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CREATING AND USING VMCANALYTICS FOR PRESERVICE TEACHERS' STUDYING OF ARGUMENTATION

By

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ABSTRACT OF THE DISSERTATION

Creating and Using VMCAnalytics for Preservice Teachers’ Studying of Argumentation

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Teacher recognition of student argumentation has been addressed by many researchers (e.g., Schwarz, 2009; Krummheuer, 1995; Bieda & Lepak, 2014; Whitenack & Yackel, 2002). Further, standards for mathematics learning emphasize the importance of including argumentation in the K-12 classroom (NCTM, 2000; CCSS, 2010). The study reported here with secondary preservice teachers to identify argumentation from video, adds to earlier, successful work using video to support the learning of preservice teachers (Sherin & Han, 2004) and to support teacher noticing (Van Es & Sherin, 2008; Star & Strickland, 2007). Further, studying video narratives has been used effectively in university courses to encourage students to engage with complex issues, collaborate, discuss, and build representations, and can help instructors gain insight into student thinking (Hmelo-Silver et al., 2013).

This qualitative case study examined the effect of a semester-long intervention with eleven preservice teachers during the spring of 2015. Episodes of student argumentation from the Video Mosaic Collaborative (VMC), an open-source video collection, were used to construct video narratives using the RUanalytic Tool (Agnew, Mills, & Maher, 2010) of students engaged in argumentation. Findings from this study indicate that video narratives supported growth in the preservice teachers' understanding
of student argumentation, indicating that for 93.3% of the events in the study, at least 45% of teachers exhibited growth and for 73.3% of the events in the study, at least 54% of teacher exhibited growth. Teachers demonstrated growth with respect to: (1) the elements of argumentation and (2) the structure of the argumentation they described; as well as (3) the use of the technical language of the formal mathematical register of argumentation they used. Other changes included making implicit argumentation explicit, using students' actual language rather than interpreting students' statements, and eliminating untrue statements about the argumentation in the events. Furthermore, there was a relationship among the categories, for example, the growth in use of technical language correlated to both growth in elements and growth in structure. Implications of the study and future research recommendations include comparing the results of other iterations of the study and using VMCAnalytics to support other mathematical practices.
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"Tell me and I forget. Teach me and I remember. Involve me and I learn."

Benjamin Franklin

Chapter 1 – Introduction

1.1 Overview

Argumentation is an important mathematical practice for the K-12 mathematics classroom. Standards for mathematics emphasize the importance of including argumentation in the K-12 classroom (NCTM 1989; 1990; 2000; CCSS, 2010). The National Council of Teachers of Mathematics Process Standards state that students from prekindergarten through grade 12 should “develop and evaluate mathematical arguments” (NCTM, 2000) and the Common Core State Standards of Mathematical Practice state that students should “understand and use stated assumptions, definitions, and previously established results in constructing arguments” (CCSS, 2010). An important component of teacher knowledge is understanding and recognizing student mathematical argumentation.

The design-based research study (Brown, 1992; Kelly, Lesh, & Baek, 2008) reported here makes use of episodes of student argumentation from the Video Mosaic Collaborative (VMC) video collection to construct three video narratives using the RUanalytic tool of students engaged in argumentation. These data made it possible to make visible what argumentation looks like in students at different grade levels.

Secondly, case studies are reported with a class of fourth-year, preservice teachers (participants) to investigate whether their study of the VMCAalytics improved their understanding of student argumentation and whether the viewing, studying, and discussion of these analytics shaped their understanding of student argumentation in
learning mathematics. Participants had the opportunity to study the video narratives described by the VMCAnalytics and to respond to guiding questions in an online discussion format. Pre- and post-assessments were designed for teachers to study a VMCAnalytic of students engaging in argumentation and to write descriptions and titles for each event, describing in detail the argumentation they observed in the video narrative. An in-depth qualitative and quantitative analysis of these data was conducted to determine if teacher knowledge of student argumentation improved after the intervention. The over-arching questions that guided the study were:

1. What does student argumentation look like in problem solving settings?
2. How can VMCAnalytic video narratives support teachers’ understanding and noticing of argumentation in student discourse? Specifically,
   a. What do teachers notice about student argumentation before the intervention?
   b. What do teachers notice about student argumentation after the intervention?

More fine-grained questions emerged as I conducted an in-depth analysis of the data:

3. After the intervention:
   a. Do teachers notice more elements of argumentation? Specifically, do they notice more claims, data, warrants, backing, counterclaims, and counterarguments?
   b. Do teachers notice more of the structure or connectedness of the argumentation? Specifically, do they notice how the elements of argumentation in each event are related to each other and the relationship among the elements in one event to the elements that were presented in prior events?
c. Do teachers use more of the formal mathematical register by using more
of the precise language of argumentation?

d. Do teachers add specificity to their descriptions by adding detail
relevant to the argumentation in the event? Do the added details clarify elements
or structure of the argumentation described (so there are fewer lines but they are
solid)?

4. If teachers' pre-assessment description included implicit elements or structure,
do they in the post-assessment make any of the implied argumentation explicit?

5. If teachers included statements in the pre-assessment that make note of
argumentation that is not actually presented in the event, do they eliminate these
statements as part of their post-assessment description?
Chapter 2 – Literature Review

The following review includes research relevant to the Video Mosaic Collaborative repository website (videomosaic.org), the RUanalytic tool, the VMCAnalytic, using video artifacts in teacher education, teacher noticing, and argumentation. Since reasoning, justification, proof-like reasoning, and proof are often included in the literature involving argumentation, my review of argumentation will include research on reasoning, justification, and proof-making literature, as they are relevant to argumentation.

2.1 Video

2.1.1 Video Mosaic Collaborative. Recently, with the advancement of next-generation internet, projects and initiatives have been developed to show the promise that next generation high speed internet has for developing applications to education, libraries, and museums (Geske & Stanchev, 2013). Wilson and Jantz (2011) report on how institutional repositories are being used in university settings to capture and preserve growing digital collections and make them widely accessible and Leonard and Derry (2013) note that these repositories and collaboratories can make available digital artifacts (including video recordings) to collaborating and non-collaborating groups for use in a variety of settings, including educational settings. Rutgers University Libraries has taken a lead role in utilizing next-generation technology (Geske & Stanchev, 2013; Otto & Ralston, 2011) by developing a powerful, community repository that "facilitates scholarly collaboration and communication" (Wilson & Jantz, 2011, p.6) and has the ability to capture the output of Rutgers University, including digital video. This parent portal houses artifacts from various collections at the university. One of these collections houses the Rutgers equine
behavioral responses videos; another is the Video Mosaic Collaborative repository (VMC; videomosaic.org) (Otto & Ralston, 2011).

Mathematical education researchers from the Robert B. Davis Institute for Learning (RBDIL) of the Graduate School of Education at Rutgers University, led by Professor Carolyn A. Maher, have studied the development of students’ mathematical reasoning in longitudinal and cross-sectional research studies for more than two decades. Students in these studies are encouraged to investigate cognitively challenging, open-ended problem tasks in learning environments that enable discovery and support and encourage student engagement (Palius & Maher, 2011). Students, working in a variety of settings, including pairs, small groups, whole class, and one on one, actively participate in making sense of mathematical ideas and developing personally meaningful representations and convincing arguments to justify their conclusions. The students defend their reasoning while working on a wide variety of problem-solving tasks over a range of mathematics topics including combinatorics, fractions, early algebra, geometry, probability, data analysis, and pre-calculus. These mathematics problems could be considered non-routine. The problem-solving sessions were captured on videotape and the more than 4,500 hours of resulting videotape footage now form a collection of data that provides documentation of how students from early elementary through secondary grades, and beyond, build knowledge of important mathematical ideas.

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1 Research on children’s mathematical learning was supported by three grants from the National Science Foundation: MDR-9053597 directed by R. B. Davis, C. A. Maher; REC-9814846 directed by C. A. Maher; and REC-0309062 directed by C. A. Maher, A. B. Powell, and K. Weber and a grant from the N.J. Department of Higher Education: 93-992022-8001. The Video Mosaic Collaborative is a research and development project sponsored by the National Science Foundation: DRL-0822204, directed by C. A. Maher with G. Agnew, C. E. Hmelo-Silver, and M. F. Palius.
It was to make these data publicly accessible to educators, teacher educators, students, and researchers (Geske & Stanchev, 2013) that researchers from the Rutgers digital library sciences, in collaboration with researchers in mathematics education, educational psychology, and with substantial funding from the National Science Foundation (NSF), developed the Video Mosaic Collaborative repository. Some of these video data have been analyzed and preserved in the form of digital clips involving critical mathematical events in accordance with the model described by Powell, Francisco, and Maher (2003). Other data have been preserved digitally in their raw video form. Still other data have been preserved in videotape form. The repository houses some of the digitized video data and related materials, including event descriptions, transcripts and student work that allow viewers to trace the development of students’ ideas over time and data are continually being added to the repository. The VMC is a free online resource that is available worldwide.

Videos from the VMC repository have been used successfully in teacher education programs to support pre- and in-service teachers' recognition of student reasoning (Maher, 2011; Hmelo-silver, Maher, Agnew, Palius, 2010), to support the professional development of experienced teachers (Berenson, 2011), and to cause a positive effect on preservice and in-service teachers' beliefs about what students can do with regards to mathematics (Maher, Landis, Palius, 2010; Maher, Palius, Mueller, 2010). For a more detailed description of how video, in general, and the videos from the VMC, specifically, were used in research studies, see section 2.1.3, *Using Video in Teacher Education Programs.*
2.1.2 RUanalytic Tool and VMCanalytics. Derry (2007), in an NSF funded report on video in educational research, argued that "cyber enabled video tools" are an effective tool in analyzing large amounts of video data into segments that are usable for teacher education and Powell, Francisco, and Maher (2003) concurred, stating that video tools support teachers' and researchers' observation of students learning and reflecting on mathematics. In light of these findings, and to make video from the rich collection of research on students' mathematical reasoning available to the broader community, Rutgers Libraries has created an award winning2 RUanalytic video annotation tool, linked to the Rutgers Libraries repositories, which has been tested and used for research over the last few years (Agnew, Mills, & Maher, 2010; Gomoll, Sigley, Winter, Hmelo-Silver, Maher, 2015; Leonard & Derry, 2013; Goldman & McDermott, 2007; Wilson & Jantz, 2011; Otto, 2012; Otto & Ralston, 2011; Sigley & Wilkinson, 2015; Maher & Sigley, 2014). The RUanalytic tool draws upon Rutgers Libraries RUcore "generic services such as search, retrieval, access, and metadata" (Wilson & Jantz, 2011, p. 14). With the RUanalytic tool, users can search the videos housed on the repositories, select segments of videos within the collection, and link together and annotate those chosen video events to create a seamless video narrative, or RUanalytic, that has a particular purpose and can promote scholarly conversation (Agnew et al, 2010; Palius & Maher, 2011; Leonard & Derry, 2013; Otto & Ralston, 2011). Decisions about what to select and with what level of granularity those selections are examined "are driven by researchers' perspectives and goals" (Leonard & Derry, 2013, p. 441). Users can interpret and annotate the events they have chosen for their RUanalytic. Each new RUanalytic, licensed by the creator, tells a

2 Otto (2012) and Otto and Ralston (2012) report that the RUanalytic tool won the NJ Library Association College and University Second/ACRL's NJ's Technology Innovation Award for 2012
unique story conceived by the user and can be used for research and educational purposes (Hmelo-Silver, Maher, Alston, Palius, Agnew, Sigley, & Mills, 2013). When researchers select video segments from a collection and use them for a specific "analytic purpose," minimally edited video, for example, the raw video on the VMC, become data (Goldman & McDermott, 2007). The use of the RUanalytic tool by different groups enables researchers and students from multiple fields to contribute and share analyses of videos hosted on different repositories linked to the tool, including the VMC repository (Leonard & Derry, 2013). An RUanalytic can be published in the repository, assigned its own persistent URL, and then referenced in the same way as any published work (Otto & Ralston, 2011). RUanalytics constructed from the video collection housed in the VMC repository are called VMCAanalytics. The VMC, together with its collaborative domain-specific (mathematics) tools, data, and metadata, is one of the emerging features of institutional repositories (Wilson & Jantz, 2011, p 14).

VMCAanalytics have been used by researchers to illustrate important ideas. Sigley and Wilkinson (2015) used VMCAanalytics in their research into the interdependency between developing mathematical understanding and the development of the formal mathematical register. Maher and Sigley, in S. Lerman's 2014 Encyclopedia of Mathematics Education, used VMCAanalytics to illustrate their definitions of the task-based interview and teaching experiment.

VMCAanalytics have been used effectively in university courses in a variety of ways. Hmelo-Silver and colleagues (2013) report that VMCAanalytics have been used to encourage university students to engage with complex issues, collaborate, discuss, and build representations, and help instructors gain insight into student thinking. Studies also
show that the VMCAnalytic has been useful as a tool in assessment, both formative and summative (Hmelo-Silver, Maher, Palius, & Sigley, 2014). Gomoll and colleagues (2015) describe how VMCAnalytics and the RUanalytic tool enabled students to apply successfully the learning course goals in two university courses and supported the application of theory to classroom observation. These researchers' findings support the idea that VMCAnalytics can be useful in other facets of education. How the VMCAnalytics can be used to support preservice teachers in their understanding of mathematical practices, such as argumentation, however, is still an open question.

2.1.3 Using video in teacher education programs and professional development.

Video has been used successfully in teacher education programs to support the learning of in-service and preservice teachers. Krummheuer (2007), using videotapes to analyze student argumentation, asserts that observing student interactions in video, rather than in real-time, promotes the identifying of new insights into classroom interactions. Star and Strickland (2007) promote using videos in teacher education because they allow preservice teachers to observe a wider variety of teachers, students, settings, pedagogical techniques, and content, than is possible in a typical field experience and with practicing teachers because it enables them to notice details of their own classroom environment that might have been missed when their attention was focused elsewhere (p. 108). Research focused on using video from teachers’ own classrooms (Borko, Jacobs, Eiteljorg, & Pittman, 2008; Sherin & Han, 2004, Brown, 1992; Lampert & Ball, 1998) in teacher education programs asserts that watching video tapes of classrooms allows observers to study learning as it actually unfolds in the complex context in which it naturally occurs (Brown, 1992; Star & Strickland, 2007). Lampert and Ball (1998) agree,
and assert that teachers are not always given the opportunity to study how students in their own classrooms learn. Videos, then, are useful for teacher learning, research, and professional development. Maher, Palius, and Mueller (2010) argue that when teachers study video, they become more aware of the complexity of learning and are more likely to focus on the process of unfolding mathematical reasoning rather than the final outcome, developing "adaptive expertise" (p. 886). Brunvand and Fishman (2006) state that:

> Much like classroom observations, video can provide insight into the complexity of teaching and the multitude of pedagogical issues that teachers face through the visual representation of classroom action accompanied by analysis, reflection, and explanations given by the teacher, students and other stakeholders (p. 152).

Sherin and van Es assert that "video has become an important tool for working with both novice and veteran teachers," (Sherin & van Es, 2005, p. 475) because teachers can gain access to classroom environments that are different from the ones they are used to, and, since video records are permanent, videotaped episodes can be watched over and over. These characteristics of videos are advantages over observing interactions in real-time. Brunvand and Fishman (2006) concur, asserting that the studying of videos (particularly videos with scaffolds) effectively enables teachers to observe pedagogical theory in real-life settings, which can aid in the development of their professional knowledge (p. 172).

Hmelo-Sliver et al. (2013) and Maher (2011) agree with Sherin and van Es (2005) and Star and Strickland (2007), that preservice teachers often do not have access to their own classroom and it is useful for students to access videos virtually. Thus, they support using videos from other sources in teacher education programs and that “existing video collections (such as the VMC) have great potential for developing resources for teacher professional development” (Hmelo-Sliver et al., 2013, p. 3,079). Maher, Landis, and
Palius (2010) report that video collections are important resources for both the professional development of in-service teachers and preservice teacher preparation programs. Maher (2011) and Maher and colleagues (2010), reporting on the use of videos, specifically videos from the VMC, emphasize that videos from collections offer teachers the opportunity to observe approaches (Maher, 2011) and knowledge (Maher et al., 2010) that they might not see in their own classrooms, specifically, approaches that allow students to engage in discussion, reasoning, argumentation, and that videos. Specifically, they assert that videos from the VMC provide a "window into an alternative setting in which communication, collaboration, and the sharing of ideas is the norm" (p. 85).

Leonard and Derry in their chapter in Luckin and colleagues' "Handbook of Design in Education Technology" (2013), also support the use of video in teacher education, calling it, "an indispensable tool in education research capturing detail and complexity in teaching and learning situations that other collection approaches cannot" (p. 439). They assert that the power of video as a data source is that it can contribute to multidisciplinary research, and that video collections, like the VMC, are useful.

Sherin (2003) describes the more than 40-year history of using video in teacher education. She explicates six uses of video in teacher education, including microteaching, modeling expert teaching, video-based cases, and field recordings. Although Sherin asserts that more empirical work needs to be done to investigate the role that videos can play in supporting teacher learning, she suggests that the affordances of video for teacher education include that “(a) video is a lasting record; (b) video can be collected, edited,
and recombined; and (c) video sustains a set of practices that are very different from teaching” (p. 11, Sherin, 2003).

Maher (2008) also suggests that video recordings have affordances for teacher education programs. She asserts that they can be effective tools for enhancing learning in mathematics teacher education. When preservice teachers view videos, they can learn how students create new knowledge and the effect that teacher moves have on student learning. Videos can suggest new pedagogical strategies. To illustrate her point, Maher reports on how a teacher in a graduate teacher education program benefited from studying videos in what might be called a “modified lesson study” (see Alston, Pedrick, Morris, & Basu, 2011). Through the analysis of video, the teacher was able to think deeply about her own mathematical ideas as well as the mathematical reasoning of students in the videos.

Palius and Maher (2011) reported on the effectiveness of video used in two intervention models in preservice and in-service teachers’ attention to students’ mathematical reasoning. In the model they used with preservice teachers, participants engaged in problem-solving tasks and then studied videos of students engaged with the same tasks. They found that the preservice participants had significant changes in the beliefs about how children learn and that participants showed growth in their identification of student reasoning from video. Maher, Landis, and Palius (2010) and Maher, Palius, and Mueller (2010) report that through the use of VMC videos, in conjunction with a modified lesson study, middle school teachers' and preservice teachers' beliefs about what students can and do and reason about with regards to mathematics were changed, showing significant positive growth. Maher, Palius, and
Mueller noted that after the intervention, preservice teachers' beliefs changed positively with regards to:

"the ability of students to solve complex problems in novel ways, value of student interaction and communication, the benefits of students offering multiple representations of ideas and approaches to solutions, the importance of students learning concepts through problem solving prior to being introduced to procedures" (p. 890).

They assert that teacher educators can use videos for preservice teachers' development and growth (p. 887). Teachers in these studies solved non-routine problems that were designed to elicit engagement in complex mathematical ideas and discussed their solutions with one another. However, results suggest that participation in the problems was not enough, and that "studying of videos of children's mathematical activity provides entry and enticement" into students' mathematical reasoning (p. 21) and was essential to producing the resulting changes in beliefs. Furthermore, in the Maher, Palius, and Mueller (2010) study, the positive growth in beliefs of the treatment group was greater than that of the comparison group who did not study the videos.

Hmelo-Silver et al. (2010) concur that videos and cyber-enabled video tools can be used to support learning. The VMC and the associated RUanalytic tool were effectively used in teacher professional development. In their study, they used video assessment to determine if teachers could recognize forms of mathematics reasoning. These researchers posit that further research that investigates how iterative interventions in different contexts can affect teacher learning and clarify what elements characterize effective interventions.

Maher, Palius, Maher, Hmelo-Silver, and Sigley (2014) support Hmelo-Silver and colleagues’ (2013) assertion that video allows preservice teachers to view examples of
students engaging in mathematical problem solving that would otherwise be inaccessible to students in teacher education programs. Maher et al. (2014) report results from studies conducted over three years with both in-service and preservice teachers, some of which are described above. Their studies included 73 preservice teachers in the experimental group and 36 preservice teachers in the control group. Their results support their hypothesis that videos available through the VMC can be used in teacher education and professional development situations to deepen awareness of students’ mathematical reasoning. Maher and her colleagues posit that more work needs to be done with teacher education programs at the elementary level. Palius and Maher (2013) also confirm these results. They conducted a similar study with graduate students in an online graduate mathematics education course. They found significant improvement in participants’ ability to recognize student reasoning after analyzing video. Maher (2011) reports that, as a result of studying videos from the VMC, teachers' recognition of the variety of forms of reasoning used by children improved.

Towers (2007) conducted a two-year study of elementary and secondary preservice teachers. In her study, she used carefully selected video excerpts from her own middle school mathematics class in mathematics education courses of which she was instructor. She traced the participants in her study from her class, which was the final semester of a two-year education program through their first full year of teaching. Her data collection included videotapes of her teacher education classes, as well as the classrooms of her participants, interviews with participants, field notes, and copies of student work. She argues that videos in teacher education support preservice teachers in a variety of ways. First, they help preservice teachers to refocus their attention from
teaching to student learning. Although there is a progression of attention toward the role of the teacher, Towers encourages her participants to begin by focusing on the students in the videos. She argues that this challenges preservice teachers to consider the perspectives of the students rather than their own perspective. Additionally, Towers argues that video can help to foster reflection on teaching practices. She found that when students moved away from looking at the student to looking at her teaching, they offered critiques of the practices that promoted deeper reflection of beliefs about teaching and learning. A third affordance that Towers reports is the offering of possibilities of “imaginative rehearsal.” This is accomplished by stopping the tape and asking participants to predict what might happen next and providing students with reflection time to imagine how they might respond to a student’s question, or how a student might respond to a teacher’s prompt. Towers reports how this technique expands preservice teachers’ conceptions of what students are able to do when, for example, they are surprised by the sophisticated response after predicting what the student might say. Towers’ findings support the findings of the previous researchers in that video can help preservice teachers attend to the “complexities and learning in classrooms” (np).

In a recent study, Martinez, Superfine, Carlton, and Dasgupta (2015) investigated the use of video to support the teacher training of two cohorts of high school preservice teachers. They concur with others about the affordances of using video in teacher education programs, stating that the videos gave these preservice teachers opportunities that more closely resembled the work of teaching. They found that videos provided their preservice teachers an opportunity to analyze the practice of teaching in ways that are different from their own teaching experiences and helped them to attend to and better
understand student thinking and explore mathematics concepts. Martinez and colleagues found that videos supported preservice teachers' development of "professional vision" or the ability "to notice and interpret significant features of classroom interactions" (p. 54), as well as supporting a more thorough analysis of teaching and the better identification of significant events in classroom interactions. The preservice teachers in their study reviewed video cases which included interviews, lesson plans, and written work, along with the videos of classroom episodes. From their analysis of pre and post assessments, Martinez et al. found that students paid more attention to student thinking and noticed a greater number of strategies, as well as more mathematical depth to the strategies they identified after the intervention than before.

In another recent study, researchers studied the affordances of using videos in conjunction with annotation tools to support preservice and in-service teachers. Lo, Lim, and Xiong (2016), in their study of in-service teachers, found that using a video annotation tool supported teachers' ability to better understand the events taking place in the classroom. Teachers in their study shifted their focus from surface features of classroom interactions to deeper pedagogical issues. They also noted that the descriptive analysis feature of the annotation tool was particularly helpful to teachers and in evaluating teachers' responses. They conclude that videos of classrooms can help teachers develop an awareness of situated contexts and that video annotation tool technology holds promise for teachers' development of reflective practices (p. 88).

2.2 Teacher Noticing

Since the advent of using video in teacher professional development and training, researchers have investigated teachers' development and growth through studying teacher
noticing from video. Math education reform encourages teachers to attend closely to students' ideas and to adapt their instruction accordingly (van Es & Sherin, 2008). Sherin and van Es (2005) and van Es and Sherin (2002; 2008) define noticing as "seeing" in different ways, (2005, p. 476). Proficiency in adjusting instruction in real time requires that teachers are able to notice and correctly interpret events in the classroom. These researchers, then, posit that the ability to notice is a key feature in developing teaching expertise because it helps teachers as they make decisions during class to change and progress classroom discussion and use what students say and do to facilitate learning. Making decisions "on the fly" is an essential pedagogical skill and if teachers are to develop a style of teaching that is adaptive to the needs of their students during any moment in a lesson, the need to be able to focus on key aspects of classroom environment and student interaction (2005, p. 476). Van Es and Sherin (2002) describe three features of teacher noticing:

1. Identifying what is important in a teaching situation;
2. Making connections between specific classroom practices and broader principles of teaching and learning; and
3. Using what they know about the specific teaching context to reason about a given situation (p. 477)

Researchers have used noticing to assess expertise in teaching (Jacobs, Lamb, & Phillip, 2010). These researchers distinguish regular noticing from professional noticing. Professional noticing is intentional and particular to a profession. In their studies, Jacobs and colleagues attended to the professional noticing of mathematical thinking of practicing kindergarten through third-grade teachers and preservice elementary teachers; discussing three aspects of mathematical thinking: "attending to children's strategies, interpreting children's understandings, and (as emphasized by van Es and Sherin (2008))
deciding how to respond on the basis of children's understandings" (p. 172). Participants watched videos of K-3 students participating in mathematical problem solving or reviewed written student work produced during a problem solving session, considered "artifacts of practice" (p. 177), and then wrote about the three aspects of mathematical reasoning they observed (attending, interpreting, and responding to students). Analysis of their data, taken from a larger study entitled "Studying Teachers' Evolving Perspectives" (STEP), suggests that teachers with more experience with children's thinking also had better professional noticing skills. In the study, the noticing skills of the preservice teachers were not as well developed as those of the practicing teachers. Their evidence from this cross-sectional study led them to argue that noticing, specifically professional noticing particular to teaching, can be used as a way to both evaluate teaching expertise, and as a conduit to supporting continued growth in the development of proficiency in teaching, both for in-service and preservice teachers. The link between experience and noticing expertise supports their assertion that professional noticing is different from everyday noticing and needs to be specifically supported.

Van Es and Sherin (2008) found that the ability for teachers to notice important aspects of instruction (what Jacobs and colleagues might call, "professional noticing") is difficult for both practicing and preservice teachers. Sherin and van Es (2005, 2008) conducted studies with both in- and preservice teachers to investigate how noticing can be developed. In line with their and others (Martinez et al., 2015; Maher et al., 2013; Hmelo-Silver et al. 2013; Brunvand & Fishman, 2006) belief that observing videos has advantages over observing interactions in real-time—that teachers can gain access to classroom environments that are different from the ones they are used to and, since video
records are permanent, videotaped episodes can be watched over and over—they used videos of classroom interactions in their studies. In one study, in-service teachers participated in a year-long video club during which they watched videotapes of their own classroom interactions, as well as the classroom interactions of other participating teachers. They met together and discussed the episodes, focusing on the question, "What did you notice?" In another study, they worked with preservice teachers, inviting them to use video editing software to analyze their own and others' teaching. In this study, preservice teachers studied classroom interactions, attending to classroom discourse. The results of these studies provide evidence that watching videos supported teachers' noticing of important pedagogical strategies. Participants talk, with regards to classroom events, changed; moving from general to specific. The researchers suggest that preservice teachers shifted their noticing from everything that was happening in the classroom, to more critical events. Both in- and preservice teachers developed new ways of noticing and interpreting events classroom episodes and the focus shifted from what the teacher was doing to what the students were doing, and, as in the Jacobs et al. study, participants commented more about the students' mathematical thinking after the intervention, and these comments became more interpretive and specific (p. 256).

In their 2007 study, Star and Strickland investigated what preservice secondary mathematics teachers noticed before and after a methods course designed to use video to improve observation skills. They agree with Jacobs et al (2010) and Sherin and van Es (2005, 2008) that noticing skills can be improved with support, and that teaching experience was positively correlated to noticing. They point out that teachers can only make sense of and reason about the classroom events they notice, so noticing is important
and should be supported (p.111). Preservice teachers in their study viewed videos of mathematics classrooms, several times, making a list of features they noticed. Their findings support others, that studying videos improved preservice teachers noticing of classroom practice.

Brunvand and Fishman (2006) have extended the idea of supporting teachers' noticing using video, through their work with "scaffolds." These researchers conducted a study with preservice elementary teachers enrolled in a methods class. In their study, participants studied video supported by scaffolds, defined as "video-editing effects that help focus attention on pre-determined segments of a video" (p. 152). The supports are used to focus teachers' attention on predetermined aspects of the classroom interaction. Results of their study supported others' results that the use of video supported preservice teacher learning, but their findings extended previous studies in the conclusion the video with scaffolds promoted more of preservice teachers' noticing than video alone. They argue, then, scaffolds better support teachers in their learning from video because the scaffolds draw teachers' attention to important content and events (p. 172).

The recent work of So, Lim, and Xiong (2016) also supports Brunvand and Fishman's (2006) findings that video scaffolds, as well as video, support teacher learning. The annotation tool that So and colleagues used in their 2016 study can be considered a "scaffold" according to Branvand and Fishman's definition (a scaffold is a video effect that helps to focus attention on selected aspects of the video). In the So et al., study, in-service teachers used an annotation tool to analyze videoed classroom interaction. Concurring with the findings of Jacobs and colleagues (2010), they found that teachers noticing shifted from surface features of the classroom interactions to deeper pedagogical
issues. They assert that annotated classroom videos help teachers develop reflected practices in situated contexts (p. 88).

2.3 Reasoning, Argumentation, and Proof

There is a strong correlation among reasoning, argumentation, and proof and often these three concepts appear together in mathematical education research (Bieda et al., 2014; Newton, Cirillo, Kosko, Staples, & Weber, 2015). Sometimes authors consider these ideas as interchangeable and sometimes they draw distinctions among them. The deep confounding of these ideas led top researchers in the field to gather at the 2015 North American Chapter of Psychology and Mathematics Education Annual Meeting and participate in the "Conceptions and consequences of what we call argumentation justification and proof" working group (Newton et al., 2015), with the purpose of advancing the field's understanding of the "interrelated objects and processes of argumentation, justification, and proof" (p. 1,343). These researchers posit that argumentation, justification, and proof are sometimes seen as complimentary, and sometimes as contradictory. They are inconsistently defined and researched because of confusion in the field and resulting in confusion in the field (Balacheff, 2002). National Council of Teachers of Mathematics (NCTM) and the Common Core Standards for Mathematics (CCSS) both call for students in K-12 education to engage in sense-making, justification, and argumentation (NCTM, 2009; CCSS, 2010), yet the perspective of the researcher, the focus of the research, and the data that are analyzed, all contribute to how these classifications are made (Newton et al., 2015).

Even the nature of the relationship between these ideas is debated. Some researchers conceptualize argumentation as the precursor to proof, or reasoning that is not
yet a proof (Conner et al., 2014), still others, such as Duval (2007) and Balacheff (1988), see the ideas a being "fundamentally different" (p. 1,345). The views are opposing, one holding that proof and argumentation are inextricably entwined, argumentation being the beginnings of proof, or proof in a less formal form; the other arguing that proof and argumentation are structurally different. Researchers' claims regarding argumentation and proof are equally dichotomous with some researchers claiming that even young children are able to write proofs (Zack, 1997; Maher and Martino, 1996; NCTM, 2000; CCSS, 2010), while other researchers claim that students at the college level struggle with writing formal proofs (Weber, 2001; Fischbein & Kedem, 1982; Healy & Hoyles, 2000). Furthermore, proofs are defined as "convincing justifications" (Balacheff, 2008), yet how convincing proofs are depend on the audience. Even mathematicians are not always convinced by proofs and are convinced by arguments not considered formal proofs (Schwarz, 2009). Ried (2001) in his Psychology of Mathematics Education conference paper about proof, proofs, proving, and probing, states that the literature on proof, argumentation, and justification includes "juxtaposing claims" that make reviewing it a "dizzying experience" (np). Newton and colleagues (2015) posit that the confusion is due to researchers defining and conceptualizing proof and argumentation differently (p. 1,346). They suggest that there are three perspectives through which researchers view argumentation, proof, and justification:

1. Proving as problem solving: This view focuses on developing instructions that teaches students how to write proofs.
2. Proving as convincing: This view focuses on the type of justifications that students find convincing and has the goal of changing students' standards for convincing to become more aligned mathematicians' standards.
3. Proving as a social activity: the focus is on what students think they are doing when they prove (pp. 1,346-1,347).
In the White Paper based on the work of the "Conceptions and consequences of what we call argumentation justification and proof" working group (Cirillo, Kosko, Newton, Staples, Weber, Bieda, Conner, Mejia-Ramos, Otten, Creager, Hummer, Singh, & Strachota, 2016), Weber and Mejia-Ramos, suggest that proof is what Lakoff (1987) called a "cluster concept" with a large number of features, none of which, in themselves, are essential to calling certain reasoning "proof." Weber calls proof: "purely deductive, highly convincing, perspicuous, within a representation system, and socially sanctioned" (p.7). In this paper, Bieda agrees that the mathematics community should not adopt one definition of proof. Mejia-Ramos, however, posits that he feels that there is no longer controversy in the field about whether proof and argumentation are incongruous:

"First, I think the field is no longer stuck in a debate about whether the informal argumentation involved in the generation of a conjecture (or the exploration of a statement to be proven) could lead students to the construction of a proof (the cognitive unity hypothesis put forward by Boero et al., 1996), or if, on the contrary, argumentation and proof in mathematics are simply and fundamentally incongruent (Duval, 1991)" (p.13).

Mejia-Ramos asserts that the question is not whether or not there is a bridge between argumentation and proof that students can traverse, but rather, what are the conditions that facilitate the movement from argumentation to proof. Based on my research focus, I have included a review of the literature concerning reasoning and proof as they relate to argumentation.

2.3.1 Reasoning and argumentation. As is evident from the previous section, and stated outright by mathematics education researchers (See, for example, Conner et al., 2014), reasoning is hard to define. Yackel and Hanna (2003) posit that reasoning is the process by which a knower comes to know and through which a person learns. Thompson (1996) describes reasoning as "purposeful inference, deduction, indication, and
association in the areas of quality and structure" (p. 267) and NCTM (2009) calls it "the process of drawing conclusions on the basis of evidence or stated assumptions" (p. 4). Others have distinguished mathematical reasoning—a set of collective practices and norms that are rooted in the discipline of mathematics (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002; Ball & Bass, 2000; 2003; Conner et al., 2014)—from the reasoning of justification (Ball et al., 2002) or mathematical argumentation (Bieda et al., 2014)—using mathematical reasoning to justify or prove mathematical claims. Whitenack and Yackel (2002) assert that "explaining and justifying are important aspects of reasoning about mathematical ideas" (p. 525) and Conner and colleagues (2014) see mathematical reasoning as "purposeful inference about mathematical entities or relationships" (p. 183) and further point out that mathematical reasoning in school mathematics can be messy and difficult to define. According to Ball et al. (2002) the reasoning of justification has two foundational principles: 1) It incorporates the body of public knowledge that includes mathematical concepts that have already been accepted by the community as true and no longer need to be proven; and 2) It utilizes “the entire linguistic infrastructure:” symbols, terms, notation, etc… that can be used in an argument or mathematical language (p. 909). The literature on argumentation supports the notion that argumentation and reasoning are inextricably linked, and it is difficult to discuss argumentation without also talking about reasoning. Argumentation has been called "a special kind of reasoning," (Pedemonte, 2006; Yankelowitz, 2009; Conner et al., 2014i; Wagner et al., 2014) and other researchers have noted that reasoning is an important part of argumentation (Krummheuer, 1995; Conner et al., 2014t; Duval, 1991; Umland & Sriraman, 2014). Toulmin, Rieke, and Janik (1984) state that arguments are trains of
reasoning (p. 12). In fact, Whitenack and Yackel (2002) state both that "reasoning involves making mathematical arguments, in particular, explaining ones' ideas to clarify those ideas for others" (p. 524) and that explaining and justifying [that is, making arguments] are important aspects of reasoning" (p. 525). Regardless of whether the relationship between argumentation and reasoning is conceptualized as argumentation as a certain kind of reasoning, or reasoning as part of argumentation, researchers are in agreement that they inevitably occur together (Toulmin et al., 1984; Whitenack & Yackel, 2002; Krummheuer, 1995; Schwarz & Linchevski, 2007; Bieda, 2010; Stylianides, 2007; Conner et al., 2014i; Conner et al., 2014t; Wagner et al., 2014; Duval, 1991; Umland & Sriraman, 2014).

2.3.2 Proof and argumentation. More literature has been produced with regards to proof than justification, argumentation, or reasoning (Newton et al., 2015). NCTM (2000) defines proof as "arguments consisting of logically rigorous deductions of conclusions from hypotheses" (p. 55) and in Cirillo and colleagues' White Paper (2016), Weber quotes Harel and Sowder (1998) and Balacheff (1987) to define proof as "an argument that convinces an individual or a community" (p. 6). Bieda (2010) concurs asserting that the purpose of proof is to convince oneself and the broader mathematical community. Mejia-Ramos notes that proof is a particular type of justification (White Paper, 2016). However, Stylianides (2007), who first introduced the term "reasoning-and-proving" (Bieda, White Paper, 2016) gives a definition of proof that is more consistent with argumentation and parallels Ball et al.'s (2002) definition of the reasoning of justification, with the addition of the social component:

Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics: 1. It uses statements accepted
by the classroom community (set of accepted statements) that are true and available without further justification; 2. It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; and 3. It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community (p. 291).

Balacheff (1988) distinguishes justification, proof, and mathematical proof when he asserts that justification is a form of discourse that establishes the validity of a statement, proof is an explanation that is accepted by the community, and mathematical proof is an explanation that is accepted by mathematicians. Francisco and Maher (2005) also distinguish justification and proof: “Justification refers to how students explain their mathematical actions and decisions. Proof is the formal and rigorous argument, which helps mathematicians explain their ideas” (p. 371). Schwarz (2009) agrees with Francisco and Maher that the role of proof is not to convince, but to communicate mathematical ideas using specific mathematical notation and language. Furthermore, in actual practice, mathematicians' reasoning processes are more aligned with argumentation than proof in that they pose problems, analyze examples, raise conjectures, generate counter examples and revise conjectures, using intuition to convince themselves of truth, before they use formal methods of proof to convince the mathematical community (Umland & Sriraman, 2014).

As stated previously, Duval (1995) and Balacheff (1988) argued that there is a distinction between argumentative reasoning and the deductive reasoning present in proof, and their work emphasizes the distance between argumentative reasoning and deductive reasoning, asserting that there is a structural gap between the two even if the language appears similar; a gap that is difficult for students to bridge. Proofs are connected by steps in which one step serves as the input condition for the next step,
whereas, argumentation consists of a series of inferences. In her discussion of justification and proof, Bieda (2010) argues that students' prior experience with the idea of proof in contexts outside of mathematics (for example, in a courtroom) conflicts with the traditional idea of the deductive reasoning-based mathematical proof. However, as posited by Mejia-Ramos (2015), Pedemonte (2007), and others (Schwarz, 2009; Conner (White Paper), 2016; Conner et al., 2014i; Conner et al., 2014t; Bieda (White Paper), 2016; Boero et al., 1996; Garuti, Boero, & Lemut, 1998), although proof and argumentation have structural differences, there is continuity between the two.

Pedemonte (2007), in her analysis of the structural and functional aspects of argumentation and proof in the context of geometry, supports the idea of the cognitive continuity (Boero et al., 1996; Garuti, Boero, & Lemut, 1998) between argumentation and proof, as well as structural differences between them. She defines cognitive unity, in the spirit of Boero et al. (1996), as the continuity between the construction of a conjecture and the construction of a proof (p. 24). Argumentation is meant to produce a conjecture during the problem-solving process. In some cases this argumentation can be organized in a logical chain of previously produced arguments to construct a proof. Pedemonte (2007) and Yankelewitz (2009), then, both assert that a mathematics proof is a specific type of argumentation. In analyzing the structure, or “logical connection between statements,” (p. 24) of proof, Pedemonte (2007) asserts that if students can achieve cognitive unity, or continuity between the construction of their conjecture and the construction of the proof, their argumentation might be more likely to produce a proof. Pedemonte also notes that argumentation and proof have a similar function in that they each have a useful role to play within discourse. Both argumentation and proof have the
following functional characteristics: 1. They are rational justifications; 2. They are meant to convince; 3. They address a universal audience. (For mathematics, the audience is the mathematical community, the classroom, the teacher, and the person constructing the argument.); 4. They belong to a field in that they are specific to a type of mathematics. (For example, the axioms in algebra argumentation are different from the axioms in geometry argumentation.) (pp. 26-27).

It is in the structural characteristics, or the logical connections between statements in the proof or argument, that Pedemonte sees differences between proof and argumentation. Although in both there is a logical connection between the statements, in proof, each step is deductive and in argumentation the steps are often inductive. If an argument has the same structure as the proof, they are structurally continuous; however, Pedemonte argues that often students’ arguments are not structurally continuous with the associated proof because it is difficult for students to produce deductive arguments when they are constructing and justifying. Hanna (1989) also refers to this psychological gap. So the transformation of an argument into a proof requires a structural change. In her study of 12th and 13th grade classes in France and Italy, Pedemonte (2007) found that generalization was a key factor in students' successful bridging from argumentation to proof. When students made generalization based on process pattern generalization—or the generalization of the process that is enacted on one case to get another case in the argument—there was more structural distance between the argument and the proof. Conner and colleagues (2014) agree, and assert that because proof is difficult for students at all levels (Weber, 2001) and students are more convinced by arguments that include inductive reasoning rather than deductive proofs, studying the collective argumentation in
the classroom can develop students' ways of reasoning, starting with their own "existing ways of reasoning" and moving to more deductive reasoning. (pp. 197-198).

Other researchers discuss proof and argumentation. Hanna (1989) explicitly connects proof and reasoning by arguing that developing the power of reasoning relates to students’ ability to construct proofs. Ball (1993) asserts that drawing conclusions that are mathematically reasonable requires students to be able to make arguments that are mathematically sound that convince themselves and others that a solution or conjecture make sense; Ball et. al. (2002) state that proofs rely on reasoning (habits of mind) to identify and logically organize assumptions. Zack (1997) asserts that, although not regarded as actual proof, inductive and deductive reasoning lay the foundation necessary for children to construct formal proofs. Yankelewitz, Mueller, and Maher (2010) note that the ability to reason is necessary for students to be able to produce convincing arguments, justify, and construct proofs. Krummheuer (1995) asserts that, although an argument is not necessarily connected with the formal logic that forms proof, argumentation is close to proof (1995; 2000). He makes the distinction between analytic argumentation and substantial argumentation (p. 236). In line with Toulmin (1958; 2003), Krummheuer (1995; 2000) asserts that analytic argumentation is associated with the formal deductive reasoning of proof. Substantial arguments, on the other hand, do not include "logical rigor" of formal deductions, but are "gradually supported" through student discussion (Krummheuer, 2000, p. 236). Because younger children's knowledge base usually does not include formal mathematical axioms and they are limited in their developmental ability to draw analytic conclusions based on deduction, the argumentation found in young students' discourse is often substantial. For children,
argumentation is more about justifying claims by producing convincing reasoning and explanation with models, then using logical deduction. Conner and colleagues (2014) agree, stating that the substantial argument explains the analytic argument and is reasoning that is more accessible to younger children. Although some researchers state that the goal of mathematics should be to teach children to create formal proofs (Hanna, 1989), more recent researchers agree that the ultimate goal of mathematics education should be to co-construct reasonable arguments in discussions (Schwarz, 2009; Krummheuer, 1995).

2.3.3 Mathematical argumentation. While proof is considered a purely mathematical construct (Conner et al., 2014), argumentation is evident in many disciplines (See, for example, Naussbaum, 2001; Chinn & Clark, 2013; Douek, 1999; and Kuhn, 1999 for examples of argumentation in science) and has a common usage that means persuasion (convincing), negotiation, and disagreement (Wagner et al., 201). Wagner et al. assert that since "the most common interpretation of an argument involves conflicting points of view, defining and describing argumentation in the classroom is not straightforward" (p. 9). A key difference between mathematical argumentation and argumentation in other disciplines is that the claims that are asserted in mathematics are considered from a positive perspective, usually based on mathematical truth according to known axioms and postulates accepted by mathematicians and where there are clear right answers. For example, the claim that 1/2 is greater than 1/3 by 1/6, though it may not be accepted as true by students and, therefore, can become the object of argumentation, is based on truths inherent in the number system; whereas, for example, the claim that it is better to get vitamins from vegetables than supplements, may include elements of
objective truth, but the claim itself is not considered objective truth. Although science used to be considered from this perspective, according to Schwarz (2009), current research in science "practices such as assessing alternatives, weighing evidence, interpreting texts, and evaluating the potential viability of scientific claims" are valued (p. 111). Thus, as Schwarz (2009) states, "the characteristics of argumentation in which people engage in different domains are quite different. This is because argumentation, and especially collective argumentation, bears domain norms according to which people reason" (p. 106) and Krummheuer (1995) agrees that argumentation is domain specific. Therefore, in my review of the literature, for the most part, I will focus on mathematical argumentation, as it is presented in mathematics education literature.

Stylianides (2006; 2007) defines a mathematical argument as “a connection sequence of assertions intended to verify or refute a mathematical claim” (p. 2) that results from a carefully designed collaborative problem solving activity (Pedemonte, 2007; Schwarz, 2009; Whitenack & Yackel, 2002; Douek, 1999). Bieda and Lepak (2014) state that argumentation is a sequence of statements constructed with the intent to convince others of the truth of a general claim and Conner, Singletary, Smith, Wagner, and Francisco (2014) add that argumentation is when someone makes a claim and supports it with evidence. Krummheuer (1995) asserts that argumentation describes any part of an exchange that expresses accountability and Douek (1999) states that the purpose of argumentation is to validate a conjecture or solution to a problem and detect

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3 Note that, although Douek (1999) reported on fourth grade students' explorations of sun shadows in science, I included her thoughts in my review. Although the data were collected in a science classroom, Douek's analysis included connections to mathematical modeling. Although grounded in science, many of her findings aligned closely with the findings in extant mathematics education literature. For example, she found that there was a productive link between context related argumentation, mathematical modeling, and conceptualization of knowledge. Additionally, she states that "argumentation is an important mechanism of personal and social production of knowledge in mathematics" (p. 92).
contradictions and limitations of models and solutions (p. 92). Mejia-Ramos (White Paper, 2016) specifies that an argument in mathematics is a series of statements for or against some mathematical claim or position and the justifications are the reasons that the claim must be true. He clarifies by stating that using the evidence of "authority" (or that "she said so") is an argument, but it is not justification (p. 12). Whitenack and Yackel (2002) argue that explanation is an essential part of argumentation, particularly using explanation to provide a justification to address challenges made to the argument (p. 525) and that challenges lead to more convincing arguments. These researchers assert that when students make mathematical arguments, they are not simply sharing their answers to problems, but they are explaining and justifying their ideas so that others can evaluate not just the answer, but also the strategy. In the *Encyclopedia of Mathematics Education* (2014), Sriraman and Umland define argumentation as a process of drawing conclusions based on a chain of reasoning in which logic plays a key role. However, they agree with Conner et al. (2014) and Krummheuer (1995) that argumentation expands proof by going "beyond proof theory (or the study of what can and cannot be proved in mathematical systems of deduction)" (p. 44) and it is more complex than proof, involving intuition. For novices, intuition plays a large part in the development of arguments, but for experts, intuition plays the lesser role of "advanced organizer." In fact, they argue that there is a difference between how mathematicians convince themselves of a truth, and how they convince others, especially others in the mathematical community. They assert that initially intuition leads experts to their initial convictions of the truth of a claim, and, once they are convinced, they use more formal methods of proof to convince the mathematical community. The Common Core State Standards for Mathematical Practice
(2010) define argumentation as using stated assumptions, definitions, and prior knowledge to reason and create arguments and to determine appropriate domains for the argument (See Mathematical Practice 3, http://www.corestandards.org/Math/Practice/ ) and NCTM (2000) requires that students communicate by justifying ideas, creating conjectures, and exploring mathematical tasks together.

It is well established that argumentation, in and of itself, is an important aspect of K-12 education and that instructional programs should cultivate a classroom environment in which students are encouraged to evaluate arguments, ask questions to clarify arguments, and develop arguments that make sense (Wagner et al., 2014; Whitenack & Yackel, 2002; Krummheuer, 2007). Both current standards (NCTM, 2000; CCSS, 2010) and researchers (Schwarz, 2009; Krummheuer, 1995; 1997; 2000; 2007; Bieda & Lepak, 2014; Conner et al., 2014; Wagner et al., 2014; Whitenack & Yackel, 2002) highlight its importance. To Krummheuer (1995, 1997, 2000, 2007) and others (Conner et al., 2014; Conner et al., 2014; Wagner et al., 2014; Whitenack & Yackel, 2002; Sriraman & Umland, 2014) argumentation is a social practice occurring when more than one person tries to convince another of a claim and the method of finding a solution to a problem with a justification is an argument. Referring to argumentation that takes place in a social setting as "collective argumentation," Krummheuer (1995) asserts that "the concept of argumentation will be bound to the interactions in the classroom that have to do with intentional explication of the reasoning of a solution during its development or after it" (1995, p. 229) and successful argumentation happens when a claim that is challenged turns into a claim that others accept. Collective argumentation, then, results when there are multiple people working together to establish a claim (p. 184, Conner et al., 2014).
The importance the literature puts on argumentation in the K-12 mathematics classroom cannot be overstated. Whitnack and Yackel conclude that students often refine their thinking as they explain and justify their ideas and argumentation encourages students to explore mathematical ideas, develop into mathematical thinkers, and become mathematical sense makers.

Researchers have found that there is a link between cognition and argumentation; specifically that when they engage in argumentation, students learn more easily and more reflectively (Krummheuer, 1995, 1997, 2000, 2007; Wagner et al., 2014). Wagner and colleagues (2014) posit it is because of argumentations' influence on deepening conceptual understanding that there is an emphasis on argumentation in the literature and the standards. Conner et al. (2014) and Bieda (White Paper, 2016) assert that discussion is useful in helping students learn mathematics, and collective argumentation is participating in discussions in a "distinctively mathematical way" (p. 401). Whitnack and Yackel (2002) and Lampert (2001) agree, and have found that students "construct new understandings" (Whitnack & Yackel, 2002, p. 525) and develop deeper understanding of concepts through argumentation and Bieda (White Paper, 2016) adds that "argumentation can build students' agency for generating and understanding mathematics" (p. 9). Argumentation causes students to "ponder the explanations behind solutions or perspectives, and requires him/her to express them in verbal, explicit communication" (Schwarz, 2009, p. 97) and encourages "clarification of contradictions and faults in one's understanding" (p. 97). Through its aim of convincing others, participants in argumentation (in the audience) learn new information about the topic of the argument as they listen and consider the validity of the ideas of others. Schwarz
(2009) asserts that argumentation, itself, "deepens" students' understanding (p. 98) and Rogoff (1990, 1998) contends that shared understanding emerges as a result of the active engagement in argumentation. Schwarz and Linchevski (2007) in their study of proportional reasoning, found that students engaged in social argumentation deepened their knowledge and Yackel (2002) asserts that argumentation can produce "the emergence of powerful mathematical concepts" that are part of what would be considered "course content" (p. 427). Douek (1999) confirms that "argumentation is an important mechanism of personal and social production of knowledge in mathematics" (p. 92).

Wagner et al. (2014) assert that argumentation is central to critical thinking and important for learning because, through argumentation, students make connections among mathematical topics (Yackel, 2002), as well as connect new concepts with prior knowledge, which makes learning more durable (Wagner et al., 2014, p. 8). The idea of strengthening the durability of knowledge through integrating new knowledge with prior knowledge is supported in the literature (See Davis, 1992). Maher, Powell, and Uptegrove (2010) concur, stating that "ideas are interconnected and extended as learners work together to make sense of each other's ideas and build convincing arguments for their solutions" (p. 3). Conner and colleagues (2014) emphasize that:

"Reasoning and proof are unequivocally accepted to be foundational to the discipline of mathematics, but engaging students in formal deductive proof at an early age may not be developmentally appropriate. Argumentation as a precursor to proof, is fundamental to the establishment of mathematical knowledge" (p. 403).

Furthermore, Schwarz (2009) and Krummheuer (1995; 1997) both assert that learning to argue as well as arguing to learn are essential aspects of education, and schools should provide contexts for students to develop argumentation, (Schwarz, 2009,
p, 95) and it is crucial for teacher educators to "foster and support student argumentation when preparing preservice teachers" (Wagner et al., 2014, p. 8).

2.3.4 **Toulmin's model to diagram mathematical argumentation.** According to Krummheuer (1995) arguments, as a method of establishing or reestablishing the truth of claims, usually contain a series of statements that take on different roles in the convincing process (p. 239). Yackel (2002) and Douek (1999) add that argumentation includes actions, tools, notation, drawings, models, numerical data, as well as verbal statements and Krummheuer (1995) asserts that arguments are often "image dependent" (p. 251). Toulmin’s (1958, 2003) model has been used to illustrate what Krummheuer calls the "functional" aspect of argumentation, or the function that each statement has within the course of the argument, especially within substantial argumentation (p. 239; See also, Pedemonte, 2007; Krummheuer 1995; 1997; 2000, 2007; Wagner et al., 2014; Conner et al., 2014; Conner et al., 2014; Hollebrans, Conner, & Smith, 2010; Rasmussan & Stephan, 2008; Yackel, 2002). The model supports the "analysis of argumentation (or AA)" (Krummheuer, 2007, p. 62) or the diagramming of the informal logic of an argument. To Conner et al., (2014) the diagramming of students' arguments is useful because the flow of the argumentation helps to determine what knowledge they have, what they are learning, and how they are thinking about ideas. They posit that the Toulmin model, "offers structure to the sometimes messy and often ill-defined construct of reasoning in school mathematics" that is included in students' mathematical argumentation (p. 196). The basic elements of this model include claims, data, and warrants. A claim⁴ is a statement that is held to be true, data are evidences, in the form of

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⁴ Note that Krummheuer (1995) refers to the statement that functions as a claim in an argument as a "conclusion." However, other researchers (see for example Conner et al. (2014), Conner et al. (2014), and
statements, facts, or information that support the claim, and a warrant is a justification for using the data as support of the claim. The claim is supported by inferring it from other statements, the data. An arguer might use the data to state a claim, or state the data and then make the claim, so their argument might have the structure, "Claim because of data," or, "Data, so claim" (Krummheuer, 1995). If the statement is accepted by the community in which it is presented, the argumentation ends and the claim is accepted. Data is essential to argumentation, so without data, there is no argumentation (Krummheuer, 1995).

If the statements that function as data are not widely accepted, the data are questioned and further support needs to be given. These supports can take two forms. If the validity of the data itself is questioned, another argument (or sub-argument) will need to be constructed to support the validity of the data. However, if data are accepted as true, but the relevance of the data to the claim is challenged, an explanation of how the data support the claim is needed. Warrants support, or "legitimatize" (p. 65, Krummheuer, 2007), the inference that is made between the data and claim, not the validity of the data themselves (Yackel, 2002). Rather, they explain explicitly what inference was made when the data were given to support the claim, answering the question, "What do these data have to do with the claim?" forming a bridge between the data and the claim.

Whether or not the inference between the claim and the data statements is accepted depends on the community (Yackel, 2002; Schwarz, 2009; Duval; 1999; Krummheuer, 1995; 1997; 2000; 2007; Ball & Bass, 2000). Additionally, although a group might accept a warrant, a teacher might challenge a statement that is accepted by the group in order to

Yackel (2002), and Toulmin (1958, 2003)) use "claim." Although Krummheuer informed my diagramming of argumentation, I have chosen to use the term claim, rather than conclusion.
extract more reasoning. Disagreements with the different elements of argumentation may lead the arguer to correct, refine, modify, replace, refute, or retract statements in the argument (Krummheuer, 1995; Whitenack & Yackel, 2002). Consensus is reached when the constructed argument is accepted by all of the participants.

Claims, data, and warrants are the basic elements of an argument. If the community does not accept the warrant, an additional element, backing, might be included. Backing explains why, in general, the warrant should be accepted has having authority, so the backing supports the validity of the warrant. According to Krummheuer (1995) and Yackel's (2002), interpretation of Toulmin's model, what are considered backing are the most "undoubted" and "unratified" ideas or statements (Krummheuer, 1995, p. 244) or "general theories," "beliefs," or "primary strategies" (Yackel, 2002, p. 425). Backing is also situational, and how irrefutable it is depends on the community and what the community accepts as being true (Yackel, 2002). What is generally accepted by mathematicians might be much different from what is generally accepted by first graders. Schwarz (2009) and Duval (1999) also support that argumentation is situated in experience. To have argumentation, there needs to be true statements that are reliable because they have been verified and doubtful statements that need to be supported or refuted. Statements that fall into these categories can be different depending on the community and Yackel (2002) adds that the statements given as data, warrants, and backing are reliant on what the group collectively understands about the mathematics concept. Ball and Bass (2000) consider what is generally accepted by the group as backing to be the public knowledge base or the shared mathematical knowledge of the group and Douek (1999) distinguishes between the institutionalized knowledge
Krummheuer (1995) summarizes the "system" of argumentation that contains these elements in this way:

- A conclusion [or claim] (C), the validity of which was doubted.
- Data (D) on which the conclusion was grounded.
- Warrants (W), which give reasons for the legitimacy of the applied inference from D to C.
- Backings (B), which support the warrants by giving categorical statements about principal convictions leading the thinking of the arguing individuals (p. 247).

The diagram in Figure 2.1 can be used to show these relationships graphically.
This argumentation structure can be thought of as, "Conclusion [claim] because of data (or Data, so [claim] conclusion) and data since warrant and warrant on account of backing" (Krummheuer, 2007). The connecting rays and line segments in the diagram in Figure 2.1 represent the reasoning and inferences in the argument (Conner et al., 2014). Additional elements may also be included in an argument: a qualifier and a rebuttal. A qualifier might be added to an argument to indicate degrees of confidence and a rebuttal clarifies the conditions under which the argument is not valid (Weber, Maher, Powell, & Lee, 2008; Stephan & Rasmussen, 2002; Toulmin, 1958; 2003). However, for the purposes of my study, I will only be attending to the elements included in the diagram. Also, I will refer to the data, warrant, and backing all as the "evidence" for an argument. Although some have criticized using Toulmin's model in K-12 education to diagram...
substantial arguments (or arguments that are not deductive) because it is too rigid and restrictive (for example, see Barrier, Mathe, & Durand-Guerrier, 2010, and Sampson & Clark, 2008), others (Krummheuer, 1995; Pedemonte, 2006; Wagner et al., 2014, Conner et al., 2014i; Conner et al., 2014; and Inglis et al., 2007) argue that the warrant does not necessarily have to eliminate uncertainty, but, in informal arguments that are convincing, the warrant can reduce the uncertainty about the validity of a claim (Conner et al., 2014; Inglis et al., 2007), and, thus, using the Toulmin layout is valid.

Krummheuer (1995) and Conner et al. (2014) have found that students' arguments can be simple (claim, data) or complex, incorporating nested sub-arguments, wherein the main claim is supported with data or evidence that must also be supported: "a developed argument … can be applicable as a commonly accepted reason in an argumentation that is more complex, [in which] several arguments can be constructed" (p. 235), leading to "infinite regress," or the securing of "doubted data by running new argumentation" in the form of nested argumentation (p. 252). Thus, when a data statement needs to be defended, it functions as both data in one argument and a claim in a sub-argument (Conner et al., 2014, p. 404). According to Krummheuer (1995), the regress concludes when the data are generally accepted by all of the participants, or when it is self-evident. Nested arguments may come before a claim, so that the argument builds from one claim to another and from left to right in the diagram. Or, the sub-arguments might result from data or warrants being questioned, in which case the data or warrants may then also become claims, and the argument builds from the top to the bottom of the diagram (Conner et al., 2014).
Researchers that analyze young children's arguments with the Toulmin model, caution that students' arguments often do not include all of the elements of argumentation (Krummheuer, 1995; Yackel, 2002) and some elements might be implicit (Conner et al., 2014). Implicit elements are not stated outright, but rather implied (p. 406). Conner et al. have observed that the statements that are part of an argument must be considered in context. Statements might be considered claims, data, warrants, or even backing depending on the audience and the context in which they are offered. Krummheuer (1995) states that the Toulmin scheme:

"is not to be misunderstood as a method for identifying the different components of that model in concrete interaction, this needs to be done by a related analysis of the interaction. The scheme merely points out the different roles utterances play in an interaction when reconstructed from the perspective of the emergence of a substantial argument" (p. 240).

He further asserts if just verbal statements are considered, "it is generally not possible to differentiate between data and warrants" (p. 243) and arguers do not necessarily order their statements to align with the structure of Toulmin's model. The teacher or researcher may question the argument for pedagogical reasons, prompting additional elements of argumentation (Bieda, White Paper, 2016), so what is generally accepted by the group might become unclear resulting in data, warrants, and backing being inferred and relative. Thus, argumentation in the classroom setting can become less organic and more contrived. Krummheuer (1995) stresses that the distinction among the elements "is often difficult to make because data and warrants are related to each other" (p. 243) and can only be discerned by "an appropriate analysis of [the whole] interaction" (p. 247). Krummheuer (1997) observed that, in young students' arguments, the calculation and justification are often not distinguishable from one another. He called this "reflexive
argumentation" (p. 2) and asserted that often students' argumentation is "opaque" (p. 5); the computation, explanation, and justification are given together, making them difficult to separate out. Yackel (2002) agrees and cautions that collective argumentation cannot be considered based on a sequence of statements that are made, but on various statements that interact with one another and, together, serve to make sense of the argument as it develops. Yackel cautions: "What constitutes data, warrant, and backing is not predetermined, but negotiated by participants as they interact" (p. 424).

2.3.5 Argumentation research in mathematics education. Little research has been conducted on the argumentation in children in mathematics education (Stylianides, 2007). However, among the researchers that have investigated these topics, there seems to be some agreement on important ideas.

Weber, Maher, Powell, and Lee (2008) studied the argumentation of middle school students in an urban informal learning after school program. They suggest that the discussion promoted students’ development of explicit arguments that included claims, data, warrants, and backing. They found that challenges from classmates resulted in debate about the validity of particular arguments and provided students with the opportunity to reevaluate their claims and revise or abandon them. The researchers analyzed the 30-minute videotaped session, coding each argument according to Toulmin’s (1958, 2003) scheme: the claim being made, the data to support the claim, and the warrant for how the data applied to the claim. They coded challenges, noting whether the challenge was to the claim, data, or warrant.

Weber and his colleagues explicate three interesting findings. First, group discussion facilitated the development of argumentation by making “implicit warrants
explicit” (See also, Conner et al., 2014) and these warrants then became objects of debate. The warrants, then, became the claim that students needed to justify which engaged them in higher-level mathematical reasoning, thus concurring with Krummheuer (1995) and Conner et al. (2014) that students' arguments can include nested or sub- arguments, where the data, warrant, or backing can become a claim. Second, as with Krummheuer (1995) and Whitenack & Yackel (2002), they found that when students’ arguments were challenged, they began to realize that a particular mode of reasoning was invalid. Finally, as students considered the warrants that they and other students used in their arguments, they began to think about generalized mathematical principles (pp. 258-259), which, according to Pedemonte (2007), is more likely to bridge the structural gap from argumentation to formal proof. The researchers also describe the role that the teacher/research played in the promoting of argumentation of the students. She required students to determine if an argument was valid, asked students to share their conclusions and provide justification, and reminded students to attend to one another’s arguments.

Ball (1993) investigated argumentation and proof-like reasoning in her own third grade classroom, in which she was the teacher/researcher. She analyzed data collected through audio and videotaped classes, interviews, student artifacts such as math notebooks, tests, and quizzes, and her own observations from her 22 diverse third graders as they worked on open-ended problems in a classroom atmosphere that emphasized exploration and inquiry. She found that students engaged in argumentation and explored proof as they worked on the tasks. They developed classroom discourse that included emphases on the necessity of providing justification. For example, students would say, “prove it to us” (p. 385). As with the students in Weber et al. (2008), Ball's students
grappled with the generalizability aspect of their arguments and critiqued the arguments of others, citing examples and counterexamples. They used mathematically accurate definitions of terms in their arguments to either prove their ideas or disprove the ideas of others.

Ball et al. (2002) also studied the mathematical reasoning of third graders using the same methodology as Ball (1993). In this study, Ball and her colleagues (2002) focused on the development of the students’ argumentation and proof abilities over the school year. They were able to generalize a progression in this development. Students began the year by justifying their answers using empirical reasoning (I know I found all of the solutions because I could not find any more) with no method of building a logical argument. They then gained an appreciation that empirical reasoning was not enough and they needed to provide a justification that their answer was correct. Students began to exhibit the understanding that, just because an assumption works for a few examples, does not mean it works for all numbers. This realization led students to begin demanding justification for generalized claims. They demonstrated the ability to use drawings and mathematically correct definitions to develop convincing informal arguments.

Additionally, Ball and her colleagues (2002) found that the task in which students engaged must necessitate the need for substantial mathematical reasoning, for example, by asking students to provide a convincing argument that the solution is correct and exhaustive. The teacher needs to make public the mathematical knowledge that can be used to construct convincing arguments and scaffold the use of precise language and knowledge when scrutinizing students’ arguments. Additionally, the teacher must create a culture within the classroom in which students are interested in other students’ ideas and
are respectful of others, as well as emphasizing the notion that arguments about truth must include justification. These findings led the researchers to conclude that the teacher’s role in the classroom is essential.

Stylianides (2006; 2007) took the perspective of the researcher who studies the classroom activity after the fact, from the outside to analyze the argumentation activities of third graders. Her data were drawn from the same longitudinal study as the Ball (1993) and Ball et al. (2002) studies described above. In her 2006 study, Stylianides analyzed episodes from that data to determine how the third graders’ arguments aligned with the four major elements (foundation, formulation, representation, and social dimension) she explicates are necessary for an argument to be considered a proof. In one episode she found that a third grade student was able to use everyday language and definitions to create a deductive argument for the conjecture that the sum of two odd numbers is even. This argument included foundation, formulation, representation, and social dimension and was intellectually honest, as well as exhibiting continuity towards a consistent view of how proof should be developed through the grades.

In her 2007 study, Stylianides analyzed base arguments (or students initial arguments) and ensuing arguments (the final convincing argument that aligned with her definition of proof) to investigate what role a teacher plays in the development of students’ arguments. As in her previous study (2006), she found that students developed arguments that aligned with her definition of proof in that they used true statements known to the community (for example, basic addition facts), included valid modes of argumentation (for example, the systematic enumeration of all possible cases, which is a mode of argumentation necessary to develop proofs by exhaustion) that were understood
by the community, and communicated appropriately using verbal language (for example, they used personally meaningful language to accurately provide a generalized explanation for the validity of a claim). Stylianides also confirmed previous assertions (Thompson, 1996; Styliandes, 2006; Zack, 1997; Maher & Martino, 1996; Yackel, 2002; Whitenack & Yackel, 2002; Wagner et al., 2014; Conner et al., 2014t; Beida, 2010) that when the teacher plays an active role in managing students’ development of a justification, by, for example, introducing students to a new mode of argumentation or a new mode of communication, students’ base arguments became more proof-like. Thus, the teacher’s understanding of argumentation and proof construction becomes very important.

Zack (1997; 1999), another teacher/researcher, studied argumentation and proof-making in fifth-grade students. Her study took place in her own inquiry-based (in the spirit of Richards, 1999) classroom. Her students solved non-routine, open-ended problems in heterogeneous groups of four or five. The classroom culture encouraged students to publically express their thinking and it was expected that students would strive to challenge the claims of others as well as defend their own assertions. Students in her class developed the understanding that an argument needed evidence, that it must make sense, and it must include an explanation of why it works. Although Zack (1997) had not set out to study argumentation and proof-like reasoning, she saw her students engaging in proof-making activities, reasoning, and argumentation and wanted to investigate further. She analyzed videotaped interviews and classroom episodes, her own observations, students’ math logs, and students’ written work. She (1997) found that the students, themselves, felt the most convincing arguments that were made were based on
generalizations. Furthermore, when she intervened and asked students to explain why they made the assertions they did, students augmented their argument to include an explanation, which is an essential part of proof (Hanna, 1995), reiterating the importance of the teacher in developing convincing argumentation (Weber et al., 2008; Krummheuer, 1995; Whitenack & Yackel, 2002). Students developed counterarguments that were used to refute claims and convince, and she observed that argumentation and the formulation of counterarguments went hand in hand. When groups insisted that other groups had to convince them their claims were not valid, counterarguments were produced. These counterarguments led to justification and the refinement of initial arguments (see Maher, 2009). When there was no disagreement, convincing arguments were not produced (see Krummheuer, 1995; Whitenack & Yackel, 2002). Zack’s (1999) results showed evidence of conjecture, refutation, and generalization and aspects of proving that arose spontaneously and in response to teacher questioning. Additionally, she found evidence of complexity and logical structure in her students’ arguments, exemplified in the use of counterarguments and logical connectives such as if, then, and but.

In a case study investigation, Maher and Martino (1996) conducted a longitudinal study of the proof-making activities of a female student from first grade through fourth grade. The goal of this study, part of a larger longitudinal study, was to help teachers create classroom environments in which students could be actively involved in the construction of mathematical ideas. Students in the larger longitudinal study, explored non-routine problems over an extended period of time; constructing models of solutions. Data that included videotapes of class sessions and interviews, student work, and researcher observations were analyzed for evidence of proof-making activity and
discourse. Researchers traced the development of reasoning and argumentation of one of the students in the larger study. In particular, they wanted to determine: 1. the use of heuristics; 2. the development of an argument specific to a part of the solution; and 3. the extension of an argument to include a full solution; i.e., was the argument generalizable? (p. 199).

Maher and Martino found evidence that this student was able to use proof-like reasoning to construct convincing arguments to validate her solutions to open-ended problems. In first grade, this student recognized and was able to verbalize the idea of needing to be convinced. The girl objected to a classmate giving an answer without explaining why the answer was true. In grade 3 she was able to use a tree diagram to develop a convincing argument about the number of different outfits (combinations) that could be made given three shirts and two pants. In grade 4, the student solved a problem involving finding all of the 5-high towers that could be built from a selection of two colors of connecting cubes. In her solution, the student was able to develop an indirect proof, specifically a proof by contradiction, to explain her answer. When she revisited the problem throughout the year, she used the strategy of solving a simpler problem, the method of simultaneously coordinating two variables, and developed a complete proof by cases to justify her solution. She used her proof by cases to confidently present and defend her argument in response to the critiques of her classmates. In the end, after initially being critical of it, her classmates were convinced of the validity of the argument.

Seven months later, in grade 5, the student’s understanding of the problem was so enduring that she was able to present an elegant written description of a modified proof.
by cases argument in which she incorporated suggestions her classmates had made the year before. Maher and Martino’s findings provide further evidence that students at young ages can participate in proof-making activities in which they construct arguments using mathematical reasoning. They also confirmed Lester’s (1980) assertion that students’ reasoning moves from less sophisticated to more sophisticated as they mature; i.e., students rely less and less on trial and error and more and more on strategies with organizational processes.

In a later study, Maher (2009) found that children as young as 8 years old were able to build convincing arguments. In this study, Maher analyzed the proof-making activities of third graders from the larger longitudinal study used in the Maher and Martino (1996) study as they worked on a problem that required them to find all of the towers four tall that could be made selecting from two colors of Unifix cubes. She noted a variety of deductive arguments that were employed by students, including argument by recursion, argument by contradiction, and case-based reasoning. In this study, Maher noted that, although teasing out the elements of the environment that were most conducive to the development of proof-making abilities in students was beyond the scope of the study, free play, collaboration, strategic questioning, revisiting the problem, and the requirement of justification were all present in the exploratory environment in which students constructed their proof-based arguments. She also confirmed the results of others' results (Stylianides, 2007; Krummheuer, 1995; Whitenack & Yackel, 2002; Zack, 1997; 1999), specifically, that more elegant and proof-like arguments emerged in response to strategic teacher questioning.
Yankelewitz (2009) conducted a qualitative case study to investigate the reasoning and argumentation used by fourth graders as they worked on fraction concepts. The students in this study explored fraction problems before formal algorithmic instruction. Data were collected through videotape recordings of classroom sessions, students’ written work, and researcher’s field notes. Yankelewitz found that students used forms of reasoning that were deductive and inductive. They made conjectures and generalizations and used different forms of reasoning to justify their generalizations. The students used direct and indirect arguments and offered counter arguments. Yankelewitz also found evidence that students used analogical reasoning to draw conclusions or make conjectures about the ideas they investigated. Students used generic reasoning, reasoning by cases, recursive reasoning, reasoning using upper and lower bounds. Yankelewitz, along with others (Alibert & Thomas, 1991), concluded that students engaged in spontaneous indirect argumentation that can eventually enable them to write indirect proof (p. 326). She also suggests that the generic reasoning students engaged in was important in transitioning to formal proof (see also, Mejia-Ramos, White Paper, 2016; Schwarz, 2009; Conner (White Paper), 2016; Conner et al., 2014i; Conner et al., 2014t; Bieda (White Paper), 2016; Boero et al., 1996; Garuti, Boero, & Lemut, 1998; Cooper et al., 2011; Pedemonte, 2007).

Evidence of justification and proof was also found by Yankelewitz, Mueller, and Maher (2010) during fourth- and sixth-grade students’ exploration of fraction concepts. The 25 fourth graders, who were part of a larger longitudinal study investigated by Yankelewitz (2009) in a suburban setting, worked in pairs. The 24 sixth grade students from an urban setting took part in an informal afterschool mathematics program and
worked in small groups of three or four. Analysis of student reasoning was based on data that included videotaped class sessions, interviews, and collected student work. For their analysis, they used the following formal definitions:

A direct proof assumes A and proves B, and an indirect proof either (1) assumes A and not B and deduces a falsehood (proof by contradiction), or (2) assumes not B and deduces not A (proof by contraposition). A proof by cases begins with an assertion of fact, saying “Either M or N” (e.g., either x is even or x is odd), where M or N must be true. It then produces two proofs, the first showing that “if M, then the statement to be proven” and the second showing, “if N, then the statement to be proven.” (p. 79)

Yankelewitz and her colleagues developed a coding system to categorize the arguments made by students. Because the focus of the study was on reasoning, student justifications were in students’ natural language and did not follow the construct of formal proof but were identified as proof if they included assumptions and conclusions of an argument according to Weber (2008). The researchers’ findings support the findings of Yankelewitz (2009) in that both the suburban fourth and urban sixth grade students used reasoning by contradiction, upper and lower bounds, cases, and generic reasoning to construct sound justifications. Notably, direct reasoning was the most common form of reasoning; however, it was always used in faulty solutions to the task. As in the Ball et al. (2002) study, students used definitions to create counterarguments and completed an exhaustive accounting of all cases. Yankelewitz et al. argue that their results support the idea that certain tasks elicit certain forms of reasoning (see Cooper et al., 2011). For example, the task that students worked on required them to find a rod with the number name one half, given that a certain other rod was one (for a more detailed description of the task, see Yankelewitz et al., 2010). The researchers noted that since, given the specific conditions of the problem, there is no rod that could be called one half, the task
elicited reasoning that included proof by contradiction. They assert, then, that introducing
tasks that elicit different forms of reasoning can lead students to become familiar with
these forms of reasoning and provide a foundation for students to be able to precisely and
effectively formulate their arguments into formal proof.

In her study of justification and proof in middle school students, Bieda (2010)
investigated the nature of students' and teachers' actions and discourse while students
engaged in a task that required them to justify and prove their solutions. She confirmed
others' findings (Maher, Landis, Palius, 2010; Bieda, 2010; Wagner et al., 2014;
Whitenack & Yackel; 2002; Yackel, 2002) that the teacher played a key role in helping
students produce arguments that included appropriate justifications and meet the
standards of proof in mathematics. Her data included classroom observation field notes of
teacher and student discourse as well as written student work. Bieda found that students
were able to participate in proving activities and that there were particular teacher moves
that corresponded to greater student proficiency in proving. She states that her results
show that students offered conjectures and provided justifications. However, in the
classroom interactions in her study, her results supported that teachers' involvement in the
justification and proving process was limited, and therefore, students did not improve
their justification and proving skills and did not learn about the standards that govern
acceptable justifications and proof:

The sparse nature of teachers' involvement in providing feedback and
instructional support during proving events suggests that the assumptions
underpinning the didactical contract between teachers and students during proof
related activity were not sufficient to support meaningful learning about the
standards for justification and proof. (p. 379)
Additionally, from her interview data, initially, teachers in her study were skeptical that all students could develop justifications; believing that only high-achieving students that are "developmentally ready" could. However, the viewing of videotapes of students engaged in justification and proof discourse convinced teachers of what was possible.

Cooper and her colleagues (2011) used a cognitive science lens to study inductive reasoning in middle schoolers to investigate how generalized deductive proofs might be developed from inductive strategies. Cooper et al. analyzed data from semi-structured interviews of 20 middle school students from sixth, seventh, and eighth grades. The researchers asked students to explore the validity of two mathematical conjectures. Students needed to determine if the conjecture was true for every number and explain how they made their decision. Students were specifically asked to generate examples as part of their solutions and explain why the conjecture was always true.

The researchers’ results showed evidence that students used both inductive reasoning through the use of examples and deductive valid proof arguments. Although many students used empirical, i.e., specific examples, as justification, slightly under half of the students included some sort of valid deductive proof. These deductive proofs took the form of narrative proofs (explanation in verbal form using deductive reasoning), visual proofs using drawings, and algebraic proofs that used formal deductive mathematical statements. Narrative proofs occurred most frequently, followed by visual, and then algebraic. Furthermore, the deductive proofs developed by students included generalization (See Weber et al., 2008; Yankelewitz, 2009; Yankelewitz, 2010; Zack 1997; 1999). Although more than half of the students used empirical reasoning, Cooper
and her colleagues concur with Yankelewitz et al. (2010) in that they stress that the fact that the study specifically had students show examples could have influenced students’ methods. Furthermore, consistent with Zack (1997), students, themselves, expressed that the general proofs were a more convincing type of argument than one that simply included a set of examples.

Cooper et al. argue that their results show that inductive, empirical methods were used to determine the truth of the conjecture. This inductive base provided a foundation upon which students were able to begin the formulation of an argument. Thus, they conjecture that inductive strategies can be important at the beginning stages of mathematical justification. Rather than seeing intuitive reasoning as something to overcome in the pursuit of mathematically rigorous proofs, these researchers feel that mathematics educators should develop ways to leverage these very natural inductive and empirical tendencies of children to support the development of deductive reasoning. These assertions agree with others who support indirect proofs as a way to provide the foundation for future proficiency with formal indirect-proof structure (Thompson; 1996; Yankelewitz; 2009; Zack, 1997; Stylianides 2007; Francisco & Maher; 2005; Bieda et al., 2014; Mejia-Ramos, White Paper, 2016; Pedemonte, 2007; Schwarz, 2009; Conner (White Paper), 2016; Conner et al., 2014i; Conner et al., 2014t; Bieda (White Paper), 2016; Boero et al., 1996; Garuti, Boero, & Lemut, 1998). Zack (1999) also affirms Cooper et al.’s findings that children should strive to maintain their personal ways of knowing as they are confronted with formal, abstract representations of mathematics in school. Proof-making and reasoning can be facilitated by making students aware of the
gap between their personal ways of knowing and the ways of knowing accepted by formal mathematics culture.

Extant research supports that argumentation is crucial to mathematics education and that the role of the teacher is essential in promoting student argumentation (Maher, Landis, Palius, 2010; Bieda, 2010; Wagner et al., 2014; Whitenack & Yackel; 2002; Yackel, 2002; Maher, 2009; Stylianides, 2007; Ball et al., 2002; Weber et al., 2006). In fact, Krummheuer (1995) describes teachers' role as "imperative" (p. 255) and Whitenack and Yackel (2002) say that the importance of the role of the teacher "cannot be overstated" (p. 526). Researchers (Thompson, 1996; Styliandes, 2006; Zack, 1997; Maher & Martino, 1996; Yackel, 2002; Whitenack & Yackel, 2002; Wagner et al., 2014; Conner et al., 2014; Conner et al., 2014; Beida, 2010) have found that teachers promote students' opportunities to explain, justify, and defend their ideas, and invite others to evaluate those ideas which results in the construction of productive argumentation and Wagner et al. (2014) and Yackel (2002) note that the questions teachers ask can prompt students to expand their arguments by requiring them to offer additional data, warrants, or backing, or to make explicit elements of argumentation that were left implicit. Furthermore, Bieda (2010) found that when teachers were minimally involved in classroom discourse, very little argumentation was produced. Yet, Wagner et al (2014) assert that preservice teachers do not have experience participating in argumentation, as defined by researchers in this review and supported by the standards (NCTM, 2000; CCSS, 2010) because they learned mathematics through traditional instructional methods. It is likely that without appropriate professional development, the same is true for many in-service teachers. Since argumentation has common meanings that do not
match the definition of mathematical argumentation, it is likely that when teachers hear that their students should engage in argumentation, they do not know what productive mathematical argumentation looks like. If they do not know what it looks like in classroom interactions, it would be difficult for them to promote it. So, if it is true that what teachers do or do not attend to in classroom interactions is foundational for future instruction and that teachers can only make sense of and reason about what they notice in the classroom (Star and Strickland, 2007), it is "critically important" to "foster and support student argumentation" (Wagner et al., 2014, p. 8) in pre- and in-service teachers and teachers' noticing of argumentation in student discourse must be supported (Whitenack & Yackel, 2002). Once teachers and preservice teachers are better able to recognize argumentation, they will have key skills that are necessary for them to begin to produce classroom environments that promote argumentation.
Chapter 3 – Methodology

3.1 Overall Design

The methodology used for this design-based (Brown, 1992; Kelly et al., 2008) research study was two-fold. For the first part of the study, I created video narratives (VMCAnalytics) that show students engaging in argumentation. For the second part of the study, I used these video narratives (VMCAnalytics) in an intervention with preservice teachers in a mathematics teacher education course designed to support their understanding of argumentation. There were two overarching questions that guided my study:

1. What does student argumentation look like in problem solving settings?
2. How can VMCAnalytic video narratives support teachers’ understanding and noticing of argumentation in student discourse? Specifically,
   a. What do teachers notice about student argumentation before the intervention?
   b. What do teachers notice about student argumentation after the intervention?

More fine-grained questions emerged as I conducted an in-depth analysis of the data.

3. After the intervention:
   a. Do teachers notice more elements of argumentation? Specifically, do they notice more claims, data, warrants, backing, counterclaims, and counterarguments?
   b. Do teachers notice more of the structure or connectedness of the argumentation? Specifically, do they notice how the elements of argumentation in each event are related to each other and the relationship among the elements in one event to the elements that were presented in prior events?
c. Do teachers use more of the formal mathematical register by using more of the precise language of argumentation?

d. Do teachers add specificity to their descriptions by adding detail relevant to the argumentation in the event? Do the added details clarify elements or structure of the argumentation described (so there are fewer lines but they are solid)?

4. If teachers' pre-assessment description included implicit elements or structure, do they, in the post-assessment, make any of the implied argumentation more explicit?

5. If teachers included statements in the pre-assessment that make note of argumentation that is not actually presented in the event, do they eliminate these statements as part of their post-assessment description?

The first part of the study, creating and publishing VMCAnalytics addressed the first question. The second part of the study, the intervention with teachers, addressed the remaining questions.

3.2 Part I

3.2.1 Creation of VMCAnalytics

Extant literature supports the effectiveness of video artifacts in teacher education (Brophy; 2003; Maher; 2008; Palius & Maher, 2011; Star & Strickland, 2007; Sherin & van Es, 2005; Brunvand & Fishman, 2006; Hmelo-Silver et al., 2013; Maher, 2011; Maher, Landis, & Palius; 2010). In line with this literature and with data collected from the Video Mosaic Collaborative (VMC), for the first part of my study, I used the RUanalytic tool to create video artifacts that show student interactions that make visible
the important structures of argumentation, including students presenting, challenging, and modifying their arguments.

To create the analytics, I searched the VMC video data collection to find episodes of students engaging in argumentation, as defined by Krummheuer (1995) and Toulmin (1958; 2003). I chose this video collection because of its rich selection of videos on student reasoning and the video annotation tool associated with the collection, the RUanalytic tool. From these episodes, I chose short events from longer clips that illustrated different elements of argumentation, for example, students making claims, providing data as evidence, using warrants, challenging and refuting claims, making counterclaims, and providing counterarguments. Once the events were chosen, I used the RUanalytic tool to produce video narratives. Each narrative included a series of video events that illustrated student argumentation. I crafted a description and title for each event of the narrative that highlighted the argumentation in that event for each analytic and relied on transcripts of the videos developed for dissertations by RBDIL researchers to include quotes from students. To give an overview of the story for each narrative, I crafted an overall title and overall description.

Once each narrative was complete, members of my dissertation committee and at least one outside expert were asked to verify that the VMCAnalytics did show mathematical argumentation consistent with the agreed upon definition. Each of the verifiers was invited to provide feedback using the new VMCAnalytic social networking tool that is currently under test. In particular, I invited them to comment on the effectiveness for showing mathematical argumentation of each full analytic, and of each video event within the analytic. I invited input and made improvements as input
suggested. The goal of these reviews was to establish the validity of each analytic for the purpose of illustrating mathematical argumentation.

After a rigorous review and revision process, three of the four annotated video narratives were submitted for publication on the VMC. Of the four VMCAnalytics I created for this study, *Fourth graders’ argumentation about the density of fractions between 0 and 1* (Van Ness, 2014), *Eighth Grader Stephanie’s Argumentation about Meaning for the Square of a Binomial using Algebraic Reasoning* (Van Ness, 2015), *Eighth Grader Stephanie’s Argumentation about Meaning for the Square of a Binomial using Geometric Reasoning* (Van Ness, 2015), and *Fraction Analytic 1* (2015, unpublished), three are published with persistent urls; the fourth, *(Fraction Analytic 1)* is in press. For an in-depth analysis of the argumentation in these VMCAnalytics, see Chapters 4, 5, 6 and 7. The published analytics can be accessed at [http://videomosaic.org/analytics](http://videomosaic.org/analytics).

### 3.2.2 VMCAnalytic as an Assessment Tool

Hmelo-Silver et al. (2014) found that VMCAnalytics can be used as an effective tool for assessing students’ understanding and Brunvand and Fishman (2006) promote the idea of supporting teacher noticing through the use of video scaffolds. So, to assess teachers’ understanding and noticing of student argumentation, I created the *Fraction Analytic 1* VMCAnalytic. Analytics typically have video event segments, accompanied by annotations that include a title for the event and a description of the event that serve to focus viewers' attention on the ideas that the creator of the analytic thinks are important. Typical analytics also have an overall title and an overall description. I designed the *Fraction Analytic 1* for assessment purposes, without titles or descriptions. This
"skeleton" analytic included video events purposely chosen to show student argumentation and an overall description with background information only. As the assessment, teachers were asked to watch the video clips in each event and provide a description and title that described the argumentation they observed in the event, both before the intervention and after the intervention. These descriptions and titles became the "artifacts of practice" in the spirit of Es and Sherin (2008, p. 177) and the data that were evaluated to determine growth.

3.3 Part II

There is extant research that supports using video (Sherin, 2003; Sherin & van Es, 2005; Maher, 2008; Palius & Maher, 2011; Maher et al., 2014; Palius & Maher, 2013; Towers, 2007) in teacher education and video narratives have been used successfully at the university level (Agnew et al., 2010; Palius & Maher, 2011; Maher et al., 2014). Therefore, for the second part of my study, I completed a design-based (Brown, 1992; Kelly et al., 2008) intervention in which I used the VMCAnalytics I created to support teachers' understanding of argumentation.

3.3.1 Participants. There were 13 participants in my study, all of whom were enrolled in a mathematics education course, of which I was the instructor, at a large northeastern university. To see a detailed syllabus from the course, see Appendix A. Of the 13 teachers, five were enrolled in the class as part of a two-year post-baccalaureate Master's program and eight were undergraduate math majors who were in the fourth year of a five-year teacher certification program. All participants were training to be middle or high school mathematics teachers.
3.3.2 Intervention Design. The intervention took place during the 15 weeks of one semester in the spring of 2015 and intervention assignments were given as part of the standard course work. Teachers were advised of the study and consented to be participants. During the semester, teachers studied video narratives that showed argumentation, discussed them online, and pre-assessment and post-assessment data were gathered. The intervention part of the class was completed almost exclusively in an online discussion forum. Brief in class episodes were occasionally dedicated to the intervention to explain the assignments and answer teachers' questions.

3.3.3 Data Sources. To gather background data, I invited teachers to respond individually in writing to the following online prompts designed to determine their ideas about argumentation in the mathematics classroom: 1. What do you think argumentation means with respect to learning math? and 2. What is your experience with argumentation in the mathematics classroom? Teachers submitted their online responses at the start of the intervention.

To gather data on teachers' understanding and noticing of argumentation, I invited them to view the *Fraction Analytic 1* and write descriptions and titles detailing the argumentation they saw in each event. They also were asked to write an overall description and give the analytic an overall title. To accomplish this, I created individual copies of the Fraction 1 analytic, introduced teachers to the RUanalytic tool, and trained them how to use it. Teachers then used tool to annotate the analytic.

After pre-assessment data were gathered, teachers were invited to study the three published VMCAanalytics that I created each of which identifies and describes instances of argumentation. I specifically defined "study" as watching the video events, as well as
reading the annotations, including the event descriptions and titles and overall description and title. This instruction was repeated verbally and in writing throughout the semester. After the viewing of each analytic, teachers responded individually online to the following:

1. Identify elements of argumentation that can be identified in this analytic.

2. What are the claims being made by the children in the arguments presented? Who is making what claim?

3. Identify evidence that the children use to support their claims.

4. For the claims presented, identify those that are:
   (a) Challenged
   (b) Modified
   (c) Refuted

5. Was the argument resolved? Explain.

Teachers were then encouraged to discuss the analytics, prompts, and responses with one another in an online forum. My role in this online forum was minimal, limited to asking clarifying questions and providing discussion prompts as necessary. The cycle of studying each VMCAAnalytic, responding to the online prompts, and online discussion took between two and four weeks, depending on the length of the analytic.

At the end of the intervention, I collected post-assessment data. Teachers were invited to revisit their responses to two argumentation questions (1. What do you think argumentation means with respect to learning math? and 2. What is your experience with argumentation in the mathematics classroom?) and make changes if they wished. I also invited them to revisit the Fraction Analytic 1. They were instructed to re-watch the
analytic and review their titles and descriptions, making revisions if necessary, to describe in detail, the argumentation they saw in each event. I encouraged teachers to revisit the VMCAnalytics viewed in class, their and their classmates' responses to the online prompts, and the online discussions.

All of these data were collected and saved. Each teacher was identified using a number from 1 to 12 to keep their identities confidential. See Appendix F for the intervention assignments given to students throughout the semester.

The multiple sources of data I collected (Creswell, 2006), including the individual responses to online prompts, the online discourse, field notes taken while monitoring online discussions, and the VMCAnalytic descriptions written by each student ensure the validity of my results. I analyzed these data in various ways. Using a comparative case study methodology (Creswell, 1998) for which each participant in the study was considered a case, I compared case studies in terms of their growth in understanding of argumentation. I also considered each event as a unit of analyses, looking across participants at the growth in each event to see if there were patterns in Events that elicited greater growth in the noticing of argumentation.

3.4 Event Description Analyses

3.4.1 Dedoose: an online, collaborative qualitative research environment. After reviewing all of the data, it became clear that the pre- and post-assessment event descriptions and title data held the most potential to determine teacher growth in terms of the argumentation they noticed. I decided to begin this analysis using an online, collaborative qualitative research environment called Dedoose (see Dedoose.com for more information about this program). Prior to coding, teachers' individual responses and
event description documents were uploaded to the site in the form of text documents so they could be coded. I completed several rounds of coding and re-coding to explore my research questions, beginning with themes that I was interested in, such as what participants identify as elements of arguments and what language they used to describe them, to guide my initial coding scheme and help me create broad categories. My initial codes were apriori, informed by Toulmin’s argumentation model to determine how the teachers' language changed. Then, as I analyzed data, I refined the grain size of my coding system to hone in on themes of interest and add codes as necessary to capture emergent themes. My coding scheme was informed by axial and selective coding sequences (Corbin & Strauss, 1990). My coding was axial in that my categories were related to their subcategories and I continually tested these relationships against my gathered data. My coding was also selective in that, as I analyze more and more of the data, I unified categories around core categories, choosing the core categories that most represent the central phenomenon of the study. Additionally, I evaluated and identified poorly developed categories and made changes accordingly.

My final coding scheme consisted of a system of various codes and sub-codes as described by Yankelewitz (2009). The top-level codes, or parent codes, were then divided into sub-codes, or child codes, and some child codes were divided into sub-codes, or grandchild codes. The structure of the codes was as follows:

Parent code

Child code (sub-code)

Grandchild code (sub-sub-code)
The online coding environment I used allowed up to five hierarchical levels of granularity. The argumentation codes focused on the Parent Code: "Elements of argumentation" and the child codes included elements such as, claim, counterclaim, counterargument, warrant, evidence, support, proof, prove, consensus. My initial coding system was refined and updated as I analyzed data, and I added new codes to capture emergent themes and deleted codes that turn out to be associated with poorly developed categories.

Once I developed my final coding scheme, I performed a multilayer analysis on the data. I worked with graduate students to establish coding reliability. I analyzed the pre and post event descriptions and titles to identify the argumentation teachers noticed by focusing on language that teachers used. I then compared pre and post event descriptions and titles for each event for each teacher to see how the use of the formal mathematical register for mathematics changed.

To ensure interrater reliability among coders using Dedoose, several tests were set up through Dedoose coding software. After completion of the first test, codes were refined and discussion of the coding system clarified the application process. Most of the discrepancies involved the actual application process, rather than contextual argumentation ideas. For example, it was unclear whether parent codes were to be applied separately or automatically, so these codes were applied inconsistently. The exception to this was the idea of consensus. The consensus codes were discussed and refined to clarify them. A second test was set up involving the same data, to be sure that the coding structure was clear. Then a third test with a new data set was conducted. Final
interrater reliability for all coders was better than 85% using Cohen’s kappa statistic
(Cohen, 1960). For a detailed discussion of the results of these analyses, see Chapter 19.

