towers that have a pattern, there can be two because there are only two colors to make up the pattern starting from the top (or bottom). That was the alternating one. (10/2 meeting transcript, line 679)

T7 said that the second argument was better than the first argument (10/2 meeting transcript, line 681).

A third example that T7 shared was different because the student chose to list the towers horizontally (10/2 meeting transcript, line 685). T7 said that the student kept them in pairs (10/2 meeting transcript, line 691) and read the following argument to the teachers:

> We think we have all the towers because if you were to find or (try to find) another group of towers, you would realize that that group of towers had already been created. It would also begin a pattern of towers and if you located a tower in the pattern that hadn't been made, then you would know that you missed one. Also the towers had a knob-like appendage at the top which you would not be able to flip the towers over to make the towers different. Example: flipping one tower over to create a reverse pattern. (10/2 meeting transcript, line 693-695)

T7 wasn't sure, but thought the student was referring to the opposite strategy (10/2 meeting transcript, line 693)

The eighth teacher (T8) then shared his students' work. Figure 4.16 shows the student's work.

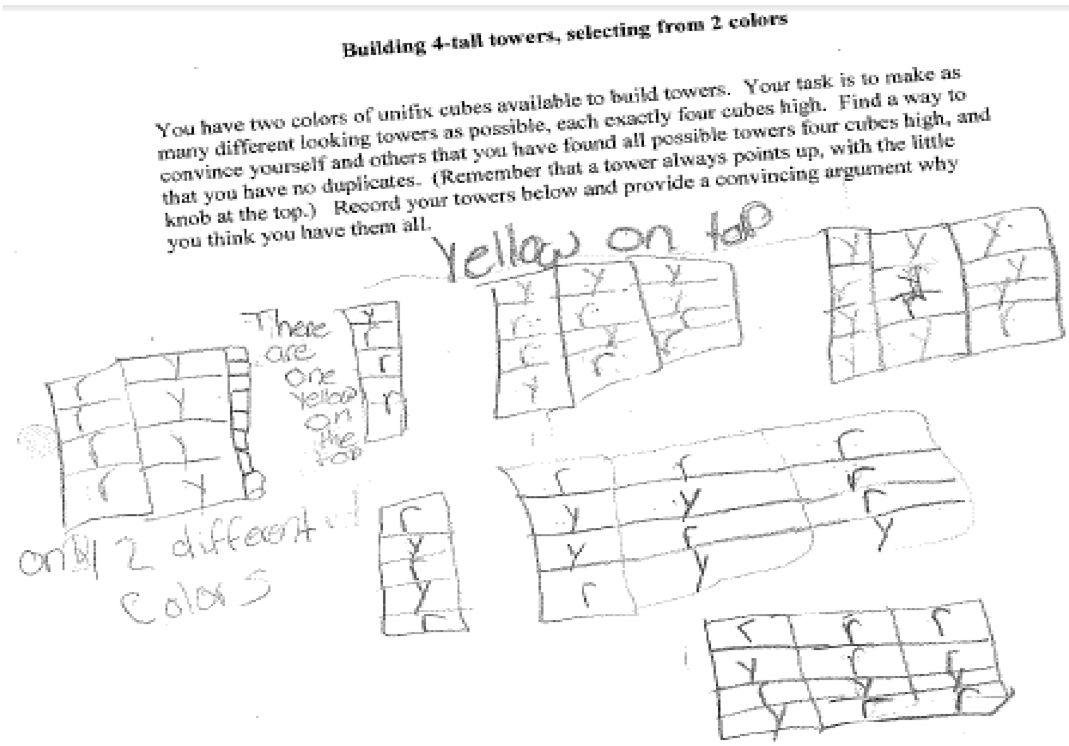



*Figure 4.16 T8's Cycle 1 Tower Drawing, Student Work Sample 1*

One of his students provided the following written argument:

1. Only 2 different colors.
2. There are one yellow on top and three red on bottom and if you move the red on the top, then there is no yellow on the top.
3.There is a block with one that has yellow red yellow red and you switch to red yellow red yellow then you won't have the yellow on the top. There is another block with yellow, yellow red, red if you move the red to the top, you have yellow on top 2 red in the middle and you have yellow on the bottom. Any other move you can get a red on top.
4. There are 3 red across and if you put red on top, then you won't have the yellow on top. If you put another red on the bottom then you get 5 cubes and there only possible be for cubes.
(10/2 meeting transcript, line 727)

T8 said that the students "started with opposite pairs and I told them that I wasn't accepting that as an answer. They needed to look at it and figure out another way that they could show me the blocks or arrange them or tell me that's all that they have" (10/2 meeting transcript, line 717). T8 said he was impressed when students rearranged her towers using the holding a constant strategy where she made towers with only yellow on the tops and then towers with only red on the top (10/2 meeting transcript, line 719). However, T8 was confused by the written work of the student's partner who wrote "we have three red across and we have 3 yellow. But if you put the red on top you won't have the yellow on top. If you put another red on the bottom, then you get 5. And there has to be four" (10/2 meeting transcript, line 741). Figure 4.17 shows the tower drawing of the partner.

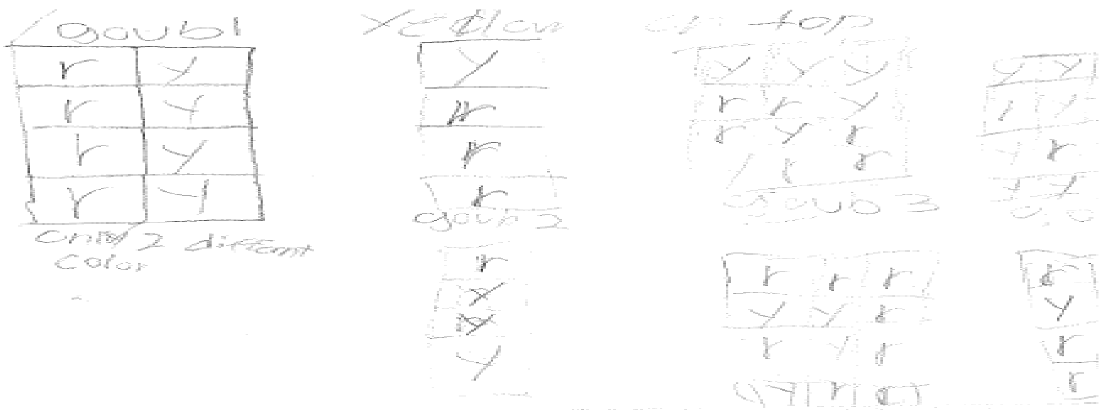


*Figure 4.17 T8's Cycle 1 Tower Drawing, Student Work Sample 2*

T8 also had two regular education eighth-grade students that volunteered to come to his mildly cognitively impaired (MCI) class to help out with the students and thought it would be interesting to share with the teachers how one student helpers would approach the task (10/2 meeting transcript, line 761). T8 read the following student's argument to the teachers while a drawing of how the student represented the towers was placed on the screen:

> Group one has the 2 reds together every time it moved to the top, middle, and end. For group 2 it just has four yellows and 4 reds on each. For group 3, it only had one yellow so the yellow cube started on top, and went down one every time and it stopped at the bottom. For group 4 all the reds were separated. For group 5, the red cube started on the top and went down one every time and stopped at the bottom. For each group, I couldn't make any more because there was no more

> possible combinations and if I added one more to it; it would be 5 (10/2/13 meeting transcript, line 787).

Figure 4.18 shows the tower drawing of one student-helper.

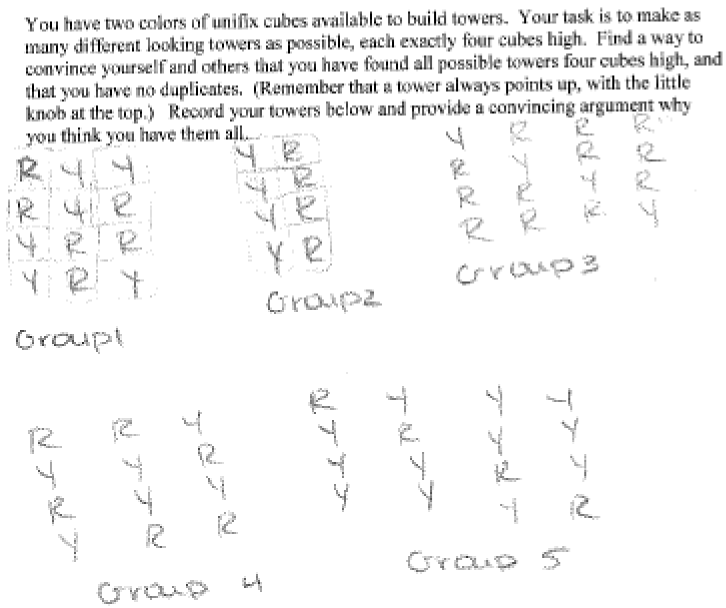



*Figure 4.18 T8's Cycle 1 Tower Drawing, Student Work Sample 3*

When the instructor asked if the argument was convincing, one teacher replied "I like how she talked about moving one down" (10/2 meeting transcript, line 789).

A ninth teacher (T9) then shared her students' work. T9 read the following student's argument to the teachers:

> The reason I think we are done with 16 combinations; are because I did it like Yellow, blue yellow, yellow So I moved the blue down one spot; yellow , yellow blue yellow and one down yellow, yellow, yellow blue, then I did blue yellow blue, blue. Then I moved the yellow down. Blue. blue yellow blue; down blue, blue, blue yellow then I did pairs of two colors; yellow, yellow blue, blue; blue, blue yellow, yellow; and yellow blue yellow blue; blue yellow, blue yellow and so on. (10/2/13 meeting transcript, line 803).

T9 shared was just a listing of towers but the student did not provide a written convincing argument as to why they knew they had all the possible towers (10/2 meeting transcript, lines 807-808).

Then T9 shared a second sample of students' work. This group of three students had one regular education student and two special education students (10/2 meeting transcript, line 817). The regular education student provided the following written argument that T9 read to the teachers:

> I made two whole towers and knew that there can only be two because there were 2 colors. Then I switch the first top color, the second color; the third color; and then the last color. I then knew that I could only do these because I switched the colors for both towers. Then I started doing 2 and 2. I took the wholes; and switched the bottom two, the top two; and the middle two. I know that these are the last ones, because there are four and if you are doing 2 and 2; then you know you can't do anymore; and finally I did the stripes. (10/2/13 meeting transcript, lines 823,827).

T9 replied that she was confused about what the student meant by stripes (10/2 meeting transcript, line 827). The instructor asked the teachers "who knows what she is talking about the stripes?" and another teacher replied "like yellow blue yellow blue" (10/2 meeting transcript, lines 828, 829).

T9 shared a third work sample by reading the following student's argument to the teachers:

> My group and I think this is all you can make because 4x4=16. There are 16 different pillars of 4. My teacher in Lloyd Road said if there is a problem like this do the amount of the blocks in one stack and times it by itself so 4 times 4 is 16. (10/2/13 meeting transcript, line 843).

This student used an invalid rule strategy that the teacher admitted she was not sure where the argument came from (10/2 meeting transcript, line 845).

The tenth teacher (T10) shared her students' work by reading the following student's argument to the teachers:

> The towers you see above are all of the possible combinations. To prove it, keep reading. My partner and I started making the 4-length tower with 3 yellows. We knew we had all of them we went on to the next one which was 3 blue. We knew we done with the 3 yellow because there was 2 blue per level and we knew there couldn't be another level because the towers had to be 4 blocks in length. We did

> the same for the 4 blue but there was one yellow per level. We moved on to make 4-length all blue towers and all yellow towers. Since there were two colors with so we knew we could only make 2 towers of it. Then we did the 2 blue and 2 yellow and knew we had them all because we tried and tried and tried to make more but it was impossible. So we knew we were done with that one. So we know and were absolutely positive that there are 16 towers. (10/2 meeting transcript, lines 855-867)

The student provided a convincing written argument for the three of one color and one of another (10/2 meeting transcript, line 855-859). However, the explanation of the two blue and two yellow towers was not convincing because “they tried and tried and tried and could not make anymore” (10/2 meeting transcript, line 865). Figure 4.19 illustrates the tower drawing and written argument of T10’s student.

A second sample that T10 shared was of a student that kept changing how he was organizing his groups (10/2 meeting transcript, line 871). T10 said that her students saw the diagonal pattern but then rearranged the towers using opposites (10/2 meeting transcript, lines 879,881). Then the same pair changed their strategy again and organized the towers by holding a constant (10/2 meeting transcript, line 883). When questioned by the instructor as to what type of proof the students used, another teacher responded by saying the students’ work was a proof by cases (10/2 meeting transcript, lines 892-893).

T10 shared a third student sample of work by putting the following argument on the screen and paraphrasing the student’s argument:

> We put them in different groups and there are no any other combinations to put in any of the groups. In group 6, there are only 3 blues and 1 yellow. In group 5, there are only 3 yellows and 1 blue. In group 4, there are 2 blue and 2 yellow in half. In group 3, they are a tower of the same colors and 2 they are in a pattern. In group 2, they are in a pattern. In group 1, tow colors are the same. We try and find more to find but no more towers. (10/2 meeting transcript, line 901)

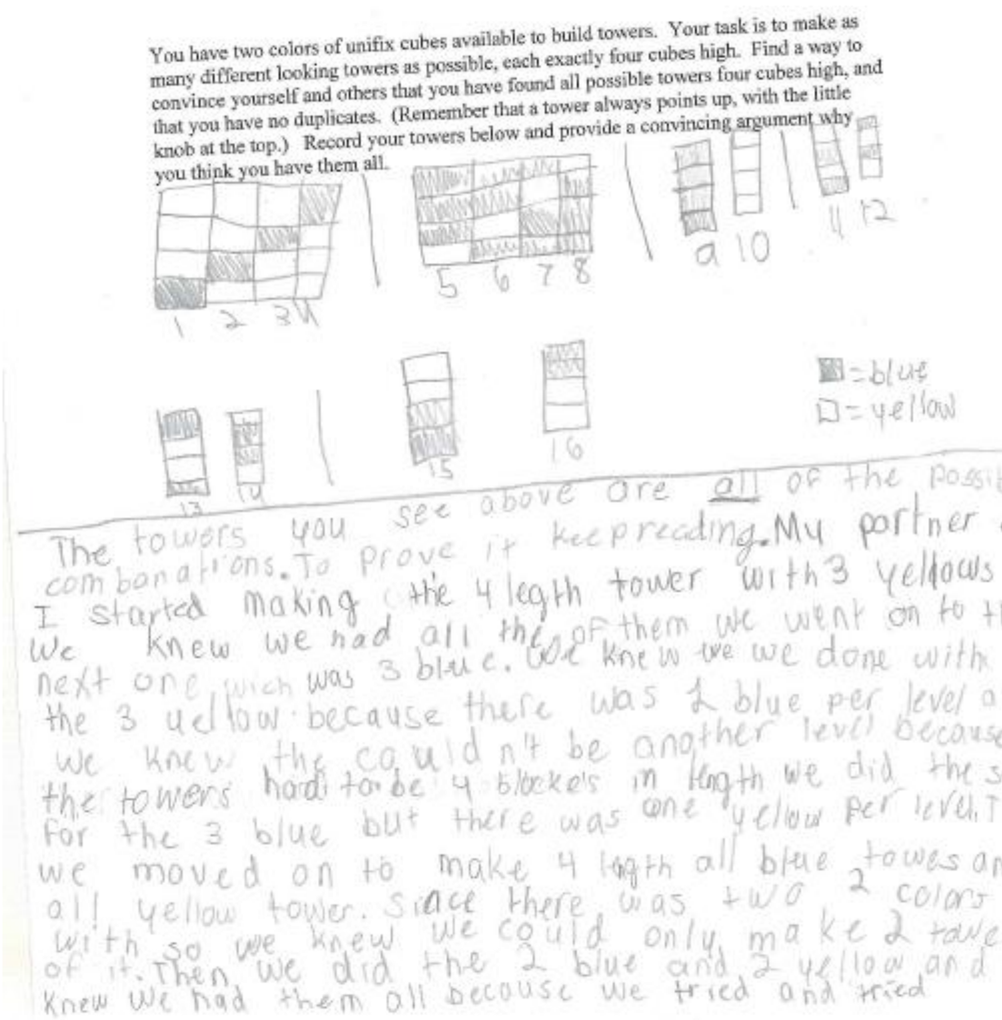



*Figure 4.19 T10's Cycle 1 Student Work Sample 1*

T10 was able to recognize the "stair case" which is defined as the elevator strategy (10/2 meeting transcript, line 913). After T10 shared her students' work, the instructor asked teachers to respond to the questions for the weekly on-line discussion.

### *4.5.2 On-line Discussion 10/3/13 to 10/8/13*

For the on-line discussion, the first two questions focused on the first cycle tasks. Teachers were asked to respond to the following questions on-line:

> 1. What kinds of questions did you ask when you had your children build 4 or 5-tall towers? Be specific - and then tell what that helped you learn about the mathematical thinking of your students.

> 2. When you implemented ***towers 4-tall*** in your classrooms, how did the strategies your children used to solve this problem look the same or different from the ones you used when you solved the ***5-tall towers*** with your colleagues?

*4.5.2.1 First Question Responses*

The first question asked teachers about the kinds of questions they asked their students when the students were building 4- or 5-tall towers. The question also asked teachers to be specific and tell what helped them to learn about the mathematical thinking of their students. Seven of the ten teachers said they asked their students about if they could arrange their towers differently or better. The responses of the seven teachers follow:

> I asked the students to explain how they arranged their towers, which was mostly in pairs of opposites. Then I would ask them if there were other ways they could arrange the towers. Some would keep them in the pairs but also arrange them in the groups according to how many of each color the towers had. (Unit 5 On-line discussion thread, line 4)

> I asked my students a variety of questions when building the 4-tall towers. Specifically, about the different arrangements they had made and how they then grouped them together. I asked why certain towers belonged in different groups and why they could not add any additional towers to each of the groups they had made. (Unit 5 On-line discussion thread, line 11)

> When I was asking the students why they were choosing to organize their blocks their way they were really helped them to become aware of their reasoning. (Unit 5 On-line discussion thread, line 14)

> I also asked students to explain to me why they were grouping towers in certain arrangements. This helped me understand how they were thinking about the problem and what relationships they saw in the blocks that they grouped together. (Unit 5 On-line discussion thread, line 22)

> When I had my students build 4-tall towers, I asked them to describe how they were grouping their towers. Most were grouping them with their opposites. This gave me insight into their thought process and how they tacked the problem. (Unit 5 On-line discussion thread, line 36)

> I asked if there a better way to organize your towers to prove to me that you made them all? Some stared at them and then said, no there is no other way to organize

> them. Some started moving the towers around and came to see the "staircase" pattern but couldn't formulate a convincing argument but they were able to say there is a pattern, which I was happy with, though it wasn't convincing. (Unit 5 On-line discussion thread, line 41)

> My students were able to create the towers and set them up as opposites. I had to ask them "Are there different ways to set up the towers? What are the ways that you could organize them?" They also had trouble explaining their reasoning about why they grouped them in different ways. I had to ask them to show me how they know there are no more ways. These 2 major parts have shown me that my students have a hard time clearly explaining steps to their thinking. They knew by creating opposite pairs that they had exhausted all options, but were not able to identify a way to organize them. (Unit 5 On-line discussion thread, line 44)

Five of the ten teachers asked their students how they knew they had found all the possible towers.

> When my students built the 4 tall towers I asked them how many towers they had. After they responded I asked if that was all of the towers and why. Most students had organized their towers in opposite pairs and explained to me that each possibility had an opposite. I asked if that was really a convincing argument as to why they had made ALL pairs of opposites and most recognized that it was not. (Unit 5 On-line discussion thread, line 1)

> I would also ask them how they knew they had all the towers. The most interesting response was from a student who said he would never know if sixteen was all the possible towers. Others would give a mathematical equation resulting in sixteen. (Unit 5 On-line discussion thread, line 4)

> When my students were building the towers, I asked them questions such as "can you make anymore?" "Do you have any duplicates?" "How can you convince me that you cannot make any more towers?" This showed me that my students need guidance of how they should be thinking during the process. In general, they have difficulties expressing their thinking. Asking them questions along the way helps them process and deduce what they are doing. (Unit 5 On-line discussion thread, line 7)

> Then I asked them how they knew they had all the towers. This was when I saw the students ran into conflict. They could not explain to me why they knew they had all the towers, other than to say that every new tower they built was a duplicate. I also recognized that my students did not understand what it meant to be a "convincing" argument. (Unit 5 On-line discussion thread, line 36)

> I then asked, do you think you have all the towers? All would say yes, once they got to 16; however, most replies to, how do you know you built them all? They

> were because we can't build anymore and they couldn't really tell me why. (Unit 5 On-line discussion thread, line 41)

*4.5.2.2 Second Question Responses*

For the second question, teachers were asked to compare and contrast the strategies used by the children and themselves when solving the 4-tall and 5-tall towers problem. Eight teachers said the children used the opposite strategy. The responses of the eight teachers follow:

> Most of my students made one tower and immediately made the opposite tower. (Unit 5 On-line discussion thread, line 1)

> Most of them [students] randomly built towers and opposites without any strategy. (Unit 5 On-line discussion thread, line 4)

> Just like my students, I used the idea of "opposites" to see if every tower had been created. Once I thought I had all 16 towers, I then started to group the towers according to similar characteristics (example: 3 red, 1 yellow). I was able to then see the relationship between the different combinations where as my students were stuck on the idea of opposites and relied on that argument for convincing me. (Unit 5 On-line discussion thread, line 11)

> For most of my students they did the problem by creating the opposite of each stack. (Unit 5 On-line discussion thread, line 14)

> This different from my student pairs who mostly used the pairs of opposites grouping which as we discussed, is not a convincing argument. (Unit 5 On-line discussion thread, line 22)

> I saw many of my students trying to make opposite pairs. (Unit 5 On-line discussion thread, line 30)

> When my partner and I first started building the towers we organized them in pairs as all of my students did the same. (Unit 5 On-line discussion thread, line 41)

> The 1st similarity is that they created opposites as the initial method to find all possible combinations. (Unit 5 On-line discussion thread, line 44)

Four teachers claimed that they solved the task using proof by cases. The responses of the four teachers that used proof by cases follow:

> I approached the problem by first doing one block of one color and four blocks of another color. Then I moved to two and three. (Unit 5 On-line discussion thread, line 1)
>
> When I completed the towers problem, I started with cases. I created all the towers one color, before moving to one of one color and three of the other. In this situation, I would change the position of the one color by one position each time. Then I created the towers that were two of each color, considering the different positions. (Unit 5 On-line discussion thread, line 4)
>
> When I did the task with my colleague, we worked though the cases of 0 red, 1 red, 2 red, etc. (Unit 5 On-line discussion thread, line 22)
>
> When I worked with my colleague, we built the towers by cases. We began with all red, 3 red, 2 red, 1 red, and 0 red. It was easier to formulate a convincing argument this way. (Unit 5 On-line discussion thread, line 36)

One teacher claimed to use a recursive strategy stating that "the popular method for building towers in both my classroom and with my colleagues was the recursive method-starting with one constant color and moving it, making all the possibilities" (Unit 5 On-line discussion thread, line 7).

**4.6 Summary**

With the completion of the aforementioned five session units, the first cycle of tasks came to an end. Throughout Cycle 1, teachers worked on the first cycle tasks, then participated in four thought-provoking on-line discussions, observed an in-district classroom visit working on the tasks, implemented the same tasks in their own classes, read literature and watched videos of other students working on the tasks, and shared their own students' work after implementing the tasks in their own classes. Teachers worked collaboratively and discussed their work as well as the work of their students. At the regional meeting on 10/2, the second cycle of tasks began with the teachers working in pairs on the pizza problem.

# Chapter 5 – Cycle 2 Session Summary and Analysis

This chapter is a summary and analysis for the second cycle of mathematical tasks. In this chapter, three session units of on-line discussion threads are analyzed. This chapter also has an analysis of the teachers doing the cycle two task at the regional meeting on 10/2/13 and the teachers sharing students' work at the in-district classroom visit and second regional meeting on 10/22/13.

## 5.1 Unit 5: Regional Meeting and On-line Discussion

Unit 5 was comprised of a regional meeting on 10/2/13 and an on-line discussion thread where teachers posted responses from 10/3/13 to 10/8/13. After the discussion on the first cycle tasks at the regional meeting, teachers worked on the Cycle 2 task.

### *5.1.1 Teachers Work on Task, Regional Meeting 10/2/13*

Teachers were asked to work in pairs to find the number of pizza combinations selecting from the following toppings: pepperoni, peppers, sausage and mushroom and convince each other of their solution (Landis, 2013). Once a solution was found, teachers were asked also to convince the instructor who was monitoring the teachers' work by circulating around the room and asking questions about the teachers' work. If successful, they were then asked to produce a written solution. If they did not successfully convince the instructor, teachers were invited to rethink their solution.

The instructor of the southern region cohort looked over the shoulders at the work of the first pair of teachers. Both teachers in this pair made a list of each pizza that could be made with the available toppings using an argument by cases (Unit 5 On-line discussion thread, lines 22, 36). One teacher in the pair (T1) wrote the topping word to identify choices for her list. The other partner (T2) began to write the topping word but then changed to write the first letter of each topping word. However, T2 stopped because she had to find a way to distinguish between peppers and pepperoni (10/2 meeting transcript, line 23).

Then the instructor of the southern region cohort stopped at a second pair of teachers to monitor their work. This pair of teachers was drawing circles to represent pizzas (10/2 meeting transcript, line 24). One teacher (T3) in the pair suggested that they use a tree diagram (Unit 5 On-line discussion thread, line 7). However, the other teacher (T4) suggested to his partner to hold one of the toppings constant and then add the other toppings to the constant topping (Unit 5 On-line discussion thread, lines 7). The strategy the second teacher pair used is defined as the controlling a variable strategy.

The instructor then stopped at a third teacher pair to monitor their work. Both teachers in this pair made a list of each pizza choice using a cases argument (Unit 5 On-line discussion thread, lines 1, 4). One teacher in the pair (T5) wrote a list using the topping word but then changed to write the first letter of each topping word with the exception of the letter E to represent the topping of pepperoni (10/2 meeting transcript, line 23). The other partner in the pair (T6) also made a list using the first letter of each topping to represent the topping using letter I to represent pepperoni. The instructor then asked the teachers "Is this connected anywhere? Can you see a connection" (10/2 meeting transcript, line 43) and left challenging the teachers to work on finding the connection.

The instructor then stopped at a fourth teacher pair. One teacher (T7) in the pair had written the words plain and all toppings at the top of her paper (10/2/13 meeting transcript, line 57). T7 then listed the rest of the pizzas with two toppings and three toppings using words numbering the pizzas to get a total of 16 pizza combinations (10/2/13 meeting transcript, line 57).

The instructor then went to a fifth pair of teachers where a cases argument was used for solving the pizza problem. One teacher (T9) wrote a list using the first letter of each topping word with the exception of the letters PR to represent the topping of pepperoni (10/2 meeting transcript, line 132). The other teacher in the pair (T10) made a list also by using the first letter of each topping word with the exception of the letter R to represent pepperoni.

T9 had 24 possible pizza combinations whereas T10 had 16 combinations. The teachers' answers were different because one teacher in the pair (T9) was counting cheese as a topping (10/2 meeting transcript, lines 79-84). The other teacher (T10) explained the following to the instructor and (T9):

> You have 5 toppings because if you count cheese it is like one of the toppings. Like yours is different because like for one topping you put the cheese where you are putting just pepperoni on it, just sausage, just mushrooms, just peppers or whatever. (10/2 meeting transcript, line 89)

Although T9 had started by counting cheese as a topping, she decided to cross out the combinations that had C's which represented cheese as a topping so that she could find the same number of pizza combinations as her partner (10/2 meeting transcript, lines 127-131).

The instructor left the fifth group and went back to the fourth group. The instructor saw a tree diagram on T8's paper and asked "Is it easier?" (10/2/13 meeting transcript, line 174). T8 replied with "Well it's going to be really big; I just know that many of my students are going to do that." (10/2/13 meeting transcript, line 175). The instructor asked the pair of teachers "Does this remind you of anything else?" and the teachers said the handshake and towers problems (10/2 meeting transcript, line 193). The instructor left the fourth group and told the pair to discuss with each other how the pizza problem reminds them of the handshake and tower problems (10/2 meeting transcript, lines 193).

The instructor then returned to the third group and saw that T6 created a chart with the letters P, M, S, and I were used to represent the toppings and X's where X represented no topping (10/2 meeting transcript, lines 195-197). When the instructor asked what her partner did, the partner replied with the following:

> So did the same thing. I had no toppings and then one topping and then two so I held a constant pepperoni and I did one of each. And then I moved to a new constant, and I put one of each in the positions and I moved to a new constant of mushrooms and we were missing one topping so we did this. (10/2 meeting transcript, lines 199-201).

The instructor responded to the pair by saying "very interesting" and then addressed the ten teachers by saying "For those of you who are finished, which most of you are…I challenge you to tell me how this problem reminds you of another problem in math. Something you have seen before something you have done before okay. And tell me what the connection is" (10/2 meeting transcript, lines 202).

After giving teachers a few minutes to discuss finding connections to the pizza problem with another problem, the instructor called for the attention of the ten teachers (10/2 meeting transcript, lines 202). Before discussing any solutions, the instructor first asked the teachers how many pizza combinations were found for a plain pie, then one-, two-, three-, and four-topping pies (10/2 meeting transcript, lines 202-211). The instructor then asked "If you looked at those numbers. It was 1, 4, 6, 4, and 1. Have you seen that before?" (10/2 meeting transcript, lines 212) and one teacher replied that it was

Pascal's Triangle (10/2 meeting transcript, lines 215). The instructor then asked the teachers "where did we see Pascal's triangle?" (10/2 meeting transcript, lines 216) and one teacher replied "the towers" (10/2 meeting transcript, lines 275). The instructor described both problems as isomorphic where "the towers problem building towers 4-tall selecting from 2 colors has the same exact mathematical structure as building pizzas selecting from four toppings" (10/2 meeting transcript, lines 234).

After this discussion, the instructor asked T6 from the third teacher pair to share her chart to solve the pizza problem (10/2 meeting transcript, lines 234). A picture of her work was taken and projected on the screen for the teachers. The chart had four rows and 16 columns but only eleven of the column cells were completed. T6 wrote the letters P, M, I, and S down the left margin next to the rows to represent the four toppings (10/2 meeting transcript, lines 234).

The instructor asked T6 to explain her chart (10/2 meeting transcript, lines 240). T6 began by saying "Well there are four possibilities. The first possibility is getting nothing so an X just means that it is not [a topping]." (10/2 meeting transcript, lines 241). The instructor asked T6 about five of the combinations she represented in her chart (10/2 meeting transcript, lines 250-268). Then the instructor asked the teachers how T6 built her towers and there was no response from the teachers (10/2 meeting transcript, line 270). So the instructor said to the teachers "She is actually keeping a constant, isn't she? She is keeping her peppers constant and she is adding the mushrooms and then she is adding the pepperoni and then she is adding the sausage." (10/2 meeting transcript, line 270).

Then the instructor decided to show the teachers four different pizza combinations using the Unifix cubes (10/2 meeting transcript, line 292-308). One of the questions that the instructor asked the teachers was “How can we build a tower that might look like that pepperoni pizza?” (10/2 meeting transcript, line 292) and one teacher replied “one red, three yellow” (10/2 meeting transcript, line 293). After the four examples were discussed, the instructor ended the meeting by reminding the teachers to implement the pizza task in their own classrooms and bring two or three samples of students’ work to discuss with the other teachers (10/2 meeting transcript, line 310-334) .

### *5.1.2 Unit 5: On-line Discussion 10/3/13 to 10/8/13*

For this discussion, teachers were instructed to watch two videos located on the VMC website: www.videomosaic.org. One video was called the Brandon Interview. The other video was called Pizza with 4 Toppings.

In the Brandon Interview video, Brandon was in the fourth grade and he shared his ideas about two problems. The following are the two problems that Brandon worked on with a partner and later discussed with the researcher:

> 1) Your group has two colors of Unifix Cubes. Work together and make as many different towers four cubes high as is possible when selecting from two colors. (Private Universe Project in Mathematics Workshops (PUP), Brandon interview [video], 1993)
>
> 2) A local pizza shop has asked us to help design a form to keep track of certain pizza choices. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms and pepperoni. How many different choices for pizza does a customer have? Find a way to convince each other that you have accounted for all possibilities. (Private Universe Project in Mathematics Workshops (PUP), Math pizza, Clip 2 of 2: Whole and Half Pizzas with Four Toppings [video], 1993)

Brandon explained his chart of 0’s and 1’s where 0’s represented no toppings and 1’s represented a topping (PUP, Brandon interview [video], 1993). Brandon also connected the pizza problem to the Towers problem (PUP, Brandon interview [video], 1993).

In the Pizza with 4 toppings video, 12 fifth-grade students worked together to solve two problems involving pizza combinations (PUP, Clip 2 of 2: Whole and Half Pizzas with Four Toppings [video], 1993). The two problems that the fifth-graders worked on are as follows:

> First problem statement: Capri Pizza has asked you to help design a form to keep track of certain pizza choices. They offer a standard “plain” pizza with cheese and tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms and pepperoni. How many choices does a customer have? List all possible choices. Find a way to convince each other that you have accounted for all possibilities. (PUP, Clip 2 of 2: Whole and Half Pizzas with Four Toppings [video], 1993)
>
> Second problem statement: Capri Pizza has asked us to help design a form to keep track of certain pizza sales. Their standard “plain” pizza contains cheese. On this cheese pizza one, two, three, or four toppings can be added to either half of the plain pie or whole pie. How many choices do customers have if they can choose from four different toppings (peppers, mushrooms, sausage and pepperoni) that can be placed on either a whole cheese pizza or half of a cheese pizza? List all possibilities. Show your plan for determining these choices. Convince us that you have accounted for all possibilities and that there could be no more. (PUP, Clip 2 of 2: Whole and Half Pizzas with Four Toppings [video], 1993)

Teachers were also assigned to read chapter 6 of Combinatorics and Reasoning (Maher, Powell, & Uptegrove, 2010) about the fifth-grade students’ justification, reasoning, notation, and strategies to solve the pizza problems. Teachers were also given on-line assignments to complete by 10/8/13 (Unit 5 on-line discussion thread, assignment page). Three questions were assigned by the instructor and one of the three questions focused on the second cycle task (Unit 5 on-line discussion thread, instructor question number 3). After watching the videos and reading the chapter, teachers were asked to respond to the following question on-line:

> What strategies did you use that were helpful when you solved the pizza problem with your colleague? Did you try things that weren't helpful in solving this problem? Be specific in telling what you tried. (Unit 5 on-line discussion thread, instructor question number 3)

The question asked the teachers about the strategies they used for solving the pizza problem. Six teachers claimed to use controlling a variable as a strategy for solving the pizza problem. The responses of the six teacher pairs to solve the pizza problem by controlling a variable as a strategy follow:

> Christine had the idea of keeping a constant. So we did all pizzas with peppers, all with mushrooms, all with pepperoni and all with sausage. We found as we eliminated an ingredient the number of possibilities were halving (just like the tower problem!). From there we decided to replicate what the towers would look like by having four possible spots. If the pizza did not occupy all of the spots with an ingredient we would put an X and if it did have an ingredient we would put the representation we came up with. (Unit 5 on-line discussion thread, line 1)

> We looked at having a constant. Starting with peppers, I listed the pizzas of one topping, two toppings, and three toppings containing peppers. I then moved to mushrooms, without using the peppers again since they had previously been listed. The pizzas with sausage were next, then the pizzas with pepperoni. Following this, I created what would look like towers, using the top block to represent peppers, second block to represent mushrooms, third for sausage, and fourth for pepperoni, placing an “x” in a position if that topping was not on the pizza. (Unit 5 on-line discussion thread, line 4)

> I definitely liked the strategy of picking a topping and having that as the constant. From there we added toppings to the constant topping. . (Unit 5 on-line discussion thread, line 7)

> As we got to three toppings it became harder to make sure we hadn’t duplicated any pizzas, so we considered holding 1 of the three toppings constant, and finding the pizza combinations that could be created by changing the other two toppings. (Unit 5 on-line discussion thread, line 22)

> We got a little confused when we got to 3 toppings, but we were able to find them all. We first found all the 3-topping pizzas with sausage and then moved on from there. So, we held the topping constant in order to find the solution. (Unit 5 on-line discussion thread, line 36)

> The strategy that we did use was keeping a topping constant and then making the possible combinations with 2 and 3 toppings. Then we would use another topping as a constant and create 2 and 3 toppings without repeating any combinations from the previous topping constant. (Unit 5 On-line discussion thread, line 44)

One teacher began the pizza problem by using the opposite strategy were she wrote plain and all toppings on the top of her paper (Unit 5 On-line discussion, line 14).. Then the teacher switched to a different strategy which was coded as 'other' and was described as any other strategy not previously defined (Unit 5 On-line discussion, line 14). Three additional teachers' responses were coded as using the 'other' strategy. The four teachers' responses follow:

> Together, we created an organized list according to the number of toppings on the pizza (plain, 1, 2, 3 or 4 toppings). We used different letters to represent the different toppings (example: P= peppers, S= sausage, M= mushrooms & R= pepperoni). After we created the list, we were able to see 16 different pizza combinations. After thinking about the problem, I attempted to use my initial thought of the tree diagram. After going back to make the tree diagram, I realized it was much harder than I had originally thought. I found myself making duplicate combinations and that it was much harder to follow and see the different pizza combinations. In the end, I think the organized list was the better approach. (Unit 5 on-line discussion thread, line 11)

> We both started off breaking the pizzas in to plain and all topping similar to beginning of the block activity. I also found that we both organized our answer so it was clear that they were no duplicates. (Unit 5 on-line discussion thread, line 14)

> The first instinct when faced with the pizza problem is to make a list. I found that writing out the entire word wasn't very efficient and my partner's labels of P, M, S, and R were much easier to use. After we determined that 16 pizzas could be made I tried to think of what my students would do and came up with a tree diagram with the four headings of P, M, S, and R and side labels of 1st pizza, 2nd pizza, and 3rd pizza (the fourth option is all four toppings so it doesn't need to be repeated at the bottom of every branch of the tree diagram). After completing the Pepper branch I realized I needed to make a list of the combinations and from that list I realized I needed to cross out the duplicates. This was a very long process and did not prove to be more efficient that listing the combinations in letter form. (Unit 5 on-line discussion thread, line 30)

> The most helpful strategy for me and my partner was to make a list of combinations of pizzas. The hardest part was coming up with an agreement as to whether cheese was a topping or not. Once we came to an agreement on that the list was simple and we both came up to the same conclusion of 16. At first we thought about a tree diagram; however, we felt that was a hard way to represent the combinations. (Unit 5 on-line discussion thread, line 41)

**5.2 Unit 6: On-line Discussion 10/9/13 to 10/15/13**

For unit 6, teachers were assigned to read an article called "Brandon's Proof and Isomorphism" (Maher & Martino, 1998). The assigned reading is about Brandon's reasoning, strategies, and notation used as he solved the tower and pizza problems and how he revised his work with each new problem (Maher & Martino, 1998). After reading the article, teachers were asked to respond to the following question on-line:

> In the chapter, Brandon's Proof and Isomorphism, we see that skillful teacher questioning can help a student think more deeply about a mathematical idea. What kinds of questions did this teacher ask to learn more about the mathematical thinking of her students? (Unit 6 on-line discussion thread, instructor question)

Teachers were asked to discuss the kinds of questions the teacher in the assigned reading asked to learn more about the mathematical thinking of her students. Six different types of questioning were defined in chapter 3 of this dissertation. The six types of questioning defined in this research are questions for explanation, justification, connection, probing, other solutions, and generalization.

Five of the types of questioning were recognized in the teachers' responses except generalization types of questions. Nine of the ten teachers described explanation questions. The responses of the nine teachers follow:

> The teacher asked questions that deepened the understanding of what the children were doing (Unit 6 on–line discussion thread, line 1)

> The teacher asked questions that required Brandon to explain his strategy of using "0"s and "1"s. (Unit 6 on–line discussion thread, line 8)

> The teacher starts out asking general questions about her students' work. She may know the answers to some of these problems, but its beneficial to ask so that the students are conscious of what they are doing and self-monitoring their problem solving. (Unit 6 on–line discussion thread, line 14)

> His teacher asked him what his 1's and 0's meant, this allowed for him to explain that 1 meant yes for that particular topping and 0 meant no. (Unit 6 on–line discussion thread, line 22)

> In the article, the teacher started by asking what the student was doing to complete the problem to find out how the student approached the problem. (Unit 6 on–line discussion thread, line 35)

> The teacher asked questions that were carefully chosen to ask for clarification about Brandon's thinking, such as "What are you doing here, Brandon?", "What does that mean?", "What I don't understand is…?" I think this helped Brandon to explain his thinking, and sometimes having a student verbalize their reasoning helps for them to be better able to clear up what they are thinking. (Unit 6 on–line discussion thread, line 38)

> First she wanted to know what Brandon was doing and what the 0's and 1's represented. (Unit 3 on–line discussion thread, line 39)

> The 2 questions that the teacher asked were: What are you doing here? What does that mean? The first question was leading the student to start thinking about how to verbalize the mathematical process that the student was using. I think we all have seen that this is a skill that is difficult for most students and needs development. The next question was to get the student to explore more deeply what they had said. Through the first 2 lesson studies I have seen that when asked to explain what they did students state the steps that they took, and don't really give justification or reasoning as to why they did something. (Unit 6 on–line discussion thread, line 40)

> The one question I think is important to ask to grasp their beginning stages of their thinking is, "Tell me what you have done so far." (Unit 6 on–line discussion thread, line 43)

Nine of the ten teachers recognized justification types of questions. The responses of the eight teachers that recognized justification types of questions in their responses follow:

> The teacher asked questions that deepened the understanding of what the children were doing & thinking. (Unit 6 on–line discussion thread, line 1)

The teacher then asked questions that encouraged Brandon to explain his reasoning and how the “0”s and “1”s related to pizza. (Unit 6 on–line discussion thread, line 8)

This teacher asks the student to clarify different things that do not make sense by asking “how do you know” or “why did you do that” or ever “I’m not sure what you are saying” rather than assuming she understands what the student means. (Unit 6 on–line discussion thread, line 14)

When asking how he knew what to do next and how he knew that he didn't have repeats, he was able to reply that if there was a 1 in the column that matched any other combination he would be able to know that he made a duplicate. (Unit 6 on–line discussion thread, line 22)

I especially thought her asking "Why?" and pushing Brandon to explain his process were important. It seems so simple to ask why but it opens up an opportunity for students to express themselves as best they can. (Unit 6 on–line discussion thread, line 27)

The teacher asked why there could be no more pairings with pepperoni once the student went through the options with that topping and moved to mushrooms. This ensured the student knew why there were no more options with pepperoni & why they could ignore the topping as they made the rest of the combinations. (Unit 6 on–line discussion thread, line 35)

Brandon’s teacher also asked him “Why?” many times so that Brandon could further explain something he had already started to explain, and add to his justification. (Unit 6 on–line discussion thread, line 38)

The 2 questions that the teacher asked were: What are you doing here? What does that mean? The first question was leading the student to start thinking about how to verbalize the mathematical process that the student was using. I think we all have seen that this is a skill that is difficult for most students and needs development. The next question was to get the student to explore more deeply what they had said. Through the first 2 lesson studies I have seen that when asked to explain what they did students state the steps that they took, and don't really give justification or reasoning as to why they did something. (Unit 6 on–line discussion thread, line 40)

Then after they have worked further on the activity asking them to explain/convince me of their work is much easier. I feel it is much easier because they will probably be more confident at this point because they answered the prior question and have more understanding of the task. (Unit 6 on–line discussion thread, line 43)

Six of the teachers recognized probing types of questions. The responses of the six teachers that recognized probing types questions follow:

> Every time the Brandon answered, the teacher pursued with a question to get more details and encourage further thinking and explanation (Unit 6 on–line discussion thread, line 1)

> The teacher then had him explain and give specific examples such as “0, 0, 0, and 1” meant a pizza with only one topping. (Unit 6 on–line discussion thread, line 8)

> When the teacher "probed further" with Brandon to get him to see a clearer explanation he was able to see that he could reorganize his findings by toppings i.e.: 1 topping, 2 topping, 3 topping, etc... (Unit 6 on–line discussion thread, line 22)

> She probed for more information but did not give away what direction Brandon should take his answer. (Unit 6 on–line discussion thread, line 27)

> From there, the questions developed based on what the student was doing. For example, when the notation of 1 and 0 was introduced, the teacher asked what they represented to understand how the student was using the notation. Then, after seeing the order of the pizzas created, the teacher asked why there could be no more pairings with pepperoni once the student went through the options with that topping and moved to mushrooms. (Unit 6 on–line discussion thread, line 35)

> She then questioned how some of the pizzas would be represented using this notation. For instance, no topping would be 0, 0, 0, and 0. (Unit 6 on–line discussion thread, line 39)

Three of the teachers recognized types of questions that exposed students to other solutions. The responses of the three teachers that recognized types of questions that exposed students to other solutions follow:

> The teacher also asks about other possible solutions or situations that the student might not have thought of, like “Could we do it this way” or “What if we did this” or “Have you considered this?” (Unit 6 on–line discussion thread, line 14)

> Then she asked Colin if his process was similar to Brandon's. (Unit 6 on–line discussion thread, line 39)

> I also liked the suggestion to have groups compare with each other to similarities and differences in their arguments. (Unit 6 on–line discussion thread, line 40)

Three of the teachers recognized connection types of questions. The responses of the three teachers that recognized connection types of questions follow:

> The teacher challenged Brandon when asking him “if the pizza problem reminded him of another problem?” This question got Brandon really thinking. He was able to recognize that the pizza problem was just like the tower problem. He then started to physically build the towers and their “opposites”. He was able to then see how the towers he created related to his chart containing “0”s and “1”s. By using proof by cases, Brandon was able to see how the two problems were related. He saw how he arranged and solved the pizza problem ended up being similar to the way he had built the different towers during the tower problem. (Unit 6 on–line discussion thread, line 8)

> This then led to the teacher asking if he thought this was similar to any other problem they did and he was able to make the connection to the tower problem. When he started building them he did opposites as when he first solved the problem but the teacher questioned his organization of the towers and Brandon stared and was able to see the relationship to the pizza problem based off of his chart. That one color, red, would be zero and yellow would be 1. He went on to explain and reorder the towers according to his chart. (Unit 6 on–line discussion thread, line 22)

> The teacher encouraged him to make a connection to a different problem he had previously worked on with “In any way does it remind you of any of the problems we’ve done?” She also asked questions that directed his attention to certain aspects of his arrangements of the colors when making connections to the towers problem, such as asking him what color he would focus on and how he was describing the “one’s” tower. (Unit 6 on–line discussion thread, line 38)

### 5.3 Unit 7: On-line Discussion 10/9/13 to 10/15/13

For unit 7, teachers were assigned three on-line questions. Teachers were asked to respond to the following questions on-line:

> 1. When you implemented the pizza task, selecting from 4 toppings, what kinds of strategies did your children use to solve the problem? Did any of their solutions look similar to the way you solved the problem with your colleagues?
>
> 2. Talk about one or two students' solutions that you thought were especially neat.
>
> 3. When asked if this problem reminds them of any other, how did your students respond? Did any of your students see the isomorphism between the two problems - building 4-tall towers selecting from 2 colors and building pizzas selecting from 4 toppings? (Landis, 2013)

***5.3.1 First Question Responses***

The first on-line question had two parts. In the first part, the question asked teachers what kinds of strategies were used by their own students. In the second part, teachers were asked how the strategies were similar to the way the teachers had approached the task.

*5.3.1.1 Part 1 First Question Responses*

Two teachers recognized the controlling for a variable strategy that their students used. The two teachers' responses follow:

> Many of them held a constant and moved forward with the remaining options. (Unit 7 On-line discussion thread, line 40)

> She had kept a constant to create her groups. (Unit 7 On-line discussion thread, line 44)

The other teachers responded by explaining the representations used to solve the pizza problem.

Various types of representations used to solve the pizza problem were mentioned by the teachers. Eight of the teachers said that their students made lists. Of these lists, three responded that the lists were organized. The three teachers that claimed their students used an organized list follow:

> Some other students began by making a list similar to that of my colleagues and mine. They came up with different combinations; however, some were organized and some were just listing. (Unit 7 On-line discussion thread, line 22)

> Some students made lists that were organized by the number of toppings, others organized by pizzas that included a particular topping, and others were completely unorganized. (Unit 7 On-line discussion thread, line 35)

> Many of my students used the strategy of an organized list in order to see the different pizza combinations. (Unit 7 On-line discussion thread, line 41)

Five teachers said that their students drew diagrams of pizzas to solve the pizza problem. The responses of the five teachers follow:

> Several other students drew and labeled the pizzas (Unit 7 On-line discussion thread, line 30)
>
> Several of my students also thought it was important to draw a pizza to go with every outcome they found. (Unit 7 On-line discussion thread, line 35)
>
> Others drew diagrams of pizzas. (Unit 7 On-line discussion, line 40)
>
> I did have one pair of students that attempted to draw the different pizza combinations by drawing circular pizzas and drawing the different toppings on each. (Unit 7 On-line discussion thread, line 41)
>
> One group of students actually drew out slices of pizza and the toppings. They quickly realized they could just write the words in the pizza instead of drawing them out so they changed to a form of notation. They had no method they just started drawing slices and whatever combination they came up with at that slices was what they used. (Unit 7 On-line discussion thread, line 44)

Six teachers said that their students used tree diagrams to solve the pizza problem. The responses of the six teachers follow:

> A few students started with a tree diagram (which I thought some might do) and quickly got frustrated with the size of it. (Unit 7 On-line discussion thread, line 19)
>
> Most students began by drawing tree diagrams and when I walked around and questioned them they realized that they began to make doubles using that strategy. Once this happened a few became frustrated and tried to figure out a different way to organize their work; however, they could not think of anything different so they just kept making a tree diagram. (Unit 7 On-line discussion thread, line 22)
>
> Some students created tree diagrams, but most who started with this organization abandoned it. Students used a variety of letters or some used abbreviations to represent the different toppings. (Unit 7 On-line discussion thread, line 28)
>
> I had two students who made tree diagrams, which I was very impressed with. (Unit 7 On-line discussion thread, line 30)
>
> Others tried to use tree diagrams. (Unit 7 On-line discussion thread, line 40)

> I even had one pair of students try using a tree diagram, but after attempting to make the diagram, they realized that they were repeating different pizza combinations and altered their strategy to an organized list. (Unit 7 On-line discussion thread, line 41)

*5.3.1.2 Part 2 First Question Responses*

For the second part of the question, teachers were asked if their students' strategies were similar to the strategies the teachers used. Three teachers said that some of their students had similar strategies (Unit 7 On-line discussion thread, lines 22, 30, 35), two teachers said that many or several had similar strategies (Unit 7 On-line discussion thread, lines 40, 41), two teachers said that few students had similar strategies (Unit 7 On-line discussion thread, lines 1, 40), and one teacher claimed one student used control for a variable which was the same as the teacher's way (Unit 7 On-line discussion thread, lines 44).

***5.3.2 Second Question Responses***

The second on-line question asked teachers to "talk about one or two students' solutions that you thought were especially neat" (Landis, 2013). Three of the teachers said that their students used letters to represent the pizza toppings. The three responses of the teachers that said their students used letters to represent the pizza toppings follow:

> I usually do not get a lot of work from him. However, with this problem, he dove right in. He used letters to describe his pizzas. (Unit 7 On-line discussion thread, lines 1).

> One pair of students used a, b, c, d, and e to represent the pizza toppings which I thought was really neat. This was a better way than I thought to do because no letter is the same. (Unit 7 On-line discussion thread, lines 5).

> One student's answer that I particularly liked was an 8th grader that drew circles and gave them letter labels to correspond to the toppings each pizza could have. He edited his work a few times and ended up organizing them in one topping pizzas, two topping, pizzas, three topping combinations, and four toppings. He used one of his partner's ideas of drawing circles and another

> partner's idea of using letters for labels to combine them to create the correct number of pizza combinations. (Unit 7 On-line discussion thread, lines 19).

Three of the teachers said that their students' solutions were neat because they used proof by cases.

> He edited his work a few times and ended up organizing them in one topping pizzas, two topping, pizzas, three topping combinations, and four toppings. He used one of his partner's ideas of drawing circles and another partner's idea of using letters for labels to combine them to create the correct number of pizza combinations. (Unit 7 On-line discussion thread, lines 19).

> I kept questioning them to look to see if they can organize their lists in a different way to make more sense out of what they are doing. After this question some were able to see they were doing one topping, two topping. (Unit 7 On-line discussion thread, lines 22).

> One pair of students that I found their solutions to be neat was a pair of girls in my class. They were quickly able to create the sixteen different pizza combinations. When asked how they created the different pizza pies and why they knew they had them all, they were able to verbally explain to me that they used a mathematical system. They demonstrated this "system" using arrows and were able to exhaust all possible combinations using this method. They simply listed all of the different pizzas that contained only one topping and then used that list to create two, three and four topping pizzas. (Unit 7 On-line discussion thread, lines 41).

Two teachers thought that their students' solutions were neat because they controlled for a variable. The responses of the two teachers follow:

> I had one student who held a constant topping when creating his lists of pizzas. He first wrote all the pizzas that contained pepperoni, starting with 4 toppings, then 3, then 2 and then just pepperoni. He did the same for sausage, but did not include and pizzas that would have pepperoni because he already listed those in his previous list. He then went to just peppers, just sausage, and plain. He only came up with 15 pizzas though. He forgot the pizza that had both peppers and sausage in his list. (Unit 7 On-line discussion thread, lines 35).

> I was so happy to see the organization in this student's problem. She started with plain and then all single toppings. Then she did all 4, then 3 toppings, then 2 toppings. She had shown me her work when there were 14 combinations. She had kept a constant to create her groups. (Unit 7 On-line discussion thread, lines 44).

One teacher thought her student's solution was neat because the student created a table. The teacher responded "The most effective use of a table came with a pair that used column headings as toppings and had four numbered rows. Students would put an X where there existed a topping." (Unit 7 On-line discussion thread, lines 28). Another teacher thought it was neat that her student used a recursive method and responded with "One strategy that is found was interesting and surprising is one of my students used the recursive method. I was surprised because I thought that using that method without a manipulative would be harder to do" (Unit 7 On-line discussion thread, lines 30). A third teacher said the following about her student's work:

> I was very surprised by some of the students' work. Many of the pairs of students considered a pizza that was half cheese and half one topping to be an option different than a pizza with that one topping, as a result they had more options and had difficulty getting all the outcomes. I also had students who considered each piece of the pizza to be a different part and therefore you could have many more outcomes because a pizza with one slice peppers and the rest sausage is different than a pizza with half peppers and half sausage. When asked if they ever order a pizza like that, the student told me his father used to have a pizzeria and would make pizzas like that. As a result, they were overwhelmed by the number of possible outcomes if each slice was looked at individually. (Unit 7 On-line discussion thread, lines 40).

### *5.3.3 Third Question Responses*

The third on-line question had two parts. In the first part, teachers were asked how their students responded when the students were asked if the pizza problem reminded them of another problem. In the second part, teachers were if their students see the isomorphism between the 4-tall towers problem selecting from two colors and the building pizzas selecting from four toppings problem.

*5.3.3.1 Part 1 Responses*

Five teachers said their students were reminded of another problem. Four of the five teachers said that their students were reminded of the 4-tall towers problem. The four teachers' responses follow:

> One student who was able to make the connection started off by saying, "I know the answer to this is 16," I asked him why he knows that and he said, "Because this is like the tower problem." I then said ok why you think that, he stated, "They are alike because the towers we had to build four high and the pizzas we get to choose from four toppings." I was convinced that he did see the isomorphism between these problems so then I asked him if they were related he would be able to build the pizzas using the towers, and he replied yes and started to build. (Unit 7 On-line discussion thread, line 22)
>
> Not many of my students made the connection between the two problems. I had a couple students who made the connection, and mainly related the two questions together because they were making different combinations in each activity. (Unit 7 On-line discussion thread, line 30)
>
> When asked what problem this was similar to, the students discussed the tower problem but none of the students mentioned the relationship between the two before being asked and they did not initially see any relationship between the two except that the number of outcomes was the same in both problems. (Unit 7 On-line discussion thread, line 40)
>
> When I asked the student discussed in number 2 if this reminded her of the tower problem, she said "Oh that's why there had to be 16 because it's like the tower problem, 4 cubes and 4 toppings. (Unit 7 On-line discussion thread, line 44)

One of the five teachers said that her students were reminded of other problems that were not related to the towers problem. The response of the teacher that said her students were reminded of other problems not related to the towers problem follows:

> When I asked my class if this problem reminded them of any other problem, several students suggested word problems that involve multiple combinations. One student gave me an example of different types of sandwiches, if you have two types of bread and three types of meat. This idea led another student to give me an example of where they had three ice cream flavors and four toppings, how many different sundaes could they make. (Unit 7 On-line discussion thread, line 41)

*5.3.3.2 Part 2 Responses*

For the second part of the question, teachers were asked if their students' saw the isomorphism between the 4-tall towers problem and the pizza problem. Three teachers said that their students saw the isomorphism between the 4-tall towers and pizza problems. The responses of the three teachers follow:

> One student who was able to make the connection started off by saying, "I know the answer to this is 16," I asked him why he knows that and he said, "Because this is like the tower problem." I asked why and he stated, "They are alike because the towers we had to build four high and the pizzas we get to choose from four toppings." I was convinced that he did see the isomorphism between these problems so then I asked him if they were related he would be able to build the pizzas using the towers, and he replied yes and started to build. (Unit 7 On-line discussion thread, line 22)
>
> Not many of my students made the connection between the two problems. I had a couple students who made the connection, and mainly related the two questions together because they were making different combinations in each activity. (Unit 7 On-line discussion thread, line 30)
>
> When I asked the student discussed in number 2 if this reminded her of the tower problem, she said "Oh that's why there had to be 16 because it's like the tower problem, 4 cubes and 4 toppings. (Unit 7 On-line discussion thread, line 44)

**5.4 Unit 8: Regional Meeting 10/22/13**

Unit 8 was comprised of an in-district classroom visit at a regional meeting on 10/22/13. Ten teachers met at a New Jersey middle school to observe an implementation of the second cycle task with the current students of one of the teachers. After the in-district classroom visit, the teachers discussed students' work for the second cycle task.

***5.4.1 Discussion of In - District Classroom Visit***

For the second in-district classroom visit, the Cycle 2 task was implemented with twenty-three sixth-grade current students of a teacher. The sixth-grade current students were sitting in individual student desks that were pushed together in pairs. They were asked to convince their partners of their solutions and write down the solutions after convincing one of the researchers.

After the current students left, the instructor held a debriefing meeting with the ten teachers from the southern region group to discuss the students' solutions from the in-district classroom implementation using an IPad to take pictures of the students' work and project it on the screen. The teachers first discussed two students' work that the instructor said was "very different, I don't think I've seen that before either" (10/22/13 meeting transcript, line 1). Figure 5.1 shows the student's work.

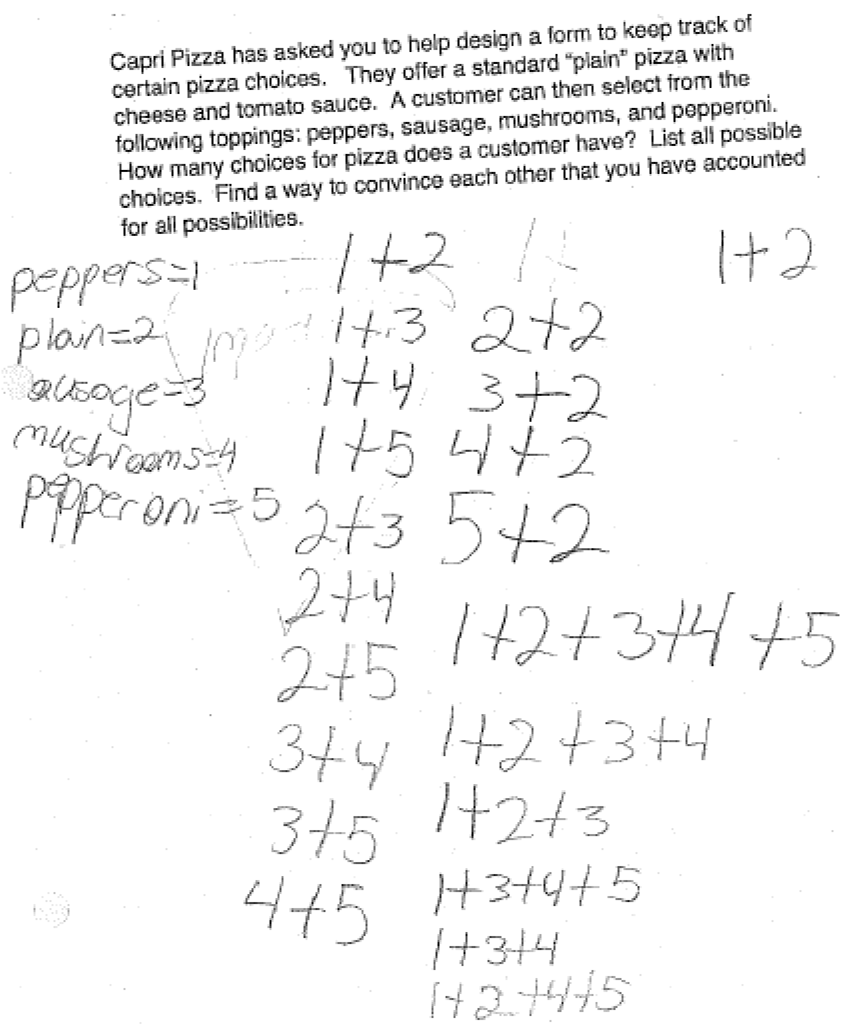

Capri Pizza has asked you to help design a form to keep track of certain pizza choices. They offer a standard "plain" pizza with cheese and tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many choices for pizza does a customer have? List all possible choices. Find a way to convince each other that you have accounted for all possibilities.



*Figure 5.1 First Discussed Student's Work from 10/22/13 Class Visit*

The students used numbers to represent the toppings. They assigned the following number to represent the toppings: peppers=1, plain=2, sausage=3, mushrooms=4, pepperoni = 5. Then the students had written addition expressions vertically on their paper such as 1+1, 1+2, 1+3, 1+4, 1+5, 2+1, 2+2, and continued the pattern (10/22/13 meeting transcript, line 1).

The instructor asked the teachers “How did it get them into trouble?” (10/22/13 meeting transcript, line 1). One of the teachers said that she had questioned one of the students in the pair by pointing to 1+1and asking “does that mean peppers and peppers” (10/22/13 meeting transcript, line 3). The teacher also said “Then she was like, no that just means it only has peppers. Then she was like, maybe I should just erase the second two and only put one two. And I said that’s a good idea” (10/22/13 meeting transcript, line 3).

The instructor asked the teachers “What other group did you find interesting?” (10/22/13 meeting transcript, line 6) and one teacher replied about a chart that was made by two boys (10/22/13 meeting transcript, line 10). The instructor asked “What got them into trouble?” (10/22/13 meeting transcript, line 13). One of teachers said “their ordering at first, because they were just doing them all random” (10/22/13 meeting transcript, line 10). The instructor said the teacher was correct and that the students were missing the three-topping pizzas because they were trying to keep track of their pizzas in their heads until one of the circulating teachers asked them to write it down (10/22/13 meeting transcript, line 15).

The instructor asked if there was another group that the teachers found interesting (10/22/13 meeting transcript, line 19). One teacher replied that one group of girls had struggled to solve the pizza problem because they had used a tree diagram and were having “a lot of doubles” (10/22/13 meeting transcript, line 22). Then, the instructor asked the teachers if there was another group that struggled (10/22/13 meeting transcript, line 25).

One teacher said that one pair of students struggled because “They counted a plain, a cheese, a sauce, and a nothing” where the nothing pizza was a pie with no red sauce (10/22/13 meeting transcript, line 25). In this case, the instructor said to the teachers that “their answer would be much bigger” (10/22/13 meeting transcript, line 52). After eliminating many combinations, the students found there were seventeen pizzas because the students decided to count a pie with no sauce as an option (10/22/13 meeting transcript, line 55).

At this point, the instructor requested to see the students’ work projected on the screen (10/22/13 meeting transcript, line 55). The student’s work that was shown first was shown in figure 5.1 above and was one of the students that used numbers as notation for the pizza toppings with addition expressions to represent each pizza (10/22/13 meeting transcript, line 1). The teachers previously discussed the student’s work at the beginning of the meeting (10/22/13 meeting transcript, line 1).

However, the instructor saw that the student erased the original work and replaced it with revised work (10/22/13 meeting transcript, line 76). The instructor stressed the importance of not allowing students to erase work because “if you didn’t see what they were doing and then you just saw this you would miss everything that they eliminated. And that really is important to know where they started and where they are going” (10/22/13 meeting transcript, line 76).

The second student's work that was shown was one of the students that made a chart (10/22/13 meeting transcript, line 81). Figure 5.2 shows the student's chart. The chart showed the four toppings on the top written as words and X's in the cells to represent a topping on the pizza (10/22/13 meeting transcript, line 81).

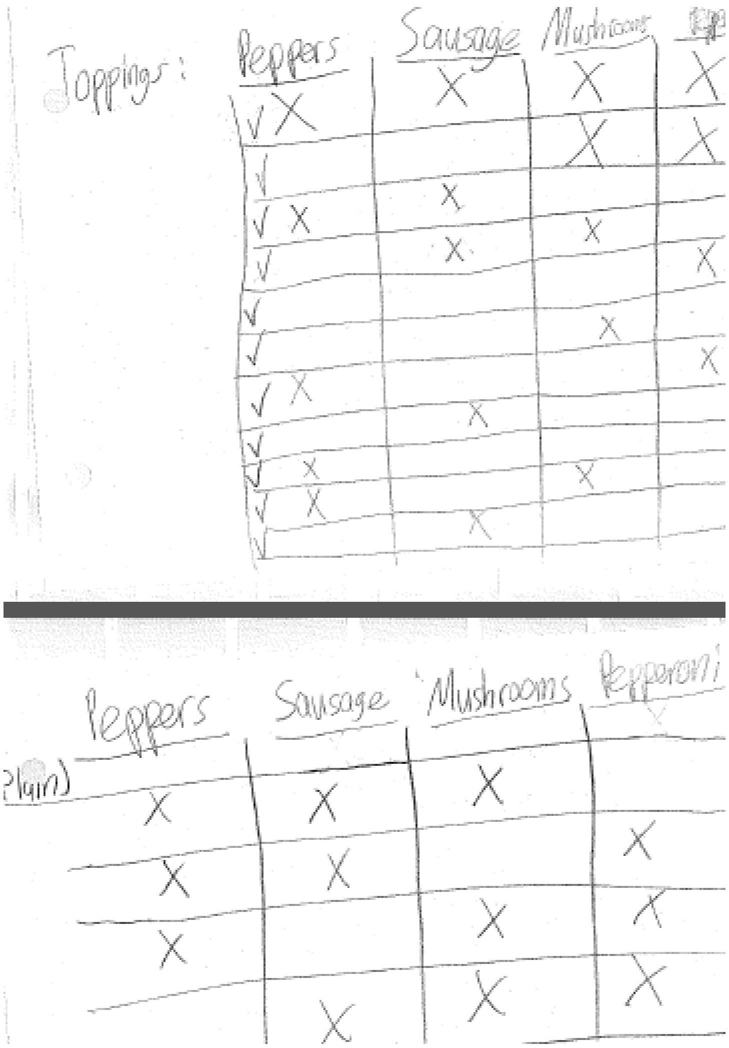


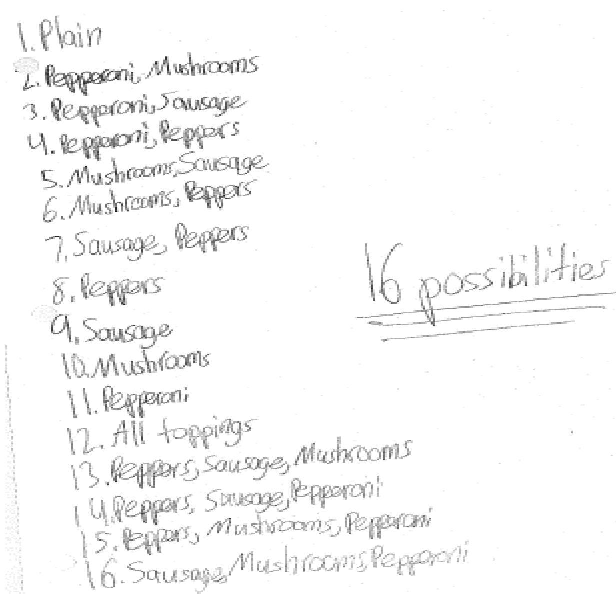

*Figure 5.2 Chart from Second Discussed Student Work from Class Visit 10/22/13*

The instructor asked the teachers if any of their students made a chart and one teacher replied "I had one but he started making this chart and I was like, this is really great like where are you going with this, explain it to me, and then all of a sudden, he was like said, no I didn't like this. And then he wrote ignore on it and started doing something completely different" (10/22/13 meeting transcript, line 83). The instructor then asked the teacher to share how the student switched the work with the teacher group (10/22/13 meeting transcript, line 84).

The student had started with a chart and then switched to an organized list using proof by cases. The student wrote 16 pizza combinations and the following argument:

> First, we looked at the pizzas with only one-topping and got four different pizzas. We know this is right because there are only four toppings. Second, we looked at the pizzas with two-toppings and we got six pizzas. We know this because we took the pepperoni and grouped it once with each of the other toppings. Then we took the mushrooms and grouped it once with the other toppings, except for pepperoni, because it was already grouped with it. Then we took sausage and only grouped it with peppers because it was already grouped with mushrooms and pepperoni. Third, we looked at the three-topping pizzas and got four pizzas. Since there are only 4 toppings, we took one topping off the pizza each time. (10/22/13 meeting transcript, lines 112, 123)

The instructor said that it was "a brilliant way to find the three-topping pizzas" (10/22/13 meeting transcript, line 128). Figure 5.3 shows the student's list.


1. Plain
2. Pepperoni, Mushrooms
3. Pepperoni, Sausage
4. Pepperoni, Peppers
5. Mushrooms, Sausage
6. Mushrooms, Peppers
7. Sausage, Peppers
8. Peppers
9. Sausage
10. Mushrooms
11. Pepperoni
12. All toppings
13. Peppers, Sausage, Mushrooms
14. Peppers, Sausage, Pepperoni
15. Peppers, Mushrooms, Pepperoni
16. Sausage, Mushrooms, Pepperoni

16 possibilities

*Figure 5.3 Student List from Second Sample Classroom Visit*

One teacher said that one student pair "started with holding peppers constant and they did all of the two-topping pizzas with peppers, but instead of moving on and continuing, they kept peppers and then did three-toppings with peppers they got and they kept peppers with four-topping." (10/22/13 meeting transcript, line 146). The instructor replied "So, they got 8, right?" (10/22/13 meeting transcript, line 147) and the teacher said "Yeah, then they totally ignored the peppers and the sausage for the two-toppings and three-toppings." (10/22/13 meeting transcript, line 148). The instructor then transitioned into facilitating a discussion where the teachers shared the pizza-problem solutions of their own students' work (10/22/13 meeting transcript, line 159-161).

### *5.4.2 Discussion of Students' Work, Regional Meeting 10/22/13*

T6 was the first to share her students' work from a sixth-grade gifted class. The student used a tree diagram to find 16 possible pizza combinations but could not provide a convincing written argument (10/22/13 meeting transcript, line 162).

The next example shared by T6 was from a girl. T6 read the following student's written argument to the teachers:

> There are four different toppings and there's four mixes. So like you could...there's a potential to have 4 things on a pizza. So, 4 times 4 is 16, and then plus one is the plain, so it's 17. (10/22/13 meeting transcript, lines 189)

The student incorrectly used the rule strategy. This student's work sparked a discussion facilitated by the instructor about mathematics that makes sense. The instructor said with this student's argument, "you will not be able to generalize pizzas with three toppings or five toppings" (10/22/13 meeting transcript, line 199).

T6 also shared another student's work that showed a list of sixteen pizza combinations using the letters PE for peppers, S for sausage, M for mushrooms, PEP for pepperoni, and P for plain (10/22/13 meeting transcript, line 200). T6 recognized that the student solved the pizza problem by controlling for a variable (10/22/13 meeting transcript, line 200). T6 read the following argument out loud to the teachers:

> We organized the choices by toppings as we went on. When we got to a new topping, we took out all of the duplicates from the other toppings. For example, we started off, out with all of the pepper combinations. There were 8 of them. When we got to the sausage, there were only 4 combinations because there were 4 duplicates from the pepper. We did the same thing for mushrooms and pepperoni. The only thing left to do was to add one plain to our list, which we added. And then they said we got 16. (10/22/13 meeting transcript, line 232).

When the instructor asked the teachers if this was a convincing argument, one of the teachers replied "it's not" (10/22/13 meeting transcript, lines 233-237). Figure 5.4 shows a list the student provided.

Plain = P    Peppers = PE    Sausage = S    Mushroom = M
Pepperoni = PEP
1 P    16 PEP
2 PE
3 PE, S
4 PE, M
5 PE, PEP
6 PE, S, M
7 PE, S, PEP
8 PE, PEP, M
9 PE, S, M, PEP
10 S, M
11 S, PEP
12 S, M, PEP
13 S
14 M
15 M, PEP

*Figure 5.4 T6's Cycle 2 Pizza Combinations List, Student Work Sample 3*

T1 was the second teacher to share her students' work. The student used the letters A through D to represent toppings and then used E to represent a plain pizza (10/22/13 meeting transcript, lines 252). T1 said the student made a list of 24 pizzas but "she didn't do any kind of pattern" (10/22/13 meeting transcript, lines 258). Figure 5.5 shows the student's work.

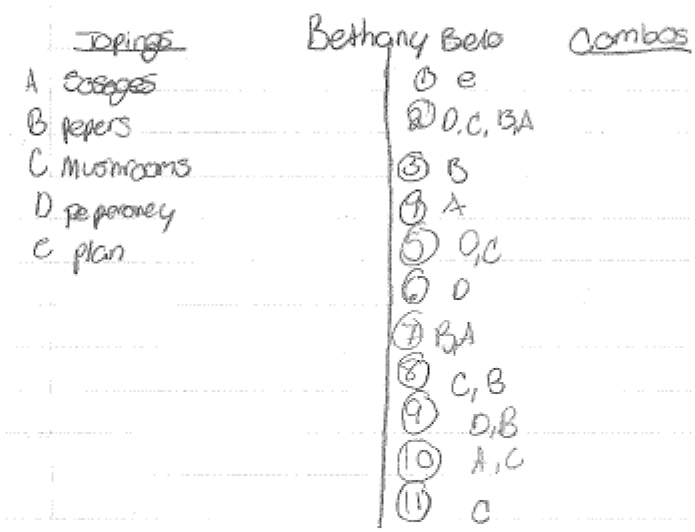



*Figure 5.5 T1's Cycle 2 Student Work Sample 1*

T1 shared a second example of students' work where the student found fourteen pizza combinations (10/22/13 meeting transcript, lines 264). The student made an organized list by writing the words of the toppings in separate boxes to represent each pizza combination. T1 recognized the student controlled for a variable by holding the mushroom topping constant (10/22/13 meeting transcript, lines 276).

T5 was the third teacher and shared two samples of students' work. In the first-shared sample, T5 said the student made a chart that had 3 columns (10/22/13 meeting transcript, lines 303). At the top of each column, the student wrote out the topping words of peppers, sausage and mushroom where three toppings were in each of the cells underneath the aforementioned topping words (10/22/13 meeting transcript, lines 303). Under the chart, the student wrote a list using Pepps for Peppers, Peppi for pepperoni, S for sausage, M for mushroom, and P for plain. T5 read the following argument out loud to the teachers:

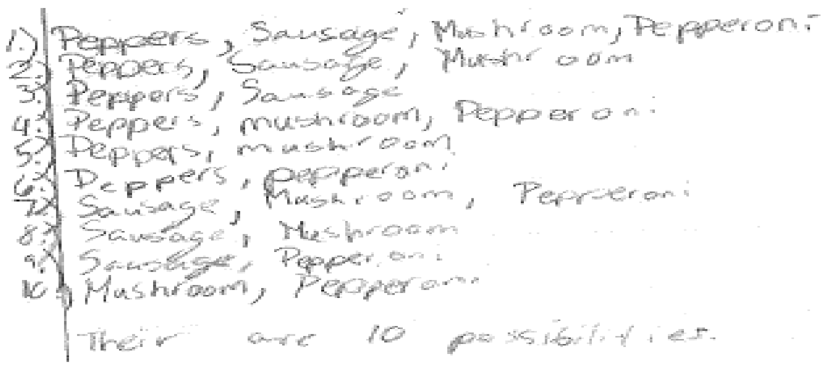
> I answered my question 16 combinations of possible pizza choices a customer can choose. I made a chart of peppers, sausage, mushrooms, and pepperoni. I did all the combinations for each of them. Then I got 7 for peppers, 3 for sausage, mushrooms, and pepperoni. I add them together to get 16. (10/22/13 meeting transcript, line discussion thread, line 307)

Thirteen possible pizzas were listed with 7 pizza combinations holding peppers as the constant; 3 holding sausage as the constant; and 3 holding mushroom as the constant (10/22/13 meeting transcript, lines 303). However, the student had written 7 + 3+ 3+ 3 = 16 possible pizzas as her solution (10/22/13 meeting transcript, lines 303).

T5 also shared a second example with the teachers by reading the following argument on the screen:

> There are 10 possibilities because from the toppings in order, I can reverse them and in the middle I can use to. If all the ten combinations are reversed then it would still be the same. (10/22/13 meeting transcript, line discussion thread, line 335)

The student made a numbered list of ten pizza combinations with the topping words written (10/22/13 meeting transcript, lines 331). Peppers are listed first 6 times, followed sausage listed first three times, and then mushroom listed first one time (10/22/13 meeting transcript, lines 331). T5 said she was impressed with this student because she liked the way “he just rearranged them” (10/22/13 meeting transcript, lines 331). Figure 5.6 shows the student’s work.



1.) Peppers, Sausage, Mushroom, Pepperoni
2.) Peppers, Sausage, Mushroom
3.) Peppers, Sausage
4.) Peppers, mushroom, Pepperoni
5.) Peppers, mushroom
6.) Peppers, Pepperoni
7.) Sausage, Mushroom, Pepperoni
8.) Sausage, Mushroom
9.) Sausage, Pepperoni
10.) Mushroom, Pepperoni

Their are 10 possibilities.

*Figure 5.6 T5's Cycle 2 Student Work Sample 2*

T8 was the fourth teacher to share his students' work. In the first example that T8 shared, the student provided a drawing of pizza slices (10/22/13 meeting transcript, lines 348). T8 recognized that the student solved the pizza problem by controlling for a variable. The drawing of the student's work is in Figure 5.7.

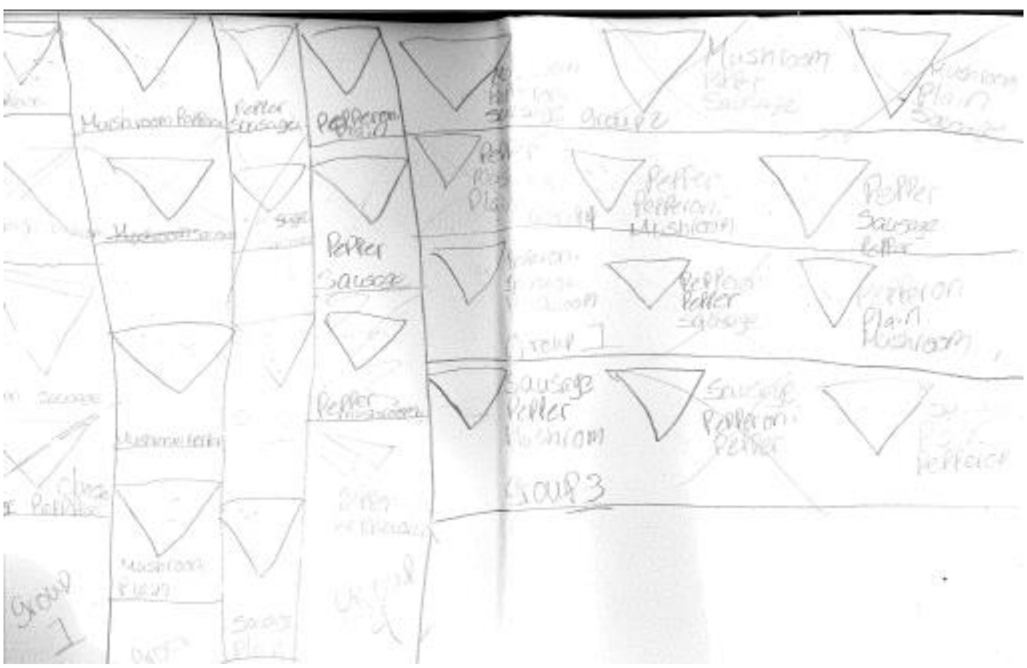



*Figure 5.7 T8's Cycle 2 Drawing, Student Work Sample 1*

The following written argument was provided by T8's student:

> Group one. I start with pepperoni, then I did two toppings; there is pepperoni in each one. If I put another one I get 3 toppings. I did 3 toppings with pepperoni in each one.
>
> Group two. I started with mushroom. Then I did two toppings; mushroom in each one. If I put another one I get three toppings. I did 3 topping with mushrooms.
>
> Group three. I start with sausage. I did two other topping. There is sausage in each one. If I put other topping I get three. I did three topping with sausage.
>
> Group four. I start with pepper. I did two other topping. There is pepper in each one. If I put other topping, I get three. I did three topping with pepper (10/22/13 meeting transcript, line discussion thread, line 362)

T8 shared a second sample of students' work. The following student's written argument was read by T8:

> Group one. I started it with pepper, then, I did two toppings with pepper with sausage mushroom and pepperoni. Three toppings are in each different topping with pepperoni, sausage, and mushroom. Four Toppings with pepperoni, pepper, mushroom.
>
> I started it with sausage. Then I did two toppings with peppers with mushrooms and pepperoni. Three toppings are in each different topping with pepperoni sausage and mushroom. Four Toppings with pepperoni, pepper, mushroom.
>
> I started it with mushroom. Then I did two toppings with pepper with sausage and pepperoni. Three toppings are in each different topping with pepperoni sausage and mushroom. Four Toppings with pepperoni, pepper, mushroom.
>
> I started it with pepperoni. Then I did two toppings with pepper with sausage and pepperoni. Three toppings are in each different topping with pepperoni sausage and mushroom. Four Toppings with pepperoni, pepper, mushroom.
> (10/22/13 meeting transcript, line discussion thread, line 362)

Figure 5.8 shows the second-shared student's drawing.

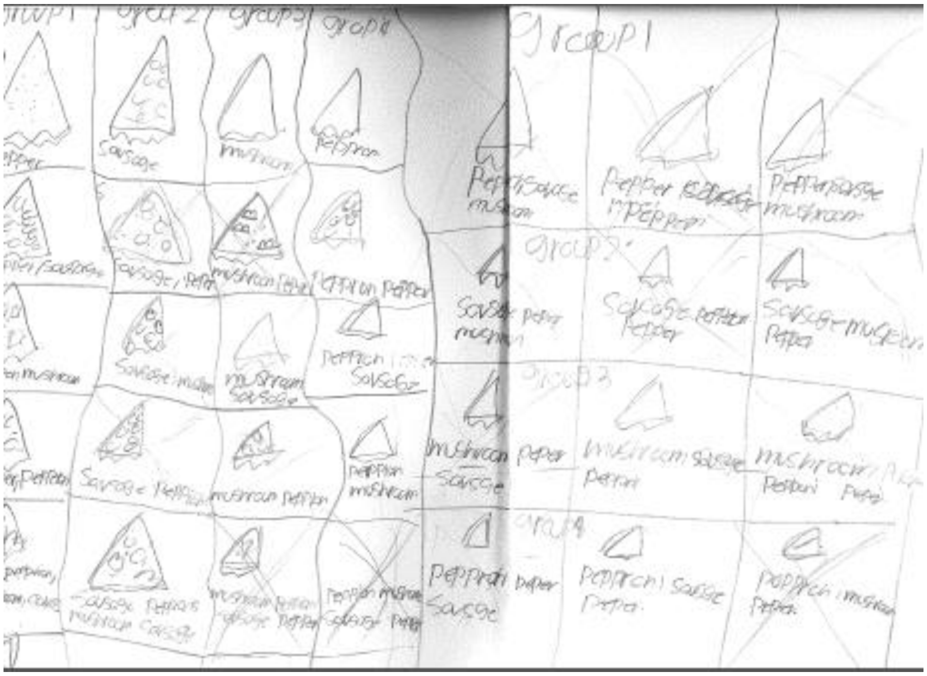


*Figure 5.8 T8's Cycle 2 Drawing, Student Work Sample 2*

The instructor asked the teachers if this argument was convincing and no teacher replied (10/22/13 meeting transcript, lines 363). The instructor then said the student was telling what she did and that "It's not clear exactly to me what she was thinking. And it could be developed but, right now it isn't convincing (10/22/13 meeting transcript, lines 363-367).

T8 also shared the work of an eighth-grade student-helper. The student-helper made an organized list of 16 possible pizza combinations using proof by cases. T8 read the following student-helper's written argument to the teachers:

> For the first 5 groups, I put the toppings by itself and then I combined all 4 of the toppings together. After that I took 1 topping, and put it with two topping and not get it to repeat. After that I took 1 topping and put it with 1 other topping and to not get it to repeat. (10/22/13 meeting transcript, line discussion thread, line 384-386)

T8 recognized that the student-helper solved the pizza problem by controlling for a variable (10/22/13 meeting transcript, line discussion thread, line 372). Figure 5.9 shows the student's list.

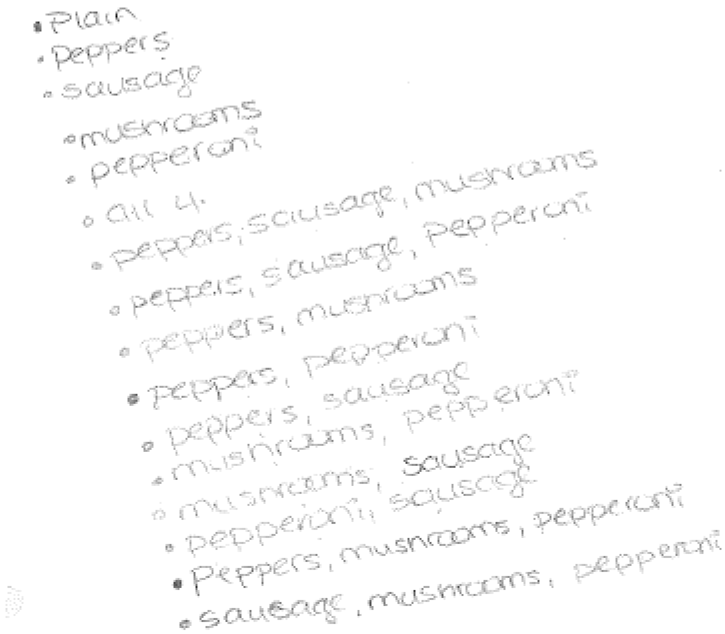



*Figure 5.9 T8's Cycle 2 Pizza Combinations List, Student Work Sample 3*

T7 was the fifth teacher to share her students' work. In the first-shared sample, T7 was surprised that the students had written 145 combinations on their paper (10/22 meeting transcript, line 491). T7 said that the student "had the idea that each slice was considered a different part" (10/22 meeting transcript, line 397). T7 asked the student "Would you go in and order a different slice on a pizza?" (10/22 meeting transcript, line 399) and T7 claimed that the student's father "use to have a pizzeria and if someone ordered that, he sure he would do it" (10/22 meeting transcript, line 399). Figure 5.10 shows the first student sample of work T7 shared. The following description is provided by T7 about what the student did to solve the pizza problem:

> He went to two with peppers and then six with each of the other four toppings because they're including plain as a topping. So when he was done with that he had 28 possibilities but that's only two toppings with peppers. after they did that, they multiplied, they had 28, they multiplied by 5, because they figured the peppers on the left column could be switched to sausage, mushrooms, plain or, uh, pepperoni [unintelligible]. So they multiplied by 5, got 140; not realizing that

> that's going to be a duplicates in there. And then they added five more pizzas on for the whole pizzas of each of those five. (10/22/13 meeting transcript, line discussion thread, line 403-409)

*Figure 5.10 T7's Cycle 2 Student Work Sample 1*

T7 shared a second student's work by describing the following written argument to the teachers:

> So this is that other student that I was talking about, um. And she broke it into quarters first and then she said, well, this quarter could have four different toppings on it, this one could have four different toppings on it, this one, and then she took that and said, well there is 13 toppings technically here and multiplied it by 4 because there is four quarters that each of them could be moved into. (10/22/13 meeting transcript, line discussion thread, line 422)

T7 replied that the student's work was "another big mess, they didn't get an answer" (10/22 meeting transcript, line 424). Figure 5.11 shows the student's work.

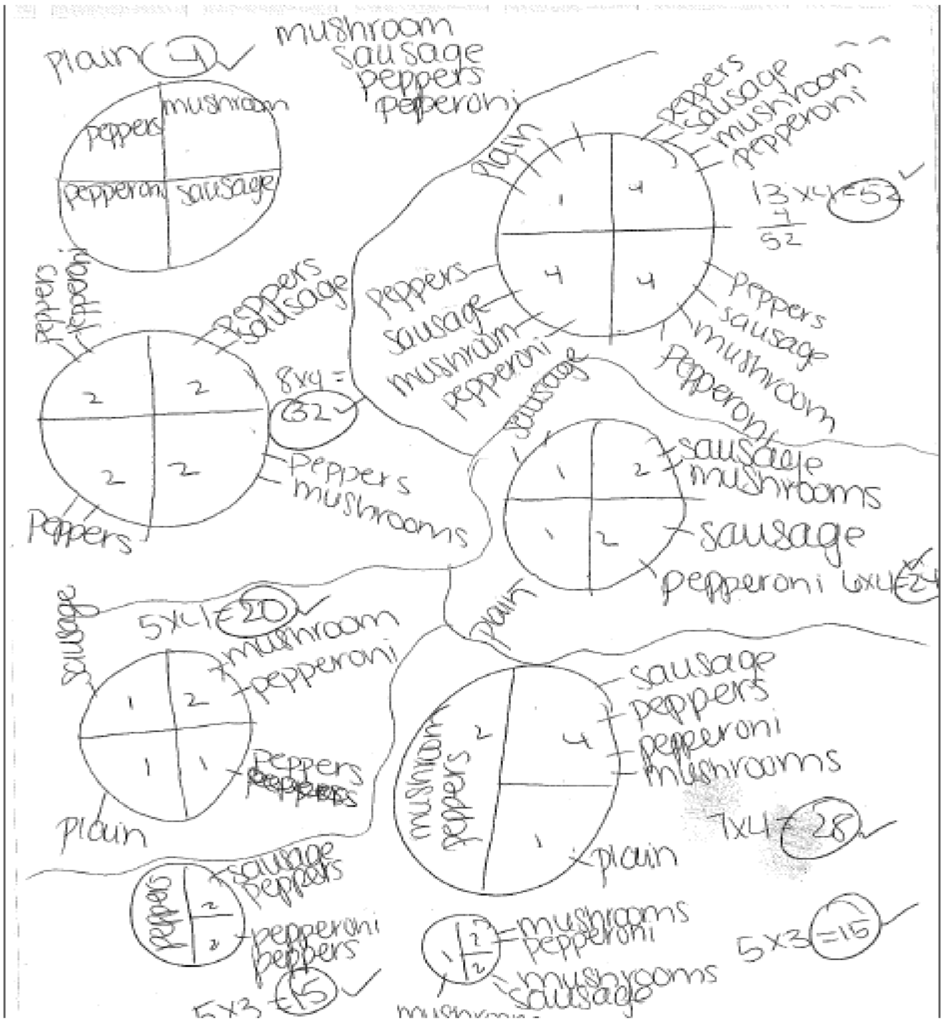



*Figure 5.11 T7's Cycle 2 Student Work Sample 2*

The third sample T7 shared was from a student who made a numbered list with 16 boxes of pizza combinations inside the boxes (10/22 meeting transcript, line 430). T7 read the following student's argument to the teachers:

> There is one plain pizza, you start with individual toppings; there's four. From this, you group them in lists such as 2-topping and 3-toppings. Then you make sure you didn't repeat a combination. There are 16 possible combinations. To check there are four original topping. Four can evenly go into 16. (10/22 meeting transcript, line 448)

T7 recognized that the student solved the pizza problem by controlling for a variable and using proof by cases (10/22 meeting transcript, lines 430-438).

T4 was the sixth teacher that shared her students' work. T4 said that this student worked with a group of three students (10/22 meeting transcript, line 464). The student began with a tree diagram but then had drawn circles (10/22 meeting transcript, line 464). T4 recognized that the student used proof by cases to make a list of the pizza combinations above the circles using letters to represent the pizza toppings (10/22 meeting transcript, line 468). Figure 5.12 shows the work of the student first shared by T4.

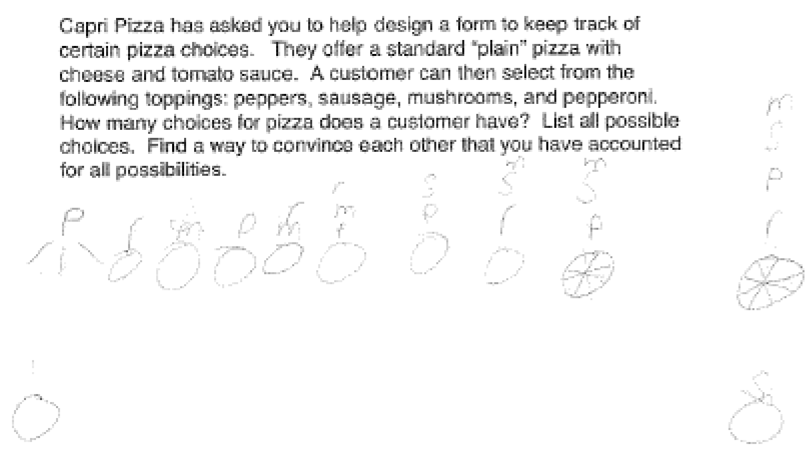



*Figure 5.12 T4's Cycle 2 Student Work Sample 1*

T4 then showed the partner's work from the first-shared sample. This student began the pizza task by listing the pizza combinations using letters to represent pizza toppings (10/22 meeting transcript, line 468). This student then revised his work using his partner's idea of proof by cases to list 16 pizza combinations (10/22 meeting transcript, line 474). Figure 5.13 shows the partner's work.

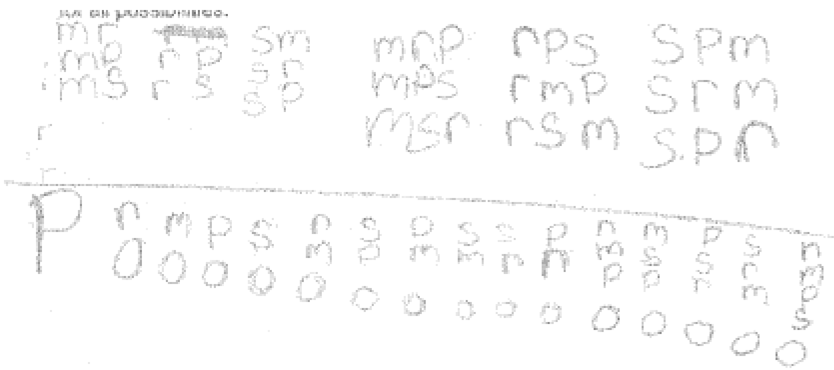


*Figure 5.13 T4's Cycle 2 Student Work Sample 2*

T3 was the seventh teacher that shared her students' work. The first sample of work that T3 shared was from a seventh-grade girl that had written a list of the topping words for each of the 16 combinations (10/22 meeting transcript, line 489). Figure 5.14 shows the work of the first-shared student sample from T3. The following written argument was placed on the screen:

1. To get all of them, we used a system.
2. The first four toppings, we went down the line using arrows.

(10/22 meeting transcript, line 448)

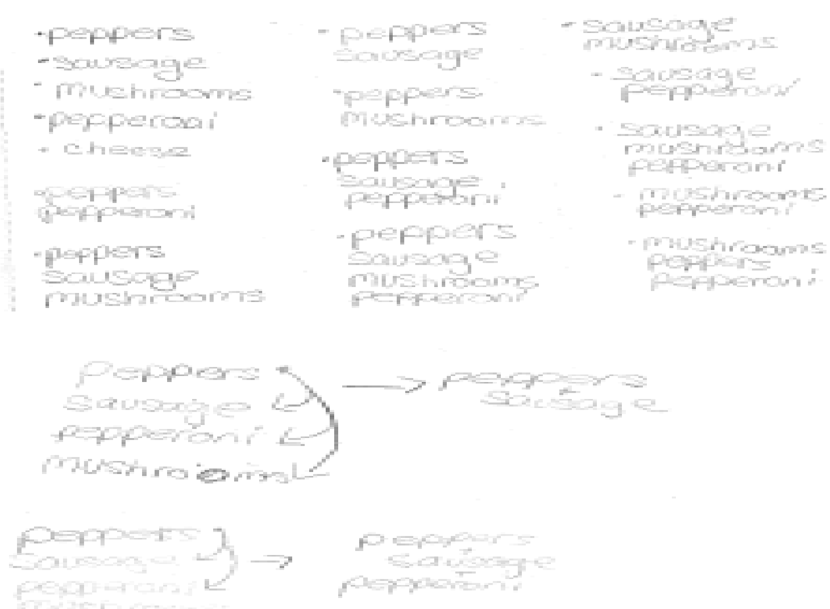



*Figure 5.14 T3's Cycle 2 Student Work Sample 1*

T3 then shared a second example with the teachers where the student listed thirteen pizza combinations by writing the topping words and using an argument by cases. The written argument provided by the student was as follows: “We know we had it because we based it on tree diagram, and then changed to list (10/22/13 meeting transcript, lines 504-505). Figure 5.15 shows T3’s student’s work.

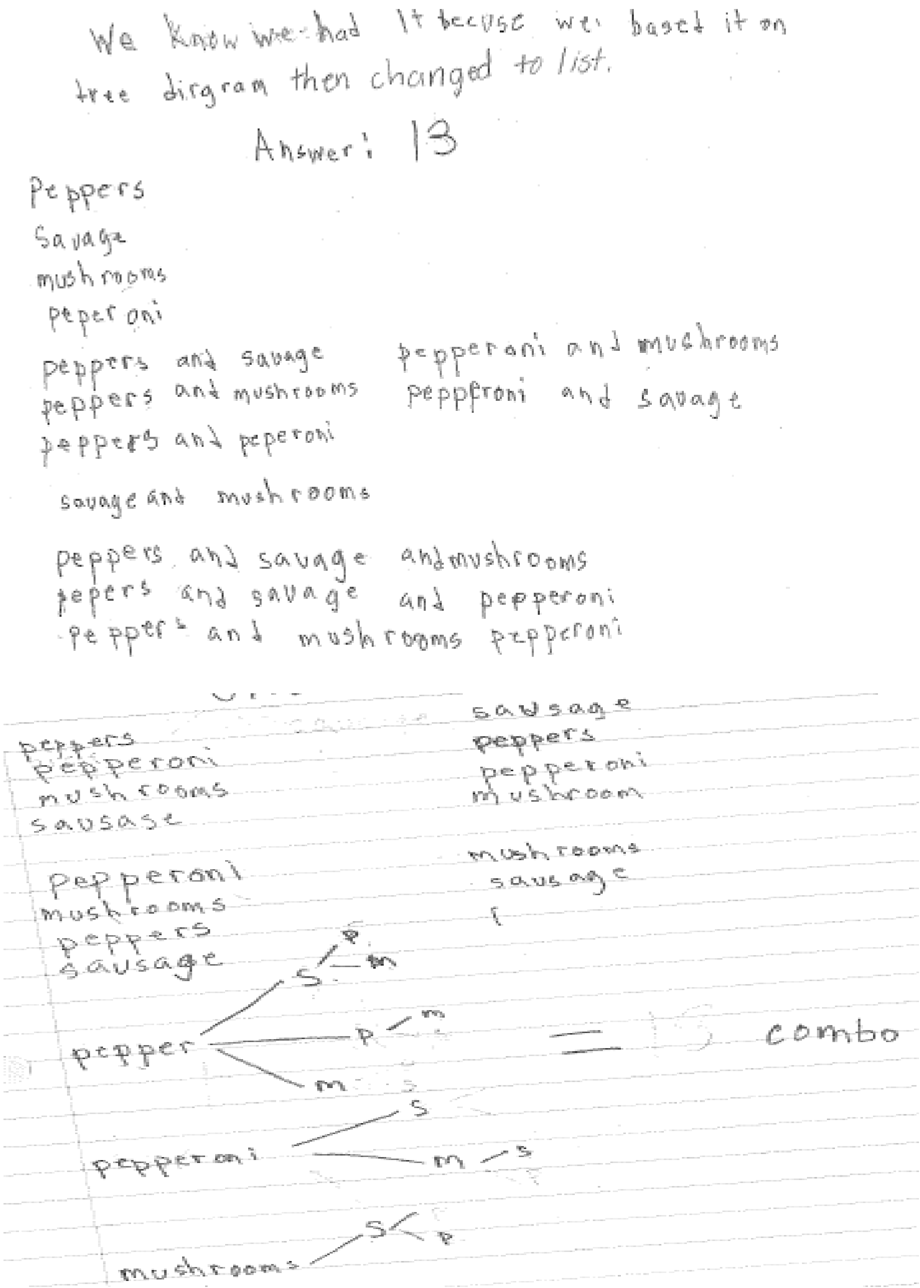



*Figure 5.15 T3’s Cycle 2 Student Work Sample 2*

T2 was the eighth teacher that shared her students' work. One seventh-grade girl made a list of 16 pizza combinations using topping words. T2 read the following written argument provided by her student:

> I know there is no more possible ways because there are only four toppings to choose from. The plain is the base of the whole thing. I used all the possible ways there are starting with one topping to two toppings to three toppings to four toppings. Then I thought I was done but I realized that I didn't have a plain pie and only a plain pie. That's how I got my answer and there was 16 ways altogether. (10/22 meeting transcript, line 532)

The instructor asked the teachers if the argument provided by the student was convincing and one teacher replied "I like her diagrams" (10/22 meeting transcript, line 534). The instructor then said "Her written work is very, very, nice. She's showing you what she got and she started to talk about that there were only 4 toppings so that could be convincing for why there are only 4 one-topping pizzas, but the rest of the groups, not so much (10/22 meeting transcript, line 535). Figure 5.16 shows the student's list.

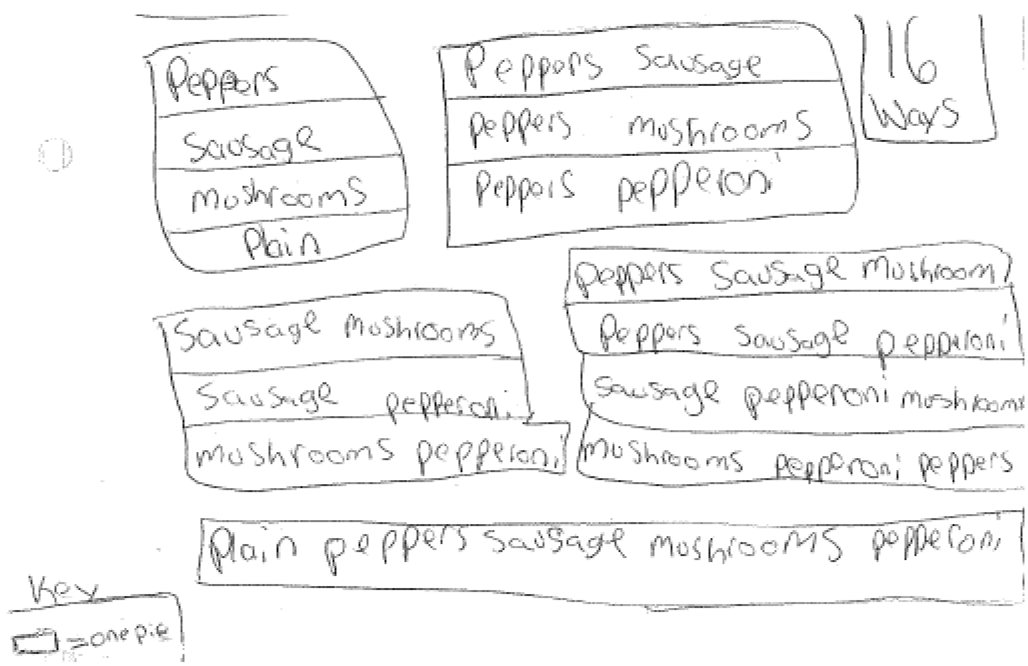



*Figure 5.16 T2's Cycle 2 Student Work Sample 1*

T2 shared a second example where the student made a connection to the 4-tall towers problem (10/22 meeting transcript, line 538). The student's written argument and diagram was placed on the screen. The diagram had 8 pairs of 4-tall towers where P was written inside the square to represent a yellow cube for plain pizza and T was written inside the square to represent a blue cube for a topping (10/22 meeting transcript, line 538). The written argument from this student was "I use the blocks for the pizza that I just did. I use blue for the topping, yellow plain only (10/22/13 meeting transcript, line 538). The instructor said that "This is the beginning of seeing a connection, isn't it? And it's neat that he is kind of saying, it either appears or doesn't appear. We just don't know which toppings are appearing" (10/22/13 meeting transcript, line 551). Figure 5.17 shows the student's work.

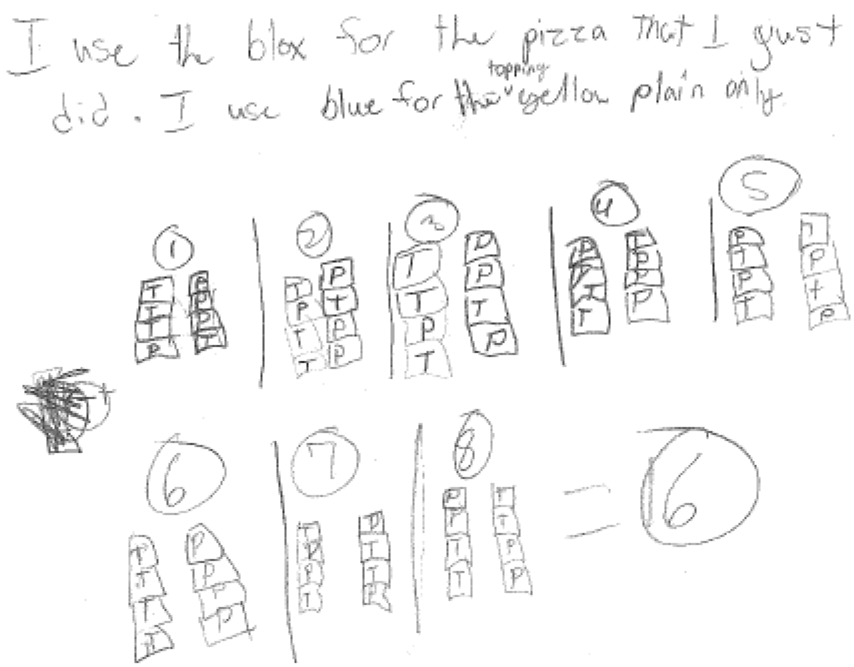



*Figure 5.17 T2's Cycle 2 Student Work Sample 2*

T10 was the ninth teacher to share her students' work. T10 read the following student's argument to the teachers:

> I think that 16 are all the possible combinations because we can't do anymore without them repeating for four-toppings. Uh, we know that all the toppings on a pizza are a choice, and that plain pizza is a choice too. That makes two pizzas total. And then there were four toppings, and though we could just put one topping per pizza, so that would make 6 pizzas total. Next we put two different toppings without repeating them again. We got 6 total pizzas for two toppings, and that makes 12 pizzas total. (10/22/13 meeting transcript, lines 572, 576)

The instructor asked the teachers if the teachers noticed the student was controlling for a variable in the two-topping pizzas and T10 replied "uh huh" (10/22 meeting transcript, lines 569-570).

Then T10 shared a second sample of students' work. This student created a table with the topping words at the tops of the columns and the rows were numbered on the left margin 1-16. In the row and column cells, there were checkmarks to represent the toppings used for the pizza combinations (10/22 meeting transcript, line 590). The instructor asked the teachers what they noticed about the chart and one teacher recognized the elevator strategy in the chart where one topping was being moved down one position in a diagonal pattern down the chart until all the positions were exhausted (10/22 meeting transcript, lines 602-603).

T10 said that this student changed his strategy to a number system (10/22 meeting transcript, line 602-603). The student's written argument was as follows: "I got 16 ways to combine toppings for a pizza pie at the restaurant. I numbered the toppings and included a plain pie" (10/22 meeting transcript, lines 628-629).

T9 was the tenth teacher to share her students' work. T9 said the student provided a web that had toppings branched out from it along with the following written argument that T9 read to the teachers:

> We believe that have found all the combinations. We believe this because first we used the web to see how much combos we could make. We also did them all because when I checked them off there were none left, because I did a combo. Once I did a combo, I took one thing off. I also added them altogether. You could do 4 by 4 because there are four toppings to pick out of, that's how I got 16. (10/2 meeting transcript, line 651)

T9 recognized the student solved the pizza problem by controlling for a variable" (10/22 meeting transcript, line 637). The student also incorrectly used the rule strategy (10/22 meeting transcript, line 651).

T9 shared a second sample of students' work with the teachers. The student created many web diagrams where each web represented a different pizza pie (10/22 meeting transcript, line 661). T9 read the following written argument provided by the student: "I got 16 combinations. I used factor trees to help me out by abbreviating the toppings and replaced the numbers with letters" (10/22 meeting transcript, line 665). Figure 5.18 shows the second student drawing shared by T9.

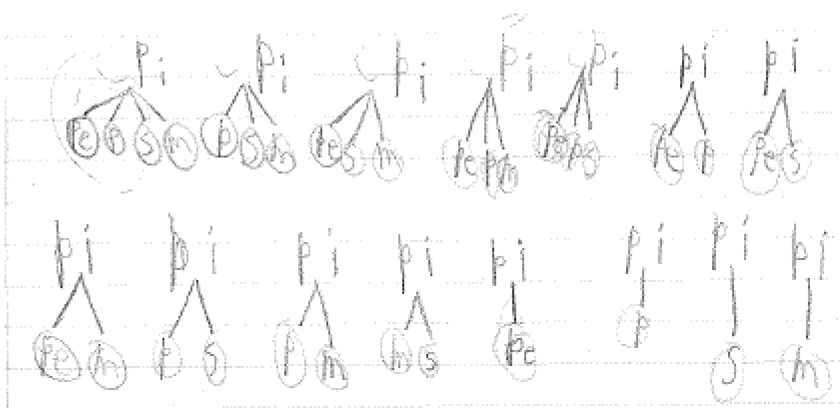


*Figure 5.18 T9's Cycle 2 Student Work Sample 2*

**5.5 Summary**

With the completion of the aforementioned three session units, the second cycle of tasks came to an end. Throughout Cycle 2, teachers worked on the second cycle tasks, then participated in three thought-provoking on-line discussions, observed an in-district classroom visit working on the tasks, implemented the same tasks in their own classes, read literature and watched videos of other students working on the tasks, and shared their own students' work after implementing the tasks in their own classes. At the regional meeting on 10/22, the third cycle of tasks began with the teachers working in pairs on the 3-tall towers problem, selecting from 3 colors and an extension problem called Ankur's challenge; where teachers worked on finding 4-all towers, selecting from 3 colors using at least one of each color (Landis, 2013).

# Chapter 6 – Cycle 3 Session Summary and Analysis

This chapter is a summary and analysis for the third intervention cycle with a new set of mathematical tasks. Three session units of on-line discussion threads are analyzed to include teachers' problem solving of the third cycle tasks at the regional meeting on 10/22/13 and the teachers sharing students' work at the in-district classroom visit and third regional meeting on 11/20/13.

## 6.1 Unit 8: Teachers Work on Third Cycle Tasks, Regional Meeting 10/22/13

Teachers were asked to convince each other of their solutions (10/22/13 Cycle 3 teachers' work transcript, lines 3-5). Once a solution was found, teachers were asked also to convince the instructor of their solution (10/22/13 Cycle 3 teachers' work transcript, lines 3-5). When successfully convincing each other and the instructor, they were asked to produce a written solution; if they were unsuccessful in convincing one of the researchers, they were invited to rethink their solution.

### *6.1.1 Building Three-Tall Towers, Selecting from Three Colors*

Teachers were given twenty-five minutes to work on the 3-tall tower problem, selecting from three colors. The instructor monitored the progress for the five pairs of teachers by circulating around the room (10/22/13 Cycle 3 teachers' work transcript, lines 3- 5). After twenty-five minutes, the instructor called for the attention of the whole group to share the teachers' solutions and said "four of the groups in the room did the problem by grouping it into 9 groups of 3" (10/22/13 Cycle 3 teachers' work transcript, line 288).

Three of the four teacher pairs had nine groups of three towers using the elevator strategy. The instructor asked one of the three teacher pairs to present their solution to the other teachers (10/22/13 Cycle 3 teachers' work transcript, line 286). The pair of teachers recognized that they used the elevator strategy for six of the nine groups of three-tall towers (10/22/13 Cycle 3 teachers' work transcript, line 295). When the instructor asked what their convincing argument was, one of the teachers in the pair (T1) said the following:

> So we said if you have the yellow and blue for example. If we had it 2 yellow and the one blue there's only 3 ways to do that. Our single cube can move to each of the positions. If we were to move that again, we would either need a fourth row or we would be repeating it. (10/22/13 meeting transcript, line 297)

T1 described two forms of argument; a recursive method and a proof by contradiction (10/22/13 Cycle 3 teachers' work transcript, line 297). Figure 6.1 shows a picture of the 3-tall towers built by the first teacher pair.

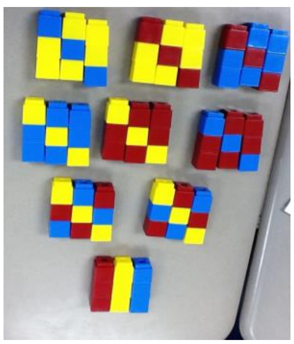


*Figure 6.1 Cycle 3 Three-Tall Arranged Towers, Teachers' Work from T7 and T8*

Then the instructor asked about the argument for the towers that looked like a "candy cane" (10/22/13 Cycle 3 teachers' work transcript, line 303). These towers are also referred to as alternating towers or towers that have one of each color (10/22/13 Cycle 3 teachers' work transcript, line 295). Two of the groups of 3-tall towers had one of each color. For the first group of the 3-tall towers with one of each color, the teacher pair used the recursive strategy of taking the yellow cube at the top and moving it down one position each time to make the next tower until all the positions were exhausted. For the second group of alternating colors, the teacher pair used the same yellow cube at the top moving down one position each time except the positions of the blue and red cubes were switched in opposite positions from the red and blue cubes of the first group of alternating colors (10/22/13 Cycle 3 teachers' work transcript, lines 291).

The second teacher in the pair (T2) said "having the yellow kind of go through each position kind of the same way that we had the one cube go through in the other positions actually moving the yellow all the way through if you move it to another place, it would go to the top" (10/22/13 Cycle 3 teachers' work transcript, lines 304,306). The instructor then replied to the teachers "now they are using a recursive argument" (10/22/13 Cycle 3 teachers' work transcript, lines 307). T1 and T2 used the Unifix cubes to show the teachers how they moved the cubes to form the other towers (10/22/13 Cycle 3 teachers' work transcript, lines 307-318).

T2 said to the teachers that he solved the three-tall towers problem differently by holding "the red constant first at the top and then I said the yellow and blue could be in two different ways. Then I had the yellow constant same thing" (10/22/13 Cycle 3 teachers' work transcript, lines 320). T2 recognized that he solved the three-tall towers problem by controlling for a variable (10/22/13 Cycle 3 teachers' work transcript, line 320).

A second teacher pair was asked by the instructor to present their solution to the others (10/22/13 Cycle 3 teachers' work transcript, lines 327). This solution of this teacher pair was unique in that they created three groups of nine towers in which the top group of nine towers were arranged with all red cubes on the top of each tower, the middle group of nine towers had all blue cubes on the top of each tower, and the bottom group of nine towers consisted of all yellow cubes on the top of each tower (10/22/13 Cycle 3 teachers' work transcript, line 290). The following solution was explained by one of the teachers (T3) in the pair:

> So we started off before we actually started building them, we made a prediction that there were going to be 27 similar to the two to the fourth power and the 3 to the third power. So we knew that there were going to be 27 so we actually started building them the same exact way that Chris and Christine did. And then we kind of got stuck and there were only 24 and we were like we're missing 3. So then I kind of looked and we decided to group them differently. And this is where we put all the…keeping the red constant, the blue constant, and the yellow constant. When we did it that way, We then saw that there were 9 that had red constant, 7 that had blue constant, and 8 that had yellow constant so that kind of gave us the 3 that we were missing. (10/22/13 Cycle 3 teachers' work transcript, line 297)

The teacher pair correctly used the rule strategy to predict how many towers there would be before they tried building the three-tall towers (10/22/13 Cycle 3 teachers' work transcript, line 297). T3 recognized that she and her partner solved the three-tall tower problem by controlling for a variable (10/22/13 Cycle 3 teachers' work transcript, line 297). Figure 6.2 shows a picture of how the second teacher pair built their three-tall towers.

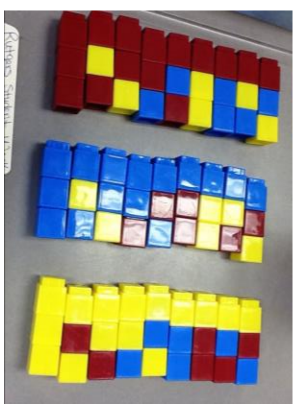


*Figure 6.2 Cycle 3 Three-Tall Arranged Towers, Teachers' Work from T2 and T3*

The third teacher pair had nine groups of three towers (10/22/13 Cycle 3 teachers' work transcript, line 352). There was a red, yellow, and a blue cube on the top of each tower for each of the nine groups (10/22/13 Cycle 3 teachers' work transcript, line 352). The following solution was explained by one of the teachers (T4) in the pair:

> We held the bottom color constant. So there are 3 different rows we can do that with because there are 3 different colors. Because you can hold the bottom row constant 3 different times. And then within the row of holding the bottom constant, we held the second one constant as well so the second one is yellow, the second one is blue, and the second one is red and we can't have any more groups of 3 because there are 3 colors. And that leaves the last row to kind of alternate between the colors. And it's red, yellow, or blue. (10/22/13 Cycle 3 teachers' work transcript, line 297)

The instructor then asked "what kind of argument did they use?" (10/22/13 Cycle 3 teachers' work transcript, line 359) and no teacher correctly recognized what kind of argument the teacher pair used.  So, the instructor said "Milin used that argument" (10/22/13 Cycle 3 teachers' work transcript, line 361) and told the teachers that the argument was inductive.  Figure 6.3 shows how the third teacher pair built their towers.

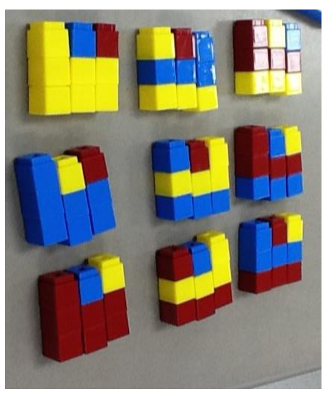


*Figure 6.3 Cycle 3 Three-Tall Arranged Towers, Teachers' Work from T4 and T6*

The fourth and fifth teacher pairs were not asked by the instructor to present their solutions (10/22/13 Cycle 3 teachers' work transcript, lines 286).  Both the fourth and fifth teacher pairs solved the three-tall tower problem with the same strategies that the first teacher pair presented (10/22/13 Cycle 3 teachers' work transcript, lines 211-258, 261-269). The fourth and fifth pair of teachers used the elevator strategy (10/22/13 Cycle 3 teachers' work transcript, lines 211-258, 261-269).

### *6.1.2 Ankur's Challenge*

Teachers were asked to find all possible 4-tall towers, selecting from three colors and using at least one of each color. The instructor monitored the progress for the five pairs of teachers by circulating around the room (10/22/13 Cycle 3 teachers' work transcript, lines 3- 5). After twenty-five minutes, the instructor called for the attention of the whole group. The instructor, having selected the work of some teacher pairs for sharing, said "I see two different ways of arranging going on. I see grouping in groups of 3, I see grouping in groups of 6, I see grouping in groups of 12. There are 3 different ways. So let's look at this is the grouping of 6" (10/22/13 Cycle 3 teachers' work transcript, line 537).

T9 and T10 were the first teacher pair to present their solution of Ankur's Challenge to the others (10/22/13 Cycle 3 teachers' work transcript, line 537). Their representation showed six groups of six towers (10/22/13 Cycle 3 teachers' work transcript, line 546). Figure 6.4 shows a picture of T9 and T10's work.

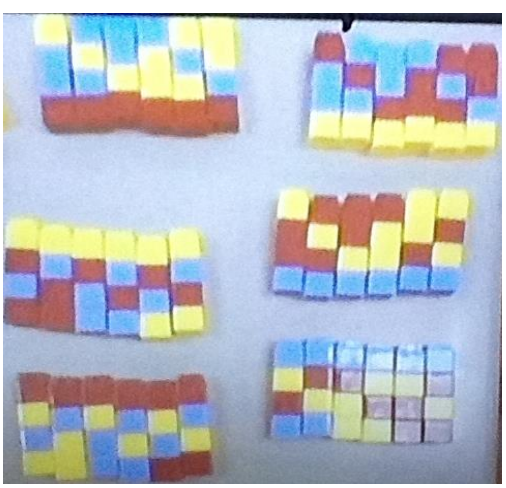


*Figure 6.4 Cycle 3 Ankur's Challenge, Teachers' Work from T9 and T10*

T9 explained the following solution to the teachers:

> We knew we exhausted all the options for yellow and the yellow because we used it in every position. Then we just added all the reds to the bottom because that we knew that took up all or used every color. We picked a color and we said let's start with red. So then we used red as the constant and we added it to the bottom and made that the constant for the bottom. Then we did the same thing with our next tower or our other 3 towers. We had two blue and one red and we knew we already exhausted all the options for blue and red so we just decided to go with the yellow on the bottom. (10/22/13 Cycle 3 teachers' work transcript, lines 552,554)

T9 recognized that she solved the Ankur's Challenge problem by using a recursive argument and by controlling for a variable. Figure 6.4 shows a picture of T9 and T10's work.

T1 and T2 were the second teacher pair to present their solution of Ankur's Challenge to the teachers (10/22/13 Cycle 3 teachers' work transcript, line 565). Their towers were arranged in three groups with each group having twelve towers (10/22/13 Cycle 3 teachers' work transcript, line 565). The top group of twelve towers was arranged with red cubes at the top of each tower, the middle group had yellow cubes at the top, and the bottom group showed blue cubes at the top (10/22/13 Cycle 3 teachers' work transcript, line 565). T1 recognized that one of the strategies the teacher pair used was controlling for a variable. The following explanation was given to the teachers by T1:

> We did all yellow tops constant on the first row. And then for the first group of 2, we did yellow constant in the middle row. And we alternated the blue and red. And we said we couldn't put another yellow in either one of those positions. (10/22/13 Cycle 3 teachers' work transcript, lines 552,554)

The instructor asked T1 "why" (10/22/13 Cycle 3 teachers' work transcript, line 572) and her partner T2 replied "because then we would only have two of the colors, rather than all 3" (10/22/13 Cycle 3 teachers' work transcript, line 573).

T1 continued with her explanation to the teachers saying “for the next group, we kept 3 rows constant so we had all 3 colors; yellow blue and red and then we just alternated the last one” (10/22/13 Cycle 3 teachers’ work transcript, lines 575,577). The instructor then asked the teachers “what argument were they using” (10/22/13 Cycle 3 teachers’ work transcript, line 578) and one of the teachers recognized their argument as being an inductive argument (10/22/13 Cycle 3 teachers’ work transcript, line 579).

T7 and T8 were the next teacher pair to present their solution of Ankur’s Challenge (10/22/13 Cycle 3 teachers’ work transcript, lines 629). This teacher pair organized their towers in groups of twelve each with three towers. The top row of twelve towers had four groups. The first group had a three-block blue row on top of a three- by three-tall subgroup of towers made using two colors with the elevator strategy. The diagonal of the first three- by three- subgroup was made of red cubes moving down one position at a time with yellow cubes in the other positions (10/22/13 Cycle 3 teachers’ work transcript, lines 629).

The middle row of twelve towers had four groups with the three-block blue row on top of the opposite colors of the top row. The bottom row of four groups had a three-block red row on top of a three by three subgroup of towers made using yellow and blue cubes where the diagonal pattern was made of yellow cubes (10/22/13 Cycle 3 teachers’ work transcript, lines 629). The following explanation was given by T7:

> We started with what we had in dealing with the previous problem and we saw that we had you know two yellow and one red. And we decided to take the blue and put it on the top and then take it and put it in the second position. And then take it and put it into the third position and then in the fourth position. Then we did the same with two red and one yellow moving the blue down. (10/22/13 Cycle 3 teachers’ work transcript, lines 629)

T7 described controlling for a variable and the elevator strategy in her explanation (10/22/13 Cycle 3 teachers' work transcript, line 629). The instructor then ended the meeting asking the teachers to watch an assigned video of Romina's Proof at the following link: http://dx.doi.org/doi:10.7282/T30P0Z85 and bring two or three samples of the Cycle 3 student samples of work to share with the teachers at the next regional meeting on 11/20/13 (10/22/13 Cycle 3 teachers' work transcript, lines 645-655).

**6.2 Unit 8: On-line Discussion 10/23/13 to 10/28/13**

For the discussion, the teachers were instructed to watch the video of Romina's Proof to Ankur's Challenge. In the Romina's Proof to Ankur's Challenge video, Romina was in the tenth-grade and she worked on the following two problems with four other tenth-grade students.

1. Choosing from two colors of Unifix® cubes, red and yellow, how many total combinations exist for towers 5 tall, that each contains two red? Convince us that you have found them all. (Private Universe Project in Mathematics Workshops (PUP), Romina's proof to Ankur's Challenge [video], 1998)

2. (Ankur's Challenge) How many towers can you build four tall, selecting from cubes available in three different colors of Unifix® cubes, so that the resulting towers contain at least one of each color? (Private Universe Project in Mathematics Workshops (PUP), Romina's proof to Ankur's Challenge [video], 1998)

Romina used the symbols 1, 0, and X to represent the three colors of Unifix® cubes and presented her solution of 36 possible towers to the other four tenth-grade students (PUP, Romina's proof to Ankur's Challenge [video], 1998). After watching the video, the teachers were asked to respond to the following questions on-line:

1. In the video you watched, Mike and Ankur come up with 39 as their solution to Ankur's challenge. What method did they use to find their solution?

2. Approaching the problem differently, Romina comes up with 36 for her solution. How does she approach solving Ankur's challenge?

> 3. If you gave Ankur's challenge to your students, do you think any of them could come up with Romina's proof? (Landis, 2013)

### *6.2.1 First Question Responses*

The first question asked teachers about Mike and Ankur's method for finding 39 towers as their solution to Ankur's Challenge (Unit 8on-line discussion thread, instructor questions). Nine teachers responded that Mike and Ankur first used a mathematical rule, where three to the fourth power gets 81 towers (Unit 8 on-line discussion thread, line 1-42). Six of the teachers responded that Mike and Ankur had 39 towers as their solution by eliminating towers that did not meet the criteria for the Ankur's Challenge problem. The responses of the six teachers follow:

> From the video, it appears they narrowed it down to 39, possibly by eliminating things that did not include all three colors and found 39. (Unit 8 on-line discussion thread, line 1)

> Then using that number they began to take away towers that did not use all three colors to satisfy the conditions of Ankur's problem. (Unit 8 on-line discussion thread, line 17)

> They followed by going back to the conditions of the problem which required all three colors to be in the tower. (Unit 8 on-line discussion thread, line 34)

> They did draw diagrams of the tower so I'm thinking that maybe they worked backwards to arrive at their answer of 39 and failed to account for all duplicates. (Unit 8 on-line discussion thread, line 36)

> They then narrowed it down because they realized that not every tower had to have all three colors. (Unit 8 on-line discussion thread, line 40)

> It appears to me that they listed their 81 towers and then eliminated the towers that only had 2 colors. They got their solution of 39, but must not have eliminated all the duplicates. (Unit 8 on-line discussion thread, line 44)

Four of the teachers mentioned the diagram that Mike and Ankur provided. The responses of the four teachers follow:

> The boys organized their towers as numbers 1, 2, and 3. They have four columns with three rows of 1 2 3 and one row of 0 0 0 that moves positions through the towers. (Unit 8 on-line discussion thread, line 8)
>
> I did notice from their diagram they had three towers that looked like:
>
> | | | |
> |---|---|---|
> | 1 | 2 | 3 |
> | 1 | 2 | 3 |
> | 1 | 2 | 3 |
> | 0 | 0 | 0 |
>
> Based off of this interpretation I concluded that they had not taken away these three towers, which consisted of two colors and not three. If they subtracted those they would have gotten to the right answer. (Unit 8 on-line discussion thread, line 17)
>
> They did not explain how they came to the answer of 39 but it appeared they used a system of numbers 0-3 for the colors and drew diagrams for the towers. (Unit 8 on-line discussion thread, line 34)
>
> I really am not sure how they arrived at the answer 39. They did draw diagrams of the tower so I'm thinking that maybe they worked backwards to arrive at their answer of 39 and failed to account for all duplicates. (Unit 8 on-line discussion thread, line 36)

### *6.2.2 Second Question Responses*

The second questions asked the teachers to respond to how Romina approached solving the Ankur's Challenge problem. Nine teachers responded that Romina solved the Ankur's Challenge problem by placing the three colors in positions until all positions were exhausted. The responses of the nine teachers follow:

> She then focused on the placement of the two cubes of the same color and realized that there were 6 arrangements for these two cubes. She then noticed that the other two positions must be filled with opposite colors, so each of her original 6 arrangements could happen twice, depending on the placement of the 2$^{nd}$ and 3$^{rd}$ colors in the tower. She then said that for the set of towers 4-tall containing 2 of a particular color, and 1 of each of two other colors, there would be 12 outcomes. Since there are 3 colors, you can repeat this process with each color and have 36 possibilities. . (Unit 8 on-line discussion thread, line 1)
>
> Romina solved the challenge problem using a drawing of towers where two cubes of the same color (labeled with 1) were held constant in two positions and the other two cubes (labeled o and x) could alternate colors. This method gave her 6 towers. Next to each tower she has the number 2 to represent 12 towers (depending on whether or not the o or the x was used for a cube. She then

multiplied 12 by 3 because there are three colors to choose to represent the 1, o, and x cubes. (Unit 8 on-line discussion thread, line 8)

She represented the two choices you would have from each tower of three colors in one box. For example:

| | | | |
|---|---|---|---|
| 1 | 1 | X<br>O | O<br>X |

This representation was two different towers. She then drew 5 more in similar to this one focusing on the one duplicate color moving positions and alternating the other two colors. She came up with a total of 6 drawings which she then multiplied by 2 since each drawing represented two towers and came up with 12 combinations for holding one color as a constant. Since there were three colors she multiplied 12 by 3 to get her resulting answer of 36. (Unit 8 on-line discussion thread, line 17)

She then is able to position those two cubes of the same color in six different positions within a tower that is 4-tall. Romina then multiplies each of the six towers by two, because she can create an additional tower by switching two of the other colors to the opposite. She then arrives at the answer of 36 total towers by multiplying 12 by 3 for the three different colors that can be contained within each tower. (Unit 8 on-line discussion thread, line 30)

Romina approached the problem with the idea that if all three colors need to be used; the fourth color will repeat, therefore she focused on the placement of two blocks the same color. She found six arrangements of towers four tall with two blocks the same color. Following this, she knew the last two blocks had to be one of each color and there were two ways to place those colors. Therefore, she had twelve possible towers (6x2) with the previous conditions. She also realized that there were three available colors that could be doubled in the tower so she took the 12 possible towers and multiplied by 3 to determine there were 36 possible towers, four blocks tall, containing three colors. (Unit 8 on-line discussion thread, line 34)

She first focused on the two of the same color and found six different arrangements with the two blocks of one color and the two other blocks the different color. She can switch the position of the two other blocks so she multiplies six by two to get twelve. Since there are three different colors, she can repeat this 3 times and multiples 12 by 3 (Unit 8 on-line discussion thread, line 36)

She then took those two colors and repositioned them giving her 6 towers. Then she did the opposite for each of the 6 which gave her 12. Because she had three colors, she knew she could do this 3 more times, giving her 36. (Unit 8 on-line discussion thread, line 37)

> She also was able to show that after moving the colors into different positions she has exhausted all possible solutions. (Unit 8 on-line discussion thread, line 40)
> She came up with 6 placements of where the 2 blocks of 1 color would be. The other 2 blocks would need to be the 2 other colors, 1 of each. So, she multiplied her 6 towers by 2 so she had 12 towers made with 2 of one color. She knew this could be replicated with both other colors, so 12*3 = 36. (Unit 8 on-line discussion thread, line 42)

### *6.2.3 Third Question Responses*

For the third question, teachers were asked if they thought their students could produce a proof like that of Romina (Unit 8 on-line discussion thread, question 3). Of the nine teachers that responded, six teachers said they thought their students would not come up with the same proof as Romina. However, two teachers reported that they believed some of their students could solve the challenging problem. The responses of the two teachers follow:

> I do not think any of my students could come up with Romina's proof. It's very simple and very complicated at the same time. I do think my students could solve the challenge problem, but it might take them more than one class period and their explanations would be basic (opposite pairs and looking for a pattern). (Unit 8 on-line discussion thread, line 8)

> If I gave Ankur's challenge to my students, many of them would probably be very confused by it. I may have a few students in my accelerated level class that would get the correct answer. (Unit 8 on-line discussion thread, line 42)

## 6.3 Unit 9: On-line Discussion 10/30/13 to 11/5/13

Teachers were assigned to read Chapter 8 of Combinatorics and Reasoning (Maher, Powell, & Uptegrove, 2010) about the tenth-grade students' solutions to the Ankur's Challenge problem. After reading the chapter, participants were asked to respond to the following guiding questions on-line:

> 1. What are some of the advantages of giving your students more than one opportunity to explain and write about their ideas? Make reference to the chapter and how it was helpful to Romina.

> 2. Explain why Romina multiplied by two when finding her solution to Ankur's challenge. (Unit 9 on-line discussion thread questions 1 and 2)

### *6.3.1 First Question Responses*

The first on-line question had two parts. For the first, teachers were asked what some of the advantages were for giving students the opportunity to explain and write about their ideas (Unit 9 on-line discussion thread, question 1). For the second part, teachers were asked to reference chapter 8 and discuss how explaining and providing a written solution was helpful to Romina (Unit 9 on-line discussion thread, question 1).

#### *6.3.1.1 Part 1 First Question Responses*

Eight of the teachers said that revising student work was an advantage for giving students more than opportunity to explain and write their ideas. The responses of the eight teachers follow:

> This will allow the students to revise their work and try to see if they fully understand the concept or if it can be presented in another format. (Unit 9 on –line discussion thread, line 1)

> I have always found that I write better papers when I create a draft, read it aloud, edit it and repeat until the paper sounds exactly right. This process is not limited to writing essays and college papers; it can be applied to open ended responses. (Unit 9 on –line discussion thread, line 14)

> I think giving students a chance to explain and write their ideas multiple times is similar to proof reading and drafting papers in language arts. Students have an opportunity to self-correct. (Unit 9 on –line discussion thread, line 21)

> It can serve as a "revise and edit" and allow them to gain a deeper understanding of the actual problem. (Unit 9 on –line discussion thread, line 35)

> It gives them a chance to refine their solutions and possibly come up with more strategies. (Unit 9 on –line discussion thread, line 37)

> For each argument-iteration, 'Romina made refinements and clarified her reasoning.' I noticed this statement within my own student as well as in Romina throughout this chapter. (Unit 9 on –line discussion thread, line 38)

> When students are asked to examine and refine their ideas they will gain a better understanding. (Unit 9 on –line discussion thread, line 39)
>
> It gives them time to revise and clarify their thinking. (Unit 9 on –line discussion thread, line 40)

Eight of the teachers said that students would develop a deeper conceptual understanding as an advantage for giving more than one opportunity to explain and write their ideas. The responses of the eight teachers follow:

> This will allow the students to revise their work and try to see if they fully understand the concept. (Unit 9 on –line discussion thread, line 1)
>
> By giving them more than one opportunity, they are able to see where their mistakes are to make it clearer and more convincing. (Unit 9 on –line discussion thread, line 3)
>
> This process is helping students develop a deeper understanding of the mathematics. (Unit 9 on –line discussion thread, line 21)
>
> The more students are asked to write and evaluate their reasoning, the more opportunities they have to develop and clarify their reasoning. (Unit 9 on –line discussion thread, line 33)
>
> Giving students more than one opportunity to explain and write down their ideas allows them to further think and explore their answer. It can serve as a "revise and edit" and allow them to gain a deeper understanding of the actual problem. (Unit 9 on –line discussion thread, line 35)
>
> The more time students have to think about their ideas, the better their thinking is about the problem. (Unit 9 on –line discussion thread, line 37)
>
> The more they explain and their classmates still don't understand they are challenged to explain it in a different way and in turn furthering their understanding of the concept. (Unit 9 on –line discussion thread, line 38)
>
> Providing students with more than one opportunity to explain and write about their reasoning allows them to analyze, critique, and further develop their ideas. (Unit 9 on –line discussion thread, line 39)

Six of the teachers mentioned that using different representations to solve the problem was also an advantage for giving students more opportunities to explain and write about their ideas. The responses of the six teachers follow:

> This will allow the students to revise their work and try to see if they fully understand the concept or if it can be presented in another format. (Unit 9 on –line discussion thread, line 1)

> Sometimes students need to explain it verbally, write it, and draw it to find a convincing argument. (Unit 9 on –line discussion thread, line 3)

> Many times in class I challenge students to solve a problem differently than how they originally did. (Unit 9 on –line discussion thread, line 21)

> The idea of giving our students multiple opportunities of explaining makes me think of all the different learning styles that we see on a daily basis in our classrooms. By allowing students multiple opportunities, it may allow them to think of the problem in a different style than how they usually learn. By giving them additional opportunities, one student may be able to see the problem verbally, visually or physically different then the first time they attempted the problem. (Unit 9 on –line discussion thread, line 35)

> The more they explain and their classmates still don't understand they are challenged to explain it in a different way and in turn furthering their understanding of the concept. (Unit 9 on –line discussion thread, line 38)

> Providing this opportunity allows multiple learning styles to be addressed. (Unit 9 on –line discussion thread, line 39)

Five of the ten teachers mentioned clarifying their solutions to others as an advantage to giving students more than one opportunity to explain and write about their ideas. The responses of the five teachers follow:

> By giving them more than one opportunity, they are able to see where their mistakes are to make it clearer and more convincing. (Unit 9 on –line discussion thread, line 3)

> The more students are asked to write and evaluate their reasoning, the more opportunities they have to develop and clarify their reasoning. (Unit 9 on–line discussion thread, line 33)

> For each argument-iteration, ‘Romina made refinements and clarified her reasoning.’ I noticed this statement within my own student. (Unit 9 on –line discussion thread, line 38)

> They will also be able to explain and reach students that may not understand a problem. (Unit 9 on–line discussion thread, line 39)

> If students are given the opportunity to explain and write their ideas, it gives them time to revise and clarify their thinking. (Unit 9 on–line discussion thread, line 40)

*6.3.1.2 Part 2 First Question Responses*

Seven of the teachers said that giving Romina more than one opportunity to explain and write about her work helped Romina to further clarify her solution to others. The seven teachers responded as follows:

> This was helpful for Romina because made refinements to her work and was also able to clarify her solution or reasoning by a different presentation. (Unit 9 on–line discussion thread, line 1)

> In chapter 8, Romina had to explain and re-explain multiple times to the boys and every time she explained it, she got better at it and was more convinced herself. After the 2nd or 3rd time of explaining it, she finally convinced Ankur. She re-wrote her example neatly and explained it again to Jeff, who then finally was convinced also. (Unit 9 on–line discussion thread, line 3)

> At first she did not explain on paper why she multiplied by 2 and 3, but after four drafts she was able to fully articulate that her reason for multiplying by 2 was to account for the combinations of Blue/Yellow and Yellow/Blue and her reason for multiplying by 3 was because there are 3 colors to choose from to construct the tower combinations. (Unit 9 on–line discussion thread, line 33)

> As seen in the article, Romina was able to change her representation and better explain her reasoning for an answer of 36 towers. Her representation changed from x, 1, and 0 to letters representing the colors by the end and her multiplication was not only shown but explained in writing. She was able to justify and provide a more convincing argument. (Unit 9 on –line discussion thread, line 35)

> When Romina explained her solution the first time, she was not very convincing as to why she was multiplying by 2 and 3. As she thought about it more, she was able to interpret her solution better. She drew it out and her argument became more convincing each time she explained it. (Unit 9 on –line discussion thread, line 37)

> She had to explain her reasoning several times and you can see her get better at the explanation each time she had to explain to those who didn't understand. For each argument-iteration, ‘Romina made refinements and clarified her reasoning.’ I noticed this statement within my own student as well as in Romina throughout this chapter. (Unit 9 on –line discussion thread, line 38)

> The chapter mentions how Romina revised her thinking several times and changed her representations in order to explain it better to the other students. (Unit 9 on –line discussion thread, line 40)

Five of the teachers said that giving Romina more than one opportunity to explain and write about her work helped Romina have a deeper conceptual understanding of the solution. The five teachers responded as follows:

> In this chapter it mentions that when a student is able to review their work they have a chance to further understand the concept. (Unit 9 on –line discussion thread, line 1)

> I believe that when Romina was given an opportunity to explain and write her work more she noticed a few mistakes and also changed her organization of her thinking which helped her make a mathematical discovery. (Unit 9 on –line discussion thread, line 21)

> In Chapter 8, Romina needed to explain her solution to Ankur, Jeff and Mike several times in order for them to not only understand, but be convinced that her solution made sense. The chapter even said that by allowing a student to review their work, it gives them a second chance to better understand their own solution, which was the case in Romina’s solution. (Unit 9 on –line discussion thread, line 35)

> She had to explain her reasoning several times and you can see her get better at the explanation each time she had to explain to those who didn't understand. For each argument-iteration, ‘Romina made refinements and clarified her reasoning.’ I noticed this statement within my own student as well as in Romina throughout this chapter. (Unit 9 on –line discussion thread, line 38)

> By allowing Romina to make revisions to her proof, she was better able to understand the problem herself. (Unit 9 on –line discussion thread, line 40)

Five of the teachers said that giving Romina more than one opportunity to explain and write about her work helped Romina refine or revise her work. The five teachers responded as follows:

> This was helpful for Romina because made refinements to her work. (Unit 9 on – line discussion thread, line 1)
>
> As Chapter 8 demonstrates, as Romina re-explained and rewrote her proof to Ankur's problem she refined her work and gave more details. (Unit 9 on –line discussion thread, line 14)
>
> I believe that when Romina was given an opportunity to explain and write her work more she noticed a few mistakes and also changed her organization of her thinking which helped her make a mathematical discovery. (Unit 9 on –line discussion thread, line 21)
>
> For each argument-iteration, 'Romina made refinements and clarified her reasoning.' I noticed this statement within my own student as well as in Romina throughout this chapter. (Unit 9 on –line discussion thread, line 38)
>
> By allowing Romina to make revisions to her proof, she was better able to understand the problem herself and realize how it would be most convincing and clear to explain to others. (Unit 9 on –line discussion thread, line 40)

Two teachers said that giving Romina more than one opportunity to explain and write about her work helped Romina to use different representations for her solution. The three teachers responded as follows:

> Her representation changed from x, 1, and 0 to letters representing the colors by the end and her multiplication was not only shown but explained in writing. She was able to justify and provide a more convincing argument. (Unit 9 on–line discussion thread, line 33)
>
> The chapter mentions how Romina revised her thinking several times and changed her representations in order to explain it better to the other students. (Unit 9 on–line discussion thread, line 40)

### *6.3.2 Second Question Responses*

Six of the teachers explained why Romina multiplied by two when finding her solution to Ankur's Challenge using words such as switching, reversing, or interchanging positions. The responses of the four teachers that used words such as switching, reversing, or interchanging positions to describe why Romina multiplied by two follow:

> Romina multiplied by two because the two different colors that aren't repeated are interchangeable, as in her chalkboard example. The one color we know has to be in the tower twice and the other two can switch, which is why she multiplied by two. (Unit 9 on–line discussion thread, line 3)

> Romina multiplied her answer by 2 to represent that her sample of 6 towers with Red being the color taking up two of the four positions in the tower was only one of two possible towers with Red in those positions. For example, Romina could have the towers R-B-R-Y and R-Y-R-B where Red is in the same positions in both towers but the Yellow and Blue cubes have switched positions. (Unit 9 on–line discussion thread, line 14)

> Romina multiplied by 2 when finding her solution because she found 6 possible combinations where the same color is duplicated. The other two positions would be the other two colors. Since the position of the other two colors can be reversed, Romina multiplied 6 by 2. (Unit 9 on–line discussion thread, line 21)

> Romina multiplied by two in her solution because she had represented two possibilities in each of the six towers she drew. She noticed that for each of the six positions for the two colors that match, there were two placement options for the non-repeated colors. Therefore, the six towers could be multiplied by two to find that there are 12 possible towers that have two of the same color. (Unit 9 on–line discussion thread, line 33)

> Romina multiplied by two when finding her solution to Ankur’s challenge because she saw that in order for the tower to be 4-cubes tall, two of the cubes would have to be the same color. She then saw that there were six different arrangements for the two colors. She then assigned an “X” and an “O” to represent the two additional colors. She then multiplied by two because there would be an opposite or a different arrangement for the two other colors within the tower. (Unit 9 on–line discussion thread, line 37)

> Romina first examined the outcomes for placing 2 of the same color in a tower and the positions that these 2 blocks could occupy. This gave her 6 different towers. To fill the other two spots in the tower she could set the alternating two colors in the unoccupied spots. She then multiplied by 2, to show what would happen if these two other colors had their positions reversed. (Unit 9 on–line discussion thread, line 40)

Four of the teachers explained why Romina multiplied by two when finding her solution to Ankur's Challenge using the word opposite. The responses of the four teachers that used the word opposite to describe why Romina multiplied by two follow:

> So she multiplied by two because for each pattern there would be an opposite or a different arrangement for the two colors in the tower. (Unit 9 on –line discussion thread, line 1)
>
> Romina multiplied by two when finding her solution to Ankur's challenge because she saw that in order for the tower to be 4-cubes tall, two of the cubes would have to be the same color. She then saw that there were six different arrangements for the two colors. She then assigned an "X" and an "O" to represent the two additional colors. She then multiplied by two because there would be an opposite or a different arrangement for the two other colors within the tower. (Unit 9 on –line discussion thread, line 35)
>
> When she made the towers she made only 6 towers but represented two combinations within each single tower; therefore, each single tower that she drew really represented 2 different towers since each one combination she came up with can have an "opposite" combination. So she multiplied the 6 single towers she drew by 2 to represent the two choices from each combination pattern. (Unit 9 on –line discussion thread, line 38)
>
> Romina multiplied by 2 to make up for the opposites that could be created by alternating the colors in the same design. (Unit 9 on –line discussion thread, line 39)

**6.4 Unit 10: On-line Discussion 11/6/13 to 11/12/13**

The instructor assigned the following on-line discussion questions for unit ten.

1. What kind of strategies did one of your students use to find solutions for the different tasks? Be specific. Did the student stay with the same strategy over the different tasks or did the student approach the problems with different strategies? Again, be specific.

2. Talk about the students' attempts to provide justifications for their answers. What kinds of convincing arguments did they use? Did they become more convincing as they did more tasks? (Unit 10 on-line discussion thread, questions 1 and 2)

### *6.4.1 First Question Responses*

The first on-line question had two parts. The first asked teachers to pick one sample of students' work from the first two tasks and report on what kind of strategies were used by that student (Unit 10 on-line discussion thread, questions 1). For the second part, teachers were asked if the student stayed with the same strategies over the first two tasks or did the student use different strategies (Unit 10 on-line discussion thread, questions 1).

#### *6.4.1.1 Part 1 First Question Responses*

The first part of the question asked teacher about the kinds of strategies used for solving the different tasks. Five teachers said that their students used the elevator strategy. The responses of the five teachers follow:

> My one pair of boys focused on moving one color down each time, the recursive argument. They didn't reach the correct answer because they didn't do this for all the colors, however, they were close. They explained to me that the color moved down each time so that's why there couldn't be any more. I knew what they were talking about but I wanted them to show me. So, I told them I wasn't convinced and to arrange the towers in a way to help me understand better and they did. They grouped them in threes. Each group showed how the one color moved down from being the first spot, to the second, to the third. (Unit 10 on–line discussion thread, line 8)

> After questioning him about whether he had all the towers, he and his partner decided to rearrange the towers. At this point, he saw the staircase and candy cane patterns. He was able to arrive at all 16 towers. (Unit 10 on–line discussion thread, line 10)

> She often referred to the towers that contained three of one color and one of the other colors as the "staircase", similar to other students. She was able to easily arrive at the 16 different towers. (Unit 10 on–line discussion thread, line 16)

> Group 5 is the staircase pattern with 3 yellows and 1 blue. Group 6 is the staircase pattern with 3 blues and 1 yellow. (Unit 10 on–line discussion thread, line 29)

> For the first tower task, my student created a "staircase pattern" and was able to form a convincing argument about the group of towers with three of one color and one of another. (Unit 10 on–line discussion thread, line 36)

Five teachers said their students used the opposite strategy. The responses of the five teachers follow:

> My student tried to match up opposites with the first task. He saw the opposites first; however he was unable to come up with all 16 towers. (Unit 10 on–line discussion thread, line 10)

> In the 4 tall towers choosing from 2 colors, he created 6 groups. The 1st group contains two colors the same in the middle and the other color on the end such as YBBY and BYYB. Group 2 is the pair of alternating towers BYBY and YBYB. Group 3 is the two towers that are solid colors BBBB and YYYY. Group 4 has the two colors split in half BBYY and YYBB. (Unit 10 on–line discussion thread, line 29)

> For the first task my student began making opposites as her strategy. (Unit 10 on–line discussion thread, line 31)

> For the 1st task, my student started with making opposite pairs. (Unit 10 on–line discussion thread, line 32)

> For the other towers four tall, the student was not able to make a convincing argument and relied heavily on the opposite reasoning. The student explained that he had achieved all of the towers because each had an opposite. (Unit 10 on–line discussion thread, line 36)

Three teachers said their students used “controlling for a variable” as a strategy. The responses of the three teachers follow:

> Then she manipulated the groups to have a constant on top. (Unit 10 on–line discussion thread, line 32)

> He and his partner kept a color constant when working with three colors instead of two. (Unit 10 on–line discussion thread, line 36)

> He also held a constant in each group for the pizza problem. For example, with the two topping pizzas he would start with all the pairs with pepper, then move on

> knowing pepper would not be used again. (Unit 10 on–line discussion thread, line 37)

One teacher said that her student used the rule strategy and made a pattern. Her response follows:

> The first assignment with the cubes my student initially chose the strategy of making a pattern. The pattern started off random and then eventually he organized the cubes to show his work clearer. As for the second problem my student automatically made the connection to the first problem. He then tried to solve it mathematically by multiplying 4 x 4. However, he was not able to justify his answer. (Unit 10 on–line discussion, line 1)

*6.4.1.2 Part 2 First Question Responses*

The second part of the first questions asked whether the student stayed with the same strategy through the first two tasks or did the student change their strategy (Unit 10 on-line discussion thread, instructor questions). Seven of the teachers mentioned that their student changed strategies from the first task to the second task. The responses of the seven teachers that said their student changed strategies from the first task to the second task follow:

> The first assignment with the cubes my student initially chose the strategy of making a pattern. The pattern started off random and then eventually he organized the cubes to show his work clearer. As for the second problem my student automatically made the connection to the first problem. He then tried to solve it mathematically by multiplying 4 x 4. However, he was not able to justify his answer. He then started with a tree diagram and then switched to a chart to solve the problem. (Unit 10 on–line discussion thread, line 1)

> My student tried to match up opposites with the first task. He saw the opposites first; however he was unable to come up with all 16 towers. After questioning him about whether he had all the towers, he and his partner decided to rearrange the towers. At this point, he saw the staircase and candy cane patterns. He was able to arrive at all 16 towers. On the second task, he began by creating a web with the word pizza in the middle. He branched off of the pizza with different toppings. (Unit 10 on–line discussion thread, line 10)

> For the first task, this student drew the different towers using two different colored highlighters (blue and yellow). She grouped the towers according to the

number of unifix cubes within each tower. She often referred to the towers that contained three of one color and one of the other colors as the “staircase”, similar to other students. She was able to easily arrive at the 16 different towers. For the second task, she made an organized list to create the 16 different pizza combinations. She used an arrow method to describe how she created the different pizza combinations. (Unit 10 on–line discussion thread, line 16)

My student did not stick with the same problem solving strategy for the two tasks. The first task building towers was much easier for him because the manipulatives provided an opportunity to make guesses and mistakes without the finality of writing it on paper. The second task proved difficult to him because he was very reluctant to write down anything he wasn't extremely sure of. When he did write something down it was a tree diagram which frustrated him quickly. He did not stick with that strategy and instead chose to list out the pizza combinations but did not have great organization. (Unit 10 on–line discussion thread, line 25)

For the first task my student began making opposites as her strategy. When being questioned about how she knew she had them all but she had a difficult time explaining it. I asked her to rearrange the towers in a different way maybe she will see a more concrete explanation, after this she was able to see there was more of a pattern by organizing it using proof by cases. She still had a hard time explaining in writing but she was able to see two different ways to approach the problem. For the second task she had a harder time coming up with a strategy since she wanted to dive into this one by making opposites again she quickly realized that wouldn’t work for this problem. She then tried a tree diagram which, after a while she realized that was difficult to do and she began getting frustrated. She then started to make a list, even with this she did not organize it right away, she finally saw that she was making 1 topping, 2 topping, etcetera… and reorganized her work to make more sense. (Unit 10 on–line discussion thread, line 31)

For the 1st task, my student started with making opposite pairs. Then she manipulated the groups to have a constant on top. She used the recursive argument to organize and solve the problem. For the 2nd task, she started by just making toppings at random. Then with the suggestion to use some type of organization she created groups by using the first topping as a constant. For example one group had mushroom as the single topping; then mushroom with each of the other toppings for a 2 topping pizza, then mushroom with 2 other toppings for a 3 topping pizza. (Unit 10 on–line discussion thread, line 32)

For the first tower task, my student created a "staircase pattern" and was able to form a convincing argument about the group of towers with three of one color and one of another. For the other towers four tall, the student was not able to make a convincing argument and relied heavily on the opposite reasoning. The student explained that he had achieved all of the towers because each had an opposite. As we moved onto the second tower problem, this student approached the problem with more strategy. He and his partner kept a color constant when working with three colors instead of two. (Unit 10 on–line discussion thread, line 36)

Two teachers claimed that their student stayed with the same strategy for the first two tasks. The responses of the two teachers that claimed their student stayed with the same strategy for the two tasks follow:

> The student who I chose used the proof by cases in the first two tasks to find his solutions. (Unit 10 on–line discussion thread, line 29)

> Looking at the work of one of my students for the first tower problem, and the pizza problem, I noticed that the student used cases for both. (Unit 10 on–line discussion thread, line 37)

### *6.4.2 Second Question Responses*

The second question had 2 parts. For the first part of the second question, teachers were asked about the kinds of convincing arguments used by their students. The second part asked teachers if the students' arguments became more convincing over the different tasks.

#### *6.4.2.1 Part 1 Second Question Responses*

Four teachers responded that their students' arguments were convincing because they used a recursive argument. The four teachers' responses follow:

> My one pair of boys focused on moving one color down each time, the recursive argument. (Unit 10 on–line discussion thread, line 8)

> For the second task, she was definitely able to verbally explain her "arrow" method to me in which she used every pizza topping in a different position. (Unit 10 on–line discussion thread, line 16)

> The convincing argument that my student used was the recursive argument. In the 1st task she moved the blocks towards the bottom to represent each possible combination and then said that there could be no more because if the blocks were moved again there would be a double. (Unit 10 on–line discussion thread, line 32)

> For this argument, the student kept a color of a block constant in the first, second and third position and repeated it for all the colors. (Unit 10 on–line discussion thread, line 36)

Two teachers responded that the convincing argument their students used was a cases argument. The responses of the two teachers follow:

> I asked her to rearrange the towers in a different way maybe she will see a more concrete explanation, after this she was able to see there was more of a pattern by organizing it using proof by cases. She still had a hard time explaining in writing but she was able to see two different ways to approach the problem. For the second task she had a harder time coming up with a strategy since she wanted to dive into this one by making opposites again she quickly realized that wouldn't work for this problem. She then tried a tree diagram which, after a while she realized that was difficult to do and she began getting frustrated. She then started to make a list, even with this she did not organize it right away, she finally saw that she was making 1 topping, 2 topping, etcetera… and reorganized her work to make more sense. (Unit 10 on–line discussion thread, line 31)

> For the tower problem, he went through each of the three cases and provided an argument for why he had them all, mentioning why you could not have more. For example, with the case for three of one color and one of the other, he mentioned that the single cube would start at the top of the tower and work down one position each time until it reached the bottom and that there could not be a fifth tower since you would need it to be five cubes tall which does not fit the criteria. For the pizza problem, he just wrote that you can start with a plain pizza, then write the options for 1 topping, 2 toppings, 3 toppings, and 4 toppings and make sure there were no repeats. (Unit 10 on–line discussion thread, line 37)

*6.4.2.2 Part 2 Second Question Responses*

Six teachers claimed that their students' justifications became more convincing as they worked on more tasks. The responses of the six teachers follow:

> I felt my students in general improved a lot from the first task. From their explanations to working together, I saw a big improvement. (Unit 10 on–line discussion thread, line 8)

> I do feel he was able to get a little more convincing with his argument; although he was not able to complete the argument. (Unit 10 on–line discussion thread, line 10)

> I have found that over the two tasks, my students have had more success verbally explaining from task one to task two. I have definitely seen a lot of growth in them from task one to task two. (Unit 10 on–line discussion thread, line 16)

> I think the frequent addition of this wording in his explanation makes the proof by cases slightly stronger than the justification for the first towers problem, though still not very convincing. (Unit 10 on–line discussion thread, line 29)

> For the second task her justification was more concrete explaining that she started with a different amount of toppings on the pizza until she reached the max which would be 4, she could not have 5 because there weren't 5 options. She also stated why opposites wouldn't work for this problem. She is beginning to get better with her explanations. (Unit 10 on–line discussion thread, line 31)

> This student did become more convincing as he completed more tasks. (Unit 10 on–line discussion thread, line 36)

Two teachers claimed that their students provided more convincing arguments with the first task than the second task. The responses of the two teachers follow:

> I feel he was more convincing with problem one than for problem two. I think as a whole all my students were more convincing for problem one. (Unit 10 on–line discussion thread, line 1)

> The written explanation for this student was better on the first tower problem than it was in the pizza problem though his verbal argument for the pizza problem was convincing. (Unit 10 on–line discussion thread, line 37)

### 6.5 Unit 11: Regional Meeting 11/20/13 and On-line Discussion

Unit 11 was comprised of a regional meeting on 11/20/13 along with an on-line discussion thread (Landis, 2013). The discussion thread gave teachers the option to post questions about their final project from 11/21/13 to 11/26/13 (Unit 11, on-line discussion thread, question 1). After the discussion of the in-district classroom visit at the regional meeting, teachers shared students' work on the third cycle of tasks (11/20/13 meeting transcript, line 1).

#### *6.5.1 Discussion of In-District Classroom Visit*

Ten teachers met at a middle school in New Jersey and brought students' work samples of the 3-tall towers problem and Ankur's Challenge. For the third in-district classroom visit, the Cycle 3 tasks were implemented with sixteen sixth-grade current students of a teacher. The students were sitting in individual student desks that were pushed together in pairs (11/20/13, meeting transcript, line 1). They were asked to convince their partners of their solutions and write down the solutions after convincing one of the researchers (11/20/13, meeting transcript, line 1).

After the current students left, the instructor held a debriefing meeting with the ten teachers from the southern region group to discuss the students' solutions from the in-district classroom implementation. An IPad was used to take pictures of the students' work and project it on the screen. The instructor asked the teachers to first look at the student's work who made a chart with three columns (11/20/13, meeting transcript, line 5). Figure 6.5 shows the student's work.

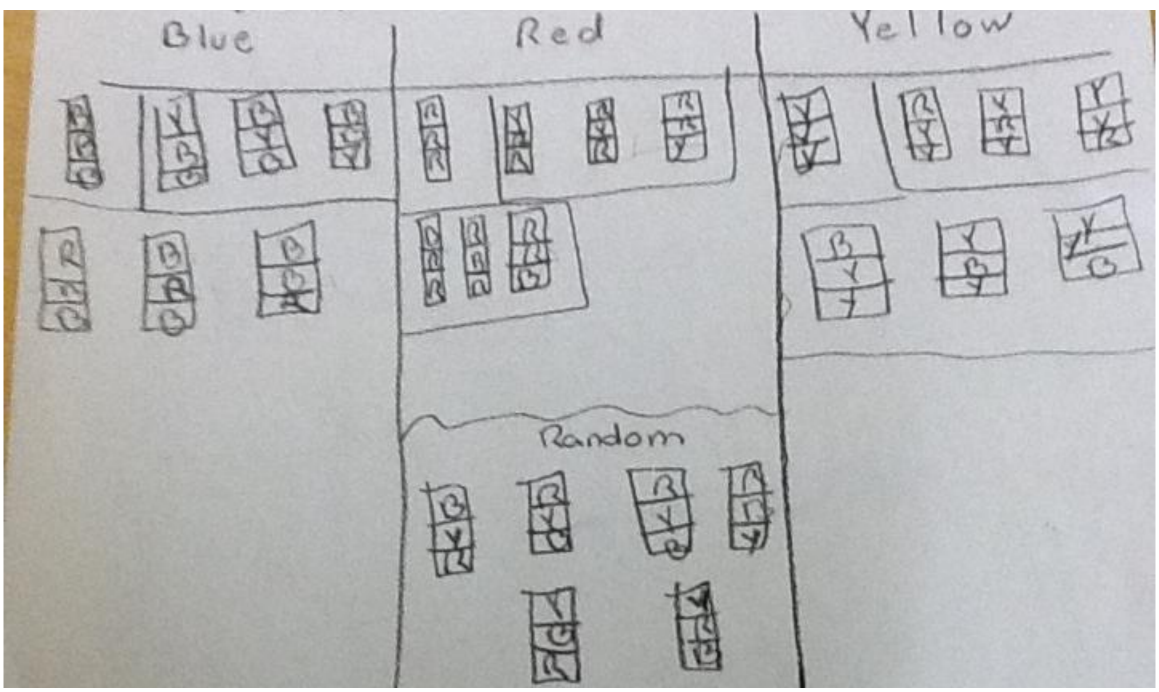



*Figure 6.5 First Discussed Student's Work from 11/20/13 Classroom Visit*

One of the teachers called the color that was used most often in each column as "dominant" (11/20/13 meeting transcript, line 13).

The first teacher (T1) read the following student's argument to the teachers:

> I think there's 27 towers are all the possible explanations because for each group there is a pattern that goes diagonal through each tower. In that pattern is one single opposite block. We have the towers below each category as red blue and yellow. The reason we have categories is because there are two main blocks in each tower. These two blocks are the same, and the blocks that made the pattern are different. So in the blue category, there will be one yellow block on a stack of two blue blocks. This helps prove our theory. We also have a random category where each tower contains one different block. So, one of our towers is red on the bottom, blue in the middle, and yellow on top. If we changed the order of the blocks, it would have a different tower we already have. (11/20/13 meeting transcript, line 21, 23)

The instructor asked the teachers "Which parts of their argument are convincing and which aren't?" (11/20/13 meeting transcript, line 26). One teacher said "It is not really convincing anywhere" (11/20/13 meeting transcript, line 27). The instructor said "What is his argument there?" (11/20/13 meeting transcript, line 30). One teacher replied that "He said he has a pattern that goes diagonal through each tower." (11/20/13 meeting transcript, line 31). The argument was a verbal description but not a convincing written argument (11/20/13 meeting transcript, line 36).

In the second sample of student work discussed, the student also used the letters B, R, and Y to represent the blue-, red-, and yellow-colored cubes respectively (11/20/13, meeting transcript, line 31). The student made five groups of three-tall towers. The student had a group that had one B, one R, and one Y written in separate squares which represented the towers made of all blue-, all red-, and all yellow-colored cubes. The student labeled the group as group two. The other four groups each had six towers. Figure 6.6 shows the student's work.

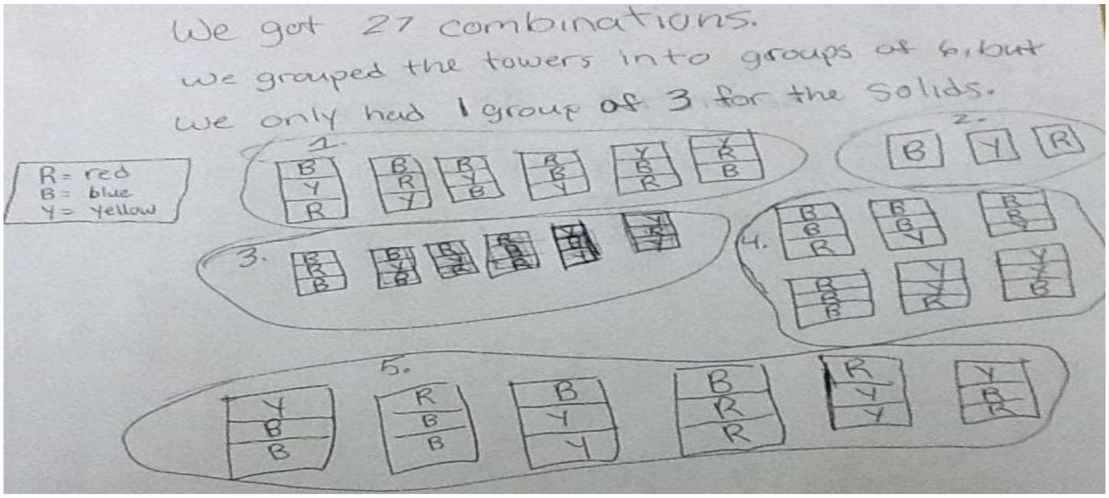



*Figure 6.6 Second Discussed Student's Work from 11/20/13 Class Visit*

The following student's argument was put on the screen for the teachers to read:

> We got 27 combinations. We grouped the towers into groups of 5, but we only had groups of 3 for the solids.
>
> Group 2: There are no more combinations for the solids because we already used all of the colors.
>
> Group 1: We used each block once in each tower and we also used one colored block twice for the top. So for example: BB<br>YR<br>RY<br>That's how we know there are no more possible ways.
>
> Group 3: We used one color twice in the middle row. Example: YB<br>RR<br>BY
>
> Group4: We used one color twice and put it on the top. For example:<br>RR<br>RR<br>YB
>
> (11/20/13, meeting transcript, line 37)

The instructor asked the teachers "how did he arrange group one? (11/20/13 meeting transcript, line 42). One teacher replied "Well like he had a pair of blues on top, and a red and yellow on the bottom." (11/20/13 meeting transcript, line 43). Another teacher said that “It looks like they were reversed on the bottom.” (11/20/13 meeting transcript, line 44). The instructor restated what the teachers expressed by saying “They kept the two with the same top and they said that the bottom could be Y and R or R and Y. Okay and there was no other way to do it. And then they kept the two tops red, and did the same thing with the other two colors. Then the two tops yellow and bottoms are the other two colors in both positions.” (11/20/13 meeting transcript, line 45).

The second group that the student drew was a B, R, and Y in squares to represent the three solid-colored towers (11/20/13 meeting transcript, lines 46-49). The instructor continued the discussion with the third group of towers where the student had the red-colored cubes in the middle of the tower and asked the teachers “what did they do with the top and the bottom?” (11/20/13 meeting transcript, lines 57-59). One of the teachers replied that the student “switched them.” (11/20/13 meeting transcript, line 60).

