HAUSDORFF DISTANCE AND CONVEXITY

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THESIS ABSTRACT

Hausdorff Distance and Convexity

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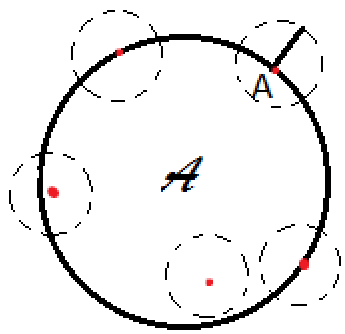
The goal of this thesis is to discuss the Hausdorff Distance and prove that the metric space $\mathcal{S}_{\mathcal{X}}$, which is the set of compact subsets of $\mathcal{X} = \mathbb{R}^n$ with the hausdorff distance is a complete metric space. In the first part, we discuss open r-neighborhoods and convexity. Proving some properties and providing examples for some definitions that are defined. This provides us with a background before we begin discussing the Hausdorff Distance. In the second part, we introduce the Hausdorff Distance and its properties. In conclusion, we go on to prove that the metric space $\mathcal{S}_{\mathcal{X}}$ is a complete metric space using everything that has been discussed previously in the thesis.

1 Introduction

1.1 r-neighborhoods

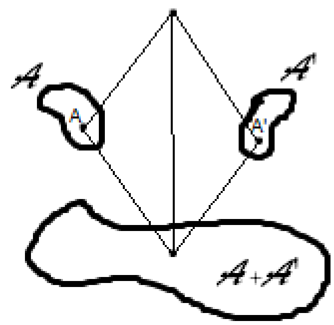
Before we define what the Hausdorff distance is, let us first begin by defining what an r-neighborhood of $\mathcal{A}$ is. Let $\mathcal{A}$ be a subset of $\mathcal{X} = \mathbb{R}^n$ then:

$$\mathcal{A}_r = \bigcup_{A \subset \mathcal{A}} B_r(A) = \{x \in \mathcal{X} | d(\mathcal{X}, \mathcal{A}) < r\} = \mathcal{A} + B_r$$

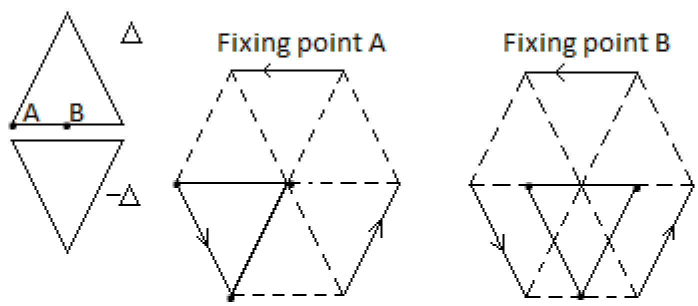


To show that these are equivalent let us first look at $\mathcal{A}_r = \bigcup_{A \subset \mathcal{A}} B_r(A) = \{x \in \mathcal{X} | d(\mathcal{X}, \mathcal{A}) < r\}$. Let $x \in \bigcup_{A \subset \mathcal{A}} B_r(A)$. This implies that $x \in B_r(A)$ for some A in A. Thus $d(x, A) < r$, which implies $d(x, \mathcal{A}) = \inf_{A \in \mathcal{A}} d(x, \mathcal{A}) < r$. This proves "$\subset$". To show "$\supset$" let $x \in \mathcal{X}$ and $d(x, \mathcal{A}) < r$. This implies that $\inf_{A \in \mathcal{A}} < r$. Thus there exists an A such that $d(x, \mathcal{A}) < r$. Therefore $x \in B_r(A)$. Before we show the equivalence involving $\mathcal{A} + B_r(A)$ we must first discuss the Minkowski sum. Let $\mathcal{A}$ and $\mathcal{A}'$ be subsets of X. Then the Minkovski sum:

$$\mathcal{A} + \mathcal{A}' = \{A + A' | A \in \mathcal{A} \text{ and } A' \in \mathcal{A}'\}$$



To help us visualize this let us say we are given two equalateral triangles, Δ and $-\Delta$. The Minkowski sum of $\Delta + -\Delta$ will give us a hexagon. Observe that if we fix a point in Δ onto the border of $-\Delta$ then translate Δ along the border we will get a hexagon. The point we pick in Δ does not matter and will still yield a hexagon.