3.4.2 Comparative qualitative analysis by event. After an analysis using Dedoose of the pre- and post-descriptions for each event, there was evidence to support that there was growth in the formal mathematical register used by the teachers. To determine what the growth was and how it changed the descriptions of the argumentation that the teachers' noted in the descriptions, I performed an in-depth analysis of each teachers' descriptions, comparing their pre and post descriptions for each of the 15 events in assessment VMCA
tic. Before this was possible, I first analyzed the argumentation that was apparent in the analytic. In the spirit of Krummheuer (1995), Toulmin (1958; 2003), Stephan and Rasmussen (2002), Conner et al. (2014), and others, I used Toulmin's layout to diagram the argumentation in the analytic and found that the Toulmin's model was a helpful tool in my analysis. The boxes in the diagrams represent elements of argumentation, for example, claims, data, warrants, and backing. The line segments and arrows represent connections among the elements, or the structure of the argument itself. For example, a student might state Claim A and Data B, but not make the connection explicit, thus, there would be no arrow between the data and the claim, see Figure 3.1.

![Image](Data B and Claim A)

**Figure 3.1. Data and claim with no connection**

However, a student might state "Claim A because of Data B" or "Data B so Claim A." The language connecting Claim A and Data B would be denoted by an arrow and the arrow denotes some structure to the argument (Figure 3.2).
Figure 3.2. Data and claim with connections

The second argument (Figure 3.2), then, would have more structure than the first (Figure 3.1). To see a detailed discussion of the argumentation in the pre-assessment video, including the diagrams and discussion, see Chapter 7.

To analyze the changes in the teacher's descriptions before and after the intervention, I took a similar approach. I used Toulmin's layout to diagram the student argumentation teachers noticed in the pre-assessment and the post assessment. Then I compared the pre-assessment text and diagrams to the post-assessment text and analyzed the differences. There were instances where teachers alluded to elements of argumentation, but the reference to the argumentation was implied. To note this, I used dashed lines. Boxes with the implied element could be dashed, or line segments or arrows connecting elements might be dashed if the structure was implied. Additionally, there were instances where teachers mentioned elements of argumentation that might be true, but were not specifically stated by the students in the analytic event. These instances were noted using gray boxes and text. Occasionally, teachers' descriptions included reference to an implied element of argumentation, but that implicit element was not included in the event. In this case, the element was denoted as gray and dashed.

Reliability for the diagrams was accomplished through review of the diagramming by other mathematics education experts. Discrepancies were discussed and changes were made accordingly. My analysis of the argumentation described by the teachers was multi-layered. It was based on the analysis of the argumentation put forth by
the students in each event of the analytic. Students used imprecise language to make their arguments and, in accordance with Krummheuer (1995), distinctions among the elements were sometimes difficult to determine. Then, I analyzed the actual description of the students' argumentation by teachers in the pre-assessment, attending very closely to the language they used, which was often imprecise. The imprecision of the language sometimes made the intentions of the teachers' statements difficult to determine. Then I compared the pre-assessment description with the post-assessment description to note and interpret changes relevant to the argumentation discussed. When teachers used more precise language, the argumentation described was often clarified. However, much of the language in the post-assessment descriptions was imprecise, and, thus, sometimes difficult to interpret. My analysis focuses on the changes from pre to post and how those changes affected the argumentation described. It is possible that other interpretations might be suggested as a result of the teachers' imprecision.

Three categories of change emerged as I analyzed these data: Category 1 (C1): Change in the elements of argumentation described by teachers as denoted by the boxes; Category 2 (C2): Change in the structure of the argumentation described as denoted by the connecting segments and arrows; and Category 3 (C3): Change in the use of the formal mathematical register, in other words, changes in the use of the technical language of argumentation. Other differences emerged and were included in a fourth category, Category 4 (C4). These changes included: making implicit argumentation explicit, connecting the argumentation in one event to the argumentation in an event that came before, eliminating statements that were not made by the students themselves, and using
more of the students' own language. My analysis and discussion of the pre and post description changes focused around these four categories, summarized below:

C1 – Elements: statements that serve as claims, counterclaims, warrants, or backing.

C2 – Structure: how the elements of argumentation are connected to one another in an event and how argumentation in one event is connected to argumentation in previous events. Connectedness also includes instances in which a claim or a counterclaim is connected either implicitly or explicitly to a previous claim or if an argument results the modification or refutation of a claim.

C3 – Technical Language: use of the formal mathematical register for argumentation, for example: using terms such as claim, evidence, support, counterclaim, counterargument, counter, agree, prove

C4 – Other: when the implicit is made explicit; when statements that are not spoken by students are eliminated, when details clarify uncertain elements, when more of the students' own words are used, when additional information that is relevant to the argumentation in the event

I described in detail the differences for each teacher's pre and post descriptions in narrative form. Then I coded their responses to obtain a quantitative analysis of the growth by first giving each post description a score of 0 or 1 for each of the four categories: elements, structure, language, other; where 0 represented no growth and 1 represented growth. I also analyzed these data to determine if there were any
relationships among the growth in different categories. A detailed discussion of the
analysis of teachers' growth in their noticing of argumentation can be found in Chapters 8 – 19.
Chapter 4 – VMCAnalytic 1: Fourth graders’ Argumentation About the Density of Fractions Between 0 and 1

4.1 Introduction

The VMCAnalytic described in this chapter was the first annotated VMCAnalytic that participants studied. They were given the instruction to "study" the VMCAnalytic and then discussed it online. "Study" was defined as watching the events and reading all of the accompanying descriptions, including the event descriptions and the overall descriptions. Teachers were reminded throughout the semester to both read the descriptions and watch the events video clips both via in class verbal reminders, and written reminders on the online discussion forum. In this chapter, I illustrate the reasoning of fourth-grade students as they try to make sense of the placement of small unit fractions between 0 and 1 on a line segment and explore the idea of density of number. We describe their argumentation as they interact with each other and a researcher who is facilitating the session. The chapter is based on the overall description and the description of each event in the VMCAnalytic itself (To view the VMCAnalytic itself, see Fourth Graders’ Argumentation About the Density of Fractions Between 0 and 1; Van Ness, 2014, located in the Video Mosaic Collaborative)

(http://dx.doi.org/doi:10.7282/T39K4CZC).

4.2 The Argumentation Session

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6 This chapter references events from the VMCAnalytic entitled “Fourth Graders’ Argumentation About the Density of Fractions Between 0 and 1”, located in the Video Mosaic Collaborative (http://dx.doi.org/doi:10.7282/T39K4CZC)
The data in this chapter were taken from the third month of a larger data set gathered as a result of a year-long research intervention involving fourth graders exploration of fractions. The events depicted here took place during the 14th of 21 sessions of the intervention and occurred on November 3, 1993 (Schmeelk, 2010; Yankelewitz, 2009). In earlier sessions, students explored various problems involving comparing fractions. These tasks began with model building using Cuisenaire rods, attending to the attribute of length. Rods, and trains made of two or more rods, were assigned variable number names. Students explored a variety of fraction problems in which the unit was defined for certain combinations of rods. In the process of building justifications for their solutions to these problems, students naturally were introduced to fraction operations and concepts such as equivalence. Prior to the research intervention, the students had no previous instruction on operations with fractions. They had prior experience identifying fraction names as parts of units from a variety of representations.

The VMCA analytic associated with this chapter includes events that were taken from the third of the four clips that capture the entire fourth-grade classroom session on November 3rd. In Clips 1 and 2, students discuss number lines, the relationships among Cuisenaire rods, a ruler, a number line, the idea of infinitely many, and how they might plot positive and negative integers and fractions on a number line. If viewers are interested in a more detailed picture of how students developed these ideas, Clips 1 and 2 are available for viewing on the Video Mosaic Collaborative Repository at videoemosaic.org (See The infinite number line, Clip 1 of 4: Naming points on the number line and The infinite number line, Clip 2 of 4: Placing integers on the number line).

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7 This research was supported by a grant from the National Science Foundation: MDR-9053597 directed by R. B. Davis, C. A. Maher.
This chapter and its associated VMCAalytic report students’ reasoning about whether or not infinitely many fractions can be placed between 0 and 1 on the number line. Claims, counterclaims, warrants, backing, qualifiers, and justifications are elements of their argumentation that we describe in events from student discourse.

4.3 Students’ Argumentation as They Consider the Density of Numbers on the Segment of the Number Line from 0 to 1

Task: Place unit fractions (1/2, 1/3, 1/4, and so on, through 1/10) on a segment of a number line labeled from 0 to 1.

Just prior to the classroom discussion that follows in this chapter, students worked on a task that involved placing unit fractions (1/2, 1/3, 1/4, and so on through 1/10) on a segment of a number line labeled from 0 to 2. Each student was asked to construct his or her individual number line with points to indicate where the fractions should be.

Figure 4.1. Placing unit fractions on a number line

The clip, in its entirety, can be found at [www.videomosaic.org](http://www.videomosaic.org) by accessing the clip *The infinite number line, Clip 3 of 4: How many numbers between 0 and 1?*
4.4 Erik challenges claim about density

After students investigated the placing of fractions on the number line, Researcher Maher made an initial claim about density. She stated: “Mathematicians claim that there are infinitely many fractions [between 0 and 1].” See Event 1.

Figure 4.2. There are infinitely many fractions between 0 and 1

This claim set the stage for the student argumentation that followed.

Figure 4.3. Students consider the claim “There are infinitely many fractions between 0 and 1”
As students considered this claim, Erik could be heard in the background of the conversation. Erik expressed doubt that this claim was true and challenged the researcher’s initial claim about density.

Researcher Maher: They also claim that there are infinitely many fractions [from 0 to 1] … Now, that is something to think about. I don’t expect you to solve that problem in your mind right now; mathematicians have worked on this for centuries.

Erik: I just don’t understand how there can be infinitely many numbers between zero to one.

Andrew: Fractions… fractions…

Erik: No, infinitely many numbers from zero to one. It doesn’t… make sense.

Figure 4.4. It doesn’t make sense!

4.5 Alan Proposes a Claim and Classmates Respond

See Event 2 and Event 3

As the conversation continued, another student, Alan, contributed to the conversation. Alan supported the researcher’s claim that there are infinitely many fractions between 0 and 1. Alan proposed a related claim that the distance between 0 and 1 on the number line can be divided into many smaller parts, even into “zillionths.”
Alan: [referring to the line segment on the board] You could divide … from zero to one into the smallest of fractions. It’s easy to imagine you could divide it into zillionths…You could divide it into zillionths and there would still be space in there.

Figure 4.5. You can divide from zero to one into the smallest of fractions

After Alan posed his claim, other students joined the discussion. Event 3 captured the discourse among Erik, Alan, and Michael. Erik was not convinced. He challenged Alan’s claim that you could divide the distance between 0 and 1 into the smallest of fractions. Erik stated that he did not think it made sense and made a counterclaim when he argued that what you can divide the number line into depends on the size of your “one whole.” Erik contradicted and challenged Alan’s claim using this counterclaim. Alan defended his claim by emphasizing that from zero to one you could divide the segment into smallest of fractions. Erik continued to argue that he did not agree, because, if, for example, the whole was divided into tenths, you could not divide it into smaller fractions.

Another student, Michael joined the conversation. Michael challenged Alan’s argument, presenting a qualifier that stated the conditions under which he believed both the initial claim that there are infinitely many fractions between 0 and 1, and Alan’s claim that the distance between 0 and 1 can be divided into zillionths, could be true. Michael
argued that one would need the longest number line in the world to show very small fractions on the number line.

Alan: You could divide it into zillionths and there would still be space in there.
Michael: If you had the longest number line in the world.
Erik: If you divide it into zillionths depending on what number you want to hold it to.
Researcher Maher: No. no. Michael, I’m talking about this little piece between zero and one.
Alan: Yeah, but you could still divide it …. 
Michael: But if you made a number line to show zillionths, then you would have to have the longest number line in the world.
Erik: Alan. Alan. That doesn’t make sense.
Alan: Yes it does.
Erik: Even if you were to divide it into zillionths is depending on how big your one whole is. If you [sic] one whole is ten you can’t divide it into zillionths.
Alan: Well, from zero to one, you could. You could divide it into zillionths.
Erik: If you one whole is ten, how could you divide it into zillionths?

Figure 4.6. You’d have to have the longest number line in the world

4.6 Evidence from Students’ Experiences and Alan’s Expanded Claim

See Event 4, Event 5, and Event 6

Alan continued to refine his argument that the segment of the number line between 0 and 1 can be divided into many smaller parts. He provided evidence for his
claim using the metaphor from his personal experience—the size of a dust particle—for his claim that the “size” of the line did not matter by asserting that, "you could have a line the size of a dust particle and you could put that on there a zillion times. You would have zillionths."

Michael referred to a pin that was smaller than a dust particle, affirming that he understood the idea of the dust particle and Erik expressed acknowledgement of this idea when he stated that there is something smaller than a dust particle—a dust bug.

Alan: Well, as I was saying before about the zillionths, you could have a line the size of a dust particle and you could put that on there a zillion times. You would have zillionths.

Michael: If you had a pin that was smaller than a dust particle, then …

Erik: Something that is smaller than a dust particle … a dust bug … a hundred dust bugs can fit into a dust particle …

The metaphor of a dust particle was a familiar idea that these students acknowledged made sense to them and other students return to throughout the class discussion.

As the discourse continued, Andrew supported Alan’s claim by using as a warrant an idea from his own experience with a microscope of observing smaller intervals. He argued that if you looked at a number line through a microscope, you could see that there is more room to plot smaller and smaller fractions, for example, one hundredth, one thousandth, and one millionth. This statement was a warrant because it supported why being able to enlarge the line justified the idea that you can divide the distance between 0 and 1 into the smallest of fractions. Alan agreed with Andrew’s statement and, in response to the suggestion that a scientist might discover a more powerful microscope;
Michael stated you could fit more numbers on the number line, suggesting that he might be able to show the smallest of fractions on the number line.

Andrew: Well, if you made a number line and you took a magnifying glass or a microscope, and put your number line under it, you would see that you have a lot of room left to put the one-hundredth, one-thousandth and the one-millionth.

Researcher Maher: Did you all hear what Andrew said?

Alan: Yes. If you did put it under a microscope it would look like you had enough room to put another zero to one in there. It would look like that. You could have it enlarged so that the line from the zero would be this big [raises hands and makes space between them about a foot]. And you would still have room there to put more.

Researcher Maher: What happens when scientists discover more and more powerful telescopes?

Michael: Then the more numbers you could fit onto one number line.

It is interesting to note that at this point in the discussion, Michael’s statement about being able to see more numbers on a number line if you used a more powerful microscope was a modification of an earlier claim in which he stated that you would need the “longest number line in the world” to show zillionths (see Event 3).
Another student, Brian, started to contribute to the argumentation. Brian agreed with Alan’s claim that you can put zillionths in the interval between 0 and 1. He stated, “So, like what Alan said, you can put zillionths in.”

Alan continued to refine his argument. He extended his claim, stating that not only can you divide the number line into “zillionths,” but you could divide it into even smaller intervals, “you could even make it smaller than that and make smaller pieces to put in there.”

4.7 Erik Presents a Counterclaim

See Event 7.

Erik presented a counterclaim that referenced the prior suggestion of using a microscope as a tool. He argued that even if you used a microscope, there really was not more space. Prior to this point in the discourse, Alan and others collectively developed an argument that if you use a microscope you can see that there is space on a number line between 0 and 1 to plot the smallest of fractions, “zillionths” and even fractions smaller
than “zillionths.” But, Erik challenged this argument with the counterclaim that when you use a microscope, you are actually not getting more space, suggesting that he interpreted the claim of Alan and others that there is enough space on a number line between 0 and 1 to plot smaller and smaller fractions, they were also claiming that when you used a microscope you “get more space.” Erik’s counterclaim contradicted what he believed to be the claim of his classmates. Erik justified what he interpreted as a claim by saying that it only looks like you are getting more space, but you are really not getting more space.

Figure 4.9. In actual reality, you are not getting more space.

Brian: So, like what Alan said, you can put zillionths in.
Researcher Maher: So you think you can put zillionths in? You are changing your mind? So you are sort of agreeing that there are lots and lots of fractions between zero and one if you had this …
Alan: Even like you could even make it smaller than that and make smaller pieces to put in there.
Erik: What I don’t understand is that if you are using a microscope to get more space, in actual reality you are not getting more space.
Researcher Maher: That is an interesting idea, isn’t it Erik?
Erik: You see, when you are using the microscope it looks like you are getting more space, but in actuality you’re not getting any more—it just looks like it; but, you are not.

4.8 Response to Erik’s Counterargument

See Event 8 and Event 9
In response to Erik’s counterclaim, Andrew, Alan, and Erik continued the argumentation by clarifying the arguments and claims being made about the microscope enabling you to see more space and the actual existence of “more space.” Andrew and Alan continued to support the idea that the microscope could help you see that there is enough space on the number line to show smaller and smaller fractions. Andrew said that when you use a microscope, you are getting more space on the number line because the human eye cannot see space that’s there and Alan explained that magnifying the number line enabled you to see how much space you have left between two points on the number line, for example, the “zillionth” and the zero.

Erik restated what he understood to be Alan’s argument to clarify the claim that was being argued: Is it that using a microscope gets you more space or that using a microscope makes it possible for you to see how much space there is? Alan stated that Erik had not represented his argument accurately. Erik again asserted that he believed that Alan was saying that using a microscope gets more space on the number line and he did not believe that to be true. Erik’s argument initiated a discussion about whether or not the microscope actually gives you more space, or just helps you see more of the space that is there.

Andrew: Well, actually you are because the human eye can’t see that, but…
Alan: When you enlarge it you can see how much space you have left between the zillionth and the zero.
Erik: Yeah, but Alan what you were saying before you were saying that when you use the microscope you get more space in that number line. That is what you were saying before.
Alan: No. That’s not what I was saying.
Erik: From what I understood, you were saying that if you use a microscope you get more space on the number line. It is not true.
Alan continued to respond to Erik’s refutation of his argument by restating the idea of the microscope, clarifying and refining his argument for Erik, and refuting the idea that his current position differs from his initial position. Alan provided further support for his argument as he explained,

If you had some really small pen … you could draw a small line in the space…but you don’t really know how much space you have left between the zillionth and the zero. You don’t really know that because you can’t see it so you look at it under a microscope you could see how much space you have left.

In light of the difference in interpretation of the claims and arguments that were being presented by the students, Researcher Maher revoiced what appeared to be the arguments being made. She stated that she interpreted what was meant by the “more space” that Alan referred to was the “same space” on the number line. She commented about Erik’s claim that Alan meant “extending” the number line, pointing out that each may have had “a different picture” in their heads and that she believed that Andrew’s picture matched Alan’s picture. Erik confirmed that Researcher Maher represented his position accurately and asserted that he believed that in the first claim made by Alan made, he was referring to an extended number line.

4.9 David’s Warrant

See Event 10

David, another student, contributed to the argumentation in support of Erik’s claim: “when you’re using a microscope it looks like you’re getting more space but in actual reality you’re not getting anymore. It just looks like you are.” David presented a warrant: You are not seeing more, you are just seeing closer. He said, “I think that you
can’t really see it too well, but if you use a microscope then you are seeing closer and it looks like you are seeing more, but you’re really not you’re just looking closer than before.”

*Figure 4.10. You’re not seeing more, you’re seeing closer*

This statement could be considered a warrant because it connected the idea of being able to see more space on the number line to being able to place more and more fractions on the interval between 0 and 1.

4.10 Debating the Idea of “More Space”

See Event 11 and 12

As the discussion progressed, Researcher Maher invited other students to contribute to the argumentation. Some students had comments to make, some did not. Several students—Audra, Jessica, Beth, Michael, and Mark—interjected their support for the claims being made.
Audra said that she agreed with Alan and Andrew because the human eye cannot see the space on the number line when it is drawn small and "if you put it under a microscope you really could see more…"

![Figure 4.11. The human eye can’t see it](image)

Jessica also agreed with Alan and Andrew because when you put the number line under the microscope, "there might be a huge space that you could fit a lot of spaces."

![Figure 4.12. The microscope helps you see the little spaces](image)

Beth further explained the idea of “more space” as she tried to clarify Alan’s statements. She argued that Alan was not saying that the space is getting larger, but rather that it was “not going to stop” and Michael tried to clarify by stating that you can think of
it as "the more you see the more space you have." Beth agreed with Michael’s assertion and made the insightful statement that this idea of infinity it was “hard to explain.”

![Figure 4.13. It’s hard to explain](image)

Mark also agreed with Alan and Andrew stating that under a microscope, you could see a large space on the number line.

![Figure 4.14. Under a microscope, you can see a large space on the number line](image)

Researcher Maher: Okay. I am interested in how some other people are thinking about it. I have not heard some of the girls thinking about this. Laura, what do you think? You are listening to this very carefully what is your opinion on this? Laura: [Smiles and shakes her head.]
Researcher Maher: Do you have an opinion?
Erik: Are you a little bit lost?
researcher maher: i don’t think so she is listening very carefully to both ideas. what do you think?

laura: [laura shakes her head left to right and mumbles.]

researcher maher: what do you think between zero and one, here? what kind of numbers do you think you are seeing there? any idea? no?

laura: [laura shakes her head left to right and opens her mouth as if to speak but remains quiet.]

researcher maher: any idea? i want to hear from some other people. what are your ideas about the numbers between zero and one? okay, audra, and then jessica, and then mark. okay, audra?

audra: i really do agree with them because…

researcher maher: with whom?

audra: with andrew and alan because the human eye can’t see it if you are making it that small so if you put it under a microscope you really could see more …

students: it would be really tiny but you could still see it …

researcher maher: jessica?

jessica: well, i think i agree with alan and andrew because you really can’t see if there are any little spaces; but when you put it under a microscope there might be a huge space that you could fit a lot of spaces

beth: he is not saying that it is getting bigger, he is just saying that it is not going to stop …

michael: oh, it is sort of like the more you see the more space you have

alan: yeah.

researcher maher: what about that? michael said that it is sort of like the more you see the more space you have? that is michael’s question; what do you think?

beth: it is hard to explain.

researcher maher: it sure is … it sure is hard to explain. kelly, were you going to say something?

kelly: [nods head]

researcher maher: no? okay. mark and then we’ll hear from jen.

mark: i think that i agree with alan and andrew because they are right you can’t see the thing but if you put it under a … microscope and if it is a really powerful one you would have a huge space there.

researcher maher: okay. amy?

amy: not sure.

students continued the discourse about space, as they refined the ideas being presented. david said:

i think that you can take the little smallest thing and then put it under a microscope and you will have a lot more space but you don’t. it looks like a lot more space but it really isn’t. you are just magnifying it.

students, including alan, agreed with this point of view.
Michael showed the evolution of his thinking from “needing the longest number line in the world” (see Event 3) to marking smaller and smaller intervals on the existing segment when he said:

It looks like you have more space and humans take advantage of it and take that really big space and mark these really, really little lines on it that you really just can’t see on it.

4.11 Alan’s Diagram

See Event 13, Event 14, Event 15.

To further refine and provide evidence for his argumentation, Alan came up to the overhead and created a diagram for the class to see. Alan drew a magnified section of the line segment in which he showed the placement of the fraction 1/100. He stated that there is space between 0 and 1/100 that existed but could not be seen with his diagram. He used the segment of the number line between 0 and 1/100 to support Andrew and David’s idea that using a microscope would enable you to see smaller fractions, as well as the claim that it enabled you to see more space.
As Alan drew a diagram and referred to the original interval between 0 and 1, he explained his drawing and refined his argument:

If this could be the zero and this could be the size of the bar [drawing a rectangle and referring to the tick mark at 0 on the original number line], then there can be your line now if you had the hundredths which would probably go somewhere in here [referring to a point on the original number line very close to 0], it would look say if it was right here [drawing a rectangle to stand for the tick mark for 1/100, a distance to the right of his 0 tick mark]. And then you would have all that space in there [drawing an arrow to refer to the space between his 0 tick mark and the tick mark for 1/100]. It looks like it, but you really don’t have that much space. That’s just if you had it [the space between 0 and 1/100] really big, that’s how much space it would look like you see. So that means you could divide this [the distance between the tick marks he drew] into halves and thirds and fourths and fifths and all of that.

Figure 4.16. That’s how much space it would look like

After Alan explained his initial diagram, Researcher Maher restated Alan’s argument by referring to it. As she restates Alan’s argument, Alan labeled the segment of the number line, indicated the placement of 0 and 1/100, and drew dashes between 0 and 1/100 indicating the placement of other, smaller fractions.

Alan continued to refine his argument using the drawing of the number line:

Yeah cause it looks like you have a lot of space but you really only have that tinsy, winsy little space in between there [pointing to the small space on the original number line between 0 and the tick mark for 1/100 that is very close to 0] I mean you could take a really small pen and you could divide this [the distance
between the tick marks for 0 and 1/100 on the magnified section of number line] up into all of these pieces. If you look at that with your regular eye you couldn’t see that so you’d have to make it bigger.

Figure 4.17. You could take a really small pen and divide it up

Alan used the segment between 0 and 1 to support his idea that you could continue to partition the space into more and more sections. He referred to one of the intervals that he drew and said, “Now if you magnified those spaces—and here would be the little bars—you could divide this space up into little tiny pieces and that you could divide up into little spaces.”

Figure 4.18. You can continue to magnify the spaces
4.12 Students Support Alan’s Argument

See Event 16, Event 17, and Event 18

After Alan’s drawings were presented, Brian supported the idea that the microscope can help you see the space on the segment by stating that humans do not have powerful enough eyes to see the space that is there. Alan continued to support his claim, describing that a very powerful microscope can help you see more and more space on the number line. He explained that, if you continue to magnify the spaces, you could divide the spaces into smaller and smaller sections.

Researcher Maher: Okay. Brian?
Brian: I have a comment about what Alan and Andrew said. You see humans don’t have powerful enough eyes to see where the zillionths are so there really is a lot of room but you don’t see it because the human eye is not as powerful as a microscope.
Michael: Oh. I get it so there is a lot of room that you can’t see.
Alan: Say in the future that you come up with this really high powered microscope you could make that zero bar from the floor to the ceiling that would maybe let you see it being that big. You could divide it up into such small pieces that when you took off the microscope you wouldn’t see anything it would be so tiny and so small that you couldn’t see it but there really is space there and if you magnify those really tiny pieces you could divide those up into spaces.
David: Then you would probably need something with a really small point to write that small.
Researcher Maher: So it sounds like the instruments get in the way, right, not the numbers.

Although several students had contributed to the discourse, at this point, Researcher Maher again invited children who had not participated in the argumentation thus far to state their ideas. In response to this invitation, James stated that he agreed with Alan and Andrew’s claim that you can place more and more numbers on the number line. He argued that it “makes sense that there is more space between the zillionth etc... etc...”
Alan continued to refine his argument about being able to cut the intervals on the number line into smaller and smaller parts. He stated that you could make the “biggest number you could think of” one and “you could go on forever with this… I mean you could keep magnifying it and magnifying it and magnifying it, dividing it, magnifying it dividing it.”

In the final moments of this session, there was consensus from the class as voiced by Brian, David, and Meredith, that an infinite number of fractions exist on a number line in the interval between 0 and 1. The argument was resolved as students gave their arguments in support of Alan’s claim. Brian said that you could use the microscope to put billions on the number line, and that it does not matter how big it is, “it could be as small as a germ and you could still put germs on it.” David brought the conversation back to dust bugs, revisiting the idea that you could put something as small as a dust bug on the number line and you could still see it with the microscope. Researcher Maher asked Gregory if he had any comments and he responded that he did not. Finally, Meredith synthesized the discussion by stating, ‘I think what he is trying to say is that if you look at
it through the microscope then there is a lot of space but if you just look at it through the human eye then there isn’t very much space in there.”

Figure 4.20. Meredith sums up the argument

Brian: You could take the number line that has so much little space between it and if you look at it with a very powerful microscope then you would be able to put billions in it; so, it doesn’t matter how big it is it could be as small as a germ and you could still put germs in it.
Researcher Maher: David.
David: I was going to say what Brain said that it could be as big as a dust bug and just …
Researcher Maher: Gregory, what do you think about all this dust bugs and things that big? Do you have any editorial comments on this discussion?
Gregory: No.
Researcher Maher: Meredith.
Meredith: I think what he is trying to say is that if you look at it through the microscope then there is a lot of space but if you just look at it through the human eye then there isn’t very much space in there.
Researcher Maher: That is a good synthesis.

4.13 Conclusion

This was the first of three VMCAalytics that were created to illustrate argumentation that teachers studied. Teachers studied the analytic individually, and then discussed it online over the course of a few weeks. In their online discussion, they answered and discussed the following five questions:
1. Identify elements of argumentation that can be identified in this analytic.

2. What are the claims being made by the children in the arguments presented? Who is making what claim?

3. Identify evidence/backing that the children use to support their claims.

4. For the claims presented, identify those that are:

   (a) Challenged

   (b) Modified

   (c) Refuted

5. Was the argument resolved? Explain.

The argumentations identified in the discussion thread for the VMCAnalytic were not discussed in class.
Chapter 5 – VMCA: Eighth Grader Stephanie’s Argumentation about Meaning for the Square of a Binomial using Algebraic Reasoning

5.1 Introduction

This chapter describes the argumentation in the second published VMCA that teachers in my study watched, *Eighth Grader Stephanie’s Argumentation about Meaning for the Square of a Binomial using Algebraic Reasoning* (http://dx.doi.org/doi:10.7282/T3FN180C). As with the first, teachers were asked to study the VMCA by watching the video events and reading the descriptions. They discussed the VMCA on the online forum. The purpose of this VMCA is to illustrate events of a student involved in argumentation. The events show forms of argument that naturally occur in a student’s mathematical arguments. The data from which the VMCA was made were taken from a larger data set collected in a longitudinal study designed to investigate students’ mathematical reasoning (Aboelnaga, 2011; Maher, 2005).

In this data set, Stephanie, an eighth-grade student enrolled in a traditional eighth-grade algebra class in a private parochial school, participated in a series of seven task-based interview sessions conducted by Researcher Carolyn Maher. The research team investigated Stephanie’s mathematical reasoning since she was in first grade. At this point in her schooling, Stephanie had changed schools and began expressing dislike for math, indicating that she wanted to make sense of the symbol manipulation rituals that

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8 This chapter references events from the VMCA entitled *Eighth Grader Stephanie’s Argumentation about Meaning for the Square of a Binomial using Algebraic Reasoning*, located in the Video Mosaic Collaborative (http://dx.doi.org/doi:10.7282/T3FN180C).

9 This research was supported by a grant from the National Science Foundation: REC-9814846 directed by C. A. Maher.
were new to her. This resulted in a teaching experiment where the goal was for Stephanie to explore and make sense of the meaning behind the rules and symbols (Aboelnaga, 2011). The data used in the VMCAAnalytic described in this chapter come from two of the seven after-school interview sessions: Interview 1 and Interview 2.

In the first four interview sessions, Stephanie explored ideas about expanding a binomial to the second power. During Interview 1, prior to Event 1 of the VMCAAnalytic, Stephanie explored the distributive property and provided a convincing argument justifying why \( a(x+y) = ax + ay \), which she restated and supported in Event 3 of the VMCAAnalytic described here. The events of the VMCAAnalytic described here show Stephanie’s exploration of the meanings of \((x+y)(x+y)\) and \((a+b)^2\).

In the sessions depicted in the VMCAAnalytic described in this chapter, Stephanie made a number of conjectures and claims about the meaning of these expressions. She tested her conjectures and claims, produced counterexamples, posed counterclaims, refuted, modified, and refined her conjectures, claims, and arguments. For the purposes of the VMCAAnalytic, conjectures were posed as questions and claims were statements made with more confidence and evidence. While the distinctions between conjecture and claim were not always clear, judgments were made distinguishing them based on factors such as the inflection of Stephanie’s voice, the context surrounding the statement, and evidence derived from making use of algebraic properties. When viewing the VMCAAnalytic, however, it was not critical to distinguish whether a claim or conjecture was posed, but rather to make visible the statements that evoked argumentation during the investigation of an idea.
Events 1 through 9 were taken from the first interview between Stephanie and Researcher Carolyn Maher. This interview took place on November 8, 1995, the fall of Stephanie’s eighth-grade year. In these events, Stephanie explored \((x+y)(x+y)\). In Event 1, Stephanie made two conjectures: 1. \((x+y)(x+y)\) might be equal to \(x^2 \cdot y^2\); and 2. \((x+y)(x+y)\) might be equal to \(x^2 + y^2\). Although Stephanie made the first conjecture, she quickly abandoned it after she made her second conjecture and did not consider this other conjecture for the rest of the exploration. The purpose of Events 2 through 9 was to trace how Stephanie investigated the reasonableness of the second conjecture: \((x+y)(x+y) = x^2 + y^2\).

The events in the VMCAnalytic described in this chapter were chosen purposefully to illustrate Stephanie’s argumentation about raising a binomial to the second power algebraically. Her exploration of these ideas went beyond what is represented here. A more detailed picture of how Stephanie developed these ideas can be found in the complete set of clips for Interviews 1, 2, 3, and 4 which are available for viewing on the Video Mosaic Collaborative Repository at www.videomosaic.org.¹⁰

5.2 Stephanie’s Initial Conjectures about the Square of a Binomial

See Events 1 and 2

In the first event of the VMCAnalytic, Stephanie began an exploration of the binomial expansion by investigating the meaning of the product of two binomials, \((x+y)(x+y)\). This event was taken from the fifth of nine clips in the first of seven

¹⁰ You can find these clips by accessing the videomosaic.org website and watching the following clips: Early algebra ideas about binomial expansion, Stephanie’s interview one of seven, Early algebra ideas about binomial expansion, Stephanie’s interview two of seven, Early algebra ideas about binomial expansion, Stephanie’s interview three of seven, Early algebra ideas about binomial expansion, Stephanie’s interview four of seven.
interviews between Stephanie and various researchers, including Researcher Maher. Earlier in this first interview and prior to this event, Stephanie gave a convincing argument about the product of a monomial and binomial, that is, \(a(x+y) = ax + ay\), using the distributive property.\(^{11}\)

In Event 1, Researcher Maher asked Stephanie to think about the meaning of \((x+y)(x+y)\). She wrote \((x+y)(x+y)\) and asked Stephanie, “Could I do this?” Stephanie suggested that it is possible but expressed uncertainty about how to begin. Researcher Maher asked her to think about what \((x+y)(x+y)\) means.

Stephanie made a preliminary conjecture: that \((x+y)(x+y)\) might be equal to \(x^2\) times \(y^2\). However, she began to talk through her knowledge of algebra. She claimed that “you’re multiplying – ‘cause you can’t combine these terms, right?” and supported this idea by stating that “they’re not the same variable” so “you have to multiply them.”

These ideas led Stephanie to make her second conjecture. She stated that she is not sure how to “get around” the fact that the terms are different and cannot be combined, conjectured, “But I’m pretty sure that if I could, the answer would be x squared plus y squared” \((x^2 + y^2)\). She expressed more confidence in this second conjecture and abandoned the initial conjecture that \((x+y)(x+y)\) equals \(x^2\) times \(y^2\).

\(^{11}\) More details about this exploration can be found by accessing the videomosaic.org website and viewing Early algebra ideas about binomial expansion, Stephanie’s interview one of seven, clips 1 through 5.
At the end of the event, Researcher Maher revoiced what Stephanie said and suggested that she test her conjecture writing on the paper: $(x+y)(x+y) = x^2 + y^2$ with a question mark over the equal sign.

In the second event, Stephanie was invited by Researcher Maher to use numbers to test her conjecture that $(x+y)(x+y) = x^2 + y^2$. She substituted 2 for $x$ and 3 for $y$ into both sides of the equation and simplifies. She realized that quantities on either side of the equation are not equivalent. She declared, “…that’s not right,” and determined that it “didn’t work.” Stephanie, thus, refuted by a counterexample the idea that $(x+y)(x+y) = x^2 + y^2$ and rejected her conjecture.
5.3 Stephanie Makes and Supports Claims using the Distributive Property

See Events 3 and 4

Prior to Event 1 of this VMCAalytic, Stephanie produced a convincing argument of why \(a(x+y) = ax + ay\). In this event, she reiterated that argument and supported it. She claimed that \(a(x+y) = ax + ay\), and gave the evidence that \(a(x+y)\) means “x plus x and y plus y, a amount of times.”

![Figure 5.3. \(a(x+y)\) means x plus y a amount of times](image)

Researcher Maher asked Stephanie if the variable, \(a\), could be considered the binomial, \((x+y)\). Stephanie stated that she thought it could be. Researcher Maher asked Stephanie what \((x+y)(x+y)\) meant, and, since she was representing the variable \(a\) as \((x+y)\), she said, “now I just see \(a\) times \(a\).”

In Event 3, Stephanie continued to explore the meaning of \((x+y)(x+y)\). She expressed uncertainty about how to multiply the binomials and represent the sum of \(x\) and \(y\). She explained, “[I] can’t even add \(x\) plus \(y\), though. Which is my problem. Like I can’t add \(x\) plus \(y\) together because, \(x\) and \(y\) are different.” Then Stephanie conjectured, expressing uncertainty, that she thought that the product, \((x+y)(x+y)\), means that \((x+y)\) is multiplied “x plus y” number of times (Figure 5.4). Researcher Maher asked Stephanie to
write down her second conjecture about the product of two binomials and asked, “Do you really believe that?” Stephanie responded more confidently, “That’s what I’m getting.”

![Image](image.png)

*Figure 5.4. \((x+y)(x+y)\) amount of times*

### 5.4 Stephanie Refines and Tests her Conjecture

See Events 5 and 6

In Event 5, Stephanie refined her conjecture for the product of two binomials indicating that \((x+y)(x+y) = x(x+y) + y(x+y)\), explaining that \((x+y)(x+y)\) means \((x+y)\), “\(x\) plus \(y\)” number times. She said that \((x+y)(x+y)\) means \((x+y)\) “\(x\) amount of times” and \((x+y)\) “\(y\) amount of times” (see Figure 5.5). Researcher Maher recorded this idea in symbols as, “\((x+y)(x+y) = x(x+y) + y(x+y)\)”.
In Event 6, Stephanie substituted numbers to test her conjecture that 

\[(x+y)(x+y) = x(x+y) + y(x+y)\] 

She substituted 2 for \(x\) and 3 for \(y\) in \(x(x+y) + y(x+y)\), the same numbers that she substituted for \((x+y)^2\) in Event 2. She computed a value of 25, declaring that it is the same value that she got previously, concluded that it worked. Researcher Maher asked her if it was always going to work and she said she thought so.

Although not shown in this event, Stephanie tested another set of numbers by substituting \(x = 4\) and \(y = 5\) in the equation \((x+y)(x+y) = x(x+y) + y(x+y)\). She found that
the quantities on either side of the equation were again equivalent. By the end of the
discussion, she expressed more confident in the statement and claimed that \((x+y)(x+y) =
x(x+y) + y(x+y)\). She indicated it makes sense that it would work all the time, but had not
yet justified the claim for all real numbers.

5.5 Stephanie Conjectures about Expanding \(x(x+y) + y(x+y)\) and Tests her
Conjectures

See Events 7, 8, and 9

Stephanie began to use the distributive property to expand \(x(x+y) + y(x+y)\). As
she expanded \(x(x+y)\), she claimed that she needed to multiply \(x\) by \(x\) and then \(x\) by \(y\). She
claimed with some confidence that \(x\) times \(x\) is \(x^2\), but expressed uncertainty about the
product of \(x\) and \(y\). She first conjectured that “it would be \(y\) to the \(x\) power.” Researcher
Maher asked her “What do you think it means, ‘\(x\) times \(y\)’?”

Stephanie used the meaning of multiplication to define the product, \(x\) times \(y\),
indicating, “Well, it’s an \(x\) amount, \(y\) number of times or \(y\) amount, \(x\) number of times”,
suggesting the commutative property. “It [the product] can go either way.” As she was
describing the meaning, she wrote \(x \cdot y\) and Researcher Maher pointed out that she wrote
it in a useful way, asking, “Do you think that’s a way to write it?” Stephanie affirmed that
it was. Note that throughout this exploration, Stephanie wrote the dot symbol for
multiplication between the \(x\) and \(y\).
Stephanie recorded a new conjecture about \(x(x+y) + y(x+y)\), indicating that the products of the monomials and binomials result in \(x^2 + xy + yx + y^2\). This new conjecture provided additional support for rejecting Stephanie’s initial conjecture, that \((x+y)(x+y) = x^2 + y^2\). Researcher Maher asked Stephanie, “So you see why your other guess didn’t work before? If what you’re doing is right – there’s your \(x\) squared, there’s your \(y\) squared, but there’s something else.” Stephanie agreed and acknowledged that the “something else” is “the \(x\) times the \(y\)” and “the \(y\) times the \(x\)”.

In Event 8, Stephanie began to simplify \(x^2 + xy + yx + y^2\). She conjectured that \(x^2 + xy + yx + y^2 = (x^2 + x + x) + (y + y + y^2)\). Researcher Maher suggested that Stephanie
test this conjecture. Stephanie substituted 2 for $x$ and 3 for $y$ and evaluated the expression to determine that its value was 23. She knew from previous work in this session that the value should be 25 and declared that it does not work. Researcher Maher confirmed that it must not be a valid step.

![Figure 5.9. Stephanie tests her conjecture](image)

Researcher Maher asked her to support her conjecture that $xy + yx$ was equivalent to $(x+x)$ and $(y+y)$. She said that she was “putting the terms together”. Researcher Maher asked Stephanie to provide more details about what terms she put together and why she put them together. At first Stephanie thought of putting all of the $x$’s together and all the $y$’s together. Then she realized that $x^2$ and $xy$, for example, are not like terms and cannot be put together. She questioned, “Oh. Is it that maybe I can’t put the $x$’s with the $x$-squared, ‘cause they’re two different terms?” Then she further refined her argument when she realized that the $xy$ is not an $x$ term and a $y$ term, but a term itself. Researcher Maher, pointing to the $xy$ term, asked, “is this an $x$?” Stephanie reasoned, “No. It’s $x$ times $y$, actually.” Stephanie rejected her conjecture that $x^2 + xy + yx + y^2 = (x^2 + x + x) + (y + y + y^2)$. She could be seen crossing out the earlier conjecture on her paper.
As a result of her algebraic reasoning in the previous events, Stephanie modified her initial conjecture that \((x+y)(x+y) = x^2 + y^2\) that she made in Event 1, and made a counterclaim. She now claimed that \((x+y)(x+y) = x^2 + 2xy + y^2\). Stephanie’s reasoning resulted in greater confidence in another conjecture that she put forth as a claim. A confident Stephanie made use of the distributive property as a warrant to back her claim that \((x+y)(x+y) = x^2 + 2xy + y^2\).

Research Maher asked Stephanie to “try some numbers and test” her new claim. Stephanie substituted 2 for \(x\) and 3 for \(y\), the same values that she used earlier in the interview to test her first conjecture that \((x+y)(x+y) = x^2 + y^2\). Recall that in Event 2, Stephanie determined that when \(x = 2\) and \(y = 3\), \((x+y)(x+y) = 25\). In this event, she substituted \(x = 2\) and \(y = 3\) into \(x^2 + 2xy + y^2\) and determined that it, too, equaled 25. Whereas, when testing her initial conjecture, she declared, “it didn’t work,” now she verified that “It worked.” She affirmed that she now believed her claim, that \((x+y)(x+y) = x^2 + 2xy + y^2\) is a true statement.
Stephanie, using meaning, definitions, and properties, created an algebraic argument for \((x+y)(x+y) = x^2 + 2xy + y^2\). Researcher Maher stated: “Actually, you’ve proved it. What you’ve just done is gone through a proof. What you’ve done here in your proof is based upon … the meaning of these things.” These properties, definitions, and meanings also provided warrants for her claim.

5.6 Stephanie Makes, Tests, and Refutes a Conjecture about \((a+b)^2\)

See Events 10 through 12

Events 10 through 14 are taken from Interview 2, which took place on January 29, 1996, about two months after Interview 1 and Events 1 through 9. In this second interview, Stephanie revisits her algebraic exploration of the square of a binominal by exploring the meaning of \((a+b)^2\).

To begin this interview, in Event 10, Researcher Maher asked Stephanie if she remembered the meaning of \((a+b)^2\). Whereas in the first interview Researcher Maher used the variables \(x\) and \(y\), in this interview she used the variables \(a\) and \(b\) in the binomial. Stephanie, recalling her previous work, said, “yeah and didn’t we distribute it so that it was like \(a^2 + b^2\)?” She wrote \((a+b)^2 = a^2 + b^2\) as her conjecture. This is a conjecture because she was questioning whether it is correct. She supported this
conjecture by suggesting that you “distribute” the exponent. Researcher Maher suggested that Stephanie tell her “what it means and test it.”

Figure 5.12. Stephanie repeats an earlier conjecture

Between Event 10 and Event 11, Stephanie used numbers to test her conjecture that \((a+b)^2 = a^2 + b^2\). What she wrote is evident on the student work that is displayed in Event 11. In this event, she explained what she did. She said that she “put numbers … in place of the letters.” She explained that she used 2 for the variable \(a\) and 3 for the variable \(b\) and that, with those values, \((a+b)^2\) had a value of 25 and that \(a^2 + b^2\) had a value of 13. Then she refuted her conjecture that \((a+b)^2 = a^2 + b^2\). She stated, "You told me um well you said, ‘What is \(a^2 + b^2\)?’ and I said that it would be a squared plus b squared. Obviously, it’s not…Because it doesn’t work out."

Figure 5.13. Stephanie refutes her conjecture
Researcher Maher restated Stephanie’s rejection of her conjecture, “Your conjecture…that a plus b in parentheses, the quantity squared is not the same as a squared plus b squared.” As Researcher Maher pointed out, in this event, Stephanie proved that her conjecture was not true “by a counterexample.”

After using a counterexample refute her conjecture that \((a+b)^2 = a^2 + b^2\) and rejecting her conjecture, stating that \((a+b)^2\) does not equal \(a^2 + b^2\), Stephanie wrote that \((a+b)^2\) did not equal \(a^2 + b^2\) and justified her statement using generalized reasoning about counterexamples. She explained that testing one set of numbers proves that \((a+b)^2\) did not equal \(a^2 + b^2\) “because if it doesn’t work once then it can’t … be true.” Researcher Maher asked Stephanie to consider her original question which was, what is \((a+b)^2\). Stephanie proved that \((a+b)^2\) is not \(a^2 + b^2\), but had not yet determined what it means.

Figure 5.14. Stephanie records the refutation of her conjecture

5.7 Stephanie Modifies and Justifies her Conjecture

See Events 13 and 14
In Event 12, Stephanie determined that \((a+b)^2\) did not equal \(a^2 + b^2\) and Researcher Maher challenged her to consider the original question which was what is \((a+b)^2\). In Event 13, Researcher Maher encouraged Stephanie to think about \((a+b)^2\) by considering the basic meaning of “something squared.” Stephanie claimed that something squared means that “you’re multiplying it by itself” and that in this case, \((a+b)\) is being multiplied by itself. The work that Stephanie did to evaluate \(a^2 + b^2\) when \(a = 2\) and \(b = 3\) was visible on the paper in this event. When evaluating the expression, Stephanie wrote that \(a^2\) was the same as \(a \cdot a\) and \(b^2\) was the same as \(b \cdot b\). Researcher Maher and Stephanie referred to this work as data to support her claim that squaring something means you are multiplying it by itself and so \((a+b)^2\) means \((a+b)\) times \((a+b)\).

![Figure 5.16. What does "something squared" mean?](image)

As a result of their conversation, Stephanie wrote \((a+b) \cdot (a+b)\) on her paper with the dot symbol for multiplication between the binomials and seemed more confident about her statement, modifying it from a conjecture to a claim. Researcher Maher encouraged Stephanie to write down her claim, that \((a+b)(a+b)\) equals \((a+b)^2\) so as “not to lose sight of what \((a+b)^2\) is supposed to represent.” Stephanie wrote \((a+b)(a+b) = (a+b)^2\).
In the final event, Event 14, Stephanie used the definition of squaring a binomial and of raising a binomial to a power to further justify her claim that \((a+b)(a+b) = (a+b)^2\). She provided a warrant to justify why the definition of raising a binomial to a power, or “multiplying it \((a+b)\) by itself”, supported the statement of equality in her claim. She said, “Because um when you square something it’s like multiplying it by like itself. And so it would be like a plus b times a plus b.” She gave examples to further justify her claim. Indicating the exponent, Research Maher asked her to consider what it would mean if the “squared” was a “3.” Stephanie explained that “You’d do it three times… a plus b times a plus b times a plus b” and if there was a 25 as an exponent, “It would be a plus b twenty-five times. Like times a plus b.”

At this point in the interview, Stephanie moved on to using the model of a square to create a geometric proof that \((a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2\). More information about Stephanie’s argumentation with regards to her geometric exploration, can be found in Chapter 6 of this volume, and in the VMCA: Eighth Grader Stephanie’s Argumentation about Meaning for the Square of a Binomial using Geometric Reasoning (http://dx.doi.org/10.7282/T3QZ2CRF).
5.8 Conclusion

This was the second of three VMCAnalytics that were created to illustrate argumentation that teachers studied. Teachers studied discussed it online over the course of two weeks. As with the first VMCAnalytic, in their online discussion, they answered and discussed the following five questions:

1. Identify elements of argumentation that can be identified in this analytic.

2. What are the claims being made by the children in the arguments presented? Who is making what claim?

3. Identify evidence/backing that the children use to support their claims.

4. For the claims presented, identify those that are:
   
   (a) Challenged
   
   (b) Modified
   
   (c) Refuted

5. Was the argument resolved? Explain.

The argumentations identified in the discussion thread for the VMCAnalytic were not discussed in class.
Chapter 6 – VMCAAnalytic 3: Eighth Grader Stephanie’s Argumentation about Meaning for the Square of a Binomial using Geometric Reasoning

6.1 Introduction

This chapter describes the argumentation in the third published VMCAAnalytic that teachers in my study watched, Eighth Grader Stephanie’s Argumentation about Meaning for the Square of a Binomial using Geometric Reasoning (http://dx.doi.org/doi:10.7282/T3QZ2CRF). As with the first and second VMCAAnalytics, teachers were asked to study the VMCAAnalytic by watching the video events and reading the descriptions and discussed the VMCAAnalytic on the online forum. As with the second VMCAAnalytic, the purpose of this VMCAAnalytic was to illustrate events of a student involved in argumentation and the student's arguments were consistent with reasoning that occurs naturally. The data set used to create the VMCAAnalytic described in this chapter was collected in a longitudinal study designed to investigate students’ mathematical reasoning (Aboelnaga, 2011; Maher, 2005).

In the VMCAAnalytic discussed in this chapter, Stephanie, an eighth grader, continued her exploration of squaring a binomial. Whereas in the second VMCAAnalytic, Eighth Grader Stephanie’s Argumentation about Meaning for the Square of a Binomial using Algebraic Reasoning, Stephanie used algebraic reasoning to develop argumentation.

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12 This chapter references events from the VMCAAnalytic entitled Eighth Grader Stephanie’s Argumentation about Meaning for the Square of a Binomial using Geometric Reasoning, located in the Video Mosaic Collaborative (http://dx.doi.org/doi:10.7282/T3QZ2CRF)

13 See the VMCAAnalytic entitled Fourth Graders’ Argumentation About the Density of Fractions Between 0 and 1, located in the Video Mosaic Collaborative (http://dx.doi.org/doi:10.7282/T39K4CZC)

14 See the VMCAAnalytic entitled Eighth Grader Stephanie’s Argumentation about Meaning for the Square of a Binomial using Algebraic Reasoning, located in the Video Mosaic Collaborative (http://dx.doi.org/doi:10.7282/T3FN180C)

15 This research was supported by a grant from the National Science Foundation: REC-9814846 directed by C. A. Maher.
regarding squaring a binomial, in this VMCAnalytic, Stephanie used geometric reasoning\textsuperscript{16}. The data from which the VMCAnalytic was taken were part of a series of seven task-based interview sessions conducted by Researcher Carolyn Maher and other researchers in a teaching experiment where the goal was for Stephanie to explore and make sense of the meaning behind the rules and symbols of algebra (Aboelnaga, 2011). The data used in this VMCAnalytic come from two of the seven after-school interview sessions: Interview 2 and Interview 4.

Stephanie explored ideas about expanding a binomial to the second power in the first four interview sessions, including providing a convincing argument justifying why $a(x+y) = ax + ay$ and using properties and algebraic ideas to explore the meaning of $(x+y)(x+y)$. During Interview 2, prior to Event 1 of the VMCAnalytic described in this chapter, Stephanie used definitions and properties to explore the meaning of $(a+b)^2$. A detailed discussion of Stephanie’s argumentation during this algebraic exploration can be found in Chapter 5 of this volume and the VMCAnalytic, *Eighth Grader Stephanie’s Argumentation about Meaning for the square of a Binomial using Algebraic Reasoning* (http://dx.doi.org/doi:10.7282/T3FN180C, Van Ness, 2105).

In the events of the VMCAnalytic described in this chapter, Stephanie made a number of conjectures and claims about the meaning of $(a+b)$ raised to the second power. She tested her conjectures and claims, produced counterexamples, posed counterclaims, refuted, modified, and refined her conjectures, claims, and arguments. As in Stephanie's algebraic exploration of squaring a binominal, conjectures are viewed as questions; claims are viewed as statements made with more confidence and judgments have been

\textsuperscript{16} For more information about Stephanie and the data set from which the VMCAnalytic described in this chapter was taken, see Chapter 5.
made distinguishing them based on factors such as inflection of Stephanie’s voice indicating confidence or lack of it and the context during which the statement is made. The focus of this chapter to make visible Stephanie's argumentation, regardless of whether she initiated that argumentation by posing a claim or a conjecture.

The data for Events 1 through 7 in the VMCAnalytic discussed in this chapter were taken from Interview 2 which took place on January 29, 1996. The data for Event 8 was taken from Interview 4 which occurred about three weeks after Interview 2. In the time that elapsed between Event 7 and Event 8, Stephanie continued to explore these ideas, as indicated in the original clips (See Early algebra ideas about binomial expansion, Stephanie’s interview two of seven: clips 5 and 6, and Early algebra ideas about binomial expansion, Stephanie’s interview three of seven: clips 1 – 7) found on the videomosaic.org website.

6.2 Stephanie Uses Geometric Models to Represent the Area of a Square with Side Length a

See Events 1, 2, 3, and 4

The data for these events were taken from the second of six clips in the second of seven interviews of Stephanie by various researchers, including Researcher Maher. Just prior to Event 1, Stephanie explored the meaning of \((a+b)^2\) algebraically and determined that \((a+b)^2 = (a+b)(a+b)\). This exploration can be accessed through viewing Early algebra ideas about binomial expansion, Stephanie’s interview two of seven: Clip 1, on the videomosaic.org website. In Event 1 of the VMCAnalytic discussed in this chapter, Stephanie considered the binomial expansion of \((a+b)^2\) as an area problem. Stephanie
expressed confusion about this idea, so Researcher Maher invited Stephanie to review some basic ideas about the meaning of area. Maher drew a square and labeled its sides with the variable $a$. Stephanie then claimed that the area of that square would be “$a$ squared” or $a \times a$. She supported her claim using the definition of area: that to find the area of a square with side of length $a$-units, you would find the product of the length of two sides.

![Figure 6.1. The area of a square with sides of length $a$ is $a \cdot a$](image)

Researcher Maher: If that were a square,
Stephanie: Yeah.
Researcher Maher: Right? And this side had length $a$.
Stephanie: Um hm.
Researcher Maher: And this side had length $a$.
Stephanie: Um hm.
Researcher Maher: If you were finding the area of a square? Remember?
Stephanie: Um.
Researcher Maher: How do you find area of a square?
Stephanie: Multiply the two sides.
Researcher Maher: Length times width. Right?
Stephanie: Um hm.
Researcher Maher: In this case or side squared? So if one side is $a$, right?
Stephanie: So it would be
Researcher Maher: And the other side is $a$, so the area is?
Stephanie: $a$ squared.
6.3 Stephanie Constructs and Modifies a Claim about the Area of a Square with Side Length \((a+b)\)

See Events 2 and 3

In Events 2 and 3, Stephanie conjectured about how she might represent a drawing of a square with a side length of \((a+b)\) units. In Event 2, Researcher Maher asked Stephanie to consider a square with side length \((a+b)\) units and how it would appear different than a square with side length \(a\)-units. Stephanie claimed that “it would have two parts.” Then Researcher Maher invited Stephanie to make a drawing. Stephanie drew a square and labeled two of the sides “A+B”\(^{17}\) as shown in Figure 6.2.

![Figure 6.2. Stephanie’s first conjecture of a square with sides \((a+b)\)](image)

Researcher Maher challenged Stephanie’s drawing stating that Stephanie skipped a step since it is unclear which part of the drawing represents the \(a\)-unit length and which part represents the \(b\)-unit length.

In response to Researcher Maher’s challenge, Stephanie said, “Oh, so you want me to section it off,” and attempted to modify her drawing indicating an \(a\)-unit length and \(b\)-unit length partitioned on the two sides of the square that had already been labeled.

\(^{17}\) Note that throughout this exploration, Stephanie used upper and lower case letters interchangeably as variables. For the purposes of this chapter, variables for the lengths of the sides of the square will be noted using lower case letters.
“A+B”. Expressing confusion, she drew and labeled a second square, partitioned into \(a\)-unit lengths and \(b\)-unit lengths on each of the four sides. If the drawing of the square was oriented with the “A+B” label at the top, the resulting square was labeled as follows: top side with the \(a\)-unit length to the left of the \(b\)-unit length; bottom side with the \(b\)-unit length to the left of the \(a\)-unit length; left side with the \(b\)-unit length above the \(a\)-unit length; and right side with the \(a\)-unit length above the \(b\)-unit length, as shown in Figure 6.3.

![Figure 6.3. Stephanie’s second conjecture of a square with sides (a+b)](image)

To clarify her representation, Stephanie used four line segments to section off her square. When Stephanie drew the line segments, she appeared unsure about her labeling and stated that she has created rectangles that are “all different sizes.” Then she questioned whether the square is “sectioned wrong?”

![Figure 6.4. Stephanie's modified diagram of a square with sides (a+b)](image)
In response to Stephanie’s uncertainty about the drawing for a square with side length \((a+b)\) units that she drew in Event 2, in Event 3, Researcher Maher suggested that Stephanie begin by drawing and labeling one side of the square. Stephanie drew a new square and labelled the top side with the \(a\)-unit length on the left, a tick mark, and the \(b\)-unit length on the right. The tick mark partitioned the lengths. She then drew a line segment down to the opposite side, thus defining the \(a\)-unit and \(b\)-unit lengths on the opposite side. She noticed that in drawing the line segment, she had determined the lengths of the opposite side of the square and labeled the sections on the bottom side of square with \(a\)-units and \(b\)-units, respectively.

![Figure 6.5. Determining lengths on the opposite side](image)

Stephanie then compared her new drawing with the previous drawing and noticed how they are different, pointing out that in her first drawing, the \(a\)-unit length on the top side was above the \(b\)-unit length on the opposite side, and the \(b\)-unit length was above the \(a\)-unit length. In her new drawing, the \(a\)-unit length on the top side was above the \(a\)-unit length on the opposite side and the \(b\)-unit length was above the \(b\)-unit length. Researcher Maher pointed out that, “when you do one side, the other automatically gets determined.”
As she considered how to partition the left and right sides of the square, she questioned whether it mattered if either the a-unit length or b-unit length came first. She partitioned the left side with a-unit and b-unit lengths. Then she drew a line segment across to the opposite side, again defining the a-unit and b-unit lengths on the opposite side and labeled the right side of the square with corresponding a-unit and b-unit lengths, as shown in Figure 6.7.

By the end of the event, Stephanie appeared more confident in the claim that her modified drawing of a square, with her new segments labeled on each side partitioned to represent a-unit lengths and b-unit lengths, more clearly modelled a square with area (a+b) units squared. Researcher Maher summarized Stephanie’s work as she stated, “You
have partitioned the square into four pieces.” Stephanie affirmed this statement. The representations that Stephanie produced in these events can be considered conjectures and claims about what a square with side lengths \(a+b\) units looks like.

### 6.4 Stephanie Finds the Area of each of the Smaller Squares in the Larger Model and Writes the Area Algebraically

See Event 4 and 5

After Stephanie makes the confident claim about the geometric representation of a square with side length \(a+b\), Researcher Maher asked Stephanie in Event 4 to again consider how to find the area of a square. Stephanie stated that you “Multiply the two, the length and the width” and Researcher Maher invited Stephanie to find the area of each of the four sections in the area model. Stephanie wrote the areas: \(ab\) (square units) in the upper left rectangle, \(bb\) (square units) in the upper right square, \(ab\) (square units) in the lower right rectangle and \(aa\) (square units) in the lower left square.

![Figure 6.8. The area of all of the sections](image)

Then Researcher Maher asked her what the area is of the original square with side \((a+b)\) units. Expressing confusion, she first conjectured that it is “\(ab\) times \(ab\)”. She quickly abandoned this idea and conjectured that it is “\(a + b\) times \(a + b\)”
Researcher Maher reviewed with Stephanie some of her previous work to explore the meaning of area of a square: to find the area of a 6 unit by 6 unit square, she multiplied 6 by 6, to get the area of the 4 unit by 4 unit square, she multiplied 4 times 4, and to get the area of a square with side length a-units, the area was a times a or a squared.

![Figure 6.9. Stephanie reviews the meaning of area](image)

Stephanie reviewed and wrote that each side length of the square she partitioned is (a+b). Then, she claimed with more confidence that the area of the large square was “a plus b times a plus b”. She supported her answer as she stated that the side lengths of the square are (a+b).

Researcher Maher: Right? So. What’s this side here? [can’t tell which]
Stephanie: Um. ab or a plus b.
Researcher Maher: That’s what you told me up in the other
Stephanie: Yeah. a plus b.
Researcher Maher: Okay. Why don’t you write a plus b on top of it, lest not we lose that idea. And what’s the side here? [the left side]
Stephanie: a plus b.
Researcher Maher: Okay. Okay. So
Stephanie: So it would be a plus b times a plus b?

Researcher Maher invited Stephanie to write down her claim that the area of the large square was a plus b times a plus b and Stephanie wrote, “(a+b)·(a+b)."
In Event 5, Researcher Maher again asked Stephanie, “What is the area of that square [referring to the partitioned square with side length (a+b) units].” Stephanie confirmed what Researcher Maher meant by asking, “in other words than a plus b times a plus b?” Then Stephanie conjectured that it would be “a plus b squared.’ She supported her conjecture by stating that there were “two [sides with length (a+b) units] of each.”

Researcher Maher: Um hm. What’s the area of that square?
Stephanie: In other words than a plus b times a plus b?
Researcher Maher: Um hm.
Stephanie: Well, doesn’t that go back to that? Then it becomes like, if a plus, wouldn’t it, wouldn’t it just be um a plus b squared?
Researcher Maher: Write that down. [Stephanie completes the algebra sentence: \((a+b)\cdot(a+b) = (a+b)^2\)] And why is it?
Stephanie: Because that’s what it was before? Because it’s um two a’s and two b? Like there’s two of each?

Researcher Maher asked Stephanie to clarify her evidence by asking, “I’m not so sure – you’re not telling me a squared plus b squared. You’re saying that this [points to \(a+b\)]
and this [points to (a+b)] twice?” and Stephanie confirmed, “yes.” During this discussion, Stephanie then wrote her claim for area of square with side length (a+b) units, algebraically, as \((a+b) \cdot (a+b) = (a+b)^2\).

6.5 Stephanie Makes a Claim about the Sum of the Areas of the Smaller Squares and Tests her Claim

Events 6 and 7

In Event 6, Stephanie computed the area of each partitioned section of the original square with side length (a+b) units and claimed that the area of the large square, i.e., \((a+b)^2\), was equal to the sum of the areas of the partitions:

Researcher Maher: All right. But now in this picture, what part of the picture represents this \([(a +b)^2]\) piece? I know what part is a plus b. You told me that it’s this side.
Stephanie: Like the whole thing?
Researcher Maher: The whole thing.
Stephanie: Yeah. The whole thing.

Stephanie supported this claim using the definition of addition as she stated that to put things together you add them [indicating that she needed to sum the areas of each smaller square to compute the area of the larger square]. Researcher Maher invited Stephanie to write the area of each piece and “not skip any steps.” Stephanie used addition to sum the areas of the smaller squares and wrote: \(aa + ab + bb + ab\) and definitions and properties to simplify the expression and got \(a^2 + 2ab + b^2\). Then she claimed, with confidence, that the area of a square with side lengths a plus b is “a squared plus two ab plus b squared.”
Figure 6.12. Stephanie claims that the area of a square with side lengths 

\[(a+b) \text{ is } a^2 + 2ab + b^2\]

In Event 7, Stephanie tested her claim that \((a+b)^2 = a^2 + 2ab + b^2\) by substituting 2 for the variable \(a\) and 3 for the variable \(b\). She determined that for these values both expressions, \((a+b)^2\) and \(a^2 + 2ab + b^2\), were equal to 25 and declared, “It works.” Researcher Maher asked “But when you claim it’s true, how many does it have to work for?” Stephanie answered, “All of them?” and exclaimed that she could not possibly test all numbers because “there’s too many numbers.” Thus, Stephanie suggested a limitation of her argument: that in order to be sure that her equivalence was true, it must be true for all numbers, and she could not possibly test all numbers because there were too many.

6.6 Stephanie Builds an Area Model to Back her Claim

See Event 8

In Event 8, Stephanie used the geometric model to refine the argument that \((a+b)^2 = a^2 + 2ab + b^2\) that she presented in Events 1 through 7. The data from this event came from Interview 4 and took place about three weeks after the data from the previous seven events which came from Interview 2. In the time that elapsed between Event 7 and Event 8, Stephanie continued to explore the idea of squaring a binomial. In this event, Stephanie spontaneously drew a square with sides of length \((a+b)\) to represent \((a+b)^2\) and
partitioned the square into four sections. She found the area of the square by summing the areas these sections. Then, describing the sum of the partitions inside the square, she explained that “Yeah. It was $a$ squared, $ab$, $b$ squared, $ab$, and it would be $a$ squared plus $2ab$ plus $b$ squared, and that’s what we figured out [that] $a$ plus $b$ squared equals.” She wrote the areas of the sections as an addition expression that she then simplified, representing the sum as the simplified expression $a^2 + 2ab + b^2$. Stephanie displayed confidence in the representation of her geometric model as she declared that “$a$ plus $b$ squared equals … $a$ squared, $ab$, …, $b$ squared, $ab$, and it would be $a$ squared plus $2ab$ plus $b$ squared.”

![Image of Stephanie writing on paper]

*Figure 6.13. Stephanie’s argument that the area of a square with side lengths $(a+b)$ is $a^2 + 2ab + b^2$*

Recalling her earlier work of the conjecture she refuted, she summed up her argumentation of how to find the product of two binomials by explaining that she first thought that $(a+b)^2 = a^2 + b^2$ but found a counterexample and rejected that conjecture. Then she determined that $(a+b)^2 = a^2 + 2ab + b^2$ using the distributive property and used the model of a square with sides $(a+b)$ units to support that claim. In this exploration,
Stephanie used the partitioning of a square into segments as a warrant to support her claim that \((a+b)^2 = a^2 + 2ab + b^2\).

Note that in the episode described in this event, Stephanie appeared to have fluency in the algebraic manipulation she completed. A more detailed picture of how Stephanie developed that proficiency can be found in the clips: “Early algebra ideas about binomial expansion, Stephanie’s interview one of seven,” as well as the VMCAAnalytic, *Eighth Grader Stephanie’s Argumentation about Meaning for the Square of a Binomial using Algebraic Reasoning* (Van Ness, 2015) which can both be found by accessing the videomosaic.org website.

Data for the events in the VMCAAnalytic described in this chapter were chosen purposefully to illustrate Stephanie’s argumentation about raising a binomial to the second power geometrically. Her exploration of these ideas went beyond what is represented here. The complete set of clips for Interviews 1, 2, 3, and 4 on the Video Mosaic Collaborative Repository at videomosaic.org illustrate in more detail how Stephanie developed these ideas: *Early algebra ideas about binomial expansion, Stephanie’s interview one of seven, Early algebra ideas about binomial expansion, Stephanie’s interview two of seven, Early algebra ideas about binomial expansion, Stephanie’s interview three of seven, Early algebra ideas about binomial expansion, Stephanie’s interview four of seven*.

6.7 Conclusion

This was the third of three VMCAAnalytics that were created to illustrate argumentation that teachers studied. Teachers studied discussed it online over the course of two weeks.
As with the first and second VMCAnalytic, in their online discussion, they answered and discussed the following five questions:

1. Identify elements of argumentation that can be identified in this analytic.
2. What are the claims being made by the children in the arguments presented? Who is making what claim?
3. Identify evidence/backing that the children use to support their claims.
4. For the claims presented, identify those that are:
   (a) Challenged
   (b) Modified
   (c) Refuted
5. Was the argument resolved? Explain.

The argumentations identified in the discussion thread for the VMCAnalytic were not discussed in class.
Chapter 7 – VMCAnalytic 4: Assessment Analytic, Fraction 1

7.1 Introduction

Götz Krummheuer (1995; 1997; 2000; 2007; 2013) and others (Pedemonte, 2007; Wagner et al., 2014; Conner et al., 2014i; Conner et al., 2014t; Hollebrans, Conner, & Smith, 2010; Rasmussan & Stephan, 2008; Yackel, 2002) have found that the Toulmin model (1958, 2003) for diagramming argumentation is a useful tool for reconstructing the informal argumentation of students in the mathematics classroom. The basic elements of an argument are the claim (Krummheuer calls it a conclusion, but others, (for example, see Conner et al. (2014), Conner et al. (2014), and Yackel (2002), and Toulmin (1958, 2003)) call it claim so I chose to use claim), the data that support the claim, and the warrant that links the data to the claim. According to Krummheuer, the claim is a conclusion that can be stated after or before the data are presented. Words like "so" and "because" can aid in mapping the relationship, if any, between the claim and the data. If the data come first, the argument would take the form of "Data, so Claim." If the claim comes first, "Claim because of Data" would describe the flow of the argumentation. In the diagrams used to map student argumentation, I use a directional arrow to indicate the flow of the argumentation; whether it goes from claim to data, or data to claim. As Krummheuer describes it, the arguer makes an intuitive leap to infer the claim from the data or to infer the data from the claim. This intuitive leap may or may not be accepted by others involved in the argumentation

When the arguer makes a leap between data and claim that is not recognized or accepted by the group, a warrant is necessary to make the argument more convincing. The warrant serves as further support for the argument when others involved in the
argumentation agree with the data, but question how these data support the claim.

Krummheuer gives the example in which a second grader claims that $4 \times 4 = 16$ and uses the data that $8 + 8 = 16$ to support his claim (1995; see p. 241, for more details about the diagramming of the second graders' argument). Although this intuitive leap might be clear to the arguer, there may be others involved in the argumentation discourse who do not question the validity of the statement $8 + 8 = 16$, but question how this addition fact supports the claim that $4 \times 4 = 16$. Toulmin (1969) explains that the warrant is a "general, hypothetical statement" (p.98) that creates a bridge over the inferential step to make the link between the data and the claim clear to others. In Krummheuer's example, the student provided a warrant that clarified the inferred step, specifically, that $4 \times 4$ is like $2 \times 4$, two times, that is, $[(2 \times 4) + (2 \times 4)]$ and $2 \times 4$ is $8$, so $4 \times 4$ is equal to $8 + 8$. The word "since" can aid in mapping the relationship between the data and claim, and the warrant, thus, "Claim because of Data, since Warrant" or "Data so Claim, since Warrant" would describe the flow of argumentation.

In some cases, the warrant might be questioned. When this happens, the arguer might provide further evidence in the form of backing. Backing provides evidence as to why the warrant should be accepted and is often a "global conviction," "primary strategy," or "collectively accepted basic assumption" (p. 244) that is accepted by the group in which the argument occurs. In other words, generally, backing does not need support because it is universally accepted by the participants as being true. Backing can be a strategy like counting (I count 16, so I know there are 16), a property that is accepted as being true, or a diagram or model that clearly and obviously illustrates a spoken idea. Backing is contextually based and may change depending on the audience. For example,
in a sixth grade classroom, the warrant given by the second grade student in the previous example might be universally accepted. However, in the second grade classroom, the fact that $4 \times 4$ is the same as two $2 \times 4$s might not be generally accepted. Backing for this warrant might be needed to make the claim convincing to these seven year olds. Backing for this claim, might, for example, entail a counter model that shows that 4 groups of 4 is the same number of counters as two groups of 2 groups of 4.

In argumentation in the classroom setting, even if the data or warrant are widely accepted, and warrants or backings are not necessary to produce convincing arguments, teachers or researchers might ask the arguer to include them to help them clarify their argument. The diagram in Figure 7.1 shows how the elements of argumentation as proposed by Toulmin (1969) and explicated by Krummheuer (1995, pp. 239 – 246) are related and gives a prototype for the mapping used for this research study.
Often when students make argumentative statements, their language is imprecise. One example that appears often in this session is the use of "bigger" rather than "greater" when referring to comparing fractions. Students use the language, 1/2 is bigger than 1/3," rather than 1/2 is greater than 1/3. The use of this natural language is not unexpected since these students are focusing on the attribute of length in the rod models they are using to build their conceptual understanding of fractional ideas, including comparison of fractions. In this session, "bigger" becomes a surrogate for "longer" or "taller" and, thus,
becomes the language they use when they compare values as well as the lengths of the rods themselves.

To differentiate between the students' natural language, and language that conforms to the formal mathematical register, I have used quotes when including students' exact words and parentheses to indicate how the natural language aligns with formal mathematical register. Language that is not in quotes or parentheses is my language, used when summarizing students' arguments when, for example, the arguments are a combination of several statements made by one student or several, or when the element is a combination of statements and gestures.

Collective argumentation occurs when students work together to develop arguments in the classroom (Krummheuer, 1995; Conner, Singletary, Smith, Wagner, & Francisco, 2014; Schwarz). An important part of understanding collective argumentation is to understand which participants are contributing which elements. To address the diagramming of different students' contributions, I have included in the diagrams, the letter of the contributor's first name according to the code above. The letter appears in brackets [letter].

Yackel (2002) and Douek (1999) assert that argumentation includes actions, tools, notations, drawings, models, numerical data, as well as verbal statements and Krummheuer (1995) states that argumentation are often "image dependent" (p. 251). Thus, when considering in the episodes in the assessment analytic, the students' Cuisenaire models (both the verbal descriptions of the models and the models themselves) are considered an essential part of the students' argumentation. When using the Cuisenaire rods, students developed some guiding principles that became shared by
the group. These principles were used frequently, often implicitly, as warrants and backing in the students' arguments. I describe them here and provide abbreviations for them in parentheses. In the diagrams that follow, the principles will be referenced by their abbreviations.

1. Any rod or train of rods can be given number names. If the rod or train is regarded as the unit it is given the number name 1. All other rods are given number names based on their relationship to the rod or train of rods with the unit rod that has the number name 1. (number names)

2. Rods and trains of rods that are the same length show the same numerical value when compared to the same unit. (same length same value)

3. If a train of n rods of the same color fit under the rod or train of rods with the number name 1, then that rod is "one nth" of the unit and the number name for one of those rods is 1/n. (partition)

4. The attribute of length is used to determine the value of a rod. When the longer and shorter rods are compared with the same unit; the longer rod represents the number with the greater value and the shorter rod represents the value with the lesser value. (longer rod)

5. If it takes a certain rod to make two rods or trains of rods the same length, then the number value of that rod represents the difference between the number two values of the other two rods. (difference)

6. If the rod or train of rods that represent the unit is partitioned into n equal pieces, each piece is 1/n of the unit.
7. If a train of two rods is the same length as a third rod, the number name for one of the rods in the train shows the difference between the number name of the other rod in the train and the number name of the third rod (if a train of rod \(a\) and rod \(b\) is the same length as rod \(c\), then the number name for \(c\) is larger than the number name of rod \(a\) by the number name of rod \(b\) and the number name for rod \(c\) is larger than the number name of rod \(b\) by the number name of rod \(a\); the difference between \(c\) and \(a\) is \(b\) and the difference between \(c\) and \(b\) is \(a\))

(difference)

8. The taller rod shows the greater value. (taller)

9. Any number name given to a rod must be equivalent in relation to a given rod that has the number name 1.

10. Only one rod or train of rods can be used as the unit rod with the number name 1.

11. If rod \(a\) with value \(a\) is shorter than rod \(b\) with value \(b\) and longer than rod \(c\) with value \(c\), then value \(a\) cannot equal value \(b\) or value \(c\). (between)

7.2 Event 1

In Event 1, a series of preliminary claims are made by the class (See Figure 7.2). None of these claims are supported, but they are the beginning of the argumentation episodes in the classroom. They can be interpreted as modifications of the same claim. Students begin to present and develop their arguments in Event 2, as they support the final claim, that 1/2 is greater than 1/3.
In Events 2 and 3, students develop argumentation that supports that \( \frac{1}{2} \) is greater than \( \frac{1}{3} \) (See Figure 7.3). Note that in the initial claim in Event 2, Jessica says that \( \frac{1}{3} \) is greater than \( \frac{1}{2} \), however, in Event 1, Laura, Jessica's partner, clearly states that \( \frac{1}{2} \) is greater than \( \frac{1}{3} \) and that is the claim they are asked to support. Furthermore, by the support they give, including Jessica's verbal argument and rod model, as well as the reaction of other students, it is likely that Jessica meant that \( \frac{1}{2} \) is greater than \( \frac{1}{3} \).

Part of the argument that \( \frac{1}{2} \) is greater than \( \frac{1}{3} \) includes the evidence that, when the number name of the orange and red rod train is 1, the dark green rod has the number name \( \frac{1}{2} \) and the purple rod has the name \( \frac{1}{3} \). Several students collectively present another argument to support the number names they gave the rods. According to Conner and colleagues (2014), students may include sub-arguments or nested arguments (Krummheuer, 1995) in their argumentation. Conner and colleagues explain that "sub-
arguments may arise in two different ways. In the first, they can be preliminary to a claim, with the argument building from one claim into another, so that the argument occurs from left to right in the diagram. In the second, a component such as data or warrant, is questioned, resulting in it also becoming a claim,” (p. 406). In the arguments Events 2 and 3, initially, as the first student presents her argument about why she thinks 1/2 is greater than 1/3, she spontaneously presents evidence to support the number names for her rods. Then, after prompting from the researcher, she and the second student, together repeat and add detail to the evidence they are presenting. In Event 3, when the third student agrees and is invited to present her argument, again, she spontaneously includes the sub-argument, presenting support for her claims that the number name for the dark green rod is 1/2 and the number name for the purple rod is 1/3. (See Figure 7.4 and Figure 7.5 for these arguments).

Throughout the diagramming of the argumentation each event, I have illustrated nested sub-arguments different ways. Using the diagrams used by Wagner and colleagues (2014) and Conner and colleagues (2014a, b) to inform my own diagramming, I have notated sub-arguments in their larger arguments by using, for example, "Claim/Data" when data for a claim is a claim that is supported in a sub-argument; "Data/Warrant," when the warrant for the main argument is also data for the sub-argument; and "Warrant/Backing" when the backing for the warrant for the main argument is also the warrant for the sub-argument (as in Figure 7.3). For clarity, I have sometimes created separate diagrams for sub-arguments (as in Figure 7.4 and Figure 7.5).

Wagner, Conner and their colleagues (2014) do not use backing in their argumentation diagrams. Rather, when a warrant is questioned, they identify it as a new
claim and notate data and warrant for that claim. Although I have used input from their model to inform my own, in the spirit of Krummheuer and Toulmin, I have included backing. In my diagrams, backing serves the same role as nested sub-arguments do in theirs. This construct can be identified by elements that are notated as "Claim/Warrant," where the warrant is not accepted by the group and becomes a claim that needs to be supported with warrants and possibly backing.

The arguments in the diagrams in Figure 7.4 and Figure 7.5 present support for students' claim that, when the orange and red rod has the number name 1, the dark green rod has the number name 1/2 and the purple rod has the number name 1/3. These arguments are sub-argument that exist within the argument developed to support that 1/2 is greater than 1/3 in Figure 7.3. It is also a line of argumentation that is repeated several times throughout the session with different color rods and different rods being given the number name 1.
Figure 7.3. Diagram of the argumentation in Events 2 and 3 supporting that 1/2 is greater than 1/3
Figure 7.4. Argumentation in Event 2 supporting the claim that the dark green rod has the number name $1/2$.
7.4 Event 4

In Event 4, the researcher asks the students to quantify how much greater $1/2$ is than $1/3$. The students use their models to show that the dark green $1/2$ rod is longer than the purple $1/3$ rod by a red rod. This claim is supported explicitly by the rod model and implicitly by the "taken-as-shared" (reference; will include a detailed explanation in my lit review) ideas that 1. Rods and trains of rods that are the same length show the same numerical value; and 2. If it takes a certain rod to make two rods or trains of rods the
same length, then the number value of that rod is the difference between the number values of the other two rods (See Figure 7.6). At this point in the argument, students have not quantified the red rod by giving it a number name.

As Event 4 continues, Audra conjectures that 1/2 is 1/3 greater than 1/3: "it's one third bigger, I think." Her statement is considered a conjecture because she says, "I think." Jessica agrees, and confirms the conjecture with evidence, turning the conjecture into a claim. The claim that 1/2 is greater than 1/3 by a red rod is thus modified to the claim that 1/2 is greater than 1/3 by 1/3. The girls collectively support this modified claim using a rod model. They line three red rods below a dark green rod and state that it "takes one third of them [the reds] to make the purple rod as long as the dark green rod." (See Figure 7.7.) Note that it is not explicitly stated in this event that the girls are giving the
red rod the number name 1/3, however as the argumentation progresses throughout the following events, it is apparent from other participants' statements that this claim has been proposed and supported. The claim that the number name for the red rod is 1/3 becomes the object of discussion and, in a following event, the claim is stated outright. For purposes of the argumentation diagramming, I have included the notion that the red rod has the number name 1/3 as a claim when it is appropriate.
Figure 7.7: Argumentation in Event 4 supporting the claim that $1/2$ is greater than $1/3$ by $1/3$

In the argumentation evident in Event 4, and outlined in the diagram in Figure 7.7, the students make the argument that $1/2$ is greater than $1/3$ because the purple $1/3$ rod...
and the red rod are the same length as the dark green 1/2 rod. Their verbal argument suggests that, at this point in the argumentation, the students are inferring that the red rod has the number name 1/3, although it is not explicitly stated:

Audra: Um, wait, it’s one third bigger, I think [organizing the dark green and red blocks together].
Jessica: I think it’s one third bigger too because if you put the red to the green
Audra: You’d see that there’s three
Jessica: You need three and if you put the purple one to it also and then it takes one third of them. [Showing the purple differs from the dark green by one red block]

The model they use to determine the number name for the red rod provides further evidence that the students are giving the red rod the number name 1/3. The model that shows a train of three red rods having the same length as the dark green rod mirrors the models that students made when they initially defined the dark green rod as having the number name 1/2 and the purple rod as having the number name 1/3 (See Figure 7.4 and Figure 7.5). In those arguments, they aligned a train of two dark green rods and a train of three purple rods with the orange and red rod train, which they gave the number name 1, and argued that a train of two dark green rods is the same length as the unit, so the dark green rod is half of the unit and has the number name 1/2 and, similarly, three purple rods are the same length as the unit, so each purple rod has the number name 1/3. Implicit in their arguments is the warrant that if n rods of the same color are the same length as the unit rod, each rod is "one-nth" of the unit and therefore has the number name 1/n (Principle 3). Taking into account the girls' previous arguments, it can be inferred, then, that in the argument presented in Event 4 that supports giving the red rod the number name 1/3, they switched the unit. Rather than defining the red rod based on the orange and red rod train as 1, as in their initial argument, they are naming it based on the dark
green rod as 1. Using the wrong unit when working with fractions is a common cognitive obstacle observed in mathematics education research (Yoshida; 2004; Steffe, 1992). Students often develop the idea of fraction as an operator before fraction as a number (Davis et al., 1993; Lambek, 1966; Kieren & Nelson, 1978; Fruedenthal, 1983; G. Davis, 1991; Schmeelk & Alston, 2010; Behr, Harel, Post, & Lesh, 1993; Mack, 2001). So, rather than naming the red rod based on the established unit, they see it as a third of the dark green rod, and thus give it the number name 1/3.

If the argument ended here, it might be argued that the girls were claiming that 1/2 is greater than 1/3 by a third of a half. However, as the events progress, their own argumentation and that of the other students confirms that, at some point during the argument, it is taken as shared that support for the claim that 1/2 is greater than 1/3 by 1/3 includes the notion that red rod is given the number name 1/3.

7.5 Event 5

In Event 5 Kelly agrees with Jessica and Laura and presents her argument. She places the light green rod, which she claims has the number name 1/2, next to the red rod, which she claims has the number name 1/3, positioning the rods vertically like towers, and claims that 1/2 is greater than 1/3 because the light green rod is bigger (in this case, because of the orientation of the rods, taller) than the red rod. Note that initially Kelly says that she agrees with students' prior argument that 1/2 is greater than 1/3 by 1/3, and begins her argument as if she is going to support it, "we showed that…one half is bigger by…" However, the argument she presents supports the claim that 1/2 is greater than 1/3: "one half [referring to the light green rod] is bigger by, because this part [pointing to the red rod] is smaller, and this is supposed to be … one third." Also, she does not provide
evidence that supports the number names she gave to the light green and red rods. It is possible, however, to infer that she followed the previous students' argumentation, and, when the students' switched the unit from the orange and red rod train to the dark green rod to give the red rod the number name 1/3 (See Figure 7.7), Kelly recognized a similar reasoning that she used to compare the fractions. Giving the dark green rod the number name 1, then, could be implicit and is included in the diagram of her argument (See Figure 7.8).

After Kelly presents her argumentation, Brian and Jessica express disagreement with Kelly's argument. At this point in the discussion, Brian does not articulate support for his disagreement, so it is not possible to determine where his disagreement lies. It is likely that it is the argument presented in Figure 7.7—that 1/2 is greater than 1/3 by 1/3, the claim that the number name for the red rod is 1/3, or Kelly's claim or evidence presented in this event. Jessica, on the other hand, does offer the beginning of a counterargument to support her disagreement. She says, "I think that’s like changing the problem because we are using the dark greens [to represent 1/2] and she [Kelly] is using the light greens [to represent 1/2]." Thus, she challenges Kelly's evidence, particularly one of the warrants, claiming that the light green rod does not have the number name 1/2. Her statement provides the data for her claim—that Kelly is giving a different rod the number name 1/2 than Jessica, Audra, and Laura did. It is interesting to note that Jessica only expresses disagreement with Kelly's naming of the light green rod 1/2 and does not mention the naming of the red rod as 1/3.
At the end of Event 5, Jessica and Laura collectively repeat the previous arguments that support their claims: that the purple rod has the number name 1/3 (Claim B, Figure 7.5), the dark green rod has the number name 1/2 (Claim C, Figure 7.4), and that 1/2 is greater than 1/3 by 1/3 (Claim 7, Figure 7.7), returning to their original premise that the orange and red rod train has the number name 1. In her argument...
supporting the claim that 1/2 is greater than 1/3 by 1/3, she states: "It would take one third of the red to equal up to the dark green," again alluding to an understanding of fraction as operator. Her statement seems imply that she may be thinking about the one red rod that makes the purple 1/3 rod as long as the dark green 1/2 rod as "one third" of the three red rods that it takes to make a train that is as long as the dark green rod. This supports the idea that the conceptualization of fractions as operators persists in students' reasoning, focusing their attention on the whole and the part, rather than the number name.

Although various counterclaims and counterarguments are presented in the events that follow, students do not challenge Kelly's argument in terms of whether it supports the claim that 1/2 is bigger than 1/3 by 1/3, or question the lack of evidence as to how she gave number names to her rods. Additionally, they do not challenge the evidence involving the switching of the units in the argument in Figure 7.7, which is a contradiction of the original assumption upon which the argument was based; that the orange and red rod train has the number name 1.

7.6 Event 6

Continuing the challenging of others' arguments begun in Event 5, Brian offers a counterargument to the claim that 1/2 is bigger than 1/3 by 1/3. His counterclaim is that 1/2 is bigger than 1/3 by 1/6. His argumentation includes his main argument and two sub-
arguments—one to support his claim that half of 1/3 is 1/6 and one that supports his claim that, when the orange and red rod train is given the number name 1, the number name for the red rod is 1/6. His main argument, the counterargument for the argument that 1/2 is greater than 1/3 by 1/3, is diagramed in Figure 7.10. This diagram includes one of his sub-arguments: that if you split one of the thirds in half, you get 1/6. As he presents his argument, he supports both his claim that 1/2 is bigger than 1/3 by 1/6 and his claim that half of 1/3 is 1/6. Note that when Brian first expresses his disagreement, he says, "Well, when they said 1/3 is bigger than 1/2 by 1/2. I think they said, is that what they said? Well, I don’t really agree." Although he says that 1/3 is greater than 1/2, analyses of his argumentation as depicted in the event, as well as in the diagrams in Figures 7.10 and 7.11, indicate that it is likely that he meant 1/2 is greater than 1/3 by 1/3.

As support for his argument that half of 1/3 is 1/6, on the rod model with the orange and red rod train unit, two dark green 1/2 rods, and three purple 1/3 rods, he uses his finger to count two imaginary lengths for each purple 1/3 rod. Since there are three purple 1/3 rods, he counts six imaginary lengths, so he concludes that half of a 1/3 rod is 1/6. When questioned by the researcher, he refines his argument (See Figure 7.11). In this sub-argument, Brian relies on two of the implicit principles of the rod models—rods and trains of rods that are the same length show the same numerical value (Principle 2) and, if a train of n rods of the same color fit under the rod or train of rods with the number name 1, then that rod is "one nth" of the unit and the number name for one of those rods is 1/n (Principle 3)—to argue that since a half of 1/3 is 1/6 (based on his previous sub-argument), and two red rods are the same length as the purple 1/3 rod, the number name
of the red must be 1/6: "... so ... the red I’m pretending is like, a half of one of the purples [1/3 rod] and you see when I split it in half it’s [the red rod], it’s one sixth."

Figure 7.10. Brain's counterargument in Event 6
Figure 7.11. Brian's argument that the red rod has the number name 1/6

The argumentation in Event 6 highlights the important role that modeling plays in students' argumentative reasoning. The rod models are key features in the building of students' arguments, often providing the warrants and backing that make the argument more sophisticated. Students use models in different ways. Sometimes models are the starting point for students' argumentation, enabling them to construct a convincing verbal
argument. At other times, students use them to add support to their verbal arguments, making their stated evidences more convincing. What becomes clear when these arguments are analyzed is that when students make sensible arguments, they often rely on their models. Brian's argumentation participation in this event exemplifies how essential models are to these students' reasoning. When Brian begins to offer his counterargument, he is sitting at his seat referring to his own rod model. Soon after he begins, he asks to go up to the front of the classroom to use the overhead rods to continue his argument, "well if you split … one of the thirds in half which would make [counting the blocks], which would make a sixth. I think it’s a sixth bigger. Like, well, [holds his rods], um should I go up there?" As soon as he comes to the overhead, he begins referring to the rod models on the overhead that can be seen by the students in the class. It is likely that Brian, as well as the other students, use the models because they feel that the models add visual support to their verbal statements, making their arguments more convincing. Research diagramming student argumentation frequently omits the backing for students' warrants (for example, see Conner and colleagues, and others to come). It is possible that, in an environment in which students are encouraged to make models as they build understanding, they can then more effectively use the models as backing to build more convincing and sophisticated arguments.

7.7 Event 7

With the exception of the faulty reasoning in Event 4 (illustrated in Figure 7), students' arguments in the first six events have been primarily based on the assumption that the unit was the orange and red rod train, which they had given the number name 1. In the following three events, students begin making claims and offering support for
those claims under the premise that that dark green rod has the number name 1. In Event 7, Kelly and Jackie begin this line of reasoning and make a few claims. They claim that, when the dark green rod has the number name 1, the light green rod has the number name 1/2 and the red rod has the number name 1/3. They also claim that 1/2 is greater than 1/3 by one white rod. They use familiar arguments to support these claims (see Figure 12 and Figure 13).

Note that Kelly and Jackie make their claims as they build a model. The model is difficult to see in this event; it is clearer in Event 8 (see Figure 13). The model they make is briefly visible, but it is obscured when they are constructing it. Although the model cannot be seen while they are talking, it can be assumed that as they describe the model verbally, they are making a rod model to show that when the dark green has the number name 1, the red has the number name 1/3 because three red rods are the same length as the dark green rod, and the light green has the number name 1/2 because two light green rods are the same length as the green rod. Although when they make their claim that 1/2 is greater than 1/3 by a white rod they do not make a model, as Jessica is speaking, one of the girls is making a model in the lower right corner of the projection screen that shows that a red and white rod train is the same length as a light green rod, lending support to their argument.

After Kelly presents her argument, Jessica challenges it, claiming that Kelly and Jackie are using a model that is not "allowed" and is "not fair." She supports these assertions by stating that they are using a "different size candy bar" referring to a metaphor for the unit developed in previous sessions (for a detailed discussion of the candy bar metaphor, see can I reference our book, or is there an article, maybe Dina's
dissertation, VMCA
c

Comparing 1/2 and 1/3: Confusion about the Unit, Van Ness
& Alston, 2015). Holding up the light green and dark green rods, Jessica explains that

half of different size units will be different sizes:

Jessica: Oh, I think they’re making a different size candy bar
T/R 1: Is that allowed?
Jessica: Um, no.
T/R 1: Why not? What’s wrong with that?
Jessica: Because it's not fair.
T/R 1: In what way it is not fair?
Jessica: Because if say you give someone half of this one [the orange and red rod train as the unit] and then one half of that one [the dark green rod as the unit] and this [dark green rod] is bigger than [light green rod; taking a light green and dark green rod in hand].
In Event 7, students continue to build models that do not match their verbal statements, but likely support their intentions. In this event, Jackie gives the light green rod the number name 1/2 and the red rod the number name 1/3 and shows that the light green rod is longer than the red rod by 1 white rod. Yet, in her verbal argument she claims that 1/3 is greater than 1/2 by a white rod, "Well, we would call this dark green one and the reds one third and the light green one half, and … we thought one third was bigger by one of these white things." This another example of the imprecision in the natural language that can be clarified through the use of models. It is clear from Jackie's
model (better seen in Event 8) that she likely means that 1/2 is greater than 1/3 by a white rod.

