SELF-SHRINKERS AND SINGULARITY MODELS OF THE MEAN CURVATURE FLOW

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ABSTRACT OF THE DISSERTATION

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This doctoral dissertation aims to generalize the uniqueness and existence results of self-shrinkers with a conical end. In addition, we study the type II singularity of Vel*á*zquez's solution to the mean curvature flow. Our results include the following:

1. Given a smooth, symmetric and homogeneous of degree one function $f(\lambda_1, \cdots, \lambda_n)$ and a properly embedded cone $\mathcal{C}$ in $\mathbb{R}^{n+1}$, we show that under some suitable conditions on f over the principal curvatures of $\mathcal{C}$, there is at most one f self-shrinker (i.e. a hypersurface Σ in $\mathbb{R}^{n+1}$ for which $f(\kappa_1, \cdots, \kappa_n) + \frac{1}{2} X \cdot N = 0$ holds, where $\kappa_1, \cdots, \kappa_n$ are principal curvatures of Σ) that is asymptotic to the given cone $\mathcal{C}$ at infinity.

2. Given a smooth, symmetric and homogeneous of degree one function $f(\lambda_1, \cdots, \lambda_n)$ satisfying $\partial_i f > 0 \quad \forall\, i = 1, \cdots, n$, and a rotationally symmetric cone $\mathcal{C}$ in $\mathbb{R}^{n+1}$, we show that there is a f self-shrinker that is asymptotic to the given cone $\mathcal{C}$ at infinity.

3. Vel*á*zquez discovered a solution to the mean curvature flow which develops a type II singularity at the origin. He showed that by performing a time-dependent rescaling of the solution around the origin, the rescaled flow converges in the C^0 sense to a minimal hypersurface which is tangent to Simons' cone at infinity. We prove that the rescaled flow actually converges locally smoothly to the minimal hypersurface, which appears to be the singularity model of the type II singularity. Moreover, we show that the mean

curvature of the solution blows up near the origin at a rate which is smaller than that of the second fundamental form. This is a joint work with N. Sesum.

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Introduction

0.1 Hypersurface and second fundamental form

Let Σ be an immersed, orientable hypersurface in a Riemannian space $\left(\mathcal{M}^{n+1}, \langle\, , \rangle\right)$, i.e. there is a manifold M^n and a parametrization

$$X : M^n \to \mathcal{M}^{n+1}$$

so that $\Sigma = X(M)$. The immersion X is called the position vector of Σ. Let N_Σ be an unit-normal vector field on Σ. Then the second fundamental form A_Σ is a defined to be 2-tensor on Σ so that

$$A_\Sigma(V, W) = \langle -D_V N_\Sigma, W\rangle = \langle D_V W, N_\Sigma\rangle$$

for any tangent vector fields V, W on Σ, where D is the Levi-Civita connection of Riemannian metric $\langle\, , \rangle$ on $\mathcal{M}$. It follows that the Hessian of the position vector X is given by

$$\nabla^2_\Sigma X(V, W) = A_\Sigma(V, W)\, N_\Sigma$$

Notice that the second fundamental form A_Σ is a symmetric 2-tensor, so at each point it has eigenvalues $\{\kappa_1, \cdots, \kappa_n\}$, which are called principal curvatures. We then define the mean curvature H_Σ to be the trace of the second fundamental form, or equivalently the sum of all the principal curvatures, i.e.

$$H_\Sigma = \operatorname{tr} A_\Sigma = \kappa_1 + \cdots + \kappa_n$$

Note that

$$\triangle_\Sigma X = H_\Sigma N_\Sigma$$

0.2 Mean curvature flow

Let $\{\Sigma_t\}_{0\leq t<T}$ be a smooth one-parameter family of hypersurfaces in a Riemannian space $\left(\mathcal{M}^{n+1},\, \langle\;,\;\rangle\right)$. That is, there is a manifold M^n and a smooth homotopy

$$X:\, M^n\times[0,\,T)\to\mathcal{M}^{n+1}$$

so that $\Sigma_t=X_t\,(M)$, where $X_t=X\,(\cdot,\,t):\, M^n\to\mathcal{M}^{n+1}$ is the position vector of the time-slice Σ_t. Assume for simplicity that M is a closed manifold (i.e. compact without boundary), then we have the following evolution formula for the area of hypersurfaces:

$$\frac{d}{dt}\int_M d\mu_t=-\int_M \langle\partial_t X_t,\, N_{\Sigma_t}\rangle\, H_{\Sigma_t}\, d\mu_t$$

where μ_t is pull-back measure of Σ_t on M. This suggests a simple and natural way to decrease the area along the flow: move hypersurfaces in such a way that

$$\langle\partial_t X_t,\, N_{\Sigma_t}\rangle=H_{\Sigma_t}$$

In other words, consider a motion of hypersurfaces for which the normal speed at each point is determined by the mean curvature. Such a motion of hypersurfaces is called a "mean curvature flow" (MCF). By reparametrizing the flow if necessary (for instance, replace X_t by $X_t\circ\varphi_t$ for some well-chosen smooth one-parameter family of diffeomorphisms $\varphi_t:\, M\to M$), we may assume that each point moves in the normal direction. In that case, we have the following equation:

$$\partial_t X_t=H_{\Sigma_t}N_{\Sigma_t}=\triangle_{\Sigma_t}X_t$$

which resembles the heat equation. For simplicity, here we only consider MCF in an Euclidean space.

There are many applications for MCF. In material science, Mullins used this flow to model the evolution of interfaces of metals. In geometry, MCF can be used to classify hypersurfaces with specific curvature conditions. In addition, since there are many similarities between the MCF and Ricci flow, people often compare these two flows, in the hope that the study of one flow can shed light on the study of the other.

0.3 Singularities of mean curvature flow

Unlike the heat equation, MCF is a quasilinear equation of the position vector and in general it may develop singularities in finite time. For instance, let's consider the evolution of a closed hypersurface by MCF in an Euclidean space. Before starting the flow, let's enclose the hypersurface by a large sphere. Then we observe the evolution of these two objects by MCF. Due to the maximum principle, the two evolving hypersurfaces must keep away from each other whenever the flows are smoothly defined. Therefore, the MCF of the given closed hypersurface must develop singularities before the sphere shrinks to a point in finite time.

Since singularities are inevitable in most of the cases, understanding their formation plays an important role in the study of MCF. In particular, we are interested to know at the first singular time, what is the sturcture of the singular set and what the flow looks like near singularities.

Ecker and Huisken in [EH] proved the following smooth estimates for the MCF. Let $\{\Sigma_t\}_{0\leq t<T}$ be a smooth MCF and P be a point. If there is $r>0$ and $0\leq\Lambda<\infty$ so that

$$\sup_{(T-r^2,T)}\ \sup_{\Sigma_t\cap B(P;r)} r\left|A_{\Sigma_t}\right|\leq\Lambda$$

then for any $m\in\mathbb{N}$, there holds

$$\sup_{\left(T-\frac{r^2}{4},T\right)}\ \sup_{\Sigma_t\cap B\left(P;\frac{r}{2}\right)} r^{m+1}\left|\nabla^m_{\Sigma_t}A_{\Sigma_t}\right|\leq C\left(n,\,\Lambda,\,m\right)$$

In particular, $\{\Sigma_t\}$ is smooth in a neighborhood of P upto time T. Therefore, singularities of MCF can be characterized by the blow-up of the second fundamental form. Moreover, by the maximum principle, if a closed MCF $\{\Sigma_t\}_{0\leq t<T}$ develops singularities as $t\nearrow T$, there holds

$$\limsup_{t\nearrow T}\ \sup_{\Sigma_t}\sqrt{T-t}\,\left|A_{\Sigma_t}\right|>0$$

Singularities are then classified according to the blow-up rate of the second fundamental form. A singular point P is said to be a type I singularity if there holds

$$\limsup_{t\nearrow T}\ \sup_{\Sigma_t\cap B(P;r)}\sqrt{T-t}\,\left|A_{\Sigma_t}\right|<\infty$$

for some $r > 0$. Otherwise, it's called a type II singularity.

In order to see what the flow looks like near a singularity, we zoom in on that point by doing a parabolic rescaling. One crucial ingredient in the analysis is Huisken's monotonicity formula (cf. [Hu2]). More precisely, let $\{\Sigma_t\}_{0\leq t<T}$ be a MCF (with polynomial volume growth) which develops a singularity at P as $t \nearrow T$, there holds

$$\frac{d}{dt}\int_{\Sigma_t} \frac{e^{-\frac{|X_t-P|^2}{4(T-t)}}}{(4\pi(T-t))^{\frac{n}{2}}} d\mathcal{H}^n(X_t) = -\int_{\Sigma_t}\left(H_{\Sigma_t} + \frac{X_t\cdot N_{\Sigma_t}}{2(T-t)}\right)^2 \frac{e^{-\frac{|X_t-P|^2}{4(T-t)}}}{(4\pi(T-t))^{\frac{n}{2}}} d\mathcal{H}^n(X_t) \tag{1}$$

Now consider the following rescaling of the flow:

$$\Pi_s = \frac{1}{\sqrt{T-t}}(\Sigma_t - P)\Big|_{s=-\ln(T-t)}, \qquad -\ln T \leq s < \infty$$

It satisfies the following equation

$$\partial_s Y_s \cdot N_{\Pi_s} = H_{\Pi_s} + \frac{1}{2} Y_s \cdot N_{\Pi_s}$$

where Y_s is the position vector of Π_s. By (1), there holds

$$\frac{d}{ds}\int_{\Pi_s} \frac{e^{-\frac{1}{4}|Y_s|^2}}{(4\pi)^{\frac{n}{2}}} d\mathcal{H}^n(Y_s) = -\int_{\Pi_s}\left(H_{\Pi_s} + \frac{1}{2}Y_s\cdot N_{\Pi_s}\right)^2 \frac{e^{-\frac{1}{4}|Y_s|^2}}{(4\pi)^{\frac{n}{2}}} d\mathcal{H}^n(Y_s)$$

In particular,

$$\frac{d}{ds}\int_{\Pi_s} \frac{e^{-\frac{1}{4}|Y_s|^2}}{(4\pi)^{\frac{n}{2}}} d\mathcal{H}^n(Y_s) \leq 0$$

Thus, the local area of the rescaled hypersurfaces are uniformly bounded. By the compactness theorem, $\{\Pi_s\}_{-\ln T\leq s<\infty}$ subconverges in the sense of Radon measure as $s \nearrow \infty$ (cf. [I]). Furthermore, by (1), any limiting hypersurfaces Π satisfies

$$H_\Pi + \frac{1}{2} Y \cdot N_\Pi = 0$$

which are called "self-shrinkers" since it generates a self-similar solution to the MCF. More precisely,

$$\tilde{\Sigma}_t = P + \sqrt{T-t}\,\Pi, \qquad t < T$$

defines a MCF for $t < T$. Roughly speaking, we can use $\left\{\tilde{\Sigma}_t\right\}_{t<T}$ to approximate the behavior of $\{\Sigma_t\}$ near the point P as $t \nearrow T$. Note that in the case when P is a type I singularity, there holds

$$\limsup_{s\nearrow\infty} \sup_{\Pi_s\cap B(O;re^s)} |A_{\Pi_s}| < \infty$$

for some $r > 0$. In fact, by the smooth estimates for MCF, all the higher order covariant derivatives of A_{Π_s} are also locally uniformly bounded in spacetime. By the compactness theorem, the $\{\Pi_s\}_{-\ln(T)\leq s<\infty}$ subconverges in the smooth topology.

0.4 Self-shrinkers

Recall that a hypersurface Σ is called a self-shrinker if it satisfies

$$H_\Sigma + \frac{1}{2} X \cdot N_\Sigma = 0$$

where X is the position vector of Σ. The classification of self-shrinkers is crucial to the study of singularities of MCF. There are some important progress in this direction. For instance, Huisken proved that a complete, mean convex self-shrinker with polynomial volume growth and bounded second fundamental form must be congruent with a generalized cylinder

$$\mathbb{S}^k\left(\sqrt{2k}\right) \times \mathbb{R}^{n-k}$$

for some $k \in \{1, \cdots, n\}$, where $\mathbb{S}^k\left(\sqrt{2k}\right)$ is the sphere in $\mathbb{R}^{k+1}$ with radius $\sqrt{2k}$. Later Colding and Minicozzi improved the above result by dropping the hypothesis of bounded second fundamental form (cf. [CM]). Moreover, they proved that generalized cylinders are actually the only "stable" hypersurfaces among all self-shrinkers.

In the case of two-dimensional self-shrinkers, Ilmanen conjectured that any complete self-shrinker with at most quadratic area growth have finitely many ends, which is either asymptotic to a cone or a cylinder. Wang has many results devoted to prove this conjecture. In particular, she proved the uniqueness of self-shrinkers with a given conical end (cf. [W]). On the other hand, Kapouleas, Kleene and Moller use the gluing method to construct complete self-shrinkers with genus and a conic end (cf. [KKM]).

Motivated by the result in [W], we would like to see under what kind of conditions can we extend the uniqueness to a more general class of flow such as a geometric flow defined by a symmetric, homogeneous of degree one function of principal curvatures? In Chapter 1, we show that if the nonlinearity of the defining function of the aforementioned flow is sufficiently small, self-shrinkers to such flow which are asymptotic to a given cone at infinity are unique.

In Chapter 2, we manage to prove the existence of self-shrinkers to the aforementioned flow with a conical end. As Kleene and Moller did in [KM], we use a fixed point argument to find a rotationally symmetric solution.

0.5 Type II singularity

To study type II singularities, Hamilton developed a rescaling process by which the rescaled flow subconverges to an eternal MCF with uniform bounded second fundamental form (cf. [M]). In the case of the mean convex MCF, by the convexity estimate (cf. [HS]) and Harnack estimate (cf. [Ha]), it can be shown that the blow-up flow is actually a translating MCF.

In order to study type II singularities in other cases, in Chapter 3 we analyze Velázquez's solution to the MCF (cf. [V]). In that case, the singularity model is given by a minimal hypersurface, which is a stationary solution to the MCF.

Chapter 1

Uniqueness of self-shrinkers to the degree-one curvature flow with a tangent cone at infinity

1.1 Introduction

Let $\mathcal{C}$ be an orientable and properly embedded smooth cone (excluding the vertex O) in $\mathbb{R}^{n+1}$. Suppose that $\mathbf{\Sigma}$ is an orientable and properly embedded smooth hypersurface in $\mathbb{R}^{n+1}$ which satisfies

$$H + \frac{1}{2} X \cdot N = 0 \quad \forall\, X \in \mathbf{\Sigma}$$

$$\varrho \mathbf{\Sigma} \xrightarrow{C^\infty_{\mathrm{loc}}} \mathcal{C} \quad \text{as } \varrho \searrow 0$$

where N is the unit-normal vector and $H = -\nabla_{\mathbf{\Sigma}} \cdot N$ is the mean curvature of $\mathbf{\Sigma}$. Then $\mathbf{\Sigma}$ is called a self-shrinker to the mean curvature flow (i.e. $\partial_t X^\perp = HN$) which is smoothly asymptotic to the cone $\mathcal{C}$ at infinity. It follows that the rescaled family of hypersufaces $\left\{\mathbf{\Sigma}_t = \sqrt{-t}\,\mathbf{\Sigma}\right\}$ forms a mean curvature flow starting from $\mathbf{\Sigma}$ (when $t = -1$) and converging locally smoothly to $\mathcal{C}$ as $t \nearrow 0$. Wang in [W] proves the uniqueness of such self-shrinkers by showing the following: suppose $\tilde{\mathbf{\Sigma}}$ is also a self-shrinker which is asymptotic to the same cone, then outside a compact set, $\tilde{\mathbf{\Sigma}}_t = \sqrt{-t}\,\tilde{\mathbf{\Sigma}}$ can be regarded as a normal graph of $\mathtt{h}_t$ defined on $\mathbf{\Sigma}_t \setminus \bar{B}_R$ for some $R > 0$; moreover, given $\varepsilon > 0$ and choose R large accordingly, there holds

$$\left|\partial_t \mathtt{h} - \triangle_{\mathbf{\Sigma}_t} \mathtt{h}\right| \leq \varepsilon \left(\left|\nabla_{\mathbf{\Sigma}_t} \mathtt{h}\right| + \left|\mathtt{h}\right|\right)$$

$$\mathtt{h}\Big|_{t=0} = 0$$

Using the idea in [ESS], Wang derives a Carleman's inequality for the heat operator on the flow $\{\mathbf{\Sigma}_t\}$, applies it to the localization of $\mathtt{h}$, and uses the unique continuation principle (see [EF], for instance) to conclude that $\mathtt{h} = 0$.

On the other hand, Andrews in [A] consider the motion of hypersurfaces in $\mathbb{R}^{n+1}$ moved by some degree one curvature. More precisely, given a smooth, symmetric and homogeneous of degree-one function $f = f(\lambda_1, \cdots, \lambda_n)$ which satisfies $\partial_i f > 0 \quad \forall\, i$, consider the following evolution of hypersurfaces:

$$\partial_t X^\perp = f(\kappa_1, \cdots, \kappa_n)\, N$$

where $\kappa_1, \cdots, \kappa_n$ are the principal curvatures of the evolving hypersurface. For instance, if we take the curvature function to be $f(\lambda_1, \cdots, \lambda_n) = \lambda_1 + \cdots + \lambda_n$, then this corresponds to the mean curvature flow. And we call an orientable C^2 hypersurface Σ in $\mathbb{R}^{n+1}$to be a "f self-shrinker" to the above "f curvature flow" provided that

$$f(\kappa_1, \cdots, \kappa_n) + \frac{1}{2} X \cdot N = 0$$

holds on Σ. Likewise, the rescaled family of "f self-shrinkers" is a self-similar solution to the f curvature flow; that is, the one-parameter family of hypersurfaces $\left\{\Sigma_t = \sqrt{-t}\,\Sigma\right\}_{t<0}$ is a f curvature flow. In the case when Σ is smoothly asymptotic to the cone $\mathcal{C}$ at infinity, the rescaled flow $\{\Sigma_t\}_{t<0}$ will converge locally smoothly to $\mathcal{C}$ as $t \nearrow 0$.

In this chapter we extend the uniqueness result of [W] to the class of f self-shrinkers with a tangent cone $\mathcal{C}$ at infinity. Based on Wang's idea of proving the uniqueness for the mean curvature flow, which works perfectly for the f curvature flow as well, we add some additional treatments for the nonlinearity of f (which is not a concern in Wang's case because the curvature function there is linear). The crucial step is to derive Carleman's inequality for the associated parabolic operator to the f curvature flow under some conditions on the nonlinearity of f , the uniform positivity of $\partial_i f$ and also some curvature bounds of $\mathcal{C}$. For this part, we are motivated by the work of Nguyen in [N] as well as Wu and Zhang in [WZ] for deriving Carleman's inequality for parabolic operator with variable coefficients.

In order to state our main result, Theorem 1.5, we have to introduce some notaions, definitions and basic assumptions. We put all of these in Section 1.2.

In Section 1.3, we essentially follow the proof of [W] to show that if Σ and $\tilde{\Sigma}$ are f self-shrinker which are asymptotic to the given cone $\mathcal{C}$ at infinity, then outside a compact set,$\tilde{\Sigma}_t = \sqrt{-t}\,\tilde{\Sigma}$ can be regarded as a normal graph of h_t defined on $\Sigma_t \setminus \bar{B}_R$ for some $R > 0$, which satisfies some parabolic equation and vanishes at time 0. We would also give some estimates on the coefficients of the parabolic operators.

In Section 1.4, we follow the idea of [ESS] for treating the backward uniqueness of the heat equation (which is also used in [W] to deal with the uniqueness of self-shrinkers of the mean curvature flow) to show that the deviation h_t would vanishes outside some compact set. We would first use the mean value inequality for parabolic equations and a local type of Carleman's inequalities to show the exponential decay of the deviation h_t as $t \nearrow 0$ as in [N]. Then we are devoted to derive a different type of Carleman's inequalities (based on the estimates of the coefficients of the parabolic operator which we derive in Section 1.3) and use it to show that h_t vanishes outside a compact set. In the end, we use the unique continuation principle to characterize the overlap region of Σ and $\tilde{\Sigma}$.

1.2 Assumptions and main results

Definition 1.1. (A regular cone)

Let $\mathcal{C}$ be an orientable and properly embedded smooth cone (excluding the vertex O) in $\mathbb{R}^{n+1}$; that is, $\mathcal{C}$ is an orientable and properly embedded hypersurface in $\mathbb{R}^{n+1}$ satisfying $\varrho\,\mathcal{C} = \mathcal{C} \quad \forall\, \varrho \in \mathbb{R}_+$ and $O \notin \mathcal{C}$.

We then define what it means for a hypersurface to be asymptotic to the cone $\mathcal{C}$ at infinity:

Definition 1.2. (Tangent cone at infinity)

A C^k hypersurface Σ in $\mathbb{R}^{n+1}$ (with $k \in \mathbb{N}$) is said to be C^k asymptotic to $\mathcal{C}$ at infinity provided that $\varrho\Sigma \xrightarrow{C^k_{\mathrm{loc}}} \mathcal{C}$ as $\varrho \searrow 0$ (see [L] for the C^k topology of hypersurfaces in $\mathbb{R}^{n+1}$). In this case, $\mathcal{C}$ is called the tangent cone of Σ at infinity.

For a given C^2 orientable hypersurface Σ in $\mathbb{R}^{n+1}$, its shape operator (or Weingarten map) $A^{\#}$ sends tangent vectors to tangent vectors and is defined by

$$A^{\#}(V) = -D_V N$$

for any tangent vector field V on Σ, where N is the unit-normal of Σ. The second fundamental form A is defined to be a 2 tensor on Σ such that

$$A(V, W) = A^{\#}(V) \cdot W$$

for any tangent vector fields V and W on Σ. The components of $A^{\#}$ and A with respect to a given local frame $\{e_1, \cdots, e_n\}$ of the tangent bundle of Σ are defined by

$$A^{\#}(e_i) = A_i^j e_j, \quad A(e_i, e_j) = A_{ij}$$

and we are used to denote $A^{\#}$ and A by their components like $A^{\#} \sim A_i^j$ and $A \sim A_{ij}$. Note that $A^{\#}$ is a self-adjoint operator with repect to the dot product restricted to the tangent space (or equivalently, A is a symmetric 2 tensor), so $A^{\#}$ is diagonalizable. The eigenvectors of $A^{\#}$ are called principal vectors and its eigenvalues are called principal curvatures, which are denoted by $\kappa_1, \cdots, \kappa_n$. The mean curvature is defined to be $H = \operatorname{tr}\left(A^{\#}\right) = \kappa_1 + \cdots + \kappa_n$, which is a linear, symmetric and homogeneous of degree-one function of the shape operator (or the principal curvatures). Here we consider a more general type of degree-one curvature.

Definition 1.3. (The degree-one curvature function)

Let $F = F(S)$ be a conjugation-invariant, homogeneous of degree-one function whose domain $\mathbf{\Omega}$ (in the space of $n \times n$ matrices) containing a neighborhood of the set consisting of all the values of shape operator $A_{\mathcal{C}}^{\#}$ of $\mathcal{C}$; besides, F can be written as a C^3 function composed with the the elementary symmetric functions $\mathcal{E}_1, \cdots, \mathcal{E}_n$ (for instance, $\mathcal{E}_1 = \operatorname{tr}$ and $\mathcal{E}_n = \det$) and $\frac{\partial F}{\partial S_i^j} > 0$ (i.e. $\frac{\partial F}{\partial S_i^j}$ is a positive matrix).

Note that by the conjugation-invariant and homogeneous property of F, we may assume that $\mathbf{\Omega}$ is closed under conjugation and homothety; that is, if $S \in \mathbf{\Omega}$, then so are RSR^{-1} and ϱS for any invetible $n \times n$ matrix R and positive number ϱ.

Also, by the condition that F can be written as a C^3 function composed with the the elementary symmetric functions, it induces a symmetric, homogeneous of degree-one function f such that

$$F(S) = f(\lambda_1, \cdots, \lambda_n)$$

whenever $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of the matrix S; the function f is defined and C^3 on an open set $\mho$ (in $\mathbb{R}^n$) containing a neighborhood of the set consisting of all the values of the pricinpal curvature vector $\left(\kappa_1^{\mathcal{C}}, \cdots, \kappa_n^{\mathcal{C}}\right)$ of $\mathcal{C}$. Likewise, we may assume that the domain $\mho$ is closed under permutation and homothety.

In fact, at a diagonal matrix $S = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$, there holds (see [A]):

$$\frac{\partial F}{\partial S_i^j}(S) = \partial_i f(\lambda_1, \cdots, \lambda_n)\, \delta_{ij} \tag{1.1}$$

$$\frac{\partial^2 F}{\partial S_i^j \partial S_i^l}(S) = \partial_{ii}^2 f(\lambda_1, \cdots, \lambda_n)\, \delta_{ij}\delta_{il} \tag{1.2}$$

$$\frac{\partial^2 F}{\partial S_i^j \partial S_k^l}(S) = \partial_{ik}^2 f(\lambda_1, \cdots, \lambda_n)\, \delta_{ij}\delta_{kl} + \frac{\partial_i f - \partial_k f}{\lambda_i - \lambda_k}\delta_{il}\delta_{kj} \quad \text{if } i \neq k \tag{1.3}$$

Since F is well-defined on conjugacy classes, (1.1), (1.2), (1.3) can be applied to any diagonalizable matrix in $\mathbf{\Omega}$. For instance, by (1.1), we have

$$\frac{\partial F}{\partial S_i^j}\left(A_{\mathcal{C}}^{\#}\right) \sim \partial_i f\left(\kappa_1^{\mathcal{C}}, \cdots, \kappa_n^{\mathcal{C}}\right)\delta_{ij}$$

where $A_{\mathcal{C}}^{\#} \sim \kappa_{\mathcal{C}}^i \delta_{ij}$ is the shape operator (and principal curvatures) of $\mathcal{C}$. Hence, by the condition that $\frac{\partial F}{\partial S_i^j} > 0$ on $\mathbf{\Omega}$, we may assume that $\partial_i f > 0 \quad \forall\, i = 1, \cdots, n$ on $\mho$.

Now let U be an open neighborhood of the set consisting of the all the shape operator $A_{\mathcal{C}}^{\#}$ of $\mathcal{C}$ at $X_{\mathcal{C}} \in \mathcal{C} \cap \left(B_3 \setminus \bar{B}_{\frac{1}{3}}\right)$ in $\mathbf{\Omega}$. Note that we may assume that U is closed under conjugation and that $\frac{\partial F}{\partial S_i^j}$ is uniformly positive on U; that is, there exist a constant $\lambda \in (0, 1]$ so that

$$\lambda \delta_j^i \leq \frac{\partial F}{\partial S_i^j} \leq \frac{1}{\lambda}\delta_j^i \tag{1.4}$$

Also, we have

$$\varkappa \equiv \sup_{X_{\mathcal{C}} \in \mathcal{C} \cap \left(B_3 \setminus \bar{B}_{\frac{1}{3}}\right)} \left| \nabla_{\mathcal{C}} \left(\frac{\partial F}{\partial S_i^j}\left(A_{\mathcal{C}}^{\#}\right)\right) \right| \tag{1.5}$$

$$= \sup_{X_{\mathcal{C}} \in \mathcal{C} \cap \left(B_3 \setminus \bar{B}_{\frac{1}{3}}\right)} \left| \sum_{k,\, l} \frac{\partial^2 F}{\partial S_i^j \partial S_k^l}\left(A_{\mathcal{C}}^{\#}\right)\left(\nabla_{\mathcal{C}} A_{\mathcal{C}}^{\#}\right)_k^l \right| \leq C\left(n, \mathcal{C}, \| F \|_{C^2(U)}\right)$$

where $A^{\#}_{\mathcal{C}}$ and $\nabla_{\mathcal{C}} A^{\#}_{\mathcal{C}}$ are the shape operator of $\mathcal{C}$ and its covariant derivative at $X_{\mathcal{C}}$, respectively; B_{ϱ} is the ball of radius ϱ in $\mathbb{R}^{n+1}$. We give a more precise estimate of $\varkappa$ in (1.164) for the case when $\mathcal{C}$ is rotationally symmetric.

Now we can define the F self-shinker:

Definition 1.4. (F self-shinker)

An oriented C^2 hypersurface Σ (excluding its boundary) in $\mathbb{R}^{n+1}$ is called a F self-shinker (or f self-shrinker) provided that F is defined on the shape operator $A^{\#}$ of Σ (i.e. $A^{\#} \in \mathbf{\Omega}$) and satisfies

$$F\left(A^{\#}\right) + \frac{1}{2} X \cdot N = 0$$

where X is the position vector, N is the unit-normal, and $A^{\#}$ is the shape operator of Σ; or equivalently, f is defined on the principal curvatures of Σ (i.e. $(\kappa_1, \cdots, \kappa_n) \in \mho$) and satisfies

$$f\left(\kappa_1, \cdots, \kappa_n\right) + \frac{1}{2} X \cdot N = 0$$

where $\kappa_1, \cdots, \kappa_n$ are the principal curvatures of Σ.

Note that the rescaled family of F self-shrinkers forms a self-similar solution to the F curvature flow. More precisely, the one-parameter family $\left\{\Sigma_t = \sqrt{-t}\,\Sigma\right\}_{-1 \leqslant t < 0}$ is a motion of a hypersurface moved by F curvature vector. That is,

$$\partial_t X^{\perp} = F\left(A^{\#}\right) N$$

where $\partial_t X^{\perp}$ is the normal projection of $\partial_t X$. Besides, for each time slice $\Sigma_t = \sqrt{-t}\,\Sigma$, there holds

$$F\left(A^{\#}\right) + \frac{X \cdot N}{2(-t)} = 0$$

We would prove the following uniqueness result F self-shrinkers with a tangent cone in Section 1.4:

Theorem 1.5. *(Uniqueness of self-shrikers with a conical end)*

Assume that $\varkappa \leq 6^{-4} \lambda^3$ *(in (1.4), (1.5)). Then for any properly embedded* F *self-shrinkers* Σ *and* $\tilde{\Sigma}$ *which are* C^5 *asymptotic to the cone* $\mathcal{C}$ *at infinity, there exists* $R = R\left(\Sigma,\, \tilde{\Sigma},\, \mathcal{C},\, U,\, \| F \|_{C^3(U)},\, \lambda,\, \varkappa\right) \geq 1$ *so that* $\Sigma \setminus B_R = \tilde{\Sigma} \setminus B_R$. *Moreover, let*

$$\Sigma^0 = \left\{ X \in \Sigma \cap \tilde{\Sigma} \,\middle|\, \Sigma \textit{ coincides with } \tilde{\Sigma} \textit{ in a neighborhood of } X \right\}$$

then Σ^0 *is a nonempty hypersurface, which satisfies* $\partial\Sigma^0 \subseteq \left(\partial\Sigma \cup \partial\tilde{\Sigma}\right)$.

Remark 1.6. In the case of [W], $F = \mathcal{E}_1$ (or equivalentsly, $f(\lambda_1, \cdots, \lambda_n) = \lambda_1 + \cdots + \lambda_n$) is a linear function, so (by (1.5), (1.2), (1.3)) $\varkappa \equiv 0$ and the hypothesis of Theorem 1.5 is trivially satisfied. On the other hand, consider

$$F = \mathcal{E}_1 \pm \epsilon \frac{\mathcal{E}_n}{\mathcal{E}_{n-1}}$$

or equivalently,

$$f(\lambda_1, \cdots, \lambda_n) = (\lambda_1 + \cdots + \lambda_n) \pm \epsilon \frac{\prod_{i=1}^n \lambda_i}{\sum_{i=1}^n \left(\prod_{j \neq i} \lambda_j\right)}$$

and take $\mathcal{C}$ to be a rotationally symmetric cone. Then by Theorem 1.5 and (1.164) in Section 1.4, the uniqueness holds when $0 < \epsilon \ll 1$.

1.3 Deviation between two F self-shinkers with the same asymptotic behaviour at infinity

Let Σ be a properly embedded F self-shrinker (in Definition 1.4) which is C^5 asymptotic to the cone $\mathcal{C}$ at infinity.

By Definition 1.2, $\varrho\Sigma$ can be arbitrary C^5 close to $\mathcal{C}$ on any fixed bounded set of $\mathbb{R}^{n+1}$ which is away from the origin (e.g. on $B_2 \setminus \bar{B}_{\frac{1}{2}}$) as long as ϱ is sufficiently small, so any "rescaled C^5 quantities" of $\Sigma \setminus \bar{B}_R$ can estimated by that of $\mathcal{C}$ for $R \gg 1$. Below we would show these in detail.

First of all, there exists $R \gg 1$ (depending on Σ, $\mathcal{C}$) such that outside a compact set, Σ is a normal graph over $\mathcal{C} \setminus \bar{B}_R$, say $X = \mathbf{\Psi}(X_{\mathcal{C}}) = X_{\mathcal{C}} + \psi N_{\mathcal{C}}$, where $X_{\mathcal{C}}$ is the position vector of $\mathcal{C}$ and $N_{\mathcal{C}}$ is the unit-normal of $\mathcal{C}$ at $X_{\mathcal{C}}$. Consequently, we can define the "normal projecton" Π (to be the inverse map of $\mathbf{\Psi}$) which sends $X \in \Sigma$ to $X_{\mathcal{C}} \in \mathcal{C}$. Moreover, by the rescaling argument, we may assume that $\mathcal{H}^n\left(\Sigma \cap \left(B_{2\mathrm{r}} \setminus \bar{B}_{\mathrm{r}}\right)\right) \leq C(n, \mathcal{C})\,\mathrm{r}^n$ for all $\mathrm{r} \geq R$ (i.e. Σ has polynomial volume growth).

