COMINUSCULE FLAG VARIETIES AND THEIR QUANTUM K-THEORY: SOME RESULTS

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A dissertation submitted to the

Graduate School—New Brunswick

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Anders S. Buch

and approved by

New Brunswick, New Jersey

May, 2017

ABSTRACT OF THE DISSERTATION

Cominuscule flag varieties and their quantum K-theory: Some results

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This thesis investigates the ring structure of the torus-equivariant quantum K-theory ring $\mathrm{QK}_T(X)$ for a cominuscule flag variety X. As a main result, we present an identity that relates the product of opposite Schubert classes in $\mathrm{QK}_T(X)$ to the minimal degree of a rational curve joining the corresponding Schubert varieties. Using this we infer further properties of the ring $\mathrm{QK}_T(X)$, one of which is that the Schubert structure constants always sum to one.

We also introduce a formula for the product of Schubert classes in $\mathrm{QK}_T(\mathbb{P}^n)$. As a corollary we establish Griffeth-Ram positivity of the Schubert structure constants for $\mathrm{QK}_T(\mathbb{P}^n)$. After a closer analysis, we conclude that the rings $\mathrm{QK}_T(\mathbb{P}^n)$ are isomorphic for all n.

Acknowledgements

Six years ago, almost to the day, a friend gave me a copy of Fulton's *Intersection Theory* as a gift. As soon as I felt ready, I asked a professor, Anders Buch, to do a reading course on it, and the rest is history. With that, I would like to start by thanking my friend Howie for giving me the book.

I thank my committee members Cristian Lenart, Chuck Weibel and Chris Woodward whose guidance and support have made my thesis experience a genuine pleasure. Extra thanks to Chuck and Chris, whose support stretches back to day one of graduate school.

Vijay Ravikumar has been a constant source of enthusiasm and inspiration, in and out of mathematics. I thank him fondly for all that he has done, and for all that he continues to do.

To my friends Lisa Bauer, Ed Karasiewicz, Charles Wolf, Zhuohui Zhang, Glen Wilson, Knight Fu, Shashank Kanade, David Duncan and Howie Nuer: thanks for sharing these amazing years with me. And to everyone at the Power Gym: thanks for the loud music and squats too.

Special thanks to Jack Hanson, for all of those tales of Suetonius. And to Katie McKeon: a special wheek-wheek! To my parents and sisters: thank you for your endless love and support.

Finally, to my advisor Anders Buch: words can't express how wonderful it has been to learn from you and work with you—how wonderful you have been to me. You have truly been the best. Thank you.

April, 2017

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Introduction

Classically, Schubert calculus addresses the enumerative geometry of points, lines, planes, etc. of complex projective space $\mathbb{P}^n$. This amounts to the intersection theory of Schubert varieties in the Grassmannians $X = \mathrm{Gr}(k, n+1)$. The classical theory culminates in the well-known Littlewood-Richardson rule for the cohomology rings $H^*(X, \mathbf{Z})$ and their Schubert classes.

Today Schubert calculus explores richer, more general cohomology theories of spaces which themselves generalise the Grassmannians X: the complex flag varieties. Two such cohomology theories are *quantum K-theory* and *torus-equivariant quantum K-theory*. Introduced in the early 2000's by A. Givental [16] and developed by Y.-P. Lee [21], (equivariant) quantum K-theory may be understood as a K-theoretic analogue of (equivariant) quantum cohomology.

For a flag variety X, a focal point of its quantum K-theory are the K-theoretic Gromov-Witten invariants of its Schubert varieties. These invariants are integers that encode the geometry of curves in X, specifically the arithmetic genus of families of rational curves in X that meet Schubert varieties X^u, X^v, X^w in general position. *Equivariant* quantum K-theory enriches this theory by having these numerical invariants take values in $\Gamma = \mathbf{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, the representation ring of a torus $T = (\mathbf{C}^*)^n$, so as to convey the natural action of T on X. We shall focus on the equivariant theory.

Equivariant quantum K-theory assembles these invariants into the structure constants of a ring. Denote this ring by $\mathrm{QK}_T(X)$. Given Schubert varieties X^u, X^v contained in X, let $[\mathcal{O}_{X^u}]$ and $[\mathcal{O}_{X^v}]$ denote their corresponding classes in $\mathrm{QK}_T(X)$. Then the product in $\mathrm{QK}_T(X)$ of these Schubert classes is expressed as follows:

$$[\mathcal{O}_{X^u}] \star [\mathcal{O}_{X^v}] \;=\; \sum_{w,d} N_{u,v}^{w,d} q^d [\mathcal{O}_{X^w}].$$

Though they are not Gromov-Witten invariants themselves, each Schubert structure constant $N_{u,v}^{w,d} \in \Gamma$ is a polynomial in K-theoretic Gromov-Witten invariants involving X^u, X^v, X^w. The formal parameter q records the degrees $d \in H_2(X, \mathbf{Z})$ of the rational curves described by the invariants. Thus the quantum product $[\mathcal{O}_{X^u}] \star [\mathcal{O}_{X^v}]$ may be understood as a generating series in q for the K-theoretic Gromov-Witten invariants of X^u and X^v. It is a goal in Schubert calculus to determine explicit formulas for these products.

These Schubert structure constants $N_{u,v}^{w,d}$ are difficult to compute; there is in fact an active area of research devoted to computing them [5], [6]. However in joint work with A. S. Buch, the following relation is established when X is *cominuscule* (a small family of flag varieties that includes the classical Grassmannians):

Theorem A. *For fixed u, v we have $\sum_{w,d} N_{u,v}^{w,d} = 1$ in Γ.*

In other words, substituting 1 for q and the Schubert classes $[\mathcal{O}_{X^w}]$ in $[\mathcal{O}_{X^u}] \star [\mathcal{O}_{X^v}]$ curiously yields the relation of Theorem A. These substitutions can in fact be described by the (non-quantum) equivariant K-theory of X. Indeed, let $K_T(X)$ be the Grothendieck ring of T-equivariant coherent sheaves on X, and let $K_T(\mathrm{pt})$ denote the equivariant Grothendieck ring of a point, which may be identified $K_T(\mathrm{pt}) \simeq \Gamma$. Then the Euler characteristic map $\chi : K_T(X) \to \Gamma$, ie. pushforward to a point, is characterised by $\chi\big([\mathcal{O}_{X^w}]\big) = 1$ on Schubert classes. Now $\mathrm{QK}_T(X)$ contains $K_T(X)$ as a subgroup; moreover there is an intermediate subring $\mathrm{QK}_T^{\mathrm{poly}}(X)$,

$$\mathrm{QK}_T(X) \underset{\text{subring}}{\supset} \mathrm{QK}_T^{\mathrm{poly}}(X) \supset K_T(X),$$

to which χ naturally extends: $\widetilde{\chi} : \mathrm{QK}_T^{\mathrm{poly}}(X) \to \Gamma$. This extension $\widetilde{\chi}$ satisfies the desired property $\widetilde{\chi}\big([\mathcal{O}_{X^w}]\big) = \widetilde{\chi}(q) = 1$. With this, Theorem A becomes equivalent to

Theorem B. *The extension $\widetilde{\chi} : \mathrm{QK}_T^{\mathrm{poly}}(X) \to \Gamma$ is a ring homomorphism.*

There is also an extension of $\chi : K_T(X) \to \Gamma$ to the entire equivariant quantum K-theory ring $\mathrm{QK}_T(X)$; this extension however takes values in $\Gamma[\![q]\!]$. Denote this $\chi : \mathrm{QK}_T(X) \to \Gamma[\![q]\!]$ as well. Theorems A and B then follow from:

Theorem C. *For opposite Schubert varieties X_u, X^v in X, we have in $\mathrm{QK}_T(X)$*

$$\chi\big([\mathcal{O}_{X_u}] \star [\mathcal{O}_{X^v}]\big) \;=\; q^{\operatorname{dist}(X_u, X^v)}$$

where $\operatorname{dist}(X_u, X^v)$ is the minimal degree of a rational curve that meets X_u and X^v.

Recent work of Buch-Chaput-Mihalcea-Perrin [6] establishes a Chevalley formula for cominuscule $\mathrm{QK}_T(X)$. This is a formula for multiplication in $\mathrm{QK}_T(X)$ by a Schubert divisor class. Together with the Chevalley formula of Buch-Chaput-Mihalcea-Perrin, the aforementioned theorems help to determine a formula for the multiplication in $\mathrm{QK}_T(X)$ when X is projective space $\mathbb{P}^n$:

Theorem D. *Let the Schubert varieties X^p of $\mathbb{P}^n$ be indexed by their codimension $0 \le p \le n$. In $\mathrm{QK}_T(\mathbb{P}^n)$, set $[\mathcal{O}_{X^{n+1}}] = q$. Then for all $1 \le p \le r \le n$ we have the recursive formula*

$$\begin{aligned}[\mathcal{O}_{X^p}] \star [\mathcal{O}_{X^r}] \;=\; &(-1)^p\Big(\frac{t_{r+1}}{t_1} - 1\Big)\cdots\Big(\frac{t_{r+1}}{t_p} - 1\Big)[\mathcal{O}_{X^r}] \\ &\qquad + \sum_{i=1}^{p}(-1)^{p+i}\frac{t_{r+1}}{t_i}\Big(\frac{t_{r+1}}{t_{i+1}} - 1\Big)\cdots\Big(\frac{t_{r+1}}{t_p} - 1\Big)[\mathcal{O}_{X^{i-1}}] \star [\mathcal{O}_{X^{r+1}}].\end{aligned}$$

The ring $\Gamma = \mathbf{Z}[t_1^{\pm 1}, \ldots, t_{n+1}^{\pm 1}]$ can be identified with the representation ring of $T = (\mathbf{C}^*)^{n+1}$, the maximal torus of $G = \mathrm{GL}_{n+1}(\mathbf{C})$. Under this identification, the monomial $\frac{t_{i+1}}{t_i}$ corresponds to $e^{-\alpha_i}$, the character of the negative simple root $-\alpha_i$ of G. Theorem D illustrates that the Schubert structure constants $N_{p,r}^{s,d}$ of $\mathrm{QK}_T(\mathbb{P}^n)$ are expressible as polynomials in $e^{-\alpha_i} - 1$. It can be used further to prove the following positivity result:

Theorem E. *For $1 \le p, r, s \le n$ and $d \ge 0$, set $e = p + r + s + d(n+1)$. Then the scaled Schubert structure constant $(-1)^e N_{p,r}^{s,d} \in \Gamma$ of $\mathrm{QK}_T(\mathbb{P}^n)$ is a polynomial with nonnegative integer coefficients in the classes $e^{-\alpha_i} - 1$.*

This property is the QK_T-analogue of the positivity conjectures of Griffeth-Ram [19] for the equivariant K-theory $K_T(X)$ of a flag variety X, proven by Anderson-Griffeth-Miller [1]. Their quantum analogues are conjectured to hold for all flag varieties X, yet remain open.

Theorem D can also be used to establish the following:

Theorem F. *There exists an isomorphism of* Γ*-algebras* $\mathrm{QK}_T^{\mathrm{poly}}(\mathbb{P}^n) \to \mathrm{QK}_T^{\mathrm{poly}}(\mathbb{P}^{n+1})$.

Theorem F is reminiscent of a similar result for the quantum cohomology ring $\mathrm{QH}^*(\mathbb{P}^n)$ of $\mathbb{P}^n$. In this setting there is an isomorphism of rings $\mathrm{QH}^*(\mathbb{P}^n) \to \mathbf{Z}[h]$ where $\mathbf{Z}[h]$ is the polynomial ring in the single generator h; thus the rings $\mathrm{QH}^*(\mathbb{P}^n)$ are isomorphic for all n. In fact, this type of result has been known for both the quantum cohomology $\mathrm{QH}^*(\mathbb{P}^n)$ and quantum K-theory $\mathrm{QK}(\mathbb{P}^n)$ of projective space; it has hitherto been unknown for their equivariant versions.

The goal of this thesis is to prove Theorems A, B, C, D, E, F. It is organised into three parts: Chapter 2 presents the joint work with Buch in proving Theorems A, B, C; Chapter 3 concerns the equivariant quantum K-theory of projective space $\mathbb{P}^n$ and the demonstrations of Theorems D, E, F. Background information and notation are given in Chapter 1.

Chapter 1

Preliminaries

1.1 Flag varieties in general

Let $X = G/P$ be a flag variety defined by a connected, semisimple complex algebraic group G and a parabolic subgroup P. Fix subgroups $T \subset B \subset P \subset G$ with T a maximal torus and B a Borel subgroup. Denote by B^{op} the opposite Borel corresponding to B, the Borel subgroup of G characterised by $B \cap B^{\mathrm{op}} = T$.

Let $W = N_G(T)/T$ be the Weyl group of G, and let $R = R^+ \cup R^-$ denote the roots, positive and negative. Let Δ denote the simple roots of G. For $\alpha \in \Delta$ the simple reflections $s_\alpha \in W$ generate W. As such every $w \in W$ can be written as a product of simple reflections: $w = s_{\alpha_1} \cdots s_{\alpha_k}$; the length $\ell(w)$ is the minimal number of terms in such a factorization of w.

The parabolic subgroup P corresponds to a subset Δ_P of Δ. If P is maximal parabolic, then Δ_P comprises all but one simple root, say, α; in this case we denote the maximal P by P_α. In any case P has its own Weyl group W_P which may be identified with $N_P(T)/T$ or equivalently as the subgroup of W generated by the simple reflections $s_\beta \in W$, for $\beta \in \Delta_P$.

The torus T and the Borel subgroups B and B^{op} act on X by left translations, and these actions bear fundamental consequences on the geometry of X. The torus T has finitely-many fixed points in X—i.e., finitely many points $wP \in X$ such that $T.wP = wP$; these points correspond bijectively to the cosets of W/W_P. Each coset has a unique representative of minimal length; let W^P denote the set of these minimal length representatives.

Under the Borel actions, the orbit-closures of these T-fixed points give rise to the

Schubert varieties of X:[1]

$$X_w := \overline{B.wP}, \text{ the } B\text{-stable Schubert variety defined by } w \in W,$$

$$X^w := \overline{B^{\mathrm{op}}.wP}, \text{ the } B^{\mathrm{op}}\text{-stable Schubert variety defined by } w \in W.$$

Although they are defined for each element of the Weyl group W, Schubert varieties X_w and X^w depend only on the coset of w in W/W_P. In fact if $w \in W^P$, then $\dim X_w = \ell(w)$ and $\mathrm{codim}(X^w, X) = \ell(w)$. This is one reason why we shall henceforth index Schubert varieties by elements of W^P.

Richardson varieties are the intersections of opposite Schubert varieties $X_u \cap X^v$, provided the intersection is nonempty. Because X is itself a B- and B^{op}-stable Schubert variety, all Schubert varieties are Richardson, but the converse is generally not true. As the intersection of T-stable varieties, Richardson varieties are closed under the action of T. They are irreducible [12], [25] with $\dim X_u \cap X^v = \ell(u) - \ell(v)$, where u, $v \in W^P$. The following additional properties of Richardson varieties will be crucial to our work:

Theorem 1.1 ([23], [24], [2])**.** *Richardson varieties are Cohen-Macaulay, normal and rational with rational singularities.*[2]

1.2 Cominuscule flag varieties

Let $\theta \in R^+$ be the highest (long) root of G. Express θ in terms of the simple roots: $\theta = \sum_{\beta\in\Delta} n_\beta \beta$ where the coefficients n_β are nonnegative integers.

Definition 1.2. A simple root $\alpha \in \Delta$ is **cominuscule** if $n_\alpha = 1$. A flag variety $X = G/P$ is **cominuscule** if $P = P_\alpha$ is a maximal parabolic whose corresponding simple root α is cominuscule.