It is interesting to note that in Event 5, Kelly gives the light green rod the number name 1/2 and the red rod the number name 1/3 in the model she construction to support Jessica, Laura, and Audra's claim that 1/2 is greater than 1/3 by 1/3. The argument that the girls present here supports the notion that in Kelly's previous argumentation, though she did not explicitly define her unit or give support for her naming of the rods, it was implicit that she had given the dark green rod the number name 1.

7.8 Event 8

In response prompting by the researcher to clarify their arguments, Jackie reiterates her argument about how they gave the light green and red rods their number names, given that the dark green rod had the number name 1. They reconstruct the model that supports the naming of the rods that was hidden in Event 7 and explain their model as they construct it, "Okay, this [the light green rod] is … a half [putting two light green rods below the dark green 1 rod] and the red is a third [putting three red rods below the light green rods]" and "you put these all together they equal up to the one [showing that three reds, two light greens both equal the dark green which is one]." Repeating their assertion from Event 7, they claim that 1/2 is greater than 1/3 by 1 white rod. Then they refine their claim stating, "We think the light green which is a half is bigger than the red … by one which is this white one. [Showing the difference between red rod and light green rod is a white]," implying that the number name they gave the white rod was 1. (See Figure 12.)
The argument offered to support the claim that 1/2 is greater than 1/3 by a white rod in Event 8 is very similar to the argument presented in Event 4 and diagrammed in Figure 6. In Event 4, the argument was developed to support the claim that 1/2 was greater than 1/3 by a red rod. In both events, students have named the rods with different number names, but make similar statements and construct similar models that are used as warrants and backing. Another similarity between the argumentation in Event 4 and Event 8 is that once students establish that one fraction is greater than another by a certain rod, they are asked to quantify the rod by giving it a number name. The argumentation in Figure 7 is similar to the argumentation in Figure 13, where students use as their data that the rod with the number name 1/2 is longer than the rod with the number name 1/3 by a third rod and the number name of the third rod is the difference, and support the use of these data through the implicit general principle (Principle 7) as the warrant: that if a train of two rods is the same length as a third rod, the number name for one of the rods in the train shows the difference between the number name of the other rod in the train and the number name of the third rod (if a train of rod \(a\) and rod \(b\) is the same length as rod \(c\), then the number name for \(c\) is larger than the number name of rod \(a\) by the number name of rod \(b\) and the number name for rod \(c\) is larger than the number name of rod \(b\) by the number name of rod \(a\); the difference between \(c\) and \(a\) is \(b\) and the difference between \(c\) and \(b\) is \(a\)). It is interesting that in both arguments, when students are asked to quantify the difference between 1/2 and 1/3 by giving the rod that represents the difference a number name, they do not use the established unit as the referent.
Figure 7.13: Argumentation in Event 8

Notice that within the main argument in Event 8—that 1/2 is greater than 1/3 by 1—there are several sub-arguments. Two nested arguments are for the claims that the light green rod has the number name 1/2 and the red rod has the number name 1/3 when the dark green rod is the unit, 1, denoted as "Claim/Warrant," in Figure 13 since the statements are claims for the sub-argument as well as warrants for the main argument.

The data for these claims are denoted Data/Backing since the statements are both data for the sub-arguments and backing for the main argument. The general principles (Principle 2 and Principle 3) are included in the diagram as well. Another nested argument is that the light green rod is longer than the red rod by one white rod. The support for that claim can be found in Figure 12. What is not supported by the arguers here is that the claim that the number name for the white rod is 1. This claim is addressed in Event 9.
7.9 Event 9

In this event, Erik challenges the girls' claim that the number name for the white rod is 1 white. Rather than challenging the claim outright, Erik conjectures that the girls meant to give the white rod the number name 1/6, rather than 1:

I think they mean that they want to call this, the dark green, 1, one whole, and they want to call this [the white rod] … one sixth. I think that’s what they’re trying to say but they just, they’re just not saying it. I think they just, they want to call it one sixth.

In response to this conjecture, the girls modify their claim to say that the number name for the white rod is 1/6, which counters their previous claim, and, together, the girls and Erik develop an argument that refutes the previous claim that the number name for the white rod is 1. The support for the new claim and the refutation of the previous claim is two-fold. First, relying on the general principles that rods and trains of rods that are the same length show the same numerical value (Principle 2), and if a train of n rods of the same color fit under the rod or train of rods with the number name 1, then that rod is "one nth" of the unit and the number name for one of those rods is 1/n (Principle 3), Erik shows that a train of six white rods is the same length as the dark green unit rod, so each white rod has the number name 1/6. This supports the claim that the white rod has the number name 1/6, but does not necessarily refute the claim that the number name for the white rod is 1. The girls contradict the claim that the white rod has the number name 1 when they state that the dark green rod already has the number name 1. This statement coupled with the implicit general principle that only one rod or train of rods can be used as the unit rod with the number name 1 (Principle 10), refutes the claim (See Figure 7.14).
Figure 7.14. Argumentation in Event 9

Figure 15 shows a summary of the argument that was developed collectively but the students supporting the claim that 1/2 is bigger than 1/3 by 1/6, given that the dark green rod has the number name 1. The backing for each of the warrants has been supported in
previous events and diagramed in previous figures. This collective argumentation resolves the argument about how much greater 1/2 is than 1/3, when the dark green rod is the unit, 1. Note that at the beginning of the argumentation, the students worked with a model based on the orange and red rod train being the unit. In the following events, the students return to that model.

**Figure 7.15. Summary of the argumentation that, when the dark green rod has the number name 1, 1/2 is greater than 1/3 by 1/6**
7.10 Event 10

In Event 10, the researcher asks students to return to the model where the orange and red rod was given the number name 1. At the beginning of the VMCAanalytic, the students used this model to argue that 1/2 is greater than 1/3 and then to quantify the difference as 1/3. When the girls presented their argument in Event 4, they claimed that when the orange and red rod train is 1, the dark green rod had the number name 1/2 and the purple rod had the number name 1/3. Recall that the girls said that 1/2 is greater than 1/3 by 1/3 because the purple and red rod train is as long as the dark green rod, and three red rods are the same length as the dark green rod (See Figure 16). Implicit to this argument was that the red rod had the number name 1/3, which suggests that they switched the unit from the orange and red rod train as 1 (when defining the number names for the other rods) to the dark green rod (when defining the number name for the red rod).

![Image 1](image1.png)

When the orange and red rod train has the number name 1, the purple rod has the number name 1/3 and the dark green rod has the number name 1/2.

![Image 2](image2.png)

The dark green 1/2 rod is the same length as the purple and red rod train, so 1/2 is greater than 1/3 by the red rod.

![Image 3](image3.png)

A train of three red rods is the same length as the dark green rod, so one red rod has the number name 1/3 (contradicting the original premise that the orange and red rod represented the unit, 1.)

*Figure 7.16. The rod models that support that 1/2 is greater than 1/3 by 1/3*

Continuing their argumentation in this event, the girls offer a similar argument, using the same evidence to claim outright that the number name for the red rod is 1/3. Jessica states:
This is one whole [the orange and red rod train], and then this is 1/3 [the purple rod] and this is 1/2 [the dark green rod] … Well, … three reds equal up to one greens and then you put the purple next to it [the dark green rod] and you need one more red, you need a red to go next to the purple [to make the purple 1/3 rod the same length as the dark green 1/2 rod], so it [the difference between 1/2 and 1/3] would be 1/3.

She again implicitly switches the unit from the orange and red rod train as 1 to the dark green rod train as 1 when giving the red rod its number name. Also implicit in this argument are several general rod principles: rods and trains of rods that are the same length show the same numerical value (Principle 2); if a train of n rods of the same color fit under the rod or train of rods with the number name 1, then that rod is "one nth" of the unit and the number name for one of those rods is 1/n (Principle 3); if it takes a certain rod to make two rods or trains of rods the same length, then the number value of that rod is the difference between the number two values of the other two rods (Principle 5); and if a train of two rods is the same length as a third rod, the number name for one of the rods in the train shows the difference between the number name of the other rod in the train and the number name of the third rod (Principle 7). However, it violates the general principles that 1. Any rod or train of rods can be given number names. If the rod or train is regarded as the unit, it is given the number name 1. All other rods are given number names based on their relationship to the rod or train of rods with the unit rod that has the number name 1 (Principle 1) and 2. Each rod can only have one number name given that a certain rod has the number name 1 (Principle 9).

7.11 Event 11

Returning to his challenge of the claim that 1/2 is greater than 1/3 by 1/3, Brian repeats one of the sub-arguments he makes in Event 6: that, when the orange and red rod train has the number name 1, the number name for the red rod is 1/6, not 1/3. He repeats
an abbreviated version of another sub-argument offered in Event 6 and illustrated in Figure 11. Brian claims that the number name for the red rod cannot be 1/3 because half of a third is a sixth, alluding the to the rod model support that accompanied his previous argument: "When you split the thirds in half and they make sixths." Implicit in this argument is that, since half of 1/3 is 1/6, a red rod must be 1/6. Note that he does not make the rod model that shows that two red rods are the same length as a purple 1/3 rod, but he holds up the rods to allude to it.

Brian then provides a different argument to support his challenge of the claim that 1/2 is greater than 1/3 by 1/3. This argument includes the notion based on the contrapositive of Principle 7: if a train of two rods is the same length as a third rod, the number name for one of the rods in the train shows the difference between the number name of the other rod in the train and the number name of the longest rod, and so if one rod quantifies the difference between two other rods, the train of the two shorter rods will be the same length as the longer rod. Specifically, in this example, if 1/2 was greater than 1/3 by 1/3, two purple 1/3 rods would have to be the same length as one dark green 1/2 rod. He puts a red rod on top of the second purple 1/3 rod to show that two purple 1/3 rods are not the same length as a dark green rod, in fact, two 1/3 purple rods are a red rod longer than the dark green 1/2 rod: "it’s [the difference between a dark green 1/2 rod and two purple 1/3 rods] that much that, that red, that red is that much bigger than one of the halves." Figure 16 shows the diagramming of his argument. Brian's argument appears to be convincing, because, at the end of this event, Jessica modifies her claim that 1/2 is greater than 1/3 by 1/3 to conjecture that she now thinks that 1/2 is greater than 1/3 by 1/3 and 1/2 is greater than 1/3 by 1/6.
7.12 Events 12 and 13

In Events 12 and 13, Erik and Brian present an argument that is meant to refute the claim that 1/2 is greater than 1/3 by 1/3, and its modification, the conjecture that 1/2 is greater than 1/3 by 1/3 and by 1/6. The foundational principle of this argument comes from Brian's previous argument from Event 11, specifically that if one rod shows the difference between two other rods, a train of the shorter two rods should be the same.
length as the longer rod. Also implicit in his argument is that if 1/2 is not greater than 1/3
by 1/3, then the conjecture that 1/2 is greater than 1/3 by 1/3 and by 1/6 is also false. Not
surprisingly, since Erik's basic argument parallels Brian's (See Event 10), Brian concurs
with Erik in this event when he says, "I kind of agree with Erik." Interestingly, when his
whole statement is considered, "I kind of agree with Erik. I think now I disagree with
them [referring to the girls]," he is implying that he agreed with something in the girls'
previous argument. It is clear from his previous challenge, that he did not agree that 1/2 is
greater than 1/3 by 1/3. However, the conjecture that was made just prior to his statement
of disagreement was that 1/2 might be greater than 1/3 by both 1/3 and by 1/6. It is likely,
then, that he agreed with this statement and that, after Erik offered his argument, he has
now changed his mind and no longer thinks this is a viable conjecture.

Erik presents some of the argument challenging the claim and conjecture in Event 12
and Brian adds to and refines the argument in Event 13. Included in their main argument
that 1/2 cannot be greater than 1/3 by 1/3 because two thirds are not the same length as
1/2, are nested arguments with the following conjectures: 1. 1/2 is as long as a third and a
half of a third (not as long as two thirds), offered by Erik; and 2. Two thirds are 1/6
longer than 1/2, offered by Brian. Erik begins his sub-argument by conjecturing that 1/2
is not the same length as two thirds as you would expect if 1/2 was greater than 1/3 by
1/2, but, rather, 1/2 is "probably [as long as] a third and a half [of a third]." The first
conjecture refines the data, which can also be seen as a claim (that 1/2 and two thirds are
not the same length). Erik then presents evidence, both verbal explanation and models, to
support this statement. Brian presents his own related conjecture, refining the data and
Erik's conjecture, by offering a subtly different conjecture as a warrant to support the
data: two thirds is 1/6 longer than 1/2. This conjecture quantifies Erik's "half of a third" and brings back into the argument the statement that, when the orange and red rod train is considered the unit, the number name for the red rod is 1/6 and the red rod is "half of a third." (See Event 6, Figures 10 and 11.) Using a model to add further evidence to his argument, Brian places a red rod on top of the second purple 1/3 rod to show that two purple rods are longer than the dark green 1/2 rod by one red rod. (See Figure 18.) Since previously Brian gave the red rod the number name 1/6 when the orange and red rod train was given the number name 1, here he quantifies Erik's "half of a third," giving it a value of 1/6:

Brian: …there's still some left over and there's still about…
Erik: A half left over [referring to the half of a third]
Brian: Yeah, there’s still, there’s still one more, there’s still one more piece left, like about a sixth left.

The result of the collaborative argumentation and the sub-arguments are included in Figure 18 with the contributors noted.
Figure 18. Brian and Erik's collective argumentation

A second argument (and sub-argument) is presented by Erik in Event 13 (See Figure 19). The claim and data in this argument are the same as in the argument diagramed in Figure 7.18, but different ideas are included in the warrant and backing. Erik offers different evidence to explain why the datum "1/2 is not the same length as two thirds" supports that 1/2 cannot be greater than 1/3 by 1/3, and therefore, 1/2 cannot be greater than 1/3 by 1/3 and by 1/6. In Erik's argument presented in Figure 7.19, he uses the notion of boundaries to argue that 1/2 cannot be 1/3 greater than 1/3 because the dark green 1/2 rod is longer than the 1/3 rod (showing that 1/2 is greater than 1/3), but shorter than two of the purple 1/3 rods, implying that 1/2 is between 1/3 and two thirds. Implicit
in this argument is the general principle that if rod $a$ with value $a$ is shorter than rod $b$
with value $b$ and longer than rod $c$ with value $c$, then value $a$ cannot equal value $b$ or
value $c$ (Principle 11). Erik offers this as a warrant for his data, as well as a claim in
itself, and supports it with warrant and backing. His argument presented in Figure 7.19
includes the boundary argument, as well as the previous argument that $1/2$ is as long as a
third and a half of a third:

Cause it’s like if you have … the dark green and it [the dark green rod] doesn’t exactly
equal up to, it doesn’t exactly equal up [to two purple $1/3$ rods]. It’s [the length of the
dark green rod] less than two thirds but it’s more than one third. It’s just about one
third and a half. So it [$1/2$] couldn’t be exactly a third bigger than it [$1/3$] and it
couldn’t be exactly two thirds or it couldn’t be exactly one third bigger. It [$1/2$] had to
be one third and a half.
Note that the counterarguments offered to challenge the claim that 1/2 is greater than 1/3 by 1/3 are varied, but they do not specifically address that the girls who formulated the argument in Event 4 switched the unit from the orange and red rod train to the dark green rod when giving the red rod a number name. The claim from Event 4 is challenged, but students do not revisit the evidence (data, warrant, or backing) to point
out the contradiction. Rather, than pointing out the flaw in the girls' argument, they
develop their own arguments that support that the claim is false and their own
counterclaims, for example that the red rod has the number name 1/6 and that 1/2 is
greater than 1/3 by 1/6.

7.13 Event 14

In Event 14 Michael states that he believes that, when the orange and red rod train is the
unit, 1, the number name for the red rod is 1/6 and presents a direct argument to support
his claim. His argument is a familiar one and implicit in it are basic principles that
students have taken as shared: when compared to the same unit rods and trains of rods
that are the same length show the same numerical value; and, if a train of n rods of the
same color fit under the rod or train of rods with the number name 1, then that rod is "one
nth" of the unit and the number name for one of those rods is 1/n:

I think it [the red rod] should be called 1/6 because if you put six reds up to one orange
with a red then it would equal, it would be the same size just, so it would be called 1/6
because reds like that. (See figure 20.)
Figure 7.20. Michael's argument that the red rod has the number name 1/6

Note that Michael's participation in the argumentation in this event illustrates how important models are to the students' arguments. Michael makes the claim "I think it [the red rod] should be 1/6 . . . because . . ." from his seat. He does not give his argument from his seat; rather, he pauses and comes to the overhead. He constructs his verbal argument and rod model simultaneously, evidence that the reasoning behind the argument and the model are inextricably linked.

Also included in Event 14 is another argument by Erik that concurs with Michael's assertion that the number name for the red rod is 1/6, given that the unit is the orange and red rod train. While Michael argues that a train of six red rods is the same
length as the orange and red rod train unit, Erik argues that three sixths are the same length as 1/2. Implicit in this argument is that there are two 1/2s in the unit, and if each 1/2 is the same length as three sixths, then there will be six sixths in the unit. Erik makes this implicit idea explicit at the end of Event 14 and the beginning of Event 15, when he places two dark green 1/2 rods in the model, one above the first three red rods in the train of six red rods, and one above the second three red rods.
7.14 Event 15

In this final event, several students contribute to the counterargument that supports that 1/2 cannot be greater than 1/3 by 1/3, and, therefore, 1/2 cannot be greater by both 1/3 and by 1/6. (See Figure 7.21.) As students present their arguments, they embed various
sub-arguments within the main argument. Meredith modifies the claim made previously by a variety of students: $1/2$ is greater than $1/3$ by $1/3$, to the claim: the difference between $1/2$ and $1/3$ is $1/3$: "...they said that it’s a third difference." She then repeats the argument made by Erik and Brian in Events 12 and 13 (See Figure 7.18) that if $1/2$ was greater than $1/3$ by $1/3$, $1/2$ would be as long as two thirds, and it is not, it is shorter, in fact, it is $1/6$ shorter. It is not unexpected that Erik and Brian concur, and, as he did in his previous arguments, Brian contributes the quantification of the number name of the red rod as $1/6$:

Meredith: I agree with Erik, Michael, and Brian because if you do call that a sixth, a sixth, and if you put the dark green up with two thirds, and you said it was, you said it...

Brian: 1/3 and a half [talking over Meredith]
Meredith: … you said, they said that was [putting a train of two purple $1/3$ rods below a dark green $1/2$ rod] it was a third difference [between $1/2$ and $1/3$], if you say that it's a third difference, this is called a third [referring to the purple rod] and then you put it there you can see the difference [putting another purple $1/3$ rod next the first, creating a train of two purple rods that is longer than the dark green rod]. But you say it was $1/3$ bigger [taking away the second purple rod], that can’t be true because $1/3$ bigger it's still [putting back the second purple rod]…

Brian: It’s about $1/6$ less. So it can’t be a third bigger.
Erik: And also, like…
Meredith: So it’s a $1/6$ bigger [putting a red rod next to the dark green rod to show that a dark green and red rod train is the same length as two purple rods].

Meredith's contribution strengthens the argumentation made in Events 12 and 13 because she uses the rod model to directly compare the dark green $1/2$ rod to the two purple $1/3$ rods, arranging them side by side. In Event 13, Brian makes a conjecture that he thinks that two thirds are $1/6$ longer than $1/2$. In this event, Meredith confirms this conjecture, turning it into a claim and provides evidence. She constructs a dark green and red rod train and shows that it is the same length as a train with two purple rods. From this model, it can be concluded that two thirds are longer than $1/2$ by $1/6$. Note, that Brian
agrees, but none of the students present a sub-argument as to why they believe the red rod has the number name $1/6$. It seems, that at this point in the argument, is has been taken as shared, at least by the students that are presenting arguments in this event. It is interesting to note that, at this point in the argumentation, students no longer provide evidence for any of the values they have assigned to the rods, and, although the premise of the argument is that the orange and red rod is the unit, 1, this assumption is not stated explicitly by the students.

Erik also continues to contribute to the argumentation in this event. He repeats his boundary argument from Event 13 (outlined in Figure 7.19). Refining his argument, he states that the dark green $1/2$ rod is longer than the purple $1/3$ rod and shorter than two purple $1/3$ rods ($2/3$) and makes a new, more concise claim: $1/2$ is less than $2/3$ but more than $1/3$. He uses this claim as data to support the counterclaim that the difference between $1/2$ and $1/3$ cannot be $1/3$ and the evidence for his claim is embedded in the main argument. He uses the rod models, as well as the general Principle 11 (if rod $a$ with value $a$ is shorter than rod $b$ with value $b$ and longer than rod $c$ with value $c$, then value $a$ cannot equal value $b$ or value $c$) to support his claim. His concluding remarks concur with Brian and Meredith's argument that, not only is $1/2$ between $1/3$ and $2/3$, but to make $1/2$ the same value as $2/3$, you would have to add $1/6$. Note that in Erik's argument, he says "light green," however, he is referencing the dark green rod, so it is likely he meant to say "dark green":

And also yeah, and also, I think because if you have the light [dark] green, the light [dark] green, it’s not bigger than, it’s not bigger than the, it’s not bigger than the umm third, it’s not bigger than two thirds. It’s bigger than $1/3$, but it’s not as big as $2/3$ so it’s less than $2/3$ but more than $1/3$. So it can’t be a third bigger. And if you have that to make it $2/3$ large, there has to be a sixth.
Figure 7.21. Argumentation in Event 15
At the end of the event, and the conclusion of the argumentation session, Michael provides a summary statement that seems to resolve the argument, "It is 1/6 in both cases." (See Figure 7.22.) Michael, then, concludes that if all of the argumentation that was presented is considered as data, they provide convincing evidence that Claim 9—1/2 is greater than 1/3 by 1/6—is true in the case where the dark green rod has the number name 1, as well as in the case where the orange and red rod train has the number name 1. Implicit in Michael's statement is that it is multiple models can be used to represent fractions, as long as the unit remains consistent, maintaining the proportional relationship of these models. (For a more detailed discussion of students' exploration of the unit, see Comparing Models and Justifying the Choice of Unit, located in the Video Mosaic Collaborative (http://dx.doi.org/doi:10.7282/T3XW4MQT) (Van Ness & Alston, 2015) and Chapter 9, Comparing Cuisenaire Models and Justifying the Choice of the Unit (Maher & Yankelowitz, in press).

In this VMCAntalytic, then, the students' counterarguments and counterclaims successfully refuted the claims that, when the orange and red rod train has the number name 1, the red rod has the number name 1/3 and that 1/2 is greater than 1/3 by 1/3. Although not apparent in this analytic, the students who made the claim that 1/2 is greater than 1/3 by 1/3 were convinced by this argumentation. For a more detailed discussion of the events that come after the ones presented here, see the VMCAntalytic Comparing 1/2 and 1/3: Confusion about the Unit, located in the Video Mosaic Collaborative (http://dx.doi.org/doi:10.7282/T3ZW1NS9) (Van Ness & Alston, 2015) and the Chapter 8, How much larger is 1/2 than 1/3? Switching the unit (Maher & Yankelowitz, in press).
Krummheuer's lens of Toulmin's argumentation model was used to guide the analysis of the argumentation presented in the events in this VMCAAnalytic. The diagrams provided useful visuals that help illustrate the elements of argumentation and the structure of those elements (how those elements are or are not connected) within the student discourse. The diagrams also highlighted important characteristics of students' arguments, such as nested arguments. The results of this analysis confirm Krummheuer (2000) assertion that the argumentation found in these young students' discourse was substantial (p. 236). The nested arguments, as described by Krummheuer (1995), were common in the argumentation included in this session. Students frequently used as data or warrants, claims or conjectures that either they, or others, had supported, as part of the main argument. For example, inherent to the argument that 1/2 was greater than 1/3, were claims and evidence as to what number names were given to the rods. Inherent to the counterargument that 1/2 is not greater than 1/3 by 1/3, are various claims and evidence regarding the characteristics of the rods (for example, Erik's sub-argument that 1/2 is between 1/3 and 2/3 as shown in Figure 7.21).
The diagrams helped to show the relationship between students' natural language, which can be unclear or imprecise, and the argumentation using formal academic language. The imprecision can be observed in various ways. For example, the students sometimes used general terms to reference specific rods or referents such as "it," "that," and "these." Students also did not always attend to precision in the language used in their verbal arguments, for example, saying that 1/3 is greater than 1/2, when their physical model and argument supported that 1/2 is greater than 1/3. Students' language in these argumentation episodes was sometimes hesitant, stopping in the middle of sentences and restarting, using words such as, "um," and "like," repeating thoughts and phrases, and not using complete sentences. Additionally, students interchanged names of fractions and rods, without clarifying that the rod has the fraction number name, or is being used to represent a value. For example, if the dark green rod was given the number name 1/2, they might, for example, refer to the dark green rod and call it "1/2," rather than a more precise reference to the rod, such as, "the rod that represents 1/2," or "the dark green 1/2 rod," or even the "1/2 rod." The following excerpts exemplify how sophisticated argumentation can be missed by imprecise language:

Erik: I don’t think you can have an answer of a third because if you have one half and if you take the one half which would be the dark green, you have the one half and then these are the thirds. How could one half be bigger than the thirds by one third? Because, and you have the half and the thirds together that the half is almost as big as two thirds, but yet the two thirds aren't exactly, are not exactly, the green, the dark green is not, the dark green is not exactly as big as two, two thirds but, two thirds, it’s the, but it’s far enough so that the two thirds are not bigger than it by one third.

In the excerpt above, Erik presents evidence using a sophisticated boundary argument that 1/2 cannot be 1/3 greater than 1/3. (See Event 12, Figure 7.19 for a detailed analysis of Erik's argument.)
Brian: I don’t- I still don’t think so, well, because, well, well, see like I said before when you split the ahh, when you split the thirds in half and they make sixths, it’s still like … See, well, because when you put it right there you see that, you see that there’s one of these, if you put one of these on top of it you might see that, that it’s that much that, that red, that red is that much bigger than one of the halves because one of these reds I’m calling is, is a sixth and anyway a half of one of these, a half of one of the thirds. But when you put it on top of one of the thirds it’s that much bigger than one of the halves.

In this excerpt, Brian offers an argument that 1/2 cannot be 1/3 greater than 1/3 because two thirds is 1/6 greater than 1/2. Embedded in this argument are the sub-arguments that the red rod has the number name 1/6 and that if 1/2 was greater than 1/3 by 1/3, 1/2 would be equal to 2/3 (See Event 11, Figure 7.17 for a detailed analysis of Brian's argument.)

Careful analysis of the students' models and verbal arguments, together, provide the context to ensure that the nuanced reasoning in these arguments is not missed. Through the analysis using the Toulmin model diagrams, it becomes apparent that, although students do not talk using formal mathematical language, nevertheless, they still construct valid arguments. Given the opportunity to express ideas and provide backing for these ideas by building models, students are engaged in productive argumentation that enhances conceptual understanding of fraction ideas in the episodes illustrated by the events in the VMCAlytic. Analyzing physical models, then, becomes essential in interpreting students' reasoning and the diagrams are a useful tool to help link students' natural talk to the elements of argumentation being offered. Thus, the diagrams help highlight the importance that use of these models had in supporting students' reasoning. Note that the models played different roles, as data, warrants, and backing, but in almost every event, the model was inextricably linked to the verbal argument. This suggests that
a challenge for teachers is to learn to attend to children’s natural language and give validity to their ideas that are expressed in their natural language. Providing tools such as rods to build models can provide a way of expressing ideas before formal language is learned.

The analysis of students' argumentation throughout this VMCA analytic also indicated that the students' arguments became more complex, often including more nested sub-arguments. The complexity is shown in the number of boxes in in the diagrams (containing the elements of argumentation) and the number of connecting arrows and segments (indicating the structure of the argumentation). Another observation is that certain ideas became “taken as shared” as sophistication in their argumentation progressed. For example, from Event 10 and following, students' arguments do not include justifications for the number name for the dark green rod or purple rod, given that the number name for the orange and red rod train was 1. They also do not explicitly note what train of rods is used to represent the unit. After the researcher asked the class to move from the model with the dark green rod as 1 to the model with the orange and red rod train as 1, the students took the values of the rods as “taken as shared,” except for the red rod, whose numerical value became an issue of debate.

Analyzing the argumentation using the Toulmin-based diagrams became an essential tool that enabled an evaluation of the extent to which teacher subjects in this study recognized student argumentation before and after the intervention. By mapping teachers' descriptions to the diagrams, it was possible to determine the details of the student argumentation that were described, and whether the descriptions became more detailed in the argumentation after the intervention. Specifically, the diagrams helped in
the evaluation of teacher growth in their descriptions of students' discourse in terms in
three aspects of argumentation: 1. understanding and use of the formal mathematical
register of argumentation; 2. recognition of the elements of argumentation in students'
arguments; and 3. recognition of the structure of the argumentation in students'
arguments. Through comparing diagramming the argumentation noted in the teachers'
descriptions of the events, I was able to qualify as well as quantify these changes. In the
results section, an analysis and discussion of these data are presented.
Chapter 8 – Teacher 1, Event Descriptions Analyses and Summary

This chapter presents the descriptions Teacher 1 (T1) wrote for the pre-assessment and post-assessment analytic to describe the argumentation in each event. The pre-assessment and post-assessment descriptions given by the teacher are presented for each event with an accompanying diagram using Toulmin's (1958, 2003) scheme. Following the descriptions for each event, I present an in-depth analysis, summarizing the argumentation described in the pre-assessment and noting the changes the teacher made from pre- to post-assessment. Words that are key to my analysis appear in red text in the teacher’s descriptions.

8.1 Teacher 1 Event Descriptions and Analyses by Event

8.1.1 Event 1

Pre-Assessment Description
Students in a fourth grade class room are presented with a problem that asks them to use candy bars to make an argument about which number is larger - 1/2 or 1/3. Laura, a student, argues that 1/2 is larger, and her classmates agree. Laura is asked to demonstrate her reasoning and consider how much larger the one number is than the other.

Post-Assessment Description
In this first event, Researcher Maher poses the initial question, "Which number is bigger, 1/2 or 1/3?" Michael and Andrew explain that this was a problem that the students had recently explored, and several hands in the classroom show that the students do believe they have an answer to the problem. Laura provides her initial claim that 1/2 is larger than 1/3. Researcher Maher than poses a second question, asking if the students believe that 1/2 is larger than 1/3, then how much larger is it? Researcher Maher calls Laura and her partner Jessica to the overhead to make a convincing argument as to why their initial claim is true.

Figure 8.1. Argumentation described by Teacher 1 for Event 1
As presented in Figure 8.1, in the pre-assessment, T1 notes a claim made by the students, that "1/2 is larger." Based on the context of the accompanying text, it can be inferred that T1 is referring to the claim that "1/2 is larger than 1/3," but the 1/3 part is implicit and not explicitly stated.

In the post-assessment, T1 changes the language to include more of the formal mathematical register, identifying the student's statement that "1/2 is larger than 1/3" as a claim and including other formal language specific to argumentation, by stating that, "Laura and her partner Jessica go to the overhead to make a convincing argument as to why their initial claim is true." The use of "convincing argument" and "initial claim" do not add structure or elements of argumentation to the post-assessment, but shows growth in the language used. Also, note that T1 makes what was implicit in the pre-assessment explicit in the post-assessment by stating that Laura's initial claim was "that 1/2 is larger than 1/3."
8.1.2 Event 2

Pre-Assessment Description
Jessica and Laura use rods on the projector to give their argument to the class as to why 1/2 is larger than 1/3. Jessica uses one orange and one red rod together and has them represent a length of 1. She then places 3 equally sized purple rods together, which equal length 1 as well, and 2 equally sized green rods which have a length of 1. The two argue that the purple rods represent 1/3 and green rods represent 1/2, so 1/2 must be larger than 1/3.

Post-Assessment Description
Jessica and Laura use several colored rods of different sizes to provide support and evidence as to why their initial claim is true, that 1/2 is indeed larger than 1/3. Jessica takes a long orange rod and a shorter red rod, places them side by side, and calls their combined length "1". She then places three purple rods below this, and assigns each purple rod the measurement "1/3" because the length of the three purple rods equals the length of the orange plus red rods. Jessica lastly places two green rods of equal length below her other measurements, and assigns these rods with the measurement "1/2". Laura states that the green rod is 1/2, and the purple rod is 1/3. Students in the class can see on the overhead projector that the green rod is longer than the purple rod.

Figure 8.2. Argumentation described by Teacher 1 for Event 2
T1 describes an argument in the pre-assessment with the claim that 1/2 is larger than 1/3. The data noted are that the purple rods represent 1/3 and the green rods represent 1/2. These are also claims. T1 describes several warrants that connect the data and the main claim, but also serve as data for the claim/data statements. The warrant for the data that the purple rods represent 1/3 is that 3 equally sized purple rods equal length 1 and the warrant for the data that the green rods represent 1/2 is that 2 equally sized green rods have a length of 1. T1 also describes as a warrant the premise that "one orange and one red rod together … represent a length of 1." Note that the use of "why" and "so" situate the statements about the rods as an argument for the claim that 1/2 is greater than 1/3.

In the post-assessment, T1 identifies that 1/2 is larger than 1/3 explicitly as a claim. Additionally, the rod model is mentioned explicitly as "support and evidence as to why" the students' initial claim was true. These statements show evidence of the growth in the use of the formal mathematical register for argumentation. Furthermore, T1 uses the data that the green rod is longer than purple rod to support the claim that 1/2 is larger than 1/3. This is an added element of argumentation and results in changing to warrants what were data in the pre-assessments. These warrants, that "the green rod is 1/2" and that "the purple rod is 1/3" are also claims. The claim/warrant statements are supported by data/backing: that the green rod has the "measurement" 1/2 and the purple rod has the "measurement 1/3. The data/backing are supported by warrants, that "two green rods of equal length below her other measurements" and "the length of the three purple rods equals the length of the orange plus red rods." With this post description, T1 shows growth in three aspects of the argumentation described.
8.1.3 Event 3

Pre-Assessment Description
Audra, another student in the class, agrees with the argument by the two girls that 1/2 is larger than 1/3. She then makes their argument even stronger by taking one piece of purple rod and one piece of green rod, and showing that the length of the green rod is longer than the length of the purple rod.

Post-Assessment Description
Researcher Maher asks whether the rest of the class agrees or disagrees with the evidence that has been provided so far. Audra speaks up, stating that she agrees. Maher asks if she can come to the projector and explain her reasoning why she agrees to the class. At the projector, Audra takes the model made by Laura and Jessica of the three different ways the girls represented the measurement "1". The pulls out one of the purple rods, representing 1/3, and one green rod, representing 1/2, and states that the class "saw that the half was bigger than the third".

Figure 8.3. Argumentation described by Teacher 1 for Event 3

In the pre-assessment description, T1 describes the claim that "1/2 is larger than 1/3" with the data that "length of the green rod is longer than the length of the purple rod" supported by the warrant using a piece of the purple rod and a piece of the green rod, implying a comparison of the lengths of the rods. T1 adds information that is relevant to
the argumentation with the statement that using the rods makes the argument given previously, "even stronger."

In post-assessment description, T1 adds structure and elements to the description, as well as more formal language. T1 uses more of the students' exact language stating that the claim was, "the half was bigger than the third." Additionally, T1 describes in more detail the warrants that support the data. Whereas in the pre-assessment T1 alluded to the model and one "piece of the purple rod" and one "piece of the green rod," in the post-assessment T1 specifically notes that the student names the rods as the 1/2 rod and the 1/3 rod. T1 also mentions the model that the previous girls used that showed, "three different ways" to represent 1. Furthermore, T1 uses "evidence" to describe the supports that have been presented by the students. These additional statements and the use of the formal mathematical register add to the argument described.

8.1.4 Event 4
Pre-Assessment Description
The teacher asks Audra how much bigger $\frac{1}{2}$ is than $\frac{1}{3}$. Audra finds red rods and places them next to the green rod ($\frac{1}{2}$). She notices that 3 red rods equal the length of the green rod. She then takes one of these red rods and places it next to the purple rod ($\frac{1}{3}$) and realizes the purple plus red rod equal the length of the green rod. Since it took one of the three red rods to make the two lengths equal, she concludes that $\frac{1}{2}$ is $\frac{1}{3}$ bigger than the number $\frac{1}{3}$.

Post-Assessment Description
Researcher Maher now asks students to consider her second question, since there have not been any counterclaims made to the argument that $\frac{1}{2}$ is larger than $\frac{1}{3}$. Audra and Jessica start to organize different rods to see how much larger $\frac{1}{2}$ is than $\frac{1}{3}$. They place purple and green rods on top of each other, and show that either one red or two white rods placed next to the purple rod equals the distance of the green rod. Researcher Maher asks the girls to assign the red rod a number value so that the question can be answered. Audra makes a conjecture that $\frac{1}{2}$ is one-third larger than the number $\frac{1}{3}$. The two girls show that three red rods are equal in length to one green rod. Since only one of these red rods must be added onto the length of the purple rod in order to equal the length of the green rod, then there must be a $\frac{1}{3}$ difference between the two numbers.
Figure 8.4. Argumentation described by Teacher 1 for Event 4

In the pre-assessment, T1 describes an argument with claim, "1/2 is 1/3 bigger than the number 1/3" and data, "the purple [1/3] plus red rod equal the length of the green rod" and "3 red rods equal the length of the green rod." T1 notes two warrants presented by the students: the model that shows "it took one of the three red rods to make the two lengths equal [the purple rod and the green rod]" supports Data 1 and the model that shows that 3 red rods are the same length as the green 1/2 rod the green rod supports Data 2. Note that in the statement "Since it took one of the three red rods to make the two lengths equal, she concludes that 1/2 is 1/3 bigger than the number 1/3," T1 is suggesting the direction of the arrows, describing a "Data so Claim" rather than a "Claim because of Data" structure. By using the word "concludes," T1 suggests that "1/2 is 1/3 bigger than the number 1/3" is the claim.
In the post-assessment, T1 adds detail that describes more argumentation than in the pre-assessment. The first added statement, "since there have not been any counterclaims made to the argument that 1/2 is larger than 1/3," includes more of the formal mathematical register, as well as two additional elements of argumentation, that there was a previous claim that "1/2 is larger than 1/3," and that no counterclaims were made. T1 then states that the students, "place purple and green rods on top of each other, and show that either one red or two white rods placed next to the purple rod equals the distance of the green rod." With this statement, T1 is describing two claims and their data, first, that "two white rods placed next to the purple rod equals the distance of the green rod" with the model as data, and second, one red rod "placed next to the purple rod equals the distance of the green rod," again with the model as data. Then T1 describes a conjecture that "1/2 is one-third larger than the number 1/3." To support this conjecture, T1 notes that "The two girls show that three red rods are equal in length to one green rod" and "one of these red rods must be added onto the length of the purple rod in order to equal the length of the green rod." Claim2 is used as data and the data for the claim is used as the warrant. "Three red rods are equal in length to one green rod," is also used as data supported by the model that shows that the three red rods are the same length as the green rod. This evidence is used to support the claim that "there must be a 1/3 difference between the two numbers [1/2 and 1/3]." T1's language, "there must be" suggests that this claim is a confirmation of the conjecture made that 1/2 is one third larger than the number 1/3." Thus, with the added detail to the post-description, T1 describes more elements, structure, and formal mathematical language for argumentation in the post-assessment than the pre-assessment.
8.1.5 Event 5

**Pre-Assessment Description**
The teacher asks if the other students agree with the arguments made. Kelly says she does, and comes up to the projector and displays a similar argument as to why 1/2 is bigger than 1/3 using different size rods than Jessica and Laura's argument. Brian announces his displeasure with the argument being made due to the fact that there are now two different colored rods being named the same quantity - both the purple and red rods have been called 1/3.

**Post-Assessment Description**
Researcher Maher asks the class if they agree with the argument presented by Audra, Jessica, and Laura. Kelly says that she agrees with the other girls, and basically repeats the same argument up at the projector using the green, purple, and red rods. Maher realizes amongst the other students that Brian is making a face of displeasure. He professes that he doesn't really agree with the claim that 1/2 is a third larger than 1/3, because two different colored rods of different lengths have both been assigned the value of one-third.
T1 in the pre-assessment description states, "Kelly says she does, and comes up to the projector and displays a similar argument as to why 1/2 is bigger than 1/3 using different size rods than Jessica and Laura's argument." This statement uses imprecise, general language to note that there is an argument that is in agreement with a prior argument that "1/2 is bigger than 1/3" and this argument is supported by a rod model that is different than model presented previously. T1 goes on to note a counterargument that seems to be related to the prior argument with the implied counterclaim that 1/2 is not bigger than 1/3 and data "two different colored rods [are] being named the same quantity," supported by the warrant that "both the purple and the red rods have been called 1/3." However, note that the claim that is being questioned here is not that 1/2 is greater than 1/3, but that 1/2 is 1/3 greater than 1/3. Thus, the part of the described argument that connects the data and warrant with the claim and the prior argument is
diagrammed in gray. Although what T1 is describing about the different colored rods is evident in the video, it is not a counterargument to the claim that $1/2$ is greater than $1/3$.

T1 makes changes to the description in the post-assessment that add to the structure and elements of argumentation described. T1 does not explicitly state the argument made and the present claim, but does add detail about the data used to support the agreement, specifically, that a model "using the green, purple, and red rods" was used.

T1 then goes on to describe the counterargument, but in the post-assessment, notes the claim that the student is disagreeing with, "$1/2$ is a third larger than $1/3$" using the formal mathematical register ("claim") and implying the counterclaim that "$1/2$ is not a third larger than $1/3$". T1's uses language that clarifies that the counterargument is not regarding the fact that $1/2$ is greater than $1/3$. T1 concludes by describing the data supporting the counterclaim, that "two different colored rods of different lengths have both been assigned the value of one-third."

**8.1.6 Event 6**
Pre-Assessment Description
Brian says that 1/2 cannot be 1/3 larger than the number 1/3, and proclaims that he believes the difference between the two numbers to be 1/6. Brian shows at the projector that 2 red rods are equal to the length of a purple rod (1/3), and half of 1/3 is 1/6, so the red must represent 1/6. He makes the argument that when Audra showed that a red rod placed with a purple rod equaled the size of a green rod, the diagram was actually displaying a difference of 1/6 between the two quantities.

Post-Assessment Description
Brian makes a counterargument to the claim that 1/2 is one-third larger than 1/3. His counter claim [sic] is that the difference between the two fractions is actually 1/6. Brian asks if he can come up to the projector, and takes one of the purple rods that the girls had given the measurement of 1/3. He says that if you cut 1/3 into two halves, it will produce two smaller rods each having a measurement of 1/6. Brian shows the class that the two red rods are equal in length to one purple rod, so the red rods must be equal to 1/6. This counters the argument made by the girls previously that the red rods also had a value of 1/3. Brian finishes his argument by placing the purple 1/3 rod and the red 1/6 rod next to each other, and shows that their combined length is equal to one green 1/2 rod.
T1 describes a variety of argumentation elements and argumentation structure in the pre-assessment description for this event. T1 notes the claim that "1/2 cannot be larger than the number 1/3." The statement, "when Audra showed that a red rod placed with a purple rod equaled the size of a green rod, the diagram was actually displaying a difference of 1/6 between the two quantities," suggests a prior argument that included a model in which "a red rod placed with a purple rod equaled the size of a green rod" was used as support. Whether this model was originally used as data or warrant and what the original claim was, cannot be determined from T1's language here. T1 uses this evidence
as data to support the claim that "the difference between the two numbers [1/2 and 1/3] is 1/6."

T1 describes another argument with the claim that the red rod represents 1/6 supported by data that two red rods are equal to the length of a purple rod and that half of 1/3 is 1/6. The use of "and" supports the idea that the two statements described are both data, rather than one data and one warrant for the data. Additionally, the use of "so" suggests the "Data so Claim" argumentation structure. T1 might be noting a connection between the claim that the difference between 1/2 and 1/3 is 1/6 and the argument that the red rod represents 1/6, but this connection is at best implied, indicated by the dotted arrow.

In the post-assessment, T1 adds detail and more of the formal mathematical register to include more elements of argumentation that are more connected, adding additional argumentation structure. T1 states that, "Brian makes a counterargument to the claim that 1/2 is one-third larger than 1/3. His counterclaim is that the difference between the two fractions is actually 1/6." The use of the formal mathematical register of argumentation in this statement situates the rest of the description as a counterargument and connects the statements that follow to each other, as well as to the argumentation that was presented previously. Thus T1 explicitly notes the prior claim that "1/2 is one-third larger than 1/3," and an explicit counterclaim "the difference between the two fractions [1/2 and 1/3] is actually 1/6," that is supported with data that the "combined length [of the purple 1/3 rod and the red 1/6 rod] is equal to one green 1/2 rod."

The claim and data in this first argument are supported by a sub-counterargument which T1 explicitly notes is countering the prior claim that "the red rods also had a value of
1/3." The counterclaim of this sub-counter argument, which also serves as the warrant for the data in the first argument, is that "the red rods must be equal to 1/6." This claim/warrant is supported by data: "if you cut 1/3 into two halves, it will produce two smaller rods each having a measurement of 1/6," "two red rods are equal in length to one purple rod," and "the purple rod has the measurement 1/3," which also serve as backing for the first argument. The data/backing, "two red rods are equal in length to one purple rod," is further supported by the warrant, that the rod model that shows that two red rods are the same length as a purple rod. Through the use of more precise language and the addition of specific details relevant to argumentation, T1 adds more formal argumentation language, more elements of argumentation, and structure to the description.

8.1.7 Event 7
Pre-Assessment Description
Jackie uses different sized rods than Jessica to make her argument as to why $\frac{1}{2}$ is larger than $\frac{1}{3}$. Her measures are correct, but Jessica does not believe this new argument to be valid. She claims that it is unfair to use a larger candy bar because then $\frac{1}{2}$ will be a different size in each one.

![Diagram of Pre-Assessment Argumentation](image1)

Post-Assessment Description
Jackie continues to argue why $\frac{1}{2}$ is larger than $\frac{1}{3}$, and show the difference between the two numbers, but now assigns different values to each colored rod than had been used previously. She calls the dark green rod 1, the light green rod $\frac{1}{2}$, and the red rod $\frac{1}{3}$. Jessica exclaims that "they're making a different size candy bar." When Researcher Maher asks if this is okay, Jessica counters Jackie and proclaims it is not okay because if you take $\frac{1}{2}$ of a 12cm candy bar, and $\frac{1}{2}$ of a 6cm candy bar, the resulting portions will be different sizes.

![Diagram of Post-Assessment Argumentation](image2)

Figure 8.7. Argumentation described by Teacher 1 for Event 7
In the pre-assessment, T1 notes the claim that "1/2 is larger than 1/3" and suggests that different sized rods than were used before are used as data for the claim. The actual claim that is being made in the event is that 1/2 is bigger than 1/3 by a white rod, so this claim is gray. T1 then describes a counterargument to this claim with the counterclaim that "it is unfair to use a larger candy bar," and data, "1/2 will be a different size in each one." (The use of the language, "the candy bar," is a reference to a metaphor established previously to mean the unit, or 1.)

In the post-assessment T1 notes more elements of argumentation and structure, as well as using more of the formal mathematical register for argumentation. The data for the claim that "1/2 is larger than 1/3" is explicitly stated in the post-assessment, specifically, that in the model being used the dark green rod has the number name 1, the light green rod has the number name 1/2, and the red rod as the number name 1/3. T1 also adds additional information related to the argumentation in the event, specifically that the student was assigning "different values to each colored rod than had been used previously."

T1 goes on to describe the counterargument, explicitly noting that the argument "counters" the prior argument, with the claim that the new model the students are using is "not okay," because of the data, "they are using a different candy bar," supported by the warrant that, "if you take 1/2 of a 12cm candy bar, and 1/2 of a 6cm candy bar, the resulting portions will be different sizes." Thus the detail and language used in the post-assessment results in more elements and structure of argumentation to be described.

8.1.8 Event 8
Pre-Assessment Description
Jackie continues her argument as to why $1/2$ is larger than $1/3$ using different rods than Jessica did. She calls the dark green rod 1, and calls the light green rods $1/2$ because two of them equal the size of the dark green. She calls the red rods $1/3$ because 3 of them equal the size of the dark green. When asked the different [sic] between $1/2$ and $1/3$, Jackie shows that one white rod makes up the difference, and names the quantity of the white rod "1" even though this would be inconsistent with her measurements.

Post-Assessment Description
Jackie justifies why $1/2$ is larger than $1/3$ using the modified values assigned to each rod. Jackie portrays her understanding of using equal proportions, and that it will not change the value of the desired answer. Jackie explains that the dark green rod is equal to the value 1. Since two light green rods combined are of equal length to a single dark green rod, the light green rods equal $1/2$. She uses a similar argument to show that the red rods are equal to $1/3$. When Researcher Maher asks how much larger $1/2$ is than $1/3$, Jackie shows that the difference between the two is one white rod. When Maher asks what number she is assigning to the white rod, Jackie answers, "One", as in one white rod.
In the pre-assessment, T1 notes the claim that 1/2 is larger than 1/3. T1 then states, "She calls the dark green rod 1, and calls the light green rods 1/2 because two of them equal the size of the dark green. She calls the red rods 1/3 because 3 of them equal the size of the dark green." With this statement, T1 may be implying that the number name for the rods supports for the claim, but the connection is not explicitly made, so the diagram indicates dashed lines. The number names for the rods, however, are stated and supported explicitly as claims in themselves. The data/claim that the light green rod is 1/2 is supported by the data/warrant that "two of them [light green rod] equal the size of the dark green." The data/claim that the red rod is 1/3 is supported by the data/warrant that "3 of them [red rod] equal the size of the dark green." The claim is also supported by the warrant/premise that the dark green rod has the number name 1.

T1 also notes the claim that the difference between the 1/2 and 1/3 is the white rod and that the name of the white rod is 1. Additional information relevant to the argumentation is included: that "naming the white rod 1 is inconsistent with her measurements."
In the post-assessment, T1 states, "Jackie justifies why 1/2 is larger than 1/3 using the modified values assigned to each rod," which makes the connection between the naming of the rods and the claim that 1/2 is larger than 1/3 explicit. The other changes to the post-assessment do not change the argumentation presented.

8.1.9 Event 9

**Pre-Assessment Description**
A student named Erik comes to the board to correct the argument made by Jackie. He says that when Jackie said that the white piece (the difference between 1/2 and 1/3) was equal to 1, she didn't mean it to be "one whole" but rather "one white piece". He then lines up 6 white pieces next to the dark green rod to show that, in this argument, the white rods equal 1/6 of the whole. His argument is now consistent with Brian's.

**Post-Assessment Description**
Erik walks to the overhead and shows that 6 white rods are equal in length to one dark green rod. He claims that this means one white rod, under the current model being used, is equal to 1/6. He states that he thinks the girls "want to call it one-sixth", in response to the girls calling the white rod "one" in the previous event. He restates the girls' argument, but just states that he believes the girls meant to assign the white rod, representing the difference between the light green and red rod, the value of 1/6. Researcher Maher wants to test the girls' understanding to see if they really understand the difference, and asks them if the difference between 1/2 and 1/3 is actually one or one-sixth. They all agree the difference is one-sixth.
In the pre-assessment, T1 describes a prior claim, that "the white piece (the difference between 1/2 and 1/3) was equal to 1" and the claim that the girls meant to say, "one white piece" not "one whole." Then another claim is noted, that "the white rods equal 1/6 of the whole" and supported by data, 6 white pieces "line up" next to the dark green rod.

In the post-assessment T1 gives more detail that adds to the argumentation described in the pre-assessment description. T1 notes, "He states that he thinks the girls 'want to call it one-sixth', in response to the girls calling the white rod 'one' in the previous event." The language, "he thinks," suggests a conjecture that the girls want to call the white rod 1/6, based on the prior claim that the white rod was 1. Additionally, T1 describes the prior claim that "the difference between the light green and the red rod" is 1, and the current claim that "the girls meant to assign the white rod, representing the difference between the light green and red rod, the value of 1/6." T1 notes a modification
at the end of the description through the statement, "They all agree the difference is one-sixth," indicating that the girls modified their claim that the white rod is one, to the claim that the white rod is 1/6.

8.1.10 Event 10

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>The teacher asks Jessica to consider a mistake that may have been made in her argument that 1/2 is 1/3 larger than 1/3, and to consider the quantity to call the red rods.</td>
</tr>
</tbody>
</table>

| Claim: "1/2 is 1/3 larger than 1/3" |

<table>
<thead>
<tr>
<th>Post-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Researcher Maher asks the students to go back and address the first argument where the orange rod plus the red rod were equal to &quot;one-whole&quot;. Maher wants the class to go back and find the original mistake that was made. Jessica appears to be confusing the values she is assigning to each colored rod, since multiple rods have been given different values in each of the two arguments. She is still claiming in this first model that 1/2 is larger than 1/3 by the fraction one-third.</td>
</tr>
</tbody>
</table>

| Claim: "1/2 is larger than 1/3 by the fraction one-third" |

Figure 8.10. Argumentation described by Teacher 1 for Event 10

T1 notes the claim that "1/2 is 1/3 larger than 1/3" in the pre-assessment.

Although detail is added in the post-assessment description, the detail does not change the argumentation described.

8.1.11 Event 11
Pre-Assessment Description
Brian still believes that the difference between 1/2 and 1/3 is 1/6, and comes back up to
the projector to show Jessica why the red rod is equal to the quantity 1/6 in her argument.
Jessica exclaims at the end the possibility that maybe 1/3 and 1/6 can both be answers.

Post-Assessment Description
Brian attempts to disprove that the difference between 1/2 and 1/3 in the first model can
be one-third. He comes back up to the projector and shows one purple rod that the girls
have assigned the value of 1/3 to. He then places a red rod on top of the purple rod, and
shows that it is half the length of the purple rod. So the red red [sic] [rod] is half of the
value of 1/3, or in other words, 1/6. Therefore 1/6 must be the difference between the two
fractions. Jessica still is not convinced that her answer of 1/3 is incorrect, and states that
"I think that they might both be answers."

Figure 8.11. Argumentation described by Teacher 1 for Event 11
In the pre-assessment, T1 describes two claims, "the difference between 1/2 and 1/3 is 1/6," "the red rod is equal to the quantity 1/6," and a conjecture, "maybe 1/3 and 1/6 can both be answers." T1's language does not indicate any connection among the statements.

In the post-assessment, T1 includes detail that describes more structure and elements of argumentation than described in the pre-assessment. T1 notes that "Brian attempts to disprove that the difference between 1/2 and 1/3 in the first model can be one-third," suggesting a prior claim that the difference between 1/2 and 1/3 is 1/3 and that his argument is a counterargument to that claim. The counterargument includes the claim that the difference between 1/2 and 1/3 is 1/6 supported by the data that the red rod is 1/6. That the red rod has the number name 1/6 is also a claim in itself. This data/claim is supported by the data/warrant that the red rod is half of the value of the 1/3, which is supported by the warrant/backing that the red rod is "half the length of the purple rod" and the purple rod has "the value of 1/3." T1 then notes that the argument leads to a modification in the claim that 1/2 is larger than 1/3 by 1/6, specifically that "they [1/3 and 1/6] might both be answers."

8.1.12 Event 12
**Pre-Assessment Description**

Erik makes his way back up to the projector to make a very clever addition argument. He states that if $1/2$ is $1/3$ larger than $1/3$, then $1/2$ should be equal to $1/3+1/3=2/3$. He lines up rods equally $2/3$ [sic] and the green rod representing $1/2$ and shows the class how their lengths are unequal. He concludes that Jessica’s argument cannot be mathematically correct for this reason.

**Post-Assessment Description**

Erik makes his way back to the projector and uses an additional argument to provide further reasoning as to why the argument made by the girls cannot be correct. He takes a dark green rod, which is equal to $1/2$, and two purple rods, each equal to $1/3$. He places the two purple rods on top of the dark green rod, and shows that their lengths are not equivalent. Therefore, the difference between $1/2$ and $1/3$ cannot be one-third. In fact, according to Erik, "the half is almost as big as two thirds."
Figure 8.12. Argumentation described by Teacher 1 for Event 12

In the pre-assessment description, T1 notes that the student "states that if 1/2 is 1/3 larger than 1/3, then 1/2 should be equal to 1/3+1/3=2/3," implying a prior claim that 1/2 is 1/3 larger than 1/3. The statement situates the rest of the description as a counterargument with the counterclaim that "Jessica's argument cannot be mathematically correct." The data for this counterclaim is that the lengths of the 2/3 and the dark green rod are "unequal" which is supported by the warrants, "the dark green rod represents 1/2," and "if 1/2 is /13 larger than 1/3, then 1/2 should be equal to 1/3+1/3=2/3." Although the fact that if 1/2 is 1/3 larger than 1/3, then 1/2 should be equal to 1/3+1/3=2/3 is true, it was not stated by the students in the event, so it appears gray in the diagram.

T1 more accurately describes the students' argumentation in the post-assessment; note that T1 does not include the incorrect warrant. T1 describes an argument with the counterclaim that the "difference between 1/2 and 1/3 cannot be one third," supported by
the same data as in the pre-assessment, but warrants that include that each purple rod is "equal to 1/3." In the post-assessment description, T1 also states, "In fact, according to Erik, 'the half is almost as big as two thirds.'" It may be that T1 intended this statement to be additional data for the counterclaim, as well as being a claim in itself. The uncertainty of the intent of this statement is reflected in the diagram by a dashed line and "data" recorded in parentheses.

8.1.13 Event 13

**Pre-Assessment Description**
Erik continues to show on the projector why the difference between 1/2 and 1/3 cannot be 1/3. He states that when the 1/2 rod is lined up next to the two 1/3 rods, the 1/2 rod is bigger than 1/3 but not as big as 2/3. It is more like "one third and a half", which would be equal to 1/6.

**Counterargument**

- **Data:** 1/2 is "one third and a half"
- **because**
- **Counterclaim:** "the difference between 1/2 and 1/3 cannot be 1/3"
- **Warrant:** "when the 1/2 rod is lined up next to the two 1/3 rods, the 1/2 rod is bigger than 1/3 but not as big as 2/3"

**Post-Assessment Description**
Erik continues his argument from the previous event. He shows that 1/2 is larger than 1/3, but not quite as big as 2/3. It is more like, "one third and a half." Erik means that half of one-third, or 1/6, is the difference between the fractions 1/2 and 1/3. Brian clarifies Erik's wording by stating that one-sixth is the quantity left over.
In the pre-assessment T1 describes a counterargument with the counterclaim that "the difference between 1/2 and 1/3 cannot be 1/3," where the prior claim that the difference between 1/2 and 1/3 is 1/3 is implicit. That 1/2 is "one third and a half," is noted as data for the counterclaim and supported by the warrant that, "when the 1/2 rod is lined up next to the two 1/3 rods, the 1/2 rod is bigger than 1/3 but not as big as 2/3. It is possible that "which would be equal to 1/6," is intended to be either data or a warrant, but the imprecision of language makes it difficult to determine what T1 meant, therefore, the statement does not appear in the argumentation diagram.

In the post-assessment, T1 includes additional elements of argumentation. Note, that the prior claim and the language that explicitly notes that the argument is a counterargument is not present in the post-assessment, however, this structure can be
implied by the statement, "Erik continues his argument from the previous event," since, in the previous event, T1 describes a counterargument.

The counterargument described here includes the claim that the difference between the 1/2 and 1/3 is 1/6 with the data that 1/2 is "one third of a half." These data are supported by warrants, "1/2 is larger than 13, but not quite as big as 2/3," "the difference between 1/2 and 1/3 is 'half of a third,'" and "half of a third is 1/6." The additional detail included by T1 adds to the argumentation in the post-assessment. Additionally, T1 describes a modification of the claim that the difference between 1/2 and 1/3 is 1/6 made by another student, specifically, that "one-sixth is the quantity left over" [between 1/2 and 1/3] and the statement in the pre-assessment, "which would equal 1/6," is clarified as meaning that "half of a third is 1/6," and therefore is included in the diagram.

8.1.14 Event 14

**Pre-Assessment Description**
Michael confirms that he agrees with Erik and Brian that 1/6 is the difference. He shows an argument with Jessica's model similar to the one that Brian gave with Jackie's model - he lines up 6 red rods and shows that they are equal in length to the orange and red rod that equal one whole. By this argument, he concludes that the red rods must be 1/6 in this model.
Post-Assessment Description
Michael shows his agreement with Erik and Brian that $1/6$ is the difference the class should be searching for. It has been made clear that the red rod is the difference between the dark green rod ($1/2$) and the purple rod ($1/3$). He compares the red rods to the model of "one whole" which is an orange rod plus a red rod. He lines up 6 red rods on top of the orange and red rods representing on-whole [sic], and shows that they are of equal length. This is his way of justifying that the red rod in this particular model is equal to $1/6$.

In the post-assessment, T1 connects the claims that the red rod has the number name $1/6$ and that the difference between $1/2$ and $1/3$ is $1/6$ through the statement, "Michael shows his agreement with Erik and Brian that $1/6$ is the difference the class should be searching for. It has been made clear that the red rod is the difference between

Figure 8.14. Argumentation described by Teacher 1 for Event 14

In the pre-assessment description, T1 notes a claim that "$1/6$ is the difference [between $1/2$ and $1/3$]." T1 describes the argument for another claim, that the "red rods must be $1/6$" with data, "6 red rods are equal in length to the orange and red rod train" and warrant, the premise that the orange and red rod train "equals one whole." T1 language does not connect the two claims.

In the post-assessment, T1 connects the claims that the red rod has the number name $1/6$ and that the difference between $1/2$ and $1/3$ is $1/6$ through the statement, "Michael shows his agreement with Erik and Brian that $1/6$ is the difference the class should be searching for. It has been made clear that the red rod is the difference between
the dark green rod (1/2) and the purple rod (1/3)." Thus, the structure of the argumentation is more connected in the post-assessment.

8.1.15 Event 15

**Pre-Assessment Description**
Brian, Erik, and Michael all finish their arguments showing that 1/6 is the difference between 1/2 and 1/3. Michael makes realization at the end that "the answer is 1/6 in both cases" which shows that the students have started to understand the concept of equal proportions. While the sizes in different models of rods may be different, the proportions in the arguments remain constant.

**Post-Assessment Description**
Meredith states her agreement with Brian, Erik, and Michael. It appears that fellow peers are starting to agree and come to a conclusion that 1/2 is bigger than 1/3 by 1/6. Meredith uses a similar argument that the boys have in pervious [sic] events to show the actual difference between 1/2 and 1/3, and explains why the answer cannot be 1/3 as the girls had argued earlier on. Michael concludes the analytic by exclaiming that the answer is 1/6 in both cases.

![Figure 8.15. Argumentation described by Teacher 1 for Event 15](image)
In the pre-assessment, T1 states two claims, "the answer is 1/6 in both cases" and "1/6 is the difference between 1/2 and 1/3." There does not appear to be an argumentation link between these claims.

In the post-assessment T1 adds detail that suggests more elements of argumentation with a more connected structure. T1 mentions the prior claim that 1/2 is greater than 1/3 by 1/3 and describes a counterargument to this claim with the counterclaim that "the answer [the difference between 1/2 and 1/3] cannot be 1/3" made by Meredith. The data, which is also a claim, "1/2 is bigger than 1/3 by 1/6" supports the counterclaim and the claim/data is further supported by a general reference to a rod model that shows "the actual difference between 1/2 and 1/3." T1 includes the statement "the answer is 1/6 in both cases," but it does not make it clear as to how this statement is connected to the rest of the argumentation. It does appear, however, that it is meant to be part of the counterargument.

8.2 Summary of Teacher 1's Growth across Events

In six of the 15 events, T1 adds detail in the post-assessment description that results in more elements of argumentation being described. In Event 2, T1 adds in the post-assessment that the green rod was longer than the purple rod. This statement results in T1 describing a more complete argument in the post-assessment with more elements and structure. In Event 4, T1 states that the students, "place purple and green rods on top of each other, and show that either one red or two white rods placed next to the purple rod equals the distance of the green rod." Using the statement and the model that was not included in the pre assessment data, T1 continues to describe two claims. The first statement is that "two white rods placed next to the purple rod equals the distance of the
green rod”; the second statement is that one red rod "placed next to the purple rod equals
the distance of the green rod." In the post-assessment for Event 7, T1 adds data not
included in the pre-assessment, namely, a statement that notes the data for the claim that
"1/2 is larger than 1/3," specifically, that in the model, the dark green rod has the number
name 1, the light green rod has the number name 1/2, and the red rod as the number name
1/3. In Events 9, 11, and 13, T1 adds language that suggests the modification of a claim.
Furthermore, in Event 9, T1 adds that "the difference between the light green and the red
rod" is 1 and "the girls meant to assign the white rod, representing the difference between
the light green and red rod, the value of 1/6," providing additional elements of
argumentation to the description.

In five of the 15 events, T1 describes more argumentation structure in the post-
assessment as in the pre-assessment descriptions. As more elements of argumentation are
mentioned in the post-assessment, the description of the argumentation structure becomes
more complex, resulting in more nesting of argumentation elements. In Event 4, more
claims are mentioned and these additional claims are used as data for other claims.
Whereas in the pre-assessment, data that are mentioned serve only one purpose within the
argumentation structure, in the post-assessment, statements serve as data/claim which
results in a warrant that serves as data. The same is true in Event 6, where, for example,
the statement that two red rods are equal in length to one purple rod serves as data in the
pre-assessment description, in the post-assessment description; it is data and backing in a
sub-argument. Also in Event 6, T1 describes a counterargument and a sub-
counterargument within that counterargument in the post-assessment that is not included
in the pre-assessment, adding more connectedness and structure to the argumentation
described. In Event 10, T1 notes that "Brian attempts to disprove that the difference between 1/2 and 1/3 in the first model can be one-third," suggesting a prior claim that the difference between 1/2 and 1/3 is 1/3 and situating his argument is a counterargument to that claim. In the post-assessment in Event 11, added details result in what was only a claim in the pre-assessment being described as data for a counterclaim, resulting in statements that could be considered data/warrant and warrant/backing. The same is true in Event 14, where T1 adds detail that results, not only in more elements of argumentation, but added structure of a nested argument where the warrant is also a claim and the backing data are supported by a premise/warrant.

In four of the 15 events, T1 uses the formal mathematical register to more precisely describe the argumentation in the post-description than the pre-assessment. In Event 1, T1 indicates,"1/2 is larger than 1/3" in both the pre and post -assessments. However in the post assessment, T1 identifies that same statement as a claim. In Event 2, T1 states in the post-assessment that the rod model is "support and evidence as to why" the students' initial claim was true. In Event 4, T1 adds to the original description that "there have not been any counterclaims made to the argument that 1/2 is larger than 1/3," which adds two elements of argumentation to what was described in the pre-assessment: that there was a previous claim that "1/2 is larger than 1/3," and that no counterclaims were made. In Event 6, T1 adds the statement that "Brian makes a counterargument to the claim that 1/2 is one-third larger than 1/3. His counterclaim is that the difference between the two fractions is actually 1/6." The use of the formal mathematical register of argumentation in this statement, specifically, the use of "counterargument," situates the rest of the description as a counterargument and connects the statements that follow to
each other, as well as to the argumentation that was presented previously. Furthermore, T1's change in language from, "Jessica does not believe this new argument to be valid," in the pre-assessment to "Jessica counters Jackie," in the post-assessment serves to explicitly situate the T1's statements in this event as a counterargument.

In seven of the 15 events, T1 shows growth in the argumentation described in the post-assessment compared to the pre-assessment in a variety of other ways. More precise language is used in the post-assessment of Event 3 that serves to clarify the argumentation described. In the pre-assessment of Event 3, T1 alluded to the model and one "piece of the purple rod" and one "piece of the green rod;" In the post-assessment, T1 noted that the student names the rods as the 1/2 rod and the 1/3 rod. In Events 13 and 14, more specific language and detail in the post-assessment descriptions result in more accurate descriptions of the arguments, specifically more warrants and backing being described. The changes that T1 made from pre- to post-assessment also serve to make implicit connections explicit. For example, in Event 8, with the added statement that "Jackie justifies why 1/2 is larger than 1/3 using the modified values assigned to each rod," T1 makes the connection between elements of the data, the naming of the rods, and the claim, that 1/2 is larger than 1/3, explicit, in comparison to the pre-assessment, in which a connection was only implied. In post assessment descriptions, T1 tends to more frequently use the students' exact language to describe the argumentation, rather than provide an interpretation of the argumentation. For example, in Event 3, T1 uses students' exact language stating that the claim was, "the half was bigger than the third" rather than, in the pre-assessment, using T1's own language, "1/2 is larger than 1/3." In Event 4, the imprecision of language in the pre-assessment suggests that the students are presenting a
counterargument to the claim that 1/2 is greater than 1/3 with implied counterclaim that 1/2 is not bigger than 1/3. However, in the post-assessment, T1 uses more precise language, reflecting more of the language used by the students, resulting in the removal of this counterargument, and the description of the counterargument that is actually being made by students, that 1/2 is not a third larger than 1/3. In Event 12, T1's description in the post-assessment better describes the event activity, resulting in T1 not including in the post-assessment description the incorrect warrant included in the pre-assessment description.
Chapter 9 – Teacher 2, Event Descriptions Analyses and Summary

This chapter presents the descriptions Teacher 2 (T2) wrote for the pre-assessment and post-assessment analytic to describe the argumentation in each event. The pre-assessment and post-assessment descriptions given by the teacher are presented for each event with an accompanying diagram using Toulmin's (1958, 2003) scheme. Following the descriptions for each event, I present an in-depth analysis, summarizing the argumentation described in the pre-assessment and noting the changes the teacher made from pre- to post-assessment. Words that are key to my analysis appear in red text in the teacher’s descriptions. Note that when "the teacher" appears in the event descriptions, it is referring to the researcher in the assessment analytic event, Carolyn Maher, not Teacher 2, the teacher participant in the study.

9.1 Teacher 2 Event Descriptions and Analyses by Event

9.1.1 Event 1

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>During this time, the teacher reviews with the students what they learned last time. Both the general lesson as well as the specific example were mentioned. The teacher asks a series of yes/no questions to understand what the students are thinking (asking them to raise their hands). Then the teacher picks 2 girls to answer and explain their answers.</td>
</tr>
</tbody>
</table>

No argumentation described.

<table>
<thead>
<tr>
<th>Post-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>During this time, the teacher reviews with the students what they learned last time. Both the general lesson as well as the specific examples were mentioned. The teacher asks a series of yes/no questions to understand what the students are thinking (asking them to raise their hands). The students start out with the claim that 1/2 is bigger than 1/3 although the students do not know which one is bigger. Then the teacher picks 2 girls to answer and explain their answers.</td>
</tr>
</tbody>
</table>

| Claim: 1/2 is "bigger" than 1/3 |

Figure 9.1. Argumentation described by Teacher 2 for Event 1
In the pre-assessment description, T2 did not mention any argumentation. In the post-assessment, T2 noticed that students stated the claim, "1/2 is bigger than 1/3." T2 also included some of the formal register for argumentation in the post-assessment by using the word claim.

9.1.2 Event 2
Pre-Assessment Description
The first girl uses rods to explain her answer. She explains that one third is greater than one half because when you take away 1/2 from a whole, what is left over is less than when you take away 1/3. Therefore 1/3 is greater. When she is asked to further elucidate what each kind of rod means, she has trouble.

Post-Assessment Description
The first girl first uses rods, and defines them to explain her answer. She, contrary to what was said in the last analytic [event], claims and explains that one third is greater than one half because when you take away 1/2 from a whole, what is left over is less than when you take away 1/3. Therefore 1/3 is greater. When she is asked to further elucidate what each kind of rod is, she has trouble.
In the second event, the students state that "1/3 is bigger than 1/2." When taken in context of Event 1 and the following events, it is likely that students were providing evidence that 1/2 is greater than 1/3 and used imprecise language when making their claim. The description for this event indicates that T2 believes that the students are now claiming that 1/3 is greater than 1/2 and describes data and a warrant that he believes the student used to support this claim. Since it is unclear from the actual event or transcript that this was the argument put forth by the students and the elements do not agree with my analyses of the argumentation (See Chapter 4, Event 2), the data and warrant boxes and text are gray. The gray information indicates that T2 meant for the statements to be elements of the students' argumentation, but the statements are not elements of argumentation actually put forth by students.

When the pre-assessment description is compared to the post-assessment description, it is evident that T2 maintains the idea that the students are defending the claim 1/3 is greater than 1/2. Note, however, that in the post-assessment, T2 adds, "contrary to what was said in the last analytic [event]." By using the language "contrary to" T2 situates the student's argument in this event as a counterargument to the claim in the first event that 1/2 is greater than 1/3, thus showing growth by adding connectivity and structure to the existing description of the argumentation. Notice also, that T2's precision of language changes. In the pre-assessment T2 states that the student "explains that one third is greater than one half;" in the post-assessment, T2 states that the student "claims and explains that one third is greater than one half," acknowledging that the student's statement is a claim.

9.1.3 Event 3
**Pre-Assessment Description**
Rather than build three wholes with different rods, this student simply takes a 1/3 rod and compares its length with a 1/2 rod. She explains that 1/3 < 1/2 because the 1/3 rod is shorter.

**Post-Assessment Description**
Audra proves her claims by showing just the 1/2 rod and just the 1/3 rod. By putting them together, she argues that 1/2 rod is clearly longer than the 1/3 rod. Her explanation differs from the previous groups' as she does not build wholes with the different rods.

---

*Figure 9.3. Argumentation described by Teacher 2 for Event 3*

In this event, the elements and structure of the argumentation the teacher describes are the same. However, in the post-assessment, T2 provides more detail in the statements. Whereas in the pre-assessment, the datum is "the 1/3 rod is shorter," in the post-assessment, the datum statement is more complete, "1/2 rod is clearly longer than the 1/3 rod." In the post-assessment, T2 uses the student's name. Additionally, when T2 states that, "Audra proves her claims," he uses the formal mathematical register to situate the argumentation described.
9.1.4 Event 4

**Pre-Assessment Description**

After explaining that $1/3 < 1/2$, the student is asked how much bigger the $1/2$ is than the one third. The students uses [sic] red blocks ($1/6$) blocks to answer this question. They see that 2 red blocks equal a $1/3$ rod and the $1/3$ rod+$1/6$rod=$1/2$. By seeing this, they claim that $1/2$ is $1/3$ greater than $1/3$. They have the right idea but not the right answer

![Diagram showing the argument structure]

**Post-Assessment Description**

After explaining that $1/3 < 1/2$, the student is asked how much bigger the $1/2$ is than the one third. The students uses red blocks ($1/6$) blocks to answer this question. They see that 2 red blocks equal a $1/3$ rod and the $1/3$ rod+$1/6$rod=$1/2$. By seeing this, they claim that $1/2$ is $1/3$ greater than $1/3$. They have the right idea but not the right answer

No change from pre to post.

*Figure 9.4. Argumentation described by Teacher 2 for Event 4*

In the pre-assessment, T2 describes an argument with the claim, "$1/2$ is greater than $1/3$" and data, "the $1/3$ rod + $1/6$ rod = $1/2$" and "2 red blocks equal a $1/3$ rod." The text of the pre-assessment and post-assessment are the same for this event.

9.1.5 Event 5
Pre-Assessment Description
The discussion continues as another student answers the question: "how much greater is 1/2 than 1/3?" At the end, one student points out that when they use blocks that are different from what they started off with, they are redefining the problem.

Post-Assessment Description
Kelly comes up to the board to explain why she agrees with the other girls. The discussion continues as another student answers the question: "how much greater is 1/2 than 1/3?" At the end, Brian points out that they are using blocks that are different from what they started off with (dark greens/light greens). When clarifying the students' answers, they say "it is one third of the red."

In the pre-assessment, T2 notes two elements of argumentation, a claim and data. T2 states, "one student points out that when they use blocks that are different from what they started off with, they are redefining the problem." The use of "when" in this statement connects the claim and the data.
The description in the post-assessment shows in growth in the elements and structure of the argumentation described from the pre-assessment. T2 states that, "The discussion continues as another student answers the question: 'how much greater is 1/2 than 1/3?'" The rest of the statements in T2's description apply to the answer to this question, i.e., that 1/2 is 1/3 greater than 1/3. Therefore, it can be noted that T2's statement implies a reference to the claim "1/2 is 1/3 greater than 1/3." Since the claim is not explicitly stated by T2, it is noted in the diagram with a dashed-line. The use of the language, "clarifying the students' answers" situates the statement, "Brian points out that they are using blocks that are different from what they started off with (dark greens/light greens)" as a counterargument to the girls' argument, requiring them to clarify. The statement, "When clarifying the students' answers, they say "it is one third of the red," also situates the statement, "it is one third of the red," as data that the girls added to support their claim.

9.1.6 Event 6
Pre-Assessment Description

The students return to the original rods and blocks made by the first girl (where the rods are lined up to create one whole in different ways). One boy realizes that half of a third is a sixth and that one third plus one sixth is one half. He argues that because two of the smaller rods add up to 1/3, the smaller rods are each a sixth (or half of one-third).

Figure 9.6. Argumentation described by Teacher 2 for Event 6

Post-Assessment Description

Brian claims that if "you split one of the thirds in half which is a sixth, then you have a sixth bigger" The students return to the original rods and blocks made by the first girl (where the rods are lined up to create one whole in different ways). Brian argues that because two of the smaller rods add up to 1/3, the smaller rods are each a sixth (or half of one-third).

From pre-assessment to post-assessment, it is evident that T2 included additional elements and showed growth in their description of the structure of the argumentation put forth by the students. In the pre-assessment, T2 describes that the student made two
claims, 1. "half of a third is a sixth;" and, 2. "one third plus one sixth is one half." The use of "and" between these statements implies that there is no connection between them. The teacher uses "because" in the next statement, making a connection evident: "He argues that because two of the smaller rods add up to 1/3, the smaller rods are each a sixth (or half of one-third)." The fact that two of the smaller rods add up to 1/3 supports the ideas that the smaller rods are each a sixth, and the smaller rods are each half of one-third. Note, the use of "or" in this statement, "the smaller rods are each a sixth (or half of one-third)" suggests that there is no connection between the statements "the smaller rods are each a sixth" and "[the smaller rods are each] half of one-third."

The changes in the post-assessment reflect more precision—T2 uses the student's name and describes the statement made as a "claim"—and additional elements of argumentation. T2 states that Brian's claims that, "if you split one of the thirds in half which is a sixth, then you have a sixth bigger," thus an additional datum and claim are given.

Note that the rest of the description is the same. However, since the remaining argumentation is supporting the idea that half of a third is a sixth, it is likely that T2 intends the original text to be additional support for the new claim and datum, in the form of warrants and backing. Since it the connection is not explicitly stated, the connection between the two arguments is notated with a dotted line. Furthermore, "Warrant/Claim" and Backing/Data labels are used for the elements indicating that the original statements could be two claims and data, or they could be considered two warrants and a backing for the new claim and data.

9.1.7 Event 7
Pre-Assessment Description
The students are asked to redefine what each rod equals. Through this, they realize that the "one half" previously defined in event 5 is not the same one half that they started with at the beginning.

Post-Assessment Description
The students are asked to redefine what each rod equals. One girl remarks "Oh, they're making different sized candy bars". Through this, they realize that the "one half" previously defined in event 5 is not the same one half that they started with at the beginning.

In the pre-assessment, T2 notes that the students claim that the one half has been defined two different ways. In the post-assessment, T2 adds an additional element of argumentation and more structure when he states that it is because the students are "making different sized candy bars" that they have defined one half differently.

9.1.8 Event 8
Pre-Assessment Description
The girls move away from the originally proposed one whole and one half. They begin to use the scale introduced in event 5 to create their argument. This time, they claim that \( \frac{1}{2} > \frac{1}{3} \) (as opposed to event 2).

![Diagram of Pre-Assessment Description]

Post-Assessment Description
Colored rods are redefined since reds are being called both \( \frac{1}{3} \) and \( \frac{1}{6} \) at the same time. The girls move away from the originally proposed one whole and one half. They begin to use the smaller scale introduced in event 5 to create their argument. This time, they claim that \( \frac{1}{2} > \frac{1}{3} \) (as opposed to event 2).

![Diagram of Post-Assessment Description]

As has been evidenced in the descriptions thus far, this teacher has a persistent belief that in Event 2 the students were claiming that \( \frac{1}{3} \) was greater than \( \frac{1}{2} \), rather than using imprecise language. This belief surfaces again in the description of this event. In the pre-assessment description, T2 states that the students are making a claim that \( \frac{1}{2} \) is greater than \( \frac{1}{3} \) and this claim is in opposition to the claim made in Event 2. The teacher states: "This time, they claim that \( \frac{1}{2} > \frac{1}{3} \) (as opposed to event 2)." The use of the language, "This time," and "opposed to event 2" suggest that the claim made here is a counterclaim to what was claimed in Event 2. Notice that the box for the prior claim is...
dashed since the claim that was made in Event 2 (that 1/3 is greater than 1/2) was alluded to by T2, but not explicitly stated.

In the post-assessment description, T2 added some additional information that is significant to the arguments being presented by the students: that the red rods, "are being called both 1/3 and 1/6 at the same time."

9.1.9 Event 9

**Pre-Assessment Description**
A student is called to explain what he means by 1/6. Previously a student had explained 1/6 as half of 1/3. This time, this student explains that because he lined 6 of the white blocks to create one whole, the white blocks equal 1/6.

**Post-Assessment Description**
Eric is called to explain and clarify what the girls are defining as one whole and one six[th]. he means by 1/6 [sic]. Previously a student had explained 1/6 as half of 1/3. This time, this student explains that because he lined 6 of the white blocks to create one whole, the white blocks equal 1/6.

*Figure 9.9. Argumentation described by Teacher 2 for Event 9*

The teacher connects a prior claim that "1/6 [is] … half of 1/3" to the argumentation event and describes that the student makes a claim and presents data. The use of "because" shows that the data and claim are linked.
In the post-assessment, T2 adds the name of the student and some additional detail, "Eric is called to explain and clarify what the girls are defining as one whole and one six[th]." With this statement, T2 includes two previous claims: that the unit, "one whole," was previous defined, and that one sixth was previously defined. This statement situates the rest of the description, which is unchanged, as a clarification of those claims, adding additional structure to the argumentation presented.

9.1.10 Event 10

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>The teacher uses the original model and asks students to reconcile their current argument with what they presented beforehand.</td>
</tr>
<tr>
<td>No argumentation described.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Post-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>The teacher uses the original model and asks students to reconcile their current argument with what they presented beforehand. More concisely, she is asking the students &quot;what is the red block?&quot; which they originally used in their explanation. Again, they say that one half is bigger than a third by a third.</td>
</tr>
</tbody>
</table>

Claim: "one half is bigger than a third by a third"

Figure 9.10. Argumentation described by Teacher 2 for Event 10

The pre-assessment description does not contain any elements of argumentation. The post-assessment description includes one element of argumentation: that the students repeat the previous claim that "one half is bigger than a third by a third."

9.1.11 Event 11
Pre-Assessment Description
The student who first proposed that \( \frac{1}{2} \) is greater than \( \frac{1}{3} \) by a sixth is still adamant that his answer is correct. He continues to use his original argument that it is greater by one sixth because the red rod is half of a third.

Post-Assessment Description
"Brian, who first proposed that \( \frac{1}{2} \) is greater than \( \frac{1}{3} \) by a sixth is still adamant that his answer is correct. He continues to use his original argument that it is greater by one sixth because the red rod is half of a third. He demonstrates that the red rod is half of a third by placing a red rod on top of a third. At the end, a student says "I think that they both might be answers"

Figure 9.11. Argumentation described by Teacher 2 for Event 11

In the pre-assessment, T2 presents the claim that "\( \frac{1}{2} \) is greater than \( \frac{1}{3} \) by a sixth" and data given by the student, that the "red rod is half of a third." The word
"because" links the data and the claim. Note that T2 mentions that the student is "still adamant that his answer is correct," but makes no mention about any opposing views, claims, or arguments, so that statement is not included in the argumentation diagram.

In the post-assessment, T2 includes the student's name, and adds detail that expands the student argumentation that was noticed. The teacher includes the original claim and data, but adds a warrant that links the data and claim, specifically, that the red rod is placed on top of a third. The teacher connects this element to the data: "He demonstrates that the red rod is half of a third by placing a red rod on top of a third."

The final statement also adds an element of argumentation as well as structure. The teacher states "a student says 'I think that they both might be answers.'" The inclusion of this statement suggests the acknowledgement of a prior claim that differs from the one being made by Brian in this event, thus situating T2's description of Brian's argumentation as a counterargument and making his claim that 1/2 is greater than 1/3 by 1/6 a counterclaim. Also implied by this statement is that the claim that "they can both be answers" is a modification of Brian's claim, as well as a prior claim. Note that "modification" and the prior claim are in dashed boxes since they are implied by T2's descriptions, but not explicitly stated.

**9.1.12 Event 12**
Pre-Assessment Description
A student begins to attempt to explain why a half is greater than a third by a third but then he ends up explaining how it is impossible for a half to be greater than one third by another third. He makes the argument that one half is not as big as two thirds.

Post-Assessment Description
Eric questions: "how can a half be bigger than a third by a third?" He continues to argues that this cannot be the case as a half is nearly two-thirds. He says that it is a half of a third bigger than a third.

The teacher notices two claims in the pre-assessment, one that "a half is greater than a third by a third" and one "it is impossible for a half to be greater than one third by another third." Although these are counterclaims, T2 does not state this, so the claims are not linked. Additionally, T2 states that the student, "makes argument that one half is not as big as two thirds." It is unclear whether T2 is connecting this statement as data for the claim that "it is impossible for a half to be greater than one third by another third." The
teacher states that the student "makes the argument that one half is not as big as two thirds," so it is possible T2 sees this as data, or evidence for the claim. If not, it is simply another claim. Because of the uncertainty, the link between the two elements and the box for "because" are dashed and the statement in question is labeled "Data/Claim."

The post-assessment includes more elements of argumentation, as well as more structure than the pre-assessment, showing growth in T2's noticing of argumentation. By including Erik's question, "how can a half be bigger than a third by a third?" T2 situates the description of the argumentation that is to follow as a counterargument to the prior claim that 1/2 is greater than 1/3 by 1/3, making the claim that 1/2 is not greater than 1/3 by 1/3 a counterclaim. However, since these elements are not explicitly stated by T2, but implied, based on T2 noting the question posed by the student, they are represented in the diagram as dashed boxes. The teacher includes an additional element of argumentation, a warrant that connects the data to the claim: "a half is nearly two-thirds." In language used by T2, "He continues to argues [sic] that this cannot be the case as a half is nearly two-thirds," the word "as" makes the link explicit.