On the other hand, fix $\hat{X}_{\mathcal{C}} \in \mathcal{C} \setminus \bar{B}_R$, $|\hat{X}_{\mathcal{C}}|^{-1}\mathcal{C} = \mathcal{C}$ is locally (near $|\hat{X}_{\mathcal{C}}|^{-1}\hat{X}_{\mathcal{C}}$) a graph over the tangent hyperplanethe $T_{|\hat{X}_{\mathcal{C}}|^{-1}\hat{X}_{\mathcal{C}}}\mathcal{C}$, so by Definition 1.2, $|\hat{X}_{\mathcal{C}}|^{-1}\Sigma$ must also be a local graph over $T_{|\hat{X}_{\mathcal{C}}|^{-1}\hat{X}_{\mathcal{C}}}\mathcal{C}$ and is C^5 close to the corresponding graph of $|\hat{X}_{\mathcal{C}}|^{-1}\mathcal{C} = \mathcal{C}$. Furthermore, we may choose a uniform constant $\rho \in (0, \frac{1}{8}]$ (depending on the dimension n, the volume and the C^3 bound of the curvature of $\mathcal{C} \cap \left(B_3 \setminus \bar{B}_{\frac{1}{3}}\right)$) so that near $|\hat{X}_{\mathcal{C}}|^{-1}\hat{X}_{\mathcal{C}}$, the graphes of $|\hat{X}_{\mathcal{C}}|^{-1}\mathcal{C} = \mathcal{C}$ and $|\hat{X}_{\mathcal{C}}|^{-1}\Sigma$ are defined on $B^n_{\rho|\hat{X}_{\mathcal{C}}|} = \left\{x \in \mathbb{R}^n \middle| \, |x| < \rho|\hat{X}_{\mathcal{C}}|\right\} \subset T_{|\hat{X}_{\mathcal{C}}|^{-1}\hat{X}_{\mathcal{C}}}\mathcal{C}$ and the C^1 norm of the local graph of $\mathcal{C}$ is small. By undoing the rescaling, it tranlates into the following: there exists $R = R(\Sigma, \mathcal{C}) \geq 1$ so that near each $\hat{X}_{\mathcal{C}} \in \mathcal{C} \setminus \bar{B}_R$, $\mathcal{C}$ and Σ can be repectively parametrized by

$$X_{\mathcal{C}} = X_{\mathcal{C}}(x) \equiv \hat{X}_{\mathcal{C}} + (x, \mathtt{w}(x))$$

$$X = X(x) \equiv \hat{X}_{\mathcal{C}} + (x, \mathtt{u}(x))$$

for $x = (x_1, \cdots, x_n) \in B^n_{\rho|\hat{X}_{\mathcal{C}}|}$, such that $\mathtt{w}(0) = 0$, $\partial_x \mathtt{w}(0) = 0$ and

$$|\hat{X}_{\mathcal{C}}|^{-1} \parallel \mathtt{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} + \parallel \partial_x \mathtt{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} \leq \frac{1}{16} \tag{1.6}$$

$$|\hat{X}_{\mathcal{C}}| \parallel \partial_x^2 \mathtt{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} + \cdots + |\hat{X}_{\mathcal{C}}|^4 \parallel \partial_x^5 \mathtt{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} \leq C(n, \mathcal{C}) \tag{1.7}$$

$$\begin{aligned} |\hat{X}_{\mathcal{C}}|^{-1} \parallel \mathtt{u} - \mathtt{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} + \parallel \partial_x \mathtt{u} - \partial_x \mathtt{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} + |\hat{X}_{\mathcal{C}}| \parallel \partial_x^2 \mathtt{u} - \partial_x^2 \mathtt{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} + \cdots \\ + |\hat{X}_{\mathcal{C}}|^4 \parallel \partial_x^5 \mathtt{u} - \partial_x^5 \mathtt{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} \leq \frac{1}{16} \end{aligned} \tag{1.8}$$

where we assume the unit-normal of $\mathcal{C}$ at $\hat{X}_{\mathcal{C}}$ to be $(0, 1)$ for ease of notation (and hence $\Pi(X(0)) = \hat{X}_{\mathcal{C}}$). Note that (1.6) is the rescale of the smallness of the C^1 norm of the local graph of $\mathcal{C}$, while (1.8) is the rescale of the small C^5 difference between the local graphes of $|\hat{X}_{\mathcal{C}}|^{-1}\mathcal{C}$ and $|\hat{X}_{\mathcal{C}}|^{-1}\Sigma$.

By Definition 1.2 and the rescaling argument, the same thing holds for each rescaled hypersurface $\Sigma_t = \sqrt{-t}\,\Sigma$, $t \in [-1, 0)$ as well. That is, outside a compact set, Σ_t

is a normal graph over $\mathcal{C} \setminus \bar{B}_R$ (with $R \gg 1$ depending on Σ, $\mathcal{C}$); besideis, near each $\hat{X}_{\mathcal{C}} \in \mathcal{C} \setminus \bar{B}_R$, Σ_t is a graph over $T_{|\hat{X}_{\mathcal{C}}|^{-1}\hat{X}_{\mathcal{C}}}\mathcal{C}$ and can be parametrized by

$$X_t(x) = X(x,\, t) \equiv \hat{X}_{\mathcal{C}} + (x,\, \mathtt{u}_t(x)) = \hat{X}_{\mathcal{C}} + (x,\, \mathtt{u}(x,\, t))$$

which satisfies

$$|\hat{X}_{\mathcal{C}}|^{-1} \parallel \mathtt{u}(\cdot,\, t) - w \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} + \parallel \partial_x \mathtt{u}(\cdot,\, t) - \partial_x \mathtt{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} + |\hat{X}_{\mathcal{C}}| \parallel \partial_x^2 \mathtt{u}(\cdot,\, t) - \partial_x^2 \mathtt{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} +$$

$$\cdots + |\hat{X}_{\mathcal{C}}|^4 \parallel \partial_x^5 \mathtt{u}(\cdot,\, t) - \partial_x^5 \mathtt{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} \leq \frac{1}{16} \tag{1.9}$$

We call $t \mapsto X(x,\, t) = \hat{X}_{\mathcal{C}} + (x,\, \mathtt{u}(x,\, t))$ is the "vertical parametrization" of the flow $\{\Sigma_t\}_{-1 \leq t < 0}$. Note that by (1.6), (1.9) and $0 < \rho \leq \frac{1}{8}$, we have

$$\frac{3}{4}|\hat{X}_{\mathcal{C}}| \;\leq\; |X(x,\, t)| \;=\; |\hat{X}_{\mathcal{C}} + (x,\, \mathtt{u}(x,\, t))| \;\leq\; \frac{5}{4}|\hat{X}_{\mathcal{C}}|$$

for $x \in B^n_{\rho|\hat{X}_{\mathcal{C}}|}$, $t \in [-1,\, 0)$; that is, $|X|$ is comparable with $|\hat{X}_{\mathcal{C}}|$. Also, we still have the following polynomial volume growth for Σ_t:

$$\mathcal{H}^n\left(\Sigma_t \cap \left(B_{2\mathrm{r}} \setminus \bar{B}_{\mathrm{r}}\right)\right) \leq C(n,\, \mathcal{C})\, \mathrm{r}^n \tag{1.10}$$

for all $\mathrm{r} \geq R$.

On the other hand, Σ is a F self-shrinker, which we can use to improve (1.9). To see this, observe that under the condtions of being a F self-shrinker and having a tangent cone $\mathcal{C}$ at infinity, the rescaled flow $\left\{\Sigma_t = \sqrt{-t}\,\Sigma\right\}_{-1 \leq t < 0}$ moves by F curvature vector and converges (in the locally C^5 sense) to the cone $\mathcal{C}$ as $t \nearrow 0$. In other words, we can define a F curvature flow $\{\Sigma_t\}_{-1 \leq t \leq 0}$ with $\Sigma_t = \sqrt{-t}\,\Sigma$ for $t \in [-1,\, 0)$ and $\Sigma_0 = \mathcal{C}$ which is continuous upto $t = 0$ (in the locally C^5 sense). Besides, near each $\hat{X}_{\mathcal{C}} \in \mathcal{C} \setminus \bar{B}_R$ (with $R \gg 1$ depending on Σ, $\mathcal{C}$), we have the vertical parametrization of the flow (as above) for $t \in [-1,\, 0]$ and the evolution of u_t satisfies (by Definition 1.4)

$$\partial_t \mathtt{u} = \sqrt{1 + |\partial_x \mathtt{u}|^2}\, F\left(A_i^j(x,\, t)\right) \quad \text{for } (x_1, \cdots, x_n) \in B^n_{\rho|\hat{X}_{\mathcal{C}}|},\ -1 \leq t < 0 \tag{1.11}$$

$$\mathtt{u}(\cdot,\, t) \xrightarrow{C^5} w \quad \text{on } B^n_{\rho|\hat{X}_{\mathcal{C}}|} \quad \text{as } t \nearrow 0 \tag{1.12}$$

where the shape operator $A_t^{\#}(x) \sim A_i^j(x,\, t)$ of Σ_t (with respect to the local coordinate frame $\{\partial_1 X_t, \cdots, \partial_n X_t\}$) is equal to

$$A_i^j(x,\, t) = \partial_i\left(\frac{\partial_j \mathtt{u}(x,\, t)}{\sqrt{1 + |\partial_x \mathtt{u}|^2}}\right) \tag{1.13}$$

It follows (by using (1.11), (1.9), (1.6), (1.7) and (1.13)) that

$$|\partial_t \mathtt{u}| = |\hat{X}_{\mathcal{C}}|^{-1}\sqrt{1+|\partial_x \mathtt{u}|^2}\left|F\left(|\hat{X}_{\mathcal{C}}|A_i^j(x,\,t)\right)\right|$$

$$\leq |\hat{X}_{\mathcal{C}}|^{-1}\left(1+\parallel \partial_x \mathtt{u}_t \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})}\right)\parallel F \parallel_{L^\infty(U)}$$

in which we use the homogeneity of F. Similarly, by differentiating (1.11) and using the homogenity of the derivatives of F, we get

$$|\hat{X}_{\mathcal{C}}| \parallel \partial_t \mathtt{u}(\cdot,\,t) \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} + |\hat{X}_{\mathcal{C}}|^2 \parallel \partial_t\partial_x \mathtt{u}(\cdot,\,t) \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} + |\hat{X}_{\mathcal{C}}|^3 \parallel \partial_t\partial_x^2 \mathtt{u}(\cdot,\,t) \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})}$$

$$+|\hat{X}_{\mathcal{C}}|^4 \parallel \partial_t\partial_x^3 \mathtt{u}(\cdot,\,t) \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} \leq C\left(n,\,\mathcal{C},\,\parallel F \parallel_{C^3(U)}\right) \tag{1.14}$$

which implies (by (1.14) and (1.11))

$$|\mathtt{u}(\cdot,\,t)-w| \leq \int_t^0 |\partial_t \mathtt{u}(\cdot,\,\tau)| \leq C\left(n,\,\mathcal{C},\,\parallel F \parallel_{C^3(U)}\right)|\hat{X}_{\mathcal{C}}|^{-1}(-t)$$

Likewise, integrate the estimates for derivatives in (1.14) to get $\forall\, t\in[-1,\,0]$

$$|\hat{X}_{\mathcal{C}}| \parallel \mathtt{u}(\cdot,\,t)-\mathtt{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} + |\hat{X}_{\mathcal{C}}|^2 \parallel \partial_x \mathtt{u}(\cdot,\,t)-\partial_x \mathtt{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} + |\hat{X}_{\mathcal{C}}|^3 \parallel \partial_x^2 \mathtt{u}(\cdot,\,t)-\partial_x^2 \mathtt{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})}$$

$$+|\hat{X}_{\mathcal{C}}|^4 \parallel \partial_x^3 \mathtt{u}(\cdot,\,t)-\partial_x^3 \mathtt{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} \leq C\left(n,\,\mathcal{C},\,\parallel F \parallel_{C^3(U)}\right)(-t) \tag{1.15}$$

which is the improvement of (1.9) by using the F self-shrinker equation (1.11).

In view of the pull-back metric $g_{ij}(x,\,t)=\delta_{ij}+\partial_i u(x,\,t)\;\partial_j u(x,\,t)$ and the associated Christoffel symbols

$$\Gamma_{ij}^k(x,\,t) = \frac{\partial_k \mathtt{u}(x,\,t)\;\partial_{ij}^2 \mathtt{u}(x,\,t)}{1+|\partial_x \mathtt{u}(x,\,t)|^2} \tag{1.16}$$

together with (1.13), (1.15), the comparablity of $|X|$ and $|\hat{X}_{\mathcal{C}}|$, (1.4), (1.5) and the continuity and homogeniety of F (and its derivatives), there exits $R\geq 1$ (depending on $\Sigma,\,\mathcal{C},\,U,\,\parallel F \parallel_{C^3(U)},\,\lambda,\,\varkappa$) such that for $X_t\in\Sigma_t\setminus\bar{B}_R$, the following hold:

$$|X_t|\,A_t^{\#}\in U \tag{1.17}$$

$$\frac{\lambda}{2}\delta_j^i \leq \frac{\partial F}{\partial S_i^j}\left(A_t^{\#}\right) = \frac{\partial F}{\partial S_i^j}\left(|X_t|\,A_t^{\#}\right) \leq \frac{2}{\lambda}\delta_j^i \tag{1.18}$$

$$|X_t|\Big|\sum_{k,l}\frac{\partial^2 F}{\partial S_i^j\partial S_k^l}\left(A_t^{\#}\right)\left(\nabla_{\Sigma_t}A_t^{\#}\right)_k^l\Big| = \Big|\sum_{k,l}\frac{\partial^2 F}{\partial S_i^j\partial S_k^l}\left(|X_t|A_t^{\#}\right)\right)\cdot\left(|X_t|^2\nabla_{\Sigma_t}A_t^{\#}\right)_k^l\Big| \leq 2\varkappa \tag{1.19}$$

$$|X_t|\,|A_t^{\#}| + |X_t|^2|\nabla_{\Sigma_t}A_t^{\#}| + |X_t|^3\,|\,|\nabla^2_{\Sigma_t}A_t^{\#}| \leq C\,(n,\,\mathcal{C}) \tag{1.20}$$

where $A_t^{\#}$ is the shape operator of Σ_t at X_t and $\nabla_{\Sigma_t}A_t^{\#}$ is the covariant derivative of $A_t^{\#}$ (with rescpect to Σ_t). Note that F is homogeneous of degree 1, $\frac{\partial F}{\partial S_i^j}$ is of degree 0 and $\frac{\partial^2 F}{\partial S_i^j\partial S_k^l}$ is of degree -1.

Now let $\tilde{\Sigma}$ to be a F self-shrinker which is also C^5 asymptotic to $\mathcal{C}$ at infinity. By the same limting behaviour, $\tilde{\Sigma}$ is C^5close to Σ (in the rescale sense) for $|X|\gg 1$, and hence it can be regarded as a normal graph of a function h defined on Σ. Later we would derive an elliptic equation which is satisfied by h. To this end, we need the following two lemmas (Lemma 1.7 & Lemma 1.9). The first one gives the decay rate of the fuction h and the difference of the shape operators between Σ and $\tilde{\Sigma}$ as $|X|\nearrow\infty$; in the second one, we estimate the coefficients of the differential equation to be satisfied by h.

Lemma 1.7. *There exits $R = R\left(\Sigma,\,\tilde{\Sigma},\,n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)}\right)\geq 1$ so that outside a compact set, $\tilde{\Sigma}$ is a normal graph over $\Sigma\setminus\bar{B}_R$ and can be parametrized as*

$$\tilde{X} = X + hN \;\; \textit{for}\; X\in\Sigma\setminus\bar{B}_R$$

where N is the inward unit-normal of Σ and h is the deviation of $\tilde{\Sigma}$ from Σ. Besides, there hold

$$\|\,|X|\,h\,\|_{L^\infty(\Sigma\setminus\bar{B}_R)} + \|\,|X|^2\nabla_\Sigma h\,\|_{L^\infty(\Sigma\setminus\bar{B}_R)} + \|\,|X|^3\nabla^2_\Sigma h\,\|_{L^\infty(\Sigma\setminus\bar{B}_R)} \leq C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)}\right) \tag{1.21}$$

$$\|\,|X|^3\left(\tilde{A}^{\#} - A^{\#}\right)\|_{L^\infty(\Sigma\setminus\bar{B}_R)} + \|\,|X|^4\left(\nabla_\Sigma\tilde{A}^{\#} - \nabla_\Sigma A^{\#}\right)\|_{L^\infty(\Sigma\setminus\bar{B}_R)} \leq C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)}\right) \tag{1.22}$$

$$\|\,|X|^3\nabla^2_\Sigma\tilde{A}^{\#}\,\|_{L^\infty(\Sigma\setminus\bar{B}_R)} \leq C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)}\right) \tag{1.23}$$

where $\tilde{A}^{\#}$ is the shape operator of $\tilde{\Sigma}$ at $\tilde{X} = X + hN$ and $\nabla_{\Sigma}\tilde{A}^{\#}$ is the covariant derivative of $\tilde{A}^{\#}$ (which can be regarded as a 2-tensor on Σ via the normal graphic parametrization) with rescpect to Σ.

Proof. Choose $R \gg 1$ (depending on Σ, $\tilde{\Sigma}$, n, $\mathcal{C}$, $\| F \|_{C^3(U)}$) so that $\Sigma \setminus \bar{B}_R$ and $\tilde{\Sigma} \setminus \bar{B}_R$ have the local graph coordinates over tangent hyperplanes of $\mathcal{C}$ with appropriate estimates for the graphes as before. That is, for each $\hat{X} \in \Sigma \setminus \bar{B}_R$, we can respectively parametrize Σ and $\tilde{\Sigma}$ locally (near $\Pi\left(\hat{X}\right) = \hat{X}_{\mathcal{C}} \in \mathcal{C}$) by

$$X = X(x) \equiv \Pi\left(\hat{X}\right) + (x,\, \mathtt{u}(x))$$
$$\tilde{X} = \tilde{X}(x) \equiv \Pi\left(\hat{X}\right) + (x,\, \tilde{\mathtt{u}}(x))$$

for $x = (x_1, \cdots, x_n) \in B^n_{\rho|\Pi(\hat{X})|}$, which satisfy (by (1.6), (1.7), (1.8) and the comparability of $|\hat{X}|$ and $|\hat{X}_{\mathcal{C}}|$)

$$|\hat{X}|^{-1} \| \mathtt{u} \|_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} + \| \partial_x \mathtt{u} \|_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} + |\hat{X}| \| \partial_x^2 \mathtt{u} \|_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} + \cdots$$
$$+ |\hat{X}|^4 \| \partial_x^5 \mathtt{u} \|_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} \leq C\,(n,\, \mathcal{C}) \tag{1.24}$$

$$|\hat{X}|^{-1} \| \tilde{\mathtt{u}} \|_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} + \| \partial_x \tilde{\mathtt{u}} \|_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} + |\hat{X}| \| \partial_x^2 \tilde{\mathtt{u}} \|_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} + \cdots$$
$$+ |\hat{X}|^4 \| \partial_x^5 \tilde{\mathtt{u}} \|_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} \leq C\,(n,\, \mathcal{C}) \tag{1.25}$$

Also, by applying the triangle inequality to (1.15), we get

$$|\hat{X}| \| \tilde{\mathtt{u}} - \mathtt{u} \|_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} + |\hat{X}|^2 \| \partial_x \tilde{\mathtt{u}} - \partial_x \mathtt{u} \|_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} + |\hat{X}|^3 \| \partial_x^2 \tilde{\mathtt{u}} - \partial_x^2 \mathtt{u} \|$$
$$+ |\hat{X}|^4 \| \partial_x^3 \tilde{\mathtt{u}} - \partial_x^3 \mathtt{u} \|_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} \leq C\left(n,\, \mathcal{C},\, \| F \|_{C^3(U)}\right) \tag{1.26}$$

By (1.26), we may assume that $\tilde{\Sigma}$ is a normal graph of h defined on $\Sigma \setminus \bar{B}_R$; that is, for each $x \in B^n_{\frac{\rho}{2}|\Pi(\hat{X})|}$, there is a unique $y \in B^n_{\rho|\Pi(\hat{X})|}$ such that

$$\Pi\left(\hat{X}\right) + (x,\, \mathtt{u}\,(x)) + h(x)\frac{(-\partial_x \mathtt{u},\, 1)}{\sqrt{1 + |\partial_x \mathtt{u}|^2}} = \Pi\left(\hat{X}\right) + (y,\, \tilde{\mathtt{u}}\,(y)) \tag{1.27}$$

or equivalently,

$$\left(x-h(x)\frac{\partial_x\mathtt{u}}{\sqrt{1+|\partial_x\mathtt{u}|^2}},\ \mathtt{u}\,(x)+\frac{h(x)}{\sqrt{1+|\partial_x\mathtt{u}|^2}}\right)=(y,\,\tilde{\mathtt{u}}\,(y))$$

where $\frac{(-\partial_x\mathtt{u},1)}{\sqrt{1+|\partial_x\mathtt{u}|^2}}$ is the unit normal N of Σ at $\Pi\left(\hat{X}\right)+(x,\,\mathtt{u}\,(x))$. In other words, h is defined implicitly by the following equation

$$\tilde{\mathtt{u}}\,(\psi(x))-\left(\mathtt{u}+\frac{h(x)}{\sqrt{1+|\partial_x\mathtt{u}|^2}}\right)=0 \tag{1.28}$$

where

$$\psi(x)=x-h(x)\frac{\partial_x\mathtt{u}}{\sqrt{1+|\partial_x\mathtt{u}|^2}} \tag{1.29}$$

defines a map from $B^n_{\frac{\rho}{2}|\Pi(\hat{X})|}$ into $B^n_{\rho|\Pi(\hat{X})|}$. Since $|h\,(x)\,|$ stands for the distance from the point $\Pi\left(\hat{X}\right)+(\psi(x),\,\tilde{u}\,(\psi(x)))$ on $\tilde{\Sigma}$ (i.e. the RHS of (1.27)) to Σ, we immediately have

$$|h(x)|\ \le\ |\tilde{\mathtt{u}}\,(\psi(x))-\mathtt{u}(\psi(x))|\ \le C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)}\right)|\hat{X}|^{-1}$$

To proceed further, first notice that for the unit normal vectors of Σ and $\tilde{\Sigma}$

$$N\,(x)=\frac{(-\partial_x\mathtt{u},\,1)}{\sqrt{1+|\partial_x\mathtt{u}|^2}},\quad \tilde{N}\,(x)=\frac{(-\partial_x\tilde{\mathtt{u}},\,1)}{\sqrt{1+|\partial_x\tilde{\mathtt{u}}|^2}} \tag{1.30}$$

respectively, we may assume, by (1.26), (1.24), that

$$\|\,\tilde{N}-N\,\|_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})}+\|\,N\circ\psi-N\,\|_{L^\infty(B^n_{\frac{\rho}{2}|\Pi(\hat{X})|})}\le\frac{1}{3}$$

which implies that for each $x\in B^n_{\frac{\rho}{2}|\Pi(\hat{X})|}$,

$$\begin{aligned}&\tilde{N}\,(\psi(x))\cdot N(x)\ge\ N(x)\cdot N(x)\ -\ |\tilde{N}\,(\psi(x))-N(x)|\,|N(x)|\\&\ge 1\ -\ \left(|\tilde{N}\,(\psi(x))-N\,(\psi(x))\,|\ +\ |N\,(\psi(x))-N(x)|\right)\ \ge\frac{2}{3}\end{aligned} \tag{1.31}$$

Let

$$\Theta\,(x,\,s)=\tilde{\mathtt{u}}\left(x-s\frac{\partial_x\mathtt{u}}{\sqrt{1+|\,\partial_x\mathtt{u}\,|^2}}\right)-\left(\mathtt{u}+\frac{s}{\sqrt{1+|\,\partial_x\mathtt{u}\,|^2}}\right)$$

then by (1.28), (1.29) and (1.31), we have $\Theta\,(x,\,h(x))=0$ and

$$\partial_s\Theta\,(x,\,h(x))=-\sqrt{1+|\,\partial_y\tilde{\mathtt{u}}\,(\psi(x))\,|^2}\ \tilde{N}\,(\psi(x))\cdot N(x)\le-\frac{2}{3}$$

Therefore, by the implicit function theorem, we have $h \in C^2\left(B^n_{\frac{\varrho}{2}|\Pi(\hat{X})|}\right)$. Besides, by doing the implicit differentiation of (1.28) (or $\Theta\,(x,\,h(x)) = 0$), we get

$$\frac{1+\partial_j\tilde{\mathtt{u}}\circ\psi\cdot\partial_j\mathtt{u}}{\sqrt{1+|\partial_x\mathtt{u}|^2}}\partial_i h \;=\; (\partial_i\tilde{\mathtt{u}}\circ\psi-\partial_i\mathtt{u}) \tag{1.32}$$

$$-\left(\partial_j\tilde{\mathtt{u}}\circ\psi\cdot\partial_i\frac{\partial_j\mathtt{u}}{\sqrt{1+|\partial_x\mathtt{u}|^2}}+\partial_j\mathtt{u}\frac{\partial^2_{ij}\mathtt{u}}{(1+|\partial_x\mathtt{u}|^2)^{\frac{3}{2}}}\right)h$$

in which we sum over repeated indicies. Note that we can use (1.32), together with (1.24) and (1.26), to estimate $\partial_x h$. For instance, for the first term on the RHS of the equation, we have

$$|\partial_i\tilde{\mathtt{u}}\circ\psi-\partial_i\mathtt{u}|\;\leq\;|\partial_i\tilde{\mathtt{u}}\circ\psi-\partial_i\mathtt{u}\circ\psi|\;+|\partial_i\mathtt{u}\circ\psi-\partial_i\mathtt{u}|$$

$$\leq C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)}\right)\,|\hat{X}|^{-2}+\sum_j\int_0^1\left|\partial^2_{ij}\mathtt{u}\left(x-\theta h\frac{\partial_x\mathtt{u}}{\sqrt{1+|\,\partial_x\mathtt{u}\,|^2}}\right)\right|d\theta\;\frac{|\partial_j\mathtt{u}|}{\sqrt{1+|\partial_x\mathtt{u}|^2}}|h|$$

$$\leq C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)}\right)\,|\hat{X}|^{-2}$$

Thus we get $\|\,\partial_x h\,\|_{L^\infty(B^n_{\frac{\varrho}{2}|\Pi(\hat{X})|})}\leq\,C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)}\right)\,|\hat{X}|^{-2}$. Similarly, doing the implicit differentiation of (1.32) and using (1.24) and (1.26) yields $\|\,\partial^2_x h\,\|_{L^\infty(B^n_{\frac{\varrho}{2}|\Pi(\hat{X})|})}$ $\leq C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)}\right)\,|\hat{X}|^{-3}$. The bounds on the covariant derivatives of h follow from the the following estimates on the pull-back metric $g_{ij} = \partial_i X\cdot\partial_j X$ and the Christoffel symbols Γ^k_{ij} in (1.16) associated with the local coordinates $x = (x_1,\cdots,\,x_n)$:

$$\delta_{ij}\,\leq\,g_{ij} = 1+\partial_i\mathtt{u}\,\partial_j\mathtt{u}\,\leq\,\frac{5}{4}\delta_{ij} \tag{1.33}$$

$$|\Gamma^k_{ij}|\;=\;\frac{|\partial_k\mathtt{u}|}{1+|\partial_x\mathtt{u}|^2}|\partial^2_{ij}\mathtt{u}|\;\leq C\,(n,\,\mathcal{C},\,F)\;|\hat{X}|^{-1} \tag{1.34}$$

where we have used (1.24). This completes the derivation of (1.21).

As for (1.22), let's first observe that the normal graph reparametrization of $\tilde{\Sigma}$ amounts to the following change of variables:

$$\tilde{X} = \Pi\left(\hat{X}\right)+(y,\,\tilde{\mathtt{u}}(y))\quad\text{with}\quad y=\psi(x)=x-h(x)\frac{\partial_x\mathtt{u}}{\sqrt{1+|\partial_x\mathtt{u}|^2}} \tag{1.35}$$

Note that from (1.35), (1.24) and (1.21), we have

$$\frac{\partial y_k}{\partial x_i}\;=\delta^k_i-\,h\cdot\partial_{x_i}\left(\frac{\partial_{x_j}\mathtt{u}}{\sqrt{1+|\partial_x\mathtt{u}|^2}}\right)-\,\partial_{x_i}h\frac{\partial_k\mathtt{u}}{\sqrt{1+|\partial_x\mathtt{u}|^2}}\;=\;\delta^k_i\,+\,O\left(|\hat{X}|^{-2}\right) \tag{1.36}$$

By taking R sufficiently large, we may assume that $\psi : B^n_{\frac{\rho}{2}|\Pi(\hat{X})|} \to \mathrm{Im}\psi \subset B^n_{\rho|\Pi(\hat{X})|}$ is a C^2 diffeomorphism and the inverse of $\frac{\partial y_k}{\partial x_i}$ satisfies

$$\frac{\partial x_i}{\partial y_k} = \delta^i_k + O\left(|\hat{X}|^{-2}\right)$$

It follows that the components of shape operators $\tilde{A}^{\#}$ of $\tilde{\Sigma}$ and $A^{\#}$ of Σ with respect to the local coodinates $x = (x_1, \cdots, x_n)$ are respectively equal to

$$\tilde{A}^j_i = \frac{\partial y_k}{\partial x_i}\frac{\partial x_j}{\partial y_l}\partial_{y_k}\left(\frac{\partial_{y_l}\tilde{\mathtt{u}}}{\sqrt{1+|\partial_y \tilde{\mathtt{u}}|^2}}\right)\Big|_{y=\varphi(x)}, \quad A^j_i = \partial_{x_i}\left(\frac{\partial_{x_j}\mathtt{u}}{\sqrt{1+|\partial_x \mathtt{u}|^2}}\right) \tag{1.37}$$

in which we sum over repeated indicies. Using the triangle inequality, combined with (1.24), (1.26), (1.35), (1.21) and (1.36), we then get from (1.37) that

$$\left|\tilde{A}^j_i - A^j_i\right| \leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) |\hat{X}|^{-3}$$

Due to (1.33), the above implies that

$$\left|\tilde{A}^{\#} - A^{\#}\right| \leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) |\hat{X}|^{-3}$$

Also, in view of $\nabla_\Sigma \tilde{A}^{\#} \sim \nabla_r \tilde{A}^j_i$, $\nabla_\Sigma A^{\#} \sim \nabla_r A^j_i$ and

$$\nabla_r \tilde{A}^j_i = \partial_r \tilde{A}^j_i - \Gamma^s_{ri}\tilde{A}^j_s + \Gamma^j_{rs}\tilde{A}^s_i, \quad \nabla_r A^j_i = \partial_r A^j_i - \Gamma^s_{ri} A^j_s + \Gamma^j_{rs} A^s_i \tag{1.38}$$

in which we sum over repeated indicies, we can similarly derive

$$|\nabla_\Sigma \tilde{A}^{\#} - \nabla_\Sigma A^{\#}| \leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) |\hat{X}|^{-4}$$

This completes (1.22).

(1.23) follows from taking one more derivative of (1.38) and use (1.37), (1.34), (1.24), (1.26) and (1.33). □

Next, we'd like to define a 2-tensor $\mathbf{a}$ on Σ (outside a compact set), which would be served as the coefficients of the differential equation to be satisfied by the deviation h. Note that by (1.17), Lemma 1.7 (in particular (1.22)), we may assume that

$$(1-\theta)\,|X|\,A^{\#} + \theta|X|\,\tilde{A}^{\#} \in U \qquad \forall\, X \in \Sigma \setminus \bar{B}_R,\ \theta \in [0, 1] \tag{1.39}$$

where $\tilde{A}^{\#}$ is the shape operator of $\tilde{\Sigma}$ at $\tilde{X} = X + hN$.

Definition 1.8. In the setting of Lemma 1.7, let's take a local coordinate $x = (x_1, \cdots, x_n)$ of Σ (outside a compact set) so that Σ and $\tilde{\Sigma}$ can be respectively parametrized as

$$X = X(x), \quad \tilde{X}(x) = X(x) + h(x) N(x)$$

where $h(x)$is the deviation and $N(x)$ is the unit-normal of Σ at $X(x)$. Then we define

$$\mathsf{a}^{ij}(x) = \sum_k \mathsf{a}^i_k(x)\, g^{kj}(x) \quad \text{with} \quad \mathsf{a}^i_j(x) = \int_0^1 \frac{\partial F}{\partial S^j_i}\Big((1-\theta)|X| A^{\#}(x) + \theta|X|\tilde{A}^{\#}(x)\Big)\, d\theta$$

and its symmetrization

$$\mathbf{a}^{ij}(x) = \frac{1}{2}\left(\mathsf{a}^{ij}(x) + \mathsf{a}^{ji}(x)\right)$$

where $g^{ij}(x)$ is the inverse of the pull-back metric $g_{ij} = \partial_i X \cdot \partial_j X$, $A^{\#}(x) \sim A^j_i(x) = -\partial_i N \cdot \partial_j X$ is the shape operator of Σ at $X(x)$, $\tilde{A}^{\#}_t(x) \sim \tilde{A}^j_i(x, t) = -\partial_i \tilde{N} \cdot \partial_j \tilde{X}$ is the shape operator of $\tilde{\Sigma}$ at $\tilde{X}(x)$ with $\tilde{N}(x)$ being the unit-normal of $\tilde{\Sigma}$ at $\tilde{X}(x)$.

Note that

$$\begin{aligned} \mathsf{a}^i_j(x) &= \int_0^1 \frac{\partial F}{\partial S^j_i}\Big((1-\theta)|X| A^{\#}(x) + \theta|X|\tilde{A}^{\#}(x)\Big)\, d\theta \\ &= \int_0^1 \frac{\partial F}{\partial S^j_i}\Big((1-\theta) A^{\#}(x) + \theta\tilde{A}^{\#}(x)\Big)\, d\theta \end{aligned}$$

since $\frac{\partial F}{\partial S^j_i}$ is homogeneous of degree 0; besides, the operator $\mathbf{a}$ is independent of the choice of local coordinates and hence defines a 2-tensor on Σ.

We have the following estimates for the tensor $\mathbf{a}$, which is based on (1.18), (1.19), (1.20), (1.22), (1.23) and the homogeneity of F and its derivatives.

Lemma 1.9. *There exits* $R = R\Big(\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \| F \|_{C^3(U)}, \lambda, \varkappa\Big) \geq 1$ *such that*

$$\frac{\lambda}{3} \leq \mathbf{a} \leq \frac{3}{\lambda} \tag{1.40}$$

$$|X|\Big|\nabla_\Sigma \mathbf{a}\Big| \leq 3\varkappa \tag{1.41}$$

$$|X|^2\Big|\nabla^2_\Sigma \mathbf{a}\Big| \leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) \tag{1.42}$$

for all $X \in \Sigma \setminus \bar{B}_R$.