The instructor then asked the teachers about the student’s fourth group (11/20/13 meeting transcript, lines 70). One teacher replied “We used one color twice and put on the top.” (11/20/13 meeting transcript, line 71). The instructor told the teachers that she was impressed with the writing of the sixth-grade student (11/20/13 meeting transcript, line 78).

The instructor then requested to see another students’ sample of work (11/20/13 meeting transcript, line 78). The following student’s argument was put on the screen for the teachers to read: “I know I got all of them because all the groups I got the same amount of 3 towers.” (11/20/13 meeting transcript, lines 80). Figure 6.7 shows the student’s work.

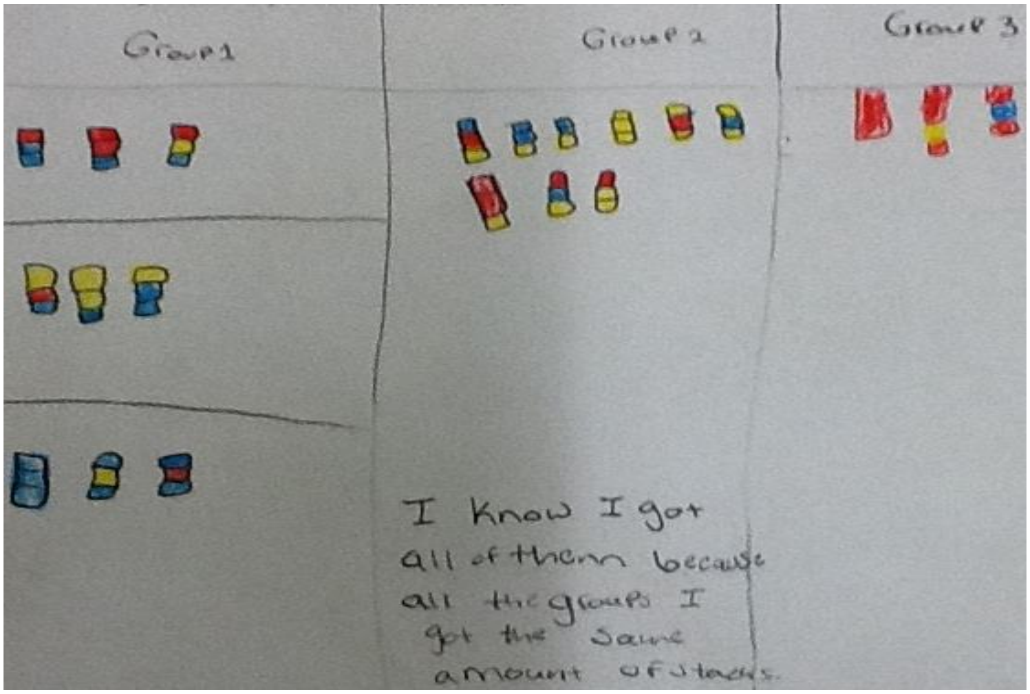



*Figure 6.7 Third Discussed Student Work from 11/20/13 Classroom Visit*

The drawing that the student provided was also put on the screen and the instructor said "The first one would be three red on top three blues on the bottom and they used an inductive argument." (11/20/13 meeting transcript, line 116). The instructor then asked the teachers what the students put in the middle (11/20/13 meeting transcript, line 116). One teacher replied "The three different colors that they had." (11/20/13 meeting transcript, line 117).

The instructor then asked the teachers "Was there any one else you wanted to talk about from today that we didn't talk about? (11/20/13 meeting transcript, line 132). One teacher wanted to discuss the extension problem (11/20/13 meeting transcript, line 133). She had watched two students that had different ideas working on the problem but had the same correct answer of 36 (11/20/13 meeting transcript, lines 151-159). The teacher said that his partner "was focusing on two colors that are not the same but not next to each other" (11/20/13 meeting transcript, lines 151) and that the student "was focusing on just having a red on the bottom and all the other colors yellow and blue but alternating" (11/20/13 meeting transcript, lines 151). The teacher recognized that the student controlled for a variable (11/20/13 meeting transcript, lines 151). The instructor ended the debriefing discussion of the in-district classroom visit with "And that they did it quickly is pretty impressive" (11/20/13 meeting transcript, line 175).

### *6.5.2 Discussion of Students' Work Samples*

For the next part of the regional meeting on 11/20/13, teachers shared students' work from their own classrooms. T10 was the first to share her students' work that had eight different groups (11/20/13 meeting transcript, line 7). The student drew a key on his paper to show that blue-colored cubes would be a white box, red-colored cubes would be a shaded box, and yellow cubes would have a diagonal stripe in the box going from the top left-hand corner of the box to the bottom right-hand corner of the box. Figure 6.8 shows the student's work.

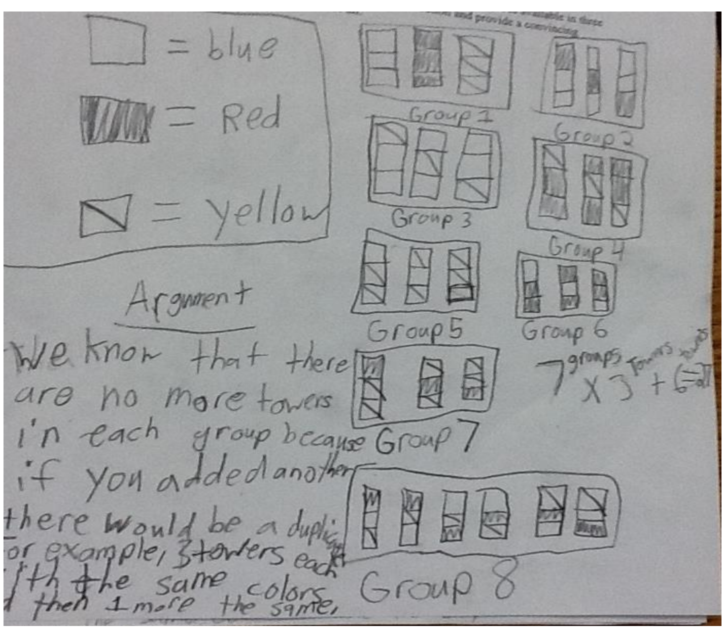



*Figure 6.8 T10's Cycle 3Three-Tall Tower Problem, Student Work Sample 1*

All eight groups of 3-tall towers were drawn using the key. The first group of towers had one tower with white boxes to represent a blue-colored tower, one tower that with shaded boxes to represent a red-colored tower, and one tower with diagonal stripes to represent a yellow-colored tower. Groups three through seven were constructed using the elevator strategy.

For groups two and seven, red –colored cubes moved from the top down one position until all positions were exhausted. For groups three and four, yellow-colored cubes moved from the top down one position until all positions were exhausted. Groups five and six had the blue-colored cubes moved from the top down one position until all positions were exhausted. For group 8, the student had six towers where the first and second towers had red-colored cubes on the top, the third and fourth towers had blue-colored cubes on the top, and the fifth and sixth towers had yellow-cubes on the top.

T10 read the following student's written argument to the teachers:

> We know there are no more towers in each group because if you added another there would be a duplicate. For example, three towers each with the same colors and then one more the same. (11/20/13 meeting transcript, line 30)

The instructor asked the teachers "What do you think of the argument?" (11/20/13 meeting transcript, line 31) and one of the teachers replied "Not good." (11/20/13 meeting transcript, line 32). The instructor agreed but pointed out that the student's strategy of controlling for a variable in group 8 was good (11/20/13 meeting transcript, line 35).

T10 also shared a second student's work that had five groups of 3-tall towers which had the letters B, Y, and R inside boxes to represent blue-, yellow-, and red-colored cubes. The first group had all the solid-colored towers; represented by three B's, three Y's, and three R's. The second and third groups had six towers each where the first and second towers had B's on the top, the third and fourth towers had Y's on the top, and the fifth and sixth towers had R's on the top. The fourth group had six towers where the first and second towers had the top and middle positions as B's, the third and fourth towers had the top and middle positions as Y's, and the fifth and sixth towers had the top and middle positions as R's. The fifth group had six towers where all three colors were used once for each tower and the first and second towers had B's on the top, the third and fourth towers had Y's on the top, and the fifth and sixth towers had R's on the top.

T10 read the following student's argument to the teachers:

> There are 27 towers of three cubes. We know this because each color has 2 combinations except for ones that are all the same color." (11/20/13 meeting transcript, line 50). In group one I know there are no other combinations because other colors would just be a duplicate. In group two, there are no other combinations because if we switched other blocks around it. It would be a duplicate. The top and bottom block are the same color so the middle block has only two options. In group 3 the bottom blocks are the same color. (11/20/13 meeting transcript, lines 50-62).

T10 recognized that her student used the strategy of controlling for a variable in one of the tower groups to solve the 3-tall towers problem, selecting from 3 colors (11/20/13 meeting transcript, lines 46, 48). The instructor said "They have the start of a good convincing argument." (11/20/13 meeting transcript, line 71).

T2 was the second teacher to share her seventh-grade students' work with the teachers (11/20/13 meeting transcript, lines 72-75). The description of T2's student's work follows:

> They did ten groups but they did 7 groups of 3. And then instead of keeping that last group of 6, they did it in pairs so they had 8, 9, and 10 are the alternating as pairs. So they had yellow as a constant and then they did the bottom. (11/20/13 meeting transcript, lines 78-82).

T2 recognized that her student also used the strategy of controlling for a variable to solve the 3-tall towers problem, selecting from 3 colors (11/20/13 meeting transcript, line 82). T2 read the following written argument provided by the student:

> I know the answer is 27. I know there is no more possible ways because in group 2, I moved the blue cube in each position way I could. There were only 3 positions because it could only be three high. I did the rest for groups 3, 4, 5, 6, and 7. In group 1 I made three different towers solid colors yellow blue and red because I only had 3 colors. For 8, 9, and 10, I kept one color on top and switched around the two underneath them. (11/20/13 meeting transcript, lines 92-102)

The instructor said that the student was "really explaining what she did" (11/20/13 meeting transcript, line 107) but was impressed with her writing (11/20/13 meeting transcript, line 111). Figure 6.9 shows the drawing of the student's towers shared by T2.

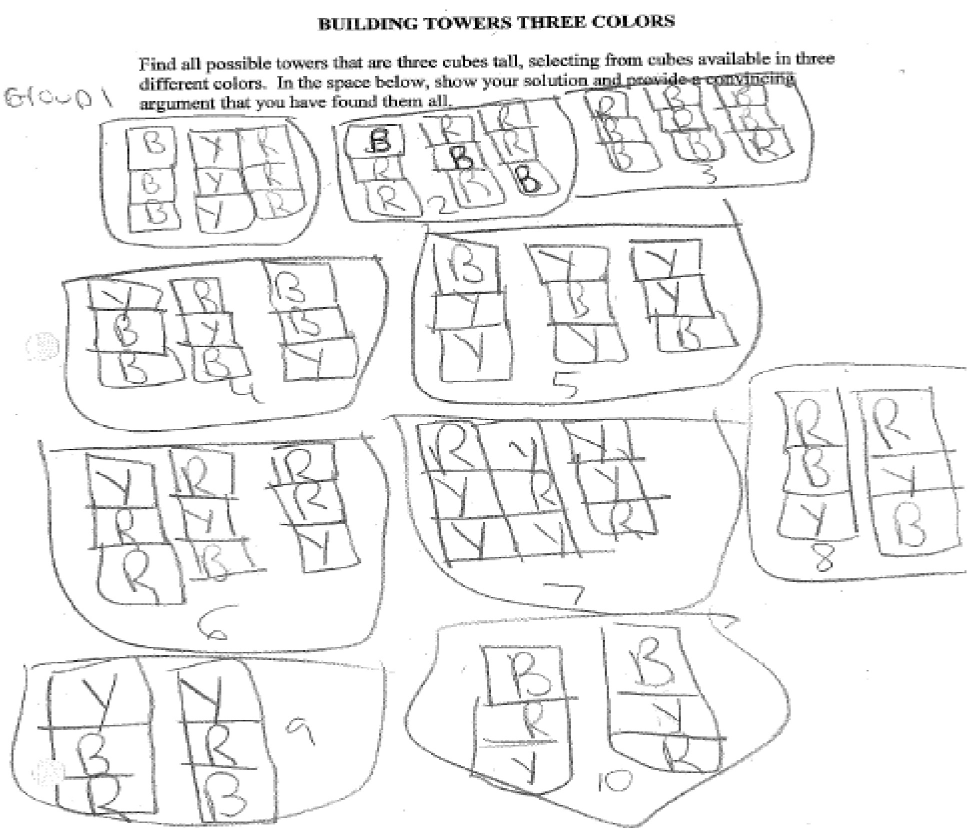



*Figure 6.9 T2's Cycle 3Three-Tall Tower Problem, Student Work Sample 1*

T2 also shared the partner's work of the first-shared sample of student work (11/20/13 meeting transcript, line 116). T2 read the following student's argument to the teachers:

> The answer is 27. There is no other way without duplicates. I know there is no other way because in group 2, I moved the blue cubes in each possible way I can. There are only 3 positions because it can only be three high. I did the rest of the groups like this. I only had 3 solid colors to choose from and that is how I got my answer. (11/20/13 meeting transcript, lines 134-136)

T2 said the partner had "a little less detail" (11/20/13 meeting transcript, line 136). The instructor told the teachers that the argument was partially convincing (11/20/13 meeting transcript, line 137).

T3 was the third teacher to present and she shared her seventh-grade resource student's work. There were three groups of towers. Each group was named for the three colored cubes. The red group had all red-colored cubes on the bottom of each of the nine towers, the blue group had all the blue-colored cubes on the bottom of each of the nine towers, and the yellow group had all the yellow-colored cubes on the bottom of each of the nine towers (11/20/13 meeting transcript, line 144).

For each group, a pattern was used be the student to construct all nine towers of each group. T3 described the example of how the student constructed the nine towers of each group as "the shape of almost like a turned L" (11/20/13 meeting transcript, line 154), checkers and a meat sandwich (11/20/13 meeting transcript, lines 160-162). Figure 6.10 and figure 6.11 show the drawing of how the student constructed the nine towers of each group.


*Figure 6.10 T3's Cycle 3Three-Tall Tower Problem, Student Work Sample 1 page 1*

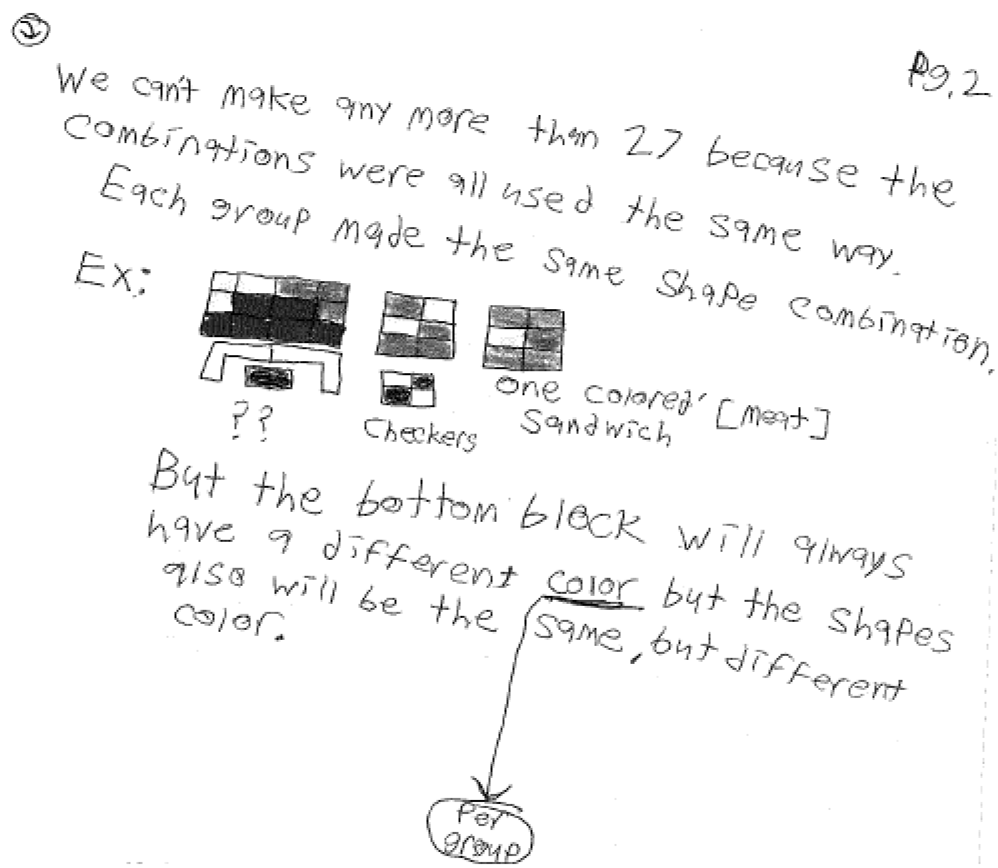



*Figure 6.11 T3's Cycle 3Three-Tall Tower Problem, Student Work Sample 1 page 2*

The following student's written argument was projected on the screen:

> The bottom block will be a group. That means the bottom block is main. I came up with 27 in all, 9 in a group, and the other, and the other. It's basically 9 times 3. We can't make any more than 27 because the combinations were all used the same way. Each group made the same shape combination. But the bottom block will always have a different color per group but the shapes also will be the same but different color. The combination is the same, but in different colors. The bottom block is the leader. When changed, it will never change its combinations. But it will change its colors. (11/20/13 meeting transcript, lines 134-136)

T3 recognized her student controlled for a variable as a strategy to solve the three-tall towers problem (11/20/13 meeting transcript, line 144).

T3 shared a second sample of work by one of her students that had five groups. The first group had 6 subgroups of 3-tall towers constructed using two colors and the elevator strategy. In the second group, the student controlled for a variable by having two towers with blue on top. The third group had two towers with yellow on top and the fourth group had two towers with red on top. The fifth group had three towers of all red-, all blue-, and all yellow-colored towers (11/20/13 meeting transcript, lines 174-188). T3 read the following written argument from the student:

> Each one has two of one color and one of one color and then moved the single one down each time. We did opposites for a total of 18 altogether. So she just did the opposites. This is the only way to have all 3 colors used once in a 3 stack high and only have blue on the top. This is the only way to have all 3 colors used once in a 3 stack high and only have yellow on the top. This is the only way to have all 3 colors used once in a 3 stack high and only have red on the top. Each one has only one solid color. (11/20/13 meeting transcript, lines 174-188)

T3 recognized that her student controlled for a variable to solve the three-tall towers problem (11/20/13 meeting transcript, lines 184-186).

T1 was the fourth teacher to present her students' work to the other teachers. The first sample T1 shared was from a seventh-grade resource student (11/20/13 meeting transcript, line 208). The student had drawn five groups of 3-tall towers and wrote an X over the fifth group (11/20/13 meeting transcript, line 190).

The first group had three towers made of boxes that had letters g, b, and r to represent the colored cubes of green, blue, and red. The first tower had a green cube in the top and middle positions with a blue cube on the bottom. The second tower had a green on top, blue in the middle, and red on the bottom. The third tower had a blue on top, green in the middle, and red on the bottom (11/20/13 meeting transcript, line 190).

The second group had three towers of all red-, all blue-, and all green-colored towers. The third group had three towers. The first tower had red at the top, blue in the middle, and green on the bottom. The second tower had all red cubes. The third tower had a red at the top and green in the middle and bottom positions (11/20/13 meeting transcript, line 190).

The fourth group had three towers. One tower had red on top, green in the middle, and red on the bottom. The second tower had green on top, red in the middle, blue on the bottom. The third tower had blue on top, green in the middle, and red on the bottom (11/20/13 meeting transcript, line 190).

However, T1 said that her student had drawn the towers differently than the written explanation that was provided (11/20/13 meeting transcript, lines 194, 202-204). The following student's argument was placed on the screen for the teachers to read:

> Group 1 was built to make the blue go down. Group 2 had all the reds going down. Group 3 made all of the greens go down. Group 4 they were all in the same category. Green was with the blue with the green; red was with the blue with the red. (11/20/13 meeting transcript, lines 196-200)

T1 also shared a seventh-grade girl's work with the teachers (11/20/13 meeting transcript, line 214). The student had drawn twenty-three towers on her paper with squares using the letters Y, G, and B to represent the yellow-, green-, and brown-colored cubes; and three more towers were drawn that were made of all green-, all brown-, and all yellow-colored cubes. The first eight towers were drawn as opposite pairs (see example 2 in student's written explanation below). The next six towers were drawn as pairs of cousins (see example 4 below). The next four towers were drawn had one of each color in the tower (see example 3 below). The next two towers were drawn as opposite pairs (see example 2). The last three towers were three towers of all green-, all brown-, and all yellow-colored towers. The following student argument was placed on the screen for the teachers to read: "Explanation: We got the amount of 23 by taking certain groups of 2, 3, or 4 based on the pattern." (11/20/13 meeting transcript, line 214). Figure 6.12 shows examples that the student provided.

Example 1: GBY<br>GBY<br>GBY

Example 2: YB<br>BY<br>BY

Example 3: GBGY<br>YGBG<br>BYYB

Example 4: GY<br>GG<br>YG

*Figure 6.12 T1's Cycle 3Three-Tall Towers, Examples from Student Work Sample 1*

The instructor said to the teachers that "It is interesting that she didn't have groups that were the same size" (11/20/13 meeting transcript, line 223).

T4 was the fifth teacher to share her students' work. The first sample T4 shared was from an eighth-grade student (11/20/13 meeting transcript, line 226). The towers were drawn as squares with the letters B, R, and Y inside the squares. The student had drawn 27 towers in three groups of nine. Nine towers had blue on top; nine towers had red on top, and nine towers had yellow on top. T4 recognized that her student controlled for a variable (11/20/13 meeting transcript, line 226). The following student's argument was placed on the screen for the teachers to read:

> How I did it was that I made it just different ways with red, blue, and yellow just with these three different colors, then when I didn't see I didn't form the tower a different way, I made it. (11/20/13 meeting transcript, lines 230)

The instructor said to the teachers "It wasn't a convincing argument but that she used the strategy of holding a constant, is a very good strategy." (11/20/13 meeting transcript, line 231).

T4 shared a second student's work. The student drew 27 towers as squares with lowercase r, capital B, and capital Y inside the squares. The towers were in ten groups. The first group had two towers. The first tower had red on top, red in the middle, and yellow on the bottom. The second tower was the same except the yellow–colored cube was changed to a blue-colored cube. (11/20/13 meeting transcript, line 232).

A second group had three towers with all blue as the middle and all red as the bottom but the tops were yellow, blue, and red. A third group had arranged the three towers with all blue as the middle and all yellow as the bottom but the tops were red, blue, and yellow. A fourth group arranged three towers with all red on the top and bottom but the middle was red, yellow, and blue. A fifth group showed three towers all blue on the top and in the middle with blue, yellow, and red on the bottom. (11/20/13 meeting transcript, line 232).

A sixth group was also made of three towers. The first tower had yellow on top, blue in the middle, and red on the bottom. The second tower had yellow on top, red in the middle and blue on the bottom. The third tower had red on the top, yellow in the middle, and blue on the bottom. A seventh group had three towers with all blue on the top, all yellow in the middle, and a red, blue, and yellow on the bottom. (11/20/13 meeting transcript, line 232).

The eighth group had two towers. The first tower had red on the top and blue in the middle and on the bottom. The second tower had yellow on top, red in the middle, and blue on the bottom. (11/20/13 meeting transcript, line 232).

A ninth group had three towers. The tops and the bottoms of all three towers were yellow and the middle was yellow, blue, and red. The last group drawn had two towers. The top and middle positions of both towers were yellow and the bottom was red and blue. (11/20/13 meeting transcript, line 232).

T4 read the following student's written argument: "I changed the colors each time duplicating patterns. For example, red, red, yellow; red, red, blue; then yellow, yellow, red; and yellow, yellow, blue; are the same pattern but different colors" (11/20/13 meeting transcript, line 238). The instructor responded to T4 with "But this is a good start." (11/20/13 meeting transcript, line 241).

T5 was the sixth teacher to share her students' work. The student wrote that her and her partner came up with an estimate of 31 towers but did not provide a convincing argument for how they decided there would be 31 towers (11/20/13 meeting transcript, line 246). The student had drawn five pairs of opposite towers. For the sixth, seventh, and eight pairs of towers, T5 recognized that the student controlled for a variable in the middle of the towers (11/20/13 meeting transcript, line 246). Although the student did not write an argument, her partner did provide a written argument (11/20/13 meeting transcript, line 252). Figure 6.13 shows the student's tower drawing.

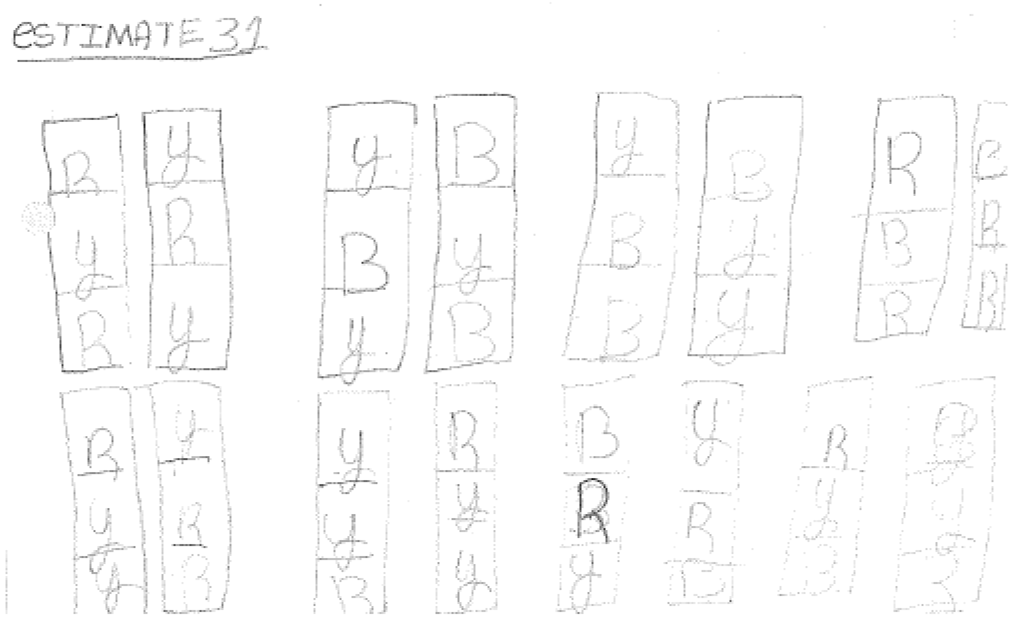



*Figure 6.13 T5's Cycle 3Drawing of Three-Tall Towers, from Student Work Sample 1*

T5 read the following partner's argument to the teachers:

> My partner came up with opposites of each other. My partner was very helpful. We had the best way to organize it and we shared it and we got to 27 towers. But when we started with 31; we had we were not thinking completely. But we were close to our estimate. (11/20/13 meeting transcript, line 254)

The instructor asked T5 if she questioned her students as to whether or not there could be another pair (11/20/13 meeting transcript, line 261). The teacher replied "They were just saying we cannot do anymore pairs so then they would be done. So I questioned that." (11/20/13 meeting transcript, line 268).

For the next sample that T5 shared, T5 read the following student's written argument: "We put them in order and we came up with three rows of nine buildings and multiplied them together and our total was 27." (11/20/13 meeting transcript, line 278).

The student had drawn three groups each with nine three-tall towers on his paper and named the groups red, yellow, and blue after the colors that were on the top of the nine towers (11/20/13 meeting transcript, line 272). The student colored the squares for the red group but then decided for the yellow and blue groups to use the letters y, r, and b to represent the yellow-, red-, and blue-colored cubes (11/20/13 meeting transcript, line 272). T5 recognized that the students controlled for a variable by having all red tops for the red group, all yellow tops for the yellow group, and all blue tops for the blue group. (11/20/13 meeting transcript, line 272). Figure 6.14 shows the student's work.

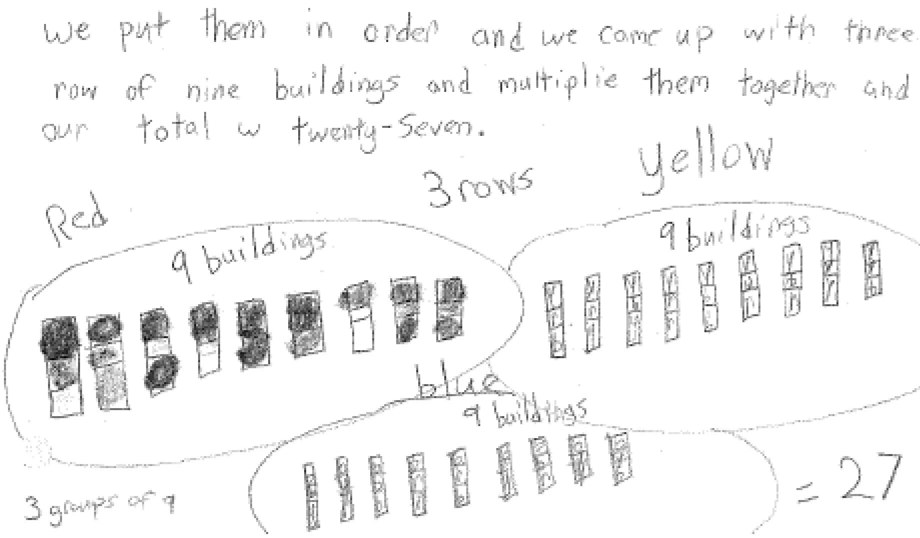



*Figure 6.14 T5's Cycle 3Three-Tall Towers, Student Work Sample 2*

Then, T8 was the seventh teacher shared his students' work. T8 shared his first sample of students' work by reading the following student's written argument to the teachers: "The cubes are red, blue, and yellow. Red has 3 cubes and blue has three cubes, and yellow has three cubes." (11/20/13 meeting transcript, line 284). The student had drawn four groups of towers. The towers had the letters r, y, and b inside squares to represent blue-, yellow-, and red-colored cubes. The first group had three towers. The towers were made of all red-, all blue-, and all yellow-colored cubes. Figure 6.15 shows the student's tower drawing for the first group.

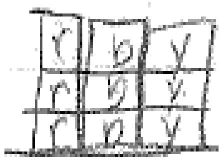


*Figure 6.15 T8's Cycle 3 Group 1 Drawing 3-Tall Towers, Student Work Sample 1*

T8 continued by reading "There are all 3 colors in each tower. There two yellow in the bottom. The red and the blue switch spots. There no way to move the red and the blue. I did the same thing for the reds and the blues on the bottom." (11/20/13 meeting transcript, line 284). The second group had 6 towers with all three colors in each tower (11/20/13 meeting transcript, line 284). Figure 6.16 shows the second group of towers drawn by the student.

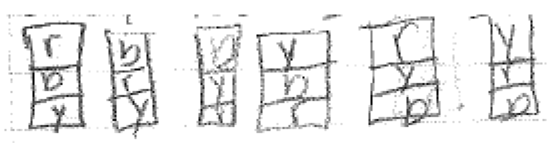


*Figure 6.16 T8's Cycle 3 Group 2 Drawing 3-Tall Towers, Student Work Sample 1*

T8 continued to read the argument: "There two blues on top and bottom. There yellow, red, in the middle. There no other color for the middle. I did the same for red and yellow. The red is in the top and bottom. The yellow is in the top and bottom." (11/20/13 meeting transcript, line 284). The third group had 6 towers. The first and second thee-tall towers had blue-colored cubes in the top and bottom positions with yellow and red in the middle. The third and fourth three-tall towers had red-colored cubes in the top and bottom positions with yellow and blue in the middle. The fifth and sixth three-tall towers had yellow-colored cubes in the top and bottom positions with red and blue in the middle (11/20/13 meeting transcript, line 284). Figure 6.17 shows the third group of towers drawn by the student.

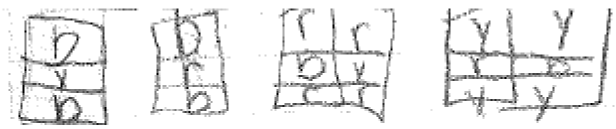


*Figure 6.17 T8's Cycle 3 Group 3 Drawing 3-Tall Towers, Student Work Sample 1*

The fourth group had twelve towers where four towers had red on top, four towers had yellow on top, and four towers had blue on top. T8 concluded the student's written argument with "This group is only two colors. The first 4 has the same color on top, and the second group there only yellow on top. The third group there only blue on top. The first group I did red yellow red, red yellow red blue, blue red, red blue. Second group is yellow, yellow, blue yellow, blue, blue yellow red, red yellow red, red. Third group is blue, blue, yellow blue yellow, yellow, yellow blue red, red blue, blue, red. T8 recognized that his student controlled for a variable (11/20/13 meeting transcript, lines 284-292). Figure 6.18 shows the fourth group of towers drawn by the student.

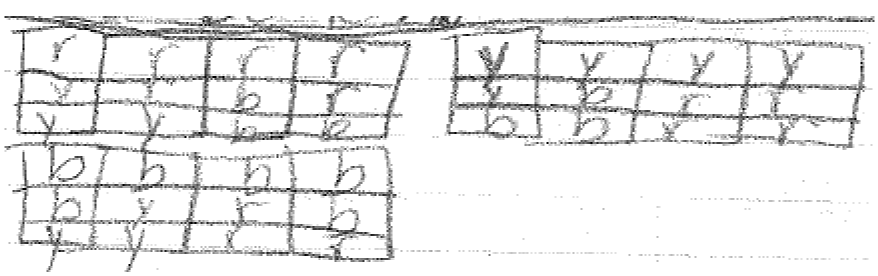


*Figure 6.18 T8's Cycle 3Group 4 Drawing 3-Tall Towers, Student Work Sample 1*

T8 also shared work from a student-helper. The student-helper had five groups of three-tall towers with the letters B, Y, and R inside squares to represent blue-, yellow-, and red-colored cubes. The first group of towers was made of all yellow-, all red-, and all blue-colored cubes. (11/20/13 meeting transcript, line 315).

The second group had 6 towers. The first tower had blue on top, yellow in the middle, and red on the bottom. The second tower had blue on top, red in the middle, and yellow on the bottom. The third tower had yellow on the top with red in the middle, and blue on the bottom. The fourth tower had yellow on top, blue in the middle, and red on the bottom. The fifth tower had red on top, blue in the middle, and yellow on the bottom and the sixth tower had red on the top, yellow in the middle, and blue on the bottom (11/20/13 meeting transcript, line 315). Figure 6.19 shows the student-helper's work.

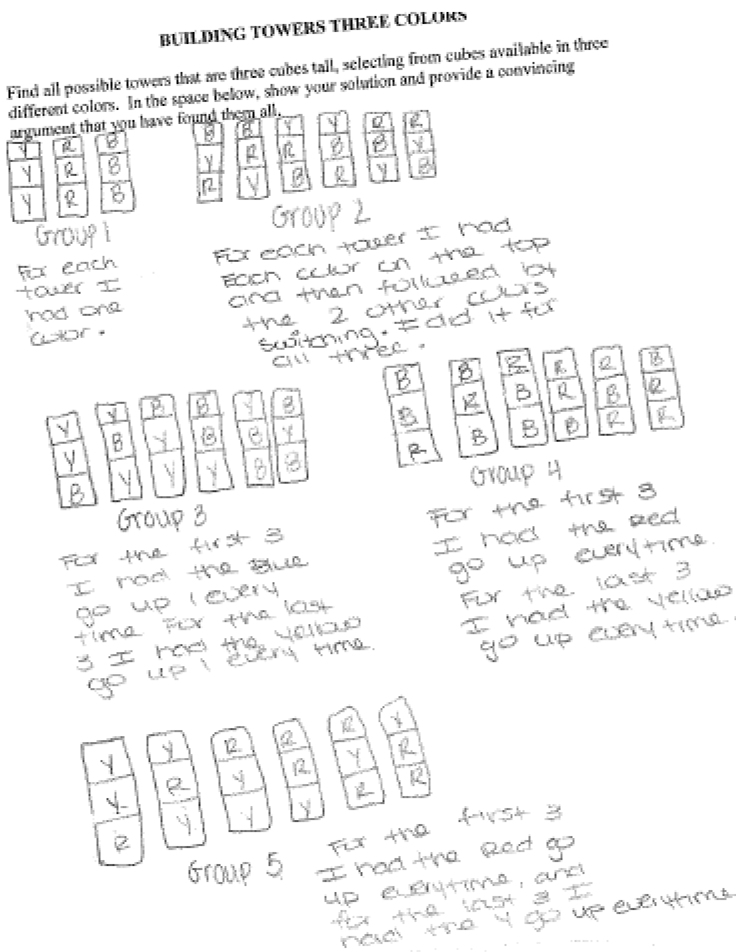



*Figure 6.19 T8's Cycle 3 Three-Tall Towers, Student-Helper's Work Sample 3*

The third group had 6 towers. The first tower had yellow on top, yellow in the middle, and blue on the bottom. The second tower had the blue in the middle position with yellow in the other positions. The third tower had the blue on the top position with yellow in the other positions. The fourth tower had blue on the top and in the middle with yellow on the bottom. The fifth tower had yellow on the top with blue in the middle and on the bottom. The sixth tower had blue on the top and bottom with yellow in the middle (11/20/13 meeting transcript, line 315).

The fourth group had 6 towers. The first tower had blue on the top and in the middle with red on the bottom. The second tower had blue on the top and bottom with red in the middle. The third tower had red on top with blue in the middle and on the bottom. The fourth tower had red on the top and in the middle and blue on the bottom. The fifth tower had red on top and on the bottom with blue in the middle. The sixth group had blue on top with red in the middle and on the bottom (11/20/13 meeting transcript, line 315).

Six towers were also in the fifth group. The first tower had yellow on the top and in the middle with red on the bottom. The second tower had yellow on the top and bottom with red in the middle. The third tower had red on top and yellow in the middle and on the bottom. The fourth tower had red on top and in the middle with yellow on the bottom. The fifth tower had red on the top and bottom with yellow in the middle. The sixth tower had yellow on top with red in the middle and on the bottom (11/20/13 meeting transcript, line 315). T8 read the following student’s argument to the teachers:

> Group 1: For each tower I had one color. Group 2: For each tower I had each color on the top and then followed and then followed by the two other colors switching. I did it for all three. Group 3: For the first 3, I had the blue go up one every time for the last 3, I had the yellow go up one every time. Group 4: For the 3, I had the red go up every time for the last 3 I had the yellow go up every time. Group 5: For the first 3, I had the red go up every time and for the last three I had the Y go up every time. (11/20/13 meeting transcript, lines 318-322)

T8 said that his student-helper told what she did and the instructor replied "That is really good that you guys are picking up. She is explaining what she did she has a very, very good strategy but she is not saying that therefore there can't be any more because I have taken that single color and put it into each of the three positions and there is no other place to put it." (11/20/13 meeting transcript, line 323).

T7 was the eighth teacher to share students' work. The student had drawn four groups of twenty-seven towers on the paper with squares using the letters Y, R, and B to represent the yellow-, red-, and blue-colored cubes. The first group had three towers with all red-, all yellow-, and all blue-colored cubes (11/20/13 meeting transcript, line 328).

The second group had 6 towers. The first tower had yellow on top and in the middle with red on the bottom. The second tower had yellow on top and in the middle with blue on the bottom. The third tower had red on the top and in the middle with blue on the bottom and the fourth group had red on top and in the middle with yellow on the bottom. The fifth tower had blue on the top and in the middle with yellow on the bottom and the sixth tower had blue on the top and in the middle with red on the bottom (11/20/13 meeting transcript, line 328).

The third group had 6 towers. The first tower had yellow on top with red in the middle and on the bottom. The second tower had yellow on top with blue in the middle and on the bottom. The third tower had blue on the top and yellow in the middle and on the bottom. The fourth tower had blue on the top with red in the middle and on the bottom. The fifth tower had red on the top with blue in the middle and on the bottom. The sixth tower had red on top with yellow in the middle and on the bottom (11/20/13 meeting transcript, line 328).

The fourth group had six towers. The first tower had blue on the top and bottom with yellow in the middle. The second tower had blue on the top and bottom with red in the middle. The third tower had yellow on the top and bottom with blue in the middle. The fourth tower had yellow on the top and bottom with red in the middle. The fifth tower had red on the top and bottom with blue in the middle and the sixth tower had red on the top and bottom with yellow in the middle. (11/20/13 meeting transcript, line 328).

The fifth group had 6 towers. The first tower had blue on top, yellow in the middle, and red on the bottom. The second tower had blue on the top with red in the middle and yellow on the bottom. The third tower had yellow on the top, blue in the middle, and red on the bottom. The fourth tower had yellow on top, red in the middle, and blue on the bottom. The fifth tower had red on the top, blue in the middle, and yellow on the bottom. The sixth tower had red on the top, yellow in the middle, and blue on the bottom. (11/20/13 meeting transcript, line 328).

The following student's argument was put on the screen for the teachers to read:

> First we did the 3 original colors. Then we did two colors on top and one on bottom and we did 2 colors on bottom and one on top. Then we did top bottom are same color and is different 6 times. Then we decided to have different colors 6 times in 6 different patterns. We came out to be 27 different towers. (11/20/13 meeting transcript, lines 336-340).

The instructor said to the teachers "The way he arranged it; it is very systematic and brilliant and he could very easily get it to a good convincing argument." (11/20/13 meeting transcript, line 343).

T7 shared another student sample where the eighth-grade student "used the same set up, but her explanation was a little clearer." (11/20/13 meeting transcript, lines 348). The student made five groups of three-tall towers. The student drew 27 towers as squares with the letters B, R, and Y inside the squares to represent the blue-, red-, and yellow-colored cubes. The towers were in five groups (11/20/13 meeting transcript, line 348).

The first group was made of three towers which were all red-, all blue-, and all yellow-colored cubes. The second group had 6 towers. The first tower had yellow on the top and red in the middle and on the bottom. The second tower had blue on the top with red in the middle and on the bottom. The third tower had red on the top and blue in the middle and on the bottom and the fourth tower had yellow on top with blue in the middle and on the bottom. The fifth tower had blue on top and yellow in the middle and on the bottom. The sixth tower had red on the top with yellow in the middle and on the bottom (11/20/13 meeting transcript, line 348). Figure 6.20 shows the drawing and explanation of the first, second and third groups drawn by the student.

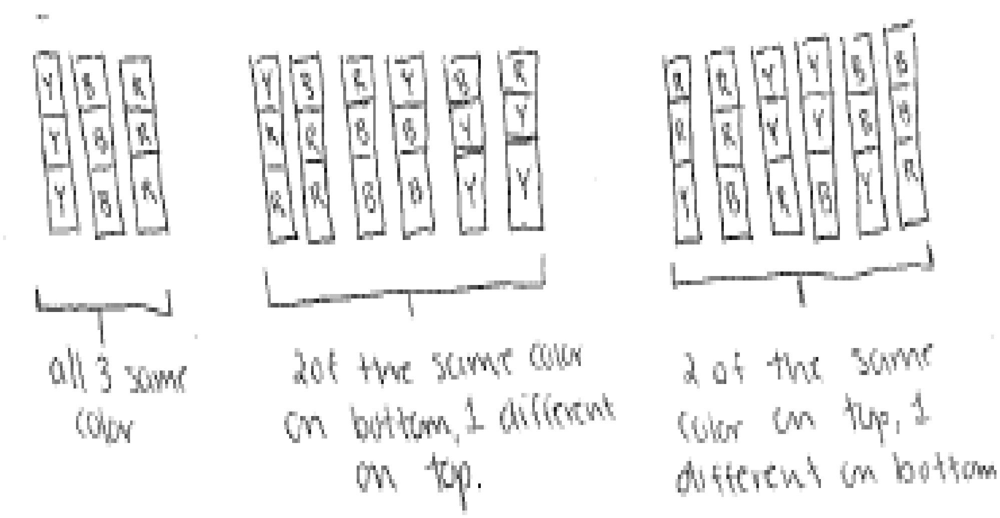



*Figure 6.20 T7's Cycle 3 Groups 1, 2, & 3Three-Tall Towers, Student Work Sample 2*

The third group had 6 towers. The first tower had red on the top and in the middle with yellow on the bottom. The second tower had red on the top and in the middle with blue on the bottom. The third tower had yellow on the top and in the middle with red on the bottom. The fourth tower had yellow on the top and in the middle with blue on the bottom. The fifth tower had blue on the top and in the middle with yellow on the bottom and the sixth tower had blue on the top and in the middle with red on the bottom. (11/20/13 meeting transcript, line 348).

The fourth group also had 6 towers. The first tower had yellow on the top and bottom with red in the middle and the second tower had blue on the top and bottom with red in the middle. The third tower had red on the top and bottom with blue in the middle and the fourth tower had yellow on the top and bottom with blue in the middle. The fifth tower had red on the top and bottom with yellow in the middle and the sixth tower had blue on the top and bottom with yellow in the middle. (11/20/13 meeting transcript, line 348). Figure 6.21 shows the tower drawing and explanation of the fourth and fifth groups.

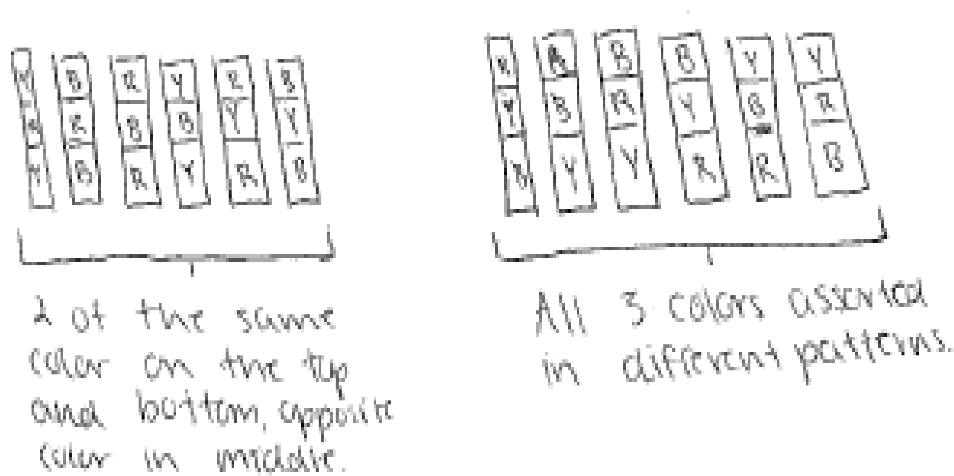



*Figure 6.21 T7's Cycle 3 Groups 4& 5Three-Tall Towers, Student Work Sample 2*

The fifth group had 6 towers with all three colors used in the tower. The first tower had red on the top, yellow in the middle, and blue on the bottom. The second tower had red on the top, blue in the middle, and yellow on the bottom. The third tower had blue on the top, red in the middle, and yellow on the bottom. The fourth tower had blue on the top, yellow in the middle, and red on the bottom. The fifth tower had yellow on the top, blue in the middle, and red on the bottom. The sixth tower had yellow on the top, red in the middle, and blue on the bottom.

T7 read the following student's argument to the teachers:

> [Towers are] all 3 same color. [Six towers had] two of the same color on bottom, one different on top. [Six towers had] two of the same color on top, one different on bottom. [Six towers had] two of the same color on the top and bottom, opposite color in middle. All 3 colors assorted in different patterns. The towers (in each group) have two similar towers. For example, if you have two reds on the bottom, you can only have a blue or a yellow on top (2 different towers). If you wanted a third tower, it would be all of the same colored cubes (red, red, red) which was already constructed. (11/20/13 meeting transcript, lines 350-352).

The instructor said to the teachers that the student had neatly shown the groups (11/20/13 meeting transcript, line 348).

T6 was the last teacher to share students' work with the other teachers. The student made seven groups of three-tall towers labeled A through G and used a key to make a drawing of the towers. The key showed yellow to be represented with a blank square, red with a shaded square, and blue with a striped square (11/20/13 meeting transcript, line 366).

Group A was made of all, all red-, all yellow-, and all blue-colored cubes. Group B had 6 towers with the red- and yellow-colored cubes. Three of the towers were constructed by moving the red block up one position until all the positions were exhausted. The other three towers were constructed by moving the yellow block up one position until all the positions were exhausted. Group C was the same as Group B except the red- and blue-colored cubes were used to construct the towers in the tower drawing. Group D was the same as groups B and C except the yellow- and blue-colored cubes were used to construct the towers in the tower drawing. (11/20/13 meeting transcript, line 370).

Group E had two towers. The first tower had red on top, blue in the middle, and yellow on the bottom. The second tower had blue on top, red in the middle, and yellow on the bottom (11/20/13 meeting transcript, line 370).

Group F had two towers. The first tower had blue on top, yellow in the middle, and red on the bottom. The second tower had yellow on the top, blue in the middle, and red on the bottom (11/20/13 meeting transcript, line 370).

Group G had two towers. The first tower had yellow on top, red in the middle, and blue on the bottom. The second tower had red on top, yellow in the middle, and blue on the bottom (11/20/13 meeting transcript, line 370).

T6 said "If you read his explanation, he says first off there were 3 towers of one color each. Next I made 9 towers of two of one color and one of another. And then did the alternate colors. After that I made 9 towers containing 3 colors each." (11/20/13 meeting transcript, lines 370-372). T6 claimed that his written argument did not match his drawing and the instructor agreed with T6 (11/20/13 meeting transcript, lines 380-381).

T6 shared a second student's work by reading the following argument to the teachers:

> First we made 3 towers each only using one color. Second we made 2 towers each with two red on the bottom and one yellow on the top and one blue on the top. Then we put two yellow on the bottom and put one of each of the other colors on top. After that, we did the opposite with two blues on the bottom. Third we put two yellows on the top, no wait then he is talking about the next group after that. (11/20/13 meeting transcript, line 394).

T6 recognized that her student controlled for a variable to solve the three-tall tower problem. (11/20/13 meeting transcript, line 398). T6 also said "This group was actually interesting too because when they first got their 27 combinations, they were confused because it was an odd number." (11/20/13 meeting transcript, lines 404). The instructor asked T6 "So were they upset when they got 27? (11/20/13 meeting transcript, line 415). T6 replied "They were at first. But then they looked at it and they organized it. And they were able to convince themselves. It was interesting because the task changed from them trying to convince me to for them to try and be able to accept that it was 27." (11/20/13 meeting transcript, lines 416,418).

**6.6 Summary**

With the completion of the three session units, the third cycle of problem solving came to an end. Throughout Cycle 3, teachers worked on the third cycle tasks, participated in three thought-provoking on-line discussions, observed an in-district classroom visit working on the tasks, implemented the same tasks in their own classes, read literature and watched videos of other students working on the tasks, and shared their own students' work after implementing the tasks in their own classes. The final meeting to discuss and reflect on the intervention occurred on 12/7/13.

# Chapter 7 – Reasoning Analysis

This chapter is an analysis of the forms of students' reasoning recognized by the teachers as well as teachers' forms of reasoning in their own problem-solving. First, the teachers' recognition of heuristics or strategies is analyzed. Second, the teachers' recognition of forms of argument is analyzed. Third, teachers' claims regarding whether or not arguments were convincing are reported. The chapter concludes with an analysis of the reasoning in the Gang of Four video pre- and post-assessments.

## 7.1 Heuristics or Strategies

This section reports teachers' recognition of heuristics in students' work during discussions at the regional meetings. The chapter also reports teachers' recognition of students' heuristics or strategies from research students' work after watching videos and reading scholarly articles from on-line discussion threads. During the intervention, teachers also used several different heuristics or strategies to work on and solve the mathematical tasks themselves.[1] Table 7.1 shows the heuristics recognized by teachers for all three cycles.

Table 7.1

*Frequencies of Heuristics/Strategies for Three Cycles*

| Heuristic/ Strategy Used | Teachers' Task Work | Research Students' Work | Class Visit Students' Work | Current Students' Work |
|---|---|---|---|---|
| Control a variable | 16 | 22 | 9 | 39 |
| Cousins | 0 | 2 | 0 | 1 |
| Elevator | 11 | 25 | 4 | 24 |
| Guess and Check | 1 | 36 | 2 | 7 |
| Opposites | 4 | 77 | 3 | 32 |
| Staircase | 2 | 2 | 0 | 0 |
| Total | 34 | 164 | 18 | 103 |

[1] A description of the heuristics/strategies is provided in chapter 3.

*Sources:* Meeting transcripts 9/7, 9/17, 10/2, 10/22, 11/20 and on-line discussion units 1-10.

### *7.1.1 Teachers' Task Work*

Teachers from the northern, central, and southern regions of New Jersey worked on the mathematical tasks. For the first cycle of tasks, eight teachers worked in pairs from the southern region cohort. Two of the pairs of teachers worked with a partner from the northern and central region for the first cycle tasks. For the second and third cycle tasks, the ten teachers from the southern region worked on the mathematical tasks in pairs.

#### *7.1.1.1 Teachers' Heuristics Used by Cycle*

For Cycle 1, thirteen strategies were used to solve the first cycle task from the six pairs of teachers. The most common strategies used by the teachers were the opposite strategy, 4 times; control for a variable, 3 times; and the elevator strategy, 3 times. Fewer common strategies used by the teachers were the staircase strategy, 2 times and the guess and check strategy, 1 time.

One teacher pair organized towers according to elevators to make ten towers. However, this pair decided to group the remaining six towers by controlling for a variable on the top of the six towers. For three of the six towers, all red cubes were on top and the other three towers had yellow on top (9/7 meeting transcript, lines 20-32). This teacher pair did not reorganize their towers. Figure 7.1 shows how the pair of teachers arranged their towers.

| | | | |
|---|---|---|---|
| YRYYY | RRR | YYY | RRRYR |
| YYRYY | RYY | RRY | RRYRR |
| YYYRY | YRY | RYR | RYRRR |
| YYYYR | YYR | YRR | YRRRR |

*Figure 7.1 Towers arranged by T9 and T10*

Another pair of teachers also began by organizing towers according to elevators to make ten towers. However, this pair decided to group the remaining six towers by controlling for a variable on the bottom of the six towers. For three of the six towers, all red cubes were on the bottom and the other three towers had yellow on the bottom. This pair also did not reorganize their work. (9/7 meeting transcript, lines 164-178). Figure 7.2 shows how the pair of teachers arranged their towers.

| | | | |
|---|---|---|---|
| YYYYR | YYR | YRR | RRRYR |
| YYYRY | YRY | RYR | RRYRR |
| YYRYY | RYY | RRY | RYRRR |
| YRYYY | RRR | YYY | YRRRR |

*Figure 7.2 Towers arranged byT6 and T7*

Another teacher pair began the first cycle task by using a combination of the guess and check and opposite strategies. In this case, the teacher pair made 9 pairs of towers from a tower and a tower using the opposite colors and checked to see if they had any duplicates (9/7 meeting transcript, lines 18-19). Later on in the session, the teacher pair reorganized their towers where the pair had built two one-tall towers. Next to the 1-tall towers, the pair built 2-tall towers by placing a red and a yellow each on top of the 1-tall towers which resulted in four 2-tall towers. Then the pair placed a red and a yellow each on top of the 2-tall towers to create eight 3-tall towers. (9/7 meeting transcript, lines 84-109). The instructor informed the teachers that they described an inductive argument (9/7/13 meeting transcript, line 90).

One pair of teachers also began by organizing towers according to elevators to make ten towers. However, this teacher pair had the remaining six towers grouped in opposite pairs with no organization. The instructor asked the teachers (T4 and T8) to find a way to organize the towers to convince her that they had found all the possible 4-tall towers. Later on when the instructor checked back with T4 and T8, the teachers had reorganized their towers by controlling for a variable on the bottom where the red cube was on the bottom of the eight towers.

Two pairs of teachers sat at the same table and each had one teacher from the southern region cohort and one teacher from the northern and central region. The fifth and sixth pairs of teachers began with the opposite strategy. Later, the both pairs reorganized their towers using the staircase strategy (9/7 meeting transcript, lines 148-157, 191-204).

Five strategies from five pairs of teachers were coded for the second cycle of tasks. All five pairs of teachers controlled for a variable when finding the three-topping pizzas for the pizza problem. One teacher pair made an organized list of pizzas written with the full topping word. The teachers both listed the no-topping pizza and the one-topping pizzas. For the two-topping pizzas, the teachers decided to control for a variable by holding peppers constant. The teachers also held peppers constant to create the three-topping pizzas. T10 had written during the on-line discussion that “As we got to the three toppings, it became harder to make sure we hadn’t duplicated any pizzas, so we considered holding 1 of the three toppings constant, and finding the pizza combinations that could be created by changing the other two toppings.” (Unit 5, on-line discussion, line 22).

Another teacher pair (T1 and T8) also made an organized list of pizzas. However, their representation was a drawing of circles to represent the pizzas. T1 worked with T8 and had written in the on-line discussion that “I definitely liked the strategy of picking a toping and having that as the constant. From there, we added toppings to the constant topping." (Unit 5, On-line discussion, line 7).

A third pair of teachers (T6 and T7) also made an organized list but represented the pizza toppings with letters. The teachers listed the plain pizza and the one-topping pizzas. For the two- and three-topping pizzas, the teachers controlled for a variable by holding peppers constant. T6 decided to reorganize their work using a chart and instructor asked T6 to share and explain her chart to the other teachers (10/2 teachers work transcript, line 202). The following was written by T6 during the on-line discussion:

> [T7] had the idea of keeping a constant. So we did all pizzas with peppers, all with mushrooms, all with pepperoni and all with sausage. We found as we eliminated an ingredient the number of possibilities were halving (just like the tower problem!). From there we decided to replicate what the towers would look like by having four possible spots. If the pizza did not occupy all of the spots with an ingredient we would put an X and if it did have an ingredient we would put the representation we came up with. As I as looking at it I noticed we did not even need to differentiate between the ingredients (M, I, P, S) when organizing this method. If you using the unifix cubes this way with one color is representing a topping and one color representing the absence of a topping. After I tried the method of keeping the first ingredient constant, I also tried to keep the second ingredient constant. I found this strategy did not work as well. It is better to organize by the number of toppings. (Unit 5, on-line discussion thread, line 1)

A fourth pair of teachers (T4 and T5) made an organized list for the second cycle task using the full topping word. They controlled for a variable by holding peppers constant for the two- and three-topping pizzas. T4 decided to do the problem again the way she thought her students would try it and made a tree diagram. T4 stopped using the tree diagram when she realized there were too many duplicates (Unit 5 on-line discussion thread, line 30).

The fifth pair of teachers (T2 and T3) also made an organized list using letters for the toppings. They had written no toppings and all toppings at the top of their papers. For the two-topping pizzas they held peppers constant. However, they spent much time trying to decide whether cheese should count as a topping. Once T2 and T3 decided not to count cheese as a topping, they easily found 16 possible pizza combinations (10/2 teachers work transcript, lines 110-141).

For the third cycle, the teachers worked on two problems. For the first problem teachers were asked to find all possible 3-tall towers that could be made selecting from three colors. The second problem was an extension of the first problem called Ankur's Challenge. For Ankur's Challenge, teachers were asked to find all possible 4-tall towers that could be made selecting from three colors, and using at least one of each color cube.

Sixteen strategies were recognized by the teachers for the Cycle 3 tasks. The most common strategies recognized were the elevator strategy, 8 times and controlling for a variable, 8 times. For the 3-tall towers problem, 6 strategies were recognized by the teachers. Three pairs of teachers used the elevator strategy to solve the 3-tall towers problem and three pairs of teachers controlled for a variable. Ten strategies were recognized by the teachers for the Ankur's Challenge problem. All five pairs of teachers used the elevator strategy and controlling for a variable to solve Ankur's Challenge.

*7.1.1.2 Summary of Teachers' Heuristics Used*

For their own problem solving, 34 strategies were used by the teachers. Teachers frequently used the strategies of controlling for a variable, 16 times; and the elevator strategy, 11 times and were used to solve the second cycle pizza problem, the third cycle three-tall tower problem, and Ankur's Challenge. Less common strategies used by the teachers strategies were opposite pairs, 4 times; and the staircase method, 2 times; and guess and check, 1 time and were only used to solve the problems for the first cycle four-tall towers problem.

***7.1.2 On-line Discussion of Research Students' Work***

In addition to the teachers working on the task themselves, the instructor assigned videos to watch and articles to read about the research students' task work. Teachers were asked to respond to questions about the research students' work in an on-line discussion thread. The teachers identified strategies from ten on-line discussion threads for the three cycles of tasks.

*7.1.2.1 Research Students' Heuristics Used by Cycle*

Four units of on-line discussion threads (Units 2, 3, 4, and two questions of Unit 5) were used by the teachers to post responses regarding the first cycle four-tall towers problem, selecting from two colors and the extension problems for predicting three-tall and five-tall towers. During the on-line discussion of the four-tall towers problem, 110 strategies were identified by the teachers. The more common identified strategies for the mathematical tasks of the first cycle were: the opposite strategy, 56 times; the guess and check strategy, 27 times; and elevator, 19 times. Less common strategies recognized by the teachers were controlling for a variable, 4 times; the cousin strategy, 2 times; and the staircase strategy, 2 times.

Three units of on-line discussion threads (Third question of Unit 5, all of Units 6 and 7) were used by the teachers to post responses regarding the second cycle pizza problem. During the on-line discussion of the pizza problem, 17 strategies were identified by the teachers within the on-line discussion. The strategies identified by the teachers during the on-line discussion of the pizza problem were: controlling for a variable, 11 times; guess and check, 3 times; and the opposite strategy, 3 times.

Three units of on-line discussion threads were analyzed for the third cycle tasks of tasks. For the third cycle, 37 strategies were recognized by the teachers. The recognized strategies for the third cycle tasks were the opposite strategy, 18 times; controlling for a variable, 7 times; the elevator strategy, 6 times; and guess and check, 6 times.

*7.1.2.2 Summary of Research Students' Heuristics*

From the on-line discussion of the research students' work, 164 strategies were identified by the teachers. Teachers frequently identified the opposite strategy, 77 times; guess and check, 36 times; the elevator strategy, 25 times; and controlling for a variable, 22 times. The opposite strategy was identified 56 times out of 77 times and the guess and check strategy was identified 27 times out of 36 times for the first cycle four-tall towers problem. The cousin and staircase strategies were identified fewer times; each having been identified by teachers twice and only for the research students' work on the first cycle four-tall towers problem.

### *7.1.3 Analysis of In-District Classroom Visits*

There were three in-district classroom visits with teachers' current students. The instructor and teachers circulated around the room to observe and ask students questions about their work. Samples of students' work from the in-district classroom visit were discussed during the debriefing meeting.

*7.1.3.1 Students' Heuristics Used from Classroom Visits by Cycle*

At the debrief meeting, five samples of students' work were selected for discussion from the first cycle tasks. From the five samples selected, seven strategies were recognized by the teachers. Three teachers recognized the opposite strategy, three teachers recognized the elevator strategy, and one teacher recognized controlling for a variable.

Five samples of students' work were discussed for the second cycle of tasks. From the five samples discussed, five strategies were recognized by the teachers. Three teachers recognized controlling for a variable and two teachers recognized the guess and check strategy.

Four samples of students' work were discussed for the third cycle of tasks. From the four samples selected, six strategies were recognized by the teachers. Four teachers recognized the strategy of controlling for a variable and one teacher recognized the elevator strategy in the students' work for solving the 3-tall towers problem, selecting from three colors. One teacher recognized that one pair of students controlled for a variable to solve the extension problem, Ankur's Challenge.

*7.1.3.2 Summary of Classroom Visit Students' Heuristics*

From the in-district classroom visits, 18 strategies were recognized by the teachers for the three cycles. The most common strategy recognized by the teachers was controlling for a variable, 9 times. Less common strategies recognized by the teachers were the elevator strategy, 4 times; opposites, 3 times; and the guess and check strategy, 2 times. The opposite strategy was only identified for students' work on the first cycle four-tall towers problem.

### 7.1.4 Current Students' Task Work

During the regional meetings, teachers shared 73 samples of students' work. The strategies recognized by the teachers for the three cycles of tasks were from the students' samples that teachers brought from their own classes to share with the other teachers. The instructor asked the teachers to share two or three samples of students' work.

*7.1.4.1 Current Students' Heuristics Used by Cycle*

Thirty-two samples of students' work were analyzed for the first cycle of tasks. For the first cycle, 51 strategies were recognized by the teachers from the students' work samples. The more common recognized strategies for the first cycle tasks were the opposite strategy, 27 times and elevator, 16 times. Less common strategies recognized by the teachers were controlling for a variable, 4 times and the guess and check strategy, 3 times; and the cousin strategy, 1 time.

Twenty-three samples of students' work were analyzed for the second cycle tasks. For the second cycle, 27 strategies were recognized by the teachers from the students' work samples. The more common recognized strategies for the second cycle tasks were controlling for a variable, 21 times and 4 times for the guess and check strategy. Less common strategies recognized by the teachers were the elevator strategy and opposite strategies each with 1 time.

Eighteen samples of students' work were analyzed for the third cycle tasks. For the third cycle, 26 strategies were recognized by the teachers from the students' work samples. The more common recognized strategies for the third cycle tasks were controlling for a variable, 14 times and 7 times for the elevator strategy. Less common strategies recognized by the teachers were opposite pairs, 4 times and the cousins, one time.

*7.1.4.2 Summary of Current Students' Heuristics*

From the current students' samples of work shared by the teachers, 103 strategies were identified by the teachers. The more common strategies recognized from the shared student samples of work for all three cycles were controlling for a variable, 39 times; opposites, 32 times; and elevator, 24 times. Less common strategies recognized by the teachers were the cousin strategy, 1 time and the guess and check, 7 times. The opposite strategy was identified by the teachers from the students' samples 27 out of 32 times; and the elevator strategy was identified 16 out of 24 times for the first cycle four-tall towers problem. Controlling for a variable was identified by the teachers from the students' samples 21 out of 39 times for the second cycle pizza problem; and 14 times during the third cycle tasks.

***7.1.5 Summary of Heuristics***

The results of heuristics analysis showed that the intervention helped teachers to progress in a formative way as the teachers used and recognized heuristics throughout the three cycles. The opposite strategy was found to be a popular strategy by teachers and students for solving the first cycle four-tall towers problem and the three-tall, five-tall towers extension problems but did not lead to convincing arguments. When solving the second cycle pizza problem, controlling for a variable was used as a strategy for all the teachers and identified from the students' samples 21 out of 27 times. Controlling for a variable and the elevator strategy were found to be popular strategies used by teachers and students for solving the third cycle three-tall towers problem and Ankur's Challenge. Teachers recognized that using the elevator strategy and controlling for a variable resulted in arguments that were more convincing in their own work as well as their students' work.

## 7.2 Forms of Argument

In addition to heuristics, teachers used forms of argument to solve the mathematical tasks. Teachers also recognized forms of argument in samples of students' work at the in-district classroom visits and at the regional meetings where teachers discussed current students' work. Teachers additionally recognized forms of argument after watching videos and reading scholarly articles about research students' task work using an on-line discussion thread. Table 7.2 shows the number of argument forms teachers recognized for all three cycles.

Table 7.2

*Frequencies of Argument Forms for Three Cycles*

| Argument Form Used | Teachers' Task Work | Research Students' Work | Class Visit Students' Work | Current Students' Work |
|---|---|---|---|---|
| Cases | 19 | 34 | 10 | 63 |
| Induction | 3 | 19 | 1 | 0 |
| Recursion | 13 | 30 | 4 | 23 |
| Contradiction | 1 | 1 | 0 | 0 |
| Rule | 2 | 18 | 2 | 11 |
| Total | 38 | 102 | 17 | 97 |

*Sources:* Meeting transcripts 9/7, 9/17, 10/2, 10/22, 11/20 and on-line discussion units 1-10.

### *7.2.1 Teachers' Task Work*

Throughout the three cycles of the intervention, teachers used forms of argument. Teachers recognized case arguments, induction, recursion, rule, and contradiction. An analysis of the teachers' forms of argument from solving the first cycle four-tall towers problem, the second cycle pizza problem, the third cycle three-tall towers problem, and Ankur's Challenge follows.

*7.2.1.1 Teachers' Forms of Argument Used by Cycle*

For the first cycle four-tall towers problem, thirteen forms of argument were used by six pairs of teachers. The most common form of argument used by the teachers to solve the first cycle four-tall towers problem was cases, 7 times. Less common forms of argument used by the teachers were recursion, 5 times and induction, 1 time. For the second cycle pizza problem, all five forms of argument used by the teachers were case arguments.

Although all the teachers used a case argument, it is also important to note that the teachers used different representations to solve the pizza problem. Three pairs of teachers drew a diagram and made an organized list. One pair of teachers made a drawing. One pair of teachers just made an organized list. Another teacher pair later reorganized their drawing to make an organized chart which was shared with the teachers during the whole-group discussion. It should also be noted that some case arguments were organized according to the number of toppings and some were organized according to the type of topping for the second cycle of tasks.

Twenty forms of argument were used by the teachers for the Cycle 3 tasks. The more common form of argument used by teachers was recursion, 8 times and cases, 7 times. Less common forms of argument were induction, 2 times; rule, 2 times; and contradiction, 1 time.

For the 3-tall towers problem selecting from 3 colors, 11 forms of argument were used by the teachers. Recursion was used 3 times and case arguments were used 2 times. Slightly less common forms of argument were induction 1 time, contradiction 1 time, and 2 times for the rule form of argument.

Eleven forms of argument were used by the teachers for the Ankur's Challenge problem. The most common forms of argument were cases and recursion each 5 times. A less recognized argument form was induction, 1 time.

*7.2.1.2 Summary of Teachers' Forms of Argument*

For their own problem solving, 38 forms of argument were used by the teachers. The most common form of argument that teachers used to solve the problems were case arguments, 19 times. Recursion was the next common form of argument used by teachers when solving the problems themselves; noted 13 times. Less common forms of argument teachers used to solve the mathematical tasks were induction, 3 times; contradiction, 1 time; and rule, 2 times.

***7.2.2 Teachers' On-line Discussion of Research Students' Work***

Teachers were also assigned videos to watch and articles to read about research students' work in addition to working on the tasks themselves. After watching the videos and reading the articles, teachers were asked to respond to questions about the research students' work in an on-line discussion thread. The data are spread over ten units of discussion threads for the three cycles of tasks.

7.2.2.1 *Research Students' Forms of Argument Used by Cycle*

Four units of on-line discussion threads were analyzed for the first cycle tasks. For the first cycle, 45 forms of argument were recognized by the teachers. The argument forms recognized by the teachers for the first cycle tasks were induction 19, times; recursion, 18 times; and cases, 8 times.

Three units of on-line discussion threads were analyzed for the second cycle of tasks. For the second cycle, 21 forms of argument were recognized by the teachers. The most commonly recognized form of argument for the second cycle tasks was cases, 20 times. A less common form of argument recognized by the teachers was recursion, only 1 time.

Three units of on-line discussion threads were analyzed for the third cycle tasks of tasks. For the third cycle, 36 forms of argument were recognized by the teachers. The recognized forms of argument for the third cycle tasks were rule, 18 times; recursion, 11 times; cases, 6 times; and contradiction, only 1 time.

*7.2.2.2 Summary of Research Students' Forms of Argument*

From the on-line discussion of the research students' work, 102 forms of argument were identified by the teachers. The forms of argument more commonly recognized from the on-line discussion threads for all three cycles were cases, 34 times; recursion, 30 times; induction, 19 times; and 18 times for the rule form of argument. A less common argument form recognized by the teachers was contradiction, only 1 time.

***7.2.3 Analysis of In-District Classroom Visits***

There were three in-district classroom visits with teachers' current students. The samples of students' work from the in-district classroom visit were discussed during the debriefing meetings. The debrief meetings were held directly after the classroom implementation.

*7.2.3.1 Students' Forms of Argument Used from Classroom Visits by Cycle*

Five samples of students' work were selected for discussion from the first cycle tasks. From the five samples selected, eight argument forms were recognized by the teachers. Four teachers recognized recursion, three teachers recognized cases, and one teacher recognized the rule form of argument.

Five samples of students' work were discussed for the second cycle of tasks. From the three samples discussed, five forms of argument were recognized by the teachers. Four teachers recognized cases and one teacher recognized rule.

Four samples of students' work were discussed for the third cycle of tasks. From the four samples selected, four forms of argument were recognized by the teachers. The argument forms recognized by the teachers were cases, 3 times and induction, 1 time.

*7.2.3.2 Summary of Classroom Visit Students' Forms of Argument*

At the debrief meetings after each in-district classroom implementation, 17 forms of argument were recognized by the teachers for the three cycles. The more common forms of argument recognized by the teachers were cases, 10 times and recursion, 4 times. Less common argument forms were induction, one time and rule, two times.

***7.2.4 Current Students' Task Work***

During the regional meetings, teachers shared 73 samples of students' work. The instructor asked the teachers to share two or three samples of students' work. Each teacher presented their students' work to the other teachers to discuss the forms of argument the teachers recognized from the students' samples for each of the three cycles.

*7.2.4.1 Current Students' Forms of Argument Used by Cycle*

Thirty-two samples of students' work were analyzed for the first cycle four-tall tower problem. For the first cycle four-tall tower problem, 41 forms of argument were recognized by the teachers from the students' work samples. The argument forms for the first cycle four-tall towers problem were cases, 23 times; recursion, 14 times; and rule, 4 times.

Twenty-three samples of students' work were analyzed for the second cycle pizza problem. For the second cycle pizza problem, 29 forms of argument were recognized by the teachers from the students' work samples. The forms of argument recognized for the second cycle pizza problem were cases, 23 times; rule, 4 times; and recursion, two times.

Eighteen samples of students' work were analyzed for the third cycle three-tall towers problem. For the third cycle three-tall towers problem, teachers recognized 27 forms of argument from the students' samples. Teachers recognized cases, 17 times; recursion, 7 times; and rule, 3 times.

*7.2.4.2 Summary of Current Students' Strategies*

There were 97 forms of argument recognized by the teachers for the three cycles of tasks from the current students' work. The more common forms of argument recognized from the shared student samples of work for all three cycles were cases, 63 times; recursion, 23 times; and rule, 11 times.

***7.25 Summary of Forms of Argument***

The forms of argument analysis showed that the intervention was helpful for enabling teachers to use and recognize forms of argument throughout the three cycles. Case arguments were found to be the most popular form of argument used by teachers and students when solving the problems in all three of the cycles. Teachers used case arguments 19 times for their own problem solving. Moreover, teachers identified case arguments from research students' work, 34 times; students' work from the classroom visits, 10 times; and current students' work, 63 times.

Recursion was used by the teachers for their own problem solving 5 times when solving the first cycle four-tall towers problem, 3 times when solving the third cycle three-tall towers problem, and 5 times when solving Ankur's Challenge. Teachers identified recursion 30 times in research students' work, 4 times in students' work from the classroom visits, and 23 times from current students' samples.

Induction was identified in research students' work 19 times regarding the first cycle four-tall towers problem and three-tall and five-tall tower extension problems. For their own problem-solving, teachers used induction three times. Teachers only identified one inductive argument from a student during a debriefing meeting about the third cycle three-tall towers problem.

Rule was identified as a form of argument in research students' work 18 times regarding the Ankur's Challenge problem. Teachers used rule as a form of argument in their own work when solving the three-tall towers problem 2 times and identified rule as a form of argument 13 times in students' work.

Teachers used contradiction as a form of argument once when solving the third cycle three-tall towers problem and identified it in a research students' work once about Ankur's Challenge. It is important to note that the teachers recognized that there were times when students used multiple forms of argument and heuristics to solve a mathematical task. Arguments or heuristics used together to solve a task are referred to as co-occurrences.

**7.3 Co-Occurring Heuristics and Arguments**

The most common co-occurrences were control for a variable and cases, 54 times. Other strong or frequent co-occurrences were: the elevator strategy and case arguments, 23 times; the elevator strategy and recursive arguments, 19 times; the opposite and case arguments, 23 times; and the opposite and recursive arguments, 11 times. Controlling for a variable co-occurred less frequently than cases with other forms of argument such as rule, 6 times; recursion, 5 times; and induction, two times. Table 7.3 summarizes the co-occurrences for the forms of argument and heuristics.

Table 7.3

*Frequencies of Co-Occurrences for Three Cycles*

| Co-occurrence | Cases | Recursion | Rule | Induction |
|---|---|---|---|---|
| Control a variable | 54 | 5 | 6 | 2 |
| Cousins | 1 | 0 | 0 | 0 |
| Elevator | 23 | 19 | 3 | 0 |
| Guess and Check | 6 | 0 | 2 | 1 |
| Opposites | 23 | 11 | 4 | 0 |
| Staircase | 0 | 2 | 0 | 0 |
| Total | 107 | 37 | 15 | 3 |

*Source:* Meeting transcripts 9/7, 9/17, 10/2, 10/22, 11/20 and on-line discussion units 1-10.

### 7.4 Teachers' Evaluation of Arguments

Throughout the intervention, teachers were asked to evaluate whether or not strategies or arguments were convincing in a variety of contexts. Teachers were asked to determine if arguments made by other teachers or themselves while working on the tasks were convincing. For the on-line discussions, teachers were asked to determine whether the research students' arguments were convincing. Teachers were also asked to determine whether or not current students provided convincing arguments from samples of their work.

It is important to note that teachers were not always completely convinced by an argument and were therefore coded as not convincing. The non-convincing arguments were coded with an additional code for being incomplete or invalid. Arguments that were not convincing were coded as incomplete if no parts, one part or some parts of the arguments were convincing; but not all parts were convincing. Arguments were coded as invalid if the arguments did not make mathematical sense for solving the problem. Table 7.4 shows the number of arguments made by the teachers regarding whether the arguments were convincing or not convincing.

*Table 7.4*

*Frequencies of Teachers' Evaluations of Arguments*

| Evaluation of Arguments | Teachers' Task Work | Research Students' Work | Class Visit Students' Work | Current Students' Work |
|---|---|---|---|---|
| Not convincing | 1 | 8 | 5 | 16 |
| Incomplete | 1 | 8 | 5 | 15 |
| Invalid | 0 | 0 | 0 | 1 |
| Convincing | 8 | 18 | 1 | 0 |

*Sources:* Meeting transcripts 9/7, 9/17, 10/2, 10/22, 11/20 and on-line discussion units 1-10.

### *7.4.1 Teachers' Evaluations of Arguments of Teachers' Own Work*

As teachers worked on the tasks themselves, the instructor asked teachers to evaluate whether arguments were convincing or not convincing. On most occasions, the instructor asked the teachers "Are you convinced [you have all towers]?" (Meeting transcript 9/7/13, lines 21& 62; 10/2/13, line 93; 10/22/13, line 342). For other times, the instructor asked "Do you think you have them all [all the towers]?" (Meeting transcript 10/2/13, lines 60 &157) or other similar questions asked about whether the arguments were convincing or not convincing (Meeting transcript 10/22/13, lines 158, 181, 298).

#### *7.4.1.1 Evaluation of Arguments by Cycle of Teachers' Own Work*

For the first cycle four-tall towers problem, the teachers paired up to build all possible four-tall towers that could be made selecting from two colors. After building the four-tall towers, teachers were asked to convince their partners that all possible four-tall towers were built without having duplicates. After convincing their partners, teachers were asked to verbally convince one of the researchers circulating around the room

(Meeting transcript 9/7/13, line 12). Teachers were then asked to record on paper the convincing argument they verbally provided to the researchers (Meeting transcript 9/7/13, lines 32, 63, 107, 144, 148).

There were two claims made for the first cycle of tasks regarding whether teachers were convinced with their own argument or other teachers' arguments. For one claim, the instructor asked the teacher pair (T9 and T10) if they were convinced by their argument for the four-tall towers problem (Meeting transcript 9/7/13, line 21). T10 described ten of the four-tall towers using a recursive argument for one red and three yellows and then three reds and one yellow. For the other six towers, T10 explained the following convincing argument to the instructor:

> First we started with two reds and two yellows. Then one yellow, moved the second red down one keeping the first red on top. There can't be another one with 2 reds on top. So then we took this red and moved it to the second position. That's one that I already had. It does because then we moved our starting red to the third position. We already had we put it in the third position because we already had a one top red. (Meeting transcript 9/7/13, lines 29-31).

For the second claim, T7 described the following argument to the instructor:

> So then I used two reds and so to approach that I kept the first red always on the bottom. So I can get only three options with where the red can go. Then I switch red to the third position, so I get all red on the bottom. So then I went to 1 yellow with 3 reds and moved my one yellow to different spots. (Meeting transcript 9/7/13, lines 39-49).

The instructor asked both T7 and her partner if they were both convinced by their arguments and they nodded their heads in agreement (Meeting transcript, 9/7/13, line 62). It should be noted that the other teacher pairs gave arguments to the instructor but the instructor found their initial arguments were not convincing. As a result, four teacher pairs were asked by the instructor to reorganize their initial work. (Meeting transcript 9/7/13, lines 133; 136; 148; 202).

During the second cycle of tasks, there was only one claim made that the argument was convincing when the instructor asked T2 “Do you think you have them all?” (Meeting transcript teachers’ work 10/2/13, lines 157) and T2 replied “Yes.” (Meeting transcript teachers’ work 10/2/13, line 158). T2 gave the following explanation to the instructor:

> Well, I mean if you do it like the tower problem and simplify it so you know what I am saying like if this is cheese, peppers, sausage. Then you could just add a mushroom or add pepperoni. Then so like the same thing here was pepper and mushroom, and added a pepperoni. So like we just added the other 4 toppings we didn’t include. We both agreed that order didn’t matter. (Meeting transcript teachers’ work 10/2/13, lines 157-164).

A second claim was made by T5 for solving the second cycle pizza problem. T5 said “I’m trying to think of how many there are possible. It is more if you are allowed to have duplicates for orders. There would be more if you said peppers pepperoni or pepperoni peppers.” (Meeting transcript teachers’ work 10/2/13, lines 53-55). The instructor asked “Do you think you have them all?” and T5 replied “I don’t know!” (Meeting transcript teachers’ work 10/2/13, lines 60-63). This argument was coded as an incomplete, non-convincing argument by the researcher.

For the three-tall towers problem, there were 5 claims that teachers were convinced by arguments. First, the instructor asked the teacher pair, T7 and T8, “Are you convinced you have them all?”(Meeting transcript teachers’ work 10/22/13, lines 93). T7 responded to the instructor “Yes, we are.”(Meeting transcript teachers’ work 10/22/13, lines 94). T7 described the following argument to the instructor:

> We started with our 3 solids of each color. So then we said we could have yellow and blue. And in that case it could be two yellow one blue or it could be two blue one yellow. And we used placements, so blue could be in 3 positions for those 3 and the yellow. Then we said instead of yellow blue we can also have yellow & red. Then we are done with our yellow. And then the only ones that are left are

> the blue and red. So we are done with all of our just 2 colors in the tower. Then we had our 3 color towers. And we had thought about each of these differently. (Meeting transcript teachers' work 10/22/13, lines 96-119)

From this point, T8 used recursion to explain the rest of the argument for solving the three-tall towers problem, selecting from 3 colors (Meeting transcript teachers' work 10/22/13, lines 122-138).

Second, T7 told the instructor about her idea of controlling for a variable (Meeting transcript teachers' work 10/22/13, line 153). T7 provided the following argument to the instructor:

> I had originally thought a red on the top a blue or yellow will alternate on the bottom. If the yellows are on top, the red and blue will switch. The third one would be the blues on top. (Meeting transcript teachers' work 10/22/13, lines 155-157)

The instructor asked T7's partner "What do you think of her argument?" (Meeting transcript teachers' work 10/22/13, line 158) and the partner replied "I think it works!" (Meeting transcript teachers' work 10/22/13, line 159).

There was a third time where a teacher pair claimed to be convinced of an argument for the three-tall towers problem. The instructor asked T2 and T3 "Can you convince me, that there are only 9 with red tops and only 9 and you can't have any more?" (Meeting transcript teachers' work 10/22/13, line 181). T2 and T3 had organized their towers in three groups of 9 three-tall towers by controlling for a variable so that either all red, yellow, or blue cubes were on the tops of the towers. The following arguments were explained by T2 and T3 to the instructor:

> T2: So if we moved it once it would be this; if we moved it twice it would be one of the yellows with a red on top.
> T3: It would be the same thing we are keeping the red on top, so we are keeping the red on top. If I move it once, the reds will be on top but then if I move it again it has two categories and that is not what I want.

> T2: Right. And then this is the same thing with this one but with blue….And then this one….The only way you can have a red on top with two of the same colors on the bottom with a yellow. And then this one has the same thing red on top with two of the same color on the bottom where the constant is either yellow or blue. You can either get a yellow or blue or a blue or a yellow. And if you were to move it this would be one of these. (Meeting transcript teachers' work 10/22/13, lines 186-199).

The instructor said to T2 and T3 that their argument was different than the others and asked the teacher pair to record their work (Meeting transcript teachers' work 10/22/13, line 208). The instructor then picked three teacher pairs to present their work (Meeting transcript teachers' work 10/22/13, line 283).

A fourth time, teachers claimed that an argument was convincing was when the instructor asked T7 and T8 "What's your convincing argument why you have every possibility of two of one color and one of the other?" (Meeting transcript teachers' work 10/22/13, line 296). T7 explained the following argument to the instructor:

> So we said if you have the yellow and blue, for example. If we had it 2 yellow and the one blue there is only 3 ways to do that. Our single cube can move to each of the positions. If we were to move that again, we would either need a fourth row or we would be repeating it. (Meeting transcript teachers' work 10/22/13, line 297).

The instructor asked "Does everyone buy that?" (Meeting transcript teachers' work 10/22/13, line 298-299) and several teachers responded yes in unison.

There was a fifth claim made by a teacher that an argument was convincing. T2 and T3 were presenting their solution of three groups of 9 three-tall towers and explained the following argument to the teachers:

> T3: So we kind of organized it with the first 3 in each group is the following color. And then the next two is where you held that particular color constant. So for the first group where the red was constant you can only then have one yellow in position 2 and position 3 which would be the second and third tower. And then

> if you do the same thing with the blue it could then only be a second position and a third position.
>
> T2: Then working with two colors but keeping red as a constant you can only have just 2 of yellow if you move it around you are going to come up with something else. So you could only have 2 yellow or 2 blue to be no repeats, no duplicates. And then the same thing with the yellow and the blue we just switched them, the position. (Meeting transcript teachers' work 10/22/13, lines 340-341).

After the explanation, the instructor asked the teachers if they were convinced by the argument and T8 shouted "Yeah!" (Meeting transcript teachers' work 10/22/13, line 342-343).

*7.4.1.2 Summary of Evaluation of Arguments from Teachers' Work*

Teachers claimed that 8 arguments were convincing while working on problems throughout the three cycles. There was only one argument that teachers found was not convincing while working on the second cycle pizza problem. This argument was coded as an incomplete, not-convincing argument by the researcher based on the teacher's comments. It is possible that teachers may not have been as confident in their ability to determine whether arguments were not convincing as it was their first meeting of the Lesson Study course.

***7.4.2 Teachers' Evaluation of Arguments of Research Students' Work***

For the on-line discussions, the instructor provided some questions that asked teachers to make claims about whether an argument from the research students was convincing. Teachers were asked to watch videos and read scholarly articles about students participating in mathematical tasks. After watching the videos and reading the articles, teachers made claims on-line in response to questions asked by the instructor about the research students' work.

*7.4.2.1 Evaluation of Arguments by Cycle from Research Students' Work*

In Unit 3, teachers were asked to watch the video of Stephanie and Dana in the third grade building 4-tall towers selecting from two colors. After watching the video, teachers were asked to respond whether the arguments of Stephanie and Dana were convincing. Five teachers wrote on-line that they were completely convinced by Stephanie's and Dana's argument (Unit 3 on-line discussion thread, line 1, 14, 41, 46, 48). Five teachers claimed that they were not convinced by Stephanie's and Dana's argument and were coded as incomplete based on the explanation provided by the teacher. The responses of the five teachers that were not convinced by Stephanie's and Dana's arguments follow:

> Stephanie's argument for having 16 towers four tall was not convincing. She only mentioned that she had them all since she was checking and could not find more. This is not convincing since there is the possibility of missing some. She did not describe any method used in creating them. Dana's argument was not convincing either since she stated that you should always assume there is more until you find out the answer. She did not give any reasoning or method to how to determine the answer or know when you have them all. (Unit 3 on-line discussion thread, line 6)

> I do not feel that Stephanie and Dana have a convincing argument. Both Stephanie and Dana believe that they have tried many different ways and are convinced that they have made all the possibilities but do not supply a convincing argument as to why they are sure. (Unit 3 on-line discussion thread, line 26)

> I really don't feel that their argument is very convincing. They really don't explain why they came to the conclusion and how they determined that all possibilities are done. At the end, Stephanie explains that if you take one red and one blue away from the tower of four they would be the same. However, she is not thinking that there would have to be more color combinations because they are working with two colors. (Unit 3 on-line discussion thread, line 30)

> I do not think Stephanie and Dana's argument is completely convincing. I think they are on the right track but they still are not sure how to explain if there are truly only 16 towers. (Unit 3 on-line discussion thread, line 34)

> I am not completely convinced by their argument. It appears that since she was "checking and checking" she used a guess and check method, but did not have a systematic way to see if she had all the possible outcomes. I do think for a third grader this is the beginning of a convincing argument, but she must extend further to find proof as to how she can justify the 16 towers that they did find. (Unit 3 on-line discussion thread, line 39)

For the Unit 4 discussion thread, teachers were asked to watch videos about Stephanie and Dana working on the 4-tall towers problem as third-graders and the 5-tall towers problem as fourth-graders as well as five other video clips showing fifth-grade students working on building towers selecting from two colors using the Guess My Towers Problem. Teachers were also asked to read the following chapters in Combinatorics and Reasoning (Maher, Powell, & Uptegrove, 2010): chapter 4, which examines strategies and representations used for solving towers problems; and chapter 5, which examines how Stephanie and their classmates built their conceptual understanding of Milin's inductive argument.

After watching the videos and reading the chapters, teachers were asked about how the children's strategies to solve the towers problem in third grade were different than the strategies used in fourth grade and which of their arguments were convincing. Three teachers made the following claims regarding research students' work that the teachers thought were convincing:

> In 4th grade, they immediately started by "doing the opposite." This was convincing to me because that is how I found the towers when I did this activity. (Unit 4, on-line discussion thread, line 1).

> I thought Dana and Stephanie's arguments of families was convincing. The way that they moved one red cube and then did the same with two red cubes stuck together moving in the same fashion as one red cube. (Unit 4, on-line discussion thread, line 11).

> The girls also noticed both elevator and staircase patterns, which helped them to find some of the possible combinations. Both of the pattern arguments were

> convincing for that portion of the towers, and definitely showed growth in their reasoning skills. (Unit 4, on-line discussion thread, line 29).

The following three teachers posited that they were not convinced regarding research students' arguments and were coded as incomplete, non-convincing arguments by the researcher:

> They used a similar strategy in fourth grade but this time they noticed the pairs that could be created. They kept the towers in pairs and again checked to see if the new towers they built were duplicates of towers they already had since they were convinced they would never know how many possible towers could be created. Their arguments are not fully developed at this grade level and need more convincing. (Unit 4, on-line discussion thread, line 7).

> When tackling this problem in third grade it sounded like Stephanie was changing the positions of the one blue cube at first and then she went to making opposites. The interesting thing when they were in $4^{th}$ grade making the opposites was that Stephanie called the opposites they were building duplicates. So I am unsure if she knew the meaning of a duplicate. I believe that in both grades the students were on their way to having a convincing argument but didn't fully get there. (Unit 4, on-line discussion thread, line 41).

> When they were in fourth grade, they started organizing their thoughts a little better. They created families of blocks based on patterns they saw. I found the fourth grade arguments much more convincing. It was clear to me when they explained the elevator pattern of the blocks moving from one position to the other. (Unit 4, on-line discussion thread, line 43).

A second question from the unit four on-line discussion asked teachers whether they found Milin's inductive argument convincing. All ten teachers wrote on-line claiming that Milin's inductive argument was convincing (Unit 4, on-line discussion thread, lines 1, 7, 11, 21, 29, 36, 41, 43, 45, 47)..

*7.4.2.2 Summary of Evaluation of Arguments from Research Students' Work*

Teachers claimed on-line that they were convinced by research students' arguments 18 times throughout the three cycles. There were 8 arguments from the research students that teachers found were not convincing. The 8 arguments were coded as incomplete, not-convincing arguments by the researcher based on the teacher's on-line response. It is possible that teachers were more confident with their ability to claim whether or not research students' arguments were convincing or not convincing on-line.

### *7.4.3 Teachers' Evaluation of Arguments of Students' Work from Class Visits*

During the debrief meeting after the in-district classroom visit, teachers were asked to make claims about whether a students' argument was convincing. The instructor asked the teachers about which students' work they wanted to discuss. Most samples of the students' work were put on the screen for teachers to discuss. However, there were times when teachers verbally discussed students' work but the students' work was not shared on the screen.

#### *7.4.3.1 Evaluation of Arguments by Cycle from Class Visits*

For the first cycle four-tall towers problem, teachers claimed that 3 arguments were not-convincing. The 3 non-convincing arguments were coded by the researcher as incomplete based on the following comments from the teachers at the meeting:

> T8: In the group 2, they were both solids. So if they were to be switched they would stay exactly the same. For example, there are four red and four yellow.
> R1: Okay. What do you think? Convincing? [Teachers responded in unison, No.] What could they have said?
> T7: This is what this group did the whole time. I don't know. You were with this group, right? All they kept saying to me was like, we switched it…it would, it would be good. So they, they just proved that they made opposites of each other, but they didn't really. I don't know if they understood the task because they didn't really say anything about how this is the most amount of towers they can make. (Meeting transcript 9/17/13, lines 165-167)
>
> T7: We started with yellow and one red. Then we moved the red down one space every time and move the yellow to the top every time. Then we did the opposite with three red and one yellow. Then we did two of each color; two red, two yellow. We moved the two red down one cube and took the one yellow on the bottom and move it to the top. We put the two yellows on top, on top of each other, and had two reds on the bottom. Two yellows on top of two reds. Oh, two reds on the bottom. Then we moved one of the reds on top of the two yellows.

> R1: Okay. It's gets hard not only to read but to understand what they're doing. Um, but that is a recursive argument. And that's a good argument, okay? Um, so that was, um, the solids here, okay; and here's the ones again with the alternating pattern. Are they convincing you?
> T3: More than the last one. (Meeting transcript 9/17/13, lines 147-149)

> T3: You can't have any other combination in this group because of the two yellow on the top, two red on bottom, and then we did the opposite; two red on top, two yellow on bottom.
> R1: Okay, is it convincing?
> T3: Well, within that group; but…
> R1: Good within that one little group two, yes they have it, but that doesn't yet convince us that they have all possible towers; all 6 of them that are two red and two yellow. (Meeting transcript 9/17/13, lines 69-72)

For the 2 of the non-convincing arguments, students had used the opposite strategy to solve the first cycle four-tall towers problem. One argument used recursion and teachers found the argument to be more convincing than the previous argument given. It should also be noted that it is possible that teachers were may have been reluctant to share their thoughts as to whether an argument was convincing or not because this was the beginning of the course. There were 2 instances where the instructor asked the teachers whether an argument was convincing, but the teachers were silent (Meeting transcripts 9/17/13, line 86; 10/22/13, line 124).

For the second cycle pizza problem, one claim was made that a student's argument was convincing during the debrief meeting. The following argument was given:

> First, we looked at the pizzas with only one-topping and got four different pizzas. We know this is right because there are only four toppings. Second, we looked at the pizzas with two-toppings and we got six pizzas. We know this because we took the pepperoni and grouped it once with each other topping. Then we took the mushrooms and grouped it once with the other toppings, except for pepperoni, because it was already grouped with it. Then we took sausage and only grouped it with peppers because it was already grouped with mushrooms and pepperonis. Third, we looked at the three-topping pizzas. (Meeting transcript in-district class visit 10/22/13, line 112).

The instructor asked if the argument was convincing and T6 replied “Yeah. They just said that they were grouping them with the other ones, and then they didn’t do because it was already there.” (Meeting transcript 10/22/13, line 114).

For the third cycle three-tall towers problem, two claims were made by teachers that the students’ arguments were not convincing and were coded by the researcher as incomplete. The following argument was shown on the screen to the teachers and read aloud by T10:

> These Two blocks are the same, and the blocks that made the pattern are different. So in the blue category, there will be one yellow block on a stack of two blue blocks. This helps prove our theory. We also have a random category where each tower contains one different block. So, one of our towers is red on the bottom, blue in the middle, and yellow on top. If we changed the order of the blocks, it would have a different tower we already have. (Meeting transcript debrief 11/20/13, line 25).

The instructor asked the teachers which parts of the argument were convincing and not-convincing and T3 responded “It is not really convincing anywhere. He gave us one example and then his partner said to… then he switched it up.” (Meeting transcript 11/20/13, line 27-29). For this sample of work, the student used a cases argument and controlled for a variable when solving the third cycle three-tall towers problem.

The second argument that a teacher said was not-convincing from the third cycle three-tall towers problem was one where the student wrote the following argument about his three groups of 9 three-tall towers: “I know I got all of them because all of the groups I got the same amount exactly.” The instructor asked if the argument was convincing and T10 replied “No.” (Meeting transcript 11/20/13, lines 126-127). For this sample of work, the student also used a cases argument and controlled for a variable when solving the third cycle three-tall towers problem.

*7.4.3.2 Summary of Evaluation of Arguments from Class Visits*

Teachers claimed that they were convinced by students' arguments from the in-district classroom visit only 1 time throughout the three cycles. There were 5 students' arguments from the in-district classroom visit that teachers found were not convincing. From the 4 not-convincing arguments, the researcher coded 5 arguments as incomplete based on the teacher's comments during the debrief meetings on 9/17/13, 10/22/13, and 11/20/13. In two of the non-convincing arguments, the opposite strategy was used and recursion was used in one of the non-convincing arguments. For the other two non-convincing arguments, students used case arguments and controlled for a variable.

***7.4.4 Teachers' Evaluation of Arguments of Current Students' Work***

During the regional meetings, teachers shared their own student's written samples of work with the other teachers. As teachers shared their written students' work, the instructor asked the teachers if they thought the students' arguments were convincing. Teachers took turns reading their students' work to the teachers while the students' work was projected on the screen.

*7.4.4.1 Evaluation of Arguments by Cycle of Current Students' Work*

For the first cycle four-tall towers problem, teachers claimed that 7 arguments were not-convincing. One of the following non-convincing arguments was coded by the researcher as invalid based on the following comments from the teacher at the meeting:

> This group was a group of boys. And they immediately went to the math of it and said that there are two colors and they have to be four high. Two times four is 8. And then once they made the first 8 blocks they realized they could do opposites. So they doubled them and made 16. (Meeting transcript students' work 10/2/13, line 326).

The instructor asked the teachers about what they thought of the argument (Meeting transcript students' work 10/2/13, line 329). T3 responded that she liked that the students used math but admitted their argument did not make sense mathematically (Meeting transcript students' work 10/2/13, lines 326-333). The other 6 arguments were coded by the researcher as incomplete and non-convincing. The 6 excerpts of arguments follow:

> T1: We couldn't make anymore because we think we made all the patterns plus we found all the blocks and [pause] we all worked together to create these patterns. So again, they couldn't really explain.
> R1: Well they are. That's their reason. Is it a good convincing argument? What do you think? Are you convinced? We're done because we couldn't make any more.
> T3: Well she said all the patterns, so…
> R1: We made all the patterns, right but does that help you?
> T3: A little bit.
> R1: It does? Are you convinced?
> T3: Well I mean for this group because of the level.
> (Meeting transcript students' work 10/2/13, lines 103-109).

> T3: They said two of the same color is touching and 1 color isn't touching. So they kind of said to me that two of the yellows are touching. Then they alternated touching Then they alternated them so that none of the same color is touching. Both colors are next to their twin.
> R1: What do you think of that I think if you wanted to get a convincing argument for the last six towers? What might you do? How do we know we have all the towers that are exactly two one color and two of the other. What would you ask them to do? Are you convinced?
> T3: That is the most convincing that I have read. (Meeting transcript students' work 10/2/13, lines 319-321).

> T4: All of them gave examples. All my students pretty much said the same thing about opposites. At least he said for this one it was going diagonally. But he didn't add anything else to it.
> R1: So what do you guys think?
> T4: It is a good start. (Meeting transcript students' work 10/2/13, lines 429-433).

> T6: this is a girl that had a really good argument when she was talking to me, but she didn't finish. Her last sentence just stops. But she starts to talk about the twos really well and this is her drawing. So she again has the stair case and then she has the opposite. And she did the twos stuck together. Um, what is she saying?
> R1: If you read what she says; but not yet. Look at her second grouping where she has 2 of one color and two of the other. What did she do?
> T6: She has a constant.
> R1: She has a constant on the top do you see that? And in the third group, there is the same thing, right? A constant is on the top. Okay.

T6: She said I used two of each color. The first one in the set has the colors together. So there's two of one color on the bottom and two of the other color on the top. (Meeting transcript students' work 10/2/13, lines 600-610).

T7: They said that there can only be two completely one cube tower (only two colors). There can be four, 3 red one yellow towers because there are only four high towers and that is the same for red. And then they said for two red and two yellow towers there can only be 2 because there are only two sides to switch to make two different towers. They were talking about two red on top and two red on the bottom. Then they said that for the towers in the center there can only be two because there are only two colors to put in center of the tower. And then they said that for the towers that have a pattern, there can be two because there are only two colors to make up the pattern starting from the top (or bottom). That was the alternating one.
R1: Right, right.
T7: So this is much better than their argument on the front. (Meeting transcript students' work 10/2/13, lines 679-681).

T8: She said group one has 2 reds together every time you move it to the top, middle, and end. I guess she was saying that the 2 reds are at the top; middle; and at the bottom. She said for group 2 it just has four yellows and 4 reds on each. For group 3, it only had one yellow so the yellow cube started on top, and went down one every time and it stopped at the bottom. For group 4 all the reds were separated. Uh, for group 5 the red cube started on the top and went down one every time and stopped at the bottom. And then at the end she said for each group, I couldn't make any more because there was no more possible combinations and if I added one more to it; it would be 5.
R1: What do you think? Are you happy with her thinking? Do you think she should have done something else? Are you not sure? Is it convincing to you? Is any of it convincing?
T3: I like how she talked about moving one down.
R1: Good. Again, you want them to be able to… uh go back to her picture. You want them to be able to let you know that there can be no more towers in that group, okay, and that there aren't any other groups or any other ways to arrange. So I think that this is the beginning of something that could be a very nice convincing argument, right? Okay.
T8: She started with opposites and I said alright that is not working for me. Do it a different way. Okay. (Meeting transcript students' work 10/2/13, lines 787-796).

For the second cycle pizza problem, teachers claimed that 7 arguments were not convincing. All 7 of the non-convincing arguments were coded by the researcher as incomplete based on the comments from the teachers during the discussion. The following are the excerpts of the teachers' responses:

> T6: We organized the choices by toppings as we went on. When we got to a new topping, we took out all of the duplicates from the other toppings. For example, we started off, out with all of the pepper combinations. There were 8 of them. When we got to the sausage, there were only 4 combinations because there were 4 duplicates from the pepper. We did the same thing for mushrooms and pepperoni. The only thing left to do was to add one plain to our list, which we added. And then they said we got 16.
> R1: What do you think of their explanation of their argument? Is it convincing?
> T6: They're just saying that they found all of them and they took out the duplicates. (Meeting transcript students' work 10/22/13, lines 232-234).
>
> T5: There are 10 possibilities from the topping in order, I can reverse them and in the middle I can use them too. If all the ten combinations are reversed then it would still be the same. I, I think he understands about the whole duplication of it. But I think he just, maybe he could have done a little bit more were as maybe just even realize, okay, and we could do a plate. We could come up with; well he has the one with the all, then to make the next set of three combinations.
> R1: So, is his argument convincing?
> T5: No. (Meeting transcript students' work 10/22/13, lines 335-337).
>
> T8: For group 2, I started with mushroom and then I did…two toppings; there are mushrooms in each one. And if I put another one I get three toppings; and she kind of went on to do that for each argument for each group. So, I don't know, what do you guys think? You think that works as an argument?
> R1: Is it convincing? She's telling you what she did, right?
> T3: I mean it could be if she just went in depth a little further. (Meeting transcript students' work 10/22/13, lines 362-366).
>
> T8: So she…the five choices is plain and the single toppings. Then I combined all four of the toppings together. After that I took one topping and put it two toppings and not to get it to repeat. After that I took one topping and put it with one other topping to get it to repeat. So I guess for the one-topping and the two-topping, she's doing the three with all the constants. But she doesn't really say that she kept a constant or anything like that.
> R1: What is her argument that she writes, is it convincing?
> T8: I don't really think that it's that convincing.
> R1: No, it's really not. Why not? She is telling you what she did. But she is not really giving an argument why there are exactly that number pizzas in each group. She is not really telling you why.

T8: I feel like that's the case most of the time. They just tell you what they do, instead of how. (Meeting transcript students' work 10/22/13, lines 386-390).

T3: They used a system. And they use it by describing arrows. So they kind of showed me that they listed each of individual toppings; so peppers, sausage, pepperoni, mushrooms. And then they went peppers sausage with the pizza, pepperoni sausage pepperoni with the pizza, pepper sausage pepperoni mushrooms and they kind of used the arrows to come up with the 16 combinations. So I thought that was pretty good. They were…understood it pretty quickly. And…they were pretty decent with their explanations. This is as far as written wise that they could give me. And that was their convincing argument with that.
R1: Okay, so is it convincing to you?
T3: I liked the way that they organized it with the arrows but there is no real argument there about why they have all of them. (Meeting transcript students' work 10/22/13, lines 491-500).

T10: Then it says, next we put two different toppings without repeating them again. We got 6 total pizzas for two toppings, and that makes 12 pizzas total.
R1: Hold on. Is that convincing…why they have two…6 two-toppings?
T10: Not really.
R1: No, their argument there falls apart. Go ahead…three-topping…
T10: We put three toppings on each pizza without repeating. It's pretty much what they said they just did them without repeating which isn't very convincing. (Meeting transcript students' work 10/22/13, lines 574-582).

For the third cycle three-tall towers problem, teachers claimed that 2 arguments were not convincing. The non-convincing arguments were coded by the researcher as incomplete based on the comments from the teachers during the discussion. The following are the excerpts of the teachers' responses:

T10: We know there are no more towers in each group because if you added another there would be a duplicate. For example, three towers each with the same colors and then one more. There is not much on top there.
R1: So what do you think of the argument?
T10: Not good.
R1: It is not, because what they did was they are saying basically you can't find any more because you will get a duplicate. That's not a good argument but interesting code.
T10: Yeah I thought their grouping was good. (Meeting transcript students' work 10/22/13, lines 30-34).

> T8: So for group one for each tower I had one color. For each tower I had each color on the top and then followed by the 2 other colors switching. I did it for all three. So for group 2 she kind of did the same thing as the first one. For group 3, for the first 3, I had the blue go up one every time. For the last 3, I had the yellow go up every time. Okay, alright, because she drew the cubes a little backwards on that. Group 4 for the first 3 I had the red go up every time then the yellow every time. So she is really just talking about how they moved but not really saying you know.
> R1: Good. That is really good that you guys are picking up. She is explaining what she did she has a very, very good strategy but she is not saying that therefore there can't be any more because I have taken that single color and put it into each of the three positions and there is no other place to put it. (Meeting transcript students' work 10/22/13, lines 318-323).

*7.4.4.2 Summary of Evaluation of Arguments of Current Students' Work*

Teachers claimed that they were not convinced by current students' arguments 16 times throughout the three cycles. From the 16 non-convincing arguments, the researcher coded 15 as incomplete based on the teacher's comments during the regional meetings on 10/2/13, 10/22/13, and 11/20/13. Only one non-convincing argument was coded as invalid by the researcher based on the teacher's comments during the discussion at the 10/2/13 regional meeting.

### ***7.4.5 Summary of Teachers' Evaluation of Arguments***

As teachers worked on the tasks themselves, teachers claimed 8 arguments were convincing and 1 argument was not convincing. For the on-line discussion of research students' work, teachers claimed 18 arguments were convincing and 8 were not convincing. During the in-district classroom visits, teachers claimed 1 argument was convincing and 5 arguments were not convincing. When teachers discussed samples of students' work, teachers claimed 16 arguments were not convincing.

Overall, teachers were convinced by 27 arguments and not convinced by 30 arguments throughout the three cycles. The data indicate that the PD intervention helped teachers to determine whether or not an argument was convincing. Teachers described 29 of the non-convincing arguments as incomplete and were coded as such by the researcher. The instructor responded with the following to a student's work shared by T8 regarding convincing arguments:

> I think again, it is a process and she is coming along in the process because in the first two tasks, I don't remember seeing all this writing, right? I think also that is very nice, even if she is just explaining what she did, she is writing. And once you get them writing, you can get them to write a convincing argument. (Meeting transcript, 11/20/13 current students' work, line 307)

Providing opportunities for teachers and students to make convincing arguments and critique whether arguments are convincing or not convincing helps teachers to attend to students' reasoning.

**7.5 Gang of Four Pre- and Post-Assessment**

After watching the Gang of Four VMC video, teachers were asked to describe each example of reasoning recognized by the children in the video; whether or not the argument is valid and/or convincing; and justify why or why not the teachers were convinced by the argument on a pre- and post-assessment. A scoring rubric by Maher, Palius, Maher, Hmelo-Silver, and Sigley (2014) was used to determine complete or partial arguments for the two cases arguments and the inductive argument that teachers' identified on the pre- and post-assessment and to measure if any changes occurred. The scoring rubric that was used follows:

> *Cases Argument 1*: Stephanie's cases argument for towers three cubes high that are selected from two colors (blue and red) resulted in a set of eight unique towers. A complete argument includes each of the following cases. Note that written responses by study participants may well be fragmentary and use much less precise language than the following:
>
> - All blue cubes or not red cubes, resulting in only one tower.
> - One blue cube and two red cubes, resulting in three unique (different) towers.

- Two blue cubes stuck together and one red cube, resulting in two unique towers.
- No blue cubes or all red cubes, resulting in one tower.
- Two blue "stuck apart" or separated by one red cube, resulting in one tower.

*Cases Argument 2*: An alternate cases argument for towers three cubes high that are selected from two colors (blue and red) proposed by several of the children resulted. Several of the cases overlap completely with the ones articulated by Stephanie. Participants may describe the organization of the third case as better (e.g., preferred, more elegant) than the way Stephanie organized her cases, which bifurcated it into the third and fifth cases in the Cases Argument 1, above.

- All blue cubes or no red cubes, resulting in only one tower.
- One blue cube and two red cubes, resulting in three unique (different) towers.
- Two blue cubes stuck together and one red cube, resulting in three unique towers.
- No blue cubes or all red cubes, resulting in one tower.

Inductive Argument: This argument may be expressed with reference to towers of a specific height, as in the two features below. It also may be expressed in general form.

- When building towers that are selected from two colors, there are exactly two unique towers of height one. With a single position in the tower, the one cube can be (say) either red or blue.
- Two unique towers of height one can be used to generate all possible towers of height two. For each tower one cube in height, two different towers can be built from it. Starting with (say) a red cube in the first position, either a red cube or a blue cube can be placed in the second position. Similarly, starting with a blue cube in the first position, either a red cube or a blue cube can be placed in the second position. The resulting four unique towers of height two is double the amount, tow, that there are of towers of height one. (And so on for n-tall).

(Maher et al., 2014, p. 46-47)

There were two forms of case arguments. The scoring rubric was used to decide whether the argument was partially or completely described for the two cases. The rubric was used in a similar way for the third inductive argument to establish the following two criteria: "with two colors of cubes, there are two possibilities for a tower of height one and that each tower then has two possible choices for the color of a cube to be addend on for a tower of height two, and so on" (Maher et al., 2014, p. 37). Table 7.5 shows a comparison of the reasoning arguments in the pre- and post-assessments.

Table 7.5

*Comparison of Reasoning Arguments in Gang of Four Assessments*

| Argument | Pre-Assessment | | Post-Assessment | |
|---|---|---|---|---|
| | *Partial* | *Complete* | *Partial* | *Complete* |
| Case Argument 1: Stephanie's Cases Argument | 3 | 3 | 4 | 6 |
| Case Argument 2: Alternative Cases Argument | 1 | 0 | 1 | 0 |
| Inductive Argument: Milin's Argument | 1 | 5 | 2 | 7 |

*Sources:* Pre- and Post-Assessment for Gang of Four Video

### *7.5.1 Gang of Four Pre-Assessment Case Arguments*

Three out of ten teachers described complete case arguments on the pre-assessment for Case Argument 1, Stephanie's cases argument. The responses of the three teachers follow:

> The argument made by Stephanie about creating towers of height 3 using 0 blue cubes, 1 blue cube, 2 blue cubes, and then three blue cubes was insightful. Stephanie leaves the blue, red, blue tower out of her list of 2 blues because she only considers towers where the blues are next to each other to fit into this category. She adds this on the end after she has built the tower of three blues. (T10, Gang of Four, pre-assessment)

> Stephanie reasoned with a pattern of how many blue blocks could be in the tower rather than focusing on the red blocks. She started by stating that there is only one possible tower with no blue blocks (which is the tower of three red blocks). She then moved on to note that there could be one blue block in the tower and showed three possible towers with one blue block noting that it could be located at the top, middle, or bottom of the tower. She also mentioned that there were no other towers with one blue block since it started at the top and moved down each time, once it reached the bottom of the tower there was nowhere else it could be placed. After, she focused on towers with two blue blocks stuck together showing that there were two possible towers, (BBR and RBB). Then moved on to three blue blocks resulting in one possible tower. Stephanie then went back to the two blue blocks and said they could also be apart resulting in one more tower (BRB). Altogether, she found eighth possible towers. Her reasoning was valid and convincing. She was able to follow a pattern to show why there was only one tower with no blue, three towers with one blue, and one tower with three blue. A concern is raised with the three towers with two blue (T7, Gang of Four, pre-assessment)

> Then Stephanie gives her reasoning. She drew out the pattern. She was trying to convince Jeff that there are 8 ways to build towers of height 3. She drew out her patterns. First, she began drawing out all the towers with 0 B. There was only 1 - RRR. Then she listed out the towers with 1 B. There were 3 - BRR, RBR, and RRB. She later explains that she used the strategy of moving the B down a level each time to make sure she did not miss any combination. Then she listed all the combinations using 2 B stuck together. There were 2 - BBR and RBB. Next she listed all the ways of 3 B. There was just 1 - BBB. Last she thought of all the ways to have 2 B not stuck together. There was just 1 - BRB. (T9, Gang of Four, pre-assessment)

Three of the teachers described partial cases arguments for Case Argument 1, Stephanie's cases argument on the pre-assessment. The partial cases arguments of the three teachers follow:

> Stephanie on the other hand created the towers according to having specific colors stacked on top of one another (blue stacked together or red stacked together) and then she considered if the colors were separated. Stephanie made a convincing argument and was easily able to articulate her reasoning to her classmates using her picture and verbal reasoning. (T3, Gang of Four, pre-assessment)

> During the second segment when Stephanie was explaining her methods toward solving the problem and she comes to the conclusion that you should follow a pattern to solve the problem, well at least her method involved a pattern. Stephanie: "Well, you are following your pattern, but my pattern goes no red, one red, this was not meant to be like that. That's not- it's in the category of one blue. I could stick that in another category, but I want this to be in the category of one blue and not in the category of opposite of this one. And then I have this one red/red/blue. So, to you - you might put that way at the end of the line but I put it right here." I think that finding a pattern with this type of problem allows for you to justify that there are absolutely no more options available. I think that Stephanie's reasoning was the most valid because she spent most of her time trying to convince Jeff that her answer was correct and that her patterns she used worked for her to get to the correct response. Throughout Stephanie's explanation she convinced the entire group that the easiest way to solve this problem was to make patterns. (T2, Gang of Four, pre-assessment)

> Stephanie: Stephanie talks about different possible combinations that she could make with the blocks that she is given. At one point in the conversation she lists the different combinations that you can make with three blocks. This is a very straight forward argument. "Stephanie: Here is one red/red/red, blue/blue/blue and then I go like red/blue/blue, blue/red/blue-"Stephanie continues to work with the colors this way and "moves" blocks down. Stephanie's argument was very unclear. I think Stephanie is doing a good job explaining that specific example;

> however, her reasoning does not convince me that she could use her method to make a prediction with x amount of blocks. (T6, Gang of Four, pre-assessment)

Also, one teacher described the following partial cases argument on the pre-assessment for the Case Argument 2, the alternative cases argument:

> The argument made by Stephanie about creating towers of height 3 using 0 blue cubes, 1 blue cube, 2 blue cubes, and then three blue cubes was insightful. Stephanie leaves the blue, red, blue tower out of her list of 2 blues because she only considers towers where the blues are next to each other to fit into this category. She adds this on the end after she has built the tower of three blues. Milin and Michelle want the blue, red, blue tower to be included in the 2 blues description, so a little discrepancy between the students thinking about the systematic list existed. Stephanie's systematic list seemed to be a valid explanation and was convincing, as Stephanie was able to convince Jeffery of the reason that she did find all 8 possible arrangements. It was clear that Jeffery understood the pattern that Stephanie created after he rationalized and created the same list of towers on his own. I am convinced by Stephanie's argument that she found all of the possible arrangements, but had similar thinking to Milin and Michelle as she was explaining. I think in her pattern, it would have been best to include the blue, red, blue arrangement in the context of 2 blues instead of at the end. Leaving this off because the blues are not stuck together could be problematic for students when following this thinking on a different problem.
> (T10, Gang of Four Pre-Assessment).

It should be noted that the teachers did not mention the words case argument on the pre-assessment.

### *7.5.2 Gang of Four Pre-Assessment Inductive Arguments*

Five of the teachers described complete inductive arguments on the pre-assessment. The responses of the five teachers that described complete inductive arguments follow:

> Milin listed or created different towers using all red, one red or two reds in different positions throughout the tower. He exhausted all possible solutions with the one, two or three reds whether they were the colors were together or separate. He was then able to see if you wanted to increase the tower size, you simply could add two additional formations for every tower already created by adding either an additional red or an additional blue. His argument made sense, but he had a hard time verbalizing it to his classmates and providing proof of why he gave a specific answer. (T3, Gang of Four, pre-assessment)

Milin steps in to continue the explanation and states that it must be 8 because if you took the 4 previous towers of 2 blocks high and wanted to create towers of three blocks high, each tower could have either a blue or a red block added to the top. Because of this the blue, blue tower could become the blue, blue, blue or blue, blue red tower. Since each of the 4 previous towers has two possible new top pieces to make a tower of 3, there are a possible 8 towers that are three blocks high. Milin's thinking is valid and convincing. (T10, Gang of Four, pre-assessment)

Milin reasoned with a pattern from the basic tower one block high which is either one blue or one red, resulting in two outcomes. He then reasoned that to build a tower two blocks high, he would need to put one more block on each of the towers that were one block tall, and since there were only two options for the color, there were two resulting towers built from each of the basic towers, resulting in four possible towers that would be two blocks high. He followed this reasoning and pattern to say there would be eight different towers three blocks high since each of the four towers that were two high would have two options for the third block, doubling the previous amount of towers. His reasoning formed a valid argument that was convincing to his audience. Since he begins with the most basic example, it is clear to see why there are only two outcomes. Building from the basic tower, he is able to clearly show why there are only two options built from each of the previous towers since he can focus on the options for the one block being added to the tower rather than having to focus on the entire tower. From this reasoning, he is able to determine that for every block added to the height of the tower, the options for the tower would double from the amount of towers, one less in height. (T7, Gang of Four, pre-assessment)

Then, Milin continued that for every new height, you could add 2 ways for each way you already had. So, when he had 1 R and 1 B, on each of those towers, you could add 1 more R and 1 more B. This doubled the number of towers you had. He continued to say with a tower of height 3; you could again put either an R or a B on top, doubling the amount of towers yet again. (T9, Gang of Four, pre-assessment)

Milin is the first student to recognize that the number of tower increases by a multiple of 2 when adding one block. This is a great observation and a good start; however, not a convincing argument. It is a valid argument; however, to be sure that the pattern holds true, the student must provide evidence of why the number of towers is increased by a factor of two for each block. Later on in the conversation, Milin does provide an explanation of why the number of towers is doubled every time you add a block. Milin says " For each one of them you could add one - no two more on because there is a black, I mean a blue, and a red -” When he says this he is saying for each way that you could make a tower, if you add a colored block you can have two towers for each that already exist by adding either a blue or a red to that tower. (T6, Gang of Four, pre-assessment)

It should be noted that the teachers did not mention the words induction on the pre-assessment. One teacher described a partial inductive argument on the pre-assessment which was as follows:

> When Milin was asked about 4 he went back to his explanation that you would be adding 1 block to each arrangement of each of the columns to equal 4 high, but that means you would multiply the amount of columns by 2 because you are adding 2 more different arrangements. (T5, Gang of Four, pre-assessment)

### *7.5.3 Gang of Four Post-Assessment Case Arguments*

On the Gang of Four post-assessment, six teachers described a complete case argument of the Cases Argument1, Stephanie's cases argument. The responses of the six teachers follow:

> Stephanie took her explanation to the next level at the group discussion. She drew her tower examples and used a proof by cases method to diagram towers with no blue cubes, exactly one blue cube, exactly two blue cubes stuck together, three blue cubes, and two blue cubes stuck apart. (T4, Gang of Four, post-assessment)

> Stephanie used a proof by cases to make her argument. She started with a tower of no blue and showed there was only one tower in that category. She followed with towers of one blue and explained that there were only three because the one blue could not go down another block. Then came the towers with two blue but she kept them stuck together which results in two towers. This caused some confusion with her classmates as they wanted her to have all the two blue together. She moved on to the one tower that had three blues and then went back to the tower with two blues but not together and showed there was only one tower in that category. (T7, Gang of Four, post-assessment)

> Stephanie uses a systematic listing strategy to solve the problem. She begins by building a tower 3-high with all red. Then, she lists towers with 2 red and 1 blue. She uses a recursive pattern moving the blue block down until she exhausted all the positions. (T9, Gang of Four, post-assessment)

> Stephanie proved her solution using proof by cases argument. She started with no blue, one blue, two blues, three blues, four blues and then all blue. She used a diagram to show her solution. (T2, Gang of Four, post-assessment)

> Stephanie begins her proof for 8 towers 3-tall using a proof by cases. She first shows a group of 0 blues which is only 1 because there is only 1 color that is not blue. Her next group is 1 blue in which she moves the blue through each position

> in a staircase pattern saying that you couldn't move the blue 1 more spot, because then you would have a tower that is 4 tall. (T10, Gang of Four, post-assessment)

> Stephanie used a proof by cases argument. She first made towers with no blue, one blue, two blue and three blue. (T6, Gang of Four, post-assessment)

Four teachers described partial cases arguments for the Case Argument 1, Stephanie's cases argument from the Gang of Four post-assessment. The responses of the four teachers follow:

> In the second video, it seemed that Stephanie started to use the constant approach which is definitely a valid and convincing argument. She started with "no blue" and continued adding one blue, two blue, and so on. I am convinced by this because by using a constant and manipulating the constant, you are able to prove that you use all possible positions on the tower. Milan also demonstrates the same argument. (T1, Gang of Four, post-assessment)

> Stephanie on the other hand, proved her argument by using proof by cases. She focused on "how many" of each color was contained in every tower. While explain her argument, Michelle jumped in to explain how he saw the "staircase" pattern where one color was the focus point in a group of towers. Michelle found it easy to follow Stephanie's argument so she went on to explain her argument in more detail. She then went on to explain to Jeff that each group of towers she created contained a specific number of either blue or yellow. She understood that you could have the same amount of either blue of yellow and then just change the position of each. This meant she could either keep the same colors toughing or not touching within the tower. In the end, Stephanie and Michelle were able to convince Jeff that there were 32 total towers that were 5-tall containing two colors. (T3, Gang of Four, post-assessment)

> Stephanie uses proof by cases. She starts out with no blue then 1 blue and so forth. She states that you could not have any more patterns because the blue because you would need to add an extra cube. She then clarifies by stating that once the blue cube is to the bottom you cannot make any more patterns. I find her argument valid also because she is explaining that once the blue is to the bottom all possibilities have been done. Although each group had a different approach to their argument I felt that they each had convincing/valid arguments. That identified how they solved the problem by stating how they knew when they were done. (T5, Gang of Four, post-assessment)

> Stephanie used a proof by cases method, by keeping 2 of the same color stuck together. On page 7-8 she was able to create a table and show how she was completing it and provide justification for why she was completing it in that way. On page 13 and 14 she explains her reasoning and proves her method to the others (So I've convinced you she repeatedly said). Her pattern is more complex than the

> others. The fact that she is able to correctly complete the table in that manner and explain it step by step shows that her reasoning is correct. And she already figured out how many outcomes there would be if there were towers of 10. This shows her understanding of the concept by her application to a more complex outcome, which would have been extremely difficult to find using a table. (T8, Gang of Four, post-assessment)

It should be noted that eight of the ten teachers specifically used the words proof by cases and then provided the description of Stephanie's case argument. Also, one teacher described the following partial cases argument on the post-assessment for the Case Argument 2, the alternative cases argument:

> Stephanie used a proof by cases to make her argument. She started with a tower of no blue and showed there was only one tower in that category. She followed with towers of one blue and explained that there were only three because the one blue could not go down another block. Then came the towers with two blue but she kept them stuck together which results in two towers. This caused some confusion with her classmates as they wanted her to have all the two blue together. She moved on to the one tower that had three blues and then went back to the tower with two blues but not together and showed there was only one tower in that category. Her argument was valid and convincing but it would have been more convincing to her classmates and others if she kept all the two blues together, even if she had the two subcategories. When her classmates tried to convince her she should keep all the two blues together she made a good argument that her pattern is different than others and that she can arrange them in different groups if she wanted to but that was not the way she was doing the problem. (T7, Gang of Four Post-Assessment)

### *7.5.4 Gang of Four Post-Assessment Inductive Arguments*

For Milin's inductive argument, seven teachers described complete inductive arguments on the post-assessment. The descriptions of the seven teachers follow:

> Milin explained that the base of the tower could be either red or blue making two options for a one-tall tower. For the second position in the tower we could have either a red of a blue which meant that each of the original one-tall towers could now create two more towers that would be two-tall. Milin kept a constant for the base the kept the base and second position constant, then kept the base, second position and third position constant, etc. Milin used inductive reasoning to explain why there could only be eight tower options that are three-tall and picking from two colors. (T4, Gang of Four, post-assessment)

> In the videos Michelle starts to draw and explain an inductive argument. She is

having difficulty so Milin helps out. Throughout the two videos I find their arguments valid. They explain that you add a color to each tower to create more. They state that as they are drawing and explaining that since you only have two colors you cannot make any more towers. When asked about the towers of four I feel their argument was also valid. They stated, "You would add a red or blue to the eight towers which would make 16." Their argument shows that you are making another group without making duplicates. Stephanie uses proof by cases. She starts out with no blue then 1 blue and so forth. She states that you could not have any more patterns because the blue because you would need to add an extra cube. She then clarifies by stating that once the blue cube is to the bottom you cannot make any more patterns. I find her argument valid also because she is explaining that once the blue is to the bottom all possibilities have been done. (T5, Gang of Four, post-assessment)

Milin solved the tower problem by using an inductive argument. He was able to convince the audience that there were 32 total towers that were 5-cubes tall containing two colors. His inductive argument explained the towers by building upon a base. He started his argument by explaining towers one tall, two-tall, three-tall, four-tall and five-tall. He started by explaining that when using towers that are only 1-cube tall, there would be two total towers. He then went on to explain that there would be four total towers for two-tall. He then followed by three-cubes tall having eight possible towers. Milin justified his answer because he understood that from each previous tower height, two additional towers could be created from one single tower. By recreating two exact towers from the previous height, that you could add one yellow and one blue on top which would then create two different new towers. (T5, Gang of Four, post-assessment)

Milin’s argument is convincing. He builds an inductive argument for multiplying the previous number of towers by 2, starting by building towers 1 tall, then 2 tall, then 3 tall, etc. explaining that to each tower, there are 2 different colors of blocks to add to the top, doubling the number of towers. (T10, Gang of Four, post-assessment)

Milin used an inductive argument.  He explained that for each of the towers of the previous height a red or blue could be added to the tower so there are two towers that can be made from each of the previous so the number of towers is doubled. He started by showing there were only two towers one tall because there were only two colors and from each of those towers, he had two options for the next block.  (T7, Gang of Four, post-assessment)

At first, Milin began by describing a pattern of multiplying by 2. This is not a valid or convincing argument because Milan does not explain clearly what he is multiplying by 2. If this was the case, 14 is a multiple of 2; however it will never be a solution to the tower problem.  As Milan is questioned further, he explains his thoughts much more clearly. He says that when you have a tower and you want to add a block to make it taller, you only have 2 colors to choose from.

> Because of this, each tower can make 2 more towers from it. Therefore, you are doubling the number of towers you have for each block you add to the height. (T9, Gang of Four, post-assessment)

> Michelle and Milin seem to have a more inductive reasoning approach toward solving this problem. That is they are rationalizing their answer of 32 for 5 high by saying if you started with 1 high you solution would be two, 2 high would be 4, 3 high would be 8, 4 high would be 16, 5 high would be 32. At first Michelle explained that doubling meant 25 would be the answer based off of what Milan said but then she began to explain and showed that you weren't doubling the towers (5x5) you are doubling their solution because you only have two colors. (T2, Gang of Four, post-assessment)

Two teachers described partial inductive arguments. The partial arguments of the two teachers are below:

> Millin also uses inductive reasoning. He is able to accurately predict different towers tall by doubling the previous. He explained this clearly about how for each tower you can add either a blue or a red, therefore doubling the towers. I think his argument is convincing. (T6, Gang of Four, post-assessment)

> Millan’s reasoning was also inductive reasoning. He stated that he had to times by 2, but when asked to explain he struggled. On page 3 he said times the towers by 2 because 1 and 2 is 2, and 2 and 2 is 2 (but then corrected and said 4). On page 6 he was able to identify Jeff’s pattern as incorrect. On pages 6 and 7 I was unable to follow his explanation, but he seemed to know how to find the answer. Jeff did seem to understand what he was saying though. Millan is able to find the correct answer, but is unable to validly explain his reasoning. On page 16 he shows that he is able to find that towers of 5 had 32 possible outcomes. So I am unsure of his reasoning being correct, but he does know how to find the correct answer. (T8, Gang of Four, post-assessment)

It should be noted that eight teachers used the word inductive in their description.

Three teachers also described recursive arguments in the post-assessment. A table that shows the teachers’ recursive arguments found from the Gang of Four pre- and post-assessment is located in Appendix K. Two of the teachers provided descriptions of Stephanie’s recursive argument and one teacher described a recursive argument by Jeff and Michelle.

### *7.5.5 Summary of Gang of Four Pre- and Post-Assessment*

On the pre-assessment, 7 teachers claimed they were convinced by Case Argument 1, Stephanie's cases argument. On the post-assessment, all 10 teachers claimed they were convinced by Case Argument 1, Stephanie's cases argument. One teacher described a partial argument for the alternative cases argument on the pre-assessment and a different teacher described a partial argument for the alternative cases argument on the post-assessment. On the pre-assessment, 4 teachers claimed they were convinced by the inductive argument. On the post-assessment, 8 teachers claimed they were convinced by the inductive argument.

## 7.6 Summary of Reasoning Analysis

Based on the results of the reasoning analysis from this research, the PD intervention was effective in helping in-service middle-school mathematics teachers to attend to students' reasoning. Teachers' recognition of reasoning was examined in the following contexts: teachers doing the tasks themselves, teachers recognizing research students' reasoning from articles and videos during an on-line discussion, teachers recognizing reasoning from current students during debrief meetings after three in-district classroom visits, and teachers recognizing reasoning from current students' samples.

Case arguments and controlling for a variable were used frequently to solve the problems in all three cycles. It should be noted that the opposite strategy was used and identified frequently for solving the first cycle four-tall towers problem and the three-tall and five-tall extension problems. However, the opposite strategy was used fewer times to solve the second cycle pizza problem, the third cycle three-tall towers problem, and Ankur's Challenge.

There were times when heuristics and forms of argument co-occurred with solving the given problem. The most frequent co-occurrences were controlling for a variable and case arguments. Other strategies and forms of argument co-occurred less frequently compared with the co-occurrence of case arguments and controlling for a variable. It should also be noted that the co-occurrences happened more frequently when solving the third cycle three-tall towers problems and Ankur's Challenge.

Teachers also determined whether arguments were convincing or not convincing. Teachers were convinced by 27 arguments from students and other teachers using forms of argument such as induction, recursion, contradiction, and case arguments from problems in all three cycles. Teachers were not convinced by 30 arguments from students and teachers. Twenty-nine teachers described the non-convincing arguments as incomplete and only 1 teacher described an invalid argument.

The Gang of Four video post-assessment revealed that all ten teachers claimed to be convinced by Case Argument 1, Stephanie's case arguments. Only one teacher described a partial argument for Case Argument 2, the alternative case argument on the Gang of Four post- assessment. Case arguments were the most common identified form of argument from the Gang of Four assessments.

Eight teachers wrote they were convinced by Milin's inductive argument on the Gang of Four post-assessment. However, all ten teachers wrote they were convinced of Milin's inductive argument during the Unit 4 on-line discussion. The data indicate that teachers were able to recognize inductive arguments. However, teachers tended to avoid using inductive arguments as the data revealed that teachers only used inductive arguments three times when solving the problems throughout the three cycles.

# Chapter 8 – Instructor Moves Summary

This chapter is an analysis of the instructor moves for the three cycles of tasks. The instructor moves are examined in three parts in the following contexts: as teachers worked on the tasks, as teachers participated in an on-line discussion threads about research students' work on the mathematical problems, and as teachers discussed current students' work samples brought by teachers during regional meetings and after the in-district classroom visits. First, the types of questions asked by the instructor are examined. Second, pedagogical practices used by the instructor are examined. The chapter concludes with a description of the representations used by the instructor. Table 8.1 summarizes the instructor moves coded by context and cycle.

Table 8.1

*Frequency of Instructor Moves by Context and Cycle*

| Moves Of Instructor | Teachers' Own Work | | | Research Students' Work | | | Class Visit Students' Work | | | Current Students' Work | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | *C1* | *C2* | *C3* | *C1* | *C2* | *C3* | *C1* | *C2* | *C3* | *C1* | *C2* | *C3* |
| *Question Type* | | | | | | | | | | | | |
| Explanation | 3 | 20 | 4 | - | - | - | 5 | 8 | 6 | 14 | 19 | 7 |
| Justification | 11 | 1 | 13 | - | - | - | 9 | - | 1 | 4 | 9 | 2 |
| Generalization | 6 | - | 1 | - | 1 | - | 1 | - | - | 2 | 4 | - |
| Connection | - | 18 | - | - | - | - | - | - | - | - | 1 | 1 |
| Probing | 16 | 37 | 39 | - | 1 | - | 27 | 12 | 19 | 28 | 21 | 9 |
| Other Solution | 9 | 2 | 13 | - | 1 | - | - | 4 | 3 | 5 | 14 | 4 |
| **Total Questions** | **45** | **78** | **70** | **0** | **3** | **0** | **42** | **24** | **29** | **53** | **68** | **23** |
| *Practices* | | | | | | | | | | | | |
| Anticipating | 3 | - | 1 | - | - | - | 8 | - | 1 | 4 | 3 | 1 |
| Monitoring | 13 | 7 | 11 | - | - | - | - | - | - | - | - | - |
| Selecting | - | 1 | 7 | - | 3 | - | 5 | 3 | 3 | - | 2 | - |
| Motivating | 9 | 3 | 31 | 9 | 8 | 9 | 9 | 7 | 11 | 29 | 28 | 51 |
| Waiting | - | 18 | 14 | - | - | - | 9 | 3 | 2 | 10 | 7 | 2 |
| Inviting | 13 | 21 | 14 | - | - | - | 24 | 7 | 5 | 21 | 33 | 8 |
| Re-voicing | 9 | 4 | 31 | - | - | - | 19 | 7 | 11 | 26 | 43 | 19 |
| **Total Practices** | **47** | **54** | **109** | **9** | **11** | **9** | **74** | **27** | **33** | **90** | **116** | **81** |

**Note: C1, C2, C3 refer to first, second, and third Cycles respectively.*
*Sources:* Meeting transcripts 9/7, 9/17, 10/2, 10/22, 11/20; On-line threads units 1-10

## 8.1 Instructor's Questions

Throughout the intervention, the instructor facilitated discussions with the teachers. During these facilitated discussions, the instructor asked different types of questions. The instructor's questions were examined in the following contexts: questions regarding teachers' own work, questions regarding research students' work, questions regarding students' work from the in-district class visit, and questions regarding current students' work brought by the teachers.

### *8.1.1 Questions Regarding Teachers' Work*

Teachers worked on three cycles of mathematical tasks throughout the intervention. As teachers worked on the three cycles of problems, the instructor asked the teachers questions regarding teachers' work. There were times when the instructor asked questions regarding teachers' own work by using Unifix cubes to represent towers or pizza combinations. At times, the instructor asked questions as teachers demonstrated towers using the Unifix cubes to show models of solutions when solving problems during the intervention.

#### *8.1.1.1 Questions Regarding Teachers' Work by Cycle*

At the initial meeting on 9/7/13, 18 types of questions were asked by the instructor as the ten teachers from the southern region of New Jersey worked on the first cycle four-tall towers problem. Seven of the questions asked by the instructor as teachers worked on the first cycle four-tall towers were justification questions about how teachers were convinced their solutions were correct (Meeting transcript 9/7/13, lines 20, 21, 30, 34, 69, 77, 112). The instructor also asked 6 probing questions as teachers worked on the four-tall towers problem. Probing questions were questions from the instructor that gave teachers the opportunity to elaborate on their work (Meeting transcript 9/7/13, lines 20, 21, 60, 86, 92, 166). The instructor also asked 2 explanation questions as teachers worked on the first cycle four-tall towers problem, which asked teachers to describe what they did to solve the problem (Meeting transcript 9/7/13, line 54, 84); 2 generalization questions where the instructor asked T2 and T3 to clarify how to find the number of possible five-tall towers (Meeting transcript 9/7/13, lines 95-103); and 1 other solution question, which exposed teachers to varied solutions (Meeting transcript 9/7/13, line 235). It is possible that the instructor asked more probing and justification problems because it was the beginning of the course.

After the teachers were provided the opportunity to work on the first cycle four-tall towers problem, the instructors of the southern region cohort and northern/central region cohort called for the entire groups' attention to discuss the teachers' solutions. From this whole-group discussion, 27 additional questions were asked by the instructor of the teachers' cohort for the southern region of New Jersey. During the discussion of the four-tall towers problem, the less common questions that the instructor asked were: explanation, 1 time; as well as justification and generalization questions each 4 times. Other solution questions were asked 8 times; and probing questions, 10 times.

Seventy-eight types of questions were asked by the instructor when the teachers were solving the second cycle pizza problem. The more common asked questions as teachers worked on the second cycle pizza problem were: probing, 37 times; explanation 20, times; and connection, 18 times. Explanation and probing questions were asked frequently to clarify and elaborate on what the teachers did to solve the pizza problem. The instructor asked less common questions such as: justification, 1 time and other solutions, 2 times as teachers worked on the second cycle pizza problem. It is possible that the instructor asked connection questions more frequently during the second cycle pizza problem because the instructor may have been hoping that the teachers would see similarities between the towers and pizza problems. For the second cycle pizza problem, the instructor asked less common questions such as: justification, 1 time and other solutions, 2 times.

As teachers worked on the third cycle three-tall towers problem, 46 questions were asked by the instructor. The more popular questions as teachers worked on the third cycle three-tall towers problem were probing 26 times, justification, 11 times and other solution, 7 times. Explanation questions were asked by the instructor 2 times as teachers worked on the third cycle three-tall towers problem.

For Ankur's Challenge, 24 questions were asked by the instructor as teachers worked on the problem themselves. The more common asked questions were probing, 13 times and other solution, 6 times. Less common asked questions were explanation and justification each 2 times and generalization 1 time.

*8.1.1.2 Summary of Questions Regarding Teachers' Work*

As teachers worked on all three cycles of tasks, the instructor asked 193 questions. More frequent questions asked by the instructor as teachers worked on the three cycles of problems were: probing, 92 times; explanation, 27 times; connection, 18 times; justification, 25 times; and other solution, 24 times. A less common type of question asked by the instructor as teachers worked on the three cycles of problems was generalization, 7 times.

### *8.1.2 Questions Regarding Research Students' Work*

The instructor posted prepared questions weekly for each Unit of the on-line discussion about research students' work. In addition to the prepared questions, the instructor responded to comments made by the teachers during the on-line discussion. At times, the instructor asked questions during the on-line discussion that were in addition to the prepared questions teachers were asked to discuss.

#### *8.1.2.1 Questions Regarding Research Students' Work by Cycle*

The instructor asked 3 questions regarding the research students' work to solve the second cycle pizza problem from Units 5, 6, and 7 of the on-line discussion threads. Regarding the second cycle pizza problem, the instructor asked the teachers on-line "What do you think of the idea of creating 3 topping pizzas by starting with the 4 toppings and eliminating one – for example, starting with peppers, mushrooms, pepperoni, and sausage – then eliminating peppers to get a mushrooms, pepperoni, and sausage pizza? I'm curious to hear what you all think about this strategy." (Unit 5, on-line discussion thread, line 28). The researcher coded this question as other solution.

A second question that the instructor asked on-line was in response to T1 about her student's work for the second cycle pizza problem. The instructor asked T1 on-line "An 'a, b, c, d, e,' notation is unique – did the students use a key so you knew what each letter represented?" (Unit 7, on-line discussion thread, line 16). This question was coded by the researcher as a probing question.

A third question was asked by the instructor in response to one of T5's on-line comments about one of her students that solved the pizza problem by multiplying 4 times 4 (Unit 7, on-line discussion thread, line 30). The question was coded by the researcher as a generalization question. The instructor asked T5 on-line "What would your student say the answer would be to pizzas, selecting from 3 toppings?" (Unit 7, on-line discussion thread, line 32).

*8.1.2.2 Summary of Questions Regarding Research Students' Work*

The instructor asked 3 questions regarding research students' work in addition to the prepared questions posted for the discussion threads. The following types of questions asked by the instructor regarding research students' work regarding the second cycle pizza problem were: probing, other solution, and generalization, each 1 time. It should be noted that questions such as "Isn't it neat to see growth when you give students an opportunity to revisit a problem?" and "Don't you think that Romina's proof is elegant?" were coded as motivating practices; not questions.

***8.1.3 Questions Regarding Students' Work from Class Visit***

There were 3 in-district classroom visits on 9/17/13, 10/22/13, and 11/20/13. The instructor asked teachers questions regarding students' work after the implementations of the three cycles of problems during the debrief meeting. A few samples of the students' work from the class visit were placed on the screen for all teachers to see and discuss.

*8.1.3.1 Questions Regarding Students' Work from Class Visit by Cycle*

On 9/17/13, 42 types of questions were asked by the instructor regarding the students' work from the in-district classroom visit. Twenty-seven of the questions asked by the instructor regarding students' work from the class visit for the first cycle four-tall towers were probing questions. Other questions that were asked by the instructor about students' work for the first cycle four-tall towers problem from the class visit were justification, 9 times; explanation, 5 times; and generalization, 1 time.

At the second in-district meeting on 10/22/13, 24 types of questions were asked by the instructor regarding students' work from the in-district classroom visit. Twelve of the questions asked by the instructor regarding students' work from the second cycle pizza problem after the in-district class visit were probing questions. Other questions asked by the instructor regarding students' work for the second cycle pizza problem from the in-district class visit were explanation, 8 times and other solution, 4 times.

At the third in-district meeting on 11/20/13, 29 questions were asked by the instructor regarding students' work from the class visit. Nineteen of the questions asked by the instructor regarding students' work for the third cycle three-tall towers problem from the in-district class visit were probing questions. Other questions asked by the instructor regarding students' work from the third cycle three-tall towers problem were explanation, 6 times; other solution, 3 times; and justification, 1 time.

*8.1.3.2 Summary of Questions Regarding Students' Work from Class Visit*

Ninety-five questions were asked by the instructor during the classroom visit debriefing meeting. The types of questions asked by the instructor were probing questions, 58 times; explanation questions, 19 times; justification questions, 10 times; other solutions, 7 times; and generalization questions, one time. Probing questions were the most common question asked by the instructor for each of the three cycles of problems.

***8.1.4 Questions Regarding Current Students' Work***

Teachers attended three regional meetings on 10/2/13, 10/22/13, and 11/20/13. At these regional meetings, teachers discussed students' work samples from their own classrooms. The instructor asked questions about current students' work as teachers presented their students' work samples.

*8.1.4.1 Questions Regarding Current Students' Work by Cycle*

For the first cycle four-tall towers problem, 53 questions were asked by the instructor regarding current students' work. The most common type of question asked by the instructor regarding current students' samples of work for solving the first cycle four-tall towers problem was probing, 28 times. Less common question types asked by the instructor regarding current students' samples of work for solving the first cycle four-tall towers problem were: justification, 4 times; explanation, 14 times; and generalization, 2 times; and other solution questions, 5 times.

Sixty-eight questions were asked by the instructor regarding current students' work for the second cycle pizza problem. The more commonly asked questions by the instructor regarding current students' work for the second cycle pizza problem were: probing, 21 times; explanation, 19 times; and other solution questions, 14 times. Less common asked questions by the instructor regarding current students' work for the second cycle pizza problem were: justification, 9 times; generalization, 4 times; and connection, 1 time.

For the third cycle, 23 questions were asked by the instructor regarding current students' work for the third cycle three-tall towers problem. The more common asked questions regarding current students' work for the third cycle three-tall towers problem were: explanation, 7 times and probing, 9 times. Less common questions regarding current students' work for the third cycle three-tall towers problem were: other solution questions, 4 times; justification, 2 times; and connection, 1 time.

*8.1.4.2 Summary of Questions Regarding Current Students' Work*

One hundred forty-four questions were asked by the instructor regarding the current students' work samples shared by each of the ten teachers throughout the three cycles of the intervention. The types of questions asked by the instructor throughout the three cycles were: probing, 58 times; explanation, 40 times; justification, 15 times; other solution, 23 times; and generalization, 6 times. The more popular questions asked by the instructor regarding teachers' current students' samples for the three cycles of problems were probing and explanation questions.

Probing questions were used by the instructor 28 out of 58 times as the instructor facilitated a discussion on the four-tall towers problem; 21 out of 58 times as the instructor facilitated a discussion on the second cycle pizza problem; and 9 out of 58 times as the instructor facilitated a discussion on the third cycle three-tall towers problem. Explanation questions were used by the instructor 14 out of 40 times for the first cycle four-tall towers problem; 19 out of 40 times for the second cycle pizza problem; and 7 out of 40 times for the third cycle three-tall towers problem. Other solution, justification, and generalization questions were asked less frequently throughout the intervention.

***8.1.5 Summary of Instructor's Questions***

The results of the questions analysis showed that the instructor's questions used throughout the intervention helped teachers to attend to their own reasoning as well as their students' reasoning. The most common type of question that the instructor asked throughout the three cycles of the intervention was probing questions, 209 times. The instructor asked probing questions 92 out of 209 as teachers worked on the three cycles of mathematical problems. The instructor also asked 1 probing question regarding research students' work in addition to the prepared questions for each on-line unit discussion, 58 probing questions regarding the students' work from the class visits, and 58 probing questions regarding the current students' work samples.

Explanation questions were asked 86 times by the instructor throughout the three cycles of the intervention. The instructor asked 27 explanation questions out of 86 questions as teachers worked on the three cycles of problems. Regarding the students' work for the in-district class visits, explanation questions were asked 19 out of 86 times. For the current students' work samples, explanation questions were asked 40 out of 86 times, which is slightly less than half of all the explanation questions asked during the intervention.

Other solution questions were asked by the instructor 54 times throughout the three cycles of the intervention. As teachers worked on the three cycles of problems, other solution questions were asked 23 out of 54 times. Other solution questions were asked 7 out of 54 times regarding the students' work from the class visits, and were asked 23 out of 54 times regarding the current students' samples of work. One other solution question was asked by the instructor during the Unit 5 on-line discussion.

Justification questions were asked 50 times by the instructor throughout the three cycles of the intervention. As teachers worked on the three cycles of problems, justification questions were asked half of time times. The other half of the justification questions were asked by the instructor 10 times regarding the students' work from the class visits, and 15 times regarding the currents students' work samples.

Connection questions were asked 20 times by the instructor throughout the three cycles of the intervention. Two connection questions were asked by the instructor during the intervention regarding current students' work samples. It should be noted that connection questions were asked 18 out of 20 times as teachers worked on the second cycle pizza problem themselves. It is possible that the instructor may have asked the majority of connection problems as teachers worked on the second cycle pizza problem because the instructor may have been hoping that teachers would recognize a connection between the first cycle four-tall towers problem and the pizza problem.

Generalization questions were asked 15 times by the instructor throughout the three cycles of the intervention. As teachers worked on the three cycles of problems, the instructor asked 7 generalization questions. One generalization questions was asked during the Unit 7 on-line discussion. Regarding the students' work from the class visits, the instructor asked 1 generalization question about the first cycle four-tall towers. The instructor asked 6 generalization questions regarding current students' work throughout the three cycles of the intervention.

The instructor's 435 questions asked throughout the intervention helped teachers to attend to students' reasoning. The role of the instructor was to facilitate teachers' discussions regarding the teachers' and students' work for the three cycles of problems. The questions asked by the instructor helped to facilitate the teachers' discussions.

## 8.2 Instructor's Pedagogical Practices

The instructor also facilitated teachers' discussions using pedagogical practices throughout the intervention. Although questioning is considered to be a pedagogical practice, this section focuses on practices other than questioning. The instructor's practices were examined in the following contexts: practices used as teachers' worked on the problems, practices used on-line regarding research students' work, practices used as teachers discussed students' work from the in-district class visit, and practices used as teachers discussed current students' work samples brought by the teachers.

### *8.2.1 Pedagogical Practices Regarding Teachers' Work*

The instructor used pedagogical practices as teachers worked on three cycles of mathematical tasks. The instructor modeled pedagogical practices to help teachers attend to their own reasoning as they worked on the problems. Pedagogical practices were used by the instructor as teachers' worked on each of the three cycles during the intervention.

#### *8.2.1.1 Practices Regarding Teachers' Work by Cycle*

Twenty-four pedagogical practices were used by the instructor as the ten teachers from the southern region of New Jersey worked on the first cycle four-tall towers problem. Thirteen of the practices used by the instructor as teachers worked on the first cycle four-tall towers problem were monitoring practices which allowed for the instructor to circulate through the room to check for teachers' understanding. Seven of the practices used by the instructor were motivating practices where the instructor used praise or encouraging words regarding the teachers' work for the first cycle four-tall towers problem. Three of the practices used by the instructor as teachers worked on the four-tall towers problem were inviting practices where the instructor exposed teachers to varied solutions. As teachers worked on the four-tall towers problem, re-voicing was used by the instructor once to restate a teacher's comment to clarify the instructor's understanding of the teachers' work.

The instructors called for the attention of the teachers to discuss their solutions after working on the first cycle four-tall towers problem. From this discussion, the practices used by the instructor were: inviting, 10 times; re-voicing, 8 times; and motivating, 2 times. The instructor also used the practice of anticipating 3 times to help teachers' predict what a student might do while working on the four-tall towers problem.

Fifty-four pedagogical practices were used by the instructor as the teachers worked on the second cycle pizza problem. The more common practices were: inviting, 21 times; waiting, 18 times; and monitoring, 7 times as teachers worked on the pizza problem. Less common practices were: re-voicing, 4 times and motivating, 3 times as teachers worked on the pizza problem. The instructor used the practice of selecting, 1 time when the instructor selected T6's chart to show all the teachers how T6 reorganized her work to find the possible pizza combinations (Meeting transcript teachers' work 10/2/13, line 234).

For the third cycle three-tall towers problem, 57 pedagogical practices were used by the instructor. The more common practices by the instructor were: re-voicing, 15 times; motivating, 13 times; waiting, 12 times; inviting, 7 times, and monitoring, 6 times as teachers worked on the three-tall towers problem. Lesson common practices by the instructor were selecting, 3 times; and anticipating, 1 time for the teachers' work on the three-tall towers problem.

For Ankur's Challenge, 52 pedagogical practices were used by the instructor. The practices that the instructor frequently used were: motivating, 18 times; re-voicing, 16 times; and inviting, 7 times. Less frequent practices used by the instructor were: monitoring, 5 times; selecting, 4 times; and waiting, 2 times.

*8.2.1.2 Summary of Practices Regarding Teachers' Work*

As teachers worked on all three cycles of tasks, the instructor used 210 pedagogical practices. More frequent pedagogical practices used by the instructor as teachers worked on the three cycles of problems were: inviting, 48 times; re-voicing, 44 times; motivating, 43 times; waiting, 32 times; and monitoring, 31 times. Less common practices used by the instructor as teachers worked on the three cycles of problems were selecting, 8 times and anticipating, 4 times. It should be noted that monitoring was only used by the instructor as teachers worked on the three cycles of tasks.

### *8.2.2 Pedagogical Practices Regarding Research Students' Work*

The instructor posted weekly comments for each Unit of the on-line discussion. For the majority of times, the instructor's comments were in response to the teachers' original posts regarding research students' work. There were also times when the instructor commented on the teachers' responses to other teachers' original posts about the research students' work. Most of the instructor's on-line comments were motivating.

#### *8.2.2.1 Practices Regarding Research Students' Work by Cycle*

Nine of the instructor's comments were coded by the researcher as motivating practices for Units 2, 3, and 4 of the on-line discussion. One motivating comment by the instructor was from the Unit 2 discussion thread. The instructor wrote on-line "Glad to hear you will hold back on giving help." (Unit 2, on-line discussion thread, line 11).

Three motivating comments by the instructor were from the Unit 3 discussion thread. The first motivating comment from the Unit 3 discussion thread was in response to T1's original post for the first question provided by the instructor on-line about how T1 noticed that "one student explained that by moving the position of the cube down each time, eventually the cube will be brought to the top and the pattern would start to duplicate." (Unit 3, on-line discussion thread, line 1). The instructor wrote on-line "What you are describing in #1 is called a recursive argument. Nice that you noticed it." (Unit 3, on-line discussion thread, line 2).

A second motivating comment from the Unit 3 discussion thread was to T4 regarding a comment T4 made about having the same students solve the tower problem again next year. T4 wrote "I am lucky enough to keep the same students from year to year (between sixth through eighth grade) so I would like to try the tower building next year and see how my students' reasoning changes." (Unit 3, on-line discussion thread, line 16). The instructor wrote on-line "You are very lucky to keep the same students – I think it is neat that you plan to let them revisit the problem next year to see how their reasoning changes." (Unit 3, on-line discussion thread, line 18). Another teacher, T2, responded to T4 that T2 also kept the same students and planned to also try the same problem with the students for the next year to "see if their reasoning changes" (Unit 3, on-line discussion thread, line 19). The instructor replied "Neat!" to T2 (Unit 3, on-line discussion thread, line 20).

Two motivating comments by the instructor were from the Unit 4 discussion thread regarding teachers' responses of research students' work for the first cycle four-tall towers problem and the three-tall and five-tall extension problems. The instructor wrote "It is so neat that your students were able to organize and reorganize their towers in so many different ways." (Unit 4, on-line discussion thread, line 35). The instructor also wrote "It sounds like you did a good pairing of your two $8^{th}$ grade boys." (Unit 4, on-line discussion thread, line 40).

Eight of the instructor comments were also coded by the researcher as motivating practices in response to the teachers' comments regarding the research students' work for the second cycle pizza problem. The instructor made 1 motivating comment on the Unit 5 on-line discussion thread; 5 motivating comments on the Unit 6 on-line discussion thread, and 2 motivating comments on the Unit 7 on-line discussion thread (Unit 5, line 26; Unit 6, line 7, 12, 13, 42; Unit 7, line 33, 43).

Seven of the instructor comments were coded by the researcher as motivating practices in response to the teachers' comments regarding the research work for the third cycle Ankur's Challenge problem. The instructor made 2 motivating comments on the Unit 8 on-line discussion thread; 3 motivating comments on the Unit 9 on-line discussion thread, and 2 motivating comments on the Unit 10 on-line discussion thread (Unit 8, lines 4, 29; Unit 9, lines 20, 27, 28; Unit 10, lines 20, 30).

It should be noted that there were 6 comments by the instructor that were coded as motivating practices but were phrased as questions. One comment was in response to T5's original post about comparing and contrasting students' work from the second grade to the third grade to solve the shirts and pants problem. The instructor wrote on-line "Isn't it neat to see growth when you give students an opportunity to revisit a problem?" (Unit 3, on-line discussion thread, line 33).

A second comment coded as motivating practices was in response to T6's original about Stephanie's family description of Milin's inductive argument regarding the three-tall towers problem selecting from 2 colors. The instructor wrote on-line "Wasn't it neat when Stephanie listened to Milin's argument and was able to explain it to others?" (Unit 4, on-line discussion thread, line 20). A third comment coded as motivating practices was in response to T8's original post of how Milin's inductive argument helped the other students to understand the three-tall towers problem, selecting from 2 colors. The instructor wrote on-line “Isn’t it neat when students listen to and learn from their classmates?” (Unit 4, on-line discussion thread, line 49).

A fourth comment by the instructor coded by the researcher as motivating practices was in response to T7’s on-line comment regarding questions asked by Brandon’s teacher while working on the pizza problem. T7’s on-line comment in response to T6’s original post follows:

> I find that I am asking students to explain things more than once to really understand what they are saying rather than assuming that I know what they are talking about as you mentioned. Many times, they are actually explaining something different than what I originally thought they were saying. (Unit 6, on-line discussion thread, line 19).

In response to T7’s comment, the instructor asked the following probing question toT7 on-line “Isn’t it neat when we listen carefully and really understand the mathematical thinking of our students?” (Unit 6, on-line discussion thread, line 20).

Two more comments were coded by the researcher as motivating practices regarding the solution to Ankur’s Challenge. For the first comment, the instructor responded to T6 on-line by writing “Don’t you think that Romina’s proof is elegant?” (Unit 7, on-line discussion thread, line 15). The second comment was in response to an original post by T7 about Romina’s proof. The instructor asked T7 on-line “Isn’t it neat how we learn from our classmates as well as our teacher?” (Unit 7, on-line discussion thread, line 35).

The instructor also used the practice of selecting three times. Selecting was coded when the instructor asked teachers on-line to share a students’ work sample at the next regional meeting. The instructor made an on-line request for T6, T9, and T10 to share one of their students' work from the second cycle pizza problem at the next meeting (Unit 7, on-line discussion thread, lines 4, 29, 38).

*8.2.2.2 Summary of Practices Regarding Research Students' Work*

Throughout the ten units of on-line discussions regarding research students' work, the instructor used 29 pedagogical practices. The most frequent pedagogical practice used by the instructor on-line for the three cycles of problems was: motivating, 26 times. There were 3 times when the instructor selected three teachers to bring a particular students' work to the next regional meeting for the teachers to discuss and the researcher coded each of the three instructor's on-line requests as selecting.

***8.2.3 Pedagogical Practices Regarding Students' Work from Class Visit***

After the 3 in-district classroom visits on 9/17/13, 10/22/13, and 11/20/13; the instructor had a debrief meeting with the teachers. At the debrief meeting, the instructor used pedagogical practices to facilitate discussions with the teachers about students' work form the classroom visits. The following pedagogical practices were used by the instructor regarding students' work from the in-district classroom visits for each of the three cycles.

*8.2.3.1 Practices Regarding Students' Work from Class Visit by Cycle*

For the first cycle four-tall towers problem, 74 pedagogical practices were used by the instructor regarding students' work from the classroom visit debrief meeting on 9/17/13. The most common pedagogical practices used by the instructor regarding the students' work of the four-tall towers problem from the in-district class visit were: inviting, 24 times; re-voicing, 19 times; waiting, 9 times; motivating, 9 times; and anticipating, 8 times. A less common practice used by the instructor regarding students' work of the four-tall towers problem from the class visit was selecting, 5 times.

Twenty-seven pedagogical practices were used by the instructor during the classroom visit debrief meeting regarding the students' work on the second cycle pizza problem. The pedagogical practices used by the instructor during the classroom debrief meeting regarding the students' work on the second cycle pizza problem from the class visit were motivating, inviting, and re-voicing each 7 times. Slightly less used practices by the instructor were selecting and waiting each 3 times regarding the students' work on the second cycle pizza problem from the second in-district class visit on 10/22/13.

For the third cycle three-tall towers problem, 33 pedagogical practices were used by the instructor regarding the students work from the in-district classroom visit on 11/20/13. The practices used by the instructor regarding the students' work on the three-tall towers problem were: motivating and re-voicing each 11 times; inviting, 5 times; selecting, 3 times; waiting, 2 times; and 1 time for anticipating. The instructor's practices helped to strengthen the instructor's role as a facilitator of knowledge.

*8.2.3.2 Summary of Practices Regarding Students' Work from Class Visits*

One hundred thirty-four pedagogical practices were used by the instructor during the classroom visit debrief meeting. The more common pedagogical practices made by the instructor regarding students' work from the three class visits were: re-voicing, 37 times; inviting, 36 times; motivating, 27 times; and waiting, 14 times. Less common pedagogical practices made by the instructor from the three class visits were: selecting, 11 times; and anticipating, 9 times.

### *8.2.4 Pedagogical Practices Regarding Current Students' Work*

The instructor met with the teachers to discuss their current students' work at the regional meetings on 10/2/13, 10/22/13, and 11/20/13. During the meetings, the instructor used pedagogical practices to facilitate discussions with the teachers about their own currents students' work. The following pedagogical practices were used by the instructor as teachers discussed their current students' work for each of the three cycles.

#### *8.2.4.1 Practices Regarding Current Students' Work by Cycle*

For the first cycle four-tall towers problem, ninety pedagogical practices were used by the instructor as the instructor facilitated a discussion with teachers regarding their own current students' work. The more frequent practices used by the instructor as teachers discussed current students' solutions of the four-tall towers problem were: motivating, 29 times; re-voicing, 26 times; and inviting, 21 times. Less frequent practices used by the instructor as teachers discussed current students' solutions of the four-tall towers problem were: waiting, 10 times and anticipating, 4 times.

One hundred sixteen practices were used by the instructor as the instructor facilitated a discussion with teachers regarding current students' work for the second cycle pizza problem. The more common practices used by the instructor as teachers discussed current students' solutions of the second cycle pizza problem were: motivating, 28 times; inviting, 33 times; and re-voicing, 43 times. Less common moves used by the instructor as teachers discussed current students' solutions of the second cycle pizza problem were: anticipating, 3 times; selecting, 2 times; and waiting, 7 times.

For the third cycle three-tall towers problem, 81 practices were used by the instructor as teachers discussed current students' solutions. The more common instructor practices used by the instructor as teachers discussed current students' solutions of the third cycle pizza problem were motivating, 51 times and re-voicing, 19 times. Less common moves were inviting, 8 times; waiting, 2 times; and anticipating, 1 time.

*8.2.4.2 Summary of Practices Regarding Current Students' Work*

Two hundred eighty-seven moves were used by the instructor as teachers discussed their own current students' work throughout the three cycles of the intervention. The more common practices used by the instructor as teachers discussed their own current students' work for the three cycles of problems were: motivating, 108 times re-voicing, 88 times; inviting, 62 times; and waiting, 19 times. Less common practices used by the instructor as teachers discussed their own current students' work were selecting, 2 times and anticipating, 8 times.

### *8.2.5 Summary of Instructor's Pedagogical Practices*

The instructor used 654 pedagogical practices to help teachers attend to students' reasoning throughout the intervention. Motivating practices were used 198 times and were found to be the most popular practice used by the instructor throughout the intervention in all three cycles. Motivating practices were used by the instructor 43 times as teachers worked on the three cycles of problems themselves, 20 times as responses to teachers' on-line comments of research students' work, 27 times when facilitating discussions with teachers of students' work from the classroom visits, and 108 times when facilitating discussions with teachers of current students' samples of work. The motivating practices used by the instructor praised teachers for their contributions and encouraged the teachers to continue working throughout the intervention.

The second most popular practice used by the instructor throughout the three cycles of problems was re-voicing and was coded by the researcher 169 times. Re-voicing was used by the instructor 44 times as teachers worked on the three cycles of problems themselves, 37 times when facilitating discussions with teachers of students' work from the classroom visits, and 88 times when facilitating discussions with teachers of current students' samples of work. Re-voicing was used by the instructor to clarify the instructor's understanding of the teachers' and students' arguments. The number of re-voicing practices used by the instructor when facilitating discussions with teachers about current students' work for the three cycles of problems was double the number of re-voicing practices as when the teachers worked on the problem themselves. It is possible that this increase is due to the larger number of students' samples discussed as compared to the samples of the ten teachers.

Inviting was used by the instructor 146 times throughout the intervention. Inviting was used by the instructor 48 out of 146 times as teachers worked on the three cycles of problems themselves, 36 out of 146 times when facilitating discussions with teachers of students' work from the classroom visits, and 62 out of 146 times when facilitating discussions with teachers of current students' samples of work. Inviting was used by the instructor to expose teachers to solutions from more than one person.

Less frequent pedagogical practices used by the instructor throughout the three cycles of the intervention were waiting, 65 times; monitoring, 31 times; selecting, 24 times; and anticipating, 21 times. Monitoring was only used by the instructor as teachers worked on the three cycles of problems themselves. Waiting, selecting, and anticipating were used fewer times by the instructor for all three cycles of problems.

## 8.3 Instructor's Representations Used

The instructor often used the teachers' and students' work samples to facilitate discussions of teachers' recognition of reasoning. However, there were times when the instructor used Unifix cubes to make towers and then asked teachers questions about how to represent towers and pizza combinations using the Unifix cubes. Other times the instructor asked the teachers to use the cubes to make towers and asked questions about the representations teachers used to make the towers.

The instructor showed the teachers Unifix cubes at the first session before teachers started the first task and said "there is a little chimney that we call here that we always want to keep facing top" (9/7/13 meeting transcript, line 10). After the teachers worked on the first cycle task, the instructor asked some of the teachers to share their work. T6 volunteered to share her work for the first cycle four-tall towers problem. As T6 explained her solution, the instructor asked her partner to hold up the towers of Unifix cubes for the other teachers to see as they shared their solution (9/7/13 meeting transcript, line 214). The following excerpt illustrates the questions asked by the instructor while teachers used the Unifix cubes to explain their argument:

> T6: So I had four of one color and I had it in one of the other color. And it could be in either the first, the second the third or the fourth position. And then I knew if I did it that way, I could reverse it and do it with the other colors as well.
> R1: Okay, before we leave, I am asking the group: for the group she just made, it looks like it is three of one color and one of another color. Could she have another tower in that group that has exactly three yellow and one red? [Teacher in unison responded 'no'.] Did she skip one? Why not? I heard a lot of no's why not?
> T6: Because the red is in each of the four positions.
> R1: Okay. She gave you the answer because the red is in each of the four positions. Okay I am going to say why couldn't she have then the red in the fifth position? There is no red in the fifth position? (Meeting transcript 9/7/13, lines 217-222).

The instructor had also asked T7 and T8 to hold up towers to illustrate their recursive argument for building three-tall towers, selecting from three colors (Meeting transcript 10/22/13, line 307). The instructor helped T7 and T8 hold up the towers as they explained their argument (Meeting transcript 10/22/13, line 309). The following excerpt illustrates the questions asked by the instructor while teachers used the Unifix cubes to explain their argument:

> R1: What is the argument for the candy cane or the 3 colored colors in a tower?
> T8: Okay, so [T7} and I did this a little bit differently. This was the idea I had, was having the yellow kind of go through each position kind of the same way that we had the one cube go through in the other positions. And it is the same concept, of actually moving the yellow all the way through. If you were to move it to another place, it would come back to the top and it would result…
> R1: We better see that. Can you get three and show us? Now they are using a recursive argument. Okay cause that's not quite as easy to see why you have them all and why there can't be another tower. So they have the 3 towers there, okay.
> T8: So as you go through the first you get the yellow to move down to the middle place.
> R1: How about if I hold 2 and you three?
> T7: Okay, so then at the end, you take the yellow one and move it back on the top we are back at the first one that he had originally there.
> R1: Alright.
> T7: Oh I thought… [toT8] Do you want to do those?
> R1: Well is that the only other way to do it? (Meeting transcript 10/22/13, lines 304-317).

The instructor also used the Unifix cubes to ask questions was to demonstrate the connection between the 4-tall tower problem and the pizza problem after T6 shared her pizza combinations chart with the other teachers (10/2/13 meeting transcript, line 290-308). The instructor had T6's chart on the screen and asked the teachers "How could we build a tower that might look like that pepperoni pizza?" (10/2/13 meeting transcript, line 292). One teacher replied "One red, three yellow." (10/2/13 meeting transcript, line 293). Using T6's chart, the instructor asked the teachers how to arrange the cubes to represent three more of the pizza combinations (10/2/13 meeting transcript, line 300-

308). It should be noted that the instructor used the Unifix cubes to demonstrate towers or had teachers use the Unifix cubes to demonstrate towers after teachers worked on the three cycles of problems.