---

[1] The Schubert varieties X_w and X^w are also known as *ordinary* and *opposite* Schubert varieties respectively, referring to the type of Borel subgroup that defines them. One word of caution: pairs of Schubert varieties X_u and X^v are also referred to as opposite Schubert varieties. It should be clear from context what is meant when we use the term "opposite" Schubert varieties.

[2] A variety Y has **rational singularities** if there exists a resolution of singularities $\pi : \widetilde{Y} \to Y$ such that $\pi_* \mathcal{O}_{\widetilde{Y}} = \mathcal{O}_Y$ and $R^i\pi_*\mathcal{O}_{\widetilde{Y}} = 0$ for $i > 0$. It is a fact that if Y has rational singularities, then all of its resolutions $\pi' : \widetilde{Y}' \to Y$ satisfy this cohomological-triviality.

The cominuscule flag varieties comprise the following spaces: Grassmannians $\mathrm{Gr}(k,n)$ of type A, Lagrangian Grassmannians $\mathrm{LG}(n,2n)$, maximal orthogonal Grassmannians $\mathrm{OG}(n,2n)$, quadric hypersurfaces of $\mathbb{P}^n$ and the two exceptional spaces known as the Cayley plane and Freudenthal variety. See Table 1.1 below. Of these, the cominuscule spaces of type A are perhaps the easiest to describe:

Example 1.3 (Grassmannians of type A)**.** Let $G = \mathrm{SL}_n(\mathbf{C})$, the group of complex $n \times n$ matrices of determinant 1. Then T is the collection of diagonal matrices in $\mathrm{SL}_n(\mathbf{C})$, B the upper triangular matrices and B^{op} the lower triangular matrices. The torus T naturally embeds into $(\mathbf{C}^*)^n$, but its rank is one smaller: $T \simeq (\mathbf{C}^*)^{n-1}$. The corresponding root system is type A_{n-1}.

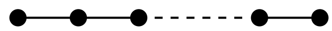
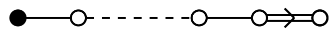
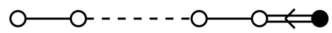
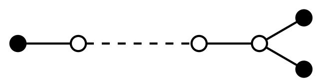
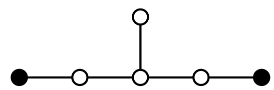
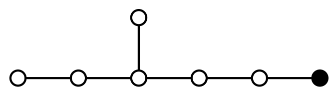
For each $1 \leq i \leq n$ let $\varepsilon_i : (\mathbf{C}^*)^n \to \mathbf{C}^*$ be projection onto the i^{th} component, and consider their restrictions to $T \subset (\mathbf{C}^*)^n$. Then the simple roots of G are $\alpha_i = \frac{\varepsilon_i}{\varepsilon_{i+1}}$ (or $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ when written additively) for $1 \leq i \leq n-1$. The highest root is precisely $\theta = \alpha_1 + \cdots + \alpha_{n-1}$ when written additively. Thus every simple root is cominuscule in type A.

The normaliser $N_G(T)$ is the collection of monomial matrices in $\mathrm{SL}_n(\mathbf{C})$. Consequently the Weyl group W is isomorphic to the permutation group S_n. The simple reflection s_{α_i} is the permutation that transposes i and $i+1$. If P_{α_k} is a maximal parabolic, then $W_{P_{\alpha_k}}$ is the subgroup of S_n generated by all transpositions $(i, i+1)$ for $i \neq k$; it can be identified with $S_k \times S_{n-k}$. The collection W^P of minimal length coset representatives of $W/W_{P_{\alpha_k}}$ then comprises all permutations $w \in S_n$ satisfying

$$w(1) < w(2) < \cdots < w(k) \quad \text{and} \quad w(k+1) < \cdots < w(n).$$

As observed earlier, all of the flag varieties $X = G/P_\alpha$ given by maximal parabolics P_α are cominuscule. If $\alpha = \alpha_k$, then $P_{\alpha_k} = BW_{P_{\alpha_k}}B$ is the stabiliser of a k-dimensional subspace of $\mathbf{C}^n$ under the natural action of G on $\mathbf{C}^n$—specifically the subspace spanned by the first k standard basis vectors of $\mathbf{C}^n$. Therefore X is $\mathrm{Gr}(k,n)$ the Grassmannian of k-planes in $\mathbf{C}^n$. In particular X is $\mathbb{P}^{n-1}$ or its dual projective space $(\mathbb{P}^*)^{n-1}$ when $k = 1$ or $n-1$.

Table 1.1: Dynkin diagrams with cominuscule roots filled-in

Type	Dynkin diagram	Flag varieties
A_n $(n \geq 1)$		$\mathrm{Gr}(k, n+1)$
B_n $(n \geq 2)$		Odd quadric hypersurfaces
C_n $(n \geq 3)$		$\mathrm{LG}(n, 2n)$
D_n $(n \geq 4)$		Even quadric hypersurfaces, $\mathrm{OG}(n, 2n)$
E_6		Cayley plane E_6/P_6
E_7		Freudenthal variety E_7/P_7

We close this section with some remarks on the singular homology of a general flag variety $X = G/P$. Each Schubert variety $X_w \subset X$ defines a Borel-Moore homology class $[X_w] \in H_*(X, \mathbf{Z})$ in singular homology; these Schubert classes in turn form an additive basis:

$$H_*(X, \mathbf{Z}) \;=\; \bigoplus_{w \in W^P} \mathbf{Z}[X_w].$$

In particular the (complex) one-dimensional Schubert classes generate $H_2(X, \mathbf{Z})$. Since these Schubert curves X_{s_β} correspond to the simple roots $\beta \in \Delta \smallsetminus \Delta_P$, we can identify

$$H_2(X, \mathbf{Z}) \;\simeq\; \bigoplus_{\beta \in \Delta \smallsetminus \Delta_P} \mathbf{Z}[X_{s_\beta}].$$

Note that when P_α is maximal parabolic, $H_2(X, \mathbf{Z}) \simeq \mathbf{Z}[X_{s_\alpha}] \simeq \mathbf{Z}$.

Elements of $H_2(X, \mathbf{Z})$ are called **(curve) degrees**; a degree d in $H_2(X, \mathbf{Z})$ is **effective** if, when d is expressed $d = \sum n_\beta [X_{s_\beta}]$, each of the coefficients n_β is nonnegative.

1.3 Equivariant K-theory

In this section we present a brief account of equivariant K-theory for a smooth projective complex variety X. For a detailed account, we refer the reader to the textbook of Chriss-Ginzburg [9] and the references therein.

Let $K^T(X)$ be the Grothendieck group of T-equivariant algebraic vector bundles E on X. The tensor product of vector bundles makes $K^T(X)$ a commutative, associative ring with 1, the multiplicative identity 1 given by the class of the trivial line bundle.

There is also $K_T(X)$, the Grothendieck group of T-equivariant coherent sheaves F on X. Connecting these two Grothendieck groups is a natural inclusion of groups $K^T(X) \to K_T(X)$ defined by $[E] \mapsto [\mathscr{E}]$, where $\mathscr{E}$ is the sheaf of sections of the vector bundle $E \to X$. In general this map is not an isomorphism, but it is when X is smooth. In this case there is an inverse map $K_T(X) \to K^T(X)$ defined by $[F] \mapsto \sum_i (-1)^i [E_i]$, where

$$0 \;\to\; \mathscr{E}_n \;\to\; \mathscr{E}_{n-1} \;\to\; \cdots \;\to\; \mathscr{E}_1 \;\to\; \mathscr{E}_0 \;\to\; F \;\to\; 0$$

is a resolution of the coherent sheaf F, with $\mathscr{E}_i$ a locally free sheaf corresponding to the vector bundle E_i. The smoothness of X guarantees, for each coherent sheaf F, the existence of such a (finite) locally free resolution. With this isomorphism we identify the groups $K^T(X)$ and $K_T(X)$; since the former is also a ring, $K_T(X)$ is a ring as well, with the ring structure inherited from $K^T(X)$ under the identification.

Let $f : X \to Y$ be a T-equivariant morphism of smooth projective T-varieties over $\mathbf{C}$. The pullback of equivariant vector bundles extends to a ring homomorphism $f^* : K_T(Y) \to K_T(X)$. Thus $K_T(X)$ is a $K_T(Y)$-algebra. As an immediate consequence we have that $K_T(X)$ is always a $K_T(\mathrm{pt})$-algebra, where $\mathrm{pt} = \operatorname{Spec} \mathbf{C}$. Indeed this follows from the fact that X comes equipped with a canonical structure morphism $X \to \mathrm{pt}$, pulling back over which yields the ring homomorphism $K_T(\mathrm{pt}) \to K_T(X)$.

Since X and Y are projective over $\mathbf{C}$, the morphism f is proper. Thus equivariant coherent sheaves F on X can be pushed forward to equivariant coherent sheaves f_*F on Y; the derived pushforwards $R^i f_* F$ are also coherent and T-equivariant on Y. All of this comes together to produce the pushforward map $f_* : K_T(X) \to K_T(Y)$ defined by

$$[F] \;\mapsto\; \sum_{i \geq 0} (-1)^i [R^i f_* F].$$

These operations are functorial as well: given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have $(gf)^* = f^* g^*$ and $(gf)_* = g_* f_*$. The projection formula also holds: given a proper

T-equivariant morphism $f : X \to Y$ and classes $[F] \in K_T(Y)$, $[G] \in K_T(X)$, we have $f_*\big(f^*[F] \cdot [G]\big) = [F] \cdot f_*[G]$. In particular this means f_* is a $K_T(Y)$-module homomorphism.

Example 1.4. Let $X = \mathrm{pt}$. Then a T-equivariant vector bundle on X is precisely a finite-dimensional $\mathbf{C}$-linear representation of T. Thus $K_T(\mathrm{pt})$ is the representation ring of T.

Notation. $\Gamma := K_T(\mathrm{pt})$.

Fix an identification of T with $(\mathbf{C}^*)^n$. By the semisimplicity of T, the irreducible representations of T form a $\mathbf{Z}$-basis of Γ; since T is abelian, these are precisely the one-dimensional representations of T. These representations in turn can be identified with characters $\alpha : T \to \mathbf{C}^*$, i.e., the one-dimensional T-representation $\mathbf{C}_\alpha$ defined by $t \cdot z = \alpha(t)z$ for $t \in T$ and $z \in \mathbf{C}$. Thus as an abelian group we have

$$\Gamma = \bigoplus_{\text{characters } \alpha} \mathbf{Z}e^\alpha.$$

On the other hand the characters of T form a lattice themselves, with a basis consisting of the projection maps $\varepsilon_i : T \to \mathbf{C}^*$ defined by $(a_1, \dots, a_n) \mapsto a_i$. Thus as a ring, Γ is the Laurent polynomial ring $\mathbf{Z}[e^{\pm\varepsilon_1}, \dots, e^{\pm\varepsilon_n}]$. We shall simply identify Γ with $\mathbf{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ by $t_i \leftrightarrow e^{\varepsilon_i}$.[3]

Example 1.5. Let $X = G/P$ be a flag variety, and consider its opposite Schubert varieties X^w, $w \in W^P$. Since we have a closed immersion $i : X^w \hookrightarrow X$, the structure sheaf $\mathcal{O}_{X^w}$ pushes forward to a (T-equivariant) coherent sheaf on X, and the higher direct images $R^p i_* \mathcal{O}_{X^w}$ vanish. Thus pushing forward along this inclusion produces the opposite Schubert class $[\mathcal{O}_{X^w}] \in K_T(X)$. These opposite Schubert classes $[\mathcal{O}_{X^w}]$ collectively form a basis for $K_T(X)$ over Γ.

The same can be said about the ordinary Schubert classes $[\mathcal{O}_{X_w}] \in K_T(X)$, $w \in W^P$. We shall make use of both bases in the sequel.

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[3] In other words Γ is the group ring of the character lattice $X(T)$ of T. Thus each character $\alpha : T \to \mathbf{C}^*$ is formally an element of Γ; we follow standard convention and denote by e^α the element in Γ corresponding to α. This exponential notation has the benefit of distinguishing the additive group structure of Γ—a purely formal operation—from the ring structure of Γ, which is inherited from the group structure of $X(T)$.

Notation. For $w \in W^P$, denote $[\mathcal{O}^w] := [\mathcal{O}_{X^w}]$ and $[\mathcal{O}_w] := [\mathcal{O}_{X_w}]$ in $K_T(X)$.

Example 1.6. Because the opposite Schubert classes form a Γ-basis for $K_T(X)$, for each $u, v \in W^P$ there exist unique $K_{u,v}^w \in \Gamma$ such that

$$[\mathcal{O}^u] \cdot [\mathcal{O}^v] \;=\; \sum_w K_{u,v}^w [\mathcal{O}^w].$$

These coefficients $K_{u,v}^w$ are the (opposite) Schubert structure constants of $K_T(X)$. They satisfy the following "positivity" property, as conjectured by Griffeth-Ram [19] and proven by Anderson-Griffeth-Miller [1]:

$$(-1)^{\ell(w)-\ell(u)-\ell(v)} K_{u,v}^w \;\in\; \mathbf{Z}_{\geq 0}[e^{-\alpha_i} - 1]_{\alpha_i \in \Delta}.$$

In others words, positivity asserts that the Schubert structure constants $K_{u,v}^w$ are, up to a predictable sign, polynomials in $e^{-\alpha_i} - 1$ with nonnegative integer coefficients, where the $-\alpha_i$ are the negatives of simple roots.

Example 1.7. When $X = \mathrm{Gr}(k, n)$ the Schubert structure constants $K_{u,v}^w$ can be computed combinatorially by using the *genomic tableaux* and *genomic jeu de taquin* of Pechenik-Yong [22].

Definition 1.8. The **sheaf Euler characteristic $\boldsymbol{\chi}$** of X is the map $\chi : K_T(X) \to \Gamma$ defined by pushforward along the structure morphism $X \to \mathrm{pt}$. On generators $[F] \in K_T(X)$ it is defined by

$$[F] \;\mapsto\; \sum_i (-1)^i [H^i(X, F)],$$

where $H^i(X, F)$ is the i^{th} cohomology group of F (which is a T-module). It is Γ-linear by the projection formula.

For an irreducible projective variety X, recall that X is **unirational** if there exists a dominant rational map $\mathbb{P}^k \dashrightarrow X$ from some projective space $\mathbb{P}^k$. Also recall that X is **rationally connected** if any two general points $x, y \in X$ are connected by an irreducible rational curve.[4] Rational varieties are unirational, and projective unirational varieties are in turn rationally connected.

[4]This means that there exists a nonempty open subset U of X such that for all distinct points x, y of U, both x and y lie in the image of a morphism $\mathbb{P}^1 \to X$.

Proposition 1.9. *Let X be a smooth, projective T-variety. If X is unirational, then $\chi([\mathcal{O}_X]) = 1$ in Γ.*

Proof. For such a variety X, all of the higher cohomology groups $H^i(X, \mathcal{O}_X)$ vanish. (In fact this holds more generally for *rationally connected* X—see [10, Cor. 4.18] for a proof involving Hodge theory.) Thus $\chi([\mathcal{O}_X])$ equals $[H^0(X, \mathcal{O}_X)]$, which, as an element of Γ, is formally a Laurent polynomial in characters of T. To identify this polynomial, we must identify the action of T on $H^0(X, \mathcal{O}_X)$.

The zeroth cohomology group $H^0(X, \mathcal{O}_X)$ coincides with the global sections of the structure sheaf $\Gamma(X, \mathcal{O}_X)$. Because X is irreducible (by smoothness) and projective, we have $\Gamma(X, \mathcal{O}_X) = \mathbf{C}$—i.e., the global regular functions on X are all constant. Consequently the T-action on $\Gamma(X, \mathcal{O}_X)$ is trivial. Thus as an element of Γ, $[H^0(X, \mathcal{O}_X)]$ is the class of the trivial character of T, i.e., $\chi([\mathcal{O}_X]) = 1$. ■

Proposition 1.9 extends to unirational varieties that are singular, provided the singularities are well-behaved (e.g., rational singularities). For this to hold equivariantly, a singular T-variety must admit a T-equivariant resolution of singularities (defined below in the proof of Corollary 1.10). This is always the case over $\mathbf{C}$—see [27]. For a singular Schubert variety, its corresponding Bott-Samelson resolution is T-equivariant [11], [3].