Proof. By (1.18), (1.19), (1.39), (1.22), the homogeneity and continuity of F (and its derivatives), there exists $R = R\left(\Sigma,\, \tilde{\Sigma},\, \mathcal{C},\, U,\, \| F \|_{C^3(U)},\, \lambda,\, \varkappa\right) \geq 1$ such that

$$\frac{\lambda}{3}\delta_j^i \leq \mathtt{a}_j^i = \int_0^1 \frac{\partial F}{\partial S_i^j}\left((1-\theta)\,|X|\,A^{\#} + \theta|X|\,\tilde{A}^{\#}\right) d\theta \leq \frac{3}{\lambda}\delta_j^i$$

$$|X|\left|\nabla_r \mathtt{a}_i^j\right| = |X|\left|\int_0^1 \sum_{k,\,l} \frac{\partial^2 F}{\partial S_i^j \partial S_k^l}\left((1-\theta)\,A^{\#} + \theta\tilde{A}^{\#}\right)\cdot\left((1-\theta)\,\nabla_r A_k^l + \theta\nabla_r \tilde{A}_k^l\right) d\theta\right|$$

$$= \left|\int_0^1 \sum_{k,\,l} \frac{\partial^2 F}{\partial S_i^j \partial S_k^l}\left((1-\theta)\,|X|\,A^{\#} + \theta|X|\,\tilde{A}^{\#}\right)\cdot\left((1-\theta)\,|X|^2\nabla_r A_k^l + \theta|X|^2\nabla_r \tilde{A}_k^l\right) d\theta\right| \leq 3\varkappa$$

Likewise, with the help of (1.20), (1.23), we can get

$$|X|^2\left|\nabla_\Sigma^2 \mathtt{a}\right| \leq C\left(n,\, \mathcal{C},\, \| F \|_{C^3(U)}\right)$$

The conclusion follows immediately. □

Now we are in a position to derive an elliptic equation satisfied by h.

Proposition 1.10. *There exits* $R = R\left(\Sigma,\, \tilde{\Sigma},\, \mathcal{C},\, U,\, \| F \|_{C^3(U)},\, \lambda,\, \varkappa\right) \geq 1$ *such that the deviation* h *satisfies*

$$\nabla_\Sigma \cdot (\mathbf{a}\, dh) - \frac{1}{2}\left(X \cdot \nabla_\Sigma h - h\right) = O\left(|X|^{-1}\right)|\nabla_\Sigma h| + O\left(|X|^{-2}\right)|h| \tag{1.43}$$

for $X \in \Sigma \setminus \bar{B}_R$*, where*

$$\nabla_\Sigma \cdot (\mathbf{a}\, dh) = \sum_{i,\,j} \nabla_i\left(\mathbf{a}^{ij}\nabla_j h\right)$$

in local coordinates and the notation $O\left(|X|^{-1}\right)$ *means that*

$$\left|O\left(|X|^{-1}\right)\right| \leq C\left(n,\, \mathcal{C},\, \| F \|_{C^3(U)}\right)|X|^{-1}$$

Proof. Fix $\hat{X} \in \Sigma \setminus \bar{B}_R$ and take a local coodinate $x = (x_1, \cdots, x_n)$ of Σ which is normal and pricipal (w.r.t. Σ) at $\hat{X} = X(0)$. That is

$$g_{ij}\Big|_{x=0} = \delta_{ij},\ \Gamma_{ij}^k\Big|_{x=0} = 0,\ A_i^j\Big|_{x=0} = \kappa_i\delta_{ij}$$

where g_{ij} is the pull-back metric, Γ_{ij}^k is the Christoffel symbols and A_i^j is the shape operator of Σ at $X(x)$. Denote the principal direction of Σ at $\hat{X}$ by

$$\partial_i X\Big|_{x=0} = e_i$$

Throughout the proof, we adopt the Einstein summation convension (i.e. summing over repeated indicies). Recall that we regard $\tilde{\Sigma}$ (outside a compact set) as a normal graph over $\Sigma \setminus \bar{B}_R$ and parametrize it by $\tilde{X} = X(x) + h(x)N(x)$. We then want to compute some geomtric quantities of $\tilde{\Sigma}$ in terms of this local coordinate at $\tilde{X}(0) = \hat{X} + hN\Big|_{\hat{X}}$. First, we compute

$$\partial_i \tilde{X}\Big|_{x=0} = \left(\delta_i^k - A_i^k h\right) \partial_k X \, + \, \partial_i h \, N\Big|_{x=0} = \left(1 - \kappa_i h\right) e_i + \nabla_i h \, N$$

$$\partial_{ij}^2 \tilde{X}\Big|_{x=0} = -\left(A_i^k \nabla_j h + A_j^k \nabla_i h + \nabla_i A_j^k \cdot h\right) e_k \, + \, \left(A_{ij} + \nabla_{ij}^2 h - A_{ij}^2 h\right) N \tag{1.44}$$

which (together with Lemma 1.7) gives the metric of $\tilde{\Sigma}$, its inverse and determinant as follows:

$$\tilde{g}_{ij}\Big|_{x=0} = \left(1 - \kappa_i h\right)^2 \delta_{ij} + \nabla_i h \, \nabla_j h = \left(1 - \kappa_i h\right)^2 \left(\delta_{ij} + \frac{\nabla_i h \, \nabla_j h}{\left(1 - \kappa_i h\right)^2}\right)$$

$$\begin{aligned} \tilde{g}^{ij}\Big|_{x=0} &= \left(1 - \kappa_i h\right)^{-2} \left(\delta_{ij} + \frac{\nabla_i h \, \nabla_j h}{\left(1 - \kappa_i h\right)^2}\right)^{-1} \\ &= \left(1 + 2\kappa_i h\right) \delta^{ij} + O\left(|\hat{X}|^{-2}\right) |\nabla_\Sigma h| \, + O\left(|\hat{X}|^{-3}\right) |h| \end{aligned} \tag{1.45}$$

$$\begin{aligned} \det \tilde{g}\Big|_{x=0} &= \left(1 - \kappa_1 h\right)^2 \cdots \left(1 - \kappa_n h\right)^2 \det\left(\delta_{ij} + \frac{\nabla_i h \, \nabla_j h}{\left(1 - \kappa_i h\right)^2}\right) \\ &= 1 - 2Hh + O\left(|\hat{X}|^{-2}\right) |\nabla_\Sigma h| \, + O\left(|\hat{X}|^{-3}\right) |h| \end{aligned}$$

and also the unit-normal of $\tilde{\Sigma}$:

$$\tilde{N}\Big|_{x=0} = \left(\det \tilde{g}\right)^{-\frac{1}{2}} \partial_1 \tilde{X} \wedge \cdots \wedge \partial_n \tilde{X} \tag{1.46}$$

$$= \left(\det \tilde{g}\right)^{-\frac{1}{2}} \left(-\sum_{i=1}^{n} \left(\nabla_i h \prod_{j \neq i} \left(1 - \kappa_j h\right)\right) e_i \, + \, \left(1 - \kappa_1 h\right) \cdots \left(1 - \kappa_n h\right) N\right)$$

$$= -\sum_{i=1}^{n} \left(1 + \kappa_i h + O\left(|\hat{X}|^{-2}\right) |\nabla_\Sigma h| + O\left(|\hat{X}|^{-3}\right) |h|\right) \nabla_i h \cdot e_i$$

$$+\left(1+O\left(|\hat{X}|^{-2}\right)|\nabla_\Sigma h|+O\left(|\hat{X}|^{-3}\right)|h|\right)N$$

By (1.44), (1.45), (1.46) and Lemma 1.7, we compute the shape operator of $\tilde{\Sigma}$at $\tilde{X}(0)$:

$$\tilde{A}_i^j\Big|_{x=0}=\tilde{A}_{ik}\,\tilde{g}^{kj}=\left(\partial^2_{ik}\tilde{X}\cdot\tilde{N}\right)\tilde{g}^{kj} \tag{1.47}$$

$$=\left(A_{ik}+\nabla^2_{ik}h+O\left(|\hat{X}|^{-2}\right)|\nabla_\Sigma h|\,+O\left(|\hat{X}|^{-2}\right)|h|\right)\left((1+2\kappa_j h)\,\delta^{kj}+O\left(|\hat{X}|^{-2}\right)|\nabla_\Sigma h|\right)$$

$$+\left(A_{ik}+\nabla^2_{ik}h+O\left(|\hat{X}|^{-2}\right)|\nabla_\Sigma h|\,+O\left(|\hat{X}|^{-2}\right)|h|\right)O\left(|\hat{X}|^{-3}\right)|h|$$

$$=A_i^j+\delta^{kj}\nabla^2_{ik}h+O\left(|\hat{X}|^{-2}\right)(|\nabla_\Sigma h|\,+\,|h|)$$

and

$$\tilde{X}\cdot\tilde{N}\Big|_{x=0}=X\cdot N\,-\,X\cdot\nabla_\Sigma h\,+h+O\left(|\hat{X}|^{-1}\right)|\nabla_\Sigma h|\,+O\left(|\hat{X}|^{-2}\right)|h| \tag{1.48}$$

Thus, in view of the F self-shrinker equation satisfied by Σ and $\tilde{\Sigma}$, we get

$$0=F\left(\tilde{A}^\#\right)-F\left(A^\#\right)\,+\,\frac{1}{2}\left(\tilde{X}\cdot\tilde{N}-X\cdot N\right)\Big|_{x=0} \tag{1.49}$$

$$=\int_0^1\frac{\partial F}{\partial S_i^j}\left((1-\theta)\,A^\#+\theta\tilde{A}^\#\right)d\theta\cdot\left(\tilde{A}_i^j-A_i^j\right)\,-\,\frac{1}{2}\left(X\cdot\nabla_\Sigma h-h\right)$$

$$+O\left(|\hat{X}|^{-1}\right)|\nabla_\Sigma h|\,+O\left(|\hat{X}|^{-2}\right)|h|$$

$$=\mathsf{a}_j^i\delta^{jk}\nabla^2_{ik}h\,-\,\frac{1}{2}\left(X\cdot\nabla_\Sigma h\,-\,h\right)+O\left(|\hat{X}|^{-1}\right)|\nabla_\Sigma h|\,+O\left(|\hat{X}|^{-2}\right)|h|$$

$$=\mathsf{a}^{ik}\nabla^2_{ik}h\,-\,\frac{1}{2}\left(X\cdot\nabla_\Sigma h\,-\,h\right)+O\left(|\hat{X}|^{-1}\right)|\nabla_\Sigma h|\,+O\left(|\hat{X}|^{-2}\right)|h|$$

$$=\langle\mathsf{a},\,\nabla^2_\Sigma h\rangle\,-\,\frac{1}{2}\left(X\cdot\nabla_\Sigma h\,-\,h\right)+O\left(|\hat{X}|^{-1}\right)|\nabla_\Sigma h|\,+O\left(|\hat{X}|^{-2}\right)|h|$$

Note that by the symmetry of the Hessian and Lemma 1.9, we have

$$\langle\mathsf{a},\,\nabla^2_\Sigma h\rangle\;=\mathsf{a}^{ij}\nabla^2_{ij}h=\frac{1}{2}\left(\mathsf{a}^{ij}+\mathsf{a}^{ji}\right)\nabla^2_{ij}h=\langle\mathbf{a},\,\nabla^2_\Sigma h\rangle \tag{1.50}$$

$$=\nabla_i\left(\mathbf{a}^{ij}\nabla_j h\right)-\left(\nabla_i\mathbf{a}^{ij}\right)\nabla_j h=\nabla_\Sigma\cdot\left(\mathbf{a}\,dh\right)+\,O\left(|\hat{X}|^{-1}\right)|\nabla_\Sigma h|$$

(1.43) follows from combining (1.49) and (1.50). $\square$

Our goal is to show that h vanishes on $\Sigma \setminus \bar{B}_R$ for some $R \gg 1$, which would be done in the next section through Carleman's inequality. For that purpose, we first observe that for each $t \in [-1, 0)$, $\tilde{\Sigma}_t = \sqrt{-t}\,\tilde{\Sigma}$ is (outside a compact set) also a normal graph over $\Sigma_t \setminus \bar{B}_R$ and it can be parametrized as $\tilde{X}_t = X_t + h_t N_t$. For the rest of this section, we would show that each $h_t = h(\cdot, t)$ satisfies a similar equation as $h(\cdot, -1)$ does in Proposition 1.10. Due to the property that $\{\Sigma_t\}_{-1 \leq t < 0}$ form a F curvature flow, it turns out that the evolution of h_t satisfies a parabolic equation. We then give some estimates for the coefficients of the parabolic equations (as in Lemma 1.9) , which is crucial for deriving the Carleman's inequality in the next section.

Now fix $t \in [-1, 0)$ and define a 2-tensor $\mathbf{a}_t$ on $\Sigma_t = \sqrt{-t}\,\Sigma$ as in Definition 1.8. First, take a local coordinate $x = (x_1, \cdots, x_n)$ of Σ_t (outside a compact set) so that Σ_t and $\tilde{\Sigma}_t$ can be respectively parametrized as

$$X_t = X_t(x), \quad \tilde{X}_t(x) = X_t(x) + h_t(x)\, N_t(x)$$

We define

$$\mathsf{a}_t^{ij}(x) = \sum_k \mathsf{a}_k^i(x, t)\, g_t^{kj}(x) \quad \text{with} \quad \mathsf{a}_j^i(x, t) = \int_0^1 \frac{\partial F}{\partial S_i^j}\left((1-\theta)\, A_t^{\#}(x) + \theta \tilde{A}_t^{\#}(x)\right) d\theta$$

and its symmetrization

$$\mathbf{a}_t^{ij}(x) = \frac{1}{2}\left(\mathsf{a}_t^{ij}(x) + \mathsf{a}_t^{ji}(x)\right)$$

where $g_t^{ij}(x)$ is the inverse of the pull-back metric $g_{ij}(x, t) = \partial_i X_t(x) \cdot \partial_j X_t(x)$, $A_t^{\#}(x) \sim A_i^j(x, t) = -\partial_i N_t(x) \cdot \partial_j X_t(x)$ is the shape operator of Σ_t at $X_t(x)$ with $N_t(x)$ being the unit-normal of Σ_t at $X_t(x)$, $\tilde{A}_t^{\#} \sim \tilde{A}_i^j(x, t) = -\partial_i \tilde{N}_t(x) \cdot \partial_j \tilde{X}_t(x)$ is the shape operator of $\tilde{\Sigma}_t$ at $\tilde{X}_t(x)$ with $\tilde{N}_t(x)$ being the unit-normal of $\tilde{\Sigma}_t$ at $\tilde{X}_t(x)$.

Then we have the following lemma, which is an analogous of Proposition 1.10 for $\Sigma_t = \sqrt{-t}\,\Sigma,\ t \in [-1, 0)$:

Lemma 1.11. *There exits $R = R\left(\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \| F \|_{C^3(U)}, \lambda, \varkappa\right) \geq 1$ such that for each $t \in [-1, 0)$, the deviation h_t satisfies*

$$\nabla_{\Sigma_t} \cdot (\mathbf{a}_t\, dh_t) - \frac{1}{2(-t)}\left(X_t \cdot \nabla_{\Sigma_t} h_t - h_t\right) = O\left(|X_t|^{-1}\right)|\nabla_{\Sigma_t} h_t| + O\left(|X_t|^{-2}\right)|h_t| \quad (1.51)$$

for $X_t \in \Sigma_t \setminus \bar{B}_R$, *where* $\nabla_{\Sigma_t} \cdot (\mathbf{a}_t\, dh_t) = \sum_{i,j} \nabla_i \left(\mathbf{a}_t^{ij} \nabla_j h_t \right)$ *and*

$$\left| O\left(|X_t|^{-1}\right) \right| \leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) |X_t|^{-1}$$

Also, we have

$$\| \, |X_t|\, h_t \, \|_{L^\infty(\Sigma_t \setminus \bar{B}_R)} + \| \, |X_t|^2 \nabla_{\Sigma_t} h_t \, \|_{L^\infty(\Sigma_t \setminus \bar{B}_R)} + \| \, |X_t|^3 \nabla^2_{\Sigma_t} h_t \, \|_{L^\infty(\Sigma_t \setminus \bar{B}_R)}$$

$$\leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) (-t) \tag{1.52}$$

Proof. Fix $t \in [-1, 0)$ and $\hat{X}_t \in \Sigma_t \setminus B_R$, then we have $\hat{X} = \frac{\hat{X}_t}{\sqrt{-t}} \in \Sigma \setminus \bar{B}_R$ and

$$\left(\nabla_{\Sigma_t} \cdot (\mathbf{a}_t\, dh_t) - \frac{1}{2(-t)} \left(X_t \cdot \nabla_{\Sigma_t} h_t - h_t \right) \right) \Big|_{\hat{X}_t} = \frac{1}{\sqrt{-t}} \left(\nabla_\Sigma \cdot (\mathbf{a}\, dh) - \frac{1}{2} \left(X \cdot \nabla_\Sigma h - h \right) \right) \Big|_{\hat{X}_t}$$

$$= \frac{1}{\sqrt{-t}} \left(O\left(|\hat{X}|^{-1} \right) |\nabla_\Sigma h| + O\left(|\hat{X}|^{-2} \right) |h| \right) \Big|_{\hat{X}_t}$$

$$= \left(O\left(|\hat{X}_t|^{-1} \right) |\nabla_{\Sigma_t} h_t| + O\left(|\hat{X}_t|^{-2} \right) |h_t| \right) \Big|_{\hat{X}_t}$$

Similarly, to derive (1.52), it suffices to rescale (1.21) to get

$$|\hat{X}_t|\, |h_t| + |\hat{X}_t|^2 |\nabla_{\Sigma_t} h_t| + |\hat{X}_t|^3 |\nabla^2_{\Sigma_t} h_t| \Big|_{\hat{X}_t}$$

$$= (-t) \left(|\hat{X}|\, |h| + |\hat{X}|^2 |\nabla_\Sigma h| + |\hat{X}|^3 |\nabla^2_\Sigma h| \right) \Big|_{\hat{X}_t}$$

$$\leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) (-t)$$

□

Next, we define the "normal parametrization" of the flow:

Definition 1.12. $X_t = X(\cdot, t)$ is called a "normal parametrization" for the motion of a hypersurface $\{\Sigma_t\}$ provided that

$$\partial_t X = F\left(A^{\#}\right) N$$

That is, each particle on the hypersurface moves in normal direction during the flow. (See also Definition 1.4)

In the derivation of the parabolic equation to be satisfied by $h_t = h(\cdot,\, t)$, we would start with a "radial parametrization" of the flow $\{\Sigma_t\}_{-1\leq t<0}$ (i.e. each particles on the hypersurface moves in the radial direction along the flow, see the proof of Propostion 1.13 for more deatails), then we make a trasition to the "normal parametrization" by using a time-dependent tangential diffeomorphism. Note that in general, the "radial parametrization" exists only for a short period of time (unlike the "vertical parametrization"), so later in the proof, we would do a "local" (in spacetime) argument, which is quite sufficient for deriviving the equation.

Proposition 1.13. *There exits* $R = R\left(\Sigma,\, \tilde{\Sigma},\, \mathcal{C},\, U,\, \parallel F \parallel_{C^3(U)},\, \lambda,\, \varkappa\right) \geq 1$ *so that in the normal parametrization of the* F *curvature flow* $\{\Sigma_t\}_{-1\leq t<0}$ *, the deviation* h_t *satisfies*

$$\mathbf{P}h \,\equiv\, \partial_t h - \nabla_{\Sigma_t}\cdot\left(\mathbf{a}(\cdot,\, t)\; dh\right) \tag{1.53}$$

$$= O\left(|X_t|^{-1}\right)|\nabla_{\Sigma_t} h| \,+\, O\left(|X_t|^{-2}\right)|h|$$

$$h(\cdot,\, t) = 0 \quad as\, t \nearrow 0 \tag{1.54}$$

for $X_t \in \Sigma_t \setminus \bar{B}_R$, $-1 \leq t < 0$, *where* $\mathbf{a}(\cdot,\, t) = \mathbf{a}_t$.

Proof. Fix $\hat{t} \in [-1,\, 0)$, $\hat{X} \in \Sigma_{\hat{t}} \setminus \bar{B}_R$, and take a local coordinate $x = (x_1, \cdots, x_n)$ of $\Sigma_{\hat{t}}$ around $\hat{X}$. Define the "radial parametrization" of the flow starting at time $\hat{t}$ near the point $\hat{X}$ by

$$X(x,\, t) \;=\; \frac{\sqrt{-t}}{\sqrt{-\hat{t}}}\, X_{\hat{t}}(x)$$

For this parametrization, we can decompose the velocity vector into the normal part and the tangential part as follows:

$$\partial_t X(x,\, t) = \frac{-1}{2\sqrt{-\hat{t}}\sqrt{-t}} X_{\hat{t}}(x) \tag{1.55}$$

$$= \frac{-1}{2\sqrt{-\hat{t}}\sqrt{-t}}\left(\left(X_{\hat{t}}(x)\cdot N_{\hat{t}}(x)\right)N_{\hat{t}}(x) \,+\, \sum_{i,\,j} g_{\hat{t}}^{ij}(x)\;\left(X_{\hat{t}}(x)\cdot\partial_j X_{\hat{t}}(x)\right)\partial_i X_{\hat{t}}(x)\right)$$

$$= F\left(A_i^j(x,\, t)\right)N(x,\, t) - \sum_{i,\,j}\frac{1}{2(-t)} g^{ij}(x,\, t)\;\left(X(x,\, t)\cdot\partial_j X(x,\, t)\right)\partial_i X(x,\, t)$$

in which we use the F self-shrinker equation of $\Sigma_{\hat{t}} = \sqrt{-\hat{t}}\,\Sigma$ (in Definition 1.4) and the homogeniety of F. Note that the normal part agrees with Definition 1.4 for the F curvature flow. Now consider the following ODE system:

$$\partial_t x_i = \sum_{i,\,j} \frac{1}{2(-t)}\, g^{ij}\left(x,\,t\right)\,\left(X\left(x,\,t\right)\cdot\partial_j X\left(x,\,t\right)\right) \tag{1.56}$$

$$x_i\Big|_{t=\hat{t}} = \xi_i,\;\; i = 1,\cdots,n$$

Let the solution (which exists at least for a while) to be $x = \varphi_t\left(\xi\right)$. In other words, φ_t is the local diffeomorphism on Σ_t generated by the tangent vector field $\frac{1}{2(-t)}X\left(x,\,t\right)^{\top}$. By (1.55) and (1.56), the reparametrization $X\left(\varphi_t\left(\xi\right),\,t\right)$ of the flow becomes a normal parametrization.

On the other hand, in the radial parametrization, $h(x,\,t) = \frac{\sqrt{-t}}{\sqrt{-\hat{t}}}h_{\hat{t}}(x)$. Thus, by (1.56) and Lemma 1.11, we get

$$\frac{\partial}{\partial t}\left\{h\left(\varphi_t\left(\xi\right),t\right)\right\} = \partial_t h\left(x,\,t\right) + \sum_{i,\,j}\frac{1}{2\left(-t\right)}\,g^{ij}\left(x,\,t\right)\,\left(X\left(x,\,t\right)\cdot\partial_j X\left(x,\,t\right)\right)\partial_i h\left(x,\,t\right)\Big|_{x=\varphi_t(\xi)}$$

$$= \frac{1}{2\left(-t\right)}\left\{-h\left(x,\,t\right) + X\left(x,\,t\right)\cdot\nabla_{\Sigma_t}h\right\}\Big|_{x=\varphi_t(\xi)}$$

$$= \nabla_{\Sigma_t}\cdot\left(\mathbf{a}\left(\cdot,\,t\right)\,dh_t\right) + O\left(\left|X_t\right|^{-1}\right)\left|\nabla_{\Sigma_t}h_t\right| + O\left(\left|X_t\right|^{-2}\right)\left|h_t\right|\Big|_{x=\varphi_t(\xi)}$$

which proves (1.53).

(1.54) follows from (1.52). $\square$

Lastly, we conclude this section by some estimates on the 2-tensor $\mathbf{a}\left(\cdot,\,t\right)$ on each time-slice Σ_t.

Proposition 1.14. *There exits* $R = R\left(\Sigma,\,\tilde{\Sigma},\,\mathcal{C},\,U,\,\|\,F\,\|_{C^3(U)},\,\lambda,\,\varkappa\right) \geq 1$ *so that for* $t\in[-1,\,0)$*,* $X_t \in \Sigma_t\setminus\bar{B}_R$*, there hold*

$$\frac{\lambda}{3} \leq \mathbf{a}\left(\cdot,\,t\right) \leq \frac{3}{\lambda} \tag{1.57}$$

$$\left|X_t\right|\left|\nabla_{\Sigma_t}\mathbf{a}\left(\cdot,\,t\right)\right| \leq 3\varkappa \tag{1.58}$$

$$\left|X_t\right|^2\left|\nabla^2_{\Sigma_t}\mathbf{a}\left(\cdot,\,t\right)\right| \leq C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)}\right) \tag{1.59}$$

$$|X_t|^2 \Big|\partial_t \mathbf{a}\left(\cdot,\, t\right)\Big| \le C\left(n,\, \mathcal{C},\, \|\, F\, \|_{C^3(U)}\right) \tag{1.60}$$

where the time derivative in the last term is taken with respect to the normal parametrization of the flow $\{\Sigma_t\}_{-1\le t<0}$.

Proof. We adopt the Einstein summation convension throughout the proof.

By using the rescaling argument and the homogenity of the derivatives of F, (1.57), (1.58), (1.59) follow from (1.40), (1.41), (1.42), respectivly. As for (1.60), note that in normal parametrization, we have

$$\partial_t \mathtt{a}^{ij}\left(t\right) = \partial_t\left(\mathtt{a}^i_k(t)\, g^{kj}_t\right) = \left(\partial_t \mathtt{a}^i_k(t)\right) g^{kj}_t + 2\mathtt{a}^i_k(t)\, F\left(A^{\#}_t\right) A^{kj}_t \tag{1.61}$$

in which we use the following evolution equation for the metric along the F curvature flow $\{\Sigma_t\}_{-1\le t<0}$ (see [A]):

$$\partial_t g_{ij}\left(t\right) = -2F\left(A^{\#}_t\right) A_{ij}\left(t\right), \quad \partial_t g^{ij}_t = 2F\left(A^{\#}_t\right) A^{ij}_t \tag{1.62}$$

By the rescaling argument, (1.17), and the homogeneity of F and its derivatives, we can estimate each term in (1.61) by

$$|X_t|^2 \Big|F\left(A^{\#}_t\right) A^{ij}_t\Big| = \Big|F\left(|X_t|\, A^{\#}_t\right)\cdot |X_t|\, A^{ij}_t\Big| \le C\left(n,\, \mathcal{C},\, \|\, F\, \|_{C^3(U)}\right)$$

and

$$|X_t|^2 |\partial_t \mathtt{a}^i_j| = |X_t|^2 \Big|\int_0^1 \frac{\partial^2 F}{\partial S^j_i \partial S^l_k}\left(\left(1-\theta\right) A^{\#}_t + \theta \tilde{A_t}^{\#}\right)\cdot\left(\left(1-\theta\right)\partial_t A^l_k + \theta \partial_t \tilde{A}^l_k\right) d\theta\Big|$$

$$= \Big|\int_0^1 \frac{\partial^2 F}{\partial S^j_i \partial S^l_k}\left(\left(1-\theta\right)|X_t|\, A^{\#}_t + \theta |X_t|\, \tilde{A_t}^{\#}\right)\cdot\left(\left(1-\theta\right)|X_t|^3 \partial_t A^l_k + \theta |X_t|^3 \partial_t \tilde{A}^l_k\right) d\theta\Big|$$

$$\le C\left(n,\, \mathcal{C},\, \|\, F\, \|_{C^3(U)}\right)\Big|\int_0^1 \left(\left(1-\theta\right)|X_t|^3 \partial_t A^l_k + \theta |X_t|^3 \partial_t \tilde{A}^l_k\right) d\theta\Big|$$

Thus, to establish (1.60), it suffices to show that

$$|X_t|^3 |\partial_t A^{\#}_t| \le C\left(n,\, \mathcal{C},\, \|\, F\, \|_{C^3(U)}\right) \tag{1.63}$$

$$|X_t|^3 |\partial_t \tilde{A}^{\#}_t - \partial_t A^{\#}_t| \le C\left(n,\, \mathcal{C},\, \|\, F\, \|_{C^3(U)}\right) \tag{1.64}$$

for all $X_t \in \Sigma_t \setminus \bar{B}_R$, $t \in [-1, 0)$.

Firstly, let's recall the evolution equation for the shape operator $A_t^\#$ in the normal parametrization along the flow (see [A]):

$$\partial_t A_i^j(t) = \frac{\partial F}{\partial S_k^l}\left(A_t^\#\right) \cdot g_t^{lm} \nabla_{km}^2 A_i^j + \frac{\partial F}{\partial S_k^l}\left(A_t^\#\right) \cdot \left(A_t^2\right)_k^l A_i^j(t) \tag{1.65}$$
$$+\frac{\partial^2 F}{\partial S_k^l \partial S_p^q}\left(A_t^\#\right) \cdot g_t^{jm} \nabla_i A_k^l(t) \nabla_m A_p^q(t)$$

which yields (1.63) by the rescaling argument, (1.20) and the homogeneity of F and its derivatives.

Secondly, we would like to compute $\partial_t\left(\tilde{A}_t^\# - A_t^\#\right)$ in the normal parametrization (of $\{\Sigma_t\}_{-1 \le t < 0}$) by using the same trick as in the proof of Proposition 1.13. Fix $\hat{t} \in [-1, 0)$, $\hat{X} \in \Sigma_{\hat{t}} \setminus \bar{B}_R$, and take a local coordinate $x = (x_1, \cdots, x_n)$ of $\Sigma_{\hat{t}}$ which is normal at $\hat{X} = X(0)$. Consider the radial parametrization of the flow starting at time $\hat{t}$ near the point $\hat{X}$ by $X(x, t) = \frac{\sqrt{-t}}{\sqrt{-\hat{t}}} X_{\hat{t}}(x)$. Then we have

$$\tilde{A}_i^j(x, t) - A_i^j(x, t) = \frac{\sqrt{-\hat{t}}}{\sqrt{-t}}\left(\tilde{A}_i^j(x, \hat{t}) - A_i^j(x, \hat{t})\right)$$

Let $x = \varphi_t(\xi)$ with $\varphi_{\hat{t}} = \mathrm{id}$ to be the local diffeomorphism on Σ_t generated by the tangent vector field $\frac{1}{2(-t)} X(\cdot, t)^\top$ as before. Then the reparametrization $X(\varphi_t(\xi), t)$ of the flow becomes a normal parametrization and we have

$$\partial_t\left(\tilde{A}_i^j(\varphi_t(\xi), t) - A_i^j(\varphi_t(\xi), t)\right)\Big|_{\xi=0,\, t=\hat{t}} = \left(\partial_t \tilde{A}_i^j - \partial_t A_i^j\right)(\varphi_t(\xi), t) \tag{1.66}$$
$$+\frac{1}{2(-t)} g^{kl}(\varphi_t(\xi), t)\left(X_t(\varphi_t(\xi), t) \cdot \partial_l X_t(\varphi_t(\xi), t)\right)\left(\partial_k \tilde{A}_i^j(\varphi_t(\xi), t) - \partial_k A_i^j(\varphi_t(\xi), t)\right)\Big|_{\xi=0,\, t=\hat{t}}$$

$$= \frac{1}{2(-\hat{t})}\left\{\left(\tilde{A}_i^j(\hat{t}) - A_i^j(\hat{t})\right) + g_{\hat{t}}^{kl}\left(X_{\hat{t}} \cdot \partial_l X_{\hat{t}}\right)\left(\nabla_k \tilde{A}_i^j(\hat{t}) - \nabla_k A_i^j(\hat{t})\right)\right\}\Big|_{\hat{X}}$$

Note that for each $t \in [-1, 0)$, by the rescaling argument and (1.22), we have

$$\| \, |X_t|^3\left(\tilde{A}_t^\# - A_t^\#\right) \|_{L^\infty(\Sigma_t \setminus \bar{B}_R)} + \| \, |X_t|^4\left(\nabla_{\Sigma_t} \tilde{A}_t^\# - \nabla_{\Sigma_t} A_t^\#\right) \|_{L^\infty(\Sigma_t \setminus \bar{B}_R)} \tag{1.67}$$
$$\le \left\{\| \, |X|^3\left(\tilde{A}^\# - A^\#\right) \|_{L^\infty(\Sigma \setminus \bar{B}_R)} + \| \, |X|^4\left(\nabla_\Sigma \tilde{A}^\# - \nabla_\Sigma A^\#\right) \|_{L^\infty(\Sigma \setminus \bar{B}_R)}\right\}(-t)$$
$$\le C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right)(-t)$$

Combining (1.66) and (1.67) to get (1.64). $\square$

1.4 Carleman's inequalities and uniqueness of F self-shrinkers with a tangent cone

This section is a continuation of the previous section. Here we still assume that Σ and $\tilde{\Sigma}$ are properly embedded F self-shrinkers (in Definition 1.4) which are C^5 asymptotic to the cone $\mathcal{C}$ at infinity, and they induce F curvature flows $\{\Sigma_t\}_{-1\leq t\leq 0}$ and $\left\{\tilde{\Sigma}_t\right\}_{-1\leq t\leq 0}$ with $\Sigma_t = \sqrt{-t}\,\Sigma$, $\tilde{\Sigma}_t = \sqrt{-t}\,\tilde{\Sigma}$ for $t \in [-1, 0)$ and $\Sigma_0 = \mathcal{C} = \tilde{\Sigma}_0$. We also consider the deviation $h_t = h(\cdot, t)$ of $\tilde{\Sigma}_t$ from Σ_t for $t \in [-1, 0]$ (we set $h_0 = 0$), which is defined on $\Sigma_t \setminus \bar{B}_R$ with $R \gg 1$ (depending on Σ, $\tilde{\Sigma}$, $\mathcal{C}$, U, $\| F \|_{C^3(U)}$, λ, $\varkappa$). For the function h, recall that we have Proposition 1.13 and Proposition 1.14. Note that the Einstein summation convension is adopted throughout this section (i.e. summing over repeated indicies).

At the beginning, we would like to improve the decay rate of h_t as $t \nearrow 0$ in (1.52) to exponential decay. To achieve that, we need Proposition 20, which is due to [EF] and [N] for different cases. The proof (of Proposition 1.19) would be included here for readers' convenience, and it is based on two crucial lemma. The first one is a mean value inequality for parabolic equations from [LSU].