### 8.4 Summary of Instructor Moves Analysis

Based on the data from this research, the instructor's moves throughout the PD intervention helped the in-service middle-school mathematics teachers to attend to students' reasoning. The instructor's pedagogical practices used and questions asked were identified and analyzed throughout the three cycles of the intervention as teachers' worked on the problems, as teachers made on-line posts regarding research students' work, as teachers discussed students' work from the in-district class visit, and as teachers discussed their own current students' work samples. Table 8.2 summarizes the instructor moves by the context in which the moves were studied.

Table 8.2
*Frequency of Instructor Moves by Context*

| Moves Of Instructor | Teachers' Own Work | Research Students' Work | Class Visit Students' Work | Current Students' Work | Total Moves of Instructor |
|---|---|---|---|---|---|
| *Question Type* | | | | | |
| Explanation | 27 | - | 19 | 40 | 86 |
| Justification | 25 | - | 10 | 15 | 50 |
| Generalization | 7 | 1 | 1 | 6 | 15 |
| Connection | 18 | - | - | 2 | 20 |
| Probing | 92 | 1 | 58 | 58 | 209 |
| Other Solution | 24 | 1 | 7 | 23 | 55 |
| *Practices* | | | | | |
| Anticipating | 4 | - | 9 | 8 | 21 |
| Monitoring | 31 | - | - | - | 31 |
| Selecting | 8 | 3 | 11 | 2 | 24 |
| Motivating | 43 | 26 | 27 | 108 | 204 |
| Waiting | 32 | - | 14 | 19 | 65 |
| Inviting | 48 | - | 36 | 62 | 146 |
| Re-voicing | 44 | - | 37 | 88 | 169 |
| Total Moves | 403 | 32 | 229 | 431 | 1095 |

*Sources:* Meeting transcripts 9/7, 9/17, 10/2, 10/22, 11/20; On-line threads units 1-10

The researcher coded 1095 instructor moves. From the 1095 instructor's moves, 403 were coded by the researcher from video transcripts of teachers working on the three cycles of problems themselves, 32 were coded by the researcher from the instructor's on-line comments, regarding research students' work for the three cycles of problems, 229 were coded by the researcher from video transcripts of the teachers' discussion of students' work from the in-district class visit, and 431 were coded by the researcher from video transcripts of the teachers' discussion of their own current students' samples of work for the three cycles of problems.

Out of the1095 instructor moves, 435 were questions asked by the instructor. The most common type of question asked from each of the contexts examined was probing, 209 times; which is slightly less than half of the questions asked by the instructor. It is possible that the instructor used probing questions frequently because these types of questions allowed teachers the freedom to elaborate on their work and the work of the teachers' students. Other types of questions used frequently by the instructor were explanation, 86 times; other solution, 55 times; and justification, 50 times. Less frequent questions asked by the instructor were connection, 20 times and generalization, 15 times. It is possible that the instructor may have asked generalization and connection questions less frequently because more time is needed for teachers' to reflect and answer generalization and connection questions as compared to the other types of questions.

Out of the 1095 instructor's moves, 660 were pedagogical practices used by the instructor. The most common practice used by the instructor was motivating practices, 204 times. When teachers worked on the three cycles of problems themselves, the researcher coded inviting, 48 times and re-voicing, 44 times which was slightly more than motivating practices, 43 times. Other pedagogical practices used frequently by the instructor were: re-voicing, 169 times; inviting, 146 times; and waiting 65 times. Less frequent practices used by the instructor were: monitoring, 31 times; selecting, 24 times; and anticipating, 21 times. It is should be noted that practices, such as monitoring, were used less by the instructor for a particular context because of the nature of the practice (i.e. Monitoring practices were used when teachers worked on the problems themselves.).

The instructor's moves were also examined by the three cycles of problems. For the first cycle, instructor moves were coded regarding the four-tall towers problem and the three-tall and five-tall tower extension problems. For the second cycle, instructor moves were coded regarding the pizza problem. For the third cycle, instructor moves were coded regarding the three-tall towers problem and Ankur's Challenge. Table 8.3 summarizes the instructor moves by cycle.

Table 8.3
*Frequency of Instructor Moves by Cycle*

| Moves | Cycle 1 | Cycle 2 | Cycle 3 | **Total** |
|---|---|---|---|---|
| *Question Type* | | | | |
| Explanation | 22 | 47 | 17 | 86 |
| Justification | 24 | 10 | 16 | 50 |
| Generalization | 9 | 5 | 1 | 15 |
| Connection | - | 19 | 1 | 20 |
| Probing | 71 | 71 | 67 | 209 |
| Other Solution | 14 | 21 | 20 | 55 |
| Total Questions | 140 | 173 | 122 | 435 |
| *Practices* | | | | |
| Anticipating | 15 | 3 | 3 | 21 |
| Monitoring | 16 | 8 | 13 | 37 |
| Selecting | 5 | 9 | 10 | 24 |
| Motivating | 53 | 45 | 100 | 198 |
| Waiting | 19 | 28 | 18 | 65 |
| Inviting | 58 | 61 | 27 | 146 |
| Re-voicing | 54 | 54 | 61 | 169 |
| Total Practices | 220 | 208 | 232 | 660 |

*Sources:* Meeting transcripts 9/7, 9/17, 10/2, 10/22, 11/20; On-line threads units 1-10

Probing questions accounted for 71 out of 140 questions asked by the instructor regarding the first cycle four-tall tower problem and three-tall and five-tall extension problems; which are slightly more than half of the questions. It is possible that the instructor asked probing questions more frequently regarding the first cycle tower problems because the first cycle problems were given in the beginning of the intervention. Other types of questions that were asked by the instructor regarding the first cycle problems were justification, 24 times; explanation, 22 times; other solution, 14 times; and generalization, 9 times.

For the first cycle problems, 220 pedagogical practices were recorded by the researcher. The most common practices used by the instructor regarding the first cycle problems were: inviting, 58 times; re-voicing, 54 times; and motivating, 56 times. Less common practices used by the instructor regarding the first cycle problems were: waiting, 19 times; anticipating, 15 times; monitoring, 13 times; and selecting, 5 times.

For the second cycle pizza problem, 173 questions were coded by the researcher. The more common questions asked by the instructor were: probing, 71 times; explanation, 47 times; and other solution, 21 times; and connection, 19 times. It should be noted that 18 of the 19 connection questions were asked by the instructor as teachers discussed their solutions of the second cycle pizza problem. It is possible that the instructor asked most of the connection questions during the discussion of the teachers' solutions of the pizza problem hoping that the teachers would see the connection between the towers problem and the pizza problem.

Regarding the second cycle pizza problem, 208 pedagogical practices were coded by the researcher. The most common practices used by the instructor regarding the second cycle pizza problem were: inviting, 61 times; re-voicing, 54 times; motivating 46 times, and waiting, 28 times. Less common practices asked by the instructor were: selecting, 9 times; monitoring, 7 times; and anticipating, 3 times.

For the third cycle three tall towers problem, 100 questions were asked by the instructor. Fifty-six of the questions asked by the instructor were probing regarding the three-tall towers problem. Other questions that were asked by the instructor regarding the three-tall towers problem were: explanation, 15 times; other solution and justification each 14 times; and connection, 1 time. For the third cycle Ankur's Challenge problem, the instructor asked the following 24 types of questions: probing, 11 times; other solution, 6 times; explanation and justification each 2 times; and generalization, 1 time.

Regarding the third cycle three-tall towers problem, 178 practices were used by the instructor. The most common practice used by the instructor for the third cycle three-tall towers problem was motivating, 82 times. More common practices used by the instructor for the three-tall towers problem were re-voicing, 45 times; inviting, 20 times; and waiting, 16 times. For Ankur's Challenge, the instructor used motivating practices 18 times. It should be noted that 102 out of 204 motivating practices were coded by the researcher for the third cycle problems.

# Chapter 9 – Teachers' Beliefs Summary and Analysis

This chapter examines teachers' changes in beliefs, if any, about the teaching and learning of mathematics. Teachers were asked to complete a 34-item pre- and post-assessment on beliefs about the teaching and learning of mathematics. Relating to this study, a subset of 22 statements of this assessment was analyzed. Some statements on the inventory were statements considered to be consistent with standards promoted by the NCTM in documents such as the Principles and Standards for School Mathematics (2000). Other belief inventory statements were considered to be inconsistent with the standards.

For this study, the teachers' beliefs are examined in four parts. First, teachers' beliefs are examined by the number of the belief statements teachers' out of the 22 belief statements subset consistent with NCTM standards regarding the teaching and learning of mathematics. Second, teachers' beliefs are examined by each category from the subset of the 22 pre- and post-assessment statements. Third, the stability of individual teacher beliefs is examined from teachers who scored 100% on the pre-assessment for beliefs consistent with the standards. Fourth, teachers' potential growth rates of the teachers' beliefs are examined from the pre-assessment to the post-assessment.

## 9.1 Teachers' Scores for Subset of Beliefs Statements

This section examines teachers' scores from the subset of the 22 pre- and post-assessment belief statements consistent with the NCTM standards. The scores are from the ten teachers from the southern region cohort who took the beliefs pre- and post-assessment beliefs in fall 2013. Table 9.1 summarizes the teachers' pre- and post-assessment scores by the percentage of the number of belief statements out of the 22 belief statements subset teachers' scored consistent with the standards (CP) and the number of the subset of belief statements teachers scored consistent with the standards (CN), inconsistent with the standards (IN), and undecided statements (UN).

Table 9.1
*Teachers' Scores for Belief Statements Consistency with the Standards*

| **Teacher** | **Pre-Assessment** | | | | **Post-Assessment** | | | |
|---|---|---|---|---|---|---|---|---|
| | *CP* | *CN* | *IN* | *UN* | *CP* | *CN* | *IN* | *UN* |
| T1 | 77 | 17 | 4 | 1 | 73 | 16 | 3 | 3 |
| T2 | 64 | 14 | 5 | 3 | 91 | 20 | 0 | 2 |
| T3 | 68 | 15 | 4 | 3 | 73 | 16 | 4 | 2 |
| T4 | 68 | 15 | 2 | 5 | 91 | 20 | 2 | 0 |
| T5 | 82 | 18 | 0 | 3 | 73 | 16 | 1 | 5 |
| T6 | 91 | 20 | 2 | 0 | 95 | 21 | 0 | 1 |
| T7 | 73 | 16 | 2 | 4 | 68 | 15 | 1 | 6 |
| T8 | 73 | 16 | 4 | 2 | 95 | 21 | 1 | 0 |
| T9 | 91 | 20 | 1 | 1 | 82 | 18 | 4 | 0 |
| T10 | 77 | 17 | 2 | 3 | 82 | 18 | 2 | 2 |
| Mean | 76.4 | 16.8 | 2.6 | 2.5 | 82.3 | 18.1 | 1.8 | 2.1 |

*Sources: Beliefs Inventory Pre- and Post-Assessment*
**CP=Percent scored out of 22 beliefs consistent with standards;*
*CN = number out of 22 beliefs consistent with standards*
*IN = number out of 22 beliefs inconsistent with standards*
*UN = number out of 22 beliefs undecided*

*9.1.1 Beliefs Pre-Assessment Score Results*

The pre-assessment scores for this cohort of teachers show that all ten teachers scored 64% (14 out of 22 beliefs) or higher for the percentage of teachers' beliefs statements out of the 22 subset statements consistent with the standards. On the pre-assessment, five teachers (T1, T5, T6, T9, and T10) scored higher than the mean for the subset of beliefs consistent with the standards. For the number of statements inconsistent with the standards, 4 teachers (T1, T2, T3, and T8) scored higher than the mean of the ten teachers. Four teachers scored higher than the mean for undecided statements. The pre-assessment scores for all ten teachers show high alignment to the standards.

*9.1.2 Beliefs Post-Assessment Score Results*

The post- assessment results for all ten teachers show a slightly higher alignment to the standards. All ten teachers scored 68% (15 out of 22 beliefs) or higher for the percentage of teachers' belief statements out of the 22 subset statements consistent with the standards. Six teachers increased their percent of the subset of beliefs consistent with the standards as compared to the pre-assessment.

The mean score of teachers' subset of beliefs consistent with the standards increased to 82.3% on the beliefs post-assessment. On the post-assessment, 4 teachers scored higher than the mean for the number of belief statements consistent with the standards, 5 teachers scored higher than the mean for the number of belief statements inconsistent with the standards, and 3 teachers scored higher than the mean for the number of undecided belief statements. In order to study the beliefs in more depth, the subset of 22 beliefs were examined by statement categories.

**9.2 Teachers' Beliefs by Statement Category**

In this section, teachers' beliefs regarding each of the categories for the subset of the 22 statements are examined. The subset of 22 statements from the pre- and post-assessment is grouped into the following six categories: expectations and abilities, mathematical discourse, concepts and procedures, manipulatives, roles of students and teachers, and differentiated instruction. Teachers' responses from the beliefs inventory were coded as consistent, inconsistent or undecided for each statement category.

A 5-point Likert scale for each statement was used ranging from strong agreement (e.g. 1) to strong disagreement (e.g. 5). Consistent (C) was coded when responses showed agreement with statements consistent with the standards or disagreement with statements inconsistent with the standards. Inconsistent (I) was coded when responses showed disagreement with statements consistent with the standards or responses that showed agreement with statements inconsistent with the standards. Undecided (U) statements were coded from teachers' responses of "3" (neutral).

### *9.2.1 Expectations and Abilities*

Four of the belief statements were categorized under expectations and abilities. Of the four beliefs statements for the category of expectations and student abilities, two statements were consistent with the standards and two statements were inconsistent with the standards. The following statements for the expectations and abilities category were:

> Q1: Learners generally understand more mathematics than their teachers or parents expect.
> Q7: All students are capable of working on complex math tasks.
> *Q13: Only really smart students are capable of working on complex math tasks.
> *Q29: Only the most talented students can learn math with understanding. (Beliefs Inventory, Appendix A & B)

The two statements not consistent with the standards are marked with asterisks.

Table 9.2 shows the number (N) and percentage (P) of belief statements consistent (C), inconsistent (I), and undecided (U) with the standards out of the subset of 22 beliefs from the pre- and post-assessments for each teacher in the expectations and student abilities category. For the pre-assessment category of expectations and student abilities, 6 teachers (T1, T2, T3, T5, T7, and T8) had 2 out of 4 belief statements as consistent with the standards; 2 teachers (T4 and T6) had 3 out of 4 belief statements as consistent with the standards; and 2 teachers (T9 and T10) had all four beliefs consistent with the standards. T1 and T2 had 2 inconsistent statements for the category of expectations and student abilities; and T3 and T5 had 1 inconsistent statement for expectations and student abilities.

Table 9.2
*Teachers' Scores from Beliefs for Expectations and Student Abilities*

| Teacher | Pre-Assessment | | | | | | Post-Assessment | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | *CN* | *CP* | *IN* | *IP* | *UN* | *UP* | *CN* | *CP* | *IN* | *IP* | *UN* | *UP* |
| T1 | 2 | 50 | 2 | 50 | 0 | 0 | 3 | 75 | 1 | 25 | 0 | 0 |
| T2 | 2 | 50 | 2 | 50 | 0 | 0 | 4 | 100 | 0 | 0 | 0 | 0 |
| T3 | 2 | 50 | 1 | 25 | 1 | 25 | 2 | 50 | 1 | 25 | 1 | 25 |
| T4 | 3 | 75 | 0 | 0 | 1 | 25 | 4 | 100 | 0 | 0 | 0 | 0 |
| T5 | 2 | 50 | 0 | 0 | 2 | 50 | 3 | 75 | 0 | 0 | 1 | 25 |
| T6 | 3 | 75 | 1 | 25 | 0 | 0 | 4 | 100 | 0 | 0 | 0 | 0 |
| T7 | 2 | 50 | 0 | 0 | 2 | 50 | 2 | 50 | 0 | 0 | 2 | 50 |
| T8 | 2 | 50 | 1 | 25 | 1 | 25 | 4 | 100 | 0 | 0 | 0 | 0 |
| T9 | 4 | 100 | 0 | 0 | 0 | 0 | 3 | 75 | 1 | 25 | 0 | 0 |
| T10 | 4 | 100 | 0 | 0 | 0 | 0 | 4 | 100 | 0 | 0 | 0 | 0 |
| Mode | 2 | 50 | 0 | 0 | 0 | 0 | 4 | 100 | 0 | 0 | 0 | 0 |

*Sources: Beliefs Inventory Pre- and Post-Assessment*

Based on the beliefs post-assessment regarding the expectations and student abilities category, the results indicate that a slight change in beliefs may have occurred for T1, T2, T4, T5, T6, and T8 as their number of belief statements increased for the category of expectations and student abilities. The results also indicate that a slight change in beliefs may have occurred for T9 as her number of belief statements decreased for the expectations and student abilities category on the post-assessment.

### *9.2.2 Mathematical Discourse*

Two of the beliefs statements were categorized under mathematical discourse. Of the two belief statements for the category of mathematical discourse, one statement was consistent with the standards and one was inconsistent with the standards. The following statements from the beliefs inventory assessments were categorized as mathematical discourse:

> Q4: It's helpful to encourage student-to-student talking during math activities.
> *Q23: Collaborative learning is effective only for those students who actually talk during group work. (Beliefs Inventory, Appendix A & B)

One inconsistent statement with the standards is marked with an asterisk.

From the pre-assessment responses, 9 teachers (all but T8) had a score of 2 out of 2 belief statements as consistent with the standards for the mathematical discourse category. On the post-assessment, one teacher (T5) had 1 out of 2 belief statements as consistent with the standards and 1 out of 2 belief statements as inconsistent with the standards. The results indicate that a slight change in beliefs may have occurred for T5 as T5's number of belief statements decreased for the mathematical discourse category. Based on the beliefs post-assessments regarding the mathematical discourse category, the results also indicate that a slight change in beliefs may have occurred for T8 as T8's number of belief statements increased for the category of mathematical discourse.

### *9.2.3 Concepts and Procedures*

Seven of the beliefs statements were categorized under concepts and procedures. Of the 7 belief statements for the category of concepts and procedures, 5 statements were consistent with the standards and 2 were inconsistent with the standards. The following statements from the beliefs inventory assessments were categorized as concepts and procedures:

> Q2: Teachers should make sure that students know the correct procedure for solving a problem.
> *Q5: Math is primarily about learning procedures.
> Q9: If students learn math concepts before they learn the procedures, they are more likely to understand the concepts.
> *Q11: Young children must master math facts before starting to solve problems.
> Q18: Learners can solve problems in novel ways before being taught to solve such problems.
> Q19: Understanding math concepts is more powerful than memorizing procedures.
> Q21: If students learn math concepts before procedures, they are more likely to understand the procedures when they learn them. (Beliefs Inventory, Appendix A & B)

Two inconsistent statements are marked with an asterisk.

Table 9.3 shows the number (N) and percentage (P) of belief statements consistent (C), inconsistent (I), and undecided (U) with the standards out of the subset of 22 beliefs from the pre- and post-assessments of each teacher for the category of concepts and procedures. From the pre-assessment responses, 5 teachers (T1, T3, T4, T7, and T10) had the mode score of 5 out of 7 belief statements as consistent with the standards for the concepts and procedures category. T5 and T8 had 6 out of 7 belief statements consistent with the standards for concepts and procedures and; T6 and T9 had 7 out of 7 belief statements consistent with the standards; and T2 had 2 out of 7 belief statements consistent with the standards.

Table 9.3
*Teachers' Scores from Beliefs for Concepts and Procedures*

| Tchr. | Pre-Assessment | | | | | | Post-Assessment | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | *CN* | *CP* | *IN* | *IP* | *UN* | *UP* | *NC* | *CP* | *IN* | *IP* | *UN* | *UP* |
| T1 | 5 | 71.4 | 1 | 14.3 | 1 | 14.3 | 4 | 57.1 | 1 | 14.3 | 2 | 28.6 |
| T2 | 2 | 28.6 | 3 | 42.8 | 2 | 28.6 | 6 | 85.7 | 0 | 0 | 1 | 14.3 |
| T3 | 5 | 71.4 | 1 | 14.3 | 1 | 14.3 | 6 | 85.7 | 1 | 14.3 | 0 | 0 |
| T4 | 5 | 71.4 | 1 | 14.3 | 1 | 14.3 | 6 | 85.7 | 1 | 14.3 | 0 | 0 |
| T5 | 6 | 85.7 | 0 | 0 | 1 | 14.3 | 6 | 85.7 | 0 | 0 | 1 | 14.3 |
| T6 | 7 | 100 | 0 | 0 | 0 | 0 | 7 | 100 | 0 | 0 | 0 | 0 |
| T7 | 5 | 71.4 | 1 | 14.3 | 1 | 14.3 | 6 | 85.7 | 0 | 0 | 1 | 14.3 |
| T8 | 6 | 85.7 | 1 | 14.3 | 0 | 0 | 7 | 100 | 0 | 0 | 0 | 0 |
| T9 | 7 | 100 | 0 | 0 | 0 | 0 | 6 | 85.7 | 1 | 14.3 | 0 | 0 |
| T10 | 5 | 71.4 | 1 | 14.3 | 1 | 14.3 | 6 | 85.7 | 0 | 0 | 1 | 14.3 |
| Mode | 5 | 71.4 | 1 | 14.3 | 1 | 14.3 | 6 | 85.7 | 0 | 0 | 1 | 14.3 |

*Sources: Beliefs Inventory Pre- and Post-Assessment*

The mode of the data for the post-assessment was 6 out of 7 beliefs consistent with the standards for the concepts and procedures category. Based on the beliefs post-assessments, the results indicate that a slight change in beliefs may have occurred for T2, T3, T4, T7, T8, and T10 as the number of belief statements for these teachers increased for the category of concepts and procedures. The results also indicate that a slight change in beliefs may have occurred for T1 and T9 as the number of belief statements for these teachers decreased for the concepts and procedures category.

### *9.2.4 Manipulatives*

Two of the beliefs statements were categorized under manipulatives. Both statements were inconsistent with the standards and marked with asterisks. The following statements from the beliefs inventory assessments were placed in the manipulatives category:

> *Q10: Manipulatives should only be used with students who don't learn from the textbook. *Q5: Math is primarily about learning procedures.
> *Q17: Manipulatives cannot be used to justify a solution to a problem. (Beliefs Inventory, Appendix A & B)

From the pre-assessment responses, 9 teachers (all but T4) had the mode score of 2 out of 2 belief statements as consistent with the standards for the mathematical discourse category.

On the post-assessment, one teacher (T5) had 1 out of 2 belief statements as consistent with the standards and 1 out of 2 belief statements as undecided with the standards. The post- assessment results indicate that a slight change in beliefs may have occurred for T5 as T5's number of belief statements decreased for the manipulatives category. Based on the beliefs post-assessments regarding the manipulatives category, the results also indicate that a slight change in beliefs may have occurred for T4 as T4's number of belief statements increased for the category of manipulatives.

### *9.2.5 Student and Teacher Roles*

Three of the belief statements were categorized under student and teacher roles. Of the three beliefs statements for the category of student and teacher roles, two statements were consistent with the standards and one statement was inconsistent with the standards. The following statements for the student and teacher roles category were:

> Q24: Students should be corrected by the teacher if their answers are incorrect.
> *Q30: The idea that students are responsible for their own learning does not work in practice.
> *Q32: Teacher questioning of students' solutions tends to undermine students' confidence. (Beliefs Inventory, Appendix A & B)

The two statements inconsistent with the standards are marked with asterisks.

Table 9.4 shows the number (#) and percentage (%) of belief statements consistent (C), inconsistent (I), and undecided (U) with the standards out of the subset of 22 beliefs from the pre- and post-assessments of each teacher for the student and teacher roles category. From the pre-assessment responses, T2 and T5 had 3 out of 3 belief statements as consistent with the standards; T1, T4, T6, and T9 had 2 out of 3 belief statements as consistent with the standards; and T3, T7, and T8 had 1 belief consistent with the standards for the student and teacher role category.

Based on the beliefs post-assessment regarding the student and teacher roles category, the results indicate that a slight change in beliefs may have occurred for T8 as the number of belief statements increased for the category of student and teacher roles. The results also indicate that a slight change in beliefs may have occurred for T1, T5, and T7 as the number of belief statements decreased for the student and teacher roles category on the post-assessment.

Table 9.4
*Teachers' Scores from Beliefs for Student and Teacher Roles*

| Tchr. | Pre-Assessment | | | | | | Post-Assessment | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | *CN* | *CP* | *IN* | *IP* | *UN* | *UP* | *CN* | *CP* | *IN* | *IP* | *UN* | *UP* |
| T1 | 2 | 66⅔ | 1 | 33⅓ | 0 | 0 | 1 | 33⅓ | 1 | 33⅓ | 1 | 33⅓ |
| T2 | 3 | 100 | 0 | 0 | 0 | 0 | 3 | 100 | 0 | 0 | 0 | 0 |
| T3 | 1 | 33⅓ | 2 | 66⅔ | 0 | 0 | 1 | 33⅓ | 1 | 33⅓ | 1 | 33⅓ |
| T4 | 2 | 66⅔ | 0 | 0 | 1 | 33⅓ | 2 | 66⅔ | 1 | 33⅓ | 0 | 0 |
| T5 | 3 | 100 | 0 | 0 | 0 | 0 | 1 | 33⅓ | 0 | 0 | 2 | 66⅔ |
| T6 | 2 | 66⅔ | 1 | 33⅓ | 0 | 0 | 2 | 66⅔ | 0 | 0 | 1 | 33⅓ |
| T7 | 1 | 33⅓ | 1 | 33⅓ | 1 | 33⅓ | 0 | 0 | 1 | 33⅓ | 2 | 66⅔ |
| T8 | 1 | 33⅓ | 1 | 33⅓ | 1 | 33⅓ | 2 | 66⅔ | 1 | 33⅓ | 0 | 0 |
| T9 | 2 | 66⅔ | 1 | 33⅓ | 0 | 0 | 2 | 66⅔ | 1 | 33⅓ | 0 | 0 |
| T10 | 0 | 0 | 1 | 33⅓ | 2 | 66⅔ | 0 | 0 | 2 | 66⅔ | 1 | 33⅓ |
| Mode | 2 | 66⅔ | 1 | 33⅓ | 0 | 0 | 2 | 66⅔ | 1 | 33⅓ | 0 | 0 |

### *9.2.6 Differentiated Instruction*

Four of the belief statements were categorized under differentiated instruction. Of the four beliefs statements for the category of differentiated instruction, three statements were consistent with the standards and one statement was inconsistent with the standards. The following statements for the differentiated instruction category were:

> Q1: Learners generally understand more mathematics than their teachers or parents expect.
> Q7: All students are capable of working on complex math tasks.
> Q13: Only really smart students are capable of working on complex math tasks.
> *Q29: Only the most talented students can learn math with understanding. (Beliefs Inventory, Appendix A & B)

One statement was inconsistent with the standards and marked with an asterisk.

Table 9.5 shows the number (N) and percentage (P) of belief statements consistent (C), inconsistent (I), and undecided (U) with the standards out of the subset of 22 beliefs from the pre- and post-assessments for each teacher in the differentiated instruction category. For the pre-assessment category of differentiated instruction, 5 teachers (T1, T6, T7, T8, and T10) had 4 out of 4 belief statements as consistent with the standards; 4 teachers (T2, T3, T5 and T9) had 3 out of 4 belief statements as consistent with the standards; and 1 teacher (T4) had 2 out of 4 beliefs consistent with the standards. T2, T3, and T9 each had 1 undecided statement for the category of differentiated instruction; and T4 had 2 inconsistent statements for differentiated instruction. It should be noted that T5 did not provide an answer for belief statement number 28 on the pre-assessment.

Table 9.5
*Teachers' Scores from Beliefs for Differentiated Instruction*

| Teacher | Pre-Assessment | | | | | | Post-Assessment | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | *CN* | *CP* | *IN* | *IP* | *UN* | *UP* | *CN* | *CP* | *IN* | *IP* | *UN* | *UP* |
| T1 | 4 | 100 | 0 | 0 | 0 | 0 | 4 | 100 | 0 | 0 | 0 | 0 |
| T2 | 3 | 75 | 0 | 0 | 1 | 25 | 3 | 75 | 0 | 0 | 1 | 25 |
| T3 | 3 | 75 | 0 | 0 | 1 | 25 | 3 | 75 | 1 | 25 | 0 | 0 |
| T4 | 2 | 50 | 0 | 0 | 2 | 50 | 4 | 100 | 0 | 0 | 0 | 0 |
| T5 | 3 | 75 | 0 | 0 | 0 | 0 | 4 | 100 | 0 | 0 | 0 | 0 |
| T6 | 4 | 100 | 0 | 0 | 0 | 0 | 4 | 100 | 0 | 0 | 0 | 0 |
| T7 | 4 | 100 | 0 | 0 | 0 | 0 | 3 | 75 | 0 | 0 | 1 | 25 |
| T8 | 4 | 100 | 0 | 0 | 0 | 0 | 4 | 100 | 0 | 0 | 0 | 0 |
| T9 | 3 | 75 | 0 | 0 | 1 | 25 | 3 | 75 | 1 | 25 | 0 | 0 |
| T10 | 4 | 100 | 0 | 0 | 0 | 0 | 4 | 100 | 0 | 0 | 0 | 0 |
| Mode | 4 | 100 | 0 | 0 | 0 | 0 | 4 | 100 | 0 | 0 | 0 | 0 |

*Sources: Beliefs Inventory Pre- and Post-Assessment*

Based on the beliefs pre- and post-assessment regarding the differentiated instruction category, the results indicate that a slight change in beliefs may have occurred for T4 and T5 as their number of belief statements increased for the category of differentiated instruction. The results also indicate that a slight change in beliefs may have occurred for T7 as the number of belief statements decreased for the differentiated instruction category on the post-assessment.

**9.3 Stability and Potential Growth of Teachers' Beliefs**

There are salient findings regarding the stability of teacher beliefs from pre- to post-assessment. Table 9.6 shows the number and percent of teachers who scored 100% consistent with the standards for each category on the pre-assessment and the number of teachers whose post-assessment score declined out of the teachers who scored 100% consistent with the standards on the pre-assessment. Table 9.6 also shows the confidence intervals indicating the probability level of getting similar results if the study were replicated. Confidence intervals (C.I.) are also provided in table 9.6 indicating the potential for getting similar results of growth on the post-assessment if the study were repeated. The confidence intervals were computed using the tool on the following website: http://statpages.info/confint.html.

Table 9.6
Pre- and Post-Assessment Stability of Teachers' Beliefs

| Belief Statement Category | Teachers with 100% on Pre-<br>% of Teachers<br>95% One Sided C.I. | Teachers with Decline on Post-<br>% of Teachers<br>95% One Sided Upper C.I. |
|---|---|---|
| Expectations and Abilities | 2 of 10<br>20%<br>≥ 3.7% | 1 of 10<br>10%<br>≤ 0.5% |
| Mathematical Discourse | 9 of 10<br>90%<br>≥ 60.6% | 1 of 10<br>10%<br>≤ 0.5% |
| Concepts and Procedures | 2 of 10<br>20%<br>≥ 3.7% | 1 of 10<br>10%<br>≤ 0.5% |
| Manipulatives | 9 of 10<br>90%<br>≥ 60.6% | 1 of 10<br>10%<br>≤ 0.5% |
| Student and Teacher Roles | 2 of 10<br>20%<br>≥ 3.7% | 1 of 10<br>10%<br>≤ 0.5% |
| Differentiated Instruction | 5 of 10<br>50%<br>≥ 22.2% | 1 of 10<br>10%<br>≤ 0.5% |
| *Sources: | Beliefs Pre- and Post-Assessment | |

Two out of 10 teachers scored 100% consistent with the standards on the pre-assessment for the following categories: expectations and student abilities, concepts and procedures, and student and teacher roles. One teacher (T9) decreased her score on the post-assessment for two categories: expectations and student abilities and concepts and procedures. Another teacher (T5) decreased the score on the post-assessment for the category of student and teacher roles. One could state with 95% confidence that if this study was replicated, at most 0.5% of a population with similar teacher participants would be expected to show post-assessment decline for the belief statement categories of expectations and abilities, concepts and procedures, and student and teacher roles.

For mathematical discourse and manipulatives, nine out of ten teachers scored 100% consistent with the standards for those statement categories on the pre-assessment. Of the 9 teachers who scored 100% consistent with the standards for manipulatives and mathematical discourse, only one teacher (T5) decreased the belief score on the post-assessment. One could state with 95% confidence that if this study was replicated, at most 0.5% of a population with similar teacher participants would be expected to show post-assessment decline for the belief statement categories of mathematical discourse and manipulatives.

Half of the teachers scored 100% consistent with the standards on the pre assessment for the category of differentiated instruction. Of the five teachers that scored 100% consistent with the standards on the pre-assessment, only one teacher (T7) decreased her score on the post-assessment. One could state with 95% confidence that if this study was replicated, at least 0.5% of a population with similar teacher participants would be expected to show post-assessment decline for the belief statement category of differentiated instruction.

It is also important to note the growth potential for the teachers' beliefs. Table 9.7 shows the number of teachers that had potential for their beliefs to grow based on their pre-assessment belief scores. Table 9.7 also shows the number and percent of teachers whose score declined on the post-assessment and the number of teachers' that received a higher score on the beliefs post-assessment. Confidence intervals are also provided in table 9.7 indicating the potential for getting similar results of growth on the post-assessment if the study were repeated.

Table 9.7: Growth Rates for Teachers' Growth Potential on the Beliefs Pre- Assessment

| Belief Statement Category | Growth Potential<br>% Teachers<br>95% One-sided C. I. | Decline on Post<br>% Teachers<br>95% One-sided C. I. | Growth on Post<br>% Teachers<br>95% One-sided C. I |
|---|---|---|---|
| Expectations and Abilities | 8 of 10<br>80%<br>≥ 49.3% | 0<br>0.0% | 6 of 8<br>75.0%<br>≥ 40.0% |
| Mathematical Discourse | 1 of 10<br>10%<br>≥ 0.5% | 0<br>0.0% | 1 of 1<br>100%<br>≥ 5.0% |
| Concepts and Procedures | 8 of 10<br>80%<br>≥ 49.3% | 0<br>0.0% | 6 of 8<br>75.0%<br>≥ 40.0% |
| Manipulatives | 1 of 10<br>10%<br>≥ 0.5% | 0<br>0.0% | 1 of 1<br>100%<br>≥ 5.0% |
| Student and Teacher Roles | 8 of 10<br>80%<br>≥ 49.3% | 0<br>0.0% | 1 of 8<br>12.5%<br>≥ .6% |
| Differentiated Instruction | 5 of 10<br>50%<br>≥ 22.2% | 0<br>0.0% | 2 of 5<br>20%<br>≥ 7.6% |
| Sources: Pre- & | Post- Assessments for | Gang of Four Video | |

The belief statement categories where 8 out of 10 teachers' belief scores had the potential to grow were: expectations and student abilities, concepts and procedures, and student and teacher roles. One could state with 95% confidence that if this study was replicated, at least 49.3% of a population with similar teacher participants would potentially show post-assessment growth for the belief statement categories of expectations and abilities, concepts and procedures, and student and teacher roles.

Moreover, teachers had higher belief scores on the post-assessment for the following belief statement categories: expectations and abilities, 6 out of the 8 teachers; concepts and procedures, 6 out of the 8 teachers; and student and teacher roles, 1 out of the 8 teachers. One could say with 95% confidence that at least 40% of a population with similar teacher participants would be expected to show post-assessment growth for the belief statement categories of expectations and abilities and concepts and procedures; and at least 0.6% of a population with similar teacher participants would be expected to show post-assessment growth for the belief statement category of student and teacher roles.

The belief statement categories where 1 out of 10 teachers' belief scores had the potential to grow were: mathematical discourse and manipulatives. One could state with 95% confidence that if this study was replicated, at least 0.5% of a population with similar teacher participants would potentially show post-assessment growth for the belief statement category of differentiated instruction. Moreover, two teachers had higher belief scores on the post-assessment for mathematical discourse and manipulatives. One could say with 95% confidence that at least 5% of a population with similar teacher participants would be expected to show post-assessment growth for each of the belief statement categories of mathematical discourse and manipulatives.

Differentiated instruction was the only belief statement category where 5 out of 10 teachers' belief scores had the potential to grow. One could state with 95% confidence that if this study was replicated, at least 22.2% of a population with similar teacher participants would potentially show post-assessment growth for the belief statement categories of mathematical discourse and manipulatives. Moreover, teachers had higher belief scores on the post-assessment for differentiated instruction having 2 out of the 5 teachers show growth on the post-assessment. One could say with 95% confidence that at least 7.6% of a population with similar teacher participants would potentially show post-assessment growth for each of the belief statement categories of mathematical discourse and manipulatives.

### 9.4 Summary of Teachers' Beliefs

The data shows there is evidence that teachers' beliefs changed slightly regarding the teaching and learning of mathematics after teachers participated in the intervention. Scores to measure teachers' beliefs were calculated as a percent based on the number of statements of agreement with the standards out of a subset of 22 beliefs statements. The teachers' mean beliefs score increased from pre- to post-assessment.

Teachers' beliefs were also measured by the following statement categories: expectations and student abilities, mathematical discourse, concepts and procedures, manipulatives, student and teacher roles, and differentiated instruction. There is evidence from the data exhibiting change in teachers' beliefs for the categories of expectations and student abilities; concepts and procedures; and student and teacher roles. The data revealed significant results for teacher growth potential in the belief statement categories of expectations and abilities and concepts and procedures, 40%. The intervention helped teachers to align their beliefs to be more consistent with the standards.

# Chapter 10 – Narratives of Teachers

This chapter is a description of ten teacher narratives. The teacher narratives describe each teacher's experiences within the intervention regarding beliefs and how the teachers attended to students' reasoning. The data sources used to examine the teachers' intervention experiences are the meeting transcripts, the on-line discussion threads, and the final projects.

## 10.1 T1

T1 completed the Beliefs pre-assessment before the start of the first meeting. Results of the pre-assessment showed that two of the statements inconsistent with the standards were in the category of expectations and abilities, one statement inconsistent with the standards was in the category of concepts and procedures, and 1 statement inconsistent with the standards was in the category of students' and teachers' roles.

T1 began the first cycle by working on finding all possible 4-tall towers that can be made selecting from two colors with another teacher from the northern and central cohort group. When the instructor went to T1 and her partner to ask how they arranged their towers, T1 and her partner arranged the four-tall towers in pairs by building one tower and then making another tower with opposite colors (9/7 meeting transcript, lines 69-70). The instructor asked T1 and her partner "How do I know that you found all the towers?" and T1 replied "opposites" (9/7 meeting transcript, line 69). The instructor asked the teacher pair to "think about rearranging the towers so you can convince me" (9/7 meeting transcript, line 82). When the instructor returned to the group about fifteen minutes later, T1 and her partner had not written their argument but had rearranged the towers using the staircase strategy.

For the unit 2 on-line discussion, T1 was the first to begin the discussion thread. T1 had written “Before doing any towers, I think students will predict that 3 high will produce 6 towers, 4 will give 8, and 5 will give 10. I teach in the resource classroom and students struggle with their facts.” (Unit 2, On-line discussion thread, line 1). T1 also made the following comments to other teachers:

> I agree that many 7th graders in the resource math setting just jump into the problem before thinking it through. I also realize that they are so quick to use addition and subtraction or very basic multiplication whenever possible. That is also why I predicted that the students would just multiply by 2 because it is a computation they are more familiar and comfortable with. (Unit 2, On-line discussion thread, line 18).
>
> I gave my 7th grade students that are in the resource room a 5th grade pretest and realized that they are much lower than I expected. (Unit 2, On-line discussion thread, line 20).

Her comments may be evidence of why T1 made statements inconsistent with the standards for the category of students’ expectations and abilities on her Beliefs pre-assessment.

T1 implemented the first cycle task in a seventh-grade resource pull-out with seven students in a forty-minute mathematics class on September 20, 2013. T1 explained that her students “loved the hands-on aspect and worked diligently in their pairs” (Final Project for T1, p. 2). However, T1 reported that the students struggled to explain and write their arguments (Final Project for T1, p. 2).

For the Unit 3 on-line discussion, T1 was the first to post her original response. She made the comment that “I learned from my colleagues and from watching the children that there are many different ways of approaching and explaining this problem.” (Unit 3, On-line discussion thread, line 1) and T1 also mentioned that she noticed many students used the opposite strategy (Unit 3, On-line discussion thread, line 1). T1 replied that "I think it is important to encourage students to explain their problems in a way that makes sense to them." (Unit 3, On-line discussion thread, line 1). For the Unit 4 discussion, T1 replied to another teacher to express concern that her "students were disinterested and frustrated." (Unit 4, On-line discussion thread, line 27). T1 also wrote that "I had them finish writing because they were almost at the point of over frustration." (Unit 4, On-line discussion thread, line 27).

At the October 2 regional meeting, T1 shared three samples of students' work with the teachers. The first sample T1 expressed concern with regards to the students understanding the task because the student pair made one big tower out of the 4-tall towers. In the second sample shared, T1 said that the students had correctly answered with 16 but only had ten towers drawn on their paper. T1 recognized the diagonal pattern in the student's work (10/2/13 meeting transcript, line 74-80). The third sample of students' work that T1 shared was of a pair that found 16 possible towers but said "We couldn't make anymore because we think we made all the patterns." (10/2/13 meeting transcript, line 103). This sparked a debate about convincing versus non-convincing arguments.

Later on at the October 2 regional meeting, T1 paired up with T8 to work on the second cycle task. T1 had written the following comment on the on-line discussion thread:

> With the pizza problem, I automatically thought "tree diagram!" but my partner reminded me we cannot do that because we will have duplicate pizzas. I definitely liked the strategy of picking a topping and having that as the constant. From there we added toppings to the constant topping. It was a little harder for me to organize my work than it was with the towers. (Unit 5, On-line discussion thread, line 7).

During the Unit 6 on-line discussion, the focus was on types of questioning. T1 said in response to another teacher that "I also agree that asking why is really important. I feel that sometimes we are in such a rush to get the curriculum done that we often don't take the time." (Unit 6, On-line discussion thread, line 29). On October 11, 2013, T1 implemented the second cycle task in her classroom. T1 wrote the following in her final project:

> Many students spent much time pondering over whether they could use extra toppings, half pies, and whether order matters in how the toppings are placed on the pizza. I felt this weakened the purpose of the activity. If I could do this activity again, I would give the student specific directions such as order doesn't matter and there cannot be half pies in this case. (Final Project for T1, p. 17)

T1's comments may be evidence for statements inconsistent with the standards for the category of expectations and abilities. For the Unit 7 on-line discussion, T1 mentioned in response to another teacher that her students did not see the connection between the towers and the pizza problem and "made a lesson of explaining the problem" (Unit 7, On-line discussion thread, line 47). T1's comment may be evidence for statements inconsistent with the standards for the category of expectations and abilities.

At the October 22 regional meeting, T1 shared two samples of students' work. One student made an unorganized list using the letters A through D to represent toppings and then used E to represent a plain pizza (10/22/13 meeting transcript, lines 252). Another student found fourteen pizza combinations using an organized list by writing the words of the toppings in separate boxes to represent each pizza combination. T1 recognized the student controlled for a variable by holding the mushroom topping constant (10/22/13 meeting transcript, lines 276).

After all teachers shared their students' samples of work, T1 paired up with T5 to work on the three-tall tower problem and Ankur's Challenge. For the three-tall towers problem, T1 and her partner made 9 groups of three 3-tall towers, For Ankur's challenge, T1 and her partner made 6 groups of 6 four-tall towers where six towers each had red, yellow, or blue held constant on the bottom and six towers each had red, yellow, or blue held constant on the top.

The Unit 8 on-line discussion focused on Ankur's Challenge. In response to one teacher, T1 said "I think my students would do the same with finding opposites and a pattern; however, the 3rd color will probably get them very frustrated." (Unit 8, On-line discussion thread, line 16). T1 also responded about whether or not her students would be able to come up with Romina's proof. T1 said "I think my students would be able to work with the towers and find the combinations but not all possibilities." (Unit 8, On-line discussion thread, line 41).

T1 also referenced Romina in the Unit 9 discussion when she responded to another teacher about giving students more than one opportunity to explain or write their reasoning. T1 said "Really good analogy! Editing papers shows mistakes and gives students a chance to fix or enhance their papers....this is exactly what Romina did with her argument." (Unit 9, On-line discussion thread, line 29).

On 10/31/13, T1 implemented the third cycle of tasks in her classroom. T1 said the following:

> I was excited to see that my students started working diligently right away and did not need me to push them to get started. They all showed growth since the first task whether it was the way they worked together, drew their explanations, or explained their thinking. (Final Project for T1, p. 26)

T1 also said “I was more comfortable letting them ponder over the problem and didn’t feel the urge to guide them” (Final Project for T1, p. 26). T1’s comments may be evidence of statements consistent with the standards for the category of teachers’ and students’ roles.

The Unit 10 discussion asked teachers to discuss the strategies used by one student over the series of tasks. T1 talked about a pair of boys that used a recursive argument to solve the problems in the third cycle of tasks (Unit 10, On-line discussion thread, line 8). In response to one of the teachers, T1 said “It was very difficult to get them to write, especially for the pizza task, because they did not understand that task as much as the towers. However, I noticed an improvement with the 3rd task as far as the writing.” (Unit 10, On-line discussion thread, line 28).

At the 11/20 regional meeting, T1 shared two students’ work for the third cycle. One student’s work T1 was concerned with because her written argument did not match the drawing of the towers (10/22/13 meeting transcript, lines 196-200). A second student’s work from T1 had four examples of different types of patterns and the student drew 23 towers based on those patterns (10/22/13 meeting transcript, line 214). T1 shared the following final reflection thoughts on the intervention:

> With the four towers, I was quick to accept arguments for fear of them reaching frustration. However, as we went on with the tasks, my students showed such improvement and I realized they were capable of more. I feel like I never give enough ‘wait’ time because I’m scared I’m going to ‘lose’ students. Now, I give more ‘wait’ time and am not so quick to give and explain the answer. (Final Project for T1, p. 27-28)

**10.2 T2**

T2 completed the Beliefs pre-assessment before the start of the first meeting. On the pre-assessment, T2 made 2 statements inconsistent with the Standards in the category of expectations and abilities and 3 statements inconsistent with the Standards in the category of concepts and procedures. T2 began the first cycle by working on finding all possible 4-tall towers that can be made selecting from two colors with T3. T2 and T3 originally arranged their four-tall towers using guess and check and the opposite strategy but then rearranged their towers using an inductive argument (9/7/13 meeting transcript, lines 19; 84-90).

For the unit 2 on-line discussion, T2 had posted “This year I am teaching 6th, 7th and 8th grade LLD students (Self-Contained). During this week of school I have been doing testing to figure out what mathematical levels they are all on. Though testing is not complete yet I have noticed that they are 2 to 3 grade levels below in their mathematical knowledge. ” (Unit 2, On-line discussion thread, line 19). T2 also said the following:

> I believe that they will just start by building all types of towers. I don't know if they would plan or discuss a plan of action before building. I'm hoping that they will notice the doubles that they build but I am not sure that they will do that. Some of my students may see the bag of cubes and just build as many towers as they can 4 high and then say that is their answer. I will need to be very specific when reading the question to them just to make sure that they understand their task before attacking it. I am very interested in seeing what they come up with. (Unit 2, On-line discussion thread, line 19).

Her comments may be evidence of why T2 made statements inconsistent with the standards for the category of students’ expectations and abilities on her Beliefs pre-assessment.

T2 implemented the first cycle task in a seventh-grade self-contained special education class with ten students in a mathematics class for 43 minutes on September 20, 2013 (Final Project for T2, p. 3). T2 said that her students “had a hard time understanding that I could not guide them” (Final Project for T2, p. 10). However, T2 reported that she wanted to encourage her students to "be more independent thinkers" (Final Project for T2, p. 10).

For the Unit 3 on-line discussion, one teacher (T5) had written that Stephanie and Dana's argument for the four-tall towers problem in third grade was not convincing. In response to T5's original post, T2 said "I actually felt their argument was convincing. Maybe I'm being too nice or easy on the girls. I can see how many don't think their argument was convincing initially but I felt when she was describing taking a cube of the bottom and creating a duplicate that showed she did understand the problem and that maybe she couldn't explain it as well because of her age. " (Unit 3, On-line discussion thread, line 31).

For the Unit 4 discussion, T2 responded to another teacher regarding one pair of students that kept changing their strategies for solving the four-tall tower problem, selecting from two colors. T2 said the following:

> I think it was so interesting that the one group you had changed their argument so many times and in the end went back to their original response because that convinced them the most. I did not have any student think of different ways. Most of them tackled the problem using opposites and then couldn't explain how they had them all. (Unit 4, On-line discussion thread, line 30).

At the October 2 regional meeting, T2 shared four samples of students' work with the teachers. In the first-shared sample, T2 recognized that her student used the opposite strategy to draw 16 towers, but explained verbally more than what they provided as a written argument (10/2/13 students' work meeting transcript, lines 139-148). In the second sample shared, T2 recognized that her student used the opposite strategy to draw 16 towers but used the letters y and b instead of coloring the squares that represented the cubes (10/2/13 students' work meeting transcript, line 183).

T2 shared the partner's paper as a third sample where the student used blue and yellow markers to draw the 16 towers but did not provide a written argument. T2 recognized the diagonal pattern in the student's work as a blue cube moved down one for each position and noted that the partners made different tower drawings (10/2/13 students' work meeting transcript, line 189). The fourth sample of students' work that T2 shared was from a student that drew the towers forming a cube but T2 said that the student had used pairs but did not build his towers the way it was drawn on his paper (10/2/13 meeting transcript, line 237).

Later on at the October 2 regional meeting, T2 paired up with T3 to work on the second cycle task. T2 had written the following comment on the on-line discussion thread:

> The most helpful strategy for me and my partner was to make a list of combinations of pizzas. The hardest part was coming up with an agreement as to whether cheese was a topping or not. Once we came to an agreement on that the list was simple and we both came up to the same conclusion of 16. At first we thought about a tree diagram; however, we felt that was a hard way to represent the combinations. (Unit 5, On-line discussion thread, line 41).

During the Unit 6 on-line discussion, the focus was on types of questioning. T2 responded with the following to two teachers:

> I completely agree with you that questioning is very important!! I feel as though I am trying to be more aware of my questioning toward my students but I am still finding it hard for them to give me clear explanations. (Unit 6, On-line discussion thread, line 2).

> I liked this type of questions that make it seem as though the teacher is confused and needs further clarification, "I'm not sure what you're saying..." I think this allows the student to feel confident that they may know something over the teacher and allows them to think further about what they did and why. (Unit 6, On-line discussion thread, line 15).

T2 had noted the significance of questioning students about their work and admitted that her students struggled to give explanations that were clear (Unit 6 On-line discussion thread, lines 2, 15).

On October 10, 2013, T2 implemented the second cycle task in her classroom. T2 wrote the following in her final project: "For the pizza task, I found it to be more difficult than the towers. I was excited to implement this task, it was more relatable" (Final Project for T2, p. 17). T2 had mentioned that a lot of her students did not like certain toppings on their pizza and had not used them to find the solution to the pizza problem. For the Unit 7 on-line discussion, T2 mentioned the following in response to another teacher:

> My students had more difficulties with this problem as well because they couldn't build something like the towers. A lot of my students did not see the duplicates right away until I asked them if they went into a pizzeria and ordered a sausage and a mushroom slice and then a mushroom and a sausage slice would they be ordering different things. They then saw they couldn't just switch the order like they did with the towers and making opposites. (Unit 7, On-line discussion thread, line 7).

At the October 22 regional meeting, T2 shared two samples of students' work. One student made an organized list using one-topping, then two-topping, three-topping, and four-topping cases (10/22/13 meeting transcript, lines 532). Another student connected the pizza problem to the towers problem by making towers and placing P in a cube to represent no topping or plain and placing T in a cube to represent a topping (10/22/13 meeting transcript, lines 538).

After all teachers shared their students' samples of work, T2 paired up with T3 to work on the three-tall tower problem and Ankur's Challenge. For the three-tall towers problem, T2 and her partner made 3 groups of 9 three-tall towers. For Ankur's

Challenge, T2 and her partner tried holding the first two cubes constant to create towers but were unable to successfully solve the problem.

The Unit 8 on-line discussion focused on Ankur's Challenge. In response to one teacher, T2 said "I did not see the answer to this problem very quickly either. Even coming up with the way to organize the towers so that you can find out what was missing was difficult, I'm curious to see if our students can solve this problem." (Unit 8, On-line discussion thread, line 5). T2 also responded about whether or not her students would be able to come up with Romina's proof. T2 said the following:

> Based off of the last two problems I gave my students, I do not believe they can. However, they always can surprise me. I do not want to set low expectations of my students so I would like to present them with this challenge and see what they can come up with. They may not come up with Romina's exact proof but they may be able to develop a unique approach of their own. (Unit 8, On-line discussion thread, line 17).

T2 responded to another teacher about giving students more than one opportunity to explain or write their reasoning with the following:

> I like how you called this opportunity a revise and edit, maybe referring it this way to our student may make them understand that we are not asking them to clarify as a punishment but as a way to make them better. (Unit 9, On-line discussion thread, line 36).

On 11/13/13, T2 implemented the third cycle of tasks in her classroom. T2 said the following: "Throughout this final task, I have seen improvement amongst most of my students." (Final Project for T2, p. 28) but expressed concerns about her students' work not matching their final answer for the problem.

The Unit 10 discussion asked teachers to discuss the strategies used by one student over the series of tasks. For Unit 10, T2 wrote the following on-line about her student's work:

> For the first task my student began making opposites as her strategy. When being questioned about how she knew she had them all but she had a difficult time explaining it. I asked her to rearrange the towers in a different way maybe she will see a more concrete explanation. After this she was able to see there was more of a pattern by organizing it using proof by cases. She still had a hard time explaining in writing but she was able to see two different ways to approach the problem. For the second task she had a harder time coming up with a strategy since she wanted to dive into this one by making opposites again. She quickly realized that wouldn't work for this problem. She then tried a tree diagram which, after a while she realized that was difficult to do and she began getting frustrated. She then started to make a list, even with this she did not organize it right away, she finally saw that she was making 1 topping, 2 topping, and etcetera and reorganized her work to make more sense. She was then able to better explain her work in the second task then in the first. For the first task she did not come up with a convincing argument for why she had all the towers. She completed the task correctly with 16 but could not justify her answer. For the second task her justification was more concrete explaining that she started with a different amount of toppings on the pizza until she reached the max which would be 4, she could not have 5 because there weren't 5 options. (Unit 10, On-line discussion thread, line 31).

In response to one of the teachers, T2 said "I feel that the convincing argument is the hardest for the students to get. I feel that once we can get them to understand the problems better their explanations will become better. They are not challenged like this in the everyday classroom but they should be." (Unit 10, On-line discussion thread, line 11).

At the 11/20 regional meeting, T2 shared two students' work for the third cycle. T2 recognized that one of her students used the elevator strategy and controlled for a variable to solve the three-tall towers problem (10/22/13 meeting transcript, lines 82). T2 also shared the partner's work who described using a recursive argument (10/22/13 meeting transcript, lines 134-136). T2 shared the following final reflection thoughts on the intervention:

> I have learned that mathematics is not just basic arithmetic and procedure, that understanding the process behind the math is very important. Since the implementation of these tasks, I have been trying to encourage the "why" behind the math. I have been trying to teach and assess my students' reasoning of math. In the beginning, my students' reasoning and mathematical thinking was little to

> none. They did not question math or even think about explaining math, they just knew what to do and figured that was good enough. Now they are working on improving their reasoning. (Final Project for T2, p. 32)

T2 also said "I find myself questioning my students more instead of leading them toward the answer. I have a different view about my teaching style." (Final Project for T2, p. 32) which is evidence thatT2 made a change in her pedagogical practices.

**10.3 T3**

T3 completed the Beliefs pre-assessment before the start of the first meeting. On the pre-assessment, T3 made 1 inconsistent statement with statements in the category of expectations and abilities, 1 inconsistent statement in the category of concepts and procedures, and 2 statements in the category of teachers' and students' roles.

T3 began the first cycle by working on finding all possible 4-tall towers that can be made selecting from two colors with T2. T2 and T3 originally arranged the four-tall towers using guess and check and the opposite strategies. The teacher pair later rearranged their towers using an inductive argument (9/7/13 meeting transcript, lines 19; 84-90). The instructor asked this pair "Which is it? Two to the fourth power to get 16, or doubled?" regarding how the pair solved the four-tall towers problem. (9/7/13 Meeting transcript, line 96). T3 replied "both" and then the instructor facilitated a discussion with the pair about writing a convincing argument about their solution (9/7/13 Meeting transcript, lines 104-107).

For the unit 2 on-line discussion, T3 had responded to another teacher (T4) with the following:

> I always find that the kids are in such a rush to get anything done. I think that my students will immediately jump right in to the building the towers through trial and error before even thinking or even considering a pattern. [T4], I definitely think it's going to be hard not to help. My resource level students are pretty needy

> and when they do not get something right away, they always ask for help. I definitely think this lesson will be challenging for my students, but also challenging for me to just sit back and watch. (Unit 2, On-line discussion thread, line 14).

Her comments may be evidence of why T3 made statements inconsistent with the standards for the category of students' expectations and abilities and teachers' and students' roles on her Beliefs pre-assessment.

T3 implemented the first cycle task in a seventh-grade pull-out resource class with 12 students in a mathematics class for 40 minutes over two days on September 24th and 25th, 2013 (Final Project for T3, p. 3). T3 said that her students liked working with the Unifix cubes and "were taking the problem very seriously and trying their very best" (Final Project for T3, p. 18). However, T3 reported that she was frustrated because she "struggled with not being able to help and guide them more" (Final Project for T3, p. 19).

For the Unit 3 on-line discussion, one teacher (T7) had written that Stephanie and Dana's argument for the four-tall towers problem in third grade was not convincing. In response to T7's original post, T3 agreed with T7 and said "Neither girl gave an explanation as to why there were 16 total towers. She could not explain her answer and felt 16 was all she could come up with due to the fact that she could not build anymore. She never mentioned any type of argument to why there were only 16 other than that she could not find anymore." (Unit 3, On-line discussion thread, line 7).

For the Unit 4 discussion, T3 responded to another teacher regarding the four-tall tower problem, selecting from two colors. T3 said the following:

> My students definitely struggled with the explanation aspect of this problem too. Several times, they voiced their opinions that they felt confused, exhausted and frustrated. With several pairs of students I was able to verbally understand some of their arguments, but to get them to write any of that down is a different story. One pair of students wrote how they could not create anymore towers because of

> duplicates and the fact that they were exhausted! (Unit 4, On-line discussion thread, line 42).

At the October 2 regional meeting, T3 shared three samples of students' work with the teachers. In the first-shared sample, T3 said that her student separated the towers in six groups. T3 recognized that her student used the elevator strategy to draw the second group of towers and then created the third group by using the opposite colors of the second group (10/2/13 students' work meeting transcript, line 315). In the second sample shared, the student wrote the towers were found by multiplying 2 times four to get 8 and the doubling the towers to get 16 (10/2/13 students' work meeting transcript, line 326). T3 said that "I thought it was pretty good that they immediately jumped to the math of it." (10/2/13 Students' work transcript, line 330) which sparked a debate about students using invalid rules to solve the four-tall towers problem selecting from two colors. T3 shared a third sample where the student had drawn three pairs of opposite towers with the purpose of using the towers to spell out the word math. (10/2/13 meeting transcript, line 389).

Later on at the October 2 regional meeting, T3 paired up with T2 to work on the second cycle task. T3 had written the following comment on the Unit 5 on-line discussion thread:

> When reading this problem, I immediately thought of a tree diagram. However, I was working with [T2] and she suggested we start with a list. Together, we created an organized list according to the number of toppings on the pizza (plain, 1, 2, 3 or 4 toppings). We used different letters to represent the different toppings (example: P= peppers, S= sausage, M= mushrooms & R= pepperoni). After we created the list, we were able to see 16 different pizza combinations. After thinking about the problem, I attempted to use my initial thought of the tree diagram. After going back to make the tree diagram, I realized it was much harder than I had originally thought. I found myself making duplicate combinations and that it was much harder to follow and see the different pizza combinations. In the

> end, I think the organized list was the better approach. (Unit 5, On-line discussion thread, line 11).

During the Unit 6 on-line discussion, T3 responded with the following to T2 about her students were struggling to write clear explanations: “They are verbally able to explain certain concepts, but then to get them to put it down on paper is a completely different story.” (Unit 6, On-line discussion thread, line 3). T3 had admitted that her students were able to give verbal arguments but struggled to provide written arguments (Unit 6 On-line discussion thread, line 3).

On October 8, 2013; T3 implemented the second cycle task in her classroom. T3 wrote that half of her students were successful at solving the pizza problem and the other half of her students were unsuccessful (Final Project for T3, p. 18). T3 said the following in her final project about the students who were unsuccessful at solving this problem:

> The other half that was unsuccessful with this task, I felt struggled to solve this problem due to the lack of manipulatives. This was an abstract problem, in which the students needed to visually see and create the different pizzas in their mind or on paper, rather than physically creating them with their hands and in front of them. If given the opportunity to complete this task again, I would definitely create the four different pizza toppings out of manipulatives.” (Final Project for T3, p. 37)

Her comments may be evidence supporting statements consistent with the standards for the manipulatives category. In response to one of the teacher’s (T1) post that T1’s students had a lot of difficulty solving the pizza problem, T3 wrote the following on-line:

> Wow, I thought the complete opposite. I thought for the most part, my students were able to solve this problem a lot quicker and more easily. I felt as though this was something that they could actually relate too since they all love pizza. I did have several groups suggest that sausage and pepperoni was a different pizza then pepperoni and sausage. However, after discussing it with them, I related it back to if they were to actually order the pizza, would they taste different if you ordered them two different ways and they immediately understood that the order did not matter. (Unit 7 On-line discussion thread, line 14)

T3 had discussed the variety of representations that the students used. Most of T3's students made a list; but one pair of students drew circular diagrams to represent each pizza combination and one student pair used a tree diagram (Unit 7, On-line discussion thread, line 41).

At the regional meeting on October 22, 2013; T3 shared two samples of students' work. One student made an organized list using a system that the student described to find the possible pizzas (10/22/13 meeting transcript, lines 489). In response to another teacher's (T5) original post, T3 described the system used by one pair of her students in the following way:

> One pair of my students used a recursive argument as well. They kept telling me they used a "system" in order to list the different pizzas. When explaining their "system" they showed me how they created the different pizzas using arrows from topping to topping. (Unit 7 on-line discussion thread, line 31)

T3 shared a second example where the student started with a tree diagram but then changed to make a list of 13 possible pizzas (10/22/13 students' work transcript, lines 504-505).

After all teachers shared their students' samples of work, T3 paired up with T2 to make 3 groups of 9 three-tall towers on the three-tall tower problem and found 28 towers for the Ankur's Challenge problem holding the same color constant for the first two positions. For the Unit 8 on-line discussion, T3 said the following:

> I have a feeling that this problem, would take my students several periods to complete. I am hoping their explanations will be more thorough now that we have been practicing more, but most often they are very basic. Their explanations either consist of what they did to arrive at their answer rather than why. I am as well expecting to hear lots of opposite and pattern explanations from my students. (Unit 8, On-line discussion thread, line 11).

Although T3 did not expect her students to come up with Romina's proof on their own, T3 said that she liked the idea of using the problem to challenge her students (Unit 8, On-line discussion thread, line 18).

On 11/13/13, T3 implemented the third cycle of tasks in her classroom. T2 said "By cycle three, I felt that they implemented more of a strategy for this task than when they had completed the first cycle. This time around, they were able to explain themselves better." (Final Project for T3, p. 55) and found more success at solving this problem. T3 also responded to another teacher (T1) about the importance of giving more than one opportunity for students to explain and write their reasoning when solving problems. T3 wrote on-line that "It is definitely important for us to address multiple ways of solving various math concepts." (Unit 9, Meeting transcript, line 9).

For the Unit 10 discussion regarding a student's work over the first two tasks, T3 wrote the following on-line about her student's work:

> For the first task, her justification was more of what and how she created the different towers versus why she had created all 16 towers and why she could not create any additional towers. She struggled to not only verbally give me an explanation, but she also struggled to write a convincing argument. For the second task, she was definitely able to verbally explain her "arrow" method to me in which she used every pizza topping in a different position. Her transition from verbally to physically being able to write down her explanation was still a struggle. I have found that over the two tasks, my students have had more success verbally explaining from task one to task two. I have definitely seen a lot of growth in them from task one to task two. (Unit 10, On-line discussion thread, line 16).

Moreover, T3 wrote an on-line response to one of the teachers that "having the actual unifix cubes physically in front of them helped my students. In task two, they had no visuals to help them create the different pizza combinations." (Unit 10, On-line discussion thread, line 2). Her comments may be evidence of why T3 made statements consistent with the standards for the manipulatives category on her Beliefs pre-assessment.

At the 11/20 regional meeting, T3 shared two students' work for the third cycle. T3 recognized that one of her students controlled for a variable on the bottom of the towers with each color making three groups of nine 3-tall towers to solve the three-tall towers problem (10/22/13 meeting transcript, line 491). T3 also shared a second student's work where the students began using a tree diagram but "realized they were having too many duplicates" (10/22/13 meeting transcript, lines 504-506) and changed to a list of thirteen possible pizza combinations. T3 shared the following final reflection thoughts on the intervention:

> As far as their reasoning and arguments are concerned, I learned that at the very beginning I needed to really push the idea of giving a convincing argument. My students definitely struggled with this, but by the third task; they knew what was expected of them. They had learned through these tasks that they needed to organize their thinking on paper and include details on what and why they did what they did. (Final Project for T3, p. 57)

T3 also said "I plan on continuing to implement similar tasks throughout the remainder of the school year as well as for years to come." (Final Project for T3, p. 32) which indicated thatT3 intended on changing her current and future pedagogical practices.

**10.4 T4**

On the beliefs pre-assessment completed before the first meeting, T4 made 1 statement, inconsistent with the Standards, in the category of concepts and procedures and another in the category of manipulatives. T4 began the first cycle by working on finding all possible 4-tall towers that can be made selecting from two colors with T8. T4 and T8 originally arranged their towers the four-tall towers using a recursive argument for finding ten towers and using the opposite strategy for the remaining 6 towers; but then later rearranged their towers by controlling for a variable of red on the bottom for eight 4-tall towers (9/7/13 meeting transcript, lines 119-124).

For the unit 2 on-line discussion, T4 had responded to another teacher (T10) about predictions for the four-tall towers problem with the following:

> My students will also start off building the towers without a reason. It's going to be difficult to hold back and not help them! But I think it will be great for them to struggle to compare and contrast and find a pattern. (Unit 2, On-line discussion thread, line 13).

T4 implemented the first cycle task in an eighth-grade general education class with 7 students in a mathematics class for 80 minutes (Final Project for T4, p. 2). T4 said that her students "were diligently working and persevering till they found all the tower combinations" (Final Project for T4, p. 9). T4 also wrote in her final project that "I have found myself asking more thought-provoking questions and doing a lot less leading towards the correct answer." (Final Project for T4, p. 9) and which indicated a change for T4 in pedagogical practices.

For the Unit 3 on-line discussion, T4 responded in the following way to another teacher (T7) about the arguments the students provided for solving the four-tall tower problem:

> No matter what group of students I listened to, I saw them all talk about opposite pairs, patterns, diagonal movements, and recursive patterns but all were unclear as to how to explain why they did what they did. This kind of reasoning is an essential mathematical practice that I need to work on with my students. (Unit 3, On-line discussion thread, line 34).

For the Unit 4 discussion, T4 responded to another teacher (T3) regarding the four-tall tower problem, selecting from two colors. T4 also had written the following:

> As I moved around the room and questioned students about their reasoning and the steps they took to construct their towers I saw that they had a very difficult

> time putting their reasoning into words – I frequently got "I don't know; I just did it," as an answer. (Unit 4, On-line discussion thread, line 36).

T4 expressed that students struggled to provide any written arguments.

At the October 2 regional meeting, T4 shared two samples of students' work with the teachers. In the first-shared sample, T4 said that her student had the towers in four groups. T4 recognized that her student used the diagonal strategy to draw the first group of towers and used the opposite colors to create the towers for the second group of towers. The third and fourth groups of towers were created using the opposite strategy of two of each color and two towers where one tower had all yellow cubes and one tower had all red cubes (10/2/13 students' work meeting transcript, line 415). In the second sample shared, T4 had described that the student had "saw the step thing going down" (10/2/13 students' work meeting transcript, line 469) regarding the elevator pattern.

Later on at the October 2 regional meeting, T4 and T5 worked together on the second cycle task. T4 had written the following comment on the Unit 5 on-line discussion thread:

> The first instinct when faced with the pizza problem is to make a list. I found that writing out the entire word wasn't very efficient and my partner's labels of P, M, S, and R were much easier to use. After we determined that 16 pizzas could be made I tried to think of what my students would do and came up with a tree diagram with the four headings of P, M, S, and R and side labels of 1st pizza, 2nd pizza, and 3rd pizza (the fourth option is all four toppings so it doesn't need to be repeated at the bottom of every branch of the tree diagram). After completing the Pepper branch I realized I needed to make a list of the combinations and from that list I realized I needed to cross out the duplicates. This was a very long process and did not prove to be more efficient than listing the combinations in letter form." (Unit 5, On-line discussion thread, line 30).

T4 found that there were too many duplicates using a tree diagram and abandoned using this representation (10/2/13 meeting transcript, lines 174-177).

During the Unit 6 on-line discussion, T4 responded with the following to T2 about how the questions asked by the teacher in the video helped Brandon with his conceptual understanding:

> Brandon had a great method for solving the problem but I think that being pushed to explain his process in detail gave him more confidence in his reasoning. Confidence is something most of my students lack and I am going to try to use this kind of questioning technique with them. (Unit 6, On-line discussion thread, line 23).

T4 expressed that probing questions helped Brandon explain his argument (Unit 6 On-line discussion thread, line 27).

On October 15, 2013, T4 implemented the second cycle task in her classroom. T4 had written "If I can avoid sharing my opinion during this discussion, my students might feel more confident in their choices just from peer reinforcement. If anything, my students will be able to talk to each other and share their thoughts." (Final Project for T4, p. 19) and T4's statements may be evidence consistent with the standards for the category of teachers' and students' roles. In response to one of the teacher's (T9) post that T9 was confused in the way students used webs to solve the pizza problem, T4 wrote the following on-line: "A lot of my student did webs too - I think it's because they knew they were looking for combinations and have be programmed to use tree diagrams instead of any other solution method." (Unit 7, On-line discussion thread, line 14).

At the regional meeting on October 22, 2013; T4 shared two samples of students' work. One student wrote circles to represent pizzas and used letters to represent toppings to list the pizza combinations. Another student began the second cycle task using a tree diagram but then stopped because the student "was running into so many problems" (10/22/13 meeting transcript, lines 464). According to T4, the student was able to fix his work after hearing the comments made to the first student (10/22/13 meeting transcript, lines 464).

After all teachers shared their students' samples of work, T4 paired up with T6 to make 9 groups of 3 three-tall towers using an inductive argument for the three-tall tower problem and successfully found 36 towers for the Ankur's Challenge problem with six groups of six 4-tall towers. For the Unit 8 on-line discussion, T4 said the following:

> I agree with holding high expectations for our students. Even though many of us have special education students they are still just as capable as anyone else of solving the problems. We may need to give them more time or let them work it out in their own unique way. Holding back has never been my strength as a teacher (I just want to be helpful always!) but I'm slowly practicing letting go and holding back. I figure it will only make my students stronger and more independent and maybe make me a little less stressed. (Unit 8, On-line discussion thread, line 21).

T4 also did not expect her students to come up with Romina's proof. However, T4 said her students have surprised her with their work on the tasks and that she might try Ankur's challenge before Thanksgiving. (Unit 8, On-line discussion thread, line 2).

In response to one of teachers (T6) regarding the importance of giving students more than one opportunity to explain and write about their ideas, T4 wrote the following on-line:

> I find myself constantly asking my students, "Do you think that's a convincing argument?" It's great because I've trained myself and them to expect more detail from their explanations. They are editing they're thinking and giving me full answers. It has also helped to open up more dialogue between students and I have found they are much more willing to work together and help each other through the problem solving process. (Unit 9, On-line discussion thread, line 25).

T4's comments are consistent with the standards for teachers' and students' roles.

Regarding a student's work over the first two tasks, T4 wrote the following on-line about her student's work:

> My student did not stick with the same problem solving strategy for the two tasks. The first task building towers was much easier for him because the manipulatives provided an opportunity to make guesses and mistakes without the finality of writing it on paper. The second task proved difficult to him because he was very reluctant to write down anything he wasn't extremely sure of. When he did write something down it was a tree diagram which frustrated him quickly. He did not stick with that strategy and instead chose to list out the pizza combinations but did not have great organization. (Unit 10, On-line discussion thread, line 25).

In response to another teacher (T5), T4 wrote the following on-line:

> I am hopeful that my students will be much more convincing in their arguments for the third task. The manipulatives definitely give them more confidence or maybe they just like playing with blocks? Either way it's a great way to get them to do math and shows us their thinking processes. (Unit 10, On-line discussion thread, line 3).

T4's comments are consistent with the standards in the category of manipulatives.

On 11/19/13, T4 implemented the third cycle of tasks in her classroom. T4 wrote in her final project that "This task proved to be more difficult for my students than the first tower problem." (Final Project for T4, p. 27) and said she hoped that other students' work would motivate some of her students to try to solve the third cycle tasks. At the 11/20 regional meeting, T4 shared two students' work for the third cycle. T4 shared that one of her students controlled for a variable by separating the towers by their color on top to solve the three-tall towers problem (11/20/13 meeting transcript, lines 230-231). T4 also shared a second student's work where T4 said that "He struggled with what he wanted to say but he did give an example." (11/20/13 meeting transcript, line 240). T4 shared the following final reflection thoughts on the intervention:

> I was pleasantly surprised and impressed by the small steps my students made towards solving the problems from each cycle but I am very concerned with the lack of ability to explain and give a convincing argument. (Final Project for T4, p. 28)

T4 also wrote that she planned on giving her students the three tasks again to compare how their thinking changed from the beginning to the end of the year (Final Project for T4, p. 28).

**10.5 T5**

On the pre-assessment, T5 made 18 statements consistent with the standards and no statements inconsistent with the Standards. T5 began the first cycle by working on finding all possible 4-tall towers that can be made selecting from two colors with a teacher from the northern and central cohort group. T5 originally arranged their towers using the opposite strategy but then rearranged the towers using a staircase strategy (9/7/13 meeting transcript, lines 192-204).

For the unit 2 on-line discussion, T5 had responded to another teacher (T10) about predictions for the four-tall towers problem with the following:

> I am going to try this activity in my resource classes, and at this point of the year I can already tell that each class as a whole is at very different levels. My concern with this project is that I feel some of my students will become frustrated and give up or just build random towers. They are used to a lot of guided instruction, so this should be interesting... (Unit 2, On-line discussion thread, line 21).

T5 implemented the first cycle task on September 20, 2013 in a seventh-grade resource class with all classified students in a mathematics class for 45 minutes (Final Project for T5, p. 2). T5 had written the following: "Even though a majority of my students had difficulty proving their point, they all had the general idea. I consider that a success because they are not used to explaining their math reasoning, and were all able to get 16 towers." (Final Project for T5, p. 7) and which is consistent with the standards in the category of expectations and abilities.

For the Unit 3 on-line discussion, T5 wrote the following from an original post about the students' arguments during the in-district classroom visit for solving the four-tall tower problem: "I really liked how some of the students reasoned their answers and organized their towers. The one group that caught my eye was the group that changed the position of each color so that they could determine the amount of towers and any duplicates." (Unit 3, On-line discussion thread, line 30). For the Unit 4 discussion, T5 responded to another teacher (T3) regarding the four-tall tower problem, selecting from two colors. T5 had written the following:

> I agree that there were elements of this activity that were confusing for my resource level students. I had to explain several times what the question was asking because some of my students could not grasp the concept at first. Also, I had to explain what a duplicate looked like, and how to draw out the towers. However, once we were into the activity, which took me several days too. The students started to get the hang of it. They all did not get the correct answer but that's ok. On a whole they all had a difficult time explaining their reasoning. I had to do a lot of prompting to help them. (Unit 4, On-line discussion thread, line 25).

T5 expressed that students struggled to explain their arguments and T5 admitted to prompting the students to help them solve the first cycle task (Unit 4, On-line discussion thread, line 25).

At the October 2 regional meeting, T5 shared two samples of students' work with the teachers. In the first-shared sample from T5, the student had arranged the towers in four groups where the first group had a red diagonal going down and the second group had a yellow diagonal going down. The other eight towers were arranged in two groups using the opposite strategy (10/2/13 Meeting Transcript, line 415). T5 recognized that her student used the diagonal strategy to draw the first group of towers and used the opposite colors to create the towers for the second group of towers. The third and fourth groups of towers were created using the opposite strategy of two of each color and two towers where one tower had all yellow cubes and one tower had all red cubes (10/2/13 students' work meeting transcript, line 415). In the second sample shared, T5 had described that the student had "saw the step thing going down" (10/2/13 students' work meeting transcript, line 469) regarding the elevator pattern.

Later on at the October 2 regional meeting, T5 and T4 worked together on the second cycle task. T5 had written the following comment in the Unit 5 on-line discussion thread: "We both started off breaking the pizzas in to plain and all topping similar to beginning of the block activity. I also found that we both organized our answer so it was clear that they were no duplicates." (Unit 5, On-line discussion thread, line 14). For the Unit 6 on-line discussion, T5 made the following original post about how the questions asked by the teacher in the video helped Brandon with his conceptual understanding:

> After the students have worked on the activity for a while the one question I think is important to ask to grasp their beginning stages of their thinking is, "Tell me what you have done so far." This question is vital because it is a springboard for their further answers. I also think this is the type of question students won't automatically feel they are doing the activity wrong, and be afraid to answer. Then after they have worked further on the activity asking them to explain/convince me of their work is much easier. I feel it is much easier because they will probably be more confident at this point because they answered the prior question and have more understanding of the task. Also, since you have some background knowledge it will be easier to continue to ask more questions because you can refer back to their previous answer. I really feel that in order to understand a child reasoning it is not so much the question you ask, but I how the question is presented to the child. I learned that through the last activity that each child may need the same type of question presented differently in order to grasp their full reasoning of the activity. (Unit 6, On-line discussion thread, line 43).

T5 also responded to another teacher (T8) that the assigned article to read called *Brandon's Proof and Isomorphism*, gave T5 ideas about how to ask students effective questions (Unit 6 On-line discussion thread, line 41).

On October 21, 2013, T5 implemented the second cycle task in her classroom. T5 had written the following in her final project:

> I really enjoyed seeing how my students chose to organize their work. I had some students use tree diagrams, listing, creating drawings, and making charts. I also think this activity was a little more difficult because there were not many manipulatives to use. So the students had a hard time telling if they had a duplicate. Many of my students did not come up with an accurate solution, but they showed improvement on their organization of their mathematical findings. (Final Project for T5, p. 13)

In response to one of the teacher's (T1) original post about difficulties that T1's students experienced, T5 wrote the following on-line: "A lot of my students had difficulty with this problem because I feel the way it is worded could be interpreted differently. Also, I feel because there were no manipulative this activity was harder." (Unit 7, On-line discussion thread, line 12). T5's comments supported statements consistent with the standards for the manipulatives category.

At the regional meeting on October 22, 2013; T5 shared two samples of students' work. One student made a chart and used topping words at the top of the chart to represent 16 pizza combinations (10/22/13 students' work meeting transcript, lines 307). Another student made a numbered list of ten possible pizza combinations and held peppers as a constant (10/22/13 meeting transcript, lines 331).

After all teachers shared their students' samples of work, T5 paired up with T6 to make 9 groups of 3 three-tall towers using a recursive argument for the three-tall tower problem and successfully found 36 towers for the Ankur's Challenge problem with six groups of six 4-tall towers. For the Unit 8 on-line discussion, T5 wrote the following in response to T2:

> I don't want to set low expectations for my students either. I also agree that they may not be able to come up with Romina's proof, but even if they came up with

> their own proof and it was not correct I would take that as a success because it is a challenging problem. (Unit 8, On-line discussion thread, line 24).

T5 also responded to another teacher (T10) on-line that she did not expect her students to come up with Romina's proof. However, T5 had written "My students may not come up [with] the solution right away, but am also surprised by their work and their solutions." (Unit 8, On-line discussion thread, line 7).

In response to one of teachers (T1) regarding the importance of giving students more than one opportunity to explain and write about their reasoning, T5 had written the following on-line: "I think it is also helpful to let students represent their work in several ways because they are able to see any mistakes and also have the chance to really understand what they have done." (Unit 9, On-line discussion thread, line 25). T5's comments are consistent with the standards for mathematical discourse and differentiated instruction.

Regarding a student's work over the first two tasks, T5 wrote the following on-line about her student's work:

> The first assignment with the cubes my student initially chose the strategy of making a pattern. The pattern started off random and then eventually he organized the cubes to show his work clearer. As for the second problem my student automatically made the connection to the first problem. He then tried to solve it mathematically by multiplying 4 x 4. However, he was not able to justify his answer. He then started with a tree diagram and then switched to a chart to solve the problem. (Unit 10, On-line discussion thread, line 1)

In response to another teacher (T3), T5 wrote the following on-line: "I have also seen growth in my students when they have to verbally explain their results. I see that they are able to elaborate a little more than in the past." (Unit 10, On-line discussion thread, line 19). T5's comments are consistent with the standards in the category of expectations and abilities.

On 11/14/13, T5 implemented the third cycle of tasks in her classroom. T5 wrote in her final project that "I thought that the organization factor in the task would be challenging for them, but they all impressed me with the variety of strategies they used." (Final Project for T4, p. 18) regarding the cycle 3 tasks. At the 11/20/13 regional meeting, T5 shared two students' work for the third cycle. T5 shared that one student estimated 31 towers with her partner but did not provide a convincing argument. In a second sample, T5 recognized that the student controlled for a variable by having three groups of nine 3-tall towers for the 3-tall problem (11/20/13 meeting transcript, line 278).

T5 shared the following final reflection thoughts on the intervention: "It is nice to see what the freedom of thinking can show me. They also have improved on explaining their work and it is starting to come naturally to them." (Final Project for T5, p. 20) concerning students' mathematical discourse. T5 also wrote that her questioning skills improved. T5 wrote the following in her final project "Often I give too much away when I question, or guide my students too much. I know that I do it, but after this class I am able to question my students without fully leading them to the answer (Final Project for T5, p. 20). T5's comments may be evidence for statements consistent with the standards for the category of teachers' and students' roles.

**10.6 T6**

On the pre-assessment before the first meeting, T6 made 1 statement, inconsistent with Standards, in the category of expectations and abilities and another statement in the category of teachers' and students' roles. At the first meeting, T6 worked on finding all possible 4-tall towers that can be made selecting from two colors with T7. T6 and T7 originally arranged the towers using the opposite strategy but then rearranged the towers by controlling for a variable (9/7/13 meeting transcript, line 52).

For the unit 2 on-line discussion, T6 had responded to T9 for predicting students' solutions for the four-tall towers problem with the following:

> You make a great point about students jumping in and creating towers randomly! I can see how this could discourage and overwhelm students. I would be weary of saying something to the student too in fear of stealing his/her ah-ha moment. I think some of my more unorganized or aloof students I might pair with a more structured student. (Unit 2, On-line discussion thread, line 26).

T6 implemented the first cycle task on September 26, 2013 in a sixth-grade gifted class with 27 students in a mathematics class for 60 minutes (Final Project for T6, p. 2). T6 had written the following:

> My students struggle with translating their mathematical thinking into words. Many students had a difficult time writing down what they were able to verbalize. My students also had a difficult time formulating a 'convincing argument'. After reading their work and observing them in class, I came to the conclusion that many of them did not truly understand what it meant to have a convincing argument. (Final Project for T6, p. 11)

T6 also wrote that she found it difficult to "not lead the students during this problem" (Final Project for T6, p. 11).

For the Unit 3 on-line discussion, T6 wrote the following in response to an original post from T3 about the comparison of students' solutions from the second to the third grade for the shirts and pants problem: "I think the way students choose to organize their thought can give us some insight to how they are thinking and can help us interpret their mathematical reasoning. Do students improve their organizational skills because their mathematical reasoning skills have improved or because they have been trained?"(Unit 3, On-line discussion thread, line 24). For the Unit 4 discussion, T6 responded to another teacher (T3) regarding the four-tall tower problem, selecting from two colors about the struggle students have when writing down their reasoning. T6 had written “I think students rarely associate writing with math and when asked to explain their thinking the students often do not see the point of doing it. It is hard to motivate students to write a reflection on their mathematical thinking or processes sometimes.” (Unit 4, On-line discussion thread, line 18).

At the October 2 regional meeting, T6 shared three samples of students’ work with the teachers. In the first-shared sample from T6, the student provided a recursive argument by showing step-by-step drawings and explanations of how this student moved the cubes (10/2/13 Meeting Transcript, line 560). T6 also shared the partner’s explanation because T6 decided the partner’s explanation for the case of two of each color was better (10/2/13 Meeting Transcript, line 565). In a third-shared sample, T6 recognized that her student used the opposite and elevator strategies to create the towers but had called the strategy a staircase (10/2/13 students’ work meeting transcript, line 596).

Later on at the October 2 regional meeting, T6 and T7 worked together on the second cycle task. T6 had written the following comment in the Unit 5 on-line discussion thread:

> [T7] had the idea of keeping a constant. So we did all pizzas with peppers, all with mushrooms, all with pepperoni and all with sausage. We found as we eliminated an ingredient the number of possibilities were halving (just like the tower problem!). From there we decided to replicate what the towers would look like by having four possible spots. If the pizza did not occupy all of the spots with an ingredient we would put an X and if it did have an ingredient we would put the representation we came up with. As I as looking at it I noticed we did not even need to differentiate between the ingredients (m, i, p, s) when organizing this method. If you are using the unifix cubes this way with one color is representing a topping and one color representing the absence of a topping. (Unit 5, On-line discussion thread, line 1).

For the Unit 6 on-line discussion, T6 responded to T3 about the representation Brandon used to solve the pizza problem: "The way a student chooses to represent his/her work can give us insight on his/her mathematical thinking." (Unit 6, On-line discussion thread, line 9).

On October 22, 2013; T6 implemented the second cycle task in her classroom. T6 wrote the following from her original post on-line:

> My students used many different strategies to solve this problem. Many partners started out creating tables. The tables were organized either by the type of topping or by the number of toppings. I thought this was an interesting strategy. There were tons of questions about the "rules" for the pizzas. My students are very creative and it was very difficult for me to not give them stipulations about duplicates or order. Most students naturally realized that the order of the toppings does not change the pizza. Some students created tree diagrams, but most who started with this organization abandoned it. Students used a variety of letters or some used abbreviations to represent the different toppings. (Unit 7, On-line discussion thread, line 28).

T6 also had written the following in her final project:

> I was doing a better job at not leading the students. I found myself asking better questions to get students to re-think things, without giving everything away. I was successful when I suggested students to re-write their pizza possibilities, or try a different organization. (Final Project for T6, p. 26)

T6's comments are consistent with the standards for the category of teachers' and students' roles.

At the regional meeting on October 22, 2013, T6 shared three samples of students' work. One student made a tree diagram to find 16 pizza combinations but did not provide a written argument (10/22/13 meeting transcript, line 162). Another student incorrectly used the rule strategy to get the answer by multiplying four times four (10/22/13 meeting transcript, lines 189). In the third-shared sample, T6 recognized that her student controlled for a variable (10/22/13 meeting transcript, line 200).

After all teachers shared their students' samples of work, T6 paired up with T5 to make 9 groups of three, 3-tall towers, using an inductive argument for the 3-tall tower problem and successfully found 36 towers for the Ankur's Challenge problem with six groups of six 4-tall towers. For the Unit 8 on-line discussion, T6 said the following regarding predictions of how T6's students will solve the third cycle task:

> I think my students will start initially by building the towers. I will be interested to see how the students organize this. In the first tower problem, the students just started building, without a real "plan of attack." I think if they approach this problem the same way they will have a difficult time building all the towers and I'm not sure if they will put much thought into a strategy or organizational method; however, I have been surprised in the past! Some of my students are capable of approaching this problem with great strategy, but that is considering if they approach it with a strategy at all. (Unit 8, On-line discussion thread, line 36).

In T6's original post regarding the importance of giving students more than one opportunity to explain and write about their reasoning, T6 had written the following on-line:

> Even if students do not make mistakes, taking an additional opportunity to explain or write an idea might prompt them to change something about their explanation and help them make a new discovery. With the first tower problem, most students just started building towers and the opposites. When given an opportunity to explain their thinking again, students realized this was not very convincing and were able to build a stronger argument. This process is helping students develop a deeper understanding of the mathematics. Many times in class I challenge students to solve a problem differently than how they originally did. (Unit 9, On-line discussion thread, line 25).

T6's comments are consistent with the standards for mathematical discourse.

Regarding a student's work over the first two tasks, T6 wrote the following on-line about her student's work:

> For the first tower task, my student created a "staircase pattern" and was able to form a convincing argument about the group of towers with three of one color and one of another. For the other towers four tall, the student was not able to make a convincing argument and relied heavily on the opposite reasoning. The student

> explained that he had achieved all of the towers because each had an opposite. As we moved onto the second tower problem, this student approached the problem with more strategy. He and his partner kept a color constant when working with three colors instead of two. This student was able to write a convincing argument explaining his strategy and justifying how he had all of the towers. In comparing these two strategies, I believe the student refined his strategy for the second tower task. I believe this was partly because of his experience with the first task. For the pizza problem, this student created a table to keep track of the types of pizza. (Unit 10, On-line discussion thread, line 1).

In response to another teacher (T1) about students knowing they should have a strategy before attempting the third cycle tasks, T6 wrote the following on-line: "I felt my students also had much more strategy approaching the task. They knew from the first task that it was not difficult to build all the towers, but to make a convincing argument was very challenging and required organization and strategy." (Unit 10, On-line discussion thread, line 9). T6's response is consistent with the standards in the category of expectations and abilities.

On 11/6/13, T6 implemented the third cycle of tasks in her classroom. T6 wrote the following in her final project: "The growth in the students from the first tower problem to this problem was apparent across the board. Students approached the problem with strategy and used their past experiences to help them formulate a convincing argument." (Final Project for T6, p. 34) and asked T6 if she would extend class time in order to finish the task. T6 wrote the following in her final project "During the first tower problem, when it came time to write, most students hesitated to start and struggled to put any thoughts on paper. During this tower problem, when it came time to write an argument, students were busy writing or collaborating with their partners." (Final Project for T6, p. 34) and this marked a change with her students' ability to write convincing arguments.

At the 11/20 regional meeting, T6 shared two students' work for the third cycle. In the first-shared sample, T6's recognized that her student used a combination of the elevator and opposite strategies with 7 groups labeled A through G (11/20/13 meeting transcript, lines 366-370). In a second sample, T6 recognized that the student controlled for a variable by first keeping two colors on the bottom constant, then kept the top two the same and changed the bottom for the 3-tall problem (11/20/13 meeting transcript, lines 394-398). T6 shared the following final reflection thoughts on the intervention:

> With practice, students improved their abilities to write a convincing argument and they also learned what is expected in a convincing argument. I watched students recognize that restating what they did was not necessarily convincing. It was evident in the third problem that students had a better grasp on being convincing. The students approached the problem with strategy (Final Project for T6, p. 35).

**10.7 T7**

Before the first meeting, T7 made 1 statement inconsistent with Standards in the category of concepts and procedures and another in the category of teachers' and students' roles on the pre-assessment. At the first meeting, T7 and T6 worked together to arrange the towers using the opposite strategy first and then rearranging the towers by controlling for a variable (9/7/13 meeting transcript, line 52).

T7 had responded to T10 for the second unit discussion about predicting students' solutions for the four-tall towers problem. T7 wrote "I agree that some students will start building towers without any strategy to ensure they have all possibilities. It will be interesting to see how the students start to arrange their towers once they build them without a method and how they notice, or lack noticing, which outcomes are missing if they did in fact miss some of the possibilities which would be expected." (Unit 2, On-line discussion thread, line 16).

T7 implemented the first cycle task on September 17, 2013 in an eighth-grade regular class with 20 students in a mathematics class for 80 minutes (Final Project for T7, p. 2). T7 had written the following: “Many students used proof by cases or the recursive argument but it took some time for them to get away from the thought of pairs of opposite towers in order to better develop their reasoning. Once students started regrouping their towers, they were able to develop better arguments.” (Final Project for T7, p. 11) and “it was a challenge not to direct students to a solution” (Final Project for T7, p. 11).

For the Unit 3 on-line discussion, T7 wrote the following in response to T6 about the way students organized their towers:

> I agree that it was interesting to see how the students organized their towers. Their argument became easier or harder based on how they grouped the towers and when they started to look at the towers in a different way, sometimes they were able to come to a more convincing argument. When I worked with my other class on the activity, many of them started to use the recursive argument as they moved the groupings around. Most of them were focusing on opposites for a while. (Unit 3, On-line discussion thread, line 29)

For the Unit 4 discussion, T7 responded to T6 about the struggle students have when writing down their reasoning as they worked on the four-tall towers problem. T7 wrote the following:

> My students also struggled with writing their argument. They could explain it to me, but when I asked them to write it they would ask what to write down and I would tell them to write exactly what they told me. Many of them just didn't want to take the time to write it on paper. (Unit 4, On-line discussion thread, line 17).

This was discussed by some of the teachers who experienced this common issue with their students (Unit 4, On-line discussion thread, lines 13, 15, 17, 18)

At the October 2 regional meeting, T7 shared three samples of students’ work with the teachers. In the first-shared sample from T7, the student provided a recursive argument verbally but T7 recognized that the student described the opposite strategy in their written argument (10/2/13 Meeting Transcript, lines 635-653). T7 shared a second example where the student used a rule strategy that gave them 16 but the written argument was not convincing (10/2/13 Meeting Transcript, lines 673-676). In a third-shared sample, T7 recognized that her student had written the towers horizontally in pairs (10/2/13 students' work meeting transcript, lines 683-687).

Later on at the October 2 regional meeting, T7 and T6 worked together on the second cycle task. T7 had written the following comment in the Unit 5 on-line discussion thread:

> [T6] and I started with the plain pizza, and then moved to the possibilities for one topping, two toppings, three toppings, and four toppings. We both developed notation for the peppers and pepperoni which differed from each other. After being encouraged to look at how this compared to the tower problem, we looked at having a constant. Starting with peppers, I listed the pizzas of one topping, two toppings, and three toppings containing peppers. I then moved to mushrooms, without using the peppers again since they had previously been listed. The pizzas with sausage were next, then the pizzas with pepperoni. Following this, I created what would look like towers, using the top block to represent peppers, second block to represent mushrooms, third for sausage, and fourth for pepperoni, placing an "x" in a position if that topping was not on the pizza. (Unit 5, On-line discussion thread, line 4).

T7 responded with the following to T2 about the teachers' questioning of Brandon's work for the pizza problem: "The more students are questioned and encouraged to explain their reasoning, the better their arguments become. I also find that questioning can help students find their own misconceptions and correct them without being told that they were incorrect." (Unit 6, On-line discussion thread, line 28).

On October 18, 2013, T7 implemented the second cycle task in her classroom. T7 had written the following in her final project about the myriad interpretations from students regarding the pizza problem:

> I was very surprised by the difficulty experienced by the students in solving this problem. I did not expect the students to analyze the problem considering different slices or sections of the pizza as being different from each other. This task showed me the effect real world experience can have on students' approaches to problems. I saw this student who had experience in a family owned pizzeria as well as in the students who considered half pizzas because they had previously ordered pizzas with one half different than the other half. I was surprised to see so many students consider half pizzas, pizzas without cheese, and plain as a topping option. (Final Project for T7, p. 23)

T7 wrote the following from her original post on-line about the plethora of representations and strategies used by her students:

> Some used lists; others drew diagrams of pizzas, while others tried to use tree diagrams. There were students who assigned a number to each of the toppings, others assigned letters; and some wrote the words out. Only a few students attempted the problem using cases. Many of them held a constant and moved forward with the remaining options. They were able to explain why a topping was no longer used after they had exhausted all the possibilities with that topping. (Unit 7, On-line discussion thread, line 40).

T7 also mentioned that some students were still struggling to write their arguments on paper; but said that for future tasks she would try having the students "dictate their reasoning as the other person in the pair records the reasoning on paper" (Final Project for T7, p. 23).

At the regional meeting on October 22, 2013, T7 shared three samples of students' work. One student had 145 pizza combinations on their paper because the student was counting each slice as having a different topping (10/22/13 meeting transcript, line 397). Another student divided the pizzas into quarters that could have different toppings in each quarter (10/22/13 meeting transcript, lines 422). T7 shared a third sample of student work where the student listed 16 possible combinations (10/22/13 meeting transcript, line 430).

After all teachers shared their students' samples of work, T7 paired up with T8 to use a recursive argument to make 9 groups of 3 three-tall towers for the three-tall tower problem. Also, T7 and T8 successfully found 36 towers for the Ankur's Challenge problem with three groups of twelve 4-tall towers separated by controlling for a variable on top with the three colors. T7 wrote the following regarding whether her students would come up with Romina's proof for Ankur's Challenge:

> I would be very surprised if my students approached the problem the way Romina did. I had not thought about this approach until another group demonstrated it at the board. Observing how they have approached the past problems, most of my students seem to take a random approach with no organization. (Unit 8, On-line discussion thread, line 34).

T7 also responded to T1's original post about the importance of using different representations in the following way: "It was interesting to see her diagrams as well as her writing and verbal explanations. It not only helps the person explaining the problem develop their reasoning more, it helps the students who learn in different ways understand the explanation better." (Unit 9, On-line discussion thread, line 12).

For the discussion about a student's work over the first two tasks, T7 wrote the following on-line:

> I noticed that the student used cases for both. For the tower problem, he had three groups of towers; all cubes one color, two cubes of each color, and three cubes of one color and one cube of the other color. He attempted the pizza problem in the same way, beginning with the plain pizza, moving to 1 topping pizzas, 2 toppings, 3 toppings, and 4 toppings. He also held a constant in each group for the pizza problem. For example, with the two topping pizzas he would start with all the pairs with pepper, then move on knowing pepper would not be used again. It was interesting to see the change in notation. For the tower problem, he used a key for red and yellow cubes and drew the towers while for the pizza problem, he wrote the toppings out and put them in parenthesis if they were on a pizza together. (Unit 10, On-line discussion thread, line 37).

In response to another teacher (T5) about students providing valid justifications for their answers, T7 wrote the following on-line:

> Many of my students also wanted to give an equation for why the answer was 16 but they were not able to justify where it was coming from other than the fact that it gave them 16. I would ask them if they knew the answer before they started because of the equation or if they came up with the equation after finding the answer. It was always the second response, so I would encourage them to explain without using the equation. (Unit 10, On-line discussion thread, line 5).

T7's response is consistent with the standards in the categories of expectations and abilities and mathematical discourse.

On 11/6/13, T7 implemented the third cycle of tasks in her classroom. T7 wrote the following in her final project: "The students had less difficulty with these two tasks than they did with the pizza problem. This is in part because of the manipulatives they were able to use and that there was less room for them to interpret the problem in different ways." (Final Project for T7, p. 32) and T7's comments are consistent with the standards in the manipulatives category.

At the 11/20 regional meeting, T7 shared two students' work for the third cycle. T7 recognized that one of her students controlled for a variable drawing four separate groups with 6 towers each plus three towers that had all solid colors. T7 also shared a second sample where the student set up the towers similarly but provided a clearer explanation (11/20/13 meeting transcript, line 348). T7 shared the following final reflection thoughts on the intervention:

> I was able to see growth in the students and their reasoning. Many students were automatically grouping the towers in different ways before being asked if it was possible and they were providing better verbal reasons. I was also able to see the positive influence on questioning students to get them to think about their work and other possibilities in mathematics. It was beneficial for students to see that mathematics is not just equations and numbers but also reasoning. (Final Project for T7, p. 33)

**10.8 T8**

On the pre-assessment before the first meeting, T8 made 1 statement, inconsistent with the Standards in the following categories: expectations and abilities, mathematical discourse, concepts and procedures, and teachers' and students' roles. At the first meeting, T8 and T4 worked together to arrange the towers using the opposite strategy for six towers as well as the recursive strategy for ten towers. Then, the teacher pair rearranged the towers by controlling for a variable (9/7/13 meeting transcript, lines 111-131). T8 made the following original post for the second unit discussion about predicting students' solutions for the four-tall towers problem:

> I think that they will pick alternating colors for the towers. I do not think they will automatically develop a pattern for moving groups of blocks throughout the tower to find all possible outcomes. I also think that they will overlook the obvious combination of making a tower with all the same color. I think that they will want to make all towers have both colors in them. (Unit 2, On-line discussion thread, line 31).

T8 implemented the first cycle task on September 23, 2013 in a Mild Cognitive Impairment Self Contained sixth-eighth grade (MCI SC 6-8) class with 9 students in a mathematics class beginning at 9:30 am (Final Project for T8, p. 2). T8 had written "They were able to show what they were doing but had trouble putting it into a clear explanation verbally." (Final Project for T8, p. 13) and it was a challenge "getting them to write their explanation down on paper" (Final Project for T8, p. 13).

T8 wrote the following original post about the way students organized their towers:

> I learned that we had come up with similar strategies as students did to organize, group, and justify answers. However, our reasoning skills allowed us to fully comprehend the task. The students were simply trying to justify their thought process. It did bother me that I had made some of the same reasoning arguments

> that students have made. I would like to think that I am more advanced than those that I teach. It must be natural for the brain to make certain patterns and groupings, whether one is a teacher or student. I think that as teacher we must guide students in increasing their ability to manipulate and justify reasoning, just as the professors did for us. I also think that students pick up reasoning skills from listening to others explanations. I know that there were many different ways to justify our answers, but I only came up with one. (Unit 3, On-line discussion thread, line 46)

T8 responded to T6 about the struggle students have when writing down their reasoning as they worked on the four-tall towers problem. T8 wrote “My students really struggled with putting their thoughts on paper. I had to pretty much have them explain it to me one step at a time and after each step have them write what they just said.” (Unit 4, On-line discussion thread, line 13). This was a common issue expressed by some of the teachers (Unit 4, On-line discussion thread, lines 13, 15, 17, 18).

At the October 2 regional meeting, T8 shared three samples of students’ work with the teachers. For the first-shared sample, T8 recognized that his student controlled for a variable (10/2/13 Meeting Transcript, line 719). T8 also shared the partner’s work where the tower drawings were similar but T8 said the written argument provided by the student was confusing (10/2/13 Meeting Transcript, line 741). T8 also shared the work of a student-helper that came during lunch to help with the students. The student-helper had five groups where the first and fourth group had two of each color, the third and fifth groups were the diagonal pattern, and the second group had two towers each of one color (10/2/13 students’ work meeting transcript, line 761).

Later on at the October 2 regional meeting, T8 worked with T1 on the second cycle task. T8 had written the following comment on the Unit 5 on-line discussion thread:

> The least helpful strategy was making a tree diagram. We started with this and quickly realized that we were repeating. The strategy that we did use was keeping a topping constant, then, making the possible combinations with 2 and 3 toppings. Then we would use another topping as a constant and create 2 and 3 toppings without repeating any combinations from the previous topping constant. (Unit 5, On-line discussion thread, line 44).

T8 responded to T1 about how the teachers' questions helped encourage Brandon to further explain his ideas for the pizza problem with the following: "I am also finding myself focusing more on the way that I question students. I am trying more to elicit their own response rather than the response that I am looking for." (Unit 6, On-line discussion thread, line 6).

On October 8, 2013, T8 implemented the second cycle task in his classroom. T8 had written the following in his final project about the pizza problem:

> They were much less successful in this problem than in any others. I think that the change to having no manipulative made the problem more challenging. The manipulatives gave them something tangible to work with, which is obviously beneficial to this type of exploration problem. (Final Project for T8, p. 27)

T8's comments are consistent with the standards for the manipulatives category. T8 responded to T5 about how one student drew circles labeled with letters in a diagram to make the pizzas

> I had a group do a similar diagram. They drew slices of pizza instead of circles representing the whole pie. I think it is a great skill to be able to create meaningful diagrams that give students ownership and connection like that. (Unit 7, On-line discussion thread, line 20).

T8 had written that he noticed "an increase in willingness and effort towards the problem when compared to how they acted with the first cycle" (Final Project for T8, p. 27).

At the regional meeting on October 22, 2013, T8 shared three samples of students' work. T8 recognized that one student had controlled for a variable in his diagram of triangles representing separate pizza combinations (10/22/13 meeting transcript, line 348). The partner had a similar diagram and argument (10/22/13 meeting transcript, lines 348). In the third-shared sample of student work, T8 had the student-helper do the problem and she listed 16 possible pizza combinations using a cases argument by controlling for a variable with peppers (10/22/13 meeting transcript, line 372). Later on in the regional meeting, T8 paired up with T7 to solve the third cycle problems. T8 and T7 used a recursive argument to make 9 groups of 3 three-tall towers. T8 and T7 solved the Ankur's Challenge problem with three groups of twelve 4-tall towers.

T8 also responded to T1's original post about the importance of giving students more than one opportunity to explain and write about their reasoning:

> Sometimes students are able to understand a peer explanation better than ours. I have been trying to provide teacher moments in my class where students that have shown understanding can go up to the board and teach other students. Everyone gets excited for this and it really helps. It gives you the ability to check in with students individually. The students really get it when they work with their peers. And those that are teaching it gain an even deeper understanding. (Unit 9, On-line discussion thread, line 12).

T8 also wrote the following comments on-line about the intervention: "As a student, math for me was always about memorizing an algorithm and knowing how to plug in numbers. We were never taught the "why" behind math; just this is how you do it." (Unit 9, On-line discussion thread, line 18).

T8 wrote the following on-line about a student's work for the first two tasks:

> For the 1st task, my student started with making opposite pairs. Then she manipulated the groups to have a constant on top. She used the recursive argument to organize and solve the problem. For the 2nd task, she started by just making toppings at random. Then with the suggestion to use some type of organization she created groups by using the first topping as a constant. For example one group had mushroom as the single topping; then mushroom with each of the other toppings for a 2 topping pizza, then mushroom with 2 other

> toppings for a 3 topping pizza. My student used a constant to group combinations in both tasks. (Unit 10, On-line discussion thread, line 32).

In response to T3 about ways to improve students' written arguments, T8 had written "This class has really allowed me to see how struggling writers can be affected by these types of questions. It is so important for us to teach writing strategies to our students in math so they feel comfortable doing so." (Unit 10, On-line discussion thread, line 22).

On 11/14/13, T8 implemented the third cycle of tasks in his classroom. In the final project, T8 had written the following:

> I was very impressed by my students' readiness to solve this problem. They told me that the problem was easy for them. They were able to apply their strategies they used from the first problem and further develop them to solve this problem. They were able to create groupings without any help. (Final Project for T8, p. 37)

T8 shared two students' work for the third cycle at the 11/20/13 regional meeting. T8 recognized that one of his students controlled for a variable in two out of four groups of the tower drawings. T8 also shared the student-helper's work where group one was three towers all the same color; the second group was six towers with one of each color cube; and groups three through five were constructed by controlling for a variable for the first two towers and then switching the bottom two positions (11/20/13 meeting transcript, line 315). T8 wrote the following reflection thoughts on the intervention:

> During the implementation of these lessons, I learned that I had completely underestimated my students. They showed perseverance throughout the projects. My students were capable of solving the problems, especially when using manipulatives. Their struggles where when they were asked to explain their reasoning and not state what they did. (Final Project for T8, p. 38)

T8 also wrote that he needed to have more writing within his lessons, but that "It is important to scaffold and support this writing so students can revisit it to analyze their thinking and further develop their writing." and is consistent with the standards in the categories of mathematical discourse and student and teacher roles (Final Project for T8, p. 38).

**10.9 T9**

On the pre-assessment, T9 made 20 statements consistent with the Standards and 1 statement inconsistent in the category of teachers' and students' roles. At the first meeting, T9 and T10 worked together to arrange the towers using the opposite strategy. Then the teacher pair reorganized their towers by controlling for a variable with six of the towers and using an elevator strategy for the other ten towers (9/7/13 meeting transcript, lines 21-31). T9 wrote the following in her original post about predicting students' solutions for the four-tall towers problem:

> In the short time I have known my students (4 days); I realize I have a diverse bunch. It is always interesting to me to see their differences. I have a bunch of students who are already very meticulous in their processes. For these students, I could see them arranging their towers in a very organized way like the girl did in the video…I also have many students who don't seem to have a good grasp on organization. I think they would just jump right in and create the towers randomly. I could see how this could be challenging for these students when trying to see if they have created all the towers. Their arguments might be harder to give. (Unit 2, On-line discussion thread, line 22).

T9 implemented the first cycle task on September 27, 2013 in a sixth-grade inclusion class with 15 students in the mathematics class for 74 minutes (Final Project for T9, p. 2). T9 had written the following:

> I learned that my students do not understand the concept of justifying their work and what it means to convince someone. Most of them tended to just write what they did instead of why they did it or ow they knew they were finished. I learned that I really needed to address what these words meant if I wanted to get clear, concise thoughts from my students. (Final Project for T9, p. 9)

T9 also wrote the following in response to T3 about students' explanations:

> I also find that explaining their reasoning is where my students tend to struggle. They seem to want to always tell you WHAT they did instead of HOW they did it or even WHY they did it. I find this skill very hard to teach. (Unit 3, On-line discussion thread, line 8)

T9 was one of the four teachers that agreed with T6 that students struggled to write down their reasoning as they worked on the four-tall towers problem. T9 wrote the following:

> I also had a problem with students putting their thoughts on paper. When I asked them to verbally convince me, I felt they did a nice job. Then when they had to write it, they just put down that they found patterns. When I told them that they verbally told me more and I wanted them to write what they said, they either could not remember what they said or thought that they did write everything down. (Unit 4, On-line discussion thread, line 12).

At the October 2 regional meeting, T9 shared three samples of students' work with the teachers. One of her students had written an argument that listed 16 towers but did not provide a convincing argument as to whether the student found all the towers (10/2/13 Meeting Transcript, lines 807-808). Another student provided a convincing argument for the first group of towers that had all the same colored cubes but did not provide a convincing argument as to why the student knew all the towers were found (10/2/13 Meeting Transcript, lines 823-827). A third student sample shared by T9 was of a student who incorrectly used the rule strategy and multiplied four times four to get 16 towers (10/2/13 Meeting Transcript, line 843).

T9 worked with T10 on the second cycle task. T9 had written the following comment in the Unit 5 on-line discussion thread:

> When solving the pizza problem, my colleague and I again used the proof by cases. We found the number of 0 topping, 1 topping, 2 toppings, 3 toppings, and 4 topping pizzas there were. I think that organized our thoughts pretty clearly. We got a little confused when we got to 3 toppings, but we were able to find them all. We first found all the 3-topping pizzas with sausage and then moved on from there. So, we held the topping constant in order to find the solution. (Unit 5, On-line discussion thread, line 36).

The following was T9's response to T1 on-line about teachers' questioning techniques:

> I agree that my questioning techniques have changed. I am much more aware of what I ask the students and how I phrase my questions to them. It is very hard for them to explain their thinking. I hope to see growth in their responses throughout the year. (Unit 6, On-line discussion thread, line 4).

T9 implemented the second cycle task in her classroom on October 11, 2013. T9 wrote the following on-line:

> I gave the pizza problem to my students today and was amazed at how many different representations they had. I even had one boy who made the connection to the towers! I was amazed. He said that a no topping pizza would be an all blue tower, and a 4-topping pizza would be an all yellow tower. It was like a breakthrough! (Unit 6, On-line discussion thread, line 11)

T9 also wrote "When I implemented the pizza task, most students began drawing webs. They had words like plain or pizza in the middle and branched out in all directions with their toppings." (Unit 7, On-line discussion thread, line 1). T9 had written the following in her final project about the plethora of interpretations the students made to solve the pizza problem:

> They could not figure out how to begin with the toppings. They wanted to make half-and-half pizzas. They claimed that was different than a two-topping pizza "mixed". A few students even wanted to make each slice a different topping. They were really making the problem much harder than it actually was. (Final Project for T9, p. 22)

T9 also mentioned that students struggled because they did not have manipulatives to use for this problem (Final Project for T9, p. 22).

T9 shared two samples of students' work at the regional meeting on 10/22/13. T9 recognized that one student had controlled for a variable in his diagram but had incorrectly used the rule argument (10/22/13 meeting transcript, line 651). Another student had found the 16 possible pizzas using 16 separate tree diagrams (10/22/13 meeting transcript, lines 665). Later on in the regional meeting, T9 and T10 used a recursive argument to make 9 groups of 3 three-tall towers for the three-tall tower problem and made 12 groups of three 4-tall towers to solve Ankur's Challenge. T9 posted that she did not think her students would come up with Romina's proof but T9 did think her students would come up with the correct solution (Unit 8, On-line discussion thread, lines 42, 43).

T9 responded to T6 about the importance of giving students more than one opportunity to explain and write about their reasoning:

> I am also asking my students if their arguments are convincing. I even ask other students if it was convincing to them. If not, I have them tell the student why it wasn't and ask them to clarify. (Unit 9, On-line discussion thread, line 26).

T9 had written the following on-line about a student's work for the first two tasks:

> My student tried to match up opposites with the first task. He saw the opposites first; however he was unable to come up with all 16 towers. After questioning him about whether he had all the towers, he and his partner decided to rearrange the towers. At this point, he saw the staircase and candy cane patterns. He was able to arrive at all 16 towers. On the second task, he began by creating a web with the word pizza in the middle. He branched off of the pizza with different toppings. It looked unorganized to me, but he was able to get 15 of the solutions. He missed one of the three topping pizzas. (Unit 10, On-line discussion thread, line 10).

In response to another teacher (T5) about how manipulatives helped to form students' arguments, T9 had written "I agree that when the students had something tangible to use their arguments seemed better. They are able to manipulate them and reorganize them a lot easier than if they have something just drawn on paper." (Unit 10, On-line discussion thread, line 7). T9 also responded to T3 about her students making convincing arguments. T9 wrote the following on-line:

> I also want to really push my students to write more convincing arguments. I have been giving them smaller tasks in class and asking them to explain their reasoning. I am finding that they do not understand the term reasoning. They think it means tell me what you did. (Unit 10, On-line discussion thread, line 18)

On 11/14/13, T9 implemented the third cycle of tasks in her classroom. In the final project, T9 had written the following:

> For each class I implemented this task in, the students' reasoning got better and better. I learned that the students really have the capacity to grow in such a short amount of time. Their responses got more detailed and in-depth. I have to admit, after the first and even the second task, I was really skeptical about how much the students would grow. (Final Project for T9, p. 33)

T9 did not attend the 11/20/13 regional meeting. However, T9 wrote the following reflection thoughts on the intervention:

> This lesson study really opened my eyes to how important it is to allow the students time to reason through the math they are learning. I feel that my students really gained self-confidence in writing about math. They went from being able to verbalize their thoughts in the beginning of the year to being able to put those thoughts to paper. I have noticed that I take more of a facilitator role in the classroom. I try to allow my students the time and space to discuss, and even argue, with their peers to determine the solutions to problems. I find myself asking if their peers are convincing them. I also ask them to defend their positions. (Final Project for T9, p. 34)

T9's comments are consistent with the standards for the category of teachers' and students' roles. T9 also had written that she became a better teacher after participating in the lesson study (Final Project for T9, p. 34).

**10.10 T10**

On the pre-assessment, T10 made 17 statements consistent with the standards and 2 statements inconsistent with the standards where 1 inconsistent statement was in the category of concepts and procedures and 1 inconsistent statement was in the category of teacher and student roles. T10 and T9 worked together to arrange the towers using the opposite strategy first. Then T10 and T9 rearranged the towers in a group of six towers by controlling for a variable and grouping the other ten towers using the elevator strategy

(9/7/13 meeting transcript, lines 21-31). T10 wrote the following in her original post about predicting students' solutions for the four-tall towers problem:

> I believe that many of my students will start building without a strategy, rather just start looking for possibilities. After some exploration, some groups may begin to see that they have pairs of towers that appear to be opposites and may look for opposites that they are missing. If students use this strategy, it may be difficult for them to give a convincing argument to explain how they know they aren't missing any complete pairs. In general, I think the students may have trouble giving convincing arguments because they are not accustomed to thoroughly explaining their thinking and reasoning in any problem solving situation. (Unit 2, On-line discussion thread, line 12)

T10 implemented the first cycle task on September 26, 2013 in a sixth-grade inclusion mathematics class with 17 students for 74 minutes (Final Project for T10, p. 4). T10 had written: "The most challenging part of this task for most of the students was writing a convincing argument." (Final Project for T10, p. 13) and that T9 found herself "restructuring activities in class to allow the students to generate explanations that were not based on telling what they did, but instead explaining why they chose a certain operation, or describing how they know their solution is complete" (Final Project for T10, p. 13).

T10 wrote the following in response to T7 about the importance of having students justify their solutions:

> Explaining the "why" is definitely very hard to teach. I think we do often ask them to explain "what" they did to solve a problem when they come up with something that doesn't quite make sense to us so that we can help them figure out where their mistakes were. They are so accustomed to describing the steps that they took, that it is difficult to provide a reason why. Our students are so focused on the solutions that they often miss the description of why something works, or more often than not, don't think that the why is important, as long as they have the correct solution. (Unit 3, On-line discussion thread, line 9)

T10 was one of the four teachers that expressed her students struggled to write down their reasoning from the four-tall towers problem. T9 wrote the following:

> Most of my students organized their towers into pairs and their argument was that they could not think of any more pairs. I did this activity with 2 of my 6th grade classes, and out of the 17 student groups I had only about 3 groups had the beginnings of a convincing argument, which they verbalized to me. Unfortunately, their writing doesn't match much of what they talked about. It would be hard for anyone else to read their conclusions and understand what they were thinking. (Unit 4, On-line discussion thread, line 15).

T10 shared three samples of students' work with the teachers at the regional meeting on October 2, 2013. One of her students had written a partially convincing argument where the argument was convincing for three of one color and one of the other; but not convincing for two of each color (10/2/13 Meeting Transcript, lines 855-867). For the second-shared sample, T10 recognized that the student used a cases argument (10/2/13 Meeting Transcript, lines 892-893). T10 also recognized the elevator strategy in a third-shared student argument (10/2/13 Meeting Transcript, line 913).

Later on, T10 worked with T9 on the second cycle task. T10 had written the following comment in the Unit 5 on-line discussion thread:

> To solve the pizza problem, we again used a proof by cases. We first found all the possibilities with 0 toppings, 1 topping, 2 toppings, etc. As we got to three toppings it became harder to make sure we hadn't duplicated any pizzas, so we considered holding 1 of the three toppings constant, and finding the pizza combinations that could be created by changing the other two toppings. (Unit 5, On-line discussion thread, line 22).

The following was T10's response to T1 on-line about questions asked by teachers:

> It is important that we don't get caught up in assuming our students understand something, and sometimes the most obvious questions can clear that up. It also helps the students think about what they are saying, and often by restating their thinking aloud, things begin to make more sense to them, or they realize there is something wrong with their reasoning. (Unit 6, On-line discussion thread, line 21).

T10 implemented the second cycle task in her classroom on October 10, 2013. T10 wrote the following on-line about the pizza problem:

> My students also had a lot of trouble with this one. In the towers problem, most groups came up with 16. With this problem, I saw a wide variety of answers! I think the lack of manipulatives definitely made this more challenging. Since many of my students made lists, there were many duplicates and many missing pizzas. (Unit 6, On-line discussion thread, line 11)

T10 also had written about the variety of representations the students used. T10 wrote the following:

> Some students made lists that were organized by the number of toppings, others organized by pizzas that included a particular topping, and others were completely unorganized. Many students made webs, with pizza at the center and all the types of pizza extending from the center. One student made a checklist similar to Brandon's in the video, but then abandoned it for a list where he found all the pizzas that had certain toppings. Several of my students also thought it was important to draw a pizza to go with every outcome they found. (Unit 7, On-line discussion thread, line 35).

T10 noted that students had difficulty solving this problem without manipulatives (Final Project for T10, p. 25). T10's comments are evidence that the statements are consistent with the standards for the manipulatives category.

T10 shared two samples of students' work on 10/22/13 during the regional meeting. T10 recognized that one student had controlled for a variable for group of the two-topping pizzas (10/22/13 meeting transcript, lines 569-570). Another student had originally made a chart but then decided to use a number system to find the possible pizza combinations (10/22/13 meeting transcript, lines 602-603; 628-629). Later on in the regional meeting, T9 and T10 used recursion to make 9 groups of 3 three-tall towers for the three-tall tower problem. T10 and T9 then made 12 groups of three 4-tall towers to solve Ankur's Challenge. T10 made an original post that she didn't think her students would come up with the proof that Romina did but had written that she had been surprised by her students' work while working on the intervention tasks (Unit 8, On-line discussion thread, line 1).

T10 also agreed with T8 about the importance of having students explain their reasoning to other students. For the Unit 9 discussion thread, T10 wrote the following:

> It's always best if a student can explain what they are doing any why to others. I think when students are asked to teach a solution method they used to the class they sometimes struggle, because they realize maybe what they are saying is not as clear as what they were thinking. When my students take on this task of explaining a new or different solution method to the class I think the student presenting learns more about their work than if they had just solved the problem and moved on to the next task. (Unit 9, On-line discussion thread, line 8).

T10's comments are consistent with the standards for the mathematical discourse category. In response to T5about how manipulatives helped to form students' arguments, T10 had written "I also think having the cubes helped my students to find the correct solution, but I don't think it necessarily helped them to justify their answers any better." (Unit 10, On-line discussion thread, line 6).

On 11/20/13, T10 implemented the third cycle of tasks in her classroom. T10 had written the following in her final project:

> The implementation of this task was easier than I had anticipated. I thought this task was going to be confusing for my students because of the number of solutions and the complexity of using three different colors of cubes. The students really surprised me with their solutions and explanations. (Final Project for T10, p. 36)

At the 11/20/13 regional meeting, T10 shared two samples of students' work. T10 recognized that her student made eight groups controlling for a variable and using the elevator strategy to solve the three-tall towers problem (11/20/13 meeting transcript, lines 7-35). The second sample shared by T10 had five groups using the same strategies as the first sample (11/20/13 meeting transcript, lines 50-62).

### 10.11 Summary of Teacher Narratives

The teacher narratives describe each teacher's experiences within the intervention regarding teachers' beliefs and how the teachers attended to students' reasoning. The teachers' on-line discussion gave teachers the opportunity to discuss the articles and videos of the research students as well as work from their current students. Teachers were also given the opportunity to discuss students' work with each other at regional meetings. Many of the teachers remarked about how their students' work frequently surprised them in a positive way.

While discussing the current students' samples, the instructor made motivating statements regarding the improvement of the teachers' abilities to effectively question students regarding students' work and determine whether arguments were convincing or not convincing (Meeting transcript 11/20/13, lines 323-325, 441). By the end of the intervention, teachers attended to students' reasoning by describing students' work in a more precise way using the terminology learned throughout the intervention when describing arguments (e.g. inductive, recursive, case) or strategies (e.g. opposite, staircase, holding a constant, guess and check).

## Chapter 11 – Findings

The purpose of the study was to examine the obstacles and successes experienced during an intervention of ten middle-school mathematics teachers from the southern region of New Jersey. The following research questions guided the study:

1) What reasoning forms do middle school mathematics teachers identify from the following:

   (a) Their solutions to given mathematical tasks during a PD intervention;

   (b) Their current students' solutions to the same mathematical tasks implemented in the teachers' own classrooms;

   (c) The research students' solutions working on the same mathematical tasks from assigned articles to read and VMC videos;

   (d) Teachers' pre and post-assessment responses of the reasoning forms used by fourth-grade students to solve mathematical tasks in the Gang of Four VMC video?

2) What pedagogical moves are used by the instructor to facilitate the teachers' knowledge construction about mathematical reasoning as teachers:

   (a) Worked on combinatorics tasks;

   (b) Attended to research students' reasoning from VMC video and scholarly articles;

   (c) Analyzed current students' written task work?

3) In what ways, if any, have the teachers' beliefs about the teaching and learning of mathematics changed?

This chapter summarizes the findings for each research question in the following three parts: the teachers' recognition of reasoning, the instructor's moves, and the teacher' beliefs regarding the teaching and learning of mathematics. For each of the three parts, themes emerged and are supported with salient discussions or events from the intervention.

The video data for this study was analyzed using the analytic model for studying video data by Powell, Francisco, and Maher (2003). Powell et al. (2003) defined critical events as occurrences which had a positive or negative contribution to the research. The critical events referred to in this research are the events where the instructor makes pedagogical moves to facilitate teachers' knowledge construction or recognition of students' reasoning and teachers' discussions or descriptions of reasoning strategies used by the teachers, their students, or by the students observed in the videos.

### 11.1 Teachers' Recognition of Reasoning

This section is a report of seminal findings from the reasoning analysis that addresses the first research question. Based on the results of the reasoning analysis from this research, the PD intervention was effective in helping in-service middle-school mathematics teachers to attend to students' reasoning. The findings from the teachers' recognition of reasoning were examined in the following contexts: teachers doing the tasks themselves, teachers recognizing research students' reasoning from articles and videos during an on-line discussion, teachers recognizing reasoning from current students during debrief meetings after three in-district classroom visits, teachers recognizing reasoning from current students' samples, and the teachers' recognizing reasoning from the Gang of Four video. Moreover, the findings from teachers' recognition of reasoning are framed around two themes: heuristics/strategies and forms of argument.

### *11.1.1 Findings from Heuristics/Strategies*

For this section, seminal findings of teachers' recognition of reasoning regarding the heuristics used when solving the three cycles of mathematical problems are reported. The intervention helped middle school mathematics teachers to use and recognize heuristics in their own solutions to the problems as well as identify strategies in research and current students' work when solving the mathematical problems.

#### *11.1.1.1 Opposites*

The strategy that was most used or recognized by the teachers was opposites, 116 times. Opposites were used or recognized 90 out of 116 times to solve the first cycle four-tall towers problem and the first cycle three-tall and five-tall tower extension problems. Four pairs of teachers used the opposite strategy when beginning to solve the first cycle four-tall towers problem but then later reorganized their towers to provide more convincing arguments (Meeting transcript 9/7/13, lines 19-204).

Teachers also noted that many of their students used the opposite strategy to begin their arguments for the first cycle four-tall towers problem (Unit 4 on-line discussion thread, lines 29-32, 36, 43). Teachers recognized the opposite strategy 35 times from students' work at the in-district class visits and from current students' samples of work brought by the teachers. Teachers also identified the opposite strategy 77 times during on-line discussions regarding research students' work.

The data show that the opposite strategy was frequently used and recognized for the first cycle towers problems but used or recognized less frequently for the second and third cycle problems. It is possible that teachers and students did not use the opposite strategy as frequently for the second and third cycle problems because providing a convincing argument was difficult using the opposite strategy.

*11.1.1.2 Control for a Variable*

The second most used or recognized strategy when solving the three cycles of problems was controlling for a variable, 86 times. Throughout the intervention, the teachers referred to this strategy as holding a constant. When solving the three cycles of problems, teachers controlled for a variable 16 out of 86 times. Teachers recognized the control for a variable strategy 48 times from students' work at the in-district class visits and from current students' samples of work brought by the teachers. Teachers also identified the control for a variable strategy 22 times during on-line discussions regarding research students' work.

For the second cycle pizza problem, teachers used or identified the control for a variable strategy 40 out of 86 times. During the third cycle, teachers used or identified the control for a variable strategy when solving the three-tall towers problem, 21 times and Ankur's Challenge, 13 times. Moreover, control for a variable was used or identified by the teachers 12 times for the first cycle towers problems. The data show that controlling for a variable was used more frequently when solving the second cycle pizza problem and the third cycle towers problems than when solving for the first cycle towers problems.

*11.1.1.3 Elevator*

The third most used or recognized strategy when solving the three cycles of problems was elevator, 64 times. When solving the three cycles of problems, teachers used the elevator strategy 11 out of 64 times. Teachers recognized the elevator strategy

28 times from students' work at the in-district class visits and from current students' samples of work brought by the teachers. Teachers also identified the elevator strategy 25 times during on-line discussions regarding research students' work. Teachers often referred to the elevator strategy as a "staircase" or "stairs" and described this strategy as a diagonal where one block was moved up each time through all the positions when solving the first cycle four-tall towers problem, the third cycle three-tall towers problem, and Ankur's Challenge (Meeting transcript 10/2/13, lines 80-91, 215, 241, 415, 429; Unit 10 on-line discussion thread, lines 10, 16, 29).

The elevator strategy was used 41 out of 64 times to solve the first cycle four-tall towers problem. To solve the third cycle three-tall towers problem, elevator was used 17 times and used 5 times for Ankur's Challenge. The elevator strategy was identified only one time to solve the second cycle pizza problem.

### *11.1.2 Findings from Forms of Argument*

For this section, the seminal findings of teachers' recognition of reasoning regarding the forms of argument used when solving the three cycles of mathematical problems are reported. The intervention helped middle school mathematics in-service teachers to use and identify forms of argument in their own solutions to the mathematical problems as well as identify strategies in research and current students' work when solving the mathematical problems.

#### *11.1.2.1 Case Arguments*

The form of argument that was most used or recognized by the teachers was case arguments, 126 times. Case arguments were used or recognized 52 out of 126 times to solve the second cycle pizza problem. All five pairs of teachers used case arguments to solve the second cycle pizza problem (Meeting transcript 10/2/13, lines 17-289). Teachers' recognized case arguments 73 times from students' work at the in-district class visits and from current students' samples of work brought by the teachers. Teachers also identified case arguments 34 times during on-line discussions regarding research students' work.

The data show that case arguments were used or recognized fewer times for solving the first- and third-cycle towers problems. It is possible that teachers and students used case arguments more frequently for the second cycle pizza problem because teachers and students may have been more comfortable organizing the pizza combinations by the different types of pizza toppings or organizing the pizza combinations by the number of pizza toppings.

Case arguments were also identified by the teachers on pre- and post-assessments after watching the Gang of Four video. On the pre-assessment, three teachers provided partial descriptions for Case Argument 1: Stephanie's Case Argument. On the post-assessment, the number of teachers that provided partial descriptions for Case Argument 1 increased to four teachers. Three teachers provided complete descriptions on the pre-assessment for Case Argument 1. The number of teachers that provided a complete description of Case Argument 1 increased to six teachers on the post-assessment. It should be noted that teachers described case arguments to the instructor but did not use the words "case" on the pre-assessment. On the post-assessment, eight of the ten teachers specifically used the words proof by cases when providing a description of Case Argument 1. The data shows that the intervention was effective in helping the middle school in-service teachers to recognize case arguments.

Case Argument 2 was identified only one time on the pre-assessment by T10 and one time on the post-assessment by T7. The data show that the intervention was not as effective in helping the middle school in-service teachers to recognize Case Argument 2, “a more elegant argument” (Maher et al., 2014, p. 37). It is possible that teachers did not find Case Argument 2 as easily recognizable as compared to Case Argument 1: Stephanie’s Cases Argument.

*11.1.2.2 Recursive Arguments*

The form of argument that was the second most used or recognized by the teachers was recursive arguments, 70 times. Recursive arguments were used or recognized 41 out of 70 times to solve the first cycle towers problem, and 26 out of 70 times to solve the third cycle three-tall towers problem. Teachers recognized recursive arguments 27 times from students’ work at the in-district class visits and from current students’ samples of work brought by the teachers. Teachers also identified recursive arguments 30 times during on-line discussions regarding research students’ work. The data shows that recursive arguments were used or recognized more times for solving the first- and third-cycle towers problems.

Recursive arguments were also identified by the teachers on pre- and post-assessments after watching the Gang of Four video. On the pre-assessment, three teachers provided descriptions of recursive arguments but did not use the word “recursive”. On the post-assessment, three teachers provided descriptions of recursive arguments on the post-assessment (T4, T6, T9, Gang of Four post-assessment). Moreover, two of the teachers used the word “recursive” (T4, & T9 Gang of Four post-assessment).

*11.1.2.3 Inductive Arguments*

Inductive arguments were also used or recognized by the teachers 23 times. Inductive arguments were used or recognized 20 out of 23 times to solve the first cycle four-tall towers problem and 3 out of 23 times to solve the third cycle tower problems. Three teachers used inductive arguments when solving the following tower problems: T2 and T3 reorganized their towers to provide a more convincing argument for the first cycle four-tall towers using an inductive argument; T4 and T6 organized their towers to solve the third cycle three-tall towers using an inductive argument; and T7 and T8 used an inductive argument to solve Ankur's Challenge. Teachers recognized only one inductive argument from a students' work at the in-district class visits on 11/20/13 (Meeting transcript 11/20/13, lines 116-122). Teachers also identified inductive arguments 19 out of 23 times during on-line discussions regarding research students' work. It is likely that the instructor's guided questions during the on-line discussion may have been a factor for the number of inductive arguments coded by the researcher.

Inductive arguments were also identified by the teachers on pre- and post-assessments after watching the Gang of Four video. On the pre-assessment, one teacher provided a partial description for Milin's inductive argument. The number of teachers that provided partial descriptions for Milin's inductive argument increased slightly to two teachers on the post-assessment. Five teachers provided complete descriptions on the pre-assessment for Milin's inductive argument. The number of teachers that provided a complete description of Milin's inductive argument increased to seven teachers on the post-assessment. It should be noted that teachers described inductive arguments to the instructor but did not use the words "inductive" on the pre-assessment. On the post-assessment, eight of the ten teachers specifically used the words "inductive arguments" when providing a description of Milin's argument.

*11.1.2.4 Rule Arguments*

Some arguments made by the teachers and the students contained mathematical sentences that worked out to be correct answers for the problem but were found using invalid mathematical rule arguments. For example, T3 described a student's work where the student took the two colors multiplied that number by 4, and then doubled the result to get 16 for the four-tall towers problem (Meeting transcript 10/2/13, line 326). The numbers worked out to the correct answer, but the reasoning behind the rule did not make logical sense and did not generalize to find the number of possible towers for different tower heights.

This identification of a rule to yield a correct answer was discussed in a paper "Rules without Reason: Allowing Students to Rethink Previous Conceptions" by Mueller, Yankelewitz, & Maher (2010). Mueller et al. (2010) reported on the reasoning of a group of sixth-grade students in the Informal Mathematical Learning Project (IML) as they worked to solve an open-ended fraction problem using Cuisenaire™ rods. The IML was an after-school project designed to give twenty-four urban and minority middle school students the opportunity to "confidently share their solutions, both correct and incorrect" (Mueller et al., 2010). For the episodes where students provided invalid rule arguments, students were given the opportunity to discuss and make claims to correct any invalid solution (Mueller et al., 2010).

Another similar study was done by Erlanger (1973) about Benny, a sixth grade boy who was considered to be excelling in his program. The program was called

Individually Prescribed Instruction (IPI) (Erlanger, 1973). When Benny was questioned about the mathematics, Erlanger (1973) found instances where Benny's reasoning did not make mathematical sense.

The instructor facilitated a discussion with the teachers about the importance of the arguments making sense mathematically (Meeting transcript 10/2/13, lines 326-378). This discussion briefly resurfaced when T5 shared her student's work that made the same argument as T3's student (Meeting transcript 10/2/13, lines 523) and again when T6 shared a student's second cycle pizza problem solution (Meeting 10/22/13 student work transcript, lines 189-199).

### *11.1.3 Summary of Teachers' Recognition of Reasoning*

Controlling for a variable was used frequently to solve the problems in all three cycles. The elevator and opposite strategies were also frequently used or identified when solving the mathematical problems. It should be noted that the opposite strategy was used and identified frequently for solving the first cycle four-tall towers problem and the three-tall and five-tall extension problems. However, the opposite strategy was used fewer times to solve the second cycle pizza problem, the third cycle three-tall towers problem, and Ankur's Challenge. It should also be noted that the elevator strategy was used or identified frequently when solving the first cycle or third cycle tower problems.

Case arguments were the most frequently used or identified form of argument used when solving the mathematical problems for all three cycles. The Gang of Four video post-assessment revealed that case arguments were the most common identified form of argument from the Gang of Four assessments. Case Argument 1: Stephanie's cases argument was identified by all ten teachers.

Milin's inductive argument was also identified by the teachers and all ten teachers wrote they were convinced of Milin's inductive argument during the Unit 4 on-line discussion. Although the data indicate that teachers were able to recognize inductive arguments, teachers only used inductive arguments three times when solving the problems throughout the three cycles.

Based on the results of the reasoning analysis, the intervention was found to be effective in helping in-service middle-school mathematics teachers improve in their attention to students' reasoning in the following contexts: (1) teachers doing the tasks themselves, (2) teachers recognizing research students' reasoning from articles and videos during an on-line discussion, (3) teachers recognizing reasoning from current students during debrief meetings after three in-district classroom visits, and (4) teachers recognizing reasoning from current students' work samples.

## 11.2 Findings from Instructor Moves

This section is a report of salient findings from the instructor moves observed during the regional meetings and debriefing meetings of the in-district classroom visits. The second research question is addressed in this section. Two themes emerged supported by critical events from the intervention as the instructor modeled questioning techniques and pedagogical practices: non-leading questioning and promoting reasoning as a process. The section concludes with a summary of the overall findings from the instructor's moves.

### *11.2.1 Non-leading Questioning*

Throughout the intervention, the instructor facilitated discussions regarding non-leading questioning techniques. One critical event occurred at the debrief meeting after the 9/17/13 in-district classroom visit. The instructor facilitated the following discussion regarding one student's work on the first cycle four-tall towers problem:

> T8: In the group 2, they were both solids. So if they were to be switched they would stay exactly the same. For example, there are four red and four yellow.
> R1: Okay. What do you think? Convincing? [Unison: No.]What could they have said?
> T7: This is what this group did the whole time. I don't know. You were with this group, right? All they kept saying to me was like, we switched it…it would, it would be good. So they, they just proved that they made opposites of each other, but they didn't really…I don't know if they understood the task because they didn't really say anything about how this is the most amount of towers they can make.
> T8: Yeah.
> R1: Okay.
> T8: For a while they were doing that, and I was like okay, you're just showing me opposites. Can you try to put it… because I think they had the one that, most get that one diagonal is going down. And I was like, can you put these together something like that? And that's where they came up with the drawing they have on the bottom.
> R1: So you led them a little bit. I would…I thought they were brilliant did that. I didn't realize that you…
> T8: No, because they were looking at it…and I was saying how can you group these?
> R1: Okay, that's good. That's fair.
> T8: And they put it together. And I said why did you group it like that? And he goes, well I have one red top and on the top.
> R1: That's good. Then that's really good. Then you weren't leading them. You just said to them, I don't see a convincing argument. Can you group these that are two of one color and two of another color, in a different way that may be able to convince me that there aren't any more [towers] and that's a good way to do [the questioning]. (Meeting transcript 9/17/13, lines 165-175).

This exchange was a critical event for emphasizing to the teachers the importance of how to effectively ask non-leading questions.

Another critical event occurred at the regional meeting on 10/2/13. The instructor was facilitating a discussion regarding one student's work brought by T3, a seventh-grade resource teacher. The following argument was placed on the screen for the teachers to read:

> There is one solid color, and 4 blocks high. There is no other way of doing this. There is only 1 yellow and 3 blues in each tower. There is a pattern, the yellow keeps moving up one. There is only 1 blue and 3 yellow in each tower. There is a pattern, the blue keeps moving up one. Two of the same color is touching, and one color isn't touching. Not one same color is touching. Both colors are next to their twin. (Meeting transcript 10/2/13, line 308)

The instructor praised the student's work and asked the teachers "What might you ask them to push them one step more for the three of one color and one of the other?" (Meeting transcript 10/2/13, line 316) and T3 replied "Why can't you go up one more step?" (Meeting transcript 10/2/13, line 317). The instructor made a motivating comment by praising T3 for coming up with an effective question (Meeting transcript 10/2/13, line 318).

Another critical event regarding non-leading questioning techniques occurred when discussing a student's work brought by T5. The student had found 18 four-tall towers grouped in opposites and the following discussion occurred:

> R1: Yeah so they used a constant. Very nice; go back to…did they tell you I have them all I have 18 here. And what might you ask them to kind of you know to let them know that they are not convincing you? What is a question you can ask them? They have 9 groups of pairs, right? And nine groups of opposites; [are there] any ideas?
> T5: Can you arrange it in a different way?
> R1: You could do that but I am saying that if that is the way they are arranging it. If you said to them, why can't you have a tenth pair of opposites? Why can't there be a tenth pair? I am not convinced there are only 9 pairs. In fact I don't even think there are 9 pairs. I am saying that if you ask them, I don't know. Could there be another pair? And that is a way for them to you know may be thinking that it is not a convincing argument and then your question can be arranged in a different way into a very good one.
> T5: They had it originally arranged straight out and when I thought they had a duplicate, I said why don't you try and pair them up because they did have had it all scrunched together but it was laying in one and so I said separate into twos but they still put it the same.
> R1: And again, remember it is so hard. But rather than say why don't you separate them into twos? Why don't you say Can you arrange them in a different way not all of them together, but can you arrange them in a different way that maybe I will be better able to figure out if you have them all because you are kind

> of like telling them to arrange it into twos. You are kind of forcing them into…going in a direction. Okay. (Meeting transcript 10/2/13, lines 511-518).

In this discussion, the instructor was modeling possible non-leading questions that could be asked to students if teachers observed their students making a similar error.

### *11.2.2 Reasoning as a Process*

The instructor also facilitated discussions helping teachers to recognize reasoning as a process. Two critical events were identified by the researcher during the debrief meeting after the first in-district classroom meeting on 9/17/13 where the instructor emphasized reasoning as a process to the teachers. One critical event was a discussion about one student's work where he explained how he arranged the towers but did not provide a reason as to why he arranged the towers.

> R1: Group 4 is this one here? [Points to student work.] Okay. Did anyone give a name to this group? Any of the students you talk to call this something? Sometimes they call it the rotating one, the candy cane one, the barber pole one… Umm, because the…
> T7: Because they alternate colors.
> R1: Yeah, okay. Cause its yellow-red-yellow-red or red-yellow-red-yellow. Okay, so let's see what they said for that group. Can someone read it that can see it?
> T1: We did red-yellow-red-yellow and for the other one we did yellow-red-yellow-red.
> R1: Convincing? Not really. Again they are telling you what they did, not why there can't be another one in that group. And that's going to be… and this is a hard thing to do and this is the beginning of the year. So I think they are doing more writing than I have seen in other groups than previous years in September. I mean, this is amazing.
> T6: Yeah, I don't think that they truly understand [Phone rings.] exactly what it means to be convincing. Like, I don't think, I don't think that they get the objective of how to be convincing. They just kind of explain to you what they do and, they just think that you're going to assume and guess because you're an adult.
> R1: But we don't want to guess and we don't want to assume. Because sometimes we will assume hoping that they're going down a path that's the right path, we're going to assume what they're doing but that may not be what they are doing. (Meeting transcript 9/17/13, lines 82-88).

In this exchange, it should be noted that the teacher did not assume that students knew how to provide convincing arguments. The instructor warned teachers about the distinction between making assumptions from students work and what students actually say or write as their argument. (Meeting transcript 9/17/13, line 88).

Another critical event that was identified was from a discussion about why students should be asked to use pen for this activity. T3 expressed her concern that her students would think that they could use pen for the rest of the year because she wanted them to use pencil when doing their mathematics (Meeting transcript 9/17/13, lines 161). The instructor replied to the teachers with the following:

> Let me tell you another reason why I wouldn't have them use pencil. Okay? Um, what students normally do when they're using pencil, is that they have an eraser. And after they do something, they erase it so you can't see it. And part of this is, they may have some good work that they're getting rid of, and all you will see is a hole in the paper. If you're working in pen and they want to get rid of something, all they do is put one line through it and you can still see what they've did. And remember, the aim of this class is not to look at the final answer but to look at the mathematical thinking as they're go through the process of solving the problem. So I would urge you that, trust that they will be able to know that this is a time I'm going to ask you to use a pen, but normally I won't do that in math class, okay. (Meeting transcript 9/17/13, line 164).

The instructor stressed the importance of being able to follow the work of the student as it emerges. By using pen, teachers can see work prior to students erasing earlier notations or rearranging their thought processes. (Meeting transcript 9/17/13, line 164).

A third critical event occurred at the first regional meeting on 10/2/13 as teachers discussed samples of students work. T2 discussed her student's sample of work for the four-tall towers problem who found the right answer of 16 possible towers and then stated "with my kids, I noticed they can explain verbally more than write. But so, his explanation he wrote on the back." (Meeting transcript 10/2/13, line 148). The instructor asked T2 to read the following student's explanation to the class: "I did the same block twice but not the same color like a pattern sort of. The reason I did it like that is because it is easier for me." (Meeting transcript 10/2/13, line 150). This critical event provides evidence that students were moving forward in the process of learning how to reason. T2's students were able to verbally explain, but they were not at the level to write convincing arguments yet.

Another critical event led to the instructor asking the teachers if any of them figured out a way to help their students out with their writing (Meeting transcript 10/2/13, line 151). T8 said that his students had formed groups and that for each group, the student explained several times about the placement of towers in the groups but then decided on a final response to T8 and T8 said to his students "write it down" ((Meeting transcript 10/2/13, line 154-156). T8, who gave the four-tall towers problem to a self-contained class of sixth-through eighth-grade mildly cognitively impaired students, shared that one of the girls started crying because the girl claimed "This is too much. I don't understand what's going on." (Meeting transcript 10/2/13, line 160). T8's response was "That's alright. Okay let me sit and help you get through this because we don't typically get to work like this so it was really…. I mean to write this much. She ended up writing like pages." (Meeting transcript 10/2/13, line 162). The instructor praised T8's strategy and how he handled the situation with his student.

T2 also shared another one of her student's work where her student drew one big block on the paper and wrote "16 combinations total and he got it right I think I have all the combinations for this worksheet." (Meeting transcript 10/2/13, line 237). T2 said that "he knew how to tackle the problem, but as far as recording, he had no clue"

(Meeting transcript 10/2/13, line 245-247). The instructor then asked the teachers "Did any of you do something that helped students with their recording?" (Meeting transcript 10/2/13, line 248).

T6 replied that she did not give her sixth-grade Pinnacle students the paper until they were done solving the problem and asked them to arrange the blocks "how you want them to look on your paper and then copy it" (Meeting transcript 10/2/13, line 257) and claimed the students did not have any problem recording their towers this way. However, T5 taught seventh-grade resource special education students and claimed her students struggled with recording their tower data. To help make the recording easier, T5 said she had one student draw representations of the towers and the partner copied the drawings onto his or her paper. T5 also remarked that her idea was similar to T8's where she had each student verbally explain the tower arrangement to her and then once they explained the tower arrangement, T5 asked the students to repeat what they verbalized. T5 said that this was done line by line having one partner record what the other person said. (Meeting transcript 10/2/13, lines 261-265).

The instructor again reiterated the importance of having students write convincing arguments at the second regional meeting on 10/22/13 in which the teachers were discussing samples of students' work regarding the second cycle pizza problem. The instructor said the following about providing help if students had special needs:

> Now remember, we talked about if your students can't write that you can be their scribe and write for them. Because you really want to see something down as to why they think they have them all. At least the beginning of a convincing argument because that's what this is all about. We're not so much interested in just that they can do the problems, we are interested in, can they provide a convincing argument. (Meeting 10/22/13 student work transcript, line 300)

Several teachers reported that their students struggled to write a convincing argument. T7 had reported that one of her students struggled to write a convincing argument for the second cycle pizza problem. T7 said that “His writing was not good, but his explanation to me was good.” (Meeting 10/22/13 student work transcript, lines 446-450). The following is the student’s written work:

> There is one plain pizza, you start with individual toppings; there’s four. From this, you group them in lists such as 2-topping and 3-toppings. Then you make sure you didn’t repeat a combination. There are 16 possible combinations. To check there are four original topping. Four can evenly go into 16. (Meeting 10/22/13 student work transcript, line 448)

The student was able to verbally provide a convincing argument to the teacher. However, the student did not provide a convincing argument in writing.

For the third cycle task, the instructor said the following to the teachers: “Push the children to the next level, wherever they are. If they haven’t been writing at all, you really want them to write a little bit, because we’re hoping to see growth from wherever they started.” (Meeting 10/22/13, student work transcript, line 513). The instructor also said “Even if they don’t have a whole solution, if their organization is here in the third task than it was in the first and second, that’s a good thing.” (Meeting 10/22/13, student work transcript, line 513).

T8 had decided to share one of his special education students’ samples of work for the third cycle three-tall towers problem. This student cried when trying to solve the first cycle four-tall towers problem seemingly overwhelmed by the problem-solving process indicated by T8 that “We don’t typically work like this.” (Meeting 10/2/13 students’ work transcript, lines 160-162). For the third cycle three-tall towers problem, T8 said that his student described the combination of colors used for each possible tower but “didn’t really say there were no other possible combinations” (Meeting transcript 11/20/13, line 306). The instructor responded to T8 about his student’s work from the first to the third cycle:

> Right, it was probably harder for her to do that. But you know I think again, it is a process and she is coming along in the process; because in the first two tasks, I don’t remember seeing all this writing, right. And I think also that is very nice. Even if she is just explaining what she did, she is writing. And once you get them writing, you can get them to write a convincing argument. (11/20, line 307)

#### *11.2.3 Summary of Instructor Moves*

Throughout the intervention, the instructor asked various types of non-leading questions. Of the 435 questions for the three cycles of mathematics problems, probing questions were the most common, 209 times. The other types of questions asked by the instructor were: explanation, 86 times; other solution, 55 times; justification, 50 times; connection, 20 times; and generalization, 15 times.

The instructor also modeled pedagogical practices to promote reasoning as a process. The most common pedagogical practice used by the instructor was motivating, 204 times. Other pedagogical practices used by the instructor were re-voicing, 169 times; inviting, 146 times; waiting, 65 times; monitoring, 31 times; selecting, 24 times; and anticipating, 21 times. The instructor’s moves throughout the intervention helped the middle school mathematics teachers to recognize reasoning in their own work as well as recognize reasoning in their students’ work.

### 11.3 Teachers’ Beliefs

This section is a report of salient findings from teachers’ beliefs. The third research question is addressed in this section. From the teachers’ beliefs expressed on the beliefs assessments, teachers’ final projects, and the teachers’ experiences from the intervention, the following two themes emerged: change in expectations and abilities with special education students and teachers' growth regarding questioning techniques and pedagogical practices.

### *11.3.1 Change in Expectations of Special Education Students*

Seven of the ten teachers participating in the intervention taught special education. Of the seven special education teachers, 3 teachers taught seventh grade resource (T1, T3, and T5), 1 teacher (T2) taught sixth-, seventh-, and eighth-grade self-contained language learning disability (LLD) students, 1 teacher (T9) taught a sixth-grade inclusion class, 1 teacher (T8) taught sixth through eighth grade self-contained mildly cognitively impaired students, and 1 teacher (T4) taught what she described as an eighth-grade general education class at an alternative academy. The instructor made the following point about special education students at the first in-district classroom visit:

> But sometimes it's very hard to tell who your special ed. students are and who your regular ed. students are. In fact in a good class you can't tell. When I would go in when I was a principal in a school and we had all inclusion, and, I would go into a class and I would try and figure it out, like, who are my children that have special needs. And, I really had trouble doing it, and that's a good thing. You don't want to be able to figure that out. And you really want to trust that the special ed. kids are going to do good stuff with this problem and they can. (Meeting transcript, 9/17/13, line 5)

Several critical events occurred that may have affected teachers' beliefs about special education students.

One critical event occurred when the instructor facilitated a discussion regarding a student sample of work brought in by T1. The following exchange occurred:

> T1: We couldn't make anymore because we think we made all the patterns plus we found all the blocks and we all worked together to create these patterns. So again, they couldn't really explain.
> R1: Well they are; that's their reason. Is it a good convincing argument? What do you think? Are you convinced? We're done because we couldn't make any more.
> T1: Well, she said all the patterns, so…

> R1: We made all the patterns, right but does that help you?
> T3: A little bit.
> R1: It does? Are you convinced?
> T3: Well I mean for this group because of the level.
> R1: No, you can't do that. And I know that a bunch of you said that online. You can't say the kids are young so we are going to expect less. If you want a convincing argument, you want a convincing argument.
> T3: I'm not saying they are young, I am saying they are special education.
> R1: No, I understand, but I'm saying online some of you said you watched the video the kids were young and we are accepting what they say and it is convincing enough. It's not. It doesn't mean that they should be doing more than this. I bet a lot of your students whether they were regular or special ed. said we are done because we tried and we tried and we can't find any more and therefore we are done and we got it. Okay. How many got that as an argument? Regular ed. and special ed. Correct? (Meeting transcript 10/2/13, lines 103-112).

This early exchange in the intervention is evidence that T3 had low expectations regarding the abilities of her special education students.

Another critical event was identified about teachers' allowing students to use colored pencils, markers, or crayons for motivation when solving the problems. The instructor facilitated a discussion with T3 and T4 who both taught special education students in fall 2013 when participating in the intervention. T3 and T4 expressed concern about their s lack of motivation at the academy from where she taught in the following discussion:

> T4: They don't do anything unless they have some sort of motivation.
> R1: Not true. We got wonderful stuff from those students.
> T4: Yeah, but they won't put anything on the paper unless they have some kind of motivation.
> T3: They need an incentive.
> T4: They need motivation to do something on paper. They won't ever touch the paper unless you give them a little incentive. Like
> R1: What would be the incentive?
> T3: You could use markers or colored pencils or whatever (Meeting transcript 9/17/13, lines 10-16).

The instructor recommended that the teachers set up a table or math center that had tools if students chose to use them and suggested not handing out colored pencils (Meeting transcript 9/17/13, line 27).

Several teachers shared work from their special education students for the third cycle three-tall towers problem that impressed the instructor (Meeting transcript 11/20/13, lines 87-97; 142-169). T2 shared a student's work that seven groups of three towers, and three groups of tower pairs that were alternating the three colors. The instructor facilitated the following discussion with T2 about her students' work:

> R1: So this is pretty neat, isn't it? Pretty neat work from special ed. [students]
> T2: Yeah they are seventh grade.
> R1: Very, very Good. Good. Did they write anything?
> T2: Yeah, she actually did.
> R1: Oh good.
> T2: She said I know the answer is 27. I know there is no more possible ways because in group 2, I moved the blue cube in each position in every way I could. There were only three positions because it could only be three high. I did the rest for groups 3, 4, 5, 6, and 7. So she just said she did this strategy.
> R1: Did she do this by herself?
> T2: Yes.
> R1: Because this is pretty impressive. Isn't it; right?
> T2: Yeah.
> R1: It is good writing.
> T2: Yeah she had definitely wanted to improve her explanation from the beginning. (Meeting transcript 11/20/13, lines 87-97).

This discussion revealed one example where T2's student improved her writing for the third cycle three-tall towers problem.

T3 was another teacher that shared samples of work from her special education students. The instructor also said the following motivating comments after discussing a few teachers' samples of students' work for the third cycle three-tall towers problem:

> So it's nice so they are really doing some good stuff. Very good, again when you have students that are struggling in the very beginning when you see progress you should feel very proud. Good. (Meeting transcript 11/20, line 189)

> So we have a lot of special ed. kids doing neat stuff. (Meeting transcript 11/20/13, line 281)
>
> So we are talking about a lot of kids that are being exposed to math which really has them thinking; which is a good thing. Because you really don't want your special ed. kids to just do arithmetic…It doesn't cut it anymore. Good. (Meeting transcript 11/20/13, line 313)

T8 reported how happy he was with the variety of strategies that his students used when solving the problems. T8 wrote the following in his final project:

> I did not expect them to come up with a constant. This is something that I did not think of at first, so I didn't think they would either. They showed me that students with low functioning levels are still able to reason at high levels if given the time and appropriate resources. It has helped me to try not to limit my students when giving assignments. (T8 Final Project, p. 57)

T9, another special education teacher, reported the following about the expectations she initially had regarding her students' abilities to reason:

> I learned that the students really have the capacity to grow in such a short amount of time. Their responses got more detailed and in-depth. I have to admit, that after the first and even second task, I was really skeptical about how much these students would grow. They completed building the towers faster and wrote much more detailed and convincing arguments. (T9 Final Project, p. 33)

### *11.3.2 Teachers' Beliefs Regarding Their Pedagogical Improvement*

Teachers' beliefs regarding their pedagogical improvement were found in the final projects, the on-line discussions, and from facilitated discussions led by the instructor. The results of the study revealed evidence of teachers' improvement regarding non-leading questioning techniques and recognition of convincing arguments.

#### *11.3.2.1 Non-Leading Questioning*

Teachers reported that they improved in their abilities to ask non-leading questions as the intervention progressed through the three cycles. T2 reported that she questions her students more instead of "leading them toward an answer" (T2 Final

Project, p. 32). T2 also wrote “The more I question my students, the better they understand the math they are given.” (T2 Final Project, p. 32) and wrote that this experience changed her view about teaching.

T4 reported similar beliefs when she wrote the following about the first cycle task: “The task gave me the opportunity to change my teaching and questioning techniques and I have found myself asking more thought-provoking questions and doing a lot less leading towards the right answer.” (T4 Final Project, p. 9) and reported that her students had family, social, emotional, and academic issues at the alternative academy. T3 wrote similar comments in her final project “I was much better at questioning my students throughout the entire [third] task. Even when asking my students specific questions, I found that they had become better at responding to me knowing what they had learned from cycle one and cycle two.” and mentioned that she allowed her students to question each other and explore problems more often (T3 Final Project, p. 57).

*11.3.2.2 Recognition of Convincing Arguments*

Teachers began to show growth during the discussion of the second cycle pizza problem in determining whether arguments were convincing or not convincing. T3 shared a student’s work sample where the students used an arrow system to list the 16 possible pizza combinations. The following written work was provided by the student:

> The first pair of 2 of all the same color is there because there are 4 blocks and all are the same color; but opposite from its partner. The second group of 2 pairs makes 4 different groups, but they link together because you can take the bottom or top and put it completely opposite of the top or the bottom. (Meeting 10/22/13 student work transcript, line 495)

T3 recognized that the organization was neat but that the written argument provided was not convincing (Meeting 10/22/13 student work transcript, line 500).

T8 was another teacher who posited that his students were not providing convincing arguments. T8 taught mildly cognitively impaired (MCI) students in fall 2013. After reading one of his students' solutions aloud to the teachers he said that she talked about how the three-tall towers were moved but did not discuss why (Meeting 11/20/13, students' work transcript, line 322). The instructor praised T8 and the other teachers with the following motivating comment:

> Good. That is really good that you guys are picking up. She is explaining what she did. She has a very, very, good strategy, but she is not saying that therefore there can't be anymore because I have taken that single color and put it into each of the three positons and there is no other place to put it. So it is good. I think you guys are showing that you have really grown. Not just your students. Good! (Meeting 11/20/13 students' work transcript, lines 323, 325)

Another teacher, T9, reported that she didn't see students' growth in their arguments until the third cycle problem. T9 wrote the following comments in her final project:

> I cannot tell you how amazed at the arguments the students were making! This lesson study really opened my eyes to how important it is to allow the students to reason through the math they are learning. (T9 Final Project, p. 34)

### *11.3.3 Summary of Teachers' Beliefs*

The findings from the teacher beliefs had two main themes: change in expectations and abilities with special education students and teachers' growth regarding questioning techniques and pedagogical practices. The intervention helped teachers to see that all students are capable of reasoning through mathematical problems. The intervention also helped teachers to believe that they could become better teachers by asking more questions and providing their students more opportunities to question each other as well as explore solutions to problems.

## 11.4 Summary of Overall Findings

Seminal findings were revealed from the study's data sources. The following two main themes emerged from the findings of the instructor's moves analysis: non-leading questioning and recognizing reasoning as a process. From the findings of the beliefs analysis, the following two themes that emerged: change in expectations of special education students and teachers' growth regarding non-leading questioning and recognition of convincing arguments. Teachers were also able to use and recognize forms of argument and strategies for solving the three cycles of problems.

# Chapter 12 – Conclusions

The results of this case study of ten teachers and one instructor revealed that (1) teachers' expectations of students' abilities increased, particularly with special education students, (2) teachers showed evidence of growth in using non-leading questioning techniques and pedagogical practices, and (3) that attending to students' reasoning is a gradual and continual process. The ten in-service middle school teachers that participated in the intervention were able to use and recognize reasoning in the following contexts: (1) teachers' using and identifying reasoning in the mathematical tasks themselves, (2) identifying reasoning from research students' work as teachers read articles and watched videos of students performing the mathematical tasks, and (3) identifying reasoning from samples of current students' work.

## 12.1 Conclusion Summary

There is evidence from this study that suggests teachers' expectations of students' abilities gradually increased, particularly with the seven teachers who taught special education students. During the Unit 2 on-line discussion thread where the instructor asked teachers to predict how their students would arrange the towers for the four-tall towers problem, some teachers made the following initial comments that suggested their expectations of their students' abilities were low:

> This school year, I am teaching 7$^{th}$ grade resource math. Based on my experience, middle-school students struggle with any type of abstract thinking especially resource level. They either jump right into something without even thinking about the concept or they don't even attempt something on their own without asking for any type of help or assistance right away. (Unit 2, on-line discussion thread, T3, line 7).

> This year I am teaching 6$^{th}$, 7$^{th}$, and 8$^{th}$ grade LLD students (Self-Contained). During this week of school I have been doing testing to figure out what mathematical levels they are all on. Though testing is not complete yet I

> have noticed that they are 2 to 3 grade levels below in their mathematical knowledge. (Unit 2, on-line discussion thread, T2, line 7).
>
> I gave my 7th grade students that are in a resource room a 5th grade pretest and realized that they are much lower than I expected. (Unit 2, on-line discussion thread, T1, line 20).
>
> It did bother me that I had made some of the same reasoning arguments that students have made. I would like to think that I am more advanced than those that I teach. (Unit 3, on-line discussion thread, T8, line 46).

The following comments from the teachers' final projects suggest that teachers were surprised or impressed by their current students' work affecting their expectations of students' abilities:

> With the four towers, I was quick to accept arguments for fear of them reaching frustration. However, as we went on with the tasks, my students showed such improvement and I realized they were capable of more. (T1, final project, p. 27)
>
> Even though they are special education students, I should not set my expectations of their abilities low. I have learned that I can challenge them and that they can surprise me and rise to that challenge. I have learned that mathematics is not just basic arithmetic and procedure, that understanding the process behind the mathematics is very important. Since the implementation of these tasks, I have been trying to encourage the why behind the math. I have been trying to teach and assess my students' understanding of the reasoning of math. In the beginning, my students' reasoning and mathematical thinking was little to none. They did not question math or event think about explaining math, they just knew what to do and figured that was good enough. Now they are working on improving their reasoning through these tasks as well as their everyday assignments. (T2, final project, p. 32)
>
> I have learned to never underestimate my students because the work they come up with can be quite surprising, both good and bad. As a special education teacher, I often have to really encourage my students to put as much effort as possible into their work. With these tasks, they were extremely motivated and enjoyed working on them. (T3, final project, p. 57)
>
> I was pleasantly surprised and impressed by the small steps my students made towards solving the problems form each cycle but I am very concerned with the lack of ability to explain and give a convincing argument. (T4, final project, p. 28)

> The students' mathematical reasoning and thinking surprised me throughout all of these problems. So many of my students solved these problems in ways I never would have predicted. Their organization and work in some cases were impressively sophisticated. Many of the ways students recorded their work, or arrived at their solution, surprised me. (T6, final project, p. 36)

> From comparing the first task and challenge problem, I was able to see growth in the students and their reasoning. Many students were automatically grouping the towers in different ways before being asked if it was possible and they were providing better verbal reasons. I was also able to see the positive influence on questioning students to get them to think about their work and other possibilities in mathematics. (T7, final project, p. 33)

> I also saw that middle school students were able to develop more advanced strategies than I had initially come up with. As we did the other problems, I was able to apply and develop the strategies I used from the previous exercise. During the implementation of these lessons, I learned that I had completely underestimated my students. They showed perseverance throughout the projects. My students were capable of solving the problems, especially when using manipulatives. (T8, final project, p. 38)

> Given the first two tasks, I was a little discouraged by some of my students' responses. I did not really see growth until after the third task. I cannot tell you how amazed I was at the arguments the students were making! This lesson study really opened my eyes to how important it is to allow the students time to reason through the math they are learning. (T9, final project, p. 34)

> I have learned that my students have strong reasoning skills when given the opportunity to explore concepts that have multiple ways to reach a solution. It is wonderful to see that even students who struggle the most were able to use their own reasoning abilities and build on ideas with a partner to explain how they arrived at a solution. The success that these students experienced as a part of these tasks made them more confident and excited to work on the tasks later in the semester. (T10, final project, p. 37)

The intervention experiences gave teachers the opportunity to see that all students, whether regular or special education, were capable of higher-level reasoning.

Maher, Landis, and Palius (2010) designed and conducted a beliefs study to examine 20 middle-school classroom and special education teachers' beliefs during a year-long intervention. The following objectives and interests were posited in the Maher et al. (2010) research:

> An objective was to track changes, if any, in teacher-held beliefs during the course of the intervention. Our expectation was that learning to attend to forms of reasoning they use in problem solving and to be more attentive to children's reasoning by studying videos might affect certain held beliefs. We were particularly interested in whether there would be differences between special education and regular classroom teachers in terms of the expectations about student learning and the conditions that teachers can create to influence children's learning. (p. 14)

For the Maher et al. (2010) research, a subset of 13 of the 34 Beliefs assessment items were aligned with the intervention had changed. The research revealed that no differences between special education and regular education teachers were found. Moreover, special and regular education teachers were reported to have changes in beliefs.

In addition to increased expectations of their students' abilities, evidence suggests that teachers improved their questioning techniques and pedagogical practices. The following teachers' comments suggest that teachers believed their questioning and pedagogical practices improved:

> I give more 'wait' time and am not so quick to give and explain the answer. (T1, final project, p. 27-28)

> I feel that through the implementation of these tasks as well as the information I have gained throughout this course has made me a better facilitator. I find myself questioning my students more instead of leading them toward the answer. I have a different view about my teaching style and I feel that this change is going to benefit all. The more I question my students, the better they understand the math they are given. (T2, final project, p. 32)

> I find that I am now asking even more questions than ever in my classroom. I am no longer guiding my students to arrive at the correct answer, but allowing them to explore and question each other. (T3, final project, p. 57)

> I was recently observed by my principal and an administrator from outside the building and they complimented me on pushing each student to continually explain what they saw and why they were answering in a specific way. (T4, final project, p. 28)

> Also, my questioning skills have improved throughout this course. Often I give too much away when I question, or guide my students too much. I know that I do it, but after this class, I am able to question my students without fully leading them to the answer. (T5, final project, p. 20)

> I have been encouraging students to write explanations more often. (T7, final project, p. 33)

> It was also evident that I need to incorporate writing into my lessons. (T8, final project, p. 38)

> I have noticed that I take more of a facilitator role in the classroom. I try to allow my students the time and space to discuss, and even argue, with their peers to determine the solutions to problems. I find myself asking the students if their peers are convincing them. I also ask them to defend their positions. This lesson study has definitely made me a better teacher. (T9, final project, p. 34)

The intervention experiences gave teachers the opportunity to improve their questioning techniques and pedagogical practices.

Another seminal finding from this study was that attending to students' reasoning is a gradual process. The following comments from the teachers' final projects suggest that teachers realized that attending to students' reasoning is a gradual and continual process:

> I learned that my students' oral reasoning was much stronger than their written reasoning. My students had a lot of difficulty explaining the pizza task, but they did show a lot of improvement by the third. (T1, final project, p. 27)

> I learned at the very beginning I needed to really push the idea of giving a convincing argument. My students definitely struggled with this, but by the third task; they knew what was expected of them. They had learned through these three tasks that they needed to organize their thinking on paper and include details of what and why they did what they did. (T3, final project, p. 57)

> I am going to give them the three cycle tasks again and then have them compare where their thinking was at the beginning of the year to where they have grown to at the end of the year. (T4, final project, p. 28)

> I think they made a better connection mathematically because they were hands-on and they could make the task their own. It is nice to see what the freedom of

> thinking can show me. They have also improved on explaining their work and it is starting to come naturally to them. (T5, final project, p. 37)
>
> Throughout the three problems we worked on in class, I can see an improvement from the beginning to the end in the students' ability to formulate a convincing argument. For the first problem, students struggled to write down an explanation. For the second problem, students did a better job writing; however, some students failed to provide a convincing argument and rather recorded what different combinations they were able to make. For the last problem, building towers three high with three colors, many students were able to write why they had created all the towers, rather than just simply listing the towers they created. With practice, students improved their abilities to write a convincing argument and they also learned what is expected in a convincing argument. I watched students recognize that restating what they did was not necessarily convincing. It was evident by the third problem that students had a better grasp on being convincing. (T6, final project, p. 35)
>
> From comparing the first task and the challenge problem, I was able to see growth in the students and their reasoning. Many students were automatically grouping the towers in different ways before being asked if it was possible and they were providing better verbal reasons. (T7, final project, p. 33)
>
> Students need to discuss math with each other. They need to build and manipulate. They need to be given the time to think about the problems they are given. I really learned that writing about math is very important. Students who have the opportunity to defend their thoughts and ideas will gain a deeper understanding of the concepts. (T9, final project, p. 34)
>
> I think my students are still struggling with the development of writing a convincing argument, but they were mostly able to give convincing verbal arguments by the completion of the third task. (T10, final project, p. 38)

The intervention experiences helped teachers recognize that attending to students' reasoning is a gradual process that requires time for students to develop and that students should be encouraged to continue strengthening their reasoning throughout their future mathematical studies.