Corollary 1.10. *Let X be a smooth, projective T-variety. Let Y be a closed, T-stable subvariety of X. If Y is unirational with rational singularities, then $\chi([\mathcal{O}_Y]) = 1$.*

Proof. Let $\pi : \widetilde{Y} \to Y$ be a T-equivariant resolution of singularities, meaning $\widetilde{Y}$ is a smooth, projective T-variety and π proper, birational and T-equivariant. As unirationality is preserved under such π, we have $\widetilde{Y}$ is unirational. Because Y has rational singularities, $[\mathcal{O}_Y] = \pi_*[\mathcal{O}_{\widetilde{Y}}]$ in $K_T(X)$; what's more, by functoriality of pushforwards,

$$\chi([\mathcal{O}_Y]) \;=\; \chi(\pi_*[\mathcal{O}_{\widetilde{Y}}]) \;=\; \chi_{\widetilde{Y}}([\mathcal{O}_{\widetilde{Y}}]),$$

where $\chi_{\widetilde{Y}} : K_T(\widetilde{Y}) \to \Gamma$ is the sheaf Euler characteristic on $\widetilde{Y}$. Thus the corollary reduces to the computation of $\chi_{\widetilde{Y}}([\mathcal{O}_{\widetilde{Y}}])$ for *smooth*, projective, unirational $\widetilde{Y}$. This is the previous Proposition 1.9. ■

Example 1.11. Because Schubert varieties are rational with rational singularities (Theorem 1.1), Corollary 1.10 says that $\chi([\mathcal{O}^w]) = \chi([\mathcal{O}_w]) = 1$ for all $w \in W^P$. Thus the sheaf Euler characteristic on X can be characterised as the Γ-linear map that evaluates all Schubert classes at 1. More generally $\chi([\mathcal{O}_{X_u \cap X^v}]) = 1$ whenever $X_u \cap X^v$ is nonempty because Richardson varieties are also rational with rational singularities.

It is a standard result that $[\mathcal{O}_u] \cdot [\mathcal{O}^v] = [\mathcal{O}_{X_u \cap X^v}]$ in $K_T(X)$.[5] Therefore by the previous remarks we have

$$\chi([\mathcal{O}_u] \cdot [\mathcal{O}^v]) = \begin{cases} 1 & \text{if } X_u \cap X^v \text{ is nonempty,} \\ 0 & \text{otherwise.} \end{cases}$$

This example shows two things: 1) For $X \neq \mathrm{pt}$, the sheaf Euler characteristic $\chi : K_T(X) \to \Gamma$ is never a ring homomorphism; 2) As both types of Schubert classes $[\mathcal{O}_w]$ and $[\mathcal{O}^w]$ form a Γ-basis for $K_T(X)$, the sheaf Euler characteristic defines a non-degenerate pairing on $K_T(X)$:

$$\begin{aligned} K_T(X) \times K_T(X) &\to \Gamma \\ ([F], [F']) &\mapsto \chi([F] \cdot [F']). \end{aligned}$$

Notation. For an opposite Schubert class $[\mathcal{O}^w]$, denote by $[\mathcal{O}^w]^\vee$ its dual under this pairing, i.e., the unique element $[\mathcal{O}^w]^\vee \in K_T(X)$ satisfying $\chi([\mathcal{O}^w] \cdot [\mathcal{O}^u]^\vee) = \delta_{wu}$. Similarly denote by $[\mathcal{O}_w]^\vee$ the dual of $[\mathcal{O}_w]$.

1.4 Equivariant quantum K-theory

Here we sketch a construction of the (small) torus-equivariant quantum K-theory ring $\mathrm{QK}_T(X)$ by using moduli spaces of stable maps into X. Our focus will be on $\mathrm{QK}_T(X)$ for cominuscule X, although everything we describe in this section holds more generally for all complex flag varieties. We refer the reader to the work of Lee [21] for a thorough

[5] Indeed, let X be a smooth, projective T-variety, and let Y, Z be closed, irreducible, T-stable subvarieties. Suppose Y and Z are Cohen-Macaulay. If $\mathrm{codim}(Y \cap Z, X) = \mathrm{codim}(Y, X) + \mathrm{codim}(Z, X)$, then $Y \cap Z$ is Cohen-Macaulay, and $[\mathcal{O}_Y] \cdot [\mathcal{O}_Z] = [\mathcal{O}_{Y \cap Z}]$ in $K_T(X)$. This can be proved using a local computation of regular sequences and their Koszul resolutions. One may also appeal to the results of S. J. Sierra [26] for more sophisticated techniques.

account of quantum K-theory, and to the notes of Fulton-Pandharipande [14] for details on moduli of stable maps into X.

Let $X = G/P_\alpha$ be a cominuscule flag variety. Recall that for such an X, the second homology group $H_2(X, \mathbf{Z})$ can be identified with $\mathbf{Z}$ by means of the identification $H_2(X, \mathbf{Z}) \simeq \mathbf{Z}[X_{s_\alpha}]$. Given an effective degree $d \in H_2(X, \mathbf{Z})$ and an integer $N \geq 0$, let $\overline{M}_{0,N}(X, d)$ denote the (Kontsevich) moduli space of N-pointed, genus-0, degree-d stable maps into X. The elements of $\overline{M}_{0,N}(X, d)$ are isomorphism classes of tuples $(f, C, p_1, \ldots, p_N)$ such that:

(1) C is a union of finitely many $\mathbb{P}^1$'s with at worst nodal singularities;

(2) The $p_1, \ldots, p_N$ are nonsingular points in C called marked points;

(3) The morphism $f : C \to X$ satisfies $f_*[C] = d$ in $H_2(X, \mathbf{Z})$;

(4) The morphism f is "stable": a component of C maps to a single point in X only if this component has at least three special points, i.e., nodes or marked points.

Note that the image of a stable map is a rational curve in X, possibly reducible.

For $1 \leq i \leq N$, denote by $\mathrm{ev}_i : \overline{M}_{0,N}(X, d) \to X$ the i^{th} evaluation map, which is defined by $\mathrm{ev}_i(f, C, p_1, \ldots, p_N) = f(p_i)$. We record the following facts about $\overline{M}_{0,N}(X, d)$ and its evaluation maps for cominuscule X:

Theorem 1.12 ([14], [20])**.** *The Kontsevich space $\overline{M}_{0,N}(X, d)$ is an irreducible, projective, normal, rational variety over $\mathbf{C}$. It has dimension $\dim(X) + \int_d c_1(T_X) + N - 3$ where $c_1(T_X)$ is the first Chern class of the tangent bundle T_X of X.*[6] *The evaluation maps $\mathrm{ev}_i : \overline{M}_{0,N}(X, d) \to X$ are flat and proper.*

The T-action of X naturally extends to an action on $\overline{M}_{0,N}(X, d)$: the torus T acts on a stable map by acting on its image in X. The evaluation maps ev_i are then compatible with this action. Therefore we have $K_T\big(\overline{M}_{0,N}(X, d)\big)$, the T-equivariant K-theory of coherent sheaves on $\overline{M}_{0,N}(X, d)$.[7]

[6] The integral $\int_d c_1(T_X)$ means the degree of the zero-cycle obtained from the pairing $d \cdot c_1(T_X) := d[X_{s_\alpha}] \cdot c_1(T_X)$ in $H_*(X, \mathbf{Z})$.

[7] Because $\overline{M}_{0,N}(X, d)$ is not smooth, $K_T\big(\overline{M}_{0,N}(X, d)\big)$ can not be identified with $K^T\big(\overline{M}_{0,N}(X, d)\big)$. This will not present any problems for us.

Definition 1.13. Let $\sigma_1, \ldots, \sigma_N \in K_T(X)$. Their ***(N*-point, degree-*d*) *K*-theoretic Gromov-Witten invariant** is the Laurent polynomial $I_d(\sigma_1, \ldots, \sigma_N) \in \Gamma$ defined by

$$I_d(\sigma_1, \ldots, \sigma_N) \;=\; \chi_{\overline{M}_{0,N}(X,d)}\big(\operatorname{ev}_1^*\sigma_1 \cdots \operatorname{ev}_N^*\sigma_N\big),$$

where $\chi_{\overline{M}_{0,N}(X,d)} : \overline{M}_{0,N}(X,d) \to \Gamma$ is the Euler characteristic map on $\overline{M}_{0,N}(X,d)$.[8]

Of particular interest to us are the $I_d\big([\mathcal{O}^u], [\mathcal{O}^v], [\mathcal{O}^w]^\vee\big) \in \Gamma$, the three-point K-theoretic Gromov-Witten invariants of $[\mathcal{O}^u]$, $[\mathcal{O}^v]$, $[\mathcal{O}^w]^\vee \in K_T(X)$ of various degrees d. Because the evaluation maps ev_i are flat, the pullbacks of Schubert classes $\operatorname{ev}_i^*[\mathcal{O}^w]$ coincide with the K_T-classes of the closed subvarieties $\operatorname{ev}_i^{-1}(X^w)$ in $\overline{M}_{0,N}(X,d)$. Thus the invariants $I_d\big([\mathcal{O}^u], [\mathcal{O}^v], [\mathcal{O}^w]^\vee\big)$ capture information about the families of rational curves meeting Schubert varieties X^u, X^v (and X^w).

We may now describe the construction of the (small) torus-equivariant quantum K-theory ring $\mathrm{QK}_T(X)$. Let $\Gamma[\![q]\!]$ denote the ring of formal power series over Γ in the single parameter q. Then $\mathrm{QK}_T(X)$ is an algebra over $\Gamma[\![q]\!]$ whose underlying abelian group is defined to be $K_T(X) \otimes_\Gamma \Gamma[\![q]\!]$. Consequently, $\mathrm{QK}_T(X)$ is free as a $\Gamma[\![q]\!]$-module with the opposite Schubert classes $[\mathcal{O}^w]$ forming a basis, where $w \in W^P$; the ordinary Schubert classes $[\mathcal{O}_w]$ also form a basis.

The ring structure $\star$ of $\mathrm{QK}_T(X)$ is *not* induced by the tensor product of $K_T(X)$ and $\Gamma[\![q]\!]$. Instead, it is the $\Gamma[\![q]\!]$-bilinear extension of the following operation on opposite Schubert classes:

$$[\mathcal{O}^u] \star [\mathcal{O}^v] \;:=\; \sum_{\substack{w \in W^P \\ d \geq 0}} N_{u,v}^{w,d} q^d [\mathcal{O}^w].$$

For each $u, v, w \in W^P$ and $d \in H_2(X, \mathbf{Z})$, the structure constant $N_{u,v}^{w,d}$ is the following recursively defined element of Γ involving three-point and two-point K-theoretic Gromov-Witten invariants:

$$N_{u,v}^{w,d} \;=\; I_d\big([\mathcal{O}^u], [\mathcal{O}^v], [\mathcal{O}^w]^\vee\big) \;-\; \sum_{\substack{r \in W^P \\ 0 < e \leq d}} N_{u,v}^{r,d-e} I_e\big([\mathcal{O}^r], [\mathcal{O}^w]^\vee\big).$$

[8]Even though $K_T\big(\overline{M}_{0,N}(X,d)\big)$ can not be identified with $K^T\big(\overline{M}_{0,N}(X,d)\big)$, the pullbacks $\operatorname{ev}_i^*\sigma$ still exist: since X is smooth, we identify $\sigma \in K_T(X)$ with vector bundles on X; these vector bundles pull back to vector bundles on $\overline{M}_{0,N}(X,d)$, which in turn define classes in $K_T\big(\overline{M}_{0,N}(X,d)\big)$.

Thus we obtain the ring structure of $\mathrm{QK}_T(X)$.

If equivariant quantum K-theory were exactly like, say, the quantum cohomology $\mathrm{QH}^*(X)$ of X, then the K-theoretic Gromov-Witten invariant $I_d\big([\mathcal{O}^u],[\mathcal{O}^v],[\mathcal{O}^w]^\vee\big)$ itself would be the structure constant $N_{u,v}^{w,d}$. This however is not the case: for $d \neq 0$, we generally have $N_{u,v}^{w,d} \neq I_d\big([\mathcal{O}^u],[\mathcal{O}^v],[\mathcal{O}^w]^\vee\big)$. The additional term $\sum_{r,e} N_{u,v}^{r,d-e} I_e\big([\mathcal{O}^r],[\mathcal{O}^w]^\vee\big)$ arises in the structure constant $N_{u,v}^{w,d}$ as a type of correction term. It ensures the following result:

Theorem 1.14 ([16])**.** *The quantum product $\star$ of $\mathrm{QK}_T(X)$ is associative.*

Thus $\mathrm{QK}_T(X)$ is a commutative, associative $\Gamma[\![q]\!]$-algebra with 1, free as a module over $\Gamma[\![q]\!]$.

Example 1.15. In the special case of $d = 0$, the structure constant $N_{u,v}^{w,0}$ of $[\mathcal{O}^u] \star [\mathcal{O}^v]$ equals $I_0\big([\mathcal{O}^u],[\mathcal{O}^v],[\mathcal{O}^w]^\vee\big)$ by definition. This in turn is the Schubert structure constant $K_{u,v}^w$ of $[\mathcal{O}^u] \cdot [\mathcal{O}^v]$ in $K_T(X)$. (See Example 1.6 for the notation.)

To see this, recall the definition of $I_0\big([\mathcal{O}^u],[\mathcal{O}^v],[\mathcal{O}^w]^\vee\big)$ as $\chi_{\overline{M}_{0,3}(X,0)}\big(\mathrm{ev}_1^*[\mathcal{O}^u] \cdot \mathrm{ev}_2^*[\mathcal{O}^v] \cdot \mathrm{ev}_3^*[\mathcal{O}^w]^\vee\big)$. Since a degree-zero stable map into X must be constant, we have $\overline{M}_{0,3}(X,0) = \overline{M}_{0,3} \times X$, where $\overline{M}_{0,3}$ is the Deligne-Mumford space of stable rational curves with three marked points. Since $\overline{M}_{0,3} = \mathrm{pt}$, we have $\overline{M}_{0,3}(X,0) = X$, and each evaluation map ev_i is the identity map on X. Thus $I_0\big([\mathcal{O}^u],[\mathcal{O}^v],[\mathcal{O}^w]^\vee\big) = \chi_X\big([\mathcal{O}^u] \cdot [\mathcal{O}^v] \cdot [\mathcal{O}^w]^\vee\big)$. By definition of χ and $[\mathcal{O}^w]^\vee$, this is precisely $K_{u,v}^w$.

Example 1.15 shows that $\mathrm{QK}_T(X)$ is a formal deformation ring of $K_T(X)$. That is, $\mathrm{QK}_T(X)$ is a ring with a grading such that $K_T(X)$ is contained in the zeroth-degree component; furthermore, for $\sigma, \tau \in K_T(X)$, the degree-zero term of $\sigma \star \tau$ in $\mathrm{QK}_T(X)$ is precisely the product $\sigma \cdot \tau$ in $K_T(X)$.

Even though the K-theoretic Gromov-Witten invariants $I_d\big([\mathcal{O}^u],[\mathcal{O}^v],[\mathcal{O}^w]^\vee\big)$ are not the structure constants $N_{u,v}^{w,d}$ of $\mathrm{QK}_T(X)$, it is still worthwhile to retain the following pairing on $\mathrm{QK}_T(X)$: for $\sigma, \tau \in K_T(X)$ define

$$\sigma \odot \tau \;=\; \sum_{\substack{w \in W^P \\ d \geq 0}} I_d(\sigma, \tau, [\mathcal{O}^w]^\vee)[\mathcal{O}^w]$$

and extend $\Gamma[\![q]\!]$-bilinearly to the rest of $\mathrm{QK}_T(X)$.