Lemma 1.15. *(Mean value inequality)*

Let $P = \partial_t - \partial_i\left(a^{ij}(x, t)\,\partial_j\right)$ *be a differential operator such that* $a^{ij}_t = a^{ij}(\cdot, t) \in C^1(B^n_1)$ *for* $t \in [-1, 0]$, $a^{ij} = a^{ji}$, *and*

$$\boldsymbol{\lambda}\delta^{ij} \leq a^{ij} \leq \frac{1}{\boldsymbol{\lambda}}\delta^{ij}$$

$$|a^{ij}(x, t) - a^{ij}(\tilde{x}, \tilde{t})| \leq L\left(|x - \tilde{x}| + |t - \tilde{t}|^{\frac{1}{2}}\right)$$

for some $\boldsymbol{\lambda} \in (0, 1]$, $L > 0$, *where* $B^n_1 = \left\{x \in \mathbb{R}^n \middle| |x| < 1\right\}$.

Suppose that $\mathrm{u} \in C^{2,1}(B^n_1 \times [-T, 0])$ *satisfies*

$$|P\mathrm{u}| \leq L\left(\frac{1}{\sqrt{T}}|\partial_x \mathrm{u}| + \frac{1}{T}|\mathrm{u}|\right)$$

for some $T \in (0, 1]$, *then there holds*

$$|\mathrm{u}(x, t)| + \sqrt{-t}\,|\partial_x \mathrm{u}(x, t)| \leq C(n, \boldsymbol{\lambda}, L) \fint_{Q\left(x,\, t;\, \sqrt{-t}\right)} |\mathrm{u}|$$

for $(x,\,t)\in Q\left(0,\,0;\,\frac{\sqrt{T}}{2}\right)$*, where* $Q\,(x,\,t;\,r)=B_r^n\,(x)\times(-r^2,\,0)$ *is the parabolic cylinder centered at* $(x,\,t)$ *and* $⨍_{\mathcal{D}}$ *means the average of a function on the domain* $\mathcal{D}$*.*

Remark 1.16. To prove the above lemma, we may consider the following change of variables:

$$(x,\,t)=\left(\sqrt{T}\,\tilde{x},\,T\,\tilde{t}\right)$$

In the new variables, the equation in Lemma 1.15 becomes

$$\left|\partial_{\tilde{t}}\mathrm{u}-\partial_{\tilde{x}_i}\left(a^{ij}\left(\sqrt{T}\,\tilde{x},\,T\,\tilde{t}\right)\partial_{\tilde{x}_j}\mathrm{u}\right)\right|\;\le\;L\left(|\partial_{\tilde{x}}\mathrm{u}|\;+\;|\mathrm{u}|\right)$$

for $\tilde{x}\in B^n_{1/\sqrt{T}}$, $\tilde{t}\in[-1,\,0]$. Then apply the standard theorem from [LSU] to the new equation.

The second lemma is a local type of Carleman's inequalities from [EFV].

Lemma 1.17. *(Local Carleman's inequality)*

Let $P=\partial_t-\partial_i\left(a^{ij}\,(x,\,t)\;\partial_j\right)$ *be a differential operator such that* $a_t^{ij}=a^{ij}(\cdot,\,t)\in C^1\left(B_1^n\right)$ *for* $t\in[-1,\,0]$, $a^{ij}=a^{ji}$, $a^{ij}\,(0,\,0)=\delta^{ij}$ *and*

$$\boldsymbol{\lambda}\delta^{ij}\le a^{ij}\le\frac{1}{\boldsymbol{\lambda}}\delta^{ij}$$

$$|a^{ij}\,(x,\,t)-a^{ij}\left(\tilde{x},\,\tilde{t}\right)|\;\le L\left(|x-\tilde{x}|\;+\;|t-\tilde{t}|^{\frac{1}{2}}\right)$$

for some $\boldsymbol{\lambda}\in(0,\,1]$*,* $L>0$*, where* $B_1^n=\left\{x\in\mathbb{R}^n\,\middle|\,|x|<1\right\}$*.*

Then for any fixed constant $M\ge4$*, there exists a non-increasing function* $\boldsymbol{\varphi}:\left(-\frac{4}{M},\,0\right)\to\mathbb{R}_+$ *satisfying* $\frac{-t}{\sigma}\le\boldsymbol{\varphi}\,(t)\le-t$ *for some constant* $\sigma=\sigma\,(n,\,\boldsymbol{\lambda},\,L)\ge1$*, so that for any constant* $\delta\in\left(0,\,\frac{1}{M}\right)$ *and function* $\mathrm{v}\in C_c^{2,1}\left(B_1^n\times(-\frac{2}{M},\,0]\right)$*, there holds*

$$M^2\int\mathrm{v}^2\boldsymbol{\varphi}_\delta^{-M}\boldsymbol{\Phi}_\delta\,dx\,dt\;+\;M\int|\partial_x\mathrm{v}|^2\boldsymbol{\varphi}_\delta^{1-M}\boldsymbol{\Phi}_\delta\,dx\,dt$$

$$\le\sigma\int|P\mathrm{v}|^2\boldsymbol{\varphi}_\delta^{1-M}\boldsymbol{\Phi}_\delta\,dx\,dt+(\sigma M)^M\sup_{t<0}\int\left(|\partial_x\mathrm{v}|^2+\mathrm{v}^2\right)dx+\sigma M\int\mathrm{v}^2\boldsymbol{\varphi}_\delta^{-M}\boldsymbol{\Phi}_\delta\,dx\Big|_{t=0}$$

where $\boldsymbol{\varphi}_\delta\,(t)=\boldsymbol{\varphi}\,(t-\delta)$ *and* $\boldsymbol{\Phi}_\delta\,(x,\,t)=\boldsymbol{\Phi}\,(x,\,t-\delta)=\frac{1}{(4\pi(-t+\delta))^{\frac{n}{2}}}\,\exp\left(-\frac{|x|^2}{4(-t+\delta)}\right)$*.*

Remark 1.18. Note that the last term on the RHS of the above inequality vanishes provided that $\mathrm{v}\big|_{t=0}=0$.

Now we state the proposition (of showing the exponential deay) and then follow [EF] and [N] to give it a proof:

Proposition 1.19. *(Exponential decay/ Unique continuation principle)*

Let $P = \partial_t - \partial_i \left(a^{ij}(x,\, t)\, \partial_j\right)$ *be a differential operator such that* $a_t^{ij} = a^{ij}(\cdot,\, t) \in C^1(B_1^n)$ *for* $t \in [-1,\, 0]$, $a^{ij} = a^{ji}$, *and*

$$\boldsymbol{\lambda}\delta^{ij} \le a^{ij} \le \frac{1}{\boldsymbol{\lambda}}\delta^{ij}$$

$$|a^{ij}(x,\, t) - a^{ij}(\tilde{x},\, \tilde{t})\,| \le L\left(|x - \tilde{x}| + |t - \tilde{t}|^{\frac{1}{2}}\right)$$

for some $\boldsymbol{\lambda} \in (0,\, 1]$, $L > 0$, *where* $B_1^n = \left\{x \in \mathbb{R}^n \middle|\, |x| < 1\right\}$.

Suppose that $\mathrm{u} \in C^{2,1}(B_1^n \times [-T,\, 0])$ *satisfies*

$$|P\mathrm{u}| \le L\left(\frac{1}{\sqrt{T}}|\partial_x \mathrm{u}| + \frac{1}{T}|\mathrm{u}|\right) \tag{1.68}$$

for some $T \in (0,\, 1]$, *and that either u vanishes at* $(0,\, 0)$ *to infinite order (see [EF]), i.e.*

$$\forall\, k \in \mathbb{N} \quad \exists\, C_k > 0 \quad s.t.\ |\mathrm{u}(x,\, t)\,| \le C_k\left(|x| + \sqrt{-t}\right)^k \tag{1.69}$$

or u *vanishes identically at* $t = 0$ *(see [N]), i.e.*

$$\mathrm{u}\Big|_{t=0} = 0 \tag{1.70}$$

Then there exit $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(n,\, \boldsymbol{\lambda},\, L) > 0$, $\boldsymbol{\alpha} = \boldsymbol{\alpha}(n,\, \boldsymbol{\lambda},\, L) \in (0,\, 1)$ *so that*

$$|\mathrm{u}(x,\, t)\,| + |\partial_x \mathrm{u}(x,\, t)\,| \tag{1.71}$$

$$\le \boldsymbol{\Lambda}\, e^{\frac{1}{\boldsymbol{\Lambda} t}}\left(\|\, \partial_x \mathrm{u}\, \|_{L^\infty(B_1 \times [-T,\, 0])} + \|\, \mathrm{u}\, \|_{L^\infty(B_1 \times [-T,\, 0])}\right)$$

for $x \in B_{1/4}^n$, $t \in [-\boldsymbol{\alpha} T,\, 0)$.

Remark 1.20. Later we would apply Propostion 1.19 under the condition (1.70) to show the exponential decay of the deviation h as $t \nearrow 0$. On the other hand, the proposition implies that under the condition (1.69), the function u in (1.68) must vanish identically at $t = 0$; in particular, it implies that u vanishes identically in the case when u is time-independent. Such phenomenon is called the "unique continuation principle" and would be used at the end of this section.

Proof. By doing some kind of change of variables like $\tilde{x} = a^{ij}(0,\,0)^{-\frac{1}{2}}\,x$, we may assume (for simplicity) that $a^{ij}(0,\,0) = \delta^{ij}$.

In the proof, we will focus on dealing with the case of (1.69), since the same argument work for the case of (1.70) with only a slight difference, which we would point out on the way of proof.

Fix a constant $M \in [\frac{4L^2(n+\sigma)}{T},\,\infty)$ (to be chosen), where $\sigma = \sigma(n,\,\boldsymbol{\lambda},\,L) \geq 1$ is the constant that appears in Lemma 1.15. Then for any $\epsilon \in \big(0,\,\min\{\frac{1}{M},\,1\}\big)$, choose smooth cut-off functions $\zeta = \zeta(x)$, $\eta_\epsilon = \eta_\epsilon(t)$ and $\eta = \eta(t)$ such that

$$\chi_{B^n_{1/2}} \leq \zeta \leq \chi_{B^n_1}, \quad \parallel \zeta \parallel_{C^2} \leq 4$$

$$\chi_{[\frac{-1}{M},\,-\epsilon]} \leq \eta_\epsilon \leq \chi_{[\frac{-2}{M},0]}, \quad \chi_{[\frac{-1}{M},0]} \leq \eta \leq \chi_{[\frac{-2}{M},0]}, \quad \eta_\epsilon \nearrow \eta \quad \text{as}\,\epsilon \searrow 0$$

$$|\partial_t \eta_\epsilon| \,\leq 2M\chi_{[\frac{-2}{M},\,\frac{-1}{M}]} + \frac{2}{\epsilon}\chi_{[-\epsilon,\,0]}$$

where $\chi_{B^n_1}$ is the characteristic function of B^n_1. Let $\mathrm{v}_\epsilon(x,\,t) = \zeta(x)\,\eta_\epsilon(t)\,\mathrm{u}(x,\,t)$ be a localization of u, which satisfies $\mathrm{v}_\epsilon\Big|_{t=0} = 0$ and convergers pointwisely to $\mathrm{v}(x,\,t) = \zeta(x)\,\eta(t)\,\mathrm{u}(x,\,t)$ as $\epsilon \searrow 0$. Besides, we have

$$|P\mathrm{v}_\epsilon| \,\leq\, L\zeta\eta_\epsilon\left(\frac{1}{\sqrt{T}}|\partial_x \mathrm{u}| \,+\, \frac{1}{T}|\mathrm{u}|\right) \tag{1.72}$$

$$+C\,(\boldsymbol{\lambda},\,L)\,(|\partial_x \mathrm{u}| \,+\, |\mathrm{u}|)\;\chi_{B_1\setminus B_{\frac{1}{2}}}(x) +\, 2LM\,|\mathrm{u}|\,\chi_{[\frac{-2}{M},\,\frac{-1}{M}]}(t) + \frac{2L}{\epsilon}\,|\mathrm{u}|\,\chi_{[-\epsilon,\,0]}(t)$$

$$\leq L\left(\frac{1}{\sqrt{T}}|\partial_x \mathrm{v}_\epsilon| \,+\, \frac{1}{T}|\mathrm{v}_\epsilon|\right) +\, C\,(\boldsymbol{\lambda},\,L)\,M\,(|\partial_x \mathrm{u}| \,+\, |\mathrm{u}|)\;\chi_E\,(x,\,t) + \frac{2L}{\epsilon}\,|\mathrm{u}|\,\chi_{[-\epsilon,\,0]}(t)$$

where $E = \Big\{(x,\,t) \in B^n_1 \times [-1,\,0)\Big|\, \frac{1}{2} \leq |x| \leq 1 \text{ or } \frac{-2}{M} \leq t \leq \frac{-1}{M}\Big\}$. Note that in the case of (1.70), it suffices to consider v (without using the ϵ cut-off) in order to make the function vanishing at $t = 0$.

Then for each $\delta \in \big(0,\,\frac{1}{M}\big)$, by Lemma 1.15 (applied to v_ϵ) and (1.72), there holds

$$M^2 \int \mathrm{v}_\epsilon^2 \boldsymbol{\varphi}_\delta^{-M} \boldsymbol{\Phi}_\delta\, dx\, dt \,+\, M \int |\partial_x \mathrm{v}_\epsilon|^2 \boldsymbol{\varphi}_\delta^{1-M} \boldsymbol{\Phi}_\delta\, dx\, dt$$

$$\leq 2\sigma L^2 \int \left(\frac{\mathrm{v}_\epsilon^2}{T^2} + \frac{|\partial_x \mathrm{v}_\epsilon|^2}{T}\right) \boldsymbol{\varphi}_\delta^{1-M} \boldsymbol{\Phi}_\delta \, dx\, dt + 2C\left(\boldsymbol{\lambda},\, L\right) \sigma M^2 \int_E \left(|\partial_x \mathrm{u}|^2 + \mathrm{u}^2\right) \boldsymbol{\varphi}_\delta^{1-M} \boldsymbol{\Phi}_\delta \, dx\, dt$$

$$+ \frac{4\sigma L^2}{\epsilon^2} \int_{-\epsilon}^0 \int_{B_1} \mathrm{u}^2 \boldsymbol{\varphi}_\delta^{1-M} \boldsymbol{\Phi}_\delta \, dx\, dt \; + \; (\sigma M)^M \sup_t \int \left(|\partial_x \mathrm{v}_\epsilon|^2 + \mathrm{v}_\epsilon^2\right) \, dx$$

By our choice of M, the first term on the RHS of the above inequality can be absorbed by its LHS. Thus, we get

$$M^2 \int \mathrm{v}_\epsilon^2 \boldsymbol{\varphi}_\delta^{-M} \boldsymbol{\Phi}_\delta \, dx\, dt \leq C\left(\boldsymbol{\lambda},\, L\right) \sigma M^2 \int_E \left(|\partial_x \mathrm{u}|^2 + \mathrm{u}^2\right) \boldsymbol{\varphi}_\delta^{1-M} \boldsymbol{\Phi}_\delta \, dx\, dt \tag{1.73}$$

$$+4\left(\sigma M\right)^M \sup_{-T \leq t \leq 0} \int_{B_1} \left(|\partial_x \mathrm{u}|^2 + \mathrm{u}^2\right) \, dx \; + \; \frac{4\sigma L^2}{\epsilon^2} \int_{-\epsilon}^0 \int_{B_1} \mathrm{u}^2 \boldsymbol{\varphi}_\delta^{1-M} \boldsymbol{\Phi}_\delta \, dx\, dt$$

Now choose an integer $k \geq M + \frac{n}{2}$, then by (1.69) the last term on the RHS of (1.73) can be estimated by

$$\frac{4\sigma L^2}{\epsilon^2} \int_{-\epsilon}^0 \int_{B_1} \mathrm{u}^2 \boldsymbol{\varphi}_\delta^{1-M} \boldsymbol{\Phi}_\delta \, dx\, dt \tag{1.74}$$

$$\leq \; \frac{4\sigma L^2}{\epsilon^2} \int_{-\epsilon}^0 \int_{B_1} \frac{C_k \left(|x| \, + \, \sqrt{-t}\right)^{2\left(M + \frac{n}{2}\right)}}{\left(\frac{-t+\delta}{\sigma}\right)^{M-1}} \frac{\exp\left(\frac{-|x|^2}{4(-t+\delta)}\right)}{\left(4\pi\left(-t+\delta\right)\right)^{\frac{n}{2}}} \, dx\, dt$$

$$\leq C\left(n,\, C_k,\, \sigma,\, M,\, L\right) \frac{1}{\epsilon^2} \int_{-\epsilon}^0 \left\{ \int_{B_1} \left(\frac{|x|^2}{-t+\delta} + 1\right)^{M + \frac{n}{2}} \exp\left(\frac{-|x|^2}{4\left(-t+\delta\right)}\right) \, dx \right\} \left(-t+\delta\right) \, dt$$

$$\leq C\left(n,\, C_k,\, \sigma,\, M,\, L\right) \frac{1}{\epsilon^2} \int_{-\epsilon}^0 \left\{ \int_0^\infty \left(|\xi|^2 + 1\right)^{M + \frac{n}{2}} \exp\left(\frac{-|\xi|^2}{4}\right) \, d\xi \right\} \left(-t+\delta\right)^{\frac{n}{2}+1} \, dt$$

$$\leq C\left(n,\, C_k,\, \sigma,\, M,\, L\right) \frac{\left(\epsilon + \delta\right)^{\frac{n}{2}+2} - \delta^{\frac{n}{2}+2}}{\epsilon^2}$$

In view of (1.74), apply the monotone convergence theorem to (1.73) by first letting $\delta \searrow 0$ and then $\epsilon \searrow 0$ to arrive at

$$\int_{B_{\frac{1}{2}} \times \left(\frac{-1}{M},\, 0\right)} \mathrm{u}^2 \boldsymbol{\varphi}^{-M} \boldsymbol{\Phi} \, dx\, dt \tag{1.75}$$

$$\leq \; C\left(\Lambda,\, L\right) \sigma \int_E \left(|\partial_x \mathrm{u}|^2 + \mathrm{u}^2\right) \boldsymbol{\varphi}^{1-M} \boldsymbol{\Phi} \, dx\, dt \, + \left(4\sigma M\right)^M \sup_{-T \leq t \leq 0} \int_{B_1} \left(|\partial_x \mathrm{u}|^2 + \mathrm{u}^2\right) \, dx$$

$$\le C\left(n,\,\Lambda,\,L\right)\left(\sigma\int_{E}\boldsymbol{\varphi}^{1-M}\boldsymbol{\Phi}\,dx\,dt\,+\,\left(\sigma M\right)^{M}\right)\left(\|\,\partial_{x}\mathrm{u}\,\|^{2}_{L^{\infty}(B_{1}\times[-T,\,0])}+\|\,\mathrm{u}\,\|^{2}_{L^{\infty}(B_{1}\times[-T,\,0])}\right)$$

Note that in the case of (1.70), we can get (1.75) directly from taking the limit as $\delta\searrow 0$ without using (1.74).

Next, we would like to estimate the first term on the RHS of (1.75). For $(x,\,t)\in E$, either $\frac{-2}{M}\le t\le\frac{-1}{M}$, in which case we have

$$\boldsymbol{\varphi}^{1-M}\boldsymbol{\Phi}\left(x,\,t\right)\le\left(\frac{-t}{\sigma}\right)^{1-M}\frac{1}{\left(4\pi\left(-t\right)\right)^{\frac{n}{2}}}\le\frac{\left(\sigma M\right)^{M-1+\frac{n}{2}}}{\left(4\pi\sigma\right)^{\frac{n}{2}}}\tag{1.76}$$

or $\frac{1}{2}\le|x|\le 1$ and $\frac{-2}{M}\le t<0$, in which case we have

$$\boldsymbol{\varphi}^{1-M}\boldsymbol{\Phi}\left(x,\,t\right)\le\left(\frac{\sigma M}{\left(-t\right)M}\right)^{M-1}\frac{M^{\frac{n}{2}}}{\left(4\pi\left(-t\right)M\right)^{\frac{n}{2}}}\exp\left(\frac{-M}{16\left(-tM\right)}\right)\tag{1.77}$$

$$=\frac{\left(\sigma M\right)^{M-1}\left(\frac{M}{4\pi}\right)^{\frac{n}{2}}}{\left(-tM\right)^{M-1+\frac{n}{2}}\exp\left(\frac{M/16}{-tM}\right)}\le\left(\sigma M\right)^{M-1}\left(\frac{M}{4\pi}\right)^{\frac{n}{2}}\left(\frac{M-1+\frac{n}{2}}{e\,M/16}\right)^{M-1+\frac{n}{2}}$$

$$\le\left(\frac{16\sigma}{\mathrm{e}}\left(M-1+\frac{n}{2}\right)\right)^{M-1+\frac{n}{2}}$$

Note that in (1.77) we use the fact that the function $\vartheta\left(\xi\right)=\xi^{M-1+\frac{n}{2}}\exp\left(\frac{M/16}{\xi}\right)$ achieves its minimum on $\mathbb{R}_{+}$ at $\xi=\frac{M/16}{M-1+\frac{n}{2}}$.

On the other hand, for any $(y,\,s)\in B_{\frac{1}{4}}\times[\frac{-1}{8M},\,0)$, the parabolic cylinder $Q\left(y,\,s;\,\sqrt{-s}\right)=B^{n}_{\sqrt{-s}}\left(y\right)\times\left(2s,\,s\right)$ is contained in $B^{n}_{1/2}\times\left(\frac{-1}{M},\,0\right)$ and hence the LHS of (1.75) is bounded below by

$$\int_{B^{n}_{1/2}\times\left(\frac{-1}{M},0\right)}\mathrm{u}^{2}\boldsymbol{\varphi}^{-M}\boldsymbol{\Phi}\,dx\,dt\,\ge\,\frac{\exp\frac{-1/4}{-8s}}{\left(4\pi\right)^{\frac{n}{2}}\left(-2s\right)^{M+\frac{n}{2}}}\int_{Q\left(y,\,s;\,\sqrt{-s}\right)}\mathrm{u}^{2}\,dx\,dt\tag{1.78}$$

Combining (1.75), (1.76), (1.77), (1.78), we conclude that for $(y,\,s)\in Q\left(0,\,0;\,\frac{-1}{8M}\right)$,

$$\int_{Q\left(y,\,s;\,\sqrt{-s}\right)}\mathrm{u}^{2}\,dx\,dt\tag{1.79}$$

$$\le\,C\left(n,\,\boldsymbol{\lambda},\,L,\,\sigma\right)\left(\frac{64\sigma}{\mathrm{e}}\left(-sM\right)\right)^{M-1+\frac{n}{2}}\left(\|\,\partial_{x}\mathrm{u}\,\|^{2}_{L^{\infty}(B_{1}\times[-T,\,0])}+\|\,\mathrm{u}\,\|^{2}_{L^{\infty}(B_{1}\times[-T,\,0])}\right)$$

Now let $\beta = \frac{1}{2}\left(\frac{64\sigma}{\mathrm{e}}\right)^{-1}$. For each $(y,\, s) \in B^n_{1/4} \times [\frac{-\beta}{4L^2(n+\sigma)}T,\, 0)$, we choose $M = \frac{\beta}{-s}$ so that $M \geq \frac{4L^2(n+\sigma)}{T}$ (and note that $\frac{-1}{8M} \leq s < 0$). By (1.79), we get

$$\fint_{Q(y,\, s;\, \sqrt{-s})} |\mathrm{u}|\, dx\, dt \;\leq\; \left(\fint_{Q(y,\, s;\, \sqrt{-s})} \mathrm{u}^2\, dx\, dt\right)^{\frac{1}{2}} \tag{1.80}$$

$$\leq C\,(n,\, \Lambda,\, L,\, \sigma) \sqrt{(-s)^{-\frac{n}{2}-1}\left(\frac{1}{2}\right)^{-\frac{\beta}{s}-1+\frac{n}{2}}} \left(\|\, \partial_x \mathrm{u}\, \|_{L^\infty(B_1\times[-T,\,0])} + \|\, \mathrm{u}\, \|_{L^\infty(B_1\times[-T,\,0])}\right)$$

$$\leq C\,(n,\, \boldsymbol{\lambda},\, L,\, \sigma)\left(2^{\frac{\beta}{4}}\right)^{\frac{1}{s}} \left(\|\, \partial_x \mathrm{u}\, \|_{L^\infty(B_1\times[-T,\,0])} + \|\, \mathrm{u}\, \|_{L^\infty(B_1\times[-T,\,0])}\right)$$

Let $\boldsymbol{\alpha} = \frac{\beta}{4L^2(n+\sigma)}$, $\boldsymbol{\Lambda} = \max\left\{C\,(n,\, \boldsymbol{\lambda},\, L,\, \sigma)\,,\, \left(\frac{\beta}{4}\ln 2\right)^{-1}\right\}$, then (1.71) follows from (1.80) and Lemma 1.15. $\square$

Combining Propostion 1.13, Proposition 1.14 and Proposition 1.19, we can show the exponential decay of h_t as $t \nearrow 0$ as in [W].

Proposition 1.21. *(Exponential decay of the deviation)*

There exist $R = R\left(\Sigma,\, \tilde{\Sigma},\, \mathcal{C},\, U,\, \|\, F\, \|_{C^3(U)},\, \lambda,\, \varkappa\right) \geq 1$, $\Lambda = \Lambda\left(n,\, \mathcal{C},\, \|\, F\, \|_{C^3(U)},\, \lambda\right) > 0$, $\alpha = \alpha\left(n,\, \mathcal{C},\, \|\, F\, \|_{C^3(U)},\, \lambda\right) \in (0,\, 1)$ *such that for* $X \in \Sigma_t \setminus \bar{B}_R$, $t \in [-\alpha,\, 0)$, *there holds*

$$|\nabla_{\Sigma_t} h| + |h| \leq \Lambda \exp\left(\frac{|X|^2}{\Lambda t}\right)$$

Proof. Fix $\hat{X} \in \Sigma \setminus \bar{B}_R$ with $R = R\left(\Sigma,\, \tilde{\Sigma},\, \mathcal{C},\, U,\, \|\, F\, \|_{C^3(U)},\, \lambda,\, \varkappa\right) \geq 1$, first we would like to show that near $\hat{X}$, there is a "normal parametrization" for the flow $\{\Sigma_t\}$ for $t \in [-1,\, 0]$.

Recall that in the begining of Section 1.3, we show that there exists a constant $\boldsymbol{\rho} = \boldsymbol{\rho}\,(n,\, \mathcal{C}) \in (0,\, 1)$ so that near $\hat{X}$, each Σ_t is the graph of the function $\mathtt{u}_t = \mathtt{u}\,(\cdot,\, t)$ defined on $B^n_{\boldsymbol{\rho}|\hat{X}|} \subset T_{\hat{X}_{\mathcal{C}}}\mathcal{C}$ for $t \in [-1,\, 0]$, where $\hat{X}_{\mathcal{C}} = \Pi\left(\hat{X}\right)$ is the the normal projection of $\hat{X}$ onto $\mathcal{C}$. Note that $|\hat{X}_{\mathcal{C}}|$ is comparable with $|\hat{X}|$. In other words, locally near $\hat{X}$, we have the following "vertical parametrization" of the flow $\{\Sigma_t\}_{-1\leq t\leq 0}$:

$$X = X\,(x,\, t) \equiv \hat{X}_{\mathcal{C}} + (x,\, \mathtt{u}\,(x,\, t))$$

Here we assume that the unit-normal of $\mathcal{C}$ at $\hat{X}_{\mathcal{C}}$ to be $(0, 1)$ for ease of notation. For this vertical parametrization, we may decompose the velocity vector into normal and tangential components as follows:

$$\partial_t X = F\left(A^{\#}(x, t)\right) N(x, t) + \sum_{i=1}^{n} \frac{\partial_i \mathrm{u}\, \partial_t \mathrm{u}}{1 + |\partial_x \mathrm{u}|^2}\, \partial_i X$$

where $A^{\#}(x, t)$, $N(x, t)$ are the shape operator and the unit-normal of Σ_t at $X(x, t)$, respectively. Note that the normal component is given by Definition 1.4 for the F curvature flow.

Next, we would like to do suitable change of variables to go from this "vertical parametrization" to the "noramal parametrization" of the flow (see Definition 1.12). For that purpose, we use the same trick as in Proposition 1.13. Let $x = \phi_t(\xi)$ with $\phi_{-1} = \mathrm{id}$ to be the local diffeomorphism on Σ_t generated by the following tangent vector field:

$$\mathcal{V}(x, t) = -\sum_{i=1}^{n} \frac{\partial_i \mathrm{u}\, \partial_t \mathrm{u}}{1 + |\partial_x \mathrm{u}|^2}\, \partial_i X \equiv -\sum_{i=1}^{n} \mathcal{V}^i(x, t)\, \partial_i X$$

That is, $\phi_t(\xi) = \phi(\xi, t)$ satisfies

$$\partial_t \phi_t = \left(\mathcal{V}^1(\phi_t, t), \cdots, \mathcal{V}^n(\phi_t, t)\right), \quad \phi_{-1}(\xi) = \xi \tag{1.81}$$

in which, by (1.9) and (1.14), we have

$$|\mathcal{V}^i| \leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) |\hat{X}|^{-1} \quad \forall\, i = 1, \cdots, n \tag{1.82}$$

Thus, by taking R sufficiently large, ϕ_t is well-defined for $\xi \in B^n_{\frac{\varrho}{2}|\hat{X}|}$, $t \in [-1, 0]$. It follows that the reparametrization $X = X(\phi_t(\xi), t)$ of the flow becomes a "normal parametrization" near $\hat{X}$ for $t \in [-1, 0]$; that is,

$$\frac{\partial}{\partial t}\left(X(\phi_t(\xi), t)\right) = F\left(A^{\#}(\phi_t(\xi), t)\right) N(\phi_t(\xi), t)$$

Let $g_{ij}(\xi, t) = \partial_{\xi_i}(X(\phi_t(\xi), t)) \cdot \partial_{\xi_j}(X(\phi_t(\xi), t))$ be the pull-back metric associated with this "normal parametrization", then by the evolution equation for the metric in [A], the homogeneity of F and the condition that $\phi_{-1} = \mathrm{id}$, we have

$$\partial_t g_{ij}(\xi, t) = -2F\left(A^{\#}(\phi_t(\xi), t)\right) A_{ij}(\phi_t(\xi), t) \tag{1.83}$$

$$= -2\Big|X\left(\phi_t\left(\xi\right),\, t\right)\Big|^{-1} F\left(\Big|X\left(\phi_t\left(\xi\right),\, t\right)\Big|\, A^{\#}\left(\phi_t\left(\xi\right),\, t\right)\right)\, A_{ij}\left(\phi_t\left(\xi\right),\, t\right)$$

$$g_{ij}\left(\xi,\, -1\right) = \delta_{ij} + \partial_i \mathtt{u}\left(\xi,\, -1\right)\, \partial_j \mathtt{u}\left(\xi,\, -1\right) \tag{1.84}$$

where the second fundamental form $A_t\left(x\right) \sim A_{ij}\left(x,\, t\right)$ is equal to

$$A_{ij}\left(x,\, t\right) = \frac{\partial^2_{ij}\mathtt{u}\left(x,\, t\right)}{\sqrt{1 + |\partial_x \mathtt{u}\left(x,\, t\right)|^2}} \tag{1.85}$$

By (1.85), (1.6), (1.7), (1.8), (1.17) and the comparability of $|X\left(x,\, t\right)|$ and $|\hat{X}|$, the ℓ^2 norm of the matrix $\partial_t g_{ij}\left(\xi,\, t\right)$ satisfies

$$|\partial_t g_{ij}\left(\xi,\, t\right)| \,\leq\, C\left(n,\, \mathcal{C},\, \|\, F\, \|_{C^1(U)}\right)\, |\hat{X}|^{-2} \tag{1.86}$$

So by (1.84), (1.6), (1.8) and (1.86), the pull-back metric $g_{ij}\left(\xi,\, t\right)$ is equivalent to the dot product δ_{ij}.

Let $\Gamma^k_{ij}\left(\xi,\, t\right)$ be the Christoffel symbols associated with the metric $g_{ij}\left(\xi,\, t\right)$, then we have

$$\partial_t \Gamma^k_{ij} \;=\; \frac{1}{2} g^{kl}\left(\nabla_i \dot{g}_{lj} + \nabla_j \dot{g}_{il} - \nabla_l \dot{g}_{ij}\right) \tag{1.87}$$

$$\Gamma^k_{ij}\left(\xi,\, -1\right) \;=\; \frac{\partial_k \mathtt{u}\left(\xi,\, -1\right)\, \partial^2_{ij}\mathtt{u}\left(\xi,\, -1\right)}{1 + |\partial_x \mathtt{u}\left(\xi,\, -1\right)|^2} \tag{1.88}$$

where $\dot{g}_{ij} \;=\; \partial_t g_{ij} \;=\; -2F\left(A^{\#}\right) A_{ij}$. Similarly, and also by (1.20), the homogeneity of the derivative of F, the equivalence of g_{ij} and δ_{ij}, we have

$$|\partial_t \Gamma^k_{ij}| \,\leq C\left(n,\, \mathcal{C},\, \|\, F\, \|_{C^1(U)}\right)\, |\hat{X}|^{-3}$$

$$|\Gamma^k_{ij}\left(\xi,\, -1\right)| \,\leq C\left(n,\, \mathcal{C},\, \|\, F\, \|_{C^1(U)}\right)\, |\hat{X}|^{-1}$$

which implies

$$|\Gamma^k_{ij}\left(\xi,\, t\right)| \,\leq C\left(n,\, \mathcal{C},\, \|\, F\, \|_{C^1(U)}\right)\, |\hat{X}|^{-1} \tag{1.89}$$

Now consider the deviation h in the local coordinates $\left(\xi,\, t\right)$, then the equation in Proposition 1.13 becomes

$$\Big|\partial_t h - \Big\{\partial_{\xi_i}\left(\mathbf{a}^{ij}\left(\xi,\, t\right)\partial_{\xi_j} h\right) + \Gamma^i_{ik}\left(\xi,\, t\right)\, \mathbf{a}^{kj}\left(\xi,\, t\right)\partial_{\xi_j} h\Big\}\Big| \tag{1.90}$$

$$\leq C\left(n,\, \mathcal{C},\, \|\, F\,\|_{C^3(U)}\right)\left(|\hat{X}|^{-1}|\partial_\xi h| \,+\, |\hat{X}|^{-2}|h|\right)$$

$$h\left(\xi,\,0\right)=0 \tag{1.91}$$

where $\mathbf{a}^{ij}\left(\xi,\,t\right)=\mathbf{a}^{ji}\left(\xi,\,t\right)$ satisfies (by Proposition 1.14 and (1.89))

$$\frac{\lambda}{C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)}\right)}\delta^{ij} \,\leq\, \frac{\lambda}{3}g^{ij}\left(\xi,\,t\right) \,\leq\, \mathbf{a}^{ij}\left(\xi,\,t\right) \,\leq\, \frac{3}{\lambda}g^{ij}\left(\xi,\,t\right) \,\leq\, \frac{C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)}\right)}{\lambda}\delta^{ij} \tag{1.92}$$

$$|\hat{X}|\Big|\partial_\xi \mathbf{a}^{ij}\left(\xi,\,t\right)\Big| \,+\, |\hat{X}|^2\Big|\partial_t \mathbf{a}^{ij}\left(\xi,\,t\right)\Big| \,\leq C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)},\,\lambda\right) \tag{1.93}$$

Thus, by (1.89), (1.92), (1.84) and (1.86), the equation (1.90) is equivalent to

$$\Big|\partial_t h - \partial_{\xi_i}\left(\mathbf{a}^{ij}\left(\xi,\,t\right)\partial_{\xi_j}h\right)\Big| \tag{1.94}$$

$$\leq C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)},\,\lambda\right)\left(|\hat{X}|^{-1}|\partial_\xi h| \,+\, |\hat{X}|^{-2}|h|\right)$$

for $(\xi,\,t)\in B^n_{\frac{\boldsymbol{\rho}}{2}|\hat{X}|}\times[-1,\,0]$.