Now returning to our equivalence, let us look at $\mathcal{A}+B_r(A)$. Notice that $\mathcal{A}+B_r(A) = \bigcup_{A\in\mathcal{A}}(A+B_r)$ which is the same as $\bigcup_{A\in\mathcal{A}} B_r(A)$ since $A+B_r = A+B_r(O) = B_r(A)$.

Therefore for any open r-neighborhood the following are equivalent:

(1) $\bigcup_{A\subset\mathcal{A}} B_r(A)$

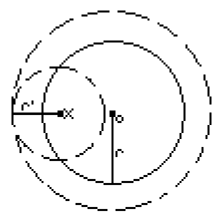
(2) $\{x \in \mathcal{X} | d(\mathcal{X}, \mathcal{A}) < r\}$

(3) $\mathcal{A} + B_r$

Now that we've defined what an r-neighborhood of$\mathcal{A}$ is le t us discuss some properties:

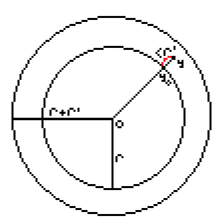
$$
\begin{aligned}
(\mathcal{A}_r)_{r'} &= \mathcal{A}_{r+r'}; r, r' > 0 & (1)\\
\bigcap_{r>o} \mathcal{A}_r &= \bar{\mathcal{A}} & (2)
\end{aligned}
$$

Let us observe why these properties are true beginning with (1). The first thing we notice is that $(\mathcal{A} + B_r) + B_{r'} = \mathcal{A} + B_{r+r'}$. So what we need to show is that $B_r + B_{r'} = B_{r+r'}$



This is the same as saying $\bigcup_{x \in B_r} B_{r'}(x) = B_{r+r'}$. To show this equality, let $y \in \bigcup_{x \in B_r} B_{r'}(x)$. This means $y \in B_{r'}(x)$ and $x \in B_r$. The distance $d(y, x) < r'$ so $|x| = d(x, O) < r$ and $|y| = d(y, O) \leq d(y, x) + d(x, O) < r + r'$ thus $y \in B_{r+r'}$.

Now let us assume $y \in B_{r+r'}$. So $|y| < r + r'$. Then y exists in some $B_{r'}(x)$. Therefore $y \in \bigcup_{x \in B_r} B_{r'}(x)$. With this shown the first property (1) is proven.



Now to prove (2). Before we begin the proof, notice:

$$\begin{aligned}\bigcap_{r>0} \mathcal{A}_r &= \{x \in \mathcal{X} | d(x, \mathcal{A}) < r \text{ for all } r > 0\} \\ &= \{x \in \mathcal{X} | d(\mathcal{X}, \mathcal{A}) = 0\}\end{aligned}$$

where $d(\mathcal{X}, \mathcal{A}) = 0$ if and only if $x \in \bar{\mathcal{A}}$. This shows "$\supset$". In order to show"$\subset$", let $\bar{\mathcal{A}}$ be closed, i.e.:

$$\bar{\mathcal{A}}_r = \bigcup_{A \in \mathcal{A}} B_r(A) = \{x \in \mathcal{X} | d(x, \mathcal{A} \leq r\} = \mathcal{A} + \bar{B}_r$$

The proof for "$\subset$" follows directly from the second definition for closed r-neighborhood. The proofs to show equality for the closed r-neighborhood is very similar to that of an open r-neighborhood so we will skip these and move on.

1.2 Convexity

Let $f : D \to \mathbb{R}$ with domain $D \subset \mathcal{X}$. We call this function convex if for any linesegment $[x, x_0] \subset D$ and $\lambda \in [0, 1]$ we have:

$$f((1 - \lambda)x_0 + \lambda x_1) \leq (1 - \lambda)f(x_0) + \lambda f(x_1).$$

It should be noted that if we reverse the inequality sign we will define concavity. Now, if $f : D \to \mathbb{R}$ is convex on a convex set $D \subset \mathcal{X}$, then the level sets $\{x \in D | f(x) < r\}$ and $\{x \in D | f(x) \leq r\}$ are convex.