9.1.13 Event 13
Pre-Assessment Description
"A student shows the girls what their answer "one half is greater than one third by one third" really looks like visually. The second boy explains that one half is between one third and two thirds so it \([1/2]\) has to be one third and a "half".

```
Data: "one half is between one third and two thirds"

Claim: "one half is greater than one third by one third"
```

```
so
```

```
Claim: "it \([1/2]\) has to be one third and a 'half'"
```


Post-Assessment Description
Brian shows the girls what their answer "one half is greater than one third by one third" really looks like visually. He teels [sic] them that if you add a third to a third, you do not get one half, you get more than a half. Eric explains that one half is between one third and two thirds so it has to be "one third and a half".

```
Data: "one half is between one third and two thirds"

Claim: "one half is greater than one third by one third"
```

```
so
```

```
Claim: "it \([1/2]\) has to be one third and a 'half'"
```

```
Warrant: "if add a third to a third, you do not get one half"
```

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Warrant: "if add a third to a third, you get more than one half"
```


Figure 9.13. Argumentation described by Teacher 2 for Event 13

The teacher describes two claims in the pre-assessment, one connected to data and the other not connected. Note that in the pre-assessment description, the first statement, "A student shows the girls what their answer 'one half is greater than one third by one third' really looks like visually." And the second statement, "The second boy explains that one half is between one third and two thirds so it \([1/2]\) has to be one third and a 'half'," are described as being stated by two different students, thus shrouding any intent to link
these statements. The link between the data and claim in the second statement, however, is evident in T2's use of the word "so."

In the post-assessment, T2's addition of the detail that the same student, Brian, makes both statements alludes to a connection between the two statements. The argumentation described by T2, then, can be interpreted as being a counterclaim to the claim made by "the girls" that "one half is greater than one third by one third." This connection is also suggested by the inclusion of the language, "if you add a third to a third, you do not get one half," implying that this counters the previous claim. Since the counterargument connection is implied, rather than explicitly stated, it is appears in the diagram as dashed lines. The teacher includes more of the student's argument in the post-assessment description, again, showing growth. In addition to the claim and the data, T2 includes the statement, "if you add a third to a third, you do not get one half, you get more than a half," which adds two warrants that connect the data to the claim: 1. if you add a third to a third, you do not get a half, and 2. if you add a third to a third, you get more than one half.

9.1.14 Event 14
Pre-Assessment Description
The last student shows that six red rods equal a whole. This means that each little red rod is a sixth of a whole. This explanation had been previously made in event 9 but at that time, they used a smaller scale model and did not use the originally proposed model.

Post-Assessment Description
Michael shows that six red rods equal a whole. This means that each little red rod is a sixth of a whole. This explanation had been previously made in event 9 but at that time, they used a smaller scale model and did not use the originally proposed model.

No change in the post-assessment description.

Figure 9.14. Argumentation described by Teacher 2 for Event 14

The teacher describes the same claim and data in both the pre- and post-assessments and links them using the language, "this means that." T2 also includes some additional information related to the argumentation, that the argument being made here in Event 14 was similarly made in Event 9 using different rods. Other than adding the name of the presenting student, the pre- and post-assessments are the same.

9.1.15 Event 15
Pre-Assessment Description
A student agrees that one half is greater than a third by one sixth by showing that one half plus a third equals two thirds.

Post-Assessment Description
A student agrees that one half is greater than a third by one sixth by showing that one half plus a third equals two thirds.

No change in the post-assessment description.

9.2 Summary of Teacher 2's Growth across Events
In eight of the 15 events, T2 makes changes from pre-assessment to post-assessment result in the description of more elements of argumentation. In Event 1, no argumentation is described in the pre-assessment, however, in the post-assessment, T2 states, "1/2 is bigger than 1/3," which notes a claim. In Event 5, the changes in the language T2 uses in the post-assessment clarify and add to the argument described in the post-assessment. The pre-assessment argument with the claim "they are redefining the problem," supported by data, "they use the blocks that are different from what they started off with," changes to an argument with the implied claim that 1/2 is greater than 1/3 supported by "it is one third of the red," and challenged by the counterclaim that...
"they are using blocks that are different from what they started off with (dark greens/light greens)." Thus, T2 describes two more elements of argumentation in the post-assessment than the pre-assessment. In Event 6, T2 states that Brian claims that, "if you split the thirds in half which is a sixth, then you have a sixth bigger," which describes an additional claim and data. In Event 7, with the statement that it is because the students are "making different sized candy bars" that they have defined one half differently, T2 describes an additional data and in Event 9, T2 notes two additional prior claims to the argumentation described. Furthermore, in Event 10, T2 does not note any argumentation in the pre-assessment, but notes a claim in the post-assessment and in Event 11, the inclusion of the question, "how can a half be bigger than a third by a third?" situates the description of the argumentation that is to follow as a counterargument to the prior claim that 1/2 is greater than 1/3 by 1/3, making the claim that 1/2 is not greater than 1/3 by 1/3 a counterclaim. Additionally, T2 describes a warrant in the post-assessment of this event that is not mentioned in the pre-assessment. The additional text added to the post-assessment description in Event 13 adds warrants to support the data that "one half is between one third and two thirds," and claim, "it [1/2] has to be one third and a 'half.'"

Changes evident in the post-assessment description add structure to the argumentation T2 describes in eight of the 15 events. In Event 2, the addition of the statement, "contrary to what was said in the last analytic [event]," situates the student's argument described in the event as a counterargument to the claim in the first event that 1/2 is greater than 1/3. In the post-assessment of Event 5, the changes in the description add structure by situating the statements made as a counterargument and in Event 6, T2
changes alter the structure of the argument described from three unrelated claims with two supported by data, to a claim supported by data, warrants, and backing. In Event 7, the description of data adds structure, as it makes a connection between the claim, "the 'one half' previously defined in event 5 is not the same one half that they started with" and the data, "they're making different sized candy bars." In Event 9, T2 adds the statement, "Eric is called to explain and clarify what the girls are defining as one whole and one six[th]," which adds structure by situating the rest of the description as a clarification of the two new claims described. In Event 11, T2 notes an implied prior claim which situates the claim that 1/2 is greater than 1/3 by 1/6 and its support as a counterargument, and the claim that "they might both be answers," as a modification of the prior claim. The changes made in Event 12 include the structure of a counterargument and in Event 13, T2 adds detail that implies the argument described in the pre-assessment is connected to the prior claim as a counterargument.

T2's use of the mathematical register showed growth from the pre- to post-assessment in four of the 15 events. In Event 1, T2 specifically identifies that "1/2 is bigger than 1/3," as a "claim" and in Event 2, in the pre-assessment, T2 states that the student "explains that one third is greater than one half;" in the post-assessment, T2 changes this to, "claims and explains that one third is greater than one half," acknowledging the student's statement specifically as a claim. In the post-event description of Event 3, T2 notes, "Audra proves her claims," using the formal mathematical register to situate the other statements in the description as an argument that supports a claim. In Event 6, T2 changes the language to use "claim" to describe the statement made by the student.
T2 showed growth from pre- to post-assessment in one other way. In Event 3, T2 changes statement with data from "the 1/3 rod is shorter," to the more complete statement, "1/2 rod is clearly longer than the 1/3 rod." In Event 8, T2 adds the statement that the red rods, "are being called both 1/3 and 1/6 at the same time," which, although not student argumentation itself, adds information that is relevant to the argumentation presented in the event.
Chapter 10 – Teacher 3, Event Descriptions Analyses and Summary

This chapter presents the descriptions Teacher 3 (T3) wrote for the pre-assessment and post-assessment analytic to describe the argumentation in each event. The pre-assessment and post-assessment descriptions given by the teacher are presented for each event with an accompanying diagram using Toulmin's (1958, 2003) scheme.

Following the descriptions for each event, I present an in-depth analysis, summarizing the argumentation described in the pre-assessment and noting the changes the teacher made from pre- to post-assessment. Words that are key to my analysis appear in red text in the teacher’s descriptions.

10.1 Teacher 3 Event Descriptions and Analyses by Event

10.1.1 Event 1

Pre-Assessment Description
The students are asked if they know which is bigger, 1/2 or 1/3. Almost all the students have an idea of what is correct. The students are asked to justify why their answer of 1/2 being bigger than 1/3 is true. They are also asked by how much bigger is 1/2 bigger than 1/3.

Post-Assessment Description
Based on their in class activity, the students are asked if they know which fraction is bigger, 1/2 or 1/3. Almost all the students have an idea of what is correct. The students are then asked if they will be able to justify why one is bigger than the other. They are also asked by how much bigger is 1/2 bigger than 1/3. The stage is set for an argumentation to take place between people who have different answers to, "by how much bigger?".

Figure 10.1. Argumentation described by Teacher 3 for Event 1
In the pre-assessment, T3 notes that the students make a claim that 1/2 is "bigger" than 1/3 in the statement: "The students are asked to justify why their answer of 1/2 being bigger than 1/3 is true."

In the post-assessment description, T3 adds detail, but rephrases the statement containing the claim to: "The students are then asked if they will be able to justify why one is bigger than the other," thus describing a new, less specific claim, that one fraction is bigger than the other. However, from the next sentence, "They are also asked by how much bigger is 1/2 bigger than 1/3," the claim that 1/2 is bigger than 1/3 is still implied. Since this claim was not explicitly stated by T3, the box is dashed.

10.1.2 Event 2
Pre-Assessment Description
Jessica is trying to prove that $\frac{1}{2}$ is bigger than $\frac{1}{3}$. When compared to a one whole rod (first row), the one whole rod that is made up of three pieces (second row) is placed next a whole rod made up of two pieces (third row), the $\frac{1}{2}$ piece is bigger than the $\frac{1}{3}$ piece so $\frac{1}{2}$ is bigger than $\frac{1}{3}$. She is trying to use the manipulatives to prove her answer and use them to back up her reasons as to why she named the row two rods one-third rods and the row three rods the one-half rods. At this point is takes her a while to understand why she named them as such.

Post-Assessment Description
Jessica is trying to prove that $\frac{1}{2}$ is bigger than $\frac{1}{3}$. When compared to a one whole rod (first row), the one whole rod that is made up of three pieces (second row) is placed next a whole rod made up of two pieces (third row), the $\frac{1}{2}$ piece is bigger than the $\frac{1}{3}$ piece so $\frac{1}{2}$ is bigger than $\frac{1}{3}$. She is trying to use the manipulatives to prove her answer and use them to back up her reasons as to why she named the row two rods one-third rods and the row three rods the one-half rods. At this point is takes her a while to understand why she named them as such. Through the demonstration in front of the class, the, Jessica and Laura try to argue their claim. [sic] through questioning, Dr. Maher is opening up discussion among the class.

Argumentation described in the post-assessment is the same except for the use of "claim."

Figure 10.2. Argumentation described by Teacher 3 for Event 2

T3 begins the pre-assessment description of this event situating the description by stating that "Jessica is trying to prove that $\frac{1}{2}$ is bigger than $\frac{1}{3}$." Thus, the statements
that follow support that claim. T3 links the claim that 1/2 is bigger than 1/3 to the data that the 1/2 piece is bigger than the 1/3 piece through the use of "so," noting that the student uses the rods or "pieces" to back up the reasons for her claim. T3 includes three warrants, one of which is the premise that the first row of rods has the "one whole rod," and includes descriptions of the rod model as backing.

T3 adds some detail in the post-assessment and identifies the statement the girls made through the use of the word claim. The detail does not add to the structure or elements of argumentation described by T3.

10.1.3 Event 3

Pre-Assessment Description
A student is asked to prove why she agreed with Laura and Jessica's presentation. She uses the one-half and one-third rods labeled by the other students, places them next to one another and states that since the one-half rod is bigger than one-third rod, 1/2 is bigger than 1/3.

Post-Assessment Description
Audra is asked to prove why she agreed with Laura and Jessica's presentation. She uses the one-half and one-third rods labeled by the other students, places them next to one another and states that since the one-half rod is bigger than one-third rod, 1/2 is bigger than 1/3. By allowing another student to come up to present, students are able to see another way to approach the problem.

No changes to the post-assessment.

Figure 10.3. Argumentation described by Teacher 3 for Event 3

In the pre-assessment, T3 describes a claim made by a student, 1/2 is bigger than 1/3" and data that support the claim, "the one-half rod is bigger than one-third rod," as well as a warrant that connects the data and the claim, "She uses the one-half and one-
third rods labeled by the other students, places them next to one another." T3 explicitly
connects the data and claim in the statement, "… and states that since the one-half rod is
bigger than one-third rod, 1/2 is bigger than 1/3," through the use of the word, "since." In
the post-assessment, T3 includes the student's name, Audra, and some detail. The detail
does not change the student argumentation that is described.

10.1.4 Event 4

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>The students presenting have a hard time keeping labels to the rods. Now to figure out how much bigger 1/2 is compared to 1/3, the girls claim that since 1/2 rod can be made up of three smaller rods of equal size, when comparing the original 1/3 rod to the 1/2 rod made up of now the smaller 1/3 rods, the 1/3 rod is missing a smaller 1/3 rod so 1/2 is 1/3 bigger than 1/3.</td>
</tr>
</tbody>
</table>

Post-Assessment Description
Jessica, Laura, and Audra, while presenting, have a hard time keeping labels of the rods. Now to figure out how much bigger 1/2 is compared to 1/3, the girls claim that since 1/2 rod can be made up of three smaller rods of equal size, when comparing the original 1/3 rod to the 1/2 rod made up of now the smaller 1/3 rods, the 1/3 rod is missing a smaller 1/3 rod so 1/2 is 1/3 bigger than 1/3.

No change in the post-assessment description.

Figure 10.4. Argumentation described by Teacher 3 for Event 4

T3 describes the argument of the students in this event has having a claim, "1/2 is 1/3 bigger than 1/3, data, "when comparing the original 1/3 rod to the 1/2 rod made up of now the smaller 1/3 rods, the 1/3 rod is missing a smaller 1/3 rod," that support the claim, and a warrant, the "1/2 rod can be made up of three smaller rods of equal size," that links
the data and the claim. The connection between the data and the claim is evident in the statement, "the 1/3 rod is missing a smaller 1/3 rod so 1/2 is 1/3 bigger than 1/3," through the use of the word "so." Other than the inclusion of the students' names, there is no change in the argumentation presented in the pre-assessment and the post-assessment.

### 10.1.5 Event 5

**Pre-Assessment Description**
Kelly presents why she agrees with the other three students and tries to justify with different colored rods. Brian disagrees because Kelly used different colored rods compared to Jessica and Laura. By using different colored rods, some students may feel that the problem has been changed and therefore the solution would change as well. As long as Kelly's argument is similar to Jessica and Laura's but uses different colored rods, the result will not be changed. The students can learn that different argumentation can lead to the same result.

**Post-Assessment Description**
Kelly presents why she agrees with the other three students and tries to justify with different colored rods. Brian disagrees because Kelly used different colored rods compared to Jessica and Laura. By using different colored rods, some students may feel that the problem has been changed and therefore the solution would change as well. As long as Kelly's argument is similar to Jessica and Laura's but uses different colored rods, the result will not be changed. The students can learn that different argumentation can lead to the same result.

No change in the post-assessment description.

*Figure 10.5. Argumentation described by Teacher 3 for Event 5*
T3 links Brian's argument with a prior claim that used different rods, situating it as a counterargument, and the statement, "the solution has been changed" as a counterclaim. T3 notes that the prior claim was in agreement, and thus linked, to an even earlier claim. T3 describes Brian's claim as having a counterclaim, "the solution would change," supported by data, "the problem has been changed," linked back to the claim through the warrant, "they are using different color rods." T3 adds detail, but the detail does not change the argumentation structure or add any argumentation elements.

10.1.6 Event 6

**Pre-Assessment Description**
Brian believed that 1/2 is 1/6 bigger than 1/3 because six small pieces make up the whole rod. When the whole rod is split in half, row 2 now has a 1/3 rod a half of 1/3 rod and the third row has one half rod. Then when you compare the half of row 2 with half of row 3, you can see that the half is bigger by a sixth because row 1 was made up of six small pieces and the 2 small pieces make up the 1/3 rod while 3 small pieces make up the half-rod. So compared to the whole rod made up of six pieces, the 1/3 rod is missing a 1/6 piece to complete a 1/2 rod.

**Post-Assessment Description**
Brian believed that 1/2 is 1/6 bigger than 1/3 because six small pieces make up the whole rod. He engages in evidence backing by stating that when the whole rod is split in half, row 2 now has a 1/3 rod a half of 1/3 rod and the third row has one half rod. Then when you compare the half of row 2 with half of row 3, you can see that the half is bigger by a sixth because row 1 was made up of six small pieces and the 2 small pieces make up the 1/3 rod while 3 small pieces make up the half-rod. So compared to the whole rod made up of six pieces, the 1/3 rod is missing a 1/6 piece to complete a 1/2 rod.
up of six pieces, the 1/3 rod is missing a 1/6 piece to complete a 1/2 rod.

The post-assessment language changed to include "evidence backing."

![Figure 10.6. Argumentation described by Teacher 3 for Event 6](image)

In the pre-assessment, T3 describes that Brian made a claim that 1/2 is 1/6 bigger than 1/3 and supports it with a variety of evidentiary statements. It is unclear as to how T3 intended to organize the statements, though the statements are linked with words such as "because," and "so." The diagram provides a suggested mapping of the argumentation, although other mappings might be suggested. The claim that 1/2 is 1/6 greater than 1/3 is ultimately supported by the statement that, "the 1/3 rod is missing a 1/6 piece to complete a 1/2 rod." The connection of these data to the claim is made through several warrants, namely that, "When the whole rod is split in half, row 2 now has a 1/3 rod a half of 1/3 rod and the third row has one half rod" and "when you compare the half of row 2 with half of row 3, you can see that the half is bigger by a sixth." This second statement is further supported by the statements that "2 small pieces make up the 1/3 rod," "3 small pieces make up the half-rod" and "six small pieces make up the whole rod." Changes in the post-assessment indicate that T3 is using more of the formal mathematical register and situating the evidentiary statements as "evidence" and "backing."
10.1.7 Event 7

**Pre-Assessment Description**
Jessica states that the other students cannot change the color of the half rod from green to white because it is not fair in the size. The green rod is bigger than the white rod so it will not work to compare using different size rods.

![Argumentation described by Teacher 3 for Event 7](image)

**Post-Assessment Description**
Jessica states that Jackie cannot change the color of the half rod from green to white because it is not fair in the size. The green rod is bigger than the white rod so it will not work to compare using different size rods.

No change in the post-assessment description.

*Figure 10.7. Argumentation described by Teacher 3 for Event 7*

The argumentation in the pre-assessment includes the claim that the "students cannot change the color of the half rod from the green to the white." Although T3 notes this statement as a claim, the students did not make this claim. In the event, it is the dark green rod and the light green rod that have been given the number name 1/2. Thus, the box and the text for the claim are gray. T3 notes that the students support this claim with the data, "it is not fair in the size," and links the data with the claim using the warrant,
that they are comparing "using different size rods." T3 notes that the student backs up this warrant by explaining that "green rod is bigger than the white rod." Note that this element is in gray because, although it is true that the green rod is longer than the white rod, the white rod has not been given the number name 1/2 in this event. Other than using the student, Jackie's, name, there are no changes from pre- to post-assessment.

10.1.8 Event 8

**Pre-Assessment Description**
The new student presenting still believes that 1/2 is bigger than 1/3 by not 1/3 or 1/6 but by 1. She has a different understanding and labeling compared to Jessica and Brian hence her interpretation is different based on her labels. Her argumentation lacks the content knowledge in the sense that instead of labeling the rod based on the pieces, she should be labeling them based on the comparison with the one-whole rod.

**Post-Assessment Description** Jackie and the other girls still believe that 1/2 is bigger than 1/3 by not 1/3 or 1/6 but by 1. She has a different understanding and labeling compared to Jessica and Brian hence her interpretation is different based on her labels. Her argumentation lacks the content knowledge in the sense that instead of labeling the rod based on the pieces, she should be labeling them based on the comparison with the one-whole rod.

**Figure 10.8. Argumentation described by Teacher 3 for Event 8**

In the pre-assessment, T3 notes that the student makes the claim that "1/2 is bigger than 1/3 by not 1/3 or 1/6 but by 1," situating her claim that 1/2 is greater than 1/3
by 1 as a counterclaim to 1/2 is greater than 1/3 by 1/6 and the counterclaim and data as a counterargument. T3 describes a general support for this counterclaim in terms of the "pieces" or rod model that the student uses. He includes some additional information relevant to the argumentation when he states that, "[the student] presenting the argument has a different understanding and labeling" and that the "argument lacks content knowledge." In the post-assessment, other than including student names, T3 does not make any changes.

**10.1.9 Event 9**

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**Pre-Assessment Description**
Eric comes up to the projector and tries to justify that the previous students [sic] does not mean that 1/2 is 1 bigger than 1/3 but actually means that 1/2 is 1/6 bigger than 1/3 because when the one pieces are lined up against their whole rod, six pieces are required. So since 1/2 was 1 1/6 piece bigger than 1/3, 1/2 is bigger than 1/3.

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**Counterargument**

- **Data:** "when the one pieces are lined up against their whole rod, six pieces are required"
- **Because**
- **Counterclaim:** they mean "1/2 is 1/6 bigger than 1/3"

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**Post-Assessment Description**
Eric comes up to the projector and tries to justify that the previous girls do not mean that 1/2 is 1 bigger than 1/3 but actually means that 1/2 is 1/6 bigger than 1/3 because when the one pieces are lined up against their whole rod, six pieces are required. So since 1/2 was one 1/6 piece bigger than 1/3, 1/2 is bigger than 1/3.

No change in the post-assessment description.

*Figure 10.9. Argumentation described by Teacher 3 for Event 9*
In the pre-assessment, T3 notes that Erik claims that the girls who claimed that "1/2 is 1 bigger than 1/3" really meant that "1/2 is 1/6 bigger than 1/3." In a sense this is both a claim (that the girls misspoke) and a counterclaim, that 1/2 is 1/6 not 1 greater than 1/3. It has been noted in the diagram as a claim counter to the prior claim, situing the rest of the elements of argumentation described as a counterargument.

T3 also states that "when the one pieces are lined up against their whole rod, six pieces are required" and that "1/2 was one 1/6 piece bigger than 1/3." It is unclear which of these statements is meant to be the warrant, and which is data. Notice, however, that T3 states:"1/2 is 1/6 bigger than 1/3 because when the one pieces are lined up against their whole rod, six pieces are required." The use of the word "because" suggests that T3 is saying that the claim (1/2 is 1/6 bigger than 1/3) is true because of the data (when the one pieces are lined up against their whole rod, six pieces are required.) T3 goes on to link the data to the claim when he states: "So since 1/2 was one 1/6 piece bigger than 1/3, 1/2 is bigger than 1/3." It might be suggested, however, that T3 intended the statement about the length of the 1/2 rod to be data and the statement about the six pieces to be the warrant. In either case, the same description of the argumentation is given in the post-assessment.

10.1.10 Event 10
**Pre-Assessment Description**
Jessica is now asked to fix the mistake he made based on other students' presentation. She now claims that since the half rod is made up of three small 1/3 pieces, the bigger 1/3 piece is missing a small 1/3s piece so the 1/2 is bigger a third. She does not realize that the small 1/3 piece is actually a 1/6 piece since she is only comparing with the half rod and not the whole rod.

**Post-Assessment Description**
Jessica is now asked to fix the mistake he made based on other students' presentation. She now claims that since the half rod is made up of three small 1/3 pieces, the bigger 1/3 piece is missing a small 1/3s piece so the 1/2 is bigger by a third. She does not realize that the small 1/3 piece is actually a 1/6 piece since she is only comparing with the half rod and not the one whole rod.

No change in the post-assessment description.

*Figure 10.10. Argumentation described by Teacher 3 for Event 10*

T3 states in the pre-assessment that the student claims that "since the half rod is made up of three small 1/3 pieces, the bigger 1/3 piece is missing a small 1/3s piece so the 1/2 is bigger [by] a third." That "1/2 is bigger [by] a 1/3" is the claim. The data, "the bigger 1/3 piece is missing a small 1/3s piece" support the claim and that "the half rod is made up of three small 1/3 pieces" provides the warrant, connecting the data and the claim. The post-assessment description is the same as the pre-assessment description.
10.1.11 Event 11

Pre-Assessment Description
After Brian goes up to reexplain why he believes 1/2 is 1/6 bigger than 1/3, Jessica suggests that maybe both her solution and Brian's solution are correct. The fact that both of these students are calling the small red piece something different is creating a confusion among them.

Post-Assessment Description
Brians tries to defend his claim that 1/2 is indeed 1/6 bigger than 1/3 but, Jessica suggests that maybe both her solution and Brian's solution are correct. The fact that both of these students are calling the small red piece something different is creating a confusion among them.

The post-assessment language changed to include "defend his claim."

Figure 10.11. Argumentation described by Teacher 3 for Event 11

T3 states that Brian made the claim that 1/2 is 1/6 bigger than 1/3 and that Jessica suggests that two claims, 1/2 is greater than 1/3 by 1/3 and 1/2 is greater than 1/3 by 1/6, might be true. The language "Jessica suggests that maybe" indicates that the statement is a conjecture because it is made with uncertainty. T3 mentions some additional information relevant to the argumentation, namely, that "both students are calling the small red piece something different."

In the post-assessment, T3 uses more of the formal mathematical register stating that Brian is trying to "defend his claim," alluding to the idea that the statements that Brian is making, although not mentioned by T3 in the description, are situated as part of an argument meant to defend the claim that 1/2 is 1/6 greater than 1/3.
10.1.12 Event 12

**Pre-Assessment Description**
Eric comes up to counter argue Jessica and Laura's work stating that how can 1/2 be 1/3 bigger than 1/3 if 2/3 is close to 1/2 but is bigger just by little bit. Now the students are starting to see that there is no way that 1/2 can be 1/3 bigger because comparing to 2/3, 1/2 should be only 1/6 bigger.

**Post-Assessment Description**
Erik comes up to counter argue Jessica and Laura's work stating that how can 1/2 be 1/3 bigger than 1/3 if 2/3 is close to 1/2 but is bigger just by little bit. Now the students are starting to see that there is no way that 1/2 can be 1/3 bigger because comparing to 2/3, 1/2 should be only 1/6 bigger. Brian also defends Erik's claim and agrees with him rather than Jessica and aura [sic].

*Figure 10.12. Argumentation described by Teacher 3 for Event 12*

In the pre-assessment, T3 describes a counterclaim, "1/2 should be only 1/6 bigger [than 1/3]" and data that support that claim, "2/3 is close to 1/2 but is bigger just
by a little bit." When the whole statement is considered, "Now the students are starting to see that there is no way that 1/2 can be 1/3 bigger because comparing to 2/3, 1/2 should be only 1/6 bigger." there in an additional idea, "comparing to 2/3" that suggests a possible warrant. Although T3's language in imprecise here, if he is referring to comparing 1/2 to 2/3, the fragment, "comparing 2/3" could be considered an warrant that connects the data to the claim. Since it is unclear what the intentions of T3 were when including that language, the warrant box and connector are dashed. Note that the use of "counter argue" situates the argument described as a counterargument, and the language, " how can 1/2 be 1/3 bigger than 1/3" indicates that it is a counterargument to the previous claim that "1/2 is bigger than 1/3."

In the post-assessment, T3 includes an additional statement, "Brian also defends Erik's claim and agrees with him rather than Jessica and aura [sic]." This statement describes additional elements and structure of the argumentation presented by the students. The language, "Brian also defends Erik's claim" indicates that Brian presented an argument that agreed with the counterargument presented by Erik. The language, "… and agrees with him rather than Jessica and [L]aura" suggests that the argument presented by Brian was a counterargument to Jessica and Laura's prior claim that 1/2 is greater than 1/3 by 1/3. T3 also included more precise language in referring to Erik's statement as a claim. Thus, in the descriptions of this event, T3 showed growth in the noticing of the students' argumentation.

10.1.13 Event 13
Pre-Assessment Description
1/3 more than 1/3 would be 2/3 so the difference [between 1/2 and 1/3] could not be 1/3 because 2/3 is more than 1/2. 1/2 is less than 2/3 but bigger than 1/3 so it [1/2] has to be between the two [1/3 and 2/3] so 1/2 is 1/3 and a 1/2 bigger. So 1/2 has to be the half of 1/3 bigger [than 1/3.]

Post-Assessment Description
Brian and Erik both are on the same page and try to prove their claim that 1/3 more than 1/3 would be 2/3 so the difference could not be 1/3 because 2/3 is more than 1/2. 1/2 is less than 2/3 but bigger than 1/3 so it has to be between the two so 1/2 is 1/3 and a 1/2 bigger. So 1/2 has to be the half of 1/3 bigger.

The post-assessment language changed to include "claim" and "prove."

Because of the imprecision of the language used, there is some uncertainty as to what T3 intended with regards to the description of the student argumentation in this event. Based on the actual events, I have used bracketed text ([ ]) to fill in missing details. Other interpretations of T3’s description could be possible if the missing text is filled in differently. Furthermore, the statement, "1/2 is 1/3 and a 1/2 bigger," is so unclear that I have not attempted to determine what T3 meant, but rather used grayed text and a gray
box to indicate that T3 is making a statement that, when read exactly, is not one the students made.

The diagram shows the mapping of T3’s description of the argumentation based on my annotations and interpretation of his statements. Two arguments are noted. In the first argument, the claim, "the difference [between 1/2 and 1/3] could not be 1/3" is supported by the data that "1/2 more than 1/3 would be 2/3," and the warrant that, "2/3 is more than 1/2" links the two. In the second argument, the claim, "1/2 has to be the half of 1/3 bigger [than 1/3]" is supported by the data that, "1/2 is between 1/3 and 2/3" and the warrant, "1/2 is less than 2/3 but bigger than 1/3" is meant to link the two. T3 uses the language, "so" and "because" to indicate the connections amongst the elements of argumentation. Note that the unclear statement,"1/2 is 1/3 and a 1/2 bigger," seems to function as a claim in this argument and the statement, "1/2 has to be the half of 1/3 bigger [than 1/3.]" might be a clearer restatement it. Because of this uncertainty, the link between the two claims is dashed and gray. In the post-assessment, T3 included more formal mathematical language with regards to argumentation, specifically, situating the argument as Brian and Erik, in agreement, trying to "prove their claim." However, T3 used the same imprecise language to describe the argumentation as in the pre-assessment.

10.1.14 Event 14
Pre-Assessment Description
Six red pieces are equivalent to one whole orange plus red rod. The small pieces would be called only 1/6 but they are equal size as the orange and red whole piece.

Post-Assessment Description
Michael now provides another reason as to why Brian and Erik are right by demonstrating that six red pieces are equivalent to one whole orange plus red rod. The small pieces would be called only 1/6 but they are equal size as the orange and red whole piece.

T3 describes two claims in the pre-assessment, that, "Six red pieces are equivalent to one whole orange plus red rod" and "the small pieces would be called…1/6." Note that the first statement is data for the second, but this connection is not made by T3. In the post-assessment, T3 adds that, "Michael now provides another reason as to why Brian and Erik are right by demonstrating that…" This statement provides context for the other statements that are made, situating "Six red pieces are equivalent to one whole orange plus red rod" as data for the claim that "the small pieces would be called…1/6," thus adding additional structure to the argumentation described.

10.1.15 Event 15
Pre-Assessment Description
Another student argues that if there is a 1/3 difference between 1/2 and 1/3, then 1/3 would exceed 1/2 and become 2/3. But we know that 1/2 is smaller than 2/3 but bigger than 1/3 so the difference has to be between the 1/3 and 2/3. The difference has to be 1/6.

Post-Assessment Description
Meredith argues that if there is a 1/3 difference between 1/2 and 1/3, then 1/3 would exceed 1/2 and become 2/3. But we know that 1/2 is smaller than 2/3 but bigger than 1/3 so the difference has to be between the 1/3 and 2/3. The difference has to be 1/6. Now more and more students are speaking up and agreeing with the three boys.

No change in post-assessment description.

In the description of the argumentation in pre-assessment, T3 notes a claim, the difference between 1/2 and 1/3 is 1/6, supported by the data that "if there is a 1/3 difference between 1/2 and 1/3, then 1/3 would exceed 1/2 and become 2/3." The warrant, "the difference has to be between the 1/3 and 2/3" connects the data and the claim, and the warrant is supported by the backing that "1/2 is smaller than 2/3 but bigger than 1/3." In the post-assessment, T3 adds the name of the student and notes that this student's argument agrees with prior students' arguments.

10.2 Summary of Teacher 3's Growth across Events
T3 clarifies the language in the post-assessment descriptions by using the formal mathematical register to describe the argumentation in five of the 15 events. In Event 2, T3 explicitly identifies the statement that "1/2 is bigger than 1/3" as a "claim" and in Event 6, identifies the supporting statements to the argument as "evidence," and "backing." In Event 3, T3 changes the statement, "After Brian goes up to reexplain why he believes 1/2 is 1/6 bigger than 1/3," from the pre-assessment, to, "Brians [sic] tries to defend his claim that 1/2 is indeed 1/6 bigger than 1/3" choosing to use the more mathematically precise "defend" and "claim." Additionally, although not explicitly stated as argumentation, this added statement alludes to the idea that the statements that Brian is making, although not mentioned by T3 in the description, are situated as part of an argument meant to defend the claim that 1/2 is 1/6 greater than 1/3. In Event 12, explicitly identifies the counterclaim, that "1/2 should be only 1/6 bigger [than 1/3]" as a "claim" and in Event 13, situates the argument described as trying to, "prove" the "claim."

In two other events, T3 shows growth in the argumentation described from pre-assessment to post-assessment. In Event 12, T3 adds the statement that, "Brian also defends Erik's claim and agrees with him rather than Jessica and aura [sic]," to the post-assessment. This statement adds an additional element, Brian's argument, as well as additional structure that this argument agrees with the argumentation described and counters a prior argument. In Event 14, T3 adds the statement that, "Michael now provides another reason as to why Brian and Erik are right by demonstrating that…” This statement provides context for the other statements that are made, identifying what was a claim in the pre-assessment description, "six red pieces are equivalent to one whole
orange plus red rod," as data for the claim that "the small pieces would be called…1/6."
The connection between these statements adds structure to the argumentation T3 describes from pre- to post-assessment.
Chapter 11 – Teacher 4, Event Descriptions Analyses and Summary

This chapter presents the descriptions Teacher 4 (T4) wrote for the pre-assessment and post-assessment analytic to describe the argumentation in each event. The pre-assessment and post-assessment descriptions given by the teacher are presented for each event with an accompanying diagram using Toulmin's (1958, 2003) scheme.

Following the descriptions for each event, I present an in-depth analysis, summarizing the argumentation described in the pre-assessment and noting the changes the teacher made from pre- to post-assessment. Words that are key to my analysis appear in red text in the teacher’s descriptions.

11.1 Teacher 4 Event Descriptions and Analyses by Event

11.1.1 Event 1

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>In this video the teacher asks the students to recall what they did during the previous day comparing the sizes of two fractions and asks a student to come to the board to try and explain why they think $1/2 &gt; 1/3$ to the class.</td>
</tr>
</tbody>
</table>

**Figure 11.1. Argumentation described by Teacher 4 for Event 1**

<table>
<thead>
<tr>
<th>Post-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>In this video the teacher asks the students to recall what they did during the previous day comparing the sizes of two fractions. The class agrees that $1/2$ is bigger than $1/3$ and the teacher asks the students to come up and convince the rest of the class.</td>
</tr>
</tbody>
</table>

T4 notes a claim in the pre-assessment description: "$1/2 > 1/3." The statement of the claim is embedded in the text, rather than explicitly stated: "...asks a student to come to the board to try and explain why they think $1/2 > 1/3$ to the class."
In the post-assessment description, T4 explicitly states the claim using more precise language: "class agrees that 1/2 is bigger than 1/3." T4 also uses more precise language to describe what students will do next, using the word "convince" rather than "explain:" "the teacher asks the students to come up and convince the rest of the class."

11.1.2 Event 2

Pre-Assessment Description
In this video the teacher asks her student to come up and use pictures to show why she claimed that 1/2 > 1/3. The teacher continues to ask her questions that determine whether she has a developed understanding of what it means to be a whole unit and what it means to be part of a whole unit.

Post-Assessment Description
In this video the teacher asks her student to come up and use pictures to show why she claimed that 1/2 > 1/3. The students then come up to the projector and use the blocks to assert their claim that 1/2 is larger than 1/3.

Figure 11.2. Argumentation described by Teacher 4 for Event 2

T4 notes the claim made by students, that "1/2 > 1/3," in the pre-assessment description. Although the post-assessment is shorter than the pre-assessment, T4 uses more precise language and includes detail that adds elements of argumentation and changes the structure of the argument noticed. T4 states: "The students then come up to the projector and use the blocks to assert their claim that 1/2 is larger than 1/3," explicitly identifying the claim, "1/2 is larger than 1/3," and noting that the student uses the block model to as data to support the claim.

11.1.3 Event 3
**Pre-Assessment Description**
In this the teacher asks the students if they agree with the argumentation that the students gave in the prior clip. When a student responds that she agrees the teacher has her justify her response to the class by going up to the front as well. She argues pictorially.

No argumentation described.

**Post-Assessment Description**
In this the teacher asks the students if they agree with the argumentation that the students gave in the prior clip. Audra responds by saying that she agrees with the girls' claim. She argues pictorially.

![Agreement with a claim]

*Figure 11.3. Argumentation described by Teacher 4 for Event 3*

In the pre-assessment T4 does not note any student argumentation. In the post-assessment, T4 shortens the description, includes the student's name, and changes the language used to include the formal mathematical register, "agreement with a claim," stating: "Audra responds by saying that she agrees with the girls' claim." The change in language adds an element of argumentation to T4's description.

**11.1.4 Event 4**
Pre-Assessment Description
Here the students up front are asked how much larger 1/2 is than 1/3. The students reason by combining their picture blocks and coming to the (incorrect) conclusion 1/3 larger.

Post-Assessment Description
Here the students up front are asked how much larger 1/2 is than 1/3. The students reason by combining their picture blocks and coming to the (incorrect) conclusion 1/3 larger. They do this because they see that three red blocks make one green block and two red blocks make up one pink block. They don't understand its relation to the whole unit though.

Figure 11.4. Argumentation described by Teacher 4 for Event 4

In the pre-assessment description, T4 notices that the students use a model with "picture blocks" [rods] to support the claim that 1/2 is larger than 1/3 by 1/3. In the post-assessment description, T4 includes more detail about the data students use to defend their claim: "They do this because they see that three red blocks make one green block and two red blocks make up one pink block." Note that the use of "because" connects the claim to these data. T4 also adds additional information relevant to the argumentation, noting that, from the students' argument, it appears that "they don't understand its relation to the whole unit." The changes in T4's description from pre to post results in the noticing of more elements of argumentation, as well as structural changes.

11.1.5 Event 5
Pre-Assessment Description
Here the teachers surveys the class and asks if 1/3 is really the difference between 1/2 and 1/3. One student agrees and one student disagrees. The student who agrees makes the argument that it was okay because they are using blocks of different colors.

![Argumentation Diagram]

Post-Assessment Description
Here the teachers surveys the class and asks if 1/3 is really the difference between 1/2 and 1/3. Kelly agrees with the girls' claim. She makes the argument that it was okay because they are using blocks of different colors.

The post-assessment language changed to include "claim."

Figure 11.5. Argumentation described by Teacher 4 for Event 5

T4 describes the agreement of one student with the prior claim that "1/3 is ... the difference between 1/2 and 1/3." The argument is the claim, it is "okay," supported by the data, "they are using blocks of different colors." "Because" connects the claim and the data. This argument, however, is not presented this way in Event 5, thus, the argumentation described by T4 is in gray text. The student who states that she agrees with the prior claim, does not say that the claim is "okay," (suggesting that she thinks the prior argument is valid), but, rather, answers affirmatively when asked if she agrees. Furthermore, the data is not a statement about the fact that, in the prior argument, the students used different color rods, but an argument that shows that 1/2 is greater than 1/3 using the light green and red rods.

In the post-assessment, T4 removes the reference to the student who disagreed, thus confirming that the rest of the description refers to the student who agrees,
eliminating the possibility that T4 meant to say, "it was not okay." T4 also adds more precise language using, "claim," explicitly identifying with what the girl is agreeing.

11.1.6 Event 6

Pre-Assessment Description
One student now comes up to the projector and displays what he thinks is the right answer. He argues that the answer is 1/6 because he notes that he can make the 1/2 panel [rod] by connecting the 1/6 panel [rod] to the 1/3 panel [rod]. He related everything back to the panel [rod] that stood for one whole unit.

Post-Assessment Description
Brian now comes up to the projector and displays what his counterclaim to the argument of the girls. He argues that the answer is 1/6 because he notes that he can make the 1/2 panel by connecting the 1/6 panel to the 1/3 panel. He related everything back to the panel that stood for one whole unit.

Figure 11.6. Argumentation described by Teacher 4 for Event 6
T4 describes a claim, "the answer is 1/6 [1/2 is 1/6 greater than 1/3]," supported by the data, you "can make the 1/2 panel [rod] by connecting the 1/6 panel [rod] to the 1/3 panel [rod]," in the pre-assessment. The reference to the unit rod, "He related everything back to the panel that stood for one whole unit," might be seen as a vague warrant meant to connect the data and claim.

T4 states in the post-assessment, "Brian now comes up to the projector and displays what his counterclaim to the argument of the girls," using more precise formal language of argumentation. Through the use of "counterclaim," T4 situates the argument presented in this event as a counterargument to the prior "argument of the girls," thus implying the prior argument for the claim that, "1/2 is one-third larger than 1/3." The changes that T4 made from pre to post add structure and elements of argumentation to the description.

11.1.7 Event 7

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**Pre-Assessment Description**
Here the students discuss why the do or don't agree with the solution provided by another student. They say that one student is using a different definition for 1 (based on a candy bar analogy) and the teacher pushes the students to keep explaining what they think that means.

Claim: "one student is using a different definition for 1"

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**Post-Assessment Description**
Here the students discuss why the do or don't agree with the claim provided by another student. They say that Brian is using a different definition for 1 (based on a candy bar analogy) and the teacher pushes the students to keep explaining what they think that means.

The post-assessment language changed to include "claim."

*Figure 11.7. Argumentation described by Teacher 4 for Event 7*
T4 notes a claim made by one student "one student is using a different definition for 1," in the pre-assessment description. In the post-assessment, T4 clarifies the language used, referencing the student by name, and using "claim."

11.1.8 Event 8

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Here the students redefine what was once 1/2 to be one whole. Once they've done that they try to show why 1/2−1/3 is (incorrectly) equal to 1/3. Through this they introduce a new tile and are asked to give a value to it.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Data: redefining what was once 1/2 to one whole</th>
</tr>
</thead>
<tbody>
<tr>
<td>Claim: &quot;1/2−1/3 is ... equal to 1/3&quot;</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Post-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Here the students redefine what was once 1/2 to be one whole. Once they've done that they try to show why 1/2−1/3 is (incorrectly) equal to 1/3. Through this they introduce a new tile and are asked to give a value to it. They are still having trouble seeing the idea of a singular unit and parts of a singular unit.</td>
</tr>
</tbody>
</table>

No change in the post-assessment description.

Figure 11.8. Argumentation described by Teacher 4 for Event 8

In the pre-assessment description, T4 mentions a claim, "1/2−1/3 is (incorrectly) equal to 1/3," noting that the claim is not correct. The data, "redefining what was once 1/2 to one whole" is also mentioned, although the link to the claim is not made. In the post-assessment, T4 adds the statement that the students, "are still having trouble seeing the idea of a singular unit and parts of a singular unit."

11.1.9 Event 9
Pre-Assessment Description
In this video another student comes up and shows the girls that their argument is really a parallel argument to what the boy Brian made. He is showing them that even though you have different representations of a whole, the answers are still proportionate.

Post-Assessment Description
In this video another student comes up and claims that the girls [sic] argument is really a parallel argument to what the boy Brian made. He is [sic] claims that even though you have different representations of a whole, the answer would still be 1/6. The girls then agree to the new claim and agree with Brian as well.

Figure 11.9. Argumentation described by Teacher 4 for Event 9
T4 notes a claim in the pre-assessment that is not stated by any student, "even though you have different representations of a whole, the answers are still proportionate."

This claim is noted in gray text. In the post assessment, T4 changes the description to include more specific argumentation language and detail. T4 states explicitly that, "the girls [sic] argument is really a parallel argument to what the boy Brian made," and "even though you have different representations of a whole, the answer would still be 1/6," are claims by the students. Furthermore, T4 states that, "The girls then agree to the new claim and agree with Brian as well," noticing, not only the girls' agreement, but that "the answer would still be 1/6" is a "new" claim. Note that the claims mentioned by T4 in the post-assessment are both claims that students made and T4 removed the claim mentioned pre-assessment that was not made by the students.
11.1.10 Event 10

**Pre-Assessment Description**
The teacher now brings the girls back to the original problem that they got wrong. She wants them, with their new understanding to go back and explain what went wrong in their original problem.

No argumentation described.

**Post-Assessment Description**
The teacher now brings the girls back to the original problem that they got wrong. She wants them, with their new understanding to go back and explain what went wrong in their original problem. They then begin to explain their newfound claim by using the [sic] blocks as reinforcement.

![Figure 11.10. Argumentation described by Teacher 4 for Event 10](image)

No argumentation is described in the pre-assessment. In the post-assessment, T4 states, "They then begin to explain their newfound claim by using the [sic] blocks as reinforcement." Through this additional text, T4 describes that a "new claim" was made by the students, and that the block model is used as data to support the claim.

11.1.11 Event 11
Pre-Assessment Description
Brian comes back up to the board to explain why the girls still aren't quite right in their explanation. To help them see why, he superimposes the pieces on top of one another. At the end the girl explains that they [sic] maybe there are two correct answers.

Post-Assessment Description
Brian comes back up to the board to explain why the girls still aren't quite right in their claim. To help them see why, he superimposes the pieces on top of one another. At the end the girls agree with Brian's claims but still believe their own, now claiming that there are two right answers.

T4 describes a conjecture in the pre-assessment description that, "maybe there are two correct answers." The use of "maybe" implies doubt, making this statement a conjecture rather than a claim. Also described is what might be considered an argument with, "the girls aren't quite right," as the claim and the model that shows "superimposed
pieces on top of one another," as the data. Due to the lack of precise language, this argument is implied at best.

T4 changes the language used and adds detail to the post-assessment, making implicit elements explicit and adding structure and additional elements to the student argumentation noticed. T4 states, "At the end the girls agree with Brian's claims but still believe their own, now claiming that there are two right answers." This statement links back to the beginning statement, "Brian comes back up to the board to explain why the girls still aren't quite right in their claim," situating Brian's argument as a counterargument to the girls' argument. If the change in the pre-assessment title: "Brian's reassertion" to the post-assessment title "Brian's [sic] reasserts his counterclaim," of the Event is taken into account, it is clear that T4 is noting Brian's claim as a counterclaim and, thus, the argument presented is meant as a counterargument.

Furthermore, with the final statement, T4 notes the statement by the girls "there may be two correct answers" as a claim that modifies their previous claim. Although T4 uses general language, noting that the "girls agree with Brian's claims but still believe their own, now claiming that there are two right answers" makes the idea that there was a previous claim explicit.

11.1.12 Event 12
Pre-Assessment Description
One boy now notes that $1/2 - 1/3$ can't possibly be $1/3$, because he sees that $1/3+1/3=2/3$, not $1/2$. And so, he says that this is why their answer doesn't make sense.

Post-Assessment Description
Erik now claims that $1/2 - 1/3$ can't possibly be $1/3$, because he sees that $1/3+1/3=2/3$, not $1/2$. And so, he asserts that this is why their answer doesn't make sense. He backs up all of his arguments with pictorial representations.

Figure 11.12. Argumentation described by Teacher 4 for Event 12

In the pre-assessment, T4 describes an argument with "$1/2 - 1/3$ cannot be $1/3$" as the counterclaim to the implied prior claim, "$1/2 - 1/3$ equals $1/3$" with data 
"$1/3+1/3=2/3$" to support the counterclaim. The language, "$1/2 - 1/3$ can't possibly be $1/3$" and "$1/3+1/3=2/3$, not $1/2$," suggest the implied claim. The more precise language in the post-assessment, "Erik now claims that $1/2 - 1/3$ can't possibly be $1/3" and "He backs
up all of his arguments with pictorial representations," clarify that Erik (rather than a boy) is claiming that \(\frac{1}{2} - \frac{1}{3}\) is not \(\frac{1}{3}\) and introduces a new element of argumentation, a warrant: "pictorial representations." The warrant is noted as further backing, linking the data and the claim.

11.1.13 Event 13

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>The one boy Eric continues to explain his position by saying that (\frac{1}{2}) has to be larger than (\frac{1}{3}) and less than (\frac{2}{3}). It has to be (\frac{1}{3}) and a half (what he is really doing is multiplication).</td>
</tr>
</tbody>
</table>

\[
\begin{array}{|c|}
\hline
\text{Claim: } \frac{1}{2} \text{ is } \frac{1}{3} \\
\text{and half} \\
\hline
\end{array}
\begin{array}{|c|}
\hline
\text{Claim: } \frac{1}{2} \text{ is larger than } \frac{1}{3} \\
\text{and less than } \frac{2}{3} \\
\hline
\end{array}
\]

<table>
<thead>
<tr>
<th>Post-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Erik continues to explain his position by saying that (\frac{1}{2}) has to be larger than (\frac{1}{3}) and less than (\frac{2}{3}). It has to be (\frac{1}{3}) and a half (what he is really doing is multiplication).</td>
</tr>
</tbody>
</table>

No change in the post-assessment description.

Figure 11.13. Argumentation described by Teacher 4 for Event 13

T4 notes two claims "\(\frac{1}{2}\) has to be larger than \(\frac{1}{3}\) and less than \(\frac{2}{3}\)," and "\(\text{It } [\frac{1}{2}]\) has to be \(\frac{1}{3}\) and a half." No relationship between these claims is noted. The post-assessment description is the same as the pre-assessment description.

11.1.14 Event 14
Pre-Assessment Description
Here Michael comes up and explains why he thinks the answer is 1/6. He says that since one whole is comprised of six 1/6's, that is how much the two quantities differ.

Post-Assessment Description
Here Michael comes up and explains his claim that the answer is 1/6. He argues that one whole is comprised of six 1/6's, that is how much the two quantities differ. He uses the tiles to demonstrate his claim.

Figure 11.14. Argumentation described by Teacher 4 for Event 14

T4 notes an argument in the pre-assessment with claim: "the answer is 1/6 [1/2 is 1/6 greater than 1/3]", data: "that's how much the quantities differ," and the warrant: "one whole is comprised of six 1/6's" linking the data and the claim. In the post-assessment T4 also notes, "He uses the tiles to demonstrate his claim," noticing an additional element of argumentation, the backing, and adding additional structure to the argumentation noticed.

11.1.15 Event 15
Pre-Assessment Description
Here the students make another subtraction argument that explains why the answer is 1/6 of the whole.

Post-Assessment Description
Here the students make another subtraction argument that explains why the answer is 1/6 of the whole. They use a comparison of the blocks to justify their final solution that $1/2 - 1/3 = 1/6$.

Figure 11.15. Argumentation described by Teacher 4 for Event 15

In the pre-assessment, T4 notes the claim, "the answer is 1/6 of the whole," and the data, "subtraction argument," as support for the claim. The language, "explains why" connects the data and the claim. In the post-assessment, T4 additionally states, "They use a comparison of the blocks to justify their final solution that $1/2 - 1/3 = 1/6." This statement situates the prior statement that "Here the students make another subtraction argument that explains why the answer is 1/6 of the whole," as part of a larger argument with the claim, "$1/2 - 1/3 = 1/6" at the center. Thus, T4 notes this claim, supported by "the answer is 1/6 of the whole" as data, the "subtraction argument" as the warrant that links the data and the claim, and the "comparison of the block" model as the backing, justifying the claim.

11.2 Summary of Teacher 4's Growth across Events
In nine of the 15 events, T4 shows growth with respect to the elements of argumentation mentioned from the pre-assessment description to the post-assessment description. In Event 2, T4 notes both data and claim in the post-assessment, whereas, only a claim is noted in the pre-assessment. In the pre-assessment of Event 3, T4 does not describe any argumentation; in the post-assessment, a claim is noted. In Event 4, T4 adds data, as well as a statement, "They don't understand its relation to the whole unit," that, while not an element of argumentation itself, is additional information relevant to the students' argumentation. The changes in Event 6 add elements of argumentation by adding the description of a counterclaim which suggests a prior claim that 1/2 is one-third larger than 1/3 and in Event 9, T4 changes the language from noting a claim that was not stated by the students to describing two claims that were stated by students. The pre-assessment in Event 10 does not describe any argumentation, but in the post-assessment T4 describes a general "new claim," and a block model as data and in Event 11, T4 describes two additional claims, one of which is a modification of the other. In Event 12, T4 adds a warrant to support the data. T4 describes an additional element of argumentation in Event 14, a tile model used as backing for the warrant, and in Event 15, through the addition of the statement, "They use a comparison of the blocks to justify their final solution that 1/2 - 1/3 = 1/6," T4 adds two elements of argumentation, a claim and backing.

T4 shows growth in seven of the 15 events with respect to the structure of the argumentation described in the pre-assessment compared to the post-assessment. In Event 2, T4 connects the claim with the data, adding to the structure of the argumentation described and in Event 4, T4 adds data that links to the claim and uses "because" to make
the connection between the data and claim explicit. In Event 6, T4's use of "counterclaim," situates the argumentation presented in the event as a counterargument to the prior "argument of the girls." In Event 10, T4 connects a claim with the data, noting the argumentation structure. In Event 11, T4 states, "At the end the girls agree with Brian's claims but still believe their own, now claiming that there are two right answers." This statement links back to the beginning statement, "Brian comes back up to the board to explain why the girls still aren't quite right in their claim," situating Brian's argument as a counterargument to the girls' argument. If the change in the pre-assessment title: "Brian's reassertion" to the post-assessment title "Brian's [sic] reasserts his counterclaim," of the event is taken into account, it is clear that T4 is noting Brian's claim as a counterclaim and, thus, the argument presented is meant as a counterargument. Furthermore, with the final statement in that the event description, T4 notes the statement by the girls "there may be two correct answers" as a claim that modifies their previous claim. In Event 14, T4 connects an additional element of argumentation, backing, to the existing argument and in Event 15, T4 adds to the structure of the argumentation described in the pre-assessment by stating, "They use a comparison of the blocks to justify their final solution that 1/2 - 1/3 = 1/6." This statement situates the prior statement that "Here the students make another subtraction argument that explains why the answer is 1/6 of the whole," as part of a larger argument with the claim, "1/2 – 1/3 = 1/6" at the center. Thus, in the post-assessment of this event, T4 notes this claim, supported by "the answer is 1/6 of the whole" as data, the "subtraction argument" as the warrant that links the data and the claim, and the "comparison of the block" model as the backing, justifying the claim.
In 11 of the 15 events, T4 shows growth with respect to the language used by including more of the formal mathematical register for argumentation. In Event 1, T4 uses more precise language of mathematics by using the word "convince" rather than "explain," and in Event 2, T4 states that the student used pictures to "show why she claimed that 1/2 > 1/3," and used "blocks to assert their claim," describing the argumentation more precisely using the formal mathematical register for argumentation. In Events 3, 5, 7, 9, 10, 11, and 14, T4 specifically identifies the students' statements as a "claims" and in Event 6, T4 uses "counterclaim," to more precisely describe the students' statements as counter to a previously stated claim. The use of "new claim," in Event 9 suggests that T4 noticed, not only the girls' agreement, but "the answer would still be 1/6" as a "new" claim. In Event 12, T4 uses "claim" and "backs up" to clarify the argumentation described, specifically noting that the "pictorial representations," are meant to "back up" the arguments presented by the students.

In five of the 15 events, T4 shows growth in argumentation described in other ways. In the post-assessments of Events 1 and 2, T4 uses language that aligns more with the language students use in the event by changing the claim from "1/2 > 1/3" in the pre-assessment to "1/2 is bigger than 1/3" (Event 1) and "1/2 is larger than 1/3," (Event 2). In the pre-assessment of Event 9, T4 describes a claim that was not stated by any of the students in the event. In the post-assessment, T4 takes out the description of that claim and adds the description of two claims that were evident in the event. In Event 11, T4 changes the description to make an implicit argument that "the girls aren't quite right," because of a model in which rod pieces are "superimposed" "on top of one another," explicit. In Event 4, T4 changes the description to give more precise details about the
elements of argumentation. In the pre-assessment of this event, the data are described as, a model with "picture blocks." In the pre-assessment, T4 more describes the rod model more precisely, "they see that three red blocks make one green block and two red blocks make up one pink block."
Chapter 12 – Teacher 5, Event Descriptions Analyses and Summary

This chapter presents the descriptions Teacher 5 (T5) wrote for the pre-assessment and post-assessment analytic to describe the argumentation in each event. The pre-assessment and post-assessment descriptions given by the teacher are presented for each event with an accompanying diagram using Toulmin's (1958, 2003) scheme.

Following the descriptions for each event, I present an in-depth analysis, summarizing the argumentation described in the pre-assessment and noting the changes the teacher made from pre- to post-assessment. Words that are key to my analysis appear in red text in the teacher’s descriptions.

12.1 Teacher 5 Event Descriptions and Analyses by Event

12.1.1 Event 1

**Pre-Assessment Description**

Dr. Maher asks students to recap what was done in their previous class. Michael explains that they were working on the problem of whether one half or one third is bigger. Andrew restates Michael’s description. Dr. Maher questions how many have worked out which is bigger, one half or one third. Laura replies, one half. Dr. Maher questions the rest of the class about their thinking and whether they can convince Dr. Davis. She also adds an additional element to the investigation by asking how many students think they know how much bigger the bigger fraction is. She calls Jessica and Laura to present at the front of the class.

![Claim: "one-half" is bigger than one third]

**Post-Assessment Description**

Dr. Maher asks students to recap what was done in their previous class. Michael explains that they were working on the problem of whether one half or one third is bigger. Andrew restates Michael’s description. Dr. Maher questions how many have worked out which is bigger, one half or one third. Laura replies, one half. Dr. Maher questions the rest of the class about their thinking and whether they can convince Dr. Davis. She also adds an additional element to the investigation by asking how many students think they know how much bigger the bigger fraction is. She calls Jessica and Laura to present at the front of the class.

No change in argumentation described.

*Figure 12.1. Argumentation described by Teacher 5 for Event 1*
In the pre-assessment, T5 notes the claim that "one-half" is bigger than one third. There is no change in the post-assessment description.

### 12.1.2 Event 2

#### Pre-Assessment Description

Jessica shows the class which rods she used but misspeaks by saying one third is bigger than one half even though she is showing that the one half rod is bigger than the one third. Dr. Maher probes for clarification by asking what number name Jessica and Laura gave the rods. Jessica needs support from Laura to formulate her thoughts and to clearly say the number names for the rods.

#### Post-Assessment Description

Jessica shows the class which rods she used but misspeaks by saying one third is bigger than one half even though she is showing that the one half rod is bigger than the one third. Dr. Maher probes for clarification by asking what number name Jessica and Laura gave the rods. Jessica needs support from Laura to formulate her thoughts and to clearly say the number names for the rods.

---

**Figure 12.2. Argumentation described by Teacher 5 for Event 2**

T5 describes a claim and general data in the pre-assessment. The claim is that "one third is bigger than one half" and the data, the rod model that shows that "the one
half rod is bigger than the one third rod." T5 includes some additional information relevant to the argumentation specifically, that the student "misspeaks by saying one third is bigger than one half even though she is showing that the one half rod is bigger than the one third."

The post-assessment description is the same as the pre-assessment description. However, when the title of the event is considered, T5 changes the title from, "Jessica and Laura explain and name their rods" in the pre-assessment, to "Jessica and Laura present their claim and name their rods." This change suggests that T5 is explicitly identifying the statement that "one third is bigger than one half," as a claim, indicating growth in the language used to describe argumentation.

12.1.3 Event 3

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Audra agrees and Dr. Maher encourages her to address the entire class and Audra places the one half and one third rods on top of each other and shows that one half is bigger.</td>
</tr>
</tbody>
</table>

\[\text{Data: the one half rod is bigger than the 1/3 rod} \quad \text{So} \quad \text{Claim: "one half is bigger [than one third]"} \]

\[\text{Warrant: the rod model that shows "one half and one third rods on top of each other"} \]

<table>
<thead>
<tr>
<th>Post-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Audra agrees and Dr. Maher encourages her to address the entire class and Audra places the one half and one third rods on top of each other and shows that one half is bigger.</td>
</tr>
</tbody>
</table>

The post-assessment language in the title changed to include "claim."

*Figure 12.3. Argumentation described by Teacher 5 for Event 3*

T5 describes in the pre-assessment an argument with claim, "one half is bigger [than one third]," data, the one half rod is bigger than the 1/3 rod, and warrant, the rod model that shows "one half and one third rods on top of each other."
The post-assessment T5's language does not change. However, when the titles of the events are compared, "Audra agrees with Jessica and Laura and justifies why" for the pre-assessment and, "Audra supports Jessica and Laura's claim and justifies why" for the post-assessment, T5 uses more precise language, specifically identifying the claim.

12.1.4 Event 4

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dr. Maher questions if Audra and Jessica figured out how much bigger one half is. Audra and Jessica demonstrated with the rods but needed prompting from Dr. Maher to give the rods number names. When deciding on the number name represented with the rods but used the dark green as one instead of the orange and red.</td>
</tr>
<tr>
<td>No argumentation described</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Post-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dr. Maher questions if Audra and Jessica figured out how much bigger one half is. Audra and Jessica demonstrated with the rods but needed prompting from Dr. Maher to give the rods number names. When deciding on the number name for the red rod, Audra represented with the rods but used the dark green as one instead of the orange and red that Jessica and Laura had been using as one whole. Jessica agrees with Audra’s conjecture.</td>
</tr>
<tr>
<td>No argumentation described</td>
</tr>
</tbody>
</table>

*Figure 12.4. Argumentation described by Teacher 5 for Event 4*

In the pre-assessment, T5 does not describe argumentation. In the post-assessment description T5 added the statement, "Jessica agrees with Audra’s conjecture," which includes more of the formal mathematical register and suggests that T5 noticed a conjecture being made in the event. However, since T5 does not state the conjecture, there is no argumentation to include in the diagram. It is interesting to note that T5 makes a change to the title of the event to capture this argumentation language as well: pre-assessment title: "Dr. Maher questions, how much bigger and Audra and Jessica present;"
post-assessment title: "Dr. Maher questions and Audra and Jessica propose their conjecture."

12.1.5 Event 5

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dr. Maher asks the class if they are convinced of Audra and Jessica’s claim and if anyone agrees. Kelly agrees and without being asked to present to class, recognizes that she needs to demonstrate why she agrees. Dr. Maher notices that Brian looks like he does not agree and asks him what he thinks. When Brian replies that he does not really agree, Dr. Maher restates affirmatively that Brian does not agree, thus facilitating the discourse. Audra and Jessica argue that they think Kelly has changed the problem. When Dr. Maher says &quot;Oh, hmmm,&quot; Jessica is prompted to justify. Dr. Maher further questions Jessica’s model to clarify her understanding and asks for confirmation of what number names she is calling the rods.</td>
</tr>
</tbody>
</table>

No argumentation described

<table>
<thead>
<tr>
<th>Post-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dr. Maher asks the class if they are convinced of Audra and Jessica’s conjecture and if anyone agrees. Kelly agrees and without being asked to present to class, recognizes that she needs to demonstrate why she agrees. Dr. Maher notices that Brian looks like he does not agree and asks him what he thinks. When Brian replies that he does not really agree, Dr. Maher restates affirmatively that Brian does not agree, thus facilitating the discourse. Audra and Jessica argue that they think Kelly has changed the problem. When Dr. Maher says &quot;Oh, hmmm,&quot; Jessica is prompted to justify. Dr. Maher further questions Jessica's model to clarify her understanding and asks for confirmation of what number names she is calling the rods.</td>
</tr>
</tbody>
</table>

No argumentation described

*Figure 12.5. Argumentation described by Teacher 5 for Event 5*

In the pre-assessment, T5 notes vague references to argumentation, describing that there is a claim with which some students agree others do not. Furthermore there are references to demonstrating "why," and justifying and the suggestion that a rod model was used, but no specifics are described, so no diagrams are included in the analysis. The only change from pre- to post-assessment is that T5 chose to use the word "conjecture" rather than "claim."
12.1.6 Event 6

**Pre-Assessment Description**
Dr. Maher asks Brian what he thinks and Brian restates what he thinks Jessica and Audra said before saying that he doesn’t agree with them. He then asks if he should go up to the front of the room to show his model. Dr. Maher summarizes that there is some disagreement and perhaps a conference is needed.

No argumentation described.

**Post-Assessment Description**
Dr. Maher asks Brian what he thinks and Brian restates what he thinks Jessica and Audra said before saying that he doesn’t agree with them. He then asks if he should go up to the front of the room to show his model. Dr. Maher summarizes that there is some disagreement and perhaps a conference is needed. Brian presents his claim by representing with the rods and showing that one half of the purple (one third) would be one sixth.

![Figure 12.6. Argumentation described by Teacher 5 for Event 6](image)

In the pre-assessment, T5 mentions vague references to agreement and a model that was perhaps used to support a claim rather than any specific argumentation. T5 adds language to the post-assessment that describes some argumentation. When combined with changes to the title, the argumentation noted includes an unspecified counterclaim supported by data "one half of the purple (one third) would be one sixth." T5 changed the pre-assessment title, "Brian disagrees and explains," to "Brian disagrees and offers a counterclaim," which situates the argumentation as a counterargument. The connection is implicit in the title, so it appears dashed in the diagram.

12.1.7 Event 7
Pre-Assessment Description
Dr. Maher reminds the class that everyone must listen to the arguments. Jackie presents her model using different rods. Jessica exclaims “oh I think they used a different sized candy bar” and describes why she thinks that is not allowed.

Post-Assessment Description
Dr. Maher reminds the class that everyone must listen to the arguments. Jackie presents her model using different rods. Jessica exclaims “oh I think they used a different sized candy bar” and conjectures why she thinks that is not allowed.

In the pre-assessment description, T5 describes an argument with an implicit claim that the other students' model is not allowed, supported by the data that the other students are using "a different sized candy bar." In the post-assessment, T5 adds the formal mathematical register for argumentation by stating, "Jessica exclaims 'oh I think they used a different sized candy bar' and conjectures why she thinks that is not allowed."

The change in language makes the implicit claim an explicit conjecture.

12.1.8 Event 8
Pre-Assessment Description
Jackie shows their model using dark green as one and concludes that one half is bigger than one third by one. When pressed by Dr. Maher to give the white a number name, Jackie isn’t able to describe it as anything other than one white. Dr. Maher checks in with her by restating what she thought she heard and Jackie agrees that that is what she said.

Post-Assessment Description
Jackie shows their model using dark green as one and conjectures that one half is bigger than one third by one which she says is the white one. When pressed by Dr. Maher to give the white a number name, Jackie isn’t able to describe it as anything other than one white. Dr. Maher checks in with her by restating what she thought she heard and Jackie agrees that that is what she said.

Figure 12.8. Argumentation described by Teacher 5 for Event 8

In the pre-assessment, T5 describes a claim that "one half is bigger than one third by one," supported by the "model using dark green as one" as data. In the post-assessment description, T5 includes more of the formal mathematical language, specifically identifying the statement that "one half is bigger than one third by one" as a conjecture. The statement, "Jackie shows their model using dark green as one and conjectures that one half is bigger than one third by one which she says is the white one," suggests another claim, specifically that the number name for the white rod is one. Thus, the additional language added to the post-assessment resulted in the use of more precise argumentation language as well as more elements of argumentation.
12.1.9 Event 9

**Pre-Assessment Description**
Erik believes the girls meant to say one sixth but just couldn’t express themselves. He gives a respectful explanation of why the white should be called one sixth and not one as they said. Dr. Maher checks in to confirm whether the girls agree that that is what they meant to say and they agree.

**Post-Assessment Description**
Erik believes the girls meant to say one sixth but just couldn’t express themselves. He gives a respectful explanation and backing of his claim why the white should be called one sixth and not one as they conjectured by lining up six whites above the dark green whole. Dr. Maher checks in to confirm whether the girls agree that that is what they meant to say and they qualify their conjecture by agreeing with Brian’s claim.
Figure 12.9. Argumentation described by Teacher 5 for Event 9

In the pre-assessment, T5 describes a claim that "the girls meant to say one sixth," and a counterclaim, "the white should be called one sixth," to a prior claim that the number name of the white rod is one. Through the statement, "Dr. Maher checks in to confirm whether the girls agree that that is what they meant to say and they agree," T5 suggests that the girls agree with the claim that the white rod has the number name on sixth and modify their claim that the number name for the white rod is one to the number name of the white rod is one sixth.

In the post assessment, T5 adds more formal mathematical register for argumentation, as well as detail to the description that notes more argumentation. T5 states, "He gives a respectful explanation and backing of his claim why the white should be called one sixth and not one as they conjectured by lining up six whites above the dark green whole." Since T5 does not mention data or warrant, the statement about the rods, intended to be backing, has been diagrammed as data. Additionally, T5 specifically identifies the statement that the number name one, a conjecture.

12.1.10 Event 10

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dr. Maher tries to probe in a way that will allow Jessica to discover her mistake in naming the red rod one third.</td>
</tr>
</tbody>
</table>

| Claim: the red rod has the number name one third |

<table>
<thead>
<tr>
<th>Post-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dr. Maher tries to probe in a way that will allow Jessica to discover her mistake in naming the red rod one third.</td>
</tr>
</tbody>
</table>

No change in argumentation described.

Figure 12.10. Argumentation described by Teacher 5 for Event 10
T5 states, "Dr. Maher tries to probe in a way that will allow Jessica to discover her mistake in naming the red rod one third," which suggests a claim that the red rod has the number name one third. The description is the same in the pre- and post-assessments.

12.1.11 Event 11

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brian tries to show again why the red is one sixth and Jessica concedes that maybe they are both right.</td>
</tr>
</tbody>
</table>

| Claim: "maybe they are both right" | Claim: "the red is one sixth" |

<table>
<thead>
<tr>
<th>Post-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brian tries to show again why the red is one sixth and Jessica concedes that maybe they are both right.</td>
</tr>
</tbody>
</table>

No change in the argumentation described.

*Figure 12.11. Argumentation described by Teacher 5 for Event 11*

T5 states, "Brian tries to show again why the red is one sixth and Jessica concedes that maybe they are both right," which suggests two claims, "the red is one sixth" (the number name for the red rod is 1/6) and "maybe they are both right," possibly alluding to the red rod having the number name 1/6 and 1/3. The description is the same in the pre- and post-assessments.

12.1.12 Event 12
**Pre-Assessment Description**
Erik describes why the girls' argument of one third cannot be correct and now with more confidence Brian agrees with Erik.

**Post-Assessment Description**
Erik describes and backs his claim as to why the girls' argument of one third cannot be correct and now with more confidence Brian agrees with Erik's claim.

---

*Figure 12.12. Argumentation described by Teacher 5 for Event 12*

In the pre-assessment, T5 states a claim, that "one third cannot be correct." Note that the precision of language makes it unclear whether the claim is that 1/3 is not the difference between 1/2 and 1/3 or 1/3 is not the number name for the red rod, or both. This claim is counter to an implicit prior claim that one third is correct.

In the post-assessment, T5 adds that Erik backs his claim, explicitly identifying the statement that "one third cannot be correct," as a claim. Additionally, when the change in title is taken into account (from "Erik explains why one third is not correct and Brian agrees" to "Erik challenges and Brian supports," it becomes clearer that T5 intended the claim that one third cannot be correct to be a counterclaim, challenging a prior claim that one third is correct. Thus, the language clarifies what was implicit and confirms the argumentation T5 intended to describe. The addition of the use of "backs," indicates that T5 indicates that backing is part of the argumentation put forth in the event, but since T5 does not give any details as to what that backing is, it is not included in the diagram.

**12.1.13 Event 13**
Pre-Assessment Description
Brian models what one third bigger would really look like and Erik reinforces Brian's arguments.

No argumentation described.

Post-Assessment Description
Brian models what one third bigger would really look like and Erik reinforces Brian's arguments.

No change in the argumentation described.

Figure 12.13. Argumentation described by Teacher 5 for Event 13

In the pre-assessment, T5 alludes to argumentation with the statement, "Brian models what one third bigger would really look like and Erik reinforces Brian's arguments." It is probable that the model that is noted is support for an argument, however, the language is imprecise, so it is uncertain what was intended. Thus, no argumentation is diagrammed. The post-assessment description is the same as the pre-assessment description.

12.1.14 Event 14

Pre-Assessment Description
Michael shows that six reds equal the orange and red whole and therefore should be called one sixth. Brian agrees.

Post-Assessment Description
Michael shows that six reds equal the orange and red whole and therefore should be called one sixth. Brian agrees.

The post-assessment language in the title changed to include "claim."

Figure 12.14. Argumentation described by Teacher 5 for Event 14

In the pre-assessment, T5 describes an argument with claim: the red rod is "called one sixth," and data: the rod model that shows that, "six reds equal the orange and red
whole." T5 did not change the post-assessment description, however, changes were made in the title. The pre-assessment title, "Michael shows why red should be named one sixth and Brian agrees," was changed to, "Michael justifies his claim why red should be named one sixth and Brian agrees." With this change, T5 explicitly identifies the statement that the red rod is called one sixth as a claim and the rod model as a justification for that claim.

12.1.15 Event 15

**Pre-Assessment Description**
Meredith agrees with the boys and shows why the girls' argument of one half being one third bigger cannot be true. Erik and Brian agree.

![Diagram](image)

**Post-Assessment Description**
Meredith agrees with the boys' claim and shows with a counterexample why the girls' claim of one half being one third bigger cannot be true. Erik further elaborates with a counterexample of why the girls' claim cannot be true and Brian agrees.

![Diagram](image)

*Figure 12.15. Argumentation described by Teacher 5 for Event 15*

T5 notes a prior claim in the pre-assessment description, that one half is one third bigger than one third and a counterclaim that one half cannot be one third bigger than one third. In the post-assessment, T5 adds detail to the description that describes additional elements and structure to the argumentation, as well as uses more of the formal mathematical register. T5 states that "Meredith agrees with the boys' claim and shows
with a counterexample why the girls' claim of one half being one third bigger cannot be true. The addition of "agrees with the boys' claim," explicitly identifies that there is a prior claim that is being agreed with. Additionally, the use of claim when talking about the "girls" specifically identifies another statement, that "one half being one third bigger than one third," as another prior claim. Furthermore, T5 adds that "Erik further elaborates with a counterexample of why the girls' claim cannot be true," which notes a counterexample as data.

12.2 Summary of Teacher 5's Growth across Events

In seven of the 15 events, T5 did not make changes to the post-assessment description. When this was the case, the title of the event is considered. In four of these instances, T5 uses the title as a way to capture or summarize the argumentation going on in the event through the use of the formal mathematical register for argumentation. In Event 2, T5 changes the title from, "Jessica and Laura explain and name their rods" in the pre-assessment, to "Jessica and Laura present their claim and name their rods." This change suggests that T5 is explicitly identifying the statement that "one third is bigger than one half," as a claim. In Event 3, when the titles of the events are compared, "Audra agrees with Jessica and Laura and justifies why" for the pre-assessment and, "Audra supports Jessica and Laura's claim and justifies why" for the post-assessment, T5 uses more precise language, specifically identifying the statement by Jessica and Laura's as a claim. In Event 11, T5 changes the title from "Erik explains why one third is not correct and Brian agrees" in the pre-assessment to "Erik challenges and Brian supports," in the post-assessment, clarifying that T5 intended the claim that one third cannot be correct to be a counterclaim and challenging a prior claim that one third is correct. In Event 14, T5
changes the title, "Michael shows why red should be named one sixth and Brian agrees," to "Michael justifies his claim why red should be named one sixth and Brian agrees," which explicitly identifies the statement that the red rod is called one sixth as a claim and the rod model as a justification for that claim.

Throughout the descriptions, T5 tends to state in the post-assessment that there are elements of argumentation, but not describe what these elements are. In Event 4, T5 does not describe any specific argumentation, however, in the post-assessment, T5 notes "Jessica agrees with Audra’s conjecture," which includes more of the formal mathematical register and suggests that T5 noticed a conjecture being made in the event as argumentation. Interestingly, T5 confirms the importance of "conjecture," to argumentation by changing the title of the event from "Dr. Maher questions, how much bigger and Audra and Jessica present;" to "Dr. Maher questions and Audra and Jessica propose their conjecture." In Event 12, T5 states in the post-assessment that, "Erik describes and backs his claim," indicating that backing is part of the argumentation put forth in the event.

There are instances in which T5 adds language to the post-description that notes more argumentation elements and structure. In Event 6, the addition of the statement, "Brian presents his claim by representing with the rods and showing that one half of the purple (one third) would be one sixth," adds two elements of argumentation, a reference to a claim and data that "one half of the purple (one third) would be one sixth." When the title is considered, T5 changed the pre-assessment title, "Brian disagrees and explains," to, "Brian disagrees and offers a counterclaim," which situates the argumentation as an implied counterargument. The use of "counterclaim" in the post-assessment title
specifically identifies that the claim made was a counterclaim. In Event 8, T5 changes the language to state, "Jackie shows their model using dark green as one and conjectures that one half is bigger than one third by one which she says is the white one," identifying the statement that "one half is bigger than one third by one" as a conjecture and suggesting another claim, specifically that the number name for the white rod is one. In Event 9, T5 adds "by lining up six whites above the dark green whole," which adds an element of argumentation, data, to support the claim. In the post-assessment of Event 15, T5 states that "Meredith agrees with the boys' claim and shows with a counterexample why the girls' claim of one half being one third bigger cannot be true." The addition of "agrees with the boys' claim," explicitly identifies that a there is a prior claim that is being agreed with and the use of claim when talking about the "girls" specifically identifies another statement, that "one half being one third bigger [than one third]," as another prior claim. Furthermore, T5 adds that "Erik further elaborates with a counterexample of why the girls' claim cannot be true," which notes a counterexample as data.
Chapter 13 – Teacher 6, Event Descriptions Analyses and Summary

This chapter presents the descriptions Teacher 6 (T6) wrote for the pre-assessment and post-assessment analytic to describe the argumentation in each event. The pre-assessment and post-assessment descriptions given by the teacher are presented for each event with an accompanying diagram using Toulmin's (1958, 2003) scheme. Following the descriptions for each event, I present an in-depth analysis, summarizing the argumentation described in the pre-assessment and noting the changes the teacher made from pre- to post-assessment. Words that are key to my analysis appear in red text in the teacher’s descriptions.

13.1 Teacher 6 Event Descriptions and Analyses by Event

13.1.1 Event 1

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>This event shows Dr. Carolyn Maher questioning the fourth grade students about an activity that they had previously worked on. The students have been thinking about which number is bigger, 1/2 or 1/3. It seems that at this point most students have come to the conclusion that indeed one is bigger and they are not equal. Additionally, the students agree that 1/2 is bigger than 1/3 but are not entirely sure how to prove it yet.</td>
</tr>
</tbody>
</table>

| Claim: 1/2 is not equal to 1/3 | Claim: one fraction is bigger | Claim: 1/2 is "bigger" than 1/3 |

<table>
<thead>
<tr>
<th>Post-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>This event shows Dr. Carolyn Maher questioning the fourth grade students about an activity that they had previously worked on. The students have been thinking about which number is bigger, 1/2 or 1/3. It seems that at this point most students have come to the conclusion that indeed one is bigger and they are not equal. Additionally, the students agree that 1/2 is bigger than 1/3 but are not entirely sure how to prove it yet.</td>
</tr>
</tbody>
</table>

No change in argumentation described.

Figure 13.1. Argumentation described by Teacher 6 for Event 1

In the pre-assessment, T6 states three claims made by the students: "indeed one is bigger [1/2 or 1/3]," "they are not equal," and "1/2 is bigger than 1/3." Note that the use
of "and" and "additionally" suggests that there is no connection among the claims. The post-assessment description is the same as the pre-assessment description.

13.1.2 Event 2

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
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</thead>
<tbody>
<tr>
<td>In this clip, we can see that the girl presenting indeed understands that 1/2 is greater than 1/3 and she is able to show this using the blocks on the projector. However, she is still unable to fully articulate her ideas and gets flustered when she is questioned by the teacher.</td>
</tr>
</tbody>
</table>

| Data: a model with "blocks" | Because | Claim: "1/2 is greater than 1/3" |

<table>
<thead>
<tr>
<th>Post-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>In this clip, we can see that the girl presenting indeed understands that 1/2 is greater than 1/3 and she is able to show this using the blocks on the projector. However, she is still unable to fully articulate her ideas and gets flustered when she is questioned by Dr. Maher. At this point the students do not seem comfortable with being questioned about their responses because they are in the mindset that you are only questioned when your response is incorrect.</td>
</tr>
</tbody>
</table>

No change in argumentation described.

Figure 13.2. Argumentation described by Teacher 6 for Event 2

In the pre-assessment, T6 states the claim "1/2 is greater than 1/3" with vague data, "she is able to show this using the blocks on the projector." The language, "she is able to show this" suggest a link between the data and claim. In the post-assessment, T6 the added language does not change the argumentation described.

13.1.3 Event 3
Pre-Assessment Description
The student introduced in this event confirms what the two girls presented had already believed to be true, that 1/2 is greater than 1/3. She also shows this by using the blocks on the projector. While this student is better able to explain her thoughts, she does not add that much depth or many new ideas to the conversation.

Post-Assessment Description
The student introduced in this event confirms what the two girls presented had already believed to be true, that 1/2 is greater than 1/3. She also shows this by using the blocks on the projector. While this student is better able to explain her thoughts, she does not add that much depth or many new ideas to the conversation.

No change in argumentation described.

Figure 13.3. Argumentation described by Teacher 6 for Event 3

T6 describes a claim: "1/2 is greater than 1/3," and vague data: a model with blocks, that agrees with "what the two girls presented had already believed to be true."

The argument, then, references a prior argument and concurs with it. The post-assessment description is the same as the pre-assessment description.

13.1.4 Event 4
Pre-Assessment Description
The approach the girls take in this event to figure out how much bigger 1/2 is than 1/3 is clever, even though it is incorrect. I like that they continued to use the manipulatives to figure out this follow up question that they are being asked. The fact that they are using these blocks allows the instructor to see where they are making their mistakes or getting confused. Here, the girls are forgetting that the green piece is actually 1/2 not 1 so they are assuming that the red pieces are 1/3 rather than 1/3 of 1/2.

Post-Assessment Description
The approach the girls take in this event to figure out how much bigger 1/2 is than 1/3 is clever, even though it is incorrect. I like that they continued to use the manipulatives to figure out this follow up question that they are being asked. The fact that they are using these blocks allows the instructor to see where they are making their mistakes or getting confused. Here, the girls are forgetting that the green piece is actually 1/2 not 1 so they are assuming that the red pieces are 1/3 of a whole rather than 1/3 of 1/2.

Figure 13.4. Argumentation described by Teacher 6 for Event 4

T6 uses language that talks about the argument about how much greater than 1/2 is 1/3, but does not describe any elements of the argument, so that argument does not appear in the diagram. However, in the statement: "Here, the girls are forgetting that the green piece is actually 1/2 not 1 so they are assuming that the red pieces are 1/3 rather than 1/3 of 1/2," T6 describes some argumentation, specifically, the claim that "the red pieces are 1/3," and implied data referring to the green piece being 1 and, as a non-specific warrant, a reference to a block model. Additional information about the
argumentation is given in this statement, as well, though intermingled with the elements of argumentation themselves. T6 suggests that the girls are forgetting that the dark green piece has been given the number name 1/2 and that the red pieces are really 1/3 of 1/2. The use of "so" links the data and the claim.

In the post-assessment the T6 adds "of a whole" to the final statement, thus stating, "the girls are forgetting that the green piece is actually 1/2 not 1 so they are assuming that the red pieces are 1/3 of a whole rather than 1/3 of 1/2." This changes the claim from the red pieces being 1/3 to the red pieces being 1/3 of a whole, and gives more clarity to the additional information, that the girls are assuming that the red pieces are 1/3 of a whole, rather than 1/3 of a half.

13.1.5 Event 5

Pre-Assessment Description
In this event, we see Carolyn Maher asking the students at the front of the class to clarify what each of the pieces represent. Even after the students clarify that they are calling the longer green piece 1/2 and the pink piece 1/3 they still believe that the green piece is 1/3 larger than the pink piece. This event was interesting because we begin to see one student in the class question the work of his classmates.

Post-Assessment Description
In this event, we see Carolyn Maher asking the students at the front of the class to clarify what each of the pieces represent. Even after the students clarify that they are calling the longer green piece 1/2 and the pink piece 1/3 they still believe that the green piece is 1/3 larger than the pink piece. This event was interesting because we begin to see one student in the class question the work of his classmates. An important part of argumentation is not only the questioning done by the teacher but also done by other students.

No change in argumentation described.

Figure 13.5. Argumentation described by Teacher 6 for Event 5
T6 describes three claims made by students, the "longer green piece" is 1/2, the "pink piece" is 1/3, and "the [dark] green piece is 1/2 larger than the pink piece." Note that no connections are made amongst these statements, so they are all considered claims. The post-assessment description is the same as the pre-assessment description.

13.1.6 Event 6

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
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</thead>
<tbody>
<tr>
<td>This event shows one of the students in the class question the work of his classmates and explains why he believes his solution is correct. Even though he says the correct answer from the start, it seems that he has some trouble articulating and explaining why he thinks this is true. However, after using the manipulatives on the projector he is eventually able to articulate an understandable response. Whether he knows his solution is correct or not, I really liked the way this student questioned his classmates because he clearly did not agree with what they were saying. Additionally, he was able to clarify his response and explain his thought process as he was asked questions by the instructor.</td>
</tr>
<tr>
<td>No argumentation described.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Post-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>This event shows one of the students in the class question the work of his classmates and explains why he believes his solution is correct. Even though he says the correct answer from the start, it seems that he has some trouble articulating and explaining why he thinks this is true. However, after using the manipulatives on the projector he is eventually able to articulate an understandable response. Whether he knows his solution is correct or not, I really liked the way this student questioned his classmates because he clearly did not agree with what they were saying. Additionally, he was able to clarify his response and explain his thought process as he was asked questions by the instructor.</td>
</tr>
<tr>
<td>No change in argumentation described.</td>
</tr>
</tbody>
</table>

Figure 13.6. Argumentation described by Teacher 6 for Event 6

T6 talks about the argumentation presented by the students in Event 6, but does not describe any elements of the argumentation. T6 mentions that the student uses manipulatives presumably to present support for a claim, but, since the claim is never stated, there are no elements to put in a diagram. The post-assessment description is the same as the pre-assessment description.
13.1.7 Event 7

Pre-Assessment Description
In this event, one of the students in the group begins to notice that they are calling two different size objects by the same "name", i.e. 1/2. One girl in the group explains that this is unfair, by relating it to sharing a candy bar. The instructor tries to further this realization by asking them which piece was labeled as 1 when they started this problem solving process. It is at this point that the girls say the dark green piece started off as representing one. However, the longer orange and small red piece were actually the pieces that represented 1 at the beginning of this problem. If I were questioning this group of girls, I would like to further this discussion because it seems to me like the girls are saying that the dark green piece is equal to 1 because they labeled the small red pieces as 1/3. Again, the red pieces would be 1/3 if the green piece was actually 1 rather than 1/2.

Post-Assessment Description
In this event, one of the students in the group begins to notice that they are calling two different size objects by the same "name", i.e. 1/2. One girl in the group explains that this is unfair, by relating it to sharing a candy bar. The instructor tries to further this realization by asking them which piece was labeled as 1 when they started this problem solving process. It is at this point that the girls say the dark green piece started off as representing one. However, the longer orange and small red piece were actually the pieces that represented 1 at the beginning of this problem. If I were questioning this group of girls, I would like to further this discussion because it seems to me like the girls are saying that the dark green piece is equal to 1 because they labeled the small red pieces as 1/3. Again, the red pieces would be 1/3 if the green piece was actually 1 rather than 1/2.

No change in argumentation described.

Figure 13.7. Argumentation described by Teacher 6 for Event 7
In the pre-assessment, T6 describes an argument where the claim is that the model that prior students made is "unfair," with that data that "they are calling two different size objects by the same 'name', i.e., 1/2. The warrant that connects the data with the claim is the relationship of "it [the model] to sharing a candy bar." T6 also states, "the girls are saying that the dark green piece is equal to 1 because they labeled the small red pieces as 1/3." This statement describes a claim and data for that claim. Because of the imprecise language, it is difficult to determine which is the claim and which is the data, so they are both labeled "Claim/Data." The use of "because," however, makes it clear that there is a link between them. T6 includes two additional pieces of information that are relevant to the argumentation, "the orange and red rod were the pieces that represented 1 at the beginning" and "the red pieces would be 1/3 if the green piece was actually 1 rather than 1/2." The post-assessment description is the same as the pre-assessment.

13.1.8 Event 8

**Pre-Assessment Description**
In this event, the girls decide to use different pieces to represent 1, 1/2, and 1/3. I think that doing this is a good approach for them to see which piece is bigger and by how much because with the pieces they were additionally using, they needed to put 2 pieces together in order to make 1. Therefore, I think this model should help the girls to see clearly which is larger 1/2 or 1/3 and it should also help them to realize that 1/2-1/3 cannot be 1/3.

| Data: a model with "pieces" | Because | Claim: "1/2-1/3 cannot be 1/3" |

**Post-Assessment Description**
In this event, the girls decide to use different pieces to represent 1, 1/2, and 1/3. I think that doing this is a good approach for them to see which piece is bigger and by how much because with the pieces they were additionally using, they needed to put 2 pieces together in order to make 1. Therefore, I think this model should help the girls to see clearly which is larger 1/2 or 1/3 and it should also help them to realize that 1/2-1/3 cannot be 1/3.

No change in argumentation described.
Figure 13.8. Argumentation described by Teacher 6 for Event 8