Let's consider the following change of variables:

$$(\xi,\,t)=\Xi\left(\bar{\xi},\,\bar{t}\right)\equiv\left(\left(\frac{\boldsymbol{\rho}}{2}|\hat{X}|\right)\bar{\xi},\;\left(\frac{\boldsymbol{\rho}}{2}|\hat{X}|\right)^2\bar{t}\right)$$

and let $\bar{h}=h\circ\psi$, $\bar{\mathbf{a}}^{ij}=\mathbf{a}^{ij}\circ\psi$. Then (1.94) and (1.91) in the new variables become

$$\Big|\partial_{\bar{t}}\bar{h}-\partial_{\bar{\xi}_i}\left(\bar{\mathbf{a}}^{ij}\left(\bar{\xi},\bar{t}\right)\partial_{\bar{\xi}_j}\bar{h}\right)\Big| \,\leq C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)},\,\lambda,\,\boldsymbol{\rho}\right)\left(|\partial_{\bar{\xi}}\bar{h}| \,+\, |\bar{h}|\right) \tag{1.95}$$

$$\bar{h}\Big|_{\bar{t}=0}=0 \tag{1.96}$$

and (1.92), (1.93) are translated into

$$\frac{\lambda}{C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)}\right)}\delta^{ij} \,\leq\, \bar{\mathbf{a}}^{ij}\left(\bar{\xi},\bar{t}\right) \,\leq\, \frac{C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)}\right)}{\lambda}\delta^{ij} \tag{1.97}$$

$$\Big|\partial_{\bar{\xi}}\bar{\mathbf{a}}^{ij}\left(\bar{\xi},\bar{t}\right)\Big| \,+\, \Big|\partial_{\bar{t}}\bar{\mathbf{a}}^{ij}\left(\bar{\xi},\bar{t}\right)\Big| \,\leq\, C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)},\,\lambda,\,\boldsymbol{\rho}\right) \tag{1.98}$$

for $\bar{\xi}\in B^n_1$, $\bar{t}\in\left[-\left(\frac{\boldsymbol{\rho}}{2}|\hat{X}|\right)^{-2},\,0\right]$.

Applying Proposition 1.19 to $\bar{h}\left(\bar{\xi},\, \bar{t}\right)$, we may conclude that there exist $\tilde{\Lambda} = \tilde{\Lambda}\left(n,\, \mathcal{C},\, \| F \|_{C^3(U)},\, \lambda\right) > 0$, $\alpha = \alpha\left(n,\, \mathcal{C},\, \| F \|_{C^3(U)},\, \lambda\right) \in (0,\, 1)$ for which the following holds:

$$|\partial_{\bar{\xi}}\bar{h}| + |\bar{h}| \tag{1.99}$$

$$\leq \tilde{\Lambda}\, \exp\left(\frac{1}{\tilde{\Lambda}\bar{t}}\right)\left(\| \partial_{\bar{\xi}}\bar{h} \|_{L^\infty\left(B^n_1\times\left[-\left(\frac{\boldsymbol{\rho}}{2}|\hat{X}|\right)^{-2},0\right]\right)} + \| \bar{h} \|_{L^\infty\left(B^n_1\times\left[-\left(\frac{\boldsymbol{\rho}}{2}|\hat{X}|\right)^{-2},0\right]\right)}\right)$$

for $\left(\bar{\xi},\, \bar{t}\right) \in B^n_{1/4} \times [-\alpha\left(\frac{\boldsymbol{\rho}}{2}|\hat{X}|\right)^{-2},\, 0)$. By undoing change of variables, (1.99) becomes

$$\frac{\boldsymbol{\rho}}{2}\,|\hat{X}|\,|\partial_\xi h| + |h| \tag{1.100}$$

$$\leq \tilde{\Lambda}\, \exp\left(\frac{|\hat{X}|^2}{\tilde{\Lambda}t}\right)\left(\frac{\boldsymbol{\rho}}{2}|\hat{X}|\, \| \partial_\xi h \|_{L^\infty\left(B^n_{\frac{\boldsymbol{\rho}}{2}|\hat{X}|}\times[-1,0]\right)} + \| h \|_{L^\infty\left(B^n_{\frac{\boldsymbol{\rho}}{2}|\hat{X}|}\times[-1,0]\right)}\right)$$

for $(\xi,\, t) \in B^n_{\frac{\boldsymbol{\rho}}{8}|\hat{X}|} \times [-\alpha,\, 0)$. Note that the pull-back metric $g_{ij}\,(\xi,\, t)$ is equivalent to the dot product δ_{ij} and that $|X\,(x,\, t)\,|$ is comparable with $|\hat{X}|$. The conclusion follows immediately. □

Next, we'd like to go from the exponential decay to identically vanishing of the deviation h outside a compact set. To this end, we have to derive a different type of Carleman's inequality on the flow $\{\Sigma_t\}_{-1\leq t\leq 0}$, which is done through two lemma. The first lemma is a modification of the integral equality in [EF].

Lemma 1.22. *Let* $(\mathrm{M},\, \boldsymbol{g}_t)$ *be a flow of Riemannian manifolds and* P *be a differential operator on the flow defined by*

$$P\mathrm{v} = \partial_t \mathrm{v} - \nabla_{\boldsymbol{g}_t}\cdot(a_t\, d\mathrm{v}) \equiv \partial_t\mathrm{v} - \nabla_i\left(a^{ij}\,(\cdot,\, t)\,\nabla_j\mathrm{v}\right)$$

where $a_t = a(\cdot,\, t)$ *is a symmetric 2-tensor on* M. *Then given functions* $G,\, \Psi \in C^{2,1}\left(\mathrm{M}\times[-T,\, 0]\right)$ *with* $G > 0$*, define a function* Φ *as*

$$\Phi = \frac{\partial_t G + \nabla_i\left(a^{ij}\nabla_j G\right) + \frac{1}{2}tr\left(\partial_t \boldsymbol{g}\right)\, G}{G} \tag{1.101}$$

$$= \partial_t \ln G + \nabla_i\left(a^{ij}\nabla_j \ln G\right) + a^{ij}\nabla_i \ln G\, \nabla_j \ln G + \frac{1}{2}tr\left(\partial_t\boldsymbol{g}\right)$$

and a 2-tensor Υ *as*

$$\Upsilon^{ij} = a^{ik}a^{jl}\nabla^2_{kl}\ln G - \frac{1}{2}\partial_t a^{ij} \tag{1.102}$$

$$+\frac{1}{2}\left(a^{ik}\nabla_k a^{jl}+a^{jk}\nabla_k a^{il}-a^{lk}\nabla_k a^{ij}\right)\nabla_l \ln G$$

It follows that for any $\mathrm{u}\in C_c^{2,1}\left(\mathrm{M}\times[-T,\,0]\right)$*, there holds*

$$\int_{\mathrm{M}}\left\{\left(2\Upsilon^{ij}-(\Phi-\Psi)\,a^{ij}\right)\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}\,+\,\frac{1}{2}\left(\partial_t\Psi-\nabla_i\left(a^{ij}\nabla_j\Psi\right)+(\Phi-\Psi)\,\Psi\right)\mathrm{u}^2\right\}G\,d\mu_t$$

$$\begin{aligned}&=\int_{\mathrm{M}}2\,P\mathrm{u}\left(\partial_t\mathrm{u}+\,a^{ij}\nabla_i\ln G\,\nabla_j\mathrm{u}+\frac{1}{2}\Psi\mathrm{u}\right)G\,d\mu_t\\&-\int_{\mathrm{M}}2\left(\partial_t\mathrm{u}+\,a^{ij}\nabla_i\ln G\,\nabla_j\mathrm{u}+\frac{1}{2}\Psi\mathrm{u}\right)^2 G\,d\mu_t\\&-\partial_t\left\{\int_{\mathrm{M}}\left(a^{ij}\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}\,-\,\frac{1}{2}\Psi\mathrm{u}^2\right)G\,d\mu_t\right\}\end{aligned}\tag{1.103}$$

where μ_t *is the volume form of* $(\mathrm{M},\,\boldsymbol{g}_t)$.

Proof. Let's begin with

$$\partial_t\left\{\int_{M}a^{ij}\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}\,G\,d\mu_t\right\}\tag{1.104}$$

$$=\int_{\mathrm{M}}\left\{2a^{ij}\nabla_j\mathrm{u}\,\nabla_i\partial_t\mathrm{u}\,G+\,a^{ij}\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}\left(\partial_t G+\frac{1}{2}\mathrm{tr}\left(\partial_t g\right)\,G\right)+\,\partial_t a^{ij}\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}\,G\right\}d\mu_t$$

in which we use the commutativity

$$\partial_t\,d\mathrm{u}=d\,\partial_t\mathrm{u},\quad d\mathrm{u}\sim\nabla_i\mathrm{u}$$

and the evolution equation of the volume form:

$$\partial_t\,d\mu_t=\frac{1}{2}\mathrm{tr}\left(\partial_t\boldsymbol{g}\right)\,d\mu_t\tag{1.105}$$

Applying integation by parts on $(\mathrm{M},\,\boldsymbol{g}_t)$, (1.104) becomes

$$\begin{aligned}&\int_{\mathrm{M}}-2\left(\nabla_i\left(a^{ij}\nabla_j\mathrm{u}\right)+\,a^{ij}\nabla_i\ln G\,\nabla_j\mathrm{u}\right)\partial_t\mathrm{u}\,G\,d\mu_t+\int_{\mathrm{M}}a^{ij}\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}\left(\partial_t G+\nabla_k\left(a^{kl}\nabla_l G\right)+\,\frac{1}{2}\mathrm{tr}\left(\partial_t g\right)\,G\right)\\&-\int_{\mathrm{M}}a^{ij}\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}\,\nabla_k\left(a^{kl}\nabla_l G\right)\,d\mu_t+\int_{\mathrm{M}}\partial_t a^{ij}\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}\,G\,d\mu_t\end{aligned}\tag{1.106}$$

By (1.101), integrating by parts twice and the symmetry of a_t, (1.106) becomes

$$-2\int_{\mathrm{M}}\left(\nabla_i\left(a^{ij}\nabla_j\mathrm{u}\right)+\,a^{ij}\nabla_i\ln G\,\nabla_j\mathrm{u}\right)\partial_t\mathrm{u}\,G\,d\mu_t+\int_{\mathrm{M}}a^{ij}\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}\,\Phi G\,d\mu_t\tag{1.107}$$

$$+\int_{\mathrm{M}}\Big\{\nabla_k a^{ij}\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}\,a^{kl}\nabla_l\ln G-\,2\nabla_j\left(a^{ij}\nabla_i\mathrm{u}\right)\nabla_k\mathrm{u}\,a^{kl}\nabla_l\ln G-\,2a^{ij}\nabla_i\mathrm{u}\,\nabla_k\mathrm{u}\,\nabla_j a^{kl}\nabla_l\ln G\Big\}\,G\,d\mu_t$$

$$-2\int_{\mathrm{M}}a^{ij}\nabla_i\mathrm{u}\,\nabla_k\mathrm{u}\,a^{kl}\nabla^2_{jl}G\,d\mu_t+\int_{\mathrm{M}}\partial_t a^{ij}\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}\,G\,d\mu_t$$

Then we recognize (1.107) (in order to make up the term $P\mathrm{u}$) to get

$$2\int_{\mathrm{M}}\Big\{\left(\partial_t\mathrm{u}-\nabla_i\left(a^{ij}\nabla_j\mathrm{u}\right)\right)\Big(\partial_t\mathrm{u}+a^{kl}\nabla_k\ln G\,\nabla_l\mathrm{u}\Big)-\,(\partial_t\mathrm{u})^2-\,2a^{ij}\nabla_i\ln G\,\nabla_j\mathrm{u}\,\partial_t\mathrm{u}\Big\}\,G\,d\mu_t \tag{1.108}$$

$$+\int_{\mathrm{M}}\Phi a^{ij}\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}\,G\,d\mu_t-\,2\int_{\mathrm{M}}a^{ij}a^{kl}\left(\nabla^2_{jl}\ln G+\nabla_j\ln G\,\nabla_l\ln G\right)\nabla_i\mathrm{u}\,\nabla_k\mathrm{u}\,G\,d\mu_t$$

$$+\int_{\mathrm{M}}\Big\{a^{kl}\nabla_k a^{ij}\,\nabla_l\ln G\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}-\,2a^{ij}\nabla_j a^{kl}\,\nabla_l\ln G\,\nabla_i\mathrm{u}\,\nabla_k\mathrm{u}+\,\partial_t a^{ij}\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}\Big\}\,G\,d\mu_t$$

By (1.102), (1.108) becomes

$$2\int_{\mathrm{M}}\Big\{\left(\partial_t\mathrm{u}-\nabla_i\left(a^{ij}\nabla_j\mathrm{u}\right)\right)\Big(\partial_t\mathrm{u}+a^{kl}\nabla_k\ln G\,\nabla_l\mathrm{u}\Big)-\,\Big(\partial_t\mathrm{u}+a^{ij}\nabla_i\ln G\,\nabla_j\mathrm{u}\Big)^2\Big\}\,G\,d\mu_t$$

$$+\int_{\mathrm{M}}\Phi a^{ij}\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}\,G\,d\mu_t-\,2\int_{\mathrm{M}}\Upsilon^{ij}\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}\,G\,d\mu_t$$

$$=2\int_{\mathrm{M}}P\mathrm{u}\,\left(\partial_t\mathrm{u}+\,a^{ij}\nabla_i\ln G\,\nabla_j\mathrm{u}+\,\frac{1}{2}\Psi\mathrm{u}\right)G\,d\mu_t-\,\int_{\mathrm{M}}\left(\partial_t\mathrm{u}-\,\nabla_i\left(a^{ij}\nabla_j\mathrm{u}\right)\right)\Psi\mathrm{u}\,G\,d\mu_t$$

$$-2\int_{\mathrm{M}}\left(\partial_t\mathrm{u}+\,a^{ij}\nabla_i\ln G\,\nabla_j\mathrm{u}+\frac{1}{2}\Psi\mathrm{u}\right)^2G\,d\mu_t+2\int_{\mathrm{M}}\left(\partial_t\mathrm{u}\,+a^{ij}\nabla_i\ln G\,\nabla_j\mathrm{u}+\,\frac{1}{2}\Psi\mathrm{u}\right)\Psi\mathrm{u}\,G\,d\mu_t$$

$$-\frac{1}{2}\int_{\mathrm{M}}\Psi^2\mathrm{u}^2G\,d\mu_t-\int_{\mathrm{M}}\left(2\Upsilon^{ij}-\Phi a^{ij}\right)\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}\,G\,d\mu_t \tag{1.109}$$

For the second term of (1.109), by the product rule and integration by parts, we get

$$-\int_{\mathrm{M}}\left(\partial_t\mathrm{u}-\,\nabla_i\left(a^{ij}\nabla_j\mathrm{u}\right)\right)\mathrm{u}\,\Psi G\,d\mu_t \tag{1.110}$$

$$=-\frac{1}{2}\int_{\mathrm{M}}\left(\partial_t\mathrm{u}^2-\,\nabla_i\left(a^{ij}\nabla_j\mathrm{u}^2\right)+\,2a^{ij}\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}\right)\Psi G\,d\mu_t$$

$$=\frac{1}{2}\int_{\mathrm{M}}\left(\partial_t\Psi\,G+\,\Psi\left(\partial_t G+\frac{1}{2}\mathrm{tr}\left(\partial_t g\right)\,G\right)\right)\mathrm{u}^2\,d\mu_t-\partial_t\left(\int_{\mathrm{M}}\frac{1}{2}\Psi^2\mathrm{u}^2G\,d\mu_t\right)-\int_{\mathrm{M}}a^{ij}\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}\,\Psi G\,d\mu_t$$

$$+\frac{1}{2}\int_{\mathrm{M}}\left\{\nabla_j\left(a^{ij}\nabla_i\Psi\right)G+\,2a^{ij}\nabla_i G\,\nabla_j\Psi+\,\Psi\nabla_j\left(a^{ij}\nabla_i G\right)\right\}\mathrm{u}^2\,d\mu_t$$

$$= \frac{1}{2}\int_{\mathrm{M}} \left(\partial_t\Psi + \nabla_j\left(a^{ij}\nabla_i\Psi\right) + \Phi\Psi + a^{ij}\nabla_i \ln G\,\nabla_j\Psi\right)\mathrm{u}^2 G\,d\mu_t - \int_{\mathrm{M}} \Psi a^{ij}\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}\,G\,d\mu_t$$
$$-\partial_t\left(\int_{\mathrm{M}} \frac{1}{2}\Psi^2\mathrm{u}^2 G\,d\mu_t\right)$$

Likewise, for the fourth term of (1.109), we have

$$2\int_{\mathrm{M}} \left(\partial_t\mathrm{u} + a^{ij}\nabla_i \ln G\,\nabla_j\mathrm{u} + \frac{1}{2}\Psi\mathrm{u}\right)\Psi\mathrm{u}\,G\,d\mu_t \tag{1.111}$$

$$= \int_{\mathrm{M}} \partial_t\mathrm{u}^2\,\Psi G\,d\mu_t + \int_{\mathrm{M}} a^{ij}\nabla_i G\,\nabla_j\mathrm{u}^2\,\Psi\,d\mu_t + \int_{\mathrm{M}} \Psi^2\mathrm{u}^2 G\,d\mu_t$$

$$= -\int_{\mathrm{M}} \left(\partial_t\Psi\,G + \Psi\left(\partial_t G + \frac{1}{2}\mathrm{tr}\partial_t g\,G\right)\right)\mathrm{u}^2\,d\mu_t + \partial_t\left(\int_{\mathrm{M}} \Psi\mathrm{u}^2 G\,d\mu_t\right) + \int_{\mathrm{M}} \Psi^2\mathrm{u}^2 G\,d\mu_t$$
$$-\int_{\mathrm{M}} \left(\nabla_j\left(a^{ij}\nabla_i G\right)\Psi + a^{ij}\nabla_i G\,\nabla_j\Psi\right)\mathrm{u}^2\,d\mu_t$$

$$= -\int_{\mathrm{M}} \left(\partial_t\Psi + \Phi\Psi + a^{ij}\nabla_i \ln G\,\nabla_j\Psi - \Psi^2\right)\mathrm{u}^2 G\,d\mu_t + \partial_t\left(\int_{\mathrm{M}} \Psi\mathrm{u}^2 G\,d\mu_t\right)$$

Combining (1.109), (1.110), (1.111) to get (1.103). □

We hereafter consider the Riemannian manifold in Lemma 1.22 to be each time-slice Σ_t with the induced metric g_t evolving (in "normal parametrization") like $\partial_t g = -2F\left(A^{\#}\right)A$ (see [A]) and the differential operator (in Lemma 1.22) to be the one in Proposition 1.13.

For the second lemma, we would choose suitable weight function G and auxiliary function Ψ in Lemma 1.22 in order to bound the LHS of (1.103) from below. The choice of G is due to [ESS] and [W]. As for Ψ, it is not shown in [W] but is used here to deal with the last term in (1.102), which comes from the nonlinear nature of F (see Definition 1.8). Note that in the linear case when $F(S) = \mathrm{tr}(S)$ (see [W]), the coefficients of the differenital operator in Proposition 1.13 becomes $\mathbf{a}^{ij} = g^{ij}$; besides, (1.102) is reduced to

$$\Upsilon^{ij} = g^{ik}g^{jl}\nabla^2_{kl}\ln G - HA^{ij}$$

The idea of using an auxiliary function for the nonlinear case is motivated by [N].

Lemma 1.23. *Assume that* $\varkappa \leq 6^{-4}\lambda^3$ *in (1.1) and (1.2). Then there exists* $R = R\left(\Sigma,\, \tilde{\Sigma},\, \mathcal{C},\, U,\, \| F \|_{C^3(U)},\, \lambda,\, \varkappa\right) \geq 1$ *so that for any constants* $M \geq 1,\ \tau \in (0,\, 1]$, *let*

$$G = \exp\left(M\,(t+\tau)\,|X|^{\frac{3}{2}} + |X|^2\right) \tag{1.112}$$

$$\begin{aligned}\Psi = &\left(\frac{3}{2}M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)^2 \mathbf{a}^{ij}\,(X\cdot\partial_i X)\,(X\cdot\partial_j X) + M|X|^{\frac{3}{2}} \\ &+\frac{1}{2}\left(\frac{3}{2}M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)\left(tr\,(\mathbf{a}) - \frac{\lambda}{3}\right) \\ &+\left(tr\,(\mathbf{a}) - \frac{\lambda}{3}\right) + \frac{3}{4}M\,(t+\tau)\,|X|^{-\frac{5}{2}}\left(tr\,(\mathbf{a})\,|X|^2 - \mathbf{a}^{ij}\,(X\cdot\partial_i X)\,(X\cdot\partial_j X)\right)\end{aligned} \tag{1.113}$$

(note that $G > 0$ *and* $\Psi \geq 0$*), there hold*

$$2\Upsilon^{ij} - (\Phi - \Psi)\,\mathbf{a}^{ij} \geq \frac{\lambda^2}{9}g^{ij} \tag{1.114}$$

$$\frac{1}{2}\left(\partial_t\Psi - \nabla_i\left(\mathbf{a}^{ij}\nabla_j\Psi\right) + (\,\Phi - \Psi)\,\Psi\right) \geq \frac{\lambda^2}{9}|X|^2 \tag{1.115}$$

for $X \in \Sigma_t\backslash\bar{B}_R,\ t \in [-\tau,\, 0)$, *where* $\mathrm{tr}\,(\mathbf{a}) = g_{ij}\mathbf{a}^{ij}$, Φ *and* Υ^{ij} *are defined in (1.101) and (1.102), respectively, with the covariant derivative is taken w.r.t* Σ_t, $\partial_t g = -2F\left(A^{\#}\right)A$, *and* $a^{ij} = \mathbf{a}^{ij}$.

Remark 1.24. In view of Proposition 1.14, the hypothesis that $\varkappa \leq 6^{-4}\lambda^3$ amounts to requiring the smallness of $|X|\,|\nabla_{\Sigma_t}\mathbf{a}|$ (compared with the ellipticity of $\mathbf{a}$). Similar hypothesis also appears in [N] and [WZ] when using Carleman's inequalities to prove the backward uniqueness of parabolic equations.

Proof. Let's start with computing the covariant derivatives of $\ln G$:

$$\nabla_i \ln G = \left(\frac{3}{2}M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)(X\cdot\partial_i X) \tag{1.116}$$

$$\begin{aligned}\nabla^2_{ij}\ln G = &\left(\frac{3}{2}M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)(g_{ij} + X\cdot N\,A_{ij}) \\ &-\frac{3}{4}M\,(t+\tau)\,|X|^{-\frac{5}{2}}\left(|X|^2 g_{ij} - (X\cdot\partial_i X)\,(X\cdot\partial_j X)\right)\end{aligned} \tag{1.117}$$

$$+2t\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)F\left(A^{\#}\right)A_{ij}$$

and its evolution

$$\partial_t \ln G = M|X|^{\frac{3}{2}}+\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)\left(X\cdot\partial_t X\right) \tag{1.118}$$

$$= M|X|^{\frac{3}{2}}+2t\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)F\left(A^{\#}\right)^2$$

in which we use the F curvature flow equation in normal parametrization (see Definition 1.12)

$$\partial_t X = F\left(A^{\#}\right)N$$

and the F self-shrinker equation for $\Sigma_t=\sqrt{-t}\,\Sigma$ (in Definition 1.4):

$$X\cdot N = 2tF\left(A^{\#}\right)$$

Thus, by (1.101), (1.116), (1.117) and (1.118), we have

$$\Phi=\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)^2\mathbf{a}^{ij}\left(X\cdot\partial_i X\right)\left(X\cdot\partial_j X\right)+\ M|X|^{\frac{3}{2}} \tag{1.119}$$

$$+\frac{1}{2}\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)\mathrm{tr}\left(\mathbf{a}\right)$$

$$+\mathrm{tr}\left(\mathbf{a}\right)+\ \frac{3}{4}M\left(t+\tau\right)|X|^{-\frac{5}{2}}\left(\mathrm{tr}\left(\mathbf{a}\right)|X|^2-\ \mathbf{a}^{ij}\left(X\cdot\partial_i X\right)\left(X\cdot\partial_j X\right)\right)$$

$$+\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)\left\{\left(\nabla_i\mathbf{a}^{ij}\right)\left(X\cdot\partial_j X\right)+\ 2tF\left(A^{\#}\right)\left(F\left(A^{\#}\right)+\mathbf{a}^{ij}A_{ij}\right)\right\}-F\left(A^{\#}\right)H$$

which, together with (1.113), implies that

$$\Phi-\Psi=\frac{\lambda}{2}\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)+\frac{\lambda}{3} \tag{1.120}$$

$$+\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)\left\{\left(\nabla_k\mathbf{a}^{kl}\right)\left(X\cdot\partial_l X\right)+2tF\left(A^{\#}\right)\left(F\left(A^{\#}\right)+\mathbf{a}^{kl}A_{kl}\right)\right\}-F\left(A^{\#}\right)H$$

By (1.102), (1.116), (1.117) and (1.120),

$$2\Upsilon^{ij}-\left(\Phi-\Psi\right)\mathbf{a}^{ij}=\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)\left(\mathbf{a}^{ik}\mathbf{a}^{jl}g_{kl}-\frac{\lambda}{6}\mathbf{a}^{ij}\right) \tag{1.121}$$

$$+\left(2\mathbf{a}^{ik}\mathbf{a}^{jl}g_{kl}-\frac{\lambda}{3}\mathbf{a}^{ij}\right)+\ \frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{5}{2}}\mathbf{a}^{ik}\mathbf{a}^{jl}\left(|X|^2g_{kl}-\ \left(X\cdot\partial_k X\right)\left(X\cdot\partial_l X\right)\right)$$

$$+\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)\left\{\mathbf{a}^{ik}\nabla_{k}\mathbf{a}^{jl}+\mathbf{a}^{jk}\nabla_{k}\mathbf{a}^{il}-\mathbf{a}^{lk}\nabla_{k}\mathbf{a}^{ij}-\mathbf{a}^{ij}\nabla_{k}\mathbf{a}^{kl}\right\}\left(X\cdot\partial_{l}X\right)$$

$$+\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)\left(2\mathbf{a}^{ik}\mathbf{a}^{jl}A_{kl}-\mathbf{a}^{ij}\mathbf{a}^{kl}A_{kl}-F\left(A^{\#}\right)\mathbf{a}^{ij}\right)2t\,F\left(A^{\#}\right)$$

$$-\partial_{t}\mathbf{a}^{ij}+\,F\left(A^{\#}\right)H\,\mathbf{a}^{ij}$$

which can be estimated from below, using (1.57), (1.58), (1.60), (1.17), (1.20) and the homogeneity of F, by

$$2\Upsilon^{ij}-\left(\Phi-\Psi\right)\mathbf{a}^{ij}\geq\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)\left(\left(\frac{\lambda^{2}}{18}-36\frac{\varkappa}{\lambda}\right)g^{ij}+O\left(|X|^{-2}\right)\right)$$
$$+\frac{\lambda^{2}}{9}g^{ij}\,+O\left(|X|^{-2}\right) \tag{1.122}$$

where the notation $O\left(|X|^{-2}\right)$ means that

$$\left|O\left(|X|^{-2}\right)\right|\,\leq\,C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^{3}(U)}\right)\,|X|^{-2}$$

Then (1.114) follows from (1.112) and the hypothesis ($\varkappa\leq 6^{-4}\lambda^{3}$) provided that $R\gg 1$ (independing of M and τ).

On the other hand, by (1.57), (1.58), (1.17), (1.20), the homogeniety of F, the hypothesis that $\varkappa\leq 6^{-4}\lambda^{3}$ (note that $\lambda\in(0,\,1]$) and $R\gg 1$ (independing of M and τ), we can estimate (1.120) from below by

$$\Phi-\Psi\geq\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)\left(\frac{\lambda}{6}\,-3\varkappa\,+O\left(|X|^{-2}\right)\right)+\frac{\lambda}{3}+O\left(|X|^{-2}\right)$$
$$\geq\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)\frac{\lambda}{9}+\frac{\lambda}{6} \tag{1.123}$$

Similarly, from the F self-shrinker equation for Σ_{t}, we can estimate the tangential component of the position vector by

$$|X^{\top}|^{2}=|X|^{2}-\left(X\cdot N\right)^{2}=|X|^{2}-\left(2t\,F\left(A^{\#}\right)\right)^{2} \tag{1.124}$$
$$=|X|^{2}-\left(2t\,F\left(|X|\,A^{\#}\right)\right)^{2}|X|^{-2}=\,|X|^{2}+O\left(|X|^{-2}\right)$$

Consequently, (1.113) can be estimated (from below), using (1.57) and (1.124), by

$$\Psi\geq\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)^{2}\mathbf{a}^{ij}\left(X\cdot\partial_{i}X\right)\left(X\cdot\partial_{j}X\right)+\,M\,|X|^{\frac{3}{2}} \tag{1.125}$$

$$\geq \left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)^{2}\left(\frac{\lambda}{3}|X|^{2}+O\left(|X|^{-2}\right)\right)+\,M\,|X|^{\frac{3}{2}}$$

Multiplying (1.123) and (1.125) to get

$$\left(\Phi-\Psi\right)\Psi \geq \left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)^{3}\frac{1}{36}\lambda^{2}|X|^{2} \tag{1.126}$$

$$+\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)^{2}\frac{1}{27}\lambda^{2}|X|^{2}+\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)\frac{\lambda}{9}M\,|X|^{\frac{3}{2}}+\frac{\lambda}{6}M\,|X|^{\frac{3}{2}}$$

To achieve (1.115), let's first rearrange (1.113) to get

$$\Psi=\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)^{2}\mathbf{a}^{kl}\left(X\cdot\partial_{k}X\right)\left(X\cdot\partial_{l}X\right)+\,M\,|X|^{\frac{3}{2}} \tag{1.127}$$

$$+\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)\left(\operatorname{tr}\left(\mathbf{a}\right)-\frac{\mathbf{a}^{kl}\left(X\cdot\partial_{k}X\right)\left(X\cdot\partial_{l}X\right)}{2|X|^{2}}-\frac{\lambda}{6}\right)$$

$$+\frac{\mathbf{a}^{kl}\left(X\cdot\partial_{k}X\right)\left(X\cdot\partial_{l}X\right)}{|X|^{2}}-\frac{\lambda}{3}$$

Then we would like to take time-derivative of (1.127) and estimate it by using Proposition 1.14, (1.17), (1.20), the homogeneity of F and its derivatives, the F self-shrinker equaiton for Σ_t (i.e. $X\cdot N = 2tF\left(A^{\#}\right)$) and the F curvature flow equation (i.e. $\partial_t X = F\left(A^{\#}\right)N$)), and also assuming that $R \gg 1$ (depending on λ). Note that we could simplify the compuation by taking "normal coodinates" of Σ_t. For instance, let's compute and estimate the time-derivative of the first term in (1.127):

$$\partial_{t}\left\{\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)^{2}\mathbf{a}^{kl}\left(X\cdot\partial_{k}X\right)\left(X\cdot\partial_{l}X\right)\right\} \tag{1.128}$$

$$=2\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)\left\{\frac{3}{2}M\,|X|^{-\frac{1}{2}}+\,\frac{3}{2}M\left(t+\tau\right)\left(-\frac{1}{2}|X|^{-\frac{3}{2}}\right)\frac{X\cdot F\left(A^{\#}\right)N}{|X|}\right\}\mathbf{a}^{kl}\left(X\cdot\partial_{k}X\right)(X$$

$$+\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)^{2}\left\{\left(\partial_{t}\mathbf{a}^{kl}\right)\left(X\cdot\partial_{k}X\right)\left(X\cdot\partial_{l}X\right)+\,2\mathbf{a}^{kl}\left(X\cdot\partial_{k}X\right)\left(X\cdot\partial_{l}\left(F\left(A^{\#}\right)N\right)\right)\right\}$$

By taking normal coordinates, we may assume that (at the point of consideration) $g_{ij}=\delta_{ij}$ (so the norm is Proposition 1.14 becomes ℓ^2 norm), $\{\partial_1 X,\cdots,\partial_n X,\,N\}$ is an orthonomal basis for $\mathbb{R}^{n+1}$, and the last term in (1.128) can be computed and estimated by

$$\partial_{l}\left(F\left(A^{\#}\right)N\right)=\frac{\partial F}{\partial S_{i}^{j}}\left(A^{\#}\right)\left(\partial_{l}A_{i}^{j}\right)N\,+\,F\left(A^{\#}\right)\left(-A_{l}^{k}\,\partial_{k}X\right)$$

$$= \frac{\partial F}{\partial S_i^j}\left(|X|\, A^{\#}\right)\left(\nabla_l A_i^j\right) N \,+\, |X|^{-1} F\left(|X|\, A^{\#}\right)\left(-A_l^k\, \partial_k X\right) = O\left(|X|^{-2}\right)$$

so (1.128) can be estimated by

$$\left(\frac{3}{2} M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)\left(3M|X|^{-\frac{1}{2}} + M \cdot O\left(|X|^{-\frac{9}{2}}\right)\right) \mathbf{a}^{kl}\,(X\cdot\partial_k X)\,(X\cdot\partial_l X)$$

$$+\left(\frac{3}{2} M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)^2 O\,(1)$$

By doing the same thing to other terms in (1.127), we arrive at

$$\partial_t \Psi = \left(\frac{3}{2} M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)\left(3M|X|^{-\frac{1}{2}} + M \cdot O\left(|X|^{-\frac{9}{2}}\right)\right) \mathbf{a}^{kl}\,(X\cdot\partial_k X)\,(X\cdot\partial_l X) \tag{1.129}$$

$$+\left(\frac{3}{2} M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)^2 O\,(1) + M\cdot O\left(|X|^{-\frac{1}{2}}\right) + \left(\frac{3}{2} M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right) O\left(|X|^{-2}\right) + O\left(|X|^{-2}\right)$$

$$\geq \left(\frac{3}{2} M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)\left(\frac{2}{3}\lambda M|X|^{\frac{3}{2}}\right) + \left(\frac{3}{2} M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)^2 O\,(1) + M\cdot O\left(|X|^{-\frac{1}{2}}\right)$$

Similarly, we can compute $\nabla_i\left(\mathbf{a}^{ij}\nabla_j\Psi\right)$ and estimate it by

$$\nabla_i\left(\mathbf{a}^{ij}\nabla_j\Psi\right) = \mathbf{a}^{ij}\nabla^2_{ij}\Psi + \left(\nabla_i\mathbf{a}^{ij}\right)\left(\nabla_j\Psi\right) \tag{1.130}$$

$$= \left(\frac{3}{2} M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)^2 O\,(1) + \left(\frac{3}{2} M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right) O\left(|X|^{-2}\right) + M\cdot O\left(|X|^{-\frac{1}{2}}\right)$$

Then (1.115) follows from (1.126), (1.129) and (1.130). $\square$

Using the above two lemma, we can derive the following Carleman's inequality on the flow $\{\Sigma_t\}_{-1\leq t\leq 0}$ (with $\Sigma_0 = \mathcal{C}$).