### 12.2 Implications and Limitations

The analysis of the research data from this case study of ten teachers and one instructor indicated that the intervention was successful in helping teachers use and recognize forms of reasoning. As teachers worked on the tasks, the teachers were able to experience what was expected of the students. This helped to prepare the teachers to recognize the forms of reasoning from research and current students. During an interview, the instructor said the following about why it was important for teachers to complete the tasks before implementing the tasks in the classroom with current students:

> I think if they didn't complete the tasks they may not even understand what the task is about. Maybe wouldn't be able to follow the reasoning or the thought process of their students. I think that they really have to be a problem solver to see the struggles that are involved and where their students might have struggles as well. I think that they can't just implement in a thoughtful way unless they had a chance to be a problem-solver themselves. (Landis, 2013b)

Teachers also compared their current students' work to the research students' work from Stephanie, Milin, Brandon, and Romina at times during the meetings and on-line discussions.

Although case arguments were the most used and recognized, recursive arguments were used and recognized frequently; particularly in the first and third cycle of tasks. However, there were only two samples of current students' work discussed where the students used a recursive argument to find the pizza combinations (10/22/13 students' work meeting transcript, lines 448; 602-603).

Inductive arguments were recognized by eight teachers on the Gang of Four video post-assessment. However, it was rare when teachers or students used an inductive argument when solving the problems from the cycles. One teacher pair used an inductive argument for the first cycle (9/7/13 meeting transcript, line 90). Two teacher pairs used an inductive argument for the third cycle tasks (10/22/13 meeting transcript, lines 361, 579-580). The instructor also mentioned that one student appeared to use an inductive argument for solving the three-tall towers problem for the third cycle (10/22/13 in-district class meeting transcript, lines 118-122). The instructor's moves throughout the intervention helped teachers to attend to students' reasoning.

The instructor moves that encouraged teachers to persevere in solving the tasks were mainly probing, explanation, and justification questions. Other frequently-used pedagogical moves were motivating, inviting, re-voicing, and monitoring. The instructor said the following during an interview about what the instructor expected teachers to do and say during the implementation of the tasks with the current students:

> I would hope they [the teachers] would be presenting the problem and then they would step back. In other words, I wouldn't want them to be leading the students down the path that they took as a problem solver or the path they thought the students should take. I would hope that they would be letting the students go down the path that they [the students] want and I would hope they [the teachers] would be listening carefully so they could try and understand the path the students were taking. I would hope that if they really didn't understand what the students were doing that they would ask the students to explain what they were doing so that they could have a better understanding of the students' thinking. (Landis, 2013b)

The expectations were modeled by the instructor as teachers worked on the tasks as well as when attending to current and research students' reasoning.

Teachers' beliefs were also examined for this study. Slight changes in the teachers' beliefs were noted from the beliefs pre-assessment to the beliefs post-assessment for two main categories: expectations and abilities and concepts and procedures. Both categories had six teachers that improved the percentage of questions that were consistent with the standards from the pre- to the post-assessment. During an interview, the instructor said the following about the knowledge teachers should have after successfully completing the course requirements:

> I would hope you would be a better listener; to really understand what your children your students were thinking and how their mathematical thoughts were

> developing. I would hope that you would be willing to know that students can learn from each other as well as from you; and give them opportunities to talk about the mathematics, argue about the mathematics. I would think that you would be a better problem-solver yourself because you would be willing to struggle and not always see that math is easy answers. I would hope that you would be able to celebrate when your students do something that maybe even would be neater or more elegant than the way you went about solving something. (Landis, 2013b)

Ten in-service teachers participated in the intervention. Due to the small sample size, the results from this study cannot be generalized. The instructor implemented the intervention with the ten teachers of the southern region but was free to choose different tasks, articles, videos, and on-line discussion questions from the instructor of the northern and central regions. The in-district classroom visits were not videotaped from any of the regions for this study. Videotaping the in-district classroom visits may have led to a more in-depth analysis for studying the teachers' recognition of students' reasoning.

It should also be noted that the intervention was implemented in a graduate-level course for one semester. Teachers were offered incentives such as waiving tuition and student fees for seven masters-level courses at Rutgers University, receiving stipends after successfully completing summer institutes, and using the courses towards a Master's degree at Rutgers (NJPEMSM, 2009). However, teachers were required to invest time and effort by attending regional meetings, responding to on-line discussions, implementing the tasks with their students, and completing final projects. Without the NJPEMSM project incentives, it is possible that this study and similar studies may have had different results.

### 12.3 Further Research Suggestions

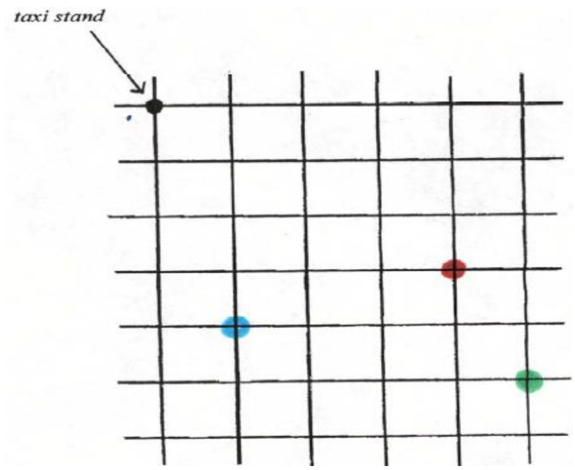
Some teachers expressed concern over the difficulty level of the tasks because seven out of the ten teachers taught special education when implementing the mathematical tasks and in some cases had mild cognitively disabled students as in the case of T8's students with Down's syndrome. At the first regional meeting, the instructor had suggested that "If you have children that really are very low functioning, this might be too challenging a problem for them and maybe building towers three-tall might have been enough of a challenge. And maybe that would be something they would be more successful with." (Meeting transcript 10/2/13, line 744). Possible further research might include adjusting the level for special education students with severe learning handicaps where control and experimental groups could be formed and compared by using the three-tall towers problem first.

Future research might include comparing the results of the other cohorts from the same instructor. In one such study, McGowan (2015) examined seven in-service teachers attention to reasoning during fall 2010 using the same instructor, mathematical tasks and contexts that were used for this case study of ten teachers. McGowan found similar patterns from his data in regard to case arguments and inductive arguments. Moreover, McGowan (2016) reported that teachers in his study "used or described recursive arguments during cycle 2 of the intervention" but teachers did not refer to a particular argument such as Stephanie's Case Argument or Milin's inductive argument (p. 167).

Currently, Wright (2015) is in the process of examining a study with teachers that were in the NJPEMSM program in 2010 that experienced the same intervention but had a different instructor. The instructor of the intervention used the first two cycles of problems that the instructor used in this study and the study by McGowan (2015). For the last problem, the instructor used the following Taxicab Problem:

> A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only 3

> times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route. What is the shortest route from the taxi stand to each point? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If it is so, how many? Justify your answer. (Maher, Powell, and Uptegrove, 2010, p. 146)




Future research might also include continuing this intervention with a new generation of in-service teachers. Videotaping and analyzing current students as they work on the tasks during the in-district classroom visits would be worth exploring for future research. Similar PD intervention models can also be offered or modified for pre-service teachers. Whatever the future may hold, it is exciting to see that more teachers are recognizing the value of attending to students' mathematical reasoning in their classes.

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# Appendix A
## Beliefs Assessment

1. Learners generally understand more mathematics than their teachers or parents expect.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

2. Teachers should make sure that students know the correct procedure for solving a problem.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

3. Calculators can help students learn math facts.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

4. It's helpful to encourage student-to-student talking during math activities.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

5. Math is primarily about learning the procedures.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

6. Students will get confused if you show them more than one way to solve a problem.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

7. All students are capable of working on complex math tasks.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|

Strongly Agree Strongly Disagree

8. Math is primarily about identifying patterns.

1 2 3 4 5
Strongly Agree Strongly Disagree

9. If students learn math concepts before they learn the procedures, they are more likely to understand the concepts.

1 2 3 4 5
Strongly Agree Strongly Disagree

10. Manipulatives should only be used with students who don’t learn from the textbook.

1 2 3 4 5
Strongly Agree Strongly Disagree

11. Young children must master math facts before starting to solve problems.

1 2 3 4 5
Strongly Agree Strongly Disagree

12. Teachers should show students multiple ways of solving a problem.

1 2 3 4 5
Strongly Agree Strongly Disagree

13. Only really smart students are capable of working on complex math tasks.

1 2 3 4 5
Strongly Agree Strongly Disagree

14. Calculators should be introduced only after students learn math facts.

1 2 3 4 5
Strongly Agree Strongly Disagree

15. Learners generally have more flexible solution strategies than their teachers or parents expect.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

16. Math is primarily about communication.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

17. Manipulatives cannot be used to justify a solution to a problem.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

18. Learners can solve problems in novel ways before being taught to solve such problems.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

19. Understanding math concepts is more powerful than memorizing procedures.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

20. Diagrams are not to be accepted as justifications for procedures.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

21. If students learn math concepts before procedures, they are more likely to understand the procedures when they learn them.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

22. Students are able to tell when their teacher does not like mathematics.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

23. Collaborative learning is effective only for those students who actually talk during group work.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

24. Students should be corrected by the teacher if their answers are incorrect.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

25. Mixed ability groups are effective organizations for stronger students to help slower learners.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

26. Collaborative groups work best if students are grouped according to like abilities.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

27. Conflicts in learning arise if teachers facilitate multiple solutions.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

28. Learning a step-by-step approach is helpful for slow learners.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

29. Only the most talented students can learn math with understanding.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

30. The idea that students are responsible for their own learning does not work in practice.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

31. Teachers need to adjust math instruction to accommodate a range of student abilities.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

32. Teacher questioning of students' solutions tends to undermine students' confidence.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

33. Teachers should intervene as little as possible when students are working on open-ended mathematics problems.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

34. Students should not be penalized for making a computational error when they use the correct procedures for solving a problem.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Strongly Agree | | | | Strongly Disagree |

# Appendix B

## Subset of 22 Beliefs on the Learning and Teaching of Mathematics

*Statements inconsistent with current standards are indicated with an asterisk.

Beliefs on Student Learning of Mathematics

| | |
|---|---|
| Q1 | Learners generally understand more mathematics than their teachers or parents expect. |
| Q7 | All students are capable of working on complex math tasks. |
| Q9 | If students learn math concepts before they learn the procedures, they are more likely to understand the concepts. |
| *Q11 | Young children must master math facts before starting to solve problems. |
| *Q13 | Only really smart students are capable of working on complex math tasks. |
| Q15 | Learners generally have more flexible solution strategies than their teachers or parents expect. |
| Q18 | Learners can solve problems in novel ways before being taught to solve such problems. |
| Q19 | Understanding math concepts is more powerful than memorizing procedures. |
| *Q23 | Collaborative learning is effective only for those students who actually talk during group work. |
| Q28 | Learning a step-by-step approach is helpful for slow learners. |
| *Q29 | Only the most talented students can learn math with understanding |
| *Q30 | The idea that students are responsible for their own learning does not work in practice. |

Beliefs on Teaching of Mathematics

| | |
|---|---|
| Q2 | Teachers should make sure that students know the correct procedure for solving a problem. |
| Q4 | It’s helpful to encourage student-to-student talking during math activities. |
| *Q6 | Students will get confused if you show them more than one way to solve a problem. |
| *Q10 | Manipulatives should only be used with students who don’t learn from the textbook. |
| Q21 | If students learn math concepts before procedures, they are more likely to understand the procedures when they learn them. |
| Q24 | Students should be corrected by the teacher if their answers are incorrect. |
| Q31 | Teachers need to adjust math instruction to accommodate a range of student abilities. |
| *Q32 | Teacher questioning of students’ solutions tends to undermine students’ confidence. |

Beliefs on Mathematics

| | |
|---|---|
| *Q5 | Math is primarily about learning procedures. |
| *Q17 | Manipulatives cannot be used to justify a solution to a problem. |

# Appendix C
# Gang of Four Video Assessment

Title: Gang of Four

Context: This episode is an assessment interview with four 4th grade students, Milin, Michelle, Jeff and Stephanie, for building all possible different towers of a particular height when selecting from two colors of unifix cubes. The children, working in pairs, had built towers four and five cubes tall during class sessions. Each of the children was subsequently interviewed individually and asked to describe how he or she had approached the tasks and to justify any solutions that had been constructed. In this group interview, the students are sharing their ideas about the towers problems, explaining and justifying their solutions to each other. While they consider towers of various heights during the session, they specifically reason about towers that are three cubes tall. Although unifix cubes were available, the children chose not to use them during the interview. The segment begins with short clips from the 4th grade classroom session to provide a background context of the students' building and organizing their towers with unifix cubes.

After viewing the video of the children explaining and justifying their approaches to the problems, please describe as **completely** as you can: (1) each example of reasoning that a child puts forth; (2) whether or not the reasoning forms a valid argument; (3) whether or not the argument is convincing; and (4) why or why not you are convinced. Give evidence from the interview to support any claims that you make. You may refer to the attached transcript as needed.

Each response will be evaluated according to the following criteria:

- Recognition of children's arguments
- Your assessment of the validity or not of children's reasoning
- Evidence to support your claims
- Whether the warrants you give are partial or complete

Appendix D

**Gang of Four Group Interview**
**Transcript for Episode**

**This episode is an assessment interview with four 4th grade students, Milin, Michelle, Jeff and Stephanie, for building all possible different towers of a particular height when selecting from two colors of unifix cubes. The children, working in pairs, had built towers four and five cubes tall during class sessions. Each of the children was subsequently interviewed individually and asked to describe how he or she had approached the tasks and to justify any solutions that had been constructed. In this group interview, the students are sharing their ideas about the towers problems, explaining and justifying their solutions to each other. While they consider towers of various heights during the session, they specifically reason about towers that are three cubes tall. Although unifix cubes were available, the children chose not to use them during the interview.**

**The segment begins with short clips from the 4th grade classroom session to provide the background context of the students' building and organizing their towers with unifix cubes.**

**Transcript**

**CM:** ……You know the towers problems?

**All:** Yeah.

**CM:** The last one we did in class - Remember what that was about?

**Jeff:** Robin Hood? That was the last one we did –

**M, M, S:** Towers of 5!

**CM:** You remember what you did with those Towers of 5?

**All:** Um-hm.

**CM:** Um-hm. Tell me about it. What was the problem?

**Jeff:** **How many –**

**Michelle:** **You had to figure out how many – how many different towers you could make for five blocks up.**

**CM:** **Any five blocks?**

**All:** **No. Two colors.**

**CM:** **Two colors. OK. And did you figure that out?**

**All:** **Yeah.**

**CM:** **And what is it? Do you remember?**

**All:** **32!**

**CM:** **You're sure of it?**

**All:** **Yeah!**

**CM:** **How can you be so sure?**

**Milin:** **We checked!**

**CM:** **How can you be so sure?**

**Jeff:** **Remember when we did all the charts - the thingies – the**

**Milin:** **And then remember –**

**Jeff;** **All the different patterns. Remember, I convinced you up in the –**

**CM:** **Yeah – in the room. OK. But I remember saying to you, Jeff, and I remember saying to you, Michelle – and to you, Stephanie - and Stephanie did try to work on towers of six and I asked all of you if you-**

**Milin:** **So did I.**

**CM:** You did, too? If you were building towers of six, how many would there be?

**Jeff:** I don't know

**Michelle:** I did some but I didn't-

**CM:** But do you know how many?

**Stephanie:** Yeah.

**Milin:** Probably 64.

**CM:** Why do you think 64?

**Milin:** Well, because there was a pattern.

**CM:** What's that?

**Milin:** You just times them by two

**CM:** Times what by two?

**Milin:** The towers by two, because one is two, and then we figured out two is two, and then, I mean four, and then –

**Jeff:** You are not making much sense!

**Michelle:** See, if you had only one block up and two colors, then you would have two towers, and we figured out that the other day that you keep on doing…

**Jeff:** Everything is opposites!

**Michelle:** …like two times two would be four and then…

**CM:** So four would be for what?

**Stephanie:** All you have to do-

**Michelle:** …four for, there would be four towers for two high.

**CM:** Okay.

**Jeff:** They are all opposites though.

**CM:** Okay well, let me hear what Michelle is saying.

**Michelle:** And then for this three high, you would have eight towers and four high, you would have twelve towers and then you keep on doing it like that.

**CM:** Do you agree with that?

**Jeff:** I don't know what you are talking about.

**Stephanie:** Well. What it is - is-

**Michelle:** Well – five high would be twenty-five and then –

**CM:** Okay, lets get a piece of paper and write down what you are saying and see if you all agree. I think Jeff hasn't been with us for a while and he doesn't know what we are talking about. But let's take one at a time. Let's just agree as we are moving along.

**Michelle:** If you had one high see there is red and blue then you would have two and then if you had –

**CM:** Okay, write that down. Two. Did you agree with that?

**Jeff:** Yeah.

**CM:** Do you know what she is talking about?

**Jeff:** There is one red and one blue so there is only one way to do it so it's two.

**CM:** One way you can do it and so it's two.

**Jeff:** Yeah, you see if you have to make towers of one and there is only two colors

**Milin:** He keeps on doing that.

**CM:** All right, let's go on.

**Michelle:** If you had two towers that would be four, because you have-

**Jeff:** Yeah I agree with that. Okay.

**Michelle:** See you would just times it. See two times two

**CM:** Okay just hold on okay write the four down. Look I don't …Can you explain to me why from two you would get to four? Milin, tell me why.

**Milin:** For each one of them you could add one – no two more on because there is a black, I mean a blue, and a red –

**Jeff:** What she is doing…

**CM:** Let her finish. Okay.

**Milin:** See. For that you just put one more for red you put a black on top and a red on top - I mean blue on top instead of black and on blue you put a blue on top and a red on top. You keep on doing that.

**CM:** Do you understand what he is talking about?

**Stephanie:** Uh- huh!

**CM:** You all understand what he is talking about?

**Jeff:** Yeah.

**CM:** All right. So - so we agree four. What happens if you're building towers three high? What did you say it would be?

**All:** It would be eight.

**CM:** Write eight down. Can you give me an argument; you don't have to do it. Why we jumped from four to eight?

**Michelle:** There's-

**CM:** Shhh. That's what Jeff wants to know

**Michelle:** There's - there's-

**CM:** Go slow. It's Jeff you are convincing not me.

**Michelle:** There is two blue. There is two here.

**Jeff:** I know that.

**Michelle:** And then we went to four so it would have to be times. Two times two equals four and four times two would equal eight.

**CM:** That doesn't help Jeff understand. He just knows that we are multiplying two times two

**Milin:** I know! I know!

**Stephanie:** All right.

**Jeff:** If this…

**CM:** Okay. One at a time.

**Jeff:** If this was like a pattern it would go two - four – six in between the eight.

**CM:** Yeah, that's what he is saying.

**Milin:** No! No!

**Stephanie:** But that's not the pattern we are working on.

**CM:** Go ahead Stephanie.

**Stephanie:** The pattern that we saw was this. For one block at a time we found two.

**Jeff:** We already got two and four
.

**Milin:** Two, four, six-

**Stephanie:** I know – two, four and then eight – Right? Two, four and then eight.

**CM:** Why eight? That's what Jeffery asked about.

**Milin:** I know.

**CM:** Go ahead. Let Milin persuade Jeff.

**Milin:** If you do that you just have to add for each one of those you have to add

**CM:** Each one of what? These four?

**Milin:** Yeah. You have to add one more color for each one

**CM:** Which way are you adding it? Where are you putting that one more color, Milin?

**Milin:** No. Two more colors for each one. See-

**CM:** So this one with red on the bottom and blue on the top.

**Milin:** You could put another blue or another red.

**CM:** You agree with that? You can put a blue or red on top and that-

**Milin:** And that will be two and then on this you could put another red or blue on top that will be four.

**Jeff:** That is the same right there.

**CM:** No, this is blue red

**Jeff:** No. Here look. It's blue oh okay, okay.

**Milin:** See. Now you see.

**CM:** Could you find what Milin is saying and now here you could put-

**Milin:** A red or a blue and same thing here

**CM:** Do you understand that?

**Jeff:** Yeah.

**CM:** So do you see how you get eight?

**End of first segment**
**Begin second segment**

**Stephanie:** Yeah, but that's what he is like, that's what is different from mine I just like took the things and went- I just took one and went –

**Milin:** And kept on-

**Stephanie:** Here is one red/red/red, blue/blue/blue and then I go like red/blue/blue, blue/red/blue-

**CM:** So, what I am hearing you say is that you're just…

**Milin:** Guessing!

**CM:** …you believe there is eight. But you say guessing. Now, why does that sound like guessing?

**Milin:** Because what if you could make more?

**Stephanie:** **Okay, this is the three high. Right? And you're convinced you can make eight?**

**Milin:** **Yep!**

**Stephanie:** **I'm convinced I can make eight.**

**CM:** **Yeah, but you haven't- he's proved to me from the four you can only make eight you can get two from this one, two from this one, two from this one and two from this one.**

**Milin:** **But could you convince her?**

**Stephanie:** **Convince who? Michelle? Him?**

**Milin:** **No. Her.**

**Stephanie:** **Her? Yeah. All right. I've done this before. Okay.**

**CM:** **Take another piece of paper if you want to, because it sounds like your approach is a little bit different**

**Stephanie:** **Alright.**

**CM:** **You've got to convince me there are eight and only eight, and no more or fewer.**

**Milin:** **You do draw big!**

**CM:** **Now Jeff this might be a little different here. Let's see what's going on here.**

**Stephanie:** **All right, first you have without any blues, which is red/red/red.**

**CM:** **Okay, no blues.**

**Stephanie:** **Then you have with one blue –**

**CM:** **Okay**

**Stephanie:** Blue/red/red or red/blue/red or red/red/blue.

**CM:** Anything else?

**Michelle:** And you would do the same pattern for-

**Stephanie:** No, not with the blue, not with one blue –

**Michelle:** You would do it, you would do it with one red and two blues? Alternate –

**Jeff:** You would alternate-

**Michelle:** You would do it the other way around.

**CM:** That's not what she is doing. Let her finish. That's what you would do. You would alternate. Let's see what Stephanie does. Maybe she's not going to do that.

**Stephanie:** Well, there's no, there's no more of these because if you had to go down another one you'd have to have another block on the bottom. But then you have with three blues – well, not with three blues. I'll go like this.

**CM:** You have no blues and now you have exactly one blue.
**Stephanie:** Now you have exactly two blues. Wait, wait . Actually, that's what I did last time I was here. I did exactly two blues.

**CM:** Okay. Let's see.

**Milin:** And then you did three blues.

**Stephanie:** All right. You could put blue/blue/red; you could put red/blue and blue.

**Milin:** You could put blue, red and blue. You could put…

**Stephanie:** Yeah, but that's not what I am doing. I'm doing it so that they're stuck together.

**CM:** **Okay.**

**Jeff:** **There should be one – there could be one with one red and then you could break it up and there's one with two reds and there's one with three reds and then…**

**Milin:** **Ah, but see – you did the same thing, but there's the blue.**

**Jeff:** **See, there's all reds and there's three reds, two reds. There should be one with one red. And then you change it to blue.**

**Stephanie:** **Well, that's not how I do it.**

**CM:** **Let's hear how Steph – we'll hear that other way; that's interesting. Okay, now, so what you've done so far is –**

**Stephanie:** **One blue, two blue.**

**CM:** **Okay, no blues**

**Stephanie:** **One blue, two blue.**

**CM:** **One blue, and two blues, but Milin just said you don't have all two blues, and you said that – why is that?**

**Stephanie:** **All right, so show me another two blues. With them stuck together, because that's what I am doing.**

**Milin:** **In that case here.**

**CM:** **Okay, so now what are you doing, Stephanie?**

**Michelle:** **What if you just had two blues and they weren't stuck together, you could –**

**Stephanie:** **But that's what I'm doing. I'm doing the blues stuck together.**

**CM:** **Okay.**

**Stephanie:** **Then we have three blues, which you can only make one of. Then you want two blues stuck apart- not stuck apart – took apart.**

**CM:** **Separated?**

**Stephanie:** **Yeah, separated. And you can go blue/red/blue –**

**CM:** **Okay, so Milin wanted to stick that in earlier I thought and Michelle right where you were doing two blues you wanted stuck-**

**Milin:** **'Cause see, look at this for two reds and one blue**

**Michelle:** **They are not stuck together here, the two reds**

**Milin:** **Yeah so, you are following no pattern.**

**Michelle:** **And you have more stuck together here.**

**Stephanie:** **Well, you are following your pattern, but my pattern goes no red, one red, this was not meant to be like that. That's not – it's in the category of one blue. I could stick that in another category, but I want this to be in the category of one blue and not in the category of opposite of this one. And then I have this one red/red/blue. So, to you - you might put that way at the end of the line but I put it right here.**

**Jeff:** **I have a question. Do you have to make a pattern?**

**Michelle:** **No.**

**Jeff:** **So, then why is everybody going by a pattern?**

**Milin:** **Because we like to.**

**Stephanie:** **It's easier. It's easier to find than just going: "Ooh, there's a pattern!"**

**Michelle:** **'Cause if you just keep on guessing like that, you're not sure if there is going to be more.**

**Stephanie:** It's easier, maybe, like Shelly and Milin's pattern was to go put this in a different category.

**Jeff:** I know their pattern.

**Stephanie:** Okay, but what I'm saying is it's easier, it's just easier to work with a pattern since it's like

**Milin:** Oh, here's another one. Let's see

**Stephanie:** Yeah, that looks good, let's put that in.

**Michelle:** Because you might have a duplicate, and then you may not know.

**Stephanie:** It's harder to check. It's harder to check just having them like come up from out of the blue.

**Milin:** Then just going like this and getting two, one

**Jeff:** How do you know there's different things in the pattern?

**Milin:** Since, see. Look at this. These are all different right?

**Jeff:** I see that

**Milin:** Yeah. See. From this, right, you can make two more. So, because – uhm – because here there 's a blue / red and then a blue-

**Michelle:** 'cause there's 'cause there's only two colors more so you know you can't make any more.

**Milin:** And there's red I mean blue/red/red And you can't make any more in this, so you go on to the next one.

**Stephanie:** All right, and these –

**Jeff:** How do you know you can't make any more from that?

**Milin:** Because there's not any more colors.

**Stephanie:** Look, okay, start here. Sorry. Start here – okay, you have the three together. The one, one blue, right? You have the one blue. How could I build another one blue?

**Jeff:** You can't.

**Stephanie:** All right, so I have convinced you that there's no more one blue?

**Jeff:** Yeah.

**Stephanie:** All right.

**Michelle:** But if you didn't have that pattern, it would be harder to convince you.

**Stephanie:** If I went I will put this one blue over here and that blue will be on another piece of paper-

**Jeff:** Yeah but-

**Stephanie:** How will –

**Jeff:** You can make a blue different from what you did if you go like this-

**Michelle:** That's if you have four.

**Jeff:** If you go like this, you can go red/red/blue or you can go blue/red/red

**Stephanie:** That's what I have.

**Jeff:** No they are all different. You can do red/blue/red –

**Stephanie:** What I am saying is this is one blue. This is one blue.

**Jeff:** Yeah, there's still all different with one blue.

**Stephanie:** Yeah-

**Milin:** No, but only on the bottom

**Stephanie:** **But I have those three. Look blue/re/red, red/blue/red, red/red/blue. But then how am I supposed to make another one once that blue got down to the last block?**

**Jeff:** **You can't.**

**Stephanie:** **Okay, so I've convinced you that there's no more one blue?**

**Jeff:** **Yeah.**

**Stephanie:** **All right, now we move on –**

**Michelle:** **Then you have to go to two blue.**

**Stephanie:** **Two blue. Here's one – right? Two blue – we have one, blue/blue/red, then we have red/blue/blue. How am I supposed to make another one?**

**Jeff:** **Blue/red/blue.**

**Stephanie:** **No, this is together. Milin gave me that same argument.**

**Michelle:** **She means, she means together –**

**Jeff:** **But the thing is does it matter that they are together?**

**Michelle:** **No, she means stuck together.**

**Stephanie:** **Stuck together, that means like –**

**Jeff:** **I know.**

**Michelle:** **Okay, so can I make any more of that kind?**

**Jeff:** **No.**

**Michelle:** **Then you have to move to three, which you can make one.**

**Stephanie:** **All right, yeah, you can only make one and then you can make**

**the three with out blue with the three red.**

**Michelle:** **And then you can make two split apart.**

**Stephanie:** **Two split apart, which you can only make one of, and then you can find that you can – you can find the opposites right in this same group. All right, so I've convinced you that there's only eight?**

**Jeff:** **Yeah.**

**CM:** **How many if you're making towers of four?**

**Michelle:** **Sixteen.**

**Milin:** **Sixteen.**

**CM:** **Do you agree Jeff?**

**Michelle:** **Because –**

**Jeff:** **Yeah.**

**Michelle:** **Because you have to add…**

**CM:** **Jeff, why do you agree? Don't let them get by so easily. This could be pressure here.**

**Michelle:** **You see, look it's because say you add a red or a blue, you can add a red or a blue here-**

**CM:** **Make a "Y" or something to show me…**

**Jeff:** **I understand because you can only – you can keep – you can make**

**Michelle:** **You can put two colors here – two colors there, two colors – and keep on going.**

**Jeff:** Yeah, you can keep on doing two colors for each one. And that's two, four, six, eight, ten, twelve, fourteen, and sixteen.

**CM:** And so that's the towers of - ?

**Jeff:** Four.

**Milin:** My guess is-

**CM:** Four.

**Milin:** Sixteen but-

**Jeff:** We already got sixteen.

**Milin:** Why did she say in the beginning the whole thing - 12?

**Jeff:** This, that you get – listen

**Michelle:** It's like, it's like-

**CM:** Why did you say twelve Michelle?

**Jeff:** Red or a blue, red or blue, red or blue-

**Milin:** Jeff, Jeff, Jeff. I know that pattern. But I want to know why she said twelve before-

**Stephanie:** Yeah Michelle, why did you?

**Jeff:** She was guessing not making patterns.

**CM:** Is that true Michelle?

**CM:** Poor Michelle. It's okay.

**CM:** You think what twelve or sixteen?

**Michelle:** Sixteen.

**CM:** **Michelle thinks sixteen.**

**CM:** **Now – Now. You made towers of five in class and what did you get?**

**Stephanie:** **Thirty-two.**

**Milin:** **Thirty-two.**

**CM:** **Does that work the same way?**

**Stephanie:** **Yeah.**

**Jeff:** **They're all multiples of two.**

**Michelle:** **If you get towers of four –**

**Stephanie:** **The hard part is to make the pattern. Like, from now, we know how to just – oh, you could give us a problem like how many in ten and we could just go –**

**CM:** **How many in ten, and you'd know the answer.**

**Stephanie:** **Yeah, I know the answer. I figured it out. It's 1,024!**

**CM:** **1,024.**

**Alice:** **Are you sure?**

**Stephanie:** **Uh-huh.**

**Milin:** **Uh-huh.**

**Jeff:** **Don't try to convince her.**

**CM:** **Try to convince me.**

**Jeff:** **No.**

**Milin:** **Okay, okay, okay**

**Jeff:** **No. No.**

**CM:** **You could do that later. However you were saying you know the answer but…**

# Appendix E
## Instructor Interview Parts 1 and 2

*Part 1: October 2, 2013 Interview in person at Carl Sandburg Middle School in Old Bridge, NJ*

1) What is your name and your role in the program?

2) How do you feel about the September 7, 2013 implementation to the teachers of the towers problem?

3) In what ways, if any are you satisfied with the progress that the teachers are making in attending to the students' reasoning?

4) In what ways, if any, are you satisfied with the progress that the teachers are making in attending to their own reasoning?

5) What are your concerns with the teachers' progress and how do you plan to deal with these concerns?

6) Why do you think research on attending to students' reasoning is important?

*Part 2: Phone interview on Sunday, November 3, 2013 11:00 am*

7) What would I see if I walked in on an implementation of a mathematical task with the students?

8) What would I see if I walked in on an implementation of a mathematical task with the teachers?

9) What would you expect the teachers to do and say during the task implementation with the children?

10) Some teachers think that there is not enough time to allow students to explore mathematical tasks to promote reasoning. What would you say to a teacher who claimed

that there is not enough time to implement mathematical tasks that are based on a constructivist perspective?

11) Based on your experience, what do you think is significant to know about the amount of time needed for teachers to attend to students' reasoning?

12) Why is it important for teachers to complete the mathematical tasks before implementing the task in the classroom with students?

13) How do you feel about the teachers' progress after completing two out of the three mathematical tasks so far?

14) If I was a teacher in New Jersey Partnership for Excellence in Middle School Mathematics (NJPEMSM) project, what skills and knowledge would you expect me to leave with after successfully completing the requirements of this project?

**Transcript:** Interview with Dr. Judith Landis (in person)

**Date:** October 2, 2013

**Place:** CS Middle School

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| 1 | 00:00:00 | PC | I suppose we should start with your name and your role. |
| 2 | | JL | My role.. I'm Judy Landis. I am the instructor for the cohort four group of the lesson study. |
| 3 | 00:00:53 | PC | How do you feel the implementation went? |
| 4 | | JL | The implementation? This year? Over the four years? What? Which? |
| 5 | | PC | Well, let's start with uh, overall the four years. |
| 6 | 00:01:09 | JL | This is the fourth year and I think the um implementation is.. is a good one. I think it's a very strong model of how to um engage teachers in not only doing math in a meaningful way but of having the students do math in a meaningful way. And getting them to look at videos of children doing the same problems that they are doing and then looking at their own students doing the same problems that they have done and talking about it. I think it's very powerful. |
| 7 | | PC | Absolutely. Um so, are you satisfied with the way the cohort has progressed? |
| 8 | | JL | Yes, Yes. |
| 9 | | PC | Okay, great! Um, how would you define attending to students' reasoning? |
| 10 | 00:02:00 | JL | How would I define it? Um, I would say that you really have to have an open ear and be willing to drop the way you are thinking about something so that you can understand the way the students are thinking about |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| | | | something. It is not always easy to do but it's important to do and um uh understanding by listening carefully, understanding the way they are thinking mathematically. |
| 11 | 00:02:30 | PC | Great! Let's see uh, in what ways, if any, are you satisfied with the progress that the teachers are making attending to their own reasoning? |
| 12 | | JL | Okay, and.. |
| 13 | | PC | In a task that they are doing. |
| 14 | | JL | Okay, say it one more time. |
| 15 | | PC | When the teachers are actually doing the task themselves. |
| 16 | | JL | Right, right. |
| 17 | | PC | Um, in what ways do you feel that that they are making progress toward the goal of attending to their own reasoning? |
| 18 | 00:03:00 | JL | I think that I would probably talk about the first three cohorts because they [cohort 4] are still at the beginning of the semester and I think that they are going to grow. I think that they are starting to realize that um uh that just working with symbols that don't make sense is not a good thing. Even if it's symbols. Um, but I think that in the past 3 cohorts, um over the course of the semester, they really um thought a lot more deeply about the mathematics and um I think they grew because they were thinking more deeply. |
| 19 | 00:03:39 | PC | Great! Um, do you have any concerns; or if you do, how do you plan to deal with those concerns while teachers are working through um their reasoning? |
| 20 | | JL | Do I have any concerns? Um… I think that uh sometimes you have um teachers like in this semester that are working with children that are special needs and |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| | | | I think some of them have limits on those kids and think that they can't do certain things. And I think breaking those beliefs is an important part of this. Of letting teachers know that if they let the children have an open-ended problem that they can solve without them telling them how to solve it. Um, they are going to be surprised that the students are going to do better things than they expect. |
| 21 | 00:04:45 | PC | Great! Last thing. Last question. Why do you think that attending to students' reasoning is important? |
| 22 | | JL | Why do I think it's important? It's really what math is all about. It's not about answers. It's about process. It's about thoughtfulness. You have to attend to what they are thinking to understand what they are thinking and um where they might be going off and how to help them go on the right path. Or how wonderful they are thinking and to celebrate the way they are thinking about something. |
| 23 | 00:05:07 | PC | Thank you very much for your time. |
| 24 | | JL | You're welcome! |

Interview Part 2: Phone interview on Sunday, November 3, 2013 11:00 am

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| 25 | 00:00:00 | JL | Hello. |
| 26 | | PC | Hello Dr. Landis. This is Phyllis. |
| 27 | | JL | Yes, hi Phyllis. How are you? |
| 28 | | PC | Hi, Good, how are you? |
| 29 | | JL | I'm good, thank you. |
| 30 | | PC | That's good. Thank you so much for agreeing to |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| | | | do this. I really appreciate it. |
| 31 | | JL | Sure thing. |
| 32 | | PC | Okay, so are you ready? |
| 33 | | JL | Yeah! |
| 34 | | PC | Are you ready for my questions? |
| 35 | 00:02:00 | JL | I'm ready. |
| 36 | 00:02:30 | PC | Okay great. Um, what would I see if I walked in on an implementation of a mathematical task with the students? |
| 37 | | JL | What would you see? |
| 38 | | PC | Right. |
| 39 | | JL | Uh, you would see students working with partners. You would see students talking to each other, arguing about you know their solution paths uh trying to convince the partners of the route they're taking and trying to explain to the partners till they understand what they are doing. Um and uh you would see a lot of animated discussion. Um, it wouldn't be a quiet classroom but it would be a good noise. Um and there would probably be lots of um materials for the students to work with, manipulatives to help them in their solution task. |
| 40 | | PC | Great, um, how would it be different if I walked in on an implementation of a mathematical task with the teachers? |
| 41 | | JL | Say that one more time. How would it look different if.. |
| 42 | | PC | If I walked in on an implementation of the task with the teachers? |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| 43 | 00:03:00 | JL | In other words, the teachers not their not the children. |
| 44 | 00:03:39 | PC | Right.  The teachers doing the task. |
| 45 | | JL | Yeah, It should be very similar to be honest.  Um I would think that the teachers even though they're teachers they're also learners and when they are doing the task um they are uh using the same uh you know uh  idea of talking with a partner and arguing with a partner and trying to explain uh what they're doing and trying to understand what their partner is doing.<br><br>I think it would look much the same um and I would hope that they would be um equally engaged uh the way the students would be. |
| 46 | 00:04:45 | PC | Okay, um, if the teachers were.. um let's say they're with the children, what would you expect the teachers to do and say during this task implementation with the children? |
| 47 | | JL | Um, when they would be the implementers? |
| 48 | 00:05:07 | PC | Right. |
| 49 | | JL | Yeah, I would hope they would be presenting the problem and then they would step back.  Um, in other words, I wouldn't want them to be leading the students down the path that they took. As uh you know a problem solver or the path they thought the students should take.<br><br>Um, I would hope that they would be letting the students um, you know, go down the path that they want and I would hope they would be listening carefully so they could try and understand the path the students were taking. Um, I would hope that if they really didn't understand what the students were doing that they would ask |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| | | | uh the students to explain um you know what they were doing so that they could have a better understanding of the students' thinking. |
| 50 | | PC | Okay, um, uh, some teachers uh have said that there is not enough time to allow students to explore mathematical tasks to promote reasoning. What would you say to a teacher who claimed that there is not enough time to implement the mathematical tasks that are based on a constructivist perspective? |
| 51 | | JL | Okay, you're you're breaking in and out. So try that question again. |
| 52 | | PC | I'm sorry. Um, what would you say to a teacher who claimed that there is not enough time to implement mathematical tasks that are based on exploration or a constructivist perspective? |
| 53 | | JL | And that that is often what they say. Um, you know I think the idea of trying to uh get them to see the benefit of letting children uh build knowledge. Um, uh and perhaps they would uh then see the worth of of spending the time doing it.cause it is far better what's gained from it than the time that's lost. For so often what the teachers do is they go over the same thing year after year after year because they've been telling the students what to do and the students have been spitting it back but they really didn't understand. So when it's you know reintroduced and then and then the next lesson in the next year um it's been re-teaching. So the idea of really letting them let the students come up with uh construction of of knowledge and and building knowledge. Uh, I think is far more valuable and the time spent is is far uh you know paid back. Now I think that that's something that is hard to tell the teachers um because I think they have to actually try it and |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| | | | experience it to see the benefits and I think they do uh see the benefits and then they believe it. But I think just telling them um doesn't really,…does not really work. |
| 54 | | PC | Right. So um, so that would be..<br><br>What do you think is the most significant thing to know in regards to time for teachers attending to students' reasoning. |
| 55 | | JL | Okay, again you're breaking out. |
| 56 | | PC | I'm sorry. |
| 57 | | JL | I hear parts of your question. |
| 58 | | PC | I'm sorry. So what do you think is significant to know then, about the time needed.for teachers to attend to students' reasoning? |
| 59 | | JL | It takes time. I mean you know I think you would be um uh misleading teachers to think that they can do this in a neat 40 minute package um which many of them unfortunately have their math class um at the middle and secondary levels. Um, I think though um when teachers uh take the opportunity um to engage themselves and then to engage their students in a longer math class where the students really can do some thoughtful mathematics. Um I think they really get excited and I think they really see the benefits. So, I think you don't have to convince them once they try it. I think there are frustrations when they are working with conditions that are hard um but then they have to be creative and figure out how to overcome those conditions so that they can build meaningful math environments for their students. |
| 60 | | PC | Okay, great! Um, uh is it….why is it important for teachers to complete the mathematical tasks |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| | | | before implementing the tasks in the classroom with the students? |
| 61 | | JL | Well, I think if they didn't complete the tasks uh they may not even understand what the task is about. Maybe wouldn't be able to follow the reasoning or the thought process of their students. Um I think um you know that they really have to be a problem solver to see the struggles that are involved and where their students um might have struggles as well.<br><br>Um, I think that they can't just implement um in a thoughtful way unless they had a chance to be a problem-solver themselves. |
| 62 | | PC | Um so uh how would you feel about the teachers' progress so far after completing two out of the three mathematical tasks? |
| 63 | | JL | I think they are making progress um and it it depends where they started from. Um, I wouldn't say that everyone is um you know at the same place nor did they start at the same place. Uh, it's a continuum. I think they are all moving along it. Um, uh and you know I think they have definitely seen uh the students do some neat things that they didn't expect uh when they gave them the opportunity to um you know to problem solve with a nice rich problem. I think that this is a very different way of approaching math for most of them. And therefore there is a learning curve. And I think an idea of being comfortable with what they are doing. But I think before this, um I don't think there was anyone who really pushed students to provide justifications and you know good arguments why they thought their answers were correct and all that. This is this is something new for them and I think the idea of really pushing students to communicate verbally |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| | | | and in writing um surprisingly to me has been um a new thing for for some of the teachers.<br><br>So, I think that there is definitely is movement in the right direction and uh you know I anticipate the third task uh their comfort level with implementing it will be even better and I expect to see some neat things when we meet at that third regional meeting at the end of November. |
| 64 | | PC | Right. Right. Uh, last question. Um |
| 65 | | JL | Okay. |
| 66 | | PC | So If I was a teacher in the New Jersey Partnership for Excellence in Middle School Mathematics project, what skills..what skills and knowledge would you expect me to leave with after successfully completing the requirements in this project? |
| 67 | | JL | Um, I would hope you would be a better listener. Um to really understand what your children your students were thinking and and how their mathematical thoughts were developing. Um, I would hope that you would um be willing to um know that students can learn from each other as well as from you. And give them opportunities to uh you know uh talk about the mathematics, argue about the mathematics. Um I would think that you would be a better problem-solver yourself because you would be willing to struggle and and not always see that math is easy answers. Um and I would hope that you would be able to celebrate when your students do something that maybe even would be neater or more elegant than the way you went about solving something. |
| 68 | | PC | Great! Well, thank you so much Dr. Landis. I really appreciate it. Um, I think it was you know |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| | | | it was a great interview! |
| 69 | | JL | Okay, good. I’m glad we got to do it. [recording stopped]. |

Appendix F Reflection Discussion Questions 12/7/13

***Questions provided and asked by Course Instructors:***

As you prepare for our session on Saturday, it is important for you to know that we will spend most of the remaining learning from each other. So, we are asking you to reflect about our activities and assuming that you have this information in your portfolio, be prepared to share and discuss with your colleagues and in the larger group your thoughts about the following:

1) Something of particular interest about solving the tasks yourself and then implementing them with the students? Are the characteristics of these tasks and the way that we used them that are consistent with the Common Core State Standards for Mathematics?
2) What, if anything, was added by studying the video clips from the Video Mosaic Collaborative (VMC)?
3) Something that was important to you about students' reasoning about mathematics? – Backed by evidence from student samples that you have selected for your portfolios. It would be particularly interesting if you have documented if one shows a particular student's activity across the tasks.
4) And what, if anything, have you observed and or learned about the role of teachers and in the importance of it for math instruction?

# Appendix G
## Cycle 1: Building 4-Tall Towers Using 2 Colors of Unifix Cubes with Two Extensions

| ACTIVITY | Date | Meeting Type | VMC Videos Assigned | Readings Assigned |
|---|---|---|---|---|
| Teachers participated in first task. | 9/7/13 | On-campus | none | none |
| Teachers made an original post using on-line guided questions and responded to the original posts of at least two other teachers between these dates. | 9/11/13 to 9/16/13 | On-line | none | none |
| Instructor implemented the first task with students at one of the regional sites and held a debriefing with the teachers after class-room implementation. | 9/17/13 | Regional | none | none |
| Teachers watched videos, read the assigned readings, and made an original post using the on-line guided questions, and responded to the original posts of at least two other teachers between these dates. | 9/18/13 to 9/24/13 | On-line | 1) PUP Math Shirts and Pants<br>2) Stephanie & Dana, grade 3<br>3) Stephanie's prediction for 3-tall towers<br>4) Meredith removes the top cube. | Chapter 3 in Maher, C.A., Powell, A.B., & Uptegrove, E. (Eds) (2010) *Combinatorics and reasoning: Representing, justifying and building isomorphisms.* New York: Springer Publications. |
| Teachers watched videos, read the assigned readings, and made an original post using the on-line guided questions, and responded to the original posts of at least two other teachers between these dates. | 9/25/13 to 10/1/13 | On-line | 1) The Meaning of "At Least"<br>2) Does the Number Double?<br>3) Milin Introduces an Inductive Argument<br>4) Stephanie and Matt Rebuild the Argument<br>5) Stephanie and Dana, grade 4 | Chapters 4 & 5 in Maher, C.A., Powell, A.B., & Uptegrove, E. (Eds) (2010) *Combinatorics and reasoning: Representing, justifying and building isomorphisms.* New York: Springer Publications. |
| Teachers implemented the first task and extension tasks with their students in their classrooms. | 9/18/13 to 9/27/13 | Regional classrooms | Aforementioned above. | Aforementioned above. |
| Teachers brought 2-3 student samples and discussed the reasoning evidenced by students' work. | 10/2/13 | Regional | No more videos assigned for Cycle 1. | No more readings assigned for Cycle 1. |

Cycle 2: Finding the number of pizzas that can be made from a choice of four toppings

| ACTIVITY | Date | Meeting Type | VMC Videos Assigned | Readings Assigned |
|---|---|---|---|---|
| Teachers participated in second task. | 10/2/13 | Regional | 1) PUP Math Brandon Interview<br>2) PUP Math Pizza with 4 Toppings | Chapter 6 in Maher, C.A., Powell, A.B., & Uptegrove, E. (Eds.) (2010) *Combinatorics and reasoning: Representing, justifying and building isomorphisms.* New York: Springer Publications. |
| Teachers read the assigned reading, and made an original post using the on-line guided questions, and responded to the original posts of at least two other teachers between these dates. | 10/3/13 to 10/8/13 | On-line | none | none |
| Teachers read the assigned reading, and made an original post using the on-line guided questions, and responded to the original posts of at least two other teachers between these dates. | 10/9/13 to 10/15/13 | On-line | none | Maher, C.A. & Martino, A. (1998). "Brandon's Proof and Isomorphism". In C. A. Maher, *Can teachers help children make convincing arguments? A glimpse into the process.* Rio de Janeiro, Brazil: Universidade Santa Ursula. |
| Teachers implemented the second task with their students in their classrooms. | 10/3/13 to 10/21/13 | Regional classrooms | Aforementioned above. | Aforementioned above. |
| Teachers watched videos, read the assigned readings, and made an original post using the on-line guided questions, and responded to the original posts of at least two other teachers between these dates. | 10/16/13 to 10/22/13 | On-line | none | none |
| Instructor implemented the 2nd task with students at regional site and met with teachers after classroom implementation where teachers brought 2-3 student samples to discuss reasoning of students' work. | 10/22/13 | Regional | No more videos assigned for Cycle 2. | No more readings assigned for Cycle 2. |

Cycle 3: Building 3-Tall Towers Using Three Colors of Unifix Cubes with Extension Activity

| ACTIVITY | Date | Meeting Type | VMC Videos Assigned | Readings Assigned |
|---|---|---|---|---|
| Teachers worked on the third cycle task. | 10/22/13 | Regional | 1) PUP Math-Romina's Proof to Ankur's Challenge | none |
| Teachers made an original post using on-line guided questions and responded to the original posts of at least two other teachers between these dates. | 10/23/13 to 10/29/13 | On-line | none | none |
| Teachers made an original post using on-line guided questions and responded to the original posts of at least two other teachers between these dates. | 10/30/13 to 11/5/13 | On-line | none | Chapter 8 in Maher, C.A., Powell, A.B., & Uptegrove, E. (Eds.) (2010) *Combinatorics and reasoning: Representing, justifying and building isomorphisms.* New York: Springer Publications. |
| Teachers made an original post using on-line guided questions and responded to the original posts of at least two other teachers between these dates. | 11/6/13 to 11/19/13 | On-line | none | none |
| Instructor implemented the third task with students at a regional site and met with the teachers after class-room implementation where teachers brought 2-3 student samples to discuss the reasoning evidenced by students' work. | 11/20/13 | Regional | none | none |
| Teachers discussed and reflected about their experiences with the course. Portfolio projects were due today. | 12/7/13 | On-campus | none | none |

# Appendix H: Instructor Moves Framework for Analysis

| |
|---|
| CYCLE:<br>TASK:<br>DATA SOURCE:<br>DATE:<br>TIME: |

| **INSTRUCTOR MOVES FRAMEWORK** | **COUNT** |
|---|---|
| **I. Representation Used** | |
| A. Manipulatives<br>1. Unifix cubes<br>2. Other | |
| B. Drawings<br>1. Tree diagram<br>2. Other | |
| C. Charts | |
| D. Symbols | |
| E. Gestures | |
| **II. Forms of Pedagogical Practice** | |
| A. Questioning | |
| 1. Explanation | |
| 2. Justification | |
| 3. Generalization | |
| 4. Connection | |
| 5. Probing | |
| 6. Other Solution | |
| B. Anticipating | |
| C. Monitoring | |
| D. Selecting | |
| E. Sequencing | |
| F. Motivating | |
| G. Waiting | |
| H. Inviting | |
| I. Revoicing | |

# Appendix I: Reasoning Strategies Framework for Analysis

| **REASONING STRATEGIES FRAMEWORK** | | **COUNT** |
|---|---|---|
| **I.** | **Heuristic/Strategy** | |
| | A. Guess and Check | |
| | B. Opposites | |
| | C. Cousins | |
| | D. Elevator | |
| | E. Staircase | |
| | F. Controlling for Variable | |
| | G. Other | |
| **II.** | **Representation Used** | |
| | B. Manipulatives<br>1. Unifix cubes<br>2. Other | |
| | C. Drawings<br>1. Tree diagram<br>2. Other | |
| | D. Charts | |
| | E. Symbols | |
| | F. List | |
| **III.** | **Form of Argument** | |
| | A. Cases | |
| | B. Induction | |
| | C. Recursion | |
| | D. Contradiction | |
| | E. Rule | |
| | F. Other | |
| **IV.** | **Participant Evaluation** | |
| | A. Convincing Argument | |
| | B. Not Convincing Argument<br>1. Incomplete<br>2. Invalid | |
| **V.** | **Researcher Evaluation** | |
| | A. Undetailed Description | |
| | B. Unidentified Participant Argument | |
| | C. Convincing Argument | |
| | D. Not Convincing Argument<br>1. Incomplete<br>2. Invalid | |

# Appendix J

## Final Project T1

### Cycle I: Class Profile

This activity took place in a 7th grade resource-pull out math class that is 40 minutes long. There are 7 students total- 5 girls and 2 boys. The majority of the students have special needs that consist of mainly mild learning disabilities; however, one student had a brain injury resulting in Tourette's syndrome. This has significantly affected her focus and ability to complete activities in one sitting. Likewise, all of my students have difficulty with written expression which will be apparent in their explanations of each activity. My math class starts at 10:45, which is right before the students' lunch period.

Cycle I: Towers of 4-tall, selecting from 2 colors

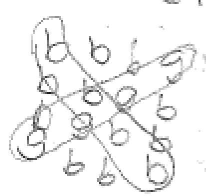
First sample: Impressive

This sample impressed me because the student attempted to use the recursive argument, although he didn't know that he was actually using that argument. Michael did not record the answer; however he did finally reach it with his partner. More importantly, he was able to explain and find an argument. He drew the towers and circled the diagonal of the blue cubes "going down" in each place. Likewise, he did this with the other color. Michael did not completely record what he had created at his desk, but he and his partner really focused on finding patterns and fixing them. I was really impressed that he was able to see this diagonal and explain to me that the color is being used in every space; therefore, that pattern could not be used again. He did not record every pattern, but he did show me.

**Building 4-tall towers, selecting from 2 colors**

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.

we kept on making 6 of 3 and we made 4 set and we found our answa



we cant make any more because there wor be dupplakits

Cycle 1: Towers 4-tall, selecting from 2 colors

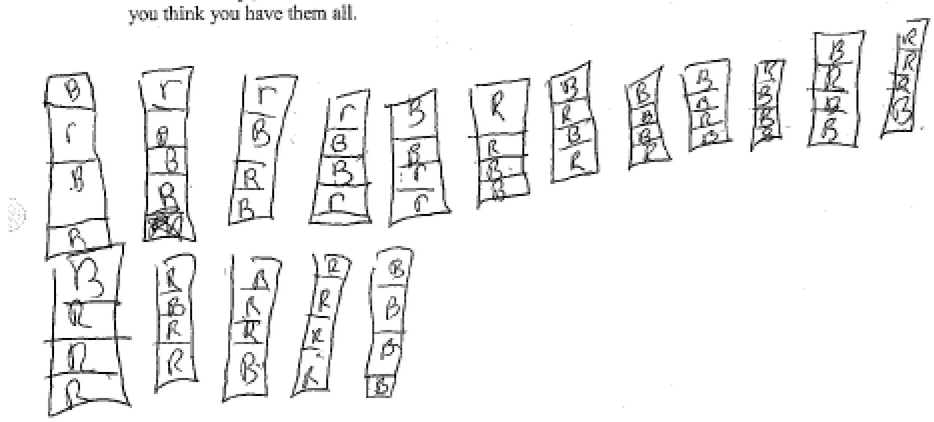
Sample 2: Surprising

This sample surprised me mainly because this is a "lower" student and I thought she did a great job of drawing her towers. Bethany has one duplicate in the set, but that is minor. I was also surprised in her explanation that she found combinations "backwards" and "opposite." I was glad she was able to recognize that not only can the towers be made by opposites, but that the constant changes when going backwards. I've never heard of a student using "backwards" as a way to describe it in any of the activities in my class, or in the classes we visited. It just goes to show how students have many different ways of seeing things.

interesting how students develop language

**Building 4-tall towers, selecting from 2 colors**

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.



We coud not crear anymore compinations becous. we used all of the comberonins backwords / oposet from the outhers. We also thougnae putting them in outher ways but it would make duplacitis. No matter how we triyed thier would be ~~be~~ duplacits.

Cycle 1: Towers 4-tall, selecting from 2 colors

Sample 3: Concerning

This sample concerned me, because the student did not grasp the concept of the assignment. Jocelyn was not using any mathematical reasoning to create designs with the cubes. At one point, she was trying to make the professional basketball team, the Laker's, sign. Likewise, she started to make a tower using all of the cubes, instead of individual towers of 4 tall. All in all, she did not follow directions, nor use any mathematical reasoning during the activity and in her explanation.

**Building 4-tall towers, selecting from 2 colors**

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.

P,y,P,y,l,<

Y was For Yellow P was For Purple
We Just ~~put~~ put to colors to~~gther~~ Mixed the
Colors, it up to 4 Cubes with diffrent
diesighns we also made the toar was
to Put ~~them~~ 7 then the 5 and then 3 and
1. We did it,

Intervention Implementation: Cycle 1

My students really enjoyed this activity. They loved the hands-on aspect and worked diligently in their pairs. I learned that it was extremely difficult for my students to explain their mathematical reasoning orally and on paper. Likewise, many did not translate what they created onto their papers which was hard for me to go back and see what they did. Luckily, I took pictures on my Ipad to refer to. One group of girls were paired in 3, which I didn't think worked well because they were all building their own towers and not communicating well. It was hard for them to see the duplicates because they had so many. If I had the opportunity, I would give my students more time to complete the activity because I felt that it was rushed in the 40 minute period. ← 40 minutes is definitely not enough time to do an activity that engages students in thoughtful mathematics

Cycle II: Class Profile

This activity took place in a 7th grade resource-pull out math class that is 40 minutes long. There are 7 students total- 5 girls and 2 boys. The majority of the students have special needs that consist of mainly mild learning disabilities; however, one student had a brain injury resulting in Tourette's syndrome. This has significantly affected her focus and ability to complete activities in one sitting. Likewise, all of my students have difficulty with written expression which will be apparent in their explanations of each activity. My math class starts at 10:45, which is right before the students' lunch period.

Cycle II: Pizza Problem

Sample 1: Impressive

This sample impressed me because of the key Bethany set up to show her pizza combinations. She created the labels A, B,C,D,E to represent the pizza toppings instead of using the actual names like the other students did. I felt that this helped her organize her work better than the other students and she could easily make changes if need be. Also, it was easier for her to visually see if there were any duplicates. All in all, I thought using A, B,C ,D, E to represent the pizza toppings was a clever way of organizing her work.

A sosages
B pepers
C Mushrooms
D peperoney
e plan

1 e
2 D, C, B, A
3 B
4 A
5 D, C
6 D
7 B, A
8 C, B
9 D, B
10 A, C
11 C

Cycle II: Pizza Problem

Sample 2: Surprising

I was surprised by Jocelyn's work because she made a huge improvement in her reasoning and drawings from the first tower problem. She did not write out her explanation; however, she orally expressed it to me. Likewise, her drawings were boxed out like pizza boxes and she numbered them. Jocelyn is probably the lowest math student in the class, so I was very surprised and impressed that she organized her work in this way. Many students included plain and extra toppings, but she stuck to the directions and created pizzas using only those toppings.

Cycle II: Pizza Problem

Sample 3: Concerning

Lesley's paper is concerning because she had many duplicates and did not notice them until I pointed them out. As you can see on the paper, she erased many. Likewise, she used "extra" toppings as a combination and included "plain" as a topping in the combinations. Lesley did not take the questioning and direction of me or her partner. When I asked her to explain what she did, she could not reason how she created the combinations.

Toppings
- Plain
- Sausage
- Mushrooms
- Pepperoni

- Peppers - Xtra Peppers
- Peppers - Sausage
- Peppers - Mushrooms
- Peppers - Pepperoni
- Pepperoni - Plain
- Pepperoni - Sausage
- Pepperoni - Mushrooms
- Pepperoni - Xtra Pepperoni

- Mushroom - Xtra Mushroom
- Mushrooms Sausage
- Sausage - Mushrooms
- Sausage - Pepperoni
- Sausage - Xtra Sau

Intervention Implementation: Cycle II

Overall, this activity was the most difficult for my students because of the ambiguity in the task. Many students spent much time pondering over whether they could use extra toppings, half pies, and whether order matters in how the toppings are placed on the pizza. I felt that this weakened the purpose of the activity. If I could do this activity again, I would give the students specific directions such as *order doesn't matter* and *there cannot be half pies in this case*. I do not think it would take away from the problem; it would just give them more direction and allow the students to focus more on the task.

it is important to let students decide whether order matters

## Cycle III: Class Profile

This activity took place in a 7th grade resource-pull out math class that is 40 minutes long. There are 7 students total- 5 girls and 2 boys. The majority of the students have special needs that consist of mainly mild learning disabilities; however, one student had a brain injury resulting in Tourette's syndrome. This has significantly affected her focus and ability to complete activities in one sitting. Likewise, all of my students have difficulty with written expression which will be apparent in their explanations of each activity. My math class starts at 10:45, which is right before the students' lunch period.

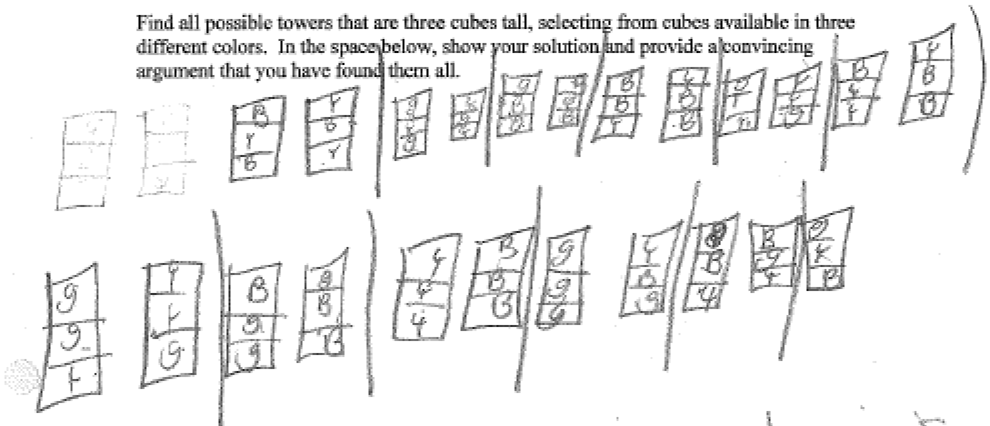
Cycle III: Towers 3-tall, selecting from 3 colors

Sample 1: Impressive

Bethany's work impressed me because of her drawing. She did not get the correct answer, but she was very close. Her drawing shows her towers broken up into pairs that are opposite of each other. I thought this was a good way of showing this task; she only missed two but was on the right track. Breaking these up into pairs clearly showed how each color was used, especially because the third color could get confusing. Her drawing showed patient planning and the use of all colors in an organized manner.

**BUILDING TOWERS THREE COLORS**

Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and provide a convincing argument that you have found them all.



We got 26 because there is
no posibul way to make more we
gruped them to show there [illegible]
we tryed to make more but it
cause there to be duplicates.

Cycle III: Towers 3-tall, selecting from 3 colors

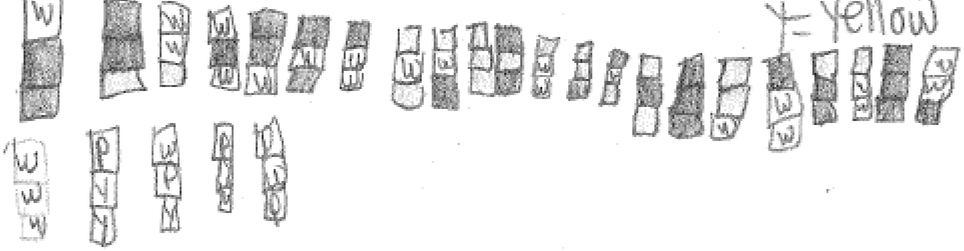
Sample 2: Surprising

I thought Victoria's work was surprising because she got the answer right! I know it is more about the reasoning, but her pictures were clear and she did explain herself, although not thoroughly. Her explanation continues on the back of the paper. Victoria struggled with the first tower task and I was surprised that this task came so easy to her. She drew all the towers and color coordinated them in her key. Victoria took what she knew from the first task and used that knowledge to complete this task with her partner, which is what I want my students to do. She learned from the first task that she could not have duplicates and that she needed to clearly draw her towers. On her desk, she had her pattern grouped by constants then did the opposite of each; however, she did not record it that way onto her paper. nice!

**BUILDING TOWERS THREE COLORS**

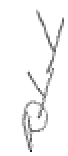
Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and provide a convincing argument that you have found them all.

P= Purple
W= White
Y= Yellow

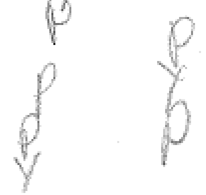


We got 27 because we worked together to find all matches.

We used a Partern and we used the [illegible]ite. We just
Put



into



and this is what we did

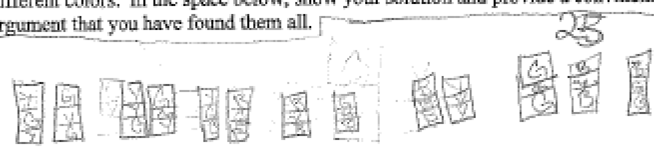
Cycle III: Towers 3-tall, selecting from 3 colors

Sample 3: Concerning

Jocelyn's work concerned me because of her mathematical reasoning. She struggled to orally explain to me how she was reaching 23 towers, even after her partner tried explaining it to her. She kind of sat back and did not contribute to the task. Jocelyn wrote that she reached 23 towers; however, her reasoning has no mathematical validity for this task. It was also very concerning that by the 3rd task she did not attempt to use the skills we used and talked about in the 2 prior tasks and did not make an attempt to give a detailed explanation.

**BUILDING TOWERS THREE COLORS**

Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and provide a convincing argument that you have found them all.




Ex:

We got the amout of 23 by coun the Other cubas

Intervention Implementation: Cycle III

I was very pleased with this task, considering I thought it was the most difficult one. I was excited to see that my students started working diligently right away and did not need me to push them to get started. They all showed growth since the first task whether it was the way they worked together, drew their explanations, or explained their thinking. I was definitely impressed by all with their progress. If I did this task again, I definitely would give them two days to complete it because I think they needed it. I was hesitant to do this because I thought the students would lose interest and forget what they had done the day before. I may be right, but it is worth a shot to give them the extra day. Also, I felt that as a teacher I grew since the first task. With this task, I was more comfortable letting them ponder over the problem and didn't feel the urge to guide them. The students responded well to this.

## Reflection

Through these three cycles, I learned that mathematics is not just about finding the answer; it's about explaining and justifying the answer. When we first completed the tasks as a cohort, I had difficulty explaining why I was done finding towers of 4 using 2 colors. I kept saying "because I can't find anymore." After doing the activities with my students, I could relate to their struggles to explain to me their reasoning. I think we are so used to being taught to produce an answer in math that it is not in our nature to explain it. By giving a convincing argument, students are encouraged to think deeply and use skills such as inductive reasoning, proof by cases, and recursive arguments. Likewise, I found that by explaining and justifying my reasoning, I was not only able to find the mistakes I was making and improve them, but also strengthen the arguments I had.

I learned that my students' oral reasoning was much stronger than their written reasoning. My students had a lot of difficulty explaining the first pizza task, but they did show a lot of improvement by the third. I was very proud of them! Often times students would explain and show me their reasoning, but their writing would not reflect their thoughts. This is something I will work with them on- having one person write as their partner dictates and vice versa. Students need to be getting credit for their hard working explanations.

As a teacher, I definitely changed from the first task. With the four towers, I was quick to accept arguments for fear of them reaching frustration. However, as we went on with the tasks, my students showed such improvement and I realized they were capable of more. I feel like I never give enough "wait" time because I'm scared I'm going to "lose" students. Now, I give more "wait" time and am not so quick to give and explain the answer. Overall, this was a great learning experience for both myself and my students.

# Appendix J

## Teacher T2

**CLASSROOM DEMOGRAPHICS**

This study took place in a 7th grade self-contained special education classroom of 10 students. The class consisted of 6 boys and 4 girls from various ethnic backgrounds including, but not limited to, Caucasian, Hispanic, African American, and Asian. The students' disabilities within this classroom are Attention Deficit Disorder, Hearing Impaired, Multiply Disabled and Autistic. The disabilities range from mild to moderate levels. The hearing-impaired students wear hearing aids and the teacher is required to wear an FM system. The functioning levels of the students are as follows: 4 students on a 2nd grade math level, 4 students on a 3rd grade math level and 2 students on a mid-4th grade level. Most students can add basic and multi-digit numbers with little difficulty and little assistance. About 85% of the class has difficulty subtracting when regrouping is involved and 95% of the class cannot multiply without the assistance of a chart. The entire class has difficulty dividing. The class period for these tasks took place from 8:35am until 9:18am over the course of a two-day period. A paraprofessional is present during this class period; however, she did not provide the students with any assistance during these tasks.

Your description really helps me understand the difficult conditions you are working under

**SURPRISED**

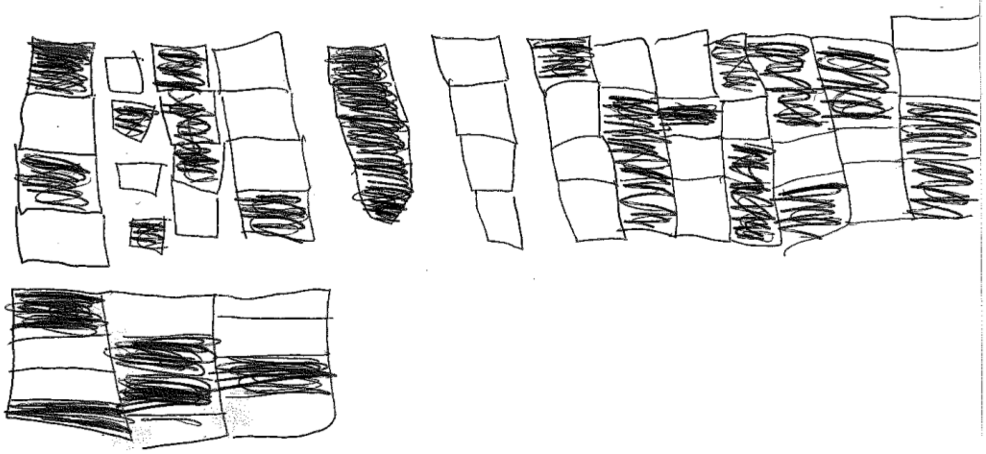
This student's work surprised me because she had a rough time starting this task. The two partners spent a lot of time discussing the task and trying to come up with a way to solve it before building. They then began like the rest of my students to build opposites of each other. They started with a solid blue tower and then a solid yellow tower; they then made alternating color towers blue, yellow, blue, yellow and its opposite. They continued to do this until they felt that they couldn't make any more without creating a duplicate. Once again I ask her to explain how she knew she had them all; she smiled at me and stated that she wasn't sure. She was convinced she could build more. I let her try to build more as I went to other groups to question their strategies. I came back and asked her if she found any more, at this point she had not. She was convinced she had them all; however, she struggled with her reasoning why. Her recordings did not show the way she actually organized her towers. She was able to rearrange them to show the moving of one colored cube into different positions; however, she did not think that was the best way to record so she went back to opposites. Again, her explanation was not convincing.

nice that you gave her time to look for more

[redacted]

**Building 4-tall towers, selecting from 2 colors**

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.



We myad as many as we
can theink of. Then we maed
showr ther was none the
samu.

**CONCERNED**

This student's work concerned me mainly because of his explanation. This student like many of my others completed this task by making opposites. I did enjoy the way he recorded his work into 8 pairs of 2; however, his explanation and reasoning was where my concerns came from. He just stated how he used "b" for blue and "y" for yellow. When he was working on this task and I was questioning his strategy he was able to tell me he was making opposites. He began making a lot of different towers. He had more then 16 at one point, I then asked him if he had them all and he stated no. He continued to make more towers; while I could clearly see there were opposites I did not let him know. I just asked him to read the task again with his partner and to look at what he is doing to make sure he is following the directions. He reread the task and stared at his towers for a bit and then asked me what a duplicate was. Even though I did go over that with the class in the beginning of the task he may of gotten off task during that time. I asked someone in the class to explain that to him and then he began pulling apart duplicate towers he had made and finally stopped at 16. When I asked him how he knew he had them all he said because the rest will be duplicates. I asked again, "how do you know they would be duplicates?" He stated that he just took all the duplicates away and that's what he was left with. He was beginning to get frustrated so I let him record his towers at this point and he tried to explain his thinking but did not do a good job at it.

[redacted] Date: 9/10/13

School: ______________ Teacher: ______________

Other Group Members: ______________

**Building 4-tall towers, selecting from 2 colors**

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.

pair 1

blue b | y yellow
yellow y | b blue
blue b | y yellow
b | y yellow

pair 2

b blue | y yellow
y yellow | b blue
b blue | y yellow
y yellow | b blue

pair 3

b blue | y yellow
b blue | y yellow
y yellow | b
b blue | y yellow

pair 4

y yellow | b blue
b blue | y yellow
b blue | y yellow
y yelloy | b blue

pair 5

b blue | y yellow
b blue | y yellow
y yellow | b blue
y yellow | b blue

pair 6

b blue | y yellow
b blue | y yellow
b blue | y yellow
b blue | y yellow

pair 7

b blue | y yellow
b blue | y yellow
b blue | y yellow
y yellow | b blue

pair 8

b blue | y yellow
y yellow | b blue
y yellow | b blue
y yellow | b blue

I solve with a block of blue and yellow.

I use b in the box and ~~goo~~ y ~~yellow~~.
I use the full name after I wrate ~~put~~ b and y.

**CYCLE I REFLECTION**

This task was very interesting to implement. I was worried at first that none of my students would come up with the correct answer due to the classroom demographics; however, I was pleasantly surprised. This is the first time my students where given an open ended task with no direction at all. They had a hard time understanding that I could not guide them in any directions at all, they kept asking me if they were right and I would just respond with you should know your right you don't need me to tell you. I was able to see that even though my students are lower functioning it doesn't mean they are incapable to be challenged and they will rise to the challenge. yes! At first, they were excited to just play with the unifix cubes but once they were given the task they went right to work. Since the implementation of this task I have been trying to challenge them more and encourage them to be more independent thinkers. This task allowed me to learn about their strengths and weaknesses as mathematical thinkers. I enjoyed questioning my students and seeing them really think about what they were doing in order to solve this task. I think it was great for my students to be challenged with a thinking task like this. One thing I wish I could've done better was grouping my students. It was early in the year and though I knew them I wasn't sure who would work best together. If I had an opportunity to do this task again I would group my students differently.

**IMPRESSED**

This student's work impressed me because of her organization and explanation. She was excited with this task just as the first one. She was determined to get the correct answer and to understand why it worked. She had a hard time understanding that the plain was what you were building up on first but by discussing it with her partner and questioning her thinking she finally decided that you can have one topping, two toppings, etc… She organized her recording in the same fashion. She indicated a key; which for each box she drew that was once pizza. She was able to show that there were 4 options for one topping which also consisted of the plain she considered as one topping, 6 options for two topping, 4 for three topping and 1 for four topping. This resulted in 16 possible ways. She was then able to verbally explain that there are only four toppings so she could not make a five topping pizza, therefore, she used all possible toppings which would mean she found all possible combinations. I then asked her if she had a pepperoni and sausage why couldn't she make a sausage and pepperoni pie? She replied by saying that the order of the toppings doesn't matter in this case because it's the same pizza. So the opposite strategy didn't work for this problem. This student impressed me the most with her thinking and explanation.

[redacted]

The Pizza Problem

Capri Pizza has asked you to help design a form to keep track of certain pizza choices. They offer a standard "plain" pizza with cheese and tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many choices for pizza does a customer have? List all possible choices. Find a way to convince each other that you have accounted for all possibilities.

Peppers
Sausage
mushrooms
Plain

Peppers Sausage
peppers mushrooms
Peppers pepperoni

16 Ways

Sausage mushrooms
Sausage pepperoni
mushrooms pepperoni

peppers Sausage mushroom
Peppers Sausage pepperoni
Sausage pepperoni mushrooms
mushrooms pepperoni peppers

Plain peppers sausage mushrooms pepperoni

Key
▭ = one pie

I Known there is know more Posable ways. Because there is only four toppins. To Chocies from the Plain is the base of the hole thing. I used all the Posaible ways there are Starting with one topping to two toppings to three toppings to foun toppings. Then I thought I was done but. I realize that I didn't have a Plaine pie and a only Plain pie. Thats how I got my answer and there was 16 ways all togeather.

## SURPRISED

This student's response surprised me due to her struggle with the first task. She had a very hard time starting this task as well; however, once she got started she got into it and became confident in what she was doing. She started off by making opposites like she did with the towers; she was making a pepper and sausage and then a sausage and pepper. I asked her if she went into a pizzeria and ordered both of those slices if she would be ordering a different slice. She replied that she would be getting the same thing and then she realized that just switching the toppings didn't work in this case. She then decided to try a tree diagram but she quickly realized she was doing the same thing with the toppings. She then asked me if she could have more than one topping on a pizza and I asked her what did she think and she smiled and ran back to her partner and started to draw up what she recorded on the paper. She really had the right idea with the one topping, two topping and so on. She was able to find all 16 ways; however, due to time she was not able to record all of her ways; she ended up being short two ways on her final copy. Her explanation improved from the first task to this one, she was able to explain a little more about what she was thinking when she was completing this task; however, it wasn't fully convincing since she didn't state how she knew she had them all.

[redacted] Date: ________

Partner: ____________________ Teacher: ____________________

The Pizza Problem

Capri Pizza has asked you to help design a form to keep track of certain pizza choices. They offer a standard "plain" pizza with cheese and tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many choices for pizza does a customer have? List all possible choices. Find a way to convince each other that you have accounted for all possibilities.

1 P.
2 Pe.
3 S.
4 M.
5 P. Pe.
6 P. M.
7 Pe
8 Pe. S
9 M. S
10 P. Pe. M.
11 P. Pe. S
12 P. M. Pe.
13 S. M. P.
14 S. M.

I got the anser 16 becuse I cept tring to get it and I thout can we put more then one toping on a pizza so I did 1 toping 2 topings 3 topings 4 topings in till I got the anser

**CONCERNED**

This student's work concerned me and surprised me all at the same time. He started making pizzas like the rest of the class but shortly after he began he called out to me that the answer was 16. I went over to him and asked him "how does he know that, if he didn't even have 16 options on his paper?" He said, "Well this is like the towers problem we did." I said, "How so?" and he stated, "Because the answer is 16." So I asked him if he could build the towers to represent the pizzas if they were related, he said "ok." So he did and then he recorded his answer. However, just as in the first task he did not explain much. I feel like his mind is in the right spot but he is having a hard time verbalizing his thinking as well as writing it. He put a "p" for plain and those were the blue cube and a "t" for topping and those were the yellow cubes. He made opposites again and made 8 groups of 2 just as he did when he made the towers the first time. However, for the pizza problem opposites would not help them come up with the right answer. He was not able to explain fully how the two problems were related and he was not able to list all of the pizza toppings as the other students did so I was not sure if he truly saw the connection or just assumed that the two tasks were related because they sounded similar to him somehow.

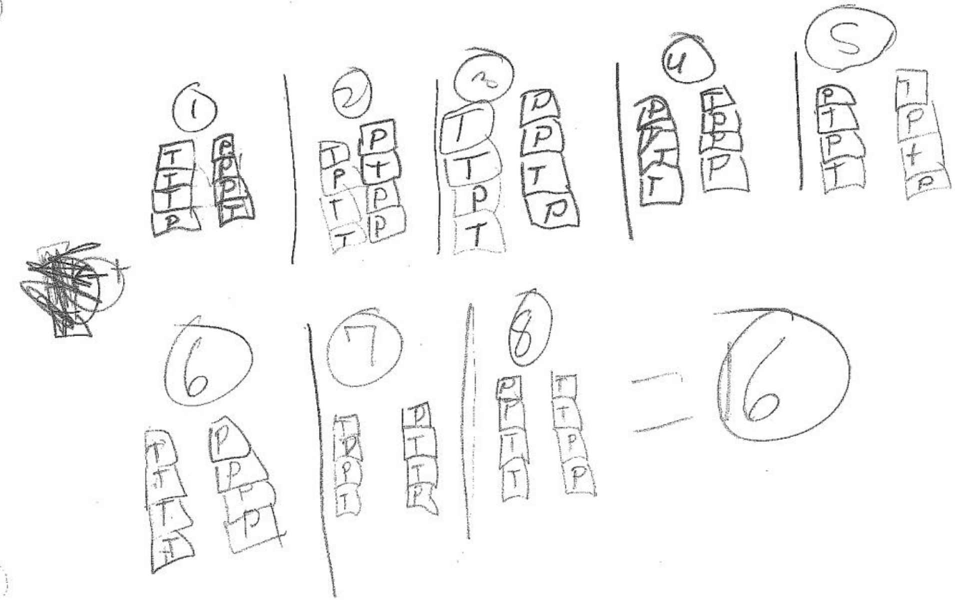
[redacted]_____________ Date: 10/11/13

_ Teacher: ____________________

The Pizza Problem

Capri Pizza has asked you to help design a form to keep track of certain pizza choices. They offer a standard "plain" pizza with cheese and tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many choices for pizza does a customer have? List all possible choices. Find a way to convince each other that you have accounted for all possibilities.

I use the blox for the pizza that I just did. I use blue for the topping yellow plain only.



**CYCLE II REFLECTION**

For the pizza task I found it to be more difficult then the towers. I was excited to implement this task being as though it was more relatable, or so I thought, for the students then the cubes. However, they had a hard time starting this task and seeing that you can have multiple toppings on one pizza. This could be because they are young and they do not normally have anything besides a plain slice or a one topping slice. A lot of the students where saying how they hated peppers; therefore it couldn't be on their pizza. Maybe that was the down fall of it being too relatable for them, they couldn't get passed the facts that they weren't actually eating any of the combinations they were just trying to give me all possible combinations available. I learned that my students work better with a manipulative and that sometimes a real-life example can back fire if their thinking is too literal. I am not quite sure what I could have done differently to make this session better without leading them. The only thing I could think of would be to explain that the combinations have nothing to do with whether they like them or not, they are just to find all possible combinations of topping available using the four options provided. Again, I changed the partners for this task but do not feel as though I fully found the right matches yet. If I could do this task again I would change partners again.

Sounds like Dana in the Shirts and pants problem – when she didn't include the yellow shirt and white pants because of her fashion sense that they didn't match!

**IMPRESSED**

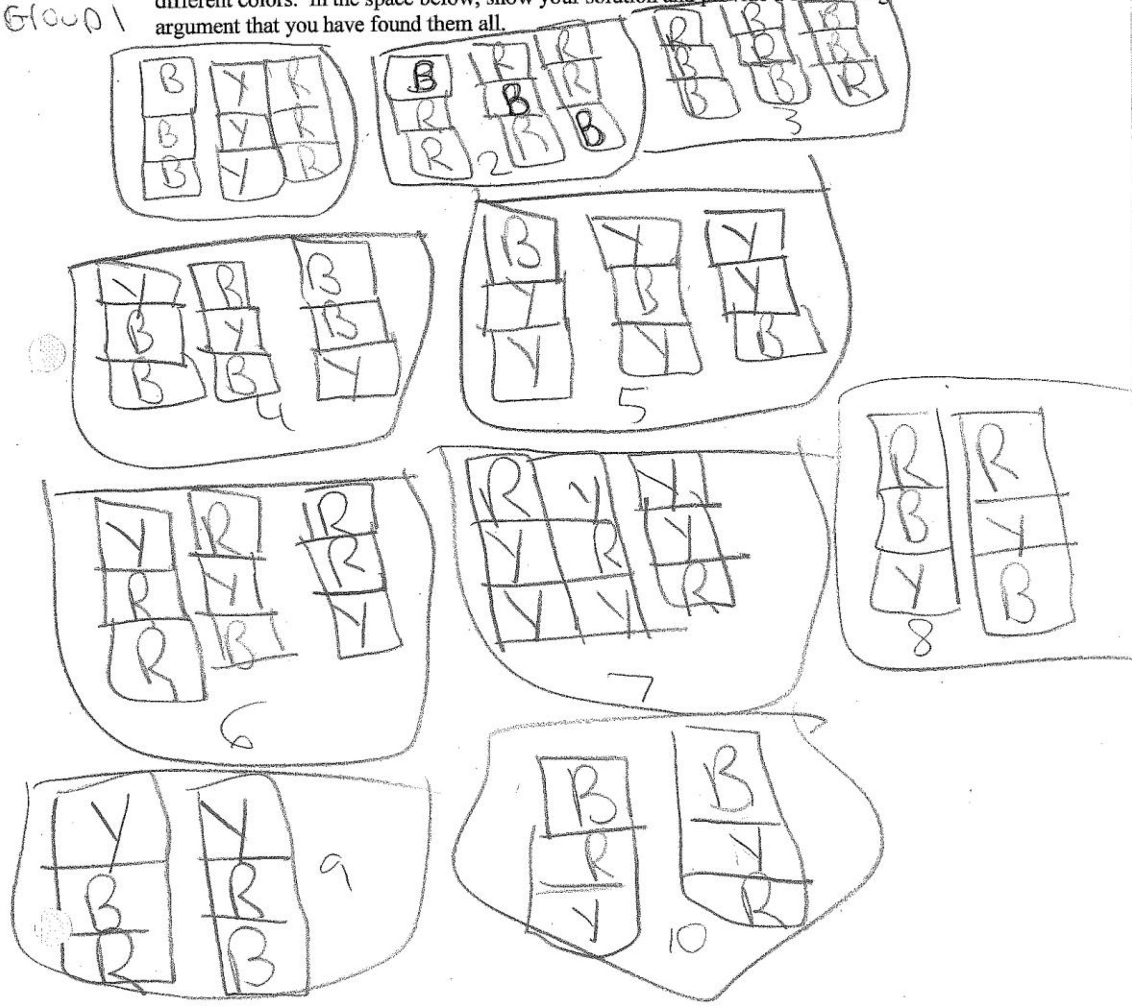
This student's work impressed me because when given this task she went right to work. She began by building her towers in different groups where she moved one cube into three different positions. She then did this for each of the three colors using each color twice into two different groups. For example; three yellow cubes and one blue cube, moving different positions and then three red cubes and one blue cube moving different positions. She then repeated this using the red twice and then the yellow twice. She organized her towers in 7 groups of 3 and three groups of two. She was able to show how one colored cube can more into all three positions until no positions are left. She explained that the one blue cube could only move in three positions because the towers are only three cubes high using the proof by cases argument. She then justified that in-group 1 the three solid towers where the only options because there were only three colors. Finally, she explained that in the three groups of two she kept a constant and switched the two bottom colors. Since there were only two colors and two positions there were no other possible ways to make the towers using three colors holding one constant on top then the two ways she found for each group. Her final answer was 27 towers. Her explanation convinced me that she understood the task and was able to grow within these three tasks.

[redacted]

**BUILDING TOWERS THREE COLORS**

Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and ~~provide a convincing~~ argument that you have found them all.

Group 1



I know the answer is 27. I know there is no more possible ways because in group 2 I moved the ~~bule~~ blue cubs in each ~~possi~~ positions way I ~~could~~ could. there was only ~~3~~ 3 positons because it can only be 3 high. ~~In group~~ I did the rest for groups 3, 4, 5, 6, and 7. In group 1 I made 3 diffrent towers solid colors yellow, bule, and red because I only had 3 colors. For 8, 9, 10 I kept one color on the top and switched around the 2 underneath them.

**SURPRISED**

This student used the keeping a constant argument. He was able to see that you can keep all of one color on top in order to make three different groups because there were three different colors. When he first started and saw this he had a different number of towers in each group. I began asking him if he thought he had them all and he stated no, when I asked him why he said because each group is a different amount and I think they should all be the same. I asked him why he thought they should all be the same and he could not answer me. He began building again looking for towers he had in one group and not another. When he finally stopped he had 3 groups of 9. He was not able to justify how he knew he had all towers for each group very well but he was able to finally organize his towers in a different way then opposites like the first task. So that surprised me since he did not see any other relationship between the towers in the first task like he did in the third.

great!

Student: [redacted] Date: 11/14/13

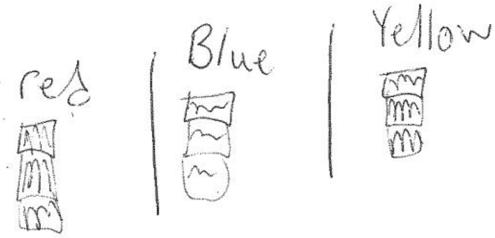
School: ______________________ Teacher: ______________________

Other Group Members: ______________________

**BUILDING TOWERS THREE COLORS**

Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and provide a convincing argument that you have found them all.

Example




I did by group. When I started I put all the red on top and the yellow, blue. I got 27 because there are nine in each group.

**CONCERNED**

This student's work was a bit of a concern. He began by organizing his towers using a constant on top. He had a method and it was the best method he had thus far; however, when asked to explain and record he did not make any sense as to what he actually made. The four towers he did record do not make sense to what he actually did and his explanation did not convince nor explain his thinking at all. Being the third and final task I expected great things as he did provide me with the best organization he had done over all three tasks; however, he did not record it the way he had them organized as well as write an explanation that made sense to what he did. This concerns me because at this point he should have gotten the hang of these tasks and truly understood how to approach them. He did approach this task a different way but based off of his final product there was no improvement like I had seen with the others in the class.

Student: [redacted]

Date: ______________________

School: ____________________________

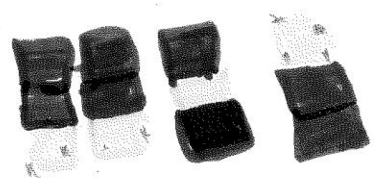
Teacher: ____________________

Other Group Members: ______________________________________________

**BUILDING TOWERS THREE COLORS**

Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and provide a convincing argument that you have found them all.

Exemply:



what I did was did red bloe yellow then I was thinking to reverssy the colon by potting bloe red yellow and then I just keep repetting the pattern.

**CYCLE III RELFECTION**

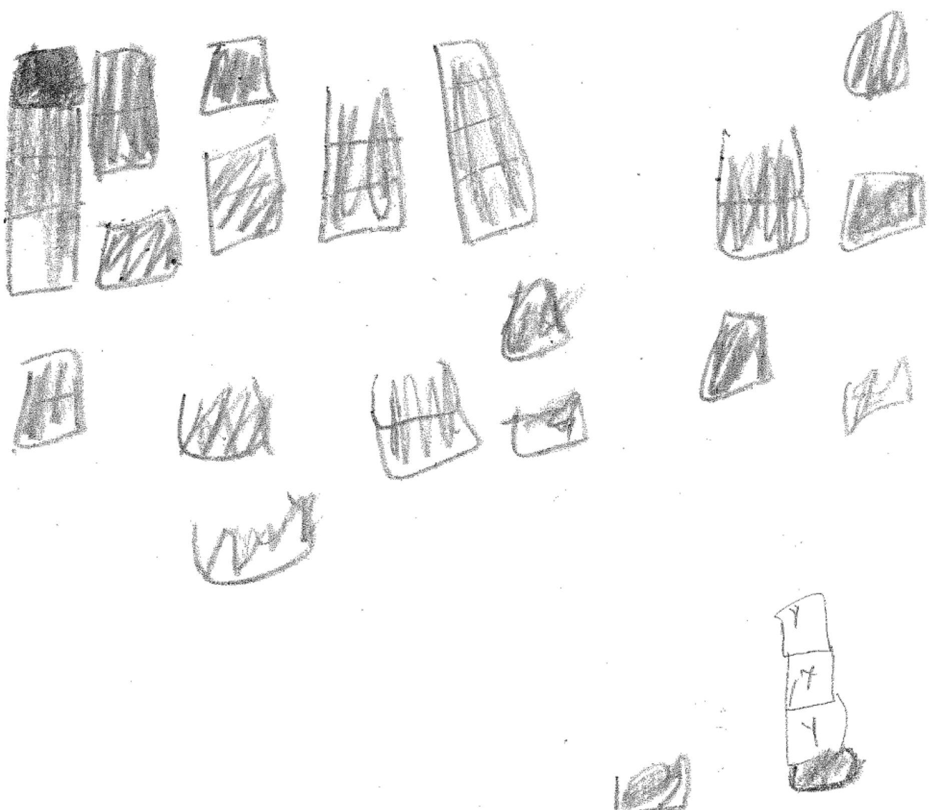
Throughout this final task I have seen improvement amongst most of my students. Surprisingly, a lot of them approached this task with a keeping a constant strategy. I found this very interesting since no one approached the first task this way at all. The grouping was better during this session, which resulted in more interesting results and more students being able to work through the problem better. I was able to group them with similar ability, which helped them tackle the task with more confidence. Everything went well with this final task; however, I find it strange that what my students came up with as answers and how they recorded it was different in some cases. The one thing I can think about doing differently would be to explain that they are to record the towers exactly how they built them on the paper.

**IMPRESSED**

This student's work impressed me because during the task she was able to see the patterns the towers made. At first she began this task by making opposites with her partner; however, once I walked around and asked her to explain how she knew she had them all, she was stumped. I then asked her if there was a way to re-organize the towers to make a more convincing argument. She thought for a bit and then began to rearrange them with one color moving to each position. She was able to verbalize that she knew she couldn't make any more because there were no more positions for the one colored cube to move. When I questioned her about the other towers that had two blue and two yellow as well as alternating blue and yellow cubes she could not give me a verbal explanation of how she knew she had them all. Even though she impressed me with her verbal explanation and the way she re-organized the towers, she had a difficult time with writing down understanding of this task.

[redacted]

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.



We went by each color and add more and more colors To build the amout of blocks right.

**CONCLUSION**

Throughout this process I have learned a great deal about my students and myself. I have learned that I do not always need to lead them in the right direction in order for them to find their way. Even though they are special education students I should not set my expectations of their abilities low. I have learned that I can challenge them and that they can surprise me and rise to that challenge. I have learned that mathematics is not just basic arithmetic and procedure, that understanding the process behind the math is very important. Since the implementation of these tasks I have been trying to encourage the "why" behind the math. I have been trying to teach and assess my students' understanding of the reasoning of math. In the beginning, my students' reasoning and mathematical thinking was little to none. They did not question math or even think about explaining math, they just knew what to do and figured that was good enough. Now they are working on improving their reasoning through these tasks as well as their every day assignments. I feel that through the implementation of these tasks as well as the information I have gained throughout this course has made me a better facilitator. I find myself questioning my students more instead of leading them toward the answer. I have a different view about my teaching style and I feel that this change is going to benefit all. The more I question my students the better they understand the math they are given.

yes!

Appendix J

Teacher T3

nice description!

# Classroom Dynamics

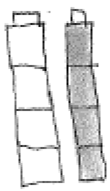
For the past four years, I have taught 7th grade Math and Problem Solving in Sayreville School District in Sayreville Middle School. Sayreville is a suburban town which is part of Middlesex County within the state of New Jersey. There is only one middle school in Sayreville, in which the school is made up of approximately 1,500 students in grades sixth through eighth. We have a six hour and thirty-four minute school day that is broken into nine, forty minute periods. In Sayreville, students have two forty- minute math periods per school day; one period of Math and one period of Problem Solving. These two classes are not necessarily two periods in a row or taught by the same teacher. Due to the class periods being shorter in length, each of the three tasks was implemented over several days.

Each of the tasks was completed in one of my 7th grade, pull-out resource setting Problem Solving classes. This particular class takes place forty minutes prior the 7th grade lunch period in which most days the students can get very antsy. This particular class is made up of 12 total students; 10 boys and 2 girls. The students not only have me on a daily basis, but there is also a paraprofessional available to them for extra support as well. This class is made up of special education students in which every student has their own IEP (Individualized Education Plan). Every student struggles with inattentiveness and the ability to retain information for long periods of time. The students struggle with basic computation, and have a hard time expressing themselves. However, throughout each of the three tasks, my students worked to the best of their ability and enjoyed working on each of these activities.

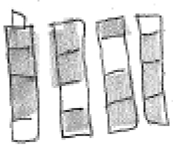
**Impressed – Cycle 1**

**Building 4-tall towers, selecting from 2 colors**

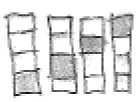
You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.



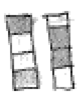
= There is one solid color, and 4 blocks high. There is no other way of doing this.



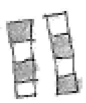
= There is only 1 yellow and 3 blues in each tower. There is a pattern, the yellow Keeps moving up. one



= There is only 1 blue and 3 yellows in eac tower. There is a pattern, the Blue Keeps movin up one



= 2 of the same color is touching, and one color isn't touching,



= not one of the same color is touching

= Both colors are next to there twin.

# Impressed

In cycle one, the work that impressed me was completed by a 7th grade female student. Evamarie was easily able to identify and create the sixteen different towers. I was really impressed with how she recorded her towers. She chose to physically draw the different towers that she had built. She then color coordinated the towers by using a blue and yellow highlighter according to the unifix cubes within each tower, based on the cubes we used in class.

She then grouped the towers into six different groups. Each group was based on the amount of each color within each tower; proof by cases. Evamarie first built the two towers that contained only one color. These two towers were the most popular towers to have been built first amongst the class. She understood that there were only two possible towers that were made up of one solid color; solid yellow and solid blue.

She then created towers that contained three blue cubes and one yellow cube. Here she demonstrated the staircase method by showing the yellow unifix cube as the focus point. The first tower started with the yellow in the bottom position with three blues on top. She then moved the yellow to the second position, then the third and finally the top. Underneath that case, she focused on the blue unifix cube and applied the staircase method with this group. During class she physically showed me that by moving the solo colored cube from the top position, to the bottom position, it would create a duplicate tower. I understood this as a recursive argument, by moving the cubes from the top position to the bottom position duplicate towers would be created.

For the next three groups, she focused on using opposites and having two of each color either touching or not touching within the tower. Here, she saw that you could have either two of the same color touching, both of the colors touching or none of the same colors touching.

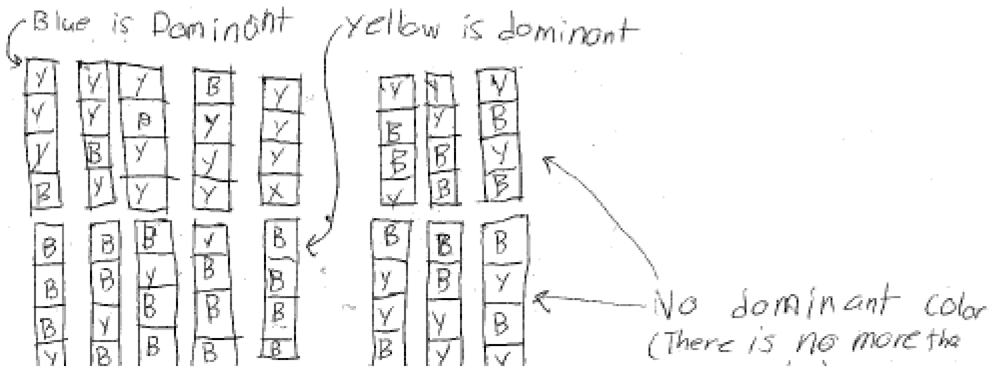
Evamarie's work really impressed because I found the way she drew the towers and grouped them was in a very organized fashion and was easy to follow. When it came time to provide a convincing argument, she gave very straightforward statements about each of the different cases. In fact, I felt as if her argument was quite convincing compared to her other classmates. I was impressed that she was able to physically see the recursive argument although she did not mention the movement of the cubes within her writing. Although her argument could have used a little more clarification, I was impressed with her overall completion in the first task.

**Surprised**

**Building 4-tall towers, selecting from 2 colors**

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.

We have 16 in all. We think we have all the Towers Because each tower has 4 Blocks init and there are 2 colors to make the tower so 2x4=8 then we realised that we can invert The colors to douBle the towers so 2x8=16 Then we couldent make any more so we think That we made all the towers.




We Put the Towers This way Becouse its a Padern, every time the Potern moves it always has one small diference.

# Surprised

In cycle one, the work that surprised me was completed by a 7th grade male student. Patrick immediately approached the problem with mathematical thought. He instantly told me, that there had to be some kind of math behind this problem; he just needed to "figure it out". Patrick and his partner Mark then started to build the towers at random. They both immediately grabbed several unifix cubes and started to build the towers through trial and error. After working for several minutes, he quickly realized that there was going to be sixteen total towers. At that point, he had only built ten different towers. I immediately asked him, where he was getting sixteen from especially when he only had ten towers built in front of him. He explained that since the towers had to be four cubes tall and there were only two colors to choose from, that 2 x 4 is 8. He then explained that since every tower has its own opposite that he then multiplied 8 x 2 to get a total of 16 towers. I then questioned him on the sixteen towers especially since he had only built ten total towers at that point. After speaking with him and his partner, they realized they needed to prove their hypothesis of sixteen towers by building the remaining towers that were missing. A few minutes later, I returned to this student and he had all sixteen towers built. He explained that he used the ten towers that he had already built to help create the six additional towers he needed. He used the idea of opposites to help build the remaining six towers.

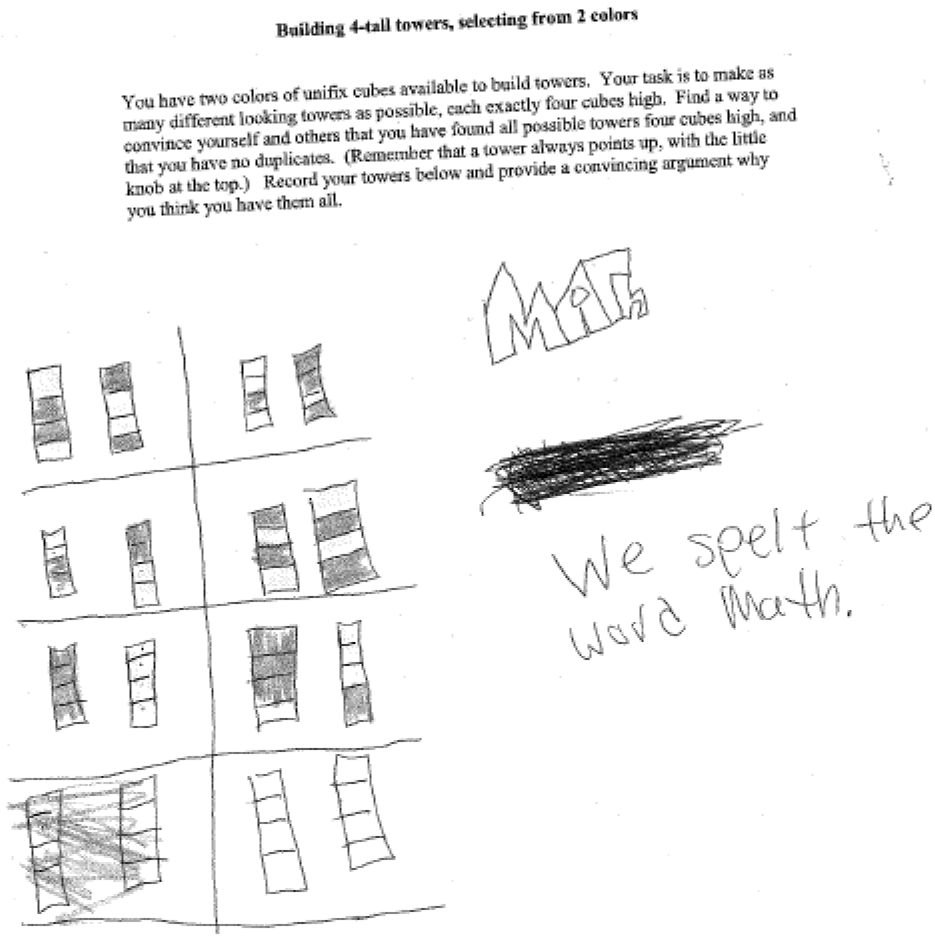
When drawing the towers, he decided to draw squares and use letters to represent the different colors. (B= blue and Y= yellow). When writing up his explanation, I was surprised with the different vocabulary that he included in his work. Just like several other students, he used the

Neat the way students use language!

idea of opposites to build several towers. However, Patrick said that he could invert each tower instead of referring to them as "opposites". He also used the vocabulary of dominance to show that one particular color was more dominant than the other in specific towers. Pat also explained dominance, in the sense that there were towers that contained the same number of blue and yellow cubes in which neither color was dominant.

Although Pat's math was incorrect in this task ($2 \times 4 \times 2 = 16$ versus $2^4 = 16$), I was surprised that he had the immediate thought that this problem could be solved mathematically. When this task was first completed, we had yet to discuss exponents in our math class. Now that we have covered exponents, it would be interesting to see if he still thought that the math was $2 \times 4 \times 2$ versus $2^4$. Overall, I was surprised that he used more sophisticated vocabulary within his argument and overall was very happy with his work in the first task.

**Concerned**



**Building 4-tall towers, selecting from 2 colors**

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.



[redacted]

We spelt the word Math.

# Concerned

In cycle one, the work that concerned me was completed by a 7th grade male student. Taylor really struggled with this task and unfortunately had a very hard time staying focused. Several times during the class, he wanted to see how tall he could build a tower before it would fall over. However, after prompting him several times, Taylor was finally able to build a total of twelve towers.

When it came to recording his combinations, he didn't seem to have a technique as of how he was organizing them. Similar to other students, he used the idea of opposites to build his towers. He drew the towers in groups of two and used blue and yellow colored pencils to represent the colored unifix cubes. After building the twelve towers, his argument, was that he was able to use all twelve towers to spell out the word "MATH". Since he used all twelve towers, and was able to successfully spell and form the word "MATH", he believed that no additional towers could be built.

Taylor's explanation concerned me, because he was unable to identify or see any kind of pattern or relationship within the towers. He struggled with creating the twelve towers and truly believed that there were only twelve, due to the word "MATH" being correctly spelt. After creating the twelve towers, Taylor and I discussed what he had built in hopes to get him to see the towers in a different light. Even with the different observations that he saw, he unfortunately was still unable to see or create any additional towers.

When it came to writing a convincing argument, I felt as if he didn't know how to even go about writing an argument other than writing what he did rather than why he did it. When I

# Intervention Implementation

Overall, I thought my students did a great job on this first task. By completing the first task of building four tall towers with two colors, I learned more about my students. It was definitely a challenge for them, but in the end they were successful. It not only taught me things about my students, but it also taught me about myself as a teacher.

From my students, I realized how much they enjoy working with the unifix cubes. I was doubtful in the beginning when we started working with them and was unsure how well they would do with the towers. However, after several minutes, I realized they were not only enjoying working with them, but they were taking the problem very seriously and trying their very best. I found that they really enjoyed this activity and the manipulatives was the key to their success.

I also learned that I needed to encourage my students to actually work with their partner to create the different towers, versus them working separately and then comparing towers with each other. I learned that they truly struggle to explain their work both verbally and physically. They had a very hard time convincing me that there were only sixteen towers. I was constantly told they could only make sixteen towers because if they had any additional towers they would be creating a duplicate.

I found that this task was not only frustrating for me as a teacher, but it was frustrating for my students as well. They struggled with the fact that I wanted more from them, then the argument of duplicates. On several occasions, I was told by my students that they were angry and exhausted! I too struggled with not being able to help and guide them more. I find that as teachers, we constantly are guiding our students too much and should let our students do more of the exploring and thinking aspect behind concepts. If we allowed them to discover more instead of immediately diving in to intervene and assist, maybe they would have more success with tasks where they are expected to explain.

If given the opportunity to do this task again, I would be more conscience of the pairs of students that I allowed to work together. Some of the pairs were really successful, whereas others did not seem to work well together. I also think that I would try to stretch this activity one more class period. Due to the time restrictions within Sayreville, this task was completed in four, forty minutes class periods. The first two forty-minute periods, the students focused on building the actual towers whereas the second two forty-minute periods they focused on their argument. By using one more class period of time, I could have implemented the extension activity to see what my students could have done with a bit harder of a challenge.

It's so much harder to do these types of "thinking" problems in a 40 minute class over several days. But I know it's not realistic to devote 4 classes to each problem. You have a challenge!

**Impressed**

The Pizza Problem

Capri Pizza has asked you to help design a form to keep track of certain pizza choices. They offer a standard "plain" pizza with cheese and tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many choices for pizza does a customer have? List all possible choices. Find a way to convince each other that you have accounted for all possibilities.

- peppers
- sausage
- mushrooms
- pepperoni
- cheese
- peppers pepperoni
- peppers sausage mushrooms
- peppers sausage
- peppers mushrooms
- peppers sausage pepperoni
- peppers sausage mushrooms pepperoni
- sausage mushrooms
- sausage pepperoni
- sausage mushrooms pepperoni
- mushrooms pepperoni
- mushrooms peppers pepperoni

① to get all of them, we use a system

② the first 4 topics, we went down the line using arows.

example:

peppers
sausage
pepperoni
mushrooms → peppers + sausage

peppers
sausage
pepperoni
mushrooms → peppers + sausage + pepperoni

# Impressed

In cycle two, the work that impressed me was completed by a 7th grade female student. After the task was read aloud, Evamarie immediately raised her hand and had several questions. She wanted to know "If cheese counted as a topping?", "If you could have more than one topping on a pizza?" and "If pepperoni and mushrooms is different than mushrooms and pepperoni?" I told her that the answers to those questions needed to be decided between her and her partner. She immediately did not like my response and wanted some kind of guidance. Although it was hard for me to not immediately respond to her by answering all three questions, I instead told her to think about when she goes to a pizza place to help her decide.

After allowing her to work with her partner Reanna for several minutes, she was able to answer all three questions on her own. I truly believe the idea of using themselves in a real pizza place allowed them to understand the problem in a much more relatable way. The strategy she used was to create an organized list. She started with five, one-topping pizzas; six, two topping pizzas; four, three topping pizzas; and one, four topping pizza. As for her argument, she explained a "system" of using arrows to go from topping to topping. She listed the four-toppings that were given in the problem in a column, and used arrows to help her create each pizza. Although her argument was not completely convincing, I was impressed with her method and how it was a different approach. I was easily able to follow her "system" and it allowed her to easily arrive at the sixteen different pizza combinations. Although her "system" did not help her prove why no additional pizzas could have been created, it did allow her to easily create the sixteen different pizza combinations that we were looking for.

## Surprised

The Pizza Problem

Capri Pizza has asked you to help design a form to keep track of certain pizza choices. They offer a standard "plain" pizza with cheese and tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many choices for pizza does a customer have? List all possible choices. Find a way to convince each other that you have accounted for all possibilities.

We know we had it becuse we based it on tree dirgram then changed to list.

Answer: 13

Peppers
Savage
mushrooms
peperoni

peppers and savage
peppers and mushrooms
peppers and peperoni

pepperoni and mushrooms
peppferoni and savage

savage and mushrooms

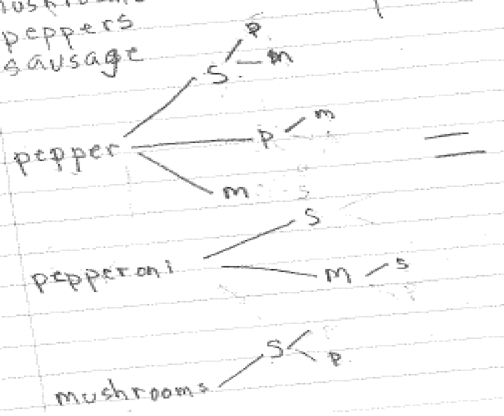
peppers and savage and mushrooms
pepers and savage and pepperoni
pepper and mushrooms pepperoni

peppers
pepperoni
mushrooms
sausase

sausage
peppers
pepperoni
mushroom

pepperoni
mushrooms
peppers
sausage

mushrooms
sausage



peppers
Savage
mushrooms
pepperoni

peppers and savage
peppers and mushrooms
peppers and pepperoni

pepperoni and mushrooms
pepperoni and savage

savage and mushrooms

peppers and savage and mushrooms
peppers and savage and pepperoni
peppers and mushrooms and pepperoni
~~peppers and mushrooms and~~

# Surprised

In cycle two, the work that surprised me was completed by a 7th grade male student. Uriel has recently been moved up a math level, into a pull-out resource setting from being in a self-contained classroom since elementary school. I have already seen a great deal from him since September and am interested to see how far he will come by the end of the school year.

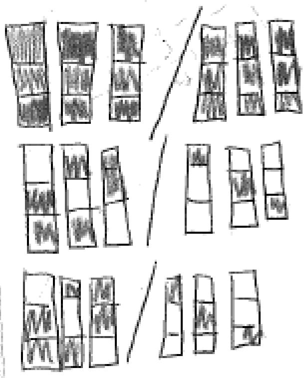
For this task, I was surprised by his work because of the strategy in which he approached the problem. When we first met in Old Bridge and worked on the Pizza Problem, I first tired solving the problem using a tree diagram as well. After working on the diagram for a few minutes, I found that it was very complicated to follow and would be difficult to see all the pizza combinations without losing track.

As for Uriel, although he only arrived at the answer of thirteen different pizza combinations, I was more or less surprised on his decision making throughout his attempt at task two. After the problem was read to him, his immediate thought was to use a tree diagram to help him create all of the different pizza combinations, similar to the thought that I had. After several branches of the tree diagram were created, he started to verbally tell me the different pizzas he had created. As he was telling me, he realized that he had created several duplicate pizza combinations and started to erase his work. He then realized he was erasing quite a bit, which helped him to see a tree diagram was not going to be the best approach to help him solve this problem.

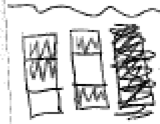
He then decided to approach the problem differently by using the strategy of making an organized list. His list contained a total of thirteen different pizzas combinations, in which he

## BUILDING TOWERS THREE COLORS

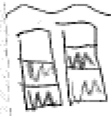
Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and provide a convincing argument that you have found them all.



= Each one has 2 of one color and 1 of one color and then moved the single one down each time. We did opposites for a total of 18 all together



= This is the only way to have all 3 colors used once in a 3 stack high and only have blue on the top



= This is the only way to have all 3 colors us once in a 3 stack high and only have yellow on the top

= This is the only way to have all 3 colors us once in a 3 stack high and only have red on the top

= Each one has only one solid color

# Impressed

In cycle three, the work that impressed me was completed by a 7th grade female student. I was very interested to see how my students would do with this task, now that they were working on the second towers task. For a third time, I chose Evamarie as my "Impressed" piece of work. It was great to see how she improved from task one to task three.

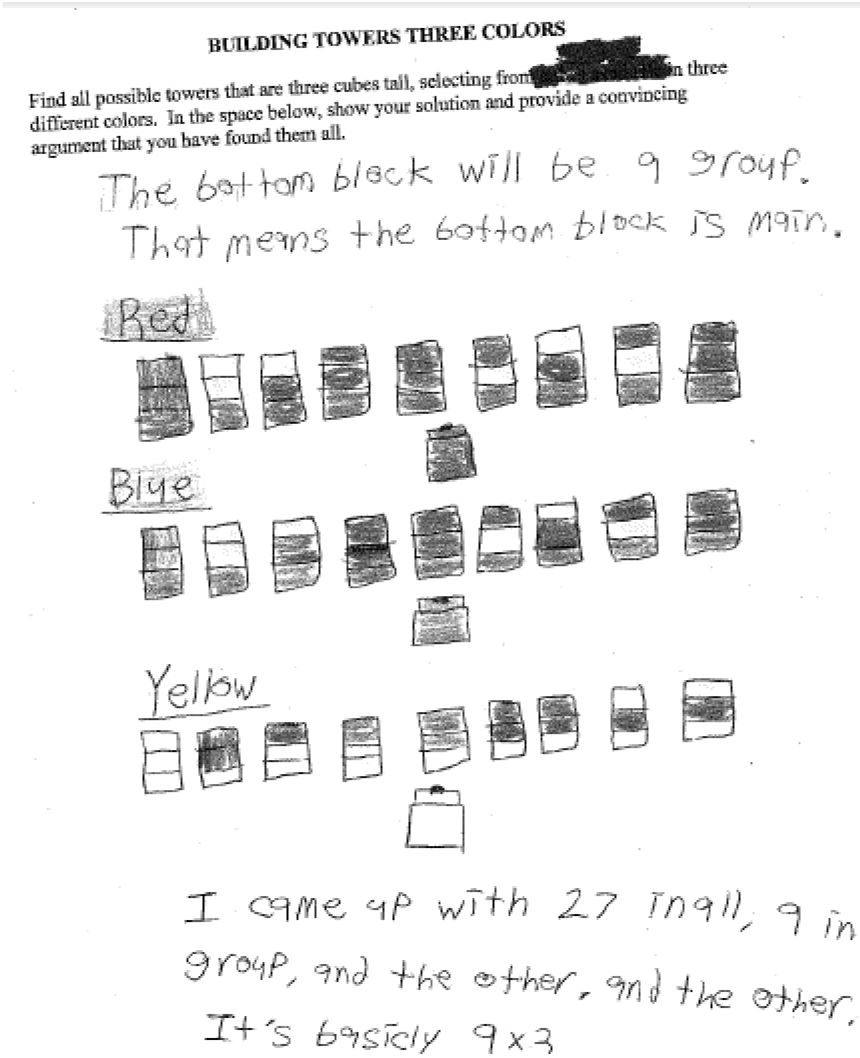
Similar to the first tower task, she grouped the towers by cases. She drew one larger group that contained a total of eighteen towers. Here she again focused on the staircase method. As you can see, in the top row she focused on the red cube going down the stairs surrounded by blue and then the blue cube going down the stairs surrounded by red. Right underneath that, she focused on the yellow cube going down the stairs surrounded by blue and then the blue cube going down the stairs surrounded by yellow. The last group of six focused on the yellow cube going down the stairs surrounded by red and then the red cube going down the stairs surrounded by yellow. She repeated this process in order for each of the three colors to be the focus cube. neat!

She then made three additional groups that contained two towers each. In these groups she kept the top cube a constant. In the first group of two she kept the cube in the first position blue and understood that she could do the opposite of the two other colored cubes. She repeated this process by then keeping the yellow cube and red cube constant in the first position. Lastly, she focused on the three solid color towers; red, yellow and blue.

As for her argument, Evamarie focused on the diagonal pattern, also known as the staircase method. I do feel like she could have gone further into detail about describing how she is certain that there are no additional towers. However, her towers were very organized and that is what impressed me the most.

Overall, I am really impressed with her efforts to always do her best. Her and her partner did a really great job. I think this also reflects on the background knowledge that they gained from completing the first cycle a few weeks prior.

**Surprised**



BUILDING TOWERS THREE COLORS

Find all possible towers that are three cubes tall, selecting from [redacted]n three different colors. In the space below, show your solution and provide a convincing argument that you have found them all.

The bottom block will be a group. That means the bottom block is main.

Red

Blue

Yellow

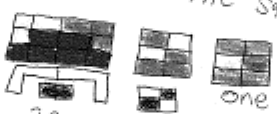
I came up with 27 in all, 9 in group, and the other, and the other. It's basicly 9x3

①

Pg.2

We can't make any more than 27 because the combinations were all used the same way. Each group made the same shape combination.

Ex:



?? Checkers one colored' [meat] sandwich

But the bottom block will always have a different color but the shapes also will be the same, but different color.

Per group

The combination is the same, but in different colors. The bottom block is the leader. When changed, it will never change its comm
But it will change its colors.

# Surprised

In cycle three, the work that surprised me was completed by a 7th grade male student. Ryan is one of my lower level pull-out resource students. He is often absent from school and has a hard time catching up and making up work and material that he has missed.

He completed cycle one a few weeks prior, but this time he worked with a different partner. Ryan immediately got to work and started building towers at random. As I watched him work, I noticed he was building the towers by always choosing to use a red cube first. I kept watching and realized he was keeping the cube in the bottom position a constant, but at that point I was unsure if he had even realized what he was doing. A few minutes later, Ryan and his partner called me over and said that they arrived at twenty-seven total towers. I immediately looked at all of the towers that they had built and was in total shock. Ryan had kept the bottom cube constant in every group. He had three groups of nine towers in front of him as well as three singular cubes. I asked what the single cubes were doing amongst the twenty-seven towers. He went on to explain that the single cubes served as a key to show which color was being kept at the bottom of each tower. He also indicated this key in his drawing.

As for his argument, he first explained it mathematically using the fact that 9 x 3 is 27. He then looked at one of the groups of nine towers individually. He was able to observe different shapes being formed within the nine towers all based on keeping the bottom cube the same color. He explained the shapes by using familiar objects; checkers, meat of a sandwich and what looked like a rotated "L". He explained that the bottom cube will always have a different color either

**BUILDING TOWERS THREE COLORS**

Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and provide a convincing argument that you have found them all.

There are 3 colors and ~~4~~ each tower needs to be 3 high which 3x3 would give me 9 in each column

# Concerned

In cycle three, the work that concerned me was completed by a 7th grade male student. Taylor was the student that I included in my concerned section in cycle one. He was only able to build twelve towers in cycle one and arrived at that conclusion because he successfully spelt the word "MATH". Although he improved quite a bit from task one to task three, I am still a bit concerned with his work.

For cycle three, Taylor was successfully able to build all twenty-seven towers. He not only built the towers at random, but he drew them randomly as well. The first three towers were the solid colored towers, but the remaining twenty-four towers were not drawn in any of kind organized fashion. He drew three groups of nine towers. After drawing all of the twenty seven towers, he immediately said that 9 x 3 is 27. I proceeded to question Taylor on the thought process he had behind his mathematical statement. He said that because there were three different colors and each tower needed to be three tall that 3 x 3 is 9. I continued to ask Taylor if that is why he drew them in groups of nine, but he claimed that he could only draw nine in a given row in order for the towers to fit on the paper. I tried to further explore the math statement he had made earlier of 9 x 3 is 27, but he could only justify the idea behind 3 x 3 is 9. When asked why you would then multiply 9 x 3 to get a total of 27, he simply just said because that is how he drew them.

This argument concerned me, because Taylor was unable to look past the way he drew the towers. All he could see was three groups of nine. We tried discussing the amount of colors within each tower and tried to see if there was any kind of obvious patterns, but Taylor always fell back on the idea that 9 x 3 is 27.

# Intervention Implementation

I really enjoyed working with the students a second time around building the three tall towers with three colors. At first I thought the three colors were going to be a challenge, but I felt like my students did really well with it. By cycle three, I felt that they implemented more of a strategy for this task then when they had completed the first cycle. This time around, they were able to explain themselves better. They were able to focus on specific characteristics within the towers rather than simply just using the idea of opposites.

Overall, this cycle went really well. I felt not only that my students were successful with this task, but I felt success in myself as well. I was much better at questioning my students throughout the entire task. Even when asking my students specific questions, I found that they had become better at responding to me knowing what they had learned from cycle one and cycle two.

If given the opportunity to complete this task again, I would approach the problem in a similar way. I again would have my students read the problem in advance and allow them some time to think about the task, prior to just jumping right into it. I feel as if my students do well when reading over things and taking some extra time to absorb the information. I would have the entire class read over the problem the day before and allow my students to think about the problem overnight before diving right in to solve the task.

# Reflection

Overall, I have really enjoyed implementing these three tasks in my classroom. I have learned more about my students and myself as a teacher through these different tasks. I also learned to never underestimate my students because the work they come up with can be quite surprising, both good and bad.

These three tasks have helped me learn more about how students really do well with patterns. I felt like all of my students really stayed organized by grouping their combinations according to specific patterns. It was really quite amazing to see how differently each of the pairs of students approached the different tasks. As a special education teacher, I often have to really encourage my students to put as much effort as possible into their work. With these tasks, they were extremely motivated and enjoyed working on them.

As far as their reasoning and arguments are concerned, I learned that at the very beginning I needed to really push the idea of giving a convincing argument. My students definitely struggled with this, but by the third task; they knew what was expected of them. They had learned through these three tasks that they needed to organize their thinking on paper and include details on what and why they did what they did.

Over the past semester of implementing the three tasks, I can honestly say that I have benefited a great deal. I find that I am now asking even more questions than ever in my classroom. I am no longer guiding my students to arrive at the correct answer, but allowing them to explore and question each other. These three tasks, made my students realize that I am not always going to give them all of the answers that in fact they need to discover things themselves.

In conclusion, I really loved working on and implementing these tasks into my own classroom. My students really thrived and I feel like I did as well. I plan on continuing to implement similar tasks throughout the remainder of the school year as well as for years to come.

Great !

# Appendix J

## Teacher T4

Cycle I: Towers 4-tall, selecting from 2 colors

Grade and Class Description

For this cycle I worked with my general education 8th grade Math class at the Long Branch Alternative Academy. The class has seven students consisting of five boys and two girls; most of which have individual issues spanning family, social, emotional and academic needs. The class period is 80 minutes (up from 45 minutes last year) and the class is held at 10:45am right after an 80 minute ELA block and right before Lunch.

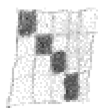
My students have low self-esteem due to many years of academic failure and their skill gaps are many, but even with these challenges, my students are just as capable of learning and demonstrating their knowledge as any other 8th grader. When I create assignments and assessments for this class I need to take into consideration much more than just academic ability. All teachers think about the whole student when planning lessons but I have the unique task of individually tailoring my instructional methods and materials to each child in my room due to their behavioral, emotional, and academic needs. Two students have very general 504 plans but I choose to treat every student as if they have their own education plan. We move at a slower pace but I am still required to cover the same curriculum using the same book, same national standards, same tests, but in less time (our class periods are slightly shorter than the middle school's and we frequently have building social and life skills activities that take students out of class). It sometimes feels like a marathon but I have gotten the knack for stripping down a topic or skill to its base form and building the skill with just the necessities – no frills and no fillers and writing tasks in small pieces. Much of my analysis of their tasks has as much to do with what is on paper as what was said and observed in class.

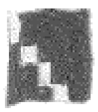
you have a challenging position

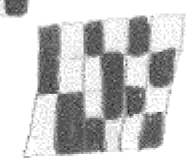
## Impressed

**Building 4-tall towers, selecting from 2 colors**

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.





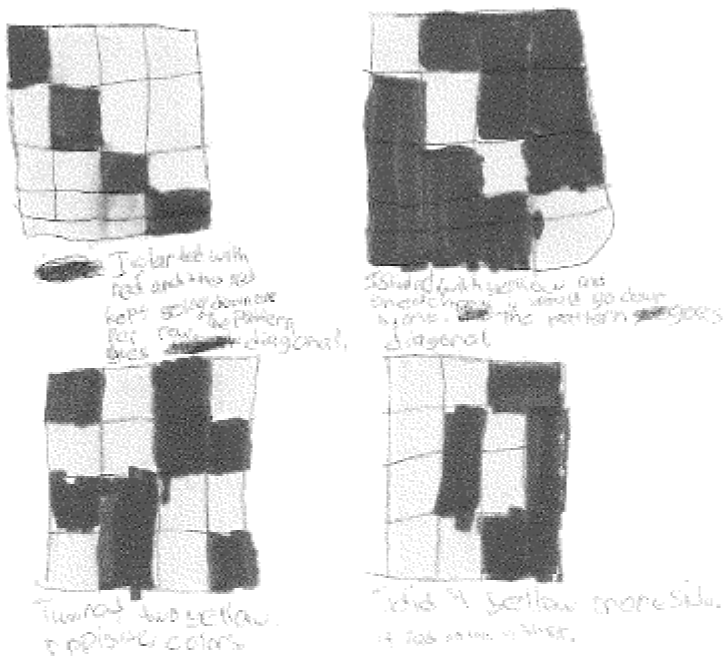


The first row i put is a1red and then I
ended with 3 yellows. The second row
i did is that i started out with 1 yellow
1red, 2 and ended with 2 yellows
The second cube i did is that the First
row is that i started out with 1 yellow
and ended with 3 reds, Then the second
row I started with 1 red the 1 yellow
and ended with 2 red

Cycle I: Towers 4-tall, selecting from 2 colors    Reflective Description – Why it impressed

I think the first cycle was filled with a lot of interesting work because it was done at the beginning of the school year. Many of my students were new to my school and unfamiliar with the policies and procedures which meant they were on their best behavior and put forth a great deal of effort. The sample that impressed me came from one of my new eighth grade students, Leslie. I had the opportunity to meet her and her mother over the summer when she came for an interview with the principal. At this meeting Leslie gave off a very negative impression: head down, no eye contact, very little communication, angry and disrespectful. It was nothing new for me but I was very cautious in my interactions with her at the beginning September. What impressed me most about Leslie's response, although incomplete, is that she did the activity. The student sample work page is not nearly adequate an example of what Leslie did and how well she communicated with her partner and with me. Writing is not her strength but she committed to the task and followed through until she was sure she could not come up with any new towers. When I probed and pressed her to think without my lead she did not rebel and shut down but instead persevered and recursive argument for her tower creation process. I am confident that if she had an extra twenty minutes, Leslie would have written a convincing argument.

## Surprised



**Building 4-tall towers, selecting from 2 colors**

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.



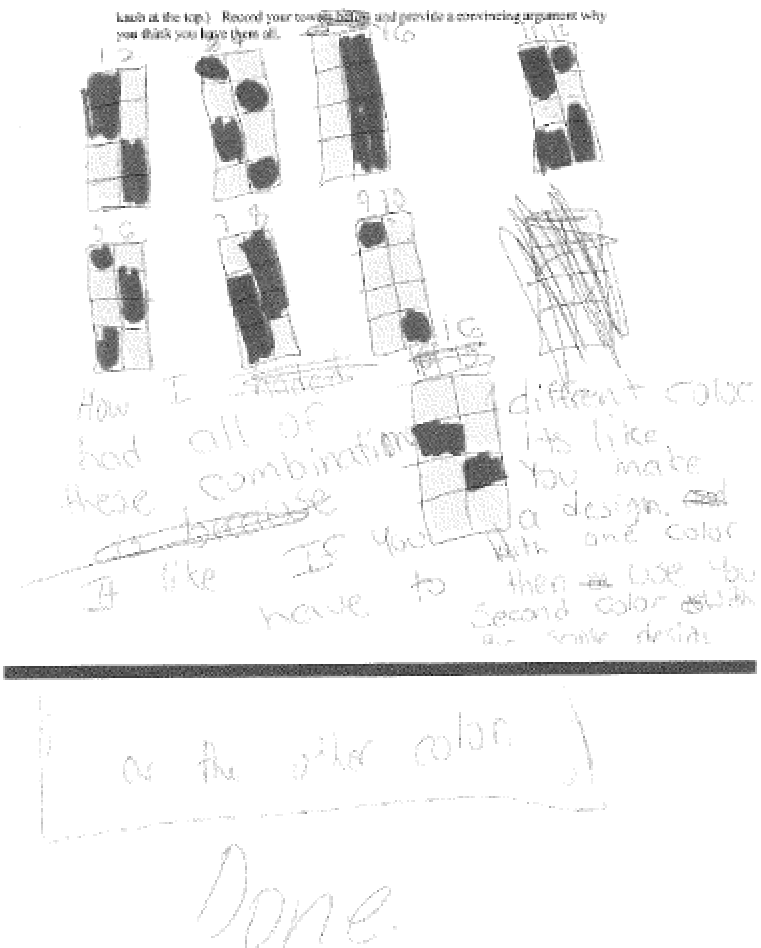
5

Cycle I: Towers 4-tall, selecting from 2 colors Reflective Description – Why it surprised

Erik's work surprised me because of the detail he included but also because of his perseverance through the task. I had Erik in my seventh grade class last year. He arrived at my school in the middle of the school year having failed the first two semesters of Math. He hated Math and as an extension barely tolerated me. He put in minimal to no effort and just skidded by with a passing grade for the year. I was incredibly surprised at the positive attitude and diligence Erik displayed during this towers task. He used a recursive argument to describe some of his cube placement but had a difficult time explaining the rest of his tower formations. What surprised me during this activity was Erik's ability to verbally communicate what he was doing. Erik clearly walked me through the steps he took to create the two-red and two-yellow towers. (This was a massive accomplishment in comparison to the previous school year's performance and I have since been able to build upon his success with the towers.) In addition to communicating with me, I observe Erik motivating his partner, Sostenes, another new student this school year. Erik's persistence through the task actually inspired others in the room and a competitive spirit took over the entire group. I was very surprised and proud of the leadership role Erik had taken on. Erik had a difficult time giving a convincing argument but I was surprised by how he grouped his towers. The lower left group is laid out in a very geometric way and when Erik verbalized his reasoning for grouping them that way demonstrated to me that he likes to look for patterns and shapes (another thing I have been able to use and build upon this school year with him).

Neat!

Cycle I: Towers 4-tall, selecting from 2 colors        Student Work – Concerned



<u>Cycle I: Towers 4-tall, selecting from 2 colors</u>　　<u>Reflective Description – Why it concerned</u>

This third sample concerned me for many reasons. I was very happy this student completed the task and was willing to document anything but his argument is scattered, disorganized and not at all convincing. When I probed further, Laron believed that his answer was the best it could be and he wanted to play with the cubes instead of reaching further for a more detailed response.

Laron wrote, *"How I had all of these combinations...It like If you have to different colors its like you make a design. with one color then use your second color with the same design (flip page) as the other color. Done."* At first I was really worried that Laron had completely missed the point but upon analyzing how he organized his tower diagram I noticed he was trying to explain his argument with proof by cases or opposite pairs. That was encouraging because at least I could see that Laron was looking for patterns and connections between the positions and colors.

However, I was extremely concerned that Laron could not explain his reasoning in words. He was also very resistant to any questioning or assistance from me or the paraprofessional in the room. He barely communicated with his partner and basically horded all of the cubes in front of himself. This task sent up so many red flags for me in regards to how I could best meet Laron's needs and prepare him for the eighth grade NJASK.

Cycle I: Towers 4-tall, selecting from 2 colors Intervention Implementation

As I stated above, I am very happy that this activity came at the beginning of the school year. It gave me the opportunity to see what my students were capable of before they got too comfortable with the class, me and each other. Most students were still on their best behavior and tried to be model students diligently working and persevering till they found all the tower combinations. This and other activities I did at the beginning of the year helped to set an energized and positive tone in the classroom. I have used my observations during the towers task to inform my teaching so I can individualize instruction to best meet each student's needs like more focus on writing clear explanations.

nice!

This task gave me the opportunity to change my teaching and questioning techniques and I have found myself asking more thought provoking questions and doing a lot less leading towards the correct answer. My students get frustrated when I say, "are you sure; how can you prove it? Explain the process and why you chose those steps." But they know that I am making them better mathematicians and not trying to torture them.

This task also gave the class the opportunity to work together. My principal has made it a building-wide goal for all students to work collaboratively (a 21st century skill) and the towers task got students working together within the first month of school. Next year I have many plans and within the first week of school I intend on using this towers task as a team building exercise and an exploration into how my students learn and express themselves. I want to engage them in something hands on and I want to observe exactly how they interact with each other and how they feel about each part of the task (manipulative, creating towers, explain in words together, diagramming, writing out the explanation). One thing I will change with the task is I will allow students two days to work: the first day is working with the manipulatives, partner discussions, and pre-writing then the second day will consist of finalizing their explanations and class presentations similar to what we have done in our regional meetings. I feel my students will benefit from the whole group discussion and presentation of different ideas.

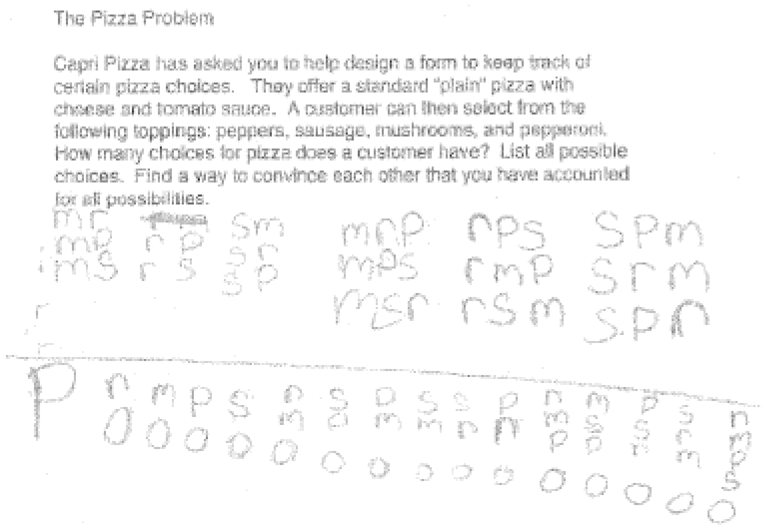
OK!

Cycle II: Pizza Problem, selecting from 2 colors Grade and Class Description

For this cycle I worked with my general education 8th grade Math class at the Long Branch Alternative Academy. The class has seven students consisting of five boys and two girls; most of which have individual issues spanning family, social, emotional and academic needs. The class period is 80 minutes (up from 45 minutes last year) and the class is held at 10:45am right after an 80 minute ELA block and right before Lunch.

My students have low self-esteem due to many years of academic failure and their skill gaps are many, but even with these challenges, my students are just as capable of learning and demonstrating their knowledge as any other 8th grader. When I create assignments and assessments for this class I need to take into consideration much more than just academic ability. All teachers think about the whole student when planning lessons but I have the unique task of individually tailoring my instructional methods and materials to each child in my room due to their behavioral, emotional, and academic needs. Two students have very general 504 plans but I choose to treat every student as if they have their own education plan. We move at a slower pace but I am still required to cover the same curriculum using the same book, same national standards, same tests, but in less time (our class periods are slightly shorter than the middle school's and we frequently have building social and life skills activities that take students out of class). It sometimes feels like a marathon but I have gotten the knack for stripping down a topic or skill to its base form and building the skill with just the necessities – no frills and no fillers that would be just for extra practice. My student get bored and frustrated easily so although I will repeat skills over and over I must continually move forward and reactivate prior knowledge at the beginning of the lesson it pertains to.

**Impressed**



The Pizza Problem

Capri Pizza has asked you to help design a form to keep track of certain pizza choices. They offer a standard "plain" pizza with cheese and tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many choices for pizza does a customer have? List all possible choices. Find a way to convince each other that you have accounted for all possibilities.

mr sm mrp rps spm
mp rp sr mps rmp srm
ms rs sp msr rsm spr

Cycle II: Pizza Problem, selecting from 2 colors Reflective Description – Why it impressed

Besides being the only student to find all of the pizza combinations, Yannick impressed me with his organization method. He began listing pizza combinations using letters: mr, mp, ms, etc. to represent mushroom-pepperoni, mushroom-peppers, mushroom-sausage, etc. This was a common starting point for most students. Yannick thought he had the correct number with eighteen different pizzas but after I asked him to explain what his letters meant he quickly found a duplicate and crossed it out. He was frustrated and not sure where to go with his ideas until he looked at his partner, Jon's, paper. Jon fooled around a lot during this task but he started hi pizza combinations by recording the labels with a circle around them. Yannick adopted this circle method and slowly organized his pizzas as one topping, two topping, three topping and four topping. He did not have enough time to write an explanation but from his list, I think he would have given a clear argument that would convince me he methodically worked to find all the possible topping combinations.

**Surprised**

The Pizza Problem

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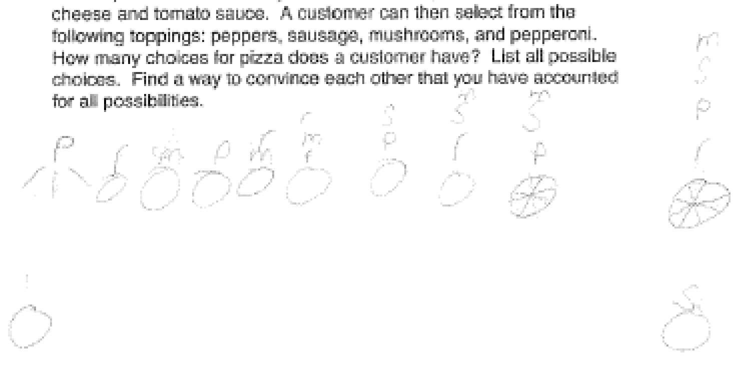


Cycle II: Pizza Problem, selecting from 2 colors Reflective Description – Why it surprised

As I stated above, Jon definitely surprised me in several ways. Firstly, he fooled around a lot during the task. I was unhappy with his performance but had to gently nudge him towards work and away from play. It took a very long time to get Jon on task but once he put pencil to paper I was happily surprised to see that he had tried two different methods. Jon began with a tree diagram with a P and three branches. He did not add to the tree diagram and when I asked him why not he told me it would have taken him too long to do a tree diagram and the problem said to list. Another part of Jon's answer that surprised me was his circles. Jon organized his work as one topping, two topping, three topping, and four topping pizzas. He did not have all of the combinations but his paper showed me he was using some sort of reasoning to find his combinations (he couldn't put it into words). Although Jon only found ten of the pizzas I think he could have found them all and given a convincing argument if given more time and less distractions (different partners, seat moved away from distracting peers, etc.)

Cycle II: Pizza Problem, selecting from 2 colors      Reflective Description – Why it concerned

Sostenes's paper concerned me for a few reasons. He was slow to start and afraid to get the wrong answer. He kept calling me over to ask, "is this it?" He followed the directions and created a list but the list is very confusing. He used the labels P, S, M, Pr, $P^3$, Pe, C, and $t^s$ but did not include a key. I figured it out later that the letters were Plain, Sausage, Mushroom, Pepper, Pepperoni, Cheese, tomato sauce. When I inquired what his list meant he had a difficult time remembering what each letter meant. I probed to see how he made the description to use tomato sauce and cheese as separate pizza choices but he just shrugged his shoulders and said, "I don't know. Is it right?" He wanted me to hand him the answer. I was very concerned because Sostenes actually had the highest grades on all Math assessments up to this point in September. His confusion with the task made me take a good look at how he performs in class and I saw how low his self-confidence was in regards to Math. I was also very concerned with his confusing pizza labels and his inability to clearly explain to me what his combinations meant.

This task provided many learning opportunities for me. I will never have my students work on a partner task like this the day after a three-day weekend. My students had very little success with this task when compared to the first cycle and I think the extra long weekend had something to do with their laziness/unfocused behavior. I learned that my students have a difficult time starting a task and need quite a bit of reinforcement once they have begun in order to motivate them to move forward. The next time I do this task with a class I plan on introducing the problem by reading it out loud to the class and inviting a discussion on how to start the problem. I know this will make the task lose some independent thought but I might be able to get students to brainstorm and share different ways of beginning the problem as a whole group before they break into partners and begin to list the pizza combinations. If I can avoid sharing my opinion during this discussion, my student might feel more confident in their choices just from peer reinforcement. If anything, my students will be able to talk to each other and share their thoughts, a 21st century skill.

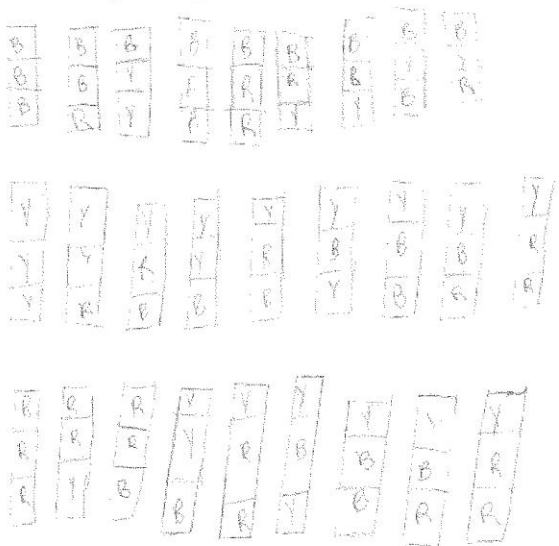
Cycle III: Towers 3-tall, selecting from 3 colors

Student Work – Impressed

Other Group Members: ______________________

**BUILDING TOWERS THREE COLORS**

Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and provide a convincing argument that you have found them all.



How I did it was that I made it just different ways with Red, blue and yellow just with these 3 different colors then I seen I didn't see I didn't [illegible] the same [illegible] different way I made it.

Cycle III: Towers 3-tall, selecting from 3 colors

Grade and Class Description

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Cycle III: Towers 3-tall, selecting from 3 colors    Reflective Description – Why it impressed

This student, Virginia, impressed me with her response because she worked on it completely on her own. She is on a special shortened day plan and works in the back of my classroom while I am wrapping up class with a younger grade. I gave Virginia the cubes and paper and asked her to try making the towers but not to write anything until she called me over and we discussed her work. At first she was frustrated and slow to start so I encouraged her by asking what she had already formed and what they represented. One impressive thing I advised her on was her organization of the towers. She had positioned her three-blue, three-yellow, and three-red in a vertical column. I asked her why they were together this way and she was able to articulate that she saw the relationship of all have three cubes of the same color. She was able to explain that the first, second and third cubes were all the same color and there couldn't be another cube of the same color because the tower would be too tall. I left her to continue exploring and was incredibly impressed with her result. She made a mistake recording her towers and duplicated half of the yellow row. Her explanation is lacking a clear argument but with time I believe I can get her to improve her writing.

## BUILDING TOWERS THREE COLORS

Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and provide a convincing argument that you have found them all.

| | | | | | | |
|---|---|---|---|---|---|---|
| RRB | RYB | YYY | RBY | BRB | RBR | BYR |
| BBB | YYB | RRR | YBY | RBR | YBR | BYY |
| RBB | YYR | YRR | BBY | RYR | BR | RYY |
| YBB | | BRR | | | | YRY |
| | | | | | | BRY |

The cubes from the bottom ~~changes~~ are the same but the cubes from the top the cubes changes. Then 3 more groups change the color to Red. 4

| | | | | | |
|---|---|---|---|---|---|
| Red | Yellow | Blue | Red | Yellow | Blue |
| Blue | Blue | Blue | Red | Red | Red |
| Blue | Blue | Blue | Red | Red | Red |

Cycle III: Towers 3-tall, selecting from 3 colors Reflective Description – Why it surprised

Leslie impressed me not with her diagrams, but with her reasoning. She has the incorrect number of towers but her explanation of: "The cubes from the bottom are the same but the cubes from the top the cubes changes. Then 3 more groups change the color to red. Red/Blue/Blue, Yellow/Blue/Blue, Blue/Blue/Blue Red/Red/Red, Yellow/Red/Red, Blue/Red/Red"

Leslie was holding the first and second cubes constant and only changed the top cube's color. Her diagram does not reflect this reasoning and much of it seems to be a mix of different patterns or opposite pairs but I was impressed with Leslie's ability to see constants in the variables (something we have been working on in equations and functions).

Cycle III: Towers 3-tall, selecting from 3 colors    Reflective Description – Why it concerned

Sostenes's response concerned me greatly. He barely worked with the cubes or talked to his partner and did not record anything. His partner, Erik, tried to organize the cubes into towers but was less motivated to work by the partner he had. When I asked Sostenes how he thought he should begin he helped up a tower of three blue cubes and said, "I don't know. Like this?" I tried not to feed him too much information and said yes keeping making more combinations. He played with the cubes and made a tower with all of the cubes until it fell on the floor. Again, like the pizza problem, Sostenes had a very difficult time working independently and that translated into his partner relationship. I purposely partnered him with Erik because they are very good friends and have the highest grades in my math class. I thought they would work well together but neither student was successful in producing the correct number of towers. I spoke to Sostenes's counselor about his apathetic mood and he is still being monitored by staff to determine why he has stopped putting forth effort in his classes.

Cycle III: Towers 3-tall, selecting from 3 colors

Intervention Implementation

This task proved to be more difficult for my students than the first tower problem. I intend on changing the procedures next time I use these tasks and will have students explain their findings after each task to the whole class. I feel that my students did not want to try very hard with this task because they weren't getting anything out of it – no grade, no final correct answer – and that frustrated them to the point that they no longer cared to try their best. I hope that reviewing what others' did will motivate them and ignite their competitiveness to get the best answer possible.

Reflection

Unfortunately, I did not see the same results as some of my classmates. I was pleasantly surprised and impressed by the small steps my students made towards solving the problems from each cycle but I am very concerned with the lack of ability to explain and give a convincing argument. None of my students demonstrated clearly how they solved each task and why their answer is the only correct answer. I am glad that these tasks came at the beginning of the school year because it opened my eyes to the exact skills I need to work on with my students.

We have been concentrating on explaining our reasoning and finding connections in the math. This eight grade class has been working with variable relationships and I have made a point to incorporate explaining the relationship with a table, graph, word form equation, variable equation, and in two sentences. It seems exhausting but my students are now able to expression relationships between graphs and equations with ease. I was recently observed by my principal and an administrator from outside the building and they complimented me on pushing each student to continually explain what they saw and why they were answering in a specific way.

If you continue to expect your students to write in math class, they will get better at it

The success with verbal answers had not translated into success with written answer yet but I am hopeful it will before the NJASK. I have learned that my students are very capable of completing these tasks but they lack the confidence and reasoning skills to make the connections in a sophisticated way. Once we work on their basic skills more, I am going to give them the three cycle tasks again and then have them compare where their thinking was at the beginning of the year to where they have grown to at the end of the year.

**Appendix J**

**Teacher T5**

## **Cycle 1:** Towers 4-tall, selecting from 2 colors

On September 20, 2013 in my grade 7, period 4, resource class I implemented the cycle 1 activity. In Old Bridge we have 45 minute math period and the class time runs from 10:40-11:26. This class consists of all classified students, and they all range in academic levels. Since our class periods are a 45 minutes, time constraint is an issue. This activity was done over a course of 2 1/2 days of class. When I implemented this activity all students were in attendance, and since it was the beginning of the year I grouped them the best that I could.

I agree - 45 minutes is not enough time to engage students in thoughtful mathematics

number of students ?

**First Sample: Impressed** by an interesting example of mathematical reasoning.

I chose Anthony's work because as soon as he was given the task he automatically reverted to multiplication. Since it was the beginning of the year I did not have a true understanding of my students mathematical abilities. Although I was able to decipher at this point that Anthony was one of my stronger math students, I was impressed that he went straight to the mathematical portion of the activity.

When he was given the material he said," Oh there are two colors and the towers have to be four high, so you multiply them." I then questioned him why do you multiply the numbers, and he was not able to explain his thoughts in more detail. I then suggested to him to start building with the unifix cubes, and that might help him come to his conclusion.

As he was working ~~was working~~ he was arranging his towers by their opposites. He would still refer to his idea of multiplication, but by seeing that each tower had an opposite this helped him realize that there are more combinations than 8. At the end of the activity he had the result of 16 towers. He knew that he was done because he was able to prove that if he made more there would be opposites. He also said that since there are 8 towers built, and their opposites then there are16 towers total. After the task was over I asked him about his thoughts on the multiplication, and he said since $4 \times 2 = 8$, and they all have an opposite then you would multiply $8 \times 2 = 16$ towers in all. Even though Anthony did not mention the concepts of exponents I was glad to see that was thinking mathematically.

we did it this way because we had
2 coders and they have to be 4 hyye
so you moltiply them. Then the can be
the opisit what you would have to add 2
more so that would be 16. So there could be
only 16 different comboyes. we orrngnisemed
by the opsitec of the combinouchens.

**Second Sample: Surprised** by the strategy or representation, or as an impressive product from an unlikely student.

For this sample I chose a boy who has a very tough time focusing on any task and requires constant redirection. In the beginning of the year Ryan also did not show at this point much effort towards school. Ryan chose to work alone on this activity, with the assistance of my paraprofessional standing in proximity of him to keep him on task. With that said, Ryan has improved a great deal since the beginning of the year, and I could not be prouder of him.

As soon as Ryan received the unifix cubes he started to create the towers immediately. He was not very organized at first, but then I noticed that he was keeping a constant throughout and that really impressed me. His constants were the top cubes organized with one blue top and one yellow top towers. Then he rearranged the 3 other cubes to create opposites of each other. However, when Ryan was asked to draw the towers for some reason he did not draw them they way he had them organized. When I asked Ryan if he was done he said yes he knew he was done because he tried to make more but then we would have the same ones. Therefore, he understood that all possibilities were created because he knew he could not have duplicates.

**Building 4-tall towers, selecting from 2 colors**

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| B | Y | B | b | b | Y | B | b | Y | Y | b | Y | b | Y |
| B | Y | Y | Y | Y | Y | Y | B | b | Y | b | Y | b | b |
| B | Y | Y | b | b | b | Y | Y | Y | b | Y | b | b | b |
| B | Y | Y | b | b | Y | b | b | Y | b | Y | Y | Y | Y |
| y | | | | | b | | | | | | | | |
| b | | | | | y | | | | | | | | |
| y | | | | | b | | | | | | | | |
| b | | | | | | | | | | | | | |

I did this why because this was the easier way to do it I knew why I was done is I stopped and tried

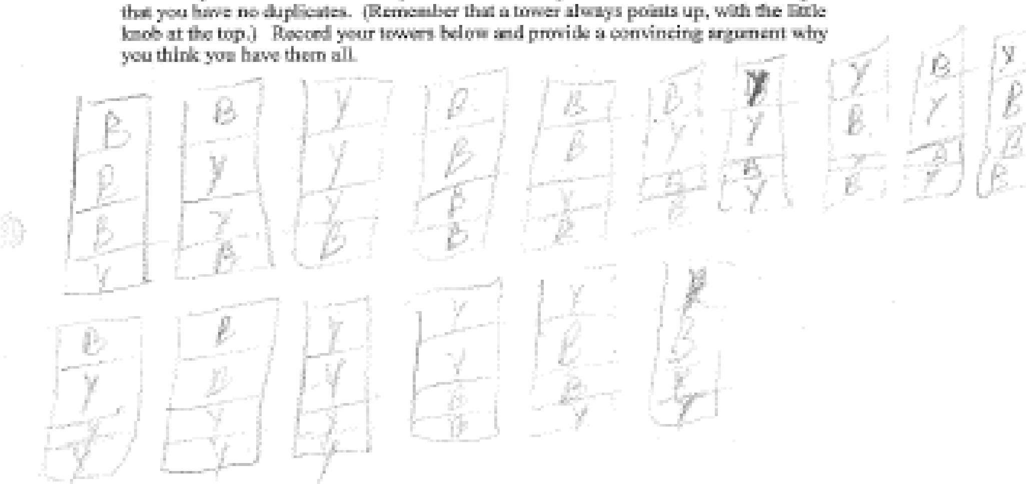
to find other ways but I couldn't.

**Third Sample: Concerned** about the student's struggles to understand the mathematical ideas.

I chose Patrick's work because throughout this activity Patrick had a difficult time grasping the concept even with his partners help. After Patrick and his partner started working on the activity I went over to see their progress. I noticed that they both still had a difficult time understanding the meaning of a duplicate. So I demonstrated to them an example and after a few demonstrations his partner understood, but Patrick was still having difficulty. Then his partner started to make some towers while Patrick was watching. He then eventually was able to help his partner after she had made several towers. Their towers were arranged in proof by cases. I was concerned because this was extremely difficult for him visually, and also had difficulty fully comprehending the activity as a whole. With my guidance and some help from his partner Patrick was able to somewhat help with this activity.

**Building 4-tall towers, selecting from 2 colors**

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.



the reason we arranged the blocks this way is because

## Cycle 1: Intervention Implementation

Since this was my first activity I was very unsure of how well this activity would work out. I also was a little concerned because I really did not know my students that well, and I was hoping I paired them successfully. For the most part I felt that I did, so that made me happy. I think that the way I approached this activity would stay the same for next time. When I introduced the activity, I was very thorough with the instructions. I made sure the students understood the task by having them repeat to me what is being asked.

This activity was very helpful because it gave me great insight to all my students that may have gone unnoticed. I would like to think that it would not have gone unnoticed for long, but this activity brought it to my attention. It was nice to hear them use their reasoning skills with their visual aides to prove their work. Even though a majority of my students had difficulty proving their point, they all had the general idea. I consider that a success because they are not used to explaining their math reasoning, and were all able to get 16 towers. The only aspect I might change is demonstrating how to draw the towers, without identifying as notation for them. I know this may sound simple, but some of my students had a hard time drawing the towers. A strategy I did use that was helpful was one



don't limit students - let them decide how they want to record their towers partner would read the order of the colors on the towers, as the other partner wrote it down.

## **Cycle 2:** Pizza Problem, selecting from 4-toppings

On October 21, 2013 in my Grade 7, period 4 ,resource class I implemented the cycle 2 activity. In Old Bridge we have 45 minute math period and the class time runs from 10:40-11:26. This class consists of all classified students, and they all range in academic levels. Our class periods are a 45 minutes, so the time constraint is an issue. This activity was done over a course of 2 1/2 days of class. When I implemented this activity all students were in attendance.

**First Sample: Impressed** by a student's mathematical reasoning.

In this activity Naziel started to list the topping horizontally, and he does not use a notation for the name of the toppings. Although when he started to write his response he had a notation, but it was not used. His approach impressed me because when he explained reasoning it was in a somewhat of a recursive way. He started listing the topping with the peppers as the constant. He listed the pizzas with peppers as the constant with all four toppings, then to three, two and one. He then did it again with sausage as the constant starting with three toppings and then to two toppings. He then started with mushrooms, but did not complete the task. He decided there were 10 possibilities. I tried to encourage him to try a different way to organize his data because I think what was frustrating him was his inability to see if there were duplicates. He decided not to approach it differently, and I was ok with that. This task was more challenging and I did not want him to become frustrated. However, I believe from what I saw in his work and his reasoning if he tried a different approach I think he would have be able to get the correct answer. I was very impressed on his approach and explanation to this activity.

Capri Pizza has asked you to help design a form to keep track of certain pizza choices. They offer a standard "plain" pizza with cheese and tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many choices for pizza does a customer have? List all possible choices. Find a way to convince each other that you have accounted for all possibilities.

Pizza = Pi
Peppers = Pe
Sausage = S
Mushroom = M
Pepperoni = Pep

Their are 10 possibilitie because from the topping in order I can reverse them and in the middle I can use to. If all the ten combinations are reverse then it would will be the same.

1) Peppers, Sausage, Mushroom, Pepperoni
2) Peppers, Sausage, Mushroom
3) Peppers, Sausage
4) Peppers, mushroom, Pepperoni
5) Peppers, mushroom
6) Peppers, Pepperoni
7) Sausage, Mushroom, Pepperoni
8) Sausage, Mushroom
9) Sausage, Pepperoni
10) Mushroom, Pepperoni

Their are 10 possibilities.

## **Second Sample: Surprised** by the strategy or representation selected, or an impressive product from an unlikely student

I chose Alexis as my student who impressed me because she did not seem confident when I was going over the activity, and looked very confused. When I went around interviewing each group I saw many different notations and strategies being used. When I came to Alexis I was not sure what she would have, and what I saw really surprised me. She was making a tree diagram where the plain pizza was the base, which is not the strategy I would have guessed she would have used. I was so happy for her because she seemed more confident and at ease. She used the tree diagram for a while, but her partner was listing his responses, so she changed to the listing method. I asked her why she changed and she said it was hard to explain to her partner. I think she was also having difficulty because of the way she had them set up, and the order of her pizza topping were hard to detect. Even though, she did not come to the correct response she understood the concept. I was very happy to see her gain some confidence in an activity that was difficult for others, and that she also did not give up.

Good!

The Pizza Problem

Capri Pizza has asked you to help design a form to keep track of certain pizza choices. They offer a standard "plain" pizza with cheese and tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many choices for pizza does a customer have? List all possible choices. Find a way to convince each other that you have accounted for all possibilities.

I started out with a branch tree because it is easier to see my combonations, then I stoped and worked with Rocco. Our combinations are

Pizza - peppers
Pizza - saugie
Pizza - mushrooms
Pizza - peperoine
Pizza - peppers, saugie
Pizza - peppers, mushrooms
Pizza - peppers pepperone

we ~~[illegible]~~
got
12
combinations

~~[illegible]~~
12
[illegible]

Pizza - peppers, mushroom, Pepperone, Sausge

Pizza - Plain

Pizza - saugie, ~~[illegible]~~ mushroom,

~~Pizza - Saugie~~ Pizza - saugie

Pizz - saugie, pepers

Pizza - Mushrooms, ~~[illegible]~~ pepperine

Pizza plan pepper

Plain - pepers, saugie, mushrooms, peperonine

Plain - pepers, saugie, mushrooms, peperonine

Plain - pepers, saugie, mushrooms, peperonie

Plain - saugie, pep, mushrooms, peperines

[illegible] group [illegible] and Rocco october 15th

plane pizza with cheese and tomatto sauce

comster choice - peppers, ~~[illegible]~~ sausage mushrooms and pepperonie | how many choces does the costster have?

**Third Sample: Concerned** about the student's struggles to understand the mathematical ideas.

Michael's work concerned me because he was unable to grasp the concept of this activity. However, his work was very organized. As I approached Michael and his partner I heard them discussing the possible choices a person can make with the four toppings. They started out making the basic selections. Then he had a hard time understanding the concept of duplicates, for example: pepperoni & sausage is the same as sausage & pepperoni. I also explained to him that we were not creating a half pie or a topping for each slice, but that too was also difficult for him to get passed. I was concerned because he was so focused on one idea that he was not able to get past it. I think in general this concerns me not just in math, but in any real life situation.

## BUILDING TOWERS THREE COLORS

Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solutions and provide a convincing argument that you have found them all.

## Cycle 2: Intervention Implementation

Well for starters this activity was very different from the first activity. I think in this activity the question is written in an ambiguous way, and it made it hard for my students to decipher what was being asked. I went over the problem, but found myself having to clarify the activity several times as I walked around. I think for next time I would go over the problem in more detail for the students.

As I walked around I feel that this activity displayed a wide variety of strategies my students used opposed to the first activity. I really enjoyed seeing how my students chose to organize their work. I had some students use tree diagrams, listing, creating drawings, and making charts.

I also think in this activity was a little more difficult because there were not any manipulatives to use. So the students had a hard time telling if they had a duplicate. Many of my students did not come up with an accurate solution, but they showed improvement on their organization of their mathematical findings.

## **Cycle 3***:* Towers 3-tall, selecting from 3 colors

On November 14, 2013 in my Grade 7, period 4, resource class I implemented the cycle 3 activity. In Old Bridge we have 45 minute math period and the class time runs from 10:40-11:26.This class consists of all classified students, and they all range in academic levels. Our class periods are a 45 minutes, so the time constraint is an issue. This activity was done over a course of 2 1/2 days of class. When I implemented this activity all students were in attendance.

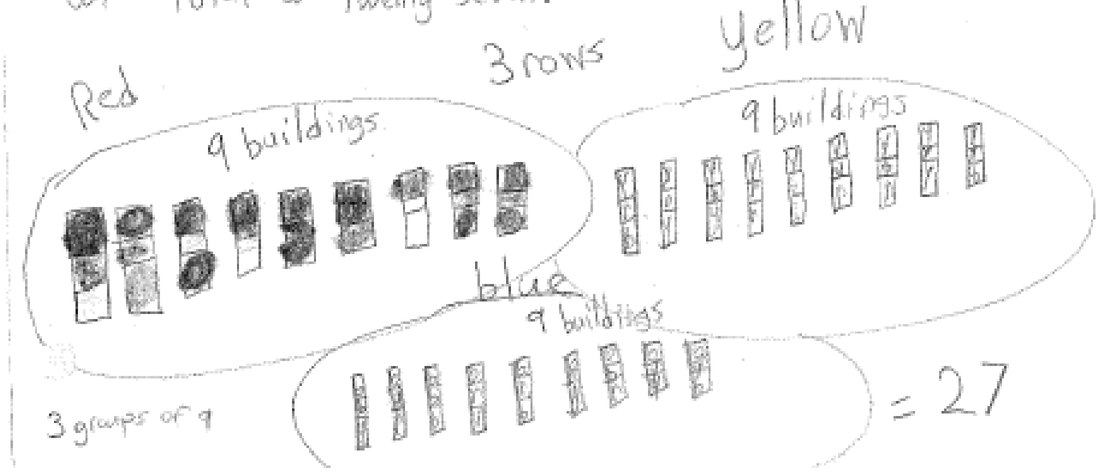
## **First Sample: Impressed** by an interesting example of mathematical reasoning.

Justin is my choice for this cycle activity. He truly impressed me with his reasoning skills and his organization of his towers. At first Justin and his partner discussed how they wanted to approach this activity. Then they started with the yellow and blue solid towers. He then started to build them in a recursive pattern using the blue and yellow colors. He then created the opposite pattern. He continued to create his towers this way for a while. When I asked him why he was arranging them this way he just said that he saw a pattern and proceeded to point to the pattern, or recursive argument. He then said he knew he also had to make the opposite pattern.

I waited a while before I went back to his group, and when I went over to them he had the towers organized by a constant of the top colors. He had two groups of nine towers and one group of ten. When he told me he was done I questioned him about the amount of towers he has in each group. Through my questioning and our discussion he was able to realize that since there are three colors that each of the three groups should have an equal amount of towers. I was very impressed that he was able to draw this conclusion with minimal assistance. That really showed me his mathematical thinking has improved since the last activity.

Neat!

we put them in order and we came up with three row of nine buildings and multiplie them together and our total w twenty-Seven.



## **Second Sample: Surprised** by the strategy or representation selected, or an impressive product from an unlikely student

I chose Nakya for the piece that surprised me because as soon as I mentioned this activity she said, "Oh boy this is going to be hard." We then discussed as a class why this activity may be harder than the last. She mentioned because you are only building them three tall and you have three colors. So at that point I was thinking that this might be hard for her conceptually to understand and organize. I knew she would give it a good effort so I was not concerned in that area.

She and her partner started with the three solid colors and then started making patterns with the three colors. They were basically making random towers and then creating their opposites. When I approached them we discussed their process, and I suggested that maybe they should try to organize their towers in some way to help them see all their possibilities. When I came back to them organized in sets of two with their opposites. I was very surprised that she was able to organize her towers in a way that displayed all possibilities, and she also made somewhat of a connection to the mathematical portion of this activity.

**BUILDING TOWERS THREE COLORS**

Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and provide a convincing argument that you have found them all.

ESTIMATE 31

**Third Sample: Concerned** about the student's struggles to understand the mathematical ideas.

The student that I chose that concerned me greatly was a student who visually had a difficult time with this activity. In her group her partner completed a lot of the towers, not because she was not helping but because visually this was hard for her see. When I went over her I saw that she was getting frustrated, so I worked on explaining the task in more detail and also giving her some examples using only two colors. She was starting to feel more comfortable with this activity because she said it was similar to the first one. When I then did a tower with three colors she could not create and opposite without getting frustrated. I would never want any of my students to feel defeated, so I had her focus on the towers with only towers with only two colors. I even told her to start with the blue and create all the blue possibilities. I went over later to check on her and she was able to do that. While she was creating the blue group, as I called it, her partner was working on the towers that included three colors. Then they both worked on the other towers that on contained two colors. She was unable to draw out the towers, so her partner read the colors to her as she wrote them down. So as you can see my concern I was still very happy that she was able to complete the part of the activity she was asked to do, and she was not frustrated doing it.

1. Plain Pizza with half Pepper and half mushrooms.
2. Plain pizza half Plain and half Sausage.
3. Plain Pizza half Sausage and half mushrooms.
4. Plain Pizza half Pepperoni and half mushrooms
5. Plain pizza half Sausage and half Pepperoni.
6. Plain pizza half Pepperoni and half Peppers
7. Plain Pizza half Pepperoni, half Sausage, and half mushrooms.
8. Plain Pizza half mushrooms, half Sausage and half Peppers
9. Plain Piza 2 Slices of Peppers

of mushrooms, and 1 slice Pepperoni.

10. Plain Pizza 2 slices of Peppers 3 Slices of Sausage, 3 Slices of mushrooms, and 2 Slices of Pepperoni.

11. Plain pizza 4 Slices of Peppers 4 Slices of Sauage, 2 Slices of mushrooms, and 1 Slice of Pepperoni.

## Cycle 3: Intervention Implementation

As we approached this activity I was a little concerned on how my students would do with this. I wasn't sure if they would understand the task as quickly as they did. With this task as I introduced the directions I read them twice as they read them from the board. I then asked a student to explain the task and then questioned them about the amount of colors needed in each tower. I think being thorough with the directions helped my students feel at ease and fully understand what they had to do. I was very surprised how well all my students did. I thought that the organization factor in the task would be challenging for them, but they all impressed me with the variety of strategies the used. Some of my students also made a mathematical connection to the activity as well. Such as: there are three colors and three high so you multiply them, and multiply them by three again because you need an opposite. I also had other variations of their mathematical perspectives. Even though none of my students made the connection to exponents, they all had a decent mathematical perspective, which made me very happy.