Chapter 2

Sums of Schubert structure constants

The motivation for this chapter's work is the following observation: consider $\mathrm{QK}_T(X)$ for X, say, a Grassmannian of type A. Then for all Schubert classes $[\mathcal{O}^u], [\mathcal{O}^v] \in \mathrm{QK}_T(X)$, if $[\mathcal{O}^u] \star [\mathcal{O}^v] = \sum_{w,d} N_{u,v}^{w,d} q^d [\mathcal{O}^w]$, then $\sum_{w,d} N_{u,v}^{w,d} = 1$ in Γ.

For example when $X = \mathrm{Gr}(2,4)$, where the Schubert varieties may be parametrised by *Young diagrams* of the 2×2 rectangle, and $\Gamma = \mathbf{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}]$,

$$\begin{aligned} [\mathcal{O}^{\yng(2,1)}] \star [\mathcal{O}^{\yng(2)}] \quad &= \quad \left(1 - \frac{t_4}{t_1} - \frac{t_4}{t_3} + \frac{t_4^2}{t_1 t_3}\right) [\mathcal{O}^{\yng(2,1)}] \ + \ \left(\frac{t_4}{t_3} - \frac{t_4^2}{t_1 t_3}\right) [\mathcal{O}^{\yng(2,2)}] \\ &\quad + \ q \left(\frac{t_4}{t_1} - \frac{t_4^2}{t_1 t_3}\right) [\mathcal{O}^{\varnothing}] \ + \ q \frac{t_4^2}{t_1 t_3} [\mathcal{O}^{\yng(1)}] \end{aligned}$$

and $\left(1 - \frac{t_4}{t_1} - \frac{t_4}{t_3} + \frac{t_4^2}{t_1 t_3}\right) + \left(\frac{t_4}{t_3} - \frac{t_4^2}{t_1 t_3}\right) + \left(\frac{t_4}{t_1} - \frac{t_4^2}{t_1 t_3}\right) + \frac{t_4^2}{t_1 t_3} = 1$.

This phenomenon translates to a statement about the sheaf Euler characteristic $\chi : K_T(X) \to \Gamma$. We prove this statement for cominuscule flag varieties: $X = G/P$ where $P = P_\alpha$ is a maximal parabolic corresponding to a cominuscule root α.

The proof in the cominuscule case follows from a characterisation of the ring structure of $\mathrm{QK}_T(X)$, due to Buch-Chaput-Mihalcea-Perrin [6]. This characterisation, in turn, rests on a 'quantum equals classical' result proved by the same authors [5]. For cominuscule X, this 'quantum equal classical' result relates the K-theoretic Gromov-Witten invariants $I_d\big([\mathcal{O}_u], [\mathcal{O}^v], [\mathcal{O}^w]^\vee\big)$ of opposite Schubert varieties X_u, X^v to the K-theory of their (generalised) curve neighborhoods. Introduced by Fulton-Woodward [15] and studied extensively by Buch-Chaput-Mihalcea-Perrin [4], [5] and Buch-Mihalcea [8], these curve neighborhoods have proven to be extremely valuable in understanding the quantum theory of flag varieties.

2.1 Curve neighborhoods

Let $X = G/P_\alpha$ be a cominuscule flag variety, and let Y be a Schubert variety of X, ordinary or opposite. Given an effective degree $d \in H_2(X, \mathbf{Z})$, the degree-d **curve neighborhood** of Y is defined to be the union of all degree-d (reducible) rational curves in X that meet Y. We denote this curve neighborhood by $\Gamma_d(Y)$. Alternatively $\Gamma_d(Y)$ can be defined using moduli of stable maps: $\Gamma_d(Y) = \mathrm{ev}_2(\mathrm{ev}_1^{-1}(Y))$ where $\mathrm{ev}_i : \overline{M}_{0,2}(X, d) \to X$ are the evaluation maps. Thus $\Gamma_d(Y)$ is the image of the one-point Gromov-Witten variety $\mathrm{ev}_1^{-1}(Y)$: the locus of genus-0, degree-d stable maps into X whose first marked point maps into Y.

Example 2.1. Any two points of $\mathbb{P}^n$ are connected by a $\mathbb{P}^1$, ie. a degree $d = 1$ curve; thus for all Schubert varieties Y, the line neighborhood $\Gamma_1(Y)$ is precisely all of $\mathbb{P}^n$. In fact we have in general that $\Gamma_d(Y) \subset \Gamma_{d+1}(Y)$. Thus $\Gamma_d(Y) = \mathbb{P}^n$ for all $d > 0$.

Recall that B is the Borel subgroup of X, and B^{op} the opposite Borel. If Y is B-stable, then so is $\Gamma_d(Y)$: for if C is any degree-d rational curve passing through Y, then so is $b.C$—for $b \in B$—as each $b.C$ passes through $b.Y = Y$. The same argument holds for B^{op}-stable Y. Remarkably, the following is also true:

Theorem 2.2 ([4])**.** *Let Y be a Schubert variety of X, ordinary or opposite. Then $\Gamma_d(Y)$ is rationally connected. In particular it is irreducible.*

Proof. The map $\mathrm{ev}_1 : \overline{M}_{0,2}(X, d) \to X$ is a locally trivial fibration [4, Prop. 2.3]; together with the rationality of $\overline{M}_{0,2}(X, d)$ (Theorem 1.12) this can be used to deduce that the fibers of ev_1 are unirational. But since $\overline{M}_{0,2}(X, d)$ is also projective, unirationality and projectiveness imply that the fibers of ev_1 are rationally connected.

Now consider the restriction $\mathrm{ev}_1 : \mathrm{ev}_1^{-1}(Y) \to Y$. This restriction is still a locally trivial fibration, which means $\mathrm{ev}_1^{-1}(Y)$ is birational to $Y \times \mathrm{ev}_1^{-1}(y)$, where $\mathrm{ev}_1^{-1}(y)$ is the fiber of some general point $y \in Y$. This product $Y \times \mathrm{ev}_1^{-1}(y)$ is rationally connected since each factor Y and $\mathrm{ev}_1^{-1}(y)$ is. Thus $\mathrm{ev}_1^{-1}(Y)$ is also rationally connected, hence $\Gamma_d(Y) = \mathrm{ev}_2(\mathrm{ev}_1^{-1}(Y))$ as well. As rationally connected varieties are irreducible, the result follows. ■

Corollary 2.3. *$\Gamma_d(Y)$ is a B-stable (or B^{op}-stable) Schubert variety whenever Y is.*

Proof. Every closed, irreducible, B-stable (or B^{op}-stable) subvariety of X is a Schubert variety. ∎

In view of these results, it is natural to try to identify the Schubert variety $\Gamma_d(Y)$ in terms of W^P, the set of minimal length coset representatives of W/W_P. For all $w \in W^P$ and for all effective $d \in H_2(X, \mathbf{Z})$, define $w(d)$, $w(-d) \in W^P$ by

$$\begin{aligned} X_{w(d)} &= \Gamma_d(X_w), \\ X^{w(-d)} &= \Gamma_d(X^w). \end{aligned}$$

Both $w(d)$, $w(-d)$ can be identified using a Hecke product on W/W_P; recent work has shed light on the combinatorics of this Hecke product apropos this geometry of X [8].

Curve neighborhoods pertain to a single Schubert variety, but the notion extends to any number of varieties. Our interest is particularly in pairs of opposite Schubert varieties:

Definition 2.4. Given an effective $d \in H_2(X, \mathbf{Z})$ and opposite Schubert varieties X_u, X^v, let $\Gamma_d(X_u, X^v)$ be the union of all degree-d (reducible) rational curves in X meeting both X_u and X^v.

Like $\Gamma_d(Y)$, this $\Gamma_d(X_u, X^v)$ is a **projected Gromov-Witten variety** [5] as it is the image of a Gromov-Witten variety: $\Gamma_d(X_u, X^v) = \mathrm{ev}_3\left(\mathrm{ev}_1^{-1}(X_u) \cap \mathrm{ev}_2^{-1}(X^v)\right)$ where $\mathrm{ev}_i : \overline{M}_{0,3}(X, d) \to X$ are the evaluation maps.

Notation. For future use let $\mathrm{GW}_d(X_u, X^v)$ denote $\mathrm{ev}_1^{-1}(X_u) \cap \mathrm{ev}_2^{-1}(X^v)$. It is the subvariety of $\overline{M}_{0,3}(X, d)$ consisting of all genus-0, degree-d stable maps whose first two marked points map into X_u and X^v respectively.

The curve neighborhoods $\Gamma_d(X_w)$, $\Gamma_d(X^w)$ are Schubert varieties, so how are projected Gromov-Witten varieties $\Gamma_d(X_u, X^v)$ related to, say, Richardson varieties? A natural candidate to consider is $\Gamma_d(X_u) \cap \Gamma_d(X^v)$, which is Richardson by Corollary 2.3, provided the intersection is nonempty. Certainly $\Gamma_d(X_u, X^v) \subset \Gamma_d(X_u) \cap \Gamma_d(X^v)$; the converse however need not be true even in simple cases:

Example 2.5. In $X = \mathbb{P}^3$, let $X_u = \mathbb{P}^1$ be the B-stable Schubert curve, and X^v the B^{op}-fixed point. Then $\Gamma_1(X_u) \cap \Gamma_1(X^v) = \mathbb{P}^3$ by Example 2.1; on the other hand $\Gamma_1(X_u, X^v)$ is isomorphic to $\mathbb{P}^2$, specifically $\mathbb{P}(e_1, e_2, e_4)$ where (e_1, e_2, e_4) is the vector space spanned by the first, second and fourth standard basis vectors of $\mathbf{C}^4$. Thus $\Gamma_1(X_u, X^v) \subsetneq \Gamma_1(X_u) \cap \Gamma_1(X^v)$.

Even though $\Gamma_d(X_u, X^v)$ is not necessarily equal to $\Gamma_d(X_u) \cap \Gamma_d(X^v)$, there is still a possibility that $\Gamma_d(X_u, X^v)$ be Richardson, or close to it. The following result of Buch-Chaput-Mihalcea-Perrin addresses this:

Theorem 2.6 ([5]). *For a cominuscule flag variety X, the projected Gromov-Witten varieties $\Gamma_d(X_u, X^v)$ are images of Richardson varieties under morphisms $G/P' \to X$ from larger flag varieties G/P'. Consequently any nonempty $\Gamma_d(X_u, X^v)$ inherits properties of Richardson varieties: in particular Cohen-Macaulay with rational singularities.*

2.2 Quantum equals classical

The projected Gromov-Witten varieties $\Gamma_d(X_u, X^v)$ also inherit properties from the Gromov-Witten varieties $\mathrm{GW}_d(X_u, X^v)$ that define them. For example $\mathrm{GW}_d(X_u, X^v)$ is either empty or unirational [26], which means $\Gamma_d(X_u, X^v)$ is too. More is true:

Theorem 2.7 ([5]). *When X is cominuscule, the restricted map $\mathrm{ev}_3 : \mathrm{GW}_d(X_u, X^v) \to \Gamma_d(X_u, X^v)$ is cohomologically trivial: $(\mathrm{ev}_3)_* \mathcal{O}_{\mathrm{GW}_d(X_u,X^v)} = \mathcal{O}_{\Gamma_d(X_u,X^v)}$, and for $i > 0$, $R^i(\mathrm{ev}_3)_* \mathcal{O}_{\mathrm{GW}_d(X_u,X^v)} = 0$.*

This last fact yields the main theorem of this section:

Theorem 2.8 (Quantum equals classical, [5]). *The following holds in $K_T(X)$:*

$$[\mathcal{O}_{\Gamma_d(X_u,X^v)}] \;=\; \sum_{w \in W^P} I_d\big([\mathcal{O}_u], [\mathcal{O}^v], [\mathcal{O}^w]^\vee\big)[\mathcal{O}^w],$$

where $I_d\big([\mathcal{O}_u], [\mathcal{O}^v], [\mathcal{O}^w]^\vee\big) \in \Gamma$ is the K-theoretic Gromov-Witten invariant of $[\mathcal{O}_u]$, $[\mathcal{O}^v]$, $[\mathcal{O}^w]^\vee$.

Proof. Because $\mathrm{ev}_3 : \mathrm{GW}_d(X_u, X^v) \to \Gamma_d(X_u, X^v)$ is cohomologically trivial,

$$(\mathrm{ev}_3)_*[\mathcal{O}_{\mathrm{GW}_d(X_u,X^v)}] \;=\; [\mathcal{O}_{\Gamma_d(X_u,X^v)}]$$

in $K_T(X)$. If we express $(\mathrm{ev}_3)_*[\mathcal{O}_{\mathrm{GW}_d(X_u,X^v)}] = \sum_{w\in W^P} a_w[\mathcal{O}^w]$ in terms of the opposite Schubert basis, where $a_w \in \Gamma$, then we must show that $a_w = I_d\big([\mathcal{O}_u],[\mathcal{O}^v],[\mathcal{O}^w]^\vee\big)$ for all $w \in W^P$.

On the other hand if $[\mathcal{O}^w]^\vee$ is the element dual to $[\mathcal{O}^w]$ under $\chi = \chi_X$, we then have

$$a_w \;=\; \chi\big((\mathrm{ev}_3)_*[\mathcal{O}_{\mathrm{GW}_d(X_u,X^v)}]\cdot[\mathcal{O}^w]^\vee\big).$$

Also, by Sierra's Kleiman-Bertini theorem [26],

$$[\mathcal{O}_{\mathrm{GW}_d(X_u,X^v)}] \;=\; \mathrm{ev}_1^*[\mathcal{O}_u]\cdot\mathrm{ev}_2^*[\mathcal{O}^v]$$

in $K_T\big(\overline{M}_{0,3}(X,d)\big)$. Together with the projection formula,

$$\begin{aligned}
a_w &= \chi\Big((\mathrm{ev}_3)_*[\mathcal{O}_{\mathrm{GW}_d(X_u,X^v)}]\cdot[\mathcal{O}^w]^\vee\Big)\\
&= \chi(\mathrm{ev}_3)_*\Big([\mathcal{O}_{\mathrm{GW}_d(X_u,X^v)}]\cdot\mathrm{ev}_3^*[\mathcal{O}^w]^\vee\Big)\\
&= \chi(\mathrm{ev}_3)_*\big(\,\mathrm{ev}_1^*[\mathcal{O}_u]\cdot\mathrm{ev}_2^*[\mathcal{O}^v]\cdot\mathrm{ev}_3^*[\mathcal{O}^w]^\vee\big)\\
&= \chi_{\overline{M}_{0,3}(X,d)}\big(\,\mathrm{ev}_1^*[\mathcal{O}_u]\cdot\mathrm{ev}_2^*[\mathcal{O}^v]\cdot\mathrm{ev}_3^*[\mathcal{O}^w]^\vee\big),
\end{aligned}$$

which by definition equals $I_d\big([\mathcal{O}_u],[\mathcal{O}^v],[\mathcal{O}^w]^\vee\big)$. ■

Corollary 2.9. *For opposite Schubert classes $[\mathcal{O}_u],[\mathcal{O}^v]\in K_T(X)$, we have in $\mathrm{QK}_T(X)$*

$$[\mathcal{O}_u]\odot[\mathcal{O}^v] \;=\; \sum_d q^d[\mathcal{O}_{\Gamma_d(X_u,X^v)}].$$

Proof. This is simply an application of Theorem 2.8 to the definition $[\mathcal{O}_u]\odot[\mathcal{O}^v] = \sum_{w,d} I_d\big([\mathcal{O}_u],[\mathcal{O}^v],[\mathcal{O}^w]^\vee\big)q^d[\mathcal{O}^w]$. ■

Lastly let $\psi : \mathrm{QK}_T(X) \to \mathrm{QK}_T(X)$ be the $\Gamma[\![q]\!]$-linear endomorphism defined on the Schubert basis by $[\mathcal{O}^w] \mapsto [\mathcal{O}^{w(-1)}]$. In other words ψ is the endomorphism that sends an opposite Schubert class $[\mathcal{O}^w]$ to the class of its line neighborhood $[\mathcal{O}^{w(-1)}]$. (It may be defined more succinctly as $\psi = (\mathrm{ev}_2)_*(\mathrm{ev}_1)^*$ where again $\mathrm{ev}_i : \overline{M}_{0,2}(X,1)\to X$ are the evaluation maps.) Together with ψ, the ideas of this section culminate in Buch-Chaput-Mihalcea-Perrin's alternate characterisation of the quantum product $\star$ for cominuscule $\mathrm{QK}_T(X)$:

Theorem 2.10 ([6]). *Let X be cominuscule. For all classes σ, $\tau \in K_T(X)$, we have*

$$\sigma \star \tau \;=\; (1 - q\psi)(\sigma \odot \tau)$$

in $\mathrm{QK}_T(X)$.