Proposition 1.25. *(Carleman's inequality)*

Assume that $\varkappa \leq 6^{-4}\lambda^3$ in (1.4) and (1.5). Then there exists $R \geq 1$ (depending on Σ, $\tilde{\Sigma}$, $\mathcal{C}$, U, $\parallel F \parallel_{C^3(U)}$, λ, $\varkappa$) so that for any constants $M \geq 1$, $\tau \in (0,\,1]$, and one-parameter family of C^2 functions $\mathrm{u}_t = \mathrm{u}\,(\cdot,\,t)$ which is compactly supported in $\Sigma_t \setminus \bar{B}_R$ for each $t \in [-\tau,\,0]$ and is differentiable in time, there holds

$$\frac{\lambda^2}{9}\int_{-\tau}^{0}\int_{\Sigma_t}\left(|\nabla_{\Sigma_t}\mathrm{u}|^2 + \mathrm{u}^2\right) G\, d\mathcal{H}^n dt \tag{1.131}$$

$$\leq \int_{-\tau}^{0} \int_{\Sigma_t} |\mathbf{P}\mathrm{u}|^2 G \, d\mathcal{H}^n dt \, + \, \frac{3}{\lambda} \int_{\Sigma_{-\tau}} |\nabla_{\Sigma_{-\tau}} \mathrm{u}_{-\tau}|^2 G\left(\cdot, \, -\tau\right) \, d\mathcal{H}^n$$

$$+ \frac{1}{2} \int_{\mathcal{C}} \Psi\left(\cdot, \, 0\right) \, \mathrm{u}^2\left(\cdot, \, 0\right) \, G\left(\cdot, \, 0\right) \, d\mathcal{H}^n$$

where $\mathcal{H}^n$ is the $n-$dimensional Hausdorff measure; $\mathbf{P}$, G and Ψ are defined in (1.53), (1.112), (1.113), respectively.

Proof. Apply Lemma 1.22 to the hypersurface Σ_t (with $\partial_t g = -2F\left(A^{\#}\right) A$), the differential operator $\mathbf{P}$ and the function u_t to get

$$\int_{\Sigma_t} \left\{ \left(2\Upsilon^{ij} - (\Phi - \Psi)\, \mathbf{a}^{ij}\right) \nabla_i \mathrm{u} \nabla_j \mathrm{u} + \frac{1}{2} \left(\partial_t \Psi - \nabla_i \left(a^{ij} \nabla_j \Psi\right) + (\Phi - \Psi)\, \Psi\right) \mathrm{u}^2 \right\} G \, d\mathcal{H}^n$$

$$= \int_{\Sigma_t} 2\, \mathbf{P}\mathrm{u} \left(\partial_t \mathrm{u} + \mathbf{a}^{ij} \nabla_i \ln G \, \nabla_j \mathrm{u} + \frac{1}{2}\Psi\mathrm{u}\right) G \, d\mathcal{H}^n - \int_{\Sigma_t} 2 \left(\partial_t \mathrm{u} + \mathbf{a}^{ij} \nabla_i \ln G \, \nabla_j \mathrm{u} + \frac{1}{2}\Psi\mathrm{u}\right)^2 G \, d\mathcal{H}^n$$

$$-\partial_t \left\{ \int_{\Sigma_t} \left(\mathbf{a}^{ij} \nabla_i \mathrm{u} \, \nabla_j \mathrm{u} - \frac{1}{2}\Psi \mathrm{u}^2\right) G \, d\mathcal{H}^n \right\} \tag{1.132}$$

By Cauchy-Schwarz inequality, the RHS of (1.132) is bounded from above by

$$\int_{\Sigma_t} |\mathbf{P}\mathrm{u}|^2 G \, d\mathcal{H}^n dt \, - \, \partial_t \left\{ \int_{\Sigma_t} \left(\mathbf{a}^{ij} \nabla_i \mathrm{u} \, \nabla_j \mathrm{u} - \frac{1}{2}\Psi \mathrm{u}^2\right) G \, d\mathcal{H}^n \right\} \tag{1.133}$$

By Lemma 1.23 and $R \geq 1$, the LHS of (1.132) is bounded from below by

$$\frac{\lambda^2}{9} \int_{\Sigma_t} \left(|\nabla_{\Sigma_t} \mathrm{u}|^2 + \mathrm{u}^2\right) G \, d\mathcal{H}^n \tag{1.134}$$

Combining (1.132), (1.133), (1.134), we get

$$\frac{\lambda^2}{9} \int_{\Sigma_t} \left(|\nabla_{\Sigma_t} \mathrm{u}|^2 + \mathrm{u}^2\right) G \, d\mathcal{H}^n \tag{1.135}$$

$$\leq \int_{\Sigma_t} |\mathbf{P}\mathrm{u}|^2 G \, d\mathcal{H}^n dt \, - \, \partial_t \left\{ \int_{\Sigma_t} \left(\mathbf{a}^{ij} \nabla_i \mathrm{u} \, \nabla_j \mathrm{u} - \frac{1}{2}\Psi \mathrm{u}^2\right) G \, d\mathcal{H}^n \right\}$$

Integrate (1.135) in time from $-\tau$ to 0 and then use (1.57) and $\Psi \geq 0$ to conclude (1.131). $\square$

Now we are ready to show that h vanishes outside a compact set. We basically follows the proof in [ESS] (which is also used in [W]).

Theorem 1.26. *Suppose that $\varkappa \le 6^{-4}\lambda^3$ in (1.4) and (1.5), then there extists $\boldsymbol{R} = \boldsymbol{R}\left(\Sigma,\, \tilde{\Sigma},\, \mathcal{C},\, U,\, \| \, F \, \|_{C^3(U)},\, \lambda,\, \varkappa\right) \ge 1$ so that the deviation $h(\cdot,\, -1)$ of $\tilde{\Sigma}$ from Σ vanishes on $\Sigma \setminus \bar{B}_{\boldsymbol{R}}$. In other words, $\tilde{\Sigma} = \Sigma$ outside the ball $B_{\boldsymbol{R}}$.*

Proof. Choose $R \gg 1$ (depending on $\Sigma,\, \tilde{\Sigma},\, \mathcal{C},\, U,\, \| \, F \, \|_{C^3(U)},\, \lambda$) so that Proposition 1.13, Proposition 1.14, Proposition 1.21, Proposition 1.25 and (1.20) hold; in particular, we may assume that for all $X \in \Sigma_t \setminus \bar{B}_R$, $t \in [-\tau,\, 0]$

$$|\mathbf{P}h| \le \frac{\lambda}{6}\left(|\nabla_{\Sigma_t} h| + |h|\right) \tag{1.136}$$

$$|\nabla_{\Sigma_t} h| + |h| \le \Lambda \exp\left(\frac{|X|^2}{\Lambda t}\right) \tag{1.137}$$

where $\Lambda = \Lambda\left(n,\, \mathcal{C},\, \| \, F \, \|_{C^3(U)},\, \lambda\right) > 0$, $\tau \equiv \min\left\{\alpha\left(n,\, \mathcal{C},\, \| \, F \, \|_{C^3(U)},\, \lambda\right),\, \frac{1}{\Lambda}\right\}$ (see Proposition 22).

For any given $M \ge 1$ and $\mathcal{R} \ge 4R + 1$, choose a smooth cut-off function $\zeta = \zeta(X)$ so that

$$\chi_{B_{\mathcal{R}-1} \setminus \bar{B}_{R+1}} \le \zeta \le \chi_{B_{\mathcal{R}} \setminus \bar{B}_R} \tag{1.138}$$

$$|D\zeta| + |D^2\zeta| \le 3$$

Note that $D\zeta$ is supported in $E = \left\{X \in \mathbb{R}^{n+1} \,\middle|\, R \le |X| \le R + 1 \text{ or } \mathcal{R} - 1 \le |X| \le \mathcal{R}\right\}$.

Let $\mathrm{u}(\cdot,\, t) = \zeta\, h(\cdot,\, t)$, then $\mathrm{u}(\cdot,\, t)$ is compactly supported in $\Sigma_t \setminus \bar{B}_R$ for each $t \in [-\tau,\, 0]$, and we have, by (1.136), (1.137), (1.138)

$$\left|\mathbf{P}\mathrm{u}\right| = \left|\zeta\, \mathbf{P}h - h\, \mathbf{P}\zeta - 2\mathbf{a}^{ij} \nabla_i \zeta\, \nabla_j h\right| \tag{1.139}$$

$$\le \frac{\lambda}{6}\left(|\nabla_{\Sigma_t} \mathrm{u}| + |\mathrm{u}|\right) + C\left(n,\, \mathcal{C},\, \| \, F \, \|_{C^3(U)}\right)\left(|\nabla_{\Sigma_t} h| + |h|\right) \chi_E$$

$$\le \frac{\lambda}{6}\left(|\nabla_{\Sigma_t} \mathrm{u}| + |\mathrm{u}|\right) + C\left(n,\, \mathcal{C},\, \| \, F \, \|_{C^3(U)},\, \lambda\right) \exp\left(\frac{|X|^2}{\Lambda t}\right) \chi_E$$

$$\mathrm{u}(\cdot,\, 0) = 0 \tag{1.140}$$

By (1.139), (1.140), Proposition 1.25 and (1.137), we get

$$\frac{\lambda^2}{9} \int_{-\tau}^{0} \int_{\Sigma_t} \left(|\nabla_{\Sigma_t} \mathrm{u}|^2 + \mathrm{u}^2\right) G \, d\mathcal{H}^n dt \le \frac{\lambda^2}{18} \int_{-\tau}^{0} \int_{\Sigma_t} \left(|\nabla_{\Sigma_t} \mathrm{u}|^2 + \mathrm{u}^2\right) G \, d\mathcal{H}^n dt$$

$$+C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)},\,\lambda\right)\int_{-\tau}^{0}\int_{\Sigma_t\cap E}\exp\left(2\frac{|X|^2}{\Lambda t}\right)G\,d\mathcal{H}^n dt \tag{1.141}$$
$$+C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)},\,\lambda\right)\int_{\Sigma_{-\tau}}\exp\left(-2\frac{|X|^2}{\Lambda\tau}\right)G\left(\cdot,\,-\tau\right)\,d\mathcal{H}^n$$

where G is defined in (1.112). Note that by the choice $\tau\leq\frac{1}{\Lambda}$, we can estimate the last two terms on the RHS of (1.141) by

$$\int_{-\tau}^{0}\int_{\Sigma_t\cap E}\exp\left(2\frac{|X|^2}{\Lambda t}\right)G\,d\mathcal{H}^n dt\,\leq\,\int_{-\tau}^{0}\int_{\Sigma_t\cap E}\exp\left(M\tau|X|^{\frac{3}{2}}-\,|X|^2\right)\,d\mathcal{H}^n dt \tag{1.142}$$

and

$$\int_{\Sigma_{-\tau}}\exp\left(-2\frac{|X|^2}{\Lambda\tau}\right)G\left(\cdot,\,-\tau\right)\,d\mathcal{H}^n\,\leq\,\int_{\Sigma_{-\tau}}\exp\left(-|X|^2\right)\,d\mathcal{H}^n \tag{1.143}$$

Consequently, by (1.142), (1.143) and noting that the first term on the RHS of (1.141) can be abosorbed by its LHS, we get from (1.141) that

$$\frac{\lambda^2}{18}\int_{-\tau}^{0}\int_{\Sigma_t}\left(|\nabla_{\Sigma_t}\mathrm{u}|^2+\mathrm{u}^2\right)G\,d\mathcal{H}^n dt \tag{1.144}$$

$$\leq C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)},\,\lambda\right)\int_{-\tau}^{0}\int_{\Sigma_t\cap E}\exp\left(M\tau|X|^{\frac{3}{2}}-|X|^2\right)\,d\mathcal{H}^n dt$$
$$+C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)},\,\lambda\right)\int_{\Sigma_{-\tau}}\exp\left(-|X|^2\right)\,d\mathcal{H}^n$$

$$\leq C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)},\,\lambda\right)\int_{-\tau}^{0}\int_{\Sigma_t\cap\left(B_{\mathcal{R}-1}\setminus\bar{B}_{\mathcal{R}}\right)}\exp\left(M\tau\mathcal{R}^{\frac{3}{2}}-\left(\mathcal{R}-1\right)^2\right)\,d\mathcal{H}^n dt$$
$$+C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)},\,\lambda\right)\int_{-\tau}^{0}\int_{\Sigma_t\cap\left(B_R\setminus\bar{B}_{R+1}\right)}\exp\left(M\tau\left(R+1\right)^{\frac{3}{2}}-R^2\right)\,d\mathcal{H}^n dt$$
$$+C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)},\,\lambda\right)\int_{\Sigma_{-\tau}}\exp\left(-|X|^2\right)\,d\mathcal{H}^n$$

The first term on the RHS of (1.144) goes away as $\mathcal{R}\nearrow\infty$; the last term is bounded from above by $C\left(n,\,\mathcal{C},\,\|\,F\,\|_{C^3(U)},\,\lambda\right)$ because of (1.10). For the LHS of (1.144), we have

$$\frac{\lambda^2}{18}\int_{-\tau}^{0}\int_{\Sigma_t}\left(|\nabla_{\Sigma_t}\mathrm{u}|^2+\mathrm{u}^2\right)G\,d\mathcal{H}^n dt\,\geq\,\frac{\lambda^2}{18}\int_{-\frac{\tau}{2}}^{0}\int_{\Sigma_t\cap\left(B_{\mathcal{R}-1}\setminus\bar{B}_{4R}\right)}\mathrm{u}^2 G\,d\mathcal{H}^n dt$$
$$\geq\frac{\lambda^2}{18}\exp\left(4M\tau R^{\frac{3}{2}}\right)\int_{-\frac{\tau}{2}}^{0}\int_{\Sigma_t\cap\left(B_{\mathcal{R}-1}\setminus\bar{B}_{4R}\right)}h^2\,d\mathcal{H}^n dt$$

Therefore, let $\mathcal{R} \nearrow \infty$ in (1.144), we arrive at

$$\int_{-\frac{\tau}{2}}^{0} \int_{\Sigma_t \setminus \bar{B}_{4R}} h^2 \, d\mathcal{H}^n dt \tag{1.145}$$

$$\leq \exp\left(-4M\tau R^{\frac{3}{2}}\right) C\left(n,\, \mathcal{C},\, \| F \|_{C^3(U)},\, \lambda\right) \left\{\exp\left(2\sqrt{2}M\tau R^{\frac{3}{2}}\right) + 1\right\}$$

Let $M \nearrow \infty$ in (1.145), we get $h_t = h(\cdot,\, t)$ vanishes on $\Sigma_t \setminus \bar{B}_{4R}$ for $t \in \left[-\frac{\tau}{2},\, 0\right]$, and hence $\tilde{\Sigma}_{-\frac{\tau}{2}} = \sqrt{\frac{\tau}{2}}\,\tilde{\Sigma}$ coincides with $\Sigma_{-\frac{\tau}{2}} = \sqrt{\frac{\tau}{2}}\,\Sigma$ outside B_{4R}, which in turn shows that $\tilde{\Sigma}$ coincide with Σ outside the ball of radius $\boldsymbol{R} = \frac{4R}{\sqrt{\tau/2}}$. □

By the previous theorem and the "unique continuation principle" in Proposition 20 (see also Remark 1.20), we have the following conclusion on the overlap region of Σ and $\tilde{\Sigma}$.

Theorem 1.27. *Under the same hypothesis of Theorem 1.26, let*

$$\Sigma^0 = \left\{X \in \Sigma \cap \tilde{\Sigma} \,\middle|\, \Sigma \textit{ coincides with } \tilde{\Sigma} \textit{ in a neighborhood of } X\right\}$$

then Σ^0 is a nonempty hypersurface and $\partial\Sigma^0 \subseteq \left(\partial\Sigma \cup \partial\tilde{\Sigma}\right)$.

Proof. Note that Σ^0 is a nonempty hypersurface follows from Theorem 1.26 and the definition of Σ^0.

Suppose that $\partial\Sigma^0 \nsubseteq \left(\partial\Sigma \cup \partial\tilde{\Sigma}\right)$, then pick $\hat{X} \in \partial\Sigma^0 \setminus \left(\partial\Sigma \cup \partial\tilde{\Sigma}\right)$ and choose a sequence $\left\{\hat{X}_m \in \Sigma^0\right\}$ converging to $\hat{X}$. Note that $N\left(\hat{X}\right) = \tilde{N}\left(\hat{X}\right)$ since $N\left(\hat{X}_m\right) = \tilde{N}\left(\hat{X}_m\right)$ for all $m \in \mathbb{N}$, where N , $\tilde{N}$ are the unit-normal of Σ and $\tilde{\Sigma}$, respectively. Thus, near $\hat{X}$, Σ and $\tilde{\Sigma}$ can be regraded as graphes of $\mathfrak{u}$ and $\tilde{\mathfrak{u}}$, respectively, over $B^n_{\boldsymbol{\varrho}} \subset T_{\hat{X}}\Sigma = T_{\hat{X}}\tilde{\Sigma}$ for some $\boldsymbol{\varrho} \in (0,\, 1)$. That is, Σ and $\tilde{\Sigma}$ can be respectively parametrized by

$$X = X(x) \equiv \hat{X} + (x,\, \mathfrak{u}(x))\,, \quad \tilde{X} = \tilde{X}(x) \equiv \hat{X} + (x,\, \tilde{\mathfrak{u}}(x)) \quad \text{for } x \in B^n_{\boldsymbol{\varrho}}$$

in which we assume that $N\left(\hat{X}\right) = \tilde{N}\left(\hat{X}\right) = (0,\, 1)$ for ease of notation. Note also that $A_i^j(0) = \tilde{A}_i^{\,j}(0)$ since $A_i^j(x_m) = \tilde{A}_i^{\,j}(x_m)$ for all $m \in \mathbb{N}$, where x_m is the coordinates of $\hat{X}_m$ (i.e. $X(x_m) = \hat{X}_m$) and

$$A^{\#}(x) \sim A_i^j(x) = \partial_i\left(\frac{\partial_j \mathfrak{u}(x)}{\sqrt{1 + |\partial_x \mathfrak{u}|^2}}\right), \quad \tilde{A}^{\#}(x) \sim \tilde{A}_i^j(x) = \partial_i\left(\frac{\partial_j \tilde{\mathfrak{u}}(x)}{\sqrt{1 + |\partial_x \tilde{\mathfrak{u}}|^2}}\right) \tag{1.146}$$

are the shape operators of Σ and$\tilde{\Sigma}$, respectively. As a result, we may assume (by choosing $\boldsymbol{\varrho}$ small if necessary) that $\tilde{A}_i^j(x)$ is so closed to $A_i^j(x)$ that the set

$$\mathfrak{U}=\left\{(1-\theta)\,A_i^j(x)+\theta\tilde{A}_i^j(x)\,\middle|\,x\in B^n_{\boldsymbol{\varrho}},\,\theta\in[0,\,1]\right\}$$

is a bounded subset of $\boldsymbol{\Omega}$ and there holds

$$\bar{\lambda}\leq\frac{\partial F}{\partial S_i^j}\left((1-\theta)\,A^{\#}(x)+\theta\tilde{A}^{\#}(x)\right)\leq\frac{1}{\bar{\lambda}}$$

for some $\bar{\lambda}\in(0,\,1]$.

From the F shrinker equation in Definition 1.4, we get

$$\sqrt{1+|\partial_x\mathfrak{u}|^2}\,F\left(A_i^j(x)\right)+\frac{1}{2}\left(\mathfrak{u}-x\cdot\partial_x\mathfrak{u}\right)=0,\quad\sqrt{1+|\partial_x\tilde{\mathfrak{u}}|^2}\,F\left(\tilde{A}_i^j(x)\right)+\frac{1}{2}\left(\tilde{\mathfrak{u}}-\,x\cdot\partial_x\tilde{\mathfrak{u}}\right)=0 \tag{1.147}$$

Substracting (1.147) and using (1.146) and the mean value theorem, we then get an equation for $\mathtt{v}=\tilde{\mathfrak{u}}-\mathfrak{u}$:

$$\mathfrak{a}^{ij}\partial_{ij}^2\mathtt{v}+\,\mathfrak{b}^j\partial_j\mathtt{v}+\,\frac{1}{2}\mathtt{v}=0 \tag{1.148}$$

with

$$\mathfrak{a}^{ij}(x)=\int_0^1\left\{\frac{\partial F}{\partial S_i^j}\left((1-\theta)\,A^{\#}(x)+\theta\tilde{A}^{\#}(x)\right)-\frac{\partial F}{\partial S_i^k}\left((1-\theta)\,A^{\#}(x)+\theta\tilde{A}^{\#}(x)\right)\frac{\partial_k\mathfrak{u}_\theta\,\partial_j\mathfrak{u}_\theta}{1+|\partial_x\mathfrak{u}_\theta|^2}\right\}d\theta \tag{1.149}$$

$$\begin{aligned}\mathfrak{b}^j(x)=&-\int_0^1\frac{\partial F}{\partial S_i^j}\left((1-\theta)\,A^{\#}(x)+\theta\tilde{A}^{\#}(x)\right)\frac{\partial_k\mathfrak{u}_\theta\,\partial_{ik}^2\mathfrak{u}_\theta}{1+|\partial_x\mathfrak{u}_\theta|^2}\,d\theta \\ &-\int_0^1\frac{\partial F}{\partial S_i^k}\left((1-\theta)\,A^{\#}(x)+\theta\tilde{A}^{\#}(x)\right)\frac{\partial_j\mathfrak{u}_\theta\,\partial_{ik}^2\mathfrak{u}_\theta+\partial_k\mathfrak{u}_\theta\,\partial_{ij}^2\mathfrak{u}_\theta}{1+|\partial_x\mathfrak{u}_\theta|^2}\,d\theta \\ &+3\int_0^1\frac{\partial F}{\partial S_i^k}\left((1-\theta)\,A^{\#}(x)+\theta\tilde{A}^{\#}(x)\right)\frac{\partial_j\mathfrak{u}_\theta\,\partial_k\mathfrak{u}_\theta\,\partial_l\mathfrak{u}_\theta\,\partial_{il}^2\mathfrak{u}_\theta}{(1+|\partial_x\mathfrak{u}_\theta|^2)^{\frac{3}{2}}}\,d\theta \\ &+\int_0^1F\left((1-\theta)\,A^{\#}(x)+\theta\tilde{A}^{\#}(x)\right)\frac{\partial_j\mathfrak{u}_\theta}{\sqrt{1+|\partial_x\mathfrak{u}_\theta|^2}}\,d\theta-\frac{1}{2}x_j\end{aligned} \tag{1.150}$$

where $\mathfrak{u}_\theta=(1-\theta)\,\mathfrak{u}+\theta\tilde{\mathfrak{u}}$. Note that (1.148) is equivalent to the following divergence form equation:

$$-\partial_i\left(\frac{\mathfrak{a}^{ij}+\mathfrak{a}^{ji}}{2}\partial_j\mathtt{v}\right)=\left(-\partial_i\left(\frac{\mathfrak{a}^{ij}+\mathfrak{a}^{ji}}{2}\right)+\mathfrak{b}^j\right)\partial_j\mathtt{v}+\frac{1}{2}\mathtt{v} \tag{1.151}$$

And by (1.149), (1.150) and (1.146), we have the following estimates for the coefficients of (1.151):

$$\frac{\bar{\lambda}}{1+\parallel \partial_x \mathfrak{u}_\theta \parallel^2_{L^\infty\left(B^n_{\boldsymbol{\varrho}}\right)}} \leq \frac{\mathfrak{a}^{ij}+\mathfrak{a}^{ji}}{2} \leq C\left(\parallel F \parallel_{C^1(\mathfrak{U})},\, \parallel \mathfrak{u} \parallel_{C^2\left(B^n_{\boldsymbol{\varrho}}\right)}\right) \tag{1.152}$$

$$|\partial_x \mathfrak{a}^{ij}| + |\mathfrak{b}^j| \leq C\left(\parallel F \parallel_{C^2(\mathfrak{U})},\, \parallel \mathfrak{u} \parallel_{C^3\left(B^n_{\boldsymbol{\varrho}}\right)}\right) \tag{1.153}$$

On the other hand, since $\hat{X}_m \in \Sigma^0$ and $\hat{X}_m \to \hat{X}$ as $m \nearrow \infty$, $\mathtt{v}$ is vanishing at each neighborhood of x_m and $x_m \to 0$ as $m \nearrow \infty$. Thus, by Proposition 1.19 and Remark 1.20, $\mathtt{v}$ vanishes on $B^n\left(x_m,\, \frac{1}{4}\left(\boldsymbol{\varrho} - |x_m|\right)\right)$ for all $m \in \mathbb{N}$, which implies that $\mathtt{v}$ vanishes on $B^n\left(0,\, \frac{1}{4}\boldsymbol{\varrho}\right)$. In other words, Σ coincides with $\tilde{\Sigma}$ in a neighborhood of $\hat{X}$, which contradicts with $\hat{X} \in \partial\Sigma^0$. $\square$

Lastly, we would like to estimate $\varkappa$ (defined in (1.5)) in the rotationally symmetric case. For that purpose, we have to first compute the covariant derivatives of the second fundamental form of $\mathcal{C}$.

Lemma 1.28. *At each point* $X_{\mathcal{C}} = (\sigma s\, \nu,\, s) \in \mathcal{C}$ (with $\nu \in \mathbf{S}^{n-1}$, $s > 0$), *pick an orthonormal basis* $\left\{e^{\mathcal{C}}_1, \cdots, e^{\mathcal{C}}_n\right\}$ *for* $T_{X_{\mathcal{C}}}\mathcal{C}$ *so that* $e^{\mathcal{C}}_n = \frac{(\sigma\nu,\, 1)}{\sqrt{1+\sigma^2}}$, *then we have*

$$A_{\mathcal{C}}\left(e^{\mathcal{C}}_i,\, e^{\mathcal{C}}_j\right) = \kappa^{\mathcal{C}}_i \delta_{ij}, \quad \textit{with } \kappa^{\mathcal{C}}_1 = \cdots = \kappa^{\mathcal{C}}_{n-1} = \frac{1}{\sigma|X_{\mathcal{C}}|},\ \kappa^{\mathcal{C}}_n = 0 \tag{1.154}$$

$$\nabla_{\mathcal{C}} A_{\mathcal{C}}\left(e^{\mathcal{C}}_i,\, e^{\mathcal{C}}_j,\, e^{\mathcal{C}}_n\right) = \frac{-1}{\sigma|X_{\mathcal{C}}|^2}\delta_{ij} = -\frac{\kappa^{\mathcal{C}}_i}{|X_{\mathcal{C}}|}\delta_{ij}, \quad \forall\, i,\, j \neq n \tag{1.155}$$

$$\nabla_{\mathcal{C}} A_{\mathcal{C}}\left(e^{\mathcal{C}}_i,\, e^{\mathcal{C}}_j,\, e^{\mathcal{C}}_k\right) = \nabla_{\mathcal{C}} A_{\mathcal{C}}\left(e^{\mathcal{C}}_i,\, e^{\mathcal{C}}_n,\, e^{\mathcal{C}}_n\right) = \nabla_{\mathcal{C}} A_{\mathcal{C}}\left(e^{\mathcal{C}}_n,\, e^{\mathcal{C}}_n,\, e^{\mathcal{C}}_n\right) = 0 \quad \forall\, i,\, j,\, k \neq n \tag{1.156}$$

where $A_{\mathcal{C}}$ *is the second fundamental form of* $\mathcal{C}$ *and* $\nabla_{\mathcal{C}} A_{\mathcal{C}}$ *is its covariant derivative. Note that* $A_{\mathcal{C}}$ *and* $\nabla_{\mathcal{C}} A_{\mathcal{C}}$ *are totally symmetric tensors (by Codazzi equation).*

Proof. Let's parametrize $\mathcal{C}$ by

$$X_{\mathcal{C}} = (\sigma s\, \nu,\, s) \quad \text{for } \nu \in \mathbf{S}^{n-1},\ s \in \mathbb{R}_+$$

and take an othornomal local frame $\left\{e_1^{\mathcal{C}}, \cdots, e_n^{\mathcal{C}}\right\}$ of $\mathcal{C}$ so that

$$e_n^{\mathcal{C}} = \frac{\partial_s X_{\mathcal{C}}}{|\partial_s X_{\mathcal{C}}|} = \frac{(\sigma\nu,\, 1)}{\sqrt{1+\sigma^2}} \tag{1.157}$$

By the general formula for the principal curvatures of hypersuface of revolution, we get

$$\kappa_1^{\mathcal{C}} = \cdots = \kappa_{n-1}^{\mathcal{C}} = \frac{1}{\sigma s\sqrt{1+\sigma^2}} = \frac{1}{\sigma|X_{\mathcal{C}}|}, \quad \kappa_n^{\mathcal{C}} = 0 \tag{1.158}$$

Since $\left\{e_1^{\mathcal{C}}, \cdots, e_n^{\mathcal{C}}\right\}$ forms a principal basis at each point, so by (1.158) we have

$$A_{ii}^{\mathcal{C}} = \kappa_i^{\mathcal{C}} = \frac{1}{\sigma s\sqrt{1+\sigma^2}} = \frac{1}{\sigma|X_{\mathcal{C}}|} \quad \text{whenever } i \neq n \tag{1.159}$$

$$A_{ij}^{\mathcal{C}} = 0 = A_{nn}^{\mathcal{C}} \quad \text{whenever } i \neq j$$

where $A_{ij}^{\mathcal{C}} \equiv A_{\mathcal{C}}\left(e_i^{\mathcal{C}},\, e_j^{\mathcal{C}}\right)$. Also, by the orthonormality of $\left\{e_1^{\mathcal{C}}, \cdots, e_n^{\mathcal{C}}\right\}$ and the product rule, the Christoffel symbols ${}^{\mathcal{C}}\Gamma_{ij}^k \equiv \left(D_{e_i^{\mathcal{C}}}^{\mathcal{C}}\, e_j^{\mathcal{C}}\right) \cdot e_k^{\mathcal{C}}$ satisfy

$${}^{\mathcal{C}}\Gamma_{ki}^j = \left(D_{e_k^{\mathcal{C}}}^{\mathcal{C}}\, e_i^{\mathcal{C}}\right) \cdot e_j^{\mathcal{C}} = -\left(D_{e_k^{\mathcal{C}}}^{\mathcal{C}}\, e_j^{\mathcal{C}}\right) \cdot e_i^{\mathcal{C}} = -{}^{\mathcal{C}}\Gamma_{kj}^i \tag{1.160}$$

Thus, from (1.159) and (1.160), we deduce that whenever $i,\, j \neq n$ or $i = j = n$, there holds

$$\nabla_k^{\mathcal{C}} A_{ij}^{\mathcal{C}} = D_{e_k^{\mathcal{C}}}\left(A_{ij}^{\mathcal{C}}\right) - {}^{\mathcal{C}}\Gamma_{ki}^j A_{jj}^{\mathcal{C}} - {}^{\mathcal{C}}\Gamma_{kj}^i A_{ii}^{\mathcal{C}} = D_{e_k^{\mathcal{C}}}\left(A_{ij}^{\mathcal{C}}\right) \tag{1.161}$$

By (1.161), (1.159) and (1.157), we get

$$\begin{aligned}\nabla_n^{\mathcal{C}} A_{ij}^{\mathcal{C}} &= D_{e_n^{\mathcal{C}}}\left(\kappa_i^{\mathcal{C}}\delta_{ij}\right) = \frac{1}{\sqrt{1+\sigma^2}}\partial_s\left(\frac{1}{\sigma s\sqrt{1+\sigma^2}}\right)\delta_{ij}\\ &= \frac{-1}{\sigma\left(1+\sigma^2\right)s^2}\delta_{ij} = \frac{-1}{\sigma|X_{\mathcal{C}}|^2}\delta_{ij} \quad \text{if } i,\, j \neq n\end{aligned}$$

which verifies (1.155).