To show this let $f : D \to \mathbb{R}$ be a convex functon on a convex set $D \subset \mathcal{X}$. If $x_0, x_1 \in \{x \in D | f(x) \leq r\}$ with $\lambda \in [0,1]$, then:

$$f(\lambda x_1 + (1-\lambda)x_0) \leq \lambda f(x_1) + (1-\lambda)(f(x_0)) \leq \lambda r + (1-\lambda)r = r$$

Thus $\lambda x_1 + (1-\lambda)x_0 \in \{x \in D | f(x) \leq r\}$. The proof for $\{x \in D | f(x) \leq r\}$ is similar.

Now let us observe that for a convex set $\mathcal{C} \subset \mathcal{X}$, the distance function $d(\cdot, \mathcal{C}) : \mathcal{X} \to \mathbb{R}$ is convex. So let $x_0, x_1 \in \mathcal{X}$. Given $\epsilon > 0$ we choose $c_0, c_1 \in \mathcal{C}$ such that $d(x_0, c_0) \leq d(x_0, \mathcal{C}) + \epsilon$ and $d(x_1, c_1) \leq d(x_1, \mathcal{C}) + \epsilon$. So using the definition of convex functions gives us $(1-\lambda)c_0 + \lambda c_1 \in \mathcal{C}, \lambda \in [0,1]$ and observe:

$$\begin{aligned} d((1-\lambda)x_0 + \lambda x_1, \mathcal{C}) &\leq d((1-\lambda)x_0 + \lambda x_1, (1-\lambda)c_0 + \lambda c_1) \\ &= |(1-\lambda)(x_0 - c_0) + \lambda(x_1 - c_1)| \\ &\leq (1-\lambda)|x_0 - c_0| + \lambda |x_1 - c_1| \\ &\leq (1-\lambda)d(x_0, \mathcal{C}) + \lambda d(x_1, \mathcal{C}) + \epsilon \end{aligned}$$

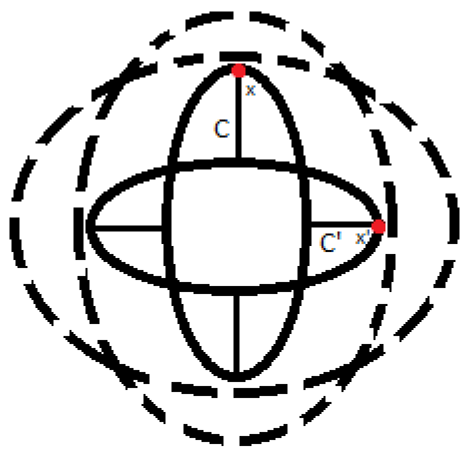
Now letting $\epsilon \to 0$, the claim follows. Also we note that open r-neighborhoods of a convex set is convex and closed r-neighborhoods of a closed compact convex set is also a closed compact convex set.

2 Hausdorff Distance

2.1 Distance between sets

Now that we have discussed r-neighborhoods and convexity, we can now discuss the distance between convex sets. Let $\mathcal{S} = \mathcal{S}_{\mathcal{X}}$ be the set of all compact subsets of $\mathcal{X}$. We define the Hausdorff distance function $d_H : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ by:

$$\begin{aligned} d_H(\mathcal{C},\mathcal{C}') &= \inf\{r \geq |\mathcal{C} \subset \bar{\mathcal{C}'}_r, \mathcal{C}' \subset \bar{\mathcal{C}}_r\} \\ &= \max(\sup_{x\in\mathcal{C}} d(x,\mathcal{C}'), \sup_{x'\in\mathcal{C}'} d(x',\mathcal{C})) \\ &= \inf\{r \geq 0 | \mathcal{C} \subset \mathcal{C}' + \bar{B}_r, \mathcal{C}' \subset \mathcal{C} + \bar{B}_r\}, \mathcal{C}, \mathcal{C}' \in \mathcal{S} \end{aligned}$$