T6 describes in the pre-assessment what could be considered an argument, with the claim that "1/2-1/3 cannot be 1/3" supported by a vague mention of a model using "pieces," presumably the rods. The post-assessment description is the same as the pre-assessment description.

13.1.9 Event 9

Pre-Assessment Description
In this event we are able to see another student in the class who disagrees with the girls presenting and offers his opinion and argument. It seems that the girls understood the problem but were misspeaking by calling the white pieces 1 rather than 1/6. As stated previously, I think reconstructing the model to use these different pieces was a smart idea so that students can use the small white pieces to see that 1/2 is 1/6 larger than 1/3. I believe that the girls were on the right track to coming up with that answer so the questioning and reasoning brought up by the boy in the class was helpful for the girls to understand their own argument.

Post-Assessment Description
In this event we are able to see another student in the class who disagrees with the girls presenting and offers his opinion and argument. It seems that the girls understood the problem but were misspeaking by calling the white pieces 1 rather than 1/6. This idea of one student clarifying what another student claims is an important part of argumentation in the classroom. As stated previously, I think reconstructing the model to use these different pieces was a smart idea so that students can use the small white pieces to see that 1/2 is 1/6 larger than 1/3. I believe that the girls were on the right track to coming up with that answer so the questioning and reasoning brought up by the boy in the class was helpful for the girls to understand their own argument.

The post-assessment language changed to include a specific comment about argumentation: "clarifying what another student claims is an important part of argumentation in the classroom."
In the pre-assessment, T6 describes an argument with "1/2 is larger than 1/3" as its claim. As data, a reference to a model with white pieces is implicit. The white pieces are mentioned explicitly, but the fact that they were used in a model is implied. As the warrant that connects the data and claim, T6 states that "the white pieces are 1/6." T6 links this statement as a claim that counters a claim made previously that the "white pieces are 1," stating, "In this event we are able to see another student in the class who disagrees with the girls presenting and offers his opinion and argument. It seems that the girls understood the problem but were misspeaking by calling the white pieces 1 rather than 1/6." Thus the warrant also serves as a counterclaim.

In the post-assessment description, T6 adds the statement, "This idea of one student clarifying what another student claims is an important part of argumentation in the classroom." The language identifies the prior statement as a claim and notes that T6 believes that the clarifying of claims is an important part of argumentation.

13.1.10 Event 10
Pre-Assessment Description
Here, the instructor tries to give the students a hint to figure out what one of the red pieces could be labeled in their original argument. If the girls understand the new argument, they should be able to translate it to the first model they were using and generalize it for any other size blocks. However, the girls are still getting confused by naming the red piece based on its relation to the green piece (which they have labeled as 1/2) rather than its relation to the orange and red piece (which they have labeled as 1).

Post-Assessment Description
Here, the instructor tries to give the students a hint to figure out what one of the red pieces could be labeled in their original argument. If the girls understand the new argument, they should be able to translate it to the first model they were using and generalize it for any other size blocks. However, the girls are still getting confused by naming the red piece based on its relation to the green piece (which they have labeled as 1/2) rather than its relation to the orange and red piece (which they have labeled as 1).

No change in argumentation described.

Figure 13.10. Argumentation described by Teacher 6 for Event 10

The argument presented by T6 in the pre-assessment is difficult to determine. Some might suggest that no argumentation is described. However, based on what was happening in Event 10, it can be suggested that the argument, as it appears in the diagram, is noted by T6. The claim is that the red rod was named based on its relationship to the green piece and the data that the green piece is 1/2 is noted as support. Another unrelated claim is made: that the "orange and red piece" is 1.
T6 provides additional information that is relevant to the students' argumentation in the event, specifically, that "the girls are still getting confused by naming the red piece based on its relation to the green piece (which they have labeled as 1/2) rather than its relation to the orange and red piece (which they have labeled as 1)." No changes were made in the post-assessment description.

**13.1.11 Event 11**

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>This event is interesting because it seems like this part of the problem has already been explained. I find it interesting though because it causes the student (who seems to understand) to find a different way to explain his work to his classmates.</td>
</tr>
<tr>
<td>No argumentation described.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Post-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>This event is interesting because it seems like this part of the problem has already been explained. I find it interesting though because it causes the student (who seems to understand) to find a different way to explain his work to his classmates. Even though he feels he as the correct solution, the other students are not agreeing so he must come up with another way to justify his claims.</td>
</tr>
<tr>
<td>The post-assessment language changed to include &quot;claim.&quot;</td>
</tr>
</tbody>
</table>

*Figure 13.11. Argumentation described by Teacher 6 for Event 11*

Although T6 talks about argumentation in the pre- and post-assessment descriptions, no specific student argumentation is mentioned. Note, however, that in the post-assessment, T6 adds, "Even though he feels he as the correct solution, the other students are not agreeing so he must come up with another way to justify his claims," using the formal mathematical register to describe what is going on in the event.

**13.1.12 Event 12**
Pre-Assessment Description
This event shows one student coming to a realization about the difference between 1/3 and 1/2. Like many other times in this analytic, it seems like he has the correct idea and is able to put the pieces together to represent the right model but is unable to clearly articulate his ideas to the rest of the class and the instructor.

No argumentation described.

Post-Assessment Description
This event shows one student coming to a realization about the difference between 1/3 and 1/2. Like many other times in this analytic, it seems like he has the correct idea and is able to put the pieces together to represent the right model but is unable to clearly articulate his ideas to the rest of the class and the instructor.

No change in argumentation described.

In the pre- and post-assessment descriptions, no specific student argumentation is mentioned.
13.1.13 Event 13

Pre-Assessment Description
Without knowing it, this student is creating an inequality by saying that 1/2 is less than 2/3 but greater than 1/3. He is questioning his classmates initial idea that 1/2-1/3=1/3 by explaining to them that in fact the piece that represents 1/2 is bigger than the piece that represents 1/3 but also not quite as large as two of those pieces put together which would equal 2/3. It also seems like this student indirectly gives the correct answer (even if he thinks it is a new or different answer that has not been said before) by saying that the piece is equal to 1/3 and 1/2 because therefore the piece remaining would be 1/6.

Post-Assessment Description
Without knowing it, this student is creating an inequality by saying that 1/2 is less than 2/3 but greater than 1/3. He is questioning his classmates initial idea that 1/2-1/3=1/3 by explaining to them that in fact the piece that represents 1/2 is bigger than the piece that represents 1/3 but also not quite as large as two of those pieces put together which would equal 2/3. It also seems like this student indirectly gives the correct answer (even if he thinks it is a new or different answer that has not been said before) by saying that the piece is equal to 1/3 and 1/2 because therefore the piece remaining would be 1/6.

No change in argumentation described.

Figure 13.13. Argumentation described by Teacher 6 for Event 13

In the pre-assessment, T6 describes an argument that is counter the prior claim that "1/2-1/3=1/3," as suggested in the statement, "He is questioning his classmates initial
The implicit counterclaim, then, is that \( \frac{1}{2} - \frac{1}{3} \) is not \( \frac{1}{3} \). T6 describes the data as "\( \frac{1}{2} \) is less than \( \frac{2}{3} \) but greater than \( \frac{1}{3} \) with two warrants, "two \( \frac{1}{3} \) pieces put together would equal \( \frac{2}{3} \)" and "the piece that represents \( \frac{1}{2} \) is bigger than the piece that represents \( \frac{1}{3} \) but also not quite as large as two of those pieces put together." A related but not connected argument is described, that "the piece is equal to \( \frac{1}{3} \) and \( \frac{1}{2} \)" supported with data, "the piece remaining would be \( \frac{1}{6} \).” Note, however, that the imprecision of language T6 uses in this last statement, "the piece is equal to \( \frac{1}{3} \) and \( \frac{1}{2} \) because therefore the piece remaining would be \( \frac{1}{6}, \)" makes it unclear what T6 intends, however, it still appears that an argument is intended. The post-assessment is the same as the pre-assessment.

### 13.1.14 Event 14

#### Pre-Assessment Description
This event shows a student clearly explaining **why** the pieces in question should be named \( \frac{1}{6} \). At this point, the student seems to have taken in all the questions being asked and information **displayed to come to this conclusion**. Additionally, at this point it is apparent that other members of the class are starting to agree with what this student is saying.

#### Data
- **Claim:** "the pieces in question" are named \( \frac{1}{6} \) [i.e., the red rods]
- **Because:** "all the questions being asked and information displayed"

#### Post-Assessment Description
This event shows a student clearly explaining why the pieces in question should be named \( \frac{1}{6} \). At this point, the student seems to have taken in all the questions being asked and information displayed to come to this conclusion. Additionally, at this point it is apparent that other members of the class are starting to agree with this new **claim** and what this student is saying.

The post-assessment language changed to include "new claim."

*Figure 13.14. Argumentation described by Teacher 6 for Event 14*
In this pre-assessment T6 uses imprecise language to describe an argument with the claim, "the pieces in question" are named 1/6 [i.e., the red rods], and data, "all the questions being asked and information displayed." The language T6 uses, however, results in the argument lacking specific details as to what information is being used as data. The language, "taken in all the questions being asked and information displayed to come to this conclusion," supports the identification of the data in the argument. In the post-assessment, T6 specifically identifies that the pieces in question (or the red rod) should be named 1/6 as a claim.

13.1.15 Event 15

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
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<tbody>
<tr>
<td>At this point in the analytic the students seem to come to both an agreement and a conclusion about what to name each piece and why. It is very impressive that the students were able to figure out all of this information without any formal training and teaching on operations with fractions and rather only by using the manipulatives, help from each other, and valuable questioning.</td>
</tr>
<tr>
<td>No argumentation described.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Post-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>At this point in the analytic the students seem to come to both an agreement and a conclusion about what to name each piece and why. It is very impressive that the students were able to figure out all of this information without any formal training and teaching on operations with fractions and rather only by using the manipulatives, help from each other, and valuable questioning.</td>
</tr>
<tr>
<td>No change in argumentation described.</td>
</tr>
</tbody>
</table>

Figure 13.15. Argumentation described by Teacher 6 for Event 15

No argumentation is described by T6 in the either the pre or post assessment description.

13.2 Summary of Teacher 6's Growth across Events
T6's descriptions tend to discuss argumentation in general, rather than specifically reference the argumentation presented by students. Argumentation elements in the analyses, rather than being stated explicitly, are inferred from general statements. T6 made changes to the descriptions in the post-assessments in six of the 15 events. In four of the six events in which changes were made, the changes result in growth in the argumentation described. In Event 4, T6 adds detail in the post-assessment that clarifies the argumentation in the event. The addition of "of a whole" to the statement "they are assuming that the red pieces are 1/3 of a whole rather than 1/3 of 1/2" changes the claim from the red pieces being 1/3 to the red pieces being 1/3 of a whole, and gives more clarity to the additional information, that the girls are assuming that the red pieces are 1/3 of a whole, rather than 1/3 of a half. In the post-assessment description of Event 9, T6 adds the statement, "This idea of one student clarifying what another student claims is an important part of argumentation in the classroom." The language identifies the prior statement as a "claim" and notes that T6 believes that the clarifying of claims is an important part of argumentation. In Events 11 and 14, T6 uses more of the formal mathematical register by referring to the statements students make as "claims."
Chapter 14 – Teacher 7, Event Descriptions Analyses and Summary

This chapter presents the descriptions Teacher 7 (T7) wrote for the pre-assessment and post-assessment analytic to describe the argumentation in each event. The pre-assessment and post-assessment descriptions given by the teacher are presented for each event with an accompanying diagram using Toulmin's (1958, 2003) scheme. Following the descriptions for each event, I present an in-depth analysis, summarizing the argumentation described in the pre-assessment and noting the changes the teacher made from pre- to post-assessment. Words that are key to my analysis appear in red text in the teacher’s descriptions.

14.1 Teacher 7 Event Descriptions and Analyses by Event

14.1.1 Event 1
Pre-Assessment Description
In this clip, researcher Carolyn Maher is asking students to recall a previous session in which the students explored fractions. Michael volunteers and explains that the students were working on the "candy bar problem" to compare the fractions 1/2 and 1/3 and to determine which fraction is bigger. The researcher asks students questions, such as "how many of you worked out which is bigger?" Laura suggests that 1/2 is bigger than 1/3, and when asked, others in the class agree with Laura. Researcher Maher asks the students to convince another researcher, Robert Davis, that 1/2 is bigger than 1/3. Now that the students have determined which fraction is bigger, researcher Maher asks "by the way, do you know how much bigger?" Finally, Jessica and Laura are asked to use the overhead to show researcher Davis how they determined which fraction is bigger. In addition, the students are asked to convince researcher Davis that 1/2 is bigger than 1/3.

Claim: "1/2 is bigger than 1/3"

Post-Assessment Description
In this clip, researcher Carolyn Maher is asking students to recall a previous session in which the students explored fractions. Michael volunteers and explains that the students were working on the "candy bar problem" to compare the fractions 1/2 and 1/3 and to determine which fraction is bigger. The researcher asks students questions, such as "how many of you worked out which is bigger?" Laura claims that 1/2 is bigger than 1/3, and when asked, others in the class agree with Laura. Researcher Maher asks the students to convince another researcher, Robert Davis, that 1/2 is bigger than 1/3. Now that the students have determined which fraction is bigger, researcher Maher asks "by the way, do you know how much bigger?" Finally, Jessica and Laura are asked to use the overhead to show researcher Davis how they determined which fraction is bigger. In addition, the students are asked to convince researcher Davis that 1/2 is bigger than 1/3.

The post-assessment language changed to include "claim."

Figure 14.1. Argumentation described by Teacher 7 for Event 1

T7 notes a claim in the pre-assessment, that "1/2 is bigger than 1/3." In the post assessment, T7 includes the same statement, but uses the formal mathematical register by calling the statement a claim.

14.1.2 Event 2
Pre-Assessment Description
At the end of the previous clip, Laura and Jessica agreed that 1/2 is bigger than 1/3. In this clip, they are showing how they came to this conclusion, and they are trying to convince researcher Davis that their conclusion is correct. Jessica and Laura are using the overhead and colored rods to show the class and researchers how they determined 1/2 is bigger than 1/3. Jessica uses purple rods to represent 1/3 and green rods to represent 1/2. Jessica argues "one third is bigger than one half [because this [pointing to the purple rod] would be one third and then this bigger piece [pointing to the dark green rod] would be one half of that." Before going in front of the class, Jessica and Laura both stated that 1/2 is bigger than one third. In the beginning of her explanation, Jessica states that "one third is bigger than one half," which contradicts her answer. However, when explaining, she points to the green rod (representing 1/2) and describes it as "the bigger piece." While it seems that Jessica and Laura know that 1/2 is bigger than 1/3 by looking at the fractions visually, Jessica's argument would not be convincing to others. Researcher Maher asks the girls to explain what the orange and red rod combination represents. Jessica initially states that the combination represents 1. When asked to restate what the orange and red rod combination represents again, Jessica whispers to Laura before trying to explain what each color rod represents. Finally, Laura and Jessica both work together to explain what each rod represents by pointing to each color and stating what number that color rod represents. The orange and red rod combination represents one, the purple rods represent 1/3, and the green rods represent 1/2. While Jessica initially led the explanation, at the end of this clip both girls identified to the class and researchers what number each color represented.

Post-Assessment Description
At the end of the previous clip, Laura and Jessica claimed that 1/2 is bigger than 1/3. In this clip, they are shown modeling these values along with the value 1. Jessica uses purple rods to model 1/3 and green rods to model 1/2. Jessica argues "one third is bigger
than one half cause this [pointing to the purple rod] would be one third and then this bigger piece [pointing to the dark green rod] would be one half of that." Before going in front of the class, Jessica and Laura both **claimed** that $1/2$ is bigger than one third. In the beginning of her explanation, Jessica states that "one third is bigger than one half," which contradicts her answer. However, while explaining, she points to the green rod (representing $1/2$) and describes it as "the bigger piece." While it seems that Jessica and Laura are **supporting their claim** that $1/2$ is bigger than $1/3$ by looking at the fractions visually, Jessica's argument would not be convincing to others. Researcher Maher asks the girls to explain what the orange and red rod combination represents. Jessica initially states that the combination represents 1. When asked to restate what the orange and red rod combination represents again, Jessica whispers to Laura before trying to explain what each color rod represents. Finally, Laura and Jessica both work together to explain what each rod represents by pointing to each color and stating what number that color rod represents. The orange and red rod combination represents one, the purple rods represent $1/3$, and the green rods represent $1/2$. While Jessica initially led the explanation, at the end of this clip both girls identified to the class and researchers what number each color represented.

![Figure 14.2. Argumentation described by Teacher 7 for Event 2](image)

The pre-assessment description notes an argument with the claim, "$1/3$ is bigger than $1/2$" supported by the data, the $1/2$ rod is the "bigger piece," and the warrants, " the purple rods represent $1/3," "the green rods represent $1/2," and the orange and red rod "combination represents 1," which is the premise for giving the rods number names. T7
notes that the students previously stated that 1/2 is greater than 1/3, but, when making the argument in this event, the students state that "1/3 is bigger than 1/2," which "contradicts" the previous claim. Furthermore, T7 explains that the argument put forth by the students supports the claim that 1/2 is greater, rather than the claim that 1/3 is greater. The connections among the elements are indicated by T7's use of language such as, "Jessica and Laura are using the overhead and colored rods to show the class and researchers how they determined 1/2 is bigger than 1/3" and through the use of the student's direct quote in which the student used "because."

In the post-assessment, the structure and elements of argumentation are the same, however, T7 changes the language to align more with the formal register of argumentation. Rather than saying, "agreed" and "stated," "claimed" is used. Additionally, T7 situates the use of the rods in the argumentation as support for the students' claim making the use of the rods as evidence for the claim even more explicit.

14.1.3 Event 3
Pre-Assessment Description
In the previous clip, Jessica used visual rods to argue that $\frac{1}{2}$ is bigger than $\frac{1}{3}$. She used color rods to represent fractions: a red and orange rod combination represent 1, purple rods represent $\frac{1}{3}$, and dark green rods represent $\frac{1}{2}$. In this clip, researcher Maher is asking the class if they agree with Jessica and Laura's argument. Audra agrees with Jessica and Laura, and is asked to go to the overhead to explain. Audra places a purple rod ($\frac{1}{3}$) above a dark green rod ($\frac{1}{2}$) and states "if you saw what the half was here and then you saw what the third was there, and you saw that the half was bigger than the third." While Audra agreed with Jessica's explanation, Audra made a clearer argument, by stating that, visually, $\frac{1}{2}$ was bigger than $\frac{1}{3}$.

![Diagram of argument structure](image)

Post-Assessment Description
In the previous clip, Jessica used rods to argue that $\frac{1}{2}$ is bigger than $\frac{1}{3}$. She used color rods to represent fractions: a red and orange rod combination represent 1, purple rods represent $\frac{1}{3}$, and dark green rods represent $\frac{1}{2}$. In this clip, researcher Maher is asking the class if they agree with Jessica and Laura's argument. Audra agrees with Jessica and Laura, and is asked to go to the overhead to explain. Audra places a purple rod ($\frac{1}{3}$) above a dark green rod ($\frac{1}{2}$) and states "if you saw what the half was here and then you saw what the third was there, and you saw that the half was bigger than the third." Audra modified and clarified the girls' claims.
The pre-assessment description includes description of the claim and argument from the previous event. This description provides the context for the description of the argumentation in the current event. Note that when describing the prior argumentation, T7 mentions the prior claim (that 1/2 is bigger than 1/3) and only part of the evidence that was put forth, that the student "used color rods to represent fractions: a red and orange rod combination represent 1, purple rods represent 1/3, and dark green rods represent 1/2." Additionally, T7 does not use language to explicitly connect the evidence statement to the claim, so, although the statement suggests that it was intended as evidence, a dashed line is used to connect the prior claim to the prior argument.
T7 states that the Audra agrees with the prior argument and Audra's argument is noted. To support the claim, "the half [rod] was bigger than the third[rod]" is given as data and the placement of the purple rod above the dark green rod in the model is the warrant that connects the data and the claim.

The reference to the number name for each rod can be seen as backing and, although not explicitly stated by T7, it can implied that T7 is suggesting that the student got these number names from the previous student's argument since T7 included these statements at the beginning of the description. Note that the statements made by the student and noted by T7: ""the third was there" and "the half was here" as well as the statements included at the beginning of the description and alluded to here, "purple rods represent 1/3" and "dark green rods represent 1/2" are included as backing, as is the idea explicitly mentioned at the beginning and implied here, that the red and orange rod "combination" represents 1. Explicit statements are shown with black text, implicit statements are shown in parentheses. The box for the "Premise/Backing" element is dashed, since there was no explicit reference to the unit in T7's description of the students' argument at the end of this event.

In the post-assessment description, T7 states, "Audra modified and clarified the girls' claims." This statement adds some of the formal mathematical register, by including the idea of modification. Additionally, it situates Audra's argument as being, not only a "clearer" agreement of the previous argument, but a modified argument, which adds to the structure of the argumentation that T7 noticed.

14.1.4 Event 4
Pre-Assessment Description
In the previous clip, Audra used colored rods to show that 1/2 was visually bigger than 1/3. In this clip, researcher Maher asks Audra if she knows how much bigger 1/2 is compared to 1/3. Audra places two white rods next to the purple rod and states "it's two." Laura notices that the dark green rod (1/2) is "one red bigger" than the purple rod (1/3). The researcher asks the girls for a numerical value for the two white rods and the one red rod: numerically, how much bigger is 1/2 than 1/3? Jessica and Audra use the red and green rods to determine that 1/2 is "one third bigger" than 1/3. Jessica places three red rods above the dark green rod to show that three red rods are the same length as 1 dark green rod. She then shows that the dark green rod is one red rod bigger than a purple rod.

Post-Assessment Description
In the previous clip, Audra used colored rods to model that 1/2 was visually bigger than 1/3. In this clip, researcher Maher asks Audra if she knows how much bigger 1/2 is compared to 1/3. Audra places two white rods next to the purple rod and states "it's two." Laura claims that the dark green rod (1/2) is "one red bigger" than the purple rod (1/3). The researcher asks the girls for a numerical value for the two white rods and the one red rod: numerically, how much bigger is 1/2 than 1/3? Jessica and Audra use the red and green rods to model and claim that 1/2 is "one third bigger" than 1/3. Jessica places three red rods above the dark green rod to show that three red rods are the same length as 1 dark green rod. She then shows that the dark green rod is one red rod bigger than a purple rod.
In the pre-assessment description, T7 makes the statement that, "Audra places two white rods next to the purple rod and states 'it's two.'" It may be suggested that this statement, made in response to "how much bigger 1/2 is compared to 1/3," is data—the model that shows "two white rods next to the purple rod"—supporting a claim: "it's two" [1/2 is two white rods bigger than 1/3]. However, since the data lacks specificity, i.e., that the purple and two white rod train is the same length as the dark green train" the connection is not specifically stated by T7, thus, statements are not connected in the diagram. T7 also states that a student "notices that the dark green rod (1/2) is "one red bigger" than the purple rod (1/3)." The statement can be seen as data for the implicit claim that 1/2 is one red rod bigger than 1/3. Note that the use of the word "notices" suggests that T7 is referring to the model the student made, and statement describes the
model, so the data is explicit and the claim is implied from the data. The implicit claim is in a dashed box.

T7 then describes the argument for the claim, "1/2 is "one third bigger" than 1/3," with the data, "the dark green rod is one red rod bigger than a purple rod" as data and the warrant, "three red rods are the same length as 1 dark green rod" as the connection between the data and the claim. Backing for the warrant is included as well, that the students make a model that shows that a train of three reds is the same length as a dark green 1/2 rod. The connections in T7's description for this argument are clearly made through the language used, for example, using the rods "to determine" the claim.

In the post-assessment description, T7 uses more formal argumentation language for some of the statements that the students make, identifying them as "claims," i.e., "Laura claims that the dark green rod (1/2) is 'one red bigger' than the purple rod (1/3)," and the use of the word "claim" in "Jessica and Audra use the red and green rods to model and claim that 1/2 is 'one third bigger' than 1/3." In the first example, the use of the word claim changes the structure of the argumentation. In the pre-assessment description, T7 stated, "Laura notices that the dark green rod (1/2) is 'one red bigger' than the purple rod (1/3)," emphasizing the model and suggesting that the statement was data for an implied claim.

In the post-assessment description, interchanging the word "notices" with the word "claim" explicitly identifies the statement as a claim, and the data (the model that shows the rods) becomes implied.

14.1.5 Event 5
Pre-Assessment Description
In the previous clip, Jessica and Audra used colored rods to show that 1/2 (dark green rods) is one third bigger than 1/3 (purple rods). They showed that 3 red rods are the same as one dark green rod and a purple rod and red rod are the same length as a dark green rod. In this clip, researcher Maher asks the rest of the class if they are convinced after hearing the girl's argument. Kelly says that she agrees with the other girls, and goes up to the overhead to show the class why. While Kelly is explaining, she uses a different set of colors that [sic] previously used. Here, Kelly uses dark green to represent 1, red to represent 1/3, and light green to represent 1/2. Kelly shows that the red rods are smaller than light green rods and states "one half is bigger by, because this part is smaller, and this is supposed to be one, one third so that's how we did it." Brian, disagrees with the girls' arguments. Audra or Jessica explains that they think Kelly is changing the problem by using different colored rods (light green instead of dark green). The researcher begins asking the girls what each rod represents for clarification. While pointing to the rods of different colors and sizes she asks "you're saying one half is bigger than one third by one third?" Jessica and Audra agree and state "yeah, you can put three of these, three reds up to one green." The girls recognize that three red rods represent 1/3 of 1/2 (the green rod). However, while the girls may see that the red rod and purple rod (1/2) are different sizes, they still describe 1/2 to be bigger than 1/3 by 1/3.
Post-Assessment Description

In the previous clip, Jessica and Audra used colored rods to model their claim that $\frac{1}{2}$ (dark green rods) is one third bigger than $\frac{1}{3}$ (purple rods). They showed that 3 red rods are the same as one dark green rod and a purple rod and red rod are the same length as a dark green rod. In this clip, researcher Maher asks the rest of the class if they are convinced after hearing the girl's argument. Kelly says that she agrees with the other girls, and goes up to the overhead to show the class why. While Kelly is explaining, she uses a different set of colors to support the claim. Here, Kelly uses dark green to represent 1, red to represent $\frac{1}{3}$, and light green to represent $\frac{1}{2}$. Kelly shows that the red rods are smaller than light green rods and states "one half is bigger by, because this part is smaller, and this is supposed to be one, one third so that's how we did it." Brian disagrees with the girls' arguments. Audra or Jessica claim that they think Kelly is changing the problem by using different colored rods (light green instead of dark green). The researcher begins asking the girls what each rod represents for clarification. While pointing to the rods of different colors and sizes she asks "you're saying one half is bigger than one third by one third?" Jessica and Audra agree and state "yeah, you can put three of these, three reds up to one green." The girls recognize that three red rods represent $\frac{1}{3}$ of $\frac{1}{2}$ (the green rod). However, while the girls may see that the red rod and purple rod...
(1/2) are different sizes, they still describe 1/2 to be bigger than 1/3 by 1/3.

The post-assessment language was changed to include, "claim."

Figure 14.5. Argumentation described by Teacher 7 for Event 5

In the pre-assessment, T7 describes the argument made in the previous event that 1/2 is greater than 1/3 by 1/3. The argument in this event is situated as an agreement to the previous argument. T7 uses the student's own words to state the claim, "one half is bigger by, because this part is smaller," which implies that the student is claiming that either, 1/2 is "bigger" than 1/3, or that 1/3 is "smaller" than 1/2. T7 notes data: the "red rods are smaller than light green rods." T7 describes the student's backing for these data, that 1. The red rod represents 1/3, the light green rod represents 1/2, and the premise, the dark green rod represents 1. Note that in the event, the student does give number names to the red and light green rods, however, she does not identify the dark green rod as the unit, thus, the text for the statement, "Here, Kelly uses dark green to represent 1," is gray in a gray box because T7 claims the student said it, but the student did not.

T7 notes that Brian disagrees with the argument presented situating the following statement as a counterargument: "Kelly is changing the problem by using different colored rods (light green instead of dark green)." Note that T7 doesn't make it explicit in the pre-assessment, but the claim is that the problem is being changed the fact that different colored rods are being used is data. T7 includes some additional information that is relevant to the student argumentation in this event, namely, "The girls recognize that three red rods represent 1/3 of 1/2 (the green rod). However, while the girls may see that the red rod and purple rod (1/2) are different sizes, they still describe 1/2 to be bigger than 1/3 by 1/3."
In the post-assessment, T7 changes the language specifically identifies statements as claims. This change is language is helpful in the statement, "Audra or Jessica claim that they think Kelly is changing the problem by using different colored rods (light green instead of dark green)," in that it explicitly identifies "Kelly is changing the problem" as the claim that the students are making." Specific language makes it clear what T7 sees as the claim in the counterargument.

14.1.6 Event 6

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
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<tbody>
<tr>
<td>In the previous clip, Brian stated that he was not convince [sic] that 1/2 is 1/3 bigger than 1/3. He restates their claim and argues &quot;Well, I don't really agree, because well if you split one of the thirds in half, which would make a sixth. I think it's a sixth bigger.&quot; From his explanation, it seems that Brian agrees with the girls that each red rods [sic] represent 1/3 of 1/2, however he explains that 1/3 of 1/2 is 1/6, not 1/3. He uses the red rods to explain his reasoning visually. He shows and explains that two of the red rods make up one purple rod (1/3). Brian shows that two red rods equal one purple rod and states that &quot;the purple is a third and the half of one third is sixth, there's sixths.&quot; Brian agrees with the others that three red rods equal one dark green rod. He compared the red rod with the purple rods (1/3) two argue that 2 red rods equal 1/3. So each red rod is 1/2 of 1/3, which is 1/6.</td>
</tr>
</tbody>
</table>
Post-Assessment Description

In the previous clip, Brian stated that he was not convinced that 1/2 is 1/3 bigger than 1/3. He restates their claim and argues "Well, I don't really agree, because well if you split one of the thirds in half, which would make a sixth. I think it's a sixth bigger." From his argument, it seems that Brian agrees with the girls that each red rod [sic] represent 1/3 of 1/2, however he explains that 1/3 of 1/2 is 1/6, not 1/3. He uses the red rods to model his reasoning. He models and explains that two of the red rods make up one purple rod (1/3). Brian shows that two red rods equal one purple rod and states that "the purple is a third and the half of one third is sixth, there's sixths." Brian agrees with the others that three red rods equal one dark green rod. He compared the red rod with the purple rods (1/3) to argue that 2 red rods equal 1/3. So each red rod is 1/2 of 1/3, which is 1/6.

No change in the argumentation described.

Figure 14.6. Argumentation described by Teacher 7 for Event 6

T7 notices several arguments in this event. T7 notices that Brian does not agree with the prior claim that 1/2 is 1/3 bigger than 1/3 and restates Brian's words, "Brian stated that he was not convinced [sic] that 1/2 is 1/3 bigger than 1/3. He restates their claim
and argues 'Well, I don't really agree, because well if you split one of the thirds in half, which would make a sixth. I think it's a sixth bigger.'" T7 then goes on to give an interpretation of Brian's "explanation." Brian's statement suggests that Brian's counterclaim is that 1/2 is 1/6 bigger than 1/3 with the data: splitting a third in half makes a sixth. However, the interpretation that T7 gives for Brian's statement differs from this, and it is that argumentation that is reflected in the diagram. T7 then notes that Brian's argument shows that he agrees with a previous argument made that "each red rods [sic] represent 1/3 of 1/2, but that Brian thinks that 1/2 of 1/2 is 1/6 not 1/3. By the statement, "however he explains that 1/3 of 1/2 is 1/6, not 1/3," it is implied that T7 believes that there was a prior claim that 1/2 of 1/3 is 1/3 and that the fact that represent 1/3 of 1/2 is data. Thus, the claim that 1/3 of 1/2 is 1/6 is a counterclaim to an implicit claim and challenges that the data supports the claim. Note, however, that, although T7 notes these as the argumentation put forth in this event and previous events, there is little evidence that the students actually made these statements. Therefore, the statements are included in the diagram as gray boxes and text.

T7's statement, "He uses the red rods to explain his reasoning visually," suggests that T7s next statements provide evidence for Brian's counterclaim, which is a continuation of T7's interpretation of Brian's statement that "if you split one of the thirds in half, which would make a sixth. I think it's a sixth bigger." However, when T7's next statements are analyzed, it is apparent that the connection is not made. Thus, in the diagram, the third argument is not connected to either of the other two. The description notes the claim that the red rod is 1/6, supported by data, that each red rod is 1/2 of 1/3, with two warrants that link the data to the claim: "2 red rods equal 1/3" and "purple rod
which implies that the number name for the purple rod is 1/3. Additionally, T7 describes backing for the warrant that 2 red rods equal 1/3, but describing the model that students made that shows, "that two red rods equal one purple rod." It might be suggested that T7 intended for there to be connections among these three arguments, but the language T7 uses does not support that interpretation. T7 changed some of the wording in the post-assessment, but the changes did not affect the argumentation described.

14.1.7 Event 7

Pre-Assessment Description
In a previous clip, Kelly showed that 1/2 was bigger than 1/3 using a light green rod to represent red rod to represent a 1/3. Prior to this, Jessica and Laura used the red rod to represent how much bigger 1/2 is than 1/3. In this clip Jackie and Kelly are explaining their representations. They used a dark green rod to represent 1, a light green rod to represent 1/2, a red rod to represent 1/3, and a white rod to represent how much bigger 1/2 is than 1/3. Jessica argued that they were using a different size candy bar, which "isn't fair." Researcher Maher asks why it isn't fair. Jessica compares the size of her 1/2 representation and Jackie's 1/2. Researcher Maher asks Jackie and Kelly "What did you call one if you're thinking of candy bars when you began the problem, you said the dark green is one" Kelly and Jackie agreed. Jessica's argument indicates that she is focusing on the size of the representation. However, while the representations differ in size, both groups started with a bar to represent 1. Their 1/2 bars both represented 1/2 of 1, even though they were different sizes.
Post-Assessment Description
In a previous clip, Kelly showed that 1/2 was bigger than 1/3 by modeling with a light green rod to model 1/2 and a red rod to model 1/3. Prior to this, Jessica and Laura used the red rod to represent how much bigger 1/2 is than 1/3. In this clip Jackie and Kelly are explaining their representations. They used a dark green rod to model 1, a light green rod to represent 1/2, a red rod to represent 1/3, and a white rod to represent how much bigger 1/2 is than 1/3. Jessica claims that they were using a different size candy bar, which "isn't fair." Researcher Maher asks why it isn't fair. Jessica compares the size of her 1/2 representation and Jackie's 1/2. Researcher Maher asks Jackie and Kelly "What did you call one if you're thinking of candy bars when you began the problem, you said the dark green is one?" Kelly and Jackie agreed. Jessica's argument indicates that she is focusing on the size of the representation. However, while the representations differ in size, both groups started with a bar to represent 1. Their 1/2 bars both represented 1/2 of 1, even though they were different sizes.
T7 describes prior argumentation in the pre-assessment. T7 notes that Kelly supported the claim that 1/2 was bigger than 1/3 with a light green rod and a red rod that represented 1/3 and that "Jessica and Laura used the red rod to represent how much bigger 1/2 is than 1/3." T7 goes on to describe claims made by the students in this event: "They used a dark green rod to represent 1, a light green rod to represent 1/2, a red rod to represent 1/3, and a white rod to represent how much bigger 1/2 is than 1/3." T7 notes the use of the rods as separate claims not connected to an argument. Although the claim that a white rod represents how much bigger 1/2 is than 1/3 is related to the prior claim that a red rod represents how much 1/2 is than 1/3, T7 does not make the connection.

T7 then notes that Jessica states that the other students are using a different size candy bar which is not fair. T7 does not give detail as to what Jessica means by, "they"
but the text suggests that T7 is alluding to the students who just made their argument with the dark green rod having the number name 1. T7 notes that Jessica gives support for her statement that it isn't fair to use different size candy bars by referring to the size of the halves, "Researcher Maher asks why it isn't fair. Jessica compares the size of her 1/2 representation and Jackie's 1/2." T7 makes an additional comment related to the argumentation: the students' bars both represented 1/2 of 1 even though they were different sizes. Thus, in the data for the last argument, it is implied that the two representations of 1/2 are different sizes.

In the post-assessment, T7 clarifies the last argument by using more specific language, "Jessica claims that they were using a different size candy bar, which 'isn't fair.'" The use of "claim" in this statement verifies that T7 believes that using a different size candy bar is not fair is a claim made by Jessica. Additionally, T7 adds some text to the data given in the first statement to clarify that the light green rod was used to model 1/2. There was no change in the structure of the argumentation.

14.1.8 Event 8

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
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<tbody>
<tr>
<td>In this clip, Jackie and Kelly are explaining why they think 1/2 is bigger than 1/3 using the scaling they showed in the previous clip: dark green represents 1, red represents 1/3, and light green represents 1/2. When asked why the light green equals 1/2, Jackie moves two light green rods under the dark green rod and says &quot;because if you put these all[both] together they equal up to the one.&quot; Jackie is talking about the fractions with respect to the dark green rod (1 whole.) Next, Jackie explains that the light green is a half and is bigger than the red &quot;by one which is this white one.&quot; Jackie observes that when one white rod is added to the red rod, the combination equals 1/2 (light green rod). Researcher Maher asks Jackie to clarify what number the white rod represents. Initially, Jackie explains that the light green is &quot;one bigger&quot; than the red rod and that her number name for the white rod is &quot;one.&quot; At the end of the clip, Jackie explains &quot;actually, I used this to um, to tell that the light green is one white bigger.&quot; While Jackie does not give a new number name for the white rod, she clarifies that the light green rod is &quot;one white bigger&quot; than the red rod. By adding &quot;white&quot; after one, it seems that she has acknowledged that the white rod does not represent 1.</td>
</tr>
</tbody>
</table>
In this clip, Jackie and Kelly are modeling why $\frac{1}{2}$ is bigger than $\frac{1}{3}$ using the scaling they showed in the previous clip: dark green represents 1, red represents $\frac{1}{3}$, and light green represents $\frac{1}{2}$. When asked why the light green equals $\frac{1}{2}$, Jackie moves two light green rods under the dark green rod and says "because if you put these all[both] together they equal up to the one." Jackie is talking about the fractions with respect to the dark green rod (1 whole.) Next, Jackie claims that the light green is a half and is bigger than the red "by one which is this white one." Jackie observes that when one white rod is added to the red rod, the combination equals $\frac{1}{2}$ (light green rod). Researcher Maher asks Jackie to clarify what number the white rod represents. Initially, Jackie claims that the light green is "one bigger" than the red rod and that her number name for the white rod is "one." At the end of the clip, Jackie modifies her claim by stating "actually, I used this to um, to tell that the light green is one white bigger." While Jackie does not give a new number name for the white rod, she modifies her claim by stating that the light green rod is "one white bigger" than the red rod. By adding "white" after one, it seems that she has acknowledged that the white rod does not represent 1.
In the pre-assessment, T7 notes a claim, that 1/2 is bigger than 1/3, and data regarding the rods—that the dark green rod represents 1, the red rod represents 1/3, and the light green rod represents 1/3. A warrant, that two light greens "equal up to one" is also noted. A second argument is noted by T7 with the claim that the light green rod is bigger than the red rod by one, supported with the warrant that when one white rod is added to the red rod, the combination equals 1/2 (light green rod) and the warrant that the white rod has the number name one. T7 includes an additional note that is relevant to the argumentation, specifically, that T7 believes that because the final statement by the student: "actually, I used this to um, to tell that the light green is one white bigger," suggests that the student no longer thought that the number name for the white rod was 1.

In the post-assessment, T7 uses the formal mathematical register for argumentation to clarify the statements that are being considered claims, specifically,
"Next, Jackie claims that the light green is a half and is bigger than the red 'by one which is this white one,'" and "Initially, Jackie claims that the light green is 'one bigger' than the red rod and that her number name for the white rod is 'one.'" Additionally, T7 adjusts the language in the last statements to suggest that, the student "modifies her claim by stating that the light green rod is 'one white bigger' than the red rod." This change in language adds an additional element of argumentation, i.e., the modification of a claim, and changes the structure of the final argument. Note that with the change in language, what was an additional comment about the argumentation became an element of argumentation linked to the argument itself.

14.1.9 Event 9

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>In this clip, Erik is describing what he believes Jackie was trying to explain in the previous clip. He mentions that they called the dark green rod 1 whole. Next, he lines up 6 white rods above the dark green rod to show that 6 white rods equal one dark green rod (1 whole.) He adds &quot;they want to call it one sixth. I think that's what they're trying to say but they just, they're just not saying it.&quot; The girls agree that the number value that they want to name the white rod is one sixth. When researcher asks how much bigger 1/2 is than 1/3, the girls respond by saying &quot;um one, one sixth.&quot; Finally, they explain that the answer cannot be one, because the dark green rod represents one.</td>
</tr>
</tbody>
</table>
Post-Assessment Description

In this clip, Erik is describing what he believes Jackie was trying to explain in the previous clip. He mentions that they called the dark green rod 1 whole. Next, he lines up 6 white rods above the dark green rod to claim and model that 6 white rods equal one dark green rod (1 whole.) He adds "they want to call it one sixth. I think that's what they're trying to say but they just, they're just not saying it." The girls agree that the number value that they want to name the white rod is one sixth. When researcher asks how much bigger 1/2 is than 1/3, the girls respond by saying "um one, one sixth." Finally, they refute their initial claim that the white rod represented one, because the dark green rod represents one.
Figure 14.9. Argumentation described by Teacher 7 for Event 9

In the pre-assessment description, with the statement, "He mentions that they called the dark green rod 1 whole. Next, he lines up 6 white rods above the dark green rod to show that 6 white rods equal one dark green rod (1 whole.) He adds "they want to call it one sixth. I think that's what they're trying to say but they just, they're just not saying it," T7's intended argumentation description is uncertain. One way to look at this described argumentation appears in the diagram. Specifically, that the claim "they want to call the white rod a 1/6" is connected to an implied previous claim that students gave the white rod the number name 1 and supported by the data that "6 white rods equal one dark green rod." The warrants: the model that shows 6 white rods lined up above the dark green rod, and the dark green rod is one whole, connect the data to the claim. However, the language of T7 puts this interpretation into question. Between the statement of data "6 white rods equal one dark green rod (1 whole.)" and the statement of the claim "they want
to call it one sixth," T7 says, "He adds." The lack of the use of connective language between these two statements, such as "so," "since," or "because" makes the connections in the diagram uncertain, and thus, represented by dashed, grayed lines.

T7 goes on to note that other students agree that the "number value that they want to name the white rod is one sixth" and these same student claim that 1/2 is bigger than 1/3 by 1/6. A final claim is noted, that "the answer cannot be one" supported by the data, "the dark green rod represents one." Again, the imprecise language of T7 when he states, "the answer cannot be one," makes the diagramming of the argumentation uncertain. If the statement refers to the claim that the number name for the white rod is 1, then a previous claim is alluded to and appears in the dashed box. If, however, the statement refers to the claim that 1/2 is greater than 1/3 by 1, then the statement is connected to another prior statement that was only implied, but never explicitly stated by T7 in this description.

T7 uses specific language in the post-assessment that includes more of the formal mathematical language of argumentation. The use of this language clarifies the argumentation that T7 is describing. For example, T7 states, "Next, he lines up 6 white rods above the dark green rod to claim and model that 6 white rods equal one dark green rod (1 whole.) He adds "they want to call it one sixth." The use of "claim" here clarifies the uncertain intent of T7's statement in the pre-assessment. The claim is that "6 white rods equal one dark green rod (1 whole.)" and the model is the data, with the backing being just that the dark green rod has the number name "one whole." The claim that the students wanted to call the white rod 1/6 is no longer linked to the argument.
T7 changes the language in the last statement, as well: "Finally, they refute their initial claim that the white rod represented one, because the dark green rod represents one." The use of the more formal mathematical register and the addition of detail make the intent of T7 more certain. T7 is referring to the number name of the white rod that cannot be 1. The claim is implied, but the prior claim is now explicit. It is also explicit that the data, "the dark green rod represents one" refutes the prior claim as well as supports the implicit counterclaim that the white rod cannot be 1.

Although argumentation diagram for the argumentation described in the pre-assessment has more boxes and connections than the diagram in the post-assessment, the imprecision of the language in the pre-assessment leads to uncertainty in the arguments being described, which is evidenced by the dashed lines. The description of the argumentation in the post-assessment is more certain.

13.1.10 Event 10

**Pre-Assessment Description**
In the previous clip, Erik argued that one half is bigger than one third by one sixth using Jessica and Kelly's representation (dark green, red, light green, and white rods.) In this clip, researcher Maher is asking for Jessica and her group to explain how much bigger 1/2 is that 1/3 using their group's representation. Here, an orange and red combination represents 1 whole, a dark green rod represent 1/2, and a purple rod represents 1/3. The girls claim that the dark green rod is one red rod bigger than the purple rod. Researcher Maher asks the girls what number value the red rod represents. She adds "now if you really understood what mistake you made here maybe you'll figure out what mistake you made up there." The girls previously gave the red rod the number name "one third." Jessica explains that three reds equal one green and "you need a red to go next to the purple, so it would be one third." Again, Jessica sees that red is one third of green. However, she has given green the number name "one half." Therefore, while green is one third of green, its numerical value is not "one third." While in the previous clip the students agreed that 1/2 is bigger than 1/3 by 1/6, their personal explanations indicate that the students do not understand the reasoning behind this claim.
Post-Assessment Description
In the previous clip, Erik argued that one half is bigger than one third by one sixth using Jessica's and Kelly's representation (dark green, red, light green, and white rods.) In this clip, researcher Maher is asking for Jessica and her group to explain how much bigger 1/2 is that 1/3 using their group's representation. Here, an orange and red combination represents 1 whole, a dark green rod represent 1/2, and a purple rod represents 1/3. The girls claim that the dark green rod is one red rod bigger than the purple rod. Researcher Maher asks the girls what number value the red rod represents. She adds "now if you really understood what mistake you made here maybe you'll figure out what mistake you made up there." The girls previously gave the red rod the number name "one third." Jessica claims that three reds equal one green and "you need a red to go next to the purple, so it would be one third." Again, Jessica sees that red is one third of green. However, she has given green the number name "one half." Therefore, while green (red?) is one third of green, its numerical value is not "one third." While in the previous clip the students agreed that 1/2 is bigger than 1/3 by 1/6, their personal explanations indicate that the students do not understand the reasoning behind this claim.

The post-assessment language was changed to include "claim."

Figure 14.10. Argumentation described by Teacher 7 for Event 10

T7, in the pre-assessment, describes a previous argument (Erik's argument about 1/2 being bigger than 1/3 by 1/6). This argument is not included in the diagram since T7 does not link that argument to the one presented in this clip. T7 notes the students' claim
that "the dark green rod is one red rod bigger than the purple rod." Then describes the argument that the students' make supporting that the red rod has the number name 1/3, using "you need a red to go next to the purple" as the data. The use of "so" connects the data and the claim and the use of "and" suggests that both statements, "red is one third of green" and "three reds equal one green" are true and provide support for the claim. Note that T7 does not link the claim that the dark green rod is one red rod bigger than the purple rod to the claim that the purple rod has the number name 1/3 in event's description, so not connection is made in the diagram.

T7 includes some additional comments that relevant to the argumentation in this event. T7 points out that the student has given the dark green rod the number name 1/2, and, while length of the red rod is one third the length of the green rod, the numerical value is not 1/3 as is being claimed by these students. T7 also comments that this argument is evidence that the students do not understand the reasoning of the other students' prior arguments. In the post-assessment, T7 uses more of the formal mathematical register by using "claim," but this change does not affect the argumentation described.

14.1.11 Event 11

Pre-Assessment Description
Students previously agreed that 1/2 is bigger than 1/3 by 1/6. However, when explaining it again, students stated that 1/2 is bigger than 1/3 by 1/3. In this clip, Brian is explaining why he believes 1/3 is not the correct answer. Brian wants to name the red rod one sixth instead of one third. He is describing a green rod that is placed under two purple rods. He shows that one red rod is half of one purple rod. He shows that two purple rods is one red rod bigger than a dark green rod. Finally, Jessica suggests that 1/3 and 1/6 might both be correct answers. Here Brian shows that adding 1/3 to 1/3 is bigger than 1/2 using the rods. He compared the size of the red rods to the size of the purple rods. In fact, one red rod is half of the size of a purple rod.
**Post-Assessment Description**

Students previously agreed that $1/2$ is bigger than $1/3$ by $1/6$. However, when explaining it again, students stated that $1/2$ is bigger than $1/3$ by $1/3$. In this clip, Brian is explaining why he believes $1/3$ is not the correct answer. Brian wants to name the red rod one sixth instead of one third. He is describing a green rod that is placed under two purple rods. He shows that one red rod is half of one purple rod. He shows that two purple rods is one red rod bigger than a dark green rod. Here, Brian was supporting his counterclaim \(1/2\) is bigger than $1/3$ by $1/6) using modeling. Finally, Jessica claims that $1/3$ and $1/6$ might both be correct answers. Here Brian shows that adding $1/3$ to $1/3$ is bigger than $1/2$ using the rods. He compared the size of the red rods to the size of the purple rods. In fact, one red rod is half of the size of a purple rod.
T7 notes two previous claims in the pre-assessment description, 1/2 is bigger than 1/3 by 1/6 and 1/2 is bigger than 1/3 by 1/3. He then states that, "Brian is explaining why he believes 1/3 is not the correct answer." Which implies, though does not make explicit, that the statements that follow support an argument that "1/3 is not the correct answer." Since two claims have been made previously and are mentioned in T7's description—the number name for the red rod is 1/3 and that 1/2 is greater than 1/3 by 1/3—it is uncertain to which "1/3 is not the correct answer" refers. The claim, then is connected by a dashed line to both. T7 then states, in the context of explaining why Brian believes that 1/3 is not the correct answer, that "Brian wants to name the red rod one sixth instead of one third."
This statement, 1. alludes to a prior claim, that the number name of the red rod is 1/3, and
2. is a counterclaim in itself as well as data for the claim that 1/3 cannot be the answer.

The next statements by T7 describe the rod model that Brian creates. The lack of
connectivity in the language here makes it unclear what connections T7 intends both
from one statement to another, and between the statements and the argument being made
by the student. Thus, there are no connectors from these elements to the data and claim.
T7 then states that "Jessica suggests that 1/3 and 1/6 might both be correct answers."
Again, since the claim that "the answer cannot be 1/3" is vague, to what T7 thinks
Jessica's statement refers is unclear. The uncertainty is represented by dashed lines. The
connection is so uncertain that it could be argued that there is no connection, and thus
should be no lines, even dashed ones. T7 then returns to Brian's argument, adding a small
amount of clarity to his prior statements. "Here Brian shows that adding 1/3 to 1/3 is
bigger than 1/2 using the rods," suggests that T7 notices that, at least some of previous
description of the rods, supports the data that " adding 1/3 to 1/3 is bigger than 1/2." The
connection is still unclear, so the connections are dashed. The final statement, "He
compared the size of the red rods to the size of the purple rods. In fact, one red rod is half
of the size of a purple rod," clarifies that the model in which the red and purple rods are
compared supports the idea that "one red rod is half of the size of a purple rod." The use
of "in fact" provides more certainty about this connection, than the previous ones.

The use of more formal mathematical language adds clarity to T7's description
and structure to the argumentation presented in the post-assessment. T7 states, "Here,
Brian was supporting his counterclaim (1/2 is bigger than 1/3 by 1/6) using modeling."
This statement clarifies T7's argumentation in several ways. First, the use of "supporting"
and "modeling" provides more certainty that the description of the rod models are support for Brian's argument. Additionally, T7 explicitly clarifies the prior vague claim that "1/3 is not the correct answer," by stating Brian's claim explicitly, "1/2 is bigger than 1/3 by 1/6." Making the claim explicit here is key in determining the structure of the argumentation T7 describes and role that T7's statements play as elements of that argumentation. Furthermore, by identifying Brian's claim as a "counterclaim," T7 situates the argument as a counterargument, and Jessica's claim that "1/3 and 1/6 might both be correct answers" as a modification of the previous claims. The precision of language used in the post-assessment adds to the structure and clarifies the role of the elements of the argumentation described by T7.

14.1.12 Event 12

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
</tr>
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<tbody>
<tr>
<td>In this clip, Erik questions the girls' argument of why 1/2 is bigger than 1/3 by 1/3. He uses one dark green rod (representing 1/2) and two purple rods (each representing 1/3) to show why their claim cannot be true. He keeps the whole (orange and red rod) and three thirds (three purple rods) in place, and removes one green rod (1/2). He states &quot;the half is almost as big as two thirds, but yet the two thirds aren't exactly...the dark green is not exactly as big as the two two thirds.&quot; Since the girls previously claimed that 1/2 is bigger than 1/3 by 1/3, Brian showed them that adding 1/3 to 1/3 results in more than 1/2. He adds &quot;two thirds are not bigger than it by one third.&quot; So while adding 1/3 to 1/3 results in a value bigger than 1/2, Brian shows that the result is not 1/3 bigger than 1/2. This shows that 1/2 is bigger than 1/3, however it is less than 1/3 bigger than 1/3. Erik's argument is convincing to Brian, and Brian states that he now disagrees with the girls. Finally, Erik concludes that the answer &quot;couldn't be exactly a third.&quot;</td>
</tr>
</tbody>
</table>
Post-Assessment Description

In this clip, Erik challenges the claim that 1/2 could be bigger than 1/3 by both 1/3 and 1/6. Additionally, he challenges the girls' argument of why 1/2 is bigger than 1/3 by 1/3. He uses one dark green rod (representing 1/2) and two purple rods (each representing 1/3) to show why their claim cannot be true. He keeps the whole (orange and red rod) and three thirds (three purple rods) in place, and removes one green rod (1/2). He states "the half is almost as big as two thirds, but yet the two thirds aren't exactly...the dark green is not exactly as big as the two two thirds." Since the girls previously claimed that 1/2 is bigger than 1/3 by 1/3, Brian modeled that adding 1/3 to 1/3 results in more than 1/2. He adds "two thirds are not bigger than it by one third." So while adding 1/3 to 1/3 results in a value bigger than 1/2, Brian models that the result is not 1/3 bigger than 1/2. This shows that 1/2 is bigger than 1/3, however it is less than 1/3 bigger than 1/3. Erik's argument is convincing to Brian, and Brian states that he now disagrees with the girls. Finally, Erik claims that the answer "couldn't be exactly a third."
In the pre-assessment description, T7 notes that the argument put forth by Erik in the event is a result of questioning the previous argument of why 1/2 is bigger than 1/3 by 1/3. This situates the argument as a counterargument and the claim, "the answer 'couldn't be exactly a third''' as a counterclaim. Some of the reasoning described by T7 is the student's reasoning, but some is inferred by T7. The inferred elements are true, but do not appear in the event, they are notated in gray text. The claim, the answer cannot be a third is supported by the data, "two thirds are not bigger than it [1/2] by one third," which is
also a claim that is supported by the students' argumentation. The student supports this
Claim/Data with statement that functions as both a warrant and data, "half is almost as
big as two thirds" and the Data/Warrant is supported by the description of the rod model,
the "dark green [rod] is not exactly as big as the two thirds," the purple rod represents
1/3, the dark green rod represents 1/2, and the premise: the orange and red rod train
represents "the whole."

T7 makes some explanatory statements to interpret the student's argument, "So
while adding 1/3 to 1/3 results in a value bigger than 1/2, Brian shows that the result is
not 1/3 bigger than 1/2," and "This shows that 1/2 is bigger than 1/3, however it is less
than 1/3 bigger than 1/3." These statements function as warrants for the data that "two
thirds are not bigger than it [1/2] by one third," and are in gray since they are true, but not
stated by the students in the event.

T7 adds detail that clarifies the argumentation in the post-assessment. T7 states,
"Erik challenges the claim that 1/2 could be bigger than 1/3 by both 1/3 and 1/6.
Additionally, he challenges the girls' argument of why 1/2 is bigger than 1/3 by 1/3."
Thus T7 situates the argument as a challenge to both the claim that 1/2 is bigger than 1/3
"by both 1/3 and 1/6," introducing an additional element of argumentation, and the claim
that 1/2 is bigger than 1/3 by 1/3. The more precise language introduces more elements of
argumentation and structure into T7's description.

14.1.13 Event 13
Pre-Assessment Description

In this clip, Brian and Erik are explaining why 1/2 cannot be exactly 1/3 bigger than 1/3. Brian uses the colored rods to explain his reasoning. He shows points to two purple rods above the green rod and explains "this would really by one third bigger and there's still some left over." Erik mentions that there is "a half left over." Brian clarifies that there is a piece of the second purple rod left over- "about a sixth left." Finally, Erik summarizes the boys' explanation. When talking about the green rod, which represents one half, he states "It's less than two thirds but it's more than one third." Erik finally states "It'd have to be one third and a half." From his explanation and reasoning, it seems that he understands fractions. If asked what he means by "a half" he would most likely describe a half of a third or one sixth.

Post-Assessment Description

In this clip, Brian and Erik are arguing why 1/2 cannot be exactly 1/3 bigger than 1/3. Brian uses the colored rods to model his reasoning. He points to two purple rods above the green rod and explains "this would really by one third bigger and there's still some left over." Erik mentions that there is "a half left over." Brian clarifies that there is a piece of the second purple rod left over- "about a sixth left." Finally, Erik summarizes the boys' argument. When talking about the green rod, which represents one half, he states "It's less than two thirds but it's more than one third." Erik finally states "It'd have to be one third and a half." The boys model and refute the girls claim that 1/2 is bigger than 1/3 by 1/3.
T7 notes the claim that "1/2 cannot be exactly 1/3 bigger than 1/3," and its argument. Although this claim is a counterclaim to a claim made previously in the VMCAAnalytic, T7 does not mention the claim or previous arguments here, therefore there no connections are noted in the diagram. The argument described by T7 has the data, "1/2 is "one third and a half [of a third]" that supports the claim and two warrants that connect the data and the claim: "1/2 is less than two thirds but it's more than one third," and there is a "sixth left over." The warrant "1/2 is less than two thirds but it's more than one third"
is supported by the model showing two purple rods above the green rod and that the green rod represents 1/2.

The warrant: "there is a 'sixth left over'" is also a claim that is supported by a sub-argument. The support for this claim includes that there is a "sixth left over," which is supported by the fact that there is some left over and half left over. T7 notes that the rod model that shows there is a piece of the second purple rod left over" provides the backing for this warrant.

In the post-assessment description, the inclusion of some of the formal mathematical register situates the argument as a counterargument. T7 states: "The boys model and refute the girls claim that 1/2 is bigger than 1/3 by 1/3." Thus, the argument is a counterargument to the claim that 1/2 is bigger than 1//3 by 1/3. Additionally, T7 further describes that the prior claim has been refuted with the student argumentation in this event. Thus, T7 adds both structure and additional elements of argumentation in the post-assessment description.

14.1.14 Event 14
Pre-Assessment Description
Prior to Erik's argument, the class agreed that a 1/2 is bigger than 1/3 by a value represented by the red rod. Some students thought that the red rod should have a number value of 1/3. Michael and Brian both disagreed. Erik's argument contradicted the girls' claim by showing that the answer could not be 1/3. Brian suggested that the number value for the red rod should be 1/6, since it is about half the size of the purple rod. In this clip, Michael is showing how he knows the number value of the red rod is 1/6. He references the designated "one whole" rod (an orange and a red rod combination.) As he is explaining, he lines up six red rods above the "whole" rod to show that 6 red rods equal one whole.

Post-Assessment Description
Prior to Erik's argument, the class agreed that a 1/2 is bigger than 1/3 by a value represented by the red rod. Some students thought that the red rod should have a number value of 1/3. Michael and Brian both disagreed. Erik's argument contradicted the girls' claim by showing that the answer could not be 1/3. Brian suggested that the number value for the red rod should be 1/6, since it is about half the size of the purple rod. In this clip, Michael is modeling his reasoning for why the number value of the red rod is 1/6. He references the designated "one whole" rod (an orange and a red rod combination.) As he is explaining, he lines up six red rods above the "whole" rod to show that 6 red rods equal one whole.

No change in the argumentation described.

Figure 14.14. Argumentation described by Teacher 7 for Event 14

In pre-assessment, T7 describes the argumentation that has happed prior to this event. T7 notes the argument in this event in which the claim is that the number value of the red rod is 1/6, supported by the data that 6 red rods equal a whole. The data is connected to the claim through the warrant. T7 describes the rod model that shows 6 red
rods lined up "above the 'one whole.'" And that the orange and a red rod combination" is "one whole." The post-assessment description describes the same argumentation.

14.1.15 Event 15

**Pre-Assessment Description**
In this clip, Erik considers the case where dark green represents one whole, light green represents 1/2, and red represents 1/3. He shows that the light green rod is "less than two thirds but more than one third. So it can't be a third." Finally, Michael mentions "It's sort of like one sixth in both cases." While before some students suggested that the cases would have different answers since the representing size of 1 differed, the students later notices [sic] that the answer is the same. This is because the physical amount may be different, however the relations were the same.

![Diagram](image)

**Post-Assessment Description**
In this clip, Erik considers the case where dark green represents one whole, light green represents 1/2, and red represents 1/3. He shows that the light green rod is "less than two thirds but more than one third. So it can't be a third." Finally, Michael mentions "It's sort of like one sixth in both cases." While before some students suggested that the cases would have different answers since the representing size of 1 differed, the students later notices [sic] that the answer is the same. Since the students have showed that the difference between 1/2 and 1/3 is 1/6 no matter the model, they have generalized their reasoning to prove the claim.
T7 describes Brian's argument as being based on "the case where dark green represents one whole, light green represents 1/2, and red represents 1/3." It is clear from the event video clip that, although Brian says, "light green," he is pointing to the dark green rod. So, the argument that T7 describes is gray since the student did not make the statements. T7 states, "Michael mentions 'It's sort of like one sixth in both cases.' While before some students suggested that the cases would have different answers since the representing size of 1 differed, the students later notices [sic] that the answer is the same." Thus, another claim is made, specifically, that "the representing size of 1 differed," it is "one sixth in both cases."

In the post-assessment T7 states, "Since the students have showed that the difference between 1/2 and 1/3 is 1/6 no matter the model, they have generalized their reasoning to prove the claim." This statement uses the formal mathematical register and
suggests another argument with a claim, "1/2 and 1/3 is 1/6" and data, "it is 'one sixth in both cases.'" T7 also notes that the claim seems to be proven using generalized reasoning and connects this summary to statement to the arguments described.

14.2 Summary of Teacher 7's Growth across Events

In five of the 15 events, T7 shows growth with respect to the elements of argumentation described in the post-assessment as compared with the pre-assessment. In Event 8, T7 adjusts the language to suggest that, the student "modifies her claim by stating that the light green rod is 'one white bigger' than the red rod." This change in language adds an additional element of argumentation, i.e., the modification of a claim and transforms what was an additional comment about the argumentation in the pre-assessment into an element of argumentation linked to the argument itself and in Event 11, changes in the description add a counterargument as well as a modification. T7's changes in Event 12 add a prior claim and an implicit counterclaim, as well as a challenge to a prior claim and in Event 13, T7 adds a counterargument and a prior claim. The changes in Event 15 add an argument with a claim and data, as well as a summary statement that, "generalized their reasoning to prove the claim."

In seven of the 15 events, T7 shows growth with respect to the structure of the argumentation described from pre- to post-assessment. In Event 3, T7 the addition of the use of the language "modified and clarified" situates Audra's argument as being, not only a "clearer" agreement of the previous argument, but a modified argument, which adds to the structure of the argumentation that T7 noticed. In Event 8, the changes T7 made to the language changed the structure of the argument, transforming what was a statement relevant to the argumentation into a claim connected to a prior claim as a modification.
Although there are more elements of argumentation mentioned in the pre-assessment compared to the post-assessment, the links among those elements are uncertain. It is difficult to determine if implied connections were intended because of the imprecision of language. In the post-assessment, the use of the formal mathematical register supports more certainty with regard to the structure of the argument by clarifying the connections that T7 makes among the elements. T7 states, "Next, he lines up 6 white rods above the dark green rod to claim and model that 6 white rods equal one dark green rod (1 whole.) then adds "they want to call it one sixth." The use of "claim" here clarifies the uncertain intent of T7's statement in the pre-assessment. The claim is that "6 white rods equal one dark green rod (1 whole.)" and the model is the data, with the backing being that the dark green rod has the number name "one whole." The claim that the students wanted to call the white rod 1/6 is no longer linked to the argument. The statement, "Finally, they refute their initial claim that the white rod represented one, because the dark green rod represents one," makes explicit the connection that the data, "the dark green rod represents one" refutes the prior claim as well as supports the implicit counterclaim that the white rod cannot be 1. Where as in the pre-assessment there are eight elements or connections that are uncertain, in the post-assessment, there are only two.

In Event 11, T7 again makes changes to the post-assessment descriptions that help to clarify the argumentation in the event. T7 states, "Here, Brian was supporting his counterclaim (1/2 is bigger than 1/3 by 1/6) using modeling." The use of "supporting" and "modeling" provides more certainty that T7 intends that the description of the rod models are support for Brian's argument. Additionally, T7's statement that Brian's claim, "1/2 is bigger than 1/3 by 1/6." explicitly clarifies the prior vague claim that "1/3 is not
the correct answer." Making the claim explicit here is key in determining the structure of the argumentation T7 describes and role that T7's statements play as elements of that argumentation. Furthermore, by identifying Brian's claim as a "counterclaim," T7 situates the argument as a counterargument, and Jessica's claim that "1/3 and 1/6 might both be correct answers" as a modification of the previous claims. The precision of language used in the post-assessment adds to the structure and clarifies the role of the elements of the argumentation described by T7. As in Event 8, the changes to the description clarify the argumentation by making the connections intended by T7 more certain.

The structure of the argumentation described in Event 12 is changed by alterations to the post-assessment. T7 states, "Erik challenges the claim that 1/2 could be bigger than 1/3 by both 1/3 and 1/6. Additionally, he challenges the girls' argument of why 1/2 is bigger than 1/3 by 1/3." Thus T7 situates the argument as a challenge to both the claim that 1/2 is bigger than 1/3 "by both 1/3 and 1/6," and the claim that 1/2 is bigger than 1/3 by 1/3. In Event 13, T7 changes the structure of the argument by including language that describes a counterargument that results in a refutation of a prior claim. This counterargument, then is linked to the prior claim. In Event 15, T7 adds structure by connecting a claim and data, as well as adding a summary statement that connects to the arguments presented.

In 13 of 15 events, T7 showed growth with respect to the use of the formal mathematical register of argumentation. In Events 1, 4, 5, 7, 8, 9, 10, 11, 12, 13, and 15, T7 uses, "claim," to specifically identify certain statements made by students as claims in an argument. In Event 2, T7 rather than saying, "agreed" and "stated," "claimed" is used. Additionally, the use of "supporting their claim," with reference to the rods explicitly
identifies the rod model as evidence for the claim. In Event 3, T7 states, "Audra modified and clarified the girls' claims." This statement adds precision of language by including the idea of modification. The change in language in Event 4 in the statement "Audra or Jessica claim that they think Kelly is changing the problem by using different colored rods (light green instead of dark green)," is helpful in that it explicitly identifies "Kelly is changing the problem" as the claim that the students are making," making it clearer as to what T7 sees as the claim in the counterargument. In Event 7, the use of "claim," verifies that T7 believes that using a different size candy bar is not fair is a claim made by Jessica. In Event 8, in addition to using "claim," to specifically identify students' statements, T7 states that one of the students, "modifies her claim," indicating more precisely not only is the statement a claim, but a modification of a previous claim. In Event 9, T7 uses, "refute their initial claim" rather than "explain that the answer cannot be one" adding precision to the argumentation described and in Event 10, T7 uses "counterclaim" and "supporting" to clarify the argumentation described. In Event 13, T7 specifies that students, "refuted," other students' claim and in Event 15, T7 uses "prove" and "generalized," in the post-assessment description.

In three of the 15 events, T7 showed growth in the argumentation described in other ways. In the pre-assessment description of Event 4, T7 states, "Laura notices that the dark green rod (1/2) is 'one red bigger' than the purple rod (1/3)," emphasizing the model and suggesting that the statement was data for an implied claim. In the post-assessment description, interchanging the word "notices" with the word "claim" identifies the statement as a claim and making an implicit element explicit. T7 also changes the description to make an implicit claim explicit in Event 8. In Event 7, T7 adds detail to
data, changing the description of the data from "using a light green rod," to "using a light green rod to model 1/2" and making the description of the data more precise.
Chapter 15 – Teacher 8, Event Descriptions Analyses and Summary

This chapter presents the descriptions Teacher 8 (T8) wrote for the pre-assessment and post-assessment analytic to describe the argumentation in each event. The pre-assessment and post-assessment descriptions given by the teacher are presented for each event with an accompanying diagram using Toulmin's (1958, 2003) scheme.

Following the descriptions for each event, I present an in-depth analysis, summarizing the argumentation described in the pre-assessment and noting the changes the teacher made from pre- to post-assessment. Words that are key to my analysis appear in red text in the teacher’s descriptions.

15.1 Teacher 8 Event Descriptions and Analyses by Event

15.1.1 Event 1

Pre-Assessment Description
In this event, we see the encouragement of argumentation by Dr. Maher. Rather than simply accept the first answer from the first student selected (Michael), she continues to ask questions that either agree or disagree with his reasoning. This allows for ideas of other students to come forward and be presented, whether correct or incorrect. No argumentation described.

Post-Assessment Description
Dr. Maher begins the session by activating the students' knowledge of the prior session, in which they compared various fractions to determine "which is bigger". We see the encouragement of argumentation - rather than simply accept the answer from the first student selected (Michael), she continues to ask questions that either agree or disagree with his reasoning. This allows for ideas of other students to be valued and given the opportunity to be presented, whether correct or incorrect. When Dr. Maher continues by asking, "Do you think you could convince Dr. Davis that one half is bigger than one third?", she stretches the students' thinking further, encouraging them to develop evidence that will support their claims. Two students, Jessica and Laura, are called to present their claim and support on the classroom overhead.

Figure 15.1. Argumentation described by Teacher 8 for Event 1
In the pre-assessment, T8 does not describe any specific argumentation. In the post-assessment description, T8 includes references to argumentation. The statement, "When Dr. Maher continues by asking, 'Do you think you could convince Dr. Davis that one half is bigger than one third?' she stretches the students' thinking further, encouraging them to develop evidence that will support their claims" suggests that T8 intended to describe the claim that "one half is bigger than one third." This claim is supported by a general reference to "evidence," that is presented on the overhead. Note that T8 includes more elements of argumentation in the post-assessment description, as well as more of the formal mathematical register with the use of "claim," and "evidence."

15.1.2 Event 2

Pre-Assessment Description
Within this event, Jessica and Laura are questioned by Dr. Maher, in hopes of clarifying their reasoning. Even though these two girls may be able to understand their approach, the questions that they are asked to answer help them work towards a more mathematically sound proof that can be understood by others.

No argumentation described.

Post-Assessment Description
Jessica and Laura proceed to use the tiles on the overhead. At first, they state that "one third is bigger than one half", but as they provide justification, they state that "because this would be one third and this bigger piece would be one half". Within this event, Jessica and Laura are questioned by Dr. Maher, in hopes of clarifying their reasoning after they explained without interruption by others. Even though these two girls may be able to understand their approach, the questions that they are asked to answer help them work towards a more mathematically sound proof that can be understood by others as well.

Data: "This would be one third and this bigger piece would be one half"

Because

Claim: "one third is bigger than one half"

Figure 15.2. Argumentation described by Teacher 8 for Event 2

T8 does not describe any argumentation in the pre-assessment. In the post assessment, T8 describes an argument with the claim that "one third is bigger than one
half," and data "this would be one third and this bigger piece would be one half." The change in the title of the event from, "Clarifying the Justification," to "Jessica & Laura Present Their Claim," identifies the statement that one third is bigger than one half explicitly as a claim. Thus, T8 demonstrates growth both in the use of the formal mathematical register of argumentation and elements of argumentation described.

15.1.3 Event 3

Pre-Assessment Description
In this event, Audra is called to share her opinion whether she agrees or disagrees with the explanation provided by Jessica and Laura in the previous event. Audra is not merely sharing her opinion, but also given the opportunity to provide her reasoning of why the justification makes sense. In this sense of argumentation, a presented idea is in the works of potentially being validated by another person, and in doing so, bringing forward new expressions of the same concept.

No argumentation described.

Post-Assessment Description
In this event, Audra is called to share her opinion whether she agrees or disagrees with the explanation provided by Jessica and Laura in the previous event. Audra states that she disagrees, and uses the placement of the pink, one third tile and green, one half tile side-by-side to prove her claim that one third is less than one half. Audra is not merely sharing her opinion, but also given the opportunity to provide her reasoning of why the justification makes sense. In this sense of argumentation, a presented idea is in the works of being validated by another person, and in doing so, bringing forward new expressions of the same idea.

Figure 15.3. Argumentation described by Teacher 8 for Event 3
In the pre-assessment description, T8 does not describe any argumentation. In the post-assessment, T8 states that, "Audra states that she disagrees, and uses the placement of the pink, one third tile and green, one half tile side-by-side to prove her claim that one third is less than one half," suggesting a counterargument to an implied previous claim or argument that one third is not less than one half. The counterclaim is that one third is less than one half and the data is "the placement of the pink, one third tile and green, one half tile side-by-side." The data and claim are made explicit by T8 through the use of the formal mathematical register for argumentation, specifically in the use of "prove" and "claim."

15.1.4 Event 4

**Pre-Assessment Description**
Within this event, we see that Audra and Jessica are able to build off of each other's responses to Dr. Maher's request to "tell me what number name you have for how much bigger it is". Here, effective teacher questioning leads to furthered group efforts, and the discovery that it is "one third" bigger.

![Claim: "1/2 is one third bigger than 1/3"]

**Post-Assessment Description**
Within this event, we see that Audra and Jessica are able to build off of each other's responses to Dr. Maher's request to "tell me what number name you have for how much bigger it is". Here, effective teacher questioning leads to further group efforts, and the discovery that the one half tile is "one third" bigger than the one third tile.

![Claim: "one half tile is 'one third' bigger than the one third tile"]

**Figure 15.4. Argumentation described by Teacher 8 for Event 4**

In the pre-assessment, T8 notes a claim that "it is 'one third' bigger," suggesting the claim that 1/2 is one third bigger than 1/3. In the post-assessment, T8 changes the claim noted, stating, "the one half tile is "one third" bigger that the one third tile." In the
pre-assessment, the claim is based on the students' statements. In the post-assessment, the claim is based on the model that students build.

### 15.1.5 Event 5

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>In this clip, it's interesting to see that Kelly, when asked if she agrees with them, does not simply remain in her seat as she responds. She proceeds to approach the overhead, and produces her reasoning for agreeing with Audra, Jessica and Laura. We see that the teacher, Dr. Maher, picks up on the fact that one student (Brian) disagrees, and uses that as a &quot;fuel&quot; to push the girls to further develop their argument. Here we see the aspect of arguments in which the person presenting the claim has the task of convincing others that their approach is correct.</td>
</tr>
</tbody>
</table>

No argumentation described.

<table>
<thead>
<tr>
<th>Post-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>In this clip, it's interesting to see that Kelly, when asked if she agrees with them, does not simply remain in her seat as she responds. She proceeds to approach the overhead, and produces her reasoning for agreeing with Audra, Jessica and Laura. We see that the teacher, Dr. Maher, picks up on the fact that one student (Brian) disagrees, and uses that as a &quot;fuel&quot; to push the girls to further develop their argument. Here we see the aspect of arguments in which the person presenting the claim has the task of convincing others that their approach is correct, allowing them to work towards developing a mathematically sound proof.</td>
</tr>
</tbody>
</table>

No argumentation described.

*Figure 15.5. Argumentation described by Teacher 8 for Event 5*

T8 does not describe any specific argumentation in either the pre- or post-assessment description. However, T8 changes the statement from, "Here we see the aspect of arguments in which the person presenting the claim has the task of convincing others that their approach is correct," to "Here we see the aspect of arguments in which the person presenting the claim has the task of convincing others that their approach is correct, allowing them to work towards developing a mathematically sound proof," which results in the use of more of the formal mathematical register for argumentation.

### 15.1.6 Event 6
Pre-Assessment Description
In this clip, we begin to see Brian's reasoning for disagreeing with the explanation presented by the girls earlier. Within his explanation, he goes into careful detail to show how certain amounts of colored tiles produce equivalent amounts of the others, and progresses in his explanation to introduce a new number name, "one sixth". Here, argumentation helps to add to the language being used within the discussion.

Post-Assessment Description
In this clip, we begin to see Brian's reasoning for disagreeing with the explanation presented by the girls earlier. Within his explanation, he goes into careful detail to show how certain amounts of colored tiles produce equivalent amounts of the others, and progresses in his explanation to introduce a new number name, "one sixth". Here, argumentation helps to add to the language being used within the discussion.

Figure 15.6. Argumentation described by Teacher 8 for Event 6

In the pre-assessment, T8 uses imprecise, generalized language to describe a counterargument to prior argument or claim with data: the tile model that shows "how certain amounts of colored tiles produce equivalent amounts of others." That T8 intends for this to be argumentation is suggested by the final statement, "Here, argumentation helps to add to the language being used within the discussion."

T8 does not add to the argumentation described in the post-assessment. However, by changing the title from the pre-assessment, "Brian's Argument," to "Brian's Claim &
Argument," in the post-assessment T8 uses formal mathematical language to note an additional element of argumentation, a claim.

15.1.7 Event 7

Pre-Assessment Description
Within this event, we witness a conflict of interpretation with the units being used between Jackie and Jessica. Rather than backing down when asked by Dr. Maher to clarify her disagreement, Jessica steps boldly to the task and is able to providing a convincing argument as to why incorrect measurements would not be acceptable. It may be greatly due to the open math learning community created within this classroom, why Jessica and the other girls are able to voice they dissension with ease rather than fear.

Post-Assessment Description
Within this event, we witness a conflict of interpretation with the units being used between Jackie and Jessica. Rather than backing down when asked by Dr. Maher to clarify her disagreement, Jessica steps boldly to the task and is able to providing a convincing argument as to why incorrect measurements would not be acceptable. It may be greatly due to the open math learning community created within this classroom, why Jessica and the other girls are able to voice they dissension with ease rather than fear.

No changes in argumentation described.

Figure 15.7. Argumentation described by Teacher 8 for Event 7

In the pre-assessment, T8 uses imprecise language to denote a claim, that "incorrect measurements" are "not acceptable," supported by a general reference to a "convincing argument," as data. No changes were made to the post-assessment description.

15.1.8 Event 8
Pre-Assessment Description
Within this clip, Dr. Maher does a great job of facilitating the conversation occurring between herself, Jackie, and the classmates who are currently listening. We can hear and observe that Dr. Maher's intention is clear - to continue to question and prod the arguments presented by the students until they can effectively and clearly convince others of their positions. Recognizing that Jackie had a change of opinion, Dr. Maher uses a technique of restating what she has understood from Jackie, and then prompting for further clarification. She makes sure to remain active in the questioning process, but does not attempt to take credit of Jackie's developing thoughts, and always gives Jackie and other students the chance to prove their points.

No argumentation described.

Post-Assessment Description
Within this clip, Dr. Maher does a great job of facilitating the conversation occurring between herself, Jackie, and the classmates who are currently listening. We can hear and observe that Dr. Maher's intention is clear - to continue to question and prod the arguments presented by the students until they can effectively and clearly convince others of their positions. Recognizing that Jackie had a change of opinion, Dr. Maher uses a technique of restating what she has understood from Jackie, and then prompting for further clarification. She makes sure to remain active in the questioning process, but does not attempt to take credit of Jackie's developing thoughts, and always gives Jackie and other students the chance to prove their points.

No change in the argumentation described.

Figure 15.8. Argumentation described by Teacher 8 for Event 8

The pre-assessment and post-assessment descriptions in Event 8 are the same. The statements, "to continue to question and prod the arguments presented by the students until they can effectively and clearly convince others of their positions," and "always gives Jackie and other students the chance to prove their points," suggest that T8 notices argumentation, but does not describe it specifically.

15.1.9 Event 9
Pre-Assessment Description
This clip highlights a particularly intriguing moment during the conversation, when another student (Erik) adds input. As Erik approaches the overhead, he begins to further validate the points that the girls have been trying to make, but also begins to introduce and use the language that Dr. Maher has been trying to gently coax from the girls' arguments. When Erik makes the distinction between "one" and "one-sixth", the girls instantly agree. Dr. Maher takes a great precaution here to not take their word for it and move on to the next activity - she questions the reason why the girls have agreed with Erik. This almost creates a moment of "meta-reasoning" as she asks for justification of why the girls consented to Erik's reasoning provided. This useful teaching technique helps greatly to enhance and continue the conversation further.

No argumentation described.

Post-Assessment Description
This clip highlights a particularly intriguing moment during the conversation, when another student (Erik) adds input. As Erik approaches the overhead, he begins to further validate the points that the girls have been trying to make, but also begins to introduce and use the language that Dr. Maher has been trying to gently coax from the girls' arguments. When Erik makes the distinction between "one" and "one-sixth", the girls instantly agree. Dr. Maher takes a great precaution here to not take their word for it and move on to the next activity - she questions the reason why the girls have agreed with Erik. This almost creates a moment of "meta-reasoning" as she asks for justification of why the girls consented to Erik's reasoning provided. This useful teaching technique helps greatly to enhance and further develop the conversations. Students can sometimes be quick to blindly agree with another student's opinion if some inclination is given by the teacher that that answer is correct. Asking the students to provide reasoning why they agree really encourages them to learn the concept versus the procedure.

No additional argumentation described.

Figure 15.9. Argumentation described by Teacher 8 for Event 9

In the pre-assessment, T8 does not describe any specific argumentation, only alluding to agreements, disagreements, and justification. In the post-assessment, T8 adds detail, but the added language does not describe any additional argumentation.

15.1.10 Event 10
Pre-Assessment Description
In this segment, Dr. Maher presents a question, then makes the statement that "if you really understood what mistake you made here, maybe you'll figure out what mistake you made up here". After making this assistive statement, she then proceeds to walk away from the overhead, and gives the students time to process what was said. It can be noted that as she walks away from the group, she does not look back or add any additional comments as the students initially express uncertainty. Walking away from the students gave them the necessary space and time to process the guided instruction provided, and find clarity in their previous argument, leading to a more convincing case.

No argumentation described.

Post-Assessment Description
In this segment, Dr. Maher presents a question, then makes the statement that "if you really understood what mistake you made here, maybe you'll figure out what mistake you made up here". After making this assistive statement, she then proceeds to walk away from the overhead, and gives the students time to process what was said. It can be noted that as she walks away from the group, she does not look back or add any additional comments as the students initially express uncertainty. Walking away from the students gave them the necessary space and time to process the guided instruction provided, and find clarity in their previous argument, leading to a more convincing case.

No change in the argumentation described.

Figure 15.10. Argumentation described by Teacher 8 for Event 10

T8 does not describe argumentation in the pre-assessment and the pre-assessment description is the same as the post-assessment description.

15.1.11 Event 11
**Pre-Assessment Description**

"I am so confused". Being an expert and facilitator of the problem, we may know this to not be true of Dr. Maher within the clip, but by feigning confusion, she allows herself to be placed on the same level of the students, almost as an unassuming peer versus an intimidating teacher who knows right and wrong. This particular technique allows Brian to open up and give a valid point of reassigning number names to the different colored rods being used. As he concludes his argument, another student, Jessica, begins to present the idea that this may be as valid as the prior solution presented. Future clips may reveal whether this belief is accepted or altered.

No argumentation described.

**Post-Assessment Description**

"I am so confused". Being an expert and facilitator of the problem, we may know this to not be true of Dr. Maher within the clip, but by feigning confusion, she allows herself to be placed on the same level of the students, almost as an unassuming peer versus an intimidating teacher who knows right and wrong. This particular technique allows Brian to open up and give a valid point of reassigning number names to the different colored rods being used. As he concludes his argument, another student, Jessica, begins to present the idea that this may be as valid as the prior solution presented. Future clips may reveal whether this belief is accepted or altered.

No change in the argumentation described.

*Figure 15.11. Argumentation described by Teacher 8 for Event 11*

T8 does not describe any specific argumentation and makes no changes between the pre- and post-assessment descriptions.

15.1.12 Event 12
**Pre-Assessment Description**
In this segment, Erik argues against the points that have been presented by the girls and Brian. In time, he is able to construct a viable argument that one half cannot be one third larger than one third. Seen here, one of the benefits of permitting argumentation within the classroom discussion is the ability to have students possibly convince other to rethink previous views and look at the problem from another perspective. This opens the door for more connections to be made, and for arguments to be refined, thus improve the mathematical problem solving and justification skills of that student.

**Post-Assessment Description**
In this segment, Erik argues against the points that have been presented by the girls and Brian. In time, he is able to construct a viable argument that one half cannot be one third larger than one third. Seen here, one of the benefits of permitting argumentation within the classroom discussion is the ability to have students possibly convince other to rethink previous views and look at the problem from another perspective. This opens the door for more connections to be made, and for arguments to be refined, thus improve the mathematical problem solving and justification skills of that student. Erik brings to the forefront reasoning that involved the use of appropriate "mathematical tools", or terms, within a discussion.

No change in the argumentation described.

*Figure 15.12. Argumentation described by Teacher 8 for Event 12*

In the pre-assessment description, T8 describes a counterargument to an implicit prior claim that 1/2 is larger than 1/3 by 1/3. The argument consists of a counterclaim that "one half cannot be one third larger than one third," and a general reference to support, "a viable argument." T8 adds detail to the post-assessment description, but this detail does not describe additional argumentation.
15.1.13 Event 13

Pre-Assessment Description
This segment continue the argument that Erik has presented, clarifying the relationship between the dark green ("one half") and light pink ("one third") pieces. We now see Brian ever more convinced by Erik's argument, and starting to work with Erik to convince Dr. Maher and the class of the correct relationship. It can be noted that Dr. Maher does not interrupt as this conversation takes place, allow their ideas to build uninhibited, unless it draws away from the focus of the problem statement. This is yet another great technique that a mathematics teacher can use as students are monitored working through a problem - a careful balance of waiting and guiding.

Post-Assessment Description
This segment continue the argument that Erik has presented, clarifying the relationship between the dark green ("one half") and light pink ("one third") pieces. We now see Brian ever more convinced by Erik's argument, and starting to work with Erik to convince Dr. Maher and the class of the correct relationship. It can be noted that Dr. Maher does not interrupt as this conversation takes place, allow their ideas to build uninhibited, unless it draws away from the focus of the problem statement. This is yet another great technique that a mathematics teacher can use as students are monitored working through a problem - a careful balance of waiting and guiding.

No change in argumentation described.

Figure 15.13. Argumentation described by Teacher 8 for Event 13

In the pre-assessment T8 describes data, "clarifying the relationship between the dark green ("one half") and light pink ("one third") pieces" that forms further support for a prior argument. The post-assessment description is the same as the pre-assessment description.

15.1.14 Event 14
Pre-Assessment Description
In this clip, Michael agrees with the statements made by Erik and Brian. He proceeds to approach the overhead, portraying the relationship he has found between the red rods and the orange one. Through this demonstration, we see a different representation presented that portrays the same point. When conversation is promoted, many of the same ideas may be voiced, but different forms can be brought forward, providing alternative routes and methods to discovering solutions in the future.

No argumentation is described.

Post-Assessment Description