By (1.161), (1.159) and noting that $|X_{\mathcal{C}}|$ is invariant along $e_k^{\mathcal{C}}$ for $k \neq n$, we get

$$\nabla_k^{\mathcal{C}} A_{ij}^{\mathcal{C}} = D_{e_k^{\mathcal{C}}}\left(\kappa_i^{\mathcal{C}}\delta_{ij}\right) = D_{e_k^{\mathcal{C}}}\left(\frac{1}{\sigma|X_{\mathcal{C}}|}\right)\delta_{ij} = 0 \quad \text{if } i,\, j,\, k \neq n \tag{1.162}$$

From (1.161) and (1.159), we have

$$\nabla_i^{\mathcal{C}} A_{nn}^{\mathcal{C}} = D_{e_i^{\mathcal{C}}}\left(A_{nn}^{\mathcal{C}}\right) = 0 \quad \forall\, i \tag{1.163}$$

Then (1.156) follows from (1.162) and (1.163). □

Combining (1.1), (1.2), (1.3) with Lemma 1.28, we conclude the following:

Proposition 1.29. *The constant $\varkappa$ defined in (1.5) can be estimated by*

$$\varkappa \leq C(n)\left(\left|\partial^2 f\left(\overrightarrow{1}, 0\right)\right| + \left|\partial_1 f\left(\overrightarrow{1}, 0\right) - \partial_n f\left(\overrightarrow{1}, 0\right)\right|\right) \tag{1.164}$$

Proof. At each point $X_{\mathcal{C}} \in \mathcal{C}$, take an orthonormal basis $\left\{e_1^{\mathcal{C}}, \cdots, e_n^{\mathcal{C}}\right\}$ for $T_{X_{\mathcal{C}}}\mathcal{C}$ so that $e_n^{\mathcal{C}} = \frac{(\sigma\nu, 1)}{\sqrt{1+\sigma^2}}$. Then by (1.2), (1.3), Lemma 1.28 and the homogeneity of the derivatives of f, we get

$$\begin{aligned}
\left|\frac{\partial^2 F}{\partial S_i^j \partial S_k^l}\left(A_{\mathcal{C}}^{\#}\right)\right| &\leq \left(\left|\partial^2 f\left(\kappa_1^{\mathcal{C}}, \cdots, \kappa_n^{\mathcal{C}}\right)\right| + \left|\frac{\partial_1 f\left(\kappa_1^{\mathcal{C}}, \cdots, \kappa_n^{\mathcal{C}}\right) - \partial_n f\left(\kappa_1^{\mathcal{C}}, \cdots, \kappa_n^{\mathcal{C}}\right)}{\kappa_1^{\mathcal{C}} - \kappa_n^{\mathcal{C}}}\right|\right) \\
&= \frac{1}{\kappa_1^{\mathcal{C}}}\left(\left|\partial^2 f\left(\overrightarrow{1}, 0\right)\right| + \left|\partial_1 f\left(\overrightarrow{1}, 0\right) - \partial_k f\left(\overrightarrow{1}, 0\right)\right|\right)
\end{aligned}$$

which implies that

$$\begin{aligned}
&|X_{\mathcal{C}}|\left|\sum_{k,l}\frac{\partial^2 F}{\partial S_i^j \partial S_k^l}\left(A_{\mathcal{C}}^{\#}\right)\left(\nabla_{\mathcal{C}} A_{\mathcal{C}}^{\#}\right)_k^l\right| \\
\leq\ & |X_{\mathcal{C}}|\frac{C(n)}{\kappa_1^{\mathcal{C}}}\left(\left|\partial^2 f\left(\overrightarrow{1}, 0\right)\right| + \left|\partial_1 f\left(\overrightarrow{1}, 0\right) - \partial_k f\left(\overrightarrow{1}, 0\right)\right|\right)\frac{\kappa_1^{\mathcal{C}}}{|X_{\mathcal{C}}|} \\
=\ & C(n)\left(\left|\partial^2 f\left(\overrightarrow{1}, 0\right)\right| + \left|\partial_1 f\left(\overrightarrow{1}, 0\right) - \partial_k f\left(\overrightarrow{1}, 0\right)\right|\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\varkappa &= \sup_{X_{\mathcal{C}} \in \mathcal{C} \cap \left(B_3 \setminus \bar{B}_{\frac{1}{3}}\right)}\left|\sum_{k,l}\frac{\partial^2 F}{\partial S_i^j \partial S_k^l}\left(A_{\mathcal{C}}^{\#}\right)\left(\nabla_{\mathcal{C}} A_{\mathcal{C}}^{\#}\right)_k^l\right| \\
&\leq C(n)\left(\left|\partial^2 f\left(\overrightarrow{1}, 0\right)\right| + \left|\partial_1 f\left(\overrightarrow{1}, 0\right) - \partial_k f\left(\overrightarrow{1}, 0\right)\right|\right)
\end{aligned}$$

$\square$

Chapter 2

Existence of self-shrinkers to the degree-one curvature flow with a rotationally symmetric conical end

2.1 Introduction

Let $\mathcal{C}^n$ be a rotationally symmetry cone, say

$$\mathcal{C} = \left\{ (\sigma s\, \nu,\, s) \,\middle|\, \nu \in \mathbf{S}^{n-1},\, s \in \mathbb{R}_+ \right\}$$

with $\sigma > 0$. Suppose that $\mathbf{\Sigma}$ is an orientable and properly embedded smooth hypersurface in $\mathbb{R}^{n+1}$ which satisfies

$$H + \frac{1}{2} X \cdot N = 0 \quad \forall\, X \in \mathbf{\Sigma}$$

$$\varrho \mathbf{\Sigma} \xrightarrow{C^\infty_{\text{loc}}} \mathcal{C} \quad \text{as } \varrho \searrow 0$$

where X is the position vector, N is the unit-normal, $H = \kappa_1 + \cdots + \kappa_n$ is the mean curvature and $\kappa_1, \cdots, \kappa_n$ are the principal curvatures of $\mathbf{\Sigma}$. Note that $\kappa_1, \cdots, \kappa_n$ are defined to be the eigenvalues of the second fundamental form A, which is a bilinear form defined by

$$A\,(V,\, W) = D_V W \cdot N$$

for tangent vector fields V and W. Then $\mathbf{\Sigma}$ is called a self-shrinker to the mean curvature flow (MCF: an one-parameter family of hypersurfaces for which $\partial_t X^\perp = HN$ holds) which is C^k asymptotic to the cone $\mathcal{C}$ at infinity. It follows that the rescaled family of hypersufaces $\left\{ \mathbf{\Sigma}_t = \sqrt{-t}\, \mathbf{\Sigma} \right\}$ forms a mean curvature flow starting from $\mathbf{\Sigma}$ (when $t = -1$) and converging locally C^k to $\mathcal{C}$ as $t \nearrow 0$. In the case when $\mathbf{\Sigma}$ is rotationally symmetric, one can parametrize it by

$$X\,(\nu,\, s) = (\mathtt{r}(s)\nu,\, s)\,, \quad \nu \in \mathbf{S}^{n-1},\, s \in (c_1, c_2)$$

for some constants $0 \le c_1 < c_2 \le \infty$. We may orient it by the unit-normal

$$N = \frac{(-\nu,\, \partial_s \mathtt{r})}{\sqrt{1 + (\partial_s \mathtt{r})^{\,2}}} \tag{2.1}$$

At each point $X \in \Sigma$, choose an orthonormal basis $\{e_1, \cdots, e_n\}$ for $T_X \mathbf{\Sigma}$ so that

$$e_n = \frac{\partial_s X}{|\partial_s X|} = \frac{(\partial_s \mathtt{r}\, \nu,\, 1)}{\sqrt{1 + (\partial_s \mathtt{r})^{\,2}}}$$

then $\{e_1, \cdots, e_n\}$ forms a set of principal directions of $\mathbf{\Sigma}$ at X with principal curvatures

$$\kappa_1 = \cdots = \kappa_{n-1} = \frac{1}{\mathtt{r}\sqrt{1 + (\partial_s \mathtt{r})^{\,2}}}, \qquad \kappa_n = \frac{-\partial_s^2 \mathtt{r}}{\left(1 + (\partial_s \mathtt{r})^{\,2}\right)^{\frac{3}{2}}} \tag{2.2}$$

As a result, $\mathbf{\Sigma}$ is a rotationally symmetric self-shrinker to the MCF if and only if

$$\left(\frac{n-1}{\mathtt{r}} - \frac{\partial_s^2 \mathtt{r}}{1 + |\partial_s \mathtt{r}|^2}\right) + \frac{1}{2}\left(s\, \partial_s \mathtt{r} - \mathtt{r}\right) = 0 \tag{2.3}$$

Kleene and Moller showed in [KM] that there exists a unique rotationally symmetric self-shrinker

$$\mathbf{\Sigma} :\; X = X(\nu,\, s) = \left(\mathtt{r}(s)\nu,\, s\right), \quad \nu \in \mathbf{S}^{n-1},\, s \in [R,\, \infty)$$

where the radial function $\mathtt{r} = \mathtt{r}(s)$ satisfies (2.3) and

$$s\Big|\mathtt{r}(s) - \sigma s\Big| \le \frac{2\,(n-1)}{\sigma}, \quad s^2\Big|\partial_s \mathtt{r} - \sigma\Big| \le \frac{2\,(n-1)}{\sigma}$$

by analyzing the following representation formula for the above ODE:

$$\mathtt{r}(s) = \sigma s$$
$$+\, s \int_s^\infty \frac{1}{x^2} \left\{ \int_x^\infty \xi \,\exp\left(-\frac{1}{2} \int_x^\xi \boldsymbol{\tau} \left(1 + (\partial_s \mathtt{r}(\boldsymbol{\tau}))^2\right) d\boldsymbol{\tau} \right) \left[\frac{n-1}{\mathtt{r}(\xi)} \left(1 + (\partial_s \mathtt{r}(\xi))^2\right) \right] d\xi \right\} dx$$

On the other hand, let $f = f(\lambda_1, \cdots, \lambda_n)$ be a C^4, symmetric and homogeneous of degree-one function defined on $\mathbf{\Omega} \subset \mathbb{R}^n$ which satisfies

$$\partial_i f > 0 \quad \forall\, i = 1, \cdots, n$$

Note that by the property of f, we may assume that its domain $\mathbf{\Omega}$ is invariant under permutation and homothety, i.e.

$$\left(\lambda_{\sigma(1)}, \cdots, \lambda_{\sigma(n)}\right),\, (\varrho\lambda_1, \cdots, \varrho\lambda_n) \in \mathbf{\Omega} \qquad \forall\, \sigma \in \mathbf{S}^n,\, \varrho > 0$$

provided that $(\lambda_1, \cdots, \lambda_n) \in \mathbf{\Omega}$, where $\mathbf{S}^n$ is the symmetric group.

Andrews in [A] considered the following evolution of hypersurfaces in $\mathbb{R}^{n+1}$:

$$\partial_t X^{\perp} = f(\kappa_1, \cdots, \kappa_n) N$$

where $\kappa_1, \cdots, \kappa_n$ are the principal curvatures of the evolving hypersurface. In particular, if we take the curvature function to be $f(\lambda_1, \cdots, \lambda_n) = \lambda_1 + \cdots + \lambda_n$, then it becomes the mean curvature flow. We call an orientable C^2 hypersurface Σ in $\mathbb{R}^{n+1}$ to be a "f self-shrinker" to the above "f curvature flow" provided that

$$f(\kappa_1, \cdots, \kappa_n) + \frac{1}{2} X \cdot N = 0$$

holds on Σ. As the MCF, the rescaled family of "f self-shrinkers" is a self-similar solution to the f curvature flow; that is, the one-parameter family of hypersurfaces $\left\{\Sigma_t = \sqrt{-t}\,\Sigma\right\}_{t<0}$ is a f curvature flow. Furthermore, if Σ is C^k asymptotic to the cone $\mathcal{C}$ at infinity, the rescaled flow $\{\Sigma_t\}_{t<0}$ converges locally C^k to $\mathcal{C}$ as $t \nearrow 0$.

In this chapter we extend the existence result in [KM] to the class of f self-shrinkers with a tangent cone at infinity. We show the following:

Theorem 2.1. *Suppose that f is $C^{\mathsf{k}+1}$ in a bounded neighborhood $\mathcal{K}$ of $\left(\overrightarrow{1}, 0\right) = (1, \cdots, 1, 0) \in \mathbb{R}^n$ with $\mathsf{k} \geq 3$. Then there exist $R = R\left(n, \mathsf{k}, \mathcal{C}, \mathcal{K}, \| f \|_{C^{\mathsf{k}+1}(\mathcal{K})}\right) \geq 1$ and $\mathsf{u} \in C_0^{\mathsf{k}}[R, \infty)$ such that*

$$\Sigma \equiv \left\{ \left(\left(\sigma s + \frac{f\left(\overrightarrow{1}, 0\right)}{\sigma s} + \mathsf{u}(s) \right) \nu,\ s \right) \Big| \, \nu \in \mathbf{S}^{n-1},\ s \in [R, \infty) \right\}$$

is a rotationally symmetric f self-shrinker which is C^{k}asymptotic to $\mathcal{C}$ at infinity. Besides, the corresponding self-similar solution to the F curvature flow is given by

$$\Sigma_t = \sqrt{-t}\,\Sigma = \left\{ \left(\left(\sigma s - t\,\frac{f\left(\overrightarrow{1}, 0\right)}{\sigma s} + \mathsf{u}_t(s) \right) \nu,\ s \right) \Big| \, \nu \in \mathbf{S}^{n-1},\ s \in [\sqrt{-t}R, \infty) \right\}$$

for $t \in [-1, 0)$, where $\mathsf{u}_t(s) = \sqrt{-t}\,\mathsf{u}\left(\frac{s}{\sqrt{-t}}\right)$ and it satisfies

$$\| s^3\, \mathsf{u}_t \|_{L^\infty[\sqrt{-t}R, \infty)} + \| s^4 \partial_s \mathsf{u}_t \|_{L^\infty[\sqrt{-t}R, \infty)} + \cdots + \| s^{k+2} \partial_s^{k-1} \mathsf{u}_t \|_{L^\infty[\sqrt{-t}R, \infty)}$$

$$\leq C\left(n, \mathsf{k}, \mathcal{C}, \| f \|_{C^{\mathsf{k}}(\mathcal{K})}\right) (-t)^2$$

$$\| s^{k+1} \partial_s^k \mathsf{u}_t \|_{L^\infty[\sqrt{-t}R,\,\infty)} \le C\left(n,\, \mathsf{k},\, \mathcal{C},\, \| f \|_{C^{\mathsf{k}}(\mathcal{K})}\right)(-t)$$

for all $t \in [-1,\, 0)$.

Note that in view of the principal curvatures of $\mathcal{C}$:

$$\kappa_1^{\mathcal{C}} = \cdots = \kappa_{n-1}^{\mathcal{C}} = \frac{1}{\sigma s \sqrt{1+\sigma^2}}, \quad \kappa_n^{\mathcal{C}} = 0$$

(see (2.2)) and the homogeneity of f, the condition that f is $C^{\mathsf{k}+1}$ in a bounded neighborhood $\mathcal{K}$ of $\left(\overrightarrow{1},\, 0\right) = (1, \cdots, 1,\, 0) \in \mathbb{R}^n$ implies that f is $C^{\mathsf{k}+1}$ in an open set containing all the principal curvatures $\left(\kappa_1^{\mathcal{C}}, \cdots, \kappa_n^{\mathcal{C}}\right)$ of $\mathcal{C}$.

In the next section, we would use a similar representation formula as in [KM] to study the ODE corresponding to the f self-shrinker equation. Then we use that, together with Banach fixed point theorem, to show the existence of f self-shrinkers.

2.2 Proof of the main result

First of all, given a hypersurface of revolution Σ in $\mathbb{R}^{n+1}$ parametrized by

$$X(\nu,\, s) \equiv (r(s)\, \nu,\, s) \quad \text{for } \nu \in \mathbf{S}^{n-1},\, s \in (c_1,\, c_2)$$

for some constants $0 \le c_1 < c_2 \le \infty$. By (2.1) and (2.2), Σ is a rotationally symmetric f self-shrinker if and only if

$$f\left(\frac{1}{r\sqrt{1+(\partial_s r)^2}}\overrightarrow{1},\, \frac{-\partial_s^2 r}{(1+(\partial_s r)^2)^{\frac{3}{2}}}\right) + \frac{1}{2}\frac{s\,\partial_s r - r}{\sqrt{1+(\partial_s r)^2}} = 0 \tag{2.4}$$

where $\overrightarrow{1} = (1, \cdots, 1) \in \mathbb{R}^{n-1}$. By the homogeneity of f, (2.4) is equivalent to

$$\wp\left(\partial_s^2 r,\, \partial_s r,\, r,\, s\right) = f\left(\frac{1}{r}\overrightarrow{1},\, \frac{-\partial_s^2 r}{1+(\partial_s r)^2}\right) + \frac{1}{2}(s\,\partial_s r - r) = 0 \tag{2.5}$$

where

$$\wp(q,\, p,\, z,\, s) \equiv f\left(\frac{1}{z}\overrightarrow{1},\, \frac{-q}{1+p^2}\right) + \frac{1}{2}(sp - z) \tag{2.6}$$

On the other hand, Σ is C^{k} asymptotic to $\mathcal{C}$ at infinity if

$$\varrho\, r\left(\frac{s}{\varrho}\right) - \sigma s \xrightarrow{C^{\mathsf{k}}_{\mathrm{loc}}} 0 \quad \text{as } \varrho \searrow 0 \tag{2.7}$$

Let

$$u\,(s) = r\,(s) - \sigma s \tag{2.8}$$

then (2.7) can be written

$$\varrho\, u\left(\frac{s}{\varrho}\right) \xrightarrow{C^{\mathsf{k}}_{\mathrm{loc}}} 0 \quad \text{as } \varrho \searrow 0 \tag{2.9}$$

which is equivalent to

$$u\,(s) = o\,(s)\,,\ \partial_s u = o\,(1)\,, \cdots,\ \partial_s^{\mathsf{k}} u = o\left(s^{1-\mathsf{k}}\right) \quad \text{as } s \nearrow \infty$$

Now we would like to get an equation of u by first plugging $r\,(s) = \sigma s + u\,(s)$ into (2.5) and then using Taylor's theorem get an expansion. To achieve that, let's define

$$\mathfrak{z}\,(u\,(s)\,,\,\theta) = \sigma s + \theta\, u\,(s) \tag{2.10}$$

$$\mathfrak{p}\,(u\,(s)\,,\,\theta) = \partial_{\mathrm{s}}\,(\sigma s + \theta\, u\,(s)) = \sigma + \theta\, \partial_s u \tag{2.11}$$

$$\mathfrak{q}\,(u\,(s)\,,\,\theta) = \partial_s^2\,(\sigma s + \theta\, u\,(s)) = \theta\, \partial_s^2 u \tag{2.12}$$

then (2.5) can be written as

$$\wp\,(\mathfrak{q}\,(u\,(s)\,,\,1)\,,\ \mathfrak{p}\,(u\,(s)\,,\,1)\,,\ \mathfrak{z}\,(u\,(s)\,,\,1)\,,\,s) = 0$$

By Taylor's theorem, (2.6), (2.10), (2.11), (2.12) and the homogeneity of f and its derivatives (with f being of degree 1, $\partial_i f$ being of degree 0 and $\partial^2_{ij} f$ being of degree -1), the above equation becomes

$$0 = \wp\,(\mathfrak{q}\,(u,\,1)\,,\ \mathfrak{p}\,(u,\,1)\,,\ \mathfrak{z}\,(u,\,1)\,,\,s) \tag{2.13}$$

$$= \wp\,(\mathfrak{q}\,(u,\,0)\,,\ \mathfrak{p}\,(u,\,0)\,,\ \mathfrak{z}\,(u,\,0)\,,\,s) + \partial_\theta \left\{\wp\,(\mathfrak{q}\,(u,\,\theta)\,,\ \mathfrak{p}\,(u,\,\theta)\,,\ \mathfrak{z}\,(u,\,\theta)\,,\,s)\right\}\Big|_{\theta=0}$$

$$+ \int_0^1 \partial_\theta^2 \left\{\wp\,(\mathfrak{q}\,(u,\,\theta)\,,\ \mathfrak{p}\,(u,\,\theta)\,,\ \mathfrak{z}\,(u,\,\theta)\,,\,s)\right\}\,(1-\theta)\,d\theta$$

$$= f\left(\frac{1}{\sigma s}\overrightarrow{1}\,,\,0\right) - \frac{\partial_n f\left(\frac{1}{\sigma s}\overrightarrow{1}\,,\,0\right)}{1+\sigma^2}\partial_s^2 u - \sum_{i=1}^{n-1}\frac{\partial_i f\left(\frac{1}{\sigma s}\overrightarrow{1}\,,\,0\right)}{\sigma^2 s^2}u + \frac{1}{2}\,(s\,\partial_s u - u) + \mathcal{Q}u$$

$$= \frac{1}{\sigma s} f\left(\overrightarrow{1}, 0\right) - \frac{\partial_n f\left(\overrightarrow{1}, 0\right)}{1+\sigma^2}\partial_s^2 u - \sum_{i=1}^{n-1} \frac{\partial_i f\left(\overrightarrow{1}, 0\right)}{\sigma^2 s^2} u + \frac{1}{2}\left(s\,\partial_s u - u\right) + \mathcal{Q}u$$

with the quadratic term $\mathcal{Q}u$ being

$$\mathcal{Q}u(s) = \left(\int_0^1 \frac{\sigma s\,\partial_{nn}^2 f \circ \omega}{\left(1+\mathfrak{p}^2\right)^2}\,(1-\theta)\,d\theta\right)\left(\partial_s^2 u\right)^2 \tag{2.14}$$

$$+\left(\int_0^1\left(\sigma s\,\partial_{nn}^2 f \circ \omega \frac{4\mathfrak{q}^2\mathfrak{p}^2}{\left(1+\mathfrak{p}^2\right)^4} + \partial_n f \circ \omega \frac{2\mathfrak{q}\left(1-3\mathfrak{p}^2\right)}{\left(1+\mathfrak{p}^2\right)^3}\right)(1-\theta)\,d\theta\right)\left(\partial_s u\right)^2$$

$$+\left(\int_0^1 \frac{\left(\sum_{i,\,j=1}^{n-1} \sigma s\,\partial_{ij}^2 f \circ \omega\right) + 2\mathfrak{z}\sum_{i=1}^{n-1}\partial_i f \circ \omega}{\mathfrak{z}^4}\,(1-\theta)\,d\theta\right) u^2$$

$$+2\left(\int_0^1\left(\sigma s\,\partial_{nn}^2 f \circ \omega \frac{-2\mathfrak{q}\mathfrak{p}}{\left(1+\mathfrak{p}^2\right)^3} + \partial_n f \circ \omega \frac{2\mathfrak{p}}{\left(1+\mathfrak{p}^2\right)^2}\right)(1-\theta)\,d\theta\right)\partial_s^2 u\,\partial_s u$$

$$+2\left(\int_0^1 \sum_{i=1}^{n-1} \frac{\sigma s\,\partial_{ni}^2 f \circ \omega}{\left(1+\mathfrak{p}^2\right)\mathfrak{z}^2}\,(1-\theta)\,d\theta\right) u\,\partial_s^2 u - 2\left(\int_0^1 \sum_{i=1}^{n-1} \sigma s\,\partial_{ni}^2 f \circ \omega \frac{2\mathfrak{q}\mathfrak{p}}{\left(1+\mathfrak{p}^2\right)^2 \mathfrak{z}^2}\,(1-\theta)\,d\theta\right) u\,\partial_s u$$

where $\mathfrak{z} = \mathfrak{z}\left(u\left(s\right),\,\theta\right)$, $\mathfrak{p} = \mathfrak{p}\left(u\left(s\right),\,\theta\right)$, $\mathfrak{q} = \mathfrak{q}\left(u\left(s\right),\,\theta\right)$ and

$$\omega = \omega\left(u\left(s\right),\,\theta\right) \equiv \sigma s\left(\frac{1}{\mathfrak{z}\left(u\left(s\right),\,\theta\right)}\overrightarrow{1},\ \frac{-\mathfrak{q}\left(u\left(s\right),\,\theta\right)}{1+\mathfrak{p}\left(u\left(s\right),\,\theta\right)^2}\right) \tag{2.15}$$

$$= \left(\frac{1}{1+\theta\frac{u}{\sigma s}}\overrightarrow{1},\ \frac{-2\theta\sigma s\,\partial_s^2 u}{1+\left(\sigma+\theta\partial_s u\right)^2}\right)$$

Rearrange (2.13) to get

$$\mathcal{L}u = \partial_s^2 u - \frac{1}{2}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1}, 0\right)}\left(s\,\partial_s u - u\right) \tag{2.16}$$

$$= \frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1}, 0\right)}\left(\frac{f\left(\overrightarrow{1}, 0\right)}{\sigma s} - \sum_{i=1}^{n-1}\frac{\partial_i f\left(\overrightarrow{1}, 0\right)}{\sigma^2 s^2} u + \mathcal{Q}u\right)$$

where $\mathcal{L}$ is a linear differential operator defined by

$$\mathcal{L} = \partial_s^2 - \frac{1}{2}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1}, 0\right)}\left(s\,\partial_s - 1\right) \tag{2.17}$$

To summarize, in order to find a rotationally symmetric F self-shrinker Σ which is C^{k} asymptotic to $\mathcal{C}$ at infinity, it suffices to solve the ODE (2.5) with the condition (2.7), which (by (2.8)) amounts to solving the problem (2.16, 2.9). We would do this by regarding (2.16) as a fixed point problem of a nonlinear map in a suitable normed space of functions where (2.9) is satisfied and the nonlinear map is a contraction. Then the existence of solutions to our problem is assured by Banach's fixed point theorem. To this end, we need two lemma.

In the first lemma, we analyze the linear differential operator $\mathcal{L}$ in (2.17). We derive a representation formula for the associated problem as in [KM], which is then used to estimate its solutions.

Lemma 2.2. *Fix $R > 0$, then for any $\boldsymbol{\eta} \in C_0[R, \infty)$ (i.e. $\boldsymbol{\eta} \in C[R, \infty)$ and $\eta \to 0$ as $s \nearrow \infty$), there is a unique $C^2[R, \infty)$ solution w to the following problem:*

$$\mathcal{L}w = \boldsymbol{\eta} \quad \textit{on } [R, \infty) \tag{2.18}$$

$$\frac{w}{s} \;\&\; (s\,\partial_s w - w) \to 0 \quad \textit{as } s \nearrow \infty \tag{2.19}$$

where $\mathcal{L}$ is defined in (2.17). Besides, w satisfies the following estimates: $\forall\, \gamma \geq 0$

$$\max\left\{\| \, s^{\gamma} w \, \|_{L^{\infty}[R,\infty)},\; \| \, s^{\gamma+1}\partial_s w \, \|_{L^{\infty}[R,\infty)}\right\} \leq 4 \frac{\partial_n f\left(\overrightarrow{1}, 0\right)}{1+\sigma^2} \| \, s^{\gamma}\boldsymbol{\eta} \, \|_{L^{\infty}[R,\infty)} \tag{2.20}$$

$$\| \, s^{\gamma}\partial_s^2 w \, \|_{L^{\infty}[R,\infty)} \leq 4 \, \| \, s^{\gamma}\boldsymbol{\eta} \, \|_{L^{\infty}[R,\infty)} \tag{2.21}$$

and $w \in C_0^{m+2}[R, \infty)$ whenever $\boldsymbol{\eta} \in C_0^{m}[R, \infty)$ for $m \in \mathbb{N}$.

Proof. Firstly, if $\mathrm{w} \in C^2[R, \infty)$ were a solution to the linear problem (2.18, 2.19), it must satisfy the following: given a sequence $\{R_j \in (R, \infty)\}_{j\in\mathbb{N}}$ such that $R_j \nearrow \infty$ as $j \nearrow \infty$, then from (2.17), (2.18) and (2.19), we get

$$\partial_s^2\left(\frac{\mathrm{w}}{s}\right) + \left(\frac{2}{s} - \frac{s}{2}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1}, 0\right)}\right)\partial_s\left(\frac{\mathrm{w}}{s}\right) = \frac{\boldsymbol{\eta}}{s} \quad \text{for } s \in [R, R_j) \tag{2.22}$$

$$\frac{\mathrm{w}(R_j)}{R_j} \;\&\; (R_j\,\partial_s \mathrm{w}(R_j) - \mathrm{w}(R_j)) \to 0 \quad \text{as } j \nearrow \infty \tag{2.23}$$

From (2.22), we get

$$\mathrm{w}(s) = s\left\{\frac{\mathrm{w}(R_j)}{R_j} - (R_j\,\partial_s \mathrm{w}(R_j) - \mathrm{w}(R_j))\int_s^{R_j} x^{-2}\exp\left(-\frac{1}{2}\int_x^{R_j}\boldsymbol{\tau}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1}, 0\right)}d\boldsymbol{\tau}\right)dx\right\} \tag{2.24}$$

$$+s\int_s^{R_j}\frac{1}{x^2}\left(\int_x^{R_j}\xi\,\exp\left(-\frac{1}{2}\int_x^{\xi}\boldsymbol{\tau}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},\,0\right)}d\boldsymbol{\tau}\right)\boldsymbol{\eta}(\xi)\,d\xi\right)dx$$

$$= s\left\{\frac{\mathrm{w}(R_j)}{R_j} - \left(R_j\,\partial_s\mathrm{w}(R_j) - \mathrm{w}(R_j)\right)\int_s^{R_j} x^{-2}\,\exp\left(-\frac{1}{4}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},\,0\right)}\left(R_j^2 - x^2\right)\right)dx\right\}$$

$$+s\int_s^{R_j}\frac{1}{x^2}\left(\int_x^{R_j}\xi\,\exp\left(-\frac{1}{4}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},\,0\right)}\left(\xi^2 - x^2\right)\right)\boldsymbol{\eta}(\xi)\,d\xi\right)dx$$

Note that in the last two terms of (2.24), we have

$$\int_s^{R_j} x^{-2}\,\exp\left(-\frac{1}{4}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},\,0\right)}\left(R_j^2 - x^2\right)\right)dx \,\le\, \int_s^{\infty} x^{-2}dx \,=\, \frac{1}{s} \tag{2.25}$$

$$\int_x^{R_j}\xi\,\exp\left(-\frac{1}{4}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},\,0\right)}\left(\xi^2 - x^2\right)\right)d\xi \tag{2.26}$$

$$\le\, \int_x^{\infty}\xi\,\exp\left(-\frac{1}{4}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},\,0\right)}\left(\xi^2 - x^2\right)\right)d\xi \,=\, 2\frac{\partial_n f\left(\overrightarrow{1},\,0\right)}{1+\sigma^2}$$

Fix $s \in [R,\,\infty)$, we take limit (as $j \nearrow \infty$) of (2.24) and use (2.23) and (2.25) to get

$$\mathrm{w}(s) = \,s\int_s^{\infty}\frac{1}{x^2}\left(\int_x^{\infty}\xi\,\exp\left(-\frac{1}{4}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},\,0\right)}\left(\xi^2 - x^2\right)\right)\boldsymbol{\eta}(\xi)\,d\xi\right)dx \tag{2.27}$$

Conversly, if we define a function w by (2.27), then $\mathrm{w} \in C^2[R,\,\infty)$ and it satisfies

$$\partial_s\mathrm{w} = \int_s^{\infty}\frac{1}{x^2}\left(\int_x^{\infty}\xi\,\exp\left(-\frac{1}{4}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},\,0\right)}\left(\xi^2 - x^2\right)\right)\boldsymbol{\eta}(\xi)\,d\xi\right)dx \tag{2.28}$$

$$-\frac{1}{s}\int_s^{\infty}\xi\,\exp\left(-\frac{1}{4}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},\,0\right)}\left(\xi^2 - s^2\right)\right)\boldsymbol{\eta}(\xi)\,d\xi$$

$$\partial_s^2\mathrm{w} = -\frac{1}{2}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},\,0\right)}\int_s^{\infty}\xi\,\exp\left(-\frac{1}{4}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},\,0\right)}\left(\xi^2 - s^2\right)\right)\boldsymbol{\eta}(\xi)\,d\xi \,+\, \boldsymbol{\eta}(s) \tag{2.29}$$

From (2.17), (2.27), (2.28) and (2.29), we immediately get (2.18). To verify (2.19), we use (2.26) and that $\boldsymbol{\zeta}$ vanishes at infinity to get

$$\frac{\mathrm{w}}{s} = \int_s^{\infty}\frac{1}{x^2}\left(\int_x^{\infty}\xi\,\exp\left(-\frac{1}{4}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},\,0\right)}\left(\xi^2 - x^2\right)\right)\boldsymbol{\eta}(\xi)\,d\xi\right)dx$$

$$\leq 2\frac{\partial_n f\left(\overrightarrow{1},0\right)}{1+\sigma^2}\left(\sup_{\xi>s}|\boldsymbol{\eta}(\xi)|\right)\int_s^\infty \frac{dx}{x^2}\to 0 \quad \text{as } s\nearrow\infty$$

$$|s\,\partial_s \mathrm{w}-\mathrm{w}| \leq \int_s^\infty \xi\,\exp\left(-\frac{1}{4}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},0\right)}\left(\xi^2-s^2\right)\right)|\boldsymbol{\eta}(\xi)|\,d\xi\to 0$$

$$\leq 2\frac{\partial_n f\left(\overrightarrow{1},0\right)}{1+\sigma^2}\left(\sup_{\xi>s}|\boldsymbol{\eta}(\xi)|\right)\to 0 \quad \text{as } s\nearrow\infty$$

Thus, (2.27) defines a solution to the linear problem (2.18, 2.19).

To sum up, there is a unique $C^2[R,\infty)$ solution to the linear problem (2.18, 2.19), which is given by (2.27). The derivatives of the solution are given by (2.28) and (2.29), respectively.