Showing the equivalence is quick. $\mathcal{C} \subset \mathcal{C}'_r$ if and only if $\sup_{x\in\mathcal{C}} d(x,\mathcal{C}') \leq r, r \geq 0$. It should also be noted that the Hausdorff distance is sysmetric, i.e.:

$$d_H(\mathcal{C},\mathcal{C}') = d_H(\mathcal{C}',\mathcal{C})$$

We should also note that the Hausdorff distance also satisfies the triangle inequality:

$$d_H(\mathcal{C}, \mathcal{C}'') \leq d_H(\mathcal{C}, \mathcal{C}') + d_H(\mathcal{C}', \mathcal{C}''), \mathcal{C}, \mathcal{C}', \mathcal{C}'' \in \mathcal{S}$$

Also $d_H(\mathcal{C}, \mathcal{C}') \geq 0$ for all $\mathcal{C}, \mathcal{C}' \in \mathcal{S}$ and by compactness of $\mathcal{C}$ and $\mathcal{C}'$ $d_H(\mathcal{C}, \mathcal{C}') = 0$ if and only if $\mathcal{C} = \mathcal{C}'$.

Let us show that the hausdorff distance is a distance on the set of all compact subsets of $\mathcal{X}$. Now in order for d_H to be a distance on $\mathcal{S}_{\mathcal{X}}$ it must satisfy the following:

1) Symmetric

2)Triangle inequality

3)$d_H(\mathcal{C}, \mathcal{C}') \geq 0$ & $d_H(\mathcal{C}, \mathcal{C}') = 0 \iff \mathcal{C} = \mathcal{C}'$

As for symmetry, this should be fairly obvious so we will skip this and move onto the triangle inequality. Let $d_H(\mathcal{C}, \mathcal{C}') \leq r$ and $d_H(\mathcal{C}', \mathcal{C}'') \leq r'$. We want to show that $d_H(\mathcal{C}, \mathcal{C}'') \leq r + r'$. It should be noted from $d_H(\mathcal{C}, \mathcal{C}')$ we get the fact that $\mathcal{C} \subset \mathcal{C}'_r$ and $\mathcal{C}' \subset \mathcal{C}_r$ and from $d_H(\mathcal{C}', \mathcal{C}'') \leq r'$ we get $\mathcal{C}' \subset \mathcal{C}''_r$ and $\mathcal{C}'' \subset \mathcal{C}'_r$. This leads us to the following results:

$$\mathcal{C} \subset (\mathcal{C}')_r \subset (\mathcal{C}''_{r'})_r = \mathcal{C}''_{r+r'} \tag{3}$$

$$\mathcal{C}'' \subset (\mathcal{C}')_r \subset (\mathcal{C}_r)_{r'} = \mathcal{C}_{r+r'} \tag{4}$$

Both (3) and (4) tells us that $d_H(\mathcal{C}, \mathcal{C}'') \leq r + r'$. Therefore the triangle inequality holds. In order to prove the final condition, let $d_H(\mathcal{C}, \mathcal{C}') = 0$ this means $\sup\limits_{x \in \mathcal{C}} d(x, \mathcal{C}') = 0$ and $\sup\limits_{x' \in \mathcal{C}'} d(x, \mathcal{C}) = 0$. If $x \in \mathcal{C}$, $d_H(x, \mathcal{C}') = 0$ then $x \in \bar{\mathcal{C}}' = \mathcal{C}'$, thus $\mathcal{C} \subset \mathcal{C}'$. The same could be said for $\mathcal{C}' \subset \mathcal{C}$. Thus d_H satisfies all the conditions and is a metric on $\mathcal{S}_{\mathcal{X}}$.