In this clip, Michael agrees with the statements made by Erik and Brian. He proceeds to approach the overhead, portraying the relationship he has found between the red rods and the orange one. Through this demonstration, we see a different representation presented that portrays the same point. When conversation is promoted, many of the same ideas may be voiced, but different forms can be brought forward, providing alternative routes and methods to discovering solutions in the future.

No changes to the argumentation described.

Figure 15.14. Argumentation described by Teacher 8 for Event 14

In the pre-assessment description, no specific argumentation is described. The post-assessment is the same as the pre-assessment.

15.1.15 Event 15
Pre-Assessment Description
Dr. Maher gives Meredith the opportunity to contribute to the conversation, as Meredith also validates the argument that Erik, Michael and Brian have all agreed to. Interestingly enough, as Dr. Maher is about to state a conclusion from the commentary that the students have provided, Michael interrupts his own deduction that the term "one-sixth" seems to transcend both cases.

Post-Assessment Description
Dr. Maher gives Meredith the opportunity to contribute to the conversation, as Meredith also validates the argument that Erik, Michael and Brian have all agreed to. Interestingly enough, as Dr. Maher is about to state a conclusion from the commentary that the students have provided, Michael interrupts with his own deduction that the term "one-sixth" seems to transcend both cases, and would be the most appropriate term to use.

In the pre-assessment, T8 states a "deduction," or claim that "one-sixth" transcends "both cases," suggesting that regardless of the model used, 1/2 is greater than 1/3 by 1/6. In the post-assessment, T8 adds, "Michael interrupts with his own deduction that the term "one-sixth" seems to transcend both cases, and would be the most appropriate term to use," suggesting another element of argumentation and a connection. What was the claim in the pre-assessment, "one-sixth' transcends "both cases," becomes data in the post-assessment that supports the claim that one sixth is "the most appropriate term to use."

15.2 Summary of Teacher 8's Growth across Events

T8 tended to add elements and structure to the post-assessment description when changes were made. In six of the eight events in which T8 made changes to the post-
assessment description, the detail resulted in more argumentation being described than in the pre-assessment. In Event 1, no specific argumentation is described in the pre-assessment, however, in the post-assessment, T8, with adding the statement, "When Dr. Maher continues by asking, 'Do you think you could convince Dr. Davis that one half is bigger than one third?', she stretches the students' thinking further, encouraging them to develop evidence that will support their claims," describes a claim that "one half is bigger than one third," supported by a general reference to "evidence," that is presented on the overhead. In Event 2, T8 does not describe argumentation in the pre-assessment, but in the post-assessment, T8 describes an argument with the claim that "one third is bigger than one half," and data "this would be one third and this bigger piece would be one half." In Event 3, no argumentation is described in the pre-assessment. In the post-assessment, T8 notes, "Audra states that she disagrees, and uses the placement of the pink, one third tile and green, one half tile side-by-side to prove her claim that one third is less than one half," suggesting a counterargument to an implied previous claim or argument that one third is not less than one half. The counterclaim is that one third is less than one half supported by data, "the placement of the pink, one third tile and green, one half tile side-by-side." In this event, T8 uses the formal mathematical register for argumentation, specifically "prove" and "claim" to make the data and claim explicit. In Event 15, T8 notes that, "Michael interrupts with his own deduction that the term 'one-sixth' seems to transcend both cases, and would be the most appropriate term to use," which changes the argumentation described in the pre-assessment. What was the claim in the pre-assessment, "one-sixth' transcends both cases," becomes data in the post-
assessment that supports the claim that one sixth is "the most appropriate term to use."
This description more accurately represents the argumentation that occurred in the event.

In three of the events (Events 2, 5, and 6), the changes in the descriptions and
event titles include more of the mathematical register of argumentation. In Event 2, the
change in the title from, "Clarifying the Justification," to "Jessica & Laura Present Their
Claim," identifies Jessica and Laura's statement, that one third is bigger than one half,
explicitly as a claim and in Event 6, T8 changes the title from, "Brian's Argument," to
"Brian's Claim & Argument," noting an additional element of argumentation, a claim. In
Event 5, T8 changes the language from, "Here we see the aspect of arguments in which
the person presenting the claim has the task of convincing others that their approach is
correct," to, "Here we see the aspect of arguments in which the person presenting the
claim has the task of convincing others that their approach is correct, allowing them to
work towards developing a mathematically sound proof." With this change, T8 uses more
precise language to express ideas about argumentation.
Chapter 16 – Teacher 9, Event Descriptions Analyses and Summary

This chapter presents the descriptions Teacher 9 (T9) wrote for the pre-assessment and post-assessment analytic to describe the argumentation in each event. The pre-assessment and post-assessment descriptions given by the teacher are presented for each event with an accompanying diagram using Toulmin's (1958, 2003) scheme. Following the descriptions for each event, I present an in-depth analysis, summarizing the argumentation described in the pre-assessment and noting the changes the teacher made from pre- to post-assessment. Words that are key to my analysis appear in red text in the teacher’s descriptions.

16.1 Teacher 9 Event Descriptions and Analyses by Event

16.1.1 Event 1

Pre-Assessment Description
There is not much argumentation, that occurs, in this segment. Instead, this segment provides background information about what will happen, in the other segments. From this segment, it is clear that students will argue that 1/2 is bigger than 1/3, and students may also determine how much larger 1/2 is, in comparison to 1/3.

Claim: "1/2 is bigger than 1/3"

Post-Assessment Description
There is not much argumentation, that occurs, in this segment. Instead, this segment provides background information about what will happen, in the other segments. From this segment, it is clear that students will argue that 1/2 is bigger than 1/3, and students may also determine how much larger 1/2 is, in comparison to 1/3. Thus, it can be determined that students will claim that 1/2 is greater than 1/3. This claim is referred as Claim 1. However, a claim has not been made, yet, about how much bigger.

Claim: "1/2 is greater than 1/3"

Figure 16.1. Argumentation described by Teacher 9 for Event 1
In the pre-assessment, T9 notes that the students make the claim that 1/2 is bigger than 1/3. In the post assessment, T9 uses technical language to identify the statement as a "claim."

16.1.2 Event 2

**Pre-Assessment Description**
In this segment, two students argue that 1/2 is greater than 1/3 by using tiles. Even though it takes a while for them to figure it out, they reach the conclusion that the larger tile(s), on top, represent 1; that the three tiles in the middle each represent 1/3, and that the two tiles on the bottom each represent 1/2. The premise of the students' argument is that the individual tiles on the bottom are larger than the individual pieces, in the middle.

**Post-Assessment Description**
In this segment, two students argue that 1/2 is greater than 1/3 by using tiles. Even though it takes a while for them to figure it out, they reach the conclusion that the larger tile(s), on top, represent 1; that the three tiles in the middle each represent 1/3, and that the two tiles on the bottom each represent 1/2. The premise of the students' argument is that the individual tiles on the bottom are larger than the individual pieces, in the middle.

No change in the argumentation described.

*Figure 16.2. Argumentation described by Teacher 9 for Event 2*

T9 notes the students' argument with its claim that 1/2 is greater than 1/3. The data described are "the individual tiles on the bottom are larger than the individual pieces, in the middle," with warrants "the larger tile(s), on top, represent 1," "three tiles in the middle each represent 1/3," and "the two tiles on the bottom each represent 1/2" connecting the data to the claim. Note that T9 uses "tile" and "piece" when referring to the rods. In the statement, "they reach the conclusion that the larger tile(s), on top,
represent 1; that the three tiles in the middle each represent 1/3, and that the two tiles on the bottom each represent 1/2," by using "conclusion" T9 identifies the modeling of 1, 1/2, and 1/2 as claims as well as warrants that support the data. The post-assessment is the same as the pre-assessment.

16.1.3 Event 3

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
<th>Post-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>The argumentation that occurs in this segment is related to evaluating the arguments of others. As shown in this segment, Audra argues that Jessica and Laura are correct by taking one 1/2 tile and showing that it is greater than one 1/3 tile.</td>
<td>The argumentation that occurs in this segment is related to evaluating the arguments of others. As shown in this segment, Audra argues that Jessica and Laura are correct by taking one 1/2 tile and showing that it is greater than one 1/3 tile. By doing this, she uses her own reasoning and relates it to what was said by Jessica and Laura, in the previous event.</td>
</tr>
</tbody>
</table>

No change in the argumentation described.

Figure 16.3. Argumentation described by Teacher 9 for Event 3

In the pre-assessment, T9 notes that the data given in this event, that "one 1/2 tile and showing that it is greater than one 1/3 tile," supports and is in agreement with the prior argument made by Jessica and Laura. The previous claim is not stated explicitly, only implied, so it appears dashed. In the post-assessment, T9 adds that, "By doing this, she uses her own reasoning and relates it to what was said by Jessica and Laura, in the previous event." This statement confirms that the reasoning presented here is connected to the argument in the previous event, but does not add to the argumentation noticed by T9.
16.1.4 Event 4

**Pre-Assessment Description**
In this segment, the students are asked how much larger the 1/2 tile is, in comparison to the 1/3 tile. Students line the tiles side-by-side, and use their designated lengths, for each tile, to incorrectly state that the 1/2 tile is 1/3 of a unit greater than the 1/3 tile.

**Figure 16.4. Argumentation described by Teacher 9 for Event 4**

In the pre-assessment description, T9 uses general, unspecific language to note that students make the claim that, "1/2 tile is 1/3 of a unit greater than the 1/3 tile." As data, T9 mentions the "designated lengths" of the rods and the rod model that shows the "tiles side-by-side" is the warrant. Although noted, no specifics are given regarding the lengths of the rods or the model.

**Post-Assessment Description**
In this segment, the students are asked how much larger the 1/2 tile is, in comparison to the 1/3 tile. Students line the tiles side-by-side, and use their designated lengths, for each tile, to make the claim that 1/2 is 1/3 greater than 1/3, which will be referred as Claim 2.

In the post-assessment description, T9 notes that students make the claim that, "1/2 is 1/3 greater than 1/3," which will be referred as Claim 2. As data, T9 mentions the "designated lengths" of the rods and the rod model that shows the "tiles side-by-side" is the warrant. Although noted, no specifics are given regarding the lengths of the rods or the model.
In the post-assessment, T9 uses more specific language, both by using the formal mathematical register, "claim," and by clarifying what claim the students are making. Rather than referencing the rods in the claim, "1/2 tile is 1/3 of a unit greater than the 1/3 tile," T9 references the fractions that are being compared, themselves, more precisely stating the claim, "1/2 is 1/3 greater than 1/3."

### 16.1.5 Event 5

#### Pre-Assessment Description
Students are asked if they agree with the argument given in the previous segment. One student, Kelly, agrees and provides an explanation that is similar to the argument given, in the previous segment. However, another student, Brian, does not agree, but he does not provide an argument, for his disagreement. By the end of the video, it appears that students understand how to find a tile that shows the difference of 1/2 and 1/3 by finding a tile to line up, next to 1/3, in order to make 1/2. However, they seem to believe that this tile also represents 1/3, when in reality, this tile is smaller than 1/3.

#### Post-Assessment Description
Students are asked if they agree with the argument given in the previous segment. One student, Kelly, agrees and provides an explanation that is similar to the argument given, in the previous segment. However, another student, Brian, does not agree, but he provides neither a counterargument nor a counterclaim, to support his disagreement. By the end of the video, it appears that Kelly, Audra, and Jessica understand how to find a tile that shows the difference of 1/2 and 1/3 by finding a tile to line up, next to 1/3, in order to make 1/2. However, they seem to believe that this tile also represents 1/3, when in reality, this tile is smaller than 1/3.

Post-assessment language was changed to include "counterargument," "counterclaim," and "support."

*Figure 16.5. Argumentation described by Teacher 9 for Event 5*

The description of the argumentation presented in the pre-assessment describes some general agreement and disagreement among the students, but no specific claims or evidence are noted so these statements are not included in the diagram. In the statement, "students understand how to find a tile that shows the difference of 1/2 and 1/3 by finding
a tile to line up, next to 1/3, in order to make 1/2. However, they seem to believe that this tile also represents 1/3, when in reality, this tile is smaller than 1/3," two related claims are noted: the tile that lines up next to 1/3 to make 1/2 is the difference of 1/2 and 1/3 and the tile that represents "the difference of 1/2 and 1/3 " is 1/3. Note that the use of the language, "it appears that students understand how to find a tile that shows the difference of 1/2 and 1/3," suggests that students learned how to use a rod to determine the difference between two other rods, but does not suggest that the students used this to make or defend a claim. Thus, since T9 does not use language that describes a relationship between the two claims, no connection appears between them. The statement, the tiles that represents the difference between 1/2 and 1/3 "is smaller than 1/3," is noted as additional information, since T9 is not noting that a student says this.

In the post-assessment, T9 includes more of the formal mathematical register of argumentation, "counterargument," "counterclaim," and support. The inclusion of these terms shows growth from pre-assessment to post-assessment, however, since no counterclaims or counterarguments are described in the event, the addition of these terms does not add to the elements or structure of the argumentation described.

16.1.6 Event 6
**Pre-Assessment Description**
This is the segment where we see and hear Brian's argument about why \( \frac{1}{2} \) is not \( \frac{1}{3} \) greater than \( \frac{1}{3} \). He argues that he can take a tile that is \( \frac{1}{2} \) of the \( \frac{1}{3} \) tile, which he figures out is the \( \frac{1}{6} \) tile, and line that up with the \( \frac{1}{3} \) tile, in order to make \( \frac{1}{2} \). By lining this up, next to the \( \frac{1}{2} \) tile, he shows that placing a \( \frac{1}{6} \) tile next to a \( \frac{1}{3} \) tile creates the same length as a \( \frac{1}{2} \) tile. Therefore, he concludes that \( \frac{1}{2} \) is \( \frac{1}{6} \) larger than \( \frac{1}{3} \).

**Post-Assessment Description**
This is the segment where we see and hear Brian's counterargument about why \( \frac{1}{2} \) is not \( \frac{1}{3} \) greater than \( \frac{1}{3} \). He argues that he can take a tile that is \( \frac{1}{2} \) of the \( \frac{1}{3} \) tile, which he figures out is the \( \frac{1}{6} \) tile, and line that up with the \( \frac{1}{3} \) tile, in order to make \( \frac{1}{2} \). By lining this up, next to the \( \frac{1}{2} \) tile, he shows that placing a \( \frac{1}{6} \) tile next to a \( \frac{1}{3} \) tile creates the same length as a \( \frac{1}{2} \) tile. Therefore, he concludes that \( \frac{1}{2} \) is \( \frac{1}{6} \) larger than \( \frac{1}{3} \), which is his counterclaim.
Figure 16.6. Argumentation described by Teacher 9 for Event 6

T9 describes an argument with the claim "1/2 is not 1/3 greater than 1/3" in the pre-assessment. The argument includes data: "placing a 1/6 tiles next to a 1/3 tiles creates the same length as a 1/2 tiles" and warrant: "a tiles that is 1/2 of the 1/3 tile… is the 1/6 tile." Note that this warrant is also a claim, although no evidence is given to support it. T9 states, "Therefore, he concludes that 1/2 is 1/6 larger than 1/3," which is a revision of the previous claim (1/2 is not 1/3 greater than 1/3) and is described as the result of the argument the student made.

In the post-assessment, more precise language is given that situates the argument described as a counterargument to a prior implied claim, that 1/2 is 1/3 greater than 1/3. The claim that "1/2 is not 1/3 greater than 1/3 is now a counterclaim. Furthermore, the revised claim, "1/2 is 1/6 larger than 1/3" is now identified specifically by T9's language as a counterclaim. Thus, the changes in the post-assessment description from the pre-
assessment description result in more structure and elements of argumentation being described by T9.

16.1.7 Event 7

**Pre-Assessment Description**
The young ladies, in this segment, began by arguing about why $1/2$ is 1 white tile greater than $1/3$. Then, when it was realized that a different tile had been used to represent 1, the conversation shifted to *whether this is possible*. Jessica said this is not possible; her argumentation is *based* on the idea that a particular tile must represent 1. Then, a *consensus* is reached about which tile represented 1, for their argument. However, no conclusion was reached about whether another tile could represent 1, if they desired to do so.

**Post-Assessment Description**
The young ladies, in this segment, began by arguing about why $1/2$ is 1 white tile greater than $1/3$. Then, when it was realized that a different tile had been used to represent 1, the conversation shifted to whether this is possible. Jessica said this is not possible; her argumentation is based on the idea that a particular tile must represent 1. Then, a consensus is reached about which tile represented 1, for their argument. However, no resolution was reached about whether another tile could represent 1, if they desired to do so.

*Post-assessment language was changed to include "resolution."*

*Figure 16.7. Argumentation described by Teacher 9 for Event 7*
In the pre-assessment description, T9 notes an argument that counters the claim that "1/2 is 1 white tile greater than 1/3." This argument has as its counterclaim, that the previous claim is "not possible," as data: "a different tile had been used to represent 1," and the warrant that connects the data and claim, "a particular tile must represent 1." The language in the description makes it clear that the argument is a counterargument in that T9 begins stating the claim that 1/2 is 1 white tile greater than 1/3, and then describing that "the conversation shifted to whether this is possible" and that "Jessica said this is not possible." The warrant is suggested through the language, "her argumentation is based on the idea that a particular tile must represent 1."

The statement "Then, a consensus is reached about which tile represented 1, for their argument. However, no conclusion was reached about whether another tile could represent 1, if they desired to do so," is noted as "additional information" because, although T9 mentions consensus and a conclusion, lack of specificity in the language makes it difficult to determine which tiles are being referenced.

The post-assessment description is the same as the pre-assessment description except for the use of the more formal mathematical register "resolution." Using "resolution" rather than "conclusion" shows growth in T9's language use, but it does not change structure or elements of argumentation noticed.

**16.1.8 Event 8**

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>In this segment, the young ladies seem to understand the visual argument, in terms of finding a tile that represents the difference between 1/3 and 1/2. They are able to find tiles that represent 1/3 of the tile that they deem as 1, as well as tiles that represent 1/2 of the tile that they deem as 1. However, they are unable to find a fraction that relates the length of the tile that represents the difference between 1/2 and 1/3 to the length of the tile that they deem as 1 unit. Therefore, they simply say that 1/2 is 1 white tile greater than 1/3.</td>
</tr>
</tbody>
</table>
Post-Assessment Description

In this segment, the young ladies seem to understand the visual argument, in terms of finding a tile that represents the difference between $1/3$ and $1/2$. They are able to find tiles that represent $1/3$ of the tile that they deem as 1, as well as tiles that represent $1/2$ of the tile that they deem as 1. However, they are unable to find a numerical value that relates the length of the tile that represents the difference between $1/2$ and $1/3$ to the length of the tile that they deem as 1 unit, even when asked to do so, by the researcher. Therefore, they simply say that $1/2$ is "one white bigger" than $1/3$, without elaborating further.
The argument in the pre-assessment is given in imprecise language that makes it difficult to determine its elements. The boxes with the data and warrants and their corresponding connectors, are dashed to represent this uncertainty. The claim, "1/2 is 1 white tile greater than 1/3," is explicitly stated. Additionally, that the students found "tiles that represent 1/3 of the tile that they deem as 1" and "tiles that represent 1/2 of the tile that they deem as 1" are both explicitly stated, but, although it seems that these statements are meant as warrants for an argument, it is uncertain. Additionally, the data for this claim, that the white rod represents the difference between 1/2 and 1/3, is implied, based on T9's statements that "the young ladies seem to understand the visual argument, in terms of finding a tile that represents the difference between 1/3 and 1/2," coupled with the statement that the student is "unable to find a fraction that relates the length of the tile that represents the difference between 1/2 and 1/3 to the length of the tile that they deem as 1 unit" and that they say, "1/2 is 1 white tile greater than 1/3." When taken together, these statements seem to imply that the students realized that the difference between 1/2 and 1/3 is the rod that represents the difference and the rod that represents the difference is the white rod. The statement, "they are unable to find a fraction that relates the length of the tile that represents the difference between 1/2 and 1/3 to the length of the tile that they deem as 1 unit," provides additional information that is relevant to the argumentation being put forth.

In the post-assessment, T9 adds some additional formal language and detail to the description. T9 uses language that is closer to the students' words when stating the claim, "1/2 is 'one white bigger' than 1/3" rather than "1/2 is 1 white tile greater than 1/3." Furthermore, in the additional information, T9 uses more precise language to explain
what students are having difficulty with, stating, "they are unable to find a numerical value that relates the length of the tile that represents the difference between 1/2 and 1/3…" using "numerical value" rather than fraction.

16.1.9 Event 9

Pre-Assessment Description
Erik comes to the front of the classroom and says that you can line 6 white tiles, together, to represent 1 unit. Therefore, the number representation for 1 while tile is 1/6. The young ladies agree that this is what they meant to say; however, they still say that 1/2 is 1 whole number greater than 1/3. After the researcher asks whether the number representation of 1 white tile is 1 or 1/6, the class reaches the conclusion that it represents 1/6.

Post-Assessment Description
Erik comes to the front of the classroom and adds his own reasoning, to the young ladies' argument, by saying that you can line 6 white tiles, together, to represent 1 unit. Therefore, the number representation for 1 while tile is 1/6. The young ladies agree that this is what they meant to say; however, they still say that 1/2 is 1 whole number greater than 1/3. After the researcher asks whether the number representation of 1 white tile is 1 or 1/6, the class reaches the conclusion that it represents 1/6.
T9 describes an argument in the pre-assessment for the claim that "the number representation for 1 [white] tile is 1/6." The data given for the claim is that "you can line 6 white tiles together to represent 1 unit." The connection is evidenced through the use of "therefore." T9 states that the ladies make the claim that "this is what they meant to say," suggesting that they are claiming that they meant to say that the white tile is 1/6 and that they make a second claim that "1/2 is 1 whole number greater than 1/3. T9 summarizes what might be considered a resolution to the argument through the statement, "the class reaches the conclusion that it [the white rod] represents 1/6."

The post-assessment description includes the statement that Erik "adds his own reasoning to the young ladies' argument." This statement adds elements and structure to the argumentation by situating the argument (claim and data) as an agreement to a previous argument, connecting the reasoning in this event to the prior argument.
**Pre-Assessment Description**
The researcher asks students to revisit how much larger 1/2 is than 1/3 when using the combination of orange and red, as 1. For this, a dark green tile represents 1/2 and a purple tile represents 1/3. Based on Jessica's argument, which concludes by saying that a red tile, which represents the difference between 1/3 and 1/2, is 1/3, it seems that she makes the mistake of thinking that the dark green tile is 1, since a red tile is 1/3 of the dark green tile. However, she lines all of the tiles, properly, as she makes her argument.

**Post-Assessment Description**
The researcher asks students to revisit how much larger 1/2 is than 1/3 when using the combination of orange and red, as 1, which was the original scale. For this scale, a dark green tile represents 1/2 and a purple tile represents 1/3. Based on Jessica's argument, which concludes by saying that a red tile, which represents the difference between 1/3 and 1/2, is 1/3, it seems that she makes the mistake of thinking that the dark green tile is 1, since a red tile is 1/3 of the dark green tile. However, she lines all of the tiles, properly, as she makes her argument. This also shows that the young ladies were unable to apply what they learned, in Events 8-9, to the original scale.

*Figure 16.10. Argumentation described by Teacher 9 for Event 10*
In the pre-assessment, T9 states, "Based on Jessica's argument, which concludes by saying that a red tile, which represents the difference between 1/3 and 1/2, is 1/3." In this statement is a claim that the difference between 1/3 and 1/2 is 1/3, data for that claim, which is a claim in itself, that a red tile is 1/3, and three warrants: 1. that the red tile represents the difference between 1/3 and 1/2; 2. That the "dark green tile is 1;" and that "a red tile is 1/3 of the dark green tile." The second two warrants, (that the dark green tile is 1 and that a red tile is 1/3 of the dark green tile) also serve as data for the claim that a red tile is 1/3. T9 also includes some additional information relevant to the argumentation in this event, specifically that thinking that the dark green tile is 1 was a "mistake." Note that T9 states, "For this, a dark green tile represents 1/2 and a purple tile represents 1/3." The imprecision of language makes it difficult to determine what is meant by this statement and what relevance it has for the argumentation in the event, so it is not included in the diagram.

In the post-assessment, T9 includes more detail in the description that leads to other additional information statements being included in the diagram. T9 clarifies the first part of the description by adding "which was the original scale," to the statement, "The researcher asks students to revisit how much larger 1/2 is than 1/3 when using the combination of orange and red, as 1." Then T9 states, "For this scale, a dark green tile represents 1/2 and a purple tile represents 1/3," which situates the statement that the dark green tile represents 1/2 and the purple tile represents 1/3 as being when the orange and red rod train is 1. These statements now become relevant to the argumentation in the event, especially in light of the statement further along in the description that notes that Jessica "makes the mistake of thinking that the dark green tile is 1." Thus these two
additional information statements are included in the diagram: the orange and red as 1 "was the original scale" and for the original scale, "a dark green tile represents 1/2 and a purple tile represents 1/3."

16.1.11 Event 11

Pre-Assessment Description
Brian still does not agree that a red tile represents 1/3; he still thinks that a red tile represents 1/6. He starts his argument off, well, by stating that he splits 1/3 in half and calls it 1/6. However, when he goes to the overhead, he uses the phrase "that much bigger" multiple times, when comparing the red tile to the larger tiles, which does not make his reasoning any clearer.

Post-Assessment Description
Brian still does not agree that a red tile represents 1/3; he still thinks that a red tile represents 1/6. He starts his counterargument off, well, by stating that he splits 1/3 in half and calls it 1/6. However, when he goes to the overhead, he uses the phrase "that much bigger" multiple times, when comparing the red tile to the larger tiles, which does not make his reasoning any clearer, as he does not define "that much bigger." Then, in the last few seconds of the event, Jessica conjectures that 1/6 and 1/3 "might both be answers." The event does not capture her reasoning, for this.

![Diagram](Figure 16.11. Argumentation described by Teacher 9 for Event 11)

T9 describes an argument in the pre-assessment that supports the claim that the red tile "represents 1/6" with data that splitting the 1/3 in half is 1/6. There is imprecise
language that might allude to a prior claim that the red tile is 1/3, but no specific connection is made. Furthermore, T9 states that the student "uses the phrase 'that much bigger' multiple times, when comparing the red tile to the larger tile, which might allude to a warrant or more data, but, again, the T9's language does not make the connection clear.

In the post-assessment, T9 clarifies the argumentation in the event by stating that the student, "starts his counterargument off..." which situates the argument described in the pre-assessment as a counterargument and the claim that the red tile is 1/6 as a counterclaim to a previous claim that the red tile is 1/3. Note that T9's use of "counterargument" allows for the inclusion of the statement that "Brian still does not agree that a red tile represents 1/3" as a prior claim that Brian is now countering. T9 adds, "Then, in the last few seconds of the event, Jessica conjectures that 1/6 and 1/3 'might both be answers.'" This statement adds additional elements and structure to the argumentation described by suggesting an additional conjecture that is a modification of the two claims already mentioned, that a red tile is 1/3 and that a red tile is 1/6. Note, however, that in the event itself, the statement by Jessica that "they might both be answers" likely refers to the difference between 1/2 and 1/3 rather than the number name of the red tile, so the arrows connecting the modified claim to the other two claims is gray. As evidenced by the diagram, there is growth in T9's post description in the language used, the elements noted, and the structure described.

16.1.12 Event 12
Pre-Assessment Description
Erik’s argument about why 1/2 cannot be 1/3 greater than 1/3 relies on the idea that if you take 2 1/3 tiles and line them up, then you get something which is larger than 1/2. Therefore, 1/2 cannot be 1/3 larger than 1/3. Then, he suggests the idea of "1/3 and a half," which can be interpreted in multiple ways. One way, which it seems that Erik means, is that "1/3 and a half" means the sum of 1/3 and 1/2 of 1/3. However, "1/3 and a half" can also mean the sum of 1/3 and 1/2.

Post-Assessment Description
Erik’s counterargument about why 1/2 cannot be 1/3 greater than 1/3 relies on the idea that if you take 2 1/3 tiles and line them up, then you get something which is larger than 1/2. Therefore, 1/2 cannot be 1/3 larger than 1/3. Then, he suggests the idea of "1/3 and a half," which can be interpreted in multiple ways. One way, which it seems that Erik means, is that "1/3 and a half" means the sum of 1/3 and 1/2 of 1/3. However, "1/3 and a half" can also mean the sum of 1/3 and 1/2. The researcher could ask what Erik means by "1/3 and a half," if she deems it necessary. Furthermore, "1/3 and a half" does not answer the question of how much larger 1/2 is, in comparison to 1/3.
In the pre-assessment, T9 notes a claim that, "1/2 cannot be 1/3 larger than 1/3" with data that "if you take 2 1/3 tiles and line them up, then you get something which is larger than 1/2." Note that the word "therefore," connects the claim and the data. In the next statements, T9 notes another claim, "that '1/3 and a half' mean the sum of 1/3 and 1/2 of 1/3." The next statement, "However, "1/3 and a half" can also mean the sum of 1/3 and 1/2," does not appear relevant to the argumentation in the event, so it is not included.

In the post-assessment, T9 uses the formal mathematical register for argumentation by noting "Erik's counterargument." The use of this language situates Erik's reasoning as a counterargument to the idea that came before and the claim that "1/2 cannot be 1/3 greater than 1/3" as a counterclaim to an implied previous claim that 1/2 is 1/3 greater than 1/3. T9 also states, "Furthermore, "1/3 and a half" does not answer the question of how much larger 1/2 is, in comparison to 1/3," which is additional information relevant to the argumentation in the event. However, note that the statement is not true. The idea that 1/2 is the same length as 1/2 and "a half of 1/3" does support the answer to the question of how much larger 1/2 is than 1/3; specifically that 1/2 is half of a third more than 1/3, and furthermore, supports that if 1/2 is a half of a third more than 1/3, it cannot be 1/3 more than 1/3. The untrue statement is noted in gray in the diagram.

16.1.13 Event 13
Pre-Assessment Description
At this point, in his argument, Erik is not exactly clear of what he is arguing. When he mentions the idea of "1/3 and a half," he could be arguing that 1/2 is 1/3 and a half, or that 1/2 is 1/3 and a half greater than 1/3. This is since he says, at one point, that "it couldn't be exactly a third bigger than it and it couldn't be exactly two thirds." Therefore, he may be arguing that 1/2 is 1/3 and a half greater than 1/3. This seems to be uncertain, at this point.

Post-Assessment Description
Brian defines "1/3 bigger" by using the tiles; however, the camera does not zoom in to the tiles, when he does this. Therefore, Brian's definition cannot be evaluated, further. Then, Erik continues to discuss his idea of "1/3 and a half," but is not exactly clear of what he is arguing. When he mentions the idea of "1/3 and a half," he could be arguing that 1/2 is 1/3 and a half, or that 1/2 is 1/3 and a half greater than 1/3. This is since he says, at one point, that "it couldn't be exactly a third bigger than it and it couldn't be exactly two thirds." Therefore, he may be arguing that 1/2 is 1/3 and a half greater than 1/3. This seems to be uncertain, at this point.

No additional argumentation described.

Figure 16.13. Argumentation described by Teacher 9 for Event 13

T9 notes an argument in the pre-assessment with the claim that "1/2 is 1/3 and a half greater than 1/3" and data "it couldn't be exactly a third bigger than it and it couldn't be exactly two thirds." However, T9 expresses uncertainty as to the student's argument. The additional information, then is gray since it is likely that Erik is making the argument presented.
Although T9 adds, "Brian defines '1/3 bigger' by using the tiles; however, the camera does not zoom in to the tiles, when he does this. Therefore, Brian's definition cannot be evaluated, further" in the post-assessment description, the language is imprecise and therefore, it does not add any additional argumentation to the description.

16.1.14 Event 14

**Pre-Assessment Description**
In this segment, Michael argues that the red tile represents 1/6 by lining six red tiles above the combination of tiles that represent 1 unit. Thus, since 6 red tiles represent 1 unit, then 1 red tile represents 1/6 unit.

**Post-Assessment Description**
In this segment, Michael argues that the red tile represents 1/6 by lining six red tiles above the combination of tiles that represent 1 unit. Thus, since 6 red tiles represent 1 unit, then 1 red tile represents 1/6 unit. Thus, he seems to support the counterclaim that 1/2 is 1/6 greater than 1/3.
In the pre-assessment, T9 notes Michael's argument with claim "the red tile represents 1/6," and data "1 red tile represents 1/6 unit." Also noted is the warrant that connects the data to the claim, "6 red tiles represent 1 unit," and backing for the warrant, the model that shows six red tiles lined up "above the combination of tiles that represent 1 unit."

In the post-assessment, T9 adds the statement, "Thus, he seems to support the counterclaim that 1/2 is 1/6 greater than 1/3." Through this statement, T9 includes more elements, language, and structure to the argumentation described. The use of "counterclaim" situates the described argument as support for a prior claim, specifically,
that 1/2 is 1/6 greater than 1/3. Furthermore, the use of counterclaim suggests that there was a claim counter to 1/2 is 1/6 greater than 1/3, thus, a previous implied claim "1/2 is not 1/6 greater than 1/3" is included in the argumentation described. Identifying the argument in the event as support for the claim that 1/2 is 1/6 greater than 1/3 suggests that T9 is implying that Michael's argument has as its implied claim, "1/2 is 1/6 greater than 1/3," and the other elements are support for that claim. Therefore, what was the claim in the pre-assessment, (that the red tile represents 1/6) becomes both data for the new claim, as well as a claim in itself (or Data/Claim) and the data and warrant for the data/claim, all take on dual roles as warrant and backing for the new claim. Thus, more argumentation is described in the post assessment than in the pre-assessment description.

16.1.15 Event 15

Pre-Assessment Description
This final segment deviates slightly from the question that has been discussed, throughout the analytic, even though it has the same concept. This segment discusses the idea of taking 2 1/3 tiles and seeing how much greater this is, in comparison, to 1/2. Then, Erik takes a 1/6 tile and places it next to the 1/2 segment, to show that 2/3 is 1/6 greater than 1/2. This can also be interpreted as a counterargument to show that 1/2 is not 1/3 greater than 1/3.

Post-Assessment Description
This final segment deviates slightly from the question that has been discussed, throughout the analytic, even though it has the same concept. This segment discusses the idea of taking 2 1/3 tiles and seeing how much greater this is, in comparison, to 1/2. Then, Erik takes a 1/6 tile and places it next to the 1/2 segment, to show that 2/3 is 1/6 greater than 1/2. This can also be interpreted as a counterargument to show that 1/2 is not 1/3 greater than 1/3. Michael then concludes the event by saying that it is 1/6 "in both cases," presumably meaning the difference between 1/2 and 1/3 and the difference between 2/3 and 1/2. This could lead to another exploration to make a connection between the two "cases."

In the pre-assessment T9 notes an argument that is a "counterargument to show that 1/2 is not 1/3 greater than 1/3," implying a prior claim that 1/2 is 1/3 greater than 1/3. The claim, "1/2 is not 1/3 greater than 1/3, is supported by data "2/3 is 1/6 greater than 1/2," which is also a claim in itself. The warrant/data for these claims are the models that show two 1/3 tiles compared to 1/2 and a 1/6 tile next to the 1/2 segment. Note that the language, "counterargument" situates the argument as counter to a prior argument and the claim, "1/2 is not 1/3 greater than 1/3," is a counterclaim that implies a prior claim that 1/2 is 1/3 greater than 1/3.
In the post-assessment T9 notes, "Michael then concludes the event by saying that it is 1/6 'in both cases,' presumably meaning the difference between 1/2 and 1/3 and the difference between 2/3 and 1/2." By this statement, T9 is adding additional structure and elements to the argumentation noted: the claim, "it is 1/6 in both cases," and the claim "the difference between 1/2 and 1/3 [is 1/6]." The link is made from the claim "it is 1/6 in both cases," to the claim that "the difference between 2/3 is 1/6 greater than 1/2 and the difference between 1/2 and 1/3 [is 1/6]." Note, that it is likely that Michael meant that 1/2 is 1/6 greater than 1/3 whether the model with the orange and red rod train or the dark green rod is the unit, rather than what is noted by T9 in the description, therefore the link between the claim "it is 1/6 in both cases" and "2/3 is 1/6 greater than 1/2" is gray.

16.2 Summary of Teacher 9's Growth across Events

In six of the 15 events, T9 adds elements of argumentation to the post-assessment description not mentioned in the pre-assessment description. T9 adds an implicit prior claim to the post-assessment description in Event 6 and in Event 9, the inclusion of the statement, "adds his own reasoning to the young ladies' argument," suggests another element of argumentation by situating the argument (claim and data) as an agreement to a previous argument. In Event 11, T9 adds a counterargument, a prior claim, and a modified conjecture and in Event 12, T9 adds a counterargument and an implied claim. The inclusion of, "Thus, he seems to support the counterclaim that 1/2 is 1/6 greater than 1/3," in the post-assessment of Event 14 suggests a counterargument that is counter to an implied prior claim. The counterclaim that "1/2 is greater than 1/3," is added, as is support for the claim and in Event 15, T9 adds two claims, "it is 1/6 in both cases," and "the difference between 1/2 and 1/3 [is 1/6]."
In six of the 15 events, the changes made to the post-assessment description change the structure of the argumentation described. The change T9 makes to the post-assessment in Event 6 adds to the structure of the argumentation by specifically identifying the argument as a counterargument that is connected to an implicit prior claim. In Event 9, T9 includes the statement, "adds his own reasoning to the young ladies' argument," adding structure to the argumentation by connecting the reasoning in this event to the prior argument. In Event 11, T9 uses "counterargument" in the post-assessment description which situates the claim and data described in both the pre- and post-assessment as a counterargument connected to a prior claim and a modified conjecture, thus describing more structure in the post-assessment than the pre-assessment. T9 also adds structure to the description of the argumentation in Event 12 with the inclusion of "counterargument." By situating the argument as a counterargument, T9 adds an implied claim, that "1/2 is 1/3 larger than 1/3" that is linked to the claim and data described. The post-assessment in Event 14 describes a different, more complex structure of argumentation than in the pre-assessment. The use of "counterclaim" situates the argument described in both the pre- and post-assessment as a counterargument and the statement of the counterclaim, specifically, that 1/2 is greater than 1/3, is not only the counterclaim, but the claim that the argument supports. Thus, what was the claim in the pre-assessment, that "the red tile represents 1/6," becomes both data and claim, the data that "1 red tile represents 1/6 unit," becomes warrant and data, and the warrant, "6 red tiles represent 1 unit," becomes backing and warrant. Furthermore, in Event 15, T9 adds structure to the argumentation by making a link from the claim "it is 1/6 in both cases," to the claim that "the difference between 1/2 and 1/3 [is 1/6]," and the counterargument. T9
also links the claim "it is 1/6 in both cases" to "2/3 is 1/6 greater than 1/2," although it is not certain whether this link was really made by the students in the event.

In eight of the 15 events, T9 shows a growth in the technical language for argumentation used in the post-assessment description compared to the pre-assessment description. In Events 1 and 4, T9 changes the language in the post-assessment to specifically identify students' statements as, "claims." In Event 5, rather than stating, "However, another student, Brian, does not agree, but he does not provide an argument, for his disagreement," T9 changes the language to, "However, another student, Brian, does not agree, but he provides neither a counterargument nor a counterclaim, to support his disagreement," using more precise language to describe the argumentation and in Event 6, T9 clarifies the argumentation described by identifying the argument as a "counterargument" and the revised claim as a "counterclaim." In the post-assessment description of Event 7, T9 uses "resolution" rather than "conclusion" identifying that resolution is part of argumentation. In Events 11 and 12, T9 uses counterargument to situate the argumentation as being counter to a prior claim and in Event 14, T9 adds, "Thus, he seems to support the counterclaim that 1/2 is 1/6 greater than 1/3."

In four out of 15 events, T9 makes other notable changes in the description of the argumentation from pre-assessment to post-assessment. In Event 2, T9 adds, that "By doing this, she uses her own reasoning and relates it to what was said by Jessica and Laura, in the previous event." Although this statement does not add elements or structure to the argumentation described, it confirms that the reasoning presented in the event is connected to the argument in the previous event. In Event 4, T9 uses more precise language to state the claim. Rather than referencing the rods, "1/2 tile is 1/3 of a unit
greater than the 1/3 tile," T9 references the fractions that are being compared, themselves, stating the claim as, "1/2 is 1/3 greater than 1/3." In Event 8, T9 uses language that is closer to the students' words when stating the claim, "1/2 is 'one white bigger' than 1/3" rather than "1/2 is 1 white tile greater than 1/3." Furthermore, T9 uses more precise language to explain what students are having difficulty with, stating, "they are unable to find a numerical value that relates the length of the tile that represents the difference between 1/2 and 1/3…" using "numerical value" rather than "fraction." In Event 10, T9 includes more detail in the description that leads to the inclusion of additional information that is relevant to the argumentation. T9 clarifies the first part of the description by adding "which was the original scale," to the statement, "The researcher asks students to revisit how much larger 1/2 is than 1/3 when using the combination of orange and red, as 1." Then T9 states, "For this scale, a dark green tile represents 1/2 and a purple tile represents 1/3," which situates the statements that the dark green tile represents 1/2 and the purple tile represents 1/3 as being when the orange and red rod train is 1. These statements now become relevant to the argumentation in the event, especially in light of the statement further along in the description that notes that Jessica "makes the mistake of thinking that the dark green tile is 1."
Chapter 17 – Teacher 10, Event Descriptions Analyses and Summary

This chapter presents the descriptions Teacher 10 (T10) wrote for the pre-assessment and post-assessment analytic to describe the argumentation in each event. The pre-assessment and post-assessment descriptions given by the teacher are presented for each event with an accompanying diagram using Toulmin's (1958, 2003) scheme. Following the descriptions for each event, I present an in-depth analysis, summarizing the argumentation described in the pre-assessment and noting the changes the teacher made from pre- to post-assessment. Words that are key to my analysis appear in red text in the teacher’s descriptions.

17.1 Teacher 10 Event Descriptions and Analyses by Event

17.1.1 Event 1
Pre-Assessment Description
This event does not allow for much student argumentation, the teacher simply asks students to share their opinions. Students support can be considered argumentation. All students agree that they are not equal. Thus if they are not equal, the argument is that one is bigger. So, when Laura answers that 1/2 is larger, the teacher asks the students if they agree. There is not much argumentation at the moment. There is not solid evidence available to support the claim yet, other than one must be bigger and the class consensus that it is 1/2.

Post-Assessment Description
Students are asked if the numbers 1/2 and 1/3 are equal or if one is larger than the other. When students are asked, all students raise their hands in agreement that they are not equal. Thus, the first element of argumentation is made: the claim is that one of the two numbers is bigger. Laura further the claim by answering that 1/2 is larger. The class supports the notion that 1/2 is larger than 1/3. There is not solid evidence available to support the claim yet, other than one must be bigger and the class consensus that it is 1/2.

Figure 17.1. Argumentation described by Teacher 10 for Event 1

In the pre-assessment, T10 noticed several claims all referring to the fractions being compared. Note that only one of the fractions, 1/2, is mentioned. The claims are that the fractions are not equal, that one fraction is bigger, and that 1/2 is the larger fraction. T10 then uses as data the claim that "one is bigger" to support the claim that 1/2 is larger, stating, "There is not solid evidence available to support the claim yet, other
than one must be bigger and the class consensus that it is 1/2." T10 also notes that the
class consensus that 1/2 is bigger. Furthermore, T10 notes a view of argumentation,
"Students [sic] support can be considered argumentation," possibly suggesting that claims
are not part of argumentation.

In the post-assessment, clarifying language, detail, and formal mathematical
register are added to the pre-assessment description. T10 states, "Thus, the first element
of argumentation is made: the claim is that one of the two numbers is bigger,"
recognizing "one of the two numbers is bigger" as a claim and identifying a claim
specifically as an element of argumentation. This suggests growth from the pre-
assessment where T10 noted that "students' support" is considered argumentation. Note
that the inclusion of both fractions, 1/2 and 1/3, serves to clarify the T10's description of
the claims. T10 adds additional structure to the arguments presented by the students in
the statement, "Laura furthers the claim by answering that 1/2 is larger." The language,
"furthers the claim," suggests that the claim that 1/2 is larger than 1/3 is a refinement of
the claim that one of the two numbers is bigger. Furthermore, in the post-assessment
description, T10 takes out the statement, "There is not much argumentation at the
moment," suggesting that T10 no longer thinks that there is not much argumentation in
the event. Thus, T10 adds additional elements of argumentation, structure, and precision
of language in the post-assessment.

17.1.2 Event 2

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
</tr>
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<tbody>
<tr>
<td>Jessica and Laura use their rods to prove to the class that 1/2 is greater than 1/3. They create a whole rod equal to 1 and show that more 1/3 rods are needed than the 1/2 rods. They prove this by first showing the class that 2 1/2 pieces make up 1 whole rod and 3 1/3 pieces equal 1 whole rod. Their argument requires modeling the 1 whole piece in rods of 1/2 and 1/3. Therefore, when the students see that more 1/3 rods are necessary to</td>
</tr>
</tbody>
</table>
Post-Assessment Description

Jessica and Laura use their rods to prove to the class that $1/2$ is greater than $1/3$. They are backing Laura’s claim that $1/2$ is larger than $1/3$. They create a whole rod equal to 1 and show that more $1/3$ rods are needed than the $1/2$ rods. They prove this by first showing the class that $2 \frac{1}{2}$ pieces make up 1 whole rod and $3 \frac{1}{3}$ pieces equal 1 whole rod. Therefore, they support their claim using a modeling technique; they represent 1 whole piece in rods of $1/2$ and $1/3$. So, when the students see that more $1/3$ rods are necessary to complete 1 whole, they can understand that $1/3$ is smaller.

Figure 17.2. Argumentation described by Teacher 10 for Event 2
T10 notes the students' argument that 1/2 is greater than 1/3 in the pre-assessment description. T10 describes the data that "more 1/3 rods are needed than the 1/2 rods [to complete 1 whole]" and connects the data to the claim by stating that, "3 1/3 pieces equal 1 whole rod" and "2 1/2 pieces make up 1 whole rod." The rod model that shows the "1 whole piece in rods of 1/2 and 1/3" is mentioned as the backing. T10 also mentions that students "understand that 1/3 is smaller," presumably than 1/2. Although T10 may intend this to be a claim students make, the students do not make this claim in Event 2. Additionally, the data that more 1/3 rods are needed than 1/2 is also not mentioned by students. Thus, the text for these two statements is gray.

In the post-assessment, T10 includes more of the formal mathematical register for argumentation. T10 clarifies the claim of the students by using more precise language "claim," and the precise language the students used, "1/2 is larger than 1/3." In the pre-assessment, T10 stated the claim as "1/2 is greater than 1/3" which is not the language the students used. Additionally, T10 clarified that the rod model was the backing for the data through the use of "support," "claim," and "backing."

**17.1.3 Event 3**
Pre-Assessment Description
Audra supports Laura and Jessica's argument by taking the pieces of 1/2 and 1/3 that make up the 1 whole and putting them on their own. So, she takes one 1/2 rod and one 1/3 rod and compares their lengths. By putting them next to each other she shows that 1/3 is shorter than 1/2. Thus, it supports the argument that 1/2 is greater than 1/3.

Post-Assessment Description
Audra presents Laura and Jessica's model in a different light, i.e. providing additional support, by taking the pieces of 1/2 and 1/3 that make up the 1 whole and putting them on their own. So, she takes one 1/2 rod and one 1/3 rod and compares their lengths. By putting them next to each other she shows that 1/3 is shorter than 1/2. However, this support would not have been enough on its own. Audra first needed the class to see that those pieces are actually 1/2 and 1/3 of a whole (which they were able to from Laura and Jessica's presentation).

Figure 17.3. Argumentation described by Teacher 10 for Event 3
In the pre-assessment, T10 describes how another student, Audra, supports the prior argument that $\frac{1}{2}$ is greater than $\frac{1}{3}$. The fact that "$\frac{1}{3}$ is shorter than $\frac{1}{2}$" provides data and the warrant is described as the student "putting the $\frac{1}{2}$ rod and the $\frac{1}{3}$ rod next to each other" and comparing their lengths.

In the post-assessment, T10 adds to the description of the student argumentation in Event 3. T10 notes that "by taking the pieces of $\frac{1}{2}$ and $\frac{1}{3}$ that make up the 1 whole and putting them on their own," the student is not only providing support (as noted in the pre-assessment) but "additional support" for the previous argument. T10 then adds another element of argumentation and structure to the description by emphasizing the importance of the naming of the rods in the previous event. The idea that "that those pieces are actually $\frac{1}{2}$ and $\frac{1}{3}$ of a whole" is backing for this argument. T10 states: "However, this support would not have been enough on its own. Audra first needed the class to see that those pieces are actually $\frac{1}{2}$ and $\frac{1}{3}$ of a whole (which they were able to from Laura and Jessica's presentation)." Thus, T10 shows growth from pre- to post-assessment in the precision of language used and the elements and structure of the argumentation described.

17.1.4 Event 4
**Pre-Assessment Description**
The students now argue that \( \frac{1}{2} \) is one red rod bigger than \( \frac{1}{3} \); they are correct. However, when the teacher asks the students what the number of a red rod is, the students incorrectly argue that it is \( \frac{1}{3} \). They argue this because they line three red rods up to equal one \( \frac{1}{2} \) rod and then see that two red rods equal a \( \frac{1}{3} \) rod. Hence, they conclude that it is \( \frac{1}{3} \) bigger. However, they did not assess the red rods out of a 1 whole. So, what they really have is \( \frac{1}{3} \) of \( \frac{1}{2} \).

**Post-Assessment Description**
Researcher Carolyn Maher now asks the students how much bigger \( \frac{1}{2} \) is than \( \frac{1}{3} \). Students have to generate a new claim and support it. The students show that \( \frac{1}{2} \) is one red rod bigger than \( \frac{1}{3} \); they are correct. However, when the teacher asks the students what the number of a red rod is, the students incorrectly state that it is \( \frac{1}{3} \). They believe this because they line three red rods up to equal one \( \frac{1}{2} \) rod and then see that two red rods equal a \( \frac{1}{3} \) rod. Thus, the students' claim is that \( \frac{1}{2} \) is \( \frac{1}{3} \) larger than \( \frac{1}{3} \). However, they did not assess the red rods out of a 1 whole. So, what they really have is \( \frac{1}{3} \) of \( \frac{1}{2} \).
In the pre-assessment description, T10 notes that students "argue that 1/2 is one red rod bigger than 1/3, stating this as a claim." Note that the word "argue" is used generally, not specifying any evidence to support an argument. T10 then presents the argument that supports that 1/2 is 1/3 "bigger" than 1/3. Note that this argument includes a sub-argument with claim: the number name of the red rod is 1/3, data: "two red rods equal a 1/3 rod," and warrant: that the rod model that shows "three red rods … equal one 1/2 rod." This sub-argument is presented as evidence for the claim: "it is 1/3 bigger." T10 includes additional information relevant to the argumentation in the event, noting that the claim that 1/2 is bigger than 1/3 by 1/3 is not correct and that students, "did not assess the red rods out of a 1 whole. So, what they really have is 1/3 of 1/2."
In the post-assessment, T10 adds elements and structure to the argumentation described, as well as using more of the formal mathematical register for argumentation through the use of more precise language. T10 states that the students need to "generate a new claim and support it," which identifies the claim that 1/2 rod is bigger than 1/2 by a red rod as a new claim. Additionally, by using "show" rather than "argue" T10 implies that a rod model was used as data for the claim "1/2 is one red rod bigger than 1/3." This adds new argumentation (elements and structure) to the argumentation noticed in the event. T10 identifies "1/2 is 1/3 larger than 1/3 as the students' claim, naming it as a "claim" and using more precise language to note it. In the pre-assessment, T10 states, "Hence, they conclude that it is 1/3 bigger," and in the post-assessment, T10 changes the language, "Thus, the students' [sic] claim is that 1/2 is 1/3 larger than 1/3," adding precision.

17.1.5 Event 5

Pre-Assessment Description
Kelly agrees with the other three girls. Even though she used the same sized rods initially, her argument was incorrect in the same manner. However, when she lined up the dark green rod to represent 1 whole and the three red rods to represent 1/3 pieces, she did not use a proper sized rod for the 1/2. The lavender rod looks like a 2/3 rod in comparison with the red rods. Therefore, her argument falls short because of course the difference in size between 2/3 and 1/3 is 1/3 to make up a whole. But it is the incorrect use of a 2/3 rod that allows her to argue this false conclusion.
Post-Assessment Description
Kelly agrees with the other three girls. Even though she used the same sized rods initially, her support was incorrect in the same manner. When Kelly she lined up the dark green rod to represent 1 whole and the three red rods to represent 1/3 pieces, she did not use a proper sized rod for the 1/2. The lavender rod looks like a 2/3 rod in comparison with the red rods. Therefore, her support is incorrectly modeled and her claim that 1/2 is larger by 1/3 appears true because of course the difference in size between 2/3 (which she called 1/2) and 1/3 is 1/3 to make up a whole. But it is the incorrect use of a 2/3 rod that allows her backing to seem correct. Note that Brian does not support the claim made by the girls.

Figure 17.5. Argumentation described by Teacher 10 for Event 5

The description of the argumentation presented in the pre-assessment includes elements that were not stated or shown in Event 5 by the student, Kelly, who agrees with the prior claim. Thus, these elements are included in the diagram, since they were stated by T10, but are in gray, since they were not stated in the event. T10 alludes to the agreement with a prior claim: "Kelly agrees with the other three girls," but does not state the prior claim, or the claim being presented in this event. Those elements, then, are dashed since they are implicit and not explicitly stated. The claim is supported by the data, "the difference in size between 2/3 and 1/3 is 1/3" and two warrants, not presented in the event but mentioned by T10, "the dark green rod to represent 1" and "three red rods...
… represent 1/3 pieces." T10 adds some additional information that is relevant to the argumentation: that the girls' conclusion that is incorrect because, "the incorrect use of a 2/3 rod that allows her to argue this false conclusion" and the student did "not use a proper sized rod for the 1/2."

T10 uses more precise language and adds an element of argumentation and structure in the post-assessment. T10 uses, "support," "claim," and "backing," specifically identifying those elements. Additionally, T10 states, "Brian does not support the claim made by the girls," adding the additional element of argumentation, and, linking it to the claim made by the girls adds structure.

17.1.6 Event 6
Pre-Assessment Description
Now, Brian has the correct response, which is that 1/2 is bigger than 1/3 by 1/6. He argues this by separating the 1/2 and 1/3 rod from the whole. He then uses the red rods to show that, yes three red rods make up the 1/2 and 2 make up the 1/3. But, he notes that the students must remember this is a 1/3 sized piece. So, he argues that you are taking 1/2 of the 1/3 to add to make the complete 1/2 rod. By dividing up the 1/3 rod into two parts, he is really multiplying 1/3 by 1/2 because it is 1/2 of 1/3 to get 1/6. His argument is justified when he lines up the two red rods to show that 2 of them make up 1/3.

Post-Assessment Description
Brian counters the girls' claim by stating a new claim: that 1/2 is bigger than 1/3 by 1/6. He tries to support his claim in the same manner in which the girls had tried to support theirs. However, Brian has the correct interpretation of the rod sizes. He argues this by separating the 1/2 and 1/3 rod from the whole. He then uses the red rods to show that, yes three red rods make up the 1/2 and 2 make up the 1/3. But, he notes that the students must remember this is a 1/3 sized piece. He explains that you must split up the 1/3 pieces in half to obtain 6 red rods that make up the whole. Then, Brian explains that you are taking 1/2 of the 1/3 to add to make the complete 1/2 rod. By dividing up the 1/3 rod into two parts, he is really multiplying 1/3 by 1/2 because it is 1/2 of 1/3 to get 1/6. His argument is justified when he lines up the two red rods to show that 2 of them make up 1/3.
Counterargument [in the title of the event]

Data: "you are taking 1/2 of the 1/3 to add to make the complete 1/2 rod"

Because

Claim: "1/2 is bigger than 1/3 by 1/6"

Warrant: "this [the purple rod?] is a 1/3 sized piece"

Claim/Warrant: "1/2 of 1/3" is 1/6

Warrant: "you ... split up the 1/3 pieces in half to obtain 6 red rods that make up the whole."

Data/Backing: "multiplying 1/3 by 1/2" [is 1/6]

Data/Backing: "2 make up the 1/3"

Warrant: the model that shows the lining up of "two red rods [to show that 2 of them make up 1/3]"
Figure 17.6. Argumentation described by Teacher 10 for Event 6

T10 describes an argument with the claim "1/2 is bigger than 1/3 by 1/6" in the pre-assessment. The argument includes data: "you are taking 1/2 of the 1/3 to add to make the complete 1/2 rod" and warrants: "three red rods make up the 1/2," "this [the purple rod?] is a 1/3 sized piece," and "1/2 of 1/3" is 1/6." The statement, "1/2 of 1/3" is 1/6, also is a claim with its own evidence, functioning as a sub-argument to the main argument. The data T10 note for this Claim/Warrant are "multiplying 1/3 by 1/2" [is 1/6] and "2 make up the 1/3." The fact that "2 make up the 1/3" is backed by the model that shows the lining up of "two red rods to show that 2 of them make up 1/3." The elements are linked through the use of the language, "by," "uses…to show," "so," and "is justified when." Notice that T10 states, "three red rods make up the 1/2," as part of the argument, but the student in the event does not say or model this. Additionally, T10 mentions the idea that if you multiply 1/3 by 1/2 the result is 1/6, which also is not mentioned by the students, thus, the text for these elements of argumentation is in gray.

More precise language by T10 in the post-assessment results in additional elements and structure being described. T10 states, "Brian counters the girls' claim by stating a new claim: that 1/2 is bigger than 1/3 by 1/6." The use of "countering" situates the argument T10 describes as a counterargument to the previous argument made by the "girls" and the claim that "1/2 is bigger than 1/3 by 1/6," then is a counterclaim to the girls' claim. T10's change in the title from pre-assessment ("Argumentation by Dividing Rods into Smaller Parts") to post-assessment ("Brian Presents a Counterargument") confirms that, in the post-assessment, T10 is noting that Brian is making a counterargument.
T10 then states, "He tries to support his claim in the same manner in which the girls had tried to support theirs." The use of the formal mathematical register here situates the argument presented as support for a claim and well as comparing it to the previous argument that the girls used to support their claim. T10 also includes an additional warrant to link the claim that "1/2 is bigger than 1/3 by 1/6" to the data, "you are taking 1/2 of the 1/3 to add to make the complete 1/2 rod," specifically, that "you … split up the 1/3 pieces in half to obtain 6 red rods that make up the whole." This statement adds additional support (in addition to "1/2 of 1/3" is 1/6) as to why the red rod has the number name 1/6. The pre- and post-assessment diagrams show the growth in the argumentation described by T10 in all three levels: elements, structure, and language.

17.1.7 Event 7
**Pre-Assessment Description**
Now the student says that their solution was incorrect because they are "making a different size candy bar." She now realizes that they were incorrectly interpreting the size of the rod in comparison with the original rods. She realizes that they are basing the size of the piece missing off a different sized candy bar than they initially used to represent 1 whole. They note that they did not call the candy bar the dark green rod, and thus in order for them to create the same size candy bar they must change their other pieces to base it off the dark green rod.

**Post-Assessment Description**
A counter is made to explain the girls' incorrect reasoning. The student says that their solution was incorrect because they are "making a different size candy bar." She now realizes that they were incorrectly interpreting the size of the rod in comparison with the original rods. Maher questions the girls, asking them what they originally used to represent 1 whole. One of the girls realizes that they are basing the size of the piece missing off a different sized candy bar than they initially used to represent 1 whole. They note that they did not call the candy bar the dark green rod, and thus in order for them to create the same size candy bar they must change their other pieces to base it off the dark green rod.
Figure 17.7. Argumentation described by Teacher 10 for Event 7

T10 notes an argument for why a solution is "incorrect," with data: they are "making a different size candy bar," warrant: they are "basing the size of the piece missing off a different sized candy bar than they initially used to represent 1 whole," and backing: "they did not call the candy bar the dark green rod." T10 notes additional information that is relevant to the argumentation in the event: "they were incorrectly interpreting the size of the rod in comparison with the original rods."

In the post-assessment, T10 starts by noting, "A counter is made to explain the girls' incorrect reasoning," which situates the rest of the argument described as a counterargument to the prior "incorrect reasoning," and the claim that "their solution was incorrect," as a counterclaim. The use of the formal language adds elements of argumentation and structure to the argument T10 describes.
17.1.8 Event 8

**Pre-Assessment Description**

The girls now based their 1/2 and 1/3 rods off of the dark green rod. They argue that the red rods make up the 1/3 pieces and the light green make up the 1/2 pieces. They continue their argument by leaving the pieces there and showing that the difference between the red and light green is 1 white piece. However, they incorrectly conclude that this means it is 1 bigger.

**Post-Assessment Description**

To present the support correctly, the girls now base their 1/2 and 1/3 rods off of the dark green rod. They show that the red rods make up the 1/3 pieces and the light green make up the 1/2 pieces. Then they leave the pieces there and show that the difference between the red and light green is 1 white piece. However, they incorrectly claim that this means it is 1 bigger. They made the mistake of incorrectly naming the white rod when giving it a number name.
In the pre-assessment description, T10 notes the claim "it is 1 bigger," suggesting that 1/2 is one bigger than 1/3. T10 notes the data, "the difference between the red and light green is 1 white piece," and the warrants that connect these data with the claim: basing "their 1/2 and 1/3 rods off of the dark green rod," "red rods make up the 1/3 pieces," and "red rods make up the 1/3 pieces." T10 adds some additional comments relevant to the argumentation in the event, specifically that the claim that 1/2 is 1 bigger than 1/3 is "incorrect." The language "based," "continue their argument," and "conclude," suggest the connections among the elements of argumentation.

In the post-assessment, T10 states, "They made the mistake of incorrectly naming the white rod when giving it a number name," implying an additional claim: that the students named the white rod 1. This statement is a claim, but also an additional warrant connecting the data "the difference between the red and light green is 1 white piece," and the claim that 1/2 is 1 bigger than 1/3. T10 also makes additional statements relevant to the students' argumentation; first, that the support that the students give for the claim is correct, and second, that the claim that a white rod has the number name 1 is incorrect.
T10 states, "However, they incorrectly claim that this means it is 1 bigger," using specific language to identify the main claim of the argument, and thus, adding elements of argumentation and more of the mathematical register in the post-assessment description.

17.1.9 Event 9

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
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<tbody>
<tr>
<td>Erik comes to the front of the class to help the girls correctly explain the point they are arguing. He explains that the girls meant that by saying 1 white piece they really meant 1/6 because 6 white pieces make it a whole piece. When asked to support his claim, Erik keeps all pieces up there and shows that 6 white pieces are the same length as 1 dark green piece. The students note that if they say that it 1/2 is one bigger than 1/3, they would be referring to the dark green piece, and thus must refer to the white piece as 1/6 (of the green).</td>
</tr>
</tbody>
</table>
Erik comes to the front of the class to help the girls correctly explain how to interpret the rods, and further the white pieces. He explains that the girls meant that by saying 1 white piece they really meant 1/6 because 6 white pieces make it a whole piece. So, he claims that 1/2 is bigger than 1/3 by 1/6. When asked to support his claim, Erik keeps all pieces up there and shows that 6 white pieces are the same length as 1 dark green piece. The students note that if they say that 1/2 is one bigger than 1/3, they would be referring to the dark green piece, and thus must refer to the white piece as 1/6 (of the green). The students agree with Erik's new claim; thus, the students state that 1/6 is the correct interpretation of the white piece.
Figure 17.9. Argumentation described by Teacher 10 for Event 9

T10 describes two arguments in the pre-assessment. The first has "the girls meant that by saying 1 white piece they really meant 1/6" as its claim. Implicit in this claim is
the prior claim that the girls said "white piece." Note that by T10's language, it is not clear if the claim is that the white rod has the number name 1, or that 1/2 is greater than 1/3 by a white rod. It is also not clear if the claim made here counters the claim the girls made. As support for the claim the teacher describes the data: "6 white pieces make it a whole piece" and the warrant that links the data and the claim, the model that "6 white pieces are the same length as 1 dark green piece." T10 states, "When asked to support his claim, Erik keeps all pieces up there," which suggests that T10 believes that the previous model that still appeared on the overhead was backing for the argument. Since T10 does not explicitly state this, the backing box is dashed.

T10 then goes on to note, "The students note that if they say that it 1/2 is one bigger than 1/3, they would be referring to the dark green piece, and thus must refer to the white piece as 1/6 (of the green)." The language of this statement is imprecise, so it is difficult to determine the second argument being described. Stated explicitly is the prior claim that "1/2 is one bigger than 1/3." It seems that T10 is describing a counterargument to this claim with the implicit counterclaim that 1/2 is not bigger than 1/3. As data, T10 notes that the one refers to the dark green piece and that the white piece is "1/6 (of the green)." No explicit link from the data to the claim is made. The vague language makes this argument implicit and, thus, the connectors and counterclaim appear dashed.

Adding the statement, "So, he claims that 1/2 is bigger than 1/3 by 1/6," in the post-assessment clarifies both Erik's argument, as well as the previously implicit second argument. The use of the language "claim" makes explicit that Erik is claiming that 1/2 is greater than 1/3 by 1/6, which is a clear counterclaim to the prior claim explicitly stated in the description, "1/2 is one bigger than 1/3." The fact that this is a counterclaim now
situates the first argument as a counterargument. The use of "so" in the statement suggests that the statements that came before are the evidence for the claim. This changes the argument structure from the pre-assessment, suggesting that the idea that the white piece is $1/6$ has become data, with the warrant the model that shows "6 white pieces make it a whole piece," and the backing "6 white pieces are the same length as the dark green rod." However, the original argument that the white piece is 6 is still nested within this counterargument, making the data both data and claim, the warrant, both warrant and data, and the backing both backing and warrant. The backing for this sub-argument is "all of the pieces up there" or the previous rod model. The data/claim that the white piece is $1/6$ is now a counterclaim to the previous claim that the white piece was 1.

The inclusion of Erik's claim that "$1/2$ is bigger than $1/3$ by $1/6$" also clarifies the second argument. T10's statement, "The students agree with Erik's new claim; thus, the students state that $1/6$ is the correct interpretation of the white piece," situates the argument as an agreement with Erik and, thus, a counterargument to prior claim that $1/2$ is greater than $1/3$ by 1. When T10 notes that students "state that $1/6$ is the correct interpretation of the white piece," it becomes clear that this claim is a modification of their previous claim that the white rod has the number name 1. Through the changes T10 made from pre- to post-assessment, more elements of argumentation are described and those elements are situated in a more connected argumentation structure with nested sub-arguments.

17.1.10 Event 10
Pre-Assessment Description

The students use their original representation of 1 whole piece. Again they make the same incorrect argument by taking off the 1/2 piece and using it as a basis. Their argument is still that 1 of the 3 red pieces that make up the dark green piece is missing to get the purple piece (1/3 rod). But, they keep forgetting that the dark green piece in this case represents 1/2 of the 1 whole.

![Diagram]

Data: "3 red pieces" make up the dark green piece

Because

Claim: "1 of the red pieces ... is missing to get the purple piece [1/3 rod]"

Warrant: "using the 1/2 piece as a "basis"

Additional info: "they keep forgetting that the dark green piece in this case represents 1/2 of the 1 whole."

Post-Assessment Description

The students use their original representation of 1 whole piece. Again they make the same incorrect claim by taking off the 1/2 piece and using it as a basis. So, they are switching their 1 whole piece to the dark green rod, but keeping the lavender rod as the 1/3 piece, although it now makes up 2/3 of the dark green piece. Their claim remains the same as their initial claim: that 1 of the 3 red pieces that make up the dark green piece is missing to get the lavender piece (1/3 rod). But, they keep forgetting that the dark green piece in this case represents 1/2 of the 1 whole. Thus, when trying to recreate the argument made by Erik with their original 1 whole piece representation, they cannot do
Figure 17.10. Argumentation described by Teacher 10 for Event 10

In the pre-assessment, T10 describes an argument in which "1 of the red pieces … is missing to get the purple piece (1/3 rod)," is supported by data: "3 red pieces" make up the dark green piece, with warrant: "using the 1/2 piece as a 'basis.'" T10 mentions additional information that is relevant to the argumentation in the event: that students "keep forgetting that the dark green piece in this case represents 1/2 of the 1 whole."

In the post-assessment, T10 adds language that helps to clarify the argument described. The statement, "Their claim remains the same as their initial claim: that "1 of the 3 red pieces that make up the dark green piece is missing to get the lavender piece (1/3 rod)," confirms that the claim noted is that "1 of the red pieces … is missing to get the purple piece (1/3 rod)" and the accompanying idea, that "3 red pieces" make up the dark green piece is being used to support the claim as data. The addition of the statement, "So, they are switching their 1 whole piece to the dark green rod, but keeping the lavender rod as the 1/3 piece," adds two warrants, the lavender rod is the "1/3 piece" and that the green rod is the 1-whole piece. The idea that the 1/2 rod is their "basis" becomes
the backing. T10 also notes two additional ideas that are relevant and clarify the interpretation of the argumentation of the students in Event 10. T10 states that the students have switched "their 1 whole piece to the dark green rod" and they cannot "recreate the argument made by Erik."

17.1.11 Event 11

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
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<tbody>
<tr>
<td>Brian has the correct idea when saying that the red piece is one of 6 pieces. He does not justify it correctly. He keeps saying that it is one half of the 1/3 but never proves that that creates 1/6. He tries putting it on top of half of the 1/3 piece to show the red piece really is the amount of the missing piece. However, his argument lacks in the fact that he does not explain that 6 red pieces make up the original 1 whole piece.</td>
</tr>
</tbody>
</table>

| Claim/Data: the red piece is "one half of the 1/3" |
| Data/Warrant: "the red piece really is the amount of the missing piece" |
| Additional info: "he does not explain that 6 red pieces make up the original 1 whole piece" |

| Additional info: The support does not justify the claim |
| Warrant/Backign: the model that shows the red piece "on top of half of the 1/3 piece" |

<table>
<thead>
<tr>
<th>Post-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brian has the correct idea when saying that the red piece is one of 6 pieces. His claim that the red piece is what can attach to the lavender (1/3 rod) to make 1/2 is correct. Also, his claim that it is one of 6 red pieces that make up the whole is correct. He does not justify it correctly. He keeps saying that it is one half of the 1/3 but never proves that that creates 1/6. He correctly puts it on top of half of the 1/3 piece to show the red piece really is the amount of the missing piece; that is a new modeling technique that was not previously attempted. However, his backing lacks in the fact that he does not explain that 6 red pieces make up the original 1 whole piece.</td>
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</table>
T10 describes an argument in the pre-assessment, to support what seems to be a claim that "the red piece is one of 6 pieces," however, the imprecision of the language makes it difficult to determine if this is meant as a claim. Furthermore, the student in Event 11 does not say or show that the red piece is one of 6 pieces. Thus, the claim and its connection to the argument are gray and dashed. The use of "justify it" suggests that the statements following are support. The description of this support includes data: "the red piece is 'one half of the 1/3,'" warrant: "the red piece really is the amount of the missing piece" and backing: the model that shows the red piece "on top of half of the 1/3 piece." Note that the statement that the red piece is "one half of the 1/3," also functions as a claim in a sub-argument that has the warrant as its data and the backing as its warrant. T10 includes additional information that is relevant to the argument, stating: the support does not actually justify the claim, and the student, "does not explain that 6 red pieces make up the original 1 whole piece." Note that both of these additional statements are
true, and it makes sense that the argument does not support the claim because the student
did not make the claim noted by T10.

Additional clarity is provided in the post-assessment description through added
detail and more precise language. T10 states "His claim that the red piece is what can
attach to the lavender (1/3 rod) to make 1/2 is correct. Also, his claim that it is one of 6
red pieces that make up the whole is correct." Note that the use of "claim" confirms that
these statements are meant as claims, thus the second claim and its connector are no
longer dashed. T10 clarifies the claim itself, changing "red piece is one of 6 pieces" to the
"red rod is … one of 6 red pieces that make up the whole" The first claim serves as an
additional element of argumentation mentioned in the post-assessment. Then T10 states,
"He does not justify it correctly." Although the pronoun "it" is used, the other language
makes is more evident that T10 intends the statements that follow to be support for what
was just stated, the claim that "it is one of 6 red pieces that make up the whole." T10 also
uses "backing" to refer to the support that was described, providing further evidence that
the statements are meant as support.

17.1.12 Event 12
Pre-Assessment Description
Erik has the correct idea because he considers the fact of 2/3. He notes that two of the 1/3 rods, which make up 2/3 is larger than the dark green (1/2) rod. By proving that their sizes are not equal he shows that it is impossible to increase the 1/3 piece by 1/3 and obtain 1/2.

Post-Assessment Description
Erik has the correct idea because he considers the fact of 2/3. So, he uses a counterexample to show that "it couldn't be exactly a third" larger. He notes that two of the 1/3 rods, which make up 2/3 is larger than the dark green (1/2) rod. By proving that their sizes are not equal he shows that it is impossible to increase the 1/3 piece by 1/3 and obtain 1/2.
In the pre-assessment, T10 describes a claim, "it is impossible to increase the 1/3 piece by 1/3 and obtain 1/2," supported by the data, "their sizes are not equal" [2/3 and 1/2] with two warrants that connect the data with the claim: two of the 1/3 rods make up 2/3 and "two of the 1/3 rods … [are] larger than the dark green (1/2) rod." T10 states, "By proving that their sizes are not equal he shows that it is impossible to increase the 1/3 piece by 1/3 and obtain 1/2," which links the claim and the data.

In the post-assessment, T10 states, "So, he uses a counterexample to show that 'it couldn't be exactly a third' larger." The use of "counterexample" situates the argument as a counterargument. The claim stated by T10, "'it couldn't be exactly a third' larger," then, is a counterclaim to an implicit prior claim that it could be exactly a third larger, or that 1/2 could be 1/3 larger than 1/3. The implicit claim is noted by dashed lines. The change in language also identifies the data as a counterexample, specifically that two thirds are not equal to 1/2. The description of the structure of the argument from pre- to post-assessment has changed. Whereas the statement, "By proving that their sizes are not equal he shows that it is impossible to increase the 1/3 piece by 1/3 and obtain 1/2," served as the initial claim in the pre-assessment description, in the post-assessment description, the statement serves as a modification of the initial counterclaim, "it couldn't be exactly a third larger." This modified claim, then is a result of the counterargument and connects back to the initial counterclaim. The change of title from pre-assessment ("Argumentation by Proving Falsehood") to post-assessment ("Erik Disclaims the Girls' Claim Using a Counterexample") supports these changes in the argumentation. The title suggests that the counterexample was meant to be part of a counterargument that challenges the girls' prior claim.
17.1.13 Event 13

Pre-Assessment Description
Erik says that it had to be 1/3 and a half. I interpret this as meaning it has to be 1/3 and the 1/2 piece of the 1/3 rod (red piece) bigger. Brian [sic] and he argue this by trying to complete the 1 whole piece by adding what the girls called a one third piece (red rod). They conclude that one of the red pieces would not finish the 1 piece and therefore would not complete the rest of the missing 1/2 piece. In addition, Erik justifies his reasoning by explaining where the 1/2 rod falls on the number scale through his view of the rods. Hence he concludes that 1/2 is bigger than 1/3 but less than 2/3, so it is impossible for the red rod to represent 1/3.

Post-Assessment Description
Brian agrees with Erik and, again, has the correct answer of 1/6. Erik says this in a different way: he claims 1/3 and a half make up the dark green (1/2) rod. I interpret this as meaning it has to be 1/3 and the 1/2 piece of the 1/3 rod (red piece) bigger. Brian and Erik support this by trying to complete the 1 whole piece by adding what the girls called
They conclude that one of the red pieces would not finish the 1 piece and therefore would not complete the rest of the missing 1/2 piece. In addition, Erik justifies his reasoning by explaining where the 1/2 rod falls on the number scale through his view of the rods. Hence he concludes that 1/2 is bigger than 1/3 but less than 2/3, so it is impossible for the red rod to represent 1/3.

Figure 17.13. Argumentation described by Teacher 10 for Event 13

T10 notes two arguments in the pre-assessment description. The first has the claim, "it [1/2] had to be 1/3 and a half" with data "one of the red pieces … would not complete the rest of the missing 1/2 piece," warrant, "one of the red pieces would not finish the 1 piece," backed up by the model that shows the student "trying to complete the one third piece (red rod)."

The second argument has the claim, "it is impossible for the red rod to represent 1/3" with data "1/2 is bigger than 1/3 but less than 2/3," warrant, "where the 1/2 rod falls on the number scale," and additional information, "I interpret this as meaning it [1/2] has to be 1/3 and the 1/2 piece of the 1/3 rod (red piece) bigger."
1 whole piece by adding a one third piece (red rod)." T10 adds an interpretation of the claim, stating the belief that by "it had to be 1/3 and a half" the student means, "it [1/2] has to be 1/3 and the 1/2 piece of the 1/3 rod (red piece) bigger."

T10 notes a second argument with claim, "it is impossible for the red rod to represent 1/3," data, "1/2 is bigger than 1/3 but less than 2/3," and a warrant described in imprecise language, "where the 1/2 rod falls on the number scale." T10 uses language such as "argue this by," "conclude," "therefore," "justifies," and "so," to connect these elements together.

The additions to the post-assessment add to the argumentation T10 described. The statement, "Brian agrees with Erik and, again, has the correct answer of 1/6" adds an additional argumentation element, the claim that "the correct answer is 1/6." The use of "claim" and "support" confirm the claim being made and that T10 is describing evidence for the claim. When T10 states, "so it is impossible for the red rod to represent 1/3," the first claim—that the answer is 1/6 (mentioned in the post-assessment, but not the pre-assessment)—is referenced since the two claims agree. Thus, additional structure, elements, and language all show growth in the argumentation noticed from the pre-assessment to the post-assessment.

**17.1.14 Event 14**

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
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<tbody>
<tr>
<td>Michael finally makes the argument that the students had been missing. Similarly to saying that 6 white pieces made up the one dark green piece, he notes that 6 red pieces make up the orange and red piece (1 whole). He does this by placing 6 red pieces on top of the 1 whole rod, which shows the class that the 1 red piece is really 1 of 6.</td>
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</tbody>
</table>
Post-Assessment Description

Michael finally backs the argument using the correct ideology and modeling support. Similarly to saying that 6 white pieces made up the one dark green piece, he notes that 6 red pieces make up the orange and red piece (1 whole). He does this by placing 6 red pieces on top of the 1 whole rod, which shows the class that the 1 red piece is really 1 of 6. Thus, the students can now see that 1/2 is 1/6 bigger than 1/3 from Erik's previous display and now Michael's additional support.

In the pre-assessment, T10 notes an argument with the claim, "the 1 red piece is really 1 of 6," data: "the 1 red piece is really 1 of 6," and warrants: "the orange and red
rod pieces is 1 whole" and the model that shows "6 red pieces on top of the 1 whole rod."
T10 includes a comment that this is "the argument the students had been missing," which
is relevant to the argumentation described.

In the post-assessment, T10 notes a concluding statement, "Thus, the students can
now see that 1/2 is 1/6 bigger than 1/3 from Erik's previous display and now Michael's
additional support." This statement adds structure and elements of argumentation. An
additional claim is noted, "1/2 is 1/6 bigger than 1/3" and supported by "Michael's
additional support" and "Erik's previous display." The phrase, "Michael's additional
support" situates the argument described by T10 as a sub-argument, part of a larger
argumentation discourse supporting the claim that 1/2 is 1/6 bigger than 1/3. This shows
growth in the argumentation that T10 described.

17.1.15 Event 15
Pre-Assessment Description
A similar argument is made to the previous one Erik had made: simply that 1/2 was not as big as 2/3, so therefore it could not be 1/3 larger than 1/3. He then notes that to get from 1/2 to 2/3 you need the 1/6 piece and models this for the class using the rods. Michael then states that it's 1/6 in both cases. Erik was able to use this reasoning because he realized the difference between 1/2 and 1/3 was the same as 2/3 and 1/2.

Post-Assessment Description
A similar counter argument is made to the previous one Erik had made: simply that 1/2 was not as big as 2/3, so therefore it could not be 1/3 larger than 1/3. Meredith then notes that to get from 1/2 to 2/3 you need the 1/6 piece and models this for the class using the rods. She shows that 2 1/3 pieces is longer in length than the 1/2 rod and that the difference between 1/2 and 2/3 is 1/6. So, the claim is that 1/2 is larger by 1/6 is made from the opposite standpoint; this is because the students show that 2/3 is bigger than 1/2 by 1/6, which is still a viable support. Michael correctly states that it's 1/6 in both cases. Erik then used this reasoning because he realized the difference between 1/2 and 1/3 was the same as 2/3 and 1/2.
Figure 17.15. Argumentation described by Teacher 10 for Event 15

In the pre-assessment T10 notes an argument that 1/2 "could not be 1/3 larger than 1/3" with data that "1/2 was not as big as 2/3." Then T10 states, "He then notes that to get from 1/2 to 2/3 you need the 1/6 piece and models this for the class using the rods." It appears that this statement is meant as warrant and backing for the claim and data.
However, the imprecision of language, i.e., "He then notes," makes it difficult to determine what T10 thinks is the role of this statement, thus, the connector between the warrant and the data/claim is dashed. There is a second argument described by T10: that "it's 1/6 in both cases," because "the difference between 1/2 and 1/3 was the same as 2/3 and 1/2." Although these are true statements made by T10, the students in the event do not state them.

Additionally, it is likely that when Michael said "it's 1/6 in both cases," he meant that 1/2 is greater than 1/3 by 1/6 when the dark green rod has the number name 1 or when the orange and red rod train has the number name 1. Therefore, all of that argument, except the claim, is gray. The additional note included by the teacher, that "Erik was able to use this reasoning because he realized the difference between 1/2 and 1/3 was the same as 2/3 and 1/2," is also gray since there is no evidence in the event that Erik used the reasoning described or realized that the differences were the same.

The use of "counterargument" in the post-assessment situates the arguments described by T10 as counter to the implicit claim that 1/2 is larger than 1/3 by 1/3. T10 states, "So, the claim is that 1/2 is larger by 1/6 is made from the opposite standpoint." This additional language used in the post-assessment adds clarity to the description and suggests that there are two counterarguments being described. The first argument has the same claim and data as in the pre-assessment, however, the claim is now situated as a counterclaim. T10 has clarified, however, that what appeared to be support for this argument in the pre-assessment, is actually a description of a second counterargument. This second counterargument has the claim: "1/2 is larger by 1/6," data: "the difference between 1/2 and 2/3 is 1/6," warrants: "2 1/3 pieces is longer in length than the 1/2 rod,"
and "to get from 1/2 to 1/3 you need the 1/6 piece." The warrants are backed by the model that shows that two thirds is one 1/6 rod longer than 1/2. T10 describes a final claim that students made as a result of these arguments, "2/3 is bigger than 1/2 by 1/6." Thus, the more formal use of the mathematical register, coupled with the added detail described in the post-assessment, result in T10's description noting a more sophisticated structure, as well as including more elements of argumentation.

17.2 Summary of Teacher 10's Growth across Events

In 13 of the 15 events, T10 describes additional elements of argumentation in the post-assessment description than in the pre-assessment description. In Event 1, T10 includes an additional claim that "one of the two numbers is bigger" and in Event 3, T10 states, "However, this support would not have been enough on its own. Audra first needed the class to see that those pieces are actually 1/2 and 1/3 of a whole (which they were able to from Laura and Jessica's presentation)" which suggests that the statement "that those pieces are actually 1/2 and 1/3 of a whole" is backing for the argument. In Event 4, T10 uses "show" rather than "argue," implying that a rod model was used as data for the "new claim" that "1/2 is one red rod bigger than 1/3." In Event 5, T10 adds, "Brian does not support the claim made by the girls," noting a disagreement, and specifically notes the claim, "1/2 is larger by 1/3," and clarifies the implicit prior claim, that 1/2 is larger than 1/3 and in Event 6, T10 notes an additional warrant to link the claim that "1/2 is bigger than 1/3 by 1/6" to the data, "you are taking 1/2 of the 1/3 to add to make the complete 1/2 rod," specifically, that "you … split up the 1/3 pieces in half to obtain 6 red rods that make up the whole." This statement adds additional support (in addition to "1/2 of 1/3" is 1/6) as to why the red rod has the number name 1/6, as well as a general
reference to a previous claim, and previous argument. The use of "counter," in the
description of Event 7 adds a counterargument to the description that challenges prior
"incorrect reasoning." In Event 8, T10 suggests that, "They made the mistake of
incorrectly naming the white rod when giving it a number name," which implies an
additional claim, that the students named the white rod 1. This statement is also a warrant
supporting the data that "the difference between the red and light green is 1 white piece."
In Event 9, T10 describes an additional prior claim, that "1/2 is one bigger than 1/3," and
a counterargument with a counterclaim, that "1/2 is bigger than 1/3 by 1/6." T10 also
describes a modified claim that "1/6 is the correct interpretation of the white piece."

In Event 10, T10 includes the statement, "So, they are switching their 1 whole
piece to the dark green rod, but keeping the lavender rod as the 1/3 piece," suggesting
two warrants, the lavender rod is the "1/3 piece" and that the green rod is the 1-whole
piece. Furthermore, the idea that the 1/2 rod is their "basis" becomes backing and T10
adds a claim in Event 11. In Event 12, a counterargument, implicit prior claim, and a
modification of a prior claim are added in the post-assessment and in Event 13, T10 adds
an additional argumentation element, the claim that "the correct answer is 1/6." In Event
14, T10's concluding statement, "Thus, the students can now see that 1/2 is 1/6 bigger
than 1/3 from Erik's previous display and now Michael's additional support," notes an
additional claim, "1/2 is 1/6 bigger than 1/3" which is supported by "Michael's additional
support" and "Erik's previous display," and T10 describes two counterarguments in Event
15, one with an implicit prior claim, a counterclaim, that "1/2 is larger by 1/6," supported
by a nested sub-argument with the claim/data that "the difference between 1/2 and 2/3 is
1/6," data/warrant that "2 1/3 pieces is longer in length than the 1/2 rod," and a concluding claim that "2/3 is bigger than 1/2 by 1/6."

In 10 of the 15 events, T10 adds structure to the argumentation described in the post-assessment than the pre-assessment. In Event 1, T10 states, "Thus, the first element of argumentation is made: the claim is that one of the two numbers is bigger. Laura furthers the claim by answering that 1/2 is larger." The use of "furthers the claim" suggests that the claim that "1/2 is larger [than 1/3]" is a refinement of the claim "one of the two numbers is bigger" and in Event 4, T10 adds implied data and connects it to one of claims noted in the pre-assessment. In Event 6, the statement, "Brian counters the girls' claim by stating a new claim: that 1/2 is bigger than 1/3 by 1/6," situates the argument T10 describes as a counterargument to the previous argument made by the "girls" and the claim that "1/2 is bigger than 1/3 by 1/6," as a counterclaim. T10's change in the title from pre-assessment ("Argumentation by Dividing Rods into Smaller Parts") to post-assessment ("Brian Presents a Counterargument") confirms that, in the post-assessment, T10 is noting that Brian is making a counterargument. Additionally, T10 states, "He tries to support his claim in the same manner in which the girls had tried to support theirs."

The use of the formal mathematical register here situates the argument presented as support for a claim and well as comparing it to the previous argument that the girls used to support their claim. T10 situates the argument as a counterargument through the use of "counter" in Event 7 and in Event 8, T10 adds an implicit warrant that connects the data "the difference between the red and light green is 1 white piece," and the claim that 1/2 is 1 bigger than 1/3. In Event 9, T10 states that Erik is claiming that 1/2 is greater than 1/3 by 1/6, which is a clear counterclaim to the prior claim explicitly stated in the description,
"1/2 is one bigger than 1/3." The fact that this is a counterclaim now situates the first argument as a counterargument. The use of "so" in the statement suggests that the statements that came before are the evidence for the claim. This changes the argument structure from the pre-assessment, suggesting that the idea that the white piece is 1/6 has become data, with the warrant the model that shows "6 white pieces make it a whole piece," and the backing "6 white pieces are the same length as the dark green rod."

However, the original argument that the white piece is 6 is still nested within this counterargument, making the data both data and claim, the warrant, both warrant and data, and the backing both backing and warrant. The backing for this sub-argument is "all of the pieces up there" or the previous rod model. The data/claim that the white piece is 1/6 is now a counterclaim to the previous claim that the white piece was 1. The inclusion in the post-assessment of Erik's claim that "1/2 is bigger than 1/3 by 1/6" also clarifies the second argument. T10's statement, "The students agree with Erik's new claim; thus, the students state that 1/6 is the correct interpretation of the white piece," situates the argument as an agreement with Erik and, thus, a counterargument to prior claim that 1/2 is greater than 1/3 by 1. When T10 notes that students "state that 1/6 is the correct interpretation of the white piece," it becomes clear that this claim is a modification of their previous claim that the white rod has the number name 1.

In Event 12, T10 states, "So, he uses a counterexample to show that 'it couldn't be exactly a third' larger." The use of "counterexample," situates the argument described in the pre-assessment as a counterargument in the post-assessment and the description of the structure of the argument from pre- to post-assessment changes. Whereas the statement, "By proving that their sizes are not equal he shows that it is impossible to increase the 1/3
piece by 1/3 and obtain 1/2," served as the initial claim in the pre-assessment description, in the post-assessment description, the statement serves as a modification of the initial counterclaim, "it couldn't be exactly a third larger." This modified claim, then, is a result of the counterargument and connects back to the initial counterclaim. The change of title from pre-assessment ("Argumentation by Proving Falsehood") to post-assessment ("Erik Disclaims the Girls' Claim Using a Counterexample") supports these changes to the argumentation. The title suggests that the counterexample was meant to be part of a counterargument that challenges the girls' prior claim. In Event 13, additional structure is added to the argumentation description that connects the claim "it is impossible for the red rod to represent 1/3," to the additional claim described in the post-assessment, that "the correct answer," is 1/6 and in Event 14, T10 uses the phrase, "Michael's additional support," situating the argument described as a sub-argument, part of a larger argumentation discourse supporting the claim that 1/2 is 1/6 bigger than 1/3. In Event 15, the use of "counterargument" in the post-assessment situates the arguments described by T10 as counter to the implicit claim that 1/2 is larger than 1/3 by 1/3. T10 states, "So, the claim is that 1/2 is larger by 1/6 is made from the opposite standpoint." This additional language used in the post-assessment adds clarity to the description and suggests that there are two counterarguments being described. The first argument has the same claim and data as in the pre-assessment, however, the claim is now situated as a counterclaim. T10 clarifies, however, that what appeared to be support for this argument in the pre-assessment, is actually a description of a second counterargument. This second counterargument has the claim: "1/2 is larger by 1/6," data: "the difference between 1/2 and 2/3 is 1/6," warrants: "2 1/3 pieces is longer in length than the 1/2 rod," and "to get
from 1/2 to 1/3 you need the 1/6 piece." The warrants are backed by the model that shows that two thirds is one 1/6 rod longer than 1/2. T10 describes a final claim that students made as a result of these arguments, "2/3 is bigger than 1/2 by 1/6."

In 12 of the 15 Events, T10 uses more of the formal mathematical register for argumentation in the post-assessment than in the pre-assessment. In Event 2, T10 identifies a student's statement as a "claim" and uses "backing" and "support" to clarify the role as support for the claim that "the rod model that shows the 1 whole piece in rods of 1/2 and 1/3" plays in the argumentation and in Event 4, T10 notes that students have to "generate a new claim and support it." In Event 5, T10 uses "support," "backing," and "claim," and in Event 6, T10 uses, "counters" and "support his claim" to more precisely describe the argumentation. T10 uses "claim," to specifically identify the claim of the argument in the post-assessment descriptions of Events 8, 9, 10, and 11. In Event 11, T10 also specifically identifies the model that shows the red piece, "on top of half of the 1/3 piece," as "backing" and in Event 12, T10 identifies data as a "counterexample." In Event 13, T10 uses "claim," and "support," to specifically identify elements of argumentation and in Event 14, T10 uses "backs," and "additional support," to identify the support for the argument. In Event 15, T10 uses "counterargument," to situate the argumentation described.

In nine of the 15 events, T10 shows growth in the argumentation described in a variety of other ways. In Event 1, T10 specifically identifies a claim as an element of argumentation, stating, "Thus, the first element of argumentation is made: the claim is that one of the two numbers is bigger." This change is notable, considering the pre-assessment view of argumentation, "Students [sic] support can be considered
argumentation," which might suggest that prior to the intervention, T10 did not think of claims as part of argumentation. In the post-assessment description of this event, T10 takes out the statement, "There is not much argumentation at the moment," suggesting that T10 no longer thinks that there is not much argumentation in the event. Additionally, T10 adds detail that makes the elements of argumentation more precise. In the pre-assessment, T10 mentions one fraction, 1/2. In the post-assessment, T10 mentions both of the fractions that are being compared, 1/2 and 1/3. Thus, the claim "1/2 is larger," takes on more meaning, specifically, that 1/2 is larger than 1/3. Additionally, for the claims that "they are not equal," and "one of the two numbers is bigger," it is clear that 1/2 and 1/3 are the fractions to which T10 is referring. Furthermore, T10 uses more precise language to clarify the class consensus, stating not just that "1/2 is larger", but that "1/2 is larger than 1/3." In the pre-assessment description of Event 2, T10 states the claim as "1/2 is greater than 1/3" whereas in the post-assessment, T10 uses the language the students used, stating that the claim is "1/2 is larger than 1/3."