At the end of the lesson I tried something new that the students responded well too. I had them walk around the room and look at everyone's towers and try to identify how they organized their towers. I then showed them mine, which they loved to see how their teacher solved the activity. All in all

I was so proud of my students throughout this activity not only because of their growth in math, but also for their determination.

## Reflection

I really enjoyed doing these activities with my students. I feel that it was a great ice breaker for the beginning of the year, and also allowed me to see another side of the students I may not have seen. These activities also allowed us to work together as a group and learn from one another.

Throughout the activities my students taught me to pay attention to details. Such as how everyone approaches/explains an activity differently. My students were able to teach me this throughout all of these activities. Implementing these three cycles of activities was about reasoning and mathematical thinking, but it also brought to light the teaching techniques and strategies we need to remember to incorporate into the classroom.

I feel that all my students were able to reason through these activities mathematically, by using various arguments. I think they made a better connection mathematically because they were hands-on and they could make the task their own. It is nice to see what the freedom of thinking can show me. They also have improved on explaining their work and it is starting to come naturally to them.

Also, my questioning skills have improved throughout this course. Often I give too much away when I question, or guide me students too much. I know that I do it, but after this class I am able to question my students without fully leading them to the answer.

# Appendix J

## Teacher T6

Detailed Class Description:

I completed the following problem with a 6th grade, "Pinnacle" math class. Pinnacle is the label for gifted and talented classes. These classes are based off of a student's performance in *all* subjects and on state testing, not only their mathematic performance or level. Because of this, I see a varying level of mathematic skill and reasoning in this class of 27 students. The class that I have selected to complete this problem with is at 12:50 PM. Class is usually over at 1:50 PM; however, for these problems we extended the class through our "enrichment period" and class was held until 2:10 PM. This is the students last period of the day. The students have three academics in the morning, activity classes and lunch before coming to math class. Although the long break could cause some students to lose focus, this group seems to be energized and excited to end the day with math class. There is one child in the class with autism who has an IEP; however, he functions normally with the class and completes partner and group work like all students.

These students are used to sitting in partners and working collaboratively in class. Students normally sit in pairs and the seating is changed frequently for students to be accustomed to working with many different people. These children spend most of the day together and take all academic class periods together. They are comfortable with challenging each other and pushing each other to reach academic expectations. I choose this class purposely because I know they will do everything they can to reason mathematically in these problems, and because they will work together in order to do so.

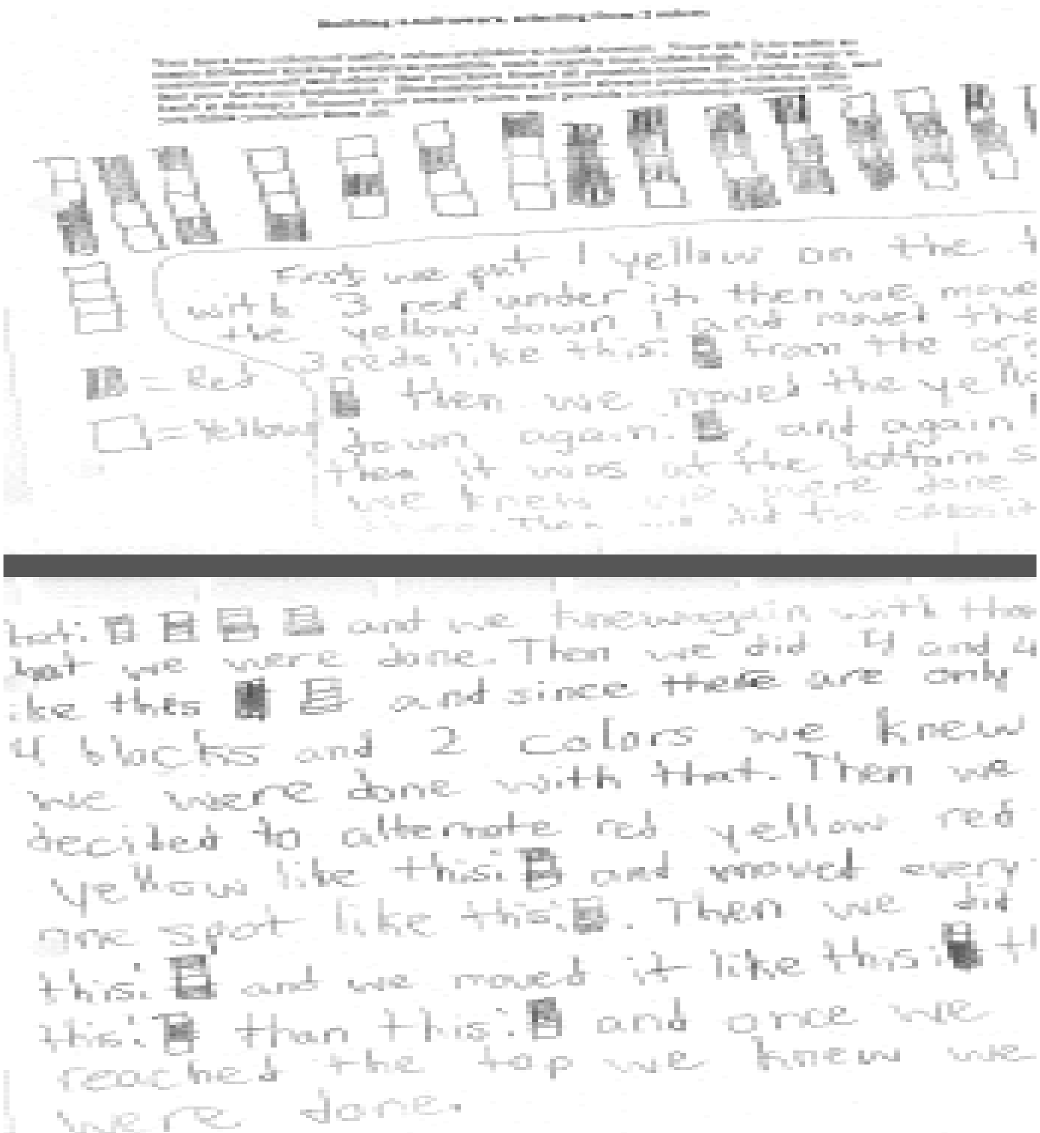
Nice description! You are fortunate to have a long block of time for math class.

Cycle I- Impressive

One student that impressed me with this problem used a combination of proof by cases and a recursive argument. The student did a good job describing the towers that were three of one color and one of another. The student talked about moving the one color cube down to all four positions. The student articulated this explanation very well in his writing. He made a new tower by moving the one color down each time, and knew that they had all of them when the cube was at the bottom.

He also stated very clearly that there were only two possible full color towers because there were only two colors. This is an obvious to the students, but I was impressed that this student thought to write the reason why there could only be 2 solid color towers.

The student started to use a recursive argument with the towers that there 2 of each color. He said he and his partner started with alternating colors and just moved everything down one to create a new color. After this; however, the student simply states the other towers that he and his partner created. Although this is not convincing, overall this student produced some impressive work. The student did a good job clearly writing his thoughts. He recorded his towers and also drew the tower he was talking about next to the description on his page. The extra drawing of the tower made his explanation easy to follow. As I mentioned in my reflection, I would have like more time to work on this problem. With some more time, this student may have been able to devise an argument for the remaining towers.



= Red
= Yellow

First we put 1 yellow on the t
with 3 red under it then we move
the yellow down 1 and moved the
3 reds like this: from the or
then we moved the yello
down again. and again
then it was at the bottom s
we knew we were done

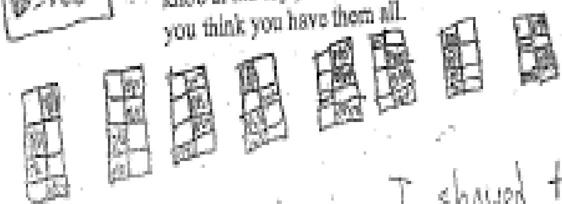
that: and we knew again with the
that we were done. Then we did 4 and 4
like this and since there are only
4 blocks and 2 colors we knew
we were done with that. Then we
decided to alternate red yellow red
yellow like this: and moved every
one spot like this: . Then we did
this: and we moved it like this: th
this: than this: and once we
reached the top we knew we
were done.

Cycle I- Surprising

I expected students to struggle with the four tall tower problem building from two colors. When I first entered the class, I struggled with understanding what it meant to truly be convincing when arguing that I had found all the towers. The student's work that surprised me was extremely honest in her recording of her convincing argument. She was able to find all 16 tower possibilities and organized these towers as opposite pairs. She had convinced herself that she had built all of the towers, yet she admitted in her writing that she was not sure how to prove that they were the only 16 towers. Many students struggled with writing their arguments and stated that they had found opposite pairs, but this student recognized that just putting the towers in pairs was not convincing. Even though she could tell it was not a good argument, she still admitted that she could not formulate mathematical reasons as to why she had created all 16. I was surprised by her confessions and also happy to see that this student recognized what was not considered convincing. This student's work shows me that she was trying hard to create a convincing argument with her towers four tall.

**Building 4-tall towers, selecting from 2 colors**


You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.



In the above drawing I showed the sixteen possible ways to make a 4
cube tall tower. I grouped them with the opposite ones to them. Th
doesn't really show that there are only sixteen ways to make t
towers. Personally, I'm not very sure how to show that, but I co
give reasons as to why I believe there are only sixteen possib
First, once my partner and I finished finding these, we tried finding ot
and just couldn't. That was very convincing, though...

Cycle I- Concerning

All of my students were able to arrive at the correct number of towers for this problem. When it came time to write a convincing argument, many students struggled. This student's work in particular concerns me because of her lack of organization and a convincing argument.

Although this student was able to build 16 towers with no duplicates, her reasoning and argument was lacking greatly. She reasoned that she had made all 16 towers because she made opposites and tried to make others and could not. This is not a convincing argument. This student was satisfied with her argument and when encouraged to try a different organization of her towers, she made no attempt. As you can see, the towers are recorded as pairs of opposites, which mirrors her attempt at a convincing argument. With the recording of pairs, it does not seem like she tried to organize them in any other matter. There were many students who choose to organize their towers as opposite pairs for this problem; however, there was some other hidden organization within their recordings. Students would record all pairs with three of one color and one or another and two of each color, hinting that there was some other thinking when creating the towers. This student's organization of the tower pairs seems more random then the other students.

**Building 4-tall towers, selecting from 2 colors**

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.

My argument is we found ~~one~~ tower and then found the opposite. Then we tried and tried again and we couldn't find anymore. Thats why were shore we found all of them.

Cycle I- Intervention Implementation

Building towers four tall from 2 different colors was the most challenging problem to implement in the classroom. I think that because it was the first problem we completed, both myself and the students struggled. After this problem, I learned that my students struggle with translating their mathematical thinking into words. Many students had a difficult time writing down what they were able to verbalize. My students also had a difficult time formulating a "convincing argument". After reading their work and observing them in class, I came to the conclusion that many of them did not truly understand what it meant to have a convincing argument. Many of the students would simply restate what they had in front of them or use the opposite argument.

It takes time to help students understand what it means to give a convincing argument - keep at it!

It was difficult for me to not lead the students during this problem. I caught myself a few times about to lead students in a certain direction with their towers, but always stopped myself. I noticed that this became easier by the end of the class period.

If I had the opportunity to complete this problem again, I would have tried to have more class time to complete it. It is difficult to adjust our schedule; however, I think the students would have benefited greatly from more time.

Although my students struggled with this problem, they put a lot of effort into this problem. None of the students became frustrated that they were not succeeding in "convincing" me, but simply worked harder to come up with a convincing argument. The students realized that it is difficult to write down your mathematical thinking, but no one gave up because it became difficult. I cannot claim that any of my students produced a "perfect" solution; however, I received some great work from my students for this problem. I was proud that the group worked outside of their comfort zones and stayed focused and motivated. great!

Detailed Class Description:

I completed the following problem with a 6th grade, "Pinnacle" math class. Pinnacle is the label for gifted and talented classes. These classes are based off of a student's performance in *all* subjects and on state testing, not only their mathematic performance or level. Because of this, I see a varying level of mathematic skill and reasoning in this class of 27 students. The class that I have selected to complete this problem with is at 12:50 PM. Class is usually over at 1:50 PM; however, for these problems we extended the class through our "enrichment period" and class was held until 2:10 PM. This is the students last period of the day. The students have three academics in the morning, activity classes and lunch before coming to math class. Although the long break could cause some students to lose focus, this group seems to be energized and excited to end the day with math class. There is one child in the class with autism who has an IEP; however, he functions normally with the class and completes partner and group work like all students.

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Cycle II- Impressed

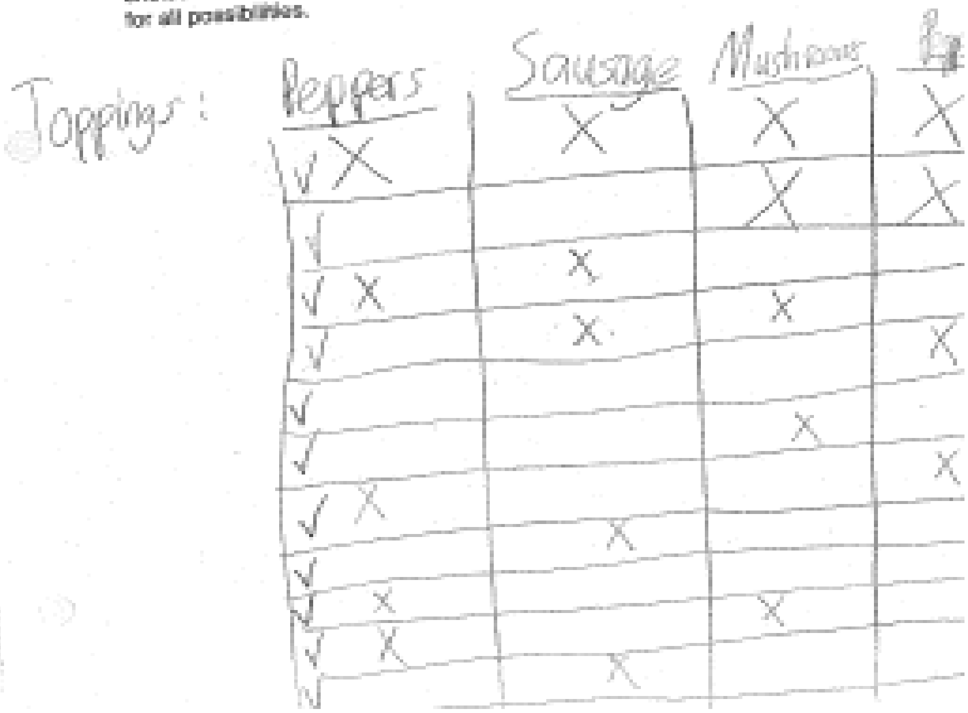
This student's work impressed me and surprised me. His ability to organize his thoughts was truly impressive; however, his method to find the pizza combinations surprised me. When we did the problem in class, we attempted to make a connection between the tower problem, building towers four high with two colors, and the pizza problem. My partner and I created an array with four positions representing each ingredient and put an X if it had the ingredient and left it blank if it did not. We only did this because we were trying to find a connection between the tower problem. When I saw this pair of students doing this method in class I was very surprised and impressed. At first, this student did not arrive at the correct answer because he did not include three topping pizzas. After he wrote out all his possible pizzas he had already created using words, he saw what he was leaving out and added it to his chart. In his chart, it is clear to see all the different combinations.

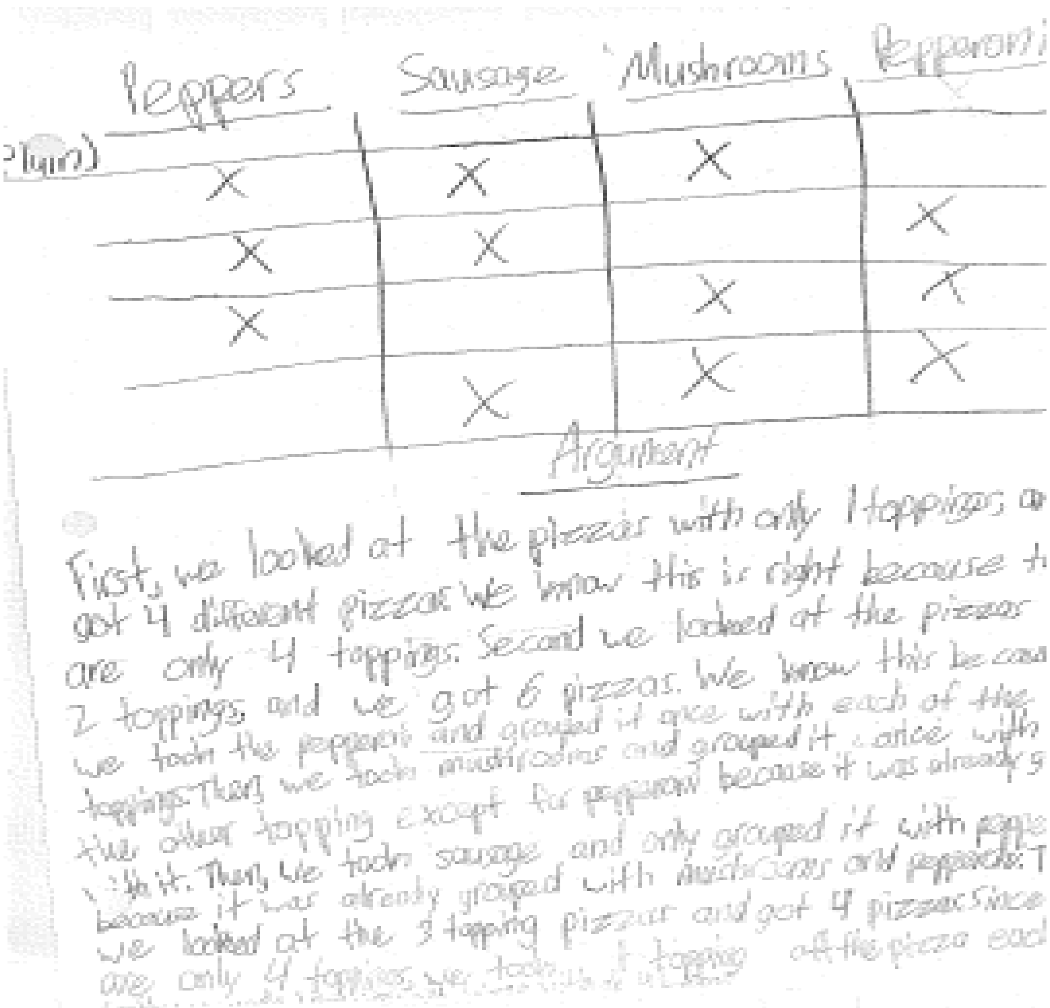
Neat!

When it came time for the student to write his convincing argument, he did a nice job explaining that he paired each ingredient with all the other ingredients, unless it had already been done. If we had more time available, I would have liked the student to re-write his chart and reorganize the pizza combinations. His chart is very random, while his explanation is not. I think that if he had a second opportunity to repeat his chart he might make more of a connection between this problem and the four tall tower problem.

The Pizza Problem

Capri Pizza has asked you to help design a form to keep track of certain pizza choices. They offer a standard "plain" pizza with cheese and tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many choices for pizza does a customer have? List all possible choices. Find a way to convince each other that you have accounted for all possibilities.





| | Peppers | Sausage | Mushrooms | Pepperoni |
|---|---|---|---|---|
| | X | X | X | |
| | X | X | | X |
| | X | | X | X |
| | | X | X | X |

Argument

First, we looked at the pizzas with only 1 toppings a[illegible] got 4 different pizzas We know this is right because t[illegible] are only 4 toppings. Second we looked at the pizzas [illegible] 2 toppings and we got 6 pizzas. We know this because [illegible] we took the pepperoni and grouped it once with each of the [illegible] toppings Then we took mushrooms and grouped it once with [illegible] the other topping except for pepperoni because it was already g[illegible] [illegible] with it. Then we took sausage and only grouped it with pepp[illegible] because it was already grouped with mushrooms and pepperoni. T[illegible] we looked at the 3 topping pizzas and got 4 pizzas Since [illegible] are only 4 toppings we took [illegible] 1 topping off the pizza eac[illegible]

1. Plain
2. Pepperoni, Mushrooms
3. Pepperoni, Sausage
4. Pepperoni, Peppers
5. Mushrooms, Sausage
6. Mushrooms, Peppers
7. Sausage, Peppers
8. Peppers
9. Sausage
10. Mushrooms
11. Pepperoni
12. All toppings
13. Peppers, Sausage, Mushrooms
14. Peppers, Sausage, Pepperoni
15. Peppers, Mushrooms, Pepperoni
16. Sausage, Mushrooms, Pepperoni

16 possibilities

Cycle II- Surprised

The following student's work surprised me for many different reasons. I would not have predicted that any of my students would use a numerical representation for all of the pizza combinations as this student did. The students represented peppers as 1, plain as 2, sausage as 3, mushroom as 4, and pepperoni as 5. After observing the student's initial method for representing the toppings I also expected the student to arrive at more than the correct number of pizzas because the student represented plain the same way as he represented one of the ingredients; however, the student recognized and accounted for this when doing his work. I think that the student realized this because he wrote out the words for the number representations. For example, 1 + 3= peppers and sausage. The student did have duplicates the first time he completed the problem, but found them and arrived at the correct number of pizzas.

Unfortunately, the student did not complete a convincing argument. He provided an explanation of why he decided to use numbers to represent the toppings. The student claimed that numbers are faster and easier to use then writing out the toppings themselves. This method did surprise me. It also surprised me that the student was able to stay with his method and arrive at the correct number of pizza combinations.

The Pizza Problem

Capri Pizza has asked you to help design a form to keep track of certain pizza choices. They offer a standard "plain" pizza with cheese and tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many choices for pizza does a customer have? List all possible choices. Find a way to convince each other that you have accounted for all possibilities.

peppers=1
plain=2
sausage=3
mushrooms=4
pepperoni=5

1+2
1+3
1+4
1+5
2+3
2+4
2+5
3+4
3+5
4+5

2+2
3+2
4+2
5+2
1+2+3+4+5
1+2+3+4
1+2+3
1+3+4+5
1+3+4
1+2+4+5

1+2

peppers=1
plain=2
sausage=3
mushroom=4
pepperoni=5

Name: ______________________ Date: 10/22/13

Partner: ______________________ Teacher: Miss Molloy

The Pizza Problem

Capri Pizza has asked you to help design a form to keep track of certain pizza choices. They offer a standard "plain" pizza with cheese and tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many choices for pizza does a customer have? List all possible choices. Find a way to convince each other that you have accounted for all possibilities.

2+1 = pepper
2+2=plain
2+3 = sausage
2+4 = mushroom
2+5 = pepperoni

26 combos

1+3= peppers sausage
1+4= peppers mushroom
1+5= peppers pepperoni
3+4= sausage mushroom
3+5= sausage pepperoni
4+5 mushroom pepperoni

1+3+4+5 = peppers, sausage, mushroom, and p
1+3+4= peppers, sausage, and mushroom
1+3+5= peppers, sausage, and pepperon
1+4+5= peppers, mushroom, and peppe

I made a key and switched my topings to numbers because for me its easyier to use numbers to count numbers, and faster to write them, then switch it back when im done.

Cycle II- Concerned

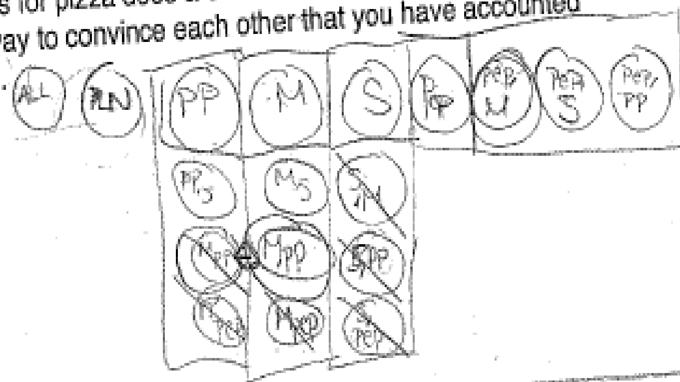
There were a few students that were not able to arrive at the correct amount of pizza combinations. Although the following student's work concerned me, I do think there is some good work in her attempt. The student started by creating a pizza with one combination and paired each of the ingredients with the other three ingredients. When it came time to write her argument, the student realized that she had duplicates and proceeded to cross the duplicates off. Although the student realized that there could be a pizza with all four ingredients, she did not recognize any of the three toping pizzas.

The student's argument was very weak. She argued that she had all of the pizzas because she started having repeats. She believe that because she was making two identical pizzas, that she must be done with all the different combinations. This misconception is what concerned me about her work.

The student's work I selected is a shy student and was paired with a lower student. The group did not react well to the problem. Both were hesitant to start and it took them a while to write anything on their paper. This would have been the only group that I might have split up and paired with other students if given the opportunity to do this problem again.

The Pizza Problem

Capri Pizza has asked you to help design a form to keep track of certain pizza choices. They offer a standard "plain" pizza with cheese and tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many choices for pizza does a customer have? List all possible choices. Find a way to convince each other that you have accounted for all possibilities.



I know that there is no more possible ways for diffrent pizzas because when I was m
combinations I was repeating the same combinations. So I had to cross out the c
That had two of them. For example, when I was making combinations for Pepperoni I h
two of mushroom and pepperoni pizzas. So I knew I had to cross them out so I i
see that I only had one of them. I did this for each one that I had double of. T
I counted all of the diffrent possible ways to make diffrent pizzas and fou
how much pizzas there was. This is how I found out how much pizzas there w

Cycle II Intervention Implementation

I thought there were many things that went well with the pizza problem. Students were actively engaged and interested in the problem. All pairs of students were working and diligently on the problem. The students were highly motivated for this problem because the cohort teachers were observing the class. The students were excited and eager to show their mathematical thinking to our visitors. I prepared the students to expect many adults in the classroom. Since it was a 6th grade class, it was beneficial to prepare them and to know what to expect so the students could focus on the math instead of the situation.

I also found that I was doing a better job at not leading the students. I found myself asking better questions to get students to re-think things, without giving everything away. I was successful when I suggested students re-write their pizza possibilities, or try a different organization or their pizza choices.

I'm not sure what I would do differently if given the opportunity to do this problem again. I thought that the pizza problem went very well and the students produced very good work. The partners that were created all worked well together. The students did write in pencil, so I might encourage students to use pen if I did repeat the problem.

Detailed Class Description:

I completed the following problem with a 6th grade, "Pinnacle" math class. Pinnacle is the label for gifted and talented classes. These classes are based off of a student's performance in *all* subjects and on state testing, not only their mathematic performance or level. Because of this, I see a varying level of mathematic skill and reasoning in this class of 27 students. The class that I have selected to complete this problem with is at 12:50 PM. Class is usually over at 1:50 PM; however, for these problems we extended the class through our "enrichment period" and class was held until 2:10 PM. This is the students last period of the day. The students have three academics in the morning, activity classes and lunch before coming to math class. Although the long break could cause some students to lose focus, this group seems to be energized and excited to end the day with math class. There is one child in the class with autism who has an IEP; however, he functions normally with the class and completes partner and group work like all students.

These students are used to sitting in partners and working collaboratively in class. Students normally sit in pairs and the seating is changed frequently for students to be accustomed to working with many different people. These children spend most of the day together and take all academic class periods together. They are comfortable with challenging each other and pushing each other to reach academic expectations. I choose this class purposely because I know they will do everything they can to reason mathematically in these problems, and because they will work together in order to do so.

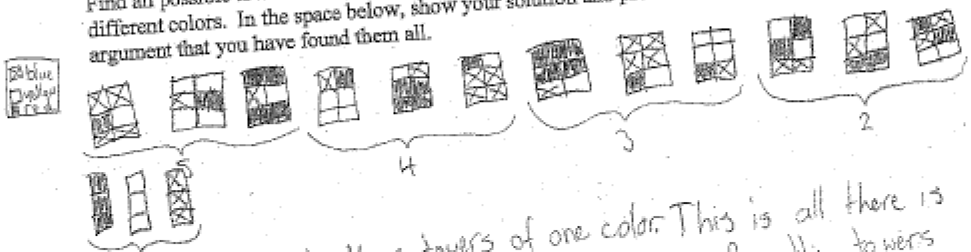
Cycle III- Impressed

— that's great!

It was difficult to pick only one students' work that impressed me for this problem. Students had a much better grasp on how to approach the problem and how to organize their thoughts to create a convincing argument. The student's work that impressed me, showed an incredible amount of growth from the first problem to the last problem. She was able to clearly make an argument and demonstrated an understanding of the different between generating and argument and generating statements about what she did in the problem. For most of the argument, this student kept a constant. She first kept a constant in the bottom position and stated that there would be two towers for each color base because "with one color and space already taken up, there are two more to switch around." Next she moved the constant to the top position and made the same argument. Then she repeated this, but kept the bottom two blocks constant and the top two blocks constant. Lastly, she kept the middle block constant and switched the top and bottom. She also explained in the beginning, her first group would be three solid blocks of color because there are only three colors to choose from. She organized her recording into 4 groups of 6 and one group of the three solid towers. I was really impressed by how simple the student made her argument, but basically making the same argument for all 5 of her groups. This was a popular argument from a lot of students and while many were able to verbalize the idea of keeping a constant and having only a certain amount of colors to put in the remaining position, this student was truly successful in translating her argument to paper. I was truly impressed and proud of this student's work. - super!

**BUILDING TOWERS THREE COLORS**

Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and provide a convincing argument that you have found them all.




To start, I made three towers of one color. This is all there is because there are only three colors. Then, for the towers with all three colors, I grouped them by what color the bases were. I know that this is all because, with one color and space already taken up, there are two more to switch around. For group 3 of the towers, I put two of the same color on top, leaving the last space open for the last color. For group 4, I grouped by the base color again, but instead with the two same colored ones on the bottom. This leaves one last space to work with that could either color. For group 5, I did one color in the middle spot and two of the same color on the top and bottom making 27 in total.

Cycle III Surprised

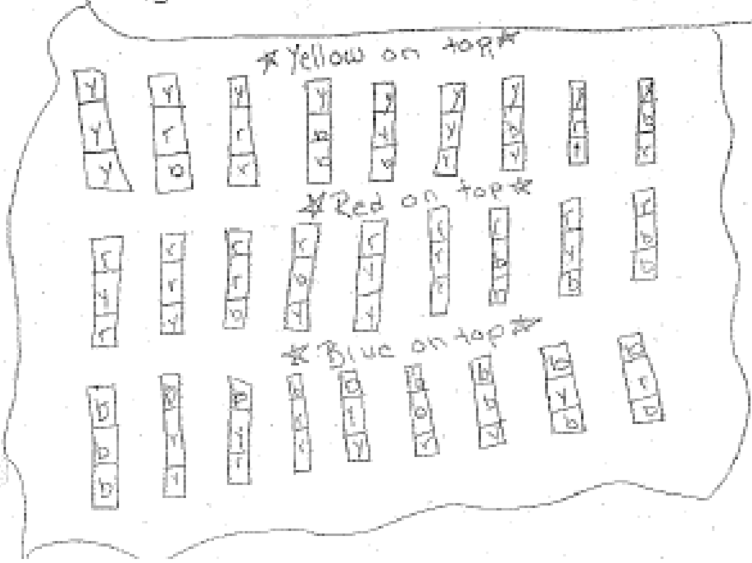
One student's work on this tower problem surprised me for a few reasons. This student is one of my lower students in the class. Although it's a gifted class, as mentioned in my detailed description of the class, I have varying levels of students. In the pizza problem and the tower task of building towers four high with 2 colors, this student was not able to arrive at the correct answer. I was very surprised that for the last problem, she arrived at the correct number of towers, organized her towers by keeping a constant and also made a numerical connection between the number of towers and the number of positions and number of colors. The student kept the top color constant and had three groups of nine. Most students who kept a constant made a separate group for the solid color towers, but this student included them in the appropriate groups. She did not seem to have any other organization after keeping the top constant, but if you look closely at her recording she does one color and another on the bottom positions and then switches the positions of the two colors for the next tower. So, although the group of nine appears random except for the top constant, it is not.

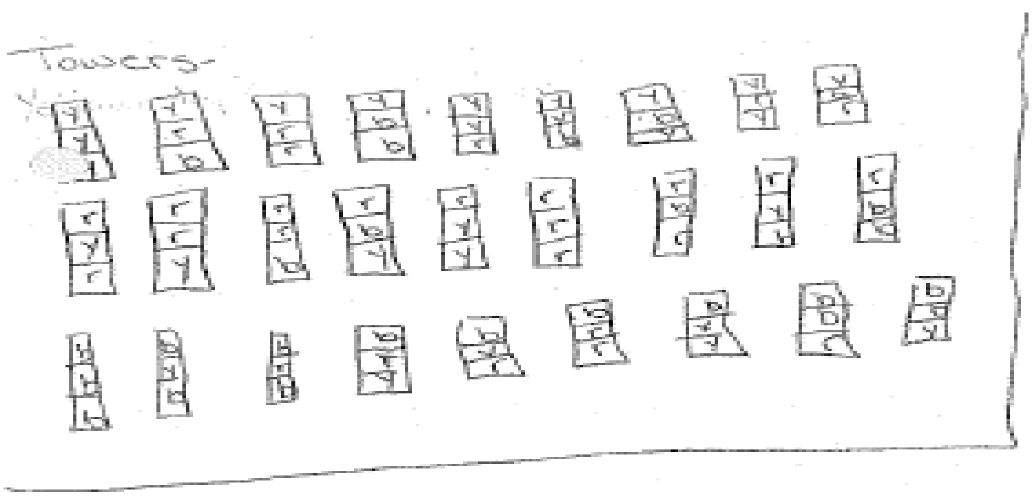
I would have encouraged this student to create an argument that supported how she and her partner organized her towers; however, I was surprised with this particular student's argument. She writes that she and her partner found an algorithm 3 to the 3rd power because there are three colors and it is three blocks high. She also recognized that 9 is a multiple of 27. I was surprised at the connections that this student made. Unlike one of my other students who also wrote about the connection of 3 X 3 X 3, she seemed to grasp more the meaning of the numbers in the algorithm.

## BUILDING TOWERS THREE COLORS

Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and provide a convincing argument that you have found them all.

We did 27 towers. We found that answer by obviously getting all the colors together. We found the algorithm by doing $3^3$ (three to the third power) and realized that was 27. We did $3^3$ because there were 3 colors 3 blocks high, and 27 is a multiple of 9. We did 9 because $3 \times 3 = 9$. We grouped them in 9 (because of $3 \times 3$) and 27 was the ~~product~~ sum. We know it is right because of the reasoning above. 27 is the number of towers!



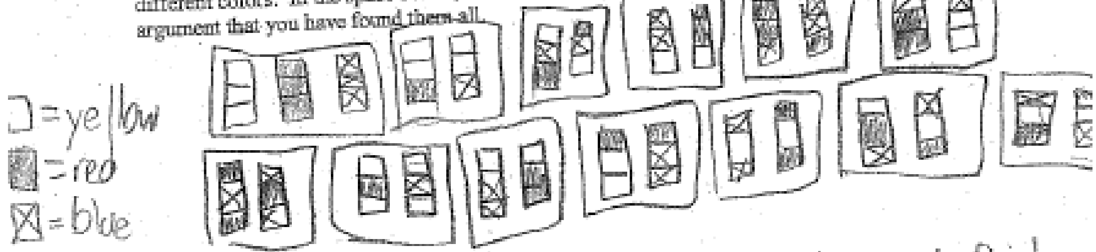
Cycle III- Concerned

For the tower problem building towers three tall, all of my students were able to arrive at the correct amount of 27 towers. This was the third problem that the students had completed that required them to create a convincing argument. Most of the students showed incredible improvement in their ability to do so. Out of all the work I received from my students, the following concerned me the most. This student was able to arrive at the correct number of towers when completing the problem; however, his convincing argument did not include any mathematical reasoning pertaining to the way he and his partner constructed the towers or to the organization he used when recording his towers. This student claimed in his argument that he arrived at the answer through multiplication. Many students drew a connection in this problem between the number of tower (27) and the three colors and three blocks high. In this student's argument he stated that he arrived at his answer by multiplying 3 by 3 by 3, for the number of blocks, the number of "how" the blocks, and the three starting colors. Although I welcome the mathematical curiosity, I do think that this student strayed away from creating convincing argument using some sort of mathematical reasoning. I do not think that the student truly understood the connection between the 27 towers and the number sentence, three cubed. I think the student fell back on using this as his explanation because he struggled with creating a convincing argument using mathematical reasoning. After completing two of the problems already, I expected students to be more prepared to write a convincing argument.

After looking at the student's organization in his recording of the towers, it did seem like the student attempted to organize the towers; however, the organization does not support any of the common reasoning methods of proof by cases, recursive, inductive reasoning or holding a constant. The student's recording of the towers leads me to believe that he struggled with coming up with a solid argument that would have supported his 27 towers. It seems that he organized them more in pairs, which as we have talked about before, is not the best method in order to prove you have exhausted all the tower options. If I had a chance to, I would question this student's argument of 3 X 3 X 3 and encourage him to organize his towers in a different grouping.

## BUILDING TOWERS THREE COLORS

Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and provide a convincing argument that you have found them all.




I got my answer by multiply it first.
3 (which is the number of blocks) times
3 (number of how the blocks) tires
3 (because of the the three starting colors
Simpilfied is 3x3x3. Or $3^3$. I got 27
which is the number of possibilities

Cycle III Intervention Implementation

I was extremely pleased with the session "building towers three colors." The growth in the students from the first tower problem to this problem was apparent across the board. Students approached the problem with strategy and used their past experiences to help them formulate a convincing argument. During the first tower problem, when it came time to write, most students hesitated to start and struggled to put any thoughts on paper. During this tower problem, when it came time to write an argument, students were busy writing or collaborating with their partners. My students took pride in completing this problem thoroughly and completely. When the class period was ending, students asked if I could call their next cycle teacher and extend our class time. I learned that my students are very capable and willing to create convincing arguments. It was amazing how with just a little experience with these problems, the students' attitudes and confidence towards the problems drastically changed.

Isn't that neat?

If I had the opportunity to do this again, I would not change much about how I did this problem. The students had enough time to complete the problem and the partner groups that I arranged worked out very well. The only thing that I would have changed throughout all 3 of these sessions was allowing students to use pen. I did not address the pen issue and students came to class prepared with a pencil, like always. Since I had limited time, I decided not to take any time passing out or having students retrieve pens, but if I had an opportunity to do it again, I would have been better prepared.

nice reflection!

Final Reflection

Throughout this semester, I learned how possible it is to incorporate sophisticated mathematics into problems that can be solved using natural mathematical reasoning. I remember vaguely learning about Pascal's triangle and it was interesting when we looked at the correlation between Pascal's triangles and the problems. I also learned about the different types of reasoning. Before this class, I might have recognized that some types of reasoning that students used were similar, but would not have categorized them the way we did. Proof by cases and keeping a constant were arguments that came naturally to me; however, a recursive argument for the tower problem was not something I would have thought of. I also did not differentiate between keeping a constant and inductive reasoning. I would have classified inductive reasoning as keeping a constant; however, I see now how you are building upon the first position.

I learned that my students struggle with writing down their mathematical thinking. Throughout the three problems we worked on in class, I can see an improvement from the beginning to the end in the students' ability to formulate a convincing argument. For the first problem, students struggled to write down an explanation. For the second problem, students did a better job writing; however, some students failed to provide a convincing argument and rather recorded what different combinations they were able to make. For the last problem, building towers three high with three colors, many students were able to write *why* they had created all the towers, rather than just simply listing the towers they created. With practice, students improved their abilities to write a convincing argument and they also learned what is *expected* in a convincing argument. I watched students recognize that restating what they did was not necessarily convincing. It was evident by the third problem that students had a better grasp on being convincing. The students approached the problem with strategy.

The students' mathematical reasoning and thinking surprised me throughout all of these problems. So many of my students solved these problems in ways I never would have predicted. Their organization and work in some cases was impressively sophisticated. Many of the ways students recorded their work, or arrived at their solution, surprised me.

# Appendix J

## Teacher T7

### Cycle 1: Building 4-tall Towers, Selecting from 2 Colors

#### The Class

The first task was implemented in a regular education eighth grade math class. There were twenty students present for the activity and none of the students have accommodations. Two of the students had completed the task previously, while they were in sixth grade, and happened to be working together. Students in this class are not afraid to ask questions and are always eager to share ideas for solving problems. Since it was the beginning of the school year, there was not much information to use to assign the students in pairs, so they were grouped according to their seventh grade NJASK math scores and the pairings worked very well. The students were respectful of each other and were able to communicate with each other.

This class is the last class of the day for the students, with the exception of a twenty minute enrichment period at the end of the day, used for extra class time amongst their academic classes. It is usually an hour long class beginning at 12:50 in the afternoon but we had the additional twenty minutes due to the enrichment period. This time proved useful for the students to complete the task and one group was able to complete the extension activity. On this day, September 17, 2013, the teachers from the NJPEMSM program were in attendance to observe the class. The students did not appear timid with the additional adults in the room and were discussing their solutions with the teachers.

You are fortunate to have a long math class!

Cycle 1: Building 4-tall Towers, Selecting from 2 Colors

Sample 1: Impressive Work

This student had the most impressive written justification of his solution in the class and was also the only student to attempt the problem using a constant for part of the explanation. His explanations was well thought about and explained clearly, giving reasons for why there are no more options, not just stating what he did for each case. He also had a clear diagram with a key explaining his notation.

He started with three blocks of one color and one block of another color and explained that the one block moves through each position in the tower and cannot go anywhere else because there cannot be a fifth block. Many students in the class mentioned this reason in their proof. His next group was the two towers of solid colors. He was the only student to explain that there were only two options because there were only two colors available to choose from for the towers, which was impressive to note.

His last group was the towers of two blocks of each color. The strategy he used in this section was different from any other student in the class. He kept the bottom block constant and moved the other block of that same color through each of the other three positions then switched the color on the bottom and did the same thing. He explained that there cannot be more than three in each of the two groups because the second block of the color kept constant on the bottom can only go in three different positions.

His explanation was clear and concise, offering proof for why there were no more options in any of the groups he created. This was an area that many of the students in the class struggled with and he was able to write his justification in a way that was convincing to the reader. His strategy was also very impressive as he thought about a constant for the group of two of each color and the different positions the blocks could take.

**Building 4-tall towers, selecting from 2 colors**

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.

We have 3 Reds 1 yellow. The yellow is on the top and moves down 1 each time. Same with 3 yellows 1 red. The Red is on the top and moves down 1 each time. You can't go any higher because that means you would have 5 blocks if it went one more higher. The four red and four yellow can't be any diffrent because

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Y | Y | [illegible] | R | R | R | R | Y | R | Y | R | Y | Y | Y |
| Y | Y | [illegible] | Y | R | R | Y | R | R | Y | Y | R | Y | R |
| Y | R | [illegible] | Y | R | Y | R | R | R | Y | Y | Y | R | R |
| R | Y | Y | Y | Y | R | R | R | R | Y | R | R | R | Y |

you can't have 5 high and
their are only 2 colors.
Their are 2 yellows and 2 reds
in these towers. the yellow starts on the
top and goes down 1 at a time.
The bottom are allways yellow. You can't
go more then 3 in a diagnal because
that would repeat. Same fur the
red. Their are also 2 reds and 2
yellows in these towers. Instead the
red starts on top and goes down
1 at a time. The bottom is always
red in all three. You can't go more
then 3 in a diagnal because that
would be a repeat

## Cycle 1: Building 4-tall Towers, Selecting from 2 Colors

### Sample 2: Surprising Work

This student's work surprised me for a few reasons. First, the written explanation was somewhat convincing but has room for improvement and second, the strategy used by the student included considering opposite towers which was a different grouping than other students in the class. This student was also able to complete the extension activity and, while some of her observations were accurate, others were not correct.

She started with a group of three of one color and one of a different color and explained how the one color block started at the top and moved down one position in the tower each time. She also explained that she would start with one tower and then switch the colors for the opposite tower. While other students attempted to solve the problem with opposites, they eventually reorganized the towers so they were not with the opposites anymore, but this student chose to keep the opposite towers together. I would have liked to see an explanation that there were only four positions for the single block so there were no more options but she did not explain why there were no more options.

The second group was for the towers of two blocks of each color. She started with a convincing argument that if the two blocks on the bottom are the same color the two blocks on the top have to be the other color and then she explained that there could be one color on the bottom then the two in the middle would have to be the other color but she did not explain that the top block would be the same as the bottom block. She also talked about the two colors alternating but did not prove why there are only two towers in this group. I was also surprised that she did not mention the solid color towers though she had them included on her diagram.

I was surprised by this work because I found the explanations that were omitted to be ones that she understands but did not write clearly. I was also surprised to read her explanation on the extension activity. She was able to come to the conclusion that there would be less than sixteen towers if they were restricted to three blocks tall rather than four blocks and her reason was that since it was shorter, there are fewer patterns to choose though her prediction of twelve towers for three blocks tall was incorrect.

**Building 4-tall towers, selecting from 2 colors**

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.

Row 1 - (3 of 1 color 1 of the other)

* only two colored blocks (brown and yellow)

Row 2 - (two of each color)

Row 1 has 10 differe towers. Starting with the top block, I chose a block, and connected it to 3 different colored blocks. The second st I did one color, the the opposite then two of the same. And continued the pro until I finished. Row two has 2 of each color in every tower. (2 brown 2 yellow) I know that if I want two of the same color on the bottom, the o way I can arrange the tower is two more opposite colored blocks on top. If I want or one color on the bottom I can only put two same colored blocks on top. Or else I woul end up with duplicated blocks. Lastly, my checker board pattern has 2 of each color alternating each color every layer.

**Building Towers 2 Colors Extension**

Without building them, make a prediction about a solution for finding all possible 3-tall towers, selecting from 2 colors. Do you think there will be more, fewer, or the same number of possible towers as you found for towers that were 4 cubes high? Why do you think that?

Being that the tower is 1 cube shorter then the "4 cube tall" tower, I would say there are less than 16 towers to make. You have less tower patterns to chose from, make

2(amount of colors) x 4(amount in the tower) = 8 to[wers]

2(amount of colors) x 3(amount in the tower) = 6 t[owers]

Without building them, make a prediction about a solution for finding all possible 5-tall towers, selecting from 2 colors. Do you think there will be more, fewer, or the same number of possible towers as you found for towers that were 4 cubes high? Why do you think that?

2(amount of colors) x 4(amount in tower) = 8 tower

2(amount of colors) x 3(amount in tower) = 6 tower

In the end, after building, I got 16 towe[rs] 4 cubes tall. Going off of my theory, I wou[ld] have 12 tower (6x2=12)

## Cycle 1: Building 4-tall Towers, Selecting from 2 Colors

### Sample 3: Concerning Work

This student's work was the most concerning to me from the class as they did not have all the solutions. The student also changed their notation while recording their work, making it harder to follow.

She started with a group of towers consisting of the tower of all one color and then the towers of three of one color and one block of another color. She explained that the one block moves up a position each time until it reached the maximum height so there are no more options. She noted the shaded block as yellow in this group. Although she put the solid tower in this group, she did not mention it. Then she moved on to the same group with opposite colors but she changed the notation of the shaded block to red. This makes the two groups look the same on paper if the key is not noted.

For her last group, she changed the notation of the shaded block back to yellow. This was the group of two of one color and two of the other color. She had the two blocks together and moved them through each possible position, starting at the top of the tower. My concern is that she missed the three towers with the two colors apart. As a result, she found thirteen towers rather than sixteen.

While the explanation she provided was clear to follow, she missed towers and her change in notation of the colored blocks made the diagrams harder to follow. She also did not provide much explanation for why she had all the blocks in the two of one color and one of another.

**Building 4-tall towers, selecting from 2 colors**

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.

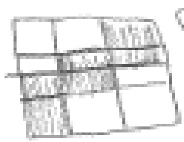
■-yellow

• Each one yellow block moves up out of the four combo, so there is no more room fo the tower to desend any more because it reached its maximum height.

■ Red

- As for this combo similar to the yellow combo. There is a pattern of one Red block moves up as the pattern continues untill the Red block cant move anymore

■yellow



Each two yellow block moves up out of the pattern it cannot go up anymore because It will breck the four tower rule

## Cycle 1: Building 4-tall Towers, Selecting from 2 Colors

### Reflection

Observing this class complete the problem and reviewing their written work, I had the opportunity to learn more about the students and how they think about problems. This proved valuable for early in the year, as I was still learning about my students. I noticed that many students would benefit from more activities requiring them to write their explanations as many had difficulty expressing their reasoning on paper. I also found it interesting to observe the different ways students attempted the problem. Many used proof by cases or the recursive argument but it took some time for them to get away from the thought of pairs of opposite towers in order to better develop their reasoning. Once students started regrouping their towers, they were able to develop better arguments.

It was especially interesting to hear students develop their reasoning. Most students started by stating what they did rather than why they have all the towers. It took some time and questioning for them to further develop their reasoning. Quite a few groups mentioned that the answer was sixteen towers based on a mathematical expression that resulted in sixteen, for example four multiplied by four. Through these conversations, I realized that many students are looking for a mathematical expression to provide them an answer rather than an understanding of the reasoning for the expression or solution. This is based on the observation that, while the expressions they mentioned did result in sixteen, they were not correct in context of the problem. When they were asked how they found the expression and if they knew it before they completed the problem, they responded that they did not know it before they built the towers but that they were content because it resulted in the correct answer. Unfortunately, their conclusions for the expression would not hold true for towers of other heights or color options. It was also interesting to hear a student mention that he did not think there was a way to know if he had all the possibilities.

I was impressed with the ability of the students to work well together in groups so early in the year. They were all interacting and discussing the problem together. I was also intrigued with the way some students were able to be flexible and explore alternate ways of arranging the towers.

It was a challenge not to direct students to a solution through questioning and to spend more time observing their work. My questioning strategies had room for growth after the first task and it was beneficial to watch it implemented as I could listen to questions and implement those questions in the future. I also found the extra time to be important for the class. It was a challenge for some of the students to finish their explanations and in the future, I would ensure they have more time to work on the written reasoning.

## Cycle 2: The Pizza Problem

### The Class

The second task was implemented on October 18, 2013, in the same class as the first task. It is a regular education eighth grade math class. There were twenty two students present for the activity and none of the students have accommodations. Students in this class are not afraid to ask questions and are always eager to share ideas for solving problems. They worked in the same groups as the first task. The groups were originally assigned based on their seventh grade NJASK math scores since there was not much other information available at the time to base the groupings. A few students had a new partner since there were some students absent for the first task and some absent for the second task. The students were respectful of each other and were able to communicate with each other.

This class is the last class of the day for the students, with the exception of a twenty minute enrichment period at the end of the day, used for extra class time amongst their academic classes. It is usually an hour long class beginning at 12:50 in the afternoon but we had the additional twenty minutes due to the enrichment period. This time proved useful for the students to complete the task.

## Cycle 2: The Pizza Problem

### Sample 1: Impressive Work

This problem proved more challenging for the students than the first task and it was difficult to find a written explanation that was completely convincing but I chose a student who had a clear representation of the work and was able to communicate some reasoning on paper. He used proof by cases to explain his reasoning and within the case for two topping pizzas, he held a constant. He was also in the first group to complete the problem so I would have liked to see a better written explanation.

The student started with the group of all toppings by themselves and included the plain pizza in this group. From there, he made the two topping pizzas. It was interesting to see how he held one topping constant until the other options were used then he would not use the constant topping again. Unfortunately, he did not explain this in writing, he just explained they were put in groups of two and did not include plain as a topping that could be paired. For the three toppings, he mentions in his written explanation that each topping would be used three times. I found it interesting that he viewed the group in this way rather than the fact that there will be four pizzas because there is one topping missing from each pizza and there are four toppings for that possibility. He concludes with the pizza with all the toppings.

While I would have liked to see a better written explanation of the work, this student had a clear representation that included writing the words rather than using abbreviations. His groups were easy to follow and he had more written as an explanation than most students in the class. This was a much more challenging problem for the students than the first task.

lain pizza
Peppers
Sausages
mushrooms
'epperoni

'eppers + Sausages
ppers + mushrooms
ppers + pepperoni

usage + mushrooms
usage + pepperoni
ushrooms + pepperoni

peroni + sausages + peppers
pers + mushrooms + pepperoni
peppers + Sausage + mushrooms
usage, mushrooms, pepperoni

ers, Sausage, mushrooms, Pepperoni

16

Combos

with the 1st group, it's all toppings by themselves. in the 2nd all toppings are together by 2, besides plain. in the 3rd, they're all together by threes, each is used 3 times. In the 4th all toppings are put togesher in one.

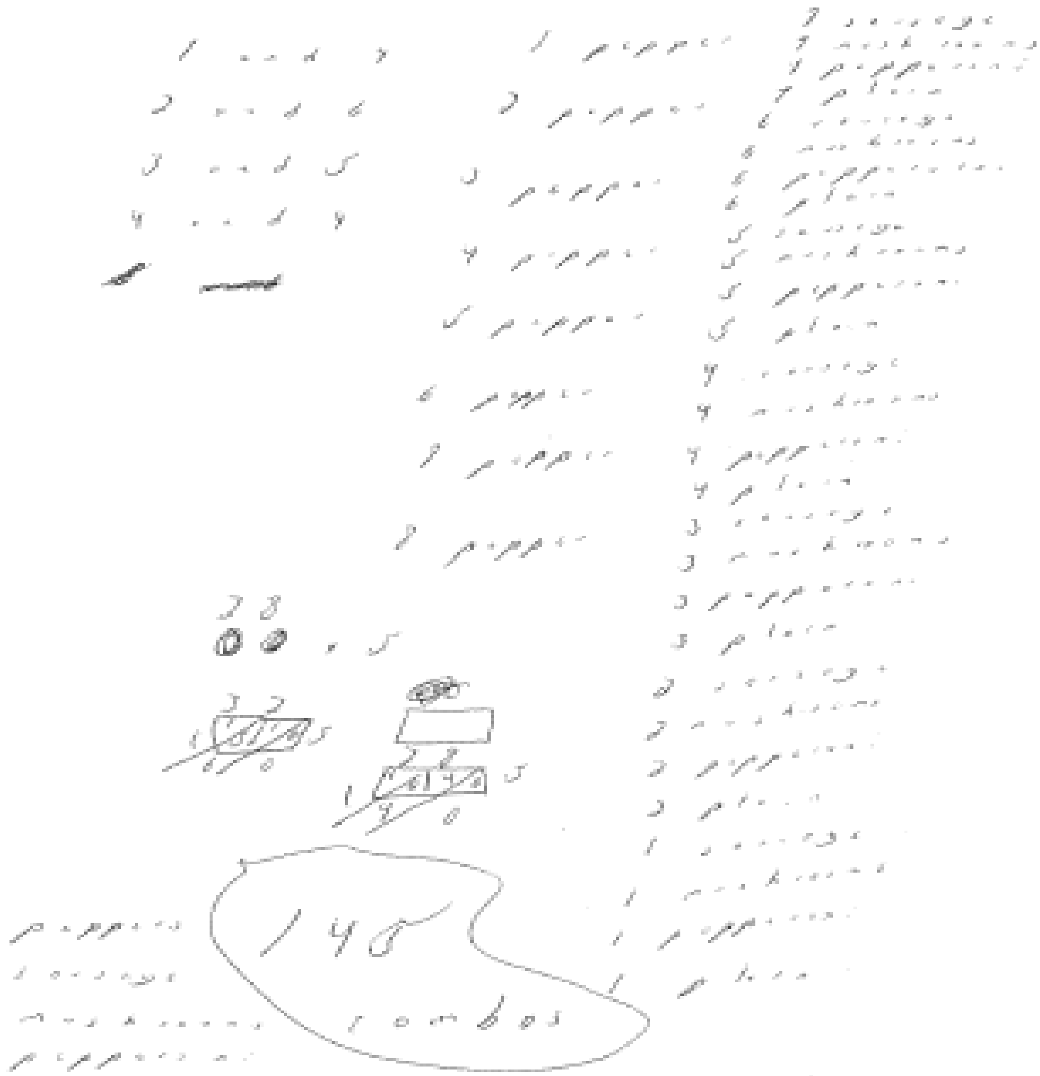
Cycle 2: The Pizza Problem

Sample 2: Surprising Work

I was very surprised by the difficulty this problem presented to the students and I was surprised with the way the students thought about the problem. The student I selected thought about each slice of the pizza as being a different possibility. This made the problem much more difficult for the pair of students. It is interesting to note that this pair included the student with the most impressive work from task one. There is not a written explanation but I was able to talk with this group extensively during the task. As a whole, the class did not have extensive written explanations for this problem.

The pair of students made a list of possibilities for each slice of pizza. They started with one slice of pepper and seven slices of a different topping choosing from sausage, mushrooms, pepperoni, and plain. Then they moved to two slices of pepper and six slices of one of the other toppings. They continued with this pattern of three slices of one topping and 5 of another up to the option of all eight slices being pepper. From here, they noticed there were 28 options of pepper paired with another topping but they did not include the all pepper pizza. They multiplied the 28 options by five to get 140 options because there were five possibilities for toppings but they did not realize that this would result in duplicates of pizzas. From there they added five more options for the five pizzas of all one topping.

When discussing with the students why they were attempting the problem this way, one of the students in the pair discussed that his family had previously owned a pizzeria and would make pizzas to order the way they were making their list. It was interesting to see how the student's experience had influenced the way he attempted the problem. This is the reason I found this solution so surprising since it also raises concerns. My concerns are that they only looked at two topping pizzas and single topping pizzas. They did not consider three or four topping pizzas and they were not aware of the duplicates they were accounting for when they multiplied. It was also surprising to note that the students in this pair had discussed whether or not a pizza with the slices arranged with different toppings were considered different options or not. For example, a pizza with four slices of one topping and four slices of a different topping could have the slices arranged in an alternating way or grouped together. I had not expected the students to think about the problem in this way and was surprised to see them think about the task in this manner. Even with questioning if they would ever order a pizza that way, they were convinced it was a possibility since the one student had that previous experience in the pizzeria. It was surprising to see the effects of previous experience develop the work of the student in this way.


148
combos

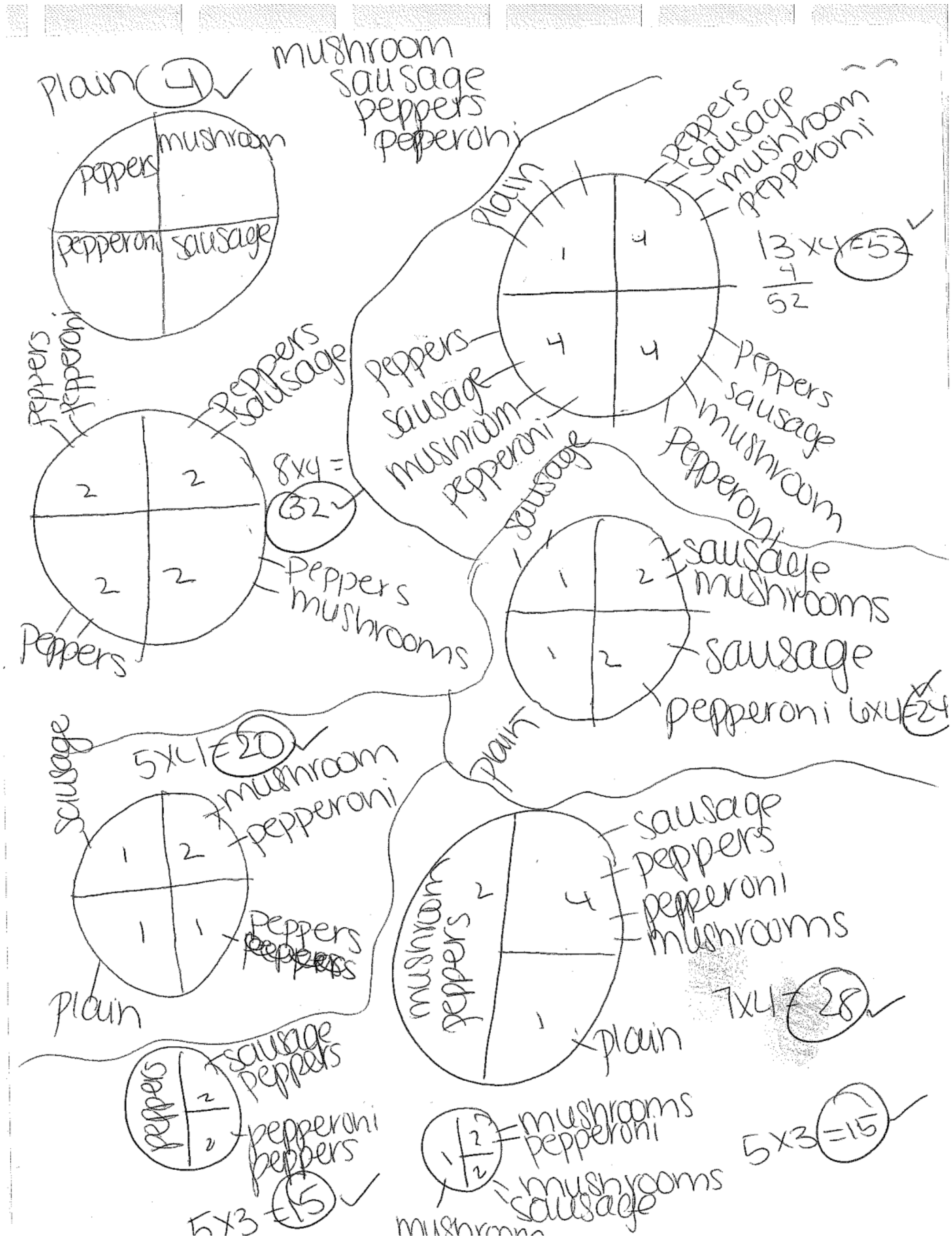
Cycle 2: The Pizza Problem

Sample 3: Concerning Work

During this task, I found more of the work to be concerning rather than surprising or impressive. I chose a student who had different representations and did not give clear reasoning as to how she attempted the problem. This student had examined the pizzas as being broken into quarters or halves with each section containing a different number of toppings.

She started on the front of the page with lists and changed to circles on the back of the page to represent the pizzas. It is difficult to follow her reasoning. She does not explain why the pizzas are divided into halves or quarters or why there are different numbers of toppings in each of the sections. She also includes numbers in each of the sections representing the number of toppings in that section. She continues with adding the numbers together in each of the sections to get the number of toppings on each pizza and she multiplies it by four or three. There is no explanation as to why she is computing numbers in this manner. It appears she is multiplying by the number of sections she broke the pizza into. There does not appear to be any structure for ensuring there are no duplicates and there is no reasoning for her work. In her written explanation, she comes to the conclusion that there are more options but she ran out of room so for now, there are 130 pizza choices.

I am concerned with the work because there is no structure to the solution and there is no explanation for why she is attempting the problem in this way. I am also concerned that she is not considering duplicates and is content with stopping at an answer because she ran out of room to continue.


Plain 4
mushroom
sausage
peppers
pepperoni
peppers
mushroom
pepperoni
sausage
plain
peppers
sausage
mushroom
pepperoni
13x4=52
peppers
sausage
mushroom
pepperoni
peppers
sausage
mushroom
pepperoni
Peppers
pepperoni
Peppers
sausage
8x4=
32
Peppers
mushrooms
Peppers
sausage
sausage
mushrooms
sausage
pepperoni 6x4=24
plain
sausage
5x4=20
mushroom
pepperoni
Peppers
plain
sausage
peppers
pepperoni
mushrooms
mushroom
peppers
7x4=28
plain
peppers
sausage
peppers
pepperoni
peppers
5x3=15
mushrooms
pepperoni
mushrooms
sausage
5x3=15

The Pizza Problem

130 possibilities

Capri Pizza has asked you to help design a form to keep track of certain pizza choices. They offer a standard "plain" pizza with cheese and tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many choices for pizza does a customer have? List all possible choices. Find a way to convince each other that you have accounted for all possibilities.

plain:

peppers – sausage
mushrooms
pepperoni

sausage – peppers
mushrooms
pepperoni

mushrooms – peppers
sausage
pepperoni

pepperoni – peppers
sausage
mushrooms
4

half – sausage pepperoni
sausage mushrooms
sausage peppers

half – mushroom pepperoni
mushroom peppers

half – peppers pepperoni
peppers mushroom

whole pizza – mushrooms
pepperoni peppers sausage

plain pizza

Capri Pizza has asked me to help design a form to keep track of certain pizza choices. They offered a standard pizza "plain" with cheese and tomato sauce. You're allowed to choose from 4 topping; peppers, sausage, mushrooms or pepperoni. I found 130 possibilities to making pizza's with all these toppins. I explain/showed all the ways I think you could make. If I didn't run out of room, I could've came up with more possibilities, but for now, I think there's 130 possibilities.

Cycle 2: The Pizza Problem

Reflection

I was very surprised by the difficulty experienced by the students in solving this problem. I did not expect the students to analyze the problem considering different slices or sections of the pizza as being different from each other. This task showed me the effect real world experience can have on students' approaches to problems. I saw this in the student who had experience in a family owned pizzeria as well as in the students who considered half pizzas because they had previously ordered pizzas with one half different than the other half. I was surprised to see so many students consider half pizzas, pizzas without cheese, and plain as a topping option.

Reflecting on what would be different next time, I realize the students could definitely benefit from additional time. As I looked through the student work, I realized many students did not have enough time to write a full explanation of their reasoning. This was in part because of the extensive options they were considering. Many students had become overwhelmed with the number of possibilities and had considered the problem to be impossible to complete. I would consider asking students to think about how options for pizzas are written on menus they have seen to see if this eliminates the confusion of considering each slice.

There were two groups who offered good reasoning for their answer of sixteen pizzas when I was listening to them explain their work but they did not explain it well on paper. This was something that was noticed in task one but was more evident in this task. It was encouraging to hear clear and concise oral explanations but the students have room for improvement in putting their reasoning on paper. In the future, I would take the advice mentioned by other teachers in the class of telling the students to dictate their reasoning as the other person in the pair records the reasoning on paper.

## Cycle 3: Building Towers Three Colors

### The Class

The third task was implemented on November 4, 2013, in the same class as the first and second tasks. It is a regular education eighth grade math class. There were seventeen students present for the activity and none of the students have accommodations. Students in this class are not afraid to ask questions and are always eager to share ideas for solving problems. They worked in the same groups as the first task. The groups were originally assigned based on their seventh grade NJASK math scores since there was not much other information available at the time to base the groupings. A few students had a new partner since there were some students absent for the first task and some absent for the third task. The students were respectful of each other and were able to communicate with each other. Since this task was implemented during a two day school week due to Election Day and Teachers' Convention, the students were able to spend Monday on the task and on Wednesday, November 6, 2013 they completed the extension activity. Nice!

This class is the last class of the day for the students, with the exception of a twenty minute enrichment period at the end of the day, used for extra class time amongst their academic classes. It is usually an hour long class beginning at 12:50 in the afternoon but we had the additional twenty minutes due to the enrichment period. This time proved useful for the students to complete the task.

Cycle 3: Building Towers Three Colors

Sample 1: Impressive Work

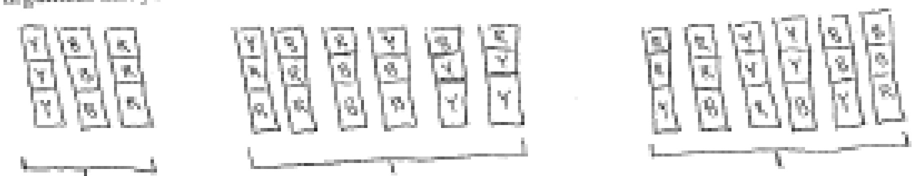
This student had a clear diagram which was easy to follow. Her notation was clear and she included descriptions of each group next to the diagram making it clear to the reader why she created the group in that way. She also had one of the more convincing arguments in the class, using proof by cases, though I would have liked to see more explanation in some areas.

She started with the towers that were all solid colors then moved to the groups with two blocks of one color and one block a different color. She had three groups of six towers that were only two colors, and used the placement of the one block that was a different color to create her groups. Her groups started with the one color block on the top, then on the bottom, then in the middle. I would have liked to see more explanation that there were only three positions for this block so there were only three groups but I was impressed with her explanation of the towers within each of these groups. She mentioned that there were two towers that she called "similar" in each group. Her reasoning is that if two blocks are the same color, there are two options for the third block since using the color that was already in the tower would result in the solid tower which was already created. Her last group was not as convincing though she did have all the possible towers. The only explanation provided was that all three colors were assorted in different ways. I did not find this to be a convincing argument.

Although there were areas for improvement, I still considered her work to be one of the most impressive in the class due to the clear organization of the towers with descriptions, and the explanation of the towers that were two colors.

**BUILDING TOWERS THREE COLORS**

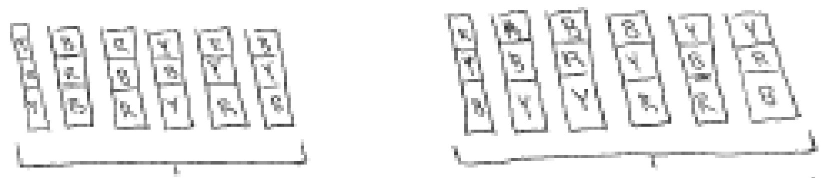
Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and provide a convincing argument that you have found them all.



all 3 same color

2 of the same color on bottom, 1 different on top.

2 of the same color on top, 1 different on bottom



2 of the same color on the top and bottom, opposite color in middle.

All 3 colors assorted in different patterns.

- the towers (in each group) have two similar towers. For example, if you have two reds on the bottom, you can only have a blue or a yellow on top. (2 different towers) If you wanted a third tower, it would be all of the same colored cubes (Red, red, red) which was already constructed.

Cycle 3: Building Towers Three Colors

Sample 2: Surprising Work

I chose a student sample from Ankur's Challenge for this section. This student had a clear diagram, color coded, and used a recursive argument, which was different than the argument used by others in the class. She started with the realization that each tower four blocks tall would have two blocks the same color and the other two colors could be alternated in position.

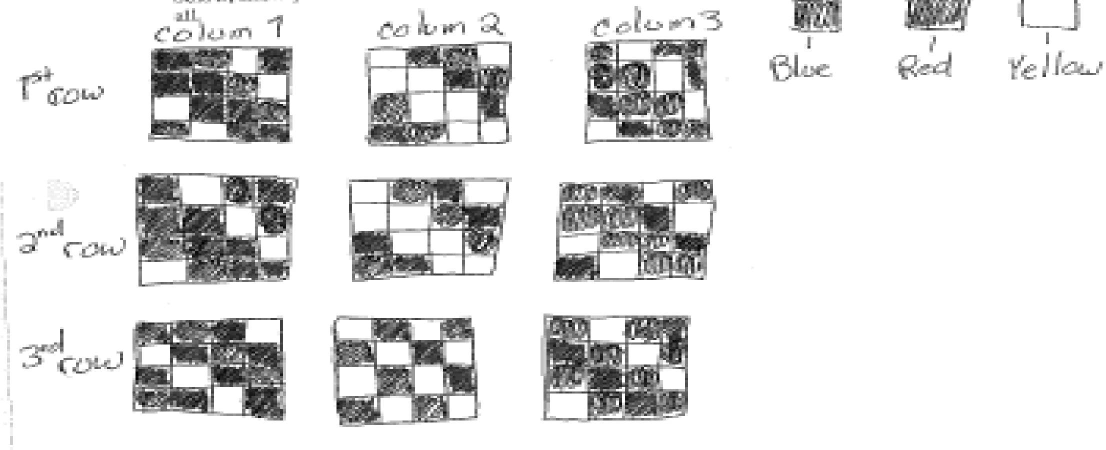
Her first row consisted of three groups of four towers each. Her explanation was that there were three colors that could be used as the double so she created a group for each with the two same colored blocks starting at the top of the tower. Then she explained how the bottom block could be moved to the top and continued in this way to end with four towers in each group. Her second row was like the first row except she noted that the two different colors on the bottom of the tower could be switched in position.

To finish, her last row consisted of groups with the two blocks of the same color when they were "not touching". She continued with the recursive argument for the three colors and concluded that there were thirty six possible towers that could be built four blocks tall with all three colors.

I was surprised by her approach to the problem since no other student used a recursive argument and she was able to clearly describe the possibilities for the starting tower in each group. Both her diagram and description were easily understood and showed a level of understanding not exhibited in some of her classmates' work.

BUILDING TOWERS THREE COLORS EXTENSION

Find all possible towers that are four cubes tall, selecting from cubes available in three different colors so that the resulting towers have at least one of each color. In the space below, show your solution and provide a convincing argument that you have found them all.




Starting of with the 1st row we start with 2 reds and then yellow than blue top to bottom. ~~Switching~~ placing the bottom color ~~o~~ on top makes a total of 4 diffent towers with those specific colors. That goes for all the 1st row just with a different color ~~dobl~~ double.

The second row is the same colors as its colum just with ~~a differe~~ the two base colors switched. This goes for the rest of the 2nd ~~colum~~ row.

The 3rd row is the collums same colors just the color with 2 is now not touching. Using the ~~pros~~ process that we used for the first 2 rows and using the appriate colum colors there will be 36 towers with 3 diffent colors 4 blocks high.

Cycle 3: Building Towers Three Colors

Sample 3: Concerning Work

Although this student came to the correct conclusion of twenty seven possible towers built three blocks tall choosing from three colors, her reasoning was incorrect and her explanation was not convincing. She did have a nicely created diagram and used color coding to make it easy to understand.

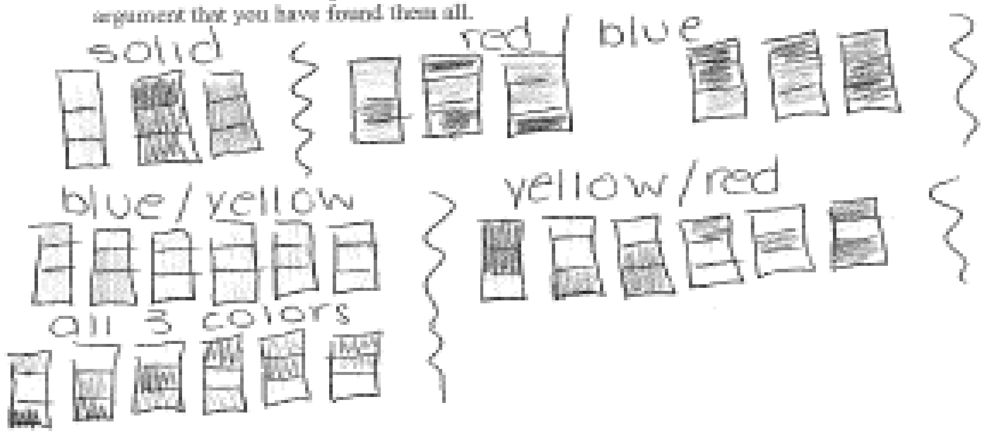
Many students wanted to have a mathematical expression for finding the answer and this student chose to say there were twenty seven towers because three blocks tall multiplied by three color options is nine and nine multiplied by three again is twenty seven. She did not offer an explanation for why she multiplied by another three to end with twenty seven and while the answer is found by computing three to the third power, which is technically what she did, she did not exhibit an understanding of the expression being three to the third power or an understanding of why the expression would be three to the third power.

I am also concerned with her explanation of the groups. The only explanation of the groups is that the possibilities are solid towers, two color towers which there are three combinations of colors, and towers with all three colors. No explanation is provided for why these are the only options or why she knows she has all the towers in each of the groups.

This student is capable of creating a better written explanation and I am concerned that she did not demonstrate her understanding of the problem better on paper. I am also concerned with the misconception of the mathematical expression. So many students think they need to have an expression to prove their work is correct rather than using reasoning and her work was an example of the misconceptions students have about reasoning in mathematical problems.

**BUILDING TOWERS THREE COLORS**

Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and provide a convincing argument that you have found them all.




There are 27 possible colors.

3 cubes tall x 3 colors = 9

9 x 3 = 27

We got solid over with.

Then switched blue and red patterns.

Then switched blue and yellow.

Then switched yellow & red.

The figured out the patterns with all 3 colors in them.

Cycle 3: Building Towers Three Colors

Reflection

I was interested in seeing how the students would do with another tower problem, compared to the pizza problem, as well as how they would complete Ankur's challenge. The students had less difficulty with these two tasks than they did with the pizza problem. This is in part because of the manipulatives they were able to use and that there was less room for them to interpret the problem in different ways, though there are multiple strategies for approaching the problem.

It was interesting to see the different strategies in the problem and the challenge. Many students used proof by cases, there was a recursive argument, and some were challenged to make an argument because they were focused on pairs of opposite towers. The students in the class were able to work well together with the exception of one group. This specific pair of students had a difficult time communicating with each other as they both wanted to do the problem on their own and were challenged with explaining their reasoning to each other. In the future, this is something I would address with them in advance. While they may have different approaches, they should be communicating with each other.

Having two days to complete the problem and challenge proved to be beneficial as it allowed the students the necessary time to work through the problem and record their work and reasoning. I would not try to complete both activities in one class period as the students would be rushed or would not have the opportunity to attempt both problems. Through watching them work on the activities, I noticed students sometimes struggle to understand reasoning when presented in a different way. We had the opportunity in class to show the different ways the students solved Ankur's challenge and some of the students were only convinced with one way and had ~~more difficulty~~ following the ~~reasoning of other students.~~ Although this was true for some students, there were also other students who were intrigued to see different ways of organizing the towers and solving the problems.

This is a skill these
students can learn to do
the more they see you
expect it of them

Final Reflection

Through the three tasks I was able to observe many different things about the students' reasoning in mathematics. First, I found students are more inclined to create equations since they are in math class rather than realizing that using written reasoning is an important part of mathematics. This reveals the need to incorporate written explanations more in mathematics classes, which unfortunately is something that can be overlooked. As a result, I have been ⇐ good! encouraging students to write explanations more often. It was also interesting to note the difference in some of the students' oral explanations in class and the reasoning they wrote on their paper. Many were able to explain their reasoning in person and had difficulty communicating the same thing in writing.

Second, I saw students who were content with the answer because they could not find any more options, rather than convincing themselves there was no other possibility based on what they know. Through questioning, students were able to develop their reasoning and have convincing arguments, but it was not something that was first observed in the class.

The third thing I observed was the different approaches students took to solve the problem. I found many students start with opposite towers and only rearranged them when they had difficulty convincing themselves they had them all and were asked if there was a different way to group them. When they rearranged them, students were more inclined to use a proof by cases approach rather than inductive or recursive. I did not have any students use an inductive approach to solving the problems.

From comparing the first task and challenge problem, I was able to see growth in the students and their reasoning. Many students were automatically grouping the towers in different ways before being asked if it was possible and they were providing better verbal reasons. I was also able to see the positive influence on questioning students to get them to think about their work and other possibilities in mathematics. It was beneficial for students to see that mathematics is not just equations and numbers but also reasoning. As a teacher, it was informative to listen to students and their thoughts about problems, why they solve them a certain way, or what convinces them. I saw that students can interpret problems differently and can, therefore, approach problems differently. It was also a learning experience in the pizza problem to realize how the students' experiences can impact their reasoning. Implementing the tasks in the beginning of the year provided valuable insight into the reasoning of my students and areas that need more focus throughout the year.

Appendix J

Teacher T8

Cycle 1 Towers 4 tall selecting from 2 colors

Two of the student's work being presented in this cycle (KC and Brenda) are in an MCI SC 6-8 grade class. This stands for Mild Cognitive Impairment Self Contained. The students are 8th graders, however recent testing performed by child study team members placed their math abilities on a 3rd grade level. They were given this work in Math class they started at 9:30 A.M. There are 9 students in this class. The third student work sample (Mary) is from an 8th grade student. She is in a General Education Math Class, and comes in my class every day during her lunch period to work with my students. The time she performed this task was during her lunch at 12:00. All 9 of my students were in class while she performed this task.

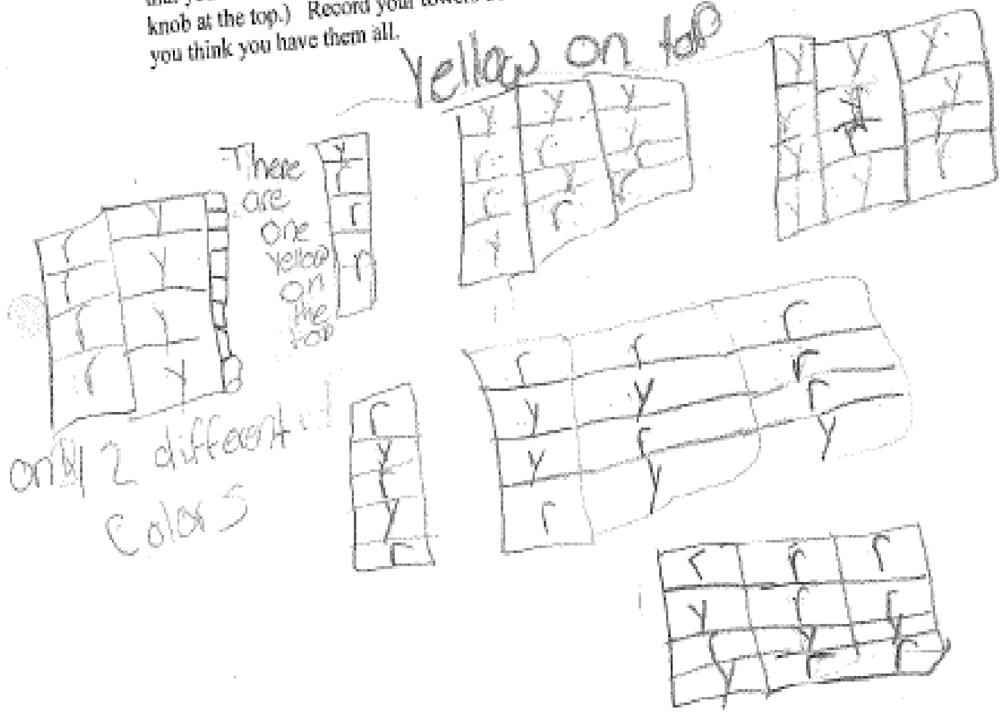
Work is impressive
for students functioning
on a 3rd grade level in math

Brenda

This student's work surprised me. She used multiple reasoning strategies to solve this problem. She started finding combinations by making opposite pairs and found all possible combinations. I questioned her to get her to try to find another way to group the towers. This student broke the towers into 3 groups and kept a constant on the top, red or yellow. One group was made up of single color towers. She broke the other 2 groups into subgroups. She did this by using the proof by cases argument. She grouped them by the amounts of yellows or reds in the tower. She then went on to exhaust all combinations for each group by using a recursive argument. The 1st sub group had only one tower. It was a yellow on top and 3 red on the bottom. She explained that there was no way to move the yellow because it would result in a red on top, which would ruin her constant. The next group was 2 yellows and 2 reds. She again tried to explain how positions could be moved within the bottom 3 positions without removing the 1 yellow cube from the top. The next group of towers had 3 yellow and 1 red. She moved started with the red cube in the 2nd position from the top and moved it one space down in each tower until it reached the bottom position. I was surprised at this student's growth throughout this project. She went from just making opposite pairs to using many valid reasoning techniques.

**Building 4-tall towers, selecting from 2 colors**

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.



Bren[illegible]

1. Only 2 different Colors

2 There are one yellow on top and three red on bottom and if you move the red on the top then there is no yellow on the top.

3 There is a block with one that has yellow red yellow red and you switch is to red yellow red yellow then you won't have the yellow on the top [illegible] another block with yellow yellow red red if you move the red toth top on top. You have yellow on top 2 red in the Middle and you have yellow on the bottom. any other move you can get a red on top

4 There are 3 red across If You Put the red on top then you wont have the yellow on top if you put the another red on the bottom then you get 5 cubs and there only post to be 4 cubs

1 for the clubs with red on top the same rules work just same color

2

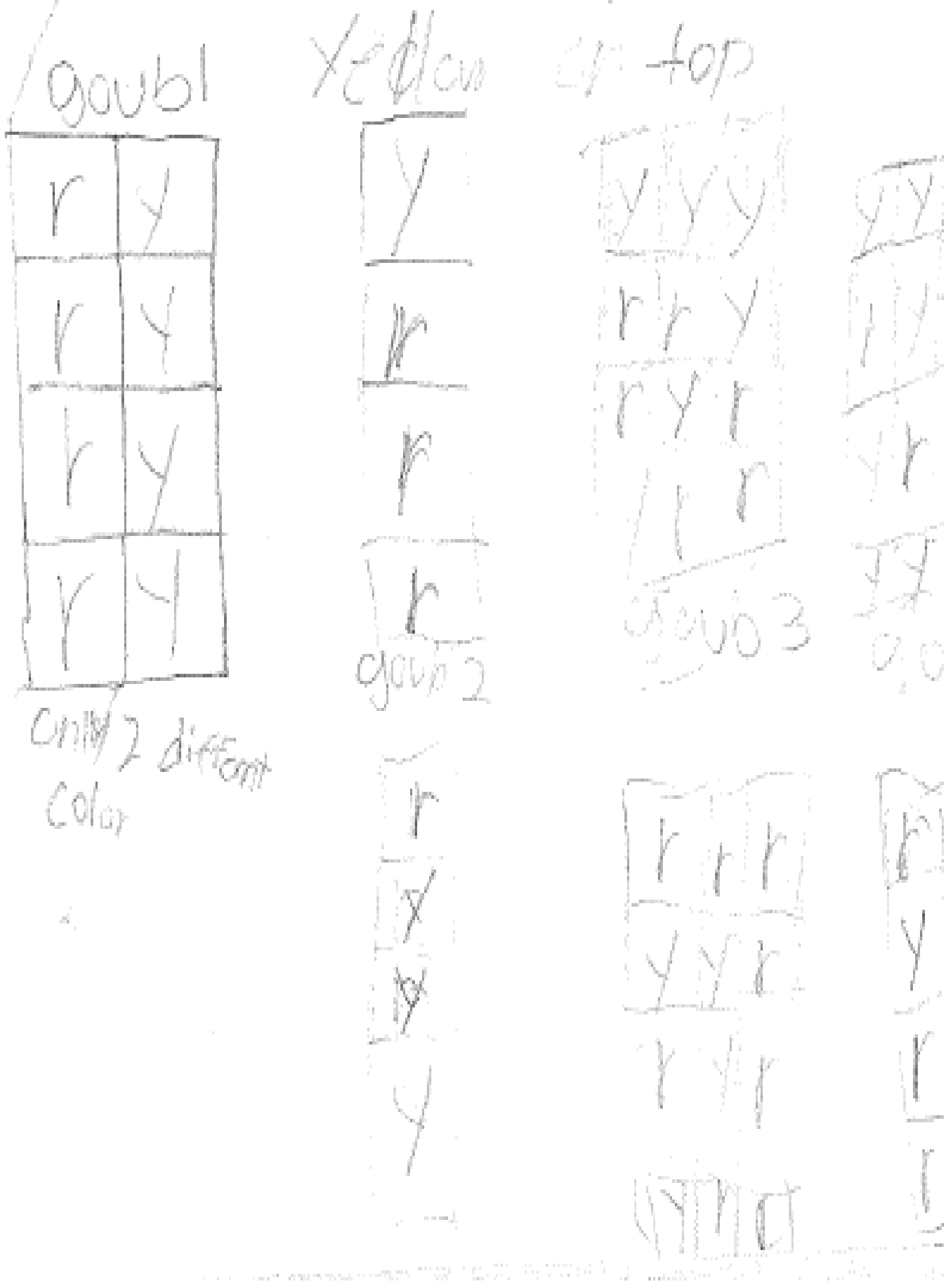
3 ~~There are 7 red and 4 yellow and there all red on the top if you put the yellow on~~ the top you won't have red on the top and you need red on the top.

KC

This student's work concerned me. This is my strongest student and I expected him to do best on this, however that was not the case. This student struggled throughout this project. He was able to make all possible combinations by making opposite pairs. When asked to come up with a different way to organize the towers he relied on his partner to do the work. He really seemed to miss the concept. Once his partner made new groups he was able to understand what she was doing and was able to explain her argument. He had no independent ideas and made no contributions to the organization and explanation of the regrouping of towers. After his group was finished I took him to the side to assess what he learned. He was able to show me the manipulation and movement of cubes to show how all possible towers were created.

**Building 4-tall towers, selecting from 2 colors**

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.



1. only 2 different color

2. There are one yellow on the top one it a red on the botton you can't have red on top on yellow cuz it only red on top on yellow cuz it only

②. one yellow and three red.

3. it 2 yellow on top and it 2 red on bottom 1 yellow on top and 2 red on the midlin 1 yellow, 1 red, 1 yellow on then one yellow on yellow 1 red. If you try a another more on the block you got a red on top or random.

④. we have 3 red cubes and we have 3 yellow but If you put the red on top you won't have to the yellow if you put the another red on top If you put the another red on the botton then you got five cube then only post to be four cube

⑧ for the cube on red on top the same rule

Mary

This student's work impressed me. She used a recursive argument. She did not start with opposite pairs. She started by making groups and moving the cubes downwards to make all possible combinations. The first group she made had 3 towers. In each tower was 2 red and two yellow. In this group she kept the 2 reds together and moved them through all possible spaces without separating. The next group consisted of towers made up of only 1 color. The 3rd group had 3 reds and one yellow. She created this group of 4 towers by moving the yellow down one space until it reached the last possible position. The 4th group was 2 reds and 2 yellows alternating. She explained this group to me by taking the 1st towers, which was red yellow red yellow, and moving the blocks down and placing the last block on the top tower. Her 5th group had towers consisting of 3 yellows and 1 red. She moved the red down 1 space until it reached the lowest spot on the tower. She excelled in this project, and needed minimal direction and assistance.

**Building 4-tall towers, selecting from 2 colors**

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.

| | | |
|---|---|---|
| R | Y | Y |
| R | Y | R |
| Y | R | R |
| Y | R | Y |

Group1

| | |
|---|---|
| Y | R |
| Y | R |
| Y | R |
| Y | R |

Group2

| | | | |
|---|---|---|---|
| Y | R | R | R |
| R | Y | R | R |
| R | R | Y | R |
| R | R | R | Y |

Group3

| | | |
|---|---|---|
| R | R | Y |
| Y | Y | R |
| R | Y | Y |
| Y | R | R |

Group 4

| | | | |
|---|---|---|---|
| R | Y | Y | Y |
| Y | R | Y | Y |
| Y | Y | R | Y |
| Y | Y | Y | R |

Group 5

Group 1 has the two reds together every time it move to the top, middle, and end. For Group 2 It just had 4 yellows, and 4 Reds. on each. For Group 3 It only had 1 yellow. so the yellow cube started on the top, and went down one every time and it stopped at the bottom. For Group 4, All the reds was separated. For group 5, the red cube started on the top, and went down one every time and stopped at the bottom.

For each group I couldn't make any more because there was no more possible combinations and if I added one more to it, it would be 5.

Intervention Implementation

What I learned most about my students during this cycle is the difficulty they have explaining their thought process. They were able to show what they were doing. But had trouble putting it into a clear explanation verbally. I had 2 sit with KC and Brenda and question them appropriately to help them verbalize their explanation instead of showing me with the blocks. The biggest challenge my students faced during this project was getting them to write their explanation down on paper. Once they were able to explain their reasoning to me I asked them to write it down and walked way so they could start working. After a few minutes I noticed that Brenda was crying. She had become so frustrated with the writing aspect of the project that she broke down. They explained that they did not know how to write down what they were saying. So I sat with them to give them the support they needed to get their thoughts on paper. I started with helping them break up their explanations by the groups. So they started writing the explanation for the single color towers. This was a good starting point for them.

I was very happy to see my students use such a variety of methods to explain their groupings. I did not expect them to come up with a constant. This is something that I did not think of at first so I didn't think they would either. This showed me that students with low functional levels are still able to reason at high levels if given the time and appropriate resources. It has helped me to try not to limit my students when giving assignments.

Cycle 2 Pizza Problem Selecting From 4-toppings

Two of the student's work being presented in this cycle (KC and Brenda) are in an MCI SC 6-8 grade class. This stands for Mild Cognitive Impairment Self Contained. The students are 8th graders, however recent testing performed by child study team members placed their math abilities on a 3rd grade level. They were given this work in Math class they started at 9:30 A.M. There are 9 students in this class. The third student work sample (Mary) is from an 8th grade student. She is in a General Education Math Class, and comes in my class every day during her lunch period to work with my students. The time she performed this task was during her lunch at 12:00. All 9 of my students were in class while she performed this task.

Brenda

This student really struggled throughout this project. I was worried because she did so well with the previous project. She started by making triangles to represent slices of pizza. The 1st she made was a plain pizza. She then started making random combinations with no organization or method. She was just coming up with different combinations. She wrote the combination on the crust, then went on to write the 1st letter of the toppings on the slice. For example, mushroom and sausage had MS written all over the slice of pizza.

She came up with a total of thirteen combinations. Her partner wasn't satisfied with the diagram and realized that they had created duplicates. Brenda was not catching on to what her partner was trying to do. Her partner started to create a table to organize the different combinations. She tried to follow his lead, but when her table is compared to his it is clear that she did not understand the organization of the table. Her writing was also not convincing. She basically just wrote what her partner said that they did.

The Pizza Problem

Capri Pizza has asked you to help design a form to keep track of certain pizza choices. They offer a standard "plain" pizza with cheese and tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many choices for pizza does a customer have? List all possible choices. Find a way to convince each other that you have accounted for all possibilities.

Group 1

I Start with Pepperoni. Then I did 2 topping. there Pepperoni in each one if I Put another one I get 3 topping. I did 3 topping with Pepperoni in each one.

Group 2

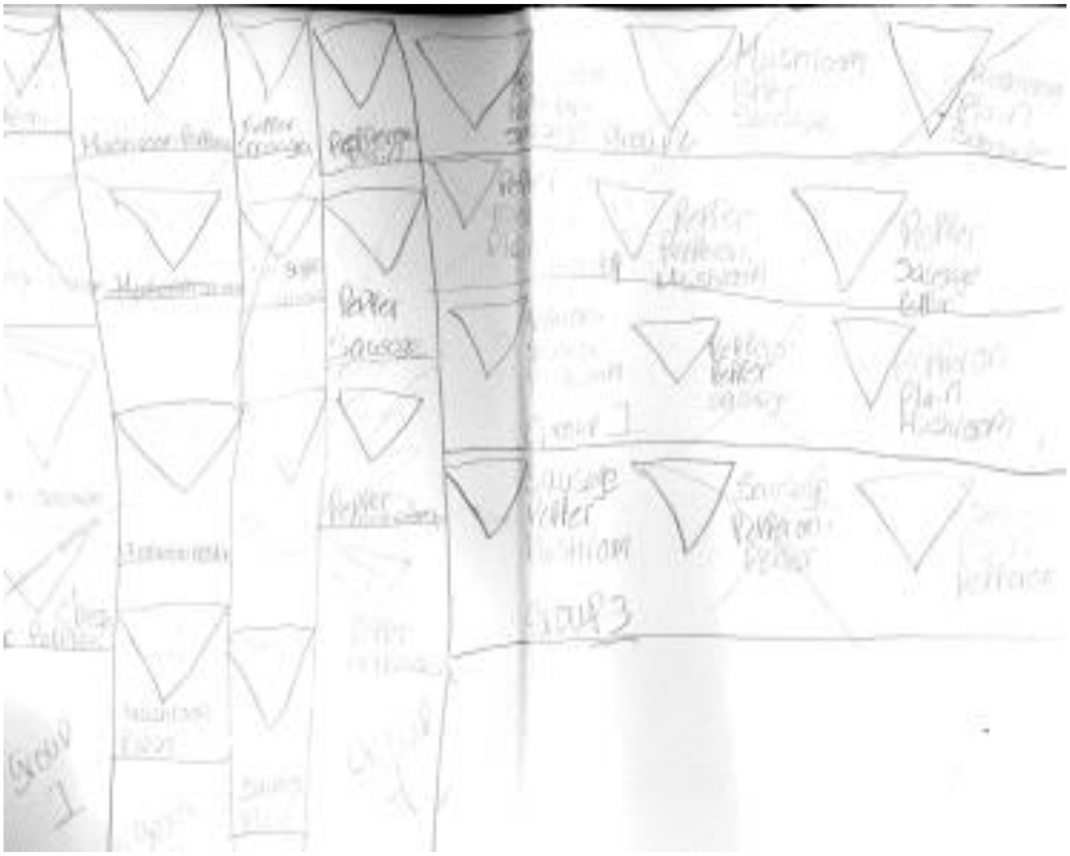
I Start with Murshroom, then I did 2 top. There Murshroom in each one. If I Put another one I Get 3 topping. I did 3 topping with Murshroom.

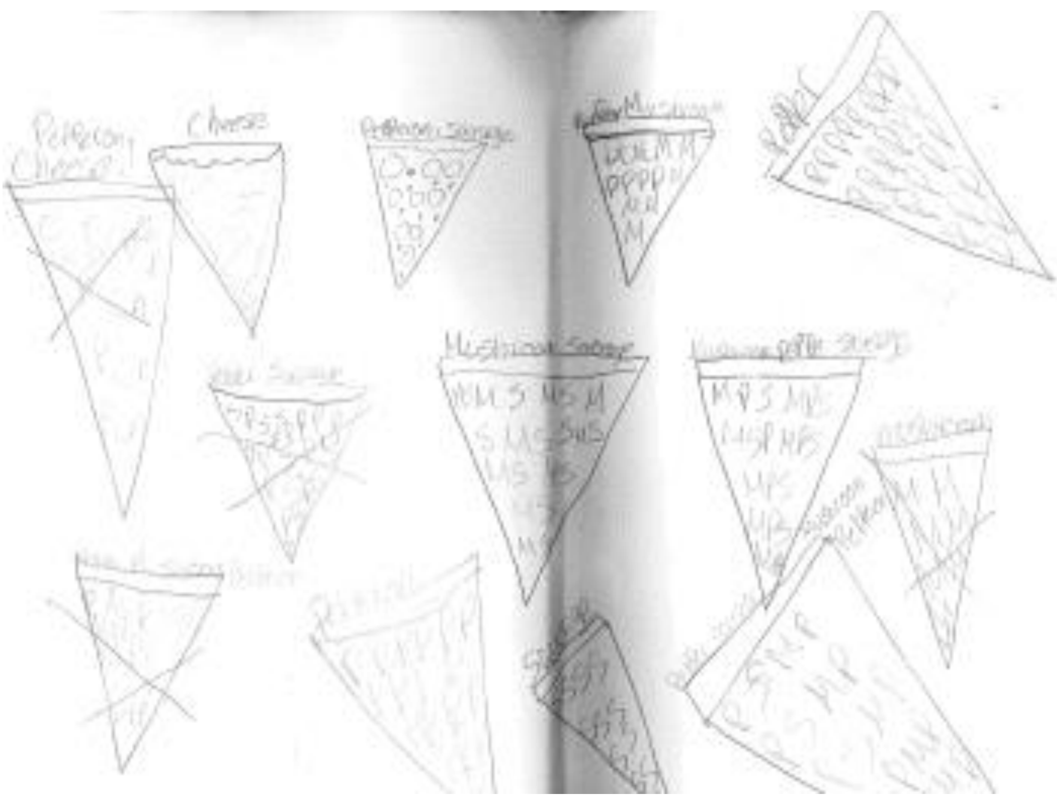
Group 3

I start with sausage. I did two other topping. There are Sausage in each one. If I Put other topping I get three. I did three topping with Sausage.

Group 4

I Start with Peper. I did two other topping. There are Pepper in each one. If I Put other topping I get three. I did three topping with Pepper.





KC

This student surprised me during this project. I was surprised because he struggled so much with the previous project. He started by making triangles to represent slices of pizza. The 1st he made was a plain pizza. He then started making 1 topping combinations. Then went on to make 2 and 3 topping combinations with no organization or method. He started drawing images of the toppings, but when he got to 2 and 3 toppings he started writing the combination on the crust and left the pizza empty.

KC then decided to make a table to organize his pizzas. He made columns for each topping. The first row in each column was a single topping. The first topping he selected was pepper. Then underneath that he did each 2 topping combination that had pepper in it. The first was pepper and sausage, the next was pepper and mushroom, the other was pepper and peperoni. The next that he made was a 4 topping pizza and pepper was the 1st topping. He did the same for each column of sausage, mushroom, and pepperoni. After this he realized he forgot plain pizza and pizza with 3 toppings. So he made groups for each of the 3 toppings and labeled them to match the appropriate group. He was satisfied that he found all possible combinations, which he did and then some. So to help him I asked "If you ordered a sausage and pepper pizza, and then ordered a pepper and sausage pizza would they be different or the same?" He said that they would be the same and said he should cross one out. Then he did the same for other duplicates. His writing did not justify his answer, it was more of an explanation of the different combinations he made.

group1

I satet it with pepper. then I did 2
tpgwith pepper with sausage mushroom and
pepperion. it 3 topping in each diffen
toppins with pepperion, sausge and mushroom
4 toppins with pepperion pepper
mushrom

I satet it with sausage. then I did
2 topping with peper with mushroom and
pepprine. it 3 tpping in each diffe
topping with pepperion sause and mushrom
4 toppins with pepperion pepper
mushrom

I satet it with mushroom. then I did
2 topping with peper with sausage and
pepperion. it 3 tpping in each diffen
toppins with pepperion sausage and
toppins with pepperion pepper
mushroom 4 toppins with pepperion pepper
mushrom

I satet it with pepperion then I did 2
topping with peper with sausge
and pepirom. it 3 tppins in each diffe
toppins with pepperiom, sausge and
mushrom. 4 toppins with pepperion, peper
and mushrom

Mary

This student's work impressed me during the development of possible choices. She simply made an organized list with no symbols or abbreviations. She started with plain. Then she went on to list all 1 topping choices. Next she wrote all 4 toppings. She then kept a constant to make combinations. She started with peppers. She did two 3 topping combinations with peppers as the constant. Then did 2 topping pizzas with peppers as the constant. She went on make 2 topping combinations with mushrooms and pepperoni as constants. She stopped at 14 and said she knew that there had to be 2 more, but was not sure what they were. When I asked her how she knew that, she said she didn't know, but she just knew that there had to be 2 more. She said she thought it had something to do with the last problem, but didn't know what it was. Eventually she found her 2 missing combinations, which were peppers mushrooms pepperoni, and sausage mushrooms pepperoni. I was also surprised when Mary wrote out her explanation. She just told me what she did. She didn't really explain why or how she developed her combinations.

The Pizza Problem

Capri Pizza has asked you to help design a form to keep track of certain pizza choices. They offer a standard "plain" pizza with cheese and tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many choices for pizza does a customer have? List all possible choices. Find a way to convince each other that you have accounted for all possibilities.

- plain
- peppers
- sausage
- mushrooms
- pepperoni
- all 4.
- peppers, sausage, mushrooms
- peppers, sausage, pepperoni
- peppers, mushrooms
- peppers, pepperoni
- peppers, sausage
- mushrooms, pepperoni
- mushrooms, sausage
- pepperoni, sausage
- Peppers, mushrooms, pepperoni
- sausage, mushrooms, pepperoni

the first 5 choices I put the toppings
tself, and then I combinated all 4
the toppings together. After that I
1 topping, and put it will 2 topping
not get it to repeat. After that
se 1 topping, and put it with 1
topping and to not get it to
t.

Intervention Implementation

I did notice an increase in willingness and effort towards the problem when compared to how they acted with the first cycle. The students had persevered through the first problem and I think that it was a confidence booster for them. However, they were much less successful in this problem than in any others. I think that the change to having no manipulatives made the problem more challenging. The manipulatives gave them something tangible to work with, which is obviously beneficial to this type of exploration problem. The students also struggled with writing convincing arguments for their work. They were not able to explain how or why they created their diagrams. They were only able to write what their combinations were and what their groupings were. I think that this problem needed to be more specific as to what was being asked of the students. I know that many teachers had problems with their student's interpretations of the problem. Some interpretations led students totally off track and hindered their learning experience throughout this cycle.

Cycle 3 Towers 3 Tall Selecting From 3 Colors

Two of the student's work being presented in this cycle are in an MCI SC 6-8 grade class. This stands for Mild Cognitive Impairment Self Contained. The students are 8th graders, however recent testing performed by child study team members placed their math abilities on a 3rd grade level. They were given this work in Math class they started at 9:30 A.M. There are 9 students in this class. The third student work sample is from an 8th grade student. She is in a General Education Math Class, and comes in my class every day during her lunch period to work with my students. The time she performed this task was during her lunch at 12:00. All 9 of my students were in class while she performed this task.

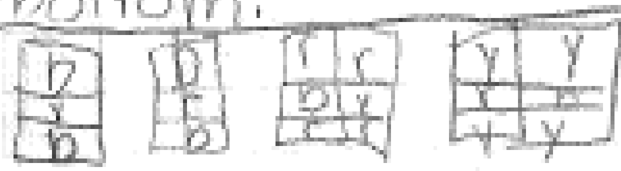
Brenda

This student's work impressed me. She had ideas for how to create and organize blocks. She said that she was using the same ideas she had learned from the 1st towers problem. She went right to making 3 towers that were only one color. Then she went on to make towers that had all 3 colors in them. She kept constants on the bottom and alternated the other colors in the top 2 positions. For example the 1st 2 towers in the group had yellow on the bottom. She alternated the blue and red cubes to exhaust the possible combinations with the yellow constant on the bottom. She did the same for keeping red and blue as constants on the bottom. Her next group had only 2 colors in the tower. She kept a constant of the same color on the bottom and the top and switched out the other colors in the middle. For example, she had 2 towers with blue on the top and bottom. One tower had yellow as the middle cube and the other had blue as the middle cube. The last group that she had was broken into 3 subgroups. Each part had a different constant on the top. She had 2 different constants on the bottom for each subgroups. Her explanation in writing was just listing the groups. She knew to organize them by colors, but really couldn't establish a convincing argument for why or how this grouping worked.

The cubesthat are red, blue, yellow.
Red has three cubesand blue has three cubes and yellow has three cubes.

Theres all three cloros in the each towers.

there two yellow in the bottom. The red and the blue switch spots. There noway to move the red and the blue. I did the same thing for the reds and the blues on the bottom.



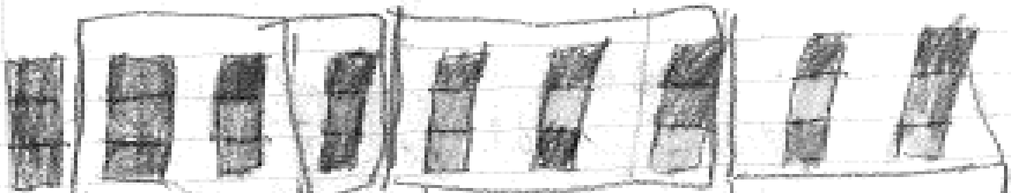
There two blues on the top and bottom. There yellow, red in the middle. There no other color for the middle. I did the same this for red and yellow the red is in the top and botton

the yellow is in the top
and bottomm

This gurop are only two
colors. The frist 4 has the
same color on top. and the
second gurop. ther only yellow
on top. The third gurop
there only blue on top.
The first gurop I did red yellow
yellow red, red yellow
red, blue blue red red
blue. Second gurop is
yellow, yellow blue yellow blue
blue yellow red, red yellow
red, red. third gurop is
blue, blue yellow, blue, yelbow
yellow, blue red, red,
blue, blue red.

KC

This student's work concerned me. He made 3 main groups by keeping a constant color on the top. He had a group of 9 with red as a constant on the top. He broke it into sub groups. He started with a tower of all red. Then he had 3 towers with just red and blue and 3 towers with red and yellow. The last 2 towers were made up of all three colors. I could see what he was trying to do and he explained it to me by moving blocks and using the recursive argument. However, when he went to draw his diagram and convincing argument he did not draw or say what he told me. His next group, which kept blue as a constant on top, only had 8 towers and he was missing the all blue tower. This was the easiest type of tower for him to find in the 1st group, so I expected him to have it again. The arrangement of the other towers was similar to that of the ones that had red as a constant, and his explanation was just telling the arrangement of the blocks. He did not write anything convincing. His last group, yellow as a constant on the top, had 11 towers. He had 3 towers that were yellow and white, which were not anything. He was also missing the all yellow tower. Again, his writing was just telling the arrangement of the blocks, not a convincing argument. I expected to see growth just as Brenda had shown, especially because he made improvements from the 1st tower problem and the pizza problem.



I put on 2
red on top next
I put bule no
button. I put red on
top, then I put
2 bule on button

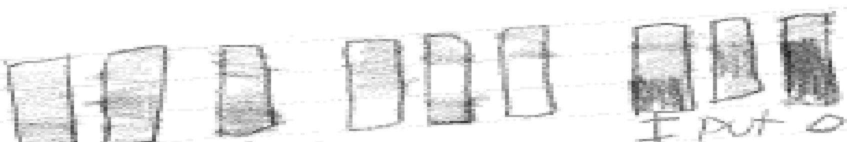
One red on top
and 2 yowllow
on button I put one red
on top next I put yowllow on
button.

I put one red
on top red
nex I put the
yowllow on midde
last I put bule
on the button

I put 1 blue
on top next
I put ~~2~~ red
on the button
I put blue on
top so I put one
red on the middle
last I put red

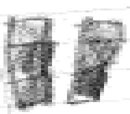
I put two
blue on top
last I put
yowllow on
the button
I put 1 blue
on top the
last I put yowllow

I put one blue
on top first next
I put the red
on the midden last
I put the yowllow
on the button.



I put the 2
yowllow on top nex
I put blue on button
I put one yowllow
on top 2 blue. on button

I put one
yowllow on
top put 2
red on button
I put 2 red on
the button one
yowllow



I put one
yowllow on top
first next I put
one red on the midder
last I put blue on
the button

Mary

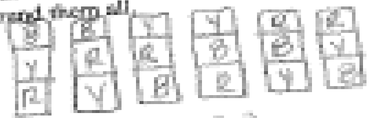
This student impressed me with her confidence and independence level during this task. She went right to work and needed minimal assistance throughout the task. She told me that she was using the same strategies that she used for the 1st problem. The first group was made up of towers only one color. For the next group she kept constants on the top and alternated the other colors in the bottom 2 positions. For example the 1st 2 towers in the group had blue on the top. She alternated the yellow and red cubes to exhaust the possible combinations with the blue constant on the top. She did the same for keeping red and yellow as constants on the top .For the next group she had towers of yellow and blue. The 1st 3 towers had 2 yellow and 1 blue. She moved the blue cube up from the bottom to the top. The next 3 had 2 blue and 1 yellow. She moved the yellow from the bottom to the top. She used the same strategy for towers made of blue and red and towers made of red and yellow. Mary used keeping a constant and the recursive argument to correctly solve this problem. I was happy to see that she was able to apply the same strategies that she used in the 1st towers problem. Her written explanations were not very convincing, however her verbal explanations were clear and detailed.

**BUILDING TOWERS THREE COLORS**

Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and provide a convincing argument that you have found them all.

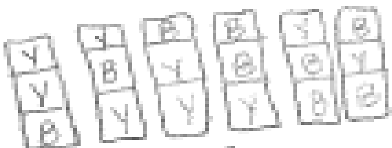
Group 1

For each tower I had one color.



Group 2

For each tower I had each color on the top and then followed by the 2 other colors switching. I did it for all three.



Group 3

For the first 3 I had the blue go up every time. For the last 3 I had the yellow go up every time.

Group 4

For the first 3 I had the red go up every time. For the last 3 I had the yellow go up every time.

Group 5

For the first 3 I had the red go up everytime, and for the last 3 I had the Y go up everytime

Intervention Implementation

I was very impressed by my student's readiness to solve this problem. They told me that the problem was easy for them. They were able to apply their strategies they used from the 1st problem and further develop them to solve this problem. They were able to create groupings without any help. They used the recursive, proof by cases, and keeping constants strategy. They changed the way that they used the strategies by keeping constants in different locations.

The students continued to struggle with writing convincing arguments for their work. They were not able to explain how or why they created their groups. They only wrote what their combinations were and what their groupings were. I was disappointed that they did not discuss their recursive strategy. They also did not explain their proof by cases reasoning. They did make note of the constants they kept. But did not explain how they had exhausted all options.

## Conclusion

I learned a lot about the actual mathematics of this project. When I did the first towers problem, I struggled to develop a way of organizing the ones that I made. I actually started by making opposite pairs. My partner then developed the strategy of keeping a constant and then I applied the recursive method to organize groups. I had not done mathematical thinking like this in quite some time. It made me hesitant at first when thinking about giving my class this problem. However, once I saw Dr. Landis' implementation in the classroom I became much more comfortable in my instruction of the material. I also saw that middle school students were able to develop more advanced strategies than I had initially come up with. As we did the other problems I was able to apply and develop the strategies I used from the previous exercise.

During the implementation of these lessons I learned that I had completely underestimated my students. The showed perseverance throughout the projects. My students were capable of solving the problems, especially when using manipluatives. Their struggles where when they were asked to explain their reasoning and not state what they did. It was also evident that I need to incorporate writing into my lessons. Teachers do incorporate writing by providing open ended responses for students. However, it is important to scaffold and support this writing and have students revisit it to analyze their thinking and further develop their writing.

Appendix J

Teacher T9

**Cycle 1: Towers 4-tall, selecting from 2 colors**

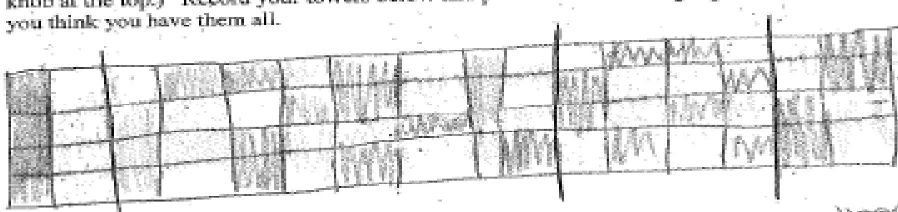
The following samples are from a sixth grade class. The class is titled Mathematics 6. It meets periods 5 and 6, which is from 10:42 to 11:56, right after lunch. The class is an inclusion class. There are 15 total students in the class. Five of the students have IEPs and two have behavioral plans. One of the students has a 504-plan for dysgraphia and usually requires a scribe. The rest of the students are regular education students. There is an in-class support teacher with me in the class.

## Work that Impressed Me – Cycle 1

This sample from Samantha impressed me because her mathematical reasoning was very clear. She clearly explains why there could only be two solid towers. Then, she moves to a recursive argument. She takes both solid towers and changes the top block. Then she moves that block down to each position until she had exhausted all the positions. Once she completes that pattern, she moves to having two of each color. She again used the recursive argument to complete her pattern.

I thought this was a very well thought out argument, especially since it was done so early in the year. Samantha was able to complete the task and give a detailed explanation as to how she knew she had all the towers. Most of my students were not able to do this so early in the year. This impressed me very much.

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.



I made a 2 whole towers: and know that there could
only be two because there were two colors. Then I
switch the first top color, the second color, the third,
color, and the last color. I then knew that I could
only do these because I switched the colors for
both towers. Then I starded doing 2 and 2. I
took the wholes and swithed the bottom 2, the
top 2, and the midde two. I know that these
were the last ones becouse there are 4 and
if your doing 2 and 2 then you know you
can't do anymore. And finaly I did the stripes

**Work that Surprised Me – Cycle 1**

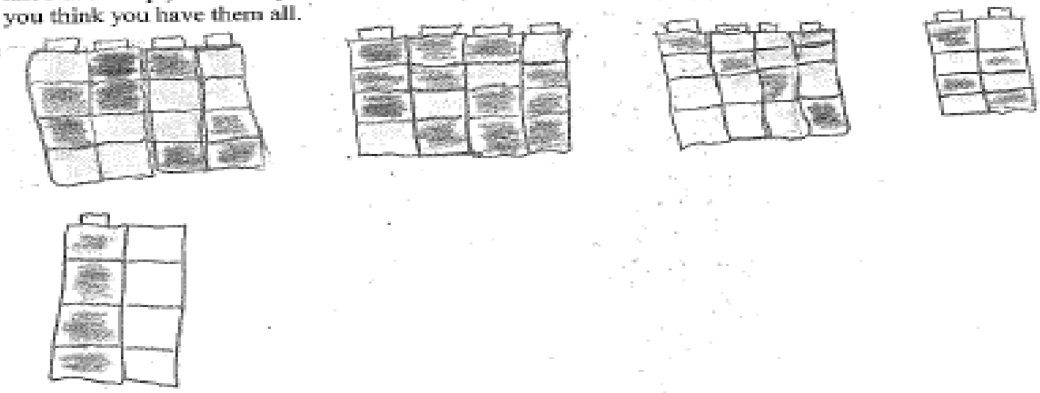
This work from Zaire really surprised me. I typically do not get much work out of Zaire unless I am sitting with him and helping him. He has trouble processing his ideas. Unfortunately, he was not able to write his argument; however, his representation of his solution really impressed me.

You can see four sets of towers. I am not able to know exactly what Zaire was thinking, but it appears to me that he also used a recursive pattern. In his first set of towers, he has two blue blocks together sandwiched between two yellow blocks. It appears that he took the top block and moved it to the bottom position. He continued with this pattern until he exhausted all the positions. The next two sets show the blocks with three of one color and one of another. He shows the "staircase" pattern that many other students saw. He then drew the "candy cane" pattern, and finally the solid blocks.

This was impressive to me because it shows much more higher order thinking than Zaire has shown to me yet. His representation is very though out and organized. It showed me that I could push Zaire farther than what he was originally showing me.

**Building 4-tall towers, selecting from 2 colors**

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.



**Work that Concerned Me – Cycle 1**

This sample from Devin really concerned me. His mathematical reasoning was not on target. He really tried to use prior knowledge that a previous teacher gave him. Unfortunately, he did not use it correctly.

Devin did get the correct number of towers and represented them well. However, his reasoning is very concerning. He said there should be 16 towers because 4 times 4 equals 16. Mathematically, this equation does not fit this problem. He can see that the towers are four blocks tall, but he could not explain where the other four came from.

This concerns me because Devin is always looking for math problems to "fit" an equation. This is not the way to approach a math problem. He should be thinking about the numbers that are actually used in the problem to try and determine the correct equation. Devin tends to jump right to equations before thinking about what the numbers actually mean.


My Group and I think this is all you Can Make because 4x4=16. There are 16 different pillars of 4. My tearch in lloyd Road said if there is a problem like this do the amount of ~~the~~ blocks in one stack and times it by it Self so 4x4=16 pillars

**Intervention Implementations for Cycle 1**

For this cycle, I learned a lot about my students. I feel that it was a great way to start off the year. It gave me the opportunity to see how my students think and reason problems out. I was also able to see where my students need help in writing about math.

I learned that my students do not understand the concept of justifying their work and what it means to convince someone. Most of them tended to just write what they did instead of why they did it or how they knew they were finished. I learned that I really needed to address what these words meant if I wanted to get clear, concise thoughts from my students. I really thought this was great for me as a teacher to see right away in the beginning of the year.

I could have paired the students up better. I strictly went on test scores to pair the students up. This did not work with this group of students given their limitations. For instance, the highest scoring student is on the autism spectrum. He did not work well with his partner who also scored very high on her testing. This is because he would not talk to her and she wanted to forge through the problem at a speedy pace because that is how she does things. The next time I pair students up, I will remember to consider other factors besides test scores. Good!

I have some students who just sat and stared at the bollocks because they were overwhelmed. I tried to encourage them to work, but it seemed to frustrate them. If I had the opportunity to do this problem again, I would tell students who get frustrated so quickly to start with creating towers of three blocks high. I believe that this is an easier problem to tackle and if solved correctly, would give these students the encouragement to try the towers four high.

**Cycle 2: Pizza Problem, selecting from 4 toppings**

The following samples are from a sixth grade class. The class is titled Mathematics 6. It meets periods 5 and 6, which is from 10:42 to 11:56, right after lunch. The class is an inclusion class. There are 15 total students in the class. Five of the students have IEPs and two have behavioral plans. One of the students has a 504-plan for dysgraphia and usually requires a scribe. The rest of the students are regular education students. There is an in-class support teacher with me in the class.

## Work that Impressed Me – Cycle 2

The student that impresses me with this problem was Samantha. Her solution was the most thought out solution that I got from my students. She begins by talking about the one-topping pizzas and how there could only be four since there were only four toppings. Then she moves to two-topping pizzas, and so on. She does not discuss it in her explanation, but looking at her list of the pizzas, you can see that she kept a constant throughout her representations and thought process. She is; however, one of the only students that discussed how if you "flip" the toppings you would get the same pizza.

This solution really impressed me because her representation was one of the most organized that I have seen. She had a really well thought out solution. Even though her explanation is not perfect, she definitely had some good ideas. Her explanation of the two-topping pizzas is very confusing, but she does list them accurately. In all, I think she really understood the problem.

I know there are only 4 with 1toppings because there are only 4 toppings. There are 7 with 2 toppings because if you list all of them you 7 and you can't flip them around because it would be the same pizza. There are 4,3 topping pizzas because you have one with peppers, sausage + mushrooms but lets say you don't like mushrooms so you change it to pepperoni. Lets say you don't like pepper so you change it to pepperoni + mushroom and sausage and you don't like mushroom so you change it to pepperoni sausage + pepper and get 4 combos. There is only 1-4 toppi[illegible] because you can't switch them around because it would be the same pizza. Also there would be just the pizza by itself. So that would bring me to the answer of 16 pizzas.

P=plain

1 P+peppers
2 P+ sausage
3 P+mushrooms
4 P+ pepperoni
5 p+ peppers+sausage
6 p+pepper+ mushrooms
7 p+pepper+pepperoni
8 p+sausage+mushrooms
9 p+sausage+ pepperoni
10 p+mushrooms+pepperoni
11 p+ pepper+sausage+mushroom
12 p+ pepper+ sausage+ pepperoni
13 p+ pepper+sausage+mushroom+ pepperoni
14 p+ pepperoni+sausage+ mushroom
15 p+pepperoni+ mushrooms+ pepper
16 p

**Work that Surprised Me – Cycle 2**

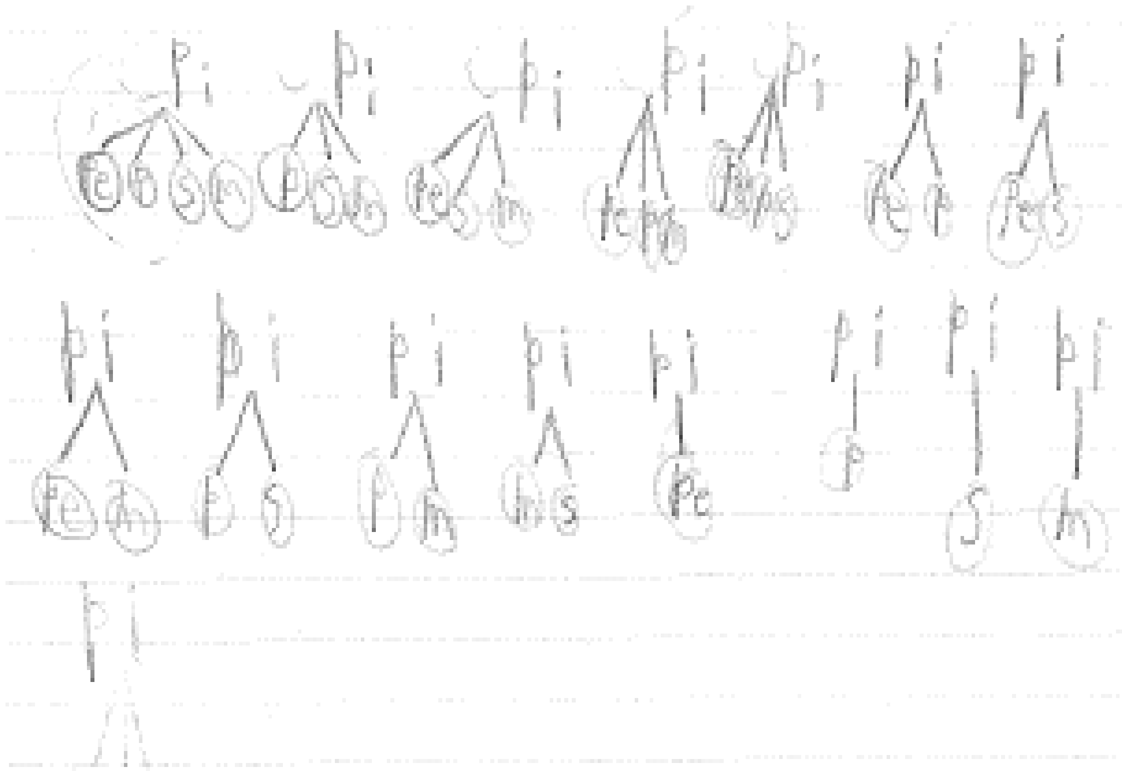
Garret's solution really surprised me because of the way he represented his pizzas. He was able to come up with the correct solution to the problem by drawing a separate tree diagram for each pizza. This really struck me because it took me a while to be able to interpret his work.

really neat!

Once I sat with him and he explained his representation to me, I really got inside his head and was able to see what he saw. He was imagining each pizza separately, instead of in groups of numbers of toppings. He began with the four-topping pizza. When he moved to the two- and three-topping pizzas, he did not keep a constant. I cannot find any order to his list. He was still able to come up with the correct number of pizzas.

What I find really interesting about this solution is that what seems like chaos to me was completely understandable to this student. He saw it and was able to interpret the solution with this representation. There is one thing I am still confused about with his explanation. He says that he replaced the numbers with letters. I still have no idea what he meant by this. I am not sure if he meant the number of toppings, how many pizzas, or something else.

I got 16 combinations
I used factor trees to help me
out by abbreiviating the toppings
and replacing the numbers with
letters




key
s= sausage
pi= pizza
p= pepperoni
pe= pepper
m= mushrooms

**Work that Concerned Me – Cycle 2**

The student who concerned me with this problem was Crystal. Crystal began by trying to draw a tree diagram. However, she did not get very far with it. In the center, she put peppers and tried to branch off from there. When I asked her about it, she really could not explain it at all. He partner suggested they try listing their pizzas. I hinted that was a good idea.

However, that did not work out much better for the two of them. What Crystal did was list all the possibilities of pizzas with peppers. Then she did the same for sausage, mushrooms, cheese, and finally pepperoni. What she did not understand was that by doing this, she had several duplicates in her list. She also did not consider that she could have three- or four-topping pizzas.

Her explanation just says that she listed all the possibilities because she listed all the toppings with all the other toppings next to them. She clearly did not comprehend the task. No matter how many times I questioned her, her answer never changed.

peppers — sausage
peppers — pepperoni
peppers — mushrooms
peppers — cheese

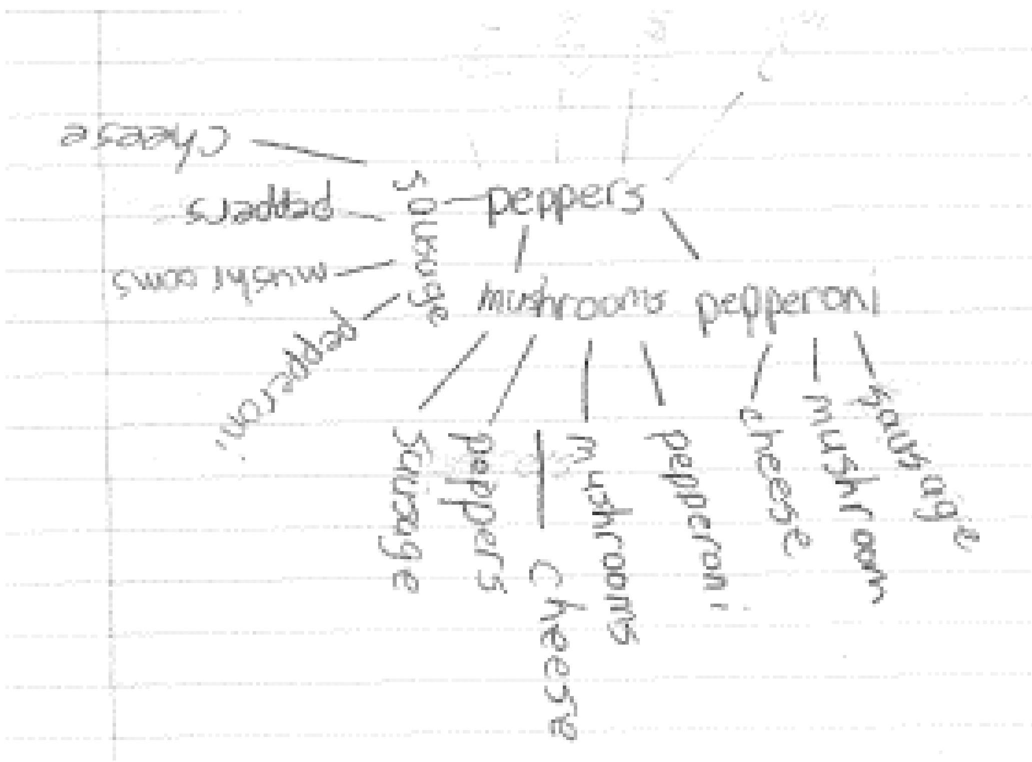
sausage — pepperoni
sausage — peppers
sausage — mushrooms
sausage — cheese

mushrooms — sausage
mushrooms — peppers
mushrooms — pepperoni
mushrooms — cheese

cheese — sausage
cheese — peppers
cheese — pepperoni
cheese — mushrooms

pepperoni — sausage
Pepperoni — mushroom
Pepperoni — cheese
pepperoni — Peppers

We have every pizza combination because we listed every pizza topping and listed the rest of the toppings next to them.


peppers
sausage
mushrooms
pepperoni
cheese
peppers
mushrooms
pepperoni
Sausage
peppers
cheese
mushrooms
pepperoni
cheese
mushroom
sausage

**Intervention Implementations for Cycle 2**

During this task, I saw many students really struggle. They could not figure out how to begin with the toppings. They wanted to make half-and-half pizzas. They claimed that was different than a two-topping pizza "mixed." A few students even wanted to make each slice a different topping. They were really making the problem much harder than it actually was. What was interesting to see with these students was how they reacted when other students claimed to be finished. They were working on tens of pizzas with no end, yet their peers had finished so fast! They could not comprehend that.

Another thing I noticed was that the students had a hard time because they only had pen and paper. There were no manipulatives to move around. I learned that students really struggle when they need to list things out. More often than not I saw students trying to create a web diagram; however, this proved to be very challenging.

What was really good about this task was that when the students found the answer, and could convince me they had all the options, they felt a great sense of accomplishment. I think because it was so challenging for them, it meant more to them when they got the answer.

I am not sure what could have been better. Perhaps somewhere in the directions I would explain more about how the toppings could be used. I felt so bad for the students who got stuck on different ways to make the pizzas. I wish I could have guided them a little more with that part of the task. I also think that for some students I would have differentiated and given them only three toppings to choose from.

**Cycle 3: Towers 3-Tall, selecting from 3 colors**

The following samples are from a sixth grade class. The class is titled Mathematics 6. It meets periods 5 and 6, which is from 10:42 to 11:56, right after lunch. The class is an inclusion class. There are 15 total students in the class. Five of the students have IEPs and two have behavioral plans. One of the students has a 504-plan for dysgraphia and usually requires a scribe. The rest of the students are regular education students. There is an in-class support teacher with me in the class.

**Work that Impressed Me – Cycle 3**

The student who really impressed me with this task was Devin. He came a very long way since the first task. He was able to write much more for this task. He kept a constant as the bottom block. He sort of describes a recursive pattern as well. He explains how he changed the middle block and then he changed the top block. He continues through until he has exhausted all the possibilities using blue as the bottom block.

Once he listed all possibilities with blue on the bottom, he says that he knows that it would be the same with yellow or red on the bottom. It is because of this that he multiplies the nine possibilities by three. Unlike the first task where he incorrectly used the equation 4 times 4 equals 16, he was able to correctly find the formula 3 times 3 times 3 here to get a total of 27 towers.

He impressed me because he showed tremendous growth from the first task to this one. He also impressed me because he used both patterns that we discussed in our class, recursive and holding a constant.

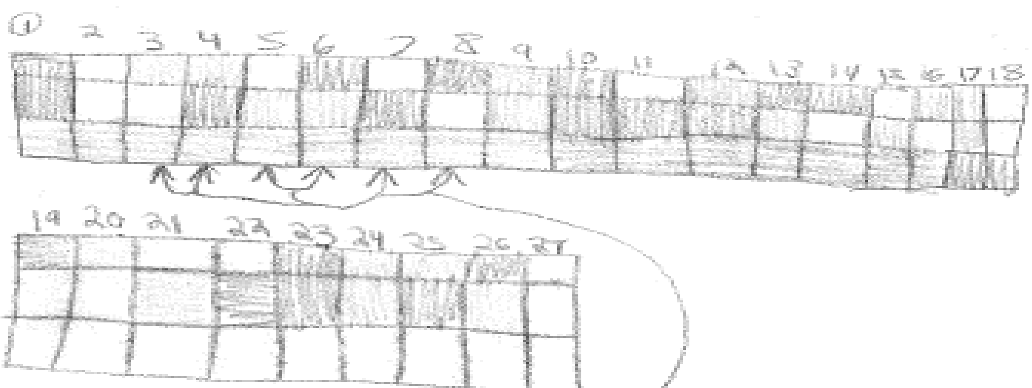
I think there are 27 Different Towers because I did A whole Solid Tower of Red and only 1 Tower Then I did all the different combination with red on the bottom and got Nine Different One's for just red on the button. Then I knew I had to do yellow and blue on the button so I did 9x3.
~~Bec~~ I can prove that their is Nine per color on the bottom by Saying the towere is 3 block high and there are 3 Different Colors So I did

3x3=9 towers Each So then 9 block each Times 3 Different colors.

9x3

towers Each

Different Colors



Numbers 3-8. I got 3 by ~~ato~~ Doing Bottom blue Top blue and middle Red. I got 4 By Putting yellow in the middle instead of Red. I got 5 by Doing The blue Two times up and Red on the top. I got 6 by Doing The same
[illegible]

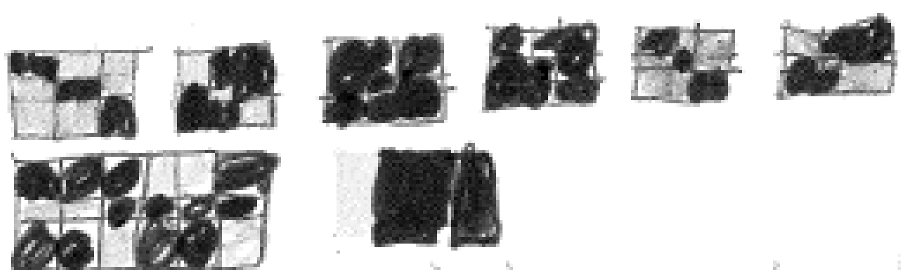
Blue on the bottom
Yellow in the ~~b~~ middle and
Red on the top. I got 8 by
Doing the same thing but
switching yellow and red from
7. I Did That all of the
same pattern but without
Blue on the buttom I would
Put Red or Yellow on the bottom

**Work that Surprised Me – Cycle 3**

Jorge really surprised me with his representation of the towers in this task. He is very reluctant to do much work for me. He does not like to work with a partner for some reason. For this task, unlike the others, I decided to pair Jorge up with a friend. This did wonders for Jorge. He opened up and actually worked!

For a student who usually gives me the bare minimum, Jorge was able to represent the towers in a very meaningful way. He was even excited when he noticed the "staircases." Once he saw the stairs for the red and yellow, he knew there were two more sets, one with blue and red and one with yellow and blue. He was very motivated to create them.

In his description, he only describes the two-colored staircases and the solid towers. He was able to create and draw the three-colored towers as well.



First, I matched up red and yellow. Then, I did the opposite to all the other colors. For example, red and blue like a stair case. Also, all three yellow, red, and blee.

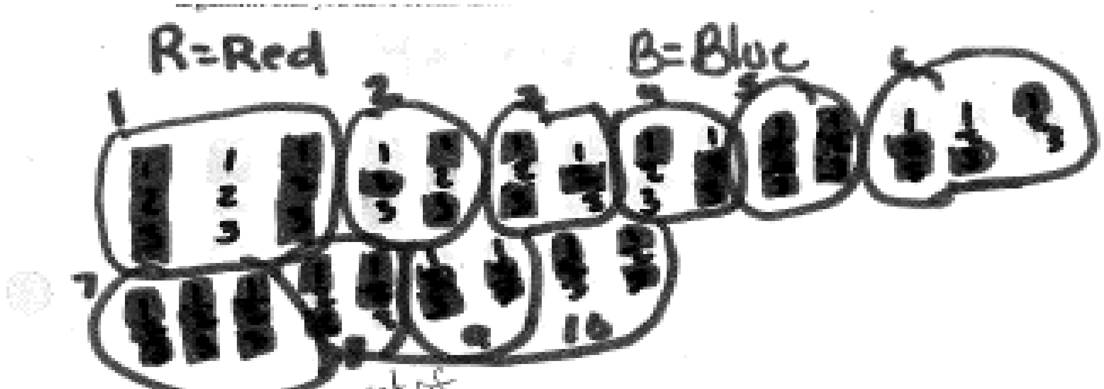
## Work that Concerned Me – Cycle 3

The work that concerned me was that of Samantha. She only got 23 towers. What concerned me was that she did not have much reasoning behind her answer.

She found 10 sets, she calls them pairs. The first pair is the three towers with solid colors. She just says they are the obvious ones. Then her explanation simply describes "opposites;" however, she does not distinguish between the opposite pairs. The last pairs she discusses are "triple pairs" and says they are the same but in different forms.

It is very difficult to follow her rational as to how she came up with her answer. It is not very convincing, especially since she did not come up with the correct answer. When I look at her drawing, I can see certain gaps. For instance, in sets 2 and 3, she made the "candy cane" type towers. She made them with red and yellow and yellow and blue, but she forgot about red and blue.

I think if Samantha would have organized her towers differently, she would have seen her missing towers. I only wish her explanation was a little more thought out. The mathematical reasoning just isn't there in this example.




All of The Blocks, set of 23 blocks.

Explanation: We know we have found all the possible ways by making 3 obvious 1's, and making 1 of each way but, we did oppisites of all of them so that made 23 blocks, & to show all of them we circled the pairs in purple.

The 1st pair is a pair because they are all the obvious 1's.
The 2nd pair is a pair because they are 2 oppisites of each other.
The 3rd pair is a pair because they are another pair of oppisites. That goes for the pairs of 4,5,8,9 & 10.
The 6th & 7th pair is a triple pair because they are all the same but in different forms.

**Intervention Implementations for Cycle 3**

This task really impressed me the most. I am not sure if it is because the students had the other two tasks to practice on or what, but they all seemed to really grasp this task. For each class I implemented this task in, the students' reasoning got better and better.

I learned that the students really have the capacity to grow in such a short amount of time. Their responses got more detailed and in-depth. I have to admit, after the first and even the second task, I was really skeptical about how much the students would grow.

I do not think I would change anything about this session. I was very impressed and pleasantly surprised with my students' responses. After doing the first two tasks, I felt like the students really dove right into this problem. The completed building the towers faster and wrote much more detailed and convincing arguments.

**Reflection Page**

I learned that there is much more to mathematics than just numbers on a page. Students need to discuss math with each other. They need to build and manipulate. They need to be given the time to think about the problems they are given. I really learned that writing about math is very important. Students who have the opportunity to defend their thoughts and ideas will gain a deeper understanding of the concepts. I also learned that it is okay to disagree with others while solving problems. For instance, with the pizza problem, everyone had a different interpretation of what it meant. As long as they could defend their choices, they could be correct.

I learned a lot about my students' reasoning and mathematical thinking. They really have the capacity to grow. Given the first two tasks, I was a little discouraged by some of my students' responses. I did not really see growth until after the third task. I cannot tell you how amazed I was at the arguments the students were making! This lesson study really opened my eyes to how important it is to allow the students time to reason through the math they are learning. Yes!

I feel that my students really gained self-confidence in writing about math. They went from being able to verbalize their thoughts in the beginning of the year to being able to put those thoughts to paper. I have noticed that I take more of a facilitator role in the classroom. I try to allow my students the time and space to discuss, and even argue, with their peers to determine the solutions to problems. I find myself asking the students if their peers are convincing them. I also ask them to defend their positions. This lesson study has definitely made me a better teacher.

Great!

# Appendix J

## Teacher T10

## Cycle 1: Building Four-Tall Towers, Selecting From Two Colors

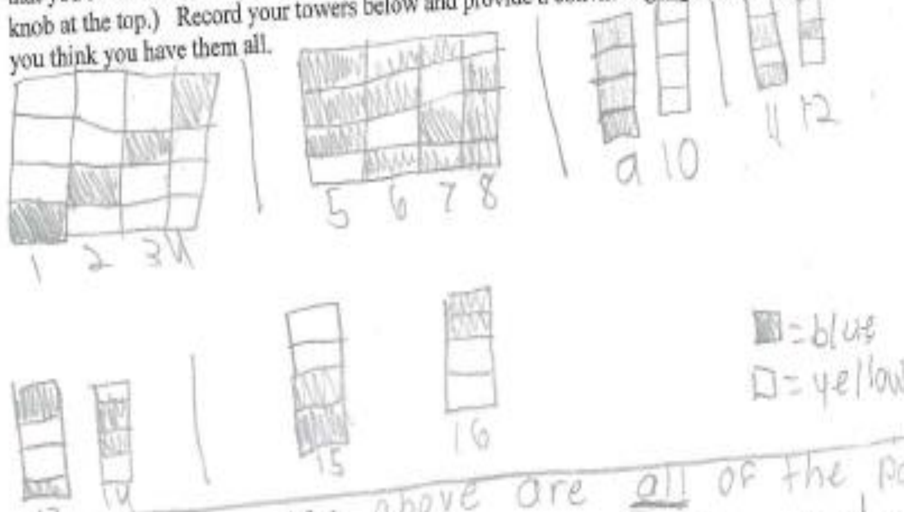
### Class Description

The class who I chose to implement the Cycle 1 task with is a regular level sixth grade math class at Matawan Aberdeen Middle School in Cliffwood, New Jersey. The class is held daily from 11:58 a.m. to 1:12 p.m. and is the students' third academic block of the day. The class is made up of 13 boys and 4 girls, with varying levels of ability. Out of the seventeen students in the class, four are basic skills students who receive additional support from a special education teacher who comes into our math period every other day for 36 minutes. In addition, several other students in the class have 504s and plans implemented through Intervention and Referral Services.

For the first cycle, the task was implemented in one class period. At the end of the period, several students expressed that they were not yet finished with their convincing arguments, and were therefore given time on a second day to complete their arguments.

Kimberly's work from the first task impressed me. She has a clear method that her and her partner used to solve the problem, by creating 6 groups of towers. The first two groups show towers that have 3 of one color and 1 of the other color. For example, her first group is of four towers, each containing three yellow and one blue. In her picture, she arranged them with the staircase pattern, showing the blue cube in the bottom position, and moving up the tower in the three towers that follow. What most impressed me about Kimberly's work was her argument that described this group of towers. Kimberly states, "We knew we were done with the 3 yellow because there was 1 blue per level and we knew the couldn't be another level because the towers had to be 4 blocks in length." This was the most convincing part of any argument that I read from the entire class. I found that in this first task, it was extremely difficult for the students to articulate a convincing argument, either verbally or written. Even though the other parts of Kimberly's argument were not convincing, such as using the opposites strategy for the towers with two of each color, this one sentence was convincing as an argument for both the group with 3 blues and 1 yellow and 3 yellows and 1 blue.

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.



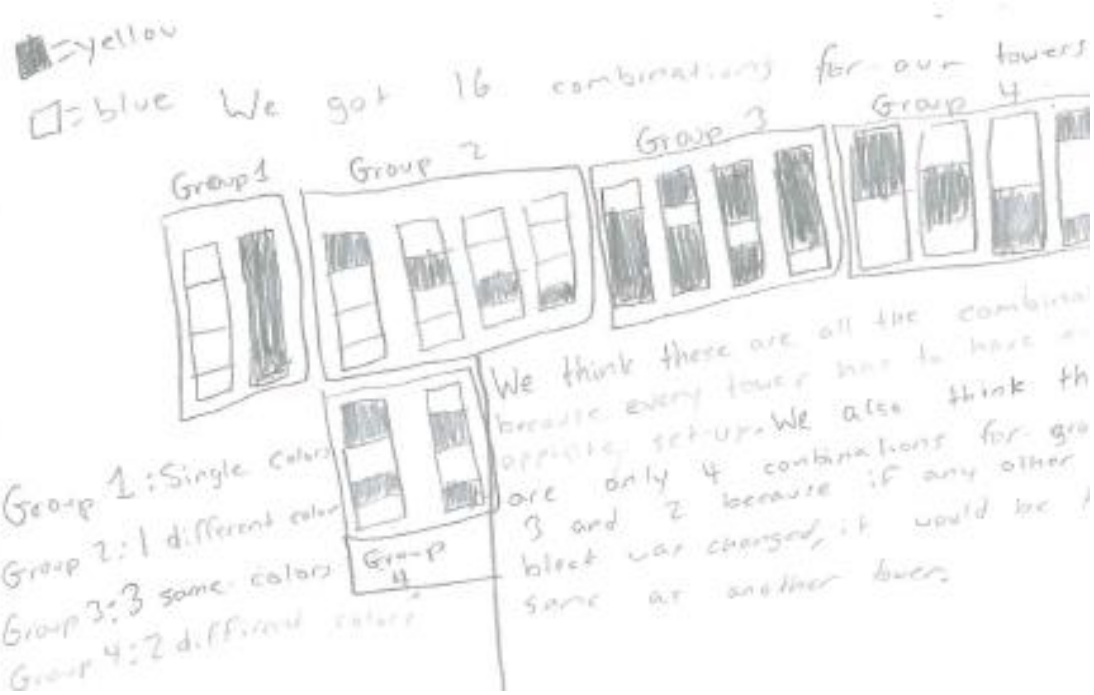

The towers you see above are all of the possib
combonations. To prove it keep reading. My partner
I started making the 4 legth tower with 3 yellows
We knew we had all the of them we went on to th
next one wich was 3 blue. We knew we we done with
the 3 yellow because there was 1 blue per level a
we knew the couldn't be another level because
the towers had to be 4 blocke's in length we did the s
for the 3 blue but there was one yellow per level. T
we moved on to make 4 length all blue towes an
all yellow tower. Since there was two 2 colors
with so we knew we could only make 2 tower
of it. Then we did the 2 blue and 2 yellow and
knew we had them all becouse we tried and tried
and tried to make more but it was impossible so we knew
we were done with that one. So we know and were abbsolutly
positive that there are 16 towers.

Harrison's work from the first task surprised me. The argument that is shown on his paper is not convincing, but while working on this task, it was clear that Harrison and his partner both thoroughly understood the task and their solution. When I first observed Harrison and his partner, they had the towers organized as they do on their paper, into 4 groups. The first group contained the solid color towers, the second group contained towers with 3 blue and 1 yellow, the third group contained towers with 3 yellow and 1 blue, and the last group contained towers with two of each color. Even though it is not recorded in their argument, their verbal argument was very convincing. For the towers with 1 blue and 3 yellow cubes, they organized this group with the blue cube in the 1st position, 2nd position, 3rd position, and 4th position. They described how they knew there could not be any more towers in this set because if they tried to move the blue cube to another spot, they would need a tower that was 5 tall. They created the same argument for 1 yellow and 3 blue cubes. They then found all the towers that had 2 blue and 2 yellow cubes. They started with 2 blues in the 1st and 2nd positions, and then moved them to the 2nd and 3rd, 3rd and 4th, and 4th and 1st positions. Because there would not be any other place to put 2 blues next to each other, they had all towers in this set. Lastly, they described the alternating towers, and the solid towers.

I was very impressed with the verbal argument that they gave me, but what most surprised me about their work was that when I returned to their group to give them a recording sheet I noticed they had decided to reorganize their towers into pairs of opposites. I asked them how they were going to convince me that there were no more opposites that could be built, and they were unsure of how to do so. I left them to think about that, and when I returned to these students, saw that they again had their towers organized in a new fashion. They now had all the towers with the top block yellow on one side, and all the towers with the top block blue on the other side. With this arrangement they held a constant, but were unable to articulate how they knew that there could be no other towers with a blue block in the 1st position or a yellow block in the 1st position.

Finally after some discussion between the pair, they decided their original organization was the easiest to describe how they knew they had all of the towers and decided to reorganize the towers how they originally had them and began recording their work and arguments. It was surprising to see one group think of so many ways to organize the towers and to hear the things that they noticed about the patterns and the ways in which they were able and unable to give a convincing argument.

Really neat!



Tommy's work from the first task concerned me. Tommy and his partner took a while to understand the task. They first created towers that were four tall and lined up next to each other to spell the word "Huskies," our school mascot. After further clarification of the task, Tommy and his partner then started to go through the towers they had created to get rid of duplicates and ended up with 14 towers. When talking to this group, they had no idea if they had found all of the towers or not and did not have any way to tell if there were more towers. I suggested that they try to group the towers in some way where they could tell if they had them all. Tommy and his partner came up with the groupings that are recorded on his paper. It appears that the groups do not have any meaning and there is not any sort of convincing argument. Tommy and his partner did not have any strategy to build towers and really did not seem to understand how to check if they had them all. Their grouping is random and doesn't show any understanding of what the task was asking, which concerned me.

**Building 4-tall towers, selecting from 2 colors**

You have two colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates. (Remember that a tower always points up, with the little knob at the top.) Record your towers below and provide a convincing argument why you think you have them all.


I think we have every Possible Combonation becaus tested out all of th combo's

## Cycle 1: Building Four-Tall Towers, Selecting From Two Colors

### Reflection for Cycle 1

I was impressed with my students work during this task. I really did not know what to expect, as it is not often in our structured curriculum that we engage the students in tasks such as this one. I often do engage the students with group work during our daily lessons, but it is typically smaller explorations that align with the topic of the lesson. I think the students really enjoyed having the opportunity to work on something outside of the curriculum with another student who is of similar ability.

This task showed me the variety of thinking and ability that the students in my class have. Many groups used the "opposites" strategy to make sure that they had pair for each tower that they created, but a few groups were more sophisticated in their reasoning. These groups generally used parts of a proof by cases, showing groups of towers with 1 of a particular color, or 3 of a particular color.

The most challenging part of this task for most of the students was writing a convincing argument. Generally students were describing pairs of opposites, but that is not convincing to describe how they know they have all of the towers. I realized that we often ask students what they did, instead of why, or how they know they have a complete solution. After the completion of the first task, I found myself restructuring activities in class to allow the students to generate explanations that were not based on telling what they did, but instead explaining why they chose a certain operation, or describing how they know their solution is complete.

# Cycle 2: The Pizza Problem

## Class Description

The class who I chose to implement the Cycle 2 task with is a regular level sixth grade math class at Matawan Aberdeen Middle School in Cliffwood, New Jersey. The class is held daily from 11:58 a.m. to 1:12 p.m. and is the students' third academic block of the day. The class is made up of 13 boys and 4 girls, with varying levels of ability. Out of the seventeen students in the class, four are basic skills students who receive additional support from a special education teacher who comes into our math period every other day for 36 minutes. In addition, several other students in the class have 504s and plans implemented through Intervention and Referral Services.

The student work in this cycle is from the same class as the student work in Cycle 1. The students were given one class period, but struggled to come up with complete solutions and written arguments as this task took them longer to figure out than the first task.

**Impressed**

Griffin's work was the most impressive of the solutions that I saw in this class. He clearly holds a constant to find the pizzas that contain pepperoni, and works his way from pepperoni pizzas with 4 toppings, to 3 toppings, followed by 2 toppings, and 1 topping. He then holds mushrooms constant, and lists the pizzas that contain mushrooms, leaving out the pepperoni pizzas that he had already listed. It was clear that he forgot the pepper and sausage pizza originally, but later included it as his sixteenth pizza. Even though Griffin does not have a convincing argument, his organization was the clearest and the easiest to follow how he knew he arrived at his solution of sixteen pizzas. He had an organized approach to solve this problem.

1. Pepperoni, mushrooms, sausage, peppers
2. Pepperoni, mushrooms, sausage
3. Pepperoni, mushrooms, peppers
4. Pepperoni, sausage, peppers
5. Pepperoni, sausage
6. Pepperoni, peppers
7. Pepperoni, mushrooms
8. Pepperoni

9. Mushrooms, sausage, peppers
10. Mushrooms, sausage
11. Mushrooms, peppers
12. Mushrooms
13. Pepper
14. Sausage
15. Plain.
16. Sausage and peppers

We know we have 16 possibilities because we listed.

yo!

The work that most surprised me for the pizza problem was Harrison's. The chart that Harrison created was an interesting way to find the solution of 16 pizzas. He explained to me the top row of the table was each of the pizzas with a single topping. In the second, third, and fourth rows of the table, he created pizzas with 2 toppings and placed them in the "correct" column based on one of the two toppings in the pizza. Other spaces are left blank, because these pizzas would be duplicates of pizzas that he already recorded. I was confused by Harrison's organization, but it made complete sense to him, and allowed him to determine the correct solution. The final row of his table shows pizzas with three toppings, which don't really fit into the organization in the table that he originally told me about. The pizza with all four toppings is off to the side. This was an original way to organize the pizzas, but I think it would be confusing for anyone else to follow, if Harrison wasn't there to explain it. His written argument is not convincing or clear, but his organization was surprising and unexpected.


I think that these are all the possible combinations because on my graph, all of the pizzas are organized in a pattern. In my graph, it shows combinations for toppings, but it also prevents from any toppings to be repeated.

Carlos's work most concerned me with this task. His work on the front of the paper suggests that he began by listing a plain pizza, and then started to list pizzas with peppers and then appears to add a topping each time. He then does the same but starts with mushrooms. It is unclear in his writing what each specific pizza is, as each topping is only separated by a comma, and perhaps each pizza is separated by the same comma. When examining Carlos's work on the back of his paper, it appears that Carlos thought there would be 4 pizzas that started with each topping. He ends up with duplicates (peppers, sausage and sausage, peppers). The idea of adding a topping each time is a good way to think about this problem, but he doesn't notice that the order of the toppings does not matter. It is troublesome that Carlos does not recognize that there is only 1 pizza that contains all four toppings.

Carlos does not form any sort of argument for this task, and it appears he may have run out of time completing his pizzas, as he has a spot numbered 1-4 on his paper that is not filled in. Carlos and his partner struggled with how to start this task and how to determine the solution.

Plain

Pepers, Pepers and sausage,
Pepers, sausages, mushrooms,
Pepers, sausages, mushroom, pepperoni

Sausages, Sausages, Pepper, Sausages, Peppers
mushrooms, Sausages, Peppers, mushroom, Peppe

Plain

1 Peppers
2 Peppers, Sausage
3 Peppers, Sausages, mushrooms
4 peppers, Sausages, mushrooms, pepperoni

---

1 sausages,
2 sausages, pepper
3 sausages, peppers, mushroom
4 sausages, peppers, mushroom, pepperoni

---

1 mushroom
2 mushroom, pepper
3 mushroom, pepper, sausages
4 mushroom, peppers, sausages, pepperoni

---

1
2
3
4

# Cycle 2: The Pizza Problem

## Reflection for Cycle 2

In general, my students found this task more difficult than the first task. I think the lack of manipulatives made it difficult for them. In the first task, they could change their organization or notice duplicates more easily. If they wanted to rethink their organization in this task, they would have to rewrite their solutions, and getting them to write is usually a struggle. This made it harder for the students to see other ways to organize their work, which may have made it easier to see if they were missing any pizzas.

Many of the student groups struggled to find the 16 pizzas and spent time using a guess and check solution method. Some of the student groups had an organized way to find the pizzas, focusing on the number of toppings, but had trouble knowing if they had all of the possibilities in that category. Many students thought it was important to draw a pizza to go with their description or include cheese and sauce in their description as well. When reading their solutions this made it more confusing to find which pizzas they were missing or where the duplicates were. Since most of the students did not organize their solutions in any way, there were not groupings, and therefore they had more difficulty writing a convincing argument.

I learned that my students better understand something when there is a hands on component. They are visual and tactile learners. A lack of manipulatives is troublesome for them, and organization is a weakness. It seems as though they do not plan out a solution method before they start, but just want to dive in and come up with a solution. I appreciate the excitement and eagerness that they took on the task with, but it should be stressed that they organize their ideas and propose a problem solving method before they start to solve this or any problem of :his type.

# Cycle 3: Building Three-Tall Towers, Selecting From Three Colors

## Class Description

The class who I chose to implement the Cycle 3 task with is a regular level sixth grade math class at Matawan Aberdeen Middle School in Cliffwood, New Jersey. The class is held daily from 11:58 a.m. to 1:12 p.m. and is the students' third academic block of the day. The class is made up of 13 boys and 4 girls, with varying levels of ability. Out of the seventeen students in the class, four are basic skills students who receive additional support from a special education teacher who comes into our math period every other day for 36 minutes. In addition, several other students in the class have 504s and plans implemented through Intervention and Referral Services.

The student work in this cycle is from the same class as the student work in Cycle 1 and Cycle 2. During this implementation, Dr. Landis and the other teachers from my cohort were present. Dr. Landis introduced the task and the students worked as many adults observed and questioning their thinking and reasoning.

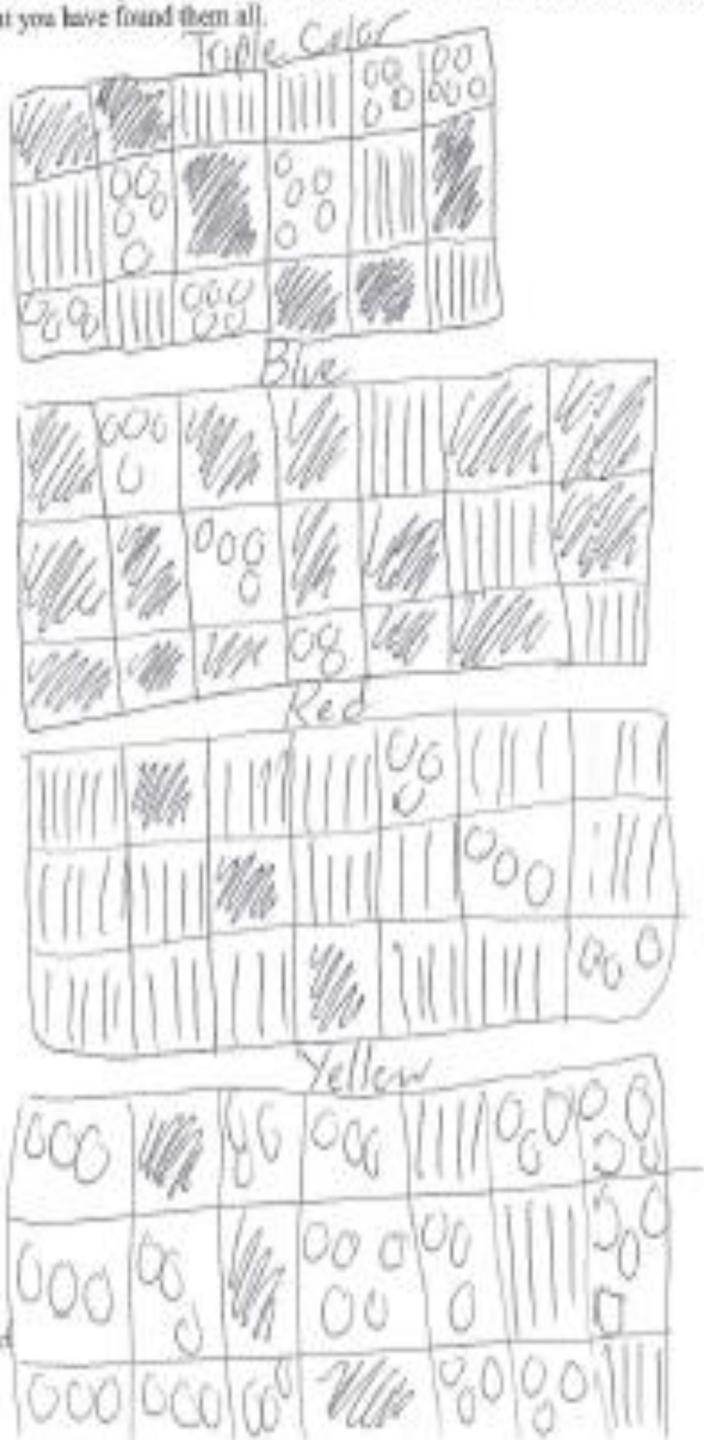
Cristian's work most impressed me for this task. Cristian created four groups of seven and one group of six. He included a key that has coloreded, striped, and dotted towers to show the three different colors. The key made it clear to see the differences between the towers that he created. Though Cristian's written explanations are not completely convincing, it was clear from speaking with Cristian and his partner that they had proof as to why there were exactly 27 towers.

At first Cristian and his partner only had the three groups of seven towers that they called the Blue, Red, and Yellow groups. Each of these groups started with the tower of that single color. They then replaced each level of that tower with one of the other two colors. For example the Blue group started with the solid blue tower followed by the yellow, blue blue tower, blue, yellow, blue tower, and the blue, blue, yellow tower. Cristian and his partner ssaid there were no other towers with blue and yellow because they moved the yellow down systematically to each position and the towers were only three cubes tall. This set was completed with the blue and red towers using the same argument as the blue and yellow towers. They repeated this argument to complete the Red and Yellow groups and arrived at 21 towers. This arrangement was very impressive to me.

For a while, Cristian and his partner were convinced there were only 21 towers, but then they realized they were missing the towers that contained all three colors. They labeled this group the "triple color" group. Cristian explained that for this group, they organized the towers by top color, and then switched the middle and bottom cubes to create two different towers. For example, the they created the blue, red, yellow tower, then held the blue constant in the top spot and switched the positions of the red and yellow cubes to create a second tower, the blue, yellow, red tower. Because there were only two other colors after the first cube was held constant, there were only two triple color towers that had blue as the first cube. They repeated this explanation for the towers with red and the first cube and yellow as the first cube. Though his writing doesn't fully explain this argument, he does describe switching the bottom and middle blocks after organizing by the top color. Their verbal argument was a convincing explanation of where the six towers in this group came from and why there were not any other towers in this group. Cristian and his partner were then convinced there were 27 towers.

Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and provide a convincing argument that you have found them all.

Key=
Blue-
Red-IIII
Yellow-




Explanation

For the Triple Color group we organized them by top color and switched around the bottom and middle.

For all the other groups, we placed the single color blocks in three different positions and also included the solid colored towers.

Aidan's work most surprised me during this task. Aidan generally struggles in math class, but is always willing to put in his best effort, as he did with this towers task. Aidan and his partner Tommy worked slowly to build the towers, without much organization. After they had several towers created, and were stuck, they created groups of towers that had the same color as the bottom block, but they were still missing a few towers. I asked them how they knew they had all the towers that fit into each category, and they started to explain about moving some of the other blocks around and finding pairs. They decided on a way to organize the blocks and Aidan was going to work on one of the groups and Tommy another. After this time, they both had organized them in different ways. After talking with some of the adults in the room, Aidan and Tommy both noticed that in each group had three towers that were had red tops, three towers with blue tops, and three towers with yellow tops. Aidan was then able to articulate that there were three towers that had red bottoms and red tops because there were three colors to put as the middle block. This argument was the most impressive solution that Aidan formed while completing any of these tasks. Though his writing doesn't support what he said during the activity, I was very surprised by his work and persistence on this towers task.

Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and provide a convincing argument that you have found them all.


I know I got all of them because all The groups got the same amount of stacks.

Jenny's work most concerned me because it is unclear from the key she used exactly which towers pertain to which combination of colors. In her key, both red and yellow are represented by an unshaded square, whereas blue is a filled in square. Later in her drawings, she starts including letters to note which color the block is, R for red, and Y for yellow. Since this labeling technique isn't used in Group 1, it is impossible to tell what the towers in Group 1 look like though an examination of her work.

Even though her recording of the towers is concerning, there are some strong elements to her explanation. She mentions grouping one of her groups (group 2) into red tops, yellow tops, and blue tops, and then switching the order of the middle and bottom cubes to create 6 towers in that set. Her other descriptions are unclear, as her key is unclear, but there does seem to be the beginnings of a convincing argument.

Find all possible towers that are three cubes tall, selecting from cubes available in three different colors. In the space below, show your solution and provide a convincing argument that you have found them all.

blue red yellow

[redacted]

What I Did was I took blue, yellow, and Red for group 1. I took B and Y and I revesed them to get my 6 combinations and for number two I did the same thing. I think I found all my combination's becowse their are only three colors and It impossibl that their could be more combination's for group 7. and for group 2. their are no more combinations because I grouped them into 3 groups and on the top are red, red / Y, Y / blue, blue / and then just reversed the colors.

# Cycle 3: Building Three-Tall Towers, Selecting From Three Colors

## Reflection for Cycle 3

The implementation of this task was easier than I had anticipated. I thought this task was going to be confusing for my students because of the number of solutions and the complexity of using three different colors of cubes. The students really surprised me with their solutions and explanations. After the pizza task where many of them struggled, the students embraced having manipulatives once again as a helpful tool to solving this problem.

During this lesson the other teachers from my cohort were present. I think the students were looking forward to sharing their solutions with new adults and were excited about the idea of impressing them with their knowledge. Even my students that regularly struggle with tasks in class were successful and able to come up with convincing verbal arguments for the ways that they arranged their towers.

I think that the student pairs used during this implementation were well organized as the students remained focused and for the majority, worked cooperatively with their assigned partner instead of just next to them. During the first two tasks, there were more groups of students who worked parallel to each other instead of cooperatively. I think the countless group activities and the implementation of these tasks have helped the students' work ethic in cooperative groups grow. They realize the sense of shared responsibility for the outcome, but also an individual responsibility to complete their part of the assignment.

# Course and Implementation Reflection

Throughout the course I learned new and interesting things about mathematical reasoning, my students thinking, and creating convincing arguments. There were times where I was impressed, confused, concerned, or surprised by my students reasoning abilities. The students truly enjoyed these tasks as it gave them an opportunity to take a break from the curriculum and work on a challenging problem that had many different ways in which they could arrive at a correct solution.

I learned specific names for the types of mathematical proofs that can be given for situations such as the tasks that we implemented. It is interesting to determine that one type of proof can extend to many problem-solving situations. Though my students may not realize they are giving a proof by cases or an inductive proof, this course had broadened my knowledge of the type of reasoning that my students are engaging in and given me ways to describe their thinking and reasoning styles.

I have learned that my students have strong reasoning skills when given the opportunity to explore concepts that have multiple ways to reach a solution. It is wonderful to see that even students who struggle the most were able to use their own reasoning abilities and build on ideas with a partner to explain how they arrived at a solution. The success that these students experienced as a part of these tasks made them more confident and excited to work on the tasks later in the semester.

When I completed the first task at Rutgers, my argument consisted of steps describing how I organized the towers to show that I had systematically found all of the possibilities. I did not completely understand that a convincing argument more focused on it takes time - keep expecting it and your students will get better at doing this!

how you know you have all of the possibilities instead of the steps that were taken to get there. I think my students are still struggling with the development of writing a convincing argument, but they were mostly able to give very convincing verbal arguments by the completion of the third task. I think that I better understand my students' struggles and frustrations as well as successes with reasoning tasks due to the implementation of these tasks. These are things that I will continue to work on with them and build so that they will be confident in their mathematical reasoning abilities and become persistent and successful problem solvers in the future. Super!