2.2 Theorem

Now that's we have defined the hausdorff distance, we can prove that the set of all compact subsets of $\mathcal{X}$, $\mathcal{S}_{\mathcal{X}}$, is a complete metric space with the hausdorff distance. Before we start this proof there are some things we must first recall. Let $\mathcal{M}$ be a metric space with distance d, then we say a sequence$\{x_n\}$ is Cauchy if for every $\epsilon > 0$ there exists an N such that; $d(x_n, x_m) < \epsilon$ for $n, m \geq N$. We also say a sequence has a limit x_0 if forall $\epsilon >)$ there exists an N such that $d(x_n, x_0) < \epsilon$ for $n \geq N$. We say $\mathcal{M}$ is complete if every Cauchy sequence converges.

The other thing we are going to need is a result from Kuratovski. Let $\{C_k\}_{k\geq 1}$ then, $\bigcap_{k=1}^{\infty} \overline{\bigcup_{l \geq k} C_l} \limsup_{k\to\infty} C_k$. The Hausdorff distance works very well with compactness, however it does not reflect convergence properties of compact sets growing without bounds. For example, $d(B_n, B_m)$. For compact sets there is no distance, only a concept of convergence due to Kuratobski.

Now, going back to $\mathcal{S}_{\mathcal{X}}$, let $\{C_k\}_{k\geq 1} \subset \mathcal{S}$ be a decreasing sequence, (i.e. $C_1 \supset C_2 \supset C_3 \supset$...). By contradiction let us assume $\mathcal{S}_{\mathcal{X}}$ was not a metric space. Then there exists an $\epsilon > 0$ such that for all $k \geq 1$ we have $C_k \not\subset C_\epsilon$, because C_k is decreasing. Let us define $A_k = C_k \setminus C_\epsilon, k \geq 1$. So A_k is compact, decreasing and non-empty. Then observe:

$$\begin{aligned}
\mathcal{A} &= \bigcap_{k=1} \mathcal{A} \neq \emptyset; \mathcal{A} \cap C = \emptyset \\
\mathcal{A} &= \bigcap_{k=1} \mathcal{A} \subset \bigcap_{k \geq 1} C_k = C
\end{aligned}$$

Which is a contradiction. Thus $\mathcal{S}_{\mathcal{X}}$ is a mtetric space. Then $\lim\limits_{k \to \infty} C_k = C := \bigcap\limits_{k \geq 1} C_k$.By Kuratovski, let us define $\mathcal{A}_k = \overline{\bigcup\limits_{l \geq k} C_l}$. Note that $\mathcal{A}_k$ is closed and bounded, therefore compact and decreasing. So the first result is that $\lim\limits_{k \to \infty} \mathcal{A}_k = \mathcal{A} := \bigcap\limits_{k \geq 1} \mathcal{A}_k$. Now we make the claim that $\lim\limits_{k \to \infty} C_k = \mathcal{A}$. First, let $\epsilon > 0$, then there exists a $k_0 \geq 0$ such that $\mathcal{A}_k$ is contained in $\mathcal{A}_\epsilon$ for $k \geq k_0$. Also $l \geq k$, thus $C_l \subset \mathcal{A}_k$, therefore$C_l \subset \mathcal{A}_\epsilon$, when $l \geq k_0$. This proves "$\Rightarrow$". In order to prove "$\Leftarrow$", let $\frac{\epsilon}{2} > 0$. Then, there exists a $K_0 \geq k_0$ such that $C_l \subset (C_m)_\epsilon$, whenever $l, m \geq k_0$, and $\mathcal{A}_k = \bigcup\limits_{l \geq k} C_l \subset (C_m)_\epsilon$. Thus, $\overline{(C_m)_{\frac{\epsilon}{2}}} \subset (C_m)_\epsilon$ and $\bigcap_{k \geq 1} \mathcal{A}_k = \mathcal{A} \subset (C_m)_\epsilon$ when $m \geq k_0$. Since $\mathcal{A} \subset (C_m)_\epsilon$, then $d_H(C_l, \mathcal{A}) < \epsilon$, $l \geq k_0$. Thus, $\mathcal{S}_{\mathcal{X}}$ with d_H is a metric space.

3 Bibliography

Toth, Gabor. Measures of Symmetry for Convex Sets and Stability. 1st ed. N.p.: Springer International Publishing, 2015. Print.