In Event 3, T10 notes that "by taking the pieces of 1/2 and 1/3 that make up the 1 whole and putting them on their own," the student is not only providing support (as noted in the pre-assessment) but "additional support" for the previous argument as noted in the post-assessment, and in the pre-assessment Event 4, T10 states, "Hence, they conclude that it is 1/3 bigger," and in the post-assessment, T10 changes the language, "Thus, the students' [sic] claim is that 1/2 is 1/3 larger than 1/3," adding precision. In Event 5, T10 makes the implicit claim that 1/2 is greater than 1/3 by 1/3 explicit in the post-assessment and in Event 8, although not argumentation itself, T10 adds statements to the event description that are relevant to the students' argumentation; first, that the support that the
students give for the claim is correct, and second, that the claim that a white rod has the number name 1 is incorrect. In Event 9, T10's language in the post-assessment makes explicit an argument that was implied in the pre-assessment. In Event 10, T10 notes two additional ideas that are relevant and clarify the interpretation of the argumentation of the students. T10 states that the students have switched "their 1 whole piece to the dark green rod" and they cannot "recreate the argument made by Erik." In Event 11, changes the language to make what was an implicit claim and connection to data in the pre-assessment, explicit in the post-assessment, and T10 clarifies the claim itself, changing "red piece is one of 6 pieces" to the "red rod is … one of 6 red pieces that make up the whole."
Chapter 18 – Teacher 12, Event Descriptions Analyses and Summary

This chapter presents the descriptions Teacher 12 (T12) wrote for the pre-assessment and post-assessment analytic to describe the argumentation in each event. The pre-assessment and post-assessment descriptions given by the teacher are presented for each event with an accompanying diagram using Toulmin's (1958, 2003) scheme. Following the descriptions for each event, I present an in-depth analysis, summarizing the argumentation described in the pre-assessment and noting the changes the teacher made from pre- to post-assessment. Words that are key to my analysis appear in red text in the teacher’s descriptions.

18.1 Teacher 12 Event Descriptions and Analyses by Event

18.1.1 Event 1
Pre-Assessment Description
The students are reminded of the problem they were working on from the previous class: use candy bars to show which is bigger, 1/2 or 1/3. Most of the students agree that 1/2 is bigger, while none believe that the two fractions are equal. The students are reluctant to give an answer to how much bigger 1/2 is than 1/3, however. Their argumentation is not apparent yet in this segment, but it is apparent that the students did not question beyond their answer of "1/2 is bigger than 1/3."

Post-Assessment Description
The students are reminded of the problem they were working on from the previous class: use candy bars to show which is bigger, 1/2 or 1/3. Most of the students agree that 1/2 is bigger, while none believe that the two fractions are equal. When the researcher asks the students a second question, "By how much is 1/2 bigger than 1/3?" the students' hands go down, for they do not know the answer. Two female students, Laura and Jessica, go to the overhead to begin explaining how they decided that 1/2 is bigger than 1/3.

Figure 18.1. Argumentation described by Teacher 12 for Event 1

In the pre-assessment, T12 notes that the class makes two claims: that 1/2 and 1/3 are not equal and that "1/2 is bigger than 1/3." In the post-assessment, T12 adds that "When the researcher asks the students a second question, 'By how much is 1/2 bigger than 1/3?' the students' hands go down, for they do not know the answer," suggesting that by lowering their hands, the class is claiming that they don't know how much bigger 1/2 is than 1/3. Thus, T12 adds an element of argumentation in the post-assessment.

18.1.2 Event 2
Pre-Assessment Description
Jessica and Laura present to the class their solution for why $1/2$ is bigger than $1/3$. They use one orange block plus one red block to equal "one whole," or 1. The girls easily put up three rods then two green rods underneath them to all equal the whole, yet one girl struggles with assigning values to each rod specifically. With the help of the other girl, Jessica agrees with Laura that one purple rod is $1/3$ and one green rod is $1/2$.

Post-Assessment Description
Jessica and Laura present to the class their solution for why $1/2$ is bigger than $1/3$. They use one orange block plus one red block, as well as three rods then two green rods underneath them. However, when asked by the teacher to explain what each color rod represents, one girl struggles with assigning values; she does explain that the orange plus red equals one - probably meant to be "one whole." With the help of the other girl, Jessica agrees with Laura that one purple rod is $1/3$ and one green rod is $1/2$.

No changes in the argumentation described.

Figure 18.2. Argumentation described by Teacher 12 for Event 2

In the pre-assessment, T12 describes an argument with the claim, "$1/2$ is bigger than $1/3$." The data described are that "one purple rod is $1/3$," and "one green rod is $1/2$." These data are also claims. These data/claims are supported by data/warrants that three purple rods are the same length as the orange and red rod train and that two green rods are the same length as the orange and red rod train when the orange and red rod train has the number name 1. T12's language, "Jessica and Laura present to the class their solution
for why $1/2$ is bigger than $1/3$," suggests that the description of the number names for the rods and the rod models is meant as support for the claim, yet the warrants/data do not really note why the claims/data support the claim. The post-assessment description is the same as the pre-assessment description.

18.1.3 Event 3

**Pre-Assessment Description**
Audra demonstrates why she agrees with Laura and Jessica's answer that $1/2$ is bigger than $1/3$ by comparing the purple rod with the green rod. Clearly, the green rod is the bigger one. Here Audra uses concise argumentation. She seems to believe that everyone in the class agrees with Laura and Jessica's assignment of halves to the green rods and thirds to the purple rods and skips to comparing the two in size.

| Data: the [dark] green rod is the bigger one [bigger than the purple rod] | because | Claim: "$1/2$ is bigger than $1/3""

Implicit warrant explicitly stated:
everyone in the class agrees with the "assignment of halves to the green rods and thirds to the purple rods"

**Post-Assessment Description**
Audra demonstrates why she agrees with Laura and Jessica's answer that $1/2$ is bigger than $1/3$ by comparing the purple rod with the green rod. Clearly, the green rod is the bigger one. Here Audra uses concise argumentation. She seems to believe that everyone in the class agrees with Laura and Jessica's assignment of halves to the green rods and thirds to the purple rods and skips to comparing the two in size.

No changes in the argumentation described.

*Figure 18.3. Argumentation described by Teacher 12 for Event 3*

In the pre-assessment, T12 describes an argument with the claim, "$1/2$ is bigger than $1/3," and data, the dark green rod is bigger than the purple rod. Then T12 notes an implicit warrant (a warrant that students use but do not mention) that everyone in the
class agrees with the "assignment of halves to the green rods and thirds to the purple rods." The students use this warrant implicitly, but T12 notes it explicitly. There is no change in the post-assessment description.

18.1.4 Event 4

**Pre-Assessment Description**
Comparing the rods, Audra determines that the green rod is one red rod bigger than the purple. Dr. Maher asks her to elaborate on what the red rod represents and what number name she can assign to it. The girls work together and see that three red rods fit into one green rod. Translating this back into the comparison of how much bigger the green is than the purple, the girls say it is 1/3 bigger because three red make one green, so one red is 1/3 green.

**Post-Assessment Description**
Comparing the rods, Audra determines that the green rod is two white rods or one red rod bigger than the purple rod. Dr. Maher asks her to elaborate on what the rods represent and what number name she can assign to it. The girls work together and see that three red rods fit into one green rod. Translating this back into the comparison of how much bigger the green is than the purple, the girls say it is 1/3 bigger because three red make one green, so one red is 1/3. The girls have correctly identified one red to be 1/3 of the green rod, yet they have not compared the red rod to the green rod and then to the whole, which is one orange rod and one red.
In the pre-assessment description, T12 describes two arguments. The first has the claim, "the [dark] green rod is one red rod bigger than the purple [rod]" with a vague reference to a rod model comparison. The second argument has as its claim, that the dark green rod is 1/3 "bigger" than the purple rod, with data, "one red is 1/3 green," or one red rod is 1/3 of the dark green rod, supported by the rod model described as both, "three red rods fit into one green rod" and "three red make one green."
In the post-assessment description, T12 notes additional elements of argumentation, including an additional argument that the dark green rod is two white rods bigger than the purple rod, using the same general reference to a rod comparison model as data. T12 also makes a notable change to the data. Rather than stating that the student says that "one red [rod] is 1/2 green," T12 states that the student says that, "one red is 1/3." This is also a claim supported by data/warrant: "three red rods fit into one green rod" or "three red make one green." T12 also adds additional information relevant to the argumentation, stating, "The girls have correctly identified one red to be 1/3 of the green rod, yet they have not compared the red rod to the green rod and then to the whole, which is one orange rod and one red." This comment suggests that T12 recognizes that there is a flaw in the students' argument.

18.1.5 Event 5

**Pre-Assessment Description**
A student Kelly says that she agrees with the girls who have said that 1/2 is bigger than 1/3 by 1/3. When asked to present to the class, Kelly incorrectly uses light green rods to demonstrate her argument, which are smaller than the dark green rods that Laura, Jessica, and Audra had been using to assign 1/2. Another student, Brian, picks up on this inconsistency. He calls it "changing the problem" when Kelly has used different rods. Brian seems to understand that once in the problem, one cannot reassign different rods for the same value. Dr. Maher continues questioning Laura, Jessica, and Audra to help them to see their own inconsistency: by calling one red rod 1/3, they too have assigned one value to two different rods. Can it possible that 1/2 is bigger than 1/3 by the very 1/3 itself?
Post-Assessment Description
A student Kelly says that she agrees with the girls who have said that 1/2 is bigger than 1/3 by 1/3. When asked to present to the class, Kelly incorrectly uses light green rods to demonstrate her argument, which are smaller than the dark green rods that Laura, Jessica, and Audra had been using to assign 1/2. Another student, Brian, picks up on this inconsistency. He calls it "changing the problem" when Kelly has used different rods. Brian seems to understand that once in the problem, one cannot use different rods for the same value. Dr. Maher continues questioning Laura, Jessica, and Audra to help them to see their own inconsistency: by calling one red rod 1/3, they too have assigned one value to two different rods.

No changes in the argumentation described.

Figure 18.5. Argumentation described by Teacher 12 for Event 5

In the pre-assessment, T12 notes an argument with claim, "1/2 is bigger than 1/3 by 1/3." This claim is supported by a model with the light green rods as data. T12 also notes the claim that the light green rod has the number name 1/2 and notes additional information relevant to the argumentation, stating, "Kelly incorrectly uses light green..."
rods to demonstrate her argument, which are smaller than the dark green rods that Laura, Jessica, and Audra had been using to assign 1/2."

T12 then describes a challenge to this argument with the claim that the students are "changing the problem," supported by the data that they are using different rods, presumably to represent 1/2. The warrant described is that, "once in the problem, one cannot reassign different rods for the same value" or the same rod cannot have two different values. T12 then notes additional information relevant to the argumentation, specifically that, "by calling one red rod 1/3, they too have assigned one value to two different rods" and this is an "inconsistency."

T12 does not make any changes to the post-assessment description that are relevant to the argumentation described.

18.1.6 Event 6

Pre-Assessment Description
Answering Dr. Maher's question, Brian explains at the board that the red rods are actually equal to 1/2 of 1/3. First he says that two red rods make one purple rod, so two in each purple rod will make six red rods in one whole, so each red rod will be equal to 1/6.

Post-Assessment Description
Brain [sic] splits the third rods in half and counts how many make up the whole - he counts 6 rods. Answering the researcher's question, Brian says that splitting a third in half
Bryan explains at the board that the red rods are actually equal to $1/2$ of $1/3$. First he says that two red rods make one purple rod, so two in each purple rod will make six red rods in one whole, so each red rod will be equal to $1/6$.

**Figure 18.6. Argumentation described by Teacher 12 for Event 6**

T12 describes an argument in the pre-assessment description with the claim that "each red rod will be equal to $1/6," supported by data, "the red rods are actually equal to $1/2$ of $1/3," and there are "six red rods in one whole." The datum that there are six red rods in 1 is supported by a warrant: the rod model that shows that two red rods make one purple rod. The support for the datum that the red rods are equal to $1/2$ of $1/3$, which is a claim in itself, is not described by T12 in this description.

The connection between the statement that, "the red rods are actually equal to $1/2$ of $1/3," and the rest of the argument is uncertain, however, the use of "first," might indicate that T12 intended for that statement to support the claim that each red rod will be
equal to 1/6. Since the connection is not certain, it is indicated with a dashed arrow. Note
that T12 uses "so" to indicate the "Data, so Claim" structure.

T12 adds detail that describes more elements and more structure to the
argumentation in the post-assessment. The addition of the statement, "Brain [sic] splits
the third rods in half and counts how many make up the whole - he counts 6 rods.
Answering the researcher's question, Brian says that splitting a third in half creates 1/6,
functions as a sub-argument for the claim that is being used as data, that the red rods are
equal to 1/2 of 1/3. The data, warrant, and backing for this claim/data, then, serve
multiple purposes. The statement that 1/2 of 1/3 is 1/6 is data and a warrant, the statement
that there are six half thirds in one is warrant and backing, and the description of the rod
model where you can split the thirds in half and count that there are six half thirds is
backing. However, there is another argument within this sub-argument, one might call it a
"sub-sub-argument," with the claim that 1/2 of 1/3 is 1/6 as its claim. Thus, the statement
that there are six half thirds in one is also data and the statement about the rod model is
also a warrant. Thus, in the post-assessment T12 described more argumentation in terms
of elements and structure than in the pre-assessment.

Furthermore, the connection between the data that the red rods are equal to 1/2 of
1/3 and the claim that the red rod has the number name 1/6 is more explicit in the post-
assessment, thus the dashed arrow connecting the elements is solid.

18.1.7 Event 7
**Pre-Assessment Description**
The girls understand now that they cannot use different rods for the same number name. They successfully explain themselves using a candy bar reference, an example relevant to their class. One half of one rod is a different size than one half of another rod.

![Diagram](image)

**Post-Assessment Description**
The girls understand now that they cannot use different rods for the same number name. One girl says "They used a different sized candy bar," referencing a previous lesson on fractions using candy bars. One half of one rod is a different size than one half of another rod.

![Diagram](image)

*Figure 18.7. Argumentation described by Teacher 12 for Event 7*

In the pre-assessment, T12 notes a claim, that "One half of one rod is a different size than one half of another rod." In the post-assessment, T12 adds, "One girl says 'They used a different sized candy bar,'" which is an additional claim. The statement, then, that "One half of one rod is a different size than one half of another rod" could be interpreted as data for this claim. However, T12 does not add language to make this connection explicit, so the connection in the diagram appears dashed. Regardless of whether the statement that "One half of one rod is a different size than one half of another rod" is thought of as data for the claim that they used a different sized candy bar, or as a claim,
T12 described an additional element of argumentation in the post-assessment that was not there in the pre-assessment.

18.1.8 Event 8

<table>
<thead>
<tr>
<th>Pre-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>While the girls previously understood that they cannot have two different sized rods be 1/2 at the same time, they have made a similar error in assigning number names to the white and green rods. Using the rods, they correctly line them up and make a convincing argument for fractions of different rods, but they neglect to line the rods back up to the whole number.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Post-Assessment Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>As the girls have made the dark green rod one whole, the researcher asks them to prove if 1/2 or 1/3 is bigger and by how much. The light green rod is a half, and they prove this by showing two light green rods equal one dark green rod. The red rods are thirds, as shown similarly. The girls say &quot;one half is one white rod bigger,&quot; but they have not yet assigned a number name to white. When asked, they give it the number name &quot;1.&quot;</td>
</tr>
</tbody>
</table>
Figure 18.8. Argumentation described by Teacher 12 for Event 8

T12 does not describe any argumentation in the pre-assessment description. Although the statement, "Using the rods, they correctly line them up and make a convincing argument for fractions of different rods," suggests that T12 sees argumentation, there is no claim mentioned for which the rods are being used to create an argument.

In the post-assessment, T12 adds specific language that makes more visible the argumentation in the event. T12 describes an argument that the light green rod has the
number name 1/2, supported by a rod model that shows that two light green rods are the same length as the dark green rod with the warrant that the dark green rod has the number name 1. Additionally, T12 states that, "The red rods are thirds, as shown similarly," which suggests another argument with the claim that the red rods have the number name 1/3 and implicit data: that three red rods are the same length as the dark green rod with the implicit warrant that the number name for the dark green is 1. T12 also includes the claims that the white rod has the number name 1 and "one half is one white rod bigger" than 1/3.

T12 begins the description stating, "As the girls have made the dark green rod one whole, the researcher asks them to prove if 1/2 or 1/3 is bigger and by how much," and ends with the comment, "The girls say 'one half is one white rod bigger,' but they have not yet assigned a number name to white. When asked, they give it the number name '1.'" The language used by T12 suggests that the statements about the argumentation in between these two statements are meant as an argument to support the claim that 1/2 is 1 bigger than 1/3. This claim is not explicitly stated, however, so it appears in the diagram with dashed lines. Since the connection between this claim and the argumentation is implicit, the connecting arrow and the box grouping the arguments are dashed. Even if this implicit connection is not considered, T12 showed growth in the argumentation described from pre- to post-description.

18.1.9 Event 9
Pre-Assessment Description
Erik helps explain the girls' reasoning. He thinks they understand that one white is really 1/6 of the whole, bringing all the rods back into the context of the problem. Through his usage of the rods, Erik successfully explains what the girls had truly meant to say - that they are using one white and mean it to equal [sic] 1/6.

Post-Assessment Description
Erik helps explain the girls' reasoning. He thinks they understand that one white is really 1/6 of the whole, bringing all the rods back into the context of the problem. Through his usage of the rods, Erik successfully explains the correct answer - that they are using one white and mean it to equal 1/6. The girls concede that he is correct, but they do not give much evidence to show why they agree with Erik.

Analysis: In the pre-assessment description, T12 notes a conjecture, that the girls meant to say that "one white is 1/6 of the whole," and a claim, that one white is equal to "1/6." The use of "he thinks…” suggests that this is a conjecture rather than a claim. In the post-assessment description, T12 adds, "The girls concede that he is correct, but they do not give much evidence to show why they agree with Erik." This statement suggests that there was a prior claim about the number name for the white rod and that the girls have modified it, based on their agreement with Erik, to the white rod having the number name 1/6. T12 does not state in this event's description what the prior claim was, but the use of
"concede" implies that there was a statement that is now changing and creates a link between the present claim and the prior one. The additional language in the post-assessment adds elements and structure to the argumentation described.

18.1.10 Event 10

Pre-Assessment Description
The girls again do not go back to the context of the problem - in this case, that the whole number base is the orange rod plus the red rod. They instead correctly say that one red rod is 1/3 of the 1/2 rod, yet they do not go further with their reasoning and neglect to assign the red rod a number name that goes along with the whole number.

Claim: "one red rod is 1/3 of the 1/2 rod"

Post-Assessment Description
Even though they agreed with Erik earlier, now the girls are again calling the red rod 1/3 of the 1/2 rod. This fact is true, but the researcher is asking them if it means 1/3 in the whole context of the problem.

No changes in the argumentation described.

Figure 18.10. Argumentation described by Teacher 12 for Event 10

In the pre-assessment, T12 notes a claim, that "one red rod is 1/3 of the 1/2 rod." In the post-assessment, T12 changes the language of the description, but the changes do not affect the argumentation described.

18.1.11 Event 11
Pre-Assessment Description
The students [sic] try to make sense of the girls' reasoning. While it is true that three red make one green, one red is not equal to the 1/3 rod. Brian's argument of trying to get people to see sixths is built [sic] on his idea that people can "see that."

Counterargument

Data: "people can see 'that'" [sixths]  
because  
Counterclaim: "one red is not equal to the 1/3 rod"

Post-Assessment Description
The students try to make sense of the girls' reasoning. While it is true that three red make one green, one red is not equal to the 1/3 rod. Brian's argument of trying to get people to see sixths is demonstrated at the board. He correctly bring [sic] the "1/3" that the girls were talking about into context of the one whole.

Counterargument

Data: model that shows sixths  
because  
Counterclaim: "one red is not equal to the 1/3 rod"

Warrant: showing the 1/3 in the context of the whole

Figure 18.11. Argumentation described by Teacher 12 for Event 11
In the post-assessment, T12 describes argumentation that suggests an implied prior claim, that the number name the red rod is 1/3 supported by data that "three red rods] make one [dark] green [rod]," with the statement, "While it is true that three red make one green, one red is not equal to the 1/3 rod." Although the data is stated explicitly, the language appears in a dashed box, since it is data for an implied claim. Furthermore, the link between the claim and the data is also dashed since it is not explicitly stated that "while it is true that three red make one green," is data for the implied claim that one red is equal to the 1/3 rod. That prior argument is connected to a counterargument with the counterclaim explicitly stated, that "one red is not equal to the 1/3 rod." This claim is supported by the statement that, "Brian's argument of trying to get people to see sixths is build [sic] on his idea that people can "see that." There is a suggestion that the data is people seeing "sixths."

In the post-assessment, T12 includes additional language that adds to the support of the counterclaim that one red rod does not have the number name 1/3. T12 states, "Brian's argument of trying to get people to see sixths is demonstrated at the board. He correctly bring [sic] the '1/3' that the girls were talking about into context of the one whole." The language that Brian demonstrates at the board implies a rod model that shows sixths. This is a more specific data, but the reference to the rod model is implied. In T12's statement is also a warrant supporting the data the rod model shows the 1/3 in context of the whole. Although the language is still imprecise, the changes to the post-assessment describe more argumentation than in the pre-assessment.

18.1.12 Event 12
Pre-Assessment Description
Erik agrees with Brian; the rods do not equal each other, nor do they both equal 1/3 when related to the whole number base. Erik and Brian reason through by using the fraction bars to show pictures.

Post-Assessment Description
Erik agrees with Brian; the rods do not equal each other, nor do they both equal 1/3 when related to the whole number base. Erik and Brian reason through by using the fraction bars to show pictures. "How can 1/2 be bigger than the 1/3 by 1/3?" Erik questions this statement from the girls and displays evidence in context of the whole to show that possibly it is "1/3 and a half" bigger."

Figure 18.12. Argumentation described by Teacher 12 for Event 12
T12 uses imprecise language in the pre-assessment to describe two arguments: one with claim that the "rods do not equal each other," and one with claim that the rods do not both "equal 1/3." T12 states, that "Erik and Brian reason through by using the fraction bars to show pictures," which suggests that both of these claims are supported by "fraction bars" that "show pictures" as their data.

T12 describes additional argumentation in the post-assessment. T12 adds, "How can 1/2 be bigger than the 1/3 by 1/3'? Erik questions this statement from the girls and displays evidence in context of the whole to show that possibly it is '1/3 and a half' bigger." Through this language, T12 notes a counterargument to the prior claim that 1/2 is bigger than 1/2 by 1/3 with the implied counterclaim that 1/2 is not bigger than 1/3 by 1/3. This counterclaim is supported by data that 1/2 is 1/3 and a half bigger, which is also a conjecture, suggested by the language, "possibly." This data is supported by a model that compares rods, "in the context of the whole." The idea that the "fraction bar" models is meant as support is explicitly stated through T12's use of "evidence." Thus the added text in the post-description describes additional structure and elements of argumentation, as well as more of the formal mathematical register for argumentation, specifically the use of "evidence."

18.1.13 Event 13
Pre-Assessment Description
Now, the boys try to use subtraction to prove their point. They take away the rods from another, and they try to explain through words the fractions of fractions involved.

No argumentation described.

Post-Assessment Description
Now, the boys try to use subtraction to prove their point. They take away the rods from another, and they try to explain through words the fractions of fractions involved. Brian takes one red rod to show 1/3 bigger than the half, and Erik shows using the fraction rods that 1/2 is less than 2/3 and more than 1/3.

![Argumentation Diagram]

Figure 18.13. Argumentation described by Teacher 12 for Event 13

In the pre-assessment, T12 alludes to argumentation ideas when stating, "the boys try to use subtraction to prove their point," but there is no claim stated and the language is so imprecise, it is unclear what is intended.

In the post-assessment, T12 adds detail that puts the pre-assessment statements in the context of argumentation. T12 describes an argument with "1/2 is less than 2/3 and more than 1/3" as its claim. The data is a general reference to subtraction, "the boys try to use subtraction to prove their point." As the warrant that supports the data, T12 notes the use of a rod model that shows 1/3 bigger than the half. With the addition of text, T12 adds structure and elements of argumentation to the post-description.

18.1.14 Event 14
Pre-Assessment Description
Michael focuses explicitly on calling the reds 1/6. By having only Michael explain only one topic, it helps to avoid confusion and intermingling of too many ideas from different people.

Post-Assessment Description
Michael focuses explicitly on calling the reds 1/6. By having only Michael explain only one topic, it helps to avoid confusion and intermingling of too many ideas from different people.

No change in argumentation described.

Figure 18.14. Argumentation described by Teacher 12 for Event 14

T12 states in the pre-assessment, "Michael focuses explicitly on calling the reds 1/6," which describes the claim that the red rod has the number name 1/6. The post-assessment is the same as the pre-assessment.

18.1.15 Event 15

Pre-Assessment Description
One student, Meredith, finally displays using the fraction rods the girls' mistake, By adding 1/3 to the 1/2, it becomes obvious that the red rod cannot equal 1/3 because that still does not match up with the 1/2

Post-Assessment Description
One student, Meredith, displays using the fraction rods the girls' mistake, She explicitly shows that since the purple rods are one third, by holding up 1 purple rod the green 1/2 rod then holding another purple rod up in the remaining space, one can see that that it too
much bigger than the 1/2. Meredith then takes a red rod and shows that this completes the two thirds rod equal to 1/2. Erik continues with saying since this red rod is smaller than the 1/2 and 1/3 and two of them equal 1/3, it has to be 1/6.

Figure 18.15. Argumentation described by Teacher 12 for Event 15

T12 in the pre-assessment notes a counterargument for the implicit prior claim that the red rod has the number name 1/3 with the counterclaim that "the red [rod] cannot
equal 1/3." To support this claim, T12 notes data that adding 1/3 to the 1/2 on the model does not match up with the 1/2.

In the post-assessment, T12 adds language that clarifies the argumentation described and notes more structure and elements of argumentation. In the argument with the counterclaim that the red rod cannot have the number name 1/3, T12 clarifies the data and adds warrant supports. The data, that two purple rods are longer than the 1/2 rod is supported by the warrants that "the purple rods are one third," and model with two purple rods beneath the dark green rod. T12 then notes another piece of data, that two purple 1/3 rods are the same length as a train of a 1/2 rod and a red rod. Although not explicitly stated, these data support the implicit claim that 1/2 is a red rod bigger than 1/3.

Furthermore, T12 describes an additional argument with the claim that the red rod has the number name 1/6, supported by data that "the red rod is smaller than 1/2 and 1/3 and "two of [the red rods] equal 1/3." The structure of this argument is "Data, so claim." The statement that the number name for the red rod is 1/6 is a modification of the prior implicit claim that the red rod has the number name 1/3. In the addition of the added text in the post-description, T12 describes more elements and connected structure of argumentation in the post-assessment than the pre-assessment.

17.2 Summary of Teacher 12's Growth across Events

In 10 of the 15 events, T12 includes language in the post-assessment that adds to the argumentation described in the pre-assessment. This language adds elements as well as structure to the argumentation noted. In Event 1, T12 describes additional elements of argumentation. In the pre-assessment description, two claims are noted. By adding the statement, "When the researcher asks the students a second question, 'By how much is
1/2 bigger than 1/3?" the students' hands go down, for they do not know the answer, suggesting that by lowering their hands, the class is claiming that they don't know how much bigger 1/2 is than 1/3," in the post-assessment description, an additional claim is noted. In Event 4, T12 notes an additional argument, including a description of the claim, data, and warrant, in the post-assessment, and in Event 7, T12 adds data to the claim described in the pre-assessment.

In Event 8, no specific argumentation is described in the pre-assessment, but in the post-assessment, two arguments and two claims are noted. In Event 9, the added statement, "The girls concede that he is correct, but they do not give much evidence to show why they agree with Erik," suggests that there was a prior claim about the number name for the white rod and that the girls have modified it, based on their agreement with Erik, to the white rod having the number name 1/6 and in Event 10, T12 adds language that describes a warrant that is not noted in the pre-assessment. In Event 12, T12 adds language that describes a counterargument to the prior claim that "1/2 is bigger than 1/3 by 1/3," describing the prior claim, an implicit counterclaim, a conjecture that is also data, and data that serve as a warrant. In Event 13, no specific argumentation is described in the pre-assessment, but the changes made in the description in the post-assessment, add an argument with a claim, data, and warrant. In Event 15, T12 describes an additional argument in the post-assessment with the claim that the red rod has the number name 1/6, supported by data that "the red rod is smaller than 1/2 and 1/3 and "two of [the red rods] equal 1/3," as well as additional supports argument that are mentioned in the pre-assessment.
In Event 6, the changes made to the post-assessment description note additional structure as well as additional elements of argumentation, by making more connections among those elements. In Event 6, the addition of the statement, "Brain [sic] splits the third rods in half and counts how many make up the whole - he counts 6 rods. Answering the researcher's question, Brian says that splitting a third in half creates 1/6," functions as a sub-argument for the claim that is being used as data, that the red rods are equal to 1/2 of 1/3. The data, warrant, and backing for this claim/data, then, serve multiple purposes. The statement that 1/2 of 1/3 is 1/6 is data and a warrant, the statement that there are six half thirds in one is warrant and backing, and the description of the rod model where you can split the thirds in half and count that there are six half thirds is backing. Furthermore, there is another argument within this sub-argument, one might call it a "sub-sub-argument," with the claim that 1/2 of 1/3 is 1/6 as its claim. Thus, the statement that there are six half thirds in one is also data and the statement about the rod model is also a warrant. In Event 9, the inclusion of "concede" implies that there was a statement that was changed and creates a link between the present claim and the prior one that did not appear in the pre-assessment.

In Event 8, the pre-assessment does not include language that describes any specific argumentation, although it suggests some general ideas. For example, T12 states, "Using the rods, they correctly line them up and make a convincing argument for fractions of different rods," suggesting that T12 sees argumentation, however there is no claim mentioned for which the rods are being used to create an argument. In the post-assessment, T12 clarifies and explicitly states argumentation that is only hinted at in the pre-assessment. T12 describes an argument that the light green rod has the number name
1/2, supported by a rod model that shows that two light green rods are the same length as the dark green rod with the warrant that the dark green rod has the number name 1. Additionally, T12 states that, "The red rods are thirds, as shown similarly," which suggests another argument with the claim that the red rods have the number name 1/3 and implicit data: that three red rods are the same length as the dark green rod with the implicit warrant that the number name for the dark green is 1. Furthermore, T12 begins the description stating, "As the girls have made the dark green rod one whole, the researcher asks them to prove if 1/2 or 1/3 is bigger and by how much," and ends with the comment, "The girls say 'one half is one white rod bigger,' but they have not yet assigned a number name to white. When asked, they give it the number name '1.'" The language used by T12 suggests implicitly that the statements about the argumentation in between these two statements are meant as an argument to support the claim that 1/2 is 1 bigger than 1/3.

Additionally, the changes that T12 makes to the post-assessment change the quality of the argumentation described, often clarifying the elements in the event. In Event 4, T12 makes a notable to change to the data. Rather than stating that the student says that "one red [rod] is 1/2 green," T12 states that the student says that, "one red is 1/3." In Event 11, T12 clarifies the data for the argument that a red rod does not have the number name 1/3 by more clearly stating that a rod model showing sixths was used and in Event 12, the use of "evidence," in the post-assessment description explicitly clarifies that the "fraction bar" models are meant to support the claims in the argumentation and in Event 15, T12 clarifies the data and adds warrant supports to the counterargument.
In Event 4, T12 states, "The girls have correctly identified one red to be 1/3 of the green rod, yet they have not compared the red rod to the green rod and then to the whole, which is one orange rod and one red." This comment suggests that T12 recognizes that there is a flaw in the students' argument and is evidence that some of the post-assessment changes add information that is relevant to the argumentation. Furthermore, in Event 6, the connection between the data that the red rods are equal to 1/2 of 1/3 and the claim that the red rod has the number name 1/6, which is implicit in the pre-assessment, is made more explicit in the post-assessment, showing that language that T12 uses in the post-assessment, also serves to make explicit elements and structure that are implicit in the pre-assessment.
Chapter 19 – Findings

19.1 Introduction

The purpose of the study was to determine if a video narrative constructed to show student argumentation could support preservice teachers' noticing and understanding of argumentation. The questions that guided the study were:

1. What does student argumentation look like in problem solving settings?

2. How can VMCAnalytic video narratives support teachers’ understanding and noticing of argumentation in student discourse? Specifically,

   a. What do teachers notice about student argumentation before the intervention?

   b. What do teachers notice about student argumentation after the intervention?

More fine grained questions included:

3. After the intervention:

   a. Do teachers notice more elements of argumentation. Specifically, do they notice more claims, data, warrants, backing, counterclaims, and counterarguments?

   b. Do teachers notice more of the structure or connectedness of the argumentation? Specifically, do they notice how the elements of argumentation in each event are related to each other and the relationship among the elements in one event to the elements that were presented in prior events?

   c. Do teachers use more of the formal mathematical register by using more of the precise language of argumentation?
d. Do teachers add specificity to their descriptions by adding detail relevant to the argumentation in the event? Do the added details clarify elements or structure of the argumentation described (so there are fewer lines but they are solid)?

4. If teachers' pre-assessment descriptions included implicit elements or structure, do they, in the post-assessment, make any of the implied argumentation more explicit?

5. If teachers included statements in the pre-assessment that make note of argumentation that is not actually presented in the event, do they eliminate these statements as part of their post-assessment description?

The first part of the study, creating and publishing VMCAnalytics addressed the first question. The second part of the study, the intervention with teachers, addressed the remaining questions. This chapter summarizes the findings for the questions answered by the second part of the study and includes examples of growth in three categories: (1) elements of argumentation, (2) structure of argumentation; (3) the use of the technical language of the formal mathematical register of argumentation; as well as a fourth miscellaneous category.

19.1 Coding with Dedoose

In my first round of data analysis, I used Dedoose, an online, collaborative qualitative research environment (see Dedoose.com for more information about this program). I uploaded teachers' pre- and post-assessment descriptions (including titles) for the 15 events in the assessment analytic in the form of PDF text documents. Data from two of the 13 participants, Teacher 11 and Teacher 13, could not be used because their data were not complete. In both cases, the teachers did not submit event descriptions and
titles for all of the 15 events in the pre-assessment. There were 30 event titles and
descriptions for each of the 11 teachers, 15 for the pre-assessment and 15 for the post-
assessment, for a total of 330 descriptions and 330 titles. I coded these descriptions and
titles using an apriori coding system that included codes for the formal mathematics
register for the elements of argumentation, informed by Toulmin’s argumentation model.
As themes emerged, I refined the coding system, developing new codes, collapsing and
deleting codes. My final coding scheme consisted of a system of various codes and
subcodes as described in Yankelewitz (2009) and shown in Table 19.1.
Table 19.1
Dedoose coding scheme

<table>
<thead>
<tr>
<th>Codes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Event title</td>
</tr>
<tr>
<td>Event description</td>
</tr>
<tr>
<td>Elements of argumentation in event descriptions</td>
</tr>
<tr>
<td>backing</td>
</tr>
<tr>
<td>challenge</td>
</tr>
<tr>
<td>claim</td>
</tr>
<tr>
<td>conjecture</td>
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<tr>
<td>consensus</td>
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</tr>
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</tr>
<tr>
<td>validity</td>
</tr>
<tr>
<td>warrant</td>
</tr>
</tbody>
</table>

19.2.1 Interrater reliability. Several coders worked together to analyze these data and several tests were set up through Dedoose coding software to determine interrater reliability. In our first test, we attained a 0.57 interrater reliability score based on Cohen’s Kappa statistic (Cohen, 1960). Codes were refined and discussion of the coding system clarified the application process. Most of the discrepancies involved the actual application process, rather than contextual argumentation ideas. For example, it was
unclear whether parent codes were to be applied separately or automatically, and, as a result, these codes were applied inconsistently. The exception to this was the idea of consensus. The consensus codes were discussed and refined for clarity.

A second test was set up involving the same data, to be sure that the coding structure was clear and a score of 0.97 was attained. Then a third test with a new data set was conducted, with a score of 0.89, which is considered very good or excellent agreement (Landis & Koch, 1977; Fleiss, 1971; Dedoose.com, 2016). An example of the interrater reliability report generated by Dedoose.com is shown in Figure 19.1.

Figure 19.1. Example of Dedoose interrater reliability report
19.2.2 Dedoose analysis. Once these data were coded, Dedoose's "analyze" features were used to look for patterns. A preliminary analysis showed that teachers used more of the formal mathematical register in the post-assessment than in the pre-assessment, as shown in Table 19.2.

Table 19.2
Frequency of formal mathematical register in pre- and post-assessment descriptions

<table>
<thead>
<tr>
<th>Formal Mathematical Register for Argumentation</th>
<th>Pre-assessment</th>
<th>Post-assessment</th>
</tr>
</thead>
<tbody>
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<td>backing</td>
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<td>challenge</td>
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<tr>
<td>conjecture</td>
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<td>0</td>
<td>2</td>
</tr>
<tr>
<td>support</td>
<td>5</td>
<td>31</td>
</tr>
<tr>
<td>use of argument, argue, argumentation</td>
<td>114</td>
<td>98</td>
</tr>
<tr>
<td>validity</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>warrant</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Additionally, teachers made changes in the titles to use more precise language to describe the argumentation in the events, as shown in Table 19.3.
Table 19.3
Increase in the use of the formal mathematical register in the titles

<table>
<thead>
<tr>
<th>Formal Mathematical Register for Argumentation</th>
<th>Pre-assessment</th>
<th>Post-assessment</th>
</tr>
</thead>
<tbody>
<tr>
<td>backing</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>challenge</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>claim</td>
<td>11</td>
<td>39</td>
</tr>
<tr>
<td>conjecture</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>consensus</td>
<td>47</td>
<td>46</td>
</tr>
<tr>
<td>convince</td>
<td>11</td>
<td>7</td>
</tr>
<tr>
<td>counterargument</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>counterclaim</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>counterexample</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>countering</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>evidence</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>modification</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>proof</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>prove/disprove</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>qualifier</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>refutation</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>support</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>validity</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

These results indicate that the use of technical language relating to argumentation, such as "claim," "counterclaim," "counterargument," "refutation," and "conjecture," of the teachers changed, becoming more precise in the use of the formal mathematical register for argumentation after the intervention. However, the question as to whether there was a notable change in recognition of the argumentation, in terms of elements and structure, was still unresolved.

19.3 Qualitative Analysis

Following the preliminary analysis with Dedoose, an in-depth qualitative analysis was performed on the pre-assessment description and post-assessment description for each teacher and each event, using argumentation diagrams adapted from Toulmin (1958, 2003) and supported by the work of Krummheuer (1995) and others (See Conner et al.,
2014; Pedemonte, 2007; Wagner et al., 2014). In addition, written narratives of the changes from pre- to post-assessment were examined. First a theoretical model was developed of the argumentation in each event of the pre-assessment VMCA.

Analytic (see Chapter 7 for a more detailed discussion of this theoretical model). Experts in argumentation and proof were consulted to validate this model and, after receiving feedback, the analysis was adjusted. For example, when there was disagreement about the role a statement or model played in the argument, the researchers discussed the statement to clarify its role. Discussion tended to center around what was considered data and warrants in the argument. Consistent with argumentation research, the research team found that statements might be considered claims, data, warrants, or backing depending on the audience and context in which they are offered and arguers do not necessarily align the order of their statements with the structure of Toulmin's model (Krummheuer, 1995). Thus, different interpretations of the same argumentation are possible. The researchers' theoretical model, then, is one way to look at the complexity of students' argumentation in the analytic and there other ways may be suggested.

Using the theoretical model as a basis, the argumentation described by teachers in the pre-assessment and post-assessment was diagrammed for each event, focusing on the changes made in the descriptions themselves. Several coders aided in the development and implementation of this system, attaining at least an 80% interrater reliability.

Once all of the argumentation described in the events was diagrammed, this researcher summarized the changes for each event and for each teacher. Then the changes were summarized over all of the events for each teacher. As suggested by Lo, Lim, and Xiong (2016), the functionality in the RUanalytic tool that allowed teachers to write
descriptions and titles associated with video clip events was useful, since with it, teachers generated the descriptions that became the data that were analyzed. As a result of this qualitative analysis, four categories of possible growth emerged. As indicated by the initial analysis, one category (Category 3) indicated that the technical language related to argumentation that teachers used changed from pre- to post-assessment, becoming more focused in the use of the mathematical register of argumentation. Three additional categories also emerged, Category 1: Growth in the number of elements of argumentation described; Category 2: Growth in the structure, or connectedness, of the argumentation described, and a miscellaneous category, Category 4. Examples of teachers' descriptions that fall into each category are included. For an in-depth discussion of the changes in each category for each teacher, see Chapters 8 – 18.

19.3.1 Examples of teacher growth in four categories. In Event 6, Teacher 9 changed the pre-assessment statement, "This is the segment where we see and hear Brian's argument about why 1/2 is not 1/3 greater than 1/3," to "This is the segment where we see and hear Brian's counterargument about why 1/2 is not 1/3 greater than 1/3," and "Therefore, he concludes that 1/2 is 1/6 larger than 1/3," to "Therefore, he concludes that 1/2 is 1/6 larger than 1/3, which is his counterclaim." The change to "counterargument" and "counterclaim" shows growth in the use of more precise language to describe the argumentation in the event (See Figure 19.2).
Teacher 9, Event 6

**Pre-Assessment Description**
"This is the segment where we see and hear Brian's argument about why 1/2 is not 1/3 greater than 1/3. He argues that he can take a tile that is 1/2 of the 1/3 tile, which he figures out is the 1/6 tile, and line that up with the 1/3 tile, in order to make 1/2. By lining this up, next to the 1/2 tile, he shows that placing a 1/6 tile next to a 1/3 tile creates the same length as a 1/2 tile. Therefore, he concludes that 1/2 is 1/6 larger than 1/3."

**Post-Assessment Description**
"This is the segment where we see and hear Brian's counterargument about why 1/2 is not 1/3 greater than 1/3. He argues that he can take a tile that is 1/2 of the 1/3 tile, which he figures out is the 1/6 tile, and line that up with the 1/3 tile, in order to make 1/2. By lining this up, next to the 1/2 tile, he shows that placing a 1/6 tile next to a 1/3 tile creates the same length as a 1/2 tile. Therefore, he concludes that 1/2 is 1/6 larger than 1/3, which is his counterclaim."

*Figure 19.2. Example of teacher growth in the use of language*

Teachers also demonstrated growth in the number of elements of argumentation that they described. In Event 4, Teacher 4 adds the statement, "They do this because they see that three red blocks make one green block and two red blocks make up one pink block," which adds specific data, that "three red blocks make one green block," and "two red blocks make up one pink block," to support the claim that 1/2 is 1/3 larger than 1/3 (See Figure 19.6) and in Event 11, Teacher 1 notes in the post-assessment that "Brian attempts to disprove that the difference between 1/2 and 1/3 in the first model can be one-third," suggesting a prior claim that the difference between 1/2 and 1/3 is 1/3 and that Brian's argument is a counterargument to that claim. The counterargument includes the claim that the difference between 1/2 and 1/3 is 1/6 supported by the data that the red rod is 1/6. That the red rod has the number name 1/6 is also a claim in itself. This data/claim is supported by the data/warrant that the red rod is half of the value of the 1/3, which is supported by the warrant/backing that the red rod is "half the length of the purple rod" and the purple rod has "the value of 1/3." Teacher 1 then notes that the argument leads to
a modification in the claim that 1/2 is larger than 1/3 by 1/6, specifically that "they [1/3 and 1/6] might both be answers." These statements add elements of argumentation to the post-assessment (See Figure 19.3).

**Teacher 4, Event 4**

**Pre-Assessment Description**
"Here the students up front are asked how much larger 1/2 is than 1/3. The students reason by combining their picture blocks and coming to the (incorrect) conclusion 1/3 larger."

**Post-Assessment Description**
"Here the students up front are asked how much larger 1/2 is than 1/3. The students reason by combining their picture blocks and coming to the (incorrect) conclusion 1/3 larger. They do this because they see that three red blocks make one green block and two red blocks make up one pink block. They don't understand its relation to the whole unit though."

*Figure 19.3. Example of teacher growth in the elements of argumentation*
Pre-Assessment Description
"Brian still believes that the difference between 1/2 and 1/3 is 1/6, and comes back up to the projector to show Jessica why the red rod is equal to the quantity 1/6 in her argument. Jessica exclaims at the end the possibility that maybe 1/3 and 1/6 can both be answers."

Post-Assessment Description
"Brian attempts to disprove that the difference between 1/2 and 1/3 in the first model can be one-third. He comes back up to the projector and shows one purple rod that the girls have assigned the value of 1/3 to. He then places a red rod on top of the purple rod, and shows that it is half the length of the purple rod. So the red rod is half of the value of 1/3, or in other words, 1/6. Therefore 1/6 must be the difference between the two fractions. Jessica still is not convinced that her answer of 1/3 is incorrect, and states that "I think that they might both be answers.""

Figure 19.4: Example of teacher growth in elements and structure

Category 2 captures teachers' growth in terms of the structure they noted. I defined structure as the "links," or "connections," teachers made among the elements of
argumentation. In Event 6, Teacher 12 changes the structure of the argumentation to include a nested sub-argument, in which additional support is connected to data and claims (See Figure 19.5). In Event 11, Teacher 1 situated the argument as a counterargument and added connectedness to the argumentation described (See Figure 19.4).

**Teacher 12, Event 6**

**Pre-Assessment Description**
"Answering Dr. Maher's question, Brian explains at the board that the red rods are actually equal to 1/2 of 1/3. First he says that two red rods make one purple rod, so two in each purple rod will make six red rods in one whole, so each red rod will be equal to 1/6."

**Post-Assessment Description**
"Brain [sic] splits the third rods in half and counts how many make up the whole - he counts 6 rods. Answering the researcher's question, Brian says that splitting a third in half creates 1/6. Brian explains at the board that the red rods are actually equal to 1/2 of 1/3. First he says that two red rods make one purple rod, so two in each purple rod will make six red rods in one whole, so each red rod will be equal to 1/6."

*Figure 19.5. Growth in the structure of argumentation teachers described*
As this researcher analyzed teachers' descriptions, other examples of growth in argumentation were noted that were less frequent than those in Categories 1, 2, and 3. This led to the development of a fourth, miscellaneous category, Category 4, that was used to capture these changes. Each of the changes classified as a Category 4 change are outlined and examples are included in the sequel.

Sometimes teachers described implicit elements of argumentation in the pre-assessment that they made explicit in the post-assessment, exemplified by Teacher 1 in Event 8. In the pre-assessment, Teacher 1 implies a connection between the naming of the rods and the claim that 1/2 is larger than 1/3, suggesting that the data support the claim. In the post-assessment, T1 states, "Jackie justifies why 1/2 is larger than 1/3 using the modified values assigned to each rod," making the connection between the naming of the rods and the claim that 1/2 is larger than 1/3 explicit. (See Figure 19.6).

<table>
<thead>
<tr>
<th>Teacher 1, Event 8</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Pre-Assessment Description</strong></td>
</tr>
<tr>
<td>&quot;Jackie continues her argument as to why 1/2 is larger than 1/3 using different rods than Jessica did. She calls the dark green rod 1, and calls the light green rods 1/2 because two of them equal the size of the dark green. She calls the red rods 1/3 because 3 of them equal the size of the dark green. When asked the different [sic] between 1/2 and 1/3, Jackie shows that one white rod makes up the difference, and names the quantity of the white rod &quot;1&quot; even though this would be inconsistent with her measurements.&quot;</td>
</tr>
<tr>
<td><strong>Post-Assessment Description</strong></td>
</tr>
<tr>
<td>&quot;Jackie justifies why 1/2 is larger than 1/3 using the modified values assigned to each rod. Jackie portrays her understanding of using equal proportions, and that it will not change the value of the desired answer. Jackie explains that the dark green rod is equal to the value 1. Since two light green rods combined are of equal length to a single dark green rod, the light green rods equal 1/2. She uses a similar argument to show that the red rods are equal to 1/3. When Researcher Maher asks how much larger 1/2 is than 1/3, Jackie shows that the difference between the two is one white rod. When Maher asks what number she is assigning to the white rod, Jackie answers, &quot;One&quot;, as in one white rod.&quot;</td>
</tr>
</tbody>
</table>

*Figure 19.6. Implicit elements made explicit*
Sometimes teachers made incorrect statements in the pre-assessment that they removed from the post-assessment exemplified by Teacher 4 in Event 9. In the pre-assessment Teacher 4 describes a claim that was not stated by any of the students in the event, "that even though you have different representations of a whole, the answers are still proportionate." In the post-assessment, Teacher 4 takes out the description of that claim and adds the description of two claims that were evident in the event, that, "the girls [sic] argument is really a parallel argument to what the boy Brian made" and "even though you have different representations of a whole, the answer would still be 1/6." (See Figure 19.7).

<table>
<thead>
<tr>
<th>Teacher 4, Event 9</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Pre-Assessment Description</strong></td>
</tr>
<tr>
<td>&quot;In this video another student comes up and shows the girls that their argument is really a parallel argument to what the boy Brian made. He is showing them that even though you have different representations of a whole, the answers are still proportionate.&quot;</td>
</tr>
<tr>
<td><strong>Post-Assessment Description</strong></td>
</tr>
<tr>
<td>&quot;In this video another student comes up and claims that the girls [sic] argument is really a parallel argument to what the boy Brian made. He is [sic] claims that even though you have different representations of a whole, the answer would still be 1/6. The girls then agree to the new claim and agree with Brian as well.&quot;</td>
</tr>
</tbody>
</table>

Some teachers used more of the students' actual language to describe the argumentation rather than their own summary of the language, exemplified by Teacher 4 in Events 1 and 2. In the post-assessments of Events 1 and 2, Teacher 4 uses language that aligns more with the language students use in the event by changing the claim from "1/2 > 1/3" in the pre-assessment to "1/2 is bigger than 1/3" (Event 1) and "1/2 is larger than 1/3," (Event 2) (See Figure 19.8).
Teacher 4, Events 1 and 2

**Event 1**

**Pre-assessment Description**
"In this video the teacher asks the students to recall what they did during the previous day comparing the sizes of two fractions and asks a student to come to the board to try and explain why they think $1/2 > 1/3$ to the class."

**Post-Assessment Description**
"In this video the teacher asks the students to recall what they did during the previous day comparing the sizes of two fractions. The class agrees that $1/2$ is bigger than $1/3$ and the teacher asks the students to come up and convince the rest of the class."

**Event 2**

**Pre-Assessment Description**
"In this video the teacher asks her student to come up and use pictures to show why she claimed that $1/2 > 1/3$. The teacher continues to ask her questions that determine whether she has a developed understanding of what it means to be a whole unit and what it means to be part of a whole unit"

**Post-Assessment Description**
"In this video the teacher asks her student to come up and use pictures to show why she claimed that $1/2 > 1/3$. The students then come up to the projector and use the blocks to assert their claim that $1/2$ is larger than $1/3$."

Figure 19.8. Using more of students' own language

Teachers sometimes made statements that were not elements of argumentation themselves, but comments that were relevant to the argumentation. Sometimes teachers made changes to these statements in the post-assessment that either added additional statements relevant to the argumentation or changed these additional statements into actual elements of argumentation. In Event 8, Teacher 2 adds the statement that the red rods, "are being called both $1/3$ and $1/6$ at the same time," which, although not student argumentation itself, adds information that is relevant to the argumentation presented in the event. (See Figure 19.9)
Teacher 2, Event 8

Pre-Assessment Description
"The girls move away from the originally proposed one whole and one half. They begin to use the scale introduced in event 5 to create their argument. This time, they claim that 1/2 > 1/3 (as opposed to event 2)."

Post-Assessment Description
"Colored rods are redefined since reds are being called both 1/3 and 1/6 at the same time. The girls move away from the originally proposed one whole and one half. They begin to use the smaller scale introduced in event 5 to create their argument. This time, they claim that 1/2 > 1/3 (as opposed to event 2)."

Figure 19.9. Including additional information that is relevant to argumentation

Sometimes teachers added detail to existing elements of argumentation that clarified these elements, as exemplified by Teacher 10 in Event 1. In the pre-assessment, Teacher 10 mentions one of the fractions being compared, 1/2. In the post-assessment, Teacher 10 mentions both of the fractions, 1/2 and 1/3. Thus, the claim "1/2 is larger" takes on more meaning, specifically, that 1/2 is larger than 1/3. Additionally, for the claims that "they are not equal," and "one of the two numbers is bigger," it is clear that 1/2 and 1/3 are the fractions to which Teacher 10 is referring. (See Figure 19.10).

Teacher 10, Event 1

Pre-Assessment Description
"This event does not allow for much student argumentation, the teacher simply asks students to share their opinions. Students support can be considered argumentation. All students agree that they are not equal. Thus if they are not equal, the argument is that one is bigger. So, when Laura answers that 1/2 is larger, the teacher asks the students if they agree. There is not much argumentation at the moment. There is not solid evidence available to support the claim yet, other than one must be bigger and the class consensus that it is 1/2."

Post-Assessment Description
"Students are asked if the numbers 1/2 and 1/3 are equal or if one is larger than the other. When students are asked, all students raise their hands in agreement that they are not equal. Thus, the first element of argumentation is made: the claim is that one of the two numbers is bigger. Laura furthers the claim by answering that 1/2 is larger. The class supports the notion that 1/2 is larger than 1/3. There is not solid evidence available to support the claim yet, other than one must be bigger and the class consensus that it is 1/2."

Figure 19.10. Details clarifying the elements of argumentation
19.4 Quantitative Analysis

A quantitative analysis was performed to determine the frequency of a Category 1 through a Category 4 change and if there were any change relationships among these categories. Table 19.4 reports the mean growth rate for the 11 teachers. Growth is reported separately for each event and category and growth for at least 54% of the teachers in any category for an event is considered significant growth. Table 19.4 indicates: (1) For 93.3% of the events in the study, at least 45% of teachers exhibited a change and (2) For 73.3% of the events in the study, at least 54% of teachers exhibited a change.

Table 19.4
Mean growth rate for events averaged across teachers

<table>
<thead>
<tr>
<th>Event</th>
<th>% Teachers With Change*</th>
<th>Teacher Change Rate &gt; 54% in a Category</th>
<th>Teacher Change Rate &gt; 45% in a Category</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C1</td>
<td>C2</td>
<td>C3</td>
</tr>
<tr>
<td>1</td>
<td>45.5</td>
<td>18.18</td>
<td>63.64*</td>
</tr>
<tr>
<td>2</td>
<td>36.36</td>
<td>36.36</td>
<td>72.73*</td>
</tr>
<tr>
<td>3</td>
<td>36.36</td>
<td>36.36</td>
<td>54.55*</td>
</tr>
<tr>
<td>4</td>
<td>27.27</td>
<td>27.27</td>
<td>54.55*</td>
</tr>
<tr>
<td>5</td>
<td>72.73*</td>
<td>72.73*</td>
<td>72.73*</td>
</tr>
<tr>
<td>6</td>
<td>36.36</td>
<td>36.36</td>
<td>54.55*</td>
</tr>
<tr>
<td>7</td>
<td>36.36</td>
<td>36.36</td>
<td>54.55*</td>
</tr>
<tr>
<td>8</td>
<td>63.64*</td>
<td>63.64*</td>
<td>63.64*</td>
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<td>12</td>
<td>45.45</td>
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</tr>
<tr>
<td>13</td>
<td>36.36</td>
<td>45.45</td>
<td>45.45</td>
</tr>
<tr>
<td>14</td>
<td>72.73*</td>
<td>72.73*</td>
<td>72.73*</td>
</tr>
<tr>
<td>Overall</td>
<td>46.67</td>
<td>44.24</td>
<td>49.09</td>
</tr>
</tbody>
</table>

*Note: C1, C2, C3, C4 refer to Category 1, Category 2, Category 3, and Category 4 respectively.

Averaging first across teachers and then events, the percent of teachers that showed growth in Category 1 (change in elements), Category 2 (change in structure), and
Category 3 (change in language) were similar (46.67%, 44.24%, and 49.09%). The change in Category 4 was lower since this was a miscellaneous category for changes that were only evident in some of the teachers' descriptions. The mean rate of change for each teacher is reported in Table 19.5.

Table 19.5
Mean growth for teachers averaged across events

<table>
<thead>
<tr>
<th>Teacher</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
<th>Overall</th>
<th>Above/Below Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>80</td>
<td>80</td>
<td>46.7</td>
<td>33.3</td>
<td>60</td>
<td>A</td>
</tr>
<tr>
<td>2</td>
<td>66.7</td>
<td>53.3</td>
<td>26.7</td>
<td>13.3</td>
<td>40</td>
<td>A</td>
</tr>
<tr>
<td>3</td>
<td>13.3</td>
<td>13.3</td>
<td>33.3</td>
<td>6.7</td>
<td>16.7</td>
<td>B</td>
</tr>
<tr>
<td>4</td>
<td>66.7</td>
<td>53.3</td>
<td>73.3</td>
<td>26.7</td>
<td>55</td>
<td>A</td>
</tr>
<tr>
<td>5</td>
<td>26.7</td>
<td>20</td>
<td>66.7</td>
<td>5.6</td>
<td>30</td>
<td>B</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>13.3</td>
<td>13.3</td>
<td>6.7</td>
<td>B</td>
</tr>
<tr>
<td>7</td>
<td>26.7</td>
<td>46.7</td>
<td>86.7</td>
<td>13.3</td>
<td>43.3</td>
<td>A</td>
</tr>
<tr>
<td>8</td>
<td>33.3</td>
<td>33.3</td>
<td>33.3</td>
<td>0</td>
<td>25</td>
<td>B</td>
</tr>
<tr>
<td>9</td>
<td>40</td>
<td>40</td>
<td>53.3</td>
<td>26.7</td>
<td>40</td>
<td>A</td>
</tr>
<tr>
<td>10</td>
<td>93.3</td>
<td>86.7</td>
<td>93.3</td>
<td>46.7</td>
<td>80</td>
<td>A</td>
</tr>
<tr>
<td>12</td>
<td>66.7</td>
<td>60</td>
<td>13.3</td>
<td>26.7</td>
<td>41.7</td>
<td>A</td>
</tr>
<tr>
<td>Overall</td>
<td>46.7</td>
<td>44.2</td>
<td>49.1</td>
<td>19.4</td>
<td>39.9</td>
<td></td>
</tr>
</tbody>
</table>

*Note: C1, C2, C3, C4 refer to Category 1, Category 2, Category 3, and Category 4 respectively.

Seven of the teachers had a mean growth rate that was above average; four of the teachers had a mean growth rate below the average.

This researcher was also interested in exploring growth among the categories. Is growth in one category related to growth in other categories? Table 19.6 summarizes the changes in each category aggregating across all event and teachers in the study. Growth in a category is represented by 1 while no growth is represented by 0. Thus, an event code of 0000 for a teacher means no growth in Category 1, 2, 3, and 4, respectively, for that event; whereas 0110 means no growth in Categories 1 and 4 and growth in Categories 2 and 3. When Category 1 and 2 changes were averaged across events and
teachers, Categories 1 and 2 both with no change occurred with a frequency of 50.6%,
Categories 1 and 2 both with change occurred with a frequency 40.8%, and either
Category 1 or 2 with change (but not both) occurred with frequency of 9.1% as reported
in Table 19.6.

*Table 19.6*

*Frequency of change in four categories aggregated across all events*

<table>
<thead>
<tr>
<th>Category 1, 2, 3, 4 Change Sequence</th>
<th>Count</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000</td>
<td>49</td>
<td>0.29697</td>
</tr>
<tr>
<td>0001</td>
<td>5</td>
<td>0.03030</td>
</tr>
<tr>
<td>0010</td>
<td>23</td>
<td>0.13939</td>
</tr>
<tr>
<td>0011</td>
<td>6</td>
<td>0.03636</td>
</tr>
<tr>
<td>0101</td>
<td>2</td>
<td>0.01212</td>
</tr>
<tr>
<td>0110</td>
<td>2</td>
<td>0.01212</td>
</tr>
<tr>
<td>0111</td>
<td>1</td>
<td>0.00606</td>
</tr>
<tr>
<td>1000</td>
<td>4</td>
<td>0.02424</td>
</tr>
<tr>
<td>1001</td>
<td>1</td>
<td>0.00606</td>
</tr>
<tr>
<td>1010</td>
<td>3</td>
<td>0.01818</td>
</tr>
<tr>
<td>1011</td>
<td>2</td>
<td>0.01212</td>
</tr>
<tr>
<td>1100</td>
<td>15</td>
<td>0.09091</td>
</tr>
<tr>
<td>1101</td>
<td>6</td>
<td>0.03636</td>
</tr>
<tr>
<td>1110</td>
<td>38</td>
<td>0.23030</td>
</tr>
<tr>
<td>1111</td>
<td>8</td>
<td>0.04848</td>
</tr>
<tr>
<td>Total</td>
<td>165</td>
<td>1.00000</td>
</tr>
</tbody>
</table>

Table 19.7 examines the interrelationship between Category 1 growth and
Category 2 growth. Using the results of Table 19.6 as input, the right most column of
Table 19.7 reveals that when the Category 1 change is 0 (in other words, no change in the
elements of argumentation described) the likelihood of the Category 2 change being 0 (in
other words, no change in the structure of the argumentation described) is 94.3%. Also,
when the Category 1 change is 1 (showing growth in the elements of argumentation
described) the likelihood of the Category 2 change being 1 (showing growth in the
structure of argumentation described) is 87.0%.
Table 19.7  
Interrelationship between growth in Category 1 and growth in Category 2

<table>
<thead>
<tr>
<th>Category 1 Change</th>
<th>Category 2 Change</th>
<th>Category 1 and 2 Paired Count</th>
<th>Overall Relative Frequency</th>
<th>Conditional Relative Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>49 + 5 + 23 + 6 = 83</td>
<td>50.3%</td>
<td>83/88 = 94.3%</td>
</tr>
<tr>
<td>1</td>
<td>2 + 2 + 1 = 5</td>
<td>3%</td>
<td>5/88 = 5.7%</td>
<td></td>
</tr>
<tr>
<td><strong>Subtotal</strong></td>
<td></td>
<td><strong>88</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>4 + 1 + 3 + 2 = 10</td>
<td>6.1%</td>
<td>10/77 = 13.0%</td>
</tr>
<tr>
<td>1</td>
<td>15 + 6 + 38 + 8 = 67</td>
<td>40.6%</td>
<td>67/77 = 87.0%</td>
<td></td>
</tr>
<tr>
<td><strong>Subtotal</strong></td>
<td></td>
<td><strong>77</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Conditional Relative Frequency is calculated as the frequency the Category 2 change is 0 (or 1) when the Category 1 change is a specified outcome.

Table 19.8 examines the interrelationship between a change in Category 3 and a change in Category 1 from the perspective that if the Category 3 change (change in the use of the technical language of argumentation) is known, what is the likelihood that the Category 1 change is a 0 or a 1. Table 19.8 reveals that if the teacher has no growth in Category 3, then the teacher’s Category 1 description has a 68.3% chance of showing no growth. If the teacher shows growth in Category 3, then the teacher’s Category 2 description has a 62.2% chance of showing growth. If the teacher has no growth in Category 3 then the teacher’s Category 2 description has a 64.1% chance of showing now growth. If the teacher shows growth in Category 3 then the teacher’s Category 2 description has a 68.1% chance of showing growth.

Table 19.8  
Interrelationship between Category 3 and Category 1

<table>
<thead>
<tr>
<th>Category 3 Teacher Description</th>
<th>Corresponding Category 1 Description</th>
<th>Growth Pair</th>
<th>Category 1 &amp; 3 Paired Count</th>
<th>Conditional Relative Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0x1x</td>
<td>23 + 6 + 2 = 31</td>
<td>37.8%</td>
</tr>
<tr>
<td>1</td>
<td>1x1x</td>
<td>5 + 38 + 8 = 51</td>
<td>62.2%</td>
<td></td>
</tr>
<tr>
<td><strong>Sub Total</strong></td>
<td></td>
<td><strong>82</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0x0x</td>
<td>49 + 5 + 2 = 56</td>
<td>68.3%</td>
</tr>
<tr>
<td>1</td>
<td>0x1x</td>
<td>4 + 1 + 15 + 6 = 26</td>
<td>31.7%</td>
<td></td>
</tr>
<tr>
<td><strong>Sub Total</strong></td>
<td></td>
<td><strong>82</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 19.9 examines the interrelationship between a Change in Category 3 and a change in Category 2 from the perspective that if the Category 3 change is known what is the likelihood of the Category 2 change being a 0 or a 1. Table 19.9 reveals that if the teacher has no growth in Category 3, then the teacher’s Category 2 description has a 63.4% chance of showing no growth. If the teacher shows growth in Category 3, then the teacher’s Category 2 description has a 68.1% chance of showing growth.

Table 19.9
Interrelationship between Category 3 and Category 2

<table>
<thead>
<tr>
<th>Category 3 Teacher Description</th>
<th>Corresponding Category 2 Description</th>
<th>Growth Pair</th>
<th>Category 2 &amp; 3 Paired Count</th>
<th>Conditional Relative Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>x01x</td>
<td>2 + 15 + 6 = 23</td>
<td>31.9%</td>
</tr>
<tr>
<td>Sub Total</td>
<td></td>
<td></td>
<td>72</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>x00x</td>
<td>49 + 5 + 4 + 1 = 59</td>
<td>63.4%</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>x11x</td>
<td>2 + 1 + 38 + 8 = 49</td>
<td>68.1%</td>
</tr>
<tr>
<td>Sub Total</td>
<td></td>
<td></td>
<td>93</td>
<td></td>
</tr>
</tbody>
</table>

The results of the analysis of Tables 19.7, 19.8, and 19.9 suggest that those teachers who grew with respect to the use of formal mathematical register for argumentation were more likely to have shown growth with the respect to the elements and structure of argumentation that they described.

19.5 Summary

It is evident from these data that teachers recognized more argumentation, in terms of structure and elements, as well as growth in the language used to describe this argumentation, noting more student argumentation more precisely after the intervention than before. Teachers noticed, for example, more claims, data, backing, counterclaims, and counterarguments after the intervention and those elements were more connected, indicating they noticed more of the structure of argumentation. Furthermore, teachers'
used more of the technical language of the mathematical register of argumentation in their descriptions. Teachers added details that showed growth in their noticing of argumentation in a variety of ways, including making implicit elements and connections explicit, using more of the students' own language rather than summarizing the language, adding comments that were relevant to the argumentation, and eliminating false statements.

These findings suggest that recognizing argumentation can be learned and that, as suggested by Brunvand and Fishman (2006), the analysis of VMCAnalytics designed to illustrate argumentation can be an effective scaffolding tool that supports this learning.
Chapter 20 – Conclusion

20.1 Implications and Limitations

The results of this research suggest that the recognition of argumentation can be supported through studying video narratives developed to show what student argumentation looks like in problem solving settings. Teachers, after studying these narratives, described more details in the students’ argumentation in terms of (1) the number of elements noticed; (2) the structure of the argumentation and; (3) the use of precise language to describe the components of the argumentation. Also, teachers added specificity to their descriptions by providing detail relevant to the structure of the argumentation in the event described and clarifying certain elements and the structure. For example, as teachers' pre-assessment description included implicit elements or structure, some made changes in the language in the post-assessment, and made the implied argumentation explicit. Also, as teachers included statements in the pre-assessment that made note of argumentation that were not actually presented in the event, some eliminated these statements as part of their post-assessment description.

The findings for this study are significant in that they support the idea that teachers' understanding and ability to notice argumentation can be learned. Considering the importance that current standards (NCTM 1989; 2000; CCSS, 2010) and extant mathematics education literature (Schwarz, 2009; Krummheuer, 1995; Bieda & Lepak, 2014; Whitenack & Yackel, 2002) place on argumentation, and in light of Wagner and colleagues assertion that it is "critically important" to "foster and support student argumentation," and Jacob, Lamb, and Phillip's (2010) and Whitenack and Yackel (2002) emphasis that professional noticing (i.e., of mathematical practices such as
argumentation) needs to be specifically supported, this is an important discovery. Van Es and Sherin (2002) suggest that it is important for teachers to make connections between specific classroom practices and broader principles of teaching. The findings in this study suggest that the intervention promoted teachers' ability to make connections between students' statements in a classroom and the broader principles of argumentation, in terms of elements, structure, and language. If it is assumed that teachers can only make sense of and reason about the classroom events they notice and that noticing should be supported (Star & Strickland, 2007, p. 111; Whitenack & Yackel, 2002), it is reasonable to assume that it would be difficult for teachers to support mathematical argumentation discourse in their classrooms if they cannot recognize student engagement in argumentation. Thus, supporting teachers' noticing of student argumentation is a logical first step toward helping them promote argumentation in their curriculum. VMCAnalytics, then, can be a valuable resource for preservice and in-service training.

Another contribution of this work is the open-source, published video narratives (VMCA nalytics) that show student engagement in argumentation. Additionally, the model of the argumentation presented in each event, including diagrams and written summary, provides a model that can be used to analyze actual student argumentation and the descriptions of that argumentation written by the teachers. The diagram and summary, taken together, provide an approach to analyze the argumentation noticed, and compare the argumentation noticed in the pre-assessment and the post-assessment, which leads to a method of determining growth.

Eleven secondary methods preservice teachers participated in the intervention. The encouraging results are limited by the small sample size. However, the detailed
qualitative analysis of the data collected over a semester course allows for tracing teacher
growth over time. The participating teachers studied events from the VMCAanalytics as
an online assignment. Hence, it is not possible to determine how much time teachers
spent studying the video data; nor it is clear whether teachers read all of the descriptive
text as well as studied the video clips, although they were instructed to do so repeatedly
throughout the semester. Perhaps lack of time in the studying the VMCAanalytics could
have contributed to the small growth for some of the teachers. Also, it was not possible to
determine accurately how long each teacher spent on watching the assessment
VMCA analytic and writing and revising the descriptions. This also might explain limited
growth for some teachers.

20.2 Areas of Further Study

Results from this study suggest areas for future research. For example, in
agreement with Hmelo-Silver and colleagues (2010) who posit that further research is
needed that investigates how iterative interventions in different contexts can affect
teacher learning and clarify what elements characterize effective interventions, analysis
of the data from other iterations of this study would determine if the findings could be
duplicated. Additionally, publication of other VMCAanalytics showing details of student
argumentation could be used to illustrate the development of student reasoning as well as
the enactment of other important mathematical practices that could be used in
interventions to determine if they, too, would support teachers' understanding and
noticing of these practices.

This research was conducted with preservice secondary mathematics teachers.
Further study could be carried out with elementary and middle-school teachers. It would
be interesting to investigate whether VMCAnalytics could be used to promote a deeper awareness of argumentation with preservice teachers with an elementary focus and in-service elementary and secondary teachers. Once teachers and preservice teachers are better able to recognize argumentation, they will have key skills that are necessary for them to begin to produce classroom environments that promote argumentation.
References:


Sherin, M. G. (2003). 1. NEW PERSPECTIVES ON THE ROLE OF VIDEO IN TEACHER EDUCATION. *Advances in research on teaching, 10*, 1-27.


Appendix A

Course Syllabus for Secondary Math Methods Spring 2015

05:300:443:01

Rutgers, The State University of New Jersey

05:300:443:01 (Index # 63348)
Methods in Teaching Secondary Mathematics
Spring 2015
Wednesday, 4:50–7:20
GSE - Room 30

Instructor: Cheryl Van Ness
Phone Number 732-932-7496
Email: cheryl.vanness@rutgers.edu

Location: Wed. 4:50-7:30, GSE-30

Office Hours:
Wednesdays, 2:00-4:00 and by appointment, 10 Seminary Place, 2nd floor

Prerequisites or other limitations:
For 05:300:443: Admission to teacher education program.

For 05:300:444: Co-requisite – 05:300:443. Students spend two days in a school each week.

Mode of Instruction:

Lecture
X Seminar
Hybrid
Online
Other

Permission required:
X No
Yes

Online Class: 3/4
Spring Break: No class on 3/18

Rutgers University welcomes students with disabilities into all of the University’s educational programs. In order to receive consideration for reasonable accommodations, a student with a disability must contact the appropriate disability services office at the campus where you are officially enrolled, participate in an intake interview, and provide documentation: https://ods.rutgers.edu/students/documentation-guidelines.

If the documentation supports your request for reasonable accommodations, your campus’s disability services office will provide you with a Letter of Accommodations. Please share this letter with your instructors and discuss the accommodations with them as early in your courses as possible. To begin this process, please complete the Registration form on the ODS website at: https://ods.rutgers.edu/students/registrations-form.
Course Description

Learning Goals:
New Jersey Professional Standards for Teachers (2014):

1. Standard One: Learner Development. The teacher understands how learners grow and develop, recognizing that patterns of learning and development vary individually within and across the cognitive, linguistic, social, emotional, and physical areas, and designs and implements developmentally appropriate and challenging learning experiences.

   i. Performances:
      (1) The teacher regularly assesses individual and group performance in order to design and modify instruction to meet learners’ needs in each area of development (cognitive, linguistic, social, emotional, and physical) and scaffolds the next level of development;
      (2) The teacher creates developmentally appropriate instruction that takes into account individual learners’ strengths, interests, and needs and that enables each learner to advance and accelerate this or her learning.

   ii. Essential Knowledge:
      (1) The teacher understands how learning occurs—how learners construct knowledge, acquire skills, and develop disciplined thinking processes—and knows how to use instructional strategies that promote student learning;
      (2) The teacher understands that each learner’s cognitive, linguistic, social, emotional, and physical development influences learning and knows how to make instructional decisions that build on learners’ strengths and needs;
      (3) The teacher identifies readiness for learning, and understands how development in any one area may affect performance in others;
      (4) The teacher understands the role and impact of language and culture in learning and knows how to modify instruction to make language comprehensible and instruction relevant, accessible, and challenging.

   iii. Critical Dispositions:
      (1) The teacher respects learners’ differing strengths and needs and is committed to using this information to further each learner’s development;
      (2) The teacher is committed to using learners’ strengths as a basis for growth, and their misconceptions as opportunities for learning;
      (3) The teacher takes responsibility for promoting learners’ growth and development; and
      (4) The teacher values the input and contributions of families, colleagues, and other professionals in understanding and supporting each learner’s development.

2. Standard Two: Learning Differences. The teacher uses understanding of individual differences and diverse cultures and communities to ensure inclusive learning environments that enable each learner to meet high standards.

   i. Performances:
      (1) The teacher designs, adapts, and delivers instruction to address each student’s diverse learning strengths and needs and creates opportunities for students to demonstrate their learning in different ways;
      (2) The teacher makes appropriate and timely provisions (for example, pacing for individual rates of growth, task breakdown, 1:1 communication, assessment, and response modes) for individual students with particular learning differences or needs;
      (3) The teacher designs instruction to build on learners’ prior knowledge and experiences, allowing learners to accelerate as they demonstrate their understandings;
      (4) The teacher incorporates tools of language development into learning and instruction, including strategies for making content accessible to English language learners and for evaluating and supporting their development of English proficiency.

1 http://www.state.nj.us/education/codes/current/title6/chap9-pdf
(6) The teacher accesses resources, supports, and specialized assistance and services to meet particular learning differences or needs and participates in the design and implementation of the IEP, where appropriate, through curriculum planning and curricular and instructional modifications, adaptations, and specialized strategies and techniques, including the use of assistive technology.

ii. Essential Knowledge:
(1) The teacher utilizes resources related to educational strategies for instruction and methods of teaching to accommodate individual differences and to employ positive behavioral intervention techniques for students with autism and other developmental disabilities;
(2) The teacher understands and identifies differences in approaches to learning and performance and knows how to design instruction that uses each learner's strengths to promote growth;
(3) The teacher understands students with exceptional needs, including those associated with disabilities and giftedness, and knows how to use strategies and resources to address these needs;
(4) The teacher understands that learners bring assets for learning based on their individual experiences, abilities, talents, prior learning, and peer and social group interactions, as well as language, culture, family, and community values.

iii. Critical Dispositions:
(1) The teacher believes that all learners can achieve at high levels and persists in helping each learner reach his or her full potential;
(2) The teacher respects learners as individuals with differing personal and family backgrounds and various skills, abilities, perspectives, talents, and interests;
(3) The teacher makes learners feel valued and helps them learn to value each other;
(4) The teacher values diverse languages, dialects, and cultures and seeks to integrate them into his or her instructional practice to engage students in learning.

3. Standard Three: Learning Environment. The teacher works with others to create environments that support individual and collaborative learning and that encourage positive social interaction, active engagement in learning, and self-motivation.

i. Performances:
(1) The teacher collaborates with learners, families, and colleagues to build a safe, positive learning climate of openness, mutual respect, support, and inquiry;
(2) The teacher develops learning experiences that engage learners in collaborative and self-directed learning and that extend learner interaction with ideas and people locally and globally;
(3) The teacher collaborates with learners and colleagues to develop shared values and expectations for respectful interactions, rigorous academic discussions, and individual and group responsibility for quality work;
(4) The teacher manages the learning environment to actively and equitably engage learners by organizing, allocating, and coordinating the resources of time, space, and learners' attention;
(5) The teacher uses a variety of methods to engage learners in evaluating the learning environment and collaborates with learners to make appropriate adjustments;
(6) The teacher communicates verbally and nonverbally in ways that demonstrate respect for and responsiveness to the cultural backgrounds and differing perspectives learners bring to the learning environment;
(7) The teacher promotes responsible learner use of interactive technologies to extend the possibilities for learning locally and globally;
(8) The teacher intentionally builds learner capacity to collaborate face-to-face and virtual environments through applying effective interpersonal communication skills.

ii. Essential Knowledge:
(1) The teacher understands the relationship between motivation and engagement and knows how to design learning experiences using strategies that build learner self-direction and ownership of learning;
(2) The teacher knows how to help learners work productively and cooperatively with each other to achieve learning goals;
(3) The teacher knows how to collaborate with learners to establish and monitor elements of a safe and productive
learning environment including norms, expectations, routines, and organizational structures;
(4) the teacher understands how learner diversity can affect communication and knows how to communicate effectively in differing environments;
(5) the teacher knows how to use technologies and how to guide learners to apply them in appropriate, safe, and effective ways.

III. Critical Dispositions:
(1) The teacher is committed to working with learners, colleagues, families, and communities to establish positive and supportive learning environments;
(2) The teacher values the role of learners in promoting each other’s learning and recognizes the importance of peer relationships in establishing a climate of learning;
(3) The teacher is committed to supporting learners as they participate in decision-making, engage in exploration and invention, work collaboratively and independently, and engage in purposeful learning;
(4) The teacher seeks to foster respectful communication among all members of the learning community.

4. Standard Four: Content Knowledge. The teacher understands the central concepts, tools of inquiry, and structures of the discipline(s) he or she teaches, particularly as they relate to the Common Core Standards and the New Jersey Core Curriculum Content Standards and creates learning experiences that make these aspects of the discipline accessible and meaningful for learners to assure mastery of the content.

I. Performance:
(1) The teacher effectively uses multiple representations and explanations that capture key ideas in the discipline, guide learners through learning progressions, and promote each learner’s achievement of content standards;
(2) The teacher engages students in learning experiences in the discipline(s) that encourage learners to understand, question, and analyze ideas from diverse perspectives so that they master the content;
(3) The teacher engages learners in applying methods of inquiry and standards of evidence used in the discipline;
(4) The teacher stimulates learner reflection on prior content knowledge, links new concepts to familiar concepts, and makes connections to learners’ experiences;
(5) The teacher recognizes learner misconceptions in a discipline that interfere with learning and creates experiences to build accurate conceptual understanding;
(6) The teacher evaluates and modifies instructional resources and curriculum materials for their comprehensiveness, accuracy for representing particular concepts in the discipline, and appropriateness for his or her learners;
(7) The teacher uses supplementary resources and technologies effectively to ensure accessibility and relevance for all learners;
(8) The teacher creates opportunities for students to learn, practice, and master academic language in their content, and
(9) The teacher accesses school and/or district-based resources to evaluate the learner’s content knowledge.

II. Essential Knowledge:
(1) The teacher understands major concepts, assumptions, debates, processes of inquiry, and ways of knowing that are central to the discipline(s) he or she teaches;
(2) The teacher understands common misconceptions in learning the discipline and how to guide learners to accurate conceptual understanding;
(3) The teacher knows and uses the academic language of the discipline and knows how to make it accessible to learners;
(4) The teacher knows how to integrate culturally relevant content to build on learners’ background knowledge;
(5) The teacher has a deep knowledge of student content standards and learning progressions in the discipline(s) he or she teaches;
(6) The teacher understands that literacy skills and processes are applicable in all content areas and help students to develop the knowledge, skills, and dispositions that enable them to construct meaning and make sense of the world through reading, writing, listening, speaking, and viewing;
(7) The teacher understands the concepts inherent in numeracy to enable students to represent physical events, work with data, reason, communicate mathematically, and make connections within their respective content areas in
order to solve problems.

III. Critical Dispositions:
1. The teacher realizes that content knowledge is not a fixed body of facts but is complex, culturally situated, and ever evolving. He or she keeps abreast of new ideas and understandings in the field.
2. The teacher appreciates multiple perspectives within the discipline and facilitates learners' critical analysis of these perspectives;
3. The teacher recognizes the potential of bias in his or her representation of the discipline and seeks to appropriately address problems of bias;
4. The teacher is committed to work toward each learner's mastery of disciplinary content and skills;
5. The teacher shows enthusiasm for the discipline(s) they teach and is committed to making connections to everyday life.

5. Standard Five: Application of Content. The teacher understands how to connect concepts and use differing perspectives to engage learners in critical thinking, creativity, and collaborative problem solving related to authentic local and global issues.

II. Performances:
1. The teacher develops and implements projects that guide learners in analyzing the complexity of an issue or question using perspectives from varied disciplines and cross-disciplinary skills (for example, a water quality study that draws upon biology and chemistry to look at factual information and social studies to examine policy implications);
2. The teacher engages learners in applying content knowledge to real-world problems through the lens of interdisciplinary themes (for example, financial literacy and environmental literacy);
3. The teacher facilitates learners' use of current tools and resources to maximize content learning in varied contexts;
4. The teacher engages learners in questioning and challenging assumptions and approaches in order to foster innovation and problem solving in local and global contexts;
5. The teacher develops learners' communication skills in disciplinary and interdisciplinary contexts by creating meaningful opportunities to employ a variety of forms of communication that address varied audiences and purposes;
6. The teacher engages learners in generating and evaluating new ideas and novel approaches, seeking inventive solutions to problems, and developing original work;
7. The teacher facilitates learners' ability to develop diverse social and cultural perspectives that expand their understanding of local and global issues and create novel approaches to solving problems;
8. The teacher develops and implements supports for learner literacy development across content areas.

II. Essential Knowledge:
1. The teacher understands the ways of knowing in his or her discipline, how it relates to other disciplinary approaches to inquiry, and the strengths and limitations of each approach in addressing problems, issues, and concerns;
2. The teacher understands how current interdisciplinary themes (for example, civic literacy, health literacy, global awareness) connect to the core subjects and how to weave these themes into meaningful learning experiences;
3. The teacher understands the demands of accessing and managing information as well as how to evaluate issues of ethics and quality related to information and its use;
4. The teacher understands how to use digital and interactive technologies for efficiently and effectively achieving specific learning goals;
5. The teacher understands critical thinking processes and knows how to help learners develop high-level questioning skills to promote independent learning;
6. The teacher understands communication modes and skills as vehicles for learning (for example, information gathering and processing across disciplines as well as tools for expressing learning);
7. The teacher understands creative thinking processes and how to engage learners in producing original work;
8. The teacher knows where and how to access resources to build global awareness and understanding, and how to
integrate them into the curriculum.

III. Critical Dispositions:
1. The teacher is constantly exploring how to use disciplinary knowledge as a lens to address local and global issues;
2. The teacher values knowledge outside his or her own content area and how such knowledge enhances student learning;
3. The teacher values flexible learning environments that encourage learner exploration, discovery, and expression across content areas.

6. Standard 5: Assessment. The teacher understands and uses multiple methods of assessment to engage learners in examining their own growth, to monitor learner progress, and to guide the teacher's and learner's decision-making.

I. Performances:
1. The teacher balances the use of formative and summative assessment as appropriate to support, verify, and document learning;
2. The teacher designs assessments that match learning objectives with assessment methods and minimizes sources of bias that can distort assessment results;
3. The teacher works independently and collaboratively to examine test and other performance data to understand each learner's progress and to guide planning;
4. The teacher engages learners in understanding and identifying quality work and provides them with effective descriptive feedback to guide their progress toward that work;
5. The teacher engages learners in multiple ways of demonstrating knowledge and skill as part of the assessment process;
6. The teacher models and structures processes that guide learners in examining their own thinking and learning as well as the performance of others;
7. The teacher effectively uses multiple and appropriate types of assessment data to identify each student's learning needs and to develop differentiated learning experiences;
8. The teacher prepares all learners for the demands of particular assessment formats and makes appropriate accommodations in 21st assessments or testing conditions, especially for learners with disabilities and language learning needs;
9. The teacher continually seeks appropriate ways to employ technology to support assessment practice both to engage learners more fully and to assess and address learner needs.

II. Essential Knowledge:
1. The teacher understands the differences between formative and summative applications of assessment and knows how and when to use each;
2. The teacher understands the range of types and multiple purposes of assessment and how to design, adapt, or select appropriate assessments to address specific learning goals and individual differences, and to minimize sources of bias;
3. The teacher knows how to analyze assessment data to understand patterns and gaps in learning to guide planning and instruction, and to provide meaningful feedback to all learners;
4. The teacher knows when and how to engage learners in analyzing their own assessment results and in helping to set goals for their own learning;
5. The teacher understands the positive impact of effective descriptive feedback for learners and knows a variety of strategies for communicating this feedback;
6. The teacher knows when and how to evaluate and report learner progress against standards;
7. The teacher understands how to prepare learners for assessments and how to make accommodations in assessments and testing 3rd conditions, especially for learners with disabilities and language learning needs.

III. Critical Dispositions:
1. The teacher is committed to engaging learners actively in assessment processes and to developing each learner's capacity to review and communicate about their own progress and learning;
(2) The teacher takes responsibility for aligning instruction and assessment with learning goals; 
(3) The teacher is committed to providing timely and effective descriptive feedback to learners on their progress; 
(4) The teacher is committed to using multiple types of assessment processes to support, verify, and document learning; 
(5) The teacher is committed to making accommodations in assessments and testing conditions, especially for learners with disabilities and language learning needs; 
(6) The teacher is committed to the ethical use of various assessments and assessment data to identify learner strengths and needs to promote learner growth.

7. Standard Seven: Planning for Instruction. The teacher plans instruction that supports every student in meeting rigorous learning goals by drawing upon knowledge of content areas, curriculum, cross-disciplinary skills, and pedagogy, as well as knowledge of learners and the community context.

I. Performance:
(1) The teacher individually and collaboratively selects and creates learning experiences that are appropriate for curriculum goals and content standards, and are relevant to learners; 
(2) The teacher plans how to achieve each student’s learning goals, choosing appropriate strategies and accommodations, resources, and materials to differentiate instruction for individuals and groups of learners; 
(3) The teacher develops appropriate sequencing of learning experiences and provides multiple ways to demonstrate knowledge and skill; 
(4) The teacher plans for instruction based on formative and summative assessment data, prior learner knowledge, and learner interests; 
(5) The teacher plans collaboratively with professionals who have specialized expertise (for example, special educators, related service providers, language learning specialists, librarians, and media specialists) to design and jointly deliver, as appropriate, learning experiences to meet unique learning needs; 
(6) The teacher evaluates plans in relation to short- and long-range goals and systematically adjusts plans to meet each student’s learning needs and enhance learning.

II. Essential Knowledge:
(1) The teacher understands content and content standards and how those are organized in the curriculum; 
(2) The teacher understands how integrating cross-disciplinary skills in instruction engages learners purposefully in applying content knowledge; 
(3) The teacher understands learning theory, human development, cultural diversity, and individual differences and how these impact ongoing planning; 
(4) The teacher understands the strengths and needs of individual learners and how to plan instruction that is responsive to those strengths and needs; 
(5) The teacher knows a range of evidence-based instructional strategies, resources, and technological tools, including assistive technologies, and how to use them effectively to plan instruction that meets diverse learning needs; 
(6) The teacher knows when and how to adjust plans based on assessment information and learner responses; 
(7) The teacher knows when and how to access resources and collaborate with others to support student learning (for example, special educators, related service providers, language learner specialists, librarians, media specialists, and community organizations).

III. Critical Dispositions:
(1) The teacher respects learners’ diverse strengths and needs and is committed to using this information to plan effective instruction; 
(2) The teacher values planning as a collegial activity that takes into consideration the input of learners, colleagues, families, and the larger community; 
(3) The teacher takes professional responsibility to use short- and long-term planning as a means of assuring student learning; 
(4) The teacher believes that plans must always be open to adjustment and revision based on learner needs and changing circumstances.
8. Standard Eight: Instructional Strategies. The teacher understands and uses a variety of instructional strategies to encourage learners to develop deep understanding of content areas and their connections, and to build skills to apply knowledge in meaningful ways.

I. Performance:
1. The teacher uses appropriate strategies and resources to adapt instruction to the needs of individuals and groups of learners;
2. The teacher continuously monitors student learning, engages learners in assessing their progress, and adjusts instruction in response to student learning needs;
3. The teacher collaborates with learners to design and implement relevant learning experiences, identify their strengths, and access family and community resources to develop their areas of interest;
4. The teacher varies his or her role in the instructional process (for example, instructor, facilitator, coach, and audience) in relation to the content and purposes of instruction and the needs of learners;
5. The teacher provides multiple models and representations of concepts and skills with opportunities for learners to demonstrate their knowledge through a variety of products and performances;
6. The teacher engages learners in developing higher order questioning skills and meta-cognitive processes;
7. The teacher engages learners in using a range of learning skills and technology tools to access, interpret, evaluate, and apply information;
8. The teacher uses a variety of instructional strategies to support and expand learners’ communication through speaking, listening, reading, writing, and other modes;
9. The teacher asks questions to stimulate discussion that serves different purposes (for example, probing for learner understanding, helping learners articulate their ideas and thinking processes, stimulating curiosity, and helping learners to question).