2.3 Euler characteristics of cominuscule quantum K-theory

As $H_2(X, \mathbf{Z}) \simeq \mathbf{Z}[X_{s_\alpha}] \simeq \mathbf{Z}$ for cominuscule $X = G/P_\alpha$, the following is well-defined:

Definition 2.11. The **distance** between opposite Schubert varieties X_u, $X^v \subset X$ is the smallest effective degree $d \in H_2(X, \mathbf{Z})$ of a rational curve in X meeting both X_u and X^v. (Here we identify $H_2(X, \mathbf{Z})$ with $\mathbf{Z}[X_{s_\alpha}]$.) Denote this distance by $\mathrm{dist}(X_u, X^v)$.

This degree $\mathrm{dist}(X_u, X^v)$ may be understood in other ways: it is the smallest degree d for which $\Gamma_d(X_u, X^v)$ is nonempty; it is also the smallest degree d for which the quantum parameter q^d appears in $[\mathcal{O}_u] \star [\mathcal{O}^v]$ in $\mathrm{QK}_T(X)$. Perhaps the earliest it has appeared in the literature is in work of Fulton-Woodward [15], in which they identify the smallest degree of q appearing in products of Schubert classes in $\mathrm{QH}^*(X)$, the quantum cohomology of the Grassmannian X of type A.

Example 2.12. For cominuscule X, if $X_u \cap X^v$ is nonempty, then $\mathrm{dist}(X_u, X^v) = 0$ because a point is none other than a degree-0 curve.

Example 2.13. Let $X = \mathbb{P}^n$. Since pairs of points in $\mathbb{P}^n$ are connected by a degree-one $\mathbb{P}^1$, we have

$$\mathrm{dist}(X_u, X^v) \;=\; \begin{cases} 0 & \text{if } X_u \cap X^v \text{ is nonempty} \\ 1 & \text{otherwise.} \end{cases}$$

Recall from Definition 1.8 the sheaf Euler characteristic $\chi : K_T(X) \to \Gamma$. Then χ is a map of Γ-modules (but rarely a map of Γ-algebras). It naturally lifts to a 'quantum-valued' sheaf Euler characteristic $\mathrm{QK}_T(X) \to \Gamma[\![q]\!]$ by extending $\Gamma[\![q]\!]$-linearly:

$$\mathrm{QK}_T(X) \;=\; K_T(X) \otimes_\Gamma \Gamma[\![q]\!] \;\xrightarrow{\chi \otimes 1}\; \Gamma[\![q]\!].$$

We abuse notation and call this extension χ as well.

As it turns out, χ detects the distance between opposite Schubert varieties X_u and X^v:

Theorem 2.14. *Let X_u, X^v be opposite Schubert varieties in X. Then in $\mathrm{QK}_T(X)$,*

$$\chi\big([\mathcal{O}_u] \star [\mathcal{O}^v]\big) \;=\; q^{\mathrm{dist}(X_u,X^v)}.$$

Proof. By Theorem 2.10 and Corollary 2.9,

$$\begin{aligned}[][\mathcal{O}_u] \star [\mathcal{O}^v] &= (1-q\psi)\big([\mathcal{O}_u] \odot [\mathcal{O}^v]\big)\\ &= (1-q\psi)\sum_{d\geq 0} q^d[\mathcal{O}_{\Gamma_d(X_u,X^v)}].\end{aligned}$$

Thus $\chi\big([\mathcal{O}_u] \star [\mathcal{O}^v]\big) = \chi(1-q\psi)\sum_{d\geq 0} q^d[\mathcal{O}_{\Gamma_d(X_u,X^v)}]$. By Corollary 1.10, the Euler characteristic χ evaluates Schubert classes at 1, and by definition ψ maps Schubert classes to Schubert classes. Since the Schubert classes form a basis, the composition property $\chi\psi = \chi$ holds, whence

$$\begin{aligned}\chi(1-q\psi)\sum_{d\geq 0} q^d[\mathcal{O}_{\Gamma_d(X_u,X^v)}] &= (\chi - q\chi\psi)\sum_{d\geq 0} q^d[\mathcal{O}_{\Gamma_d(X_u,X^v)}]\\ &= \sum_{d\geq 0} q^d\chi\big([\mathcal{O}_{\Gamma_d(X_u,X^v)}]\big) - \sum_{d\geq 0} q^{d+1}\chi\big([\mathcal{O}_{\Gamma_d(X_u,X^v)}]\big)\\ &= q^{\mathrm{dist}(X_u,X^v)}\chi\big([\mathcal{O}_{\Gamma_{\mathrm{dist}(X_u,X^v)}(X_u,X^v)}]\big).\end{aligned}$$

Since $\Gamma_{\mathrm{dist}(X_u,X^v)}(X_u,X^v)$ is unirational and $\chi\big([\mathcal{O}_Y]\big) = 1$ for Y unirational with rational singularities, we have $\chi\big([\mathcal{O}_u] \star [\mathcal{O}^v]\big) = q^{\mathrm{dist}(X_u,X^v)}$. ∎

Example 2.15. Theorem 2.14 puts into larger context the following fact about Richardson varieties in $K_T(X)$:

$$\chi\big([\mathcal{O}_u]\cdot[\mathcal{O}^v]\big) \;=\; \begin{cases} 1 & \text{if } X_u\cap X^v \text{ nonempty, i.e., } \mathrm{dist}(X_u,X^v)=0,\\ 0 & \text{if } X_u\cap X^v \text{ empty, i.e., } \mathrm{dist}(X_u,X^v)>0.\end{cases}$$

As we have seen earlier in Chapter 1, equivariant K-theory understands $\chi\big([\mathcal{O}_u]\cdot[\mathcal{O}^v]\big)$ to be 1 or 0 depending on whether $X_u\cap X^v$ is empty. Theorem 2.14 instead casts this into a larger enumero-geometric framework involving curve distances. This example further illustrates the degree-zero nature of the classical theory in quantum K-theory.

Theorem 2.14 has immediate consequences on the Schubert structure constants of $\mathrm{QK}_T(X)$. To discuss these consider

$$\mathrm{QK}_T^{\mathrm{poly}}(X) \;:=\; K_T(X) \otimes_\Gamma \Gamma[q],$$

where $\Gamma[q]$ is the polynomial ring in the parameter q. Then $\mathrm{QK}_T^{\mathrm{poly}}(X)$ is free as $\Gamma[q]$-module with basis of opposite Schubert classes $[\mathcal{O}^w]$; the ordinary Schubert classes $[\mathcal{O}_w]$ form another basis.

Theorem 2.16 ([4])**.** *$\mathrm{QK}_T^{\mathrm{poly}}(X)$ is a subring of $\mathrm{QK}_T(X)$.*

In other words the Schubert structure constants $N_{u,v}^{w,d}$ vanish for sufficiently large d.

Consider now the following schema of maps.

$$\begin{array}{ccc}
\mathrm{QK}_T(X) & \xrightarrow{\;\chi\;} & \Gamma[\![q]\!] \\
\cup & & \cup \\
\mathrm{QK}_T^{\mathrm{poly}}(X) & \dashrightarrow & \Gamma[q] \\
\cup & & \cup \\
K_T(X) & \xrightarrow{\;\chi\;} & \Gamma
\end{array}$$

Thus the induced map $\chi : \mathrm{QK}_T^{\mathrm{poly}}(X) \to \Gamma[q]$ may be viewed either as an extension of $\chi : K_T(X) \to \Gamma$ or as a restriction of $\chi : \mathrm{QK}_T(X) \to \Gamma[\![q]\!]$ to $\mathrm{QK}_T^{\mathrm{poly}}(X)$.

In either case, because $\mathrm{QK}_T^{\mathrm{poly}}(X)$ consists of polynomials in q, there is also the map $\widetilde{\chi} : \mathrm{QK}_T^{\mathrm{poly}}(X) \to \Gamma$ defined by the composition

$$\mathrm{QK}_T^{\mathrm{poly}}(X) \quad = \quad K_T(X) \otimes_\Gamma \Gamma[q] \quad \xrightarrow{\chi} \quad \Gamma[q] \quad \xrightarrow{q \mapsto 1} \quad \Gamma.$$

Then $\widetilde{\chi} : \mathrm{QK}_T^{\mathrm{poly}}(X) \to \Gamma$ is a map of $\Gamma[q]$-modules, provided q acts on Γ as the identity. It is yet another lift of $\chi : K_T(X) \to \Gamma$, one with an unexpected algebraic property:

Theorem 2.17. *The map $\widetilde{\chi} : \mathrm{QK}_T^{\mathrm{poly}}(X) \to \Gamma$ such that $\widetilde{\chi}\big([\mathcal{O}^w]\big) = \widetilde{\chi}(q) = 1$ is a ring homomorphism.*

Proof. As $\widetilde{\chi}$ is Γ-linear by definition, we need only verify that $\widetilde{\chi}$ is multiplicative on all of $\mathrm{QK}_T^{\mathrm{poly}}(X)$. For this it suffices to show that $\widetilde{\chi}$ is multiplicative on products of the form $[\mathcal{O}_u] \star [\mathcal{O}^v]$, as each type of Schubert class $[\mathcal{O}_w]$ and $[\mathcal{O}^w]$ forms a $\Gamma[q]$-basis for $\mathrm{QK}_T^{\mathrm{poly}}(X)$. But by Theorem 2.14,

$$\begin{aligned}
\widetilde{\chi}\big([\mathcal{O}_u] \star [\mathcal{O}^v]\big) &= \widetilde{\chi}\big(q^{\mathrm{dist}(X_u, X^v)}\big) \\
&= 1 \\
&= 1 \cdot 1 \\
&= \widetilde{\chi}\big([\mathcal{O}_u]\big) \cdot \widetilde{\chi}\big([\mathcal{O}^v]\big).
\end{aligned}$$

■

Corollary 2.18. *Let u, $v \in W^P$. For $[\mathcal{O}^u], [\mathcal{O}^v] \in \mathrm{QK}_T(X)$, let $N_{u,v}^{w,d}$ be the elements of Γ such that*

$$[\mathcal{O}^u] \star [\mathcal{O}^v] = \sum_{w,d} N_{u,v}^{w,d} q^d [\mathcal{O}^w]$$

in $\mathrm{QK}_T(X)$. Then $\sum_{w,d} N_{u,v}^{w,d} = 1$.

Proof. Because $\mathrm{QK}_T^{\mathrm{poly}}(X)$ is a subring of $\mathrm{QK}_T(X)$ with the same structure constants, we may prove the claim for $\mathrm{QK}_T^{\mathrm{poly}}(X)$ instead. Since $\widetilde{\chi} : \mathrm{QK}_T^{\mathrm{poly}}(X) \to \Gamma$ is a Γ-linear ring homomorphism, we have $1 = \widetilde{\chi}\big([\mathcal{O}^u]\big) \cdot \widetilde{\chi}\big([\mathcal{O}^v]\big)$ equals $\widetilde{\chi}\big([\mathcal{O}^u] \star [\mathcal{O}^v]\big) = \sum_{w,d} N_{u,v}^{w,d}$. ■

Chapter 3

A formula for quantum K-theory of projective space

In this final chapter, we prove a recursive formula for the products $[\mathcal{O}^u] \star [\mathcal{O}^v]$ in $\mathrm{QK}_T(\mathbb{P}^n)$. With this formula we establish Griffeth-Ram positivity of the structure constants $N_{u,v}^{w,d}$ of $\mathrm{QK}_T(\mathbb{P}^n)$. This was originally conjectured by Griffeth-Ram [19] for the equivariant K-theory ring $K_T(G/B)$ of a complete flag variety G/B; their conjecture was eventually proven by Anderson-Griffeth-Miller [1] for $K_T(G/P)$ for all flag varieties G/P. A similar positivity property is expected to hold for $\mathrm{QK}_T(G/P)$, but this conjecture currently remains open in general.

We also establish an isomorphism between the equivariant quantum K-theory rings of $\mathbb{P}^n$ and $\mathbb{P}^{n+1}$. This may be understood as the equivariant K-theoretic analogue of the fact that, for any projective space $\mathbb{P}^n$, the quantum cohomology ring $\mathrm{QH}^*(\mathbb{P}^n)$ is isomorphic to the polynomial ring $\mathbf{Z}[h]$.

Our work builds on the Chevalley formula of Buch-Chaput-Mihalcea-Perrin. Their Chevalley formula describes products $[\mathcal{O}^{s_\alpha}] \star [\mathcal{O}^v]$ involving the (opposite) Schubert divisor class $[\mathcal{O}^{s_\alpha}]$. In the notation of Section 3.1, the Chevalley formula states:

Chevalley formula ([6])**.** *For $1 \leq r \leq n$ we have in $\mathrm{QK}_T(\mathbb{P}^n)$*

$$\widetilde{\mathcal{O}}^1 \star \widetilde{\mathcal{O}}^r \;=\; \begin{cases} \left(\dfrac{t_{r+1}}{t_1} - 1\right)\widetilde{\mathcal{O}}^r \;+\; \dfrac{t_{r+1}}{t_1}\widetilde{\mathcal{O}}^{r+1} & \textit{if } r < n; \\[2ex] \left(\dfrac{t_{n+1}}{t_1} - 1\right)\widetilde{\mathcal{O}}^n \;+\; (-1)^{n+1}q\dfrac{t_{n+1}}{t_1} & \textit{if } r = n. \end{cases}$$

3.1 The recursive formula for $\mathrm{QK}_T(\mathbb{P}^n)$

It will be convenient to identify $\mathbb{P}^n$ as a quotient of $G = \mathrm{GL}_{n+1}(\mathbf{C})$ instead of $\mathrm{SL}_{n+1}(\mathbf{C})$. We still have the torus T as the set of diagonal matrices in G, but now $T \simeq (\mathbf{C}^*)^{n+1}$. The Borel subgroups B and B^{op} remain the upper and lower triangular matrices. The

$(n+1)$-many characters $\varepsilon_i : T \to \mathbf{C}$ defined by $(a_1, \ldots, a_{n+1}) \mapsto a_i$ generate the character lattice of T; when written additively the simple roots are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \le i \le n$. With this we have $\mathbb{P}^n = G/P_{\alpha_1}$.

The opposite Schubert varieties X^p are the B^{op}-stable linear subspaces of $\mathbb{P}^n$. In terms of homogeneous coordinates $[x_1 : \cdots : x_{n+1}]$ they can be identified as $X^p = Z(x_1, \ldots, x_p)$, the zero locus of the first p-homogeneous coordinates. Thus the X^p can be indexed by their codimension p in $\mathbb{P}^n$, where $0 \le p \le n$. Then $\mathrm{QK}_T(\mathbb{P}^n)$ has a $\Gamma[\![q]\!]$-basis of opposite Schubert classes $[\mathcal{O}^p]$ where $\Gamma = \mathbf{Z}[e^{\pm\varepsilon_1}, \ldots, e^{\pm\varepsilon_{n+1}}]$.[1]

Notation. Identify $\Gamma \xrightarrow{\sim} \mathbf{Z}[t_1^{\pm 1}, \ldots, t_{n+1}^{\pm 1}]$ by $e^{\varepsilon_i} \mapsto t_i$. In this notation, the negative simple root $e^{-\alpha_i}$ corresponds to $\frac{t_{i+1}}{t_i}$.