Appendix K

*Descriptions of Recursive Arguments from Gang of Four Assessments*

| Teacher | Pre-Assessment | Post-Assessment |
|---|---|---|
| T4 | Jeff is seen lining up all of the towers into a pattern that shows a yellow block moving diagonally across the towers. | When Jeff and Michelle’s work was displayed I noticed their towers were arranged in what looked like a recursive pattern (a checkerboard looking) red dominant towers with one yellow cube arranged in a diagonal pattern across it. |
| T9 | Then Stephanie gives her reasoning. She drew out the pattern. She was trying to convince Jeff that there are 8 ways to build towers of height 3. She drew out her patterns. First, she began drawing out all the towers with 0 B. There was only 1 - RRR. Then she listed out the towers with 1 B. There were 3 - BRR, RBR, and RRB. She later explains that she used the strategy of moving the B down a level each time to make sure she did not miss any combination. Then she listed all the combinations using 2 B stuck together. There were 2 - BBR and RBB. Next she listed all the ways of 3 B. There was just 1 - BBB. Last she thought of all the ways to have 2 B not stuck together. There was just 1 - BRB. | Stephanie uses a systematic listing strategy to solve the problem. She begins by building a tower 3-high with all red. Then, she lists towers with 2 red and 1 blue. She uses a recursive pattern moving the blue block down until she exhausted all the positions. |
| T6 | "Stephanie: Here is one red/red/red, blue/blue/blue and then I go like red/blue/blue, blue/red/blue "Stephanie continues to work with the colors this way and "moves" blocks down. Stephanie's argument was very unclear. I think Stephanie is doing a good job explaining that specific example; however, her reasoning does not convince me that she could use her method to make a prediction with x amount of blocks. | Stephanie talks about moving the blue blocks by position and convinces her peers when she shows them how she has moved it to each possible position. |

*Sources:* Pre- and Post-Assessments for Gang of Four Video

| | |
|---|---|
| **Description: Transcript of teachers working together on the four-tall towers problem; 9/7/13**<br>**Advisor: Professor Carolyn Maher**<br>**Location: Graduate School of Education, Rutgers University** | **Author: Phyllis J. Cipriani**<br>**Verified by: Victoria Krupnik**<br>**Date Verified: 1/3/16**<br>**Page 1 of 24** |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| 1 | 00:00:00 | R1 | Okay! Um, probably what we want to do very, very briefly is just tell you a little bit about ourselves. Um I will tell you about me and Alice will tell you a little bit about her. Then we are going to get right into the task which I think you will enjoy doing. And it will be the same task that you will be doing with your students during the month of September.<br>Right now, I am quote retired, I say retired because here I am teaching.<br>But I was a school administrator for many years. I was a principal in Holmdel and Colts Neck. I was a director of math and science in Highland Park.<br>I ended up as an assistant superintendent in Holmdel<br>I am happily retired and working at Rutgers.<br>So that's me, Alice… |
| 2 | 00:53:00 | R2 | My trajectory started off with Judy and we got our degrees the same year.<br>And worked together all those years and like all of you now, I started as a teacher myself. Um, first as a school teacher and then middle school as a principal.<br>In a school in Princeton, New Jersey<br>And somehow morphed from doing that to do work here at Rutgers into the program here. And so I have been here many years. And also officially retired but same as Judy like working on projects like this and with some of our research projects<br>Many of you have looked at our video tapes that came out of our research of longitudinal studies that were some at Judy's school in Colts Neck some of them in a town called Kennilworth where Robert Sigley who is back there was there and he began was an elementary student and just can't seem to go! [laughter]<br>He is finishing up his doctorate here in math education. And so one of the challenges to you as you get the video tapes is to look for him [laughter]<br>They call him "Bobby" so that will be something to do!<br>We will talk more about that later. But this is so nice for |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| | | | Judy and me to get to know you people and to get to visit with you in your schools to and<br>Learn together about what your students are doing as we study together what... |
| 3 | 00:03:22 | R1 | I think you will find that this is going to be a wonderful class for you in terms of looking at your own learning and looking at the learning of your own students. I know it is my favorite thing to do is working with teachers.<br>Looking at your own learning<br>I think Alice too<br>Do we have unifix cubes? |
| 4 | 00:03:42 | R3 | Yes |
| 5 | | R1 | How many of you have worked with unifix cubes in your classroom? Better question: How many of you have not worked with unifix cubes? Okay, alright not a problem. Um, I think what you will do you is you will get to experience what you can do with those today; here. And in your schools. Am I right Marjory, they should have unifix cubes all the districts? |
| 6 | 00:04:08 | R3 | I think so. I will have to talk with Alice in the morning I am not sure about the district of Berkeley Heights. |
| 7 | | R1 | How about Matawan-Aberdeen? |
| 8 | | R1 | Oh, Matawan-Aberdeen? Your district was not in the PEMSM project last year. |
| 9 | 00:04:31 | R3 | I will have to check with Linda and Amy. |
| 10 | 00:04:35 | R1 | Okay. Okay. Every district should have unifix cubes. This is the fourth year of the PEMSM project. So, you want to search down for the teacher who has the unifix cubes and then you want to borrow them so that you can do the tasks. Okay, these are…can I borrow a set?<br>[instructor used the cubes to show teachers]<br>These are called unifix cubes. They are cubes that link together and when you get the bag you can check it out yourself. But they snap apart and they snap together. And there is a little chimney that we call here that we always want to keep facing top. Okay, because the task is going to involve you building towers of unifix cubes. Do we have the task? |
| 11 | 00:05:37 | R3 | We do not have the task. |
| 12 | 00:05:39 | R1 | We don't have the task. We will have to verbally say it. Okay. What you are going to do is you are going to select from the two colors in this bag and you are going to build with a partner so you are going to pair off as many towers that are exactly four cubes tall that you can build using or Selecting from these two colors. Now the task is beyond |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| | | | that once you build the towers, this is the key thing. You want to convince your partner that you built all the towers that are possible without having duplicates.<br>If you have convinced your partner, you are going to kind of contact Alice or me or I don't know if Marjory is going to be circulating. |
| 13 | 00:06:24 | R3 | I will be here for a minute. |
| 14 | 00:06:27 | R1 | But whoever is walking around the room, You are going to pull one of us over and you are going to convince us that you have found all the possible towers that you can have four cubes tall selecting from these two colors.<br>Alright. And once you can convince us, were going to ask you to write down your solution. Okay, So you are going to record your solution and your convincing argument and then when everyone is done, we are going to share your solutions with one another and you are going to see all the different ways that you have found for solutions to this task.<br>Before you begin is there any question of what the task is asking you? |
| 15 | 00:07:07 | UCT | You said four high, right? |
| 16 | 00:07:10 | R1 | Four tall, Yep, Okay you can begin. [Teachers begin to work-noisy!] |
| 17 | 00:07:15 | R2 | Your solution and your way of convincing us you know is probably going to depend on your towers… so make sure your towers are written for right now….[inaudible based on noise] |
| 18 | 00:08:01 | | [Teachers begin to work in pairs.] [R1, R2, R3 circulate to monitor work of the pairs of teachers for 35 minutes. One camera follows R1.] [Multiple conversations take place. Some partial conversations follow, most of which are inaudible until the entire group is called together to discuss which begins on line 205]. |
| 19 | | R1 | That's good you are checking for duplicates (saying to first group R1 stopped at). [This group has the towers organized using the opposites reasoning strategy][Gives thumbs up!] |
| 20 | 00:10:02 | R1 | [asks question to second group stopped at] How many did you find? How do you know you have them all? [response is inaudible] |
| 21 | 00:11:06 | R1 | Okay so let me see if I understand.. Okay and if you look at this one here, okay. So you are pretty convinced you have them all here? And….[elevator strategy used] I guess…So this was a little trickier. Talk to me how do I know you have all those? It is not clear for me. |
| 22 | 00:11:30 | UCT | So first we started with two reds and two yellows one of a |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| | | | color. |
| 23 | | R1 | Okay. |
| 24 | | UCT | Then we have one yellow and [inaudible]. Using the same technique ... two reds and two yellows then we put three reds |
| 25 | 00:11:40 | R1 | Okay. |
| 26 | 00:11:47 | UCT | So then we took this red and moved it to the second position. |
| 27 | | R1 | Okay, so let me see. So you said you moved it to the second position. |
| 28 | | R1 | So you are pretty convinced you have them all here? (GH nods yes). This is a little tricky [getting the two of one color] Talk to me, how do I know you have all of those? |
| 29 | | GH | First we started with two reds and two yellows. Then one yellow, moved the second red down one keeping the first red on top. There can't be another one with 2 reds on top. So then we took this red and moved it to the second position. That's one that I already had. |
| 30 | | R1 | Oh, I see, got it. So those were the reds in the second position. Then you did all the possible towers that you could do that. But I do not see the second red in this position. Where is it? Does this fit in this group? |
| 31 | 00:12:23 | UCT | It does because then we moved our starting red to the third position. We already had we put it in the third position [inaudible] because we already had a one top red. |
| 32 | 00:12:38 | R1 | I see got it. Okay, okay. I think you would give me a similar argument for these. So why don't you, it looks pretty neat. These are going back in the bag so Why don't you record how you did your towers on paper and write me a convincing argument that you just gave me. |
| 33 | | UCT | Okay. |
| 34 | | R1 | [speaking to the third group stopped at] What are you guys doing? Could we put the red in another different position? |
| 35 | | UCT | There can only be four blocks in the towers. |
| 36 | | R1 | Okay, okay. Now the question seems pretty silly to you, but when you ask it to your students, it is a good question to ask. Towers 4 tall, why can't there be another position? |
| 37 | | MM | It also gets them thinking about what if there was another position. |
| 38 | | R1 | Absolutely, absolutely. Okay good. So keep going |
| 39 | | CD | So then I used 2 reds |
| 40 | | R1 | Okay |
| 41 | | CD | And so to approach that I kept the first red always on the bottom. |
| 42 | | R1 | Pretty neat |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| 43 | | CD | So I can only get 3 options with where the red can go. |
| 44 | | R1 | Okay. |
| 45 | | CD | Then I switch red to the third position. |
| 46 | | R1 | Ahhh. |
| 47 | | CD | So I get all the red on the bottom |
| 48 | | R1 | Okay. |
| 49 | | CD | [inaudible response] So then I went to 1 yellow with 3 reds and moved my one yellow to different spots. |
| 50 | 00:14:21 | R1 | Very neat. Do either of you teach algebra? |
| 51 | | CD | I do. |
| 52 | 00:14:30 | R1 | Holding a constant is what you did. That is a really good strategy. The way you solved it. You should really Point that out. |
| 53 | | CD | [inaudible response] I can see a lot of children doing that. |
| 54 | | R1 | Actually you do that and we talk about that in every class. This is something elementary school children don't see it right away do as well to and that is holding constants. And it makes the problem simple. Okay, good. Now you did something different? |
| 55 | | MM | Well I did the same thing. I also did two reds stuck together and then two yellows stuck together and then separated on the top. [inaudible details] |
| 56 | | R1 | Okay. Got it. |
| 57 | 00:15:37 | MM | Two reds stuck together and two yellows on top. Two yellows stuck together and put them together. And then they keep going…[inaudible] |
| 58 | | R1 | Okay, okay. |
| 59 | | | Yellow in first. Yellow in the second. Red in the third. |
| 60 | 00:16:00 | R1 | Okay, so how many towers did you find? Because you found it a little different than what she did. |
| 61 | | UCT | 16. |
| 62 | | R1 | You both got 16. You both convinced of it? |
| 63 | | R1 | Yes, you are. Okay, What I want you to do and you both are going to record differently because these look different than what she did. Record on paper exactly how your towers look and both of you write a convincing argument like what you just told me. So I know that you found them all, okay! Good job! |
| 64 | | R1 | [fourth group stopped at] Oh, okay. [inaudible] |
| 65 | | UCT | So we know once we found the opposite. [inaudible] Then we started grouping them together. With 2 yellows and 2 reds are. |
| 66 | | UCT | We were doing the opposite and no other way. |
| 67 | | R1 | Let me ask you a question [inaudible] |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| 68 | | UCT | We don’t have 3 colors. |
| 69 | | R1 | No, so that is pretty convincing. That means the only colors are exactly two colors. How do I know that you found all the towers? What do you call this? |
| 70 | | UCT | Opposites. |
| 71 | | R1 | Opposites, okay. |
| 72 | | UCT | We should show them next to each other. We have a yellow on top.[inaudible] |
| 73 | | R1 | You have a tower [inaudible] |
| 74 | | UCT | Originally that is how we did it. [inaudible-Showing Dr. Landis how they formed it] We went to the stack because we couldn’t go any more down. We have a yellow on top. [inaudible] We already had yellow. Instead of the yellow.. |
| 75 | 00:17:21 | R1 | I see.[she asks a question-inaudible] |
| 76 | | UCT | So we did it with the other guys. All our reds; second one yellow.[inaudible] Second one has all our reds and then third one yellow and the fourth one doesn’t have it. |
| 77 | | R1 | Can there be a yellow in the top position? |
| 78 | | UCT | No |
| 79 | | R1 | Okay, let me tell you if you said yeah..The second position, then the third…[inaudible]. |
| 80 | | UCT | Well we already did all of our colors. |
| 81 | | TD | Oh yeah we did. |
| 82 | | R1 | What’s confusing is let me tell you what is confusing to me. You say you have them here so you didn’t have to repeat them here. I want you to think about rearranging the towers so that you can convince me because I wasn’t working with you, so you can convince me that you have all the towers there are and there are no more. Okay, because I am not totally convinced yet. Call me back when you got it. |
| 83 | | TD | Okay. |
| 84 | | R1 | [R1 returns back to the first group stopped at] Look at this. What is that? |
| 85 | | UCT | We did it this way and we came up with 16 and then. Two one one high. |
| 86 | | R1 | Where did you get 2? |
| 87 | | UCT | Well, one high. We were trying to justify. |
| 88 | 00:18:52 | R1 | Ahhhhhhhh Okay, that’s very interesting. Okay. |
| 89 | 00:20:21 | UCT | So we are adding two additional ones Every time you are adding an extra position. So for the ones that are one high, when you add a second block, you have a yellow, you could add either a yellow or red again; and to that red you add another yellow again. [inaudible] So you are adding two each time. |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| 90 | 00:20:49 | R1 | That's beautiful. What you are doing is called an inductive argument. Remember that. Inductive argument is what you just explained to me is inductive. And what I want you to do is When we share solutions, explain how you can either put a yellow or a red here and a yellow or red here. Yeah, and that's really neat and you can carry that from here to here. Very very nice, okay. So does this look familiar? You said something about doubling it? |
| 91 | | MC | You're doubling it. Two times two is four and that should be two squared. |
| 92 | | R1 | Now you said double and then you said it was squared? Is that what you said? |
| 93 | | UCT | Yeah, Exponents. |
| 94 | | R1 | Exponents, okay. |
| 95 | | NL | Well you could…There are two colors and there are five so two to the fifth is 32. |
| 96 | | R1 | So Which is it? Two to the fourth to get 16 or doubled? |
| 97 | | MC | Both. |
| 98 | | R1 | How about if I build towers 5 tall? How many would you get? |
| 99 | | NL | 32 |
| 100 | | R1 | And how did you get 32? |
| 101 | | MC | Right you add 2 every time. |
| 102 | | R1 | Okay, what if you did 2 to the fourth to get 16. How would you get 32 with five towers? |
| 103 | | NL | There are 2 different colors and 2 to the fifth power are 32. |
| 104 | | R1 | OH. So now you are telling me you are using exponents and…That's pretty good.[inaudible response] |
| 105 | | R1 | Now again, [inaudible] you may or may not get that far. Okay, and there are people in this room that may not be able to see the link between the number of towers Based upon how many colors and how tall so that is really neat. |
| 106 | | UCT | Yeah! |
| 107 | | R1 | What I want you to do is somehow make a convincing argument on paper in writing. Like how do you know that there is not another member of this group? How do you know that there is not another member of this group? How do you know there isn't another member of this group and how do you know there isn't another member of that group How much did you get? |
| 108 | | UCT | 16 |
| 109 | | R1 | Five, ten, fifteen, I only see 15. Could there be another one? Write a convincing argument for why each group can't have another member. |
| 110 | 00:23:34 | R1 | [R1 goes to the fifth group] how about you guys. You think |

| Line | Time | Speaker | Transcript |
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| | | | you have them, explain to me what you did and Why can't there be another tower with one yellow? |
| 111 | | CP | We started with red and yellow… |
| 112 | | R1 | Could there be another one? |
| 113 | | UCT | No because there are only 2 colors. |
| 114 | 00:24:25 | R1 | That is pretty convincing. Okay, keep going. |
| 115 | | CP | Then we started with the yellows and we took one red |
| 116 | | R1 | Okay. Now when you have opposite pairs and that is a good strategy but what you are doing is pretty confusing. But tell me why can't you have another red in the fifth position? |
| 117 | | CP | Because then it wouldn't be four high. |
| 118 | 00:24:30 | R1 | Now This sounds like a silly question to you but it won't be silly to your students. And you are going to want them to say to you; you are only doing them four tall so there is no fifth position; So then, this is very convincing that you have all the possible towers that have exactly 3 yellow and one red. Okay and I bet you can give me a similar argument. But I am not sure what this is all about and why are they partnering and how do I know you have all of them? |
| 119 | | UCT | We have opposite pairs so we have two red here [pointing to two red blocks over two yellow blocks], two red, two yellow and then we just flipped it. |
| 120 | | R1 | Okay. |
| 121 | | UCT | And we did the same thing here, we move red down and Started with one yellow and started to take the two red down |
| 122 | | R1 | Okay. |
| 123 | | UCT | To this position so it is kind of like a rebuilding by moving the reds down. |
| 124 | 00:25:45 | R1 | So what you are talking about is called a recursive argument. Moving the positions okay. |
| 125 | | UCT | Then we had this same thing and then we had opposite pairs. [inaudible] |
| 126 | | R1 | Okay |
| 127 | | UCT | Red here red here and yellow here[inaudible] |
| 128 | 00:26:31 | R1 | MMhh |
| 129 | | UCT | And it matched up here. |
| 130 | | R1 | I can see this, But you know it is not easy for me to see that. I can see this, and I can see this, but I think you need to convince me of this. |
| 131 | | CP | That was one of the things we struggled with. We were like how can we set it up? If we take each of the opposite pairs and compare them. |

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| 132 | | R1 | But then I would say to you: How do I know that there isn't another pair that goes into this group? So what I am going to do is challenge you to do is to see if you can rearrange them and call me back when you are done. |
| 133 | 00:27:30 | R1 | [Back to first group] in the fourth position; there is no other way to put a single red or yellow so you know you have them all. So it is a convincing argument, not just telling what you did but why. Okay, good. |
| 134 | | R1 | [R1 goes to a fifth group in the room] Do either of you teach algebra? |
| 135 | | UCT | No, pre-algebra. |
| 136 | | R1 | You held a constant, that is what you did. which is a very nice strategy. [inaudible] Talk about how you solved it and what you did. |
| 137 | | UCT | The second set is bottom yellow. |
| 138 | | R1 | What I wanted you to do is record on paper how the yellow is.. I don't |
| 139 | | UCT | I tried to start with a pattern. |
| 140 | | R1 | Ahhh. Okay. You are not allowed to have 2. Okay alright. Are you convinced with hers? |
| 141 | | UCT | Yeah. |
| 142 | | R1 | Do you understand what she did? Because she did it. |
| 143 | | UCT | Yeah, I tried to do .. the pattern that I tried to do; a pattern and It looked very strategic |
| 144 | | R1 | What I want you to do is record your solution exactly the how you did it here and write a convincing argument as to why you have them all. Not how you got them, but why there aren't any more towers. |
| 145 | | JLB | Now when you say record it do you mean like draw the diagram? |
| 146 | | R1 | However you need because these are going back in the bag. Whatever you need to do. Have it on paper so that your teacher still knows how you did it |
| 147 | | R1 | [R1 returned to third group and reads solution] You are telling me what you did. How do I know why those groups are complete. [inaudible responses] That's very nice how you are using the notation. |
| 148 | | R1 | [R1 returned to fourth group] Let's see how your writing is. It is neat how you recorded it. You recorded it vertically. You didn't write red or yellow. You wrote Y and R. that is very nice. You are not going to tell them to do it. I like the way your towers are built. Some of your children will write it very differently so you want them to write their own. Two towers that's how I know they were |

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| | | | exhausted. Good. |
| 149 | | R1 | This is nice but you are not going to say what you did. |
| 150 | | R1 | I like the way you are showing me the groups. |
| 151 | | R1 | Good. |
| 152 | | R1 | Did you write a convincing argument? |
| 153 | | UCT | Not yet. |
| 154 | 00:32:37 | R1 | Each person has to do their own writing. Everyone has to write their own, no, no no! |
| 155 | | R1 | [inaudible]When you do this task with the children. Each child even though they are in pairs, is going to have a sheet to record their solution. Because even though they are working together, you are going to be very surprised…what they write may be very different. And even the way they solve it may be very different. |
| 156 | | UCT | Okay. |
| 157 | 00:33:14 | R1 | Okay now what I want to do is see your towers on paper somehow. These go back in the bag so I want to see how you solved it. |
| 158 | | UCT | Okay. |
| 159 | 00:33:26 | R1 | Why don't you get all towers down first. And all four of you, nobody is on I phones. Everyone is doing it right? |
| 160 | | UCT | Are you checking? |
| 161 | | R1 | Am I checking what? |
| 162 | | R1 | (Back to group 2) Do you know if anyone in your district has the cubes? |
| 163 | | UCT | I only have a package of 300 so that is not going to be enough. |
| 164 | 00:33:43 | R1 | And How many in your class? |
| 165 | | R1 | No, no no, pairs? |
| 166 | | R1 | Good, and do you have a convincing argument? |
| 167 | | CP | Yeah we changed it up a little bit. |
| 168 | | R1 | You did? how did you change it? What did you do? |
| 169 | | CP | Do you want to start? [talking to his partner] |
| 170 | 00:34:07 | JLB | I wanted just red on the bottom. |
| 171 | | R1 | Okay. Neat |
| 172 | | JLB | Okay and then I did this red on the bottom so I could have 3 red, and then what if I wanted to just put in one yellow, then I could have a red here and a yellow and a yellow. and still red on the bottom, one red in, one red in, and no red. |
| 173 | | R1 | Neat! Absolutely, if you have convinced me of this, then you will convince me of that. Now this is neat. Now what you've done here..That's called controlling for a constant. And that is a very very neat strategy. Some of your kids will do it. Some of your kids will actually put the top cube |

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| | | | red. |
| 174 | | R1 | Exactly, exactly. Sometimes what your kids will do when working with 5 tall towers is hold the middle cube constant. Holding a constant. It is a very strong strategy. |
| 175 | | CP | So it is like a tree diagram is kind of the same what we did here. Well No it is not the same concept as a tree diagram but… |
| 176 | | R1 | Okay, It is similar. Not really because this is holding a constant which is a neat strategy. When you looked at this and you told me about it. That you are looking at cubes 3 tall. How many positions did you put the yellow in?. |
| 177 | | CP | 3 |
| 178 | | R1 | So that this is a much more convincing argument. Very nice. Both of you record it not one. You want to show your…Both of you want to record. And when you do it with the children, even both must record. The recording and what they write, they look different. And you want to see each child's work. |
| 179 | 00:36:15 | UCT | Do you suggest giving the kids instructions verbally or actually writing it down? |
| 180 | | R1 | No actually you will be given the problem. Some of the children are able to do it with the towers but not record it. [inaudible]The process will be to build first then record. Okay. |
| 181 | | UCT | Are you collecting it?[inaudible] |
| 182 | | R1 | No, we are not collecting it. Okay, no no no. What you have written You are taking with you. |
| 183 | | UCT | But it will help me guide them through..? |
| 184 | | R1 | No you are not going to guide them. Because they may solve it differently than you. |
| 185 | | UCT | But I am saying for the explanations. My kids are…They might need a little help with the wording. |
| 186 | | R1 | They may not do a lot. You are going to let them know what you liked about what they did. Each time they are going to do more about the writing. |
| 187 | | UCT | Okay. |
| 188 | | R1 | Just for them, not for us as the grownups. So that is why I wanted you both to do it. We are going to talk more about how do you get kids to write about this and how do you get them to write a convincing argument. |
| 189 | | UCT | Okay |
| 190 | | R! | Okay, that's great! |
| 191 | | | [R1&R2 together monitor the work of one group] [multiple conversations-inaudible] [R1 calls everyone together to discuss as an entire group] |

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| 192 | 00:38:21 | R2 | You were saying that you changed..But this one.. |
| 193 | | UCT | You have it upside down. |
| 194 | | R2 | Oh! |
| 195 | | R1 | I was wondering what you were doing, Alice! [giggle]Okay. |
| 196 | 00:38:49 | R2 | Here is this one and there are four different places I can change it. |
| 197 | | R1 | Okay. |
| 198 | | R2 | [inaudible] but what happened here? And then she said I can change that [showing staircase strategy] And then you said I can do that and start over again and change the bottom, okay and then[inaudible]. And back where we started. And so she did that sort of a notation.[inaudible] |
| 199 | 00:39:38 | R1 | Got it, okay. |
| 200 | | R2 | But this is such an interesting perspective. |
| 201 | | R1 | MMhh. Good, tell me more about this. |
| 202 | | VB | I started to do opposites but then I had to figure out where…[knocked blocks over] |
| 203 | | R1 | Wooo!! What about towers 5 tall? [inaudible response] |
| 204 | | R1 | When children do opposites, How do they know that there is not another pair? Okay, that is good. |
| 205 | 00:40:20 | | [R1 calls everyone together to discuss as an entire group] Alright, can I have your attention? I have seen and Alice and Marjory we have seen some really neat solutions. And not all of you have solved the problem in the same way. And that is good. A lot of good strategies And they are not all the same. When your children do this, they also will do things that perhaps that you have not done. And Your eyes are going to have to be open to see whether or not mathematically makes good sense. Okay, and it is kind of fun. This will be the fun part of your teaching because you will be actually be learning new things from your students. And my best teaching was done when I was in a fourth grade class with very, very bright children and they were explaining things in a way that I hadn't seen it. And sometimes their solutions I thought were even neater than what I did. And that's fun so you are going to celebrate mathematical thinking of your own, right now as we look at different solutions you've done.<br>Now when you do it with your children, I do not recommend that you do it the way we are going to do it here as grown-ups. I am hoping you are going to have either a projection camera?<br>How many of your schools have that tool? Lucky people. How many of you have overhead projectors that if you |

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| | | | don't have a projection camera?<br>Is there any of you that don't have either of those tools?<br>Wooo! Okay. |
| 206 | 00:40:21 | UCT | We have smartboards. |
| 207 | 00:41:50 | R1 | Smartboards excellent. Okay. So I am saying that as long as you have a way for children to share their solutions other than talking about it. Because it is going to be very hard when we talk about the solutions here for all of you to see what they have done. We are going to do it because we are grown-ups and we can do it. But when you are with your students you are to want somehow their towers and solutions to be shown so everyone can see it. Okay. |
| 208 | 00:42:12 | R2 | You know there are lots of ways to do that. If you are a low-tech class, one really good way is to have tag board like what's there on the wall and have them put up paper on the wall and do whatever and have them walk around. You are all teachers you know how to do all these things and the looking at the different solutions. The chartboards are wonderful! |
| 209 | 00:42:44 | R1 | Absolutely. Absolutely. I know it can be done. Sayreville I remember 3 years ago didn't have the technology and I remember they did use the chart paper and it was very successful.<br>Okay, so what we want to do is we want to first look at what you have done okay, how you solved it and we want to hear your convincing arguments and then we will talk about how you're going to do this with your children in school. Okay.<br>Alice, so which group should we start with? Do we have a starting point? |
| 210 | 00:43:20 | R2 | Sure. What for all of you, sort of what was the first strategy in here? |
| 211 | 00:43:30 | MM | Just to start with all one color. I think. And then move on from that. |
| 212 | | R2 | Yeah, moving starting from one. How did you do yours? |
| 213 | | MM | I started with all one color. And then I added in one of the other color and I knew that it could be in either the<br>The first the second the third or the fourth position. |
| 214 | 00:43:50 | R1 | You know what I am going to do…. Your partner I am going to ask you to hold up a group as she is talking about it. Because then the rest of the people will understand because It may not be the way that you solved it. Okay. Can everyone see what she is holding up? |
| 215 | | UCT | MMhh. |
| 216 | | R1 | Okay, keep talking. |

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| 217 | 00:44:05 | MM | So I had four of one color and I had it in one of the other color. And it could be in either the first, the second the third or the fourth position. And then I knew if I did it that way, I could reverse it and do it with the other colors as well. |
| 218 | 00:44:16 | R1 | Okay, Before we leave, I am asking the group: for the group she just made, it looks like it is three of one color and one of another color. Could she have another tower in that group that has exactly three yellow and one red? |
| 219 | 00:44:33 | unison | No |
| 220 | | R1 | Did she skip one? Why not? I heard a lot of no's why not? |
| 221 | | MM | Because the red is in each of the four positions. |
| 222 | | R1 | Okay. She gave you the answer because the red is in each of the four positions. Okay I am going to say why couldn't she have then the red in the fifth position? There is no red in the fifth position? |
| 223 | 00:44:52 | unison | They are only 4 towers. |
| 224 | | R1 | Oh, the towers only go 4 tall...Now again you are going to say: Why am I asking you such a ridiculous question? Your children will not think it is so ridiculous. And you are going to want to push them to really be thinking about why they are saying that they have all the towers that are exactly three of one color and one of another. Keep going. |
| 225 | 00:45:12 | R2 | To follow up with that as you are working with your kids, they when you say why not why not. They think you are asking for some really complicated answer. When what you are really wanting is what they know. Which is Hey look lady, there is only four.. [laughter] for instance many of you said there was these 2 [holding up all red towers and all yellow towers] but how many were there? |
| 226 | | unison | 2 |
| 227 | | R2 | Are there any others that are only all one color? |
| 228 | | unison | No |
| 229 | | R2 | Why not? |
| 230 | | unison | There are only two colors. |
| 231 | 00:46:02 | R2 | There are only two colors. Kids are going to think you are stupid. But what is so important is for them to realize the way they are thinking is logical. |
| 232 | 00:46:13 | R3 | The other key point of what Alice and Judy said is that the idea of 4 positions and two colors, means well what do those ideas represent? Are they the additions of the problem? So any time we are doing a math problem you know one of the big ideas is what is the problem asking you to do, what are the rules or constraints of the problem? And so asking a question that is getting a student to |

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| | | | address means that the student knows what they are solving.<br>Because someone could come up with a different answer and might be solving a different problem and it would be important to know that. |
| 233 | 00:46:58 | R1 | Also your children your students need to know that when you are asking them a question, that it doesn't mean that they are wrong, it means you are trying to understand how they are thinking. Throughout this semester, they will get to understand that. That when you question them. You are curious to know how they are thinking. |
| 234 | 00:47:19 | R2 | It's okay to not understand what they are doing. I must have been over here with Victoria for how long? And it was because…[laughter] it was that she was organizing her towers in a way that I had not really thought about before. |
| 235 | 00:47:46 | R1 | How many of you at one point organized your towers like this [holding up elevator pattern] Did any of you? I thought I saw somewhere… Yeah…Yeah. Many different ways to organize. |
| 236 | 00:48:00 | R2 | But I would like…Given that you just held that up. Because when Victoria was doing this, I saw it. But I didn't quite understand what what her logic was. Can you help me? |
| 237 | 00:48:15 | VB | I just thought it was easier for me to kind of be more organized with it. Knowing that I had all red. Now I have to add a yellow and there are only four blocks and So then I had to take away one red and the same thing there so that I had to take Two yellows so I knew I had to take away a red block that would be 4 total I knew there were more solutions. To where I started<br>This is where my issue was because then this is where I got a little disorganized so I started rearranging them all and go back…<br>But lauren helped me and she was like just do the opposite right away. So I was making it more difficult on myself so that way was easier.<br>But then once I got to the end I had to go and double check what I already had it because we rearranged it. |
| 238 | 00:49:02 | R2 | So this arrangement, gave you 8 of them. I think. Really logically and what Victoria said that it took me the longest time to understand what she was saying is. Look it! there are four spots that I can change a red for a yellow[holding up staircase pattern with yellow on bottom and red on top]. And yet this one then there are four spots and I can change this red for the yellow then there are four spots and I can |

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| | | | change this red for a yellow and then after this one, what did you think was next tower? |
| 239 | | Unison | All yellow. |
| 240 | | R2 | Absolutely, it was all yellow. And then she started again and what do you think was next? |
| 241 | | UCT | Red at the bottom. |
| 242 | | UCT | Mmhh. |
| 243 | | R2 | See, Isn't that fun?[laughter] It was fun for me because it was a whole new idea. |
| 244 | 00:50:04 | R1 | Let's go back to hearing the rest of your solution. Okay. |
| 245 | | MM | The next thing that I realized was that I have<br>Two of each of the colors. And I had two of each of the colors where the two colors were next to each other. So they had the position right next to each other. Red red yellow yellow. |
| 246 | | R1 | Can you hold it up? |
| 247 | | R1 | Okay, So now she's working with two of one color and two of the other.<br>That was harder to do wasn't it? Harder to do and Harder to get a convincing argument why you had them all. |
| 248 | 00:50:32 | MM | So then if I had two of the same colors stuck together two times. I could separate one of the colors and keep the other color stuck together. And the only place.. |
| 249 | | R1 | Let her hold it up. |
| 250 | | MM | So then the only place that the two separated ones could go so that my one color was stuck together and my other two colors were separated. Were the first and the fourth position because if they occupied any of the other positions it would be the same thing as the first one. And then the last one was to separate all of the colors so that they were alternating positions and they weren't stuck together at all and we got 3 different towers and then you can repeat that process just doing the reverse colors, which I don't really have. |
| 251 | 00:51:16 | R1 | Okay, okay. So they got how many towers with two of one color and two of the other? |
| 252 | | MM | Well 3 and then you multiply by two so you get 6. |
| 253 | | R1 | Total 6, okay. |
| 254 | | R2 | But don't you use an opposite strategy to find the others? |
| 255 | 00:51:30 | MM | Yeah, I knew if I did something one way, I could just do the reverse for the other colors. |
| 256 | | R2 | Okay, I am having a hard time with your explanation. Can you tell me once more what you did there? |
| 257 | | MM | So I started… |

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| 258 | | R1 | Hold one up at a time. |
| 259 | 00:51:48 | MM | I started off with two colors of each and having them stuck together each so that they are occupying positions right next to each other. Then I kept one of the colors stuck together so they are still occupying the same 2 positions. And the other colors I didn't want stuck together. The only positions that they can occupy that way so that these stay stuck together. And Then I can have them so that all of them are not all stuck together are the first and the fourth position. And then I can have them so that they are not all stuck together and there is a specific way that they have to go as well. Well two specific ways that you can go for that. So you have one two one two. |
| 260 | | R2 | Okay, I don't understand why when you have the two of them that were stuck together, there were two on top or whatever they were. And then they moved down to the middle put the reds on the top. Okay, and then that one down there by your hand. Why can't you put two reds on the bottom? |
| 261 | 00:52:54 | MM | You can. |
| 262 | | MM | I did the opposite. |
| 263 | | R1 | She did 3 and then she did the other 3. So she will have two red at the bottom… |
| 264 | | R2 | I understand that. But What I am interested in.is sort of Two different strategies that you are working on together. You have the two and the two. But For those of us who looked at that video then in, why not go all the way down but you said no, no, no that is not what I am doing [laughter]. |
| 265 | 00:53:32 | R1 | And that actually is…Stephanie does that too.<br>By the way, when you watched the video did Stephanie already talk about stuck together and stuck apart? |
| 266 | | Unison | Yes |
| 267 | 00:53:48 | R1 | Okay I thought it was interesting that they were using stuck together said that they are stuck apart. So that was kind of neat. Alright, There was a group I don't remember where you are that actually held a constant; I saw either the top or the bottom one color. And this group did it back there. Can you have your partner hold up the cubes and you tell what you did. Because this is going to be a very different way of organizing the towers. |
| 268 | 00:54:26 | UCT | First we started with all of one color and then we decided to just keep the bottom ones consistently red; not to change that. So then we went and said, well this one has no yellows so let's just use one yellow at a time and there is |

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| | | | three positions that one yellow can occupy, keeping the bottom one red. |
| 269 | | R1 | Can you show us those three positions? |
| 270 | | UCT | So we had it in the second and the third and the fourth; but we are still keeping the bottom red. |
| 271 | | R1 | So does that look at all similar to what they did? The way they had three of one color and one of the other and they moved it to each of the four positions. They only have 3 positions why to move it. |
| 272 | 00:55:08 | unison | The bottom is constant. |
| 273 | | R1 | Ahhh, Keeping the bottom constant is a really neat strategy. |
| 274 | | R2 | Can I just stop for a second? What's the difference between that structure and between what Roberta and her partner did? |
| 275 | 00:55:26 | UCT | The top is constant. |
| 276 | | R1 | Does it matter if the top is constant or the bottom is constant? Absolutely not! In fact, You will actually see students when they build towers that are five tall. They may hold the middle cube constant which blew my mind the first time I saw it. But it's just fine. It doesn't matter which they are holding constant. It just makes the problem a simpler. |
| 277 | | R2 | But you can't change. |
| 278 | | R1 | Once you start a strategy you are stuck with it. But it doesn't matter which cube you are holding constant. Keep going |
| 279 | | UCT | So then we had the 3 towers which is one yellow. So we wanted to use 2 yellows. And do the same thing<br>Second and third or the second and fourth or the third and and fourth. |
| 280 | 00:56:25 | UCT | The last choice was still keeping the bottom red but having the other 3 yellow that only go first, second and third once we had those… |
| 281 | | R1 | How many did you get when you had red as a constant on the bottom.<br>How many towers did you find? |
| 282 | | UCT | 8 |
| 283 | | | Rest of the group how many towers do you think they had yellow as the bottom constant? |
| 284 | | unison | 8 |
| 285 | | R1 | And that is a really powerful thing for children if they are holding constants. Sometimes you will see that they will have a group of 8 towers with red on the bottom and a group of 9 towers with yellow on the bottom. And they are |

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| | | | not disturbed by it. And sometimes they are disturbed by it. Because they think that it should be the same. But<br>It will be interesting to watch and see that if your students are forming groups, that they are going to know that the groups should have the same number.<br>Not your job to tell them they have the same number but to see are they disturbed that they are getting 8 with red on the bottom and<br>9 with yellow on the bottom and vice versa. Good okay. And a convincing argument…Once students convince you they have all the towers with the red bottom, they really will be able to convince you they have all the towers with the yellow bottom.<br>Good. Was there another strategy Alice, that you saw? |
| 286 | 00:57:50 | R2 | Well, Just the group that was over here had two or three different ones. The first one did it very much like you all did over there. You did the red and the yellow as opposites and then you did one color |
| 287 | 00:55:08 | R1 | Okay. |
| 288 | | R2 | And so Your first one You did the red and the yellow as opposites and then you did one color… |
| 289 | | CT | One color, two colors, three colors, no colors. |
| 290 | | R2 | Well, But you didn't do that in the beginning. The first thing you did. This group, that group, Opposites to each other and then if I am not mistaken, you did the same thing.<br>But that was your first thing? |
| 291 | 00:58:40 | CT | Yeah. |
| 292 | | R2 | That was your first strategy? |
| 293 | | CT | Yeah. |
| 294 | | R2 | And then you got 16, what did you do? |
| 295 | 00:58:50 | CT | So, As a way to sort of prove we got all the combinations; we were thinking about what was really happening. We thinking about it as like a base of two colors starting with a red and a yellow. We basically created a tree diagram flat on the table so that from our the red and the yellow we put a red and a yellow for each.<br>So that you could see there was two blocks in the tower with four combinations. |
| 296 | | R1 | There was another group that did that. Right? In the back there. Uhh Huh |
| 297 | | R2 | You actually built yours? |
| 298 | | UCT | Yeah. We built one tower high, two tower high, three tower high…. |
| 299 | 00:59:58 | R2 | This took me forever to see. Oh don't you see if you start |

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| | | | with this one and trace it up. if you look at this, if you can show each four tower, this one, this one, this one, this one… And so you could see them all but you kind of had to imagine whereas those back there and I think there were other couple of more places. You could actually build each level. Because it could either be and so that helped you to move to the next level; So that convinced you |
| 300 | 01:00:40 | R1 | So if you are telling me that towers four tall have how many towers? |
| 301 | | unison | 16 |
| 302 | | R1 | 16. What would you think towers 5 tall would have? |
| 303 | | unison | 32 |
| 304 | | R1 | And 6 tall? |
| 305 | | Unison | 64 |
| 306 | | R1 | Now, How do you get those numbers? |
| 307 | | UCT | You are doubling the outcome. |
| 308 | | R1 | You are doubling the outcome. What do you mean the outcome? |
| 309 | | UCT | So say okay so if I have 5 tall towers, 5 times 2 is ten but you are not doubling the position you are doubling the outcome. So you would have 2, then 4 times 2 is 8. |
| 310 | 01:01:38 | R1 | So that is your outcome, Can anyone say that another way? |
| 311 | | UCT | We started off by doubling the outcome. And then we looked at it in the form of exponents. We saw that if you kept the base, Exponents change depending on how high and then we saw that four cubes high you would get |
| 312 | | R1 | So… |
| 313 | | UCT | You would get 16 results. |
| 314 | | R1 | Okay, okay. Now when you have children that think they're multiplying or doubling rather than exponents. When they are told to build towers 5 tall, What do you think that they could possibly predict as their solution? How many towers do you think they will find? |
| 315 | | unison | 25 |
| 316 | | R1 | They might say 25 because when they built four tall they got what? |
| 317 | | unison | 16 |
| 318 | | R1 | Sixteen is 4 times 4, isn't it? So logical for five tall, five times five; if they do that, and they start using the strategy of a tower and it's opposite; what happens with 25? |
| 319 | 01:02:41 | unison | They get an odd number. |
| 320 | | R1 | Odd number and then they go... I have actually worked with middle school children that said well wait a minute I have to revise that. It can't be five times five, it must be |

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| | | | five times five minus one. |
| 321 | 01:03:00 | R2 | And you know why they did that. Because when you ask these kids what about 3 tall. They automatically say 9. And that is easy to disprove and it is 3x3 minus one and so that one is 8 so then they say that the next one must be 24 because you are adding 8 you know every time. and so this is when you |
| 322 | | R1 | And it is interesting the next thing after you do building towers 4 tall; is you are going to have a sheet &<br>Both Alice and I have on eCollege & we are going to show you how to get to eCollege if you haven't yet figured it out.<br>But what you are going to do is you are going to then have a sheet that says to students:<br>I don't want you to build, but I want you to predict. How many towers do you think you're going to have if you are building 3 tall. And why?<br>Most of them I predict aren't going to come up with 8<br>They might come up with 9, and the 9 will be because of 3 times 3. Okay<br>How many towers do you think you are going to get if you build 5 tall.<br>So that would be a prediction with supporting answers.<br>And then you might um,<br>In some of your classes, not all, some of you may be done in your period. I hope you all have an hour math class. |
| 323 | | | No. |
| 324 | | | If you don't have an hour math class and If you can't get your handy dandy principal to arrange it so that you can have the one hour math class. if that is impossible, then You are going to have to do this over two days.<br>If you do it over two days, it will just be a little more of a problem for you.<br>What you will have to do is once the children build them the first day; you don't want them to start over.<br>What you will do is you will have a baggie for each group and you will take their towers and<br>If they arranged them in pairs let's say, you are going to Tape around the pair and stick it in the bag. If they arranged in groups of four or five; you are going to take their group and<br>Put masking tape around it and then stick it in the bag.<br>That way when they come back the second day they know what they built the first day and<br>They don't have to start from scratch. Okay. |

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| | | | The idea of having a longer class really is helpful and I will tell you it is much easy not to revisit because<br>One day they can forget everything they did the day before |
| 325 | 01:06:00 | R2 | It will take 10-15 minutes to get them back up to the spot that they were. |
| 326 | | R1 | So It is harder for you. Um, if you are in a school that is really locked; you might have to do it in two days. However, I know that our math classes were not an hour long. They didn't do that back in those days. But yet I was able to creatively work with the teacher.<br>Maybe to combine related arts with a math class. Maybe I was able to back it into a lunch time…I was able to do something that allowed them to have that extra block of time. |
| 327 | 01:06:41 | R2 | It is harder. |
| 328 | | R1 | It is, but even with some of the middle schools. Some were able to be flexible. Some of the schools. Sometimes your principal has enough on their plate and<br>It takes work. But sometimes you will have an administrator that will be willing to work with you. It will make it easier for you to do it in one block, and not in two. |
| 329 | 01:07:13 | R2 | Judy, can we talk about the understanding? |
| 330 | | R1 | Absolutely. |
| 331 | | R2 | I am sure that those of us as we sit around the room, As we listen to what each other are saying some of the ideas but you don't understand others. And as you are working with your kids, your goal is for them to start thinking about these ideas but not necessarily to get to where we are even right now.<br>But over this semester we are going to be working on a series of tasks and they are related to each other. And Some of these ideas we will return to. So that by December, you say oh that's what they were talking about. But what I would like is if…Could you describe a little bit of this organization that you did? This is also what you do when they are done |
| 332 | 01:02:41 | UCT | Alright so we started again with the one cube high tower and we saw that we didn't have no reds or one red. So there are two possibilities and 2 combinations when we have one cube high tower.<br>When we moved onto two cube kind, we can have<br>No red, one red, which is four combinations and two red. So altogether we have 4 different combinations for 2 cube high towers. |
| 333 | | R1 | Interesting she is not saying yellow she is saying what? |

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| 334 | | unison | No red. |
| 335 | | R1 | No red |
| 336 | | UCT | Okay and then again we started with no red, 1 red,… |
| 337 | | R2 | Notice they had the elevator. |
| 338 | | UCT | Three red. When we looked at the previous combination, we saw that when we add two plus one, we would get three possibilities for the following combinations. I don't know if I am explaining it. |
| 339 | | R1 | You are doing a very nice job. A very nice job, it is not easy. |
| 340 | | UCT | And then the last one for the four cubes high, is no red, one red, two red, three red, and four red. And again These numbers build up from the previous combinations. |
| 341 | | R2 | Could you explain that? |
| 342 | | UCT | So for example if I have<br>No red here for the 3 cube high and one red here if I put them together I have four combinations. And then for the four cube high I have four combinations of one red |
| 343 | | R1 | How about if I take 3 cubed high and I take these [yellow with red elevator] & these [red with yellow elevator] and put them together; where do you see that in the four cubes. Ahh. |
| 344 | | UCT | It would be 1,4,6,4,1 |
| 345 | | R1 | Does anyone know what she is talking about? Have you ever heard of that? |
| 346 | | UCT | Pascal's triangle. |
| 347 | | R1 | Oh my gosh. Pascal's triangle. How many of you have heard of that? Pretty neat stuff. Now your children I am sure being in middle school probably haven't heard of this. They don't have to know everything that you know. But some of them might have parents that have told them about Pascal's triangle. Embedded in this problem is pascal's triangle. So that's pretty neat stuff. Good. |
| 348 | | R2 | What you see, they were able to predict. So you are going to build the five, and you would have exactly one and then you would have one plus 4 which is 5.<br>Which was ten which is exactly two red. And then you know if you have those, The mirror image so that 6 plus 4<br>And I don't know about Judy, but on the discussion grid there's going to be a question as to why does it work so it is really interesting<br>And as we move through the semester. And he also had 16 here and you also had 16 which is 1 plus 4 plus 6 plus 4 plus 1when you had them like this.<br>Are they the same 16? |

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| 349 | | | Yes |
| 350 | | R2 | How can you be sure? But that is a question that is not so obvious to the kids. |
| 351 | | R1 | You did a phenomenal job. Cubes have to go back in the bags in stacks of ten and then we will break up into 2 groups. |

R1- Dr. Judy Landis
R2- Dr. Alice Alston
R3- Dr. Marjory Palius
UCT- Unidentified Cohort Teacher
Any initials-identified cohort teacher

| Description: Transcript of in-district classroom visit debrief meeting<br>Advisor: Carolyn Maher<br>Location: TRIE, NJ<br>Date: September 17, 2013 | Author: Phyllis J. Cipriani<br>Verified by: Simone Grey<br>Date Verified: Summer 2015<br>Time Length: 56:44<br>Page 1 of 23 |
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| 1 | 00:00:00 | R1 | We had a meeting in the beginning of the school year and we talked about um, you know what kind of questions you want to use. Um, and we talked about having them, you see how today it was really good. You see how we had the problem written out and if you give out a paper, that has the problem, they are going to start writing on the paper. And they won't go to the unifix cubes which is going to take… it will not be good for them to do that. So what you want to do is don't give out the papers until you see that the students have worked with the unifix cubes. Built the cubes and actually talked to each other about as convincing an argument as they can get. Now these kids we pushed them far and they're 8th graders. And they also are regular ed. and they also are…Is there any inclusion, here? |
| 2 | 00:00:50 | UCT | There was one. |
| 3 | | R1 | Okay, okay but they are …what |
| 4 | | UCT | [unintelligible] Sometimes she got confused with the directions. The way he explained it. It was interesting the way he did it. |
| 5 | 00:01:07 | R1 | But sometimes it's very hard to tell who your special ed. students are and who your Regular ed. students are.<br>In fact in a good class you can't tell. When I would go in when I was a principal in a school and we had all inclusion, um and, I would go into a class and I would try and figure it out, like, who are my children that have special needs. And, I really had trouble doing it, and that's a good thing. You don't want to be able to figure that out. And you really want to trust that the Special Ed. kids are going to do good stuff with this problem and they can. Um, who was concerned about giving colored pencils? |
| 6 | 00:01:45 | T3 | I was. |
| 7 | | R1 | Okay. |
| 8 | | UCT | [Unintelligible.] |
| 9 | | R1 | And it's okay. We've had children, high school students from The Academy at Long Branch. Who was that? That's you. [Pointing] And, ah, at first, those teachers say, Oh my g. And not only resource but its children with really… |

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| 10 | 00:02:02 | T4 | They don't do anything unless they have some sort of motivation. |
| 11 | | R1 | Not true. We got wonderful stuff from those students. Um… |
| 12 | | T4 | Yeah, but they won't put anything on the paper unless they have some kind of motivation. |
| 13 | | T3 | [Unintelligible.] They need an incentive. |
| 14 | | T4 | They need motivation to do something on paper. They won't ever touch the paper unless you give them a little incentive. Like… |
| 15 | | R1 | What would be the incentive? |
| 16 | | T3 | You could use markers or colored pencils or whatever [unintelligible]. |
| 17 | | R1 | Well if it…okay, um, that wasn't the case with the 2 teachers from there. |
| 18 | | T4 | They were from the high school section. |
| 19 | | R1 | Yeah. |
| 20 | | UCT | We are from the middle school section? |
| 21 | | R1 | That is different? |
| 22 | | UCT | That is different kids. |
| 23 | | R1 | Okay. |
| 24 | | UCT | Very different. |
| 25 | 00:02:37 | R1 | Okay. What I would say, if you have a resource room, if you have a very special situation, at best what I would do is, I would have a table setup with these are tools, if you want to use them you can. I wouldn't hand out the colored pencils… |
| 26 | | UCT | [Unintelligible.] Just make them available. |
| 27 | | R1 | Exactly, exactly. Because, um, you kind of lock set them into thinking that's the only way you can do it. And if you look today, how many different ways did you see that they were, um, recording their towers? What were some of the ways that you saw that were different than…um? No one had colored pencils, so obviously you know no one did that. What did you see? What was one way they used? |
| 28 | | UCT | Well, I saw [unintelligible] just write it all out. Well, they were just writing out exactly what they did. And they didn't use any pictures until the end. |
| 29 | 00:03:26 | R1 | Okay. |
| 30 | | UCT | But most kids were like writing, drawing pictures first and then writing their explanation. |
| 31 | | R1 | Okay. |
| 32 | | UCT | So thought that was different. |
| 33 | | R1 | Okay. But what they drew at the end, what did they do? |

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| 34 | | UCT | Boxes. |
| 35 | | R1 | Okay. So they actually made something that looked like a unifix cube. And um, did they put anything in the little cubes? |
| 36 | | UCT | Shaded. They shaded it. |
| 37 | | R1 | They shaded it, okay. And I actually saw some of the people who were shading; they made you a code. In case you couldn't figure out that what was here was... What colors were they? |
| 38 | 00:03:54 | UCT | Yellow and red. |
| 39 | | R1 | One was yellow. One was red. They really made it very easy for you. Did you see another way that students recorded the towers? |
| 40 | | T4 | Some of them just used the letters. [Unintelligible] |
| 41 | | R1 | Exactly, and they didn't have any boxes. And they used an R for the red and a Y for yellow. They even included a little [unintelligible] in case you couldn't figure it out that the Y was yellow and the R was red. Um, and that's really neat. I mean that is a nice strategy. Almost kind of like using a notation, algebraic notation where you take the first letter of the word and then, represents the color that they use.<br>Did you see any other way that they did it? [Pause.] Okay. Everyone today that used the R and the Y, were they writing the towers vertically or horizontally? |
| 42 | 00:04:49 | unison | [Multiple responses.] Vertically. |
| 43 | 00:04:50 | R1 | They were writing it vertically; and that's more common. However, I have seen students take a tower that's a, uh, green, red, and yellow, and they decided that they read from left to right, so they're going to build, or they are going to write or record their towers, from left to right.<br>So if they had this tower, they would not record it this way. [Demonstrating.] They would record it this way [turned 4-tall tower to side] [Unintelligible] Okay, here's the top of the tower. So they would write on their paper: R-R-R-Yellow. Okay? As long as they always have the tower pointing the same way, this is fine. They understand it; it might be hard for you. I've had teachers that when they saw it written this way they said, oh my god I can't think of it that way, I have to think this way, okay. So, uh, there are lots of variations.<br>Um, if you have students that, um uh, you know some teachers asked should we get them graph paper, should we get them rulers, again no! Because you're going to lock them into…What if they don't want to build this and shade |

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| | | | it, or build it or put R-R-R-Yellow? [Demonstrating.]<br>You want to give them the freedom to, to do it the way they want to do it, okay. You don’t want to lock step them. |
| 44 | 00:06:14 | UCT | But if they ask for it? |
| 45 | 00:06:16 | R1 | I would…I…if they ask, you say, you know, if there any tool in your room normally, do you have a math center where you have certain tools? No? But you might want to for this say, well here’s a whole bunch of stuff, if you want, you don’t have to use any of it, but if you want to it’s here if you want to come and take it. And don’t give it out. Okay. All right? Um, I really think trust those Special Ed. students. I have seen some of the neatest solutions from children with special needs, um, because they think outside the box sometimes. And their solutions are, um, creative and, um, I think it will be interesting for you to trust that they can. Now, what happens when you have children that are working and they look really stuck? They look like they are getting frustrated. Do not make them cry. Okay? That’s possible that you can do that. Younger children especially, when I’ve done this in the elementary school, um. Sometimes they reach such a frustration level, uh, they…they actually, uh, they’re releasing it by crying.  We never want to push a student to cry, okay.<br>It will be your job to sense when you push them as far as you can. Alright? And when you think you’ve pushed them, even if they don’t’ have a solution, even if they don’t get 16 towers four -tall; let’s say they only find 12. If they’ve gone as far as you as the teacher at this point in the school year think they can go, then you say, okay, I want you to record the towers you found and write me a convincing argument. [Unintelligible.]<br>Um, I think that if they do something good, you’re going to celebrate it. It may be their notation is really good. It may be that the way they record the towers was, you know, drawing the four cubes or shading for this color and leaving a blank for this color.  It might be that they used the first letter of the color to show what it. You’re going to celebrate what they do. If they have a solid tower that’s all red and a solid tower that’s all yellow, and they can tell you there’s no more solid towers because I’ve used up both colors, and you only gave me two colors, so what do you want me to do? You know, and then you celebrate that. Because that’s a convincing argument for a piece of the problem. Alright. Um, let’s look at some of the solutions. And let’s talk about what these children did [unintelligible] |

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| | | | and what you think you saw. We didn't, uh, make transparencies of everything because there are too much transparencies. |
| 46 | 00:09:40 | R1 | Are you cold? Oh my gosh. Okay. Alright, let's see. Should we kill some of the lights so that it'll be a little easier to read it? Okay. Alright, now this was G------ working with E---, where were they sitting? |
| 47 | | UCT | They were the two in the back end of the [unintelligible]. |
| 48 | | R1 | Okay so, did anyone get a chance to talk to G------ and E---? Or did anyone get to look at G------ and E---, the two that had been doing this before? Okay, so it'll be…and if you didn't, now is your time to really look. Can you see how they grouped? Is this clear enough? Your eyes are better than mine. So I want you to take this time to look at the groupings that they did and see if you can figure out how they grouped. [Extensive pause.] |
| 49 | 00:10:37 | UCT | He did singles, two colors, two colors and opposites. |
| 50 | | R1 | Okay, so basically, you're calling singles… |
| 51 | | UCT | Like all one color. |
| 52 | | R1 | All one color, okay. |
| 53 | | UCT | And they did one way and then the opposite. So that's why there are two in each group. |
| 54 | | R1 | Okay so you're saying this is a tower and it's opposite. This is…is this also a tower and its opposite? |
| 55 | | UCT | MH. |
| 56 | | R1 | And, is this? |
| 57 | | UCT | Yes. |
| 58 | | R1 | What about group 5? |
| 59 | | UCT | Group 5 is that diagonal like. I just meant diagonal was all one color. [Unintelligible. Multiple responses.]…One…one different color. [Unintelligible.] |
| 60 | | R1 | Good. So it's a group of 3 of one color, and one of the other. And the single color goes up on the diagonal. And what about group 6? |
| 61 | | UCT | That's the opposite of the fifth group. [Unintelligible.]. |
| 62 | | R1 | It is, okay. Also a group of three and one. Okay. How many of you got a good convincing argument for the groups that had two of one and two of the other? Or was that where the children had trouble? |
| 63 | | unison | [Multiple responses.] |
| 64 | | R1 | It's easy to make a convincing argument for the 3 and 1. Right? Especially when they put a single cube going up in every slot. It is easy to make an argument for the solid towers. This is the harder part, isn't it? Okay. So what did they say for, let's look at group 2…well let's start with |

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| | | | group 1. What did they say? Can someone read it? [Pause.] |
| 65 | | UCT | Alright, group 1 says, you can only have four blocks and then you have one yellow in the first one then the second one is the alternate color [unintelligible]. |
| 66 | | R1 | Okay. Is that convincing? What might they have said to really let you know why they have all the towers in that group? |
| 67 | | UCT | There are only two colors. |
| 68 | | R1 | Yeah, there are only two colors to choose from, and one was all red and one was all yellow. So they, they must've left a piece out, okay. I know they know it though. Alright, so how about group 2? [Pause.] Someone else…can you, can you see it? |
| 69 | | UCT | You can't have any other combination in this group because of the two yellow on the top, two red on bottom, and then we did the opposite; two red on top, two yellow on bottom. |
| 70 | | R1 | Okay, is it convincing? |
| 71 | | UCT | Well within that group but [unintelligible]. |
| 72 | | R1 | Good within that one little group two, yes they have it, but that doesn't yet convince us that they have all possible towers; all 6 of them that are two red and two yellow. What did they say for group 3? [pause] |
| 73 | 13:31:00 | CP | We did two of each color, red on top,<br>& two yellow in the middle, and one red on the bottom. For the other one, we did one yellow on top, two red in the middle, and one yellow on the bottom. |
| 74 | | R1 | And students do a good job of telling you what they did, okay. You're going to see that happens a lot. You don't really want to know what they did; you want to know why they think there aren't any more in that group. Okay? So, is their argument or is their writing for group 3 convincing? [Pause.] |
| 75 | 00:14:10 | CP | Sure, I think the right combination. |
| 76 | | R1 | Umm… |
| 77 | | CP | 1-2-1. The one color, then two of the same [unintelligible]. One being the same as the other. |
| 78 | | R1 | Okay, so that is group 3 here. So they have the two there and they made those there so. Was there any other way that they could put the two in the middle the same with the opposite color on the top? |
| 79 | | CP | No. |
| 80 | | R1 | No. Okay, so, within this group you can say they exhausted the possibilities? Umm…let's see, do we have the…. |
| 81 | 00:14:52 | UCT | Yes, we have it. |

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| 82 | | R1 | Okay, good. So we're going to have to flip back. This is now. Let's look at the group before we do this. Group 4 is this one here? [Point to student work.] Okay. Did anyone give a name to this group? Any of the students you talk to call this something? Sometimes they call it the rotating one, the candy cane one, the barber pole one… Umm, because the… |
| 83 | 00:15:20 | T7 | Because they alternate colors. |
| 84 | | R1 | Yeah, okay. Cause its yellow-red-yellow-red or red-yellow-red-yellow. Okay, so let's see what they said for that group. [Pause.] Can someone read it that can see it? |
| 85 | 00:15:38 | T1 | We did red-yellow-red-yellow and for the other one we did yellow-red-yellow-red. |
| 86 | | R1 | Convincing? [Phone rings.] Not really. Again they are telling you what they did, not why there can't be another one in that group. And that's gonna be… and this is a hard thing to do and this is the beginning of the year. So I think they are doing more writing than I have seen in other groups than previous years in September. I mean, this is amazing. |
| 87 | 00:16:08 | T6 | Yeah, I don't think that they truly understand [Phone rings.] exactly what it means to be convincing. Like, I don't think, I don't think that they get the objective of how to be convincing. They just kind of explain to you what they do and, they just think that you're going to assume and guess because you're an adult. |
| 88 | | R1 | But we don't want to guess and we don't want to assume. Because sometimes we will assume hoping that they're going down a path that's the right path, we're going to assume what they're doing but that may not be what they are doing. So let's see if they have a better chance at getting group 5; which is the diagonal having three of one color and one of the other. Someone read group 5? Oh, was that the group? |
| 89 | | UCT | No. |
| 90 | | R1 | No. Which one was group 5? |
| 91 | | UCT | Group 5 was the one [unintelligible]. |
| 92 | 00:16:57 | R1 | Oh it is? Okay. That's what we want to…let's see if they did a better job convincing us on that. Because that should be easier to convince. |
| 93 | | UCT | For this group we started with 3 red on top and one yellow on the bottom. Then we moved the yellow block up one, which gave us two red, one yellow and then another red on the bottom. For the next one, we did one red, and then a yellow, and two more red. For the last one, we did one |

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| | | | yellow on the top and three red on the bottom. |
| 94 | | R1 | Okay. Again they're telling you what they did. What could they say here that would make it a convincing argument? What could they say for group 5? [Bell rings.] |
| 95 | | UCT | [Unintelligible response.] that one color that moves in the tower. |
| 96 | | R1 | Good, okay. And, did they exhaust all four positions? And if they exhausted all four positions there would be no other place to put a single red cube in one red and three yellow. That's a convincing argument, okay. Um, there are only four positions in tower four tall and they the single red cube in each of the four positions. That's a convincing argument; more than telling you what they did. Okay? Because you can see what they did. I mean if you look at their paper…is this them? Uh, was it L-----? |
| 97 | | UCT | No. |
| 98 | | R1 | You're not going to look at that one then. What was their name? |
| 99 | | UCT | G------ and E---. |
| 100 | | R1 | G------. Alright, let's see. If you look at their picture you can actually see, we're not going to be able to look at it because I can't find it. I see G------'s hard copy. If you looked at their towers…here's G------…and you saw this, you can see what they did. They don't have to tell it to, you know, what they are doing. But you want them to be convincing you by saying, I have a single red cube, I'm putting in each position, I am exhausting all my four positions with a red cube in each, therefore, I can't do any other tower with three yellow and one red. That's a convincing argument. Do you see the difference?<br>Okay. Alright, so that was that group. Is…what would you say about these students? What parts of their argument were convincing to you? Maybe they didn't write it. Okay, but what did you find convincing in the way they grouped? |
| 101 | | UCT | [Unintelligible]. That's not it. |
| 102 | | R1 | That's not her again,where is G------? |
| 103 | | UCT | I think it is the one you had in your hand. |
| 104 | | R1 | Say it again, this one. Oh, it's on the bottom, okay. Which groups are convincing even though they didn't verbally convince you in writing? Which groups are convincing to you? |
| 105 | 00:19:58 | CP | 1, 5, and 6 |
| 106 | | R1 | 1,5, and 6 are pretty convincing. Okay. What about 2, 3, and 4? |
| 107 | | CP | Sorry, I have to take this phone call [leaving the class]. |

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| 108 | | R1 | Sure, sure. |
| 109 | | UCT | I think four is pretty convincing. I think it’s just alternating colors. So there’s no other way [unintelligible]. If you moved one to the top, you’d have the same tower as the second one. |
| 110 | 00:20:21 | R1 | And if they said that then you’re right, that would be convincing. Um, who was there when they saw, uh, the students taking, uh, a cube from the bottom, and they moved it up? |
| 111 | | UCT | Right here. |
| 112 | | R1 | Okay. And when you do that, when you take a cube from the bottom and move it up to the top to build another tower, take a cube from the bottom and move it up to the top; that’s called a recursive argument. Okay. It’s powerful. Um, you want to be looking for that in your students. You follow? If you didn’t see them, do you know what I am talking about? |
| 113 | 00:20:54 | UCT | Yeah, the group up here was [unintelligible] as well. |
| 114 | | R1 | Good. |
| 115 | | UCT | The group I had worked with too. They had all the ones with the twos, so it would be I guess for 2, 3 and 4, they had all the same ones on the bottom. And then they took the cube off the bottom and they put it on the top, and they found out that, you just had a different grouping of all the combinations, you have beforehand. So… |
| 116 | | R1 | Good. Okay. And that’s a powerful argument, not all groups did it but I’m glad that a bunch of you got to see some groups. If you have a group of two and two and they took the top cube off and put it on the bottom, this is a different tower of two and two. They take the top cube off and put it on the bottom, different one. They take the top cube off, put it on the bottom, another one [demonstrating using the unifix cubes]. What happens when you take the top cube off and put it on the bottom? |
| 117 | 00:21:46 | CP | It goes back to where It started... |
| 118 | | R1 | It starts repeating. And therefore, they’ve exhausted. And some students did that for two of the groups. Um, for groups 3 and 4, that’s how they did that. No that’s 3 and 4; 2 and 3. And that’s how they made their convincing argument. Who saw, uh, okay…students… |
| 119 | | CP | These two girls right here did it. |
| 120 | | R1 | Okay. |
| 121 | | CP | And then they had the two alternating ones, and they were like, oh these don’t fit. Where are we… |
| 122 | | R1 | Okay. And how could they have convinced you with the |

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| | | | alternating ones that there aren't any more for that either? |
| 123 | | UCT | The same way. |
| 124 | | R1 | Exactly. So, say what you're saying. Same way meaning? [Demonstrating.] If they had this as one alternating, what would they do? |
| 125 | | UCT | Take the top, put it on the bottom, and now they have a different one. But if they took the top and put it on the bottom again, they would go back to the other one so there's no more. |
| 126 | 00:22:38 | R1 | Okay. So they really could've used the recursive argument, um, for the alternating ones too. Okay, so that's really good. Alright, um. This is impressive. I don't know if you are aware, um, in previous years, um, we saw kids very resistant to write. Um, and I think you're doing more in your schools to write so much in the beginning of the school year. I don't think it hurt that we had so many adults in the room too. It's going to be harder, I warn you, it's going to be harder when you are alone in your room. Um, especially if you have a whole class with as many students as were in here. Uh, in a resource room it will be easier. You'll have less students. It'll be easier to see what they're doing. In a class of 20 or 30 students, when you have 10 or 15 groups, don't try to get to all 10 or 15 groups in one class. You will drive yourself crazy and you won't hear anything! What I would suggest you do is, stay with three or four, maybe five, groups at most. Stay and listen. Question them. Try to understand their thinking and then make sure you, um, record who you went to because the next lesson, don't go to the same five groups. Go to five different ones. And then last lesson that we do in November, you go to the groups that you haven't seen yet. That way you get to see everyone, but you see them with a meaning, rather than running around and not hearing anything. Okay. Here is a different group and a different recording. Okay and if a student in your class uses this and uses a key, I would point it out as a good thing when you start to share their solutions. Um, because I think it's really neat when they do this. Um, okay. So, how did they do it? What do we see here? Groups…we will call it 1 and 2.[Pause] |
| 127 | 00:24:45 | UCT | Opposites. |
| 128 | | R1 | Well, one group is the opposite of the other. And? |
| 129 | | UCT | The one block…[Multiple unintelligible responses.] |
| 130 | | R1 | Yeah, kind of the same, um, strategy where they're exhausting every position. So that's that. How about, what |

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| | | | going on over here? |
| 131 | 00:25:05 | UCT | It’s kind of like the same idea, but only moving the two blocks at a time. And that one and one is the opposite of the first. |
| 132 | | UCT | Yeah. |
| 133 | | R1 | Opposite of… |
| 134 | | UCT | [Multiple responses] The first one…the top one is from the bottom of each time. |
| 135 | 00:25:19 | R1 | Ah! So you think here, you’re seeing that they took the top cube and moved it down to the bottom to get the second one. |
| 136 | | UCT | Yeah. Yeah. Well then the other top cube. |
| 137 | | R1 | The top to go down to the bottom. |
| 138 | | UCT | Um huh. |
| 139 | | R1 | And the top, so here the top, go down…so did this group…it was Lauren… |
| 140 | | UCT | That was, those were the two girls that were here. |
| 141 | | R1 | And did they use a recursive argument? |
| 142 | | unison | [Multiple responses.] No, it wasn’t Lauren. |
| 143 | | R1 | In the corner where you were sitting? Did they use a recursive argument? |
| 144 | | CP | Yeah, those girls. |
| 145 | | R1 | Well that’s what they did. Okay. Did they write about it? That’s interesting. Mm…Can anyone read that? |
| 146 | | UCT | We started with yellow and…yellow and one red |
| 147 | 00:26:03 | UCT | We started with yellow and one red. Then we moved the red down one space every time and move the yellow to the top every time. Then we did the opposite with three red and one yellow. Then we did two of each color; two red, two yellow. We moved the two red down one cube and took the one yellow on the bottom and move it to the top. We put the two yellows on top, on top of each other, and had two reds…on the bottom..hold on. Two yellows on top of two reds. Oh, two reds…on the bottom. Then we moved one of the reds on top of the two yellows. |
| 148 | 00:26:49 | R1 | Okay. It’s gets hard not only to read but to understand what they’re doing. Um, but that is a recursive argument. And that’s a good argument, okay? Um, so that was, um, the solids here, okay; and here’s the ones again with the alternating pattern. Are they convincing you? |
| 149 | 00:27:16 | unison | [Multiple responses.] T3:More than the last one…yeah…yeah. |
| 150 | | R1 | Say that again. |
| 151 | | UCT | Much more than the last one. |
| 152 | | R1 | Okay. [Sneezing.] Alright, a recursive argument is very |

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| | | | convincing. Bless you. |
| 153 | 00:27:27 | R1 | Let's see. This is Steven…this is Steven and Christian, and where were they? |
| 154 | | UCT | They were the two in the back corner. |
| 155 | 00:27:36 | R1 | The back corner over there. And Christian had to go out for a little bit, and he came back. Did anyone get to watch the two boys in the back corner? Yeah, okay. So let's see what they did. And you can read this one; this is nice. By the way, I probably would recommend that you ask the students to work either in a pen, ballpoint pen, or if you have a black felt tip pen. Because if you're going to want to be looking at the work on an overhead projector, or even putting it on a screen, you have to be able to read it. You saw how hard it is to read some of it. Um, you know, some of their handwriting, I mean, I don't think my handwriting would be much better, but it's too light. You don't really…you can't read it. So even though you probably work in pencil when they do math. For this, I would have them work in pen, okay. |
| 156 | | UCT | If we're not making overheads, can we just let them use pencil? |
| 157 | | R1 | No. Because you're final project for this class, you're going to be making a book of the student work, and you're going to have to Xerox papers if you want to keep copy for yourself. |
| 158 | 00:28:49 | UCT | You can adjust the darkness though…[Multiple responses.] |
| 159 | | UCT | I mean, my kids… |
| 160 | | R1 | Two reasons |
| 161 | | T3 | Like always want to use pen in math. So if I allow them to use pen, then they going to think for the rest of the year they can use pen. |
| 162 | | R1 | No, no. Kids are pretty, they understand if you say, I'm asking you to use pen for this because I want to be able to read it and I am sharing with my colleagues from Rutgers. |
| 163 | 00:29:11 | T4 | But if they don't want to break their classroom habits, then it's okay. We can copy it [unintelligible]. I know how to work a copier if anyone needs help. |
| 164 | | R1 | Let me tell you another reason why I wouldn't have them use pencil. Okay? Um, what students normally do when they're using pencil, is that they have an eraser. And after they do something, they erase it so you can't see it. And part of this is, they may have some good work that they're getting rid of, and all you will see is a whole in the paper. If you're working in pen and they want to get rid of something, all they do is put one line through it and you |

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| | | | can still see what they've did. And remember, the aim of this class is not to look at the final answer but to look at the mathematical thinking as they're go through the process of solving the problem. So I would urge you that, trust that they will be able to know that this is a time I'm going to ask you to use a pen, but normally I won't do that in math class, okay. Alright, so we were looking at this? Okay, could someone read it? And let's see what Steven and Christian used as an argument. [Pause.] |
| 165 | | CP | In the group 2, they were both solids. So if they were to be switched they would stay exactly the same. For example, four red and four yellow. |
| 166 | | R1 | Okay. What do you think? Convincing? [Unison No.]What could they have said? |
| 167 | | CDR | This is what this group did the whole time. I don't know. You were with this group, right? All they kept saying to me was like, we switched it…it would, it would be good. So they, they just proved that they made opposites of each other, but they didn't really…I don't know if they understood the task because they didn't really say anything about how this is the most amount of towers they can make. |
| 168 | | CP | Yeah. |
| 169 | | R1 | Okay. |
| 170 | | CP | For a while they were doing that, and I was like okay, you're just showing me opposites. Can you try to put it… because I think they had the one that, most kids get the one diagonal going down. And I was like, can you put these together something like that? And that's where they came up with the drawing they have on the bottom. |
| 171 | | R1 | So you led them a little bit. I would…I thought they were brilliant did that. I didn't realize that you… |
| 172 | | CP | No, because they were looking at it…and I was saying how can you group these? |
| 173 | | R1 | Okay, that's good. That's fair. |
| 174 | | CP | And they put it together. And I said, why did you group it like that? And he goes, well I have one red top and on the top. |
| 175 | | R1 | That's good. Then that's really good. Then you weren't leading them. You just said to them, I don't see a convincing argument. Can you group these that are two of one color and two of another color, in a different way that may be able to convince me that there aren't any more. And that's a good way to do it. Um, take a look at what they did here. What did they do? |

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| 176 | | T8 | They held a constant. |
| 177 | | R1 | They held a constant. Remember we talked how powerful that was. Okay, and by holding the constant here, what did they do here? [Pause.] Does it look like the three and the one, okay, where they exhausted the positions? They only have three positions now. Okay. And in these positions, they are going to take the dark cube, red cube, and put it in each one. And they know now that these are the only towers you can make that are, what is it, 2 red and 2 yellow, right? The one group. Did they have another group like that? Cause they had, there should be 6 of them, right? Up, they do. Okay. No, that's not them. |
| 178 | | T8 | Yeah. |
| 179 | | R1 | That is them? |
| 180 | | UCT | It looks like them, but it's not them. |
| 181 | | R1 | It the…that for the three and the one, okay. Three of one color. |
| 182 | | UCT | Like that one says for the two groups of four. So for their, and they had just put their argument that there are two groups like this, just opposites or something. |
| 183 | | R1 | Okay, now if they convince you of this, have they convinced you of the, if this is red, a single red…If they make a convincing argument for this, will you be convinced that the towers that are exactly three red and one yellow are the same argument? Would you be convinced or would you make them go through it? You're shaking your head. What? |
| 184 | | UCT | I'd be convinced. |
| 185 | | R1 | I would be convinced also, okay. Because the argument for this is the same as the argument, if it's a single red going down or single yellow going down. Did they talk about the other group here? Maybe they have and we didn't read it. Someone read it. Let's see if they have both groups.[pause] |
| 186 | | UCT | In the two groups of three, there are 3 of each color at the top top. There are four layers in the second, third and fourth rows. There's at least one of each color, whether it be red or yellow, you can't put the top layer of one of them anywhere else in the group because each row has at least one of each color. |
| 187 | | R1 | What did they mean? Do you understand what they are saying? [Extensive pause.] |
| 188 | | UCT | [Multiple response.] I think that it would be the same as another one. |
| 189 | | UCT | I think they are talking about the group of them. Like the first row is three and the second row is 2 and one so they |

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| | | | are going from left to right and from right to left talking about it. |
| 190 | | R1 | What do you mean? I’m not sure I follow. |
| 191 | | UCT | Like, they put them together with 3 towers. Now we’re talking about them as if they are one unit. |
| 192 | | R1 | Okay. |
| 193 | | UCT | Another group did this too. |
| 194 | | R1 | Okay. |
| 195 | | UCT | So they look for patterns I think, and then try to stick them together, and use that to explain. |
| 196 | | R1 | Sure, but what do they mean, you can’t put the top layer on one of them anywhere else in the group because each row has at least one of each color. |
| 197 | | CP | I think they’re trying to keep that constant. |
| 198 | | R1 | They are trying to keep a constant. Okay, so that’s the top row. |
| 199 | | CP | Yeah. |
| 200 | | R1 | It’s like a foreign language. Who could translate this into English? Cause I think they really meant something. Do you understand what they meant? Or do you, we should say, do we think we understand what they meant? Because the only way we’ll know for sure what they meant is if we asked them. What you think they meant? You can’t put the top layer of one of them anywhere else in the group, because each row has at least one of each color. |
| 201 | | UCT | Yeah cause then it would match the other one. It would be a repeat.[unintelligible] |
| 202 | | R1 | Okay, what do you mean? |
| 203 | | UCT | The first row, if you took that one, brought it to the second row. I think they’re assuming that we know that there’s another group. I don’t know, that’s how I took it. Assuming that there’s another group with the opposites. So that would equal and that would be a duplicate. |
| 204 | | R1 | Okay, so that is one way to interpret it. Okay, Any other interpretations? |
| 205 | | UCT | [Unintelligible] Well what they’re saying with the first one, it seems like what they’re saying is, if you take, like for example, the first tower on the left there. It seems like they're saying, if you take that shaded one at the top and you put it down one, now that second row going horizontally now has two, whereas the third row and the fourth row only have one. That seems to me more of what they are talking about taking the top row and… |
| 206 | | R1 | That’s another interpretation. Do you follow what she’s saying? Can someone say what she said in another way? |

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| | | | Because it's complicated. I am kind of interpreting it the same way that you are. Did you follow what she said? Did anyone follow what she said? [Pause.] It's hard, isn't it? Huh? To try and understand what kids are thinking? |
| 207 | | UCT | Especially if they are talking. And they are not talking about towers individually. They are talking about the position across three towers. So they're not even focusing on one tower. |
| 208 | | R1 | That's right. |
| 209 | | UCT | So they're not even focusing on one tower. |
| 210 | | R1 | Right. |
| 211 | | CP | So they are moving it across the row instead of the column? |
| 212 | | UCT | Well they're saying that the, like the bottom in the second and third level in all those, only has one red. |
| 213 | | R1 | Ah, so it's … |
| 214 | | UCT | Horizontally. Until you get to the top. Then there's 3 red. So what they're saying is if that top one was moved down somewhere else that breaks your pattern, because now your horizontal rows are going to have more than one red [unintelligible]. |
| 215 | | R1 | Okay, so horizontal rows is what she is saying. If you move this red that's on the left down one, okay. There'll be two red in this horizontal row. This row only has one red. This one has one red. So again, we don't know for sure. It's another interpretation. But I think it's…it's a good guess that they're saying that you can't move the top row into another position. Otherwise you're going to have more than on red when you look at it across. Okay? Do we know that's really what they meant? No. We really don't. Not, nor do you know that's what they meant. The only way you what a student really means, is if you interview them and talk to them. Okay? And sometimes that's possible and sometimes it's not when you are a teacher in a class with a lot of students. Okay, alright. How about this one? Olivia. I love the way they, uh, recorded it in terms of…do you see how they broke it into groups for you? That's really nice. Okay. And now she's talking and they labeled. So this is really makes it easy for you as a teacher to know which argument goes with which group. Alright? Group A can't have any more towers because all of the red blocks are in a different position. Okay. So that's really the start of a real nice convincing argument. There are only 4 positions, the red block is in each of the positions, we've exhausted it. Okay, same with B. It's the opposite argument of A. |

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| | | | Group C was just the standard four blocks of each color and we just switched the colors. What do you think? |
| 216 | | UCT | I think that the way she grouped it, like was more convincing than the way she wrote it. |
| 217 | | R1 | Okay, alright. And how do you think she grouped it? How do you think she formed those towers? And again, we're guessing. |
| 218 | | UCT | Like, Try to follow the A and B pattern diagonally it looks like, but with two colors, except for when she got to the last one. |
| 219 | | R1 | And what did she do to go from this one to this one? |
| 220 | | UCT | She moved using a recursive argument. She moved the bottom to the top. |
| 221 | | R1 | Very good. And that probably is a really good guess as to what happened for these towers. She might've been using a recursive argument. She might've been taking the two together, and moved them down the way she did the one. |
| 222 | | UCT | That's what she did. [Multiple responses. Unintelligible.] |
| 223 | | R1 | Good. Okay, you actually saw it. Perfect! Okay, so the we don't have to guess. How about group D? We just alternated the colors. Again, that is what they did. But what do we want them to say? We want them to say why there is no else in that group. And they might use a recursive argument to show that if they move again, they just get back the…the first tower that they had. Group E. Made a tower of the full color, and there's only two colors. What do you think? |
| 224 | | UCT | That one's good. |
| 225 | | R1 | It's pretty good, right? Okay, so no other possibility, alright. Nice stuff. Alright, let's see what did you see…I saw one group that did towers and opposites. Which group was that? [pause] Did anyone else see a group that had pairs as opposed to? |
| 226 | 00:41:23 | unison | [Multiple responses.] Yeah, I think That group over there did that. |
| 227 | 00:41:26 | R1 | Okay, okay. And that is a strategy. It's not as sophisticated as that, and it's also harder to convince you that they have them all. Okay. We're not going to read it but let's see, how did they group them? Are they in pairs or are they actually not in pairs? |
| 228 | | UCT | Yeah, they're opposites. |
| 229 | | R1 | They're in pairs of opposites. Good question to ask when they make 8 pairs, um, and they have all 16…a good question to ask is, why isn't there another pair with an opposite? Like, why can't there be 17? You know. Um… |

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| 230 | | UCT | [unaudible response] |
| 231 | 00:42:05 | R1 | What was their argument? |
| 232 | | UCT | Well he had started talking about the fact that no matter what tower you build, there's 2 colors; you are always going to be [inaudible]. There will be an odd number of towers. |
| 233 | 00:42:17 | R1 | Okay, good good and that's nice. |
| 234 | | UCT | She was just saying if they wanted more they would have a higher tower that they knew that if they wanted more towers they knew what they had to do. |
| 235 | | R1 | Good, good, and those are good. I am looking for the girl who actually here that got to do the prediction. She was here right? |
| 236 | 00:42:42 | UCT | Yeah. |
| 237 | | R1 | Okay, Megan was…Did anyone follow Megan? She was sitting right here. |
| 238 | | UCT | Yeah, I did |
| 239 | | R1 | Okay. She did some fascinating stuff. And she had a theory that I have never seen before it is not actually a theory that is going to pan out to work. But, it is awfully nice when students come up with a mathematical theory and then go on to… Can we get her paper? This is hard to read and um you are going to read it for us. She is making a prediction now. In your class, if you are working with students with this problem And you have time this is the next thing that you go to. These problems are all listed for you um the worksheets; it's called an extension. You are asking them to tell you without building how many towers they think they are going to get if you ask them to build 3 tall. Do you think they are going to be the same as four tall? Do you think it will be less? Do you think it will be more? And you had interesting comments on the web. Some of you said that you think that the children will definitely say less than 3-tall because there are less towers high, less combinations. Who was it that said they thought the Children might say it is the same? And your reason was? |
| 240 | 00:44:11 | UCT | Well, some of them might just think oh you just take the one layer off you will have the three towers tall.[inaudible] |
| 241 | | R1 | Yeah. Now do you know no one predicted that. Towers 3 tall would have more towers than towers four tall. Do you think students might do that? [long pause] If you ask them how many towers 3 tall, would it be more than towers four tall? How many of you think might say yes? [pause] How many don't think they will do that? [pause] Okay, well |

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| | | | they do do that! Because you are going to watch a video of Meredith. And Meredith did think that towers 3 tall would have more towers than four tall and you'll get to see why. Okay, and um she is not the only person who has done that, I have been in other classes where they actually predicted more. Can you read for us her prediction for.. |
| 242 | 00:45:11 | UCT | She says Being that the tower is one cube shorter than the four cube tall tower; I would say there are less than 16 towers; We have less tower patterns to choose from. And then she said If we do 2 the amount of colors times four the amount in the towers you are going to get 8. And then she says 2 which is the amount of colors times 3 which is the amount in the tower, you are going to get 6 towers. |
| 243 | 00:45:37 | R1 | Okay, That's real confusing. I was there and I understood her. What she is saying is this: if we build the towers four tall which we did; okay we had four cubes tall, right and there were two colors. So she said when I build towers four tall I had the four cubes times the 2 colors; that's 8. But, when I actually built the towers four tall, I got how many? |
| 244 | 00:46:08 | unison | 16 |
| 245 | | R1 | So she is saying, her theory is .. is gonna be the height of the tower times the two colors; 4 times 2 is 8. But double it because you get 16 when they are four tall. Okay, That was her theory. So telling …her name is?... |
| 246 | | UCT | Megan. |
| 247 | | R1 | Megan's theory. What would she guess then the towers 3 tall to be? |
| 248 | 00:46:34 | UCT | 12 [inaudible response] |
| 249 | | R1 | Good and how did she get it? |
| 250 | | UCT | The amount of colors times 3 cubes to get 6 and then double it. |
| 251 | | R1 | Okay, now that is pretty neat, right? So what would you do next for someone like Meredith? What would you ask her to do? Megan, so she could… |
| 252 | 00:46:51 | UCT | Build them! |
| 253 | | R1 | Build them. Now when she builds them she is not going to find 12. And then she is going to have a disequilibrium. Because she is going to say, wait a second, I am only finding 8 so maybe my theory isn't good. |
| 254 | | UCT | What was her reason to double it just to make it..? [multiple responses] |
| 255 | 00:47:10 | R1 | No, no, no. It was just to make it work. How many have you have students, I mean I did, and know what the answer is. and work the numbers so that it comes to the |

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| | | | answer doing crazy things with adding or subtracting, multiplying or dividing just to that they force the answer to be 16.<br>She knew that she had four cubes high. So that was the 4. She knew that she had two colors.<br>Now what did you do with 2 colors and 4 cubes tall, how did you get 16? |
| 256 | 00:47:44 | UCT | 2 to the fourth [inaudible]. |
| 257 | | R1 | You got 2 to the fourth. You took 2 for the number of colors raised to the fourth power the height of the tower. And two to the fourth gave you 16. Now she didn't know to do that. She did four times two, giving her 8, and she said no the answer is not 8. The answer is 16, so I'll double it. Okay. And that is exactly what she did. |
| 258 | | UCT | But instead of having her try and build the other towers, can't you challenge her to try and make sense more of the numbers first. |
| 259 | 00:48:16 | R1 | How would she make sense? |
| 260 | | UCT | But I think she, when she was working in pairs for the entire time. |
| 261 | | R1 | Yeah. |
| 262 | | UCT | So I think she had 8 groups on her desk. She was still holding 2 groups together. That 8 probably made sense to her because she had 8 groups so maybe that's really what she meant by 8 groups. Even when she was explaining the ones that had 2 of the same color and 2 different colors; she was focused on the Pairs staying together. So I think that for her she probably said oh 8 groups here. And that is the 8 she is talking about. |
| 263 | 00:48:51 | R1 | Maybe, maybe not. When she explained it to me, she didn't point that. She might have been thinking that but when she verbally actually told me because I was asking her. Um I was saying… she said to me it was the height of the tower, times 2 because there are two colors. That was her reasoning. How would you try and get her to make sense of it? I'm curious. |
| 264 | | UCT | Um, [slight pause] I would just ask her where that second 2 is coming from? |
| 265 | | R1 | Okay. Okay. |
| 266 | | UCT | It doesn't make sense, It doesn't fit so then.. she would hopefully realize that extra 2 that doubling 8 towers doesn't make sense. So then you can try and challenge her well; can we make sense of the number of colors and kind of take away the two and the number of blocks that we have and mathematically make the numbers work. |

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| 267 | 00:49:50 | R1 | Okay, what you're trying to do is what many of you talked about in the dialogue that you had on line. It was to get them to understand 2 to the fourth as being the way you get the answer for towers four tall.<br>You don't want to do that.<br>Your students one, may not know exponents.<br>Your students two, may not be ready to hear that. And many of the<br>You guys are very good that you got that this was connected to exponents. I have worked with other teachers over the years.<br>They always thought that it was just doubling and that you got your answer by just doubling and that is what they got before.<br>So, I am saying You got it because you were ready to get it. Your students may not be ready and just by you leading them is is not what you want to do. Okay.<br>So, I think, Probably its safer if you have her build 3 tall towers and she can't find the 12, that may Force her to come up with another plan, another strategy. Another way, another theory. Okay.<br>Um, I had I think I told you I had students in middle school that I asked them to build towers 5 tall and Make a prediction for 5 tall. And they didn't use her theory. What would her theory be for five tall?<br>How many towers would she expect to get? |
| 268 | 00:51:14 | unison | 20 |
| 269 | 00:51:15 | R1 | 20. Okay, most of the students that I have seen in middle school, that have done this. They predict 25 okay, because 16 is four times four right? So 25 is 5 times 5. Again if you had less time to let them build the towers they would see that they get more than 25. Once they get more than 25, then they have to readjust their theory. Okay and that is a strong way to do it. I know you would like them to see what you know. But they may not be ready. Okay and getting them to understand that this is really exponents, towers 5 is two to the fifth, towers ten is 2 to the tenth; um that's not where we are going, okay. So it is okay for them to come up with a theory, and then have some disequilibrium when you ask them to build the towers. Okay, Questions? You guys did a good job and the students were phenominal, really phenominal. And you should celebrate what they did do. They may not have gotten a full convincing argument, but it is hard to do that. And if they don't get it now in September, in October they |

| Line | Time | Speaker | Transcript |
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| | | | will be better, and in November they will be better. Okay. Let me give you a heads up about your final project for the course. Even though we will talk about it again later. Know that the final project, is going to be for you to collect, work samples from the 3 tasks that you will be doing at Rutgers as adult learners. And it will be the three tasks they do when we go in to do the three classroom implementations. So you are going to want to be having student work first of all that you can read, and second of all that is indicative of what your students can do. And again, it doesn't have to be picture perfect excellent work with a complete solution and a complete convincing argument. Very few students are going to do that in the beginning of the year. Okay, you might have an outlier. I might be telling you they won't and you will find someone that does. But you also are going to have um picking sampels of work that surprised you, um which means that it could have been a student who couldn't do something and they did it. Okay, you are going to pick work that impressed you which may be a student who gives a convincing argument. And you are going to pick student work that troubles you. Troubles you because the students don't have any mathematical sense in what they are doing and what their arguments are; don't make any sense mathematically. Okay, while you are looking through the student work each time, you are going to be looking for those three kinds of work. Work that impressed you, work that surprised you, work that troubled you. And at the end, you are going to be having that for each of the three classroom implementations. Okay, so when you are looking at your student work, you are going to be bringing to the next time we meet, you are going to be bring samples of student work. I am only asking you to bring 2 or at most 3. Okay so you are really going to have to look through your class and you will have the option to pick what you want to bring to share. You might pick it because you are thrilled that the student that you thought didn't have anything in their head, did wonderful work. You might bring it because you don't understand at all what the student is saying or what they are thinking or what they drew or what their argument is. And you are going to bring it so that all of us can talk about it, and figure out what the student is maybe saying. And again I say maybe because we won't know for sure, unless we talk to the student. Okay, Um or you may bring work that |

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| | | | troubles you because it doesn’t make any mathematical sense and you really want to say I gotta help this student because they are saying something that doesn’t make sense. I have had students as a teacher, I began as a middle school teacher. I had students that took a word problem and plucked the numbers out okay and did some operations with it So if the numbers were big numbers they said I am not going to do any division I will add it and get my answer. Sometimes the answer of what they did is the right answer. But if their reasoning is wrong do you celebrate a right answer? No, in fact you all know that the state test now say that work is important, process is important and if they don’t have the right process, it doesn’t matter that the answer is correct. if they got it the wrong way, we don’t celebrate it. Okay, so when we meet next time, okay you are going to bring 2 to 3 pieces of student work. You don’t have to put them on overheads. We should be able to share it without overheads. Any questions? [pause] Any questions on the implementation? [pause] |

R1: Dr. Judy Landis
UCT: Unidentified Cohort Teacher
Initials: Identified Cohort Teacher

| Description: Transcript of current students working together on the pizza problem.<br>Advisor: Professor Carolyn Maher<br>Location: Carl Sandberg Middle School, Old Bridge, NJ<br>Date: October 2, 2013 part one | Author: Phyllis J. Cipriani<br>Verified by: Simone Grey<br>Date Verified: Summer 2015<br>Page 1 of 50 |
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| Line | Time | Speaker | Transcript |
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| 1 | 00:00:00 | R1 | It could be work you are troubled about. It could be work that you are celebrating that you did not expect to do whatever they did. Alright now, how are we going to show this up? |
| 2 | | TD | Do you want me to go first? I already have this. |
| 3 | | R1 | Sure, excellent.[slight pause] This school has good technology. I know that some of your students don't. But you are going to have the uh… projection camera. |
| 4 | | TD | Yeah. |
| 5 | | R1 | It is hard. How many of you had trouble making a convincing argument. They will get better, I can promise you that. Um, I have been doing this for many years now and We had a lot of skeptics. Teachers in the beginning that said my kids will never get this. But by the end of the semester, you are going to see progress. Now wherever they start, they are going to make progress from that point. So if they are doing very little now, you are going to celebrate that they are able to do more a few months from now. Okay, but let's take a look at what they are doing now. |
| 6 | 00:01:07 | TD | Okay, so, I had only 8 students; one was absent, |
| 7 | | R1 | Okay. |
| 8 | | TD | one was absent, |
| 9 | | R1 | Okay. |
| 10 | | TD | So I had a group of 3 and then two pairs. And to be honest I was really only impressed with one pair. This is the one that troubled me this pair because they couldn't even like they were making a pattern with the towers themselves. I should… I'll show you a picture of what they did. Okay, So this is what they were doing. They were trying to make a pattern. So first they were doing the colors like trying to match them up and then they started thinking that they could make a pattern actually with the towers. |
| 11 | | R1 | Okay. So they had towers four tall. |
| 12 | | TD | Yeah, they were doing. But they weren't getting the concept at all. It was like they were getting frustrated, I was getting frustrated. |

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| 13 | | R1 | Before you take it off, Take a look at their towers, they are upside down, okay but they are all facing the same way. |
| 14 | | TD | They weren't upside down, I was upside down. |
| 15 | | R1 | Oh, you were upside down. Okay so They are all facing the same way so we can look at them, And take a look and see any patterns in the way that they grouped at all or is it random grouping? [pause] |
| 16 | | UCT | Well there's 18 there. |
| 17 | | R1 | There are 18, okay. Are there duplicates? |
| 18 | | UCT | There have to be. |
| 19 | | R1 | There have to be. Good. Which ones, um … |
| 20 | | UCT | Oh yeah these two right here. |
| 21 | | R1 | Okay. |
| 22 | | UCT | And I feel like that's what happened so many times that every time I went over, they were like we're done and I had to like say do you have any duplicates? Do you have any duplicates? Um, that was the one thing they kept doing. |
| 23 | | R1 | Okay. Do they have…. Take a look at the different kinds they did. If you look at the towers that are one yellow and 3 blue, how many towers do they have that are exactly one yellow and 3 blue? |
| 24 | | UCT | One… |
| 25 | | UCT | 6, 6 |
| 26 | | CP | Yeah. |
| 27 | | UCT | 3 |
| 28 | 00:03:21 | R1 | You see 3, someone said 6. |
| 29 | | unison | 6 [multiple responses inaudible] |
| 30 | | UCT | Wait There is One yellow, 3 blue, 2, 3, 4, 5, and 6 pointing to the screen. |
| 31 | | R1 | Why don't you point it out the six of them? And of the six… Are there supposed to be 6? |
| 32 | | UCT | 1-2-3-4-5-6 |
| 33 | | R1 | Okay, okay, so if there are 6, you are saying they are only supposed to be 4? What are the duplicates there? |
| 34 | | CP | 2 bottom. |
| 35 | | UCT | The ones on the top and the ones on the bottom |
| 36 | | R1 | Okay, okay And the next row? |
| 37 | | UCT | That one |
| 38 | | R1 | So they know they are making the combinations here they are getting the different groupings. What is the problem with this? Why are they….Why will they have a difficult time..? |
| 39 | | UCT | The way they grouped them. |

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| 40 | 00:04:16 | R1 | Okay, Good, alright and there isn't a grouping where they put them together in any systematic way so they don't even have opposites they don't have you know pairs so I think . Let's see what they said though. |
| 41 | | TD | Okay. [slight pause] Okay y is for yellow and p is for purple. We just put the colors together and mixed it up four cubes with different designs. We also made a tower the tower was to put 7 then five then three then one we did. |
| 42 | | R1 | Okay, alright. What did you do when they did that? |
| 43 | | TD | Um… I just kept….well…I was walking around and I just kept complementing them saying like it was like they were making it but they couldn't really explain what they were doing. |
| 44 | 00:05:08 | R1 | Okay, Suggestions from the class: What would you do if a group did something like this in your class? What would you do? How would you react? What would you do to get them, without leading them, Did they have good work? There is a lot of good stuff that was really good. So we don't want to say that this is garbage. You know.<br>How do you redirect them? [long pause] |
| 45 | 00:05:37 | UCT | Try to help them organize it differently. |
| 46 | | R1 | Okay, Good! |
| 47 | | UCT | Maybe they can see either a pattern or which ones are duplicates. |
| 48 | | R1 | Good okay and if you said to them, you know can you show me a different way to arrange your, your towers and that would be fair you're not leading them. See if they would regroup them and maybe they might find duplicates. So, when you have something like this although this page I am really glad that you made a picture of what they actually did. Cause What they actually did is what you want to celebrate. This is not what you celebrate. |
| 49 | 00:06:10 | UCT | Yeah, exactly, that's what I have found with all of them, they have a really tough time writing down their thoughts. So |
| 50 | | R1 | But it's not just… |
| 51 | | UCT | That's why I am glad I took a picture of it. |
| 52 | | R1 | Good, that's good. I think what you want to do is When they get sidetracked. Some of you said were saying that they were making letters or words. |
| 53 | 00:06:27 | UCT | Yeah! |
| 54 | 00:06:28 | R1 | This is interesting because this is the first year that I have ever heard that from the teachers in the program. This is novel. Which districts did Letters and arranging? Matawan-Aberdeen, |

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| 55 | | UCT | Toms River. |
| 56 | | R1 | Toms river. |
| 57 | | UCT | Sayreville |
| 58 | | R1 | Sayreville Every district practically, Old Bridge. So This is a new twist. Maybe there is something in the water this year. But I think what you want to do is get them back and find the good in what they are doing. Because There was an awful lot of good in what your students had. Do you have another paper? |
| 59 | | TD | Yeah. This is another, her partner Jocelyn. |
| 60 | | R1 | Okay. |
| 61 | | TD | Okay, They did pretty good. But um On their paper they only drew ten towers. But when I was walking around I have a picture of theirs too. They were the ones that really seemed to get it. They could see… |
| 62 | | R1 | Okay. They had more than ten towers? |
| 63 | | TD | They had 16. |
| 64 | | R1 | Okay. |
| 65 | | UCT | So, I guess when they were drawing I don't know why. |
| 66 | | R1 | Well they had…How many of your students had difficulty recording the towers? |
| 67 | | UCT | Yeah like the actual.. |
| 68 | | TD | and like that girl, she, that was her biggest problem. |
| 69 | 00:07:37 | R1 | Now, This is a wonderful way of recording, um, I know that not everyone or every group probably used Bs and Gs to do their… |
| 70 | | UCT | They had brown and green. |
| 71 | | R1 | Well I think it's great! |
| 72 | | UCT | I liked how they saw the diagonal. |
| 73 | | R1 | Very nice, very good. |
| 74 | | TD | And they saw that okay, that's where it ended and then they have a green diagonal. |
| 75 | | R1 | And isn't that wonderful? |
| 76 | | UCT | Yeah |
| 77 | | R1 | okay so what do we have there |
| 78 | | TD | He said We kept on making a set of three and then we made a set of 4 and we found our answer. We can't make any more because then they would be duplicates. |
| 79 | | R1 | Okay |
| 80 | | TD | But He drew what he was doing. They found the diagonal of both colors. |
| 81 | | R1 | Okay, now They found a diagonal. What do they have there they have a …really, what are they doing in that grouping? What is that a grouping of? On the bottom. |

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| 82 | | UCT | Three of one color. |
| 83 | | UCT | Yeah. |
| 84 | | R1 | Okay, So it was three blues with one green. And really there they are showing you …. |
| 85 | | UCT | Yeah the green going up. |
| 86 | | R1 | The green going up.. |
| 87 | | UCT | And they had a word they called it stairs and the stairs are going down. |
| 88 | | R1 | Okay so they actually they gave a word they called it stairs. Now This is wonderful. Absolutely something you would want this group to share with the others. Ah, Also, How about in their listing of the ten towers? Um, Did they have any way of organizing those ten? |
| 89 | | UCT | First, it says first they were opposites, so you get 2. Then there they have their diagonal. |
| 90 | 00:09:01 | R1 | Good.Okay, good |
| 91 | | UCT | They have their diagonal with the green going up. |
| 92 | | R1 | Good. Good. |
| 93 | | UCT | And then….[multiple conversation-inaudible] |
| 94 | | R1 | And then they kind of lost out there but There is a lot of good stuff happening in this and what you would want them to do even to celebrate with the rest of them how they used just the first initial of the color of the block. They didn't draw they didn't use colors, they didn't write out blue green or blue brown or green brown. They just used the first letter and that's… |
| 95 | | UCT | Yeah |
| 96 | | R1 | Remember, We talked about that using a symbolic notation. So, those are things that you would want to say Isn't that a neat way to record the towers? But not everyone did it that way. |
| 97 | | UCT | Yep |
| 98 | | R1 | Good! |
| 99 | | TD | And then the last pair was a group of 3 which I had a problem with the group of three girls but. It was too much. |
| 100 | | R1 | Okay. |
| 101 | | TD | They were all trying to do it at once. They couldn't communicate you know as well with being with two. They had soooo many duplicates. |
| 102 | | R1 | Okay. |
| 103 | | TD | You know, I just felt like It took longer for them to do it. But they did finally find out 16. Um, they said We couldn't make anymore because we think we made all the patterns plus we found all the blocks and [pause]we all worked together to create these patterns. So again, they couldn't |

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| | | | really explain. |
| 104 | | R1 | Well they are..That's their reason. Is it a good convincing argument? What do you think? Are you convinced? We're done because we couldn't make any more. |
| 105 | 00:10:42 | T1 | Well She said all the patterns, so.. |
| 106 | | R1 | We made all the patterns, right but does that help you? |
| 107 | | T3 | A little bit. |
| 108 | | R1 | It does? Are you convinced? |
| 109 | | T3 | Well I mean for this group because of the level. |
| 110 | | R1 | No, you can't do that. And I know that a bunch of you said that online. You can't say the kids are young so we are going to expect less. If you want a convincing argument, you want a convincing argument. |
| 111 | 00:11:07 | T3 | I'm not saying they are young, I am saying they are special ed. |
| 112 | | R1 | No, I understand, but I'm saying online some of you said you watched the video the kids were young and we are accepting what they say and it is convincing enough. It's not. It doesn't mean that they should be doing more than this. I bet a lot of your students whether they were regular or special ed. said we are done because we tried and we tried and we can't find any more and therefore we are done and we got it. Okay. How many got that as an argument? [pause] Regular ed. and special ed. Correct? |
| 113 | 00:11:42 | T6 | A lot of my kids said that but they at least recognized that it wasn't good enough. |
| 114 | | R1 | Okay. |
| 115 | | UCT | Like even this one girl that I have it is not one that I am going to share. |
| 116 | | R1 | Okay. |
| 117 | | UCT | But she wrote We tried all the ways and we know that we can't make anymore but I know that is not very convincing though. |
| 118 | | R1 | Good |
| 119 | | UCT | And then she put dot dot dot. |
| 120 | | R1 | Okay, I think that is great. It isn't convincing but that may be where some of the students are now. Okay, I'm not talking just special ed. Regular ed also. Okay, but it isn't convincing but this is neat notation too. Look at how they actually drew the unifix cubes and they put the letter of the color in each one. So that would be another thing to share. Good. |
| 121 | | UCT | And that's all the pairs, that's all. |
| 122 | 00:12:27 | R1 | Nice, okay you got some good work there. Who else wants to go? You are all going so who wants to go next? |

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| 123 | | UCT | Do you just want to take a picture of it? |
| 124 | | UCT | With the Ipad? |
| 125 | | UCT | Yeah. |
| 126 | | R1 | I guess that is what most people will do. |
| 127 | | UCT | Yeah, |
| 128 | | UCT | Are we supposed to bring copies of them? |
| 129 | | R1 | No No because she said she would be able to take a picture and put it up. Um, The next school we go to is Beachwood, right? |
| 130 | | UCT | Yeah. |
| 131 | | R1 | Do we have a document camera or an elmo? |
| 132 | | MM | I have the same thing, I have an Ipad so we will just take a picture. Yeah, As long as its wireless. |
| 133 | | R1 | Yeah, alright. Um, I guess we will just do it. Because it is a pain in the neck to make overheads, right. |
| 134 | | UCT | If, I just do it half…I can just take a picture. |
| 135 | | R1 | No, you can't all look at it. |
| 136 | | UCT | No, Instead I have my I pad cable so I can just use that. |
| 137 | | R1 | That would be good. Okay. |
| 138 | | NL | Okay, um alright so this is one of my students, he is a seventh grader. Um,I teach special ed. but |
| 139 | | R1 | Okay |
| 140 | 00:13:47 | UCT | So the one over here he realized after he drew them all that he drew this one already. So he went back and crossed that out. |
| 141 | | R1 | How nice, Good! |
| 142 | | UCT | But he did have 16. He did get the right answer. |
| 143 | | R1 | Very nice. And isn't it great that you can see what he crossed out. Remember we talked about the idea of not using a pencil which you can erase and then you can't see what the change was. It is nice to see that he recognized these too. |
| 144 | | UCT | Because He had counted these and got 17 but he knew he had 16 in front of him. |
| 145 | | R1 | Good, good |
| 146 | | NL | So then, I asked him what do you think you did? |
| 147 | | R1 | Good, excellent |
| 148 | 00:14:19 | NL | So, with My kids I noticed they can explain verbally more than write. But so his explanation he wrote on the back. |
| 149 | | R1 | Okay, What did he write, can you read it to us? |
| 150 | | NL | Yeah I can read it. He said I did the same block twice but not the same color like a pattern sort of. The reason I did it like that is because it is easier for me. |
| 151 | 00:14:40 | R1 | Okay, so that's what he wrote. Now, many of you had |

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| | | | trouble with the kids writing. Who had something that they did that helped with that, Did anyone figure out what they could do? There were a couple of you what did you do? |
| 152 | 00:14:54 | CP | What I did was, Once they were actually able to formulate their explanations. |
| 153 | | R1 | Good. |
| 154 | | CP | They had it in different groups. So, for each group, They would explain it to me. |
| 155 | | R1 | Okay |
| 156 | | CP | They actually had to explain to me probably about five times I would say altogether. But after the last one, they were like This is what I want to tell you and say tell me for this group and then write it down. |
| 157 | 00:15:17 | R1 | Excellent |
| 158 | | CP | A lot of them needed help with the spelling or words or like pretty much sentence for sentence saying to me and I would help them get it down. |
| 159 | 00:15:27 | R1 | Good okay and that is a very neat strategy what he did. He said Tell me what your justification for this group of towers and then when they would tell him he said that’s great get it down. Okay. |
| 160 | | CP | They went through and told me the whole thing. And I was like Okay now write it. And One of my girls I put it on the post she actually started crying. She was like this is too much. I don’t understand what’s going on. |
| 161 | 00:15:49 | R1 | We don’t want them to be crying. |
| 162 | | CP | That’s alright. Okay let me sit and help you get through this because we don’t typically get to work like this so it was really…. I mean to write this much. She ended up writing like pages. |
| 163 | 00:16:06 | R1 | That’s great! That’s great. |
| 164 | | CP | She just needed that. |
| 165 | | R1 | And how many of you... it should be everyone is doing state testing for regular and special ed. Do they need to write on this? |
| 166 | | unison | Yep! |
| 167 | 00:16:18 | R1 | Of course they do! Okay. So It’s not an option that math doesn’t require writing. Math has to have writing. Math has to have communicating. That’s one of the standards, you know where you communicate your…your mathematical thoughts. So your strategy is a very, very good one.<br>You want to ask them Tell me what, how you did this, how you grouped these together, why you think you have them all. |

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| | | | And Probably the easiest one is if you look at the first two towers. Okay.<br>For those that are having trouble with the rest. For the first two towers, can you tell me why you think you have them all in that group, if that is a group.<br>And I think they could tell you that. What would they say? |
| 168 | 00:17:00 | CP | That was..That was my starting point. They were like those were the first ones they wanted to talk about. |
| 169 | | R1 | Good, and what did they tell you for those two towers? |
| 170 | | CP | They are the same. |
| 171 | | UCT | There is only two colors. |
| 172 | | R1 | There is only two colors, good. Get that down on paper. Okay. |
| 173 | | UCT | Yeah. |
| 174 | 00:17:20 | R1 | Then they already have a justification for the solid towers that are all of one color. Okay, they already have it and they can get that down if they say we made two towers, one that's yellow and one that's blue; whatever the colors were and we are done because there are only two colors. So we have all the towers in that group. And that would be convincing.<br>And All of your students probably could do that even if they couldn't do the rest. |
| 175 | 00:17:45 | T2 | Oh yeah they definitely could. Mine, like with the questioning and everything. |
| 176 | | R1 | Good. Good |
| 177 | | T2 | The problem with mine is, like they would say it and get it. So then I would say, write it, and then they would just look at me. |
| 178 | 00:17:56 | R1 | Yeah. |
| 179 | | T2 | So Getting them to actually write it was definitely the hardest thing. But the…Verbally talking to them most of them got it. |
| 180 | 00:18:02 | R1 | Okay, well That's good that is what you want to…even if they can't write everything you want to get them to write something.<br>And your strategy is the one I would use. Tell me about the first two towers. That looks like….if that is a group. If that is a group,<br>Why is that group complete and there aren't there anymore? Okay And then they write it. That's good. Now did they have the towers on the first row; I'm looking at towers Three four five six, seven and eight What kind of towers are those? |
| 181 | 00:18:36 | T2 | Um, Well it was funny. Because He listed them in |

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| | | | opposites but he didn’t draw them that way. |
| 182 | | R1 | Oh…kay. |
| 183 | | T2 | So when he did them, he started doing them opposites but then I guess between him and his partner They were getting confused about what they were drawing and that is probably why he didn’t stick with it. But um they did originally do opposites. |
| 184 | | R1 | Okay. So they were in pairs. |
| 185 | 00:18:56 | T2 | They were in pairs, yeah. |
| 186 | | R1 | So what you want to do is have them show on paper that these are in pairs. It is hard to tell that now, because I already made a group of 6 but that is not what they did. So they can just put a line between the groups and then you will know how it was on the table. |
| 187 | 00:19:11 | T2 | Yeah. His partner did. But he didn’t. |
| 188 | | R1 | Oh, good. Excellent! Look at that neat key down there, right. How many of your students did a key? [Pause]And some of them and that would be something that you want your students to share with the rest, nice! |
| 189 | 00:19:32 | T2 | Yeah, so that was one and I pretty much showed them that each one recorded it differently. So I thought that was interesting. |
| 190 | | R1 | Okay, good. Yep. |
| 191 | | T2 | This pair. He is actually like one of my lower students |
| 192 | | R1 | Okay. |
| 193 | | NL | Out of the group and he is a seventh grader too; but him and his partner were able to tackle this right away. They got 16. Again they had a hard time explaining. They said The tower we were building is four inches high and it is y and b. |
| 194 | | R1 | Okay well they are saying four cubes high for four inches. That’s fine. |
| 195 | | UCT | Yeah. A cube was an inch to them so.. |
| 196 | | R1 | Okay and maybe it is. Y and B would be the colors? |
| 197 | | UCT | Yellow and blue. |
| 198 | 00:20:05 | R1 | Yeah so now did they group it in any particular way. Can you see? |
| 199 | | UCT | Yeah they did like pairs. This was all blue and yellow all yellow and blue. This was the opposites. |
| 200 | | R1 | Okay, So they did opposites there. |
| 201 | | UCT | Same thing, yeah but they numbered them. |
| 202 | | R1 | Okay, So again you want to put it on paper lines between the groups to show how they grouped them on the table, okay, good. |
| 203 | 00:20:28 | NL | Well they didn’t actually group them. They grouped them |

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| | | | like this but they actually had them all on the line. |
| 204 | | R1 | Oh they were… |
| 205 | | NL | That’s probably why they drew them like that. |
| 206 | | R1 | Okay |
| 207 | | NL | Because Even though they did opposites. They had them all against the table like that. |
| 208 | | R1 | Okay, okay. |
| 209 | | NL | So that was another way they recorded it [shows yellow and blue colored towers] and then one of my students was like Wouldn’t it be easier to use yellow and blue crayons. So this pair Jenn and [unintelligible] wanted to use the crayons. She didn’t write it for me.[unintelligible] So this pair Jenna and…she definitely knew what she was doing [unintelligible] |
| 210 | 00:20:51 | PA system | Announcement: Good afternoon, It is now 4:00 all students going home please go to door 23… |
| 211 | | R1 | That’s a good thing. Right okay. |
| 212 | | UCT | She did it anyway. She did enjoy it. A lot of my kids enjoyed it but talking to her; she definitely knew what she was doing; but she couldn’t she had a hard time just writing. But verbally like questioning her she did well. |
| 213 | 00:21:14 | R1 | What’s good about what she did? Or She had the towers arranged what do you see? |
| 214 | | UCT | The blue diagonal. |
| 215 | | UCT | Yeah, the blue diagonal, yeah. |
| 216 | | R1 | She has a blue going down in each position. Uh huh. |
| 217 | | UCT | That’s funny why she didn’t do it on the bottom. |
| 218 | | R1 | No she didn’t |
| 219 | | UCT | With the yellow |
| 220 | | R1 | Isn’t that interesting, right. |
| 221 | | UCT | Yeah. |
| 222 | | R1 | Um, And I think um, that what again, you want to do is when they are sharing, you might want her is that how she had those four together. Were they… |
| 223 | 00:21:43 | T2 | These five in a way. She got.. |
| 224 | | R1 | She put 5 together. Okay. |
| 225 | | T2 | But then the blue ones, The ones with the yellow in different positions; She didn’t have them grouped like that. |
| 226 | | R1 | Okay |
| 227 | | UCT | Even the colored one, The dark was shaded |
| 228 | 00:21:54 | R1 | Maybe |
| 229 | | UCT | Yeah she might have been able to. |
| 230 | | R1 | Maybe…and you might want here to share with the others why she grouped it that way. Can anyone get a convincing |

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| | | | argument that those are all the towers that could form that group? And why can't you have a blue in the fifth position? Remember what I asked you? It is a good question to ask the kids too. Alright so you are working with self-contained special ed. |
| 231 | | NL | Yes. And this was an eighth grader. |
| 232 | | R1 | And you are self-contained special ed.? |
| 233 | | NL | Seventh grade Yeah, resource. |
| 234 | | R1 | Resource, Okay, great. |
| 235 | | NL | My last one is the most interesting one. |
| 236 | | R1 | Oh good. |
| 237 | 00:22:31 | NL | Okay. This is not how he built his towers at all. When I asked him to just write me an explanation as to why he even drew it this way. He just kind of froze and just stared at me and was like I don't know why. This is just the way I see it.<br>So He wrote 16 combinations total and he got it right I think I have all the combinations for this worksheet. |
| 238 | 00:22:49 | R1 | How did he build the towers? What did they look like? |
| 239 | | NL | They were pairs, I think. A lot mostly all my students got either opposites or pairs. |
| 240 | | R1 | Okay, okay. |
| 241 | | NL | Only like Estrello found the staircase-looking positions. But um His partner I didn't include his partner because he copied him. He really had no clue really what was going on. |
| 242 | 00:23:14 | R1 | Well I don't know did he know what he was doing? I'm not sure I understand. |
| 243 | | NL | Well with the towers he knew, |
| 244 | | R1 | Okay. |
| 245 | | NL | Well like he knew how to tackle the problem |
| 246 | | R1 | Okay, good, Good. |
| 247 | | NL | but as far as recording he had no clue. |
| 248 | | R1 | Okay and Again, what you want to try and do as best you can. Because it is hard for some students to do the recording. It is hard to know which towers they already put down on the paper and which ones still have to go um and probably what did any of you do something that helped students with their recording? |
| 249 | | T6 | Well, I didn't give them the paper until they were really completely done which was good. |
| 250 | | R1 | Good |
| 251 | | T6 | But also kind of not good because I don't think some of them had enough time to write their argument out. |
| 252 | 00:23:55 | R1 | Okay |

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| 253 | | T6 | Um. And I think If they had a little bit more time. |
| 254 | | R1 | Okay. |
| 255 | | T6 | They might have been able to make a better argument. |
| 256 | | R1 | Okay. |
| 257 | | T6 | Um. I just told them to you know put your blocks how you want them to look on your paper and then copy it and that's what they did. They didn't really have a problem with it at all. |
| 258 | | R1 | They didn't have a problem. Okay, Anyone have students that did have a problem and how did you get around it? |
| 259 | | VB | Um. First my students were a little confused on how to draw them and started tracing the whole thing. |
| 260 | | R1 | Okay. |
| 261 | | VB | Um Then they started drawing them and they would lose track where what they were drawing. And then they would start drawing them and they would lose track. So I showed some of them like how to draw these. Take one put it on paper and then put it away. |
| 262 | | R1 | Okay. |
| 263 | | VB | And I found it easier if one kid drew it and they copied from the other kid's paper. Two of them doing it they were getting too confused, it was too much with working with partners. |
| 264 | | R1 | Okay. |
| 265 | 00:24:42 | VB | Also with them verbalizing. I did what [T8] said I had each of them, explain to me. But once they explained it to me, I said okay, now repeat this. And We did it line by line by what with the partner recording what the other person said. |
| 266 | | R1 | Okay, okay. |
| 267 | | VB | Because I do a resource room too. And with expressing, they had a hard time to do that they could tell me a second later but they forget it. |
| 268 | 00:25:03 | R1 | Yeah. It is hard to do that. So that you had a couple of things you did that seemed to work for you and that's good. Um, if you have students that have their towers standing up, okay on the table and they are having trouble recording, as they do you put the tower down, once they record. And If they are both recording, they both have to be patient. You can't have one putting towers down and the other one still recording. But You have them working together that way, which you know is helpful. |
| 269 | 00:25:34 | GH | I had one group that was convinced there were 32 towers. |
| 270 | | R1 | Were they doing 5 tall? |
| 271 | | GH | No. They were convinced that if they stood this way or laid down they were two separate towers. |

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| 272 | | R1 | Okay. |
| 273 | | GH | No matter what I said or did they… |
| 274 | | R1 | Okay.  Interesting. |
| 275 | | GH | it would not sink through to them because the nob was still at the top. |
| 276 | | R1 | Okay. |
| 277 | | GH | No matter how they were lying, they were convinced. |
| 278 | | R1 | Okay, Did you put them next to each other? |
| 279 | 00:26:01 | GH | I did |
| 280 | | R1 | And? |
| 281 | | GH | and they said but it doesn't matter. One is laying down.[laughter] |
| 282 | | UCT | One of mine did that. |
| 283 | | UCT | It doesn't matter. Yeah. |
| 284 | | UCT | It was weird, I don't know how to get around that. |
| 285 | | R1 | Uh huh. |
| 286 | | GH | And Another one said they put three up and then put some that way and then said This is four tall but it is four tall but they are not connected. So you can't make that claim. |
| 287 | | R1 | Okay so it was the idea of them thinking outside the box which is sometimes a good thing and sometimes gets you in trouble. |
| 288 | 00:26:31 | MM | One of my groups kind of like made the blocks so that were only partially connected and arched it.  They were like Does this count as a tower? But they knew that it didn't count as a tower. |
| 289 | | R1 | Okay. |
| 290 | | MM | I think they were just avoiding writing their argument down so they were trying all these different things.  I looked at them and I was like do you think that is a different tower? |
| 291 | 00:26:51 | NL | I feel like A lot of mine did that They were completely trying to avoid the explanation part by continuing to find more, there's more.. |
| 292 | | R1 | Well but there may be more but you don't want to cut them off if they really think there is. But Z------…? |
| 293 | | NL | Z------, yeah. |
| 294 | 00:27:06 | R1 | What I would probably want to do is if he had ten towers there, you say I love what you did. And that's telling them to that you need to see it again. Announcement: This is door 23…. I need to see that on paper. |
| 295 | | NL | I did. I said How am I going to look at your paper and know this was 3 yellow and one blue and this is 3 blue and one yellow. He was like it is right there and okay and I was like .. |

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| 296 | | R1 | Uh huh., cause you have to see it. |
| 297 | | NL | But this doesn't help me. I had him try it again on paper. |
| 298 | | R1 | Uh huh. |
| 299 | | NL | And on the back he did come up with something, he said I drew the picture because I wanted to do something fun. He crossed that out and that's the way I want to make it. |
| 300 | 00:27:48 | R1 | Okay, now I think that you have most students that if you are trying to explain to them you really want to see what's on their desk because you like what they did.<br>Most of them will do it. Obviously Zachary didn't but um… Okay, thank you good job. Okay. Are we passing around the I pad? |
| 301 | | UCT | No, mine doesn't connect. |
| 302 | | R1 | It doesn't connect. Okay. |
| 303 | | UCT | It is different and I thought I had it. I have one it's not working… |
| 304 | 00:28:18 | R1 | Before you begin, I want you to say what grade your teaching, and whether it is regular ed., special ed., resource room, inclusion…[long pause] |
| 305 | 00:28:47 | MC | Okay. Alright I am in Sayreville, and I teach seventh grade resource. |
| 306 | | R1 | So, This is pull out |
| 307 | | MC | Yeah, pullout resource. I have 12 kids, 2 girls, ten boys. This is actually a group of girls. They probably did the best out of the entire class. Um,<br>They arranged the blocks basically by the different patterns that they saw. And …When it came to explanations, it was really more about what was in the blocks not why there was you know the reason of why there was only 16. Uh, So if you look at the first one, there is one solid color and four blocks high and there is no other way of doing this. |
| 308 | 00:29:21 | R1 | And why? Why is there no other way of doing it? Because there are only two colors okay so that….Before you go, look at what they did and read through and let's talk about it. [Long pause to read the following argument: There is one solid color, and 4 blocks high. There is no other way of doing this. There is only 1 yellow and 3 blues in each tower. There is a pattern, the yellow keeps moving up one. There is only 1 blue and 3 yellow in each tower. There is a pattern, the blue keeps moving up one. Two of the same color is touching, and one color isn't touching. Not one of the same color is touching. Both colors are next to their twin.] |
| 309 | | UCT | I think that it is very good. |
| 310 | | R1 | I think that it is very good too. Remember this is what |

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| | | | grade resource? |
| 311 | | T3 | Seventh grade resource. |
| 312 | | R1 | Seventh grade resource room. This is Spectacular, isn't it! |
| 313 | | UCT | I like that they said she said twins. |
| 314 | | R1 | They did it together. |
| 315 | | T3 | So these two were the stair case thing that they called. This one I thought was interesting because they said that there was a pattern and they saw that The yellow one keeps moving up one and then the opposite they just said that was that The blue one keeps moving up one. And they kept referring to that as a staircase. |
| 316 | 00:30:11 | R1 | And that's very good. Isn't it? What might you ask them to push them one step more for the three of one color and one of the other? |
| 317 | | T3 | Why can't you go up one more step? |
| 318 | 00:30:22 | R1 | Good why can't you go up one more spot and then if they got that, that would be a really nice convincing argument. Keep going. |
| 319 | | UCT | So the next group, um, they said two of the same color is touching and 1 color isn't touching. So they kind of said to me that two of the yellows are touching Then they alternated them so that none of the same color is touching. Both colors are next to their twin. |
| 320 | | R1 | What do you think of that I think if you wanted to get a convincing argument for the last 6 towers. What might you do? How do we know we have all the towers that are exactly two one color and two of the other. What would you ask them to do? Are you convinced? |
| 321 | | UCT | That is the most convincing that I have read. |
| 322 | | R1 | This is excellent work. Okay, I would really say…You know what I am curious did they do this project last year with their teacher. |
| 323 | | UCT | This particular girl is new to my school. |
| 324 | | UCT | Rianna I had her last year. |
| 325 | | R1 | And she did not. Okay because This is really superb work. Whether it is regular ed or special ed. Right, remember when took 6 towers that were two of one color and two of the other and we said Can you group them another way. And That might give a possibility of a convincing argument. Okay, Very, very nice. That's neat! |
| 326 | | MC | This group was a group of boys. And they immediately went to the math of it and said that there are two colors and they have to be four high. Two times four is 8. And then once they made the first 8 blocks they realized they could do opposites. So they doubled them and made 16. |

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| | | | [Actual student argument follows: We have 16 in all. We think we have all the towers because each tower has 4 blocks in it and there are 2 colors to make the tower so 2x4=8. Then we realized that we can invert the colors to double the towers so 2x8=16. Then we couldn't make any more so we think that we made all the towers. We put the towers this way because it's a pattern. Every time the pattern moves it always has one small difference.] |
| 327 | 00:31:08 | R1 | Interesting that was the one you wrote about, right? |
| 328 | | UCT | Yeah. |
| 329 | | R1 | What do you think of their argument? That there were two colors and four blocks high. Two times four is 8. Yeah, two times four is 8. But what do you think of that argument? Two times four is 8 and then you double it because they are opposites. |
| 330 | 00:31:28 | T3 | I thought it was pretty good that they immediately jumped to the math of it and not just…. |
| 331 | | R1 | Well but Does it make sense? |
| 332 | | T3 | No |
| 333 | | R1 | What mathematically… |
| 334 | | T6 | Yeah One of my kids said that it was oh well there is 16 because it's four times four and then so I asked him how does that make sense though? Why would that work and he said oh well there is four blocks. |
| 335 | 00:31:51 | R1 | Right. |
| 336 | | UCT | And then he was like Then the other four and he realized that it didn't work So then he went to this. Oh never mind it is two times four and then everything has enough and then he did the same thing 2 times 4 is 8 times 2 is 16 for 4 high. |
| 337 | | UCT | That's what he said. First he realized that there are two colors that are four high and he then built the 8 and then they kind of looked at it and were like but wait you can switch them by doing like the opposite. And then they were like but we can just double the 8 to get 16. |
| 338 | 00:32:17 | R1 | Okay now go back to four towers. Say it again. |
| 339 | | UCT | That doesn't work for other ones, though. |
| 340 | | R1 | Let's think about it. Let's take a different height. Let's take the towers that are 3 tall? Okay, Does that work? The 3 tall? |
| 341 | | MC | No. because then he would think that there are 12. |
| 342 | | R1 | And they aren't. So Again, just the way the student says I got it. It's four times four. And Four times four is 16, isn't it alright? So then what are five tall? Oh that's five times |

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| | | | five.  Oh wait a second, that's 25 and it's a tower and its opposite.  So there has to be an even number.  And remember I told you that my middle school students when I did this the very first time, at Monroe township; they said oh I know what it is that's five times five minus one, okay. Students can take numbers and they can you know make the numbers into some mathematical sentence that is a correct sentence.  But does it make any sense? It really doesn't. So the 2 times yeah 2 times four is 8; but then if you do powers 3 tall, 2 times 3 is 6, gets doubled is 12. And how many towers were there 3 tall? |
| 343 | | UCT | 8 |
| 344 | | R1 | There were only 8. |
| 345 | | UCT | I think they know it doesn't make sense but at least with that, they could rationalize where those numbers are two colors four high.  So I know that doesn't mean you get the other towers. But I think from the students, at least they are saying subtracting one, they have no meaning. |
| 346 | | UCT | Those numbers do have meaning in the problem. |
| 347 | | R1 | But they don't |
| 348 | | UCT | But they are not doing what they are supposed to. |
| 349 | 00:33:51 | R1 | They have meaning but just the way 4 times 4 gives you the correct answer of 16 for towers that are four tall, that is not how you get the towers four tall.  Do you Remember how we got them? |
| 350 | | UCT | Well, I know that but  I am just saying….they know that it doesn't make sense |
| 351 | | UCT | Two colors and four like The numbers are coming from somewhere. 4 times 4 towers; whereas the other ones… |
| 352 | | R1 | The numbers are coming from somewhere. But Is that why you are getting towers? |
| 353 | | unison | No. |
| 354 | | R1 | No |
| 355 | | UCT | But what I am saying from a student's point of view every number does have a meaning in this problem. [multiple responses at once-inaudible] |
| 356 | | R1 | Say it again. |
| 357 | | MC | I think maybe if he had done the one high, two high, three high and did the whole process, |
| 358 | | R1 | Right. |
| 359 | | MC | it would have probably then they would have realized Oh well 2 times 8 times 2 is not going to get 16 but that doesn't work for when we did the one tall, the two tall, where here we only did the four tall, |
| 360 | | R1 | Okay. |

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| 361 | | MC | So I think for this, the math does make more sense to them because we are only doing the four tall. |
| 362 | | R1 | Let me go back. Cause you are saying something very powerful. Carolyn Maher is so against programming students. Did you see that on the video yet? Where Carolyn Maher comes on and she talks about how we don't want to do towers one tall, then towers two tall, then towers three tall, then towers four tall. |
| 363 | | UCT | No. |
| 364 | | R1 | Okay, you didn't see it yet. Okay. You might be seeing it later. Um, She is..Carolyn Maher is a professor at Rutgers. Have you met her? |
| 365 | | unison | No. |
| 366 | 00:35:15 | R1 | No? She is in some of your videos when she is interviewing students. She is one of the people that worked in Kenilworth. She said the reason why we don't want to do that is we don't want to program students and it would be very easy to say first you start with one tall, then we go to two tall, then we go to three and then we… we are able to do that….<br>But If you start with 4 tall right away they have to develop a way of thinking about it that makes mathematical good sense without being programmed. |
| 367 | 00:35:49 | MC | But I feel like in the video that is what they did do. Like the start with showing the one tall, then building the two tall and then adding the red and the yellow. |
| 368 | | R1 | Are you talking about Milin? |
| 369 | | MC | Yeah. |
| 370 | | R1 | Okay, But Milin was not the teacher doing it. Milin that's the way he thought about it. That was the student who said the way I see the problem. And He was using an inductive argument, remember we talked about that. When you start with one tall and then you can put two colors a red or a yellow on top of it. The student is coming up with that not the teacher. The teacher is not saying let's all start with towers one tall today. And then let's build, 2,3,4. So, It is different, okay. This also, even though, yes If they said to you I know there are two colors. Yes there are that's good and I know we are building them four tall, yes we are, but two times four is 8 doubled is 16. Doesn't really make mathematical sense in getting your answer here. The way that four times four doesn't make sense in getting the answer. Yes, it works but it doesn't make sense. |
| 371 | 00:37:00 | T6 | We know that right but could we say okay I like what you are thinking. Like Can we test your theory? |

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| 372 | | R1 | Good! I love it! I love it. |
| 373 | | UCT | If you have three towers high. What would you say? |
| 374 | | R1 | Good. And now build towers 3 tall. |
| 375 | | UCT | So you would take the next two and then three tall and see how many you got…. |
| 376 | | R1 | Good. That is beautiful What she is saying is let's have you tell me Make a prediction if this is the way you are thinking about 4 tall, make a prediction for 3 tall and they would come up with 12 And then What you are saying is beautiful, I love it! |
| 377 | | UCT | So you are not telling them, right. |
| 378 | | R1 | No you are not. You are not telling them. And actually what you are doing is you are giving them the opportunity to get some disequilibrium. I say it is going to be 12 towers 3 tall. Wait a second I can't find 12 I am only finding 8 so maybe I better revise my theory. Okay, very good, that's nice, That's very very nice. Okay. |
| 379 | 00:37:51 | UCT | The only other thing I saw in this one that was really cool was he used vocab that I was kind of surprised about. Like He used the word dominant. |
| 380 | | R1 | Nice. |
| 381 | | UCT | Where blue is dominant |
| 382 | | R1 | Nice. |
| 383 | | UCT | Where yellow is dominant. |
| 384 | | R1 | And What did he mean by dominant? |
| 385 | | UCT | That it was..In this case, Blue is dominant because it is the only one that is up there. For this one, he is saying like yellow is dominant because it is all yellow. He even wrote, instead of opposite, he wrote invert. |
| 386 | | R1 | Invert? Okay neat. so he is using language. Interesting I have seen kids use the word dominant. And they meant that if you have most of it…But You know I think that once they define it for you. That's their language. They are developing language. Um, So good! |
| 387 | 00:38:32 | UCT | And then the last one of mine is the one I wrote on-line. |
| 388 | | R1 | Okay. |
| 389 | | MC | This group only came up with 12 towers and after they created the 12 they arranged them so that it spelled out the word math. So they literally had put the towers in the shape of an M, A, T, and H. They literally had none left over. So that is where they left off. |
| 390 | 00:38:53 | R1 | Now, again I would encourage you do not celebrate their creativity when it is spelling out the word math or making a T or building a cube. Um, That is….I mean Sometimes we do want them to think outside the box, okay, but we |

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| | | | want it to be mathematically sound and spelling out math is very cute, but it is not where we are going. |
| 391 | 00:39:20 | MC | Yeah, that’s what I kind of said to them. Do you really think that’s the purpose of this? |
| 392 | | R1 | Okay, Good! |
| 393 | | MC | They were like no, but that is what we came up with, Alright so I let them keep going. |
| 394 | | R1 | How are they arranging their towers? Someone in the group. |
| 395 | | UCT | Opposites.[multiple responses] |
| 396 | | R1 | It looks like opposites also. Doesn’t it? Okay. And if you have opposites, I know someone on the group online said. And remember it is so hard. It is hard. It is almost impossible because with 16, you can find the 16 by doing opposites. How many of you have done 25 towers of? Sorry towers of 5 tall…you actually built the towers five tall? |
| 397 | 00:40:06 | UCT | I haven’t started yet. |
| 398 | | R1 | Okay, Did any of you yourself build towers 5 tall? Do we have cubes in the…, [T1], |
| 399 | | TD | I can get them. |
| 400 | | R1 | you might want to when we are done here. |
| 401 | | TD | Okay. |
| 402 | 00:40:19 | R1 | If you have them, or If you yourself build towers 5 tall with the strategy of using opposites, it ain’t so easy finding all the towers. It really isn’t. Okay it is kind of hard. So, I think that and forget about the um you know getting a convincing argument. It would be too hard to find them. Okay, good. [long pause] Okay, good. Great, Let’s see what we got here. Okay, so what district, what grade? |
| 403 | | T4 | This one.is .So I am from Long Branch, the behavioral school. And I have the non-classified students but most of them probably should be classified as emotionally disturbed. |
| 404 | | R1 | Okay |
| 405 | | JLB | Um This student I was kind of impressed with him because last year when I had him in seventh grade and now he is in eighth grade. |
| 406 | | R1 | Okay. |
| 407 | | JLB | Um, In seventh grade, he did nothing. |
| 408 | | R1 | Okay. |
| 409 | | JLB | It was almost like he was mute. The only time I ever heard him was when he was cursing in like a whisper. |
| 410 | | R1 | Okay. |
| 411 | | JLB | So This year he came in with a completely different |

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| | | | attitude. He actually wants to do well in math. |
| 412 | | R1 | Nice, Good. |
| 413 | | JLB | So this was a good activity for him to do in the beginning of the year. |
| 414 | | R1 | Good |
| 415 | | JLB | He didn't organize his work. Him and his partner did not organize the work. They had them altogether. [The following student's work is projected on the screen: I started with red and the red kept going down one per row. The pattern goes diagonal. I started with yellow and on each one it would go down by one. The pattern goes diagonal. Two red and two yellow are opposite colors. I did 4 yellow on one side and 4 red on the other.] |
| 416 | | R1 | Okay |
| 417 | | UCT | And When I questioned him he kind of answered my question about why he can't go another red up and everything. |
| 418 | | R1 | Right. |
| 419 | | UCT | He did answer my question |
| 420 | | R1 | Okay. |
| 421 | | UCT | And then when I gave the paper to write it, he just started doing something different, And He started picking the towers out of the 16.[inaudible] A lot of my kids got very upset with having to do it in pen. They were crying for white out constantly because the kids I have don't like to make a mistake. |
| 422 | | R1 | Right |
| 423 | | UCT | So They don't like to try at all. |
| 424 | | R1 | Right. |
| 425 | | UCT | The fact that they were trying and it was pen and they did something wrong. I had a lot of kids that liked walked out. Um So he knew he made a mistake here with the five. |
| 426 | | R1 | Right |
| 427 | | UCT | But you can tell he didn't bring the line all the way down. He was trying to make up for the fact that this was the only one. |
| 428 | | R1 | Sure, sure. |
| 429 | | JB | Um And all of them gave examples, all my students pretty much said the same thing about opposites. At least he said for this one it was going diagonally. |
| 430 | | R1 | Okay. |
| 431 | | UCT | But he didn't add anything else to it. |
| 432 | | R1 | So what do you guys think? [pause] |
| 433 | | UCT | It is a good start. |
| 434 | | R1 | It's a very good start. |

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| 435 | | UCT | It is a good start. |
| 436 | | R1 | It’s a very good start. I mean there he did group it with the threes and ones and that’s very good. He is using a proof by cases. That’s what this is. When you have it grouped by all the towers that are three of one color and one of another; all the towers that are two of one color, two of the other. |
| 437 | | UCT | Yeah. |
| 438 | | R1 | All the towers that are solid. |
| 439 | | UCT | There are outer and inner. Then outer and inner. |
| 440 | | R1 | Right, okay.so he is doing a proof by cases. Is it complete? No, does he have it written down what the argument is; No. but his recording is beautiful and I bet if you questioned him. |
| 441 | 00:44:08 | JB | I did question him. He did say it verbally. But.. |
| 442 | | R1 | Nice. |
| 443 | | UCT | But speaking for him, it was overwhelming just talking I think for him. |
| 444 | | R1 | Okay. |
| 445 | | JB | This is what he had. |
| 446 | | R1 | Okay, okay. Now if you have a child with special special needs. Um, you might want to even you know After they say it, you might want to be the recorder and write down the words he is saying, you know, Not editing. Also the idea for all children. They don’t like to make mistakes. They want to get rid of whatever it is that that they don’t want you seeing because it’s not… But you do want to see where they started and where they are going. So if you let children know that it is okay to put a line through it and then I am not even going to look at it. Although you will look at it. This is nice |
| 447 | 00:44:54 | JB | Yeah. Another thing that motivated my kids was I constantly tell them what I am doing in class. |
| 448 | | R1 | Good. |
| 449 | | JB | So I told them I was coming here. |
| 450 | | R1 | Nice. |
| 451 | | JB | I was getting early dismissal so I told them I was coming here. |
| 452 | | R1 | Okay, okay, Good. |
| 453 | | JB | That I think also motivated them. |
| 454 | | R1 | Nice. |
| 455 | | JB | And that was this girl. |
| 456 | | R1 | Did you tell them that you were going to share their work? |
| 457 | | UCT | Yeah |
| 458 | | R1 | and how proud you are... |

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| 459 | | UCT | Yeah. |
| 460 | | R1 | That's great. That's really good. |
| 461 | 00:45:15 | JLB | Yeah. This girl. She is in my six-seven class. She is a repeat sixth grader and she is 14 years old and she has been left back twice now. She has been trying really hard now so it is kind of nice and She she just couldn't put it into words at all really but she did get at least this down. The following student's work:<br>[We started with yellow and then we moved them down and moved them down until yellow gets on the bottom or the same thing but the red going down.] |
| 462 | | R1 | Yep! |
| 463 | | UCT | She did say For the same thing but red going down. |
| 464 | | R1 | Isn't that nice. So she actually saw that three yellow and one red would be the same as three red and One yellow. |
| 465 | | UCT | Yeah. |
| 466 | | R1 | Very nice. |
| 467 | | UCT | Yeah because she was working hard.[inaudible] |
| 468 | | R1 | Very nice. Good, that's great. |
| 469 | | JLB | She doesn't have all of them here but. She kind of colored it funky but...she did what she told me she did but she didn't write it down. She saw the step thing going down. |
| 470 | | R1 | Okay, okay, And it looks like she has y's for the yellow. |
| 471 | 00:46:06 | JLB | Yeah. |
| 472 | | R1 | You see that? That I would celebrate and I would let her show the others how she can use the letter y to show the yellow. |
| 473 | | JLB | Yeah |
| 474 | | R1 | Right. |
| 475 | | JLB | Yeah, She did that and she didn't want to do the writing. So she asked for the markers. So then that started the whole trend of I want the markers too. |
| 476 | | R1 | Right. Right. |
| 477 | | JLB | so they can avoid the writing a little bit. They all wrote something but…. |
| 478 | | R1 | Okay, And you are going to want to push them for the second task a little bit more and help them if they need help writing with using your strategy tell me what you think for this group; write it down. Or your strategy which is you know asking them to tell you and then you record it. Good! Okay. |
| 479 | 00:46:50 | UCT | That's it! |
| 480 | | R1 | Okay. |
| 481 | | UCT | You know it's funny my kids did this activity. I told them I am going to Rutgers. They loved it! And then We did |

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| | | | something a couple days later on integers with cards. Oh this is so much fun, Is this from Rutgers too? I'm like ah no. |
| 482 | 00:47:10 | R1 | Isn't that nice that they think this is fun. And you know what it is fun! I think that I can remember children in my school When they were working on these kinds of problems, when it was lunch time they said no, no, no, we don't want to go to recess. And you say, Oh my gosh, like That is unusual. Stay in from recess to do math. Okay. How many of you have let the children share their strategies yet or have you not gotten to that yet?<br>What you might want to do, because they are probably curious to know that did I do something good, Am I on the right path? As you can fit it in, it would be good to let not the whole class share, because they will be bored out of their mind, but pick a few to share, it doesn't have to be the best and brightest with complete solutions pick some that have an interesting notation, pick some that have a code, pick some that have part of a convincing argument. |
| 483 | | VB | Okay I teach seventh grade um resource. |
| 484 | | R1 | Okay |
| 485 | 00:48:22 | VB | So a lot of my students did groups but they did do opposites. |
| 486 | | R1 | Okay. |
| 487 | | VB | in the beginning though with a lot of my students I had to actually show them what I meant by duplicates but some like thought they are all duplicates because they all have blue and yellow in them. |
| 488 | | R1 | Okay. |
| 489 | | VB | So I had to explain to them look at the pattern. |
| 490 | | R1 | Okay. |
| 491 | | VB | So, this group. They were two of my lower functioning students so I was pretty impressed that they were able to do it. Um They said the reason why we arranged the blocks this way is because we think it was easier the way we did it. But It helped us a lot better. But I kept saying why? [laughter] I was like how do you know you can't do anymore? |
| 492 | | R1 | Right. |
| 493 | | UCT | A lot of them I think understood that you would make duplicates but they could not come out and say it. One of the kids said, if I could add one that would change it, which I was surprised. |
| 494 | | R1 | If I add one to the what, height? |
| 495 | | UCT | Yeah to the height. Well that would be different. Again, |

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| | | | we are only doing four tall; well not this group. |
| 496 | | R1 | Right, right, right. |
| 497 | 00:49:22 | UCT | Um, So, I was pretty happy with them. It was really more the girl who was doing it. |
| 498 | | R1 | So, she was taking the lead. Are they about the same ability? |
| 499 | | UCT | Yeah, yes they are |
| 500 | | R1 | Okay. Okay. |
| 501 | 00:49:42 | UCT | But like They were all about the same ability. But she usually did a little more generally [inaudible] but then when she started to get the ball rolling he was able to get on it. |
| 502 | | R1 | Okay, notice how she chose to use the blue to color in the b and then said I don't even have to that anymore because the key is telling you blue. So, that is the reason why I told you don't necessarily give them colored pencils or crayons or markers because P and S didn't need it and they were able to show you what the towers looked like. Good. |
| 503 | | VB | Okay so with this group again. I found also with my kids. A lot first came up with 14 and then they said 18. |
| 504 | | R1 | Okay. |
| 505 | | VB | So, it was weird. A lot of …It was common in all my classes. And then um with them I believe they did get 18; and I told them double check maybe you have duplicates because I think this may be perceptually hard for them to see it. |
| 506 | | R1 | Okay. |
| 507 | | VB | Even when they separated it and they rearranged them a couple of times. It was very difficult for them to see the opposites but again they were telling me since they didn't see there were duplicates. They were kind of saying to me too well [inaudible] I moved them over because they were in the same spot. So it says, I believe that I am done because we made the opposites from all the towers I have made.<br>I built the towers from the way I can make them as many ways as I can.<br>I have 18 towers and at the end I put the pairs in 2s two each<br>So I was done,<br>I thought it would be the fast way to organize the towers, so it would be easier.. I thought the opposites were easier to handle than to just wing it.[laughter] |
| 508 | | R1 | To… |
| 509 | | UCT | To handle than to just wing it.[laughter] |

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| 510 | | R1 | Yeah, I love it! |
| 511 | | UCT | Like I said a lot of them had their way [inaudible] I had one group of boys that had arranged it to where like they had the top colors blue yellow on top. |
| 512 | | R1 | Yeah so they used a constant. Very nice. Go back to,did they tell you I have them all I have 18 here. And what might you ask them to kind of you know to let them know that they are not convincing you? What is a question you can ask them? They have 9 groups of pairs, right? 9 groups of opposites. Any ideas? |
| 513 | | UCT | Can you arrange it in a different way? |
| 514 | | R1 | You could do that but I am saying that if that is the way they are arranging it. If you said to them, Why can't you have a tenth pair of opposites? Why can't there be a tenth pair? I am not convinced there are only 9 pairs. In fact I don't even think there are 9 pairs. I am saying that if you ask them, I don't know Could there be another pair? And that is a way for them to you know may be thinking that it is not a convincing argument and then your question can be arranged in a different way into a very good one. |
| 515 | 00:52:39 | VB | They had it originally arranged straight out and |
| 516 | | R1 | Okay. |
| 517 | | VB | and when I thought they had a duplicate. I said why don't you try and pair them up because they did have had it all scrunched together but it was laying in one and so I said separate into twos but they still put it the same. |
| 518 | | R1 | And Again, remember It is so hard. But Rather than say why don't you separate them into twos. Why don't you say Can you arrange them in a different way not all of them together, but can you arrange them in a different way that maybe I will be better able to figure out if you have them all. Alright because you are kind of like telling them to arrange it into twos. You are kind of forcing them into.. um..you know…going in a direction. Okay. |
| 519 | 00:53:20 | VB | This one I don't know why [unintelligible]. But I will get to the explanation. |
| 520 | | R1 | Okay. |
| 521 | | VB | These two boys they thought mathematically right away. |
| 522 | | R1 | Okay. |
| 523 | | VB | They were able to come up with it they knew it had to be an even number. So like two is even so it has to be four high, so it has to be an even amount of towers. We did it this way because we had two colors and they had to be four high, So you multiply them. Then they can be the opposite of what you have to add two more so then you get 16. So |

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| | | | there could be only 16 different combinations and it looks like cowboys![giggling].<br>We organized them by opposites and combinations. So with them they did start with the math right away and once they got started; they were able to do classwork and then I could see them freeze a little bit; but then<br>They were able to check to see which ones they had. |
| 524 | 00:54:12 | R1 | Is this the same argument that um..which teacher had this argument? Four times 2 is 8. |
| 525 | 00:54:22 | T3 | I think that was mine. |
| 526 | | R1 | Is that what they are saying? 4 times 2 is 8 times 2 is 16. |
| 527 | | UCT | Yeah they told me it is four high, there's only two colors and that's 8. I know they have to be even. And then they were kind of talking well like this one is the figure; then They have to have the opposite of that one. |
| 528 | 00:54:42 | R1 | Right right right. |
| 529 | | UCT | So, I think they started with the math but then when they started building. I think they kind of threw it to the wayside and then they were kind of figuring it out as they were building. |
| 530 | | R1 | But again, there is math here. I am not saying there isn't math. But I am saying the mathematics is not really what is going to give you the solution. Because it doesn't work with towers that are three tall, five tall, 8 tall;<br>It only happens to work here just the way that four times four works here because four times four is 16 and two to the fourth is 16. Okay? Okay, good. [pause] I can remember um, I started as a middle school math teacher. I can remember very clearly when students would be able to force the numbers to get them the solution. And the solution happened to be the right answer. But the mathematics made no sense. And I had to try and let them know that I am not impressed when they get you know some Gobbeldegook that turned into an answer that happens to be correct when the process is wrong. When they do the state testing if the process is wrong do they get credit if the answer is right? |
| 531 | | UCT | No. |
| 532 | | R1 | No. They may get something but they don't get credit for a wrong process. So you don't want to complement them on a wrong way of thinking. Okay, Looks like we got a lot of writing here, huh. It is hard when you are one person in the room and you have a whole class. It's hard to see what's going on. You'll get better at doing it. But It's not easy. |
| 533 | 00:56:27 | UCT | In my one class with my para. She wanted to say more but |

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| | | | I'm like no you can't. But it's hard You see them getting frustrated you want to help them. |
| 534 | 00:56:35 | R1 | Absolutely. It is hard for you. But you have this course and this training. She is not in the class so of course she is going to want to. Um, Okay. |
| 535 | 00:56:46 | MM | So, I did a lot of the one and also the partner because it was kind of interesting how they recorded their drawings differently and some of them had a good argument for one part and the other partner had the better argument for the other part. |
| 536 | | R1 | Okay, Good. |
| 537 | | MM | So this first one, um He has some organization to his picture but I think that his picture was kind of an afterthought. Um and He really wanted to get his explanation down first. |
| 538 | | R1 | Okay. |
| 539 | 00:57:16 | MM | His first explanation is for the three with one color and one of the other color. He says First we put one yellow on the top and three reds under it. Then we moved the yellow down one and moved the three reds like this. And he drew the little picture from the original, he drew the little picture; then we moved the yellow down again and again. |
| 540 | | R1 | Okay. |
| 541 | | MM | And Then we were at the bottom so that we knew we were done with those. Then we did the opposite. |
| 542 | | R1 | Okay, so That's nice, isn't it? He is showing you how he is exhausting each of the four positions. |
| 543 | | MM | He said Then we did the opposite. |
| 544 | | R1 | Uh huh. |
| 545 | | MM | Like that. |
| 546 | | R1 | Okay |
| 547 | | MM | And we knew again with these we were done. |
| 548 | | R1 | Okay |
| 549 | | MM | Then we did 4 and 4 like this. And those are all red and all yellow. And since there are only four blocks and two colors we knew that we were done with that. |
| 550 | | R1 | What do you think of that? [pause] That's nice, isn't it? Now These are convincing arguments. This is really good. |
| 551 | | UCT | I like how they drew the blocks next to it. |
| 552 | | R1 | Yeah, isn't that nice, Right? |
| 553 | | UCT | And then,I think This is where he started to get rushed. |
| 554 | | R1 | Okay. |
| 555 | | UCT | So then We decided to alternate the red-yellow, red-yellow like this. |
| 556 | | R1 | Right. |

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| 557 | | UCT | and moved everything one spot. So he moved Like he moved that red down one.  That's what he is saying. |
| 558 | | R1 | Okay what is that called? |
| 559 | | UCT | Recursive |
| 560 | | R1 | Someone said it,… a recursive argument. |
| 561 | | UCT | Yeah. And then we did like this; and we moved it like this; and then so he is starting to get rushed. |
| 562 | | R1 | Nice, Okay, nice. |
| 563 | | UCT | That is why I included his partners' work because he actually has a better explanation for when they get to the twos and the twos. |
| 564 | | R1 | Okay, And that is a hard one to write a convincing argument for. |
| 565 | | MM | Yeah but..and his organization on the picture is a little bit better too. |
| 566 | | R1 | Okay. The other one I thought the organization was good. Didn't you? I mean he had…He had… well, does he or not? |
| 567 | | MM | The way he grouped it. |
| 568 | | R1 | It was his description that was good. |
| 569 | | UCT | Yeah. |
| 570 | | MM | Like I said, I think their drawing was kind of an afterthought. |
| 571 | | R1 | Okay, okay. |
| 572 | | MM | He was more concerned with writing… |
| 573 | | R1 | Uh huh. |
| 574 | | MM | This was his partner. |
| 575 | | R1 | Okay. |
| 576 | | MM | And The beginning part is the explanation of the one and the three so we don't have to go over that. |
| 577 | | R1 | sure. |
| 578 | | MM | Then we took two reds and two yellows. We put the two reds on top of the two yellows. |
| 579 | 00:59:31 | R1 | Okay. |
| 580 | | MM | Um After that we took the two reds and put them in between the two yellows |
| 581 | | R1 | Okay. |
| 582 | | UCT | Like if these are the… |
| 583 | | R1 | On the top… |
| 584 | | UCT | two reds on top. |
| 585 | | R1 | Uh huh. |
| 586 | | UCT | And the two reds in between… |
| 587 | | R1 | Okay. |
| 588 | | UCT | Um If you make the opposite of the last two so here are the |

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| | | | opposite of the last two and put them next to each other in a certain way then it will be a pattern of two.<br>Then you can put two of each color in a pattern of red yellow red yellow and yellow red yellow red.<br>Lastly you can only have four blocks in a tower and there are only two colors so you can have a tower of only red and only yellow.<br>So they were partners and I thought that they did pretty well. |
| 589 | 01:00:10 | R1 | What do you think? Are you convinced? That's pretty neat stuff. Now I see a star on the top, were they the ones that were in the Rutgers project before? |
| 590 | 01:00:10 | UCT | Oh No, I was just going through and reading them and starring the ones I wanted to show. |
| 591 | | R1 | Okay [laughter] good I got it! Okay. |
| 592 | 01:00:31 | UCT | The next group that I included actually said 20 towers. But the way that they grouped them was pretty interesting. So, His picture is down here but I will show you in a second his partner's picture was a little clearer cause he writes bigger. And They got this staircase pattern for the 3 and the one and they wanted to do the same thing with the two in the middle; but they actually wound up getting duplicates and I don't think that they saw that. So what he said was What we did we made the basics, all red, all yellow |
| 593 | 01:01:09 | R1 | Do you like the language? We made the basics. |
| 594 | | UCT | Then we did all the combos of one. Turns out all the combos of one were also 3. So I think he means one block of 3. |
| 595 | 01:00:20 | R1 | And that is what he means. Right. He is talking about all the towers that are exactly one of one color and three of the other. |
| 596 | | UCT | It was three because there are four blocks in a tower. Um (1=3) he put. That means that there are going to be 3 different color blocks. Next we did the twos. We had two of each color places back to the explanation of the four color towers since there are only four blocks in a tower so there are no other combos except for four.<br>So they didn't do a really good job of writing it down but so they do their two plain ones and they have their stair case with the one.<br>What they were trying to do, was do a staircase with 2,2,2 And then they broke it apart. But They did the opposite too so they wound up getting duplicates.<br>I thought it was interesting to that they tried to take the argument from the 3 and one. Since a lot of them got that |

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| | | | easily. And then move on to the bottom. And these last two are kind of just like the ones that don't belong. |
| 597 | | R1 | Like the candy cane… |
| 598 | | UCT | Yeah. |
| 599 | | R1 | ..or the barber pole. Did your kids give that a name like the alternating? No name? Okay. |
| 600 | | MM | And then I have one more. And this is a girl that had a really good argument when she was talking to me, but she didn't finish. |
| 601 | | R1 | Okay |
| 602 | | UCT | As you can see her last sentence just kind of stops. |
| 603 | | R1 | Okay. |
| 604 | | MM | But She starts to talk about the twos really well and this is her drawing. So she again has the stair case and then she has the opposite. |
| 605 | | R1 | Right. |
| 606 | | UCT | And she did the twos stuck together. Um, what is she saying? |
| 607 | 01:02:59 | R1 | If you… read what she says; but not yet. Look at her second grouping where she has 2 of one color and two of the other. What did she do? [slight pause] |
| 608 | | MM | She has a constant. |
| 609 | | R1 | She has a constant on the top do you see that? And in The third group, same thing, right? Constant on the top. Okay, |
| 610 | 01:03:21 | UCT | She said I used two of each color. The first one in the set has the colors together. So there's two of one color on the bottom and two of the other color on the top. |
| 611 | | R1 | Say that one more time. Two of one color on the bottom…? |
| 612 | | MM | Two of one color on the bottom and two of one color on the top. |
| 613 | | R1 | Okay. |
| 614 | | UCT | The second one of the group one only broke one color apart. With one color on top, two colors in the middle and one color on the bottom. |
| 615 | | R1 | Okay |
| 616 | | UCT | And that is where she stops her explanation. |
| 617 | 01:03:50 | R1 | Do we remember what some of you did? That you had a constant on the top and then how did you arrange the bottom three blocks? She has it done differently than what you did. What did you do to make your convincing argument? |
| 618 | | UCT | We did it the same as this [pointing to what the student |

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| | | | did] We had the red all on the top and it was the same thing. |
| 619 | | R1 | That's right. |
| 620 | | UCT | and it was the same thing. |
| 621 | | R1 | So the middle block there um which has the red on the bottom. would be in your left position, the one on the right which is the red in the middle; would be in the second position and the red on the top would be third. |
| 622 | 01:04:25 | UCT | Yeah. |
| 623 | | R1 | And that would be really good to point out how they are using a constant. |
| 624 | | UCT | And then a lot of my students…I had one that I was going to share with the same thing as you but there is really no point. 2 times the 4 times the two again. A lot of my students I had said that. |
| 625 | 01:04:45 | R1 | Students want to show you that they are using symbolic notation to get an answer.<br>And sometimes it's good but sometimes it's not good when mathematically it doesn't really lead to a good solution. Even by accident; okay.<br>You are going to see later in the semester um, we are going to do a problem, that I guarantee someone in this room is going to want to use symbolic notation.<br>And sometimes it is going to help you and sometimes it is going to hurt you. It depends on whether you remember the right symbolic notation. Okay, alright, good. Very nice. You are showing some good samples of work and again, is there anyone who has the whole you know work with a convincing argument. No, but there is a lot of things here that are working towards a good argument. And nice recordings. Nice grouping and nice notation. Holding constants all these things are things you want your students to do and to point out to others. If you have a class where they are ready to hear about holding a constant, then other children might say. Ooo, you know next time when I do this problem, I'm going to use Sara's strategy because I like holding the constant, it made it easier. If you have a student who really isn't ready to hear it, don't worry, it will go in one ear and out the other and they won't hear it. Okay. |
| 626 | 01:06:17 | CDR | So I had another 8th grade class um track one. |
| 627 | | R1 | Track one means what? |
| 628 | | CDR | Normal…It is the higher level. |
| 629 | | R1 | It is a regular ed. |
| 630 | | CDR | Yeah. It is a track one, but then I have a track 2. |

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| 631 | | R1 | Okay, and Is there something above than track 1? |
| 632 | | T7 | Yeah, there is pinnacle. |
| 633 | | T6 | That was what mine sixth grade pinnacle one was. |
| 634 | | R1 | Okay, good. |
| 635 | | T7 | Um so this was a student who had explained it to me using a recursive argument. |
| 636 | | R1 | oh |
| 637 | 01:06:49 | T7 | But when they wrote it. It doesn’t come out that clearly if they used it or not. |
| 638 | | R1 | Okay, okay. |
| 639 | | T7 | It took me a little while to follow what they had done. So when they drew their towers out, they drew them in the pairs. |
| 640 | | R1 | Okay. |
| 641 | | T7 | So they could show their opposites. |
| 642 | | R1 | Yep. |
| 643 | 01:07:03 | T7 | So all those arrows are showing that they took the top block and moved it down which one it would give them. |
| 644 | | R1 | Oh, so that’s interesting. |
| 645 | | T7 | But its’ not the easiest to follow when they do it that way. |
| 646 | | R1 | Sure. Sure. |
| 647 | | T7 | But they said the first pair of two of all the same color is there because there are four blocks and all are the same color but opposite from its partner. And he said the second group of two pairs make four different groups so they are talking about the four pairs of opposites. |
| 648 | | R1 | Mmhh. |
| 649 | | T7 | But they link together because if you take the bottom or top and put it completely opposite of the top or the bottom one, it would make a different tower. |
| 650 | 01:07:41 | R1 | Mmhh |
| 651 | | T7 | Which wasn’t exactly what they were saying to me. |
| 652 | | R1 | Right |
| 653 | | T7 | They were explaining if they Take the top and put it on the bottom |
| 654 | | R1 | Sure. |
| 655 | | T7 | Then they started talking about switching. |
| 656 | | R1 | Okay. |
| 657 | | T7 | Then they said they can only do this process 8 times. Technically you only do it four times. |
| 658 | | R1 | Uh huh. |
| 659 | 01:07:58 | T7 | Before it starts to repeat itself so their argument when they said it to me was much more convincing than when they started writing it. |

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| 660 | | R1 | Okay, okay. |
| 661 | | T7 | Um, then they said that the third group of two pairs make 3 different groups. But they only link together twice. if you switch the top and bottom one with the opposite, it would be completely different towers and you can only do this with four towers. The last 2 towers are a set of two different colors mixed twice and switching them would just duplicate itself further. That is why there can only be 16 different towers without any duplicates. Following their argument on paper was much more difficult than listening to them. |
| 662 | | R1 | Than what they said to you. Okay, okay. That sometimes does happen. If you go back they were trying to use a recursive argument. Cause if you look,…By the way, Don't you like the way they have group one group two group three and then they have written down about the first group, the second… so it makes it easier to follow their reasoning for each group. That is something you want to point out. Other groups may not have done that.<br>If you look at the third group, tower number one on the left side and then there is an arrow going towards… |
| 663 | 01:09:11 | UCT | The Second to last one on the left |
| 664 | | R1 | Second to last one. |
| 665 | | UCT | Then they don't link it to the next one |
| 666 | | R1 | Yeah. |
| 667 | | UCT | because it was tied to another one |
| 668 | | R1 | Sure, sure. |
| 669 | | UCT | and they didn't do that |
| 670 | | R1 | Yeah. So, but, I hear what you are saying. Notice how they drew their towers. Okay, they put a little dot or a little squiggle to show whether it was red or black or whatever the colors were. You know, And that is nice too. They said they don't have to color the whole thing, why waste my time. You can tell which are red or which are blue or black, right? So again they developed their notation and that is a nice way to do it. Good. |
| 671 | 01:09:54 | T7 | So, This group had started with a mathematical argument with numbers. |
| 672 | | R1 | Okay. |
| 673 | | CDR | I had a lot of my students doing that and then I would ask them well Did you know that there were 16 to begin with? and they said no but I was forcing them to put work there. They had written there are four blocks high, 2 colors, 16 combos; and 64 total blocks which are all divisible by each other. |

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| 674 | | R1 | So what do you think of that? [pause] That's some math also.<br>What do they mean 64 blocks? What are they really saying? |
| 675 | | UCT | The cubes. |
| 676 | | R1 | They are talking about the cubes, yeah. Does that mean anything for this problem? Not really. But Notice how they crossed out but you can still read it. That is really good that you can still read it. |
| 677 | | UCT | So then they fixed their argument. |
| 678 | | R1 | Okay. |
| 679 | | CDR | They said that there can only be two completely one cube tower (only two colors). There can be four, 3 red one yellow towers Because there are only four high towers and that is the same for red. And then they said For two red and two yellow towers there can only be 2 Because there are only two sides to switch to make two different towers. They were talking about two red on top and two red on the bottom.<br>Then they said that for the towers in the center there can only be two because there are only two colors to put in center of the tower. And then they said that<br>For the towers that have a pattern, there can be two because there are only two colors to make up the pattern starting from the top (or bottom). That was the alternating one. |
| 680 | | R1 | Right right. |
| 681 | | UCT | So this is much better than their argument on the front. |
| 682 | | R1 | Absolutely absolutely, good. |
| 683 | 01:11:34 | UCT | Then,the last one was, she had a lot of difficulty with this one. When she wrote all hers out she wrote them in pairs. |
| 684 | | R1 | Now notice how she is recording her towers? What is unusual about that? [pause] |
| 685 | | UCT | It is horizontal. |
| 686 | | R1 | Say it again. |
| 687 | | UCT | It's horizontal. |
| 688 | | R1 | It's horizontal. Remember, I might have mentioned that. I have seen this before. Some teachers have the hardest time reading it this way. Because we don't think horizontal when we are thinking of building towers. Most of us think vertical. Okay, And uh…But Yet this child absolutely is thinking and having no trouble. Now The only problem with horizontal is they always have to make sure that the top of the tower is always pointing the same way. So if the tower is yellow, yellow, yellow, red. And yellow is on the |

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| | | | bottom. Yellow yellow yellow red. It means the left side always has to be the bottom of the tower.<br>If they start flipping it, then it is going to be hard for them to see the towers clearly. But this is absolutely fine and what also is unusual? |
| 689 | 01:12:46 | UCT | They wrote out the words. |
| 690 | | R1 | They wrote out the words. Okay. But this is how these children see it. And I think it is interesting for us to know that they don't all see it the way we see it. Good. |
| 691 | | UCT | So, Then she had kept them in pairs. She hadn't grouped them in any way. She had a hard time finding an argument. |
| 692 | | R1 | Okay. |
| 693 | | UCT | So she put a question mark and said We think we have all the towers because if you were to find or (try to find) another group of towers, you would realize that that group of towers had already been created. It would also begin a pattern of towers and if you located a tower in the pattern that hadn't been made, then you would know that you missed one. So I am not exactly sure what towers she was thinking of but I think she was talking about the opposites pattern. |
| 694 | 01:13:27 | R1 | Uh huh |
| 695 | | T7 | Also the towers had a nob-like appendage at the top [laughter] Which you would not be able to flip the towers over to make the towers different. Example, flipping one tower over to create a reverse pattern. [inaudible] all the possible towers are created; unless you see the same type pattern. |
| 696 | 01:13:43 | R1 | Okay that is cute. with the question mark there is that she is not 100% sure there. And I want to ask you she is saying that we think we have them all because if you found another one it would be a duplicate. Basically is what she is saying. Is that convincing? Not really. Okay, It is an argument that a lot of students use, though. But it is not a convincing one. Good, very nice. I love the way you are showing different ways for students to record their work. That's nice, good. How many of you have trouble reading that? [pause] |
| 697 | 01:14:20 | UCT | Well, I think it's a little blurry too. |
| 698 | | R1 | It is. It is. But I don't mean that. I mean reading that it is horizontal. |
| 699 | | UCT | Oh yeah.[inaudible] |
| 700 | | R1 | I had a teacher last year actually that couldn't read that and said that I can't read that. I have to change that so that I can see it. Okay, nice, very nice. Next. [pause] So it is |

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| | | | interesting that students don't always need to be thinking of the problem the way you are. Also not only in the way they solve it but also in the way that they record. Because I think a bunch of you would have never thought to write it with words horizontally.[long pause multiple conversations-inaudible] |
| 701 | | R1 | Remember this is the first time. Okay. |
| 702 | 01:15:46 | T8 | Alright I'm in Long Branch, I have sixth through 8th grade self-contained MCI that means mildly cognitively impaired. When I started doing this, I started with my whole class of nine students.<br>And my para blurted out there is supposed to be 16, right. And I am like you got to be kidding me. [laughter] Every kid heard it. Every kid heard it.<br>However, Only two of my students, were still even after being given the answer<br>Were still actually able to understand the task at hand and solve it. |
| 703 | 01:16:18 | R1 | Uh huh |
| 704 | 01:16:19 | CP | The other ones just kind of played with the blocks and some of them just made them over and over and over. so I really wasn't able to use some of the students work…Once I realized that seven students work I would not be able to use. [inaudible because teacher is talking during announcement] |
| 705 | | | Announcement: Good evening it is now 5 o clock… |
| 706 | 01:16:55 | CP | So what I had done…they were the ones who were still building and actually had found the 16. So then I was like okay, I gave my para something else to do with the other ones and I am going to sit with these guys. |
| 707 | | R1 | And don't do that again. Who else had the para who wasn't so helpful? Was that you? |
| 708 | | UCT | It's not that she wasn't trying to be helpful. She was trying to be helpful. |
| 709 | | CP | She was just trying to help them to do it. |
| 710 | | R1 | So you are both at a different school. So what you want to do if you have another teacher in your room, whether it is a para or another teacher; you got to pull them aside before you begin and explain that this is for your class and you are not really wanting them… |
| 711 | | T5 | I did. |
| 712 | | R1 | You did. |
| 713 | | CP | She knew she wasn't supposed to help them but she did anyway. |
| 714 | | T5 | Yeah, She still didn't think it was really helping them. |