II. Essential Knowledge:
1. The teacher understands the cognitive processes associated with various kinds of learning (for example, critical and creative thinking, problem solving, invention, and memorization and recall) and how these processes can be stimulated;
2. The teacher knows how to apply a range of developmentally, culturally, and linguistically appropriate instructional strategies to achieve learning goals;
3. The teacher knows when and how to use appropriate strategies to differentiate instruction and engage all learners in complex thinking and meaningful tasks;
4. The teacher understands how multiple forms of communication (oral, written, nonverbal, digital, and visual) convey ideas, foster self-expression, and build relationships;
5. The teacher knows how to use a wide variety of resources, including human and technological, to engage students in learning;
6. The teacher understands how content and skill development can be supported by media and technology and knows how to evaluate these resources for quality, accuracy, and effectiveness.

III. Critical Dispositions:
1. The teacher is committed to deepening awareness and understanding the strengths and needs of diverse learners when planning and adjusting instruction;
2. The teacher values the variety of ways people communicate and encourages learners to develop and use multiple forms of communication;
3. The teacher is committed to exploring how the use of new and emerging technologies can support and promote student learning;
4. The teacher values flexibility and reciprocity in the teaching process as necessary for adapting instruction to learner responses, ideas, and needs.

9. Standard Nine: Professional Learning. The teacher engages in ongoing individual and collaborative professional learning designed to impact practice in ways that lead to improved learning for each student, using evidence of student achievement, action research, and best practice to expand a repertoire of skills, strategies, materials, assessments, and ideas to increase student learning.
l. Performance:
(1) Independently and in collaboration with colleagues, the teacher uses a variety of data (for example, systematic observation, information about learners, and research) to evaluate the outcomes of teaching and learning and to adapt planning and practice;
(2) The teacher actively seeks professional, community, and technological resources, within and outside the school, as supports for analysis, reflection, and problem-solving.

ii. Essential Knowledge:
(1) The teacher understands and knows how to use a variety of self-assessment and problem-solving strategies to analyze and reflect on his or her practice and to plan for adaptations/adaptations;

iii. Critical Dispositions:
(1) The teacher takes responsibility for student learning and uses ongoing analysis and reflection to improve planning and practice;

11. Standard Eleven: Ethical Practice. The teachers acts in accordance with legal and ethical responsibilities and uses integrity and fairness to promote the success of all students.

l. Performance:
(2) The teacher advocates, models, and teaches safe, legal, and ethical use of information and technology including appropriate documentation of sources and respect for others in the use of social media;
(3) The teacher promotes aspects of students’ well-being by exercising the highest level of professional judgment, and working cooperatively and productively with colleagues and parents to provide a safe, healthy, and emotionally protective learning environment;
(4) The teacher maintains the confidentiality of information concerning students obtained in the course of the educational process and dispenses such information only when prescribed or directed by Federal and/or State statutes or accepted professional practice;
(5) The teacher maintains professional relationships with students and colleagues;

ii. Essential Knowledge:
(1) The teacher understands how personal identity, worldviews, and prior experience affect perceptions and expectations, and recognizes how they may bias behavior and interactions with others;
(4) The teacher knows and understands strategies to foster professional and productive relationships with students and colleagues.

III. Critical Dispositions:
(1) The teacher recognizes that an educator’s actions reflect on the status and substance of the profession;
(2) The teacher upholds the highest standards of professional conduct both as a practitioner in the classroom and as an employee vested with the public trust;
(3) The teacher recognizes, respects, and upholds the dignity and worth of students as individual human beings, and therefore deals with them justly and considerately;
(4) The teacher recognizes his or her obligation to the profession of teaching and does not engage in any conduct contrary to sound professional practice and/or applicable statutes, regulations, and policy.

Council for the Accreditation of Education Professionals (2013)²:

Standard 1: CONTENT AND PEDAGOGICAL KNOWLEDGE
1.1 Candidates demonstrate an understanding of the 10 InTASC standards at the appropriate progression level(s) in the following categories: the learner and learning, content, instructional practice, and professional responsibility.

² http://raspop.files.wordpress.com/2013/09/final_board_approved1.pdf
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1.3 Providers ensure that completers apply content and pedagogical knowledge as reflected in outcome assessments in response to standards of Specialized Professional Associations (SPA), the National Board for Professional Teaching Standards (NBPTS), states, or other accrediting bodies (e.g., National Association of Schools of Music – NASM).

1.4 Providers ensure that completers demonstrate skills and commitment that afford all P-12 students access to rigorous college- and career-ready standards (e.g., Next Generation Science Standards, National Career Readiness Certificate, Common Core State Standards).

Standard 2: CLINICAL PARTNERSHIPS AND PRACTICE

2.3 The provider works with partners to design clinical experiences of sufficient depth, breadth, diversity, coherence, and duration to ensure that candidates demonstrate their developing effectiveness and positive impact on all students' learning and development. Clinical experiences, including technology-enhanced learning opportunities, are structured to have multiple performance-based assessments at key points within the program to demonstrate candidates' development of the knowledge, skills, and professional dispositions, as delineated in Standard 1, that are associated with a positive impact on the learning and development of all P-12 students.

Course Catalogue Description

Reviews the status of secondary mathematics teaching in the United States, the reform movement of the 1990s, and current thinking about issues of concern to practicing teachers. Encourages development of personal style and approach to teaching high school mathematics. Topics include instructional planning, assessment, individual differences, cultural and gender differences, and teaching styles.

Other description of course purposes, context, methods, etc.:

The course, Methods in Teaching Secondary Mathematics, is designed for candidates who will become middle and/or high school teachers of mathematics. It is expected that candidates will increase their knowledge of Common Core State Standards (CCSS), the PARCC assessments, and SGOs. The course will encourage the development of personal style in teaching. The goal of this course is for students to develop detailed knowledge about the development of learners' mathematical reasoning and ways to build instruction based on that knowledge. The course will provide a repertoire of pedagogical techniques and routines related to the above as well as a base to reflect upon one's role as a mathematics teacher within a community. Specifically, students will:

1. Gain introductory knowledge of the field of mathematics education with a focus on learning and teaching mathematics at the secondary level.
2. Learn about mathematical structures underlying strands of problem tasks from the algebra/counting/Combinatorics/probability strands of the 25+ years of longitudinal and cross-sectional research on mathematical learning conducted by researchers at Rutgers.
3. Be introduced to research on how learners engage with open-ended, challenging tasks as they build justifications of their solutions to problems.
4. Learn about forms of learner's mathematical reasoning through studying videos and VMC/Analytica from the Video Mosaic Collaborative (VMC) collection stored at Rutgers (see www.videomosaic.org).
5. Learn about research on learning and teaching through readings and the study of videos of learners engaged in doing mathematics.
6. Reflect on knowledge gained from the study of mathematical learning from videos to current teaching practices.
7. Will learn about the NCTM and Common Core State Standards and learn to recognize enactment of these standards from video and actual practice.

**Required Texts:**


Membership in the National Council of Teachers of Mathematics (NCTM.org)

**Additional Readings will be provided electronically through Sakai:**


**Recommended Resources:**

**Grading Policy**

**Letter Grade Equivalents:**
The grading will be as follows-
- A = 100-90%
- B+ = 89-87%
- B = 86-80%
- C+ = 79-77%
- C = 76-70%
- D = 69-60%
- F < 60%

**Academic Integrity Policy:**
Any violation of academic honesty is a serious offense and is therefore subject to an appropriate penalty. Refer to [http://academic-integrity.rutgers.edu/integrity.shtml](http://academic-integrity.rutgers.edu/integrity.shtml) for a full explanation of policies.

**Course Requirements**
Attendance (this policy is separate from the participation grade)
You are allowed ONE absence, which I will assume is for a good reason. Beyond that, your final grade will be reduced as indicated (unless, of course, you have a doctor's note or other documentation indicating a bona fide reason): 2 absences—reduction of a half grade; 3 absences—reduction of 1 full grade; 4 absences—failing grade in course. Again, if it is an excused absence, you are responsible for contacting me, getting the course materials, and making up for the class in order to receive the participation points.

I. Class Participation (20%)
You are expected to participate in class. Each week you will have readings and you will need to be prepared to discuss the content of the readings and ask questions in class. Aside from the readings, we will be engaging in discussions, group work, and individual activities in class. Your engagement in the course determines how successful the class will be and how much you will learn. You can earn a maximum of 5 points each class for in-class participation, individual assignments, completing the readings, and completing group work. If you miss a class for an excused absence, you can make up the points by doing out-of-class activities. Classes missed for unexcused absences are not able to be made up.

Use of cell phones, laptops, and other electronic devices for non-class related activities is strictly prohibited during class sessions; non-compliance will result in a reduction of Class Participation credit.

II. Online Discussion Forum (25%): You are expected to participate in the online discussion forum. Each week you will need to be prepared to discuss and respond to guiding questions posed on Sakai. The purpose of the online discussions will be to provide a forum for discussion and reflection of mathematical argumentation and the role that it might play in the mathematics classroom. More guidelines will be provided as needed.

III. Task Development, Implementation, and Videotape (20%): Each student will develop, present, facilitate, monitor, and videotape a mathematical task with a small group of students (or, if necessary, peers). The goal of the presented task should be to enact a productive mathematics discussion among students (Chapters 3 & 4 of 5 Practices for Orchestrating Productive Mathematics Discussions by Smith & Stein should provide helpful guidance and models). Students will also produce a 1-3 page summary reflection paper and videotape recording of the implementation that will both be submitted to TeachScape.

Part 1 - Preparation (10%, 1-3 pages): In class presentations starting 3/11
B - Anticipating – Anticipate all the ways in which students are likely to solve the task and the errors that they might make.
C - Questioning – Based on what you anticipate will happen, write down questions you could ask about students about their approaches that could help them make progress on the task. Ask higher-level questions that go beyond gathering information or leading students through a procedure... think about how your questions will scaffold
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thinking to enable students to think harder and more deeply about the mathematical ideas in the lesson.

D - Monitoring sheet — Create a Monitoring sheet that you could use to record data during your lesson.

Part 2 – Implementation: By Friday 4/10
Using your task, teach a problem-solving session with the students in your practicum, a group of volunteer students, or, if necessary, a small group of your peers. Use the monitoring questions developed from your presentation on 3/11, with revisions as appropriate. Videotape your implementation.

Part 3 – Task video and written summary reflection (due 4/22; 10%): You will select a problem-solving task covered in class and implement the task with a small group in your practicum classroom. You will videotape your implementation. You will write a 1-3 page paper describing your implementation. Your paper should include a description of the task, a summary of how the implementation went, a description of the reasoning you observed in the students, and anything you might change for the next implementation. The video and summary reflection will be uploaded to the Teachscape website. More info will be given in class.

IV. Lesson Plan (due 4/1; 20%): You will create a lesson plan that shows evidence of your pedagogical development in terms of instructional planning, assessment, individual student differences, and teaching style. The goal for this lesson plan is to develop a mathematically engaging lesson that can address the needs of secondary school mathematics students. The elements, rubric, and requirements for this lesson plan will be provided in class. Your lesson plan should be uploaded to both our course Sakai website as well as the Teachscape site. The Lesson plan can, but does not have to be, based on the Task you developed.

V. Sample Student Growth Objectives (due 4/15; SGO) (5%): Develop sample SGOs and upload to the appropriate Sakai folder.

VI. Reflection Paper (due by May 8th; 10%): Write a 1-3 page paper reflecting on the readings, analytics, and activities. Tie in your experiences in your practicum to your paper.

Successful completion of the course requires that actively engage in all activities and submit all assignments. This process requires that you:

- Attend all class sessions; arrive on time; remain for the entire period
- Be prepared for each class by having thoughtfully completed all readings and studied assigned videos from the Video Mosaic Collaborative, VMC (www.videomosaic.org)
- Participate fully in all class and online activities
- Upload Middle Phase Lesson Plan and video excerpt of your task implementation to Teachscape
- Submit all other assignments on time.
SUBMISSIONS to Teachscope
As a student at the Graduate School of Education, you will be submitting different artifacts based on which program, and which phase of the curriculum in which you are currently enrolled. For this course, you will be submitting two artifacts for the “middle phase.”
1. A Lesson Plan = April 1, 2015
2. Video and Reflection = April 22, 2015

Summary of Requirements
I. Class participation/attendance 20%
II. Online Discussion Forum 25%
III. Task Development 20%
IV. Lesson plan 20%
V. Individual SGO 5%
VI. End of Class Reflection Paper 10%

Course Schedule By Week
***The following schedule is tentative – adjustments will be made as necessary***

<table>
<thead>
<tr>
<th>Week 1</th>
<th>Class Activity:</th>
</tr>
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<tbody>
<tr>
<td>1/21/2015</td>
<td>1. Introduction and course overview</td>
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<tr>
<td></td>
<td>2. Introduction to Practicum placement</td>
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<td></td>
<td>3. Introduction to Sakai, NCTM, CCSS</td>
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<tr>
<td></td>
<td>4. Overview of Robert B. Davis Institute for Learning (REDIL) and its video collections, Introduction to the Video Mosaic Collaborative (VMC) Repository and its collectors, Introduction to the VMCAnalytic</td>
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<tr>
<td>Assignment:</td>
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</tr>
<tr>
<td></td>
<td>1. Visit the Common Core (CCSS)</td>
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<td></td>
<td>2. Get student membership to NCTM</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Week 2</th>
<th>Class Activity: Algebra Tasks – Guess My Rule</th>
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<tbody>
<tr>
<td>1/28/2015</td>
<td>1. Read pages 1-33 in Accessible Mathematics</td>
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<td></td>
<td>2. Read pages 1-16 in Great Ways to Differentiate Mathematics</td>
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<tr>
<td>Assignment:</td>
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</table>

Week 3 | Class Activity: Tower of Hanoi |
<table>
<thead>
<tr>
<th>Date</th>
<th>Activity</th>
<th>Assignment</th>
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<tbody>
<tr>
<td></td>
<td>Assignment:</td>
<td>1. Read pages 17-62 in <em>Great Ways to Differentiate Mathematics</em></td>
</tr>
<tr>
<td>Week 4</td>
<td><strong>Class Activity:</strong> Tower of Hanoi—recursive vs explicit (closed form)</td>
<td></td>
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<tr>
<td>2/11/2015</td>
<td>arguments</td>
<td></td>
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<tr>
<td></td>
<td>Assignment:</td>
<td>1. Read pages 1-19 in <em>Practices for Orchestrating Productive Mathematics Discussions</em></td>
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<tr>
<td></td>
<td></td>
<td>2. Watch Tower of Hanoi videos</td>
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<tr>
<td></td>
<td></td>
<td>a. <a href="http://www.youtube.com/embed/mRwpwonbypg">www.youtube.com/embed/mRwpwonbypg</a></td>
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<td></td>
<td></td>
<td>something is missing here, the other one</td>
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<tr>
<td>Week 5</td>
<td><strong>Class Activity:</strong> Tower of Hanoi—recursive vs explicit (closed form)</td>
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<tr>
<td>2/18/2015</td>
<td>arguments</td>
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<tr>
<td></td>
<td>Introducing the Analytic tool</td>
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<td></td>
<td>2. Read pages 20-42 in <em>Practices for Orchestrating Productive Mathematics Discussions</em></td>
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<td>3. Read pages 34 - 59 in <em>Accessible Mathematics</em></td>
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<tr>
<td>Week 6</td>
<td><strong>Class Activity:</strong></td>
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<tr>
<td>2/25/2015</td>
<td>1. Algebra Tasks—Use algebra blocks and the area model to multiply</td>
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<tr>
<td></td>
<td>and divide polynomials, factor, complete the square. What about negative</td>
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<tr>
<td></td>
<td>numbers?</td>
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<td></td>
<td>2. Remember (a-b)^2</td>
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<tr>
<td></td>
<td>First journal due for 444</td>
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<tr>
<td></td>
<td>Assignment:</td>
<td>1. Evaluate the tasks for CCSS</td>
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<tr>
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<td></td>
<td>2. Read pages 61-93 in <em>Practices for Orchestrating Productive Mathematics Discussions</em></td>
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<td>3. Prepare task; be ready to present in two weeks</td>
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<tr>
<td>Week 7</td>
<td><strong>Online Activity:</strong></td>
<td>How might a student understand these ideas?</td>
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<tr>
<td>Date</td>
<td>Event</td>
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<tr>
<td>3/11/2015</td>
<td><strong>Week 8</strong>&lt;br&gt;<strong>Class Activity:</strong> HAPPY PI DAY: 3/14/15 9:26:53&lt;br&gt;<strong>Task Presentations due (5-10 minutes per student)</strong>&lt;br&gt;Second journal due for 444&lt;br&gt;<strong>Assignment:</strong>&lt;br&gt;No Assignment. Happy Spring Break! (work on Lesson plan)&lt;br&gt;(Read about the background of SGOs, talk with cooperating teacher about SGOs, get sample.)</td>
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<tr>
<td>3/18/2015</td>
<td><strong>Spring Break</strong></td>
<td></td>
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<tr>
<td>3/25/2015</td>
<td><strong>Week 9</strong>&lt;br&gt;<strong>Class Activity:</strong> Student growth objectives (SGOs) – Mr. Tobin – AP Calculus teacher, Dunellen HS&lt;br&gt;<strong>Working Session:</strong> Work in groups to develop SGOs for given content area; use online resources&lt;br&gt;Third journal due for 444&lt;br&gt;<strong>Assignment:</strong>&lt;br&gt;1) Work on SGOs&lt;br&gt;2) Work on Lesson plan&lt;br&gt;3)</td>
<td></td>
</tr>
<tr>
<td>4/1/2015</td>
<td><strong>Week 10</strong>&lt;br&gt;<strong>Class Activity:</strong> Counting and Probability Tasks (Towers, Pizzas, Ankar’s Challenge)**</td>
<td></td>
</tr>
</tbody>
</table>
Fourth journal due for 444.  
Assignment:  
1) Work on SGOs  
2) Implement your task by Friday, 4/10 |
<table>
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</thead>
<tbody>
<tr>
<td>Week 12 4/15/2015</td>
<td>Class Activity: PARCC analysis; look at website and complete sample activities. Work in pairs to choose a task and solve it. Analyze the task for its alignment with CCSS Standards. Evaluate the task for its measure of mathematical learning. Individual sample SGOs due</td>
</tr>
</tbody>
</table>
| Week 13 4/22/2015 | Class Activity: Calculus tasks  
Video and reflection due (upload to Teahscape)  
Assignment:  
1) Research the Danielson framework  
2) Ask cooperating teacher about the teacher evaluation framework the school uses. |
| Week 14 4/29/2015 | Class Activity: Teacher evaluation / Danielson framework; Calculus tasks  
Assignment:  
Class Reflection paper due by Friday, May 8th.  
Class Reflection paper due Friday May 8th |
| Reading Days Tues. May 5th and Wed. May 6th |  |
| Exams Thurs. May 7 through Wed. | Good luck on Exams ☺  |
## Appendix B

### Transcript for Assessment Analytic: Fraction Analytic 1

**Event 1**

<table>
<thead>
<tr>
<th>Time</th>
<th>Participant</th>
<th>Transcript</th>
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<tbody>
<tr>
<td>5.091</td>
<td>T/R 1:</td>
<td>Tell me what you did the last time Dr. Martino was here. What was the problem you were working on? [Michael raises his hand] Anybody want to tell me and tell Dr. Davis? You were working on a problem I think in class together I think you were in groups, weren’t you? Do you all want to think for a moment and maybe discuss with your partner to help you remember what you were working on? [Michael’s hand is still up] Michael?</td>
</tr>
<tr>
<td>5.092</td>
<td>Michael:</td>
<td>We were working on the candy bar problem. Like, with like which is bigger a half or one third and we were using candy bars to show that.</td>
</tr>
<tr>
<td>5.093</td>
<td>T/R 1:</td>
<td>Okay, so you were working on which is bigger, one half or one third. Andrew?</td>
</tr>
<tr>
<td>5.094</td>
<td>Andrew:</td>
<td>Yeah, we were working on, we had to write about um and we had to do an example on it. and um to see if which is bigger, one half or one third.</td>
</tr>
<tr>
<td>5.095</td>
<td>T/R 1:</td>
<td>How many of you worked out which is bigger? One half or one third? [several hands go up] How many of you think they are the same? [all hands go down] How many of you think one is bigger? [several hands go up again] Which is bigger? One half or one third. Laura.</td>
</tr>
<tr>
<td>5.096</td>
<td>Laura:</td>
<td>One half.</td>
</tr>
<tr>
<td>5.097</td>
<td>T/R 1:</td>
<td>You say one half is bigger. What do the rest of you think? Do you think one half is bigger? [several students provide affirmation] Do you think you can convince Dr. Davis that that’s the case? [several hands go up] Can you convince Dr. Davis that one half is bigger than one third. By the way, do you know how much bigger? How many of you think you know how much bigger it is? Okay, that’s the second question. Okay, I really would like someone to come up. Jessica maybe and Laura can come up to the overhead and show Dr. Davis how you decided which is bigger. And see if you can convince us of your result. [Jessica and Laura come to overhead].</td>
</tr>
</tbody>
</table>
Event 2

| 5.0.98 | Jessica: | Well, um, one third would be just this piece here [she points to the purple rod] and one half of that would be [she sets up two dark green rods] and one half would be this [one dark green rod] and one third is bigger than one half cause this [purple rod] would be one third and then this bigger piece [dark green rod] would be one half of that. And- |
| 5.0.99 | T/R 1: | Can you tell me what number name you’re calling the orange and the red rod? |
| 5.0.100 | Jessica | Um, one. |
| 5.0.101 | T/R 1 | You’re calling the orange and red rod one? Can you say that again, what number names gave to each of those rods so I can hear from back here? |
| 5.0.102 | Jessica | [whispers to Laura] You say. Um, this would be, this, we’re counting this as one whole [orange and red train] and I think this [dark green rod] has two and this [purple rod] has, wait, um. Um [giggling], um I can’t we called it, yeah, [Laura helps her out] I think this one was one- |
| 5.0.103 | Laura: | That was one third |
| 5.0.104 | Jessica: | this was one third, and this was one half. |
| 5.0.105 | Laura: | One half. |

Event 3

| 5.0.106 | T/R 1: | What do the rest of you think? What do you think? Audra what do you think of what... the two young ladies built up there? |
| 5.0.107 | Audra: | I agree because- |
| 5.0.108 | T/R 1: | Want to speak to the class [asking her to go up front] |
| 5.0.109 | Audra: | [comes to overhead] I agree because if you saw what the, um half, was here and then you saw what, no, what the half was here and then you saw what the third was there, and you saw that the half was bigger than the third. |
Event 4

5.0.110 T/R 1: How many of you agree with the argument that a half is bigger than a third with the argument that was made here? Ok, did you figure out how much bigger?

5.0.111 Audra: It’s two, two [places two white rods next to purple rod]

5.0.112 Jessica: It’s a red bigger, but

5.0.113 T/R 1: Ok, you’re saying it’s a red bigger or two white ones but that’s what I see you have built there, but I would like you to tell me what number name you have for how much bigger it is.

5.0.114 Audra: Um, wait, it’s one third bigger, I think [organizing the dark green and red blocks together]

5.0.115 Jessica: I think it’s one third bigger too because if you put the red to the green

5.0.116 Audra: You’d see that there’s three

5.0.117 Jessica: You need three and if you put the purple one to it also and then it takes one third of them. [Showing the purple differs from the dark green by one red block]

Event 5

T/R 1: [Child in front row has his hand up] I don’t know if they have convinced Dr. Davis. Um, but I am wondering if they have convinced you? Kelly. What do you think, do you agree with this? [Kelly stands up, and comes to the front of class]

Kelly: Um, yes.

T/R 1: You agree with them.

Kelly: Well, if you have, um, a red, if you hold, um, well, if you have, um, well we used these and we went like, and then we like held reds up and we showed that um, that um, one half is bigger by, because this part is smaller, and this is supposed to be one, one third so that’s how we did it.

T/R 1: Brian you are making a face, what do you think? Do you agree with them?

Brian: Not really.

T/R 1: Brian doesn’t agree with you

Audra or Jessica: I think that’s like changing the problem because we are using the dark greens and she [Kelly] is using the light greens.

T/R 1: Oh, hmm.

Jessica: If you take this and it has three thirds [Brian builds the model on his desk]

T/R 1: Let me make sure I understand this. You’re calling this one, right? And you’re calling this one third.

Jessica: And we’re calling this
Event 6

5.0.134 T/R 1: *their argument is right because I'm very confused. What do you think?*

5.0.135 Brian: *Well, when they said one third is bigger than one half by one third, I think they said, is that what they said? Well, I don't really agree, because well if you split, if you split one of the thirds in half which would make [counting the blocks], which would make a sixth. I think it's a sixth bigger. Like, well, [holds his rods], un should I go up there?*

5.0.136 T/R 1: *Sure, ladies can you make a little space here for Brian. Maybe you need to have a little conference here, we have some disagreement. *

5.0.137 Brian: *[He goes to the overhead.] Well, see for um, when they said it was one half bigger, if you split a third in half it'd make a sixth, like one, two, three, four, five, six. Like, like pretending they were, like pretending they were split in half. If you split one of these in half and you have three of them up there they'd make, they'd make six and any way, and when you split them in half right in the middle over there it's kind of like that, it's kind of like this, there was this was, that was the one third [points to a purple rod] and that was the one half [points to the dark green rod] on the bottom and so it's just like this and the red I'm pretending is like, is like, is a half of one of the purples and you see when I split it in half it's, it's one sixth and, and it equals, and it equals up to a green *

5.0.138 T/R 1: *I'm hearing you say Brian that the number name for red is one sixth and the reason why is—*

5.0.139 Brian: *Well, I mean a red. I'm considering a red one sixth [Dr. Maher: yeah] because two of these [red rods] equals, see they're two, they're two sixths, two halves of one purple and the purple is a third and the half of one third is sixth, there's sixths.*
Event 7

<table>
<thead>
<tr>
<th>T/R 1:</th>
<th>Ok we will hear Kelly and Jackie and we will hear Brian's again. Brian said it and I know some of you heard it, I heard it. But I would like you all to listen to these arguments.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jackie:</td>
<td>Well, we would call this dark green one and the reds one third and the light green one half, and we thought that we thought one third was bigger by one of these white things. [Her model is using 6 cm as the unit.]</td>
</tr>
<tr>
<td>Jessica:</td>
<td>Oh, I think they're making a different size candy bar</td>
</tr>
<tr>
<td>T/R 1:</td>
<td>Is that allowed?</td>
</tr>
<tr>
<td>Jessica:</td>
<td>Um, no.</td>
</tr>
<tr>
<td>T/R 1:</td>
<td>Why not? What's wrong with that? In what way it is not fair?</td>
</tr>
<tr>
<td>Jessica:</td>
<td>Because if you give someone half of this one [12 cm²] and then one half of that one [6 cm²] and this is bigger than [takes a light green and dark green rod in hand].</td>
</tr>
<tr>
<td>T/R 1:</td>
<td>Ok so what do you ladies think? Are you making different size candy bars? What are calling the candy bar when you started the problem? What was one? What did you call one if you're thinking of candy bars when you began the problem?</td>
</tr>
<tr>
<td>One of the girls:</td>
<td>The dark green...</td>
</tr>
<tr>
<td>T/R 1:</td>
<td>Is that what you built when you went up there, you said the dark green is one? Is that what you said?</td>
</tr>
<tr>
<td>One of the girls:</td>
<td>Yeah...[the girls look at each other in agreement]</td>
</tr>
</tbody>
</table>

Event 8

| 5.0.155 | T/R 1: | Ok then use the—okay if you're calling dark green one then I want to hear your argument which is bigger a half or a third and by how much? |
| 5.0.156 | Jackie: | Okay, we think that a half is bigger than the third. |
| 5.0.157 | T/R 1: | Okay you think a half is bigger than one third and you're calling the dark green one? Did you change your mind? |
| 5.0.158 | Jackie: | Yeah, and we think light green is a half [of the 6 cm model]. |
| 5.0.159 | T/R 1: | Well show me your argument now and tell me which is bigger a half and a third and by how much? |
| 5.0.160 | Jackie: | Okay, this is a half [light green] and the red is a third. |
| 5.0.161 | T/R 1: | Can you show me why that's a half? |
| 5.0.162 | Jackie: | Because if you put these all together they equal up to the one...[Showing that three reds, two light greens both equal the dark green which is one] and we think the light green which is a half is bigger than the red by, by one which is this white one. [Showing the difference between red and LG is a white] |
| 5.0.163 | T/R 1: | Ok, I see that you switched what you made, um, your model, uh, but you showed me that one half is still bigger than a third and you still
<table>
<thead>
<tr>
<th>Line</th>
<th>Character</th>
<th>Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0.164</td>
<td>Jackie</td>
<td>Yeah.</td>
</tr>
<tr>
<td>5.0.165</td>
<td>T/R 1</td>
<td>Did you say that?</td>
</tr>
<tr>
<td>5.0.166</td>
<td>Jackie</td>
<td>Yeah, the green, the light green is one bigger than the red. And the red is one bigger, the light green is one bigger</td>
</tr>
<tr>
<td>5.0.167</td>
<td>T/R 1</td>
<td>And what number name are you calling the white?</td>
</tr>
<tr>
<td>5.0.168</td>
<td>Jackie</td>
<td>One</td>
</tr>
<tr>
<td>5.0.169</td>
<td>T/R 1</td>
<td>You all agree with that?</td>
</tr>
<tr>
<td>5.0.170</td>
<td>Jackie</td>
<td>Actually, I used this to um, to tell that the light green is one white bigger.</td>
</tr>
</tbody>
</table>

**Event 9**

<table>
<thead>
<tr>
<th>Line</th>
<th>Character</th>
<th>Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0.178</td>
<td>Erik</td>
<td>[walks to the overhead] I think they mean that they want to call this, the dark green one, one whole, and they want to call this, yeah, like you line all the whites up to it which I think should be six and they want to call it one sixth. I think that’s what they’re trying to say but they just, they’re just not saying it. I think they just, they want to call it one sixth.</td>
</tr>
<tr>
<td>5.0.179</td>
<td>T/R 1</td>
<td>I don’t see six of them up there.</td>
</tr>
<tr>
<td>5.0.180</td>
<td>Erik</td>
<td>Well however many are up there that what they are trying to say.</td>
</tr>
<tr>
<td>5.0.181</td>
<td>Jessica</td>
<td>Yeah because I think they meant</td>
</tr>
<tr>
<td>5.0.182</td>
<td>Erik</td>
<td>I think you meant to say not one whole but one sixth.</td>
</tr>
<tr>
<td>5.0.183</td>
<td>T/R 1</td>
<td>Is that what you meant to say?</td>
</tr>
<tr>
<td>5.0.184</td>
<td>Girls</td>
<td>Yeah</td>
</tr>
<tr>
<td>5.0.185</td>
<td>T/R 1</td>
<td>So you’re saying then you all agree, that’s what, you all really wanted to call the little white one, one sixth and not one? When you call the light green one? So I’m a little concerned now? Are you agreeing with Brian or disagreeing with Brian that the number name that you would give for how much bigger one half is than one third? How much? One half is how much bigger than one third?</td>
</tr>
<tr>
<td>5.0.186</td>
<td>Girls</td>
<td>Um, one, one sixth</td>
</tr>
<tr>
<td>5.0.187</td>
<td>T/R 1</td>
<td>Is it one or one sixth?</td>
</tr>
<tr>
<td>5.0.188</td>
<td>Girls</td>
<td>One sixth.</td>
</tr>
<tr>
<td>5.0.189</td>
<td>T/R 1</td>
<td>You’re sure it’s one sixth?</td>
</tr>
<tr>
<td>5.0.190</td>
<td>Girls</td>
<td>Yeah</td>
</tr>
<tr>
<td>5.0.191</td>
<td>T/R 1</td>
<td>Why can’t it one?</td>
</tr>
<tr>
<td>5.0.192</td>
<td>Girls</td>
<td>Because that’s be um, the dark green.</td>
</tr>
</tbody>
</table>
| 5.0.193| T/R 1     | The dark green is one? I understand when-
### Event 10

<table>
<thead>
<tr>
<th>Time</th>
<th>T/R 1:</th>
<th>Jessica:</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0.199</td>
<td>I still want to go back to the problem Brian was helping them with the problem up there. I still wonder if we can solve this one because you started with this other one and you said that the orange and red [together] are one, right? Isn't that what you said?</td>
<td>This is one whole, and then this is one third and this is one half. [pointing to the three different rod lengths]</td>
</tr>
<tr>
<td>5.0.200</td>
<td>T/R 1: Right, and you said it's bigger by the red, right? And the question was, what number name do you give to the red? Now if you really understood what mistake you made here maybe you'll figure out what mistake you made up there.</td>
<td>[girls whisper to each other]</td>
</tr>
<tr>
<td>5.0.201</td>
<td></td>
<td>Jessica:</td>
</tr>
</tbody>
</table>
| 5.0.202 |                                                                       | Well, we and we, um, named, well, three reds equal up to um, one greens and then you put the purple next to it and you need one more red, you need a red to go next to the purple, so it would be one third. |}

### Event 11

<table>
<thead>
<tr>
<th>Time</th>
<th>T/R 1:</th>
<th>Brian:</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0.203</td>
<td>They're still telling me that one half is bigger by— one half is bigger than one third by one third. Can anyone tell me what's going on here? I am so confused.</td>
<td>I don't- I still don't think so, well, because, well, well, see like I said before when you split the ahh, when you split the thirds in half and they make sixths, it's still like [He goes to the overhead.]</td>
</tr>
<tr>
<td>5.0.204</td>
<td>T/R 1: So Brian is giving the red rod a different number name, he's not calling it a third he's calling it a sixth. They don't believe that though, they still want to call it a third. Someone has to-</td>
<td></td>
</tr>
<tr>
<td>5.0.205</td>
<td></td>
<td>Brian: See, well, because when you put it right there you see that, you see that there's one of these, if you put one of these on top of it you might see that, that it's that much that, that red, that red is that much bigger than one of the halves because one of these reds I'm calling is, is, is a sixth and anyway a half of one of these, a half of one of the thirds. But when you put it on top of one of the thirds it's that much bigger than one of the halves.</td>
</tr>
<tr>
<td>5.0.206</td>
<td></td>
<td>Jessica: Well, I think they might both be answers.</td>
</tr>
<tr>
<td>5.0.207</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Event 12

<table>
<thead>
<tr>
<th>Time</th>
<th>Speaker</th>
<th>Text</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0.207</td>
<td>Jessica</td>
<td>Well, I think they might both be answers.</td>
</tr>
<tr>
<td>5.0.208</td>
<td>T/R 1</td>
<td>You think it can be a third and a half? How many think they could be a third and a half? How many of you don’t think it could be a third and a sixth? How many of you disagree?</td>
</tr>
<tr>
<td>5.0.209</td>
<td>Erik</td>
<td>I don’t think you can have an answer of a third because if you have one half [he goes to the overhead] and if you take the one half which would be the dark green, you have the one half and then those [purple rods] are the thirds. How could one half be bigger than the thirds by one third? Because, and you have the half and the thirds together that the half is almost as big as two thirds, but yet the two thirds aren’t exactly, are not exactly, the green, the dark green is not, the dark green is not exactly as big as two, two thirds but, two thirds, it’s the, but it’s far enough so that the two thirds are not bigger than it by one third.</td>
</tr>
<tr>
<td>5.0.210</td>
<td>Brian</td>
<td>I kind of agree with Erik. I think now I disagree with them [referring to the girls].</td>
</tr>
<tr>
<td>5.0.211</td>
<td>Erik</td>
<td>I don’t really think that if you have this [a purple rod] that you could have one third bigger than it [Brian - yeah] because it’s got to be one third and probably a third and a half.</td>
</tr>
<tr>
<td>5.0.212</td>
<td>Brian</td>
<td>Yeah, he’s right.</td>
</tr>
<tr>
<td>5.0.213</td>
<td>Erik</td>
<td>It couldn’t be, it couldn’t be exactly a third.</td>
</tr>
</tbody>
</table>

### Event 13

<table>
<thead>
<tr>
<th>Time</th>
<th>Speaker</th>
<th>Text</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0.212</td>
<td>Brian</td>
<td>Yeah, he’s right.</td>
</tr>
<tr>
<td>5.0.213</td>
<td>Erik</td>
<td>It couldn’t be, it couldn’t be exactly a third.</td>
</tr>
<tr>
<td>5.0.214</td>
<td>Brian</td>
<td>Cause one third bigger, this would be one third bigger like that to the end over there. That would actually be like this [showing with the dark green and purple pieces], this would really be one third bigger and there’s still some left over and there’s still about</td>
</tr>
<tr>
<td>5.0.215</td>
<td>Erik</td>
<td>A half left over.</td>
</tr>
<tr>
<td>5.0.216</td>
<td>Brian</td>
<td>Yeah, there’s still, there’s still one more, there’s still one more piece left, like about a sixth left.</td>
</tr>
<tr>
<td>5.0.217</td>
<td>Erik</td>
<td>Cause it’s like if you have, if you have the like dark green and it doesn’t exactly equal up to, it doesn’t exactly equal up. It’s less than two thirds but it’s more than one third. It’s just about one third and a half. So it couldn’t be exactly a third bigger than it and it couldn’t be exactly two thirds or it couldn’t be exactly one third bigger. It had to be one third and a half.</td>
</tr>
</tbody>
</table>
### Event 14

<table>
<thead>
<tr>
<th>Michael:</th>
<th>Umm, I think it should be called one sixth because [he goes to the overhead] because if you put six reds up to one orange [arranges six reds under the orange w/red rod train] with a red then it would equal, there would be, there would be, it would be the same size just, so it would be called one sixth because reds like that.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brian:</td>
<td>Yeah, I agree with Michael and Erik</td>
</tr>
<tr>
<td>T/R 1:</td>
<td>So, so Brian, Michael is offering another way of thinking about that red as being one sixth. You thought about the red as being one sixth to make a half of a third and Michael is saying that red is one sixth</td>
</tr>
<tr>
<td>Erik:</td>
<td>Yeah, Michael is right because it takes three sixths to equal one half, and if</td>
</tr>
</tbody>
</table>

### Event 15

<table>
<thead>
<tr>
<th>T/R 1:</th>
<th>I see Meredith has been wanting to say something</th>
</tr>
</thead>
<tbody>
<tr>
<td>Meredith:</td>
<td>I agree with Erik, Michael and Brian because if you do call that a sixth, a sixth, and if you put the dark green up with two thirds, you said it was, you said it was, um, they said that it's a third difference, if you did a third difference, this is called a third and then you put it there, you see negative, [Meredith placed a red rod next to the dark green rod] You said it was one third bigger, that can't be true because one third bigger</td>
</tr>
<tr>
<td>Erik and Brian:</td>
<td>Yeah</td>
</tr>
<tr>
<td>Brian:</td>
<td>It's about one sixth less. So it can't be a third bigger.</td>
</tr>
<tr>
<td>Erik:</td>
<td>And also, like</td>
</tr>
<tr>
<td>Meredith:</td>
<td>So it's one sixth bigger</td>
</tr>
<tr>
<td>Erik:</td>
<td>And also yeah, and also, I think because if you have the light green, the light green, it's not bigger than, it's not bigger than the, it's not bigger than the umm third, it's not bigger than two thirds. It's bigger than one third, but it's not as big as two thirds so it's less than two thirds but more than one third. So it can't be a third bigger. And if you have that to make it two thirds large, there has to be a sixth.</td>
</tr>
<tr>
<td>T/R 1:</td>
<td>Well that is really something, uh I think --</td>
</tr>
<tr>
<td>Michael:</td>
<td>It's sort of like one sixth in both cases.</td>
</tr>
</tbody>
</table>
Appendix C

VMCAnalytic 1: Fourth graders’ Argumentation About the Density of Fractions Between 0 and 1

http://dx.doi.org/doi:10.7282/T39K4CZC

Transcript for Event 1

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>RT1</td>
<td>Okay. So let’s go back to what your assignment was. We were trying to figure out what sort of happens in between and just as we said these keep on going and there are infinitely many, what we are going to be learning over the next maybe six or seven years of math that even in between zero and one there are also infinitely many numbers.</td>
</tr>
<tr>
<td>2</td>
<td>Students</td>
<td>There are?</td>
</tr>
<tr>
<td>3</td>
<td>RT1</td>
<td>Yeah</td>
</tr>
<tr>
<td>4</td>
<td>Erik</td>
<td>Infinitely many?</td>
</tr>
<tr>
<td>5</td>
<td>RT1</td>
<td>We have infinitely many. Now, I want you to think about that a little bit and that’s my statement. Mathematicians claim that between zero and one there are also infinitely many numbers.</td>
</tr>
<tr>
<td>6</td>
<td>Alan</td>
<td>Infinitely between zero and one?</td>
</tr>
<tr>
<td>7</td>
<td>Students</td>
<td>[mumbles]</td>
</tr>
<tr>
<td>8</td>
<td>RT1</td>
<td>They also claim that there are infinitely many fractions. I want you to think about that. I used to, when my son was your age he used to think about those things.</td>
</tr>
<tr>
<td>9</td>
<td>Jessica</td>
<td>In fractions or just...</td>
</tr>
<tr>
<td>10</td>
<td>Andrew</td>
<td>One million</td>
</tr>
<tr>
<td>11</td>
<td>Students</td>
<td>[murmuring]</td>
</tr>
<tr>
<td>12</td>
<td>RT1</td>
<td>Infinitely many fractions. Now, that is something to think about. I don’t expect you to solve that problem in your mind right now; mathematicians have worked on this for centuries.</td>
</tr>
<tr>
<td>13</td>
<td>Erik</td>
<td>I just don’t understand how there can be infinitely many numbers between zero to one.</td>
</tr>
<tr>
<td>14</td>
<td>Andrew</td>
<td>Fractions... fractions.</td>
</tr>
<tr>
<td>15</td>
<td>A student</td>
<td>I know why.</td>
</tr>
<tr>
<td>16</td>
<td>Erik</td>
<td>No, infinitely many numbers from zero to one. It doesn’t</td>
</tr>
<tr>
<td>17</td>
<td>RT1</td>
<td>Alan</td>
</tr>
<tr>
<td>18</td>
<td>Erik</td>
<td>Make sense.</td>
</tr>
</tbody>
</table>

Transcript for Event 2

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>RT1</td>
<td>Alan.</td>
</tr>
<tr>
<td>18</td>
<td>Erik</td>
<td>Make sense.</td>
</tr>
<tr>
<td>19</td>
<td>Alan</td>
<td>You can divide that line into the smallest of fractions. You could divide it into zillionths.</td>
</tr>
<tr>
<td>20</td>
<td>Students</td>
<td>Yeah</td>
</tr>
<tr>
<td>21</td>
<td>RT1</td>
<td>Did you hear what Alan said? Erik, did you hear what Alan said? You want to say that one more time, Alan?</td>
</tr>
</tbody>
</table>
### Transcript for Event 3

<table>
<thead>
<tr>
<th></th>
<th>Alan</th>
<th>You could divide it into zillionths and there would still be space in there.</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Michael</th>
<th>If you had the longest number line in the world.</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Erik</th>
<th>You could divide it into zillionths, you could divide it into zillionths depending on what number your one whole is.</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th></th>
<th>RTI</th>
<th>No. no. Michael, I’m talking about this little piece between zero and one.</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Alan</th>
<th>Yeah, but you could still divide it ...</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Michael</th>
<th>But if you made a number line to show zillionths, then you would have to have the longest number line in the world.</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Erik</th>
<th>Alan. Alan. That doesn’t make sense.</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Alan</th>
<th>Yes it does.</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Erik</th>
<th>Even if you … The only way you could divide into zillionths is depending on how big your one whole is. If your one whole is ten you can’t divide it into zillionths.</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Alan</th>
<th>Well, from zero to one, you could. You could divide it into zillionths.</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Erik</th>
<th>If you one whole is ten, how could you divide it into zillionths?</th>
</tr>
</thead>
</table>

### Transcript for Event 4

<table>
<thead>
<tr>
<th></th>
<th>Michael</th>
<th>Made up of lots and lots and lots and lots of infinitely many tiny little points and I can’t put them all in so I just draw this line. [Andrew raises hand]. What do you think, Alan?</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Alan</th>
<th>Well, as I was saying before about the zillionths, you could have a line the size of a dust particle and you could put that on there a zillion times. You would have zillionths.</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Michael</th>
<th>If you had a pin that was smaller than a dust particle, then ...</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Erik</th>
<th>Something that is smaller than a dust particle … a dust bug … a hundred dust bug can fit into a dust particle ...</th>
</tr>
</thead>
</table>

### Transcript for Event 5

<table>
<thead>
<tr>
<th></th>
<th>RTI</th>
<th>Andrew? What’s your answer?</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Andrew</th>
<th>Well, if you made a number line and you took a magnifying glass or a microscope, and put your number line under it, you would see that you have a lot more room left to put the one-hundredth, one-thousandth and the one-millionth.</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Alan</th>
<th>[Inaudible]</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th></th>
<th>RTI</th>
<th>Did you all hear what Andrew said?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>------</td>
<td>-------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>49.</td>
<td>Alan</td>
<td>Yes. If you did put it under a microscope it would look like you had enough room to put another zero to one in there. It would look like that. You could have it enlarged so that the line from the zero would be this big [raises hands and makes space between them about a foot]. And you would still have room there to put more.</td>
</tr>
<tr>
<td>50.</td>
<td>RT1</td>
<td>What happens when scientists discover more and more powerful telescopes?</td>
</tr>
<tr>
<td>51.</td>
<td>Michael</td>
<td>Then, the more numbers you could fit onto one number line.</td>
</tr>
</tbody>
</table>

**Transcript for Event 6**

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<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>62.</td>
<td>RT1</td>
<td>Brian?</td>
</tr>
<tr>
<td>63.</td>
<td>Brian</td>
<td>So, like what Alan said, you can put zillionths in.</td>
</tr>
<tr>
<td>64.</td>
<td>RT1</td>
<td>So you think you can put zillionths in? You are changing your mind? So you are sort of agreeing that there are lots and lots of fractions between zero and one if you had this ...</td>
</tr>
<tr>
<td>65.</td>
<td>Erik</td>
<td>Yeah...</td>
</tr>
<tr>
<td>66.</td>
<td>Alan</td>
<td>Even like you could even make it smaller than that and make smaller pieces to put in there.</td>
</tr>
</tbody>
</table>

**Transcript for Event 7**

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<table>
<thead>
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</thead>
<tbody>
<tr>
<td>16</td>
<td>Erik</td>
<td>What I don't understand is, if you're using a microscope to give more space in actual reality you're not getting more space.</td>
</tr>
<tr>
<td>17</td>
<td>Maher</td>
<td>Yeah that's an interesting idea Erik.</td>
</tr>
<tr>
<td>18</td>
<td>Erik</td>
<td>You see when you're using a microscope it looks like you're getting more space but in actual reality you're not getting anymore. It just looks like you are.</td>
</tr>
</tbody>
</table>

**Transcript for Event 8**

<p>| | | |</p>
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>71.</td>
<td>Andrew</td>
<td>Well, actually you are because the human eye can't see that, but</td>
</tr>
<tr>
<td>72.</td>
<td>Alan</td>
<td>When you enlarge it you can see how much space you have left between the zillionth and the zero.</td>
</tr>
<tr>
<td>73.</td>
<td>Erik</td>
<td>Yeah, but Alan what you were saying before you were saying that when you use the microscope you get more space in that number line. That is what you were saying before.</td>
</tr>
<tr>
<td>74.</td>
<td>Alan</td>
<td>No. That's not what I was saying.</td>
</tr>
<tr>
<td></td>
<td></td>
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<tr>
<td>----</td>
<td>----</td>
<td>-----</td>
</tr>
<tr>
<td>75</td>
<td>Erik</td>
<td>From what I understood, you were saying that if you use a microscope you get more space on the number line. It is not true.</td>
</tr>
</tbody>
</table>

**Transcript for Event 9**

| 76 | Alan | What I mean is, look, if you had some really small pen, if they come-up with it, you could draw a small line in the space your, but you don’t really know how much space you have left between the zillionth and the zero. You don’t really know that because you can’t see it so you look at it under a microscope you could see how much space you have left. |
| 77 | RT1  | Yeah, it might be, Erik, when you were thinking more space you were thinking of extending it … |
| 78 | Erik | Yeah, the first time the way he said it that’s what …. |
| 79 | RT1  | That’s not what I heard Erik, he drew a picture yesterday and what I heard yesterday, Alan saying is that he was [Alan nodding head yes] still talking about that same space. Both of you had a different picture in your heads about the kind of space; and, you were kind of talking about the picture in your head and Alan was talking about the picture in his head and I think Andrew’s picture matches Alan’s picture and Jessica’s picture, but I’m not sure. |

**Transcript for Event 10**

| 26 | David | Well I think that, that you can’t really see it too well but if you use a microscope then you’re seeing it closer it looks like you’re seeing more. But you’re really not, you’re just looking closer. |
| 27 | Alan | There really is more space… |

**Transcript for Event 11**

<p>| 98 | RT1  | Any idea? I want to hear from some other people. What are your ideas about the number between zero and one? Okay, Audra, and then Jessica and then Mark. Okay, Audra? |
| 99 | Audra | I really do agree with them because |</p>
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>100.</td>
<td>RTI</td>
<td>With whom?</td>
</tr>
<tr>
<td>101.</td>
<td>Audra</td>
<td>With Andrew and Alan because the human eye can’t see it if you are making it that small so if you put it under a microscope you really could see more …</td>
</tr>
<tr>
<td>102.</td>
<td>Students</td>
<td>It would be really tiny but you could still see it …</td>
</tr>
<tr>
<td>103.</td>
<td>RTI</td>
<td>Jessica?</td>
</tr>
<tr>
<td>104.</td>
<td>Jessica</td>
<td>Well, I think I agree with Alan and Andrew because you really can’t see if there are any little spaces; but when you put it under a microscope there might be a huge space that you could fit a lot of spaces</td>
</tr>
<tr>
<td>105.</td>
<td>Beth</td>
<td>He is not saying that it is getting bigger, he is just saying that it is not going to stop …</td>
</tr>
<tr>
<td>106.</td>
<td>Michael</td>
<td>Oh, it is sort of like the more you see the more space you have</td>
</tr>
<tr>
<td>107.</td>
<td>Alan</td>
<td>Yeah.</td>
</tr>
<tr>
<td>108.</td>
<td>RTI</td>
<td>What about that? Michael said that it is sort of like the more you see the more space you have? That is Michael’s question; what do you think?</td>
</tr>
<tr>
<td>109.</td>
<td>Beth</td>
<td>It is hard to explain.</td>
</tr>
<tr>
<td>110.</td>
<td>RTI</td>
<td>It sure is … It sure is hard to explain. Kelly, were you going to say something?</td>
</tr>
<tr>
<td>111.</td>
<td>Kelly</td>
<td>[Nods head]</td>
</tr>
<tr>
<td>112.</td>
<td>RTI</td>
<td>No? Okay. Mark and then we’ll hear from Jen.</td>
</tr>
<tr>
<td>113.</td>
<td>Mark</td>
<td>I think that I agree with Alan and Andrew because they are right you can’t see the thing but if you put it under a …</td>
</tr>
<tr>
<td>244</td>
<td>Mark</td>
<td>microscope and if it is a really powerful one you would have a huge space there.</td>
</tr>
</tbody>
</table>

**Transcript for Event 12**

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<thead>
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</thead>
<tbody>
<tr>
<td>247</td>
<td>Maher</td>
<td>Okay, you are not sure about that you are going to think about it some more. Okay, David?</td>
</tr>
<tr>
<td>248</td>
<td>David</td>
<td>I think that you can take the little smallest thing and then put it under a microscope and you will have a lot more space but you don’t. It looks like a lot more space but it really isn’t. You are just magnifying it.</td>
</tr>
<tr>
<td>249</td>
<td>Students</td>
<td>Yeah. So it looks like you have more space but you really don’t.</td>
</tr>
<tr>
<td>250</td>
<td>Michael</td>
<td>It looks like you have more space and humans take advantage of it and take that really big space and mark these really really little lines on it that you really just can’t see on it.</td>
</tr>
</tbody>
</table>

**Transcript for Event 13**

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<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>254</td>
<td>Alan</td>
<td>Up here [walks up to overhead]. If like this could be the</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>255</td>
<td>RT1</td>
<td>So, zero and this could be the size of a bar and there could be your line. Now, if you had the hundredths which would probably go somewhere in here it would look say if it were right here. And, then you would have all that space in there. It looks like it, but you really don’t have that much space. It’s just that if you had it really big that is how much space you would think you could see.</td>
</tr>
<tr>
<td>256</td>
<td>Alan</td>
<td>So that means you could divide this into halves and thirds and fourths and fifths and all of that.</td>
</tr>
</tbody>
</table>

**Transcript for Event 14**

| 31 | Maher | So you’re telling me, let me see if I understand this. The rest of you will you help me with this? You’re telling me that this bar over here that is marking zero right? Okay Michael is marking it over there, but this bar that’s marking zero you’ve magnified because you’ve used a very powerful microscope. And so if you’re telling me now it would be really hard to place at but it may look like…its so close to zero you can’t even mark it once you’ve magnified if you have all this extra space in between. That’s interesting. |

**Transcript for Event 15**

<p>| 32 | Alan | Yeah cause it looks like you have a lot of space but you really only have that tiny winny little space between there. I mean you could take a really small pen and you could divide this up into all of these pieces. If you look at that with your regular eye you couldn’t see that so you’d have to make it bigger. |
| 259 36:24 | RT1 | Laura, does that help? Does that make sense? Be sure Laura understands what you are saying. |
| 260 36:39 | Alan | Yeah, now that’s their [inaudible] you could divide that space up into all little lines. Now if you magnified those spaces—and here would be the little bars—y you could divide this space up into little tiny pieces and that you could |</p>
<table>
<thead>
<tr>
<th>Time</th>
<th>Participant</th>
<th>Transcript</th>
</tr>
</thead>
<tbody>
<tr>
<td>268</td>
<td>Brian</td>
<td>I have a comment about what Alan and Andrew said. You see humans don’t have powerful enough eyes to see where the zillionths are so there really is a lot of room but you don’t see it because the human eye is not as powerful as a microscope.</td>
</tr>
<tr>
<td>269</td>
<td>Michael</td>
<td>Oh, I get it so there is a lot of room that you can’t see.</td>
</tr>
<tr>
<td>270</td>
<td>Alan</td>
<td>Say in the future that you come up with this really high powered microscope you could make that zero bar from the floor to the ceiling that would maybe let you see it being that big. You could divide it up into such small pieces that when you took off the microscope you wouldn’t see anything it would be so tiny and so small that you couldn’t see it but there really is space there and if you magnify those really tiny pieces you could divide those up into spaces.</td>
</tr>
<tr>
<td>272</td>
<td>RT1</td>
<td>So it sounds like the instruments get in the way, right, not the numbers—What do you think James? You are so quite back there James, Amy and Jakki what are you thinking through all this discussion?</td>
</tr>
<tr>
<td>273</td>
<td>James</td>
<td>It really does make sense that there is more space in between the zillionths etc. etc. So I agree with Alan mostly.</td>
</tr>
<tr>
<td>274</td>
<td>Alan</td>
<td>The biggest number you could think of you could make one and so on; you could go on forever with this... I mean you could keep in magnifying it and magnifying it and magnifying it, dividing it, magnifying it dividing it.</td>
</tr>
<tr>
<td>275</td>
<td>41:02</td>
<td>Brian</td>
</tr>
<tr>
<td>276</td>
<td>RT1</td>
<td>David</td>
</tr>
<tr>
<td>277</td>
<td>David</td>
<td>I was going to say what Brian said that it could be as big as a dust bug and just ...</td>
</tr>
<tr>
<td>278</td>
<td>RT1</td>
<td>Gregory, what do you think about all this dust bugs and things that big? Do you have any editorial comments on this discussion?</td>
</tr>
<tr>
<td>279</td>
<td>Gregory</td>
<td>No.</td>
</tr>
<tr>
<td>280</td>
<td>RT1</td>
<td>Meredith</td>
</tr>
<tr>
<td>-----</td>
<td>-----</td>
<td>-----------</td>
</tr>
<tr>
<td>281</td>
<td>Meredith</td>
<td>I think what he is trying to say is that if you look at it through the microscope then there is a lot of space but if you just look at it through the human eye then there isn't very much space in there.</td>
</tr>
<tr>
<td>282</td>
<td>RT1</td>
<td>That is a good synthesis.</td>
</tr>
</tbody>
</table>
Appendix D

Transcript for VMCAnalytic 2: Eighth Grader Stephanie’s Argumentation about Meaning for the Square of a Binomial using Algebraic Reasoning
http://dx.doi.org/doi:10.7282/T3FN180C

Transcripts for VMCAnalytic: Eighth Grader Stephanie’s Argumentation about Meaning for the Square of a Binomial using Algebraic Reasoning

(first Stephanie analytic)

Interview one of seven, clip 5 of 9

Clip Transcript for Event 1

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>5</td>
<td>R1</td>
<td>Okay. Um. Now – could I do this? [Dr. Maher writes ((x+y)(x+y))].</td>
</tr>
<tr>
<td>6</td>
<td>Stephanie</td>
<td>You probably could. I don’t know how, but you probably um</td>
</tr>
<tr>
<td>7</td>
<td>R1</td>
<td>What do you think it means?</td>
</tr>
<tr>
<td>8</td>
<td>Stephanie</td>
<td>It means that – um – (x) plus (y) times – OH! Could you just do it (x) squared times (y) squared?</td>
</tr>
<tr>
<td>9</td>
<td>R1</td>
<td>What do you think this means?</td>
</tr>
<tr>
<td>10</td>
<td>Stephanie</td>
<td>It means that you’re multiplying – ‘cause you can’t combine these terms, right?</td>
</tr>
<tr>
<td>11</td>
<td>R1</td>
<td>I’ll buy that.</td>
</tr>
<tr>
<td>12</td>
<td>Stephanie</td>
<td>So…</td>
</tr>
<tr>
<td>13</td>
<td>R1</td>
<td>Why can’t you, by the way?</td>
</tr>
<tr>
<td>14</td>
<td>Stephanie</td>
<td>‘Cause they’re not the same variable.</td>
</tr>
<tr>
<td>15</td>
<td>R1</td>
<td>Okay.</td>
</tr>
<tr>
<td>16</td>
<td>Stephanie</td>
<td>Uh. Because you can’t combine them, um you have to multiply them by – okay – you’re supposed to multiply these. But you can’t combine these either.</td>
</tr>
<tr>
<td>17</td>
<td>R1</td>
<td>Um hm</td>
</tr>
<tr>
<td>18</td>
<td>Stephanie</td>
<td>So – but you can’t exactly take this (inaudible)</td>
</tr>
<tr>
<td>19</td>
<td>R1</td>
<td>Um. It’s interesting, isn’t it?</td>
</tr>
<tr>
<td>20</td>
<td>Stephanie</td>
<td>I can’t figure out how to get around it. But I’m pretty sure that if I could, the answer would be (x) squared plus (y) squared.</td>
</tr>
<tr>
<td>21</td>
<td>R1</td>
<td>Why don’t you put a question mark here and let’s test it.</td>
</tr>
<tr>
<td>22</td>
<td>Stephanie</td>
<td>Okay.</td>
</tr>
<tr>
<td>23</td>
<td>R1</td>
<td>Okay. Your conjecture – Stephanie’s conjecture – this is (x) squared plus (y) squared. Test it. Try some numbers and see.</td>
</tr>
</tbody>
</table>

Interview one of seven, clip 5 of 9

Transcript for Event 2

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</thead>
<tbody>
<tr>
<td>23</td>
<td>R1</td>
<td>Okay. Your conjecture – Stephanie’s conjecture – this is (x) squared plus (y) squared. Test it. Try some numbers and see.</td>
</tr>
</tbody>
</table>
Interview one of seven, clip 5 of 9

Transcript for Event 3

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<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>Stephanie</td>
</tr>
<tr>
<td>25</td>
<td>R1</td>
</tr>
<tr>
<td>26</td>
<td>Stephanie</td>
</tr>
<tr>
<td>27</td>
<td>R1</td>
</tr>
<tr>
<td>28</td>
<td>Stephanie</td>
</tr>
</tbody>
</table>

<p>| | |</p>
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</tr>
</thead>
<tbody>
<tr>
<td>31</td>
<td>R1</td>
</tr>
<tr>
<td>32</td>
<td>Stephanie</td>
</tr>
<tr>
<td>33</td>
<td>R1</td>
</tr>
<tr>
<td>34</td>
<td>Stephanie</td>
</tr>
<tr>
<td>35</td>
<td>R1</td>
</tr>
<tr>
<td></td>
<td>Stephanie</td>
</tr>
<tr>
<td>36</td>
<td>R1</td>
</tr>
<tr>
<td>37</td>
<td>Stephanie</td>
</tr>
<tr>
<td>38</td>
<td>R1</td>
</tr>
<tr>
<td>39</td>
<td>Stephanie</td>
</tr>
<tr>
<td>40</td>
<td>R1</td>
</tr>
<tr>
<td>41</td>
<td>Stephanie</td>
</tr>
<tr>
<td>42</td>
<td>R1</td>
</tr>
<tr>
<td></td>
<td>Stephanie</td>
</tr>
<tr>
<td>43</td>
<td>R1</td>
</tr>
<tr>
<td>45</td>
<td>Stephanie</td>
</tr>
<tr>
<td>46</td>
<td>R1</td>
</tr>
<tr>
<td>47</td>
<td>Stephanie</td>
</tr>
<tr>
<td>48</td>
<td>R1</td>
</tr>
<tr>
<td>49</td>
<td>Stephanie</td>
</tr>
<tr>
<td>50</td>
<td>R1</td>
</tr>
<tr>
<td>51</td>
<td>Stephanie</td>
</tr>
<tr>
<td>52</td>
<td>R1</td>
</tr>
<tr>
<td>53</td>
<td>Stephanie</td>
</tr>
<tr>
<td>----</td>
<td>-----------</td>
</tr>
<tr>
<td>54</td>
<td>R1</td>
</tr>
</tbody>
</table>

**Interview one of seven, clip 5 of 9**

**Transcript for Event 4**

<table>
<thead>
<tr>
<th>62</th>
<th>R1</th>
<th>Okay. Well. So. Now. But – you’re thinking of a times x plus y, right?</th>
</tr>
</thead>
<tbody>
<tr>
<td>63</td>
<td>Stephanie</td>
<td>I can’t even add x plus y, though. Which is my problem. Like I can’t add x plus y together. ‘Cause they’re different.</td>
</tr>
<tr>
<td>64</td>
<td>R1</td>
<td>Okay. But if this were a times, you would imagine x plus y a times in your head?</td>
</tr>
<tr>
<td>65</td>
<td>Stephanie</td>
<td>Um hm.</td>
</tr>
<tr>
<td>66</td>
<td>R1</td>
<td>x plus y, x plus y, x plus y. But now, this is not a, right?</td>
</tr>
<tr>
<td>67</td>
<td>Stephanie</td>
<td>Right.</td>
</tr>
<tr>
<td>68</td>
<td>R1</td>
<td>This is x plus y. So how many times are you imagining x plus y in your head?</td>
</tr>
<tr>
<td>69</td>
<td>Stephanie</td>
<td>Once. Right now, just because there’s not an a amount of times. And it’s x plus y, x plus y amount of times.</td>
</tr>
<tr>
<td>70</td>
<td>R1</td>
<td>Is it x plus y, x plus y amount of times? Okay. I’m asking how many? - This is your x plus y.</td>
</tr>
<tr>
<td>71</td>
<td>Stephanie</td>
<td>Okay.</td>
</tr>
<tr>
<td>72</td>
<td>R1</td>
<td>Alright?</td>
</tr>
<tr>
<td>73</td>
<td>Stephanie</td>
<td>Yeah.</td>
</tr>
<tr>
<td>74</td>
<td>R1</td>
<td>You have a bunch of them.</td>
</tr>
<tr>
<td>75</td>
<td>Stephanie</td>
<td>Yeah.</td>
</tr>
<tr>
<td>76</td>
<td>R1</td>
<td>How many of them do you have?</td>
</tr>
<tr>
<td>77</td>
<td>Stephanie</td>
<td>I have x plus y times x plus y; so I have it x plus y amount of times, but I don’t know.</td>
</tr>
<tr>
<td>78</td>
<td>R1</td>
<td>Okay. Don’t lose that idea.</td>
</tr>
<tr>
<td>79</td>
<td>R1</td>
<td>Okay.</td>
</tr>
<tr>
<td>80</td>
<td>R1</td>
<td>Why don’t you just get that idea? Make sure of it. Write it down, ‘cause that’s a that’s a good thing to hold on to. - - You have x plus y; x plus y amount of times. That’s pretty good. - - Do you really believe that?</td>
</tr>
<tr>
<td>81</td>
<td>Stephanie</td>
<td>That’s what I’m getting.</td>
</tr>
</tbody>
</table>

**Interview one of seven, clip 5 of 9**

**Transcript for Event 5**
<table>
<thead>
<tr>
<th>88</th>
<th>R1</th>
<th>Alright. Now. What makes this kind of messy, 'cause you’re thinking about this $x$ plus $y$ amount of times. It was nice when you — you thought $x$ amount of times was bad enough — but that was sure easier than —</th>
</tr>
</thead>
<tbody>
<tr>
<td>89</td>
<td>Stephanie</td>
<td>Yeah.</td>
</tr>
<tr>
<td>90</td>
<td>R1</td>
<td>$x$ plus $y$. Right?</td>
</tr>
<tr>
<td>91</td>
<td>Stephanie</td>
<td>Yeah.</td>
</tr>
<tr>
<td>92</td>
<td>R1</td>
<td>So can you break down the way you think about it in terms of $x$ plus $y$ amount of times? Do you have to think about it $x$ plus $y$ amount of times?</td>
</tr>
<tr>
<td>93</td>
<td>Stephanie</td>
<td>No. If I break down, I can think of it $x$ amount of times and $y$ amount of times.</td>
</tr>
<tr>
<td>94</td>
<td>R1</td>
<td>Okay. Very interesting. So — so you can think of $x$ plus $y$ — you could have $x$ amount of times, right?</td>
</tr>
<tr>
<td>95</td>
<td>Stephanie</td>
<td>Um hm</td>
</tr>
<tr>
<td>96</td>
<td>R1</td>
<td>And you could have it $y$ amount of times. Isn’t that right?</td>
</tr>
<tr>
<td>97</td>
<td>Stephanie</td>
<td>Yes.</td>
</tr>
<tr>
<td>98</td>
<td>R1</td>
<td>Is that a way to think about it?</td>
</tr>
<tr>
<td>99</td>
<td>Stephanie</td>
<td>Oh! Yeah.</td>
</tr>
</tbody>
</table>
### Interview one of seven, clip 6 of 9

#### Transcript for Event 6

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td><strong>Stephanie</strong></td>
<td>Alright, should I put in like some numbers?</td>
</tr>
<tr>
<td>13</td>
<td><strong>R1</strong></td>
<td>Well, see what you —</td>
</tr>
<tr>
<td>14</td>
<td><strong>Stephanie</strong></td>
<td>Oh, well otherwise—</td>
</tr>
<tr>
<td>15</td>
<td><strong>R1</strong></td>
<td>Yeah, that's — Yeah, put in some numbers. Sure, that's a great idea.</td>
</tr>
<tr>
<td>16</td>
<td><strong>Stephanie</strong></td>
<td>Alright, I'll just put in like a number for x. So I'll make x two.</td>
</tr>
<tr>
<td>17</td>
<td><strong>R1</strong></td>
<td>Well, put in a number for y and x.</td>
</tr>
<tr>
<td>18</td>
<td><strong>Stephanie</strong></td>
<td>Alright.</td>
</tr>
<tr>
<td>19</td>
<td><strong>R1</strong></td>
<td>That's interesting.</td>
</tr>
</tbody>
</table>
|    |        | [Stephanie writes: 
|    |        | \(2 + 3 \) + \(3 + 3\)  
|    |        | \(4 \times 6 + 6 + 9\)  
|    |        | \(10 + 15\)  
|    |        | \(25\)]                                                       |
| 20| **Stephanie** | Six plus nine equals ten plus (inaudible) twenty-five.         |
| 21| **R1**  | Is that what you were supposed to get before?                  |
| 22| **Stephanie** | Yes.                                                           |
| 23| **R1**  | You like that, huh?                                            |
| 24| **Stephanie** | Um hm.                                                         |
| 25| **R1**  | So it worked at least for two numbers? Does that mean it's always going to work? |
| 26| **Stephanie** | It might. I —                                                  |
| 27| **R1**  | But does that mean it always is gonna work?                    |
| 28| **Stephanie** | I think so. I think that's allowed. To do it like that.        |
| 29| **R1**  | Does that make sense? What did you?                            |
| 30| **Stephanie** | Yeah.                                                          |

### Interview one of seven, clip 7 of 9

#### Transcript for Event 7

<p>| | | |</p>
<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td><strong>R1</strong></td>
<td>Now let's — um — could you make this simple with your Distributive Law?</td>
</tr>
<tr>
<td>8</td>
<td><strong>Stephanie</strong></td>
<td>Yes.</td>
</tr>
<tr>
<td>9</td>
<td><strong>R1</strong></td>
<td>Do you think you can — do you know enough — what does it mean to write (x) times (x) plus (y)?</td>
</tr>
<tr>
<td>10</td>
<td><strong>Stephanie</strong></td>
<td>Oh. Can I — ?</td>
</tr>
<tr>
<td>11</td>
<td><strong>R1</strong></td>
<td>What does that mean: (x) times the quantity (x) plus (y)?</td>
</tr>
<tr>
<td>12</td>
<td><strong>Stephanie</strong></td>
<td>Well, (x) times — so. Wait. That's — It — See if it was just (x) times (x) I could do an (x)-squared.</td>
</tr>
<tr>
<td>13</td>
<td><strong>R1</strong></td>
<td>Well, it is. You have (x) times (x).</td>
</tr>
<tr>
<td>14</td>
<td><strong>Stephanie</strong></td>
<td>Yeah, but I can't do it with (y); 'cause (y)-squared is different than (x)-squared.</td>
</tr>
<tr>
<td></td>
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<tr>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>15</td>
<td>R1</td>
<td>Okay. But this piece you think is x-squared?</td>
</tr>
<tr>
<td>16</td>
<td>Stephanie</td>
<td>I can do it.</td>
</tr>
<tr>
<td>17</td>
<td>R1</td>
<td>x times x.</td>
</tr>
<tr>
<td>18</td>
<td>Stephanie</td>
<td>Yeah.</td>
</tr>
<tr>
<td>19</td>
<td>R1</td>
<td>Well, do that.</td>
</tr>
<tr>
<td>20</td>
<td>Stephanie</td>
<td>It would just be – do you want me to write x times x or x-squared?</td>
</tr>
<tr>
<td>21</td>
<td>R1</td>
<td>x-squared.</td>
</tr>
<tr>
<td>22</td>
<td>Stephanie</td>
<td>x-squared. okay.</td>
</tr>
<tr>
<td>23</td>
<td>R1</td>
<td>Okay.</td>
</tr>
<tr>
<td>24</td>
<td>Stephanie</td>
<td>But here it would be x to the y power.</td>
</tr>
<tr>
<td>25</td>
<td>R1</td>
<td>Let’s think about that. What are you saying here? You’re trying to guess what x times y is, right?</td>
</tr>
<tr>
<td>26</td>
<td>Stephanie</td>
<td>Yeah.</td>
</tr>
<tr>
<td>27</td>
<td>R1</td>
<td>So let’s get a paper to conjecture. You can conjecture here.</td>
</tr>
<tr>
<td>28</td>
<td>Stephanie</td>
<td>Okay.</td>
</tr>
<tr>
<td>29</td>
<td>R1</td>
<td>How do you think you would write – what do you think it means ‘x times y’?</td>
</tr>
<tr>
<td>30</td>
<td>Stephanie</td>
<td>Well, it’s an um x amount y number of times or y amount x number of times. It can go either way.</td>
</tr>
<tr>
<td>31</td>
<td>R1</td>
<td>So. Well. Look at what you just wrote.</td>
</tr>
<tr>
<td>32</td>
<td>Stephanie</td>
<td>Um hm.</td>
</tr>
<tr>
<td>33</td>
<td>R1</td>
<td>Do you think that’s a way to write it?</td>
</tr>
<tr>
<td>34</td>
<td>Stephanie</td>
<td>Well, yeah. You can write it like that. I’m just saying –</td>
</tr>
<tr>
<td>35</td>
<td>R1</td>
<td>Yeah. That’s fine. I like it that way. Okay. (Stephanie writes: $x^2 + y^2$)</td>
</tr>
<tr>
<td>36</td>
<td>R1</td>
<td>Okay. So you see why your other guess didn’t work before? If what you’re doing is right – there’s your x-squared, there’s your y-squared, but there’s something else.</td>
</tr>
<tr>
<td>37</td>
<td>Stephanie</td>
<td>Yeah. I understand.</td>
</tr>
<tr>
<td>38</td>
<td>R1</td>
<td>See that. What is that something else?</td>
</tr>
<tr>
<td>39</td>
<td>Stephanie</td>
<td>It’s the x times the y.</td>
</tr>
<tr>
<td>40</td>
<td>R1</td>
<td>Or – what’s next?</td>
</tr>
<tr>
<td>41</td>
<td>Stephanie</td>
<td>Or the y times the x. Or –</td>
</tr>
</tbody>
</table>

Interview one of seven, clip 7 of 9
Transcript for Event 8
<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>47</td>
<td>Stephanie</td>
<td>Alright. Because then – alright – it would be x plus x plus x plus x just so that it’s easier for me - y plus y plus y-squared. (Stephanie writes: $x^2 + x + x + y^2 + y^2 + y^2$)</td>
</tr>
<tr>
<td>48</td>
<td>R1</td>
<td>So you’re conjecturing that this is the same as this?</td>
</tr>
<tr>
<td>49</td>
<td>Stephanie</td>
<td>Yeah. Because you’re just putting all the</td>
</tr>
<tr>
<td>50</td>
<td>R1</td>
<td>Let’s try it with numbers and see if that makes sense – what you’re conjecturing.</td>
</tr>
<tr>
<td>51</td>
<td>Stephanie</td>
<td>Alright.</td>
</tr>
<tr>
<td>52</td>
<td>R1</td>
<td>What does that mean?</td>
</tr>
<tr>
<td>53</td>
<td>Stephanie</td>
<td>That means like –</td>
</tr>
<tr>
<td>54</td>
<td>R1</td>
<td>Try some numbers. Try easy numbers. (Stephanie writes: $2^2 + 2 + 2 + 2^2 + 2^2$)</td>
</tr>
<tr>
<td>55</td>
<td>Stephanie</td>
<td>And that’s two squared, that’s four, plus two, six, eight, plus three, plus three, that’s six, plus nine is fifteen. That works! (she writes: $8 + 15$) No. It doesn’t. That’s twenty-three.</td>
</tr>
<tr>
<td>56</td>
<td>R1</td>
<td>That gives you twenty-three</td>
</tr>
<tr>
<td>57</td>
<td>Stephanie</td>
<td>Yeah.</td>
</tr>
<tr>
<td>58</td>
<td>R1</td>
<td>So something isn’t working here, huh?</td>
</tr>
<tr>
<td>59</td>
<td>Stephanie</td>
<td>No.</td>
</tr>
<tr>
<td>60</td>
<td>R1</td>
<td>So that might not be a valid step.</td>
</tr>
<tr>
<td>61</td>
<td>Stephanie</td>
<td>No.</td>
</tr>
<tr>
<td>62</td>
<td>R1</td>
<td>Okay. So. I’m kind of curious. What did you want to do with this thing here?</td>
</tr>
<tr>
<td>63</td>
<td>Stephanie</td>
<td>Well, because – well – when we add the um-</td>
</tr>
<tr>
<td>64</td>
<td>R1</td>
<td>You have x-squared plus xy plus yx plus y-squared.</td>
</tr>
<tr>
<td>65</td>
<td>Stephanie</td>
<td>It was just putting the terms together.</td>
</tr>
<tr>
<td>66</td>
<td>R1</td>
<td>What terms were you putting together?</td>
</tr>
<tr>
<td>67</td>
<td>Stephanie</td>
<td>Well, the x and the – oh. Is it that maybe I can’t put the x’s with the x-squared, ‘cause they’re two different terms? Would that make a difference?</td>
</tr>
<tr>
<td>68</td>
<td>R1</td>
<td>Okay. Where’s the x?</td>
</tr>
<tr>
<td>69</td>
<td>Stephanie</td>
<td>Right here and here. [points to the xy and yx]</td>
</tr>
</tbody>
</table>
|70 | R1 | But is this an x?
Interview one of seven, clip 8 of 9
Transcript for Event 9

<table>
<thead>
<tr>
<th>Time</th>
<th>R1</th>
<th>Stephanie</th>
</tr>
</thead>
<tbody>
<tr>
<td>130</td>
<td>R1</td>
<td>Okay. So what did we end up – What’s the simplest way? Can you that’s what we started with.</td>
</tr>
<tr>
<td>131</td>
<td>R1</td>
<td>Well. Okay. Now we didn’t eliminate these.</td>
</tr>
<tr>
<td>132</td>
<td>R1</td>
<td>See what you believe any more.</td>
</tr>
<tr>
<td>133</td>
<td>R1</td>
<td>x to the second.</td>
</tr>
<tr>
<td>134</td>
<td>R1</td>
<td>You believe that still. Okay.</td>
</tr>
<tr>
<td>135</td>
<td>R1</td>
<td>But what we got was plus 2 xy plus y to the second.</td>
</tr>
<tr>
<td>136</td>
<td>R1</td>
<td>Oh.</td>
</tr>
<tr>
<td>137</td>
<td>R1</td>
<td>That’s what we have now.</td>
</tr>
<tr>
<td>138</td>
<td>R1</td>
<td>You believe that? Try some numbers and test it.</td>
</tr>
<tr>
<td>139</td>
<td>R1</td>
<td>You believe that? Try some numbers and test it.</td>
</tr>
<tr>
<td>140</td>
<td>R1</td>
<td>You like that, huh? Isn’t that wonderful?</td>
</tr>
<tr>
<td>141</td>
<td>R1</td>
<td>It looks a lot easier than this.</td>
</tr>
<tr>
<td>142</td>
<td>R1</td>
<td>Um. Okay. So you believe that this is the same as this?</td>
</tr>
<tr>
<td></td>
<td>R1</td>
<td>(indicates: (x^2 + y^2 + 2xy + z^2))</td>
</tr>
<tr>
<td>143</td>
<td>R1</td>
<td>Yes.</td>
</tr>
<tr>
<td>144</td>
<td>R1</td>
<td>Actually, you’ve proved it. What you’ve just done is gone through a proof. What you’ve done here is your proof is based upon you know the meaning of these things.</td>
</tr>
<tr>
<td>145</td>
<td>R1</td>
<td>We could keep testing lots of numbers?</td>
</tr>
<tr>
<td>146</td>
<td>R1</td>
<td>Um hm.</td>
</tr>
<tr>
<td>147</td>
<td>R1</td>
<td>Actually, you’ve proved it. What you’ve just done is gone through a proof. What you’ve done here is your proof is based upon you know the meaning of these things.</td>
</tr>
<tr>
<td>148</td>
<td>R1</td>
<td>We could keep testing lots of numbers?</td>
</tr>
</tbody>
</table>

Interview two of seven, clip 1 of 6
Transcript for Event 10

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
</table>
| 1 | R1 | Okay. Um. Let’s see. Maybe you can rebuild it. Okay? Um.  
|   |   |  
|   |   | [takes paper and pen. Writes \((a+b)^2\)] Do you remember what that means?  
| 2 | Stephanie | Um. I – this is yeah and didn’t we distribute it so that it was like \([writes \ a^2 + b^2]\)?  
| 3 | R1 | Okay. Do you want to test it? [Stephanie makes a noise.] Tell me what it means and test it.  

Interview two of seven, clip 1 of 6

Transcript for Event 11

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
</tr>
</thead>
</table>
| 26 | Stephanie | This. [draws a line around \((a+b)^2\)] Like if you distribute um if you put two and three in here.  
| 27 | R1 | So – you’re putting, why are you putting the two and the three in there? Tell me again.  
| 28 | Stephanie | ‘Cause you asked me to put numbers in-  
| 29 | R1 | So  
| 30 | Stephanie | -in place of the letters  
| 31 | R1 | So so what so the two is being used for  
| 32 | Stephanie | \(a\) and the three is \(b\).  
| 33 | R1 | Three is for \(b\). And when you did that you have  
| 34 | Stephanie | Um. Well, this  
| 35 | R1 | This to be twenty-five.  
| 36 | Stephanie | Turns out to be five and then five squared is  
| 37 | R1 | Okay. And when you did it, when you, you said, what is this? [points to a^2 + b^2 that Stephanie wrote earlier]  
| 38 | Stephanie | Oh. You told me um well you said ‘what is this?’ [the \((a^2 + b^2)\)] and I said that it would be like \(a\) squared plus \(b\) squared. Obviously, it’s not.  
| 39 | R1 | Ah ha.  
| 40 | Stephanie | Because it doesn’t work out.  
| 41 | R1 | Okay. So. So then in in your testing it  
| 42 | Stephanie | [Stephanie chuckles.]  
| 43 | R1 | Your conjecture  
| 44 | Stephanie | Yeah.
### Interview two of seven, clip 1 of 6

**Transcript for Event 12**

<table>
<thead>
<tr>
<th>Line</th>
<th>R1</th>
<th>Stephanie</th>
</tr>
</thead>
<tbody>
<tr>
<td>51</td>
<td>R1</td>
<td>So, so why don’t you write down what you just said- that a that this ((a+b)^2) is not equal to this ([points\ in\ the\ vicinity\ of\ a^2 + b^2]) is something you just found. Why don’t you write out what ((a+b)^2) is?</td>
</tr>
<tr>
<td>52</td>
<td>Stephanie</td>
<td>So like [pause] is not equal to um [writing]</td>
</tr>
<tr>
<td>53</td>
<td>R1</td>
<td>Would you have to test something else to prove it’s not equal? If you show it doesn’t work once is that is that okay?</td>
</tr>
<tr>
<td>54</td>
<td>Stephanie</td>
<td>Well, yeah. Because if it doesn’t work once then it can’t like be true.</td>
</tr>
<tr>
<td>55</td>
<td>R1</td>
<td>Okay, So so you proved in essence then that this is not true. So the question was, I go back to my original question.</td>
</tr>
<tr>
<td>56</td>
<td>Stephanie</td>
<td>[chuckling] What is that?</td>
</tr>
<tr>
<td>57</td>
<td>R1</td>
<td>What is it, right?</td>
</tr>
<tr>
<td>58</td>
<td>Stephanie</td>
<td>Yeah.</td>
</tr>
<tr>
<td>59</td>
<td>R1</td>
<td>Okay. So I’ll let you struggle a little bit and think about that.</td>
</tr>
</tbody>
</table>

### Interview two of seven, clip 1 of 6

**Transcript for Event 13**

<table>
<thead>
<tr>
<th>Line</th>
<th>R1</th>
<th>Stephanie</th>
</tr>
</thead>
<tbody>
<tr>
<td>61</td>
<td>R1</td>
<td>That about you know what it means. Think about meaning.</td>
</tr>
<tr>
<td>62</td>
<td>Stephanie</td>
<td>[Stephanie inaudible]</td>
</tr>
<tr>
<td>63</td>
<td>R1</td>
<td>And maybe maybe what might help you—think about what what you know about meaning in the simplest way, to think about what this could be in meaning. What does (a + b), that quantity squared, mean?</td>
</tr>
<tr>
<td>64</td>
<td>Stephanie</td>
<td>It means that you [chuckles] it means like—well—1</td>
</tr>
<tr>
<td>Line</td>
<td>Speaker</td>
<td>Text</td>
</tr>
<tr>
<td>------</td>
<td>---------</td>
<td>-------------------------------</td>
</tr>
<tr>
<td>65</td>
<td>R1</td>
<td>What does something squared mean?</td>
</tr>
<tr>
<td>66</td>
<td>Stephan</td>
<td>It means that</td>
</tr>
<tr>
<td>67</td>
<td>R1</td>
<td>Try something</td>
</tr>
<tr>
<td>68</td>
<td>Stephan</td>
<td>you're multiplying it by itself</td>
</tr>
<tr>
<td>69</td>
<td>R1</td>
<td>Oh, Okay. So what is being</td>
</tr>
<tr>
<td>70</td>
<td>Stephan</td>
<td>a plus b.</td>
</tr>
<tr>
<td>71</td>
<td>R1</td>
<td>So so tell me what you just – let’s number these pages. Because I know what will happen. This is number one and today’s date is the twenty-ninth.</td>
</tr>
<tr>
<td>72</td>
<td>Stephan</td>
<td>Twenty-ninth</td>
</tr>
<tr>
<td>73</td>
<td>R1</td>
<td>Okay. This is for my benefit.</td>
</tr>
<tr>
<td>74</td>
<td>Stephan</td>
<td>Um hm.</td>
</tr>
<tr>
<td>75</td>
<td>R1</td>
<td>‘Cause I – This is what we know. So this - you can be numbering them now. Um, So so you know what a plus b</td>
</tr>
<tr>
<td>76</td>
<td>Stephan</td>
<td>Yeah.</td>
</tr>
<tr>
<td>77</td>
<td>R1</td>
<td>So moving from meaning</td>
</tr>
<tr>
<td>78</td>
<td>Stephan</td>
<td>Oh. What does it like</td>
</tr>
<tr>
<td>79</td>
<td>R1</td>
<td>So write down what you think it means. You know what a squared means. You clearly know what a squared means.</td>
</tr>
<tr>
<td>80</td>
<td>Stephan</td>
<td>Well, yeah</td>
</tr>
<tr>
<td>81</td>
<td>R1</td>
<td>You believe that a squared, if a is two, is the same as two times two?</td>
</tr>
<tr>
<td>82</td>
<td>Stephan</td>
<td>Yes.</td>
</tr>
<tr>
<td>83</td>
<td>R1</td>
<td>You know that. Right? And b squared here is the same as three times three. That you believe?</td>
</tr>
<tr>
<td>84</td>
<td>Stephan</td>
<td>Yes.</td>
</tr>
<tr>
<td>85</td>
<td>R1</td>
<td>Okay. So what does a plus b, that quantity squared, what does that mean?</td>
</tr>
<tr>
<td>86</td>
<td>Stephan</td>
<td>a plus b times a plus b?</td>
</tr>
<tr>
<td>87</td>
<td>R1</td>
<td>So why don’t you write that down? What that means; a plus b quantity squared. [pause] Okay,</td>
</tr>
</tbody>
</table>
88  Stephanie  Oh! Okay.
88  Stephanie  Oh! Okay.
89  R1  Right?
90  Stephanie  This is this what we did last time (inaudible).
91  R1  I don’t know. Does it look familiar to you?
92  Stephanie  Yeah, but we used x and y.
93  R1  Oh! Does it matter?
94  Stephanie  No.
95  R1  Okay. Could we use w and r?
96  Stephanie  Yeah.
97  R1  Do you prefer to use x and y?
98  Stephanie  No. This is fine. [chuckling]
99  R1  Is a and b okay? Okay, I didn’t really do that deliberately to throw you off.
100 Stephanie  No. I just – that’s what I remembered.

Yellow text is not in the clip

101 R1  Okay. So. Uh. It might be useful, um, Stephanie -- to write down that this \([a+b]/(a+b)\) equals this thing \([it appears that the researcher is pointing to the (a+b)^2]\) or you know – not to lose sight of what this is supposed to represent.
102 Stephanie  Oh.
103 R1  You know what I’m saying. As a as a whole sentence. Because that you absolutely believe, right?
104 Stephanie  Um hm.

Interview two of seven, clip 1 of 6
### Transcript for Event 14

<table>
<thead>
<tr>
<th>Line</th>
<th>Person</th>
<th>Transcript</th>
</tr>
</thead>
<tbody>
<tr>
<td>105</td>
<td>R1</td>
<td>You believe that?</td>
</tr>
<tr>
<td>106</td>
<td>Stephan ie</td>
<td>Yes.</td>
</tr>
<tr>
<td>107</td>
<td>R1</td>
<td>And why do you believe that? Why is that true?</td>
</tr>
<tr>
<td>108</td>
<td>Stephan ie</td>
<td>Because um when you square something it’s like multiplying it by like itself? And so it would be like a plus b times a plus b.</td>
</tr>
<tr>
<td>109</td>
<td>R1</td>
<td>Okay. So. Um. Here you have squared.</td>
</tr>
<tr>
<td>110</td>
<td>Stephan ie</td>
<td>Um hm.</td>
</tr>
<tr>
<td>111</td>
<td>R1</td>
<td>And you have two factors of what you’re squaring. You have a plus b as a factor two times. Right?</td>
</tr>
<tr>
<td>112</td>
<td>Stephan ie</td>
<td>Um hm.</td>
</tr>
<tr>
<td>113</td>
<td>R1</td>
<td>‘Cause it’s squared.</td>
</tr>
<tr>
<td>114</td>
<td>Stephan ie</td>
<td>Yes.</td>
</tr>
<tr>
<td>115</td>
<td>R1</td>
<td>And if I had a three here? <em>[indicates the exponent]</em></td>
</tr>
<tr>
<td>116</td>
<td>Stephan ie</td>
<td>You’d do it three times.</td>
</tr>
<tr>
<td>117</td>
<td>R1</td>
<td>What would you do three times?</td>
</tr>
<tr>
<td>118</td>
<td>Stephan ie</td>
<td>a plus b times a plus b times a plus b.</td>
</tr>
<tr>
<td>119</td>
<td>R1</td>
<td>times a plus b [<em>simultaneously with Stephanie’s last ‘a plus b’</em>] Okay. And you get twenty-five times?</td>
</tr>
<tr>
<td>120</td>
<td>Stephan ie</td>
<td>It would be a plus b twenty-five times. Like times a plus b.</td>
</tr>
</tbody>
</table>
Appendix E

Transcript for VMCAalytic 3: Eighth Grader Stephanie’s Argumentation about Meaning for the Square of a Binomial using Algebraic Reasoning

http://dx.doi.org/doi:10.7282/T3FN180C

<table>
<thead>
<tr>
<th>Transcripts for VMCAalytic: Eighth Grader Stephanie’s Argumentation about Meaning for the Square of a Binomial using Geometric Reasoning</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Transcript for Event 1</td>
<td></td>
</tr>
<tr>
<td>Interview two of seven, clip 2 of 6</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>R1</td>
</tr>
<tr>
<td>2</td>
<td>Stephanie</td>
</tr>
<tr>
<td>3</td>
<td>R1</td>
</tr>
<tr>
<td>4</td>
<td>Stephanie</td>
</tr>
<tr>
<td>5</td>
<td>R1</td>
</tr>
<tr>
<td>6</td>
<td>Stephanie</td>
</tr>
<tr>
<td>7</td>
<td>R1</td>
</tr>
<tr>
<td>8</td>
<td>Stephanie</td>
</tr>
<tr>
<td>9</td>
<td>R1</td>
</tr>
<tr>
<td>10</td>
<td>Stephanie</td>
</tr>
<tr>
<td>11</td>
<td>R1</td>
</tr>
<tr>
<td>12</td>
<td>Stephanie</td>
</tr>
<tr>
<td>13</td>
<td>R1</td>
</tr>
<tr>
<td>14</td>
<td>Stephanie</td>
</tr>
<tr>
<td>15</td>
<td>R1</td>
</tr>
<tr>
<td>16</td>
<td>Stephanie</td>
</tr>
<tr>
<td>17</td>
<td>R1</td>
</tr>
<tr>
<td>18</td>
<td>Stephanie</td>
</tr>
<tr>
<td>19</td>
<td>R1</td>
</tr>
<tr>
<td>20</td>
<td>Stephanie</td>
</tr>
<tr>
<td>21</td>
<td>R1</td>
</tr>
<tr>
<td>22</td>
<td>Stephanie</td>
</tr>
<tr>
<td>23</td>
<td>R1</td>
</tr>
<tr>
<td>24</td>
<td>Stephanie</td>
</tr>
<tr>
<td>25</td>
<td>R1</td>
</tr>
<tr>
<td>26</td>
<td>Stephanie</td>
</tr>
<tr>
<td>27</td>
<td>R1</td>
</tr>
<tr>
<td>28</td>
<td>Stephanie</td>
</tr>
<tr>
<td>29</td>
<td>R1</td>
</tr>
</tbody>
</table>
Interview two of seven, clip 3 of 6

Transcript for Event 2
MISSING FROM VMC

Interview two of seven, clip 3 of 6

Transcript for Event 3
MISSING FROM VMC

Interview two of seven, clip 4 of 6

Transcript for Event 4

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>R1</td>
<td>Isn’t that right? And, um, you know how to find the area of a square. How do you find the area of a square?</td>
</tr>
<tr>
<td>4</td>
<td>Stephanie</td>
<td>Multiply the two, the length and the width?</td>
</tr>
<tr>
<td>5</td>
<td>R1</td>
<td>Yeah. Or a rectangle, you know how to do that, right? So you should be able to find the area of each of these four pieces.</td>
</tr>
<tr>
<td>6</td>
<td>Stephanie</td>
<td>Yeah.</td>
</tr>
<tr>
<td>7</td>
<td>R1</td>
<td>Go for it! [Stephanie writes ab in the upper left rectangle, bb in the upper right square, ab in the lower right rectangle and ab in the lower left square.] Okay. So, what’s the area of the square? The big one?</td>
</tr>
<tr>
<td>8</td>
<td>Stephanie</td>
<td>[Stephanie grunts.]</td>
</tr>
<tr>
<td>9</td>
<td>R1</td>
<td>The one you started with?</td>
</tr>
<tr>
<td>10</td>
<td>Stephanie</td>
<td>Um, ah times ah.</td>
</tr>
<tr>
<td>11</td>
<td>R1</td>
<td>No. What’s the</td>
</tr>
<tr>
<td>12</td>
<td>Stephanie</td>
<td>Oh. a</td>
</tr>
<tr>
<td>13</td>
<td>R1</td>
<td>You’ve done four pieces.</td>
</tr>
<tr>
<td>14</td>
<td>Stephanie</td>
<td>plus b times a plus b. Or</td>
</tr>
<tr>
<td>15</td>
<td>R1</td>
<td>I’m going to try my question again.</td>
</tr>
<tr>
<td>16</td>
<td>Stephanie</td>
<td>Okay.</td>
</tr>
<tr>
<td>17</td>
<td>R1</td>
<td>Let’s go back to some of these other things. [sorts through some of the papers on the desk] Okay. When this was six and this was six</td>
</tr>
<tr>
<td>18</td>
<td>Stephanie</td>
<td>Um hm.</td>
</tr>
<tr>
<td>19</td>
<td>R1</td>
<td>You found the area inside, right?</td>
</tr>
<tr>
<td>20</td>
<td>Stephanie</td>
<td>Um hm.</td>
</tr>
<tr>
<td>21</td>
<td>R1</td>
<td>Which was what?</td>
</tr>
<tr>
<td>22</td>
<td>Stephanie</td>
<td>Um. Thirty-six. Or</td>
</tr>
<tr>
<td>Line</td>
<td>Character</td>
<td>Text</td>
</tr>
<tr>
<td>------</td>
<td>-----------</td>
<td>------</td>
</tr>
<tr>
<td>23</td>
<td>R1</td>
<td>How did you get that?</td>
</tr>
<tr>
<td>24</td>
<td>Stephanie</td>
<td>How did I get that? I multiplied six times six.</td>
</tr>
<tr>
<td>25</td>
<td>R1</td>
<td>You you really did. You didn’t count them. I know you multiplied. You said six is the length of this side.</td>
</tr>
<tr>
<td>26</td>
<td>Stephanie</td>
<td>Yeah.</td>
</tr>
<tr>
<td>27</td>
<td>R1</td>
<td>times the length of this side. And for this one you said the area was</td>
</tr>
<tr>
<td>28</td>
<td>Stephanie</td>
<td>Um. Sixteen.</td>
</tr>
<tr>
<td>29</td>
<td>R1</td>
<td>Because you took.</td>
</tr>
<tr>
<td>30</td>
<td>Stephanie</td>
<td>I multiplied</td>
</tr>
<tr>
<td>31</td>
<td>R1</td>
<td>(inaudible) And this one you said the area was a squared. Because you took</td>
</tr>
<tr>
<td>32</td>
<td>Stephanie</td>
<td>a and I multiplied it by a.</td>
</tr>
<tr>
<td>33</td>
<td>R1</td>
<td>Right? So. What’s this side here? [can’t tell which]</td>
</tr>
<tr>
<td>34</td>
<td>Stephanie</td>
<td>Um. ab or a plus b.</td>
</tr>
<tr>
<td>35</td>
<td>R1</td>
<td>That’s what you told me up in the other</td>
</tr>
<tr>
<td>36</td>
<td>Stephanie</td>
<td>Yeah. a plus b.</td>
</tr>
<tr>
<td>37</td>
<td>R1</td>
<td>Okay. Why don’t you write a plus b on top of it, lest not we lose that idea. And what’s the side here? [the left side]</td>
</tr>
<tr>
<td>38</td>
<td>Stephanie</td>
<td>a plus b.</td>
</tr>
<tr>
<td>39</td>
<td>R1</td>
<td>Okay. Okay. So</td>
</tr>
<tr>
<td>40</td>
<td>Stephanie</td>
<td>So it would be a plus b times a plus b?</td>
</tr>
<tr>
<td>41</td>
<td>R1</td>
<td>Why don’t you write that down? a plus b times a plus b. [Stephanie writes a + b * a + b] Don’t you need some parentheses in there? [Stephanie inserts parentheses so it now reads (a + b) * (a + b).] Does it matter?</td>
</tr>
<tr>
<td>42</td>
<td>Stephanie</td>
<td>Mm. I don’t know. Um. I guess it just tells you to do that first.</td>
</tr>
</tbody>
</table>

Interview two of seven, clip 4 of 6

Transcript for Event 5

<table>
<thead>
<tr>
<th>Line</th>
<th>Character</th>
<th>Text</th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>R1</td>
<td>in-a-minute. But it’s you said a plus b times a plus b, right? Equals</td>
</tr>
<tr>
<td>46</td>
<td>Stephanie</td>
<td>Um hm.</td>
</tr>
<tr>
<td>47</td>
<td>R1</td>
<td>Put an equal. [Stephanie does.] Equals what? If I know the length of this side and I know the length of this side, what part will give me the area?</td>
</tr>
<tr>
<td>48</td>
<td>Stephanie</td>
<td>What part will give you the area?</td>
</tr>
<tr>
<td>49</td>
<td>R1</td>
<td>Um hm. What’s the area of that square?</td>
</tr>
<tr>
<td>50</td>
<td>Stephanie</td>
<td>In other words than a plus b times a plus b.</td>
</tr>
<tr>
<td>51</td>
<td>R1</td>
<td>Um hm.</td>
</tr>
<tr>
<td>52</td>
<td>Stephanie</td>
<td>Well, doesn’t that go back to that? Then it becomes like, if a plus, wouldn’t it, wouldn’t it just be um a plus b squared?</td>
</tr>
<tr>
<td>53</td>
<td>R1</td>
<td>Write that down. [Stephanie completes the algebra sentence: (a+b) * (a+b) = (a-b)/2]. And why is it?</td>
</tr>
<tr>
<td>54</td>
<td></td>
<td>Because that's what it was before? Because it's um two a's and two b's. I like there's two of each?</td>
</tr>
<tr>
<td>55</td>
<td>R1</td>
<td>Okay. So a plus b. I'm not sure - you're not telling me a squared plus b squared. You're saying that this [points to (a-b)] and this [points to (a+b)] twice.</td>
</tr>
<tr>
<td>56</td>
<td></td>
<td>Yes.</td>
</tr>
</tbody>
</table>

Interview two of seven, clip 4 of 6

Transcript for Event 6

<p>| 57 | R1   | All right. But now in this picture, what part of the picture represents this [(a+b)/2] piece? I know what part is a plus b. You told me that it's this side. |
| 58 | Stephanie | Like the whole thing? |
| 59 | R1   | The whole thing. |
| 60 | Stephanie | Yeah. The whole thing. |
| 61 | R1   | Okay. So this whole area is what this is equals. Let's write it out. What is the whole thing? You have pieces of it |
| 62 | Stephanie | Um hm. |
| 63 | R1   | So it's the whole thing. That means, this piece [the a1 a] |
| 64 | Stephanie | and this piece [the top left a1 b] and this piece [the bottom right a2 b] |
| 65 | R1   | Okay. So |
| 66 | Stephanie | All together. |
| 67 | R1   | All together, when you |
| 68 | Stephanie | Yes. |
| 69 | R1   | Talk about things all together, what do you do? |
| 70 | Stephanie | You add them. |
| 71 | R1   | You add them. So it's this piece, plus this piece, plus this piece, plus this piece, plus this piece. [indicates the pieces in the same order as before] |
| 72 | Stephanie | You want me to add them. |
| 73 | R1   | I want you to write this piece, plus this piece, plus |
| 74 | Stephanie | Okay. |
| 75 | R1   | this piece, plus this piece, and not skip any steps. [Stephanie writes: a * a + a * b + b * b + a * b. ] You have four terms? |
| 76 | Stephanie | Yes. |
| 77 | R1   | Okay. Let's simplify them. Equal |
| 78 | Stephanie | Just put it like back down here? |
| 79 | R1   | Just put the equal underneath that and let's simplify. |
| 80 | Stephanie | All right. |</p>
<table>
<thead>
<tr>
<th>Line</th>
<th>Speaker</th>
<th>Text</th>
</tr>
</thead>
<tbody>
<tr>
<td>81</td>
<td>R1</td>
<td>Is there another way you can write a times a?</td>
</tr>
<tr>
<td>82</td>
<td>Stephanie</td>
<td>a squared. [writes a²]</td>
</tr>
<tr>
<td>83</td>
<td>R1</td>
<td>Okay.</td>
</tr>
<tr>
<td>84</td>
<td>Stephanie</td>
<td>Plus it could be b squared, 'cause there's a</td>
</tr>
<tr>
<td>85</td>
<td>R1</td>
<td>Put that at the end.</td>
</tr>
<tr>
<td>86</td>
<td>Stephanie</td>
<td>Okay. So a squared plus ab (inaudible) plus b squared.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[writes a²]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>simplify that. Couldn't it be ab squared?</td>
</tr>
<tr>
<td>87</td>
<td>R1</td>
<td>Okay. So what you have here: a squared plus ab</td>
</tr>
<tr>
<td>88</td>
<td>Stephanie</td>
<td>Yeah.</td>
</tr>
<tr>
<td>89</td>
<td>R1</td>
<td>plus ab</td>
</tr>
<tr>
<td>90</td>
<td>Stephanie</td>
<td>Um hm.</td>
</tr>
<tr>
<td>91</td>
<td>R1</td>
<td>plus b squared.</td>
</tr>
<tr>
<td>92</td>
<td>Stephanie</td>
<td>Yes. That would be two ab or</td>
</tr>
<tr>
<td>93</td>
<td>R1</td>
<td>You have ab and you have another ab</td>
</tr>
<tr>
<td>94</td>
<td>Stephanie</td>
<td>Yes.</td>
</tr>
<tr>
<td>95</td>
<td>R1</td>
<td>so you have two ab, so write that down.</td>
</tr>
<tr>
<td>96</td>
<td>Stephanie</td>
<td>a squared plus two ab plus b squared. [writes: a² + 2ab + b²] w/ii</td>
</tr>
<tr>
<td>97</td>
<td>R1</td>
<td>Hmm. What did you just do?</td>
</tr>
<tr>
<td>98</td>
<td>Stephanie</td>
<td>I'm simplified it?</td>
</tr>
<tr>
<td>99</td>
<td>R1</td>
<td>Okay. So what is this a squared plus two ab plus b squared represent?</td>
</tr>
<tr>
<td>100</td>
<td>Stephanie</td>
<td>This. [puts her hand over the (a+b) square]</td>
</tr>
<tr>
<td>101</td>
<td>R1</td>
<td>The area of the square?</td>
</tr>
<tr>
<td>102</td>
<td>Stephanie</td>
<td>Yes.</td>
</tr>
<tr>
<td>103</td>
<td>R1</td>
<td>With what side? What length side? [pause]</td>
</tr>
<tr>
<td>104</td>
<td>Stephanie</td>
<td>Well, it represents like the area of the square.</td>
</tr>
<tr>
<td>105</td>
<td>R1</td>
<td>This, what particular square? What is the length of the side of that square?</td>
</tr>
<tr>
<td>106</td>
<td>Stephanie</td>
<td>Oh, a plus b.</td>
</tr>
<tr>
<td>107</td>
<td>R1</td>
<td>a plus b. Now, a plus b is the length of the side.</td>
</tr>
<tr>
<td>108</td>
<td>Stephanie</td>
<td>Um hm.</td>
</tr>
<tr>
<td>109</td>
<td>R1</td>
<td>The area you told me in simplified form -- you said the area is a plus b quantity squared.</td>
</tr>
<tr>
<td>110</td>
<td>Stephanie</td>
<td>Um hm.</td>
</tr>
<tr>
<td>111</td>
<td>R1</td>
<td>But didn't we start this whole visit here</td>
</tr>
<tr>
<td>112</td>
<td>Stephanie</td>
<td>With (inaudible)</td>
</tr>
<tr>
<td>113</td>
<td>R1</td>
<td>to try and figure out what a plus b quantity squared meant?</td>
</tr>
<tr>
<td>114</td>
<td>Stephanie</td>
<td>Yes.</td>
</tr>
<tr>
<td>115</td>
<td>R1</td>
<td>And now you're telling me it's a squared plus two ab plus b squared.</td>
</tr>
<tr>
<td>116</td>
<td>Stephanie</td>
<td>[hesitantly] Yeah.</td>
</tr>
</tbody>
</table>

Interview two of seven, clip 5 of 6
Transcript for Event 7

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>R1</td>
<td>Things about the area a square... you told me that ( a + b ) quantity square equals ( a ) squared plus two ( ab ) plus ( b )</td>
</tr>
<tr>
<td>6</td>
<td>Stephanie</td>
<td>Yes.</td>
</tr>
<tr>
<td>7</td>
<td>R1</td>
<td>That's what you, I believe, were working on for this last hour and fifteen minutes.</td>
</tr>
<tr>
<td>8</td>
<td>Stephanie</td>
<td>Yes.</td>
</tr>
<tr>
<td>9</td>
<td>R1</td>
<td>Okay. So if your arithmetic work is correct, I, you should be able to test some numbers – at least to see if you don't get a counter example right away.</td>
</tr>
<tr>
<td>10</td>
<td>Stephanie</td>
<td>So you want me to test numbers?</td>
</tr>
<tr>
<td>11</td>
<td>R1</td>
<td>What do you think? Wouldn't you be inclined to test</td>
</tr>
<tr>
<td>12</td>
<td>Stephanie</td>
<td>Oh. Well, yeah</td>
</tr>
<tr>
<td>13</td>
<td>R1</td>
<td>some numbers.</td>
</tr>
<tr>
<td>14</td>
<td>Stephanie</td>
<td>I didn't know</td>
</tr>
<tr>
<td>15</td>
<td>R1</td>
<td>for ( a )'s and ( b )'s and see what happens?</td>
</tr>
<tr>
<td>16</td>
<td>Stephanie</td>
<td>All right. So let me do some really easy numbers. Um. If</td>
</tr>
<tr>
<td>17</td>
<td>R1</td>
<td>Try a very try a easy number. That's a good idea.</td>
</tr>
<tr>
<td>18</td>
<td>Stephanie</td>
<td>Yeah. So</td>
</tr>
<tr>
<td>19</td>
<td>R1</td>
<td>Especially this time of day.</td>
</tr>
<tr>
<td>20</td>
<td>Stephanie</td>
<td>( a ) is two and ( b ) is three.</td>
</tr>
<tr>
<td>21</td>
<td>R1</td>
<td>That's what you did before.</td>
</tr>
<tr>
<td>22</td>
<td>Stephanie</td>
<td>Yeah. So it would be</td>
</tr>
<tr>
<td>23</td>
<td>R1</td>
<td>You've got half of it done already.</td>
</tr>
<tr>
<td>24</td>
<td>Stephanie</td>
<td>[talking under her breath as she writes] Two is four, plus two times two time three plus three squared, that's a nine (inaudible) ( (Steps	ext{hie has written: } 2^2 \times 3 \times 3) ) ( \square \times 3^2 ); beneath that she wrote: ( 4 + 12 + 9 ); beside her work she added ( 16 + 9 ) and got ( 25 ); [pause] Twenty-five. It works.</td>
</tr>
<tr>
<td>25</td>
<td>R1</td>
<td>It worked for that example.</td>
</tr>
<tr>
<td>26</td>
<td>Stephanie</td>
<td>Yeah.</td>
</tr>
<tr>
<td>27</td>
<td>R1</td>
<td>But when you claim it's true, how many does it have to work for?</td>
</tr>
<tr>
<td>28</td>
<td>Stephanie</td>
<td>All of them?</td>
</tr>
<tr>
<td>29</td>
<td>R1</td>
<td>All of them. Yeah.</td>
</tr>
<tr>
<td>30</td>
<td>Stephanie</td>
<td>(inaudible)</td>
</tr>
<tr>
<td>31</td>
<td>R1</td>
<td>Could you possibly test all of them?</td>
</tr>
<tr>
<td>32</td>
<td>Stephanie</td>
<td>No-o! [laughs] There's too many numbers. Um. Do you want me to try again?</td>
</tr>
</tbody>
</table>
Interview four of seven, clip 1 of 9

Transcript for Event 8

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>Stephanie</td>
<td>And um, that it would be nine, and because it was like three by three, three squared. And we did a couple of those. And then, um, [pause], we- you asked me if it was um, if one side was [writing] a plus b [writing]</td>
</tr>
<tr>
<td>10</td>
<td>R1</td>
<td>Oh yes, I remember that one.</td>
</tr>
<tr>
<td>11</td>
<td>Stephanie</td>
<td>Then what it would be.</td>
</tr>
<tr>
<td>12</td>
<td>R1</td>
<td>Yeah.</td>
</tr>
<tr>
<td>13</td>
<td>Stephanie</td>
<td>And um, if the small part’s a and the big part’s b [draws square divided into parts representing ((a+b)^2)]</td>
</tr>
<tr>
<td>14</td>
<td>R1</td>
<td>Mhm. [pause, Stephanie writes] did you figure out what all those pieces were?</td>
</tr>
<tr>
<td>15</td>
<td>Stephanie</td>
<td>Yeah. It was a squared, ahem, b squared, ah, and it would be a squared plus 2ab plus b squared, and that’s what we figured out then. [pause, writes] a plus b squared equals.</td>
</tr>
<tr>
<td>16</td>
<td>R1</td>
<td>Oh, okay, right. And the original conjecture what a plus b squared equaled you were testing.</td>
</tr>
<tr>
<td>17</td>
<td>Stephanie</td>
<td>Yes.</td>
</tr>
<tr>
<td>18</td>
<td>R1</td>
<td>And originally, what did you conjecture?</td>
</tr>
<tr>
<td>19</td>
<td>Stephanie</td>
<td>Um-</td>
</tr>
<tr>
<td>20</td>
<td>R1</td>
<td>What most people-</td>
</tr>
<tr>
<td>21</td>
<td>Stephanie</td>
<td>I think it was a squared plus b squared.</td>
</tr>
<tr>
<td>22</td>
<td>R1</td>
<td>Yeah, lots of students</td>
</tr>
<tr>
<td>23</td>
<td>Stephanie</td>
<td>And that was wrong.</td>
</tr>
<tr>
<td>24</td>
<td>R1</td>
<td>conjecture that, right, so-</td>
</tr>
<tr>
<td>25</td>
<td>Stephanie</td>
<td>Yeah.</td>
</tr>
</tbody>
</table>
Appendix F

Journal Prompts

11/22/2016

589

Secondary Methods 443 > Forums

Forums

NEW FORUM ORGANIZE TEMPLATE SETTINGS STATISTICS & GRADING WATCH

Forums

Journal Entry 4 9 unread of 15 messages  Topic Settings  More

Describe your ideas about argumentation and how they have changed, please read full description.

Hide Full Description

Argumentation happens when someone wants to convince someone else that a claim is true. Both NCTM Principles and Practices (2002) and the Common Core State Standards of Mathematical Practice (MP 3, 2010) emphasize the importance of argumentation in mathematics, thus it is essential for teachers to understand what argumentation is and what it looks like in the mathematics classroom.

Part 1

This semester, you have studied analytics that showed mathematical argumentation. Before studying these analytics, you considered two questions, the first of which was:

1. What do you think argumentation means with respect to learning math?

In this Journal entry, respond to the question above. In your response, consider if your thoughts about argumentation have changed as a result of studying the analytic, and if so, how. As before, answer use as much detail as possible in your answer, referring to the analytics you studied if appropriate.

Part 2

At the beginning of the semester I shared an analytic with you with events illustrating a fourth-grade problem-solving session about fractions. This analytic had no titles or descriptions for the events and only a general overall description. I asked you to:

Describe in as much detail as possible the argumentation you notice in each event of this analytic by:

a. writing a title that summarizes the argumentation you see in the event

https://sakai.rutgers.edu/portal/site/sakai/2016-40eb-81bd-130d9a3545f?oid=0516bdfc-91f8-403a-a301-1e4e7770141discussionForum/forumsOnlyIdForum...
b. writing a description for each event that details the argumentation you see in the event.

2. Then, write an overall description of the analytic, including a summary of argumentation across the events.

I would like you to revisit the analytic you annotated now that you have studied other analytics created to show argumentation. Based on the other analytics you studied, edit the event titles, event descriptions, and the overall description to best articulate the argumentation shown in the analytic.

As before, you can refer to the transcripts (which are still posted in Resources on Sakai) when necessary to add detail to your descriptions, to use names to refer to individual children, and to clarify the dialog if it is hard to hear.


Journal Entry 3  New Topic  |  Forum Settings  |  More
Study the analytic, answer, and then discuss the questions. Read the full description.

> Hide Full Description

Access the first analytic about argumentation by using the following link:

https://rucore.libraries.rutgers.edu/analytic/share?h=fe885985592a4a9e22c4d104a4199ac

Read through all of the descriptions, including the overall description at the bottom of the screen.

Then respond to each of the questions in this forum. Read your classmates responses and comment on them. As usual, be as specific as possible.

Question 5 – Resolution?

0 unread of 17 messages  |  Topic Settings  |  More

see full description

> Hide Full Description

5. Was the argument resolved? Explain.

Question 4 – how did the claims change?

0 unread of 17 messages  |  Topic Settings  |  More

see full description

> Hide Full Description

4. For the claims presented, identify those that are:

(a) Challenged
(b) Modified
(c) Refuted
Question 1 — Elements of argumentation

1. Identify elements of argumentation that can be identified in this analytic.

Question 2 — What are the claims?

2. What are the claims being made by the children in the arguments presented? Who is making what claim?

Question 3 — Evidence Backing

3. Identify evidence/backing that the children use to support their claims.

Journal Entry 2

Study the analytic, answer, and then discuss the questions. Read the full description.

Question 4 — How did the claims change?

4. For the claims presented, identify those that are:
   (a) Challenged
   (b) Modified

Question 5 — Resolution?

5. Was the argument resolved? Explain.
(c) Refuted

Question 3 — Evidence Barking 0 unread of 19 messages  Topic Settings | More
See full description
▼ Hide full description
3. Identify evidence/backing that the children use to support their claims.

Question 2 — What are the claims? 0 unread of 24 messages  Topic Settings | More
see full description
▼ Hide full description
2. What are the claims being made by the children in the arguments presented? Who is making what claim?

Question 1 — Elements of argumentation 0 unread of 26 messages  Topic Settings | More
See full description
▼ Hide full description
1. Identify elements of argumentation that can be identified in this argument.

Journal 1 and Online Class Forum  New Topic | Forum Settings | More

Online class 3-4-15 - Questions about 5 Practices pp. 61-94 0 unread of 90 messages  Topic
Settings | More
There are three conversations about the reading. Read full description
▼ View Full Description

Online class 3-4-15 - Questions about Algebraic thinking analytic 0 unread of 65 messages
Topic Settings | More
There are four conversations about the analytic. Read full description.
▼ View Full Description

Journal Entry 1 0 unread of 14 messages  Topic Settings | More
Describe your ideas about argumentation, please read full description.
▼ View Full Description

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