For reasons that shall become apparent soon, it will be convenient to introduce the following "alternating" Schubert classes:

Notation. For each $0 \le p \le n$, let $\widetilde{\mathcal{O}}^p := (-1)^p[\mathcal{O}^p]$ in $\mathrm{QK}_T(\mathbb{P}^n)$.[2] These alternating Schubert classes $\widetilde{\mathcal{O}}^p$ form a $\Gamma[\![q]\!]$-basis for $\mathrm{QK}_T(\mathbb{P}^n)$ since the Schubert classes $[\mathcal{O}^p]$ already form a basis.

Let $\widetilde{\mathcal{O}}^{n+1} := (-1)^{n+1}q$ in $\mathrm{QK}_T(\mathbb{P}^n)$ as well. We shall sometimes denote this by $\widetilde{q}$. We emphasise that no Schubert variety X^{n+1} actually exists in $\mathbb{P}^n$ that corresponds to this class $\widetilde{\mathcal{O}}^{n+1}$; we nevertheless make the convention in $\mathrm{QK}_T(\mathbb{P}^n)$ because algebraically, the quantum parameter q functions as though it were this missing Schubert class $\widetilde{\mathcal{O}}^{n+1}$, at least according to the Chevalley formula. Indeed, with this convention the Chevalley formula succinctly becomes

$$\widetilde{\mathcal{O}}^1 \star \widetilde{\mathcal{O}}^r \;=\; \Big(\frac{t_{r+1}}{t_1} - 1\Big)\widetilde{\mathcal{O}}^r + \frac{t_{r+1}}{t_1}\widetilde{\mathcal{O}}^{r+1}$$

for $1 \le r \le n$.

We now arrive at the main theorem of this chapter:

---

[1]Recall from Example 1.4 that Γ is the group ring of the character lattice of T. The characters $\alpha : T \to \mathbf{C}^*$ are formally elements of Γ, which we denote in exponential notation by e^α.

[2]Note $\widetilde{\mathcal{O}}^0 = [\mathcal{O}^0] = [\mathcal{O}_{X^0}] = 1$ in $\mathrm{QK}_T(\mathbb{P}^n)$ since the opposite Schubert variety X^0 is precisely $\mathbb{P}^n$.

Theorem 3.1 (Recursive formula)**.** *Let $1 \le p \le n$. Then for all $p \le r \le n$, we have in* $\mathrm{QK}_T(\mathbb{P}^n)$

$$\begin{aligned}\widetilde{\mathcal{O}}^p \star \widetilde{\mathcal{O}}^r \;&=\; \Big(\frac{t_{r+1}}{t_1}-1\Big)\cdots\Big(\frac{t_{r+1}}{t_p}-1\Big)\widetilde{\mathcal{O}}^r \\ &\qquad + \sum_{i=1}^{p} \frac{t_{r+1}}{t_i}\Big(\frac{t_{r+1}}{t_{i+1}}-1\Big)\cdots\Big(\frac{t_{r+1}}{t_p}-1\Big)\widetilde{\mathcal{O}}^{i-1}\star\widetilde{\mathcal{O}}^{r+1}.\end{aligned}$$

Observe that when $p = 1$, this recursive formula reduces to the Chevalley formula of Buch-Chaput-Mihalcea-Perrin. Also note that Theorem 3.1 provides a recursive formula for $[\mathcal{O}^p]\star[\mathcal{O}^r]$. Moreover, if we write $[\mathcal{O}^p]\star[\mathcal{O}^r] = \sum_{s,d} N_{p,r}^{s,d} q^d [\mathcal{O}^w]$ and $\widetilde{\mathcal{O}}^p\star\widetilde{\mathcal{O}}^r = \sum_{s,d} \widetilde{N}_{p,r}^{s,d}\, \widetilde{q}^d\, \widetilde{\mathcal{O}}^s$, then the structure constants are related by $\widetilde{N}_{p,r}^{s,d} = (-1)^e N_{p,r}^{s,d}$ where $e = p+r+s+d(n+1)$.

We prove Theorem 3.1 in Section 3.6 below. We shall present consequences of the theorem first.

3.2 First consequences of the formula

We begin by addressing Griffeth-Ram positivity of the structure constants $N_{p,r}^{s,d}$ relative to the Schubert classes $[\mathcal{O}^p]$. Let Γ_+ denote $\mathbf{Z}_{\ge 0}[\frac{t_2}{t_1}-1,\ldots,\frac{t_{n+1}}{t_n}-1]$, the set of elements in Γ that are polynomials in the classes $\frac{t_{i+1}}{t_i}-1$ with nonnegative integer coefficients. This set Γ_+ is closed under addition and multiplication. In particular $a+b \ne 0$ and $ab \ne 0$ whenever a and b are nonzero elements of Γ_+.

Lemma 3.2. *For $1 \le p < r \le n$, we have the identity in Γ*

$$\frac{t_{r+1}}{t_p}-1 \;=\; \sum_{i=1}^{r-p} s_i\Big(\frac{t_{p+1}}{t_p}-1,\ldots,\frac{t_{r+1}}{t_r}-1\Big),$$

where $s_i\big(\frac{t_{p+1}}{t_p}-1,\ldots,\frac{t_{r+1}}{t_r}-1\big)$ is the i^{th} elementary symmetric polynomial in $\frac{t_{p+1}}{t_p}-1$, $\ldots$, $\frac{t_{r+1}}{t_r}-1$. Consequently, for $p \le r$, the polynomials of the form $\frac{t_{r+1}}{t_p}-1$ and $\frac{t_{r+1}}{t_p}$ are elements of Γ_+.

Proof. The identity follows from induction on $r-p$. ∎

Theorem 3.3 (Griffeth-Ram positivity for $\mathrm{QK}_T(\mathbb{P}^n)$)**.** *For $1 \leq p, r, s \leq n$ and $d \geq 0$, let $e = p + r + s + d(n+1)$. Then the scaled Schubert structure constant $(-1)^e N_{p,r}^{s,d} \in \Gamma$ of $\mathrm{QK}_T(\mathbb{P}^n)$ is an element of Γ_+.*

Proof. By the remark following Theorem 3.1, we have $\widetilde{N}_{p,r}^{s,d} = (-1)^e N_{p,r}^{s,d}$ where the $\widetilde{N}_{p,r}^{s,d}$ are the structure constants relative to the alternating classes $\widetilde{\mathcal{O}}^p$. By Theorem 3.1 and induction, each of the $\widetilde{N}_{p,r}^{s,d}$ are polynomials in $\frac{t_{j+1}}{t_i}$ and $\frac{t_{j+1}}{t_i} - 1$ with *nonnegative* integer coefficients. The result now follows from Lemma 3.2. ■

Let $K_T(\mathbb{P}^n)$ be the torus-equivariant K-theory ring of $\mathbb{P}^n$. Since $\mathrm{QK}_T(\mathbb{P}^n)$ is a formal deformation of $K_T(\mathbb{P}^n)$,[3] Theorem 3.1 immediately restricts to a formula for $K_T(\mathbb{P}^n)$. The only modification needed is to change our convention $\widetilde{\mathcal{O}}^{n+1} = (-1)^{n+1}q$ in $\mathrm{QK}_T(\mathbb{P}^n)$ to $\widetilde{\mathcal{O}}^{n+1} := 0$ in $K_T(X)$:

Theorem 3.4. *In $K_T(\mathbb{P}^n)$, for all $1 \leq p \leq r \leq n$,*

$$\widetilde{\mathcal{O}}^p \cdot \widetilde{\mathcal{O}}^r = \Big(\frac{t_{r+1}}{t_1} - 1\Big) \cdots \Big(\frac{t_{r+1}}{t_p} - 1\Big)\widetilde{\mathcal{O}}^r + \sum_{i=1}^{p} \frac{t_{r+1}}{t_i}\Big(\frac{t_{r+1}}{t_{i+1}} - 1\Big) \cdots \Big(\frac{t_{r+1}}{t_p} - 1\Big)\widetilde{\mathcal{O}}^{i-1} \cdot \widetilde{\mathcal{O}}^{r+1}.$$

Proof. On the one hand, the degree-zero piece of $\widetilde{\mathcal{O}}^p \star \widetilde{\mathcal{O}}^r$ is precisely $\widetilde{\mathcal{O}}^p \cdot \widetilde{\mathcal{O}}^r$. On the other hand, by Theorem 3.1, the degree-zero piece of $\widetilde{\mathcal{O}}^p \star \widetilde{\mathcal{O}}^r$ is that of each term $\widetilde{\mathcal{O}}^{i-1} \star \widetilde{\mathcal{O}}^{r+1}$, which is the corresponding K_T-theoretic product $\widetilde{\mathcal{O}}^{i-1} \cdot \widetilde{\mathcal{O}}^{r+1}$. ■

In [18], Graham-Kumar establish a formula for the Schubert structure constants of $K_T(\mathbb{P}^n)$. If we write $\widetilde{\mathcal{O}}^p \cdot \widetilde{\mathcal{O}}^r = \sum_s \widetilde{K}_{p,r}^s \widetilde{\mathcal{O}}^s$ for the structure constants relative to the alternating Schubert classes $\widetilde{\mathcal{O}}^s$, then their formula [18, Thm. 6.14] states

$$\widetilde{K}_{p,r}^s = \frac{t_1 \cdots t_s}{t_1 \cdots t_p t_1 \cdots t_r} \left[\frac{\prod_{i=1}^{p}(1 - yt_i)\prod_{i=1}^{r}(1 - yt_i)}{\prod_{i=1}^{s+1}(1 - yt_i)}\right]_{p+r-s}.$$

Here, the formula should be computed in the formal power series ring $\Gamma[\![y]\!]$ where y is a formal parameter; then $[\quad]_{p+r-s}$ denotes the coefficient of y^{p+r-s} in the resultant

[3] See Example 1.15 and the remarks that follow it.

expression. With this formula, Graham-Kumar develop a recurrence relation [18, Thm. 6.14] among the structure constants $\widetilde{K}_{p,r}^{s}$. Theorem 3.4 recovers this relation.

Returning to $\mathrm{QK}_T(\mathbb{P}^n)$, our next result concerns the vanishing of the structure constants $\widetilde{N}_{p,r}^{s,d}$:

Proposition 3.5. *Let p, r be such that $1 \leq p \leq r \leq n$. Set $\widetilde{\mathcal{O}}^p \star \widetilde{\mathcal{O}}^r = \sum_{s,d} \widetilde{N}_{p,r}^{s,d}\, \widetilde{q}^d\, \widetilde{\mathcal{O}}^s$.*

(1) *If $p + r \leq n$, then the structure constant $\widetilde{N}_{p,r}^{s,d}$ is nonzero precisely when $d = 0$ with $r \leq s \leq r + p$.*

(2) *If $p+r > n$, then the structure constant $\widetilde{N}_{p,r}^{s,d}$ is nonzero precisely when $d = 0$ with $r \leq s \leq n$, and when $d = 1$ with $0 \leq s \leq k - 1$ where k is such that $p + r = n + k$.*

Consequently, $\widetilde{\mathcal{O}}^p \star \widetilde{\mathcal{O}}^r = \widetilde{\mathcal{O}}^p \cdot \widetilde{\mathcal{O}}^r$ if and only if $p + r \leq n$; equivalently, quantum terms appear in $\widetilde{\mathcal{O}}^p \star \widetilde{\mathcal{O}}^r$ if and only if $p + r > n$.

Proposition 3.5 can also be phrased as follows: the smallest Γ-submodule of $\mathrm{QK}_T(\mathbb{P}^n)$ containing $\widetilde{\mathcal{O}}^p \star \widetilde{\mathcal{O}}^r$ is either the Γ-module generated by $\widetilde{\mathcal{O}}^r$, $\widetilde{\mathcal{O}}^{r+1}$, $\ldots$, $\widetilde{\mathcal{O}}^{r+p}$ when $p + r \leq n$, or the Γ-module spanned by $\widetilde{\mathcal{O}}^r$, $\widetilde{\mathcal{O}}^{r+1}$, $\ldots$, $\widetilde{\mathcal{O}}^n$, $\widetilde{q}$, $\widetilde{q}\,\widetilde{\mathcal{O}}^1$, $\ldots$, $\widetilde{q}\,\widetilde{\mathcal{O}}^{k-1}$ when $p + r = n + k$.

Proof of Proposition 3.5. To prove Case (1), we induct on p. Suppose $p + r \leq n$. If $p = 1$, then $1 \leq r \leq n - 1$, and the result can be verified by directly inspecting each $\widetilde{\mathcal{O}}^1 \star \widetilde{\mathcal{O}}^r$ with the Chevalley formula.

For general p, Theorem 3.1 states that $\widetilde{\mathcal{O}}^p \star \widetilde{\mathcal{O}}^r$ is a Γ-linear combination of $\widetilde{\mathcal{O}}^r$ and $\widetilde{\mathcal{O}}^{i-1} \star \widetilde{\mathcal{O}}^{r+1}$ for $1 \leq i \leq p$. Thus, with the exception of $\widetilde{N}_{p,r}^{r,0}$, the structure constants $\widetilde{N}_{p,r}^{s,d}$ are determined by the structure constants $\widetilde{N}_{i-1,r+1}^{s,d}$ of the $\widetilde{\mathcal{O}}^{i-1} \star \widetilde{\mathcal{O}}^{r+1}$. Since each $(i-1)+(r+1) \leq p+r \leq n$ and $i-1 < p$, the result follows from induction on p. (Note, Griffeth-Ram positivity is tacitly used here to guarantee that cancellation among the $\widetilde{N}_{i-1,r+1}^{s,d}$ does not occur. Thus $\widetilde{N}_{p,r}^{s,d}$ is nonzero for $d = 0$ with $r \leq s \leq r+p$ as claimed.)

Case (2) can be proved in a similar manner, as $\widetilde{\mathcal{O}}^p \star \widetilde{\mathcal{O}}^r$ is still a Γ-linear combination of $\widetilde{\mathcal{O}}^r$ and $\widetilde{\mathcal{O}}^{i-1} \star \widetilde{\mathcal{O}}^{r+1}$ for $1 \leq i \leq p$. The only difference is that there will be two cases to consider here: when $(i-1) + (r+1) \leq n$, and when $(i-1) + (r+1) > n$. Induction will apply to the latter situation, whereas Case (1) above will apply to the former. ∎

3.3 A closer look at $\mathrm{QK}_T^{\mathrm{poly}}(\mathbb{P}^n)$

Proposition 3.5 allows us to take a closer look at the ring structure of $\mathrm{QK}_T(\mathbb{P}^n)$. Although the results of this section hold over $\mathrm{QK}_T(\mathbb{P}^n)$, it is more natural to work in the subring $\mathrm{QK}_T^{\mathrm{poly}}(\mathbb{P}^n)$. Recall that $\mathrm{QK}_T^{\mathrm{poly}}(\mathbb{P}^n)$ is the $\Gamma[q]$-subalgebra of $\mathrm{QK}_T(\mathbb{P}^n)$ whose underlying additive group is $K_T(\mathbb{P}^n) \otimes_\Gamma \Gamma[q]$; in other words, it is the subring of elements that are polynomials in q (hence also in $\widetilde{q}$). As the alternating Schubert classes $\widetilde{\mathcal{O}}^p$ and their products $\widetilde{\mathcal{O}}^p \star \widetilde{\mathcal{O}}^r$ all lie in $\mathrm{QK}_T^{\mathrm{poly}}(\mathbb{P}^n)$, it follows that Theorem 3.1 and all of the previous section's results hold for $\mathrm{QK}_T^{\mathrm{poly}}(\mathbb{P}^n)$.