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| 715 | | T8 | Um so, Being that the students knew they were supposed to find 16 and just weren't. Really I mean you're talking some of them have down's syndrome so it was really over some of their heads. |
| 716 | 01:17:56 | R1 | Sure |
| 717 | | T8 | But for these two, I was pleasantly surprised when we sat there and worked through the problems. Um, they started with opposite pairs and I told them that I wasn't accepting that as an answer. They needed to look at it and figure out another way that they could show me the blocks or arrange them or tell me that's all that they have. |
| 718 | 01:18:10 | R1 | Mmhh. |
| 719 | | T8 | Um, So it was a boy and a girl. The girl had taken them all and kept a constant. She rearranged it with all yellows on top and all reds on the bottom. |
| 720 | | R1 | I see that over here. |
| 721 | | CP | I was like so happy when I saw her do that. So I was like okay, what else can you tell me about this. And then The boy picked up and this block here and said this one has 3 reds in it. I said okay, well what about that? So there are no other ones with 3 red in it. So then The girl ran with that and did the 2 reds, kind of once they got that concept they were counting down and They were able to do it group it like that. Um.. |
| 722 | 01:18:50 | R1 | So they were doing a proof by cases or the beginning of a proof by cases? |
| 723 | | CP | Yeah. They were grouping them by how many…with a constant and then how many were in the bottom three. |
| 724 | | R1 | Okay. |
| 725 | 01:19:04 | CP | So, this is their drawing. They did have a hard time at first they weren't sure what to do or how to set it up or actually drawing it. They figured out to draw the pictures. So I will go on to his explanation. Um, Okay for the first,…Do you want me to show you this or show you his picture? Picture? |
| 726 | 01:19:27 | R1 | Yeah, do that that would be helpful. |
| 727 | | CP | Alright for the first one, he said only two different colors so that was for this group here and I was satisfied.<br>Um, then for the next one, he said there are one yellow on the top and then three red on the bottom um<br>You can't have red on the top on yellow because.. I am sorry his spelling is really bad so I'm trying to figure out what he wrote… [CP is reading from the following argument:<br>1.Only 2 different colors. |

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| | | | 2.There are one yellow on top and three red on bottom and if you move the red on the top, then there is no yellow on the top.<br>3.There is a block with one that has yellow red yellow red and you switch to red yellow red yellow then you won't have the yellow on the top. There is another block with yellow, yellow red, red if you move the red to the top, you have yellow on top 2 red in the middle and you have yellow on the bottom. Any other move you can get a red on top.<br>4.There are 3 red across. If you put the red on top. Then you won't have the yellow on top. If you put another red on the bottom then you get 5 cubes and there only possible be for cubes. ] |
| 728 | | R1 | And you know we are not worried about the spelling okay if we can read it we are fine. We don't care if it is spelled wrong. |
| 729 | | CP | You can't have red on top of yellow because it is only one yellow and three red. |
| 730 | 01:20:06 | R1 | Okay. |
| 731 | | CP | So that was his explanation. |
| 732 | | R1 | Okay. |
| 733 | | CP | Um, When I said how do you know you can't make another combination like that? He had actually taken the block and like maneuvered it around and ended up with a red on top so they were able to see that. Um, For the next one, he said two yellow on top 2 red on the…on the bottom. One yellow one…So what he started to do was actually like just identify each cube in his explanation. That was kind of their explanation too was going through it. So what they did was identify each one so I am going to skip reading it. |
| 734 | | R1 | Okay |
| 735 | | CP | And then he said if you try another move one block you get a red on top. |
| 736 | | R1 | Was he using a recursive argument? |
| 737 | | CP | Yeah. I mean That's where it started here. |
| 738 | | R1 | Okay. |
| 739 | | CP | When He started to actually manipulating the one block. |
| 740 | | R1 | Okay. |
| 741 | 01:21:03 | CP | To see if they could make another; um then the next one was really confusing and he said. We have three red across and we have 3 yellow. But if you put the red on top you won't have the yellow on top. If you put another red on the bottom, then you get 5. And there has to be four. So I |

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| | | | liked it here that he said there was 5 cubes, that was something but the 3 yellow across and 3 red did not really make much sense to me. Um, And then he said for the cube in red on top the same thing. |
| 742 | 01:21:38 | R1 | Okay. You know, I am thinking now how many of you have, you are saying some of them were down's syndrome |
| 743 | | CP | Yes |
| 744 | | R1 | Okay, If you have children that really are very low functioning. This might be too challenging a problem for them and maybe building towers three-tall might have been enough of a challenge. And maybe that would be something they would be more successful with. |
| 745 | 01:22:00 | CP | Okay. I will try that with them because they were so confused. |
| 746 | | R1 | That would be good. I would be curious to know if they had more success with that. I would do it with everyone. Um, you have an option. If you saw the groupings that you did, didn't work, you can change it up for the next task. You don't have to keep them if it didn't work, okay. If it did work, you can keep them. |
| 747 | 01:22:27 | CP | I know that if I hadn't been sitting with these 2 students for the entire time. |
| 748 | | R1 | Right. |
| 749 | | CP | They wouldn't have had their explanations written down like that. |
| 750 | | R1 | Right. |
| 751 | | CP | So, that is just something else I wanted to throw out there. I know a lot of people don't get that opportunity to sit there with two students. |
| 752 | | R1 | Right. |
| 753 | | CP | For a block period which is for like two hours to do and I did. So I was very lucky in that sense. I was actually able to sit there and watch these guys for the whole process. |
| 754 | | R1 | Right, And that is good to we talked about staying with a group long enough to really hear what they are thinking and saying. Good. |
| 755 | 01:23:01 | CP | So this is a girl that he worked with. |
| 756 | | R1 | Yeah. |
| 757 | | CP | Um, Her drawings were more systematic than his. |
| 758 | | R1 | Okay. |
| 759 | | CP | She had kind of moved them down whereas his were kind of just at random. So Part of me was saying okay, she really seems to be grasping the concept and he is just following what she is saying, and then when I got to her written explanation, um I could really see that was evident. |

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| | | | So the first one she said only two different colors and again, they were talking out their explanations together. and then they put them down one by one and then she actually went through and listed them.<br>There are one yellow on top. Three red on the bottom.and if you move the red on top there is no yellow on the top. For the next one, she goes through and lists out all the blocks again, so like that was like their starting point saying all the options that they had um and then she said any other move you can get a red on top so that was pretty much sums up all their arguments. she would go through and list out each one in the group and say if you move one any other way then you will have the same possibility. Um, so I wanted to have more stuff to work with since I only had the two kids in the one group and they both kind of had the same thing. So I had this group of girls to come in my class during their lunch period and kind of help out with the kids. |
| 760 | | R1 | Mhh |
| 761 | | CP | So I thought it would be interesting to give them the activity and see how they do with it. |
| 762 | | R1 | Okay. |
| 763 | | CP | This is an 8th grader. |
| 764 | | R1 | Okay |
| 765 | | CP | It is not a high functioning class but just |
| 766 | | R1 | Regular ed |
| 767 | | CP | Regular ed. |
| 768 | | R1 | Uh huh. |
| 769 | 01:24:44 | CP | So this is Mary and her partner Precious came in and worked on this. And theirs is totally different than what the kids had and I liked it and I thought this is something different. |
| 770 | | R1 | Take a look everyone at group one. How did they arrange group one?[pause} |
| 771 | | UCT | Two red, two yellow |
| 772 | | R1 | Say again. |
| 773 | | UCT | Two of each color. |
| 774 | | R1 | It is two of each color and more than that what else is it? |
| 775 | | UCT | Two reds together. |
| 776 | | R1 | Yeah the two reds are stuck together in each of those three, right? How about Group 2, we said that is the basics, right? Is that what they said? Okay, Group 3, what’s that? |
| 777 | | UCT | The opposite of group 5 |
| 778 | | R1 | The opposite of that is group 5, what is group 4? |
| 779 | 01:25:29 | UCT | Reds and yellows are separated. |

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| 780 | | R1 | Yep! Reds are separated there. So interesting way of grouping, huh. And that's kind of neat.. |
| 781 | | CP | That's what like Stephanie did in the video right. |
| 782 | | R1 | Stuck together, stuck apart. Um, what you want them to do now this is neat that you did it with kids that aren't even your kids. That it is kids that are helping your kids. |
| 783 | 01:25:50 | CP | Yeah |
| 784 | | R1 | What you want to do is, once kids have a grouping, you want to say well how do you know that there isn't another tower that can fit in this group. So that there would be a convincing argument. Okay, good. |
| 785 | | CP | I have her written argument. |
| 786 | | R1 | Okay. |
| 787 | 01:26:11 | CP | Alright So she said group one has 2 reds together every time you move it to the top, middle, and end. I guess she was saying that the 2 reds are at the top; middle; and at the bottom. She said<br>For group 2 it just has four yellows and 4 reds on each. For group 3, it only had one yellow so the yellow cube started on top, and Went down one every time and it stopped at the bottom.<br>For group 4 all the reds were separated. Uh, for group 5 the red cube started on the top and went down one every time and stopped at the bottom. And then at the end she said For each group,<br>I couldn't make any more because there was no more possible combinations and if I added one more to it; it would be 5. |
| 788 | 01:26:53 | R1 | What do you think? [pause] Are you happy with her thinking? Do you think she should have done something else? Are you not sure? [pause] Is it convincing to you? Is any of it convincing? [pause] |
| 789 | 01:27:25 | T3 | [unintelligible response] I like how she talked about moving one down. |
| 790 | | R1 | MMhh. Good. Again, you want them to be able to… uh go back to her picture |
| 791 | | CP | Yeah |
| 792 | | R1 | You want them to be able to let you know that there can be no more towers in that group. Okay. Um And that there aren't any other groups or any other ways to arrange. So I think that this is the beginning of something that could be a very nice convincing argument. Right? Okay. |
| 793 | | CP | She started with opposites |
| 794 | | R1 | Yep |
| 795 | | CP | And I said alright that is not working for me. |

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| 796 | | R1 | Do it a different way. Okay. |
| 797 | | CP | Find another way. |
| 798 | | R1 | Good, good, okay neat. Okay, What you want to do while we are waiting for the next thing to be photographed you want when you are doing o the next task; to make sure that the pairings…both students are you know recording; where both students are um you know handing in a piece of paper that shows their work because you will find that it may not always be the same as the partner. They may be working together but thinking differently. |
| 799 | | UCT | Should we keep them in the same pairs? |
| 800 | | R1 | Oh Yes you should go in the same class |
| 801 | | UCT | No, I said should we use the same pairs? |
| 802 | | R1 | Oh, if you like the pairs, yes you can keep the same pairs. If you didn't like them, because the kids were arguing or one did the work and one just sat there and did nothing. Switch it up. But you must go into the same class that you did the first task with. Okay. |
| 803 | 01:29:05 | GH | Okay, so um This group which I didn't get to see, um I forgot to tell her to use a pen. Um But she wrote down the reason I think we are done with 16 combinations; are because I did it like Yellow, blue yellow, yellow So I moved the blue down one spot; yellow yellow blue yellow and one down yellow, yellow, yellow blue, then I did Blue yellow blue blue. Then I moved the yellow down. Blue Blue Yellow blue; down blue blue blue yellow then I did pairs of two colors; yellow yellow blue blue; blue blue yellow yellow; and yellow blue yellow blue; blue yellow Blue yellow and so on…. |
| 804 | 01:29:47 | R1 | And so she is telling you what she did. But she is not convincing you. |
| 805 | | GH | But when I spoke to her, she was very clear about how she knew each one um I think towards the end of the period the time was running out and so she…she probably could have written a lot more than this. |
| 806 | 01:30:04 | R1 | Okay |
| 807 | | GH | But basically she just listed all of her towers. |
| 808 | | R1 | And that is a common thing where students just tell you what you can see when you are looking at their diagram. What you want to try and get them to do is some of you had where they grouped it and made a convincing argument for each group. Good. |
| 809 | 01:30:26 | GH | Now with this one, This is probably the best answer I got. |
| 810 | | R1 | Okay. |
| 811 | | GH | From one of my students. |

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| 812 | | R1 | And tell us again what grade? |
| 813 | | UCT | Sixth grade and um this is…. |
| 814 | | R1 | Regular ed. |
| 815 | | GH | This is an in-class-support class. |
| 816 | | R1 | In class, okay. |
| 817 | | UCT | Yeah, this actually has one regular ed student and 2 special ed. students for each grouping. |
| 818 | | R1 | Okay. |
| 819 | | UCT | It is my only grouping of 3. |
| 820 | | R1 | Okay. |
| 821 | | UCT | So um this is the regular ed student talking right now |
| 822 | | R1 | Okay. |
| 823 | | UCT | I made two whole towers and knew that there can only be two because there were 2 colors. Then I switch the first top color, the second color; the third color; and then the last color.<br>She was moving it in pairs. She moved these down to here, down to here, and then down to there.<br>So that was the three and one like staircase. |
| 824 | 01:31:13 | R1 | So that is interesting. Say that one more time. |
| 825 | | UCT | So first she switched the top color; and left the three next to these [pointing to the towers all blue and all yellow to the left]. And then she went the second and then the third, and then the fourth. So she moved them but with her opposites together so she didn't separate. |
| 826 | | R1 | Interesting, okay. |
| 827 | | UCT | Um I then knew that I could only do these because I switched the colors for both towers. Then I started doing 2 and 2. I took the wholes; and switched the bottom two, the top two; and the middle two; Now, I couldn't really understand in the drawing what she was talking about that very much confused me. I know that these are the last ones, because there are four and if you are doing 2 and 2; then you know you can't do anymore; and finally I did the stripes. Which I don't know what she is talking about. |
| 828 | 01:32:05 | R1 | Who knows what she is talking about the stripes? |
| 829 | | UCT | Like yellow blue yellow blue. |
| 830 | | R1 | The alternating |
| 831 | | UCT | OH! |
| 832 | | R1 | The barbershop |
| 833 | | UCT | But that wasn't the last one. |
| 834 | | R1 | You're right. |
| 835 | | UCT | That's why I was confused. |
| 836 | | R1 | Yeah. Yeah. Her writing is confusing. Cause it is not |

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| | | | matching the top of it. The top is hard to follow. Absolutely. |
| 837 | | UCT | Yeah. |
| 838 | | R1 | Uh Huh. |
| 839 | 01:32:35 | UCT | Now um I thought the next one was very interesting because he was in the group; but what he wrote was completely different and had nothing to do with what she said. |
| 840 | | R1 | Okay. |
| 841 | | UCT | So he, grouped them a little differently than hers. Like He wrote… |
| 842 | | R1 | Ahh…. Look at his explanation. |
| 843 | | GH | My group and I think this is all you can make because 4x4=16. There are 16 different pillars of 4. My teacher in Lloyd Road, which is our elementary school, said if there is a problem like this do the amount of the blocks in one stack and times it by itself so 4 times 4 is 16. |
| 844 | | R1 | Oh, I bet that teacher didn't say that. |
| 845 | | UCT | Well I found it interesting because they were the same group had the same conversation and I don't know where this came from? |
| 846 | 01:33:07 | R1 | How many times do the students take what they remember you said which may not be what you said; that's what the teacher last year said. Right, um if you go back there again 4 times four is 16. That's a good mathematical sentence. It just has nothing to do with the problem. Okay, Good. Okay, nice. Good for you that you are picking up this stuff. You know that I think that it's very hard for students to see, um that you are You want them to connect symbolically. Pictorial with concrete work. You want them to do that. But You want it to be the right symbolic. You don't want it to be just some symbolic that is not making sense.<br>Interesting stuff. Good. |
| 847 | | UCT | It was like she was trying to bring back previous knowledge. |
| 848 | 01:34:02 | R1 | Which is good, that is a good thing, to bring back previous knowledge or to connect it to previous knowledge. But you have to connect it to something that was really correct. Okay this is nice.<br>Do You see how the towers are numbered and that they are also divided into 2 and 3 groups. So this is much easier for you as a teacher to look at and for them then to write an explanation than if they had one big log of towers. Good. |
| 849 | 01:34:32 | UCT | Um, I think that she just remembered to show the towers |

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| | | | [inaudible]. |
| 850 | | R1 | Also Did she talk about them? |
| 851 | | UCT | No, she doesn't refer to the numbers at all. |
| 852 | | R1 | Okay. |
| 853 | | UCT | But I think she just did it to show it. |
| 854 | | R1 | Okay. |
| 855 | | UCT | So it says The towers you see above are all the possible combinations. To prove it keep reading. My partner and I started making the four length tower with 3 yellows um once we knew we had all of them, we went on to the next one; which is 3 blue. So I think they were talking about how they worked here and had 3 of one color and one of the other. |
| 856 | | R1 | Yep! |
| 857 | | UCT | And then we did all the opposites. |
| 858 | | R1 | Good |
| 859 | | UCT | And then she said we knew we were done with the three yellows because there was one blue per level and we knew that there couldn't be another level because the towers had to be four blocks in length. |
| 860 | | R1 | This is very good, isn't it? And what..what grade? |
| 861 | | T10 | This is 6th grade regular. |
| 862 | | R1 | 6th grade regular ed. That is very impressive for them to write that. You didn't help them, did you? |
| 863 | | T10 | No. |
| 864 | | R1 | That's very impressive. |
| 865 | | T10 | Um…We did the same for the 3 blue but there was one yellow per level. Then we went on to the four lengths all blue towers and all yellow towers. Since there was two two colors to work with so we knew we could only make two towers with it. |
| 866 | 01:35:42 | R1 | Are you impressed, this is good stuff. |
| 867 | | T10 | The last part, then we did the two blue and two yellow and we knew we had them all because we tried and tried and tried and could not make anymore.[laughter] So we knew we were done with that one. We are absolutely positive that there are 16 towers. |
| 868 | | R1 | Now isn't that interesting. Perfect, perfect, perfect, and then it fell apart. Okay, But then, what they do…But you see that argument from students we tried and we tried and that we kept looking and looking and now we know we have them all. So she doesn't give a good convincing argument for the 2 and 2 but the others are wonderful. |
| 869 | 01:36:23 | UCT | Yeah she started out really great. |
| 870 | | R1 | Absolutely. Wonderful![inaudible response] Absolutely |

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| | | | okay. |
| 871 | | T10 | Alright. This kid Chris, he was in my group of kids that I wrote about on the posts where they kept changing how they had the groups. |
| 872 | 01:36:41 | R1 | Yes, How many of you had groups that kept rearranging the towers to show you different ways? |
| 873 | | UCT | Yeah. |
| 874 | | R1 | You did. Who else?.. [hands are raised] You did. You did. Good. That is very very good. I mean If they can be flexible in their thinking to keep changing the way their towers are arranged it's really good. |
| 875 | | UCT | So his explanation wasn't that great |
| 876 | | R1 | Okay. |
| 877 | | T10 | but the discussion I had with them was a lot better. |
| 878 | | R1 | Good. |
| 879 | | UCT | So um this is exactly when they went through right away with their partners. They had to group it this way and they didn't go do opposite pairs at all. They went with where They saw this diagonal pattern, they wanted to work with that. Then they tried to do something similar with the ones that were two yellow and two blue but they couldn't really put it together. |
| 880 | | R1 | That's a hard one to do. |
| 881 | | UCT | Um, So I was you know helping them with it and we were talking about it and they got to the ones that were 2 of each color and they were so stuck. So I was like alright, keep thinking about it and I will come back to you. When I came back to them, they had took all the opposite pairs and I am like what happened? [laughter] You just had a few of those left. And they were like oh. Okay, so then I was asking them how do you know that there isn't any more pairs or opposites? |
| 882 | | R1 | Right. |
| 883 | 01:37:49 | UCT | And they were like I don't know I can't tell you. Great keep working on what you were before or visit something new. So I went to the other group and came back and they had them all lined up. They had all the one color constant, and the other color constant, with the two things divided. So They were looking at them, well like how about this? Great, so what can you tell me about this? |
| 884 | | R1 | Yeah. |
| 885 | | UCT | And They were looking at it and they were like well I don't know. They didn't even realize that they had the constant. |
| 886 | | R1 | Uh huh. |
| 887 | | UCT | Well coming from my view, like I am standing up, what do |

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| | | | I see when I look at the top of your towers.[laughter] And they were like oh, there's all the reds and there's all the yellows. But they still couldn't get how that could help them |
| 888 | | R1 | Okay. |
| 889 | 01:38:27 | UCT | So then I was like well what do you think you were both convinced by. Well definitely the beginning. So they went back to this [pointing at work on projector]. |
| 890 | | R1 | Okay. |
| 891 | | UCT | But then they still got very stuck.[inaudible] |
| 892 | | R1 | And this is still very nice. Talk to me what type of proof is this? When they arrange by towers that have exactly one of one color and three of another color. Towers that have exactly Two of each color towers that have four of each color…what kind of proof, do you remember what it is called? |
| 893 | 01:38:55 | UCT | Proof by cases. |
| 894 | | R1 | Proof by cases, good. And you have to get familiar with what these informal proofs are called. Okay, we talked about a bunch of them today. Recursive argument was one, Do you remember the one that Milin did? What did he do? Milin… |
| 895 | | UCT | Inductive? |
| 896 | | R1 | Inductive argument was another; proof by cases was another okay. Good. |
| 897 | | UCT | So their explanation wasn't really good but their discussion was. |
| 898 | | R1 | Excellent, and I like that you saw that they could rearrange it a different way. |
| 899 | | UCT | Yeah and I didn't even ask them to do that. They just did that on their own. |
| 900 | | R1 | Alright very nice. |
| 901 | | UCT | And then this is just one other one. [The following student argument is shown with the projector: We put them in different groups and there are no any other combinations to put in any of the groups. In group 6, there are only 3 blues and 1 yellow. In group 5, there are only 3 yellows and 1 blue. In group 4, there are 2 blue and 2 yellow in half. In group 3, they are a tower of the same colors and 2 they are in a pattern. In group 2, they are in a pattern. In group 1, tow colors are the same. We try and find more to find but no more towers.]<br>The student is in a group of 3 and He is really quiet. He hardly says anything. And He was working by himself on all of it and the other two were doing their own thing. |

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| 902 | | R1 | right |
| 903 | 01:39:46 | T10 | Then I asked them all to work on it together.  So he really had the best understanding. So it was kind of the first time he was really talking about what he was doing with other students. |
| 904 | | R1 | Good. Good. |
| 905 | | T10 | Because he is very very quiet. |
| 906 | | R1 | Uh huh. |
| 907 | | UCT | So I thought that was a good because it gave him an opportunity to actually talk about it and share with other kids. |
| 908 | | R1 | Very nice. |
| 909 | | T10 | So He also made groups and the letters are a little hard to read. They have these are The first few groups they have here.are the ones that are 2 color. |
| 910 | | R1 | Okay. |
| 911 | | T10 | Then you have the ones that are… |
| 912 | | R1 | Three and one. |
| 913 | | T10 | Yeah, 3 and 1.  And then they have they have these like have the stair case. |
| 914 | | R1 | Okay. Okay. |
| 915 | | T10 | So, he goes to talk about it how he put them in different groups.  And there wasn't any other combinations to put into the group.  In group 6 there are only 3 blues and 1 yellow.  So he doesn't really describe moving the pieces but when I talked to them that was what they were talking about. The same thing for group 5. In group four there are 2 blue and 2 yellow and they are split in half, so he has them where they are together; |
| 916 | | R1 | Right. |
| 917 | | UCT | So it was the two blue and two yellow on top and the 2 yellow and 2 blue. I guess taking the solid towers. |
| 918 | | R1 | Sure. |
| 919 | 01:40:56 | UCT | In group 3 they are towers of the same color and 2 they are in a pattern.  They were not sure what the pattern was. |
| 920 | | R1 | Okay. |
| 921 | | UCT | Their writing wasn't that great but definitely in their group had good arguments. |
| 922 | 01:41:06 | R1 | Very good. And how many of you need to think by yourself before you talk to someone else. Alright, some of your students are that way too. Before they can share and talk ideas with someone else they have to think first by themself. So you have to respect that and that is very nice. I think you have good stuff. I think that you can be very proud for the first time around. But your students did some |

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| | | | really good stuff. And you want to celebrate what they did. |

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| **Description: Transcript of teachers working together on the pizza problem.**<br>**Advisor: Professor Carolyn Maher**<br>**Location: CS Middle School, Old Bridge, NJ**<br>**Date: October 2, 2013 part two** | **Author: Phyllis J. Cipriani**<br>**Verified by: Simone Grey**<br>**Date Verified: Summer 2015**<br>**Page 1 of 17** |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| 1 | 00:00:00 | R1 | Pair off with a partner and now you are going to be a problem-solver. And you will have a problem to solve. I want you to read it when you get it and I want someone to explain what they think the problem is asking of you, okay. Um, you want to work with someone so make sure you are sitting next to someone you can work with. And after you read it, if you think you understand the problem tell me what you think the problem is asking you to do. [long pause] [some inaudible conversations] Okay, what do you think? What is oh, I heard something…why is it okay? |
| 2 | | UCT | First I was like okay just like thinking…1,2 but then Bri said she was like you don't know how many toppings you have so I was just like You don't know if it is one topping or two topping, 3 topping… |
| 3 | | R1 | Okay, lets…why don't we go back again. What is the problem asking of you? |
| 4 | | UCT | How many combinations you have? |
| 5 | | R1 | Of what? What do you have? |
| 6 | | UCT | Toppings |
| 7 | | R1 | You are making pizzas. Are you making pizzas choosing from what? |
| 8 | | UCT | The different toppings. |
| 9 | | R1 | And how many toppings? |
| 10 | | UCT | 4 toppings |
| 11 | | R1 | Choosing from 4 toppings. And your job is to make as many pizzas as you can selecting from the four toppings. Okay. Any questions on what your job is to do? |
| 12 | | UCT | Are we allowed to duplicate? |
| 13 | | R1 | Are you allowed to duplicate-what do you think? |
| 14 | | UCT | Does the order of the toppings matter? |
| 15 | | R1 | Well that is something you and your partner will have to decide. Okay. Do you think you understand what you need to do? |
| 16 | | UCT | Yeah! |
| 17 | 00:02:18 | R1 | Begin. Work so that we can share, okay.[R1 begins to walk around monitoring]. [teachers working in pairs on the pizza problem for 17 minutes-multiple conversations-inaudible] |

| Line | Time | Speaker | Transcript |
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| | | | [Some group conversations are recorded before instructor calls for everyone's attention] [R1 circulates and monitors work of the five groups.] |
| 18 | | R4 | Are you both from Toms River? |
| 19 | | UCT | [first group] so sausage and mushrooms? |
| 20 | | UCT | Uh huh. |
| 21 | | UCT | And sausage and peppers. |
| 22 | | UCT | Two peppers? Oh peppers and mushroom. |
| 23 | | UCT | What? What did you say? [R1 looks over their shoulders at their work.] You can have sausage, pepper, mushroom. [Group one is solving the problem using proof by cases 0,1,2,3,4. Both teachers are making a list of each pizza. One partner is spelling the entire word, the other partner starts to write the full word, and begins to write the first letter of each topping, then realizes that she needs to distinguish between peppers and pepperoni.] |
| 24 | | R1 | [second group R1 stops to discuss work][The teachers have drawn circles to represent pizzas.] |
| 25 | | CP | No, Peppers, roni, sausage…. No, right? Is there going to be one less? |
| 26 | | R1 | [R1 stops at third group] Hmm, what do we have here? What is this here? |
| 27 | | UCT | How many four topping pizzas, three topping, two topping you can make. |
| 28 | | R1 | Interesting. Very interesting. Do you see what she did? Did you do the same thing? |
| 29 | | CDR | Yeah, I didn't number them. Well I said toppings from topping to topping. |
| 30 | | R1 | What? The same two? |
| 31 | | CDR | The same what? |
| 32 | | R1 | She said there is one topping here. |
| 33 | | CDR | Right. |
| 34 | | R1 | Then she said there was four here. |
| 35 | | CDR | Then 6 then 4 then 1. |
| 36 | | R1 | Yeah. Did you ever see that anywhere before? Those numbers.1,4,6,4,1 |
| 37 | | UCT | Oh it's the Fibonacci numbers! |
| 38 | | R1 | Close! Try again! |
| 39 | | CDR | Oh yeah, I know what you are talking about. |
| 40 | | MM | Oh, it's Pascal's triangle. |
| 41 | | R1 | Have you seen Pascal's triangle before? |
| 42 | | MM | Yeah, for the towers. |
| 43 | | R1 | Is this connected anywhere? Can you see a connection? [R1 waits with slight pause] |

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| 44 | | CDR | Well yes and no. Like Originally, there was 13 but it's not… |
| 45 | | R1 | I challenge you to find a connection. Because Your children may find a connection Because they don't look alike. I am challenging you to find a connection between the problems. Because they don't look alike, do they? See if you can make a connection. Do you need…We don't have the blocks here do we? [Video shows CDR's work where she began a list writing the topping word but then changed to write the first letter of each topping word. CDR decided to use the letter E to represent the topping of pepperoni.] |
| 46 | | MM | Can we reorganize them? |
| 47 | | R1 | [fourth group R1 stops to discuss work] Are they the same or different? |
| 48 | | R1 | Why did you do that? |
| 49 | | UCT | Cause I was going to do like that. Yeah. |
| 50 | | R1 | No the RONI won't cause you are telling me If it is on purpose. They put something in where Students will have to invent notation. And you did invent notation, Good. |
| 51 | | UCT | Wait! |
| 52 | | R1 | Do you have all the…. |
| 53 | | UCT | We have. I'm trying to think of how many there are possible. |
| 54 | | R1 | Yeah but I don't think that's all. I think it's more. |
| 55 | | UCT | It is more if you are allowed to have duplicates for orders. There would be more if you said peppers pepperoni or pepperoni peppers |
| 56 | | R1 | Well okay, so I am asking If you do a peppers, pepperoni pizza and a pepperoni, peppers pizza…are they the same pizza or are they different? |
| 57 | | UCT | The same. [Video showed JLB's work where the words plain and all toppings were written at the top. This is the opposite strategy.] |
| 58 | | R1 | So you think it is the same. Then you wouldn't want to do it. Okay. |
| 59 | | UCT | So no halfs or quarters. ALL or nothing.[laughter] |
| 60 | | R1 | Okay, okay. So I will ask you again, do you think you have them all? |
| 61 | | UCT | I don't know! |
| 62 | | R1 | Did you try and try and try and couldn't find anymore? [laughter] |
| 63 | | UCT | We think that now [laughter]. |
| 64 | | R1 | Okay. Okay. |

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| 65 | | UCT | I figure you can have a plain no cheese.[inaudible] |
| 66 | | R1 | [R1 goes to group 5] Can you go to a pizza store and order that pizza? |
| 67 | | NL | I have had done that plenty of times where I ordered no cheese. |
| 68 | 00:09:37 | R1 | Okay, okay. What does it say here? [inaudible] Did you work in a pizza store? |
| 69 | | NL | Yeah, that is probably why! |
| 70 | | R1 | Okay. And you know what your students are also going to have some questions. |
| 71 | | NL | I know what it is saying you get a standard and then a topping. |
| 72 | | R1 | Okay, okay so what are they calling a standard plain pizza? |
| 73 | | UCT | Cheese and sauce. |
| 74 | | R1 | Okay. |
| 75 | | UCT | And then I read then you can select the toppings on top of the cheese. |
| 76 | | R1 | What do you think? |
| 77 | | NL | I think you can go either way. I mean I do see it that way but I just think you can look at the toppings a different way. |
| 78 | | R1 | Okay so this problem isn't like the other. So in other words, when you did your…. |
| 79 | | UCT | She did cheese all over included and I took cheese out. So that's why I have more than she does. |
| 80 | | R1 | How many do you have? |
| 81 | | UCT | I have 16 |
| 82 | | R1 | That's 25? |
| 83 | | UCT | Well this one doesn't count. |
| 84 | | R1 | Okay so you have 24. Alright, So the way You are getting more than her. |
| 85 | | UCT | Yeah some of hers you are counting 2 times. |
| 86 | | R1 | How many toppings do you have if you count cheese as a topping? |
| 87 | | MC | You have 5 toppings because if you count cheese it is like one of the toppings. Like yours is different because like for one topping you put the cheese where you are putting pepperoni just pepperoni on it. Just sausage. Just mushrooms, just peppers or whatever… |
| 88 | | R1 | And what are you saying? |
| 89 | | MC | I am saying just cheese and then you can have just cheese and pepperoni is a two topping. |
| 90 | | R1 | As a 2-topping? So You are counting cheese as a topping also? |

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| 91 | | UCT | Yeah. |
| 92 | | UCT | Also the cheese its, the same one I have. |
| 93 | | R1 | Okay, so I am still still confused here. You have this cheese is it a topping or not a topping? |
| 94 | | UCT | Can we count it as a topping? |
| 95 | | R1 | Yes you could. Okay. |
| 96 | | UCT | That is why it is in the front. |
| 97 | | R1 | Okay so, So that is a group, cheese, is that what you are saying? |
| 98 | | UCT | Yeah, Just cheese. |
| 99 | | R1 | Okay, just cheese. Okay, okay. |
| 100 | | UCT | So then I counted it as one topping which would really just be… |
| 101 | | R1 | So these your… are these your one topping pizzas? |
| 102 | | UCT | Yeah but I counted it as like your first one is just cheese plain; with I guess you can put really no toppings. |
| 103 | | R1 | So.. This one means what? That’s why I’m confused. |
| 104 | | UCT | That it is just one item on the pizza. Just cheese, nothing else; Plain; a plain pie. |
| 105 | | R1 | Okay and now you’re saying this looks like this I think. |
| 106 | | UCT | It is. Yeah. |
| 107 | | R1 | So these are your one topping pizzas. |
| 108 | | UCT | Yeah |
| 109 | 00:12:30 | R1 | And these are your one topping pizzas? |
| 110 | | UCT | I guess technically if you don’t count cheese as a topping. |
| 111 | | R1 | Are we counting cheese or are we not counting cheese? |
| 112 | | UCT | Well but just a plain cheese counts as one of your options. Because You can choose to not add any topping. |
| 113 | | R1 | Oh that’s right. Okay. |
| 114 | | UCT | That’s why she is saying it. |
| 115 | | R1 | Okay, well you have it like that too. |
| 116 | | UCT | I am just like listening and it sounds like a cheese pie and a pepperoni, a cheese pie and a sausage, ...[inaudible] |
| 117 | | R1 | But these are your one topping pies. |
| 118 | | UCT | Yeah. |
| 119 | | R1 | So are these your one topping pies? |
| 120 | | UCT | Yeah, I guess you can say that. Yes. |
| 121 | | R1 | Okay and what are these? |
| 122 | | UCT | 2 topping. |
| 123 | | R1 | And these are…? |
| 124 | | UCT | 3 and then 4 toppings |
| 125 | 00:13:25 | R1 | Okay so you Both are doing it 1,2,3,4 |
| 126 | | UCT | Yeah. |
| 127 | | R1 | but how come you have different answers. Oh. You don’t, |

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| | | | oh did you cross these out. |
| 128 | | UCT | Yeah, she crossed them out. |
| 129 | | R1 | Ahhh, Okay you got rid of some stuff. Okay. |
| 130 | | UCT | Well I mean If Cheese was not a topping I am not going to write all these so I got rid of it. |
| 131 | | R1 | Okay you got rid of the C. |
| 132 | | NL | I am missing one here.[video shows work of NL which is proof by cases counting cheese as a topping and using the first letter of each topping word to represent the topping with the exception of PR to represent pepperoni.] |
| 133 | | R1 | Ahh so you are missing a 3 topping. Okay. |
| 134 | | UCT | Wait so it can it just be… |
| 135 | | R1 | No, How many 3 toppings do you have? |
| 136 | | UCT | 6 |
| 137 | | R1 | Now which are your 3 toppings I am a little confused here. |
| 138 | | UCT | I am counting it now. |
| 139 | | R1 | Okay. |
| 140 | | NL | Mine are not very organized. She is more organized. |
| 141 | 00:14:03 | R1 | No, You are both okay. But that's okay. |
| 142 | | UCT | 3 toppings, 2 toppings would be 6. |
| 143 | | R1 | So she has… |
| 144 | | UCT | Oh yeah, 1,2,3,4,5,6… |
| 145 | | R1 | Okay, what's 3 toppings? |
| 146 | | UCT | 4 |
| 147 | | R1 | Okay.[Teachers are counting their 3 topping pizzas.] |
| 148 | | UCT | PRM |
| 149 | | UCT | You have PMR? Oh that's right you used R for pepperoni. |
| 150 | | UCT | Yeah I used R for pepperoni. |
| 151 | | R1 | So you both have 4. Okay. And so your… these are for 3 topping pizzas? These Over here. |
| 152 | | UCT | MMhh. |
| 153 | | R1 | Okay, And your four topping pizzas? |
| 154 | | UCT | Just [Teacher circles one pizza with symbols representing all 4 toppings] |
| 155 | | R1 | Only one. Okay, alright. Interesting, okay so…you have 16? |
| 156 | | UCT | Yeah. |
| 157 | | R1 | Do you think you have them all? [slight pause] |
| 158 | | UCT | Yes. |
| 159 | | R1 | Can You say it with conviction or…. |
| 160 | | UCT | Well, I mean if you do it like the tower problem and simplify it so you know what I am saying like if this is cheese, peppers, sausage, then you could just add a mushroom or add pepperoni. |

| Line | Time | Speaker | Transcript |
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| 161 | | R1 | Oh that's interesting. |
| 162 | | UCT | And then so like the same thing here was peppers and mushrooms, peppers, mushrooms and added a pepperoni. So like we just added the other 4 toppings we didn't include. |
| 163 | | R1 | Okay. |
| 164 | 00:15:52 | UCT | We both agreed that order didn't matter. |
| 165 | | R1 | That it didn't matter. Okay, so it was sausage pepperoni is the same as a pepperoni sausage. Okay. Some of your students might not be sure about that. |
| 166 | | UCT | No, they probably won't. |
| 167 | | R1 | And if they are not, you are going to ask them to talk to each other and come up with what they think. I have a question, you got 16 pizzas. Does this problem remind you of any problem that you did? |
| 168 | | UCT | towers |
| 169 | | R1 | Well or anything that you have done before… |
| 170 | | UCT | There is an organized list problems. |
| 171 | | R1 | Okay. |
| 172 | | UCT | That is just what I thought of while doing this problem and she wanted to do a tree diagram. |
| 173 | 00:17:00 | R1 | Ahh, Interesting. Um, it would be interesting to see what happens when you do that. Um. Okay, um.. How can you um… talk with each other…how did this remind you of the tower problem, okay and then we will talk about it as a group. [leaves fifth group and goes back to fourth group-other conversations occurring] |
| 174 | | R1 | MMMhh, I see you did a tree diagram, is it easier? |
| 175 | | UCT | Well it's going to be really big, I just know that many of my students are going to do that. |
| 176 | | R1 | MMHhh. |
| 177 | | UCT | They are totally going to do that, I just know it. And um, then they are going to have their lists.[inaudible] |
| 178 | | R1 | MMhh. |
| 179 | | UCT | We thought about a table but we don't know what to do with the 2 topping. |
| 180 | | R1 | MMHH. |
| 181 | | UCT | And then you have to figure out what to do with the duplicates. |
| 182 | | R1 | And that would be confusing. Right. |
| 183 | | UCT | [inaudible] So I think a lot will have the lists where they have to write out and…[inaudible] |
| 184 | | R1 | AAhhh. Okay. |
| 185 | | UCT | Are we going to have to do this in class? |

| Line | Time | Speaker | Transcript |
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| 186 | | R1 | We will talk about that. And you are right when you do it this way, it is a lot easier. Alright, question for both of you…How many toppings… uhm how many pizzas did you get? |
| 187 | | UCT | 16 |
| 188 | | UCT | Does anybody want anything else? |
| 189 | | R1 | Does this remind you of anything else? |
| 190 | | UCT | Yeah, the handshake problem. |
| 191 | | R1 | MMHH Well Talk with each other, how is it like that? |
| 192 | | UCT | Like the towers[response is inaudible] |
| 193 | | R1 | You have 2 things that you are telling me about. It is reminding you of the Handshake problem and it is reminding you of the towers problem. Talk with each other, How does this remind you of these problems? [R1 monitors next group]. |
| 194 | | R1 | MMMhhh. |
| 195 | | MM | So I am taking this and then you have four options and then I am putting an x. Video shows MM's chart with the letters P, M, S, and I were used to represent the toppings and X's where X represented no topping |
| 196 | | R1 | Interesting, very interesting and what does the x stand for? |
| 197 | | MM | No topping, so this is like plain. |
| 198 | | R1 | Ahhhh, I got it, pretty neat! And what did you do? |
| 199 | 00:19:04 | UCT | So did the same thing. I had no toppings and then one topping and then two so I held a constant pepperoni and I did one of each. |
| 200 | | R1 | MMMhhh. |
| 201 | | CDR | And then I moved to a new constant, and I put one of each in the positions and I moved to a new constant of mushrooms and we were missing one topping so we did this. |
| 202 | 00:19:40 | R1 | MMhh, mmhhh. Very interesting. Very interesting. Um for those of you who are finished, which most of you are…I challenge you to tell me how does this problem remind you of another problem in math. Something you have seen before something you have done before okay. And tell me what the connection is. [giggling] [slight pause] Let's uhhh. Can we maybe put up some of the work; um let's talk about a solution. I saw many of you did it by talking about How many toppings the pizza had and that is how you organized your pizza and You talked about a plain pie. Did all of you decide that a plain pie was one of the choices? |

| Line | Time | Speaker | Transcript |
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| 203 | | unison | yes |
| 204 | | R1 | Okay. Then you talked about one topping pies. How many pies could you make that had exactly one topping? |
| 205 | | unison | 4 |
| 206 | | R1 | 4 okay. Then you talked about 2 topping pies. And how many pies could you make that had exactly two toppings? |
| 207 | | unison | 6 |
| 208 | 00:21:02 | R1 | 6, and Then you talked about 3 topping pies, how many? |
| 209 | | unison | 4 |
| 210 | | R1 | 4, And then you talked about a pie that had all 4 toppings and there were? |
| 211 | | unison | 1 |
| 212 | | R1 | Okay, and if you looked at those numbers. It was 1,4,6,4,1 Have you seen that before? |
| 213 | | UCT | yes |
| 214 | | R1 | Where? |
| 215 | | UCT | Pascal's triangle. |
| 216 | | R1 | Pascal's triangle. Remember that?[slight pause] Where did we see Pascal's triangle? [slight pause] Very recently… |
| 217 | | UCT | The towers. |
| 218 | | R1 | With the towers problem, when you were building towers 4 tall selecting from 2 colors, How many towers did we get?[slight pause] |
| 219 | | unison | 16 |
| 220 | 00:21:40 | R1 | How many pies did we get? |
| 221 | | unison | 16 |
| 222 | | R1 | And we also saw the 1,4,6,4,1. How did we see that? Um, can I…. |
| 223 | | UCT | Yeah, you can just pull it up. |
| 224 | | R1 | Okay. I am going to write right here! Okay, Remember when we were building… The pizza you are telling me has 1,4,6,4,1 right? For the plain pie, one topping, two topping, 3 topping, 4 topping. Think of the towers problem. Building towers four-tall selecting from two colors. Okay. What did we get with that? [slight pause] |
| 225 | | UCT | 16 |
| 226 | 00:22:21 | R1 | How? Talk to me about the towers. |
| 227 | | UCT | Well one would be all yellow. |
| 228 | | R1 | All yellow. |
| 229 | | UCT | If you were starting with yellow, that would be all yellow. |
| 230 | | R1 | Okay |
| 231 | | UCT | Then you move on to Three yellow, one red. |
| 232 | | R1 | Okay. |
| 233 | | UCT | And then 2 red, 2 yellow. And then the one yellow, 3 red. |

| Line | Time | Speaker | Transcript |
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| | | | And then no yellow. |
| 234 | 00:22:45 | R1 | Isn't that interesting? Okay. So it looks like this problem, structurally is the same as the pizza problem. The towers problem building towers 4 tall selecting from 2 colors has the same exact mathematical structure as building pizzas selecting from four toppings.<br>Do the problems look alike without looking at the mathematical structure? They look very different, don't they? Building towers and building pizzas. They do not look the same.<br>But they have the same mathematical structure. Do you have any idea what that is called? When 2 problems have the same mathematical structure?[slight pause]<br>Isomorphic problems. Okay, They are isomorphic. It is a good word for you to know. Do your kids have to know it? No, okay, but it is a good word for you to know and it is also a good thing for you to know that those problems have the same mathematical structure. Now I saw Toms river was starting to come up with how you can actually say those problems are the same. Okay, Can we copy your work and just show it up and see if we can build on it? |
| 235 | | UCT | I have to turn the projector back on. |
| 236 | | R1 | Did you shut it down? |
| 237 | | UCT | Yeah. |
| 238 | | R1 | Okay. Can we put it back on? |
| 239 | | UCT | Uh huh. |
| 240 | | R1 | Great! Thank you. I can tell you there is going to be a young 5$^{th}$ grader named Brandon that you are going to meet in video next week. Brandon was my student. He was in my school. I was the principal at colt's neck. And Rutgers came into my school for ten years. I had just completed my doctorate at Rutgers. And um I thought I would like to have Rutgers involved in my school. And they came in and they did workshops and They were in classrooms and they did interviews with individual children. In fourth grade, Brandon was in the lowest math track. We were still tracking this was way back in the 90s when tracking was the thing; now I hear some of your schools are still tracking, but it kind of gives me a little bit of a jolt. But elementary school But I am hoping that none of the elementary schools in your district are tracking Because whatever the teachers did and they were very good teachers, whatever they did to decide who was in the top group and who was in the second and who was in the bottom, they put Brandon in the bottom group. When you |

| Line | Time | Speaker | Transcript |
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| | | | watch Brandon on video,<br>I think you are going to be floored about what he was doing. Okay, I think you are going to have to watch the video several times to really understand what he was doing and yet he was in the bottom math group. So I want you to very carefully think about who your students are because I think you may have a Brandon in your class and some of them may make the same connection Brandon did. Now let's see what is up here. Oh my gosh, this is interesting. Talk to me about this.[slight pause] |
| 241 | | MM | Um, Well there's four possibilities. The first possibility is getting nothing so an x just means that it is not [a topping].. |
| 242 | | R1 | Okay so these are your four toppings. |
| 243 | | MM | Yeah |
| 244 | | R1 | And Notice this little I here. |
| 245 | | MM | That's pepperoni. |
| 246 | 00:26:02 | R1 | Okay. When you do this problem with your students and you will be, they are going to have to invent notation. I saw one group did an R for pepperoni, |
| 247 | | UCT | Yeah. |
| 248 | | R1 | Right? Anyone else do a different thing for pepperoni? than I or R? |
| 249 | | UCT | I used E |
| 250 | 00:26:21 | R1 | E for pepperoni. Okay. And it doesn't matter what you use. Some people may even want to use PP which is a little confusing because you end up with so many Ps that you don't know which is the P and which is the PP. But this is inventing notation. Do not change the problem so that they don't have to invent notation. We want them to invent notation. Ah, Okay so this is the pizza that has…? |
| 251 | | MM | Just cheese |
| 252 | | R1 | this is the plain pie. Okay, what pizza is this? |
| 253 | | UCT | Peppers. |
| 254 | | R1 | Now that is the pizza that has just peppers so it is a single topping pizza. Um, While they are doing that, can we ask,… you said there were unifix cubes? |
| 255 | | UCT | Yeah. |
| 256 | | R1 | Is it easy to get them? |
| 257 | | UCT | I just have to go to a different room. |
| 258 | | R1 | Is it close by or not? |
| 259 | | UCT | Kind of do you want me to get them? |
| 260 | | R1 | If you could. Yeah, we don't need a lot one bag is enough, okay. And what is this? |
| 261 | | MM | That just means that you are not getting that topping. |

| Line | Time | Speaker | Transcript |
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| 262 | 00:27:37 | R1 | So it is a 2 topping pizza |
| 263 | | MM | Oh yeah sorry that one is peppers and mushrooms. |
| 264 | | R1 | And this one is a..? |
| 265 | | MM | Peppers, mushrooms, and pepperoni pizza. |
| 266 | | R1 | Interesting okay, and this is..? |
| 267 | | MM | Peppers, mushrooms, pepperoni, and sausage. |
| 268 | | R1 | So it is a 4 topping pizza. |
| 269 | | MM | Yeah. |
| 270 | | R1 | So, okay, The way she is building her pizzas is how? [slight pause] She is actually keeping a constant, isn't she? She is keeping her peppers constant and she is adding the mushrooms and then she is adding the pepperoni and then she is adding the sausage. How many of you built your pizzas that way? [pause] |
| 271 | | MM | I didn't build my pizzas that way. |
| 272 | | R1 | You didn't but that is how you are recording it. Okay. |
| 273 | | MM | Well I just decided to change it up. |
| 274 | | R1 | Good. |
| 275 | | MM | Like the top I did one topping of peppers, and then I did all of the possible toppings of peppers with 2. |
| 276 | 00:28:13 | R1 | Nice. |
| 277 | | MM | Well, No that is at the very top. I am talking about… |
| 278 | | CDR | All the way. Above it. |
| 279 | | MM | Above it. |
| 280 | | R1 | Oh up here. Okay. |
| 281 | | MM | The first time I did it I actually just did peppers. And then peppers with another topping and changed the other topping. And then peppers with 2 other toppings. |
| 282 | | R1 | Okay |
| 283 | | MM | I changed those and then peppers with 3 other toppings and I think I actually like that way better. |
| 284 | | R1 | Ahh. Now look at this. This is a very different way of organizing then. Those of you that did one topping pizzas, 2 topping pizzas, 3, isn't it and Notice what she got. When she did all the toppings with the pizzas with peppers, how many did she get? [slight pause] |
| 285 | | UCT | 8 |
| 286 | | R1 | And when she built the ones with sausage she got…. |
| 287 | 00:28:53 | UCT | 4, 2 and one and then 3 ingredients, 2 ingredients,one. |
| 288 | | R1 | Isn't that neat? I think you are going to see students that are going to do it that way as well. Okay, this is yeah I know I have seen it all 4 years in this… different cohorts. so I think you are going to see it also. Okay so here you got your pizzas here alright. And do we have them all or |

| Line | Time | Speaker | Transcript |
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| | | | not really? |
| 289 | | MM | Well no, I didn't finish. |
| 290 | | R1 | Okay, so Let's just take with what we have and the unifix cubes are….here, just one bag is perfect, thank you so much. |
| 291 | | UCT | I just brought a whole tub. |
| 292 | 00:29:48 | R1 | Okay, all I need is….This is good. I'll just take 2. That's great, thank you. Okay, If we are going to look at…um the pizzas here. Like here is a pepperoni pizza. Okay. How could we build a tower that might look like that pepperoni pizza. Any ideas? |
| 293 | | UCT | 1 red, 3 yellow. |
| 294 | | R1 | Say it again. |
| 295 | | UCT | One red three yellow. |
| 296 | | R1 | 1 red, 3 yellow Okay, so what is the red? [holds up red cube] [pause] |
| 297 | | UCT | pepperoni |
| 298 | | R1 | And what is the yellow?[pause] |
| 299 | | UCT | Cheese. |
| 300 | | R1 | Good. Plain. [using the unifix cubes to show 1 red on top, 3 yellow] So this is a one topping pizza that is a pepperoni pizza. Okay. Yellow meaning the topping doesn't appear, red meaning the topping does appear and it is the pepperoni here. How would you build this pizza? What would it look like? |
| 301 | | unison | 2 red, 2 yellow |
| 302 | | R1 | Okay. [R1 using unifix cubes to show 2 red on top, 2 yellow on bottom] Oops, gotta get it on top. Alright So the toppings that appear are the reds and the ones that don't appear are the yellow. Okay, How about the third one? |
| 303 | | unison | 3 red and one yellow |
| 304 | | R1 | Okay, [building the tower of 3 red on top and yellow on bottom]You are going to see Brandon doing this, okay. And he is going to be working with a chart that is similar to this, but not exactly like it. What he actually does is he uses the numbers 1 and 0; okay; 1 being the topping appears. 0 being not on the pizza. After he did that, the interviewer said oh I know, His father must be a programmer. You know, Binary, right? 1s and 0s Guess what? His father wasn't a programmer. His father had nothing to do with anything that was math. So very Very interesting that Brandon came up with his way of recording rather than putting x's or if it didn't appear; he put 0. If it did appear, he put a 1. Alight, um What would be this pizza? What would it look like? [slight pause] |

| Line | Time | Speaker | Transcript |
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| 305 | 00:31:56 | UCT | One red 2 yellow 1 red. |
| 306 | | R1 | Say it again. |
| 307 | | UCT | One red 2 yellow 1 red. |
| 308 | | R1 | One red 2 yellow 1 red. Okay, so If you watched the video, you are going to see how Brandon makes the connection between the two problems. Alright and it is pretty neat stuff.<br>I really encourage you if it means that you can't follow him; Watch it again, because it is brilliant. Um, You want to know what Brandon is today? He is probably older than some of you. I think he is 30 at this point and remember he was in the lowest math group. Today he is a veterinarian. Pretty bright, okay. So I think when you are looking at your students, sometimes we have an image in our head of what we think they should do or what they should be to be bright. But sometimes the kids who really are bright, don't shine in the same way. They may not be so good at a quick…you know spitting back quickly, but they may be really brilliant if you give them time to think and be thoughtful and to show you what they really know. Okay how are we going to bring this into your class. Because this is the next task that you are going to do with the children. You are going to take again, pairs; if you can pair except for maybe one triple. Don't do a lot of triples, you saw you had trouble with triples. Take pairs. If you liked your pairs last time keep them. If you didn't like them, switch them up. Keep the same class. Because remember we are trying to look at growth over the course of the semester and If you switch the class that you go into, you are not going to be able to see growth because you won't know where they were last time. So go into the same class you were in, give the task,<br>Do not just let them read it but have it on the overhead, the way we did with the first task.<br>Have a student read it and then have them talk about what the problem means.<br>They are going have all the same issues that you had and then some. They are going to say oh:<br>Is a pepperoni, sausage pizza different than a sausage pepperoni pizza? Because they are going to think order matters. They have heard about permutations versus combinations.<br>And you are going to say ask your partner what does he think? Okay. And let them battle out whether they think they are the same or different. Ah, |

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| | | | You are going to have some children jump right away into a tree. They are going to run into big trouble. Because it's going to give them lots of duplicates and they are going to get confused. Do not send them down the path of 1 topping pizza, 2 topping pizza, 3 topping; don't do that. Cause then you are going to tell them how you did it; and how they can do it easily. Let them you know work on their own how they want to solve the problem. Um By the way when you did 1 topping, 2 topping; 3 topping pizzas, 4 topping; what were you doing? What kind of proof? |
| 309 | | UCT | Proof by cases. |
| 310 | | R1 | Good<br>Proof by cases. Okay, You are going to want them to uh, work with each other. You are going to want them to find the number of pizzas. If they get done, what are you going to have them do? You are going to have them write a convincing argument.<br>Why do you think you have them all? Have them write. Encourage them to write. Okay. Why do you think there aren't anymore? Okay. Again if they did it by a proof by cases, and<br>If you say well you only have one four topping pizza?<br>Do you think you have them all? Yeah, because there are only 4 toppings. There is only one way to get a four topping pizza. Okay. Um<br>Again if they get done, and you have had them have written the convincing argument ask them does this problem remind you of any other?<br>Now someone actually said the handshake problem, and then you want to see if they can find the connection there. Some of them will say oh those building towers. See if they can find the connection between the two. Again, they may not know Pascal's triangle. But they may say oh my gosh there is 16 for the number of pizzas here and there is 16 in the number of towers.<br>Maybe that is the connection. Maybe they might..<br>Did any of your children see 1,4,6,4,1 when they did towers? I doubt it, okay so if they didn't see it there; they probably won't be able to make that connection<br>But ask them what kind of connections they see. you might be surprised. Maybe some of them will think back to the towers and be able to do what Brandon did and actually see how those problems are isomorphic or how they have the same structure.<br>Question on how you are going to implement? [pause} |

| Line | Time | Speaker | Transcript |
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| | | | Okay now remember at the end of the semester, you are going to do a project that is going to be asking you to collect student work from the 3 tasks that you will be doing this semester. So you are looking for work that surprised you, that impressed you or that puzzled you. Those are the kinds of things you are looking for. So when you come next time again, you are going to bring 2 or 3 work samples from this pizza problem okay, that we can talk about. Okay, Question? |
| 311 | 00:37:19 | UCT | So when you come on the 22nd, you are teaching the class? Is that the next lesson that we are doing? |
| 312 | | R1 | No, no, that's good. what we are doing now is you can implement… you already saw me do an implementation . You have already done one implementation yourself. You go ahead any time after today you can implement this problem. When you see it fits into your schedule. Our next regional meeting is which is going to be first is going to be followed by the classroom implementation. I will be doing the implementation of the problem with your class. However you should do it with a different class. Okay. Alright. |
| 313 | | UCT | Okay, Yeah. |
| 314 | | R1 | Um and which is class that you are writing about? Which is the class that you did the implementation last time? |
| 315 | | UCT | The class that you will be doing. |
| 316 | | R1 | So then, you will just use that work when you do it. Okay. In other words, I want you to have the practice of doing an implementation so |
| 317 | | UCT | Just use their work. |
| 318 | | R1 | Do it with another class yes, because you want to keep the same class for all three. Okay, yeah. |
| 319 | | UCT | So can I use the class that everyone came into, then? |
| 320 | 00:38:33 | R1 | Yes. Is that the class that you are going into now? |
| 321 | | UCT | Yes I would like to use that class because I have that extra 20 minutes. |
| 322 | | R1 | Absolutely you can. And what you did today, um that's fine. Because you had implemented it on your own. |
| 323 | | UCT | With a different class. |
| 324 | | R1 | That's fine. That's fine. Any questions on implementation? Or on anything. Or okay. |
| 325 | 00:38:50 | UCT | So we have to do it before we see you next time. |
| 326 | | R1 | You have to it before because when you see me next time, you have to bring your work and you are going to be talking about it. So you absolutely have to do it before you see me. Okay. |

| Line | Time | Speaker | Transcript |
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| 327 | | UCT | Do we have to figure out about the duplicates[inaudible] |
| 328 | | R1 | No, they have to figure out… you are not telling them. You are listening, and you are questioning, but you are not telling them. |
| 329 | | UCT | But some of them might think that with the order of the toppings, should we tell them that. |
| 330 | | R1 | Wait, Stop one minute. We have 5 more minutes, because we started late. And this is a very good question. Some of your students and someone here came up with Can you have half a pizza of this and a half of that. Who was that? |
| 331 | | UCT | Oh it was me, I said all or nothing. |
| 332 | | R1 | But do you know what, Some of your kids are going to say. I know I can go into a pizza store and I can get half a pie of this and half of that and if they do that, they will get a much more complicated problem. But but…<br>Don’t stop them, let them go. They will get a different answer and It will be a lot harder but if that is the direction they want to go then let them do it. Okay.<br>You said also what was your question about the…? |
| 333 | 00:40:07 | UCT | No I said You can get extra toppings so I might get pepperoni twice. |
| 334 | 00:41:17 | R1 | Another group said can we have Double pepperoni. Right I don’t remember who it was but they said can we have double pepperoni? The question is Again you did it without having any doubling stuff. If they change the problem because they are thinking outside the box or they are thinking for reality. You worked in a pizza place and she said, You can get a pie without cheese. Okay. Some of them will get confused and say well is the cheese pie a separate pie or is cheese the topping um you are going Let them battle out What they think the parameters are. But the parameters are what you did which is really that you have a cheese pie which is your plain pie. And that is one of the pies. And then you have 4 toppings to choose from, to make your other pies. You are not going to tell them because if you tell them you are making the problem you know simpler for them. You are going to let them struggle a little bit. Struggling is good. Let them work with their partner. You can listen. You don’t want to lead them in the wrong direction but don’t lead them in your direction. Any other questions? Alright have a good couple of weeks. |

R1 - Dr. Judy Landis
R4 - Phyllis Cipriani
UCT - Unidentified Teacher
Initials - identified teacher

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| **Description: Transcript of debrief meeting from in-district classroom visit of the pizza problem.**<br>**Advisor: Professor Carolyn Maher**<br>**Location: TRIE**<br>**Date: October 22, 2013 part one** | **Author: Phyllis J. Cipriani**<br>**Verified by: Simone Grey**<br>**Date Verified: Summer 2015**<br>**Page 1 of 9** |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| 1 | 00:00:00 | R1 | [The two students used numbers to represent the toppings (peppers=1, plain=2, sausage=3, mushrooms=4, pepperoni = 5). Then students had the following on their paper:<br>1+1<br>1+2 2+2 just plain peppers=1<br>1+3 plain=2<br>1+4 sausage=3<br>1+5 mush=4<br>2+3 pepperoni=5<br>2+4<br>2+5<br>3+4<br>3+5<br>Did anyone not see that group? [slight pause and No response.] Ok, everyone did see that them. And that was a very interesting notation, wasn't it? Very different. I don't think I've seen that before either. Did that get them into trouble? How did it get them into trouble? |
| 2 | | UCT | Yes. |
| 3 | | UCT | Well, at one point...she...the one...the girl...um..[inaudible]...she put one plus one, two plus two. And I was like.. well..does that mean...I think she gave like two peppers. Does that mean peppers and peppers? Then she was like no, that just means it only has peppers. Then she was like, maybe I should just erase the second two and only put one two. And I said, that's a good idea. And then she did went back because she had five plus five, and five just meant mushrooms [unintelligible]. |
| 4 | | R1 | Ok. Ok, so their codes worked but also got them into trouble. Um...did they end up finding the 16 pizzas? |
| 5 | | UCT | Yes. I think so. |
| 6 | 00:00:50 | R1 | Ok. Ok. What other group did you find interesting?[Teachers point to a corner of the room.] |
| 7 | | UCT | There was a... |
| 8 | | UCT | Right over here. |
| 9 | | R1 | Those two boys. What did those two boys do? |
| 10 | 00:01:00 | UCT | They made the chart and put the X's. In each... |

| Line | Time | Speaker | Transcript |
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| 11 | | R1 | How many of you saw the chart with the X's? [Teachers raise hands.] |
| 12 | | UCT | In each row was a pizza. |
| 13 | | R1 | In each row was a pizza. Now what got them into trouble? |
| 14 | | UCT | Their ordering at first, because they were just doing them all random. And then once they re-organized it [inaudible]. |
| 15 | 00:01:18 | R1 | Yes. And actually...um...uh...I thought it was very clever...um...they were checking themselves. They had 11 pizzas in their chart. And they were checking themselves to say what did they have. And they had a...uh...oh well they said well I have a mushroom and pepperoni pizza. I have a mushroom and pepper pizza. They were trying to keep all this in their head as to what that they had.<br>And Margaret went over to them and simply said to them, you know why don't you write down what you are saying, rather than trying to keep it in your head. And that helped them by writing down what pizzas they found they...what did they figure out?<br>What had they forgotten? Were you there when…They forgot the three topping pizzas. Okay, so, then they were able to add those in.<br>But that was…[Margaret enters.] We just gave you a compliment. I thought your comment to the two boys that were doing the chart with the X's was really, um uh, very, very good. You didn't lead them. But as they were trying to figure out what pizzas they had on their chart or didn't have. You said, why don't you write that down as opposed to keeping that in your head. |
| 16 | | UCT | Yeah. |
| 17 | | R1 | And that's how when they found out they were missing the three toppings. So that, that was really good. |
| 18 | | UCT | Yeah. Thank you. |
| 19 | 00:02:37 | R1 | What other group did you follow that you thought was interesting? |
| 20 | | UCT | The girls back here had a little bit of difficulty at first actually. |
| 21 | | R1 | Okay. |
| 22 | | UCT | So, I was watching how they were organizing and they would start with sort of like a tree diagram in a way. They would write plain and stem from that 3 toppings. And I would ask I asked them how many pizzas that represented and they'd said one. It's a plain base With those 3 toppings. And then...they've gone...But Then it also represents just the plain pizza because it said plain. And But then it also represented just each of those types. So |

| Line | Time | Speaker | Transcript |
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| | | | they had a lot of difficulty following that and then they found out that as they were working through it, it and they started crossing things off because then they had a lot of doubles talking that it was double and it was harder for them to find all of the doubles to write. So, their organization was very challenging. |
| 23 | 00:03:24 | R1 | Okay, okay. |
| 24 | | UCT | They did end up getting it but it took a while. |
| 25 | | R1 | It was a struggle to get it. Okay. Any other group have a struggle? Who followed the group that was here? It was Paul and...who was Paul’s partner? |
| 26 | | R1 | With the curly hair. |
| 27 | | UCT | Oh, that was John and Caitlyn. |
| 28 | | R1 | John and Caitlyn. Anyone follow John and Caitlin? |
| 29 | | TC | We were all trying. |
| 30 | | UCT | [Several students speaking at once.] I thought that theirs was really..uh. …Just for a little bit…. I saw theirs but I didn’t... |
| 31 | 00:03:49 | R1 | Okay. What did...what were they doing that got them into big trouble? |
| 32 | | UCT | They counted a plain, a cheese, a sauce, and a nothing. But nothing meant nothing. |
| 33 | | R1 | Okay. |
| 34 | | UCT | According to them. Like what, they didn’t actually count the nothing. |
| 35 | | UCT | Like white pizza. And they were even saying... |
| 36 | | UCT | They didn’t say the white to me. |
| 37 | | UCT | They were even kept saying, well, no one’s going to order that. So let’s get rid of that. |
| 38 | | R1 | Right. Good, So they had 64... |
| 39 | | UCT | They went from like 64...had like 19...I was’m like how did they even narrow this down.. |
| 40 | | UCT | Then they had like 16, then 17. |
| 41 | | CP | Yeah, but at the...at the same time though, it’s seemed like a valid argument because people order pizzas without cheese or pizzas without sauce. So, like when… |
| 42 | | R1 | It..It changes the problem. |
| 43 | | CP | He explained it to me he was like okay...so, they had the 16 lined up and then they were like, underneath they had the plain and they had the option. He goes, well I am going to multiply each group by 3 because I can do each combination as it is with cheese and sauce or just with cheese or and just with sauce. |
| 44 | 00:04:35 | R1 | Right. |

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| 45 | | CP | And I'm like...that actually...that works for me because its... |
| 46 | | R1 | Now, When did you go into a... a pizza store and order just cheese, no sauce pie? Has that...is that real... |
| 47 | | UCT | Lots of times. |
| 48 | | R1 | You really can do that? |
| 49 | | UCT | I have. I have. |
| 50 | | R1 | You have? Oh my gosh. |
| 51 | | UCT | When I tried to make it a little bit easier for them. I said that that might be like a specialty thing so that they might not want to include it on their menu. Just like But there is always a lot of things we can do that we don't include on the menu. |
| 52 | 00:05:00 | R1 | You know I think that was real world. And if that really does exist and...and we can follow their reasoning; of course their answer would be much bigger;. many more choices than if you just counted that plain pie as the base for adding all the four toppings. Um, they did decide though on their own that they were going to eliminate all those things, but they ended up with 17 pizzas. Do you see that? And what did they get as 17th pizzas? You have? |
| 53 | | UCT | I didn't see their work. I heard them say 17 though. |
| 54 | | UCT | I heard them say that to you but I didn't... |
| 55 | | R1 | And the reason why they had 17 is, they gave up the splitting the cheese and the...uh sauce on everything, but at the very end they said, well, we could have a plain pie with cheese and sauce or we could have just a white pizza. |
| 56 | | UCT | Oh. |
| 57 | | R1 | I guess that's your pizza without the sauce. |
| 58 | | UCT | Oh okay. |
| 59 | | R1 | So they didn't give it up entirely and that's why 17 would be a correct answer if they were considering getting a...a...a plain...a white pie. Interesting, huh? Good.<br>Could we see some of the work? You're gonna take a picture? |
| 60 | | UCT | Yeah. |
| 61 | 00:06:13 | R1 | Great, okay. Neat stuff really did happen in this room. Um...really neat stuff. And I think that because there were so many of us, it gave you an opportunity to really listen to what they were saying.<br>Was your questioning any better? |
| 62 | | R1 | Is that...is that provided by the school? |
| 63 | | UCT | No |
| 64 | | R1 | Did you feel more secure with your questioning? Did you feel still like you were wanting to lead 'em? [unintelligible |

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| | | | responses] |
| 65 | | UCT | Not entirely. |
| 66 | | R1 | Not entirely. Okay so you felt... |
| 67 | | UCT | I think because we did...well because I did the problem, that half the time I felt like it was a little bit...I kind of anticipated somethings. |
| 68 | | R1 | Good, good. Okay. Alright let's see.<br>Can we make that bigger? |
| 69 | | UCT | No that is as big as it goes. |
| 70 | | R1 | That is as big as it gets. I hope your eyes are good! Umm, and here's the [unintelligible]… |
| 71 | | UCT | I can...but I can...uh...sometimes there's a delay with this. |
| 72 | | R1 | Okay, okay. |
| 73 | | UCT | Because our wireless isn't...I'm trying to like zoom in but it's not really letting me. |
| 74 | | R1 | Uh..use the other one. Well, uh...Let's focus on this one. Woops! |
| 75 | | UCT | Oh, which one? |
| 76 | | R1 | Let's focus on either one. Okay so, here we got this one. This is the one we talked about. This is really interesting. Now, what I noticed this group did, um, they copied over their work. Okay. They started all over because they were...they didn't want to hand in a paper that had the cross-outs. And I think you want to get your kids not to worry about cross-outs because if you didn't see what they were doing and then you just saw this you would miss everything that they eliminated. And that that really is important to know where they started and where they are going. Um...what they also tried to do, I said to them, that I really see their code and I see what they have, but I really don't know what pizzas they are. Could they tell it to me in English, I said. And um,… |
| 77 | | UCT | This is..he had this down too.. |
| 78 | | R1 | Right. And then he started to write what a 1 + 3 + 4 + 5one-plus,-two plus three pizza was with it using the words. okay. Um, but I...I encourage... Like here is the original paper and he didn't like it that he had cross-outs on it. |
| 79 | | MM | Ooops, sorry. This one. |
| 80 | | R1 | That one. Okay. But I really think that...You don't want your kids wasting time with crossing out and writing over. And you know, this is not a published work where you need to have it really looking beautiful and this and that. So you want them to just be able to get more time to write their thinking and their arguments down as opposed to |

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| | | | copying over. Okay. Alrighty! |
| 81 | | MM | And then Hayden he did and Phil were working over here and they started doing this right away.[On video a chart is shown with four toppings on the top written as words and X's in the cells to represent a topping on the pizza.] |
| 82 | | R1 | And isn't that interesting. Did any of your students do that? [Student raises hand.] Okay, you had uh someone student in your class do it too. |
| 83 | | UCT | I had one but...he started making this chart and I was like, Oh snap this is really great like where are you going with this, explain it to me, and then all of a sudden, he was like said, naw I didn't like this. And Then he wrote ignore on it and started doing something completely different. |
| 84 | | R1 | Interesting. Interesting. Can you show us what he switched to? |
| 85 | | UCT | He actually went.. |
| 86 | | R1 | Can you show us what he switched to? |
| 87 | | UCT | Yeah, he went to a number system. |
| 88 | 00:09:22 | R1 | Interesting. Very interesting. Okay. |
| 89 | | UCT | This is the back. So that was...Yeah. |
| 90 | | R1 | So now, go back to that...that first chart. I think you hit it on the head when you said they didn't really have organization when they put their pizzas down. They went from a four topping...to a two-topping...to a two-topping...to a three-topping. So, it was hard for them to know, did we have everything in each group. And that's why your question to them was really spot on, asking them to, you know, to write down what they found.<br>And where did they write it? [Big pause.] |
| 91 | | UCT | Hold on. Sorry. Here it is. |
| 92 | 00:10:13 | R1 | Alright. So there they wrote the different pizzas down. Now, are they organized in terms of into groups? |
| 93 | | UCT | Yeah |
| 94 | | R1 | Okay so what do they have first? |
| 95 | | UCT | It says plain at the top. |
| 96 | | UCT | It's hard to see. |
| 97 | | R1 | Okay, and then after the plain they did?… |
| 98 | | UCT | Pepperoni and mushrooms. |
| 99 | 00:10:30 | R1 | So they did two toppings. |
| 100 | | UCT | Two-topping then one-topping then three-toppings. |
| 101 | | UCT | Yeah. |
| 102 | | R1 | Okay. And then the fourth topping was last? |
| 103 | | UCT | No. |
| 104 | | UCT | No., well their fourth topping was 12 because they...they |

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| | | | didn't have their 3 toppings. |
| 105 | | R1 | Okay. Okay. |
| 106 | | UCT | So they thought it was 12 at first. |
| 107 | | UCT | Okay and then added the 3 toppings. |
| 108 | | UCT | Yep! |
| 109 | | R1 | So that's really neat. What kind of argument did they give? Did they write an argument? |
| 110 | | UCT | Umm, Yeah. |
| 111 | | R1 | Okay can you read it for us? |
| 112 | | MM | First, we looked at the pizzas with only one-topping and got four different pizzas. We know this is right because there are only four toppings. Second, we looked at the pizzas with two-toppings and we got six pizzas. We know this because we took the pepperoni and grouped it once with each other topping. Then we took the mushrooms and grouped it once with the other toppings, except for pepperoni, because it was already grouped with it. Then we took sausage and only grouped it with peppers because it was already grouped with mushrooms and pepperonis. Third, we looked at the three-topping pizzas... |
| 113 | 00:11:27 | R1 | Stop, stop, stop. What do you think about that argument for two-topping pizzas? Is it convincing? Did you follow what that...what they said? |
| 114 | | MM | Yeah. They just said that they were grouping them with the other ones, and then they didn't do because it was already there. |
| 115 | | R1 | So, they were holding a constant, right. And the first constant they held was the peppers. And how many pizzas did they find with peppers... peppers and...how many pizzas? |
| 116 | | UCT | 3 |
| 117 | | R1 | Three, uh huh. Okay, and then, after they exhausted 3 pizzas that had peppers and one other topping, what did they go to? |
| 118 | | UCT | Sausage? |
| 119 | | UCT | Sausage? And grouped it … |
| 120 | 00:12:14 | R1 | Twice, good. And then they went to…? |
| 121 | | UCT | Mushrooms. |
| 122 | | R1 | Mushrooms...and it only had one possibility because the others already appeared. So that is really nice; holding constant and all. That's really very, very nice. They're not just saying, which some of your students did say, we know we have all two topping pizzas because if we try any other we'll get a duplicate. We'll...we'll have something we already have. This is a lot more convincing than that. Do |

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| | | | you agree? Okay. Alright! What else did they say? Three-toppings. |
| 123 | | UCT | Um..third we looked at the three-topping pizzas and got four pizzas. Since there are only 4 toppings because we took one topping off the pizza each time. |
| 124 | 00:12:57 | R1 | What do you think of that? [Pause.] Pretty neat, huh? How many of you..., someone um... I responded to someone on the line; I don't remember who it was anymore; but someone wrote it was hard for the students to find 3 topping pizzas. |
| 125 | | UCT | I think I wrote that. I did and I said there wouldn't be any more. |
| 126 | | R1 | Okay. And I said, what do you... |
| 127 | | UCT | They kept forgetting one. |
| 128 | | R1 | Yeah..uh uh...elimination. There were a bunch of, uh, pairs in this room that said, I'm going to elimination one topping from each of the three-topping pizzas. Isn't that a brilliant way to find the three-topping pizzas? Because how many of you, if you could remember back, had having trouble finding the three-topping pizzas? Either with a recursive argument that you used or just by, you know, concentrating on what you were doing. It was hard to find all four three-3 topping pizzas, right? But the idea of what these students did is, they started with four and for each pizza they eliminated one. One of the toppings. |
| 129 | | UCT | And then um...the group up here too had a good argument about how they knew that they had them all. And they were counting how many times like an ingredient should be in each one of the chunks. |
| 130 | 00:14:04 | R1 | Neat. And, what did they find? |
| 131 | | R1 | What were they saying? That it should...Yeah, that it should appear 3 times. |
| 132 | | UCT | 3 |
| 133 | | UCT | Yeah. |
| 134 | | UCT | It should appear 3 times. |
| 135 | | UCT | It should appear 3 times. |
| 136 | | R1 | Do we have that anywhere? |
| 137 | | UCT | Yeah. It's... |
| 138 | | R1 | Because that, that is neat. |
| 139 | 00:14:22 | R1 | See how well they actually…[inaudible] So they were actually saying, let me see if I can find something that helps me or feel secure that I have if can find all the possible, um uh, that I have the right answers. Did any of your students that you looked at in your class or here say, I'm going to get all the toppings with, um, that had |

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| | | | peppers. And they got 8 of 'em? And then I was going to get the ones with...no? No one did it? I thought someone did? |
| 140 | | UCT | No. |
| 141 | | UCT | I think... |
| 142 | | UCT | I had...I had a student do it. |
| 143 | | R1 | Okay. And then the next one. |
| 144 | | R1 | There's someone...there's a group in here that did that as well. |
| 145 | | R1 | Tell us what they did 'cause some. Someone may not have seen it. |
| 146 | 00:15:00 | CDR | So they started with holding peppers constant and they did all of the two-topping pizzas with peppers, but instead of moving on and continuing with three toppings, they kept peppers and then did three-toppings with peppers they got and they kept peppers with four-toppings. |
| 147 | 00:15:17 | R1 | So, they got 8...8, right? |
| 148 | | UCT | Yeah. Then they totally ignored the peppers and the sausage for the two-toppings and three-toppings. |
| 149 | | R1 | Do you remember how many pizzas they got with sausage? |
| 150 | | UCT | Four |
| 151 | | UCT | I think it was four. |
| 152 | | UCT | Four. |
| 153 | | R1 | And then they went to another topping and this time they ended up with...2. Right? Um, so it would be 8, 12, 14, a plain and a four-topping. So, that..that was pretty neat. Do you see where they have that? |
| 154 | | UCT | I am not really sure if they wrote it actually. [unintelligible] I don't think they really wrote what they were saying, but when they were saying it, they were looking at their groupings. And they were saying that they should be each one grouped. |
| 155 | | R1 | Each something. |
| 156 | | UCT | Three times. |
| 157 | | R1 | Three times…[unintelligible] And that is pretty neat. Right? Pretty neat that they know to be looking. |
| 158 | | UCT | Yeah. That and they also said that they took one away. |
| 159 | 00:16:14 | R1 | Good. Good. Yes. So the elimination method really works nicely. Good. Was there anything else we wanted to look at from your group? |
| 160 | | UCT | Uh, well I did it...do you want me to do my other class, too then? |

| Description: Transcript of Teachers' own student samples of work.<br>Advisor: Carolyn Maher<br>Location: TRIE<br>Date: October 22, 2013 part two | Author: Phyllis J. Cipriani<br>Verified by: Simone Grey<br>Date Verified: Summer 2015<br>Page 1 of 28 |
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| 161 | | R1 | Sure. Sure |
| 162 | | T6 | So when I did it, uh, I did it with a similar sixth grade class. Pinnacle. It's called Pinnacle. It's gifted and talented. Just like you saw. Same amount of kids, grouped together in partners. Um so, this was an interesting thing that I saw because I was surprised that they actually got the right number of combinations. Because they made like a tree diagram; and it starts with peppers and then It branches out to the three other toppings and then it branches out again and again. And like each...It was like a directional thing. But he was getting confused when he was telling me about it. So he started to write out all of the different possibilities after I told him that that might be a good idea, um. But I… It is just a weird way to write it, and then still get the right answer, 'cause like he was saying peppers is one pizza, and um, these are another one pizza on their own. And then peppers and sausage. And peppers and mushroom. And peppers and pepperoni.<br>And then he did these 3, and these 4, and these 3, and these 3, but then you can go backwards to like mushrooms and peppers with the pizza.<br>So it was really confusing when he was saying it to me, but he got the right number of combinations. |
| 163 | 00:17:43 | R1 | He was able to do it. |
| 164 | | T6 | And when he wrote them down, he was able to do it. So it was like directional. It was moving in all different types of [unintelligible]. |
| 165 | | R1 | Very interesting. How many of you find that is easy to follow? |
| 166 | | UCT | Not really? |
| 167 | | T6 | He was even…he was even when he was telling me about it, he goes, I know I did it! I just...so... |
| 168 | | R1 | Yeah. Yeah. Yeah. Now, what are those numbers? 4 and 11, 15 and 01 and 16. See it? |
| 169 | | T6 | Oh um, those were like his... I don't know because when that's when he got confused when I asked him about these numbers. Because he was saying there is four here, and |

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| | | | then there's certain amounts that have all 3. |
| 170 | | R1 | Okay, okay. |
| 171 | | UCT | And then I was asking him if that's what he meant with these numbers. |
| 172 | | R1 | Okay, okay. |
| 173 | | UCT | I think this was him tallying when he was counting them for himself. |
| 174 | 00:18:27 | R1 | It is confusing, isn't it? Now, he was able to write down what he had and that helped you to know what he had. Good. Do you have one other? |
| 175 | | MM | Okay so, this was another girl who organized it with her one-topping pizzas, and her twos, her threes and her fourths. And I just thought her think and her partner were interesting because, right off the bat, before they even started doing anything they said, I think it is going to be 4 times 4, and actually, he said I think it is going to be 5 times 5 gonna be 25 because... |
| 176 | | UCT | The girl here had 25 also. |
| 177 | 00:19:04 | UCT | Right before...but before...like...we were talking about how they...like the block problem, how they try and make sense of the answer 16 after they get it. But this was before he even got anything. |
| 178 | | R1 | He was making a prediction. |
| 179 | | UCT | Yeah, he made a prediction that it was five times five because he was counting plain as a topping. |
| 180 | | R1 | Right, right. Okay. |
| 181 | | UCT | So then it was five times five. And then he fixed it to be 16, but then they also said, oh, but it's going to be an odd number because that plain one. |
| 182 | | R1 | Ahhh, interesting. |
| 183 | 00:19:30 | UCT | So they thought it was going to be an even number and then the plain one was going to be odd. So they were like, really thrown off when their answer was even at the end. |
| 184 | | R1 | So they didn't like 16. They wanted an odd answer because the plain pizza... |
| 185 | | UCT | They thought that was something, like, that was added in. |
| 186 | | R1 | Interesting. Very interesting. |
| 187 | 00:19:45 | R1 | Now When they did four times four, um is that why we got 16? No okay. |
| 188 | | UCT | Another group did that back here did. |
| 189 | | TD | Yeah, but her reason is, you probably can't read it, but she said, there's four different toppings and there's four mixes. So like you could...the puh...there's a potential to have 4 things on a pizza. So, 4 times 4 is 16, and then plus one is the plain, so it's 17. |

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| 190 | | R1 | Now but uh...mathematically, is that why we're getting 16? |
| 191 | | UCT | No. |
| 192 | | UCT | No. |
| 193 | | R1 | No. Okay. So that again we had that confusion with the towers when they got 16 as the answer for towers four tall. And If their reasoning was four times four give you your answer, what did they say for three tall? |
| 194 | | UCT | Nine. |
| 195 | | R1 | 9. And, but when you build 3 tall you don't get 9, do you? What do you get? |
| 196 | | UCT | 8. |
| 197 | | R1 | You get 8. And why do you get 8? How do you get 8? |
| 198 | | UCT | Two cubed. |
| 199 | | R1 | Two to the third power, okay. So here, the same problem we have, if they're just multiplying 4 by 4, yeah you get 16, but it's not going to help you know any other problem. You won't be able to generalize pizzas with three-toppings or five-toppings. Okay. Good. |
| 200 | 00:21:00 | UCT | [Video shows student work of P=plain, S = sausage, M= mushroom, PE = peppers, and PEP = pepperoni.]So this one I included because, um, they kept a constant. So they kept, uh, peppers constant at first and added things, and then they moved on to sausage and added, and kept sausage constant. And then they did mushrooms, and then I think the sixteenth one was maybe one they forgot. But um, they got their, oh no, that's just pepperoni. PP. Sorry. |
| 201 | | R1 | Okay, okay. |
| 202 | | UCT | So um...so yeah they ...It's a little confusing 'cause they used capital P and a capital E for peppers. And a capital P, capital E, capital P for pepperoni. But they kept something constant they whole time, and then they realized that it was getting smaller as you went because they were accounting for their duplicates. |
| 203 | | R1 | And here...here...this is good. This is what we were talking about before. These are all the pizzas that have pepper in them. How many pizzas did they get? |
| 204 | | UCT | Well, they got 8. |
| 205 | | R1 | Yeah, can you count? Can you see 'em everybody? Okay, there are 8. |
| 206 | | R1 | 1-2-3-4-5-6-7-8. Then they said, I'm going to move to a different topping. I'm going to move to pepper uh, |
| 207 | | UCT | Sausage. |
| 208 | | R1 | Pizzas that have sausage in them. How many pizzas, not you, how many pizzas have sausage in 'em? |

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| 209 | | UCT | 4. |
| 210 | | R1 | 4. 1-2-3-4. Okay. Why only 4? Why can't there be a fifth? |
| 211 | | UCT | Because there's only one topping? |
| 212 | | R1 | Well, no. Where...where...what would happen if we tried to build another pizza that had sausage? |
| 213 | | UCT | You have a repeat. |
| 214 | | R1 | Where would we find it? |
| 215 | | UCT | A repeat. |
| 216 | | R1 | It would be a repeated. We already would be up in here. Okay. And you don't want to repeat. So then they said, well now I'm going to go down to mushroom, pizzas that have mushrooms. And how many did you get? |
| 217 | | UCT | She has all mushrooms. |
| 218 | | UCT | 2. |
| 219 | | R1 | 2. Okay, so yuh had...yuh had 8 here, then you had, where is it? You had...4, you had 2...and...oh wait, you had another one here. Where...what is that one belong to? |
| 220 | | UCT | That's just pepperoni. |
| 221 | | R1 | Oh that's a single. |
| 222 | | UCT | Yeah. |
| 223 | | R1 | Okay. So it's a pepperoni. |
| 224 | | UCT | Yeah 'cause they held the one ingredient constant the whole time, and by the time they got to the pepperoni, it was just pepperoni. |
| 225 | | R1 | Oh, and that is...and that is confusing; the PEP and PE. |
| 226 | | UCT | That is what I said. I said I think that's a pepper one that they forgot, and then I said not it's...that's pepperoni. |
| 227 | | R1 | Yeah, yeah okay. So they were doing what we were talking about before, holding constants and exhausting, you know, everything in one category… |
| 228 | | UCT | Yeah. |
| 229 | | R1 | Nice. |
| 230 | | UCT | The...the...the group, this was a group of 3. And they jumped straight to that too. So it would have been interesting if they did the Tower Problem but we didn't. |
| 231 | | R1 | Good, And what was their argument? |
| 232 | | MM | Um, we organized the choices by toppings as we went on. When we got to a new topping, we took out all of the duplicates from the other toppings. For example, we started off, out with all of the pepper combinations. There were 8 of them. When we got to the sausage, there were only 4 combinations because there were 4 duplicates from the pepper. We did the same thing for mushrooms and pepperoni. The only thing left to do was to add one plain to our list, which we added. And then they said we got 16. |

| Line | Time | Speaker | Transcript |
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| 233 | 00:22:58 | R1 | What do you think of their explanation of their argument? Is it convincing? |
| 234 | | MM | They’re just saying that they found all of them and they took out the duplicates. |
| 235 | | R1 | Right, right. |
| 236 | | UCT | It’s not... |
| 237 | | R1 | Not…not totally convincing. Okay. Alright good, though. Nice...nice stuff. Let’s let someone else go ‘cause we wanna have lots of time for you to work on the problem. Okay. |
| 238 | | R1 | So you’re looking for two pieces of work from your class; either work that you didn’t understand, work that you thought was really impressive, or work that surprised you. |
| 239 | | UCT | Some girls. |
| 240 | | UCT | So none of my kids got the right answer. I only have 8 students, so in pairs, it was four. |
| 241 | | R1 | If we move that forward, would we make it bigger? |
| 242 | | UCT | No, if we move it back. |
| 243 | | R1 | Move it back would it make it make it bigger? |
| 244 | | UCT | It gets bigger but it’s kind of [unintelligible]. |
| 245 | | R1 | Blurry? |
| 246 | | UCT | Yeah, it’s... |
| 247 | | R1 | Okay, okay. |
| 248 | | UCT | I don’t know why it’s so light either but... |
| 249 | | R1 | Woops! Now we see feet. |
| 250 | | UCT | I don’t think it’s going to make it that much better. |
| 251 | | R1 | Okay. Alright, then let’s leave it. Okay. So let’s uh, go ahead and tell us about your students. |
| 252 | | UCT | Okay, so as I was saying nobody got it. It was like a huge…I mean they were, for a while, they were going on about, you know...you know, plain is different, you know, plain is different than, you know, they wanted to make a pizza with plain and then a topping. Or they were saying that sausage first, and then mushrooms on top, is different than having mushrooms and sausage, and it was just this whole, you know, as we talked about, and I even wrote it down. So, the first, this one I thought this one was interesting because I, I actually wrote this, that she...Do you have? Oh. Well first of all, she wrote out, she made a key, A-B-C-D-E. So E was plain, A was a topping. |
| 253 | 00:26:10 | R1 | Pretty neat. Almost like the 1-2-3-4, right? |
| 254 | | TD | I thought that was really clever of her to do that. So she went through and she got to 24, you know, and then I had to kind of guide her and we, you know, asked her questions, her and her partner. And then she realized that |

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| | | | she had duplicates. So, she erased them and she rewrote it...oops...nicely. And then, she got 11. And we didn't get to explain... |
| 255 | | R1 | She got 11? |
| 256 | | UCT | Yeah. |
| 257 | | R1 | So she... |
| 258 | | UCT | And she didn't do any kind of pattern. They just kind of... |
| 259 | | R1 | Okay, randomly found them. |
| 260 | | UCT | Yeah, randomly. |
| 261 | | R1 | Finding them randomly might work for some students, but it's hard. And if you gave them five-toppings, it would be really hard. |
| 262 | | UCT | Yeah. |
| 263 | | R1 | Yeah, so...um...no convincing argument? |
| 264 | | UCT | No, Everything was pretty much oral as I went around and it was like an hour, and they just didn't get to it. But, you know, like I...I... having four groups at least I can get around and talk to all of them a lot. And then the other, this one, this girl. She really didn't have any order either. But she, she found 14. She was a little closer, but then I noticed that she had duplicates in there. Mushroom. Peppers. Sausage. |
| 265 | | R1 | Can we move that down so that we can see the top? |
| 266 | | UCT | No. |
| 267 | | UCT | No. |
| 268 | | R1 | No? Can't even do that. Okay. |
| 269 | | UCT | We can move it up so. But it'll just make it smaller. |
| 270 | | UCT | [Inaudible] It'll probably get clearer. |
| 271 | | R1 | Oh, that's better. |
| 272 | | UCT | Yeah. |
| 273 | | R1 | Okay, so...do you see...look at how she did it. What was the order that she wrote her pizzas in? |
| 274 | | UCT | I see she really stuck to mushrooms a lot. I mean. |
| 275 | | R1 | Okay. |
| 276 | | UCT | It's like mushroom. Mushroom. Mushroom. Mushroom. Mushroom. So that was, I guess, kind of her constant. But then she wound up having a duplicate here. Um... |
| 277 | | R1 | Okay. Now she's going down the way she's numbered it. She starts with the three-topping pizza, but then what does she do? |
| 278 | | UCT | Go to two-topping. |
| 279 | | R1 | She goes to the two-topping. And how many two-topping pizzas does she find? |
| 280 | | UCT | 4. |

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| 281 | | R1 | That's really... |
| 282 | | UCT | 5 |
| 283 | | R1 | Well, 2-3-4-5. She found four in a row, right? That's pretty good. That's pretty good. Okay, and then what did she do? She then went... |
| 284 | | R1 | She went to just one-topping. |
| 285 | | UCT | One-topping. |
| 286 | | R1 | And she found how many one-topping? |
| 287 | | UCT | 3..and then plain. |
| 288 | 00:28:22 | R1 | She only found...okay so, she found 3. She skipped one of those. Okay. Then she has the plain. And now she went back and she found another two-topping. Is that different than the one she has? |
| 289 | | UCT | Yes. |
| 290 | | R1 | So that's pretty neat, right? And then what did she do? |
| 291 | | UCT | She tried making three. |
| 292 | | R1 | Three-toppings? |
| 293 | | UCT | Yup! |
| 294 | | R1 | And it looks like she found the single pizza that she had forgotten. Is 13 a [unintelligible]? |
| 295 | | UCT | Oh yeah. Pepperoni. Yup! |
| 296 | 00:28:47 | R1 | So, I think there's a lot of good stuff here. And I think, um, you want to try and get a student like this to talk about...uh ah...obviously she knew that she had to find more two-topping pizzas. She had to find another single topping pizza. And to push her to the next level of, how do you know you have all the one-topping pizzas? You had 3 and you weren't satisfied, you knew there was another one. How did you know you there was going to be another one? |
| 297 | 00:29:13 | UCT | Yup! |
| 298 | | R1 | Okay, okay, next [big pause] |
| 299 | | UCT | Okay. |
| 300 | 00:29:36 | R1 | Now remember, we talked about if your students can't write that you can be their scribe and write for them. Because you really want to see something down as to why they think they have them all. At least the beginning of a convincing argument. Because that's what this is all about. We're not so much interested in just that they can do the problems, we are interested in, can they provide a convincing argument? |
| 301 | | UCT | Okay. |
| 302 | | UCT | Let me just scroll [unintelligible]. |
| 303 | 00:30:05 | VB | [Video of work shows a chart with 3 columns with peppers, sausage and mushroom at the top of the columsn. With 3 toppings in the cells underneath the aforementioned |

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| | | | words. A list of 13 possible pizzas is under the chart with 7 pizza combinations holding peppers as the constant; 3 holding sausage and 3 holding mushroom. Student wrote 7+3+3+3=16]<br><br>Okay so, I, she got the right, um, answer. She got 16. Think I only had like one or two kids that did get 16. A lot of my kids struggled with... putting like T1 said, like sausage mushroom, mushroom sausage, or putting...I had kids that are like, well if it's 12 slice pizza, then I can put one topping on each slice so...I had a lot of big numbers, let's just say.. |
| 304 | 00:30:26 | R1 | Okay. |
| 305 | | VB | But she had, she did get it right and I liked because in here, she did this chart in the beginning and she had it very well organized and she counted up and then showed her addition. But then on the next page, it's almost like she reorganized it different times. I think this was part of her way of proving it but also part of her way to see if she had any duplicates or not. |
| 306 | | R1 | Okay, now help us. Why does she have 7 underneath the first column? |
| 307 | | VB | Um, [intelligible] I think that she found. What did she say? Oh, it's in my hand. Um, I answered my question 16 combinations of possible pizza choices a customer can choose. I made a chart of peppers, sausage, mushrooms, pepperoni. I did all the combinations for each of them. Then I got 7 for peppers, 3 for sausage, mushrooms, and pepperoni. I add them together to get 16. So I think maybe what she's...I'm not really sure why she would have the 7 there. Unless.. |
| 308 | | R1 | Does anyone know why she had the 7 there? |
| 309 | | UCT | Maybe seven of the pizzas, peppers are in there. |
| 310 | | UCT | Yeah, yeah. |
| 311 | | R1 | Do you see 7 pizzas with pepper? It's hard for me to read it. Can you read that? |
| 312 | | R1 | Okay on this page, um you see the seven [unintelligible]... |
| 313 | | R1 | Okay. |
| 314 | | R1 | Not in that chart, but on this one you see, on that bottom part with the pictures. |
| 315 | | R1 | On the bottom. Okay. So she has 7 pieces with peppers, okay. And then what did she get? |
| 316 | | UCT | Then she had, then she had the sausage, that she's showing that she has the 3 pizzas with the sausage. |
| 317 | 00:31:47 | R1 | Got it. Got it. |

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| 318 | | VB | And then the same thing she did with the mushrooms. So then, I don't know where her like, her plain [unintelligible]. Maybe she just didn't put that in there. |
| 319 | | R1 | So they, she has 7 and 3 and 3 is not 16. So, what is she…what…how many pizzas should there be that have peppers? |
| 320 | | UCT | 8 |
| 321 | | R1 | 8. Okay, so she's missing one there. How many pizzas should there be that have sausage? |
| 322 | | UCT | 4 |
| 323 | | R1 | 4. So she's missing one there. And mushrooms? |
| 324 | | R1 | 2. And then there is an everything pizza and a plain pizza…so. But she's doing a lot of good stuff. Okay. She's missing some things, but um..and I am not sure how she ends up with 16 when you add 7 and 3 and 3, unless she's adding… |
| 325 | | UCT | Then the…yeah, the three – the pepper, sausage – maybe those separate. |
| 326 | | R1 | Yeah, I'm not sure. |
| 327 | | UCT | In that box, you see, maybe that's what she's… |
| 328 | | R1 | Okay |
| 329 | | UCT | Her boxes, really the fourth column [unintelligible]. On the front she had four columns. |
| 330 | | R1 | Ah, okay. You think that's it? Okay, okay. Alright. |
| 331 | | UCT | [Video shows a list of ten pizza combinations with the topping words written. Peppers is listed first 6 times, followed by three sausage listed first and then mushroom listed first.] Okay, and the next…really only get ten possibilities, but I liked what he did. He started off with, he started off with the three and then kind of just rearranged, worked it down. So he had pepper-sausage, mushroom-pepper, and he started off with the four there. And then he went to pepper, sausage, mushroom…peppers sausage, but then he did peppers, mushroom, pepperoni. So, I liked how he kind of the three just rearranged them. He only came out with ten, but I did…I was pretty impressed when he did this. I didn't think that. |
| 332 | | R1 | But, that is neat. That's a neat way of thinking about it. He's eliminating one topping each time. Mhm. Now…any writing about why he thinks he has them all? |
| 333 | | UCT | Yeah. There are ten possibilities because a topping in order I can reverse them, and in the middle I can use, use them. If all the ten combinations are reversed, then it will still be the same. Then he's saying like if I, you know, have the same or duplicate. |

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| 334 | | R1 | What do you think of that argument? (Pause.) Read it again. I don't think…we need to… |
| 335 | | T5 | There are 10 possibilities from the topping in order, I can reverse them and in the middle I can use them too. If all the ten combinations are reversed then it would still be the same. I, I think he understands about the whole duplication of it. But I think he just, maybe he could of done a little bit more were as maybe just even realize, okay, we could do a plate. We could come up with, well he has the one with the all, then to make the next set of three combinations. |
| 336 | 00:34:30 | R1 | So, is his argument convincing? |
| 337 | | T5 | No. |
| 338 | | R1 | No, okay. So, but I think what you want to do is you want to point out what's good about it. And you found good stuff in it. So you want to share, when you're sharing, share what's good and then see if you can get them to go to the next level. Good. |
| 339 | | UCT | I'll go. [Teacher gets up.] Oh. [Unintelligible.] Alright. |
| 340 | | R1 | Oh, I see a lot of pizza there. |
| 341 | | UCT | [Several teachers speaking at once.] Wow!...Woah! |
| 342 | | R1 | Yeah! |
| 343 | | R1 | Wow. How many of your students did slices of pizza? |
| 344 | | UCT | [Several teachers speaking at once.] I had a kid…I had a lot of kids draw it… |
| 345 | | R1 | Or half-pizzas? How many did half-pizzas? |
| 346 | | UCT | [Several teachers speaking at once.] I had whole…I had half…I had some do slices. |
| 347 | | R1 | And that makes a much more complicated problem. It's, it's a problem that can be solved. Much more difficult than solving this problem. Oh my, it looks like little sailboats. |
| 348 | | CP | Yeah! It was uh...okay so, when I gave my kids the problem, the first thing they started doing was just drawing slices of pizza. And it was just kind of at random, whatever they came up with in their head. Um, both the students were doing it kind of at the same time, they were almost like racing against each other. And uh, so that's why you see like the Xs there. Because once they had them, I was like alright guys, well this is like a mess, isn't it? And they're like yeah, okay.<br>And I go uh, can you do it another way to show me, and organize these so that you know you have all the options? So they started comparing answers, they had more than this one sheet; I just want to show you [unintelligible]. |
| 349 | | R1 | Okay, okay. |
| 350 | | CP | I had originally given them the problem like the paper that, |

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| | | | you know, we handed out in class today.<br>But because they started drawing it and their squares were, I mean you saw how big this, they did it on this big piece of paper. I didn't want to limit 'em, I said here's as much paper as you want. I said have at it. |
| 351 | | R1 | Mhm. |
| 352 | | CP | Um, so then they sat together and they organized 'em by group. Um, they did keep a constant. There wasn't a lot of order, um, as to like why they them 'em it in certain spots. And they did have doubles. Um, and actually, as you'll see over here, you can't see up top but this is labeled Group One, this is Group Two, Group Three and Group 4. |
| 353 | | R1 | So, the group goes down. So there are 1-2-3-4-5-6. |
| 354 | 00:37:11 | CP | Yes, so you have the…you have the single toppings here you [pointing to board], then you have the two toppings. |
| 355 | | R1 | Okay. |
| 356 | | CP | Down here [pointing to board] plain kind of just got thrown in [unintelligible]. And then, as I go, what about all the 3 toppings pizzas that you did? They forgot to put them in. Um, I think part of it might have been because they ran out of room on the bottom of the page. So they just started making those groups here, so like this is all peppers [pointing to the board. And then they labeled it Group 1. And this is all sausage as the first topping; so they labeled it Group 2. And the same with mushrooms. And the same with, uh, pepperoni.<br>Um, so then they had that, and I was like of like, well isn't, if you go to the store and order sausage and peppers, you go behind him and order peppers and sausage, aren't you going to get the same pizza?<br>And they're like, oh, okay. So then they started going through and crossing out the doubles. Um, and I think they did get… they got close to 16. I don't think they got exactly 16. I am trying to see where they're four topping pizza was, they did have it somewhere. Where is it?<br>Okay. Here it is. On the bottom. So they did four toppings with each one as a constant. So like peppers was first with the four toppings [pointing to the board], then sausage was the first there. So they did end, I think they did have 16 altogether.<br>Um, their explanation I found it interesting. I brought the…the girl from…this…the boy's drawing. But the girl… |
| 357 | | R1 | The girl's writing. |
| 358 | | CP | The girl is a bit of better writer. May I share her theory? |

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| 359 | | R1 | Ok. Read us what she wrote. |
| 360 | | CP | It's a lot easier for me to read. |
| 361 | | R1 | Good, absolutely |
| 362 | | CP | [Students written argument:<br>Group one. I started with pepperoni, then I did two toppings; there is pepperoni in each one. If I put another one I get 3 toppings. I did 3 toppings with pepperoni in each one.<br>Group two. I started with mushroom. Then I did two toppings; mushroom in each one. If I put another one I get three toppings. I did 3 topping with mushrooms.<br>Group three. I start with sausage. I did two other topping. There is sausage in each one. If I put other topping I get three. I did three topping with sausage.<br>Group four. I start with pepper. I did two other topping. There is pepper in each one. If I put other topping, I get three. I did three topping with pepper.]<br><br>Um, so…it's…her arguments, uh, Group one. I started with pepperoni, then I did two toppings; there is pepperoni in each one. She's basically talking about two toppings now where pepperoni is the first one. Then she says, if I put another one I get 3 toppings. And I was like, alright. It kind of almost sounded like the same argument that she had made for, uh, the towers argument.<br>Where she was like, if I put another one, then I get something different. So I like that she was kind of taking her previous knowledge from the argument and applying it but I don't really think it, kind of, I mean I guess it works; but I don't really think it went with what they were doing there. Uh, then she did, said I did 3 toppings with pepperoni in each one.<br>For group 2, I started with mushroom and then I did…two toppings; there is mushroom in each one. And if I put another one I get three toppings.<br>And she kind of went on to do that for each argument for each group. So, I don't know, what do you guys think? You think that works as an argument? |
| 363 | 00:39:43 | R1 | Is it convincing? [slight pause]She's telling you what she did, right? |
| 364 | | UCT | [Several teachers speaking at once.] Yeah. |
| 365 | | UCT | Yeah. |
| 366 | | UCT | I mean it could be if she just went in depth a little further. |
| 367 | | R1 | If she…if…also…also…isn't, when you have an argument by a student, it's not our job to interpret what we think they mean. |

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| | | | Um, because we really don't know what they mean until they tell us what they mean, okay, And even if it look, we could guess! Um, there are times that I'll look at student work and I'll say, oh I think that he was thinking this, but I don't really know. The only way I know whether he was thinking this is by asking the student was he thinking this. What was he thinking? And same here. It's not clear exactly to me what she was thinking. And it could be developed but, right now what it is, isn't convincing. Okay. |
| 368 | | CP | Alright, and then my other, uh, the other student that I work with, the 8th grader who comes in and helps out in the class, um, I gave it to her, I don't know if you guys can see all that well. |
| 369 | | R1 | No. Read to us. |
| 370 | | CP | She started with plain and then she did single toppings, and then she did all four. |
| 371 | | R1 | Okay. |
| 372 | | CP | Um, then she went to 3 toppings. She kept peppers as a constant and she only had two. |
| 373 | | R1 | Okay. |
| 374 | | CP | Then she continued with two toppings as peppers as a constant. Then she went to mushrooms with two toppings and mushrooms as the constant. Then pepperoni, um, she just had one. Then she ended with 14, and I was just like, well, do you think you have all of them? And she said no. So okay, how…what do you think…how many are you going to have? And she said 16. |
| 375 | | R1 | Now, why? |
| 376 | | CP | Well, that's what I said. She goes well, because 4 times 4 is 16. |
| 377 | | R1 | Okay. |
| 378 | | CP | And I was like, what does that have to do anything with the problem? And she's like, well yeah, I guess that really doesn't have to do with the problem. But I just know that there's gonna be 16. |
| 379 | | R1 | Did she find two more? |
| 380 | | CP | She did. She wrote the two…she wrote the two with the three toppings on the bottom. But I was like, but how did you know there was gonna to be 16. She's like I don't really know, she like, I just knew that… |
| 381 | | R1 | Okay. |
| 382 | | CP | I don't know if she was making the connection. |
| 383 | | R1 | She was making a prediction based upon maybe something that was four times four. What…where is her convincing argument? What did she write? |
| 384 | | CP | She said, uh, for the first five choices I put the toppings by themself and then I combined all four.<br>I guess for the one topping and the two topping, and for the three She doesn't really say but She held it constant. |
| 385 | | R1 | Okay. |
| 386 | | CP | So she…the five choices is plain and the single toppings. Then I |

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| | | | combined all four of the toppings together. After that I took one topping and put it two toppings and not to get it to repeat. After that I took one topping and put it with one other topping to get it to repeat. So I guess for the one-topping and the two-topping, she's doing the three with all the constants. But she doesn't really say that she kept a constant or anything like that. |
| 387 | | R1 | What is her argument that she writes, is it convincing? |
| 388 | | T8 | I don't really think that it's that convincing. |
| 389 | | R1 | No, it's really not. Why not? She is telling you what she did. But she is not really giving an argument why there are exactly that number pizzas in each group. She is not really telling you why there are exactly that many. |
| 390 | | T6 | I feel like that's the case most of the time. They just tell you what they do, instead of how. |
| 391 | | R1 | It's hard, yeah. But, there were some kids here though that actually had more convincing arguments in the class that we looked at. Which group had more of a convincing argument? At least you saw pieces of it, right. You saw some kids that were telling you about the one-topping pizzas, pretty convincing. Four-topping pizzas, pretty convincing. The plain pizza, pretty convincing. Again, we ran into trouble with the two-topping and the three-topping. So okay, who's next? Yep. Okay. It's hard to write a convincing argument, isn't it? Alright. How many of you would know how to write a convincing argument for the two-topping pizzas? [Pause.] Harder than writing it for a four-topping, right? Harder than writing it for the one-topping. Oh my…we can't read it. Help us. |
| 392 | | R1 | Ok…this…we can zoom in on this, right? |
| 393 | | UCT | It's not work…I had [unintelligible]. |
| 394 | | R1 | It works better than this? |
| 395 | | UCT | This was the class that you had all come into the last time. And they had a lot of difficulty with this problem. I had a lot of huge numbers and a lot of students that didn't finish. |
| 396 | | R1 | Okay. Ooo! 145, huh? |
| 397 | | CDR | And that's not even their final answer. [Laughter] So…this group had the idea that each slice was considered a different part. |
| 398 | | R1 | Ah…okay. |
| 399 | | UCT | So when they were, I asked them about it, and I said, would you go in and order a different slice on a pizza? And the one boy responded, well my father use to have a pizzeria and if someone ordered that, he sure he would do it. And I said, okay, and I kind of let them go with it. |
| 400 | | R1 | Right |
| 401 | | UCT | So on the back here, it is a little hard to see but, what they wrote up here was, he said, okay, I gonna start with one slice peppers and then 7 slices of sausage. Or 7 slices of mushroom. 7 slices of plain. Whatever it was. |

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| 402 | 00:45:56 | R1 | Ahhhh, wow! |
| 403 | | UCT | Then he went to two with peppers and then six with each of the other four toppings because they're including plain as a topping. |
| 404 | | R1 | Oh my, oh my! |
| 405 | | UCT | So when he was done with that he had 28 possibilities but that's only two toppings with peppers. |
| 406 | | R1 | Now, what do you think of that? What do you do with that when you have a student who does that? What did you do? |
| 407 | | UCT | I, well, I was asking them why they were doing it and they just kept saying, well, because my father would do it that way. |
| 408 | | R1 | They convinced you, right, that his father owned a pizza store. This is a much, much, much harder problem that they have created. |
| 409 | | UCT | Oh, absolutely [unintelligible]. The one boy was, he was just like I can't believe you are making me do this. This is so lame…I won't fit it on the page, so. Then after they did that, they multiplied, they had 28, they multiplied by 5, because they figured the peppers on the left column could be switched to sausage, mushrooms, plain or, uh, pepperoni [unintelligible]. So they multiplied by 5, got 140; not realizing that that's going to be a duplicates in there. And then they added five more pizzas on for the whole pizzas of each of those five. |
| 410 | | R1 | Wow, wow. |
| 411 | | UCT | And they totally ignored 3 toppings and 4 toppings, and…it was a mess. |
| 412 | 00:46:10 | R1 | This actually is more complicated than I've seen. I've actually…in Sayreville, when we came into the class there were two boys last year who, uh, decided to make half pizzas. Because you can order a pie that is half pepperoni and half sausage, right? So what they did was, they made a much more complicated problem again, because they wanted each half of the pie to have the combinations of… |
| 413 | | T7 | A lot of my students did that. And actually what the next one is going to look like. The next one, I think, is even more complicated than this. |
| 414 | | R1 | Did you have anyone who, who didn't make a complicated mess? |
| 415 | | T7 | Very few [bell sounds]. I think I had about two groups I think that finally finished and had 16 and actually had good reasoning. |
| 416 | | R1 | Can we see one of those? |
| 417 | | T7 | Sure, I can pull it up. |
| 418 | | R1 | Okay. Yeah. Because you know I think that, um, it's unusual to have a whole class of this. |
| 419 | | | Announcement: Anyone taking a late bus, please come to the front of the building |
| 420 | | T7 | So this one… |
| 421 | | R1 | And as a teacher, you probably pull your hair out, and say Oh my God! What do I do with this? Yuh know. In Sayreville, it |

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| | | | was one group that went off the other…because they were independent thinkers they were thinking outside the box. Which is, it sounds like you might have a whole class of outside the box and…poor teacher. [Chuckles.] Sometimes it's good when they think outside the box. It really is. |
| 422 | | T7 | So this is that other student that I was talking about, um. And she broke it into quarters first and then she said, well, this quarter could have four different toppings on it [pointing to board], this one could have four different toppings on it, this one, and then…and then she took that and said, well there is 13 toppings technically here and multiplied it by 4 because there is four quarters that each of them could be moved into. |
| 423 | | R1 | Right. Right. |
| 424 | | T7 | So that was another big mess. They didn't get an answer. |
| 425 | | R1 | These are bright children? |
| 426 | | T7 | This is a normal level 8th grade class. |
| 427 | | R1 | This is normal? Well, they should be Pinnacle because usually it's your Pinnacle students who think outside the box. How many of you went down this road? None of you, right! Where you came up with dividing the pizzas, either in quarters. Oh, you did? |
| 428 | | UCT | [Multiple conversations happening – inaudible] I thought about it, but then you came over and I was like, alright, I think I'm going too far into this, so I went back. |
| 429 | | R1 | Okay. [Multiple conversations - inaudible] I think a lot of us were just like too, and then…Okay, okay. Because these are really…was this someone who went down? |
| 430 | | UCT | [video shows a numbered list of 16 boxes with possible pizza combinations inside the boxes.]So this is someone who had all 16. So he started this first section up here [pointing to board] is his one-toppings ones. |
| 431 | | R1 | Right. |
| 432 | | UCT | And then he went to his two-toppings, keeping a constant peppers. |
| 433 | | R1 | Nice. |
| 434 | | UCT | Then moving to the sausage, then mushrooms. |
| 435 | | R1 | Okay. |
| 436 | | UCT | Then he did the three-toppings. |
| 437 | | R1 | Right |
| 438 | | UCT | Again keeping peppers and sausage, then getting rid of the sausage and so on. Then he listed all his 16 down here [pointing to board]. |
| 439 | 00:48:42 | R1 | And you know, I think it would probably not be a bad idea for the student like this who went down the straight and narrow path, made the problem uncomplicated, to let him show the others what he did and let them react to it. |
| 440 | | UCT | Right. He had to explain it to his partner though... |
| 441 | | R1 | Ah. |

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| 442 | | UCT | His partner had started with listing it as well but she didn't take into account her duplicates so she ended up with 32. |
| 443 | | R1 | Okay. Did you, um, get any justification, here? Any? |
| 444 | | CDR | From him? Yeah, he explained it to me not… |
| 445 | | R1 | No, but did he write anything? |
| 446 | | CDR | Yes. His writing was not good, but his explanation to me was good. |
| 447 | | R1 | Okay. |
| 448 | | CDR | His writing just said, there is one plain pizza, you start with individual toppings; there's four. From this, you group them in lists such as 2-topping and 3-toppings. Then you make sure you didn't repeat a combination. There are 16 possible combinations. To check there are four original topping. Four can evenly go into 16. |
| 449 | | R1 | Okay. |
| 450 | | T7 | So his reasoning on paper was not good. |
| 451 | 00:49:35 | R1 | Not good…but what did he tell you? What did he say to you? |
| 452 | | UCT | When he talked me through it, he said I started with my one plain and went to my one-toppings. I know there's only four because four toppings. |
| 453 | | R1 | So he did say that… |
| 454 | | UCT | Yeah, he went to his two-toppings and he held the constant of peppers; he knows he can't use it. But when he went to write it, all that disappeared. |
| 455 | | R1 | Okay. So what you want to try and do is, again what we had said, when he's saying something that resembles a good convincing argument, you say, write that down. So at least you get the beginning of some part of a convincing argument, even if the whole thing isn't there. |
| 456 | | CDR | Right. |
| 457 | | R1 | Good, good. Wow. You got a challenge, huh! |
| 458 | | CDR | Oh, yeah! They had a lot of difficulty with that. |
| 459 | | R1 | Okay, okay. How did they…and this is the class that did the towers problem? |
| 460 | | UCT | Yes, this is the same class that you all saw. |
| 461 | | R1 | And the towers they didn't have that problem with... |
| 462 | | CDR | No, they didn't. But this one, they just…they thought a lot more into this problem than they did. And that's why that first group, they were shocked when they said…when they heard that another group had already gotten the answer and listed all of 'em. They couldn't wrap their minds around how they did it, and they kept trying to tell the class that they missed… |
| 463 | | R1 | How they did it so quickly. Yeah, yeah. Okay. Interesting. Alright. |
| 464 | | JLB | Alright, mine are really quick because my kids didn't want to write an explanation. Um, they did a lot of talking during this and then because it was about pizza. |

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| | | | And we do, um, high-level pizza activities every other Friday. They were off topic pretty much the whole time because they were thinking about pizza; because they were high level. But this first one is one of my students Y___. He started a tree diagram and was running into so many problems. And, um, Lesley was also in his group, she was doing the same thing and she was flirting with Yaniq so they didn't really get very far. Um, it's really funny because he doesn't realize she's flirting with him. Um, but the third, the other kid in the group was John, and John started doing…he…it's like he started copying them with the tree diagram to start with the plain pizza, and then he started going off on his own. He just started drawing circles and I asked him why, and why would he labeled some of them with the spokes and some not. And he said, he didn't know. Um, but he did the pepperoni, the mushrooms, the peppers, and then sausage he forgot it, so it came down at the bottom [pointing to the board]. Then he did a combination of two, then a combination of three, then a combination of two, then a three, then a three and then all. |
| 465 | | R1 | Okay. |
| 466 | | T4 | And then I asked him why he organized his work the way he did, he's like, I don't know. And I said, do you think that you're done? He's like, probably not, I don't really care. He was very like…so once I talked to that group and I said, wow John, this looks really interesting then Y___ all of a sudden asked for another piece of paper. |
| 467 | | R1 | Okay. |
| 468 | | UCT | [video shows Y's work where he began by listing pizza combinations using letters for the pizza toppings. Then Y decided to use his partner's idea of using proof by cases.]Um, he started making a list first, he didn't like his list because he knew he had duplicates. And then he started doing what John did, but he actually organized his work in the one, two, three, four…list. |
| 469 | | R1 | Okay. |
| 470 | | UCT | So…at least I got one right answer in the class. [Laughs] |
| 471 | | R1 | Well, you know, I…I think that's really nice that Yaniq decided that he would take [pointing to the board]… |
| 472 | | UCT | He tried three different ways before he got it. |
| 473 | | R1 | That's good. That's really good. |
| 474 | | UCT | Yeah. And he, and he listened to the comments that I made to John…and made sure that, he actually took it seriously. He took what John did and realized John had 1-2-3, and then he fixed it so that it was all the twos, all threes. |
| 475 | | R1 | Isn't that nice? |
| 476 | | UCT | Yeah. |
| 477 | | R1 | So that they were building upon what each, you know, their partner said. That's good. |
| 478 | | UCT | Yeah, and then John copied Yaniq, so I had to put John second |

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| | | | here. |
| 479 | | R1 | Did you ask them how does he know he has all the one-topping pizzas? |
| 480 | | UCT | Yeah, and they were like, I don't know, we're just going to copy. He did know either. He didn't realize either. They were very all over the place. They just didn't know…yeah. But I was happy that at least he, he was able to articulate - This was all the one-toppings, two-toppings, three-toppings… |
| 481 | | R1 | Okay. But that's good. |
| 482 | | UCT | He…he used his example from the book [unintelligible]…it said that I had a duplicate here so…he was able to. |
| 483 | | R1 | Good. Okay. And, you know, you might want to…maybe the easiest question is, how do you know there is only one pizza with all four toppings. |
| 484 | | UCT | Yeah. |
| 485 | | R1 | Alright. And they may be able to answer that. And if you can get 'em just to do a convincing argument for something that simple, that's a start. You know, don't expect them to write a convincing argument for everything. But you want them to be writing something. And if they can't write 'cause they have special needs, it's okay to write for them; if you're writing their words. Okay. Alright. [Extensive pause.] |
| 486 | | R1 | So what is that white box? It's not a projection camera? |
| 487 | | UCT | That's just a projector. |
| 488 | | R1 | It's just a projector. So you can't…so the Elmos, there are none in the school, huh? The Elmo? Okay. |
| 489 | 00:54:43 | MC | Okay. This is a group of girls that I had and they…were actually were my only group that came up with the 16 possibilities. Um, they first started listing them out using just the one-toppings. And when I went back and they had the 16. |
| 490 | | R1 | Hmm. |
| 491 | | T3 | They showed me that, or what they said rather, was that they used a system. And they use it by describing arrows. So they kind of showed me that they listed each of individual toppings; so peppers, sausage, pepperoni, mushrooms. And then they went peppers sausage with the pizza, pepperoni sausage pepperoni with the pizza, pepper sausage pepperoni mushrooms and they kind of used the arrows to come up with the 16… |
| 492 | | R1 | Okay |
| 493 | | T3 | Uh, combinations. So I thought that was pretty good. |
| 494 | | R1 | Uh huh. |
| 495 | | MC | They were…understood it pretty quickly. And…they were pretty decent with their explanations. This is as far as written wise that they could give me. Written argument: The first pair of 2 of all the same color is there because there are 4 blocks and all are the same color; but opposite from its partner. The second group of 2 pairs makes 4 different groups, but they link together because you can take the |

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| | | | bottom or top and put it completely opposite of the top or the bottom. |
| 496 | | R1 | Uh huh. |
| 497 | | MC | And that was their convincing argument with that. |
| 498 | | R1 | Okay, so is it convincing to you? [Extensive pause.] Blank looks. Does that mean you're asleep or does that mean…? |
| 499 | | | [Multiple conversations.] |
| 500 | | T3 | I liked the way that they organized it with the arrows but there is no real argument there about why they have all of them. |
| 501 | | R1 | Exactly, okay. But it is a neat way to find the pizzas. Not as convincing as to, do you have them all? Okay, but that, that's neat. |
| 502 | | UCT | And then the next one, actually I wanted to go this first… |
| 503 | | R1 | Okay. |
| 504 | | UCT | So this group, um, actually started with a tree diagram and they thought that was their best way to start it. And then you can kind of see that, they kind of erased it. Cause after they started to make the tree diagram, they realized they were having duplicates. And they kind of looked at me and said; this isn't going to work anymore. So I said, okay, well can we do it another way?<br>[Announcement: Inaudible.] |
| 505 | 00:56:33 | UCT | So after they realized that they were making duplicates they thought that they could use another way; that's when they changed to do a list, which is …this. And they kind of just organized them at random and they only came up with 13. |
| 506 | | R1 | Okay, that random organization is not a good one. Right? Yeah. |
| 507 | | UCT | Yeah, but at least, I kind of liked that they started with the tree diagram… |
| 508 | | R1 | And moved away from it. |
| 509 | | UCT | And saw that they were duplicates and realized that wasn't working for them, so then they went to the list. But with the list they only came up with 13. |
| 510 | | R1 | Okay, and how many of you, when you started with a tree diagram, said uh uh, this isn't working so good. |
| 511 | | UCT | I know that I did. |
| 512 | | UCT | Yeah. This group in the front did the same thing. They started with a tree diagram and then they realized it wasn't gonna work. |
| 513 | | R1 | Okay, good. Alright. [Extensive pause.] So again, the third task, you want to push the children to the next level wherever they are, okay. If they haven't been writing at all, you really want them to write a little bit, because, uh, we're hoping to see growth, you know, from wherever they started. If they, even if they don't have a whole solution, if their organization is there in the third task than it was in the first and second, that's a good thing. |

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| 514 | | NL | Okay. So I pretty much had the same difficulties as everybody else. |
| 515 | | R1 | Okay |
| 516 | | UCT | Um, but through constantly, like this girl Jenna asked me like a million questions. And it was really hard for me to tell her, like I can't really answer it. But I just kept like stating, just read the question .What is they problem asking you? And she'd be like, can we have no cheese? And I was like, what does the problem say? So I just kept saying that because I didn't know what else to say half the time because her questions were really like…I would have led her too much and she would've…she would have gotten the answer from me. |
| 517 | | R1 | Okay, okay. |
| 518 | | UCT | But eventually, she was like, can we have more than one topping? I said, ask your partner, I am not sure. |
| 519 | | R1 | Good. |
| 520 | | NL | Can we have two? So on and so forth. So, eventually she figured it out. |
| 521 | | R1 | Okay. |
| 522 | | UCT | So she came up with 16. And she organized it like, she made me a little key, I don't know if you can see it over here [pointing to board]. But one box, she said was one pie. |
| 523 | | R1 | One box is a pie, okay. |
| 524 | | UCT | So like, one rectangle is a pie. |
| 525 | | R1 | Okay, neat. Uh huh, okay. |
| 526 | | UCT | One box is a pie neat |
| 527 | | R1 | Very nice. It's a neat organization, isn't it? What grade is this? |
| 528 | 00:58:56 | UCT | She's seventh. |
| 529 | | R1 | Seventh, okay. Uh huh. Did she write? |
| 530 | | UCT | She actually did write an explanation. Um, more than she did the first time. |
| 531 | | R1 | Okay. |
| 532 | | UCT | She said, I know there is no more possible ways because there are only four toppings. The plain is the base of the whole thing. I used all the possible ways there are, starting with one-topping to two-toppings to three-toppings to four-toppings. Then, I thought I was done but I realized that I didn't have a plain pie and only a plain pie. And that's how I got my answer and there were 16 ways altogether. |
| 533 | | R1 | Okay, convincing? Not convincing?[pause] |
| 534 | | UCT | I like her diagrams. |
| 535 | | R1 | Her, her written work is very, very, nice. She's showing you what she got and she started to talk about that there were only 4 toppings. So that could be convincing for why there are only 4 one-topping pizzas, but the rest of the groups, not so much. |
| 536 | | UCT | Yeah. |
| 537 | | R1 | Good. Okay. |

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| 538 | | UCT | [video shows The student's written argument and diagram was placed on the screen. The diagram had 8 pairs of 4-tall towers where P was written inside the square to represent a yellow cube for plain pizza and T was written inside the square to represent a blue cube for a topping. The written argument from this student was "I use the blocks for the pizza that I just did. I use blue for the topping, yellow plain only" And then, this guy actually surprised me a little bit. Um, he was the only one to say, this is like the towers. |
| 539 | | R1 | And why did he think that? |
| 540 | | UCT | He came out with it right away. And then I asked him why, and he's like, I don't know. He's like it's 16. And I was like, well, why is it 16? And he's like, I don't know. I said, well, why don't you just play around with it, so...Initially, he didn't write this. I didn't bring his, like, original because he wrote on a scrap piece of paper. He was just making a diagram, just like tree diagram, then he made a list. But then, when he realized that he was convinced it was for the towers, he couldn't explain to me why. I said, well, can you build them? So I gave him the towers, and I was like, alright, well, you have yellow and blue, how does that relate to your toppings? And then he couldn't really explain. But he figured that drawing it would make more sense. So he…his only explanation was this little bit. He is a hearing impaired student, so his language isn't the greatest. |
| 541 | | R1 | Okay. |
| 542 | | UCT | He said, I used the blocks for the pizza that I just did. I used blue for the topping and yellow for the plain. So then the first one is like TTTP, and the second was PPPT. |
| 543 | | R1 | Okay so TTTP, meaning? |
| 544 | | UCT | Three toppings, and…three toppings and blank; he said it was plain, but it would be just like 3 toppings. |
| 545 | 01:01:04 | R1 | Meaning that the fourth topping, whatever it was, didn't appear in the pizza. |
| 546 | | UCT | Yes. |
| 547 | | R1 | That's kind of neat, huh? |
| 548 | | UCT | Yes. And he did kind of the opposite thing like he did initially solving the towers; and just fill it in with Ts and Ps. |
| 549 | | R1 | Right. |
| 550 | | UCT | To meaning three-topping, or one-topping, or two-topping. So, I thought, he was just convinced. And then, the one thing that he did say to me, I actually wrote it in the post, was that he thought it was related to the towers because they had to choose from four toppings. And I was like, okay. And he's like, and for the towers you build four high, so those fours relate somehow. And I was like, oh okay, so then that's when I told him to build, at that point but… he ran out of time; so he just finished whatever he finished. |

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| 551 | | R1 | Yeah. Okay, okay. This is the beginning of seeing a connection, isn't it? Alright. Um, and it's neat that he is kind of saying, it either appears or doesn't appear. We just don't know which toppings are appearing. |
| 552 | | UCT | Yeah. |
| 553 | | R1 | But that's…that's neat. Good. Okay. [Extensive pause.] It's really hard to get them not to erase if they're using pencil. And you really want to see why they are changing, and what thinking they started with, and where they're going. And that why I urge you to let them use, uh, you know, pens. That they can't erase. Or if they have to use pencil, a line through it, no erasing. Because you want to know where they went, where they started, and where they're going. |
| 554 | | UCT | My kids were surprised when I said you could use pen. Like some of them didn't want to. |
| 555 | | R1 | Right. |
| 556 | | UCT | But some of them were like…yeah? Really? Okay! |
| 557 | | R1 | Yeah. Okay. |
| 558 | 01:02:52 | UCT | Uh, so Daniel started out with this web looking thing. |
| 559 | | R1 | Hmm. |
| 560 | | UCT | I had a few kids do this. They kind of just drew pizza, and then just drew all of their pieces off of it. |
| 561 | | R1 | Hmm. |
| 562 | | UCT | And I kind of asked him like, how do you know you have all of them? What's your organization? He tried a tree diagram, but it didn't really work out. So, him and his group decided, they realized they all had these webs, but some had different amount than others, because they had missed things. So then they went for this organization, which was much nicer. And found [Laughter.], zero-toppings, one-topping, two-topping, three-topping, and four-topping. So they were able to take their work from their webs and reorganize it into this and find the 16. |
| 563 | | R1 | So that's good! How many of you think that you could interpret that web? |
| 564 | | UCT | I was having a lot of trouble. |
| 565 | | R1 | I don't think I could. Yeah… |
| 566 | 01:03:32 | UCT | And I mean, I asked them about it, and they were saying like, oh well, we first…you know…this is what they did. They wrote down the one-topping, and they wrote down the two-toppings, but in their web they were kind of just trying to fit in things later. So it got really unorganized and hard to follow. |
| 567 | | R1 | Okay, okay. Look how neat their listing is. |
| 568 | | UCT | The, the list got really awesome. |
| 569 | | R1 | Do you notice how they're keeping constants peppers in those two-topping pizzas? |
| 570 | | UCT | Um huh. |
| 571 | | R1 | First the pepper pizzas, and the one that have pepperoni, and the one with sausage. [Someone sneezes.] Bless you. Very nice. Did |

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| | | | they write a convincing argument? |
| 572 | | UCT | They wrote that, I think that 16 is all the possible combinations because we can't do anymore without them repeating for four-toppings. Uh, we know that all the toppings on a pizza is a choice, and that plain pizza is a choice too. That makes two pizzas total. And then there were four toppings, and thought we could just put one topping per pizza, so that would make 6 pizzas total. |
| 573 | 01:04:23 | R1 | Hold on! Hold on! Read that last sentence. |
| 574 | | UCT | Then there were 4 toppings, and thought we could just put one topping per pizza. And that would make 6. So they so far explained the all and the none. That's two. And all the one-topping pizzas. |
| 575 | | R1 | And then 4 one-topping pizzas. Okay. Everyone following so far? Okay, now let's see how they get complicated. |
| 576 | | UCT | Then it says, next we put two different toppings without repeating them again. We got 6 total pizzas for two toppings, and that makes 12 pizzas total. |
| 577 | | R1 | Hold on. Is that convincing…why they have two…6 two-toppings? |
| 578 | | T10 | Not really. |
| 579 | | R1 | No.Their argument there falls apart. Go ahead…three-topping… |
| 580 | | UCT | Uh, three-toppings [unintelligible] we put three toppings on each pizza without repeating. It's pretty much what they said, they just did them without repeating. |
| 581 | | R1 | Okay. |
| 582 | | UCT | Which isn't very convincing. |
| 583 | | R1 | Okay, okay. |
| 584 | | UCT | I know one of the kids in this group, I don't think it was Daniel, but his partner, was talking about, for the three-toppings, how they eliminated one. And that's how he came up with his way to write it down. |
| 585 | | R1 | And isn't that neat? Yeah. Cause I don't think that any of you thought to eliminate a topping, but isn't that a neat strategy? To find those three-topping pizzas, which could be hard, right. Good. Very nice. |
| 586 | | UCT | And my other student, Scott. So Scott was first really concerned about creating the menu because it says, uh, it says, they ask you to help design a form to keep track of certain pizza choices… |
| 587 | | R1 | Got it. Got it. |
| 588 | | UCT | So he was really concerned about making a form that you could fill out for the pizza. [Laughter.] |
| 589 | | R1 | Okay. |
| 590 | | LC | [Video shows This student created a table with the topping words at the tops of the columns and the rows were numbered on the left margin 1-16. In the row and column cells, there were checkmarks to represent the toppings used |

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| | | | for the pizza combinations] So that was his first work. And then, he realized this wasn't getting anywhere, so his next idea was the chart. Which I was like, oh great work! So he was doing the chart… |
| 591 | | R1 | Look at that |
| 592 | | UCT | And he had it in a nice organized system where he was going through it. |
| 593 | | R1 | Yes! |
| 594 | | UCT | I was just so excited. And then, he got to somewhere and then he just kind of messed up. And his partner didn't really like the chart idea, and his partner was writing out lists of words. Um… |
| 595 | | R1 | Uh huh. |
| 596 | | UCT | And so then, Scott was like, well no, I don't want to do this anymore. I want to do what he's doing. So that's why he wrote "ignore" on the side. So that's no longer his work. He didn't want you to look at that. |
| 597 | 01:06:16 | R1 | We don't want to ignore it though. It's…it's…it's really neat. What do you notice in his chart? |
| 598 | | UCT | I mean, he started to do like, he tried to do one-topping, and then two-topping, but he got confused somewhere along the way. |
| 599 | | R1 | Okay, so…when he did one-topping, what do you notice? [Walks to the board.] |
| 600 | | CP | It's just like with the towers. |
| 601 | | R1 | It's just like the towers. In what way? |
| 602 | | CP | They would setup the block moving down each space. |
| 603 | | R1 | Remember how, if you had a single blue cube and three yellows, the blue would go up each spot and that's how you know you had all the possible towers that were exactly three yellow and one blue. Well, look at that. [Pointing to the board.] He has a four single toppings by doing that as well. What does he do here? Is there any other organization when he's building the two-topping pizzas? What does he do? |
| 604 | | UCT | He definitely starts to move that pair together. And then, he kind of messes up after that… |
| 605 | | R1 | Well, I don't know if he messed up. Is there any other way to put two together? |
| 606 | | UCT | [Multiple conversations.] Oh, he split them….Oh yes! He took the last one and put it to the beginning…. Yeah. |
| 607 | | R1 | It might be. Now again, we may be giving him, uh, credit for something that really he didn't do. But, it certainly looks to me that is intentional. He can't move it anymore, so he moves, one over to the front. And he has here [Pointing to the board.]…okay. And now, they're not together anymore. And they're not together anymore. |
| 608 | | UCT | I mean, the one in the tenth row, he already has in the seventh row. |
| 609 | | R1 | In the tenth row is…[Point to the board.] |
| 610 | | UCT | So, that was a bit of a repeat. |

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| 611 | | R1 | Yes! And it looks like they erased it though. |
| 612 | | UCT | Yeah, cause he was going through it later… and |
| 613 | | R1 | Right, right. |
| 614 | | UCT | And then he was realizing, there were some things he already had. But what I thought was kind of neat was that, he left all this space, and he put the one at the bottom because he knew at the end, he'd have everything. And he kind of like, left a lot of room to work. But then, he decided…yeah. |
| 615 | | R1 | That's really nice, though. This is really, really neat it, cause kind of looks similar to, uh, what Brandon was doing, right? Except Brandon was used zeroes and ones. |
| 616 | | UCT | So then when he decided he didn't want to do this because his partner was doing something else, he went to the number system. |
| 617 | | R1 | Okay. |
| 618 | | UCT | And he actually only had, um, he didn't have that 5 for plain. He only had one through 1 through 4 for the toppings. |
| 619 | | R1 | Okay |
| 620 | | UCT | And he came up with 15 pizzas. And he was so convinced that there were 15. Not without a plain. He decided to |
| 621 | | R1 | Okay. |
| 622 | | UCT | And as I said, his partner had a list with all the words, so they were comparing back and forth to see, to make sure they both had the same things written down. And the last on that his partner has was a plain pizza. And he was like, well Scott, what about a plain? And the look on Scott's face was just like, what? Like, you can have a plain pizza? He had not even at all considered that. So he decided to add it to his key, and then make a 5 for plain. But… |
| 623 | | R1 | Interesting, very interesting. What do you think? I…we had never seen this before today, and look here it is again; using numbers for the pizza toppings. It kind of eliminates the problem of the peppers and pepperoni, which cause a big problem, right? But when you put everything to be a number, and they gave you the code, what the numbers are, it makes it a much, you know, easier way to handle. Interesting too, the way they got their pies. Do you see? [Walking to the board.] I don't know if you can read it? But I'm up close here. Where you start with 1-2-3-4, and they eliminate. |
| 624 | | UCT | Yeah. Some he took away, some he just added. |
| 625 | | R1 | And they have 1-2-3, and they eliminate. 1-2 and they eliminate. 1. It's kind of neat, huh? Good. Interesting stuff. |
| 626 | | UCT | His argument was not good. |
| 627 | | R1 | Okay. What did he say? |
| 628 | | UCT | He just wrote, I got 16 ways to combine toppings for a pizza pie at the restaurant. Uh, that's because I numbered the toppings, including a plain pie. |
| 629 | | R1 | Okay. And you're right. It's not a good argument but wonderful |

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| | | | work.<br>That's what I love doing. |
| 630 | | UCT | He went through a lot with that… |
| 631 | | R1 | He went through a lot of work, and some really neat stuff. Okay, last one…but not least. Do you find it interesting to see what the kids are doing? Interesting to kind of play detective and figure out what they're thinking, and how they're thinking. I know. That's, that's what I love doing. My best part of teaching was when I was able to get into the mind of the students who thought differently than I did |
| 632 | | R1 | Especially when you don't think they are capable of it, and you actually see that they get. It's pretty cool stuff. |
| 633 | | UCT | And then you actually see they are capable. |
| 634 | | UCT | Yeah. They might not be able to write it down in words, but if they can come close to explaining it, I mean, that's pretty cool. |
| 635 | | R1 | Absolutely. |
| 636 | | | [Conversation not related to lesson study. Not included.] |
| 637 | 01:11:27 | UCT | Okay, so this first one, I could not follow when she was doing. Um, so she made this web, she has a standard pizza in the middle, and then she goes all around. And I really didn't understand this, and I wasn't until I was just sitting there with Lori that I was like, ok look! All the mushroom pizzas are together. |
| 638 | | R1 | Uh, huh. |
| 639 | | UCT | And then all her pepper pizzas are together. |
| 640 | | R1 | Okay. |
| 641 | | UCT | And then there's sausage and pepperoni. |
| 642 | | R1 | Interesting. |
| 643 | | UCT | And I didn't notice that until just today. |
| 644 | | R1 | Okay. |
| 645 | | UCT | And she numbered them…7…she had 7 with mushroom, 4 with peppers, 3 with sausage and 1 with pepperoni. |
| 646 | | R1 | Okay. |
| 647 | | UCT | Um, and she originally had 15 here because that add to 15. I don't know what she considered to be her $16^{th}$. |
| 648 | | R1 | Did she have the all four topping pizza anywhere? |
| 649 | | UCT | Um, from this I couldn't tell. And then I said, this really make no sense, is there any way you can re-write that so that when I read it later, I can see. And she listed her combinations, and she does. She starts with standard pizza and then she has the four topping pizza here. And you can see, she kept mushroom, and then the peppers, and the sausage, and then the pepperoni. Um, but…oh yeah, the standard pizza, I was gonna say there's no plain but there is, that's the standard pizza. |
| 650 | | R1 | Okay. |
| 651 | | UCT | So she did get it from that crazy web. And, and what she wrote was, We believe that have found all the combinations. We |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| | | | believe this because first we used the web to see how much combos we could make. We also did them all because when I checked them off there were none left, because I once…I did a combo…once I did a combo, I took one thing off. I also added them altogether. You could do 4 by 4 because there are four toppings to pick out of, that's how I got 16. But she did mention that she was taking one thing off each time. Um, but that's, that was her argument. |
| 652 | | R1 | Okay, okay. Did she mean she was taking one item off each time… |
| 653 | | UCT | One item off each time. |
| 654 | | R1 | Or was she taking it off the web and then putting it in her list? |
| 655 | | UCT | Good question. |
| 656 | | R1 | Yeah, because those are two very different interpretations. Um, again. We need more of an argument but, nice the way they're getting, generating the lists. There's a lot of organization there. |
| 657 | | UCT | Yeah. |
| 658 | | R1 | Good. |
| 659 | | UCT | And then, the next student, um, he is high level on the spectrum and I paired him with someone, but he doesn't work well with others. So they kind of just worked side by side. |
| 660 | | R1 | Okay, okay. That's okay. |
| 661 | | UCT | And so he…wrote this. And this was very confusing. He had so many different webs; I couldn't figure it out. And then when I talked to him, he said, each of those webs is a different pie. |
| 662 | | R1 | Okay. |
| 663 | | UCT | So, um, PI stood for his pizza. [Pointing to the board.] So this was a plain pizza, and this was a pizza with mushrooms, a pizza with sausage. And he had a key down here. You can't really see it. And he did get all 16, but he started with a 4 pizza, the four-topping, and then went to three, and then two…one, and then none. |
| 664 | | R1 | Is that interesting. Okay, okay. |
| 665 | | UCT | So he kind of did it backwards, but the most I could get from writing, he doesn't write very much, was, I got 16 combinations. I used factor trees to help me out by abbreviating the toppings and replacing the numbers with letters. I'm not really sure what he meant by that… |
| 666 | | R1 | Okay, okay. And that's interesting notation, isn't it? It helped him, because he really didn't use factor tree, right? |
| 667 | | UCT | No. |
| 668 | | R1 | Because each one was a separate pizza, and that's why it worked for him. |
| 669 | | UCT | Yeah. |
| 670 | | R1 | Just need a convincing argument, huh? You know, it's a hard thing to do. And hopefully, some of your children will get closer to doing it as the semester goes on. Okay, are you ready to do a problem? |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| 671 | | UCT | Sure. |
| 672 | 01:14:38 | R1 | Okay, break into pairs. |

R1-Dr. Judy Landis
UCT- Unidentified Cohort Teacher
Initials – Identified Cohort Teacher

| Description: Transcript of teachers doing Cycle 3 towers 3-tall with 3 colors & Ankur's challenge<br>Advisor: Carolyn Maher<br>Location: TRIE<br>Date: October 22, 2013 part 3 | Author: Phyllis J. Cipriani<br>Verified by: Simone Grey<br>Date Verified: Summer 2015<br>Page 1 of 36 |
|---|---|

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| 1 | 00:00:00 | R1 | Thank you for bringing them. |
| 2 | | UCT | You're welcome. They are what? |
| 3 | | R1 | Okay, you are going to have 3 different colors and they are coming now. And can someone tell me what the problem means?[pause] |
| 4 | 00:00:27 | UCT | Building towers three-high selecting from 3 different colors, instead of two, now. |
| 5 | | R1 | You got it! And then writing a convincing argument why you think you have them all. Okay, everyone know what to do? Then begin. [teachers work together in pairs on Cycle 3 task of building towers 3-tall with three colors for 30 minutes-begins monitoring groups] [R1 monitors the teachers on line 86 calls for whole group attention][the conversation between CDR and CP is filmed by the second camera] |
| 6 | | UCT | We need paper! |
| 7 | | R1 | Say that again. Oh, I think you do need paper. Let me give it to you. |
| 8 | | UCT | I have paper. |
| 9 | | R1 | No they didn't get the problem, I'll do that. |
| 10 | | CP | Does there have to be one color in each tower? |
| 11 | | R1 | Question:ask your partner. What does your partner think? |
| 12 | | CP | She thinks I should ask you![laughter] |
| 13 | | R1 | Good answer! But try again! |
| 14 | | R4 | That's funny! |
| 15 | | R1 | Okay. [video shows NL and MC working on the 3-tall towers. Video also shows LC and GH laughing when instructor asked why they were laughing LC said that GH wanted the tower diagonal to go up and LC wanted the tower to go down. Instructor said isn't it interesting that it wouldn't bother you but it bothered her.] [multiple inaudible conversations] |
| 16 | | CDR | They were all mixed up in the bags. |
| 17 | | R1 | They should…they actually should. |
| 18 | | R4 | You are from Toms River, right? |
| 19 | | CDR | Yes. |
| 20 | | R4 | You are not Tara, you're..? |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| 21 | | CDR | Christine |
| 22 | | R4 | Christine. We were in your class the first time. |
| 23 | | CDR | Yeah. |
| 24 | | R4 | Do you mind if I focus on you guys is that alright? |
| 25 | | CDR | No that's fine. To CP, maybe I should have asked you first! CDR drops the cubes and says maybe not we can't even keep the cubes together! [teachers work in pairs-small] We have 1 yellow do you want to put it on the top? |
| 26 | | CP | Do you want to…. |
| 27 | | CDR | On the bottom? And a yellow on the bottom? |
| 28 | | CP | I'll do like the yellows and the reds. |
| 29 | | CDR | Oh, you mean like that one?[building towers with unifix cubes] I thought you…[laughter] Sorry, I thought you were doing these… 2 reds. We just need one with the yellow on the bottom. The 2 reds and the yellow on the bottom. That much, alright. |
| 30 | | CP | See I would do it with the yellows on top. |
| 31 | | CDR | Oh, okay. |
| 32 | 00:02:50 | CP | How would you do it? |
| 33 | | CDR | I would have done, First position, second position, and then third position. |
| 34 | | CP | Third position, Okay. |
| 35 | | CDR | Alright But we can do it that way. |
| 36 | | CP | No that's fine. |
| 37 | | CDR | Alright however you want to work it. I am so confused right now. |
| 38 | | CP | So that is first position, second position, third… |
| 39 | | CDR | Right, third. And then the yellows are in the… |
| 40 | | CP | And then we have twos, |
| 41 | | CDR | …first, second, third. so that is one and two. No, cause that is all red, then we have two red one yellow; One red two yellow and Then we do yellow. I think we are good. Now we are going to do red with blue and then we need to do the yellow the same way. |
| 42 | | CP | Shouldn't it be 12? |
| 43 | | CDR | [begins to count] Wait, what? |
| 44 | | CP | Four tall times 3 is 12. |
| 45 | | CDR | There should be 8. So if we just focus.. |
| 46 | | CP | 8 that's right. |
| 47 | | CDR | Yeah, there should be 8, okay. |
| 48 | | CP | Okay. |
| 49 | | CDR | Alright, so now we need to do…do you want to do the reds and the blues the same way? And I will do the yellows and the blues the same way? |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| 50 | | CP | Okay. |
| 51 | 00:04:05 | R1 | So, did you answer the question for each other? [R1 moves to CP and CDR].[CDR & CP use unifix cubes to build towers] |
| 52 | | CDR | Yes. |
| 53 | | CP | Yes [R1 moves to a different group and circulates]. |
| 54 | | R1 | Okay. |
| 55 | | CDR | Here's the reds…I need the yellows, thank you. Okay so you are doing red and I am doing yellow.[CDR & CP using unifix cubes to build towers] [pause] we are doing 3 tall using 3 colors. |
| 56 | | CP | You are fast! [small group conversation between CP & CDR] |
| 57 | | CDR | Thank you. I don't know why, I guess cause I am use to doing it that way. Or are you use to doing it that way. We have all the twos then. So Red and Yellow. |
| 58 | | CP | We should have 6 of the two color combinations. |
| 59 | | CDR | Right. |
| 60 | | CP | 6 of each |
| 61 | 00:05:33 | CDR | Red yellow, Red blue; yellow blue; okay; so we can do all 3 colors. |
| 62 | | CP | Okay. |
| 63 | | CDR | So let's start with this maybe red as the constant on the top. So we can have Red, yellow, blue or we can have red blue yellow; |
| 64 | | CP | Okay. |
| 65 | | CDR | We have 6 but we have to use all the colors. Right, so it would be yellow blue blue yellow. |
| 66 | | CP | Yeah we have to change it. |
| 67 | | CDR | Right, so yellow is the constant on top. So we should do Yellow… what are you doing? |
| 68 | | CP | Yellow, red blue |
| 69 | | CDR | So then yellow red |
| 70 | | CP | Yellow on top,Then blue yellow red. |
| 71 | | CDR | [laughter] They are going to have so much difficulty with this! [counting]…24,25,26,27. |
| 72 | | CP | Nice! |
| 73 | | CDR | I think we're good! Alright. |
| 74 | | CP | Get ready to convince. |
| 75 | | R4 | Would it be alright if you laid them flat on the desk? |
| 76 | | CDR | Sure. |
| 77 | | R4 | It makes it easier to take the picture of it. Of course it helps to press the button! There we go! |
| 78 | | CDR | Oh, nice! |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| 79 | | R4 | Thank you. |
| 80 | | CDR | Thank you. There's yellow [hands yellow unifix cubes to CP] |
| 81 | | CP | Alright. |
| 82 | | CDR | We always start listing it I guess. |
| 83 | 00:07:41 | CP | Is there another way we can do this? |
| 84 | | CDR | Yeah I guess we can switch these. That's all I would do. Well, We did all of our solids, then we did red and yellow or we can say yellow and blue then yellow and red; and then last do blue red; So that is all our pairs. And each time…So that each time we changed our position and like here we kept our constant red. |
| 85 | | CP | Okay. |
| 86 | | CDR | She might have us….I mean The other way you could do it is the recursive thing where if you are looking at this one and then the blue goes to the top so then it is this one and then move blue down so it looks like that and then this yellow would move and move them like that. That's the other way. |
| 87 | | CP | I like the recursive way. |
| 88 | | CDR | The recursive what? |
| 89 | | CP | I feel like that is good. |
| 90 | | CDR | Yeah. So then the question is we need another set of these if we do it recursively. |
| 91 | | CP | Oh like this [using unifix cubes to build towers] |
| 92 | | CDR | Right. Alright, it works. I don't know if my kids could do this. Give this problem to my kids? I'm like….I don't know how they would do. The first one they had fun with and they enjoyed it. But there were only Eleven groups that actually got the answer but Everyone else had like 100; 1000. [unintelligible].. Do you remember the kid last time with the pizza? and it was his group that came up with 8 different slices. So… |
| 93 | 00:11:25 | R1 | Let's see what you got. Are you convinced you have them all? [returning to CDR and CP to monitor their progress.] |
| 94 | | CDR | Yes we are. |
| 95 | | R1 | You are, okay, convince me. |
| 96 | | CDR | Alright, so…do you… alright so… we started with our 3 solids of each color; |
| 97 | | R1 | Right. |
| 98 | | CDR | Then we moved to our groups of 2. So we figured. |
| 99 | | R1 | What do you mean? [R1 sits down next to CP & CDR] |
| 100 | | CDR | Two colors. |
| 101 | | CP | So like this and this. [shows R1 using the cubes]. |
| 102 | | R1 | Two colors. Okay the top two rows are 2 colors. |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| 103 | | CDR | So then we said we could have yellow and blue. And in that case it could be two yellow one blue or it could be two blue one yellow. |
| 104 | | R1 | Got it. |
| 105 | | CDR | And we used placements, so… |
| 106 | | R1 | Okay. |
| 107 | | CDR | Blue could be in 3 positions |
| 108 | | R1 | Nice |
| 109 | | CDR | For those 3 and the yellow |
| 110 | | R1 | Okay. |
| 111 | | CDR | And then we said instead of yellow blue we can also have yellow & red. |
| 112 | | R1 | MMHH |
| 113 | | CDR | Then we are done with our yellow. |
| 114 | | R1 | MMhh. |
| 115 | | CDR | And then the only ones that are left are the blue and red. |
| 116 | | R1 | Okay. |
| 117 | | CDR | So we are done with all of our just 2 colors in the tower. |
| 118 | | R1 | Okay. |
| 119 | | CDR | Then we had our 3 color towers. And we had thought about each of these differently. |
| 120 | | R1 | Okay. |
| 121 | | CDR | So do you want to go? [speaking to CP] |
| 122 | 00:12:19 | CP | Sure. So for this one we kind of did the same concept as these towers. |
| 123 | | R1 | Okay. |
| 124 | | CP | We add the one yellow. |
| 125 | | R1 | Oh that’s interesting, uhhuh. |
| 126 | | CP | And then it worked itself down. |
| 127 | | R1 | Okay. |
| 128 | | CP | And then we just did each place. |
| 129 | 00:12:30 | R1 | Now how do you know you have all the towers that can be in this grouping? |
| 130 | | CP | Well |
| 131 | | R1 | By putting the yellow down. |
| 132 | | CP | If you take this one, first if you put the yellow down then there would be 4 towers tall; and that wouldn’t work |
| 133 | | R1 | Okay you are saying you can’t have, you can’t have |
| 134 | | CP | Yeah and if you tried because we were doing where you just take one and put it back on top to the corresponding one again. |
| 135 | | R1 | AHHHhh. I see. So you used a recursive argument to form this. Okay, okay, alright. Interesting. |
| 136 | | CP | And It works the same for this one except for the blue and |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| | | | the red are switching positions. |
| 137 | 00:13:00 | R1 | Got it. |
| 138 | | CP | So if you took these and we used our recursive argument and moved this one again… |
| 139 | | R1 | How about if you did that again?[pause] |
| 140 | | CP | Do it again? |
| 141 | | R1 | Yeah. |
| 142 | | CDR | Then your red one would be on top. |
| 143 | | R1 | Yeah. |
| 144 | | CP | Oh, did I do that wrong? |
| 145 | | R1 | Yeah.[laugh] |
| 146 | | CP | Okay, where do I go since I messed myself up. |
| 147 | | CDR | So you put your yellow and then you do your red again and then your red is on top. |
| 148 | | R1 | Yeah, put the red on top. And then what happens? |
| 149 | | CP | Then you end up with the middle one. |
| 150 | 00:13:26 | R1 | Okay. Okay. Good very nice. Okay. |
| 151 | | CDR | What I had originally thought about. |
| 152 | | R1 | Yes. |
| 153 | | CDR | I had thought of um…a constant on top |
| 154 | | R1 | Okay. |
| 155 | | CDR | so I had originally thought a red on the top a blue or yellow will alternate on the bottom,<br>If the yellows are on top, the red and blue will switch. |
| 156 | | R1 | MMhh. MMhh. Okay. |
| 157 | | CDR | The third one would be the blues on top. |
| 158 | 00:13:46 | R1 | What do you think of her argument? |
| 159 | | CP | I think it works. |
| 160 | | R1 | I think both are good. Okay, Record your towers on your paper and write your convincing argument. Good. [R1 moves to a different group and the conversation is audible.] |
| 161 | 00:14:12 | R1 | Where is my group that is arranging it differently, but I bet you changed it didn't you? |
| 162 | | UCT | Is it us? |
| 163 | | R1 | Yes, it is you. [laughter] It is you.. |
| 164 | | UCT | What did we do? |
| 165 | | R1 | You are arranging it differently than everyone. Um at least unless you are… |
| 166 | | UCT | That was just to help us count. |
| 167 | | R1 | And how did that help? |
| 168 | | UCT | Well she saw something that I didn't do and then she told me to do it this way [laughter]. |
| 169 | | R1 | Okay, okay, now; |
| 170 | | UCT | So I looked at all of them that had the red on top. |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| 171 | | R1 | Okay, and how many….? |
| 172 | | UCT | And there were 9. |
| 173 | | R1 | Okay. |
| 174 | | UCT | And then We looked at the blues that were on top and we only had 7 and 8 yellow. |
| 175 | 00:14:47 | R1 | Good. Ahh. And that bothered you. |
| 176 | | UCT | Yeah. Because that gave us the 24 and we were trying to get 27. |
| 177 | | R1 | And you were hoping to get 27? And you had the 9 here and said… |
| 178 | | UCT | We are missing The 2 from the blue the one from the yellow. |
| 179 | 00:15:00 | R1 | And did you find them? |
| 180 | | UCT | Yeah. |
| 181 | | R1 | Okay, now Can you convince me, that there are only 9 with red tops and only 9 and you can't have no more? if you convince me of this, I won't make you convince me of this. Okay. |
| 182 | | UCT | Okay so you want to know why there are only 9 red. |
| 183 | | R1 | Exactly right. I want to know why there are only 9.[huge pause] |
| 184 | | NL | So if you were to move it, you would move it twice. |
| 185 | | MC | I know but I am just saying it worked. |
| 186 | | NL | Yeah, so if we moved it once it would be this; if we moved it twice it would be one of the yellows. |
| 187 | | MC | MMhh. |
| 188 | | NL | With a red on top. |
| 189 | | MC | It would be the same thing we are keeping the red on top, so we are keeping the red on top. If I move it once the reds will be on top but then if I move it again it has two categories and that is not what I want. |
| 190 | | NL | Right. And then this is the same thing with this one but with blue….And then this one….The only way you can have a red on top with two of the same colors on the bottom with a yellow. And then this one has the same thing red on top with two of the same color on the bottom where the constant is either yellow or blue. |
| 191 | | R1 | Why can't you have red on the top and red on the bottom? [pause] |
| 192 | | NL | Well that would be two colors together. I'm saying… |
| 193 | | R1 | Yeah, well why can't you have red; and then two more red? |
| 194 | | MC | We already have 3 red. |
| 195 | | R1 | Ahh, okay. Now, Again, I asked you that, but I really knew |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| | | | the answer. But that is what you want to do to your students; ask them to be sure that they got it. How about these? |
| 196 | | NL | The same argument. |
| 197 | 00:16:50 | MC | With the red on top? |
| 198 | | R1 | MMhh. |
| 199 | | NL | You can either get a yellow or blue or a blue or a yellow. And if you were to move it this would be one of these…. |
| 200 | | MC | So it will be this one. |
| 201 | | R1 | You want to keep the red on top. I heard you tell me that. Is there any other way to put a blue and a yellow on the bottom with red on top? Other than these two |
| 202 | | MC | No because we would only have 2 colors |
| 203 | | NL | There's only three positions. |
| 204 | | MC | Yeah, there is only three positions if there were four positions, then yes. |
| 205 | | CDR | I started writing yellows and reds first. |
| 206 | | CP | You know what I did? I just went… |
| 207 | | CDR | Down and I know I should have done that first. |
| 208 | 00:17:50 | R1 | Really neat. Your argument is different than everyone elses. I want you to record it. |
| 209 | | MC | How about that we are different! |
| 210 | | R1 | I want you to record it as groups of nine with solid color tops. Okay and then write a convincing argument for one of the three groups. Okay and then write a Convincing argument for this and that there are no more in this group. Now you want to write a convincing argument that is convincing. Okay; a convincing argument that is convincing. Do not tell me how you grouped them. Tell me why there are no more in the groups. Okay. [pause] |
| 211 | | GH | I feel like I wrote the same thing five times. |
| 212 | | R1 | [R1 goes to another group] You mean for each of these? |
| 213 | | GH | Yeah. |
| 214 | | R1 | Well that is because the argument is very similar. |
| 215 | | GH | Yeah. Even this one too. |
| 216 | | R1 | How is that similar? |
| 217 | | GH | Because all three are on the bottom one each time. |
| 218 | | R1 | So you used a recursive argument? |
| 219 | | GH | That is how we did all of them. |
| 220 | | R1 | Okay, okay. But here, is this a recursive argument? |
| 221 | | LC | Yeah because this is the bottom, the middle and the top |
| 222 | | R1 | And what other argument can you use here? |
| 223 | | GH | Stair case {actually is elevator but GH calls it staircase}Red in every position. |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| 224 | | R1 | You have one red and 2 yellows there are 3 positions. You have the red in each of the 3 positions. Is there any other place to put one red? |
| 225 | | UCT | No |
| 226 | | R1 | So that is another argument. Okay. |
| 227 | | UCT | Right, But if we say we took the bottom and moved it to the top and exhausted it until. |
| 228 | 00:18:59 | R1 | That's a recursive argument. |
| 229 | | GH | Yeah. |
| 230 | | R1 | It's another argument but I am saying that when it looks like this… you don't even have to use a recursive argument. |
| 231 | | GH | Right. |
| 232 | 00:19:09 | R1 | Okay, good. You are doing nicely on this. How do you think your students will do? |
| 233 | | unison | Not good! Laugh [unaudible multiple responses] |
| 234 | | R1 | Not sure? |
| 235 | | MC | I think it will be very hard for them. |
| 236 | | R1 | Did any of you besides this group make a prediction of how many towers you were going to find before you found them? |
| 237 | | TD | We did, we said 27. |
| 238 | | R1 | And why did you go 27? |
| 239 | | TD | Because we did three to the third. |
| 240 | | R1 | And why did you do three to the third? |
| 241 | | TD | The other one we did two to the third. |
| 242 | | R1 | You did! And what is the base? |
| 243 | | UCT | The colors the three colors.[inaudible multiple conversations at once] |
| 244 | | R1 | Good. Good. Good. And I predict that some of your students will do the same. If they knew that the towers were two to the fourth, I think some of them will be able to say like you did that these are three to the third. [pause] If they didn't do two to the fourth and they did four times four, aint gonna help them. |
| 245 | | UCT | But the way that I am explaining it, it makes sense of it that you get 32. |
| 246 | | R1 | Why? |
| 247 | | MM | Because there's three colors and there's three rows and each time…we kept an odd number on the first and second row. |
| 248 | | R1 | Okay. Okay. |
| 249 | | MM | So for every time we can have the bottom row change color you can change the second row three times. |

| Line | Time | Speaker | Transcript |
|---|---|---|---|
| 250 | | R1 | Okay. |
| 251 | | MM | So each time you change the second row, there is that alternating with the 3 colors. |
| 252 | | R1 | Now it is interesting. You have 9 groups of 3 and you have it different than the others. That's really neat. That's really neat. Very very neat. Okay, so I see 3 different ways that you've done this. Um, I hope we are going to get these up on with uh taking a picture of it. Cause some of your stuff is really really good. Get that convincing argument down. Okay. Cause if you can do it, you'll believe that some of your students will be able to do it too. [pause] [laugh] Did you get it? The angle is off… |
| 253 | | NL | I gotta weird angle but… |
| 254 | | R1 | Yeah; As long as we can see the top constant. |
| 255 | | NL | Yeah. That is why I am taking another one. Good, good. |
| 256 | | R1 | How did you get.. convince me that you had them all? |
| 257 | | UCT | I am just drawing them out [inaudible response] |
| 258 | | R1 | Good. Good. Now when your students are recording the towers do not let them pull out their Iphone and take a snapshot. [multiple conversations-inaudible]. |
| 259 | | UCT | I don't let them be on their phone cause I will take them. |
| 260 | | R1 | Okay. [laughter] [multiple inaudible low conversations] |
| 261 | | R1 | Are you still working? |
| 262 | | UCT | Yeah we are still working on it. We just realized and we were rearranging it. |
| 263 | | R1 | Why are you rearranging? |
| 264 | | VB | We have like the one yellow and two blue; and here is one yellow and two blue…all the combos just making it look organized. [multiple conversations occurring at once] |
| 265 | | R1 | I see. Okay. Okay. So you were trying to exhaust yellow and blue and the yellow and the red. |
| 266 | | VB | Yeah. Either way. |
| 267 | | R1 | So you were trying to exhaust…Yellow and the blue and the red the yellow and the blue and... Okay! I gotcha! I gotcha! Now you feel better about it? |
| 268 | | UCT | Yeah, now if I show that to them, they will be fine. |
| 269 | | VB | I just wrote that we used the recursive method and we moved each one into the other ones. |
| 270 | | R1 | And did you do that for… all of them? |
| 271 | | UCT | yeah. |
| 272 | | R1 | Do you need a… |
| 273 | | VB | We did it for these ones but for These we kept a constant. |
| 274 | | R1 | Okay so this you didn't do recursive but you kept the tops and then you said that the bottom are going to be the other |

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| | | | two colors so there is only this way to do it. Going back to here, you really don't need to do a recursive argument. You have a single blue in each of the 3 positions. Right? There is no other place to put that single blue. There are only 3 spots. Towers are only 3 tall. Right? That's convincing in itself. You have exhausted all the possibilities. |
| 275 | | VB | Okay. |
| 276 | | R1 | Okay. [inaudible] No, no, no! |
| 277 | 00:20:39 | CP | [multiple conversations at the same time as above]What did you say for this one? |
| 278 | | CDR | I said yellow and blue can be compared as 2 yellow and one blue or one blue and 2 yellow. And the single color can be moved to the other positions in the towers. Meaning that there are 6 towers that are yellow or blue. So basically that is the other color blue. Because we have 3 spots here and three spots there. And then for the other ones we kept yellow and paired it with red and followed it. We could have put red constant on the top; the red constant in the middle; and the red constant on the bottom. I don't know why I just thought of that by talking to myself! |
| 279 | 00:22:19 | R1 | When we..if we took a picture… |
| 280 | | UCT | It would be easier if we took it of the towers. |
| 281 | | R1 | Yeah, What we will do is we will take a photo of it so everyone can see it. |
| 282 | | UCT | Of the towers. |
| 283 | | R1 | Yeah of the towers. Remember you are writing a convincing argument. You are really trying to show why you have all the possibilities and there are no more. [slight pause]. So we want Your group to show it. We want your group to show it. And We want one of the other 3 which are all the same. Okay. Can we get you since you are done to take a picture of your towers and you are going to be talking about giving us a convincing argument as to why you arranged them the way they are. [Ipad says Siree not available]OOooo, Siree. |
| 284 | | CDR | She has a password on hers I have to wait and get it from her. |
| 285 | | R4 | So who is going to show theirs? |
| 286 | 00:23:43 | R1 | Okay. [pause] So it will be interesting for you to watch whether your students do it differently that the way you did it. [multiple conversations] Good. [R1 to R4] We got this group to show and we got this group to show and you have to take a picture of this one. Okay, alright. Good, Alright. You are flipping em! Okay. Alright. So now your ah..use |

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| | | | of camera is not a bad thing for your students, but don’t let it replace the students need to record. That’s fine, that’s fine. Okay. Can we have…what is the 3 groups? Do you want to start? |
| 287 | | CP | If you want us to. |
| 288 | | R1 | Sure. Alright, this is the way 3 of the groups that are sitting here did the problem okay and it was the most common solution. Alright. [slight pause] So you whizzed through this but there will be an extension coming up that I don’t think you are going to whiz through easily. Okay, okay, while it is warming up. You are going to see, most of you um four of the groups in the room did the problem by grouping it into 9 groups of 3. Okay, Or a variation sometimes you left the the…candy cane ones; you left them in groups of 2. Okay, but it was mostly 9 groups of 3. However, there was one group in this room, our Sayreville teachers [Sayreville teachers holler woo woo!] who thought in a different way and how did you group it? In groups of …? |
| 289 | | UCT | 9 |
| 290 | 00:26:16 | R1 | Three groups of 9 by holding the top cube constant alright I have seen students do it that way. I also have seen students do it by holding the bottom color constant. And on a real variation, I have seen the outside of the box thinker hold the middle cube constant. Okay, so expect all of the above. Okay. Oh, this is much better, isn’t it, huh? Okay, much better. Can we get it.. Oh that’s great. Alright. Okay so now you have the cubes, we are really looking for a convincing argument. Okay we are ready! |
| 291 | 00:26:36 | CP | [video shows the nine groups of 3 towers organized using the elevator strategy] Okay. The first one is pretty clear. It is the bottom one. Solid colors. Three different colors, 3 towers solid colors. Do you want to go with the twos? |
| 292 | 00:26:59 | CDR | Sure. So then what we did was that we could have one color and then we have two different colors in it. So we said the first one would be yellow and blue. And we could either have it with the 2 yellow and one blue or we could have it with the two blue and the one yellow. |
| 293 | | R1 | Good! Keep pointing! Good! |
| 294 | | CP | Okay. |
| 295 | | CDR | So what we did in each of those groups, is we held our single cube and we put that in each of the 3 positions. So we had that. Then we said okay, instead of yellow and blue we can keep the yellow and pair it with the red this time. There were two yellow one red, or Two red one yellow. |

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| | | | We took our single cube and moved it through to each position. Yellow was paired with both of our colors So our last one which was red and blue and again we kept our single cube and moved that into 3 positions, then we were done with this. |
| 296 | 00:27:43 | R1 | Before you leave the two of one color and one of the other. What's your convincing argument why you have every possibility of 2 of one color and one of the other? If you convince us of one group you don't have to go through the other groups. |
| 297 | 00:28:00 | CDR | Alright, So we said if you have the yellow and blue for example.<br>If we had it 2 yellow and the one blue there is only 3 ways to do that. Our single cube can move to each of the positions. If we were to move that again, we would either need a fourth row or we would be repeating it. |
| 298 | 00:28:18 | R1 | Does everyone buy that? And that's convincing in itself. You don't have to do recursive. Okay, that is a convincing argument. |
| 299 | | unison | MMhh [unison response] |
| 300 | 00:28:28 | UCT | The only other option would be the yellow and the blue would be if there are now 2 blue and one yellow and it is the same argument for that. |
| 301 | | R1 | Same argument for all the top group. |
| 302 | | CDR | Right. |
| 303 | | R1 | What is the argument for the candy cane or the 3 colored colors in a tower? |
| 304 | 00:28:47 | CP | Okay, so Christine and I did this a little bit differently. This was the idea I had, was having the yellow kind of go through each position kind of the same way that we had the one cube go through in the other positions. |
| 305 | | R1 | Okay. |
| 306 | 00:29:00 | CP | And it is the same concept, of actually moving the yellow all the way through. If you were to move it to another place, it would come back to the top and it would result…. |
| 307 | 00:29:08 | R1 | We better see that. Can you get three and show us? [used unifix cubes to illustrate] Now they are using a recursive argument. Okay cause that's not quite as easy to see why you have them all and why there can't be another tower. So they have the 3 towers there, okay. |
| 308 | | CP | So as you go through the first you get the yellow to move down to the middle place |
| 309 | | R1 | How about if I hold 2 and you three?[offering and then holding the towers] |
| 310 | | CDR | Okay, so…. |

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| 311 | | CP | Go ahead. |
| 312 | | CDR | Then at the end, you take the yellow one and move it back on the top we are back at the first one that he had. |
| 313 | | R1 | Okay |
| 314 | | CDR | Originally there. |
| 315 | | R1 | Alright. |
| 316 | | CDR | Oh I thought…[laughter] Do you want to do those? |
| 317 | | R1 | Well is that the only other way to do it? |
| 318 | 00:29:54 | CDR | Well so then the other way, well then so, what we did was we kept the yellow there but we switched the red and the blue and did the same thing. Now what I originally thought about it. Um, I had done it differently. |
| 319 | 00:30:06 | R1 | Okay. |
| 320 | | CDR | I held a constant at the top so I had held the red constant first at the top and then I said the yellow and blue could be in two different ways then I had the yellow constant same thing. |
| 321 | | R1 | So you made pairs, you didn't make triples. |
| 322 | | CDR | Right. |
| 323 | 00:30:20 | R1 | And I saw that with a bunch of you. And that's a good way to do it. Because if you have the top red, and you're using all 3 colors, the bottom has to be yellow and blue. And then it can either be yellow or blue or blue and yellow. Good. Okay, questions? Alright, nice. I hope you wrote that what you said because what you said was good. Good! Another group that did it differently. |
| 324 | | UCT | I have to take a picture. |
| 325 | 00:30:48 | R1 | Okay. Um, okay, good. Now these are going to be the towers that are solid color on the top. Again be prepared your students might want to put the solid color on the bottom or the solid color in the middle. The first time I saw the solid color in the middle I went Wooo; I wasn't expecting it. Isn't that nice? Whoa, okay. Okay. Can we move it down a little bit? |
| 326 | | UCT | I can take another picture. |
| 327 | 00:31:32 | R1 | Yeah, take another picture. It is blurry. Isn't that nice, the way…you might want to do that when talking about your children and stuff even though you are going to have them record, ooo that is good. Woops, that is not as good. Okay, isn't that nice? Really nice. Okay. Woops, it's gone. Okay. She is annoyed with your phone. It didn't bother me. So you are going to make sure your students record because none of you drew the cubes, did you? How many of you used an R Y B, most of you. Okay and I bet a lot of your students will too. Okay. Keep going.[Asked next group to |

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| | | | explain their solution] |
| 328 | 00:32:18 | MC | So we started off before we actually started building them, we made a prediction that there were going to be 27 similar to the two to the fourth power and the 3 to the third power. So we knew that there were going to be 27 so we actually started building them the same exact way that Chris and Christine did. And Then we kind of got stuck and there were only 24 and we were like we're missing 3. So then I kind of looked and we decided to group them differently. And this is where we put all the…keeping the red constant, the blue constant, and the yellow constant. When we did it that way, We then saw that there were 9 that had red constant, 7 that had blue constant, and 8 that had yellow constant so that kind of gave us the 3 that we were missing. |
| 329 | 00:32:59 | R1 | And that bothered you. And that's the key, your students will have the same thing. They like…Remember how they found 3 of each topping across all the pizzas.<br>If they are building towers with red tops they want if they have 9 there, they are going to want theirs to be the same number for the blue tops and the yellow tops so that when you didn't have it, even if you didn't know 27,<br>I think it would cause you some agita that you didn't have the same number in each group. Okay keep going. |
| 330 | 00:33:28 | UCT | So then we kind of just looked to see of what of the 9 towers that did have the right constant. Where we were missing The two from the blue constant and the one from the yellow being constant.<br>And then we were kind of able to create the 9 different towers in each of the different groups. |
| 331 | 00:33:44 | R1 | Okay. |
| 332 | | MC | And then we kind of looked and then we kind of organized… |
| 333 | | R1 | Did you see one she didn't see? |
| 334 | 00:33:53 | NL | Well I saw the constant but the one we were missing I couldn't figure out. |
| 335 | | R1 | Okay |
| 336 | | NL | So she was able to pick it up before I could. |
| 337 | | R1 | okay |
| 338 | | NL | And then I saw what she was doing. |
| 339 | | R1 | Okay. Okay. |
| 340 | 00:34:08 | MC | So we kind of organized it with the first 3 in each group is the following color. And then the next two is where you held that particular color constant. So for the first group where the red was constant you can only then have one yellow in position 2 and position 3 which would be the |

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| | | | second and third tower. And then if you do the same thing with the blue it could then only be a second position and a third position. |
| 341 | | NL | Then working with two colors but keeping red as a constant you can only have just 2 of yellow if you move it around you are going to come up with something else. So you could only have 2 yellow or 2 blue to be no repeats, no duplicates. And Then the same thing with the yellow and the blue we just switched them, the position. |
| 342 | 00:34:47 | R1 | Are you convinced? |
| 343 | 00:34:51 | CP | Yeah! |
| 344 | | R1 | It’s pretty neat, isn’t it? Now if your students do this kind of a thing, Don’t torture them to convince you of the blue tops and the yellow tops after they convinced you with the red tops. Because the argument should be the same. |
| 345 | | UCT | Right.[inaudible] |
| 346 | | R1 | Alright, very nice. So that was different than the 3 groups of 9. What were you going to say? |
| 347 | | UCT | I was saying that It was driving me nuts that one solid and then you have two checkered, two checkered and then you go back to 2 solid. |
| 348 | | NL | Yeah and I moved it around when we were there but we didn’t record it that way. I said that these should have been over here. It would have made more sense. But we had already recorded it. |
| 349 | | UCT | No I know. |
| 350 | | R1 | Interesting. Interesting. Okay. Now the last group they have 9 groups of 3 but Watch how they arranged them. They arranged them differently than the group before. |
| 351 | 00:35:47 | MM | It was kind of similar I guess than the other group…. |
| 352 | | R1 | [video shows work of the teacher pair had nine groups of three towers where each of the three colors were on top of the towers]Okay. Give them a minute to look and watch how they arrange nine groups of three. Now, Interesting. Now I think that for some of you, who built your nine groups of 3 with moving a single cube down, you are going to say well what did they do here. This is confusing but I’m hearing you are telling me that this is…. |
| 353 | | UCT | Well I was more convinced that way |
| 354 | | R1 | You saw it better this way. |
| 355 | | JLB | but Margaret said wait let’s hold something constant it and then it makes more sense this way. |
| 356 | | MM | Yeah, so What we did was we held the bottom color constant. So there are 3 different rows we can do that with Because there are 3 different colors. Because you can hold |

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| | | | the bottom row constant 3 different times. And then within the row of holding the bottom constant, we held the second one constant as well so the second one is yellow, the second one is blue, and the second one is red and we Can’t have anymore groups of 3 because there’s 3 colors. |
| 357 | 00:36:47 | R1 | MMhh |
| 358 | | MM | And that leaves the last row to kind of alternate between the colors. And it’s either red, yellow, or blue…. |
| 359 | | R1 | Very nice. Very nice. What kind of argument did they use? |
| 360 | | UCT | Recursive? |
| 361 | | R1 | Noooo, remember Milin? Milin used that argument. It was an inductive argument. It really is when you think about it. Think about this as your bottom. Okay, This is your base, okay. On this base in the second position we have 3 colors to choose from so the second row can either be yellow blue or red. Is that right, okay.<br>So now let’s just look at this one. The third row on top of this cube. What could you put? You have 3 choices again. What could you put? |
| 362 | | UCT | Yellow red blue |
| 363 | | R1 | And on top of this row? |
| 364 | | UCT | Yellow red blue |
| 365 | | R1 | And on top of this row? |
| 366 | | UCT | Yellow red blue |
| 367 | | R1 | Inductive argument. |
| 368 | | TD | And then when I was writing it, it actually made sense that it was going to be 3 times 3 times 3 because for each constant row at the bottom, there can be 3 colors for each one of those, there is 3 for the different second row and then for each one on top of that 3 colors. |
| 369 | | R1 | Do you see the inductive argument? Yes? |
| 370 | | UCT | Yes! |
| 371 | | R1 | Some of you do, some of you aren’t answering me. Okay, but it is pretty neat, isn’t it? Okay And that’s really really powerful in terms of a proof. Okay, very nice proof. Good job! Okay, Did you write it as an inductive argument your proof? Can you read what you wrote? |
| 372 | 00:38:47 | MM | Um, Yeah, I said… |
| 373 | | R1 | Nothing like putting you on a spot! |
| 374 | | MM | well I didn’t use complete sentences! |
| 375 | | R1 | That’s okay. |
| 376 | | MM | I said I keep the bottom block constant and then you keep the second block constant and there are 3 different ways that you can change that. Well actually I kind of drew out |

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| | | | one, like I drew out the first group of 3 and I said keep the bottom block constant yellow, keep the second row constant, or second block constant yellow and then alternate the red blue & yellow for the third spot so there's 3 different towers here because<br>There is only one spot you are changing and there's three colors. And then I said:<br>Repeat with changing the second row to blue first one is still yellow, still 3 towers, for the same reason, and then you can repeat this one more changing the second row to red. It says there are 3 colors, the first row is staying constant the second row is constant and<br>The top is alternating in 3 colors<br>And then I just got lazy and said you can repeat this process changing the color of the bottom row<br>Two additional times and you can do this three total times because there are 3 total colors. |
| 377 | 00:39:47 | R1 | And that is just fine. Remember what I said, I am not going to beat your students over the head! Nor am I going to beat you over the head. If you have provided a good inductive convincing argument for the yellow base then I guarantee you that you are going to be able to do it for the other 2 colors bases. Okay, That's great! Alright Ready for the challenge! How many of you have students in your class that would say: Oh we did this problem but<br>What if we changed it up a little bit, okay Well I would say that your class would do that. That came up with all those crazy solutions. Well in Rutgers, they did this problem with I believe they were high school students and Ankur was one of the students.<br>And he said I have another problem. Okay, I would like to know, what would you get if you build the towers four tall selecting from 3 available colors so that the resulting towers have at least one of each color. Okay, he changed the problem up. And Carolyn Maher was the person in the room with the student and she said oh that is a neat problem! Let's do Ankur's challenge. So let's do it! Okay, here you go. [R1 hands out paper, teachers begin to work in pairs on Ankur's challenge - multiple inaudible conversations]. |
| 378 | 00:41:04 | CP | There has to be one of each color? |
| 379 | | R1 | Well read the problem and make sure you understand the problem before you begin it. teachers work on Ankur's problem in pairs or small groups for 30 minutes. R1 calls for the attention of the group on line]. [multiple |

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| | | | conversations occurring] Margaret can we open the door now? |
| 380 | | CDR | He said there was going to be a fire [drill]. |
| 381 | | MM | Really? [laughter] |
| 382 | | CP | We will do the top group on top and the bottom group on the bottom. |
| 383 | | CDR | Alright, sounds good. |
| 384 | | R1 | Read the problem and see what it says. |
| 385 | | R1 | [R1 circulates and stops at the pair of CP and CDR.] Look at all those towers! |
| 386 | | CP | Pumping them out! |
| 387 | | R1 | Pumping them out! |
| 388 | | CP | So We took the ones we already had and we and we doubled them. So the color could go on the top of it or it could go on the bottom of it. |
| 389 | | R1 | Aahhh. |
| 390 | | CDR | So these are the ones we had from last time and we added a color to the top and the bottom. |
| 391 | | R1 | Interesting. |
| 392 | | CDR | Which originally I hadn't realized until they could also go on the top. So it was the same thing here. |
| 393 | | R1 | Okay, very interesting. So you used what you had rather than start over. |
| 394 | | CDR | Right. |
| 395 | | R1 | Did you eliminate any of your towers? |
| 396 | | CDR | The solid towers 4 tall towers |
| 397 | | R1 | The solid towers. |
| 398 | | CDR | Yeah because We only had one other option. |
| 399 | | R1 | Got it. Okay. Okay. [monitoring work of CP & CDR moving towers around to organize groups] |
| 400 | | CP | This is like a mess! |
| 401 | | CDR | I know [laughs]. Alright so, we have…. |
| 402 | | CP | We can kind of use our same number and put them on the top and the bottom. |
| 403 | | CDR | We can do that [CP moves blocks up that have all blue as a constant on top] I'm thinking do we still have options. Two yellow and one blue and a red; okay so should have two yellow and one blue. But the red can also be…[inaudible] we have the elevator with two yellows and one blue. Do we? Oh yes we do. Two yellow one blue. The red in the second position. Okay so we do have that. |
| 404 | 00:45:04 | CP | Do you want to make constants? |
| 405 | | CDR | Sure. [laughter] I am so against… never mind we'll do it! |
| 406 | | R1 | You don't like constants? |

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| 407 | | CDR | I do to a degree. Like the whole one long line of all reds. |
| 408 | | R1 | [laughter] That bothered you? |
| 409 | | CDR | A little bit, yeah. |
| 410 | | R1 | Okay. |
| 411 | | CDR | But it's alright, I can follow it. |
| 412 | | CP | So what do you want to do? |
| 413 | | CDR | Let's do all red on the top and then all yellow on top; okay let's do all red on top and then… well let's just put red on the top and then reorganize. [R1 moves to a different group.] |
| 414 | 00:48:44 | CP | Otherwise I would be lost. |
| 415 | 00:48:46 | CDR | Yeah I know I agree I think I am lost as well. [CP says something but it is inaudible] So then what I would do is I would keep…, |
| 416 | | CP | A constant? |
| 417 | | CDR | I would move, yeah the second row and all the reds. We move this here. The second row there is no other reds. Second row with the blues… Then we have to reorganize these somehow. Oh we have doubles! We have another double. Alright so, This makes sense because we only have yellow or blue. Then we have this here. Let's keep the third row constant yellow we can have yellow or blue [CP says something inaudible]; |
| 418 | | CP | We are missing one. |
| 419 | | CDR | Right so you are saying we are missing one here because we have red blue and we keep the third one yellow we can still have a red on the bottom. So we need that one. Red blue yellow and this red. Red blue yellow red…[CDR builds this tower with the unifix cubes] |
| 420 | | CP | Okay, I don't know how it works at all... |
| 421 | | CDR | I don't know why. Well it did sort of okay so here is the red that are on top and the red in the center and we only have yellow so we have to use yellow as the only option; then we kept red and blue we said we could have… |
| 422 | | CP | Red blue and yellow. |
| 423 | | CDR | Oh right so then we kept the red with the blue with the yellow. Then we have a red; we are keeping the blue; we have red, yellow. We could also have another one in here. We could have Red, then blue, then red, Oh no we can't because we can't have a blue there, it has to be a yellow. This goes red blue red it has to be a yellow. So that is our only option there, then we have red blue blue then it has to be yellow. So that is our only option there. |
| 424 | | CP | Okay. |
| 425 | | CDR | So are we okay so far? |

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| 426 | | CP | Yeah. |
| 427 | | CDR | So red yellow, then let's keep them together.[CDR moves tower with red yellow blue blue next to tower with red yellow blue yellow] So we have 2 blues and blue yellow so here is one tower that we are missing. We are missing Red yellow blue blue |
| 428 | | CP | Wait, you have red yellow blue blue there. |
| 429 | | CDR | Oh I am sorry I mean red yellow blue red. |
| 430 | | CP | Right. |
| 431 | 00:49:59 | CDR | Red yellow blue red [CDR builds the tower using unifix cubes], so that much is here. Then we have red yellow keep it yellow it has to be blue; red yellow red blue okay so that looks good so then we have basically… |
| 432 | | CP | Alright you got 2,3, 2,3, like the… |
| 433 | | CDR | Well I just…I kept this was red in the third row; for the third row since these were both reds at the top the bottom had to be yellow. These were reds and yellows at the top; the bottom had to be blue.[laughter] Alright but I think that yeah,… we have our reds down; |
| 434 | | CP | Are we going to have to write the towers down? |
| 435 | | CDR | I don't think so. |
| 436 | | CP | So maybe just the ones we were missing? |
| 437 | | CDR | Probably. So now we need to do the yellows group. Yellow and yellow are these. Yellow and blue so those have… |
| 438 | | CP | It should be like the same thing. |
| 439 | | CDR | Yellow blue 2 reds; yellow.. yellow blue red blue; yellow blue.. Oh yeah here and then we have our two colors have to be red on the bottom. |
| 440 | | CP | Why don't we have 3 color constants up here. |
| 441 | | CDR | Yellow, blue,…That's what we are missing. Yellow blue red yellow; yellow blue red yellow; okay. Then here's our yellow blue blues. This one is holding yellow constant. |
| 442 | | CP | Would you hand me the yellow. |
| 443 | | CDR | Sure. |
| 444 | | CP | Yellow red blue yellow. |
| 445 | | CDR | Yellow red blue yellow. We kept yellow; then we put red; here we have 2 in the middle. Then we have that; alright. Two blues; then the blue and the yellow I guess. |
| 446 | | CP | Then the red. |
| 447 | | CDR | Then the red so we have 2 yellow. Alright we have our alternating red, with yellow. And the yellow constant and it has to be red on the bottom. |
| 448 | | CP | So you are working with 2 constants on the top and the bottom. |

| Line | Time | Speaker | Transcript |
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| 449 | | CDR | Right. The blue stay constant, the yellow stay constant. Then we did red constant with this then we did blue constant but it would have to only be red. [inaudible] or it would have to be the same color…then the yellow with the red; I keep forgetting that that's what we're doing. So then blue constant with red constant. |
| 450 | | CP | Red and yellow. |
| 451 | | CDR | Yellow constant. Yellow red and then we take blue; red yellow blue. Blue constant then red constant, red constant. It has to be yellow. Blue constant, red constant; blue constant, it has to be yellow. And we should have… all of them.[pause] 12. |
| 452 | | CP | So for pretty much every one except for the first one, all these are 3 constants.[counts towers] |
| 453 | | CDR | Right. Right. I thought about it a little bit differently like that's true; I was, I was thinking red constant at the top; then we had red constant and the alternating which is true. And then the red constant and the blue constant; then I thought..these are my 3 constants so I switched the third row to be a red constant; and this is my only option with this color here [pointing to yellow unifix cube]. Then I switched my third row to a blue constant with that being yellow. So I took like this and I figured yellow in my third row, red in my third row and blue in my third row, and one of my options left. So then this already had 3 colors; but these only had 2 colors. I have to put that 3rd color there. |
| 454 | | CP | Okay. |
| 455 | | CDR | But either way it works. I just find it more easier to use this constant because I don't know why you would move it down…. |
| 456 | | CP | Because there are 4 rows and 3 colors |
| 457 | | CDR | Right. |
| 458 | | CP | So that's the difference so like basically… I guess either way, it works. |
| 459 | | CDR | Cause then why isn't there a red and a color constant? Right because the red constant that is why they're down here. We have all the red constant. Then we have red blue and yellow constant. Then we have our yellow constant red constant blue constant in the third row. Those are the options but either way... Wow! Could you imagine doing this? What grade do you have? |
| 460 | | CP | 6th |
| 461 | 00:58:17 | R1 | [R1 returns to CDR,CP pair]. Let's see what you got. |
| 462 | | CDR | We found double as we started to do that. |
| 463 | | R1 | Ahhh, okay. So you got those red tops, yellow tops, blue |

| Line | Time | Speaker | Transcript |
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| | | | tops. He convinced you? |
| 464 | | CDR | Yeah! |
| 465 | 00:58:25 | R1 | Ah, You are pretty persuasive there. |
| 466 | | CDR | So we started with the red constant in the top row. |
| 467 | | R1 | okay |
| 468 | | CDR | And then we moved to keeping the second row constant as well. |
| 469 | | R1 | Okay. |
| 470 | | CDR | So then if the second row is red. |
| 471 | | R1 | Yep |
| 472 | | CDR | We have to have a yellow and a blue there is no other way to do it. |
| 473 | | R1 | Good, good. |
| 474 | 00:58:48 | CDR | If the second row is blue; let's also keep the third row constant. |
| 475 | | R1 | Okay. |
| 476 | | CDR | If it is yellow I already have all 3 colors so I can use any color I want. |
| 477 | | R1 | Good. |
| 478 | | CDR | If I make my third row red, I need to have yellow because it has to be a yellow here. |
| 479 | | R1 | Okay. |
| 480 | | CDR | And if my third row is a constant blue, then I have to have a yellow again. |
| 481 | | R1 | Okay. |
| 482 | | CDR | So then over here we have red; this time the second row we already did red; we already did blue; we are going to keep a constant yellow. |
| 483 | | R1 | Okay. |
| 484 | | CDR | Make the third row blue. |
| 485 | | R1 | Right. |
| 486 | | CDR | Since you have 3 colors those are my only options there. |
| 487 | | R1 | Good. |
| 488 | | CDR | If I keep If I change my third row instead of being blue to yellow…. |
| 489 | | R1 | Yep. |
| 490 | | CDR | I have to have the blue. If I make it red it has to be blue. |
| 491 | 00:59:20 | R1 | Good, good. And the same with the others. Very nice. Okay, I am not going to torture you to record this. You are going to Take a picture of it. Okay and I want you to then write a convincing argument on why you have every one and just focus on the red tops because the argument is the same for the yellow tops and the blue tops |
| 492 | | CDR | Right. |

| Line | Time | Speaker | Transcript |
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| 493 | | R1 | Okay. [R1 walks away to monitor another group] |
| 494 | 00:59:42 | CP | This is scary-looking. [CP takes a picture][R1 moves to another group.] |
| 495 | | CDR | You know what, when we have teachers convention, we have off on Tuesday so we are only going to see the kids Monday, and they take their quarterly next Friday so when am I going to do this with them? [CDR takes picture with iphone].[inaudible conversation between CP & CDR]. |
| 496 | | R1 | If you can find a convincing argument. Okay. How are you doing here? |
| 497 | | CDR | Margaret did you ever find out who had yours here? When are you thinking about doing this? |
| 498 | | MM | I don’t know. |
| 499 | | CDR | Other schools did it; Kelly did it and she had to get it from somewhere. |
| 500 | | MM | Yeah they have a set in the district. |
| 501 | | CDR | Someone said every school has one.[inaudible conversations] |
| 502 | | R1 | But now you gotta figure out is there another way to rearrange them. |
| 503 | | UCT | When I rearrange them there is a group of nine. |
| 504 | | R1 | Okay, okay, okay. Good idea. I think you might be on to something. Okay. |
| 505 | | UCT | I am going to try to see if I have all the combinations. |
| 506 | | R1 | Oh good. Good, that is a good strategy. By the way, I really like the way you presented the problem to your colleagues. That was neat. That was really neat. |
| 507 | | UCT | Yeah, I had this woman. She said well it was fun to play with the blocks. |
| 508 | | R1 | So they gave you a hard time! Worse than the students. |
| 509 | | UCT | I think I am on to something. |
| 510 | | R1 | You are on to something. Oh she is on to something, I gotta hear. How many do you have here with 3 bottoms? |
| 511 | | UCT | Ah, 1,2,3,4,5,6,7,8,9,10 but we have 30 altogether. I am thinking there may be different ones. |
| 512 | | R1 | You think maybe more. Okay. So You got a convincing argument and you got a picture of your towers. |
| 513 | | UCT | Yeah. I took a picture. I didn’t put it on Margaret’s. |
| 514 | | R1 | You want to put it on hers then you are going to show what you did. [R1 returns to CP & CDR and looks over their shoulder] |
| 515 | | MM | It is just a website. You are free to sign up and they don’t even email you back that you signed in which is cool cause a lot of like things you can do with the students. |
| 516 | | R4 | Can I get a shot of your written work? To see what you |

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| | | | have? |
| 517 | | MM | I guess. I didn't really do much. |
| 518 | | R4 | Thanks. |
| 519 | | MM | Um the ones with the silver lining have grape. |
| 520 | 01:05:13 | R1 | Good. Good. About 5 more minutes on this and then we are going to share solutions. [multiple conversations continue…] Good, good. Now you got 12. |
| 521 | | UCT | Yeah I am trying to figure out… |
| 522 | | R1 | Okay you are trying to sort it out. |
| 523 | | UCT | I don't remember which one I found! |
| 524 | | R1 | When you share your solution you are going to talk about why….[inaudible] |
| 525 | | UCT | So far, Blue is constant on the bottom. And blue is constant in the second spot then a yellow or a red. And with the blue yellow and red there are only 2 colors. Blue is constant and then yellow is second. You can have a yellow or a red. Here you can have a blue for the third color… This is way too long… |
| 526 | | R1 | No that's very good though, it is very good and it's an inductive argument. Right? [inaudible] |
| 527 | | UCT | I'm trying to think how they would do it. |
| 528 | | R1 | Well you are not going to need to do that. |
| 529 | | UCT | I know but I think about how they think. |
| 530 | | R1 | That's good. |
| 531 | | MM | We did these ones with blue and blue and we did the next ones with blue and blue too. But there are more. |
| 532 | | UCT | There are more so that is what I am thinking. |
| 533 | | MM | So we should have nine right? |
| 534 | | UCT | So I have to try and match them up a little bit. Blue red blue yellow; blue red blue yellow. Then we don't have that one anywhere. Okay Then we are going to have blue red yellow red. Then red… okay there's 2,4,6, 8, 9,12 so there's 36. |
| 535 | | R1 | What do you think? She is making a prediction. |
| 536 | | MM | Good. |
| 537 | | R1 | Very nice, very nice. What I want you to do is Get a picture of it of the top middle and bottom and then I want you to convince others. Okay, alright. You are ready guys? Who is ready to present? Okay come on up. They got theirs very fast um and I am not sure which strategy they used but they will share it with us. And I see two different ways of arranging going on. I see grouping in groups of 3, I see grouping in groups of 6, I see grouping in groups of 12. There are 3 different ways. So let's look at this is the grouping of 6. It is not as easy a problem, is it? Was it |

| Line | Time | Speaker | Transcript |
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| | | | easy to make a prediction? |
| 538 | 01:09:15 | unison | No |
| 539 | | R1 | Uh uh. What made it hard to make a prediction? |
| 540 | | UCT | This is it. You need to get 3 colors all at one time. |
| 541 | 01:09:27 | R1 | That Each tower had to have each of the 3 colors. That made it a little tricky, right? If we didn't put that restriction in, if I just told you to build towers that are 4 tall selecting from 3 colors. How many would you get? |
| 542 | 01:09:48 | UCT | Four tall so Three to the four so 81 |
| 543 | | R1 | 81, right? But You are not going to get 81. Why do you only going to get 36?[pause] |
| 544 | | UCT | Because you are getting rid of the ones that have only 2 colors or one color. |
| 545 | 01:10:00 | R1 | Exactly, okay they have to have all 3 colors. Alright, So take a look here. This is a way of arranging it. Talk to us about it. |
| 546 | | UCT | Okay, So we have groups of 6. [Video shows a picture on the screen with six groups of six four tall towers. Three of the groups had constant colors of red, then yellow, then blue positioned at the bottom of the towers for three groups of towers. The other three groups had constant colors of yellow, red and blue positioned at the top of the towers for the other three groups.] |
| 547 | | R1 | Good. |
| 548 | 01:10:14 | TD | Um. Well, First of all, Since there's four cubes & 3 colors and you have to use all the colors you know that only 2 cubes will be the same color. So we basically just stuck the reason why we pretty much got it really fast is like that is because we stuck with what we just did. So here we knew we exhausted all the options for yellow and the yellow because we used it in every position. Then we just added all the reds to the bottom because that we knew that took up all or used every color. |
| 549 | 01:10:40 | R1 | Do you see what they did? They didn't start over. They took what they had from the other problem. And what they had in the other problem is … |
| 550 | | TD | The two blue and the yellow. yeah |
| 551 | | R1 | Two blue and the one yellow or the 2 yellow and the one blue. Once they had that,…Keep going. |
| 552 | 01:10:58 | TD | So then we just took, we picked a color and we said let's start with red. So then we used red as the constant and we added it to the bottom and made that the constant for the bottom. Then we did the same thing with our next tower or our other 3 towers. We had two blue and one red and we knew we already exhausted all the options for blue and red |

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| | | | so we just decided to go with the yellow on the bottom. |
| 553 | 01:11:18 | R1 | Isn’t that kind of ingenius? Kind of nice, huh? Go ahead. |
| 554 | | TD | And then we did the same to the other colors and we just added blue to the bottom so now we have red on the bottom, yellow on the bottom, blue on the bottom. |
| 555 | | R1 | Okay. |
| 556 | | TD | then we were left with our little candy cane ones that we went with. |
| 557 | 01:11:36 | UCT | And then like for our 3 color ones, our multicolor ones. Basically we kind of you know we had these 2 let’s just say the Blue, yellow, red and the blue, red yellow. We rebuilt the top of that making sure that we could use Two blue at the bottom, 2 yellow at the bottom and two red at the bottom. So then, if you look at it Blue, red yellow red and the inside is kind of flipped. So we kind of used one color as the constant and used it with 2 different colors there. |
| 558 | 01:12:04 | R1 | So it is kind of like…I heard students call that the sandwich. They have the sandwich here here is the bread and here the bread is the same on top and bottom. And the only middle if you are using all three colors can either be yellow and red or red and yellow. Right? Now you have a sandwich but you have 2 different kinds of bread, right? But again what can your middle be? |
| 559 | 01:12:30 | UCT | Yellow and red. |
| 560 | | R1 | And here? |
| 561 | | UCT | Yellow and red. |
| 562 | | R1 | Okay. Keep going. |
| 563 | | UCT | Um, Well, then we just did it for each one cause We had the multicolor ones with the yellow on the top, the multicolor ones with the red and the blue… so we just did the same here for each one. |
| 564 | | UCT | We just added the two colors for each. |
| 565 | 01:12:49 | R1 | Questions for this group. [someone sneezes] Bless you! Okay very good. Okay, Another group. [someone sneezes] Bless you! Let’s see the red top.[Video shows 3 groups of 12 towers, 12 towers have red at the top, 12 have yellow at the top and 12 have blue at the top.] |
| 566 | 01:13:02 | CDR | Keep a constant but I don’t usually keep one constant all the way I don’t know through. |
| 567 | | R1 | So which group do you want to talk to and they now have 3 groups of 12 which will also get 36. |
| 568 | | CDR | Which are all broken into smaller groups. |
| 569 | | R1 | Good, so which which tops do you… let’s talk about the yellow tops since they are in the middle. |

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| 570 | | CDR | Alright. Do you want to? [CDR asks CP] |
| 571 | 01:13:21 | CP | Okay, sure. Alright So we did all yellow tops constant on the first row. And then For the first group of 2, we did yellow constant in the middle row. And We alternated the blue and red. And we said we couldn't put another yellow in either one of those positions. |
| 572 | 01:13:37 | R1 | Why? |
| 573 | | CP | Because then we would only have two of the colors, rather than all 3. |
| 574 | | R1 | Okay, makes sense. |
| 575 | | CP | For the next group, we kept 3 rows constant so we had all 3 colors |
| 576 | | R1 | Uh huh. |
| 577 | | CP | yellow blue and red and then we just alternated the last one. |
| 578 | 01:13:54 | R1 | And what were they doing? What argument were they using? |
| 579 | | UCT | Inductive. |
| 580 | | R1 | An inductive again. Cause they said that the fourth addition could either be red, blue, or yellow one of those 3 colors. |
| 581 | | CP | Right. |
| 582 | | R1 | Keep going. |
| 583 | 01:14:09 | CDR | So basically These five right here are all within the same group. But they are broken down into smaller groups. And the reason is we kept the yellow constant. |
| 584 | | R1 | Right. |
| 585 | | CDR | We kept the blue constant. |
| 586 | | R1 | Right. |
| 587 | | CDR | Here though, we have the 3 reds constant. Then we moved and said what if we moved one to the third row and changed it to a yellow constant it would have to have the red. |
| 588 | 01:14:26 | R1 | Good. |
| 589 | | CDR | So we are done there. And once the third one changes to a blue constant, we have to have the red. So we are done there. And then from there We said okay, still keep the yellow constant but this time change it instead of the yellow we did the blue and we are going to do red now and then we changed our third row constant it could be blue with any of the colors it could be red, but we have to have the blue or yellow and we have to have the blue again. |
| 590 | 01:14:49 | R1 | What do you think of it? Pretty neat, huh? Very nice. |
| 591 | | CP | I changed mine up a little bit. |
| 592 | | R1 | Yeah? |

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| 593 | 01:14:57 | CP | From my explanation. |
| 594 | | R1 | Okay. |
| 595 | | CP | Because it was just…I don’t know if it was right. |
| 596 | | R1 | What did you change? |
| 597 | | CP | I switched these 2 groups around. |
| 598 | | R1 | Which two groups? |
| 599 | | CP | These 2 and these 3. |
| 600 | | R1 | Okay. Okay. |
| 601 | | CP | I did these 3 first. |
| 602 | | R1 | Okay |
| 603 | | CP | And what I did was basically just say I alternated |
| 604 | 01:15:14 | R1 | Uh huh Uh huh. |
| 605 | | CP | The constant keeping all 3 rows constant. And for this group, I labeled it as the first row constant, the second row constant, and the third row constant. |
| 606 | | R1 | Okay. |
| 607 | | CP | And alternated out the middle two. |
| 608 | | R1 | Uh huh. Uh huh. |
| 609 | 01:15:27 | CP | You can’t uh… what was I going to say? I just finished writing it down and so it was right on my paper. |
| 610 | | R1 | Okay |
| 611 | 01:15:34 | CP | If you put another piece in there and there is only 2 options instead of 3 because if you put the… where am I?....If you put a red in here.. |
| 612 | 01:15:47 | R1 | Okay. |
| 613 | | CP | We already have that one right here and if you put a blue…you can’t put a blue, it is already there! So basically if you put a red in the third row, it would have already shown up. |
| 614 | | R1 | In that one. |
| 615 | | CP | In that one, yeah. |
| 616 | 01:16:01 | R1 | Good. Very good. And that’s neat. That is To look and to see you know you are thinking of a lot of dimensions. You are thinking of What do you have to have? Have All 3 colors and do you already have it and how can you change it up, it’s really nice. |
| 617 | | CDR | Yeah, Cause at First we ended up with doubles. |
| 618 | | R1 | Okay. |
| 619 | | CDR | Not that many. |
| 620 | | R1 | Okay, alright. |
| 621 | 01:16:20 | CDR | We were building off of what we already have. |
| 622 | | R1 | Uh huh, uh huh and you do have when you try and build off of what you already have you could end up with duplicates that you then have to eliminate. Good, very nice. |

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| | | | Which was another group that did it differently? Was it your group? |
| 623 | 01:16:35 | UCT | Yeah, but our work is not on the…. |
| 624 | | R1 | Okay, well let's get it on the camera. The power of the Ipad! Looks nice. [brief pause to take pictures using the Ipad] Okay, let's see what we got. Now we don't have 3 groups of 12, we don't have 6 groups of 6, we have.. what do we have? |
| 625 | 01:17:20 | UCT | 12 groups of 3. |
| 626 | | R1 | 12 groups of 3. Who else did 12 groups of 3? Anyone? What do you have? |
| 627 | | UCT | We kept changing it around [laughter] |
| 628 | | R1 | [laughter] Okay, so let's talk about 12 groups of 3. |
| 629 | 01:17:36 | UCT | [Video shows: This teacher pair organized their towers in groups of twelve each with three towers. The top row of twelve towers had four groups. The first group had a three-block blue row on top of a three- by three-tall subgroup of towers made using two colors with the elevator strategy. The diagonal of the first three- by three-subgroup was made of red cubes moving down one position at a time with yellow cubes in the other positions.]<br><br>Alright so we started with what we had in dealing with the previous problem and we saw that we had you know two yellow and one red. And we decided to take the blue and put it on the top and then take it and put it in the second position. And then take it and put it into the third position and then in the fourth position. |
| 630 | 01:17:55 | R1 | What do you think of that? That's kind of neat, isn't it? Huh? Now did the red and the yellow you still have two yellows and one red [cough] Okay, in each of them. Isn't that a neat way to do it? Okay and you did the same with the others, didn't you? |
| 631 | | UCT | Then we did the same with two red and one yellow moving the blue down. |
| 632 | 01:18:16 | R1 | Uh huh. |
| 633 | | UCT | And then we realized the only one we don't have 2 of is the blue. |
| 634 | | R1 | Right. |
| 635 | | UCT | So had we just chose 2 blue and 1 yellow. |
| 636 | | R1 | Sure. Sure. |
| 637 | | UCT | We could have did two blue and one red and then we just moved the opposite color down. |
| 638 | 01:18:31 | R1 | Very nice. Isn't that neat? And isn't it interesting how you all thought about it differently but got the same answer. |

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| 639 | | UCT | We couldn't convince or write an argument for it though. |
| 640 | | UCT | Well I didn't even think that about the fact that we were going to have 2 of one color and one of each of the others until I saw this. |
| 641 | | R1 | Uh huh. |
| 642 | | UCT | Which is always true but this actually shows it. |
| 643 | | R1 | Isn't that neat? |
| 644 | 01:18:53 | UCT | For the other tower that we started with, I think maybe for if like you said If we had two blue and one red. We realized that after we built these that if we built these we had the same thing two of the same color. |
| 645 | 01:19:06 | R1 | Now what you just discovered, Romina is a student who discovered it in solving Ankur's challenge. And she did something called Romina's proof. Which talks just about what you are talking about.<br>And you are going to see a video of it. Okay, so what I want you to do is<br>Start rearranging the cubes back into stacks of ten by color. And as you are doing it, multi task, I am going to talk to you about how to implement this with the students alright. because this is going to be your next task. Not this one, alright Ankur's challenge is not what you are starting with. You are going to start with the problem that is building towers 3 tall selecting from 3 colors. Okay? And that may be all that you do. However, Some of you some have some very bright students that might [someone sneezes] bless you [another sneeze] bless you!...that might be able to take the challenge and do Ankur's challenge after they built the other one and provided a convincing argument. If they don't do that, do not go onto Ankur's challenge. Okay, you got that! Now, your job is to find the cubes number one. Okay, how many of you know where the cubes are in your building? Well you do because you just got them. You do because you have them. You know where they are? |
| 646 | | UCT | I know. |
| 647 | | R1 | And<br>How about the rest of you? Um, do you know where they are? |
| 648 | | UCT | I used Christine's. |
| 649 | | R1 | Okay so You may need to do that again. Um<br>And Sayreville is going to find theirs. |
| 650 | | UCT | We have blue and yellow we just have to find the other color. |
| 651 | | R1 | Good. What you are going to do is you are going to get big baggies. Because if you take the time from what we did |

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| | | | today, to give out the cubes, the period will be half over when you have a whole class. We only have 10 people. Okay, So you are going to get big baggies. And in each baggy, you are going to put 10 stacks of yellow, or whatever 3 colors you have, ten stacks of blue, and ten stacks of red. So you will have 100 of each color. You will have 300 pieces in each bag, okay ten stacks of each color. But you can do it because actually some of your schools had it arranged that way. I am not sure whether they are anymore. Um if you can't find really good baggies, then you will have to just put 2 colors together and one separate. And then you know rather than having to dig out of the bag a third color just have 100 cubes of a third color and give it to the students. Okay, you are going to have the students do the problem. You are going to have them convince you. And you are going to have them write. Please have everyone write. Okay. |
| 652 | | UCT | I'm trying! |
| 653 | | R1 | I know and you are doing a good job. It's not easy. |
| 654 | 01:21:59 | UCT | Yeah. |
| 655 | 01:22:00 | R1 | It's something new for you as well as for them. Alright. But I think that they can write something. Even if they can't write everything. Okay, and remember we are not looking for them to tell you what they did. We are looking for them to convince you why they think they have the right answer. Okay [someone sneezes] why there aren't any more towers. Bless you. Questions on implementation? Okay, And there were 2 bags on the... We are going to meet on the 20th of November. Okay, so you need to you have a lot of time; if you um…You want to get it done a few days before we meet. Because you want to bring the student work that you want to talk about and remember the aim is you really want to have student work that impressed you, surprised you or disturbed you. |

R1- Dr. Judy Landis
R4- Phyllis Cipriani
UCT - Unidentified Cohort Teacher
Initials - Identified Cohort Teacher

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|---|---|
| **Description: Transcript of Regional Meeting – Debriefing of classroom implementation**<br>**Advisor: Carolyn Maher**<br>**Location: MAMS, NJ**<br>**Date: November 20, 2013** | **Author: Phyllis J. Cipriani**<br>**Verified by: Simone Grey**<br>**Date Verified: Summer 2015**<br>**Page 1 of 12** |

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|---|---|---|---|
| 1 | 00:00:00 | R1 | [class was 16 sixth-grade students sitting in pairs at student desks] Why don't we talk about you know the class we just did. And Harrison is up there with Michael. |
| 2 | 00:00:05 | LC | Harrison was sitting over there. |
| 3 | 00:00:06 | R1 | the corner. |
| 4 | 00:00:07 | LC | Yeah. |
| 5 | 00:00:08 | R1 | Okay, okay. Um I think…, Did you get a chance to see them, Harrison and Michael?<br>[video shows the student's work described below:<br>Top of the columns student had written blue, red, and yellow. B to represent blue cubes, R to represent red cubes and Y to represent yellow cubes. Under the blue column, 2 rectangles with seven 3-tall tower. In the first rectangle, there were four 3- tall towers. The first all blue-colored cubes. The second YBB The third tower BYB and BBY for the fourth tower.<br>The second rectangle under first rectangle with R as the diagonal through three towers and B for other positions The towers under the red and yellow columns were similar to the towers under the blue column except for the diagonal positions. In the red column, red was dominant; in yellow column; yellow was dominant ]<br><br>Okay, take a minute to look at how they arranged their…look how nice they did their um… recording. And that that's not always easy. I saw two groups having a hard time the group that we are going to talk about here had a very hard time recording. One of them wanted to do letters. Um…<br>The boy on the left facing me like um, on on the right, |
| 6 | 00:00:41 | LC | That's Aidan. |
| 7 | 00:00:42 | R1 | Aidan wanted to just put a letter for the color but his partner is his name? |
| 8 | 00:00:50 | LC | Tommy |
| 9 | 00:00:51 | R1 | Tommy had the wisdom of let's use colored pencils. And When he used colored pencils,<br>It took them forever um to do the recording. But let's take a look. [pause] |

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| 10 | 00:01:03 | UCT | I like randoms. [laughter] |
| 11 | 00:01:04 | R1 | Yeah, isn't that neat? Their vocabulary is really interesting and nice. This group also had another interesting thing I think this was the group that said the blue category, uh, meaning if there were two blues if it was mostly blue that was the blue category. Um, so How did they arrange, how did they form their groups? How many in each group? How many groups? Anyone? |
| 12 | 00:01:39 | UCT | They used a main color. |
| 13 | 00:01:40 | UCT | They used a dominant color. |
| 14 | | R1 | Okay, good. Alright. They used a dominant color. And randoms really meant what? |
| 15 | | unison | All 3 of them. |
| 16 | | R1 | Yeah, okay Alright so each group had how many? |
| 17 | | UCT | 7 towers. |
| 18 | | R1 | Okay And the reason why they had 7 was. ..? |
| 19 | | UCT | They stuck to solid colors. |
| 20 | | R1 | They stuck to solid towers in the predominant color the dominant color. Alright, If they didn't do that, you would have seen four groups of six with 3 solid color towers. Okay, and within the blue group what was their convincing argument? Did they write? Can you read for us? Wow! |
| 21 | 00:02:22 | LC | I think there's 27 towers are all the possible explanations because for each group there is a pattern that goes diagonal through each tower. In that pattern is one single opposite block. We have the towers below each category as red blue and yellow. The reason we have categories is because there are two main blocks in each tower. |
| 22 | 00:02:40 | R1 | Okay, we can.. we understand what he is saying he is telling you what is he is saying? Two main blocks… |
| 23 | | UCT | Two main blocks in that color. |
| 24 | | R1 | Okay, Keep going. |
| 25 | 00:02:51 | UCT | These Two blocks are the same, and the blocks that made the pattern are different. So in the blue category, there will be one yellow block on a stack of two blue blocks. This helps prove our theory. We also have a random category where each tower contains one different block. So, one of our towers is red on the bottom, blue in the middle, and yellow on top. If we changed the order of the blocks, it would have a different tower we already have. |
| 26 | 00:03:18 | R1 | Okay, so are you..which parts of their argument are convincing and which aren't? |
| 27 | | MC | It is not really convincing anywhere. |
| 28 | | R1 | Okay. Cause… |
| 29 | | UCT | He gave us one example and then his partner said to.. then |

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| | | | he switched it up. |
| 30 | 00:03:36 | R1 | Okay. Absolutely. Okay, how about are you I mean we know what he mean s when he is talking about the other groups with the blue category, red category. But Are you convinced there? What is his argument there? |
| 31 | | LC | He said he has a pattern that goes diagonal through each tower. |
| 32 | 00:03:55 | R1 | Right. |
| 33 | | LC | So, like he went… |
| 34 | | R1 | Okay, So he is telling you and verbally they actually said you have the other color in each of the positions. |
| 35 | | UCT | Right |
| 36 | 00:04:00 | R1 | Only 3positions…They verbalized all that. It would have been nice if they put some of that in their writing. But they are using a diagonal strategy and it is very very convincing so that is really nice. Good. |
| 37 | 00:04:30 | LC | [student work shown on video: B=blue; R=red; Y=yellow; 5 groups: argument on screen: Group 2: There are no more combinations for the solids because we already used all of the colors.<br>Group 1: We used each block once in each tower and we also used one colored block twice for the top. So for example: BB<br>YR<br>RY<br>That's how we know there are no more possible ways.<br>Group 3: We used one color twice in the middle row. Example: YB<br>RR<br>BY<br>Group 4: We used one color twice and put it on the top. For example: RR<br>RR<br>YB ]<br>This is Carlos and… |
| 38 | | R1 | Okay, Carlos was sitting where? |
| 39 | | UCT | Over in that corner. |
| 40 | | R1 | Yeah. Okay. Alright so take a look at Carlos's work and see how they arranged it and what strategy they used. [long pause] Can you distinguish between the B's and the R's? Sometimes that's hard. [pause] |
| 41 | 00:04:59 | UCT | Well he did the same as the last group but this time he kept the blues, the yellows, and reds of the solid towers by themselves. So that is why there is 6, four groups of 6. |
| 42 | | R1 | Okay, and in that…look at group one, how did he arrange |

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| | | | group one? [unintelligible response or question] Yeah. |
| 43 | | UCT | Tops are the same.[multiple responses] Well like he had a pair of blues on top, and a red and yellow on the bottom. |
| 44 | | UCT | Yeah. It looks like they were reversed on the bottom. |
| 45 | | R1 | Uh huh, okay and I think that is actually what I remember them hearing that is what they did. So even though they grouped them they didn't use the diagonal. Um, They kept the two with the same top and they said that the the bottom could be Y and R or R and Y. Okay and<br>There was no other way to do it. And then they kept the two tops red, and did the same thing with the other two colors. Then the two tops yellow and<br>bottoms are the other two colors in both positions.<br>So even though it looks the same, they should have a different argument, shouldn't they? Okay, let's see what their argument is. [pause] Can you read it for us?<br>Oh group 2. So Which was group 2? |
| 46 | 00:06:09 | UCT | Well I think group one, was the solids. |
| 47 | | R1 | So let's do group one. Yep it is. |
| 48 | | UCT | We used each block once in each tower and we used |
| 49 | | UCT | The solid towers. |
| 50 | | R1 | Perfect we are just going over student solutions. |
| 51 | | UCT | We used each block once in each tower and we also used one colored block twice for the top. |
| 52 | 00:06:26 | R1 | Okay so that is confusing but they are giving us an example to help us understand.<br>So, what is the example? |
| 53 | 00:06:39 | UCT | They kept the top block the same and |
| 54 | | UCT | Okay. |
| 55 | | UCT | And switched the bottom two |
| 56 | | UCT | And switched the bottom |
| 57 | | R1 | Good And that is the strategy they used. Um, what..and group 3, what are they saying? |
| 58 | 00:06:45 | UCT | Use one color twice in the middle. So I guess twice would mean twice for both two towers. |
| 59 | | R1 | That is exactly right. They are showing you which is really good so that we don't have to guess what they mean. Okay, and Then what? So what did they do? They have the red in both middle positions and what did they do with the top and the bottom? |
| 60 | 00:07:08 | UCT | Switched them. |
| 61 | 00:07:09 | UCT | Switch the yellow and the blue. |
| 62 | | R1 | Good. They have the other two colors and they are flipping them again. So they are using..it is very clever, isn't it? They are using The same strategy, they are |

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| | | | keeping one color constant and then they are using the other 2 colors in the other 2 remaining positions.It's Neat, isn't it? Did any of your students do that? Nobody? |
| 63 | | UCT | Use That strategy? |
| 64 | | R1 | Did anyone use that strategy? It is a neat strategy. |
| 65 | 00:07:39 | UCT | I kind of think my kids that it is the opposite. |
| 66 | | R1 | The opposite? |
| 67 | | UCT | Like he kept two red on top and two red down the bottom and he called it like a sandwich. |
| 68 | | R1 | Absolutely. |
| 69 | | UCT | And then he switched the yellow and the blue and called that the meat of the sandwich. |
| 70 | | R1 | That's neat, that's very neat and I heard sandwich before and it is a sandwich and absolutely, that is a good strategy too. Okay, Anything else. And they are showing you group 4. What did they do there? |
| 71 | 00:08:03 | UCT | We used one color twice and put on the top. |
| 72 | | R1 | Okay. |
| 73 | | UCT | And this one they actually did twice in one tower. |
| 74 | | R1 | Uh huh. Uh huh Language is hard but what did you think of their writing? |
| 75 | 00:08:19 | P | You can be honest. |
| 76 | | R1 | The principal is here but just ignore him! Okay. Were you impressed with how they wrote? |
| 77 | | UCT | Yeah I think they did a lot better verbally than they wrote it out. |
| 78 | | R1 | I think their writing is pretty darn good. I know some struggled more than others. But I say their writing... Remember, this is sixth grade. They are little. They are not seventh and $8^{th}$ graders, they are little and they did really, I say their writing for $6^{th}$ grade was pretty good. Okay, let's see another paper. |
| 79 | 00:08:52 | UCT | The last one. |
| 80 | | R1 | [Student's written argument on screen: I know I got all of them because all the groups I got the same amount of 3 towers.] Okay, Perfect, oh these were my friends! Right? |
| 81 | | UCT | Yeah. |
| 82 | | R1 | Okay, These were the two boys that were over here. And Laurie said, they happened to not be strong students. And who was here I know they kind of and some of you were here watching them.and they kept struggling. Was it you? |
| 83 | 00:09:12 | CDR | Yeah. I watched them for a while in the beginning. |
| 84 | | R1 | Talk to us. Tell us what you saw. |
| 85 | | CDR | Well towards the end when I had gone over there later I had watched them for a while in the beginning and they |

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| | | | were just kind of going through them randomly and they started putting them in groups but by the end; the boy on the left, Tommy |
| 86 | 00:09:28 | R1 | Yeah. |
| 87 | | CDR | He had started saying okay let's do, they organized them so that there were all the yellow on the bottom or all red on the bottom or all blue on the bottom. He said<br>Let's do all the ones with just the one color on the bottom and a different group with 2 colors on the bottom and then the different color on the top. |
| 88 | 00:09:42 | R1 | Right. |
| 89 | | CDR | Okay, he was telling Aidan. Let's do all the other ones like that But then when Tommy went to do his own.. |
| 90 | | R1 | Right. |
| 91 | | CDR | He did it differently. |
| 92 | | R1 | Yes. |
| 93 | | CDR | He did all the ones with um red on the bottom and he didn't care if it had two colors or one color. |
| 94 | 00:10:00 | R1 | Right. |
| 95 | | CDR | And so his argument was different than Aidan's because Aidan was telling him what he had done the first time and They were really getting confused with that. |
| 96 | | R1 | Yeah and also. |
| 97 | | CDR | And then they kept making doubles again. |
| 98 | | R1 | Yeah. Yeah. |
| 99 | | CDR | Then he said oh well it could have been red on top not realizing that he had them in another group. |
| 100 | | R1 | Exactly. And that… |
| 101 | | LC | And you started to help them with that. And like go though it. |
| 102 | 00:10:21 | R1 | Right |
| 103 | | LC | Aidan started doing what Tommy had. |
| 104 | | R1 | right |
| 105 | | LC | And Tommy was confused by it. |
| 106 | | R1 | Right. |
| 107 | | LC | But that is what you had written down![laughter] How is it confusing to you? |
| 108 | | R1 | Well, I think also. Remember how hard it is. Think about you, not your students, think about you when you were doing this problem with a partner. And you r partner was thinking about it differently than you. [laughter] right. You are laughing because remember it wasn't so easy. |
| 109 | 00:10:46 | UCT | We were doing it in a different way. |
| 110 | | R1 | Uh huh, and to get into the mindset of doing it a different |

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| | | | way than the way you are thinking is hard. It really is hard and you have to let go of your own way of thinking about it and try and understand what your partner is doing. They had a hard time with that. But I was thrilled that they were able to at the end get it to a point where they agreed they were going to use one strategy which was to keep a solid color. |
| 111 | 00:11:16 | UCT | They kept a solid color on the bottom and then the top. They changed the colors in the middle. |
| 112 | | R1 | Yes and that was again another really neat thing and they came up with this. And I thought it was clever. And I wish they didn't used colors because you really can't tell. Can you see? Oh you don't even have the paper do you?. |
| 113 | | LC | Oh yea it's over here. |
| 114 | | R1 | Okay, If this were the problem, they took one color, the same color, and put it on the top. Then they had a sandwich and then they had a different color on the bottom. |
| 115 | 00:11:47 | UCT | There you go. |
| 116 | | R1 | Oh this is better okay because you can't see it up here. So they used In group one, they used it was the blue bottom group and the first one would be three red on top three blues on the bottom and they used an inductive argument. So what do you think did they put in the middle? They are using an inductive argument. What did they put in the middle? |
| 117 | 00:12:11 | UCT | The three different colors that they had. |
| 118 | | R1 | Exactly. They put each of the colors they could choose from so they put a red, yellow, and blue in the middle. Isn't that a neat strategy? I think it's neat. I don't think you look impressed but I am impressed. Okay,and then they did the same thing all along. Okay, this is again, this is all the bottoms are the blue, The bottoms are always blue. And They even carried it a step further. The bottoms are always blue Again, they are using an inductive argument. So What could the top be? What colors could the top be? |
| 119 | 00:12:49 | UCT | Red yellow or blue. |
| 120 | 00:12:51 | R1 | Red, yellow, or blue. So they are using an inductive argument even for having chose the tops. And then the middle could be… |
| 121 | | UCT | Red, yellow or blue |
| 122 | | R1 | Red yellow or blue And that's what the middles are and that is how they found all nine towers in the blue bottom group. Isn't that brilliant? These are from the weaker students in the class? I think it is brilliant, I really do; I |

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| | | | think that it is very very systematic and they carried it through in groups 2 and 3. Now what was hard for them was writing. In fact, one of the boys really, really had a hard time writing. Who was the one who went on to do something else? one went to the bathroom and the other one had trouble writing. |
| 123 | 00:13:33 | LC | It was probably Tommy. |
| 124 | | R1 | Okay, so…And what did he write? |
| 125 | | LC | I know I got all of them because all of the groups I got the same amount exactly. |
| 126 | | R1 | Okay, Now that was interesting too. They had at first when they finally said okay we are going to use a solid bottom. Okay so they said Okay I am using a solid bottom. In the blue bottom group I have 11, In the yellow bottom group I have 9. And In the red bottom group I have ten. And I can't remember who it was, I think Tommy said it didn't bother him he was fine with it and Aidan said it bothered him so they kept looking and then they found duplicates. So, they got rid of the 11 group and made it a 9. They got rid of the ten group and made it 9. And that is what he is saying now. He thinks he is good because all the groups have the same number of towers. So Is it convincing? |
| 127 | 00:14:32 | UCT | No |
| 128 | 00:14:33 | R1 | No, not really, okay. But is it a good thing for him to notice. Absolutely. Could we…Do we know…you just have the paper up of his partner. I think Aidan wrote more than Tommy. Um But this is really neat like it is not often do students use an inductive argument. It's a harder argument to see. So that They are using it to get their groups or their towers. I think is really brilliant. |
| 129 | 00:15:00 | UCT | Nah, He has the same thing. |
| 130 | | R1 | He has the same thing. Okay so they had trouble writing. While they were doing it though…They used a very …they could have had a very convincing argument. Cause what they did was brilliant. Okay, Is this all the ones from today? |
| 131 | 00:15:15 | LC | Yeah |
| 132 | | R1 | Okay, Was there any one else you wanted to talk about from today that we didn't talk about? |
| 133 | | CDR | The extension problem. |
| 134 | | R1 | Yeah. |
| 135 | | CDR | The one group in the center, Kyle and Ronnie where the food is. The students that where here. |
| 136 | | R1 | Right here oh, they were there; where the food is. Okay. |

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| 137 | | CDR | They went through the extension problem without much help. |
| 138 | 00:15:36 | R1 | Really. |
| 139 | | CDR | They went through and they first started to that, they realized there were going to be two of the same color. |
| 140 | | R1 | Okay. |
| 141 | | CDR | And they started to put them together in the top two or the bottom two and put them in the middle. |
| 142 | | R1 | Okay |
| 143 | 00:15:48 | CDR | And they alternated the other colors. So they would do like whatever the other two colors were |
| 144 | | R1 | Sure |
| 145 | | CDR | and they switched them around. |
| 146 | | R1 | Okay. |
| 147 | | CDR | And then they went to putting them on the top and the bottom. |
| 148 | | R1 | Okay. |
| 149 | | CDR | In the same color And then they..It was interesting to watch because Ronnie had all that done and Kyle had started to work on the other ones where none of the two colors were next to each other. |
| 150 | 00:16:10 | R1 | Okay |
| 151 | | CDR | But he was thinking about it totally differently instead of focusing on two colors that are not the same but not next to each other, he was focusing on just having a red on the bottom and all the other colors yellow and blue but alternating. |
| 152 | | R1 | Ohhhh! |
| 153 | | CDR | So he was missing some at first. |
| 154 | | R1 | Okay |
| 155 | | CDR | He was getting some duplicates and |
| 156 | | R1 | right |
| 157 | 00:16:27 | CDR | And then when I asked him He said he was thinking about it differently that Ronny was. And I said well that's okay, Well let's think about how can we maybe build off that. And eventually he realized what he had to do. |
| 158 | | R1 | Okay |
| 159 | | CDR | As soon as he clicked He just went right through and he ended up with all 36. |
| 160 | | R1 | That's amazing. That really is. |
| 161 | | CDR | And they did it pretty quickly too. |
| 162 | | R1 | Did they have groups? |
| 163 | | CDR | They had em in the groups. |
| 164 | | P | They had a great teacher. [laughter] |

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| 165 | | R1 | Yeah! Were they groups of 12? |
| 166 | | CDR | No, they had them …Originally they had them in groups of 4. |
| 167 | | R1 | Okay |
| 168 | | CDR | Two red on the bottom, or two red on the top. Two of 'em two red on the bottom, two of them. |
| 169 | | R1 | Okay |
| 170 | | CDR | And then as they were explaining to me, they realized oh well I can take the two that are together in the middle and move those over there so they had a group of 6. |
| 171 | | R1 | Okay. |
| 172 | | CDR | So you have 6. |
| 173 | | R1 | 6 versus 6. Did they have 6 versus 6? |
| 174 | | CDR | Yeah. Yeah. |
| 175 | 00:17:14 | R1 | Yeah, very neat;yeah.. nice. And that they did it quickly is pretty impressive. I can remember that you know that problem can cause trouble very easily. Right I think some of you struggled with some of it right? When we met because it is not easy<br>It is not a trivial problem at all. It is quite a complicated problem and there is no numerical easy trick to find the answer like when you are doing the towers 3 tall selecting from 3 colors. And you get 27. Why is that? What mathematical…. |
| 176 | 00:17:47 | UCT | Oh, Exponents. |
| 177 | 00:17:14 | R1 | It's three to the third power, right? Three colors, three high. But when you are doing it where you put in that tricky sentence where you want each tower to have at least one of each color; it changes the idea. You can't get an easy way to get the answer. Unless you are Romina and then you come up with something, right! Good. Okay, Let's take a look at some of your students' work. We have that too! Laurie, thank you so much. This makes it so much easier isn't to look at the work when it is already up. Okay. |
| 178 | 18:34 | P | Does that help?[lights get shut off]. |

| Description: Transcript of Regional Meeting, Students' Samples<br>Advisor: Professor Carolyn Maher<br>Location: MAMS, NJ<br>Date: November 20, 2013, part 2 | Author: Phyllis J. Cipriani<br>Verified by: Simone Grey<br>Date Verified: Summer 2015<br>Page 1 of 24 |
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| 1 | 00:00:00 | R1 | Good. Okay, Let's take a look at some of your students' work. We have that too! L-----, thank you so much. This makes it so much easier isn't to look at the work when it is already up. Okay. [Video of clock shows 2:00 pm exactly] Does that help?[lights get shut off]. I am just afraid that some people may fall asleep. Okay, whose work is this? |
| 2 | 00:00:09 | T10 | That is Mine. |
| 3 | 00:0010 | R1 | Good. So why don't we let you talk about it. |
| 4 | | T10 | I can't see at all right now. |
| 5 | | T10 | They created 7 different groups. |
| 6 | | R1 | Okay |
| 7 | | T10 | So actually 8 groups. Sorry. |
| 8 | | R1 | Okay, Give us a minute to look at the 8 groups. Does it go up and down? |
| 9 | | T10 | No. |
| 10 | | R1 | Let's see, don't you like their code? |
| 11 | | T10 | Yeah, that's actually what I thought was the neatest part about what they did. I like the code they used. Like, in all the individual groups they created, You can see the pattern. So I thought that was nice. |
| 12 | | R1 | Okay. And you can. What patterns do you people see? Talk about a group and what pattern are you picking up. |
| 13 | | UCT | Group one is where they are moving the yellow down in each original. |
| 14 | | R1 | Good, okay. Yep. |
| 15 | | UCT | I think the groups are labeled below it, which is a little bit confusing. |
| 16 | | UCT | Blue and two are yellow. |
| 17 | | UCT | One is labeled group one and one is labeled group 2. |
| 18 | | R1 | Okay, Okay, Okay, okay, if that is what they are doing. That's good. What other things do you people see? |
| 19 | | UCT | Go ahead. |
| 20 | | UCT | They did the same thing across group 4. They just switched the dominant color. |
| 21 | | R1 | Okay. |
| 22 | | UCT | The only one that was different for me was when they did all 3 colors. One was blue and one was red. |

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| 23 | | R1 | Good. |
| 24 | | UCT | They did that for two of them. |
| 25 | | UCT | They seem to keep the.. |
| 26 | | UCT | The only one that was different was 8 but that was where they had all 3 colors. |
| 27 | | R1 | Okay |
| 28 | | UCT | I think they kept the top color the same and reversed what the bottom colors were. |
| 29 | | R1 | Okay, nice. Very nice and what was their argument? |
| 30 | | T10 | We know there are no more towers in each group because if you added another there would be a duplicate. For example, three towers each with the same colors and then one more. There is not much on top there. |
| 31 | | R1 | So what do you think of the argument? |
| 32 | | UCT | Not good. |
| 33 | | R1 | It is not, [laughter] because what they did was they are saying basically you can't find any more because you will get a duplicate. That's not a good argument. But interesting code. |
| 34 | 00:02:34 | UCT | Yeah I thought their grouping was good |
| 35 | | R1 | Absolutely. And their strategy is good too. That group 8 to keep a constant on the top. Yeah Is it better? |
| 36 | | R4 | for the camera. |
| 37 | | R1 | Okay, alright okay. Can you all see with the light on? |
| 38 | | LC | So this is another one of my students but it is light and hard to read. |
| 39 | | R1 | It is. Maybe you can read to us. What was their argument? |
| 40 | | LC | He had 5 different groups. He had one group that was all the solid towers. |
| 41 | | R1 | Okay. |
| 42 | | LC | And then their other group had the same color top and bottom so their group twos have blue red blue and blue yellow blue |
| 43 | | R1 | Okay. |
| 44 | | LC | The second ones in that group had yellow top and bottom |
| 45 | | R1 | Okay. Neat. |
| 46 | | LC | The third group over here. Um, they have one color on top and then two of the same color. |
| 47 | | R1 | Okay. |
| 48 | 00:03:34 | LC | Above it. And The fourth group they had two of the same color on top and the opposite color below. Or a different color below. Here they have all 3 colors so they have the same color on top and they switched the bottom . |
| 49 | | R1 | Okay |

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| 50 | | LC | They said there are 27 towers of three cubes. We know this because each color has 2 combinations except for ones that are all the same color. |
| 51 | | R1 | Say that one more time. We have… |
| 52 | | LC | Each color has 2 combinations except for ones that are all the same color. |
| 53 | | R1 | Okay Okay. |
| 54 | | LC | Because I think they are talking about These two combinations, these two and these two combinations are together. |
| 55 | | R1 | Okay, okay. Yes. |
| 56 | 00:04:12 | LC | In group one I know there are no other combinations because other colors would just be a duplicate.<br>In group two, there are no other combinations because if we switched other blocks around it. It would be a duplicate. The top and bottom block are the same color so the middle block has only two options. |
| 57 | 00:04:27 | R1 | Now that is very good. That one sentience is very good. Okay, Read that again. |
| 58 | | LC | So for this one. |
| 59 | | R1 | Yeah. |
| 60 | | LC | The top and the bottom block are the same color so the middle block only has two options. |
| 61 | | R1 | That's very nice. Good. |
| 62 | | LC | In group 3 the bottom blocks are the same color |
| 63 | | R1 | Okay, the bottom blocks are the same color. |
| 64 | | LC | I think they were talking about it. |
| 65 | | R1 | Okay. |
| 66 | | LC | So there are only 3 choices for the top blocks. I don't know… |
| 67 | | R1 | Okay, It is getting harder to read what it is but…They are starting to zero in on a good argument. That's good. |
| 68 | 00:05:09 | LC | [inaudible] For groups 4 and 5 they don't have any combinations for the same reason as before; But 4 has the top and middle block the same and they only have 3 choices for the bottom block but really it doesn't apply there. |
| 69 | | R1 | Okay. |
| 70 | | LC | And 5 has all different colors so all the combinations; they had something. |
| 71 | 00:05:28 | R1 | They did. They have the start of a good convincing argument. That's good. Better than saying we you know if we look for more there will be duplicates. That's not convincing. But what they are doing is the start of a convincing argument. That's good. Okay. And who |

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| | | | belongs to this? |
| 72 | | NL | Oh that’s mine. |
| 73 | | R1 | However you want to do it. |
| 74 | 00:05:58 | NL | Um..You want me to start explaining it? |
| 75 | | R1 | Yeah, yeah. Tell us what your students did. |
| 76 | | NL | Okay, she grouped them similar to Laurie’s first student. |
| 77 | | R1 | Okay |
| 78 | | NL | They did ten groups but they did 7 groups of 3 |
| 79 | | R1 | Okay. |
| 80 | | NL | And then instead of keeping that last group of 6, they did it in pairs so they had 8,9, and 10 are the alternating as pairs. |
| 81 | | R1 | So they were doing all 3 colors. |
| 82 | | NL | Yeah. So they had Yellow as a constant and then they did the bottom. |
| 83 | 00:6:30 | R1 | Good, good. And when they did the first groups when there were three in the group. What was their strategy? |
| 84 | | NL | Well this is how they kind of solved the Towers of a single color. So they went to that right away. |
| 85 | | R1 | Now that’s very good and that is very nice. Again refresh me you are special ed? |
| 86 | | NL | MMhh |
| 87 | | R1 | So this is pretty neat, isn’t it? Pretty neat work from special ed. |
| 88 | | NL | Yeah they are seventh grade. |
| 89 | 00:06:59 | R1 | Very, very Good. Good. Did they write anything? |
| 90 | | NL | Yeah, she actually did. |
| 91 | | R1 | Oh good. |
| 92 | | NL | She said I know the answer is 27. I know there is no more possible ways because in group 2, I moved the blue cube in each position in every way I could. There was only three positions because it could only be three high. I did the rest for groups 3,4,5,6,and 7. So she just said she did this strategy… |
| 93 | | R1 | Did she do this by herself? |
| 94 | | NL | Yes. |
| 95 | | R1 | Cause this is pretty impressive, isn’t it? Right? |
| 96 | | NL | Yeah. |
| 97 | | R1 | It is good writing. |
| 98 | | NL | Yeah she had definitely wanted to improve her explanation from the beginning. |
| 99 | 00:07:32 | R1 | Good, Okay. |
| 100 | | NL | Then she wrote in group one I made three different towers solid colors yellow blue and red because I only had 3 colors. |

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| 101 | | R1 | Okay |
| 102 | | NL | For 8, 9, and 10, I kept one color on top and switched around the two underneath them. |
| 103 | | R1 | For 8, 9, and 10 |
| 104 | | NL | So that is the alternating ones. |
| 105 | | R1 | Yep! Okay |
| 106 | | NL | So she said for each of them, two like red on top and then switched the bottom. |
| 107 | | R1 | Good, good. So she is really explaining what she did and keep going or was that it? |
| 108 | | NL | That was it. |
| 109 | | R1 | Good. What do you think of that? Special ed.! |
| 110 | | UCT | Impressive! |
| 111 | 00:08:03 | R1 | Very impressive, isn’t it? Very, very impressive, nice and Even her writing.. |
| 112 | | NL | She was a hard-working student so she was determined. |
| 113 | | R1 | She did a real good job. |
| 114 | | NL | She did. |
| 115 | | R1 | Okay, what’s next? |
| 116 | | NL | And then I accidentally chose her partner. I didn’t even realize I did. |
| 117 | | R1 | Okay |
| 118 | | NL | But again she used a bunch of colors. She liked to use colors and highlighters so I let her go for it. |
| 119 | | R1 | Okay. |
| 120 | | UCT | But so the solid colors are the reds. |
| 121 | | R1 | Yep |
| 122 | | NL | And the Everything else is blue and yellow. |
| 123 | | R1 | Yep. |
| 124 | | NL | So She did the same organization as Jenna. |
| 125 | | R1 | Okay |
| 126 | 00:08:34 | UCT | In group 6, she messed up. She made a blue and it was supposed to be red but she didn’t fix it. That one. |
| 127 | | R1 | Okay. |
| 128 | | UCT | She usually doesn’t write much at all… she actually…they wrote their explanation separate. This is what… |
| 129 | | R1 | And this.. It’s kind of color here is very helpful to see patterns and to see what she did in terms of strategy. |
| 130 | | NL | Yeah |
| 131 | | R1 | The color actually stands out. I don’t know how it does when it’s on paper. |
| 132 | | NL | Because she like made it nice and big, I said Yeah, I liked it. It was just very appealing. [laughter] |
| 133 | 00:09:06 | R1 | Good. Okay and what did she write? |

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| 134 | | NL | She wrote: The answer is 27. There is no other way without duplicates. I know there is no other way because In group 2, I moved the blue cubes in each possible way I can. There is only 3 positions because it can only be three high. So, I guess like maybe Jenna helped here a little bit with this. |
| 135 | | R1 | Okay. |
| 136 | | NL | I, I… The rest of the groups. I guess she said I did the rest of the groups like this. I only had 3 solid colors to choose from and that is how I got my answer. So hers is a little less detail than Jenna's. |
| 137 | 00:09:35 | R1 | Yeah, but Still that is pretty good, isn't it? Again not complete but really good pieces of a convincing argument . Very nice work. Good. Okay, who belongs to.. |
| 138 | 00:09:51 | MC | This is mine. |
| 139 | 00:09:53 | R1 | Okay. |
| 140 | | MC | [Video shows student's argument projected on the screen: We can't make any more than 27 because the combinations were all used the same way. Each group made the same shape combination. But the bottom block will always have a different color per group but the shapes also will be the same but different color. The combination is the same, but in different colors. The bottom block is the leader. When changed, it will never change its combinations. But it will change its colors.]<br>This is my student. He did three groups of 9 and I was really surprised actually when I went over to him. |
| 141 | | R1 | MMMhhh. |
| 142 | | T3 | He is Probably one of my lower level students. |
| 143 | | R1 | Okay. |
| 144 | | MC | And he arranged it in three groups of nine by keeping the bottom cube a constant. |
| 145 | | R1 | Yeah I can see that. Isn't that neat? |
| 146 | | MC | Which is what he did because he put one single cube in front and he used colors too to show which is constant. |
| 147 | | R1 | Nice. |
| 148 | | MC | But then he wrote his explanation real quick because he was the one who was talking about the sandwich. |
| 149 | | R1 | Yep. |
| 150 | | MC | But then I got I came up with 27 in all. One group and then the other and then the other. which is basically 9 times 3. |
| 151 | | R1 | Okay |
| 152 | 00:10:35 | MC | So this is when he showed I mean he talked about like the different shapes. So if you look at the first example and it |

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| | | | is right there on the bottom of the four towers. |
| 153 | | R1 | Okay |
| 154 | | MC | And made the shape of almost like a turned L which are both yellow and a turned blue. |
| 155 | | R1 | Now I am confused with that. |
| 156 | | MC | She is using the cross on the screen to show you, Yeah, right there. |
| 157 | | R1 | Ohhhhh! I see |
| 158 | | MC | Yeah like those are all yellow and those are all blue and He Kept the bottom red and he didn't know what to call that. |
| 159 | | R1 | Very interesting, he is seeing patterns, huh? |
| 160 | | MC | So then in the next group of towers, he kept the bottom two red and finished the rest with yellow and blue and He calls them checkers. |
| 161 | | R1 | Neat. |
| 162 | | MC | In the last two which he kept the top red and the bottom red and the yellow and blue in the middle and He called it the sandwich meat. |
| 163 | | R1 | Isn't that nice? I.. |
| 164 | | MC | But the bottom block will always have a different color per group but the shapes also will be the same but different color. The combination is the same, but in different colors. The bottom block never changes the bottom block will be the leader. |
| 165 | 00:11:38 | R1 | Isn't that nice language? And this is also special ed? Right? |
| 166 | | MC | Yeah |
| 167 | | R1 | And what grade? |
| 168 | | MC | Seventh grade resource. |
| 169 | | R1 | 7$^{th}$ resource so you are talking pullout and this is wonderful work. Very nice. I really think it's neat the way they are developing language, checkers, and sandwiches, and that is kind of nice. Good. Okay. |
| 170 | | MC | And then my next one. She actually was…pretty fast with this. |
| 171 | 00: | R1 | Okay |
| 172 | | MC | So she arranged them similar to the group of 6. She had a diagonal one and the chained them. |
| 173 | | R1 | Yep. |
| 174 | | MC | So for that she wrote: Each one has two of one color and one of one color and then moved the single one down each time. We did opposites for a total of 18 altogether. So she just did the opposites. This is the only way to have all 3 colors used once in a 3 stack high and only have blue on the top. This is the only way to have all 3 colors used once in a 3 stack high and only have yellow on the top. This is |

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| | | | the only way to have all 3 colors used once in a 3 stack high and only have red on the top. Each one has only one solid color. |
| 175 | | R1 | Mmhh. |
| 176 | | MC | Then the next..she made 3 groups of 2; and for each argument she wrote: This is the only way to have all 3 colors used once in a 3 stack high and only have blue on the top. |
| 177 | | R1 | Okay. So that is down at the bottom. |
| 178 | | MC | It is the second middle one. Where the ones are crossed out. |
| 179 | | R1 | Okay, okay. |
| 180 | 00:12:44 | MC | [inaudible] You can't see them here. She kept the top two blue |
| 181 | | R1 | Yep. |
| 182 | | MC | Then she changed the red then yellow. |
| 183 | | R1 | Okay. |
| 184 | | MC | And the next one she did the same thing. She kept yellow at the top and switched. |
| 185 | | R1 | Okay. |
| 186 | | MC | Then the third group she had the same with red on top and switched the bottom two. |
| 187 | | R1 | Okay. |
| 188 | | MC | And then the last group she wrote each one has only one solid color. |
| 189 | 00:13:03 | R1 | Okay, So it's nice so they are really doing some good stuff. Very good. Again when You have students that are struggling in the very beginning when you see progress you should feel very proud. Good. Okay. |
| 190 | 13:21 | UCT | [video shows student work: four groups 1 crossed out. Group 1: ggb; gbr, bgr Group 2 rrr, ggg, bbb Group 3: rbg,rrr,rgg and Group 4: rgr,grb, bgr]This one was mine, the student I had did not get the answer but.. |
| 191 | | R1 | It doesn't matter, we are looking process. |
| 192 | | UCT | The ones that did get the answer, I couldn't do any more. |
| 193 | | R1 | Okay. |
| 194 | | UCT | He actually recorded wrong. But I chose and I remember him getting it He was doing the recursive argument. |
| 195 | | R1 | Okay, nice |
| 196 | 13:41 | UCT | He was saying group one the blue goes down, Which it does go down.[inaudible] |
| 197 | | R1 | Okay |
| 198 | | UCT | Group 2 has all the reds going down and again he is not following what he put; what he actually had. |

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| 199 | | R1 | Okay. |
| 200 | | TD | Group 3 I made all the greens go down. Group 4 they were all in the same category. Green was with the..green was with the blue..was with the red was with the blue. I think he means the candy cane.[fifth group crossed out] |
| 201 | 00:14:11 | R1 | Yeah one of each color. |
| 202 | | UCT | So again, he didn't copy it right and neither did his partner. But then he said |
| 203 | | R1 | What he had on his paper on the desk |
| 204 | | MC | What he had on his desk; he did not record it right. He didn't explain it. |
| 205 | | R1 | Yeah, yeah. No that is hard to do so. Yeah, but that is hard to do. But what he did write is nicely done. Is this also special ed.? |
| 206 | 14:33 | UCT | Yeah, resource pull out. |
| 207 | | R1 | Yeah, and what grade? |
| 208 | | UCT | Seventh. But as a whole I think they did a lot better on this one. |
| 209 | | R1 | Good |
| 210 | | UCT | They knew they needed a strategy. They knew they had to have a pattern. |
| 211 | | R1 | Good. |
| 212 | | UCT | They all tried their best. |
| 213 | | R1 | Very good. Okay That's great. |
| 214 | 14:52 | TD | [video shows student work and following written argument: Explanation: We got the amount of 23 by taking certain groups of 2, 3, or 4 based on the pattern. Ex. 1:GGG,BBB,YYY. Example 2: YBB>BYY Ex. 3: GYB,BGY,GBY,YGB and example 4: GGY>YGG]And then She, she got 23. But all she did missed the four so then she had to take it off and take opposites of four. But everything else is…If you look down at this. We got the amount 23 by taking certain groups of two, three or four based on the pattern; so if you look at example one, you know it is just green green green; blue blue blue yellow yellow yellow. That she is counting that as three. |
| 215 | 00:15:30 | R1 | Okay. |
| 216 | | TD | And then um… example 2 No I'm sorry, example 2 is yellow blue blue; blue yellow yellow and she shows that I guess she pairs them together. She did the opposite and those two would be paired together. |
| 217 | | R1 | Okay, |
| 218 | | TD | And then 3 I just read that example one and three are put together. and if you look at example 3 she had those four paired together. So she paired them in groups of 2, 3, and |

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| | | | 4. |
| 219 | 15:46 | R1 | Okay, interesting. |
| 220 | | TD | Yeah |
| 221 | | R1 | Interesting. Okay. |
| 222 | | TD | But she didn't write it like that again. |
| 223 | | R1 | Okay, It is interesting that she didn't have groups that were the same size. |
| 224 | | TD | Yeah, that was different that most of them. |
| 225 | | R1 | Uh huh. Okay, good. And who is this? |
| 226 | 16:14 | JLB | [video shows student work : three groups of nine Blue on top yellow on top red on top] Yes. I chose this one because she actually had to do it on her own. She had been absent a lot. She is on a special schedule. She comes in to, she is in 8th grader but she comes in to my 6-7 class. She did this all on her own. And She messed up on the bottom with the last row. But the way she had it on her desk she had the red on top and was she started combining and her recording doesn't make as much sense as she goes down the row. |
| 227 | 00:16:48 | R1 | Okay so she used a constant on the top. |
| 228 | | JLB | Yeah, And then she was rushing; she had to get to gym. so that's why she didn't record them very well. |
| 229 | | R1 | Okay. |
| 230 | | JLB | Then she said How I did it was that I made it just different ways with red, blue and yellow. Just with these 3 different colors, then When I didn't see I didn't form the tower a different way. I made it.<br>There wasn't much of a good strategy. |
| 231 | 00:17:13 | R1 | Okay it wasn't a convincing argument but that she used the strategy of holding a constant, is a very good strategy. Very good strategy. Excellent Okay. |
| 232 | | JLB | [video shows student work with ten towers: G1: rry,rrb G2: ybr,bbr,rbr G3: rby; bby, yby; G4: rrr, ryr,rbr G5:bbb,bby, bbr G6: ybr,yrb,ryb G7: byr, byb, byy G8:rbb, yrb G9: yyy, yby,yry, G10: yyr, yyb<br>And then the other one I have mine are very short. They don't like to write at all.<br>This one is Y----. Y---- got it very quickly but he had a hard time putting it on paper. |
| 233 | 00:17:43 | R1 | To record the towers on paper. |
| 234 | | JLB | Yeah, yeah. There were times when he recorded like an opposite pair. Then there was other times where he recorded three together where he kept the bottom constant the middle constant the Top constant.<br>He was kind of all over the place with his reasoning. |
| 235 | | R1 | MMhh |

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| 236 | 00:17:59 | JLB | And then when it came time to write his explanation all he said was I changed the colors each time duplicating patterns. He did write more for his explanation. Did you copy the back? |
| 237 | | UCT | Yeah! |
| 238 | | JLB | He put for example, and he actually explained that he did red…red, red, yellow; red, red, blue; then yellow yellow red and yellow yellow blue they are the same pattern but using different colors so he was on the right track. |
| 239 | 00: | R1 | Absolutely. |
| 240 | | JLB | He struggled with what he wanted to say but he did give an example. |
| 241 | | R1 | But this is a good start. |
| 242 | | JLB | but he did give an example. |
| 243 | | R1 | You know And I think that's wonderful cause some of these you don't know what they mean. that he gives you an example so you know what he means. And I think that your students are all at different places. You can't expect them to all end up at the finish line when they all didn't start at the same place. That they are moving toward the finish line is what you want and that is good. This is movement. Good. Okay. |
| 244 | 00:19:00 | VB | Okay, that's mine. So this group At first they made an estimate of 31. They wanted to try that out. |
| 245 | | R1 | [laughter] Okay. They liked the number. |
| 246 | | VB | Yeah, why did you guys choose 31, and they were like well I don't know. So With them the way they had drew it wasn't the way they had it on their desk. But a lot of times some of them would have the same middle constant and then they had just the tops constant.. |
| 247 | | R1 | Okay. |
| 248 | | VB | Just looking at the picture, they had the tops constant. They just flipped the pattern. So they actually had 12 and then there were three on top. You know, the three on top. |
| 249 | | R1 | Uhhohh interesting. 12 groups of 2 or 2 groups of 3? 12 groups of 2 or 3? |
| 250 | | VB | Three on top. They kind of were trying to keep it the opposites but like I said he had them in a constant; and they wanted to separate it. That's what he told me; split them up. |
| 251 | | R1 | Okay. |
| 252 | | VB | I don't think she had an explanation but I have her partner's explanation. |
| 253 | | R1 | Good let's hear that. |
| 254 | | VB | Me and my partner came up with opposites of each other. |

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| | | | My partner was very helpful. We had the best way to organize it and we shared it and we got to 27 towers. but when we started with 31; we had we were not thinking completely [chuckle]. But we were close to our estimate. So They don't really explain but a lot of my kids when they did this; they kind of like did the all 3 colors 3 times 3 is 9 and then times it by three so I would ask where did you get the 3? |
| 255 | | R1 | Good for you! |
| 256 | 00:20:33 | VB | And they were like I don't know, so a lot of my kids tried to put that together but had no idea how to get that. |
| 257 | | R1 | Okay, And that is very good that you questioned them where is that other 3; what is it? |
| 258 | | VB | Yeah. |
| 259 | | R1 | Good. Now what happened with..when they did the two groups they had pairs. |
| 260 | | VB | Yeah. |
| 261 | | R1 | Did you question them why can't there be another pair. Rather than 12 pairs or a 13th pair. Because when you are doing pairs it is hard to see that you have them all. How did they know they had them all? |
| 262 | | VB | Well And then, these two cracked me up because one day…we did it in two days. One day they built them and the second day they finally organized the way they were done with it. |
| 263 | | R1 | Okay. |
| 264 | 21:15 | VB | Well all they kept saying is how do you know you can't do anything different here. |
| 265 | | R1 | okay |
| 266 | | VB | And then they would build them and just lay them across the desk to see if they had another. |
| 267 | | R1 | Okay. |
| 268 | | VB | Then they were just saying we cannot do anymore pairs so then they would be done. So that I questioned that. |
| 269 | | R1 | Okay. |
| 270 | 21:39 | VB | But my next group. |
| 271 | | R1 | Okay |
| 272 | 21:39 | VB | [video of student work shows 3 groups of nine with red, yellow and blue on top. Red group is colored with markers, yellow and blue groups have letters in boxes]There we go. This I thought was cool. With them, they held the constant at the top. So they had the nine blues the nine yellows. These two guys are pretty low functioning so I was pretty proud when they did that. |
| 273 | | R1 | Absolutely. |

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| 274 | | VB | And they had one group of ten and two groups of nine. So I tried to questioning them , well you have 3 colors don't you think you should have at least 3. And they were…And I was prompting them and they were able to figure out that they had one too many. Just to be sure of themselves they rebuilt the towers and scanned to check. I had a lot of kids do that |
| 275 | | R1 | Okay, that is a strategy to find duplicates. |
| 276 | | VB | I think because it was 3 colors it was harder to find the duplicates. |
| 277 | | R1 | Uh huh. |
| 278 | | VB | So we put them in order and we came up with three rows of nine buildings and multiplied them together and our total was 27. |
| 279 | 00:22:27 | R1 | Okay, It is nice that they have groups of 9. I think it's easier in a group of nine if they arranged it, easier to make a convincing argument when you have 12 pairs. So you keep a constant that is wonderful. These are all special ed. |
| 280 | | VB | Yes. |
| 281 | | R1 | So we have a lot of special ed kids doing neat stuff. |
| 282 | | VB | MMhh. |
| 283 | | R1 | Good |
| 284 | 00:22:50 | CP | Alright, That's me! [Video shows following student argument: There's all 3 colors in each tower. There two yellow in the bottom. The red and the blue switch spots. There no way to move the red and the blue. I did the same thing for the red and the blues on the bottom. There are two blues on the top and bottom. There yellow, red, in the middle. There no other color for the middle. I did the same for the red and yellow. The red is in the top and bottom G1: rrr,bbb,yyy G2: rby,bry,byr,ybr,ryb,yrb G3: byb,brb,rbr,ryy,yry,yby G4:ryr,rry,rbb,rrb, yyb,ybb,yrr,yrr,bby,byy,brr,bbr.]<br>so Um, So for the first group the student said the cubes that are red, blue, and yellow. The red has 3 cubes, blue has 3 cubes, and yellow has 3 cubes. So she is still working on clearly explaining those groups but she definitely got it right away. |
| 285 | | R1 | Uh huh. |
| 286 | | CP | For the next one, she said uh..there is all 3 colors in each of the towers. |
| 287 | | R1 | Okay |
| 288 | 00:23:19 | CP | And then what she did was she kept a constant on the bottom. She said I put yellow on the bottom. The red and blue switch spots. I told her she could write the same |

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| | | | thing for the other colors. |
| 289 | | R1 | Okay.Uhuh, Okay. |
| 290 | | CP | Do I really have to write this out over and over again? I said that it was good. |
| 291 | | R1 | Okay |
| 292 | 00: | CP | If you scroll down a little more, that's good. Alright. So for this one. There are two blues on the top and bottom. So she kept a constant on the top and on the bottom. |
| 293 | | R1 | Yep! |
| 294 | 00:23:53 | CP | There yellow, red, in the middle. There no other color for… the middle. |
| 295 | | R1 | in the middle |
| 296 | | CP | I don't know why I am trying to read that. |
| 297 | | R1 | Okay. |
| 298 | | CP | So I did this for red and yellow. Red is on the top and the yellow is on the top and bottom. |
| 299 | | R1 | Yellow is in the top and the bottom and she changed the middle position. So she is telling you what she did. It is nice that she tells you that there is no other color to put in the middle. Cause you are already putting the two colors and it is a sandwich on the top and bottom. |
| 300 | | CP | Yeah. |
| 301 | 00:24:25 | R1 | Good. |
| 302 | | CP | So For the next one. She said this group is only 2 colors. The first four have the same color on top. The second group there is a yellow on top. And for the third group there is a blue on top. |
| 303 | | R1 | Okay. |
| 304 | 00:24:45 | CP | She is talking about three different groups and the colors she gets are up there. |
| 305 | | R1 | Yep. |
| 306 | | CP | The first group I did red yellow, yellow red. Red yellow red blue.<br>She really just kind of went on to explain she didn t really say there were no other possible combinations,. |
| 307 | | R1 | Right. Right It was probably harder for her to do that. But you know I think again, it is a process and she is coming along in the process. Because in the first two tasks, I don't remember seeing all this writing. Right And I think also that is very nice. Even if she is just explaining what she did, she is writing. And once you get them writing, you can get them to write a convincing argument. |
| 308 | 00:25:27 | CP | Yeah. This actually she did on the second day. We ran out of time on the first period. |
| 309 | | R1 | Okay. |

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| 310 | | CP | So.. |
| 311 | | R1 | Yep. |
| 312 | | CP | So she did a good job. |
| 313 | | R1 | Absolutely. Absolutely. Good. And this is also special ed. Who are the regular ed teachers here? 1,2,3 and the rest are special ed. So we are talking about A lot of kids that are being exposed to math which really has them thinking which is a good thing. Because you really don't want your special ed kids to just do arithmetic…It doesn't cut it anymore. Good. |
| 314 | 00:26:02 | CP | Alright so this one is the student that comes to my class room and helps out. |
| 315 | | R1 | Look how neat that student is.[video shows student work: G1:yyy,rrr,bbb G2:byr,bry,yrb,ybr,rby,ryb G3: yyb,yby,byy,bby,ybb,byb G4:bbr,brb,rbb,rrb,rbr,brr G5: yyr;yry;ryy;rry;ryr;yrr |
| 316 | | CP | She flew through this. I was like shocked by what she did. |
| 317 | | R1 | Okay. |
| 318 | | CP | So for group one for each tower I had one color. For each tower I had each color on the top and then followed by the 2 other colors switching. I did it for all three. So for group 2 she kind of did the same thing as the first one.[Students argument video shows.] |
| 319 | | R1 | MMhh. |
| 320 | | CP | For group 3, for the first 3, I had the blue go up one every time. For the last 3, I had the yellow go up every time. |
| 321 | | R1 | MMhh. |
| 322 | | CP | Okay, alright, I was looking at because she drew the cubes a little backwards on that. Group 4 for the first 3 I had the red go up every time then the yellow every time.. So she is really just talking about how they moved but not really saying you know why. |
| 323 | | R! | Good. That is really good that you guys are picking up. She is explaining what she did she has a very, very good strategy. but she is not saying that therefore there can't be any more because I have taken that single color and put it into each of the three positions and there is no other place to put it. |
| 324 | | CP | Yeah |
| 325 | 00:27:21 | R1 | So it's good. I think you guys are showing that you have really grown. Not just your students. Good. |
| 326 | | CP | For the last one group 5 I had the red go up every time and for the last 3, I had one yellow go up every time. |
| 327 | | R1 | Okay, okay. |
| 328 | 00:27:43 | CDR | [Video shows student work: G1:rrr,yyy,bbb G2: |

| Line | Time | Speaker | Transcript |
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| | | | yyr,yyb,rrb,rry,bby,bbr G3: yrr,ybb,byy,brr,rbb,ryy G4: byb,brb,yby,yry,rby,ryr, G5: byr,bry,ybr,yrb,rby,rybThat is mine. I had two students do it the same way but their arguments were different. |
| 329 | | R1 | Were they partners? |
| 330 | | CDR | No. they weren't |
| 331 | | R1 | Okay. |
| 332 | 27:56 | CDR | So This is someone who had a lot of trouble with the pizza problem. He was one of the ones who thought that it had endless possibilities because you could have every slice different, |
| 333 | | R1 | Okay. |
| 334 | | CDR | So coming into this one. He was able to make all the towers. |
| 335 | | R1 | Okay. |
| 336 | 28:13 | CDR | So he started off first we did the 3 original colors. Then he did two colors on the top and one on the bottom six times. But he never explained why it was 6 times. |
| 337 | | R1 | Okay. |
| 338 | | CDR | The we did 2 colors on bottom and one on top 6 times. Then we did the top bottom are same colors is different 6 times. |
| 339 | | R1 | Uh huh. |
| 340 | | R1 | Then we decided to do all 3 different colors 6 times in 6 different patterns and we came out to be 27 towers. though he explained a little bit about what he did, he doesn't go why it was 6 or anything like that. |
| 341 | | R1 | Right. |
| 342 | 28:48 | CDR | But the next girl says |
| 343 | | R1 | Before you leave it. The way he arranged it; it is very systematic and brilliant and he could very easily get it to a good convincing argument. |
| 344 | | CDR | Right he has everything organized when you look at it. |
| 345 | | R1 | Absolutely. |
| 346 | | CDR | And you see that throughout but his explanation |
| 347 | | R1 | Absolutely. |
| 348 | 00:29:09 | CDR | [Video shows work of student: G1:yyy,bbb,rrr G2: yrr,brr,rbb,ybb,byy,ryy G3:rry,rrb,yyr,yyb,bby,bbr G4:yry,brb,rbr,yby,ryr,byb G5: ryb,rby,bry,byr,ybr, yrb So the next one is a different group and this girl used the same set up. But her explanation was a little bit clearer. |
| 349 | | R1 | Okay, let's hear what she says. |
| 350 | 29:34 | CDR | She writes a little bit about each one. All 3 are the same color. Then she says there are Two of the same color on bottom, one different on top. Then two of the same color |

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| | | | on top, one different on the bottom. Then she says 2 of the same color on the top and bottom, opposite color in the middle. Then she says all 3 colors assorted in different patterns. |
| 351 | | R1 | Okay |
| 352 | | CDR | And then she goes on to explain that a little more. She says the towers (in each group) have two similar towers. For example, if you have 2 reds on the bottom, you can only have a blue or a yellow on top (2 different towers). If you wanted a third tower, it would be all of the same colored cubes (red, red, red) which was already constructed. |
| 353 | | R1 | Very nice writing. |
| 354 | | CDR | Yeah, I would have liked to see her explain 3 different colors more. [bell sounds] Because that would have been nice. |
| 355 | | R1 | Yeah |
| 356 | | CDR | But for her other ones. Her explanation was much better than the previous ones. |
| 357 | | R1 | Very good. Very good. It is very nice too the way she has the groups so neatly shown. |
| 358 | | CDR | Yeah |
| 359 | | R1 | And she is telling you how she got the groups. |
| 360 | | CDR | Right. |
| 361 | | R1 | And now what we want to see more of is the bottom of why the groups are complete. |
| 362 | | CDR | right |
| 363 | | R1 | Good, very nice though. And that is 8th grade? |
| 364 | | CDR | 8th grade regular. |
| 365 | 00:30:00 | R1 | Okay, Good. |
| 366 | | MM | Okay this is mine. The first student that I chose I was a little confused because if you scroll down to where the work is and his drawing. And you can see way down at the bottom of the page you see the key which is blank for yellow, the shaded one is red and striped is… |
| 367 | | R1 | Checkerboard. |
| 368 | | MM | Is blue. So when he did it, when he actually drew it out he has all these different groups and he is Moving the one color down and exhausting that. And then he is doing the two at the bottom constant for the last groups. |
| 369 | | R1 | MMhh |
| 370 | | MM | [Video shows groups A: 3 colors B: red and yellow elevator C: elevator with blue and red D: elevator with blue and yellow E: rby, bry F: byr,bry G: yrb, ryb ] |

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| | | | But then if you read his explanation, he says first off there were 3 towers of one color each. Next I made 9 towers of two of one color. |
| 371 | | R1 | 9 towers of 2 but he really didn't |
| 372 | | MM | And one of another. And then did the alternate colors. After that I made 9 towers containing 3 colors each. |
| 373 | | R1 | Yes[someone sneezes] |
| 374 | | MM | But he was basically saying… He started off explaining his picture |
| 375 | | R1 | Okay. For group A |
| 376 | | MM | Yes. But then on the back; he starts talking about he has three groups of 9. |
| 377 | | R1 | Yeah. |
| 378 | | MM | So I'm thinking maybe he did them a different way. And then changed…A lot of the students got it quickly so they were exploring different ways of organizing it. |
| 379 | | R1 | Okay |
| 380 | | MM | And I think what he did was explain how he built it for three groups of 9 and then when he recorded it didn't match. |
| 381 | | R1 | Okay. But that doesn't match what he recorded. Does that looking at his We saw his code; checkerboard; plain; how does that sit with you? Are you able to see this clearly? Where is it on here? |
| 382 | | MC | Well Laurie's [inaudible] |
| 383 | | R1 | Yes, Right. And sometimes Does he need to come in? |
| 384 | | UCT | I think it is difficult too when there is two of the same then they kind of bled together. |
| 385 | | R1 | Absolutely. But right here, so even though he has a code. Sometimes having it so detailed makes it harder for another reader to understand what he's done. Good, you know it is Interesting that you are seeing that maybe he had arranged it differently before because<br>This is a good arrangement but not the one he spoke about. |
| 386 | | MM | Then the next student has a really long explanation. |
| 387 | | R1 | Okay |
| 388 | | MM | But the way he organizes them was first he did all 3 of the same color. And his next group was keeping the two on the bottom constant so red, red, yellow yellow, blue blue. |
| 389 | | R1 | MMhh. |
| 390 | 00: | | And then He was able to verbalize pretty well why the towers in that group are that there are only two other colors you can have and you can't do an y more 4-tall towers in that group |
| 391 | | R1 | Right. |

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| 392 | | MM | So those went were all done. |
| 393 | | R1 | Is that what he actually wrote?  Read to us what he actually wrote. |
| 394 | | MM | First we made 3 towers each only using one color. Second we made 2 towers each with two red on the bottom and one yellow on the top and one blue on the top. Then we put two yellow on the bottom and put one of each of the other colors on top.  After that, we did the opposite with two blues on the bottom.  Third we put two yellows on the top, no wait then he is talking about the next group after that. So he didn't really explain… |
| 395 | 00:34:19 | R1 | Okay. Get it down. Okay, okay. |
| 396 | | MM | The next group him and his partner decided to keep the top one constant and change the bottom two. |
| 397 | | R1 | Okay |
| 398 | 00:34:33 | MM | Yellow on the top   And then in the next group   they basically did the opposite almost of the second group. Where he kept the top 2 the same and changed the bottom And then In the last one he did the bottom one constant and changed the top two. |
| 399 | | R1 | Okay so you saw a lot of different kinds of strategies. There are a lot of good strategies. |
| 400 | 00:35:01 | | Announcement: there will be a math teacher  meeting in the media center today  at 2:45. |
| 401 | 00:35:14 | R1 | Good okay. That's good you get out of the meeting! Good! But there is a lot of writing here but you are saying is that what he told you did not go down on paper. |
| 402 | 00:35:23 | MM | Yeah, him and his partner actually were a little to into writing the explanation. They would like say a sentence to each other before they would let each other write it down. |
| 403 | | R1 | Okay |
| 404 | 00:35:34 | MM | I think if they had written their thoughts. But this group was actually interesting too because when they first got their 27 combinations.  They were confused because it was an odd number. |
| 405 | | R1 | Ahhh okay. |
| 406 | 00:35:49 | MM | They had remembered that Each one had an opposite, even though that wasn't a good strategy. Everyone should have a pair |
| 407 | | R1 | Absolutely. |
| 408 | | MM | So that didn't sit with them well at first. |
| 409 | | R1 | Did that happen with some of your kids? Where they thought it should be an even number because of the partner and it's opposite? |

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| 410 | 00:36:05 | UCT | Some of my kids thought double 16 because there were 16 last time. |
| 411 | | R1 | Oh okay. |
| 412 | | UCT | But we had a Different height last time and different amount of colors. |
| 413 | | R1 | Right right. |
| 414 | | VB | I think someone here also said that too. That we just got 16. |
| 415 | | R1 | Yeah. Yeah. I heard that before where they want it to be even. So were they upset when they got 27? |
| 416 | | MM | Um, They were at first. But then they looked at it and they organized it.<br>And they were able to convince themselves. It was interesting because the task changed from them trying to convince me |
| 417 | | R1 | Right |
| 418 | | MM | to for them to try and be able to accept that it was 27. |
| 419 | | R1 | Okay, okay. |
| 420 | | MM | I had a lot of students too who jumped right away to three to the third power. |
| 421 | | R1 | That's good. Why did they say that? |
| 422 | | MM | Well I actually, I didn't include…. |
| 423 | | R1 | That's great. Did any of your students connect it to 3to the third? |
| 424 | | UCT | I had some students too that had 3 groups of 9 |
| 425 | | R1 | Okay. |
| 426 | | MM | I had one girl and maybe I should have maybe showed you this one but she had 3 groups of 9 keeping the top constant Top yellow; top red, top blue. |
| 427 | | R1 | Yep Yep! |
| 428 | | MM | She said um, we found the algorithm and we realized that was 27. We did 3 to the 3 because there were 3 colors 3 blocks high. |
| 429 | | R1 | That is very good. This is 7$^{th}$ grade? |
| 430 | | MM | sixth grade. |
| 431 | | R1 | Sixth grade! Very impressive! |
| 432 | | MM | This is not really one of my top students either so I was surprised that she said that to back up further that she knows that 27 is a Multiple of 9 and can have 3 groups of 9. |
| 433 | | R1 | Okay. Okay. That was very interesting that she knew something about exponents when they did the towers 4-tall, did she know that was two to the fourth.? |
| 434 | | MM | Well a lot of people knew that but they weren't actually |

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| | | | able to explain where the two and where the four came from. |
| 435 | | R1 | Okay. |
| 436 | | MM | This was the first time that I heard them make the correlation between the colors and how many high. |
| 437 | | R1 | Okay. |
| 438 | | MM | A lot of them said you would get 16 but they really didn’t have any idea how they were getting it. |
| 439 | | R1 | Okay. Okay. Interesting, very interesting. Okay, good. |
| 440 | | UCT | That’s the end. |
| 441 | | R1 | That’s the end. Good everyone shared. Good. Nice stuff. I hope you see progress because there was definite progress. I think you have to do is know where they started and therefore where they are now. Good questioning going on when I was not only talking to students but I was listening to you talk to the students. And There were good questions that were being asked. And again it is very hard and you are going to have to keep telling yourself Don’t lead them, don’t push them a certain way. Because you saw today they went ways that maybe you wouldn’t have gone. And that’s nice. You want them to do that. Okay? Any questions or comments on the student work? Yeah? |
| 442 | | UCT | I actually had my students walk around. |
| 443 | | R1 | Okay |
| 444 | | UCT | At the end we were done, and they thought that was kind of cool cause each group shared what they thought. |
| 445 | | R1 | Okay. That is nice. That’s good. Very good. Any other comments on the work? Okay, why don’t I show you what the final project will be. |

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| **Description: Transcript of 2nd on-campus Meeting – northern, central and southern regions**<br>**Advisor: Professor Carolyn Maher**<br>**Location: Rutgers Graduate School of Education**<br>**Date: December 7, 2013** | **Author: Phyllis J. Cipriani**<br>**Verified by: Victoria Krupnik**<br>**Date Verified: Fall 2015**<br>**Page 1 of 39** |

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| 1 | 00:00:00 | R2 | Whatever works for you. Um, I know for my 2 regions, did you send these to yours as well?… |
| 2 | 00:00:09 | R1 | Mmhh [shook head no] |
| 3 | 00:00:10 | R2 | Just in this last day or so. I sent you an email with 4 things to think about. And I don't expect you spent a great deal of time on this [laughter]. For the people from the southern region, hey forget it! [laughter] And so I think for that reason, um that we are going to do a sort of back and forth, and what I am going to do since Judy's voice is not letting her… |
| 4 | 00:00:45 | R1 | I have laryngitis. |
| 5 | 00:00:46 | R2 | [laughter] So when she says something, it is really important that we all listen. But What I think I'll do if it is okay with you all, just take the four questions that I was thinking and if Judy agrees and if Marjory agrees, that perhaps were a way to talk about what we have been through together over the last several months. Um I am going to read one of them and then I am going to ask each of you in your groups, your group being your table. Um and in some of your groups there is at least an outlier from another region. Um I think everywhere there is because there you are! [laughter]<br>And so your job is to not, not let the rest of the group sort of dominate inside so that you so can say what do you think about that, really? And In each of your groups…<br>In Each of your groups because as Judy and I was just saying; what we really want because Judy and I was saying, what we really want to do is hear from all of us together so in each of your groups there should be at least one of you or several of you take down notes of the major points that you're making. Because we will all come together and ask for sharing. Does that make sense? Okay, alright. Especially you guys But all of you might want to jot down what it is that I am going to say. [multiple conversations]<br>Okay! Before I do give it to you the email that I wrote to the northern and central region the other night um said something like this… |

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| | | | It said that uh today that we will be asking you to reflect about the activities of this topics in lesson study course, and you can refer, please do, to specifically to any…<br>You can make specific reference to anything that's in your portfolio because as we share ideas it makes much more sense if it is done by student A and student B and it is really an example of what you are trying to talk about. Does that make sense? So in your jotting down, make sure you have some samples. Anyway, after we have given you about ten minutes to talk at your group, then we are going to share with all of us. But the first one that I want you to think about is of particular interest about your own doing of mathematics. Does that make sense? What was important or what was frustrating. Whatever, about your doing the task yourself and then the connection between your doing the task yourself and your implementing the tasks with your students. This is really thinking about the mathematics. It is a little bit about you as the implementer, cause this is something you might have known, whatever or how do you bridge that. But you are focusing on the math. Now what I went on to say is,<br>Thinking about the 3 or 4 tasks that I think we have all done.<br>We have all shared in the towers, 4, 3, 5 with 2 colors selecting from the 2 colors with all that math. We have all done some renderings of the pizza problems, the four um the four toppings with the whole, the halves, the halves just with 2 toppings, the halves with 4.<br>So when you are thinking about the mathematics, you might be thinking about some of those differences.<br>And then I think everybody worked with three colors of unifix cubes ™. You may have done different things. I know that in our two groups we did some different things. In some cases it was just building with the kids and ourselves building towers with 3 colors and others and I think most everybody, uh, uh, at least with some of their kids, and<br>I think of course with themselves did Ankur's challenge of you know of how many are there with one of each color. So I think those are shared tasks. One of my groups also explored the taxicab problem.<br>So if you sort of want to show off you can talk about that as well. But I think the ones that we all shared which were the three color ones.<br>So talk about that. |

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| | | | Now what I ask you and what I want you to think about. Are there any characteristics and what are they about these tasks and about the way that we used them with ourselves and our kids.<br>That are consistent with what we were trying to talk about with the Common Core Standards. And in general the common core for content and the common core for practices. Of here you can also as we did and really share the frustration that if there is some constraints about what is going on with common core.<br>But for most of what I would like you to talk about right now<br>Given that the Common core state standards, and in particular the mathematical standards for practice, giving it the benefit of the doubt that<br>This is really what we are trying to do. Let's talk about the consistences right now.<br>Okay questions? Okay, talk to each other.<br>[teachers discuss in their groups for ten minutes] |
| 6 | 00:10:35 | RW | [some teachers discussing -inaudible] Verbally they could do it. But they were looking at each other like… are we done, is this what you want? There was a lot of pushing them and pushing them…to put what they did mathematically in writing… yeah |
| 7 | 00:10:55 | KK | I had the same thing. They were okay with building the towers and everything like that but when I said write what they did they were like…[using hand motion to show blank face] …what? even to draw something or make a diagram to pick up the pencil and write something down….it was really hard for them to write something down. |
| 8 | | MC | But even when they wrote something, I found they just wrote what they did. Or how they did it. |
| 9 | | KK | Right, I started with this color… |
| 10 | | MC | Yeah! Writing what they did and not why they did it.<br>[inaudible conversations] |
| 11 | | LC | There were so many personality conflicts, all this stuff and you would not think it was a regular class. |
| 12 | | KK | Maybe because it was fun and there were more teachers. |
| 13 | | NL | Maybe in a perfect world, there is a one to one ratio [laughter] |
| 14 | 00:13:22 | R4 | Do you think these tasks helped you to promote attending to students' reasoning? |
| 15 | | RW | Um.. |
| 16 | | R4 | What do you think? |
| 17 | | RW | I think yes but I don't think it's practical to do regularly |

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| | | | because students fall behind and we have tests that my district paid for the marking period and it is hard to balance. To be open enough to drag out a one problem concept for like 2 to 3 days of a lesson and then have to deal with the stress of having to catch them up to prepare for test items that would be on the marking period like with the marking period benchmarks. So I think it does help but if I were to do this more often, So like, I think the focus has to would have to change on the amount I will be able to teach each marking period. Like with common core, it's less information and you are supposed to go deeper. But in my district, we still got a pacing guide that says we should cover all this and prepare them for you know. So okay in our district what happens is like they'll say if they ask all the students. Like on the benchmark, they say these students during this period missed one to 3, you should have done this or these projects. So it is very scripted and you have very little room to… |
| 18 | 00:15:11 | GH | In our district, they want us to cover the entire curriculum before the NJASK. |
| 19 | 00:15:12 | RW | See? |
| 20 | 00:15:13 | GH | So we have 7 weeks after the NJASK we do everything we need to cover. Everything is so rushed to get it done by that time. |
| 21 | | RW | Yeah, by the state test. That's something that needs to be worked out. |
| 22 | | NL | Ours is more structured. So we don't have math, we have problem-solving. So math is more structured, we do have something we need to follow but problem-solving is kind of like a free for all so this is perfect for our class. |
| 23 | 00:15:39 | KK | So do you have that every other day, problem-solving? |
| 24 | | NL | No, every day. |
| 25 | | KK | Wow! |
| 26 | | NL | But they don't have a block. It's like period 1 will be math and period 8 will be problem-solving. |
| 27 | | MC | And they could have 2 different teachers. |
| 28 | | NL | Yeah. |
| 29 | | KK | Wow! |
| 30 | | NL | They usually do. |
| 31 | | KK | Do you like that? |
| 32 | | MC | No. Problem-solving has no text book, there's no workbook, so it's literally you have to find something for them to do for an hour. |
| 33 | | NL | So you have to do something that will last for almost a week. |

| Line | Time | Speaker | Transcript |
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| 34 | | KK | Wow! |
| 35 | 00:16:04 | RW | How long is your math class? |
| 36 | | MC | It's only 40 minutes. So it's 40 minutes of math, 40 minutes of problem-solving. |
| 37 | | NL | So at least we kind of had the freedom to not worry about falling behind in problem-solving but we did fall behind in math. |
| 38 | | MC | Yeah it was easy to stretch it out in 40 minutes. |
| 39 | | RW | What grade? |
| 40 | | MC | 7th grade. |
| 41 | | KK | Our superintendent at our district, we have 4 middle schools and he says everybody should be within a couple of days with each other if you are teaching the same thing. So every time I took, a day away to teach this, I felt like I gotta catch my kids up because now I am behind everybody else. And we have the same thing, benchmarks. So I had to make sure that the curriculum is covered before they have the benchmark test or else my kids are at a disadvantage. So, It is hard. It is hard to keep that balance. Okay, now I have to do 2 lessons at once because we spent a day playing with blocks. Which I saw value in doing the tasks, they really got a lot out of it. But it was stressful for me to try and fit everything in. |
| 42 | 00:17:00 | R4 | So what grade level do you teach? |
| 43 | | KK | Um, I did it mostly with my 6th grade honors. |
| 44 | | R4 | 6th grade honors. |
| 45 | | KK | I only have one group of them so granted I only have one class that I would fall behind in. [inaudible] |
| 46 | | R4 | What I am saying is I am hearing like time issues and...um pacing guide issues, um and then you said something about testing… that you were concerned with about testing. |
| 47 | 00:17:27 | RW | Yeah, I guess cause for them being so concerned with making sure that the student should have that support to do better on the standardized test than the year before, so like a child got 221 and then gets 232. So in our district what they do is they use the benchmark as a scale or kind of like try and predict how they will do on the state test. So since they are so concerned with the test, teachers are making sure they are ready for the standardized test In our district we use the benchmark as a scale of since they are so concerned, teachers teach to the test. These are the skills are concerned because I had to prepare for the benchmark the whole marking period what I have been doing. We don't get the test until the week before but they give us the skills that will be on the test and an overview of how to |

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| | | | prep the students but teachers are concerned about the benchmark because it is 30 percent of their grade. Making sure they are ready for the standardized test In our district we use the benchmark as a scale of since they are so concerned, teachers teach to the test. These are the skills |
| 48 | | LC | 30% wow! |
| 49 | | RW | 30! |
| 50 | | LC | Of the marking period? |
| 51 | | RW | Of the marking period. So if they bomb the benchmark it drops their grade dramatically. So all the teachers are concerned and make sure they prepare their students for the benchmark because the problem is if the student was not prepared, then they say, I got A's all marking period, how did I end up with a B minus or a C on my report card? |
| 52 | 00:19:06 | KK | I have 6th grade but it is only out of if there is say 19 points available, it is counted as a quiz out of 13. By the end of the marking period, it counts as a small percent of their total grade. So it's not really a big deal. |
| 53 | 00:19:20 | R4 | And yours is a test grade? |
| 54 | | RW | It counts as a separate category. In our district, the marking period grade is 20% classwork, 30% projects, 30% benchmarks, 20% like doing tests and quizzes. So it is a 50-50 balance of how they do on tests and how they do in the classroom. So you figure, if a student has an A on the last day of the marking period, and you give him this benchmark, you can imagine what they will say. [inaudible] Like then the kids say, Ms. you didn't teach us this. So I make sure that every skill on the benchmark that you went over and over. |
| 55 | 00:20:20 | R4 | And what grade level? |
| 56 | | RW | District wide, same test for the whole district grade level. |
| 57 | | LC | They give you a different one every year? |
| 58 | | RW | Yep! Every year because if the child came from wherever, and we have a supervisor and sometimes they don't do a great job as you can imagine. So the math supervisor will put a job description over like a benchmark review. That's why being a teacher is so frustrating because they keep updating, adding and changing things and it is always changing. |
| 59 | 00:20:58 | R4 | What grade do you teach, Roberta? |
| 60 | | RW | 5th grade. |
| 61 | | R4 | 5th grade. Tough job! |
| 62 | | RW | Yeah, and that is 5th grade we teach everything, math, science, social studies, language arts, reading, writing |

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| 63 | | R4 | The superheroes are right here at this table. |
| 64 | | RW | That's true. |
| 65 | | R4 | The math teachers of the world are the superheroes because you got to do a lot! Believe me I am talking because I am one of you! I also teach high school so I know.. the stresses. Thanks for your input. Hi everybody, do you want to trade tables? |
| 66 | 00:21:52 | R2 | Okay, can we all have your attention. Okay, What I think I will do is sort of jump around from group to group. And um and maybe start right here with this group and whoever is your spokesperson or if you are helping each other. Would you begin this first group by sharing with all of us. What I want you first to do is restate what you think we were supposed to talk about. |
| 67 | 00:22:37 | AT | Well, we started and we kind of took your question and divided it in half and we only got through the first half. We talked about us doing the problems versus the students doing the problems during implementation. Um and we really focused on talking about that. Um like for example, we discussed using a tree diagram versus not using a tree diagram. And then we also talked about just right now before you stopped us was the pizza diagrams of how we interpreted the pizza versus how our students interpreted pizza and how like life experience could really really impact how they notice what they think about. Um so for example one of my students said well sausage and pepperoni is different than pepperoni and sausage because if you say sausage first, it really means you want more sausage than pepperoni. [laughter] Or if you say pepperoni first then you want more pepperoni than sausage. So like and you were saying about that too. |
| 68 | 00:23:35 | CDR | Yeah, I had a student who thought through each slice of the pizza as a different option. And he was telling me that each slice could be a different topping and if the slices were in different patterns, those are also different options. And when I asked him if he would go into a restaurant and order something like that he responded with my father owned a pizzeria and if someone came in and ordered it like that um, we would do it. So his experience to him was yeah those are all different options. And he was so frustrated with this problem. Because he thought there was going to be thousands and thousands of options and he would never get through it. So, it was definitely a challenge for him. And it was a challenge for me to you know kind of watch what he was doing and try to |

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| | | | understand how his experiences impacted that so much. I just… I really learned from the pizza problem that the experiences that they have, are going to make them interpret these problems so differently than how we would expect them to interpret it. |
| 69 | 00:24:33 | R2 | You know that is really interesting. Because from what I am hearing, I would like if we stop right there and let other people respond to that.<br>The issue that I think is emerging here and this is something that the Common Core standards practice really talks about is embedding what you are doing in mathematics, in real situations and what I am hearing you say is an okay thing that there are real issues that come up with that. They came up with you a little bit with you as adults working but you have enough classroom experience to push that away.<br>But that it really made a difference in interpretation of the problem. Is that what you are saying? |
| 70 | 00:25:28 | CDR | Yeah |
| 71 | | R2 | Moreso than any of the other problems. |
| 72 | | CDR | Right. For the other problems, they had concrete solutions and they kind of could work through it. But for this problem, it was a little more interpretation. |
| 73 | 00:25:42 | R2 | Okay so what you have really done here is you connected your own thinking about the math. You talked a little bit about…about differences with your kids.<br>And I would say you also said. Okay, these are something we think we have a value in doing.<br>But that there are issues having to do with it and the pizza problem becomes sort of a metaphor for us to think about. Um, others? Did anyone else talk about that? |
| 74 | 00:26:13 | TD | Yeah. We said the same thing. We said that we thought that it would be more effective if the directions were more specific for them. And like if it said you can’t do this, you can’t do that. I know that part of this is working through that but<br>I felt that my kids spent a lot or most of their time trying to figure that out. Rather than actually working on the task and trying to figure it out. |
| 75 | | R2 | This was just the pizzas and not the others? |
| 76 | 00:26:36 | TD | Yeah, just the pizzas. |
| 77 | | R2 | What about anybody else? What about this notion of embedding problems in real situations? Does this mean that it is hard to do? Or that you have to be so specific that it is not real anymore? |

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| 78 | 00:27:02 | NV | Well you have to be either really specific or you have to be willing to accept that there is a solution that you may not come up with yourself. And um, Emily, Jen, and I, we teach in the same district and we talk about that in our common planning. When we are grading or checking responses for open-ended type questions, you can't anticipate what students will give you or what they come up with but it.. it doesn't necessarily mean that their response is wrong. |
| 79 | 00:27:40 | R2 | I think that is the answer to a totally different question! |
| 80 | 00:27:45 | MM | Yeah I totally agree with what Natalie just said. Um I think that if you make what they are supposed to say too black and white you are kind of stunting their reasoning skills and their problem-solving skills. If they come up with something that is totally out there. It should be acknowledged and celebrated and then talked about how oh maybe that's not what we meant but I think it is still logical thinking which is math in a sense. |
| 81 | 00:28:13 | R2 | Yeah. Now what you are doing is sort of jumping ahead except I don't think, I don't think so and I think it has to do with some of the role of the teacher and the role of instructor. What I am hearing is this real tension between Okay.<br>We know there are particular mathematical ideas that they want to get and so we don't want to lose that. But that if there is anything to this notion of the kids thinking and building. We have to hear what they are saying and go from there. And this was a pretty good example of that. Others of you, this notion, this is something I would like you to talk a little bit more in small groups Is in terms of the role of the teacher,<br>This notion of listening which is what you were talking about Helena and what you are talking about: hear what they are saying. I know in our groups, that I pushed you all to do the shirts and pants in the very beginning because I think it is another example where the context really can…. You know what I mean. |
| 82 | 00:29:37 | MM | One of the students said the colors that matched. |
| 83 | | R2 | Yeah. |
| 84 | | MM | And that is an example. |
| 85 | 00:29:46 | UCT | And also students who knew it was 6 combinations but put there was one combination that yellow pants and blue shirt didn't match so… It was a different color they said. [multiple conversations- inaudible] |

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| 86 | 00:30:00 | R2 | Well and if you remember in the clip the little boy the first half of it he would only accept you know he got 2 pairs of pants therefore he can only have 2 outfits and they have to go with the colors they go with. Do you remember? Then he invented another pair of pants so he had 3. Uh and so they were little kids so they exaggerated a little bit but for me so frequently an outfit is layering the clothes which we learn as teachers but your notion, Margaret, of what your saying is that it is the part of the teacher to listen and understand. How about over here? Did y'all talk about this characteristic? |
| 87 | 00:31:19 | KK | One of my students interpreted it as two pairs of pants meaning 4 instead of 2. For him, you wear a pair of pants, he didn't get it. |
| 88 | | R2 | Very interesting. And this is a little bit different from the people who are talking about all these different possibilities for pizzas.<br>This is really that math practice thing about precision about how you hear and then what you do with mathematical language starting with the boy who was told one thing but said it was 2.<br>Any other of let's see now let's talk about how did you respond a little bit what did you say about this person? |
| 89 | 00:32:06 | RW | Um, well we were discussing we said that um here what we did when we worked through the problem and<br>A lot of us were shocked about how well the students worked in class because they struggle with getting it.<br>The information that was written down on a paper so that probably took 2 to 3 sessions to work through the process in terms of motivating them to get through the problem.<br>Cause they have to do the problem and then get their thoughts on paper um but that was something we brought up in our discussion. And then The second thing we brought up in our discussion was because of timing how we get concerned about falling behind in terms of our district responsibilities and having to pay for that. |
| 90 | 00:32:54 | R2 | We had a conversation about that the other day. |
| 91 | 00:32:56 | RW | Yeah. And how That was a kind of struggle.<br>Because the type of problems we have are good because they're open-ended because it is forcing them to do mathematical reasoning and skills that help with students mathematically. But then when you are balancing The other concerns are for your district, if you are dealing with pacing guides;<br>You don't want your students to fall behind and be at a |

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| | | | disadvantage. But when the benchmarks come up, so we were discussing that and then the last thing brought up was How… we noticed that when we did this, the type of pacing that took place in the classroom that we were visiting in and because of the amount of adult to student interaction that is so much different than our experiences doing it alone in our classroom. Because that one on one conversation and the questioning that each student was allowed to give or to receive helped push the process along the way we expected to see it without all the adults around. So when you go into that session, you leave with Wow those kids did all that! [laughter] And in one session they got through their thoughts, they wrote it down, they defended it, they were able to express it. And then when you go back to your class and say Now wait a minute, [laughter] what happened? Like how come my kids can't do nothing [laughter], I can't see that and you start thinking there is only one of me [laughter] and all of them. And you can't kind of like foster the same amount of motivation as it did. So I think the…you said that the benefit of experiencing a lesson study is what's going to be helpful. It shows you that if given the right resources and the right setting that students can also do that but it takes the resources and appropriate setting. And then the class that did so well, is usually like a problem class so it shows that when you have a lot of teachers there... |
| 92 | 00:35:11 | R2 | When there is more individual attention they will listen? |
| 93 | | RW | Yeah it's like a whole new class. It fixes the attitude when a group of teachers comes in [laughter] for math classes. |
| 94 | | R2 | Well I guess the advocacy is for this kind of lesson study. Collegial working together either within schools or across schools in the district that does this every now and then. |
| 95 | | RW | Yeah. |
| 96 | | R2 | I thought that all the classes we went to they were excited that we were there. And I never saw and even in the interviews that we did altogether with the teachers and the little guys. They thought that was great stuff. And so I think that what you are saying is that we do learn from that and it would be good to continue it But the number of issues that you are faced with are a lot. |
| 97 | 00:36:13 | RW | MMHH! [laughter] |
| 98 | | R2 | A lot. Yeah, Natalie. |
| 99 | 00:36:18 | NV | Just to go off what Roberta said. I know we kind of discussed it with our trip to Edison, but um, I personally am a little disappointed in the lack of the alignment that |

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| | | | this course had with the content of the common core standards and with the university. |
| 100 | 00:36:36 | R2 | The content? |
| 101 | | NV | The content. But with a university as big as Rutgers, that it is coming down from the state to the teachers. I would have thought that the tasks that were given, would have been better aligned to what we were teaching our students. The tasks were discrete math. Discrete math is not something that is covered in common core standards. So what we were asked to do is take a day out of our classroom to visit a class in addition to taking a day away from our curriculum to teach something or to do something with our kids that they are not being tested on. So in moving forward, in the future, my recommendation, well it is just me, but would be that the task that you give be better aligned so that it is a seamless transition from one day to the next so that whatever they used. That whatever they learned the day before so they can connect it directly to the task that they are working on. But My kids were working on ratio and proportion and then all of a sudden, we would walk in and there is towers. Then we leave ratio and proportions and go into expressions and equations and . Now they are dealing with pizza toppings. So although you can connect little pieces, it's not so direct for these kids who are struggling day to day. And then to take 2 days out of the classroom. It's a little frustrating um and with everything that a teacher has to do, including teaching, grading, SGOs. I mean it's a lot so…. |
| 102 | 00:38:22 | | Yeah, I think that you are saying the same thing or a different variation of with what Roberta said. I know that we but I don't know if Judy in your groups had. We had a lengthy discussion about this in our last one, and responding I would like to hear what other people think [laughter] |
| 103 | 00:38:49 | RW | Initially, I felt the same exact way You know what happened when you give the problem and for the first 15 minutes, they are looking at you asking you what am I supposed to do. You know and not being able to make the connection from what they are learning in class. I felt the same way. But then I took a step back and said, I can't look at it for the content I looked at it as a process in terms of them coming across a familiar question which you know is going to take when they take the state tests whenever it is, and look at it as a problem-solving strategy as opposed to the content. |

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| 104 | 00:39:22 | NV | I agree with you that it is mathematical practices across the board. |
| 105 | | RW | Right, right, right. |
| 106 | | NV | But the task in my opinion should have been connected to a content that we were teaching. So that they can incorporate those mathematical standards or mathematical practices with the content. It was too separate for me. |
| 107 | 00:39:46 | RW | Yeah, yeah, but I think for me it helped me give them the lesson I pulled out for them, and was able to give back to them because they were frustrated at first, was that it was important to be able to document what you can verbalize. So that was my big lesson. The content, like we didn't even get to talk about Pascal's triangle. I didn't even touch none of that. Because they weren't there cognitively but what I did use the lesson for was the opportunity to give them feedback on the importance of being able to work through a problem, come up with some type of verbal reasoning and then get that down on paper. Because out of my whole class, like one group was able to achieve it and I used the opportunity to highlight that and say you know when you problem-solve this is your goal, think about it, work with the problem, talk to your members and then get down in writing. So I think I saw the lesson more as a piece of mathematical practice than for mathematical content. |
| 108 | 00:40:52 | R2 | You know I think and I would like to hear from other people on this and I know that some of us are more vocal on this though I don't know why because we are all teachers! [laughter] Please say what you would like to say at the risk of not hearing it because I want to know what you would say. Solariz- |
| 109 | 00:41:08 | SO | Although I didn't feel comfortable deviating from the pacing guide for the same reasons as Roberta, cause we work in the same district. And we have to give benchmarks on certain dates and we are expected to cover a certain amount of material so that the students will be prepared for it. I actually enjoyed the task and I think it was something different that the kids weren't use to doing in the classroom. And they enjoyed it and I think they always looked forward to that one day that month they were going to do something new that they didn't do in the classroom. And I think they did a great job at doing it. |
| 110 | 00:41:41 | R2 | So were talking about both sides of an issue. What about from this group? |
| 111 | | UCT | Kind of like what Roberta was saying, if you have a kid |

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| | | | that can say and write how to explain the tower problem. There is no question on the NJASK or PARCC that comes up that is going to be that in depth where they are going to have to explain step by step by step by step especially with the 8 teachers in the room pushing them constantly and saying like no, but I don't know what that means. A couple times with Dr. Alston was talking to the kids, I'm like this kid is going to throw something. [laughter] So in the end I said that this kid explained it seven times and then he explained it to me again and then he went and he wrote it after all that and then typed it up. So if he was able to do that he is going to soar through any standardized test, because what we put them through is nothing compared to an open-ended question on the state assessment. |
| 112 | 00:42:28 | UCT | I definitely see all those things so I get it! Graphic functions, then we did towers in my algebra course. So I get it that it is a huge transition, but um and if it's more aligned but my kids do this all the time. We do constantly different things. Like with our benchmarks we have to show that because we teach an algebra course in the 8$^{th}$ grade, they still have to take an 8$^{th}$ grade standardized test. So I have to find a way to teach them algebra as well as constantly incorporate other kinds of mathematics for the Common Core. So for me it is kind of nice because that was a part of the second half of class, incorporating the different kinds of tasks and I loved the idea of sticking through a common theme, and from here on out I am going to try to figure out a way to do that. |
| 113 | 00:43:13 | R2 | When you say common theme, in terms of the task? |
| 114 | | UCT | The task. Where they are building and building. So I now moved to ratios and proportions and we are using that as an opener for the next month and I am focusing on that as my open-ended questions. |
| 115 | 00:43:26 | R2 | That is really interesting. Hopefully what we are learning from each other from our own experiences and from this is oh okay, I have this… this issue, how is it that we can learn from the good here and also you guys are just look at you! You are a really wonderful bunch of teachers but you are really different from each other. I mean really different from each other. Some of you are special ed teachers, some of you are general ed., some of you are high school teachers, some of you are 6$^{th}$ grade teachers. Um and so there is sort of a challenge to come up with common tasks that can work for this.<br>Which is different from the challenge that I am hearing |

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| | | | Melissa say which is now what did we learn from this? And how do we now so that we don't have to have that sort of grip of whatever subject that a task might work and um I remember… In that particular class and also in the high school task that Alice and Bhupinder were doing. So what you take from this, like say you do proportions, is kind of interesting. What else? Anybody? Yeah. |
| 116 | 00:44:57 | TD | I can.. I am seeing both sides of it. Now that I am thinking about it. But I just think it might have been more effective if and just nice to have another resource you know to teach something that is in the curriculum like say area and perimeter; something they always see. They could have done something hands on with that, I just think it would be like so much more effective for them to do something that they are going to use. |
| 117 | 00:45:19 | R2 | Okay then that's your task; your challenge. To make sure you do this. And ya'll are going to be talking about this to each other you have another course together and you are going to be sharing ideas about this. Because we are moving on in time, obviously these are fantastic these questions. I would like you to take just a couple of minutes at your group and then come back because to hear the feedback that I have gotten especially from you two is something we struggle with all the time. Um and and and it is a tough one and and we need to make in terms of resourcing each other as we go. What I'd like you to do now is to just Talk 3 minutes about the following question: what if anything was added to the course by studying the video clips from the VMC? Does that make sense? Is that a simple question? [teachers talk] |
| 118 | | UCT | The videos that we were doing, I think that they were not challenging but I liked the videos. |
| 119 | | A | I mean the videos are good. They definitely help you see that like so for example… I am mixing stuff up now…. But were there tree diagrams on the videos for the pizza problem? |
| 120 | | UCT | Yeah. |
| 121 | | A | I think they were more spread out. |
| 122 | | UCT | I just remember Brandon with the table. I thought here was like another video. Was there other videos or two other videos that…. |
| 123 | | UCT | There was one where he was arguing about how to get organized. |

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| 124 | | UCT | I don't think they did. |
| 125 | | UCT | No I think they were drawing circles. |
| 126 | | UCT | Yeah. |
| 127 | | BK | [New group] We both ended up being the only ones from high school. We were not able to do it with solid classes like algebra, it was like our enrichment class that we were able to do it. So we still are following a pacing guide for that. But every now and then, we would use this as a break and that way before we start the next thing. But otherwise it would have never fit in. [inaudible conversations] |
| 128 | 00:48:50 | UCT | It takes too much time. |
| 129 | | R4 | So you are saying when it doesn't fit in, you are saying because it is not tested on a standardized test like the ASK or the HSPA? |
| 130 | | UCT | It's not in the curriculum at all. It is discrete math. |
| 131 | | R4 | Okay, but is it tested on the standardized test that is required? |
| 132 | | NV | It is not a tested area. It is not a tested area. |
| 133 | | R4 | It's not a tested area. |
| 134 | | NV | Our students have to take the PARCC assessment and it is not on the PARCC Assessment. |
| 135 | | R4 | Would it fit under patterns? |
| 136 | | NV | No there is no patterns in 7$^{th}$ grade. |
| 137 | | R4 | Not in 7$^{th}$ grade? |
| 138 | | NV | It is not a major cluster area. |
| 139 | | R2 | Are all of you teaching 7$^{th}$ grade? Or is there… Okay so you 2, okay you 3. |
| 140 | | NV | [inaudible at first] Patterns are not a focus. It is hard to find out where it fits in. If you are teaching this, then it fits in somewhere over here. There is no collaboration. |
| 141 | | R4 | There is a disconnect between… |
| 142 | | NV | Absolutely. |
| 143 | | R4 | The standards that you have to teach and what's being expected of you in the test. |
| 144 | | TD | And it would be great if we had another resource for us to use. Like this could be a great resource for us to use like in our middle school teaching. |
| 145 | | R4 | Like what type of resource to attend to…? |
| 146 | | TD | Something that would have to do with the curriculum. |
| 147 | | R4 | What type of resource would help you to attend to? |
| 148 | | TD | Well like I said before something with perimeter and area so that oh my gosh, we could do that next year. Or it could open my eyes as a teacher to do more things in the curriculum. |

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| 149 | | VB | I think that the first task was a good way to get to know your students. That was a good icebreaker. |
| 150 | 00:50:01 | TD | Yeah it would be good for the beginning of the year. |
| 151 | | VB | Yeah like I would do that every year. |
| 152 | | TD | Yeah or at the end of the year. |
| 153 | | BK | Yeah like even my kids were like oh we get to play with the blocks today, yay! Okay, fine. But then the next day we are like where did we leave off before and it was like they couldn’t remember what we had done and forgot what we did. So it was like…[inaudible conversations] |
| 154 | | UCT | Yeah that’s what mine were like. |
| 155 | | UCT | before and it was like they could remember the pizza problem but forgot what we did. So it was like…[inaudible conversations] |
| 156 | | R4 | I know. |
| 157 | | UCT | What was the question? |
| 158 | | BK | She said Video-how did we find the videos helpful? |
| 159 | | NV | In the beginning actually. |
| 160 | | UCT | [multiple responses] Brandon’s video. |
| 161 | | NV | Brandon’s video was good. |
| 162 | | R4 | What about Brandon’s video made it good? |
| 163 | | NV | Because it was a completely different way of solving the problem. So it was… |
| 164 | | UCT | Like the 1 and 0 |
| 165 | | NV | Binary system like what he had to use to solve the problem. And I was amazed that this kid could think on that level. And it seemed logical and right but I think the logic in the video in particular with the first one with Milan, Jenna, Michelle, and Stephanie. Very confusing very hard to follow. |
| 166 | | TD | Because they were saying okay yeah this, no this. And I am looking at the transcript trying to see what they are saying. |
| 167 | | NV | Yeah it is not clear. |
| 168 | | TD | And they’re like ah no… |
| 169 | | BK | But also then I noticed Jeff in high school and you can see that lack of motivation go down, and there is a difference when you do this in elementary school they are like yay! Kids in high school are like twirling their hair. They learn so much along the way that even then… even then you ask them what do they recall…. |
| 170 | 00:52:33 | R2 | [R2 calls for group attention] What did you all say about the videos? |
| 171 | 00:52:45 | UCT | We were pretty much saying that we understood the problem and the different viewpoints from the videos as far |

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| | | | as understanding. But it did really change our view as far as instructing and implementing but I felt that I was more amazed when we went into these classrooms and saw firsthand that watching the videos. The videos helped me see a different point of view or a different perspective though. |
| 172 | 00:53:12 | R2 | Yeah. And so what you're saying is the same thing I heard over here that Roberta, Laurie and others said that somehow throughout the course of the year and to me why I feel that lesson study is pretty fabulous is that being able to share experiences in classrooms with each other is important. Oh okay. |
| 173 | 00:53:39 | DH | One of the things I liked is my kids didn't believe me that second graders could solve the towers problem. But I was able to show them on the video that they were able to. They was like Are those kids geniuses? [laughter] They are at a second grade level. I just told them that they were really focused on the problem. I don't remember what grade they were in. but they were like… |
| 174 | 00:54:00 | R2 | They should have…The first one they were third graders when they first actually did it. So you were actually right.so what you did was to use the video to actually convince them that they could do it. |
| 175 | | DH | And then I watched the Romina video and like I kept watching it and I had to watch it 2 or 3 times to really see. But then when the people sitting in my group that were watching this task like they were able to ask like I felt like I was asking Romina when I was able to ask them well why did you do it this way or what did you mean? It was nice to have the connection, of seeing people do it in your room the also do it in the same way that the people in the video did it. |
| 176 | | R2 | So what you are talking about now, is what you learned from the video as compared to your own … |
| 177 | | DH | Like with the conversation from the other people. |
| 178 | | R2 | What did others, back in Solaris' group back in there-what did you say? |
| 179 | | UCT | Well we were talking mainly about Ankur's challenge and how that video really got our attention because it was such a different approach to solving that problem. Like I would not have thought of using 0's and 1's to show the use of the colors and the way he came up with the chart. |
| 180 | | SO | No it was Brandon! |
| 181 | | UCT | Oh I'm sorry I apologize. It was the video of Brandon. |
| 182 | | R2 | No but I am glad that you caught that because to me both |

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| | | | of my ecollege groups the discussion about the Brandon interview video tape, um is always good. |
| 183 | 00:55:47 | UCT | We were also talking about the first video when they were very young.<br>How it is really hard to understand students and what they are trying to say to each other.<br>We don't know if it is because they are at such a young age or how they were trying to communicate with each other and if the language wasn't there to try to get across.<br>It was hard to follow even with the transcripts to match to go along with the video.<br>So um…I had to watch it a couple of times. |
| 184 | 00:56:13 | R2 | The issue that you are underscoring right there.  I know because I worked with the little guy Milin, Do all of you remember Milin?  Both in the gang of 4; which was your assessment video.  And then in the guess my tower, that series of em, where Milin seems to have a strategy that he has really got, but then he has a really hard time communicating and I think that is what you are saying. |
| 185 | 00:56:46 | JU | Yeah and I think that if it wasn't something that I was able to see.  I would have a hard time understanding what that…because we all have problems with different points of view; and if someone were trying to explain it like to a student at the table with him; I wouldn't be able to understand his approach.  So that I could see it was more connected.  It was hard to follow the video. |
| 186 | 00:57:06 | R2 | But you know not only does it say gosh that it is hard to follow but it doesn't say like what Margaret was saying when we were talking a little while ago was that somehow what we come away from this as teachers is that is part of what we have to do.  We have to figure out how to get into that kid's head.  And how hard that is.  When the other thing that all of you or at least in our group have indicated is that one of the biggest struggles is to get them to clearly communicate what they are thinking.  As you are saying verbally, and as other people had said, when we ask them to write things down.  Don't you think?  What else, what did you all think about the video? |
| 187 | 00:57:59 | CP | Uh, we said we thought it was helpful just to see like examples of questioning for the students and to see their responses and to kind of have an idea you know what to expect.  Or to see all the different possibilities of what you are going to get as a teacher with all the different students. You know see all their different examples of their work. Different methods that they used to solve the problems. |

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| 188 | 00:58:24 | UCT | So we talked about a couple of more things. That it was good to see the problem worked out before. And kind of to get an idea of what to expect. But then we were also impressed with that you saw these kids and they were and their solutions and their responses were so amazing that we kind of wanted our classrooms in conducting that and then when we didn't get that we were like Uhhh [laughter]. Our kids were not as good as the kids in the video. So I think that kind of made it like it really um raised our expectations then when they weren't met, we kind of felt like we did something wrong. Or I don't know. But I mean it was good to see different points of view that I suppose some of us probably didn't do on our own. |
| 189 | 00:59:02 | MC | We thought it kind of would be helpful if um I kind of wanted to see the videos where the kids were not successful [laughter]. To see how the teacher dealt with the kids struggled. I mean not all the kids are going to get it and that was the case. I mean I teach special ed. and some of those kids were you know… completely lost. So I guess to see videos where kids were not successful and to see how that teacher responded and how they walked through it without leading them on probably would have probably been helpful. |
| 190 | 00:59:33 | R2 | Um sure. Point well taken. And what is interesting is that I am trying to sort through in my head the video clips. But there is a little bit of that and I think that it may be of the high-end of what you are saying. But in those guess my towers, that…that…that… string of clips where you realize that if you had seen Stephanie in the gang of four in the beginning, you would think that she must be the most brilliant student to hit the table. And then when you see the first clip in the guess my towers, and you realize she's clueless. And that for a little bit of it but I know what you are saying a little bit of it of helping me to realize that for a lot of people these ideas it is not just once around. Uh which back to what Melissa was saying earlier on is this notion of problems over time. Even if they are not directly connected with what you do the day before or the day after. But that there is some connection among them. That they can see. |
| 191 | 01:01:00 | UCT | I have never done that before. But through the readings that's you know what I read. And I am trying it. So I like the idea but I think that the building is good for the amount of time. then I always yeah this is open-ended this works. And you know there was no rhyme or reason, there wasn't |

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| | | | much thought other than. but then they were all good open-endeds so why not because I have no idea what the state is going to be testing them on. |
| 192 | 01:01:24 | R2 | Did you do it the day before Thanksgiving? |
| 193 | 01:01:26 | UCT | Oh no, every day we do it. Every day open-ended. |
| 194 | 01:01:30 | R2 | Yeah I think you raised what I hope other people agree with what Melissa is saying if that these problems are important that and this is also back to what Natalie said, they are totally non-contextualized with the curriculum in general or if they don't become um sort of a metaphor to stay with other problems and where people says this is a lot like that towers thing and that is really good. Hopefully the video clips and the strand will match as well. Okay, moving right along. Uh, the third question that I want you all to work together with is I want you now to think about your own portfolios or whatever type of productions they are, and for those of you who didn't do this, you might learn, I think that there is a few people at each table who are from the central and northern region who did this. I want you to think about something that was important to you. And go around the circle to whatever works with your group that was important to you about your how your students were reasoning about the mathematics that they were doing. Does that make sense as a question? And I want you to back it with examples and I want you Gina to say my kid Joey used this kind of strategy which was really interesting to me because it showed me something about his reasoning. Does that make sense? And for those of you and I know for our two groups I pushed you to do this, um if there was a student that you tracked across the different problems. Also think a little about were there differences? I mean was a kid who could not write down the first sentence at the beginning, did they begin to be able to record or did they stay that way? Do you understand? So talk! [teachers begin to talk in groups] |
| 195 | 01:04:00 | A | [one group] So I will just read the question. Something that was important to you about students' reasoning about mathematics? – Backed by evidence from student samples that you have selected from your portfolios. |
| 196 | | UCT | It was interesting when we did the first tower task how most of my students had to do them in pairs and that was their strategy. Some of them made a tree diagram but only after they made the pairs to like kind of prove it. Then when we did the 3-tall towers, everybody was like they like were kind of over doing opposites. They just kind of |

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| | | | like they had…,I don't know did it so much faster. Well they were all like to the point that well it is 3 times 3 times 3. Well why? Why is that? Because they told me like that the two threes were for the numbers of colors and one was for the height, but they couldn't tell me the third three. And then they tried to explain about exponents but it was the first time we did them so they had no idea and we didn't really talk about four-tall towers at all, or exponents like we didn't talk about that at all. [inaudible] |
| 197 | | CDR | I had a student who did the same thing, 3 times 3 times 3 where one 3 was the height but then she did 3 times 3 is 9 and then 9 times 3 is 27 with no explanation. |
| 198 | | UCT | Right. |
| 199 | | KK | [New group] So he set it equal to 36 and he started to try to back up but he wouldn't even build the towers. He was so focused on making it notation. And did that from the first one, and with the pizza problem he tried to do it. He was so unsuccessful and so stuck on that he wouldn't make anything. |
| 200 | 01:05:30 | GH | I wrote about one student who went from being completely wrong to really getting it quickly. |
| 201 | | KK | Mine were so stuck on their strategies because like their so…the honors kids are still taught that there is a set way to do it and this is how you have to do it. So he was stuck on that he wasn't going to try and find anything. I got to find the answer and write it down. |
| 202 | | GH | This kid was so stuck with equations, and I showed him page 24 of the book and he went on e-bay and bought it. He needed to have it all the time.I bet if he did this again, he would get it. I thought he would realize and get it but It was just weird. |
| 203 | | KK | The first tower problem he did, he was writing four squared that was what was convincing to him. So then I said, well what if it is 3 towers? Then it would be three-squared. So it would be 9, so I said show them to me. And he came up with these and he said I guess I must be missing one. Like he was so….no he said it has got to be 9, look here is my paper. He was so stuck on it. He could not get past it. But he is the same kid that we were talking about you know perimeter of a triangle and he is trying to tell people that it is going to be the sum of the two lengths and the hypotenuse. Like he tries to make everything more difficult. Well what if it is not a right triangle? Well. I am like no there is no well, add up all the 3 sides, it is that simple! He tries to make everything harder. And this was |

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| | | | impossible for him to do. He just couldn't think outside of I need a set of rules. He couldn't do it. |
| 204 | | GH | You know I wonder because it was four towers…So my student had to use the four. But then it was 3 towers 3 colors, 3 to the third. Like it just fit in his thinking. I wonder if it was just luck. |
| 205 | | KK | Yeah, like whether he really realized it. |
| 206 | | GH | He was able to build em and explain it clearly. And he kept the bottom constant. And so he got 9, and then he said well there are 3 colors so it has got to be 3 times 3 equals 9, and times 3 is 27. So he was able to communicate that. |
| 207 | | KK | Right. [inaudible]. He was never going to get passed that. I didn't know what he was talking about so It will be a mystery forever. I was okay with that. |
| 208 | | RW | Well I made it hard on myself because I chose taxicab. |
| 209 | | UCT | What was that? |
| 210 | | RW | So taxicab was is like a problem where you had to find all the streets. |
| 211 | | NV | [new group] [inaudible] So at first, one day I was out, they asked can we do the blocks again? So I tell them that we are learning, when we come back, we can show them what we did the day before. So they find it is funny that we do the work before we give it to the kids. |
| 212 | | UCT | My students said you did this project? What did you get? Tell me the answer and I was like no! |
| 213 | | BK | I was like I don't know what the answer is. |
| 214 | | R4 | Do you think it is necessary to work it out before um..? |
| 215 | | NV | Absolutely. Absolutely. |
| 216 | | R4 | What are some reasons why? |
| 217 | | NV | I think that it gives you a good outlook to where your kids may have common misconceptions that may be brought up during solving what are the possible solutions like I said earlier. Solutions that a kid may say that may not be something that I thought of but when the cohort sits down that day and solves it together, somebody else might bring it to my attention. So I am walking in not blinded by my thoughts. Right. |
| 218 | | R4 | So did doing the problem beforehand help you attend to the reasoning of your students? |
| 219 | | NV | I think for me I think so. I can't speak for everybody else but … |
| 220 | | VB | It also helps you with questioning. And if students had a problem with it. |
| 221 | | TD | Yeah, right. And not lead them. |

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| 222 | | VB | Maybe a student will struggle with what you went through. |
| 223 | | TD | Like if we did something in class that we didn't do ourselves they wouldn't do it.[inaudible] [laughter] |
| 224 | | R4 | Did your students find it meaningful…the tasks? |
| 225 | | VB | I just think that they thought it was a fun activity but it wasn't related to what we were learning so they just thought it was fun. Like a free day [inaudible] |
| 226 | | BK | I had a lot of trouble with the tasks because in my district, I teach 9th grade now. So we were like okay, towers? No. with the videos presented; the kids remember the cubes but they don't remember any sort of a method or technique. |
| 227 | | UCT | Right Yep! |
| 228 | | NV | That was the whole issue with the video like I was saying earlier. You see Stephanie as a small child that is very engaged, and questioning and she's energetic and you see another video of her when she is in high school and they ask her oh what about this and she says I don't know. I don't remember. |
| 229 | | BK | Yeah! When I got to high school, I didn't feel so bad anymore! |
| 230 | | R4 | Do you teach high school? |
| 231 | | BK | Yeah. |
| 232 | | R4 | What grade? |
| 233 | | BK | Grade 9. I used the tasks with…not with my main algebra kids I used them with my enrichment classes only because that was the only place I could squeeze problems like these. Especially with the lower kids, and like I said they remember using cubes but they do not remember anything else about the task and it was a distractor. |
| 234 | | R4 | Now is enrichment class another class besides their math class? |
| 235 | | BK | It is. |
| 236 | | R4 | It is. |
| 237 | | BK | So it is in addition to you know their algebra block but if they score don't score a proficient on the NJASK, they are placed into this supplemental class. |
| 238 | | R4 | Oh, so it is a supplemental class. |
| 239 | | BK | They are required to take it. [inaudible] They get placed in it based on their test score. They lose an elective. |
| 240 | | R4 | And you only can do math in there or can you do other subjects? |
| 241 | | BK | Only mathematics, Right. |
| 242 | | R4 | Okay. |
| 243 | | BK | This is fine but like I said. I can use these problems when I finish a chapter like writing equations or graphing |

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| | | | equations. I can use this as like a day in between. Otherwise you are doing it in the middle of it won't make sense. |
| 244 | | R4 | So you are saying you are trying to find a way to connect these tasks to what's being taught in the curriculum or tested. |
| 245 | | BK | Basically yeah. |
| 246 | | R4 | So Pascal's triangle is not something…. |
| 247 | | BK | I would not come across that in enrichment class. No. I could have done this with my algebra but I have a block class that I meet every other day and that pacing guide is a lot more rigorous but I would not be able to fit something in like this every day. Hopefully I have kids that were in both classes and when we get to something like Pascal's triangle, they might be able to make the connection. |
| 248 | | R4 | I completely understand. I also teach high school. So, I understand your position. |
| 249 | | BK | Motivation is lacking for sure. So when I saw Stephanie as a high school student that was very pleasing because I didn't feel bad. |
| 250 | | R4 | Okay, anything else that this group wants to add that maybe we can find a way to help you with resources or whatever… |
| 251 | | JU | The tasks that were given, we are a 6,7,8, high school, there are different teachers in our cohort. And maybe they can make like a pool of tasks we can choose from that are more like aligned to maybe what we do. Depending on whether you are from September to December or January to March for Fall and Spring semesters that lies with Common Core. Because if you teach the same grade, We are supposed to be doing the same thing at about the same time. So if there was more tasks that the cohort can choose from, then they can pull it and it would fit more seamlessly into the curriculum. So you can have it be part of the curriculum and have an open-ended discussion so that is my suggestion. |
| 252 | | R4 | Okay. Good. Those are all great suggestions. Thank you! |
| 253 | 01:09:22 | R2 | I think we will start with the group in the back by Solariz [group called back] Just start talking…remember the question: something that was important to you about students' reasoning about mathematics with any examples that you might have. |
| 254 | | VB | I don't know if mine is an example but I was just impressed with my one group of students are pretty low functioning how they organized their 3 groups which they |

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| | | | were very disorganized and when I came over, they had the constant, um the top one, like blue blue, red yellow. So I was pretty impressed with that because they figured out how to…had a disorganized mess and they put it together like that. And then they could see like that had 2 groups of 9 and one group of ten. And I didn't really prompt them that much. Well if you have two groups of 9 and you have 3 colors. Why do you think we have this extra group of ten? And they were able to come up with if they had a duplicate. They didn't really give me a detailed explanation. But I was just proud of them that they could figure it out on their own. |
| 255 | | R2 | And they began with a strategy of holding a constant. |
| 256 | | VB | Yeah. |
| 257 | | R2 | How about other people? |
| 258 | 01:10:40 | NV | Yeah. Solaris and I kind of came up with the same thing. We would agree once the other one started talking. It was interesting for us to see is that when you looked at how the kids organized the towers when they were completing… It was either both tower activities, the first and the third. But how they grouped certain towers that they believed belonged in a…yeah together and again it goes back to what I said earlier. It may not be how I grouped them but once you ask them what was your thinking with this group? Why is this a group and why is this particular tower not a part of the group? And they offer that explanation, they were right. It was just a different approach than what others or including myself would have come up with. |
| 259 | 01:11:35 | R2 | I think that is really interesting. And maybe somebody… I can't remember if it was the interview in Brunswick or Edison or if it was the interview in wherever the other one was [laughter] but in one of those 2 interviews, when we asked the kids, was it yours? Or was it yours? When they asked the kids to compare tasks, they were saying that this one which was Ankur's challenge was so much easier for them. Was that in your group? And in your school? And it blew my head because they were saying it was so much easier than the four tall with 2 colors, selecting from 2 colors. And they said this one was so much easier. After they had gone through all this stuff that had been so challenging to you in terms of solving Ankur's challenge. And they found it easier for exactly that reason because all of a sudden well not all of a sudden; but over what they had been doing. They had begun to think about strategies |

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| | | | like what you were just describing. Okay we are going to hold these red ones constant and work with them that way. Which they didn't do for the first one. Okay, that's really interesting. How about this group? What did you learn? |
| 260 | 01:13:14 | UCT | My group thought that it was interesting how they did the first tower task, most of my students either didn't organize or just did opposites. Um and then obviously like with some teacher prompting they reorganized them to a way that they could prove that there weren't anymore<br>But then when I did the 3 tall with the 3 colors, they got it so quickly and they immediately went to multiplication. They didn't have the exponents but they were doing 3 times 3 times 3, and they could explain to me what two of the those three 3's stood for. We hadn't talked about the towers at all and so it was interesting to me that they made this huge jump. That I don't know where it happened, but it was really good thinking then I had two students that could actually explain to the class the exponents behind it. So it was very interesting because we hadn't discussed it at all. Somehow they pulled it out where they did it the first time. So… |
| 261 | 01:14:08 | R2 | What else? |
| 262 | 01:14:10 | ES | It was neat to go back and look at the kids work. And kind of at that point see some connection between what they did on the first task and the second task; and the first task to the third. And really see the way there were commonalities and how they were um…approaching it. Um we have this new rotate and drop schedule in our school and my kids coined this phrase when using our rotate strategy. So they had figured it out the first time, and then they just kept talking and talking about that because of the tasks. |
| 263 | 01:14:45 | R2 | That is really interesting. Can you explain what the rotate strategy was? |
| 264 | 01:14:50 | ES | Yeah. So, for our schedule, you know there is 8 classes but 2 drop every day. One in the morning and one in the afternoon, and then they pick up the next day at the top and then they just keep moving so you never have the same kids at the same time of day. And so their classes are constantly moving and they made that connection to the towers that this yellow block on the top is now in the middle and is now at the bottom. And now it is back at the top, and I can't move it again because they are four-tall or you would be back at the top again. |
| 265 | 01:15:26 | R2 | Did ahh.. you know that is really interesting. And in so |

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| | | | many of the classes where other, I don't know if Judy,…whether it was yours…so many of the students really solved those towers both with 3 colors and also with especially with 2 colors. Uh as what we call rotational groups. And so they would lump em together. They would say oh don't you see these four towers that if you take the top one off and put it on the bottom, or the other way, whatever and you keep doing that. You get back to where you started which means you can't have any more in that group. And then they move to another one, and they do the same thing. So what you just gave me as information as maybe it is because of your schedule … |
| 266 | 01:16:17 | ES | And they told it to me. They said well this is like our schedule. Remember, so like for the third task … remember we talked about that. [laughter] It is the rotation strategy. I am like okay! [laughter} |
| 267 | 01:16:31 | R2 | There's the metaphor! What about you guys? |
| 268 | 01:16:35 | HS | Um, I saw growth, between the first tower task and the…the third tower task as well. It wasn't quite as much growth as Brittany's saw with her kids but when they first did um the tower activity. They just went at it and they built them and used opposites to check and make sure that they had them all and there were no duplicates. By the time they got to the third task. Um which was towers as well but then I had students who were creating groups. Um and like Natalie said they weren't grouped the way I would group them but they were able to name the groups and explain like this is why these belong together and they were able to use that as a strategy for checking. So it is nice to see that you know we went from very loose organizational strategy and just using uh opposites to check to having now I have an organization system that will help me to confirm that I have them all and there are no duplicates. |
| 269 | 01:17:29 | R2 | What about organizational structures for the pizza problem? Was it among all your students was it different from the others? |
| 270 | 01:17:44 | DH | Well for the first ones for the pizzas. I had a lot of kids just started just trying to make pies and like drawing circles for pies. After like 5 or 10 minutes, they were kind of like this is dopey and I am not doing it. Because all they really were doing is they were just writing out a pie and writing pepperoni, sausage mushrooms and they realized they didn't have to keep making a circle for the pie over and over again. But I mean Solaris do you want to talk about |

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| | | | the connection? Because I think everyone should hear about this. We had this wonderful student in Solaris' class. |
| 271 | 01:18:16 | SO | Um I had a Brandon in my class who… |
| 272 | 01:18:18 | DH | Way smarter than Brandon! [laughter] |
| 273 | 01:17:21 | SO | Who was able to make a connection between the pizza task and the towers problem and I was definitely blown away because I said it's not the same problem. It's not related at all. And he is like…yes it is! And he was able to um to explain it to probably almost every single teacher that was there and that's why Dan said before he probably wanted to throw something because he had to explain it over and over because we were so impressed that we wanted him to share with everyone. But he visualized um the toppings being the towers and having a certain topping be um a certain position in the tower. And then the colors being either being the actual topping being that they were being the absence of the topping and he was able to explain that and that was great to see him make that connection which is why I featured him in my portfolio! |
| 274 | 01:19:18 | DH | It was awesome Solariz because we got to the point where Dr. Alston asked where is pepperoni and sausage? And he was like here. Where is sausage mushroom? And it was kind of like duh.. they are right here! What's wrong with you guys?[laughter] We would be like uh this one? And he was like no, that one. But he was flying through it. It was amazing. |
| 275 | 01:19:46 | NV | I am trying to think who's class we were in. |
| 276 | 01:19:59 | HS | It was me! Where he made the pyramid? |
| 277 | | NV | No. Where the kid made he invented his own pizzas and he combined it was mush pepperoni. [Ohhh-multiple unison responses and conversations inaudible] Yeah. It was the way he was organizing it. |
| 278 | | UCT | It might have been mine. I didn't write a paper so he didn't have the solution. |
| 279 | 01:20:19 | R2 | Oh and we were looking at his work. It was when we were studying pizzas. |
| 280 | | HS | Yeah he made up his own pizzas and a name. |
| 281 | | R2 | At this point, before I move back over here in terms of that I have my two high school teachers, Alice and Bhupinder, how would you respond to this or any of the questions? |
| 282 | 01:20:44 | A | The pizzas? Towers? |
| 283 | 01:20:45 | R2 | No the question about kid's reasoning strategies and any changes? |
| 284 | | A | Um…I'll be honest, my kids did not show like any reasoning. I really had to explain… |

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| 285 | 01:21:00 | BK | We both mutually decided how to use these tasks because being at the high school level, and throwing it into one of our regular classes it was too hard to you know put it in the pacing guide. So Alice and I both teach an enrichment class which I guess if you want to think about the kids who have not passed the state test the year before. They get an additional class. |
| 286 | 01:21:21 | R2 | So it is not really enrichment. It's remedial. |
| 287 | | BK | Well I guess so. So we were able to use the tasks like in between chapters like starting or ending a chapter before starting the next one. So that way it would not mess up the whole constant flow. But um yeah obviously we had like the lower level kids so the reasoning with them was not the same as what we saw on the videos or what we saw in the regular general ed. classes. Like I used the first few tasks, shirts and pants and the first towers task and maybe one student in a class of 18 was able to get the answer. |
| 288 | 01:22:00 | A | I agree. |
| 289 | 01:22:01 | BK | So the reasoning was a lot different. I had to work on them with their organization skills as a way to guide them through their problems. Because they didn't understand the convincing part. Like they were like How can I use organization to convince you about a math problem. And I was like no you are proving to me that you have taken care of all the possibilities. And that's when they started realizing like okay so this is the way I should go. So at one point what I did with the pizza problem with them and that was the complicated one, it was four toppings, and he said you could do it on a whole pie or a half a pie or the mix and matching of all that. And I showed them the way we did it together in class in terms of our organization. And thereafter, their towers their reasoning got better with organizing it. But prior to that, being that they were the lower level kids. They struggled a lot. Even shirts and pants, like shirts and pants hey did tree diagrams and as soon as we started pizza problem, they wanted to do tree diagrams. And it got too overwhelming for them to even realize what they were doing. |
| 290 | 01:23:08 | R2 | So even with the ability to select an appropriate strategy….is something uh…and yet doesn't it make you feel uh sort of surprised that they came up with it. |
| 291 | 01:23:23 | UCT | A little bit. |
| 292 | | R2 | So here we have talked about kids that are from that strata of class who surprised you and did very, very well. And yet something is happening or something is not happening |

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| | | | that that there is discovery. |
| 293 | 01:23:45 | A | They really get they get frustrated I think like just our lower levels and we are not at the same school district but we have the same type of classes. Um, they get very frustrated when you expand a problem at all. Like in any way shape or form. So like if I gave them, everything from the very beginning. And they created whatever they could, they came up with whatever they could most of the time it was just random. And then I would say to them, about half-way through the class like to the whole class I would say okay well you guys came up with this stuff and you know how do you know. And I was trying to get them to come up with a reason. And they were like I don't know, I just did that is what came to mind. And then I would say okay well that's fine I let that go. What about grouping them into categories? That's too much. No way I can't do that. I can't put them in categories they are all different towers. And that would be their response. So even with that of just the simple idea of making the towers and thinking about well how are they related from one I created was too much for them. And I really had to push to get them to write a sentence. Like even just a sentence. They were like no, that is not that! |
| 294 | 01:24:59 | BK | Also I think the last class that we visited was Karen's and her first class was an honors class and then her second class was a mixed class with special ed. Even then I think I pointed out that the honors kids were able to get their you know their logic into words on paper. Whereas the next class, they drew pictures maybe but that was it. There was no sort of writing involved. So I guess that part of it continues even for I guess the lower level kids which I guess in my population includes special ed. kids as well. So that kind of there is a reoccurring theme for all of them. |
| 295 | 01:25:35 | R2 | Well that is really something important for us to think about. And it really addresses ah ah you know core curriculum standards are after. |
| 296 | 01:25:50 | KK | Yeah I definitely see that. And I know but we did Ankur's challenge with the honors kids and then we just did the regular the 3 colors um for the regular or general ed. students and they all got it pretty much but then when we said write up what you found, they were like staring at the paper and they didn't know how to even start to put their ideas on paper. |
| 297 | 01:26:17 | R2 | I forget what grade you are. |
| 298 | 01:26:19 | KK | $6^{th}$. They were just well here you know we did it. So for |

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| | | | them to write down an explanation of what they did or how they did it. It was like overwhelming for them. So I think that's going to be a test for them. You know with the common core to actually explain why it works. |
| 299 | | R2 | Do they even know what that means? Because I remember some of the little guys in your class. They were so little and they really had done such a good job. But for them to to… have enough confidence about themselves to say Oh, this is very obvious. You know I think rotational was one of them. Some of them said can you just put an arrow. What else while we are over here? As we go now, because we only have about 5 more minutes. And we have already done this. The last question that I asked you all was sort of about the role of the teacher and the decisions of the teacher and the tasks. We have been talking about it all morning but we are specifically talking about that right now. What other kind of…yes right here? |
| 300 | 01:27:29 | CP | One of the things um that I saw was growth of confidence and perseverance through the problem-solving process and in particular with the writing process of the explanation. Luckily I have a very small class. And I was able at one point to sit with my two students who were excelling in solving the problem but really couldn't get their thoughts into words. Um so I was able to sit there, kind of help them get each idea down to kind of get them started. Um I mean one of um.. my girls struggled so bad that she actually started crying at one point. Because she got so frustrated with it. Um but then to go to the third problem, the other tower problem, she didn't even really ask for much help with the writing. She kind of because I was able to sit there with her and give her such guidance with that one the first time. |
| 301 | 01:28:23 | R2 | For the first one. |
| 302 | 01:28:24 | CP | Getting her thoughts into words and figuring out how to write what she is trying to say and explain. She was able to do it that much better the third time to where she didn't really ask for much help and she was able to organize it and describe them group by group. For how she grouped them and I was really happy to see that. |
| 303 | 01:28:43 | GH | I think I saw tremendous growth in just their understanding of the words, convincing argument. Write a convincing argument, justify the reasoning, explain your way of thinking to them in September was this is what I did step by step. And then the last task they could say I knew there were only 3 solid towers because there were 3 colors. Like |

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| | | | they were able to justify the reasoning and it was that progression that I saw of just the understanding of what that meant to convince us. |
| 304 | 01:29:14 | R2 | And if only those 2 high school classes had some other opportunity in the years moving up where other than always being hit by the word, explain. Which means …. You were going to say….. |
| 305 | 01:29:37 | RW | Um, I was going to say that I did not experience some of the teachers' tremendous growth because of which problem I selected. But I think that from listening to the other teachers reflections, I realize the importance of some type of continuity. Because if I were to have chosen Ankur's challenge as opposed to the taxicab problem, I think my students may have experienced that but the problems that I chose, every session was uh like frustrating because it was a new problem. For the same type of dynamics that I experienced that I had with the first tower problem, I experienced with pizza, I experienced with taxicab. And so I am like okay this is just a challenge across the board. After it was all over and I took a step back and I am thinking that I should have chosen the Ankur's challenge because they may have been able to make that leap or to express themselves clear in their writing instead of all these problems being new. Because my thinking was that out of these three, One of them are going to click and they are going to get the ah ha moment. But… |
| 306 | 01:30:51 | R2 | But you are saying is the importance of building. |
| 307 | 01:30:53 | RW | Yeah… |
| 308 | 01:30:54 | R2 | Is important. But what is so fascinating and I think all of us have talked about this a little bit is that the strand of problems are all connected. but where our students get…on any one of them, may not be a jumping off place for the taxicab. Do you understand what I am saying? |
| 309 | | RW | MMhh. |
| 310 | | R2 | And I think most of us to a certain point, and those of you taking the intro course you will learn the taxicab. And it is the same structure. It has… it is isomorphic to…. with that new word that we all learned… both the towers with 2 colors and the pizzas with 4 toppings, and Pascal. But if your kids are at a certain place and that, when you talk about the role of the teacher. |
| 311 | | RW | Right. |
| 312 | | R2 | You are doing the kind of reflection. |
| 313 | | RW | Yeah, If I had to do it all over again I would have left it |

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| | | | out. |
| 314 | | R2 | But you know what I would do? |
| 315 | | RW | What? |
| 316 | 01:32:06 | R2 | Unless you think that the other thing is something that you are just going to throw out, come back to it. Come back and do the 3. [laughter] but in February sometime or maybe next school year. I haven't heard from this table at all about... |
| 317 | | RW | Next year. |
| 318 | 01:32:19 | R2 | but in February sometime or maybe next school year. I haven't heard from this table at all about... |
| 319 | 01:32:33 | MM | I just want to make a comment about somebody said that the students didn't think it was math writing down their reasoning. Cause I find that I did the problems with gifted and talented students. And I found that they struggled as well with writing down their reasoning. They all saw it was four times four and then they said they found the 16 towers by doing four times four. That's not how they got the 16 towers but I think that they grasp onto a process because we kind of trained them to think math is just plugging and chugging. And here is an equation, solve the problem, don't explain your reasoning, get the right answer. And now when we try and incorporate more with the Common Core, more of the mathematical practices, students are use to math being and especially the gifted and talented students I think that they… I think that is why they like math so much because it's just doing something that… |
| 320 | 01:33:33 | R2 | Can I ask you as teacher questioning...when they all see that four times four [laughter] what did they say with 3 colors? |
| 321 | 01:33:44 | MM | Um they said it was 3 times 3 times 3. |
| 322 | | R2 | No, no, no. I am talking about when you make.. I mean that is the reason we start with towers four tall because almost always they say four times four is 16 which is the reason that you don't start with the little ones and get bigger. Bu then when you can say what about selecting from two colors when you are building them 3 tall? Almost every child probably said 9 and they did it in every class we went to said 9. And then you say well find them. And then they come up with saying there are 8. But then they say it has to be an even number. Because of the opposites which is great and so it is 8. And then when you say what about when there are 5 tall? They all still want to say 25, well it has got to be even, so 24. And then you begin to build so that they can see that four times |

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| | | | four…you understand what I am saying? And so just to reflect back on the nature of the tasks and the nature of the questioning in it. And I am so glad that you brought that up! What else from this group? |
| 323 | 01:35:15 | UCT | Well I noticed that when my students feel success in doing the tasks. Because when we did the last task with Ankur's challenge, I hadn't heard a single question, am I doing this correctly? For the first task, is this correct? For the second task pizza is this correct? For the last one, Ankur's challenge, okay this is what we did and this is why we did it. |
| 324 | 01:35:41 | UCT | Wow![some hand-clapping] |
| 325 | 01:35:42 | R2 | What a wonderful way to finish! But I know in both of the central and northern regions that we were so surprised at the number of different methods those kids had in Edison and in Lyndon and more than we did. When we did it, we made it a very fruitful problems and which is what does the task have to do with it. The fact is that almost all of them got the 36 relative but for them the 36 wasn't the answer. But this is really great! Okay so in our final 5 minutes Marjory is going to talk to you a little bit about. |
| 326 | 01:36:44 | R3 | Sure, I have a bunch of things to cover so. Okay. You know that there is a post test that you all need to do. |