The set $\left\{\widetilde{q}^d \widetilde{\mathcal{O}}^p\right\}$ of monomials with $d \geq 0$ and $0 \leq p \leq n$ form a basis for $\mathrm{QK}_T^{\mathrm{poly}}(\mathbb{P}^n)$ over Γ. Moreover this basis is totally ordered under lexicographic ordering; denote this ordering by $\preceq$. Then relative to this ordering, Proposition 3.5 states that each product $\widetilde{\mathcal{O}}^p \star \widetilde{\mathcal{O}}^r$ is a nonzero Γ-combination of monomials $\widetilde{q}^d \widetilde{\mathcal{O}}^s$ satisfying $\widetilde{\mathcal{O}}^r \preceq \widetilde{q}^d \widetilde{\mathcal{O}}^s \preceq \widetilde{\mathcal{O}}^{r+p}$ when $p + r \leq n$, or $\widetilde{\mathcal{O}}^r \preceq \widetilde{q}^d \widetilde{\mathcal{O}}^s \preceq \widetilde{q}\widetilde{\mathcal{O}}^{k-1}$ when $p + r = n + k$ for some positive k. Thus, each product $\widetilde{\mathcal{O}}^p \star \widetilde{\mathcal{O}}^r$ has a well-defined leading monomial $\widetilde{\mathcal{O}}^{r+p}$ or $\widetilde{q}\widetilde{\mathcal{O}}^{k-1}$ with a corresponding leading coefficient $\widetilde{N}_{p,r}^{r+p,0}$ or $\widetilde{N}_{p,r}^{k-1,1}$.

Definition 3.6. For each monomial $\widetilde{q}^d \widetilde{\mathcal{O}}^p$ in $\mathrm{QK}_T^{\mathrm{poly}}(\mathbb{P}^n)$, with $d \geq 0$ and $0 \leq p \leq n$, define its **degree** to be the integer $\deg\left(\widetilde{q}^d \widetilde{\mathcal{O}}^p\right) = d(n+1) + p$. In particular $\deg(\widetilde{q}) = n + 1$. In general, for any element $f \in \mathrm{QK}_T^{\mathrm{poly}}(\mathbb{P}^n)$, define its **degree** to be the degree of its leading monomial.

Clearly $\widetilde{q}^d \widetilde{\mathcal{O}}^p \preceq \widetilde{q}^e \widetilde{\mathcal{O}}^r$ implies $\deg(\widetilde{q}^d \widetilde{\mathcal{O}}^p) \leq \deg(\widetilde{q}^e \widetilde{\mathcal{O}}^r)$; the converse is also true. In particular we have $\widetilde{q}^d \widetilde{\mathcal{O}}^p = \widetilde{q}^e \widetilde{\mathcal{O}}^r$ if and only if $\deg(\widetilde{q}^d \widetilde{\mathcal{O}}^p) = \deg(\widetilde{q}^e \widetilde{\mathcal{O}}^r)$. We omit the elementary proof, but record this as a proposition as it will be fundamental to our analysis of $\mathrm{QK}_T^{\mathrm{poly}}(\mathbb{P}^n)$:

Proposition 3.7. *For monomials $\widetilde{q}^d \widetilde{\mathcal{O}}^p$ and $\widetilde{q}^e \widetilde{\mathcal{O}}^r$ in $\mathrm{QK}_T^{\mathrm{poly}}(\mathbb{P}^n)$, we have $\widetilde{q}^d \widetilde{\mathcal{O}}^p \preceq \widetilde{q}^e \widetilde{\mathcal{O}}^r$ if and only if $\deg(\widetilde{q}^d \widetilde{\mathcal{O}}^p) \leq \deg(\widetilde{q}^e \widetilde{\mathcal{O}}^r)$.*

Proposition 3.8. *In $\mathrm{QK}_T^{\mathrm{poly}}(\mathbb{P}^n)$, we have $\deg(\widetilde{\mathcal{O}}^p \star \widetilde{\mathcal{O}}^r) = p + r$. Thus for any product $\widetilde{\mathcal{O}}^{p_1} \star \cdots \star \widetilde{\mathcal{O}}^{p_m}$, we have $\deg(\widetilde{\mathcal{O}}^{p_1} \star \cdots \star \widetilde{\mathcal{O}}^{p_m}) = p_1 + \cdots + p_m$.*

Proof. This follows immediately from Proposition 3.5. ■

Proposition 3.9. *The leading coefficient of* $\widetilde{\mathcal{O}}^p \star \widetilde{\mathcal{O}}^r$ *is a unit in* Γ. *Consequently, the leading coefficient of* $\widetilde{\mathcal{O}}^{p_1} \star \cdots \star \widetilde{\mathcal{O}}^{p_m}$ *is always a unit in* Γ.

Proof. When $p = 1$, we have $\widetilde{\mathcal{O}}^1 \star \widetilde{\mathcal{O}}^r = \left(\frac{t_{r+1}}{t_1} - 1\right)\widetilde{\mathcal{O}}^r + \frac{t_{r+1}}{t_1}\widetilde{\mathcal{O}}^{r+1}$, and the leading coefficient $\frac{t_{r+1}}{t_1}$ is a unit. For general p, the leading coefficient of $\widetilde{\mathcal{O}}^p \star \widetilde{\mathcal{O}}^r$ is the leading coefficient of $\frac{t_{r+1}}{t_p}\widetilde{\mathcal{O}}^{p-1} \star \widetilde{\mathcal{O}}^{r+1}$ by Theorem 3.1 and Proposition 3.5. Thus the result follows from induction. ■

Let $\mathbf{Z}_{\geq 0}$ denote the set of nonnegative integers with the usual linear ordering $\leq$.

Proposition 3.10. *The assignment* $\left\{\widetilde{q}^d\widetilde{\mathcal{O}}^p\right\} \to \mathbf{Z}_{\geq 0}$ *defined by* $\widetilde{q}^d\widetilde{\mathcal{O}}^p \mapsto \deg \widetilde{q}^d\widetilde{\mathcal{O}}^p$ *is a bijection of totally ordered sets.*

Proof. By Proposition 3.7, the assignment $\widetilde{q}^d\widetilde{\mathcal{O}}^p \mapsto \deg \widetilde{q}^d\widetilde{\mathcal{O}}^p$ preserves orders, hence is injective. Now $\deg\left(\widetilde{q}^d\widetilde{\mathcal{O}}^p\right) = d(n+1)+p$ by Proposition 3.8, where $d \geq 0$ and $0 \leq p \leq n$. Thus surjectivity follows from the fact that every integer is uniquely expressible as a quotient of $n+1$ with remainder. ■

Corollary 3.11. *The collection of elements* $\left(\widetilde{\mathcal{O}}^n\right)^{\star d} \star \widetilde{\mathcal{O}}^p$ *forms a basis for* $\mathrm{QK}_T^{\mathrm{poly}}(\mathbb{P}^n)$ *over* Γ, *where* $d \geq 0$ *and* $0 \leq p \leq n-1$.

Proof. The idea is straightforward: show that the $\left(\widetilde{\mathcal{O}}^n\right)^{\star d} \star \widetilde{\mathcal{O}}^p$ form a "triangular" basis for $\mathrm{QK}_T^{\mathrm{poly}}(\mathbb{P}^n)$ over Γ. (The standard monomials $\widetilde{q}^d\widetilde{\mathcal{O}}^p$ form a "diagonal" basis.)

The product $\left(\widetilde{\mathcal{O}}^n\right)^{\star d} \star \widetilde{\mathcal{O}}^p$ is a Γ-linear combination of monomials $\widetilde{q}^e\widetilde{\mathcal{O}}^s$ satisfying $\widetilde{q}^e\widetilde{\mathcal{O}}^s \preceq \widetilde{q}^m\widetilde{\mathcal{O}}^\ell$ where $\widetilde{q}^m\widetilde{\mathcal{O}}^\ell$ is the leading monomial of $\left(\widetilde{\mathcal{O}}^n\right)^{\star d} \star \widetilde{\mathcal{O}}^p$. By Proposition 3.8, we have the equalities $\deg \widetilde{q}^m\widetilde{\mathcal{O}}^\ell = \deg\left(\widetilde{\mathcal{O}}^n\right)^{\star d} \star \widetilde{\mathcal{O}}^p = dn + p$, and this degree uniquely determines the leading monomial $\widetilde{q}^m\widetilde{\mathcal{O}}^\ell$ (Proposition 3.7 again). Since every nonnegative integer can be uniquely expressed as $dn + p$ for some $d \geq 0$ and $0 \leq p \leq n-1$, we can conclude from Proposition 3.10 that the $\left(\widetilde{\mathcal{O}}^n\right)^{\star d} \star \widetilde{\mathcal{O}}^p$ correspond bijectively to the standard $\widetilde{q}^m\widetilde{\mathcal{O}}^\ell$ by way of their leading monomials. Thus the $\left(\widetilde{\mathcal{O}}^n\right)^{\star d} \star \widetilde{\mathcal{O}}^p$ are linearly independent over Γ.

To show that they span $\mathrm{QK}_T^{\mathrm{poly}}(\mathbb{P}^n)$, we must check that the leading coefficient of each $\left(\widetilde{\mathcal{O}}^n\right)^{\star d} \star \widetilde{\mathcal{O}}^p$ is a unit in Γ. This follows from Proposition 3.9. ■

3.4 Final consequence: An isomorphism theorem

The results of the previous section bring us closer to presenting the final major consequence of Theorem 3.1. This final result relates the equivariant quantum K-theory rings of $\mathbb{P}^n$ and $\mathbb{P}^{n+1}$. To state this precisely, we establish some notation for $\mathbb{P}^{n+1}$.

All of the previous notation for $\mathbb{P}^n$ and its torus T hold analogously for $\mathbb{P}^{n+1}$ with the obvious adjustments. For instance, let T' denote the torus acting on $\mathbb{P}^{n+1}$; it may be identified with $(\mathbf{C}^*)^{n+2}$ in which case its representation ring $\Gamma' = K_{T'}(\mathrm{pt})$ is isomorphic to $\mathbf{Z}[t_1^{\pm 1}, \ldots, t_{n+2}^{\pm 1}]$ under $e^{\varepsilon_i} \mapsto t_i$. The opposite Schubert varieties X^p are again indexed by their codimension in $\mathbb{P}^{n+1}$, with one Schubert variety for each $0 \le p \le n+1$. (Note the additional Schubert variety X^{n+1} here.)

If $\mathrm{QK}_{T'}(\mathbb{P}^{n+1})$ denotes the T'-equivariant quantum K-theory ring of $\mathbb{P}^{n+1}$, then we have the alternating Schubert classes $\widetilde{\mathcal{O}}^p := (-1)^p[\mathcal{O}^p]$ as before, where $0 \le p \le n+1$. However we denote the quantum parameter in $\mathrm{QK}_{T'}(\mathbb{P}^{n+1})$ by Q to distinguish it from the quantum parameter q of $\mathrm{QK}_T(\mathbb{P}^n)$. With this, we can again make the convention $\widetilde{\mathcal{O}}^{n+2} := (-1)^{n+2}Q$ as the imaginary "$(n+2)^{\mathrm{nd}}$-alternating Schubert class" of $\mathrm{QK}_{T'}(\mathbb{P}^{n+1})$; we shall also denote it by $\widetilde{Q}$.

There is a natural inclusion of rings $\Gamma \to \Gamma'$ defined by $t_i \mapsto t_i$ for $1 \le i \le n+1$. We use this to endow $\mathrm{QK}_T(\mathbb{P}^n)$ with the structure of $\Gamma'[\![q]\!]$-algebra:

Notation. Set $\mathrm{QK}_{T'}(\mathbb{P}^n) := \mathrm{QK}_T(\mathbb{P}^n) \otimes_{\Gamma[\![q]\!]} \Gamma'[\![q]\!]$. We refer to this as the quantum K-theory ring of $\mathbb{P}^n$, equivariant over T'. The ring structure is induced by the tensor product. Thus $\mathrm{QK}_{T'}(\mathbb{P}^n)$ is a $\Gamma'[\![q]\!]$-algebra, free on the alternating classes $\widetilde{\mathcal{O}}^p$ as a module over $\Gamma'[\![q]\!]$.

We similarly denote $\mathrm{QK}_{T'}^{\mathrm{poly}}(\mathbb{P}^n) := \mathrm{QK}_T^{\mathrm{poly}}(\mathbb{P}^n) \otimes_{\Gamma[q]} \Gamma'[q]$. Then $\mathrm{QK}_{T'}^{\mathrm{poly}}(\mathbb{P}^n)$ is a subring of $\mathrm{QK}_{T'}(\mathbb{P}^n)$. (See Theorem 2.16.)

Consider the rings $\mathrm{QK}_{T'}^{\mathrm{poly}}(\mathbb{P}^n)$ and $\mathrm{QK}_{T'}^{\mathrm{poly}}(\mathbb{P}^{n+1})$. As a Γ'-module, $\mathrm{QK}_{T'}^{\mathrm{poly}}(\mathbb{P}^n)$ has a basis given by the monomials $\widetilde{q}^d\widetilde{\mathcal{O}}^p$, where $d \ge 0$ and $0 \le p \le n$; similarly $\mathrm{QK}_{T'}^{\mathrm{poly}}(\mathbb{P}^{n+1})$ has a Γ'-basis given by the monomials $\widetilde{Q}^d\widetilde{\mathcal{O}}^p$ for $d \ge 0$ and $0 \le p \le n+1$.

Theorem 3.12. *Let $\varphi : \mathrm{QK}_{T'}^{\mathrm{poly}}(\mathbb{P}^n) \to \mathrm{QK}_{T'}^{\mathrm{poly}}(\mathbb{P}^{n+1})$ be the Γ'-linear map defined by $\widetilde{q}^d\widetilde{\mathcal{O}}^p \mapsto (\widetilde{\mathcal{O}}^{n+1})^{\star d} \star \widetilde{\mathcal{O}}^p$. Then φ is an isomorphism of Γ'-algebras.*

Analogous versions of Theorem 3.12 already exist for quantum cohomology $\mathrm{QH}^*(\mathbb{P}^n)$ and quantum K-theory $\mathrm{QK}(\mathbb{P}^n)$; it has not been known whether they hold for their equivariant versions, until now.

Theorem 3.12 is not stated for the full rings $\mathrm{QK}_{T'}(\mathbb{P}^n)$ and $\mathrm{QK}_{T'}(\mathbb{P}^{n+1})$ for pathological reasons: for example $1+\widetilde{q}+\widetilde{q}^2+\cdots$ is an element of $\mathrm{QK}_{T'}(\mathbb{P}^n)$ whose image under φ would be undefined. One way to circumvent these issues may be to work with $\widehat{\Gamma}'$ instead, where $\widehat{\Gamma}'$ is the completion of the ring Γ'. We shall not pursue this here.

Proof of Theorem 3.12. We must verify that φ is multiplicative and bijective. For the first task, it suffices to show that φ is multiplicative on products of basis elements $\widetilde{q}^d\widetilde{\mathcal{O}}^p \star \widetilde{q}^e\widetilde{\mathcal{O}}^r$; in fact, since $\widetilde{q}$ acts freely on $\mathrm{QK}_{T'}^{\mathrm{poly}}(\mathbb{P}^n)$, we need only verify this for the products $\widetilde{\mathcal{O}}^p \star \widetilde{\mathcal{O}}^r$. But this follows immediately from the recursive formula of Theorem 3.1, and from our convention that $\widetilde{\mathcal{O}}^{n+1} := \widetilde{q}$ in $\mathrm{QK}_{T'}^{\mathrm{poly}}(\mathbb{P}^n)$.

To show that φ is an isomorphism, we show that φ maps a basis for $\mathrm{QK}_{T'}^{\mathrm{poly}}(\mathbb{P}^n)$ bijectively onto a basis for $\mathrm{QK}_{T'}^{\mathrm{poly}}(\mathbb{P}^{n+1})$. Indeed, φ sends the basis element $\widetilde{q}^d\widetilde{\mathcal{O}}^p$ to the element $\left(\widetilde{\mathcal{O}}^{n+1}\right)^{\star d} \star \widetilde{\mathcal{O}}^p$. By applying Corollary 3.11 to these $\left(\widetilde{\mathcal{O}}^{n+1}\right)^{\star d} \star \widetilde{\mathcal{O}}^p$ in $\mathrm{QK}_{T'}^{\mathrm{poly}}(\mathbb{P}^{n+1})$, we obtain the desired result. ■

3.5 Preliminaries to the proof of the recursive formula

Recall that the Schubert variety X^1 equals $Z(x_1)$, the zero set of x_1 where $[x_1 : \ldots : x_{n+1}]$ are homogeneous coordinates of $\mathbb{P}^n$. It is a smooth effective Cartier divisor of $\mathbb{P}^n$ with ideal sheaf J. This sheaf J is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-1)$ as a coherent sheaf on $\mathbb{P}^n$.

Since X^1 is T-stable, its structure sheaf $\mathcal{O}_{X^1}$ is canonically a T-equivariant sheaf. The sheaves J and $\mathcal{O}_{\mathbb{P}^n}(-1)$ also have T-equivariant structure, but they are not the same even though they are isomorphic as coherent sheaves. The sheaf $\mathcal{O}_{\mathbb{P}^n}(-1)$ obtains its T-equivariant structure from the tautological line bundle $\mathscr{U}$ on $\mathbb{P}^n$: $\mathscr{U}$ is a subbundle of the trivial bundle $\mathbb{P}^n \times \mathbf{C}^{n+1}$, and the natural action of T on $\mathbb{P}^n \times \mathbf{C}^{n+1}$ restricts to an action on $\mathscr{U}$. Thus $\mathcal{O}_{\mathbb{P}^n}(-1)$ obtains its T-equivariant structure as the sheaf of sections of the T-equivariant line bundle $\mathscr{U} \to \mathbb{P}^n$.

The T-equivariant structure of J is built on this. Given the character $-\varepsilon_1 : T \to \mathbf{C}^*$

and its corresponding one-dimensional T-representation $\mathbf{C}_{-\varepsilon_1}$,[4] let $E_{-\varepsilon_1}$ denote the T-equivariant line bundle $\mathbb{P}^n \times \mathbf{C}_{-\varepsilon_1}$; this is the trivial line bundle equipped with the non-trivial T-action of $-\varepsilon_1$ on its fibers. If we also denote its corresponding invertible sheaf by $E_{-\varepsilon_1}$, then we have $J = E_{-\varepsilon_1} \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$ as T-equivariant sheaves. This can be checked by comparing the T-action locally on J and $\mathcal{O}_{\mathbb{P}^n}(-1)$.

Thus we obtain the following short exact sequence of T-equivariant sheaves:

$$0 \;\to\; E_{-\varepsilon_1} \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \;\to\; \mathcal{O}_{\mathbb{P}^n} \;\to\; \mathcal{O}_{X^1} \;\to\; 0.$$

Therefore in $K_T(\mathbb{P}^n)$ we have

$$1 \;=\; [E_{-\varepsilon_1} \otimes \mathcal{O}_{\mathbb{P}^n}(-1)] + [\mathcal{O}^1] \;=\; e^{-\varepsilon_1}[\mathcal{O}_{\mathbb{P}^n}(-1)] + [\mathcal{O}^1] \;=\; \frac{1}{t_1}[\mathcal{O}_{\mathbb{P}^n}(-1)] + [\mathcal{O}^1].$$

Thus $[J] = \frac{1}{t_1}[\mathcal{O}_{\mathbb{P}^n}(-1)]$ and $[\mathcal{O}^1] = 1 - [J]$, which means $\widetilde{\mathcal{O}}^1 = [J] - 1$.

Notation. $J := [J]$ in $K_T(\mathbb{P}^n)$. In other words, we shall abuse notation and let J denote both the ideal sheaf on $\mathbb{P}^n$ and its class in $K_T(\mathbb{P}^n)$.

Though it is presented as a formula for $\widetilde{\mathcal{O}}^1$, the Chevalley formula for $\mathrm{QK}_T(\mathbb{P}^n)$ can just as well be cast as a formula for multiplication by J:

Proposition 3.13. *For $1 \leq p \leq n$ we have in $\mathrm{QK}_T(\mathbb{P}^n)$*

$$J \star \widetilde{\mathcal{O}}^p \;=\; \frac{t_{p+1}}{t_1}\big(\widetilde{\mathcal{O}}^p + \widetilde{\mathcal{O}}^{p+1}\big).$$

Consequently $\widetilde{\mathcal{O}}^p = \big(\frac{t_1}{t_p}J - 1\big) \star \widetilde{\mathcal{O}}^{p-1}$. (Recall the convention that $\widetilde{\mathcal{O}}^{n+1} := (-1)^{n+1}q$.)

Because coefficients of the form $\frac{t_{r+1}}{t_i}\big(\frac{t_{r+1}}{t_{i+1}} - 1\big)\cdots\big(\frac{t_{r+1}}{t_p} - 1\big)$ frequently appear in Theorem 3.1, we introduce the following:

Notation. Let $1 \leq p \leq r \leq n$. For $1 \leq i \leq p$ define $T_{(i,p)} = \big(\frac{t_{r+1}}{t_i} - 1\big)\cdots\big(\frac{t_{r+1}}{t_p} - 1\big)$ and $T_{(p+1,p)} = 1$. Note $T_{(p,p)} = \big(\frac{t_{r+1}}{t_p} - 1\big)$. Technically we should write $T_{(i,p,r+1)}$ to indicate the dependence on r; however we omit it from the notation as we shall only be working with a fixed r in the sequel.

[4] See Example 1.4 for the notation.

In this condensed notation, the recursive formula 3.1 becomes

$$\widetilde{\mathcal{O}}^p \star \widetilde{\mathcal{O}}^r \;=\; T_{(1,p)}\widetilde{\mathcal{O}}^r \;+\; \sum_{i=1}^{p} \frac{t_{r+1}}{t_i} T_{(i+1,p)} \widetilde{\mathcal{O}}^{i-1} \star \widetilde{\mathcal{O}}^{r+1}.$$

For convenience we record two properties $T_{(i,p)}$ which follow immediately from the definitions:

Lemma 3.14. *For $1 \leq i \leq p$,*

(a) $\quad T_{(i,p)} \;=\; \left(\frac{t_{r+1}}{t_i} - 1\right) T_{(i+1,p)}$

(b) $\quad T_{(i,p)} \;=\; T_{(i,p-1)}\left(\frac{t_{r+1}}{t_p} - 1\right).$

3.6 The proof of the recursive formula

Proof of Theorem 3.1. We must show that for all $1 \leq p \leq r \leq n$, we have

$$\widetilde{\mathcal{O}}^p \star \widetilde{\mathcal{O}}^r \;=\; T_{(1,p)}\widetilde{\mathcal{O}}^r \;+\; \sum_{i=1}^{p} \frac{t_{r+1}}{t_i} T_{(i+1,p)} \widetilde{\mathcal{O}}^{i-1} \star \widetilde{\mathcal{O}}^{r+1}$$

in $\mathrm{QK}_T(\mathbb{P}^n)$. We do this by induction on p.

When $p = 1$, the recursive formula reduces to the Chevalley formula of Buch-Chaput-Mihalcea-Perrin, so there is nothing to prove. For general p, we have by Proposition 3.13 and induction,

$$\begin{aligned}
\widetilde{\mathcal{O}}^p \star \widetilde{\mathcal{O}}^r \;&=\; \left(\frac{t_1}{t_p} J - 1\right) \star \widetilde{\mathcal{O}}^{p-1} \star \widetilde{\mathcal{O}}^r \\
&=\; T_{(1,p-1)}\left(\frac{t_1}{t_p} J - 1\right) \star \widetilde{\mathcal{O}}^r \\
&\qquad + \sum_{i=1}^{p-1} \frac{t_{r+1}}{t_i} T_{(i+1,p-1)} \left(\frac{t_1}{t_p} J - 1\right) \star \widetilde{\mathcal{O}}^{i-1} \star \widetilde{\mathcal{O}}^{r+1} \\
&=\; T_{(1,p-1)}\left(\left(\frac{t_{r+1}}{t_p} - 1\right)\widetilde{\mathcal{O}}^r + \frac{t_{r+1}}{t_p}\widetilde{\mathcal{O}}^{r+1}\right) \\
&\qquad + \sum_{i=1}^{p-1} T_{(i+1,p-1)} \left(\left(\frac{t_{r+1}}{t_p} - \frac{t_{r+1}}{t_i}\right)\widetilde{\mathcal{O}}^{i-1} \star \widetilde{\mathcal{O}}^{r+1} + \frac{t_{r+1}}{t_p}\widetilde{\mathcal{O}}^{i} \star \widetilde{\mathcal{O}}^{r+1}\right). \qquad (*)
\end{aligned}$$

The proof now becomes a matter of simplifying the expression. We shall show that the coefficient of each $\widetilde{\mathcal{O}}^{i-1} \star \widetilde{\mathcal{O}}^{r+1}$ in $(*)$ agrees with the recursive formula:

- When $i = 1$, we have the term $\widetilde{\mathcal{O}}^0 \star \widetilde{\mathcal{O}}^{r+1} = \widetilde{\mathcal{O}}^{r+1}$. Its total coefficient in $(*)$ is

$$\frac{t_{r+1}}{t_p} T_{(1,p-1)} + \Big(\frac{t_{r+1}}{t_p} - \frac{t_{r+1}}{t_1}\Big) T_{(2,p-1)}.$$

  By Lemma 3.14, this simplifies to $\frac{t_{r+1}}{t_1} T(2,p)$ which is precisely the coefficient of $\widetilde{\mathcal{O}}^{r+1}$ in the recursive formula.

- For all other $2 \leq i \leq p-1$, the coefficient of $\widetilde{\mathcal{O}}^{i-1} \star \widetilde{\mathcal{O}}^{r+1}$ in $(*)$ simplifies as follows:

$$\begin{aligned}
&\frac{t_{r+1}}{t_p} T_{(i,p-1)} + \Big(\frac{t_{r+1}}{t_p} - \frac{t_{r+1}}{t_i}\Big) T_{(i+1,p-1)} \\
&\overset{\text{Lemma 3.14(a)}}{=} \quad \frac{t_{r+1}}{t_p}\Big(\frac{t_{r+1}}{t_i} - 1\Big) T_{(i+1,p-1)} + \Big(\frac{t_{r+1}}{t_p} - \frac{t_{r+1}}{t_i}\Big) T_{(i+1,p-1)} \\
&= \quad \frac{t_{r+1}}{t_i}\Big(\frac{t_{r+1}}{t_p} - 1\Big) T_{(i+1,p-1)} \\
&\overset{\text{Lemma 3.14(b)}}{=} \quad \frac{t_{r+1}}{t_i} T_{(i+1,p)},
\end{aligned}$$

  which is the coefficient of $\widetilde{\mathcal{O}}^{i-1} \star \widetilde{\mathcal{O}}^{r+1}$ in the recursive formula.

- Finally when $i = p-1$, the shift in the indices in $(*)$ yields the extra term $\frac{t_{r+1}}{t_p} \widetilde{\mathcal{O}}^{p-1} \star \widetilde{\mathcal{O}}^{r+1}$. This is precisely the $i = p$ term of the recursive formula. ■

Bibliography

[1] D. Anderson, S. Griffeth and E. Miller, *Positivity and Kleiman transversality in equivariant K-theory of homogeneous spaces*. J. Eur. Math. Soc. (JEMS) **13** (2011), no. 1, 57–84.

[2] M. Brion, *Lectures on the geometry of flag varieties. Topics in cohomological studies of algebraic varieties, 33–85*. Trends Math., Birkhäuser, Basel (2005).

[3] M. Brion and S. Kumar, *Frobenius splitting methods in geometry and representation theory, Progress in Mathematics 231*. Birkhäuser Boston, Inc., Boston, MA, (2005).

[4] A. S. Buch, P.-E. Chaput, L. Mihalcea and N. Perrin, *Finiteness of cominuscule quantum K-theory*. Ann. Sci. Éc. Norm. Supér. (4) **46** (2013), no. 3, 477–494.

[5] ———, *Projected Gromov-Witten varieties in cominuscule spaces*. Submitted, available on arXiv:1312.2468.

[6] ———, *A Chevalley formula for the equivariant quantum K-theory of cominuscule varieties*. Submitted, available on arXiv:1604.07500.

[7] A. S. Buch and L. Mihalcea, *Quantum K-theory of Grassmannians*. Duke Math. J. **156** (2011), no. 3, 501–538.

[8] ———, *Curve neighborhoods of Schubert varieties*. J. Differential Geom. **99** (2015), no. 2, 255–283.

[9] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*. Reprint of the 1997 edition. Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, (2010).

[10] O. Debarre, *Higher-dimensional algebraic geometry*. Universitext, Springer-Verlag, New York, (2001).

[11] M. Demazure, *Désingularisations des variétés de Schubert généralisées*. Ann. Sci. Éc. Norm. Supér. (4) **7** (1974), 53–88.

[12] V. V. Deodhar, *On some geometric aspects of Bruhat orderings, I: A finer decomposition of Bruhat cells*. Invent. Math. **79** (1985), no. 3, 499–511.

[13] W. Fulton, *Intersection theory. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. vol. 2, Springer-Verlag, Berlin, (1998).

[14] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology. Algebraic geometry—Santa Cruz 1995, 45–96, Proc. Sympos. Pure Math., 62, Part 2*. Amer. Math. Soc., Providence, RI, (1997).

[15] W. Fulton and C. Woodward, *On the quantum product of Schubert classes*. J. Algebraic Geom. **13** (2004), no. 4, 641–661.

[16] A. Givental, *On the WDVV equation in quantum K-theory. Dedicated to William Fulton on the occasion of his 60th birthday.* Michigan Math. J. **48** (2000), 295–304.

[17] T. Graber, J. Harris and J. Starr, *Families of rationally connected varieties*. J. Amer. Math. Soc. **16** (2003), no. 1, 57–67.

[18] W. Graham and S. Kumar, *On positivity in T-equivariant K-theory of flag varieties*. Int. Math. Res. Not. IMRN **2008**, Art. ID rnn093.

[19] S. Griffeth and A. Ram, *Affine Hecke algebras and the Schubert calculus*. European J. Combin. **25** (2004), no. 8, 1263–1283.

[20] B. Kim and R. Pandharipande, *The connectedness of the moduli space of maps to homogeneous spaces*. Symplectic geometry and mirror symmetry (Seoul, 2000), World Sci. Publ., River Edge, NJ, (2001), 187–201.

[21] Y.-P. Lee, *Quantum K-theory. I. Foundations*. Duke Math. J. **121** (2004), no. 3, 389–424.

[22] O. Pechenik and A. Yong, *Equivariant K-theory of Grassmannians*. Submitted, available on arXiv:1506.01992.

[23] S. Ramanan and A. Ramanathan, *Projective normality of flag varieties and Schubert varieties*. Invent. Math. **79** (1985), no. 2, 217–224.

[24] A. Ramanathan, *Schubert varieties are arithmetically Cohen-Macaulay*. Invent. Math. **80** (1985), no. 2, 283–294.

[25] R. W. Richardson, *Intersections of double cosets in algebraic groups*. Indag. Math. (N.S.) **3** (1992), no. 1, 69–77.

[26] S. J. Sierra, *A general homological Kleiman-Bertini theorem*. Algebra Number Theory **3** (2009), no. 5, 597–609.

[27] O. E. Villamayor U., *Patching local uniformizations*. Ann. Sci. École Norm. Sup. (4) **25** (1992), no. 6, 629–677.