Now given $\gamma\geq 0$, we would like to verify (2.20). Note that we may assume $\| s^\gamma\boldsymbol{\eta}\|_{L^\infty[R,\infty)}<\infty$; otherwise there's nothing to prove. Then for each $s\in[R,\infty)$, by (2.27), (2.28) and (2.26), we have

$$s^\gamma|\mathrm{w}(s)| \leq s\int_s^\infty \frac{1}{x^2}\left(\int_x^\infty \xi\,\exp\left(-\frac{1}{4}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},0\right)}\left(\xi^2-s^2\right)\right)\xi^\gamma|\boldsymbol{\eta}(\xi)|\,d\xi\right)dx$$

$$\leq s\int_s^\infty \frac{1}{x^2}\,2\frac{\partial_n f\left(\overrightarrow{1},0\right)}{1+\sigma^2}\left(\sup_{\xi\geq s}\left(\xi^\gamma|\boldsymbol{\eta}(\xi)|\right)\right)dx \tag{2.30}$$

$$= 2\frac{\partial_n f\left(\overrightarrow{1},0\right)}{1+\sigma^2}\sup_{\xi\geq R}\left(\xi^\gamma|\boldsymbol{\eta}(\xi)|\right)$$

and

$$s^{\gamma+1}|\partial_s\mathrm{w}(s)| \leq s\int_s^\infty \frac{1}{x^2}\left(\int_x^\infty \xi\,\exp\left(-\frac{1}{4}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},0\right)}\left(\xi^2-s^2\right)\right)\xi^\gamma|\boldsymbol{\eta}(\xi)|\,d\xi\right)dx$$

$$+\int_s^\infty \xi\,\exp\left(-\frac{1}{4}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},0\right)}\left(\xi^2-s^2\right)\right)\xi^\gamma|\boldsymbol{\eta}(\xi)|\,d\xi \tag{2.31}$$

$$\leq 4\frac{\partial_n f\left(\overrightarrow{1},0\right)}{1+\sigma^2}\sup_{\xi\geq R}\left(\xi^\gamma|\boldsymbol{\eta}(\xi)|\right)$$

For (2.21), we can get it from (2.17), (2.18), (2.30) and (2.31) as follows:

$$s^\gamma|\partial_s^2\mathrm{w}(s)| = s^\gamma\left|\frac{1}{2}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},0\right)}\left(s\,\partial_s\mathrm{w}\,(s)-\mathrm{w}\,(s)\right)+\boldsymbol{\eta}(s)\right|$$

$$\leq \frac{1}{2} \frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},\, 0\right)} \left(s^{\gamma+1}|\partial_s \mathrm{w}(s)| \,+\, s^{\gamma}|\mathrm{w}(s)|\right) \,+\, s^{\gamma}|\boldsymbol{\eta}(s)|$$

$$\leq\, 4 \sup_{\xi \geq R} \left(\xi^{\gamma}|\boldsymbol{\eta}(\xi)|\right)$$

Lastly, from (2.29), we can see that $\mathrm{w} \in C_0^{m+2}[R,\, \infty)$ as long as $\boldsymbol{\eta} \in C_0^{m}[R,\, \infty)$ for $m \in \mathbb{N}$. □

Next, let's consider a normed space which we are going to work with. Fix $R > 0$, define a norm

$$\| \, v \, \| \equiv \max\Big\{ \, \| \, s\, v \, \|_{L^\infty[R,\,\infty)},\ \| \, s^2 \partial_s v \, \|_{L^\infty[R,\,\infty)},\ \cdots,\ \| \, s^{\mathsf{k}} \partial_s^{\mathsf{k}-1} v \, \|_{L^\infty[R,\,\infty)}$$

$$,\ \| \, s^{\mathsf{k}} \partial_s^{\mathsf{k}} v \, \|_{L^\infty[R,\,\infty)} \, \Big\} \tag{2.32}$$

and a vector space

$$\Im = \Big\{ v \in C_0^{\mathsf{k}}[R,\, \infty) \Big| \ \| \, v \, \| < \infty \Big\} \tag{2.33}$$

Note that $v \in \Im$ if and only if $v \in C_0^{\mathsf{k}}[R,\, \infty)$ and it decays at infinity with the following rate:

$$v = O\left(s^{-1}\right),\ \partial_s v = O\left(s^{-2}\right), \cdots,\ \partial_s^{\mathsf{k}-1} v = O\left(s^{-\mathsf{k}}\right),$$

$$\partial_s^{\mathsf{k}-1} v = O\left(s^{-\mathsf{k}}\right) \quad \text{as } s \nearrow \infty$$

For instance, $s^{-1} \in \Im$. Also, $\Im$ with the norm $\| \cdot \|$ is a Banach space. In the second lemma, we estimate $\mathcal{Q}v$ in (2.14) for $v \in \Im$.

Lemma 2.3. *Given $M > 0$, there is $R = R\,(\mathcal{C},\, \mathcal{K},\, M) \geq 1$ such that for any $v \in \Im$ with $\| \, v \, \| \leq M$, we have*

$$\omega\,(v,\, \theta) \in \mathcal{K} \quad \forall\, \theta \in [0,\, 1] \tag{2.34}$$

and $\mathcal{Q}v \in C_0^{\mathsf{k}-2}[R,\, \infty)$ satisfing

$$\max\Big\{ \, \| \, s^5 \mathcal{Q}v \, \|_{L^\infty[R,\,\infty)},\ \| \, s^6 \partial_s \mathcal{Q}v \, \|_{L^\infty[R,\,\infty)}, \cdots,\ \| \, s^{\mathsf{k}+2} \partial_s^{\mathsf{k}-3} \mathcal{Q}v \, \|_{L^\infty[R,\,\infty)}$$

$$,\ \| \, s^{\mathsf{k}+2} \partial_s^{\mathsf{k}-2} \mathcal{Q}v \, \|_{L^\infty[R,\,\infty)} \, \Big\} \,\leq\, C\left(n,\, \mathsf{k},\, \mathcal{C},\, \| \, f \, \|_{C^{\mathsf{k}}(\mathcal{K})},\, M\right) \tag{2.35}$$

Moreover, if we take $\tilde{v} \in \Im$ with $\| \tilde{v} \| \leq M$, there holds

$$\max \Big\{ \| s^5 \left(\mathcal{Q}v - \mathcal{Q}\tilde{v} \right) \|_{L^\infty[R,\,\infty)},\ \| s^6 \left(\partial_s \mathcal{Q}v - \partial_s \mathcal{Q}\tilde{v} \right) \|_{L^\infty[R,\,\infty)}, \cdots ,\ \| s^{\mathsf{k}+2} \left(\partial_s^{\mathsf{k}-3} \mathcal{Q}v - \partial_s^{\mathsf{k}-3} \mathcal{Q}\tilde{v} \right) \|_{L^\infty[R$$

$$(2.36)$$

$$,\ \| s^{\mathsf{k}+2} \left(\partial_s^{\mathsf{k}-2} \mathcal{Q}v - \partial_s^{\mathsf{k}-2} \mathcal{Q}\tilde{v} \right) \|_{L^\infty[R,\,\infty)} \Big\} \leq C \left(n,\, \mathsf{k},\, \mathcal{C},\, \| f \|_{C^{\mathsf{k}+1}(\mathcal{K})},\, M \right) \| v - \tilde{v} \|$$

Note that the Banach space $(\Im, \| \cdot \|)$ is defined in (2.32, 2.33), $\omega(v,\, \theta)$ is defined by (2.15) and $\mathcal{Q}v$ is defined by (2.14).

Proof. Let $R \geq 1$ be a constant to be determined. For each $\Lambda \geq 2R$, consider the following change of variables:

$$s = \Lambda \xi \tag{2.37}$$

$$\overleftrightarrow{v}(\xi) = \Lambda\, v\,(\Lambda \xi) \tag{2.38}$$

for $\xi \in \left[\frac{1}{2},\, 1\right]$.

From (2.32), the condition $\| v \| \leq M$ implies that

$$\| \overleftrightarrow{v} \|_{C^{\mathsf{k}-1}\left[\frac{1}{2},\,1\right]} \equiv \max \left\{ \| \overleftrightarrow{v} \|_{L^\infty\left[\frac{1}{2},\,1\right]}, \cdots ,\ \| \partial_\xi^{\mathsf{k}-1} \overleftrightarrow{v} \|_{L^\infty\left[\frac{1}{2},\,1\right]} \right\} \leq M$$

$$\| \partial_\xi^{\mathsf{k}} \overleftrightarrow{v} \|_{L^\infty\left[\frac{1}{2},\,1\right]} \leq M\Lambda \tag{2.39}$$

And (2.15) is translated into

$$\omega\left(v\left(s\right),\, \theta\right) = \left(\frac{1}{1 + \Lambda^{-2}\theta \frac{\overleftrightarrow{v}}{\sigma\xi}} \vec{1},\ \frac{-2\Lambda^{-2}\theta\sigma\xi\, \partial_\xi^2 \overleftrightarrow{v}}{1 + \left(\sigma + \Lambda^{-2}\theta\, \partial_\xi \overleftrightarrow{v}\right)^2} \right) \equiv \overleftrightarrow{\omega}\left(\overleftrightarrow{v}\left(\xi\right),\, \theta\right) \tag{2.40}$$

Thus, by (2.40) and (2.39), we can verify (2.34) by choosing $R \gg 1$ (depending on $\mathcal{C},\, \mathcal{K},\, M$).

Also, let's consider the following change of variables:

$$\overleftrightarrow{\mathfrak{z}}\left(\overleftrightarrow{v}\left(\xi\right),\, \theta\right) \equiv \Lambda^{-1} \mathfrak{z}\left(v\left(s\right),\, \theta\right) = \sigma\xi + \Lambda^{-2}\theta\, \overleftrightarrow{v}\left(\xi\right) \tag{2.41}$$

$$\overleftrightarrow{p}\left(\overleftrightarrow{v}\left(\xi\right),\, \theta\right) \equiv \mathfrak{p}\left(v\left(s\right),\, \theta\right) = \sigma + \Lambda^{-2}\theta\, \partial_\xi \overleftrightarrow{v} \tag{2.42}$$

$$\overleftrightarrow{q}\left(\overleftrightarrow{v}\left(\xi\right),\, \theta\right) \equiv \Lambda^3 \mathfrak{q}\left(v\left(s\right),\, \theta\right) = \theta\, \partial_\xi^2 \overleftrightarrow{v} \tag{2.43}$$

$$\overleftrightarrow{\mathcal{Q}v}(\xi) = \Lambda^5\, \mathcal{Q}v\left(\Lambda\xi\right) \tag{2.44}$$

From (2.14), we can write out (2.44) in terms of (2.40), (2.41), (2.42) and (2.43) as

$$\overleftrightarrow{\mathcal{Q}v}(\xi) = \left(\int_0^1 \frac{\sigma\xi\, \partial^2_{nn} f \circ \overleftrightarrow{\omega}}{\left(1 + \overleftrightarrow{p}^2\right)^2} \, (1-\theta)\, d\theta \right) \left(\partial^2_\xi \overleftrightarrow{v}\right)^2 \tag{2.45}$$

$$+ \left(\int_0^1 \left(\frac{\sigma\xi\, \partial^2_{nn} f \circ \overleftrightarrow{\omega}}{\Lambda^4} \frac{4\overleftrightarrow{q}^2 \overleftrightarrow{p}^2}{\left(1 + \overleftrightarrow{p}^2\right)^4} + \frac{\partial_n f \circ \overleftrightarrow{\omega}}{\Lambda^2} \frac{2\overleftrightarrow{q}\left(1 - 3\overleftrightarrow{p}^2\right)}{\left(1 + \overleftrightarrow{p}^2\right)^3} \right) (1-\theta)\, d\theta \right) \left(\partial_\xi \overleftrightarrow{v}\right)^2$$

$$+ \left(\int_0^1 \frac{\left(\sum_{i,\,j=1}^{n-1} \sigma\xi\, \partial^2_{ij} f \circ \overleftrightarrow{\omega}\right) + 2\overleftrightarrow{\mathfrak{z}} \sum_{i=1}^{n-1} \partial_i f \circ \overleftrightarrow{\omega}}{\left(\overleftrightarrow{\mathfrak{z}}\right)^4} \, (1-\theta)\, d\theta \right) \left(\overleftrightarrow{v}\right)^2$$

$$+2 \left(\int_0^1 \left(\frac{\sigma\xi\, \partial^2_{nn} f \circ \overleftrightarrow{\omega}}{\Lambda^2} \frac{-2\overleftrightarrow{q}\,\overleftrightarrow{p}}{\left(1 + \overleftrightarrow{p}^2\right)^3} + \partial_n f \circ \overleftrightarrow{\omega} \frac{2\overleftrightarrow{p}}{\left(1 + \overleftrightarrow{p}^2\right)^2} \right) (1-\theta)\, d\theta \right) \partial^2_\xi \overleftrightarrow{v}\, \partial_\xi \overleftrightarrow{v}$$

$$+2 \left(\int_0^1 \sum_{i=1}^{n-1} \frac{\sigma\xi\, \partial^2_{ni} f \circ \overleftrightarrow{\omega}}{\left(1 + \overleftrightarrow{p}^2\right) \left(\overleftrightarrow{\mathfrak{z}}\right)^2} \, (1-\theta)\, d\theta \right) \overleftrightarrow{v}\, \partial^2_\xi \overleftrightarrow{v}$$

$$- 2 \left(\int_0^1 \sum_{i=1}^{n-1} \frac{\sigma\xi\, \partial^2_{ni} f \circ \overleftrightarrow{\omega}}{\Lambda^2} \frac{2\overleftrightarrow{q}\,\overleftrightarrow{p}}{\left(1 + \overleftrightarrow{p}^2\right)^2 \left(\overleftrightarrow{\mathfrak{z}}\right)^2} \, (1-\theta)\, d\theta \right) \overleftrightarrow{v}\, \partial_\xi \overleftrightarrow{v}$$

Note that from (2.45), (2.40), (2.34), (2.41), (2.42), (2.43), (2.38) and (2.37), we have

$$\overleftrightarrow{\mathcal{Q}v} \in C^{\mathsf{k}-2} \left[\frac{1}{2},\, 1\right]$$

and by (2.39) it satisfies

$$\| \overleftrightarrow{\mathcal{Q}v} \|_{C^{\mathsf{k}-3}\left[\frac{1}{2},1\right]} \le C\left(n,\, \mathsf{k},\, \mathcal{C},\, \| f \|_{C^{\mathsf{k}-1}(\mathcal{K})},\, M\right) \tag{2.46}$$

$$\| \partial^{\mathsf{k}-2}_\xi \overleftrightarrow{\mathcal{Q}v} \|_{L^\infty\left[\frac{1}{2},1\right]} \le C\left(n,\, \mathsf{k},\, \mathcal{C},\, \| f \|_{C^{\mathsf{k}}(\mathcal{K})},\, M\right) \Lambda \tag{2.47}$$

Note that in (2.47) we have used the fact that $\mathsf{k} \ge 3$ and $\partial^{\mathsf{k}-2}_\xi \overleftrightarrow{\mathcal{Q}v}$ is linear in $\partial^{\mathsf{k}}_\xi \overleftrightarrow{v}$. Similarly, we can define $\overleftrightarrow{\tilde{v}}$ and $\overleftrightarrow{\mathcal{Q}\tilde{v}}$ in the same fashion and then use the mean value theorem to get

$$\| \overleftrightarrow{\mathcal{Q}v} - \overleftrightarrow{\mathcal{Q}\tilde{v}} \|_{C^{\mathsf{k}-3}\left[\frac{1}{2},1\right]} \le C\left(n,\, \mathsf{k},\, \mathcal{C},\, \| f \|_{C^{\mathsf{k}}(\mathcal{K})},\, M\right) \| \overleftrightarrow{v} - \overleftrightarrow{\tilde{v}} \|_{C^{\mathsf{k}-1}\left[\frac{1}{2},1\right]} \tag{2.48}$$

$$\| \partial^{\mathsf{k}-2}_\xi \overleftrightarrow{\mathcal{Q}v} - \partial^{\mathsf{k}-2}_\xi \overleftrightarrow{\mathcal{Q}\tilde{v}} \|_{L^\infty\left[\frac{1}{2},1\right]} \tag{2.49}$$

$$\leq C\left(n,\,\mathsf{k},\,\mathcal{C},\,\|\, f\,\|_{C^{\mathsf{k}+1}(\mathcal{K})},\,M\right)\left(\|\,\partial_\xi^{\mathsf{k}}\overleftrightarrow{v}-\partial_\xi^{\mathsf{k}}\overleftrightarrow{\tilde{v}}\,\|_{L^\infty\left[\frac{1}{2},1\right]}+\Lambda\,\|\,\overleftrightarrow{v}-\overleftrightarrow{\tilde{v}}\,\|_{C^{\mathsf{k}-1}\left[\frac{1}{2},1\right]}\right)$$

Finally, undoing change of variables for (2.46), (2.47), (2.48) and (2.49) leads to the conclusion. □

Now we are ready to show the existence of the problem (2.16), (2.9), which yields the solution to (2.5), (2.7) via the formula (2.8).

Theorem 2.4. *There exists* $R=R\left(n,\,\mathsf{k},\,\mathcal{C},\,\mathcal{K},\,\|\,f\,\|_{C^{\mathsf{k}+1}(\mathcal{K})}\right)\geq 1$ *and* $u\in\Im$ *such that*

$$\mathcal{L}u=\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},\,0\right)}\left(\frac{f\left(\overrightarrow{1},\,0\right)}{\sigma s}-\sum_{i=1}^{n-1}\frac{\partial_i f\left(\overrightarrow{1},\,0\right)}{\sigma^2 s^2}u+\,\mathcal{Q}u\right)\qquad on\ [R,\,\infty)$$

where $\Im$ *is a subspace of* $C_0^{\mathsf{k}}[R,\,\infty)$ *defined in (2.33),* $\mathcal{L}$ *is the linear differential operator defined in (2.17) and* $\mathcal{Q}$ *is a nonlinear operator defined by (2.14). It follows that* $r\,(s)=\sigma s+u\,(s)$ *solves*

$$f\left(\frac{1}{r}\overrightarrow{1},\,\frac{-\partial_s^2 r}{1+(\partial_s r)^{\,2}}\right)+\,\frac{1}{2}\,(s\,\partial_s r-r)=0\quad on\ [R,\,\infty)$$

$$\varrho\, r\left(\frac{s}{\varrho}\right)-\sigma s\xrightarrow{C^{\mathsf{k}}_{loc}}0\quad as\ \varrho\searrow 0$$

Moreover, we have the asymptotic formula $u\,(s)=\frac{f\left(\overrightarrow{1},0\right)}{\sigma s}+\mathsf{u}(s)$ *with the error term* $\mathsf{u}\in C_0^{\mathsf{k}}[R,\,\infty)$ *satisfying*

$$\|\,s^3\,\mathsf{u}\,\|_{L^\infty[R,\,\infty)}+\|\,s^4\partial_s\mathsf{u}\,\|_{L^\infty[R,\,\infty)}+\cdots+\|\,s^{\mathsf{k}+2}\partial_s^{\mathsf{k}-1}\mathsf{u}\,\|_{L^\infty[R,\,\infty)}\leq C\left(n,\,\mathsf{k},\,\mathcal{C},\,\|\,f\,\|_{C^{\mathsf{k}}(\mathcal{K})}\right)$$

$$\|\,s^{\mathsf{k}+1}\partial_s^{\mathsf{k}}\mathsf{u}\,\|_{L^\infty[R,\,\infty)}\leq C\left(n,\,\mathsf{k},\,\mathcal{C},\,\mathcal{K},\,\|\,f\,\|_{C^{\mathsf{k}}(\mathcal{K})}\right)$$

Remark 2.5. We find the asymptotic formula $u\,(s)\simeq\frac{f\left(\overrightarrow{1},0\right)}{\sigma s}$ by doing iteration of (2.5), (2.7). More precisely, let $r_0\,(s)=\sigma s$ and define $r_1\,(s)$ to be the solution of

$$f\left(\frac{1}{r_0}\overrightarrow{1},\,\frac{-\partial_s^2 r_0}{1+(\partial_s r_0)^{\,2}}\right)+\,\frac{1}{2}\,(s\,\partial_s r_1-r_1)=0$$

$$\varrho\, r_1\left(\frac{s}{\varrho}\right)-\sigma s\xrightarrow{C^{\mathsf{k}}_{\mathrm{loc}}}0\quad\text{as }\varrho\searrow 0$$

Then $r_1\,(s)=\sigma s+\frac{f\left(\overrightarrow{1},0\right)}{\sigma s}$.

Proof. Let $M > 0$ and $R \geq 1$ be constants to be chosen, and take R sufficiently large (depending on $\mathcal{C}$, $\mathcal{K}$, M) so that Lemma 2.3 can be verified. Let $\boldsymbol{B} = \left\{ v \in \Im \middle| \parallel v \parallel \leq M \right\}$ be the closed ball of radius M in the Banach space $(\Im, \parallel \cdot \parallel)$ defined in (2.32), (2.33).

By Lemma 29, for each $v \in \boldsymbol{B}$, we can define $\mathcal{F}v$ to be the unique solution to the following problem:

$$\mathcal{L}\left(\mathcal{F}v\right) = \frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},\, 0\right)} \left(\frac{f\left(\overrightarrow{1},\, 0\right)}{\sigma s} - \sum_{i=1}^{n-1} \frac{\partial_i f\left(\overrightarrow{1},\, 0\right)}{\sigma^2 s^2} v + \mathcal{Q}v \right) \quad \text{on } [R,\, \infty) \tag{2.50}$$

$$\frac{\mathcal{F}v(s)}{s} \;\&\; \left(s\,\partial_s \mathcal{F}v - \mathcal{F}v\right) \to 0 \quad \text{as } s \nearrow \infty \tag{2.51}$$

Since $\mathcal{Q}v \in C_0^{\mathsf{k}-2}[R,\, \infty)$, $\mathcal{F}$ maps $\boldsymbol{B}$ into $C_0^{\mathsf{k}}[R,\, \infty)$. In fact, we would show that $\mathcal{F}v \in \boldsymbol{B}$ and $\mathcal{F}$ is a contraction on $\boldsymbol{B}$ if we choose M and R appropriately.

First of all, let's consider

$$\mathsf{v}(s) = \mathcal{F}v(s) - \frac{f\left(\overrightarrow{1},\, 0\right)}{\sigma s} \tag{2.52}$$

Then we have $\mathsf{v} \in C_0^{\mathsf{k}}[R,\, \infty)$ (since $\mathcal{F}v \in C_0^{\mathsf{k}}[R,\, \infty)$). Also, by plgging (2.52) into (2.50, 2.51), we get

$$\mathcal{L}\mathsf{v} = -2\frac{f\left(\overrightarrow{1},\, 0\right)}{\sigma s^3} + \frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},\, 0\right)} \left(-\sum_{i=1}^{n-1} \frac{\partial_i f\left(\overrightarrow{1},\, 0\right)}{\sigma^2} \frac{v}{s^2} + \mathcal{Q}v \right) \tag{2.53}$$

$$\frac{\mathsf{v}}{s} \;\&\; \left(s\,\partial_s \mathsf{v} - \mathsf{v}\right) \to 0 \quad \text{as } s \nearrow \infty \tag{2.54}$$

By Lemma 29 (with $\gamma = 3$) and Lemma 30, (2.53, 2.54) implies that

$$\max\left\{ \parallel s^3 \mathsf{v} \parallel_{L^\infty[R,\, \infty)},\, \parallel s^4 \partial_s \mathsf{v} \parallel_{L^\infty[R,\, \infty)} \right\} \tag{2.55}$$

$$\leq\; 8\frac{\partial_n f\left(\overrightarrow{1},\, 0\right)}{1+\sigma^2} \frac{f\left(\overrightarrow{1},\, 0\right)}{\sigma} + 4\sum_{i=1}^{n-1} \frac{\partial_i f\left(\overrightarrow{1},\, 0\right)}{\sigma^2} \parallel s\, v \parallel_{L^\infty[R,\, \infty)} + 4\frac{\parallel s^5 \mathcal{Q}v \parallel_{L^\infty[R,\, \infty)}}{R^2}$$

$$\leq C\left(n,\, \mathsf{k},\, \mathcal{C},\, \parallel f \parallel_{C^{\mathsf{k}}(\mathcal{K})},\, M\right)$$

$$\parallel s^3 \partial_s^2 \mathsf{v} \parallel_{L^\infty[R,\, \infty)} \leq C\left(n,\, \mathsf{k},\, \mathcal{C},\, \parallel f \parallel_{C^{\mathsf{k}}(\mathcal{K})},\, M\right) \tag{2.56}$$

Besides, if we take $\tilde{v} \in \boldsymbol{\mathcal{B}}$ and define $\tilde{\mathsf{v}} = \mathcal{F}\tilde{v} - \frac{f\left(\overrightarrow{1}, 0\right)}{\sigma s}$, then similarly we have

$$\mathcal{L}\tilde{\mathsf{v}} = -2\frac{f\left(\overrightarrow{1}, 0\right)}{\sigma s^3} + \frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1}, 0\right)}\left(-\sum_{i=1}^{n-1}\frac{\partial_i f\left(\overrightarrow{1}, 0\right)}{\sigma^2}\frac{\tilde{v}}{s^2} + \mathcal{Q}\tilde{v}\right) \tag{2.57}$$

$$\frac{\tilde{\mathsf{v}}}{s} \;\&\; \left(s\,\partial_s\tilde{\mathsf{v}} - \tilde{\mathsf{v}}\right) \to 0 \quad \text{as } s \nearrow \infty \tag{2.58}$$

By doing subtraction of (2.53, 2.54) with (2.57, 2.58), we get

$$\mathcal{L}\left(\mathsf{v} - \tilde{\mathsf{v}}\right) = \frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1}, 0\right)}\left(-\sum_{i=1}^{n-1}\frac{\partial_i f\left(\overrightarrow{1}, 0\right)}{\sigma^2}\frac{v-\tilde{v}}{s^2} + \left(\mathcal{Q}v - \mathcal{Q}\tilde{v}\right)\right) \tag{2.59}$$

$$\frac{\mathsf{v}-\tilde{\mathsf{v}}}{s} \;\&\; \left\{\,s\,\partial_s\left(\mathsf{v}-\tilde{\mathsf{v}}\right) - \left(\mathsf{v}-\tilde{\mathsf{v}}\right)\right\} \to 0 \quad \text{as } s \nearrow \infty \tag{2.60}$$

which yields, by Lemma 2.2 (with $\gamma = 3$) and Lemma 2.3, that

$$\max\left\{\| \, s^3\left(\mathsf{v}-\tilde{\mathsf{v}}\right) \|_{L^\infty[R,\infty)},\ \| \, s^4\left(\partial_s\mathsf{v} - \partial_s\tilde{\mathsf{v}}\right) \|_{L^\infty[R,\infty)}\right\} \tag{2.61}$$

$$\leq\ 4\sum_{i=1}^{n-1}\frac{\partial_i f\left(\overrightarrow{1}, 0\right)}{\sigma^2}\ \| \, s\left(v-\tilde{v}\right) \|_{L^\infty[R,\infty)} + 4\frac{\| \, s^5\left(\mathcal{Q}v - \mathcal{Q}\tilde{v}\right) \|_{L^\infty[R,\infty)}}{R^2}$$

$$\leq\ C\left(n,\, \mathsf{k},\, \mathcal{C},\, \| \, f \, \|_{C^{\mathsf{k}+1}(\mathcal{K})},\, M\right)\ \| \, v - \tilde{v} \, \|$$

$$\| \, s^3\left(\partial_s^2\mathsf{v} - \partial_s^2\tilde{\mathsf{v}}\right) \|_{L^\infty[R,\infty)} \leq\ C\left(n,\, \mathsf{k},\, \mathcal{C},\, \| \, f \, \|_{C^{\mathsf{k}+1}(\mathcal{K})},\, M\right)\ \| \, v - \tilde{v} \, \| \tag{2.62}$$

Next, differentiate (2.53), (2.59) and use the formula

$$\partial_s\mathcal{L} - \mathcal{L}\partial_s = -\frac{1}{2}\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1}, 0\right)}\partial_s \tag{2.63}$$

and also (2.55), (2.56), (2.61), (2.62) to get

$$\mathcal{L}\left(\partial_s\mathsf{v}\right) = 6\frac{f\left(\overrightarrow{1}, 0\right)}{\sigma s^4} + \frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1}, 0\right)}\left(\frac{1}{2}\partial_s\mathsf{v} - \sum_{i=1}^{n-1}\frac{\partial_i f\left(\overrightarrow{1}, 0\right)}{\sigma^2}\partial_s\left(\frac{v}{s^2}\right) + \partial_s\mathcal{Q}v\right) \tag{2.64}$$

$$\frac{\partial_s\mathsf{v}}{s} \;\&\; \left(s\partial_s^2\mathsf{v} - \partial_s\mathsf{v}\right) \to 0 \quad \text{as } s \nearrow \infty \tag{2.65}$$

$$\mathcal{L}\left(\partial_s \mathsf{v} - \partial_s \tilde{\mathsf{v}}\right) = \frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},\, 0\right)} \left(\frac{1}{2}\left(\partial_s \mathsf{v} - \partial_s \tilde{\mathsf{v}}\right) - \sum_{i=1}^{n-1} \frac{\partial_i f\left(\overrightarrow{1},\, 0\right)}{\sigma^2} \partial_s \left(\frac{v-\tilde{v}}{s^2}\right) + \partial_s \left(\mathcal{Q}v - \mathcal{Q}\tilde{v}\right) \right) \tag{2.66}$$

$$\frac{\partial_s \mathsf{v} - \partial_s \tilde{\mathsf{v}}}{s} \;\&\; \left\{ s\left(\partial_s^2 \mathsf{v} - \partial_s^2 \tilde{\mathsf{v}} -\right) - \left(\partial_s \mathsf{v} - \partial_s \tilde{\mathsf{v}}\right) \right\} \to 0 \quad \text{as } s \nearrow \infty \tag{2.67}$$

By the gradient estimates in Lemma 2.2 (with $\gamma = 4$) Lemma 2.3, (2.55) and (2.61), (2.64, 2.65) and (2.66, 2.67) yield that

$$\| \, s^5 \partial_s^2 \mathsf{v} \, \|_{L^\infty[R,\,\infty)} \le \left\{ 24 \frac{\partial_n f\left(\overrightarrow{1},\, 0\right)}{1+\sigma^2} \frac{f\left(\overrightarrow{1},\, 0\right)}{\sigma} + 2 \, \| \, s^4 \partial_s \mathsf{v} \, \|_{L^\infty[R,\,\infty)} + 4 \sum_{i=1}^{n-1} \frac{\partial_i f\left(\overrightarrow{1},\, 0\right)}{\sigma^2} \, \| \, s^2 \partial_s v \, \|_{L^\infty[R,} \right. \tag{2.68}$$

$$\left. + 8 \sum_{i=1}^{n-1} \frac{\partial_i f\left(\overrightarrow{1},\, 0\right)}{\sigma^2} \, \| \, s \, v \, \|_{L^\infty[R,\,\infty)} + 4 \frac{\| \, s^6 \partial_s \mathcal{Q} v \, \|_{L^\infty[R,\,\infty)}}{R^2} \right\}$$

$$\le C\left(n,\, \mathsf{k},\, \mathcal{C},\, \| \, f \, \|_{C^{\mathsf{k}}(\mathcal{K})},\, M\right)$$

$$\| \, s^4 \partial_s^3 \mathsf{v} \, \|_{L^\infty[R,\,\infty)} \le C\left(n,\, \mathsf{k},\, \mathcal{C},\, \| \, f \, \|_{C^{\mathsf{k}}(\mathcal{K})},\, M\right) \tag{2.69}$$

$$\| \, s^5 \left(\partial_s^2 \mathsf{v} - \partial_s^2 \tilde{\mathsf{v}}\right) \|_{L^\infty[R,\,\infty)} \le \left\{ 2 \, \| \, s^4 \left(\partial_s \mathsf{v} - \partial_s \tilde{\mathsf{v}}\right) \|_{L^\infty[R,\,\infty)} + 4 \sum_{i=1}^{n-1} \frac{\partial_i f\left(\overrightarrow{1},\, 0\right)}{\sigma^2} \, \| \, s^2 \left(\partial_s v - \partial_s \tilde{v}\right) \|_{L^\infty[R,} \right. \tag{2.70}$$

$$\left. + 8 \sum_{i=1}^{n-1} \frac{\partial_i f\left(\overrightarrow{1},\, 0\right)}{\sigma^2} \, \| \, s\left(v - \tilde{v}\right) \|_{L^\infty[R,\,\infty)} + 4 \frac{\| \, s^6 \left(\partial_s \mathcal{Q} v - \partial_s \mathcal{Q} \tilde{v}\right) \|_{L^\infty[R,\,\infty)}}{R^2} \right\}$$

$$\le C\left(n,\, \mathsf{k},\, \mathcal{C},\, \| \, f \, \|_{C^{\mathsf{k}+1}(\mathcal{K})},\, M\right) \| \, v - \tilde{v} \, \|$$

$$\| \, s^4 \left(\partial_s^3 \tilde{\mathsf{v}} - \partial_s^3 \mathsf{v}\right) \|_{L^\infty[R,\,\infty)} \le C\left(n,\, \mathsf{k},\, \mathcal{C},\, \| \, f \, \|_{C^{\mathsf{k}+1}(\mathcal{K})},\, M\right) \| \, v - \tilde{v} \, \| \tag{2.71}$$

Continue this process until we arrive at

$$\mathcal{L}\left(\partial_s^{\mathsf{k}-2} \mathsf{v}\right) = \partial_s^{\mathsf{k}-2} \left(-2 \frac{f\left(\overrightarrow{1},\, 0\right)}{\sigma s^3}\right) + \frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},\, 0\right)} \left(\frac{\mathsf{k}-2}{2} \partial_s^{\mathsf{k}-2} \mathsf{v} - \sum_{i=1}^{n-1} \frac{\partial_i f\left(\overrightarrow{1},\, 0\right)}{\sigma^2} \partial_s^{\mathsf{k}-2} \left(\frac{v}{s^2}\right) + \partial_s^{\mathsf{k}-2} \mathcal{Q} \right. \tag{2.72}$$

$$\frac{\partial_s^{\mathsf{k}-2}\mathsf{v}}{s} \;\&\; \left(s\partial_s^{\mathsf{k}-1}\mathsf{v} - \partial_s^{\mathsf{k}-2}\mathsf{v}\right) \to 0 \quad \text{as } s \nearrow \infty \tag{2.73}$$

$$\mathcal{L}\left(\partial_s^{\mathsf{k}-2}\mathsf{v} - \partial_s^{\mathsf{k}-2}\tilde{\mathsf{v}}\right) = \frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},\,0\right)}\left(\frac{\mathsf{k}-2}{2}\left(\partial_s^{\mathsf{k}-2}\mathsf{v} - \partial_s^{\mathsf{k}-2}\tilde{\mathsf{v}}\right) - \sum_{i=1}^{n-1}\frac{\partial_i f\left(\overrightarrow{1},\,0\right)}{\sigma^2}\partial_s^{\mathsf{k}-2}\left(\frac{v-\tilde{v}}{s^2}\right)\right) \tag{2.74}$$
$$+\frac{1+\sigma^2}{\partial_n f\left(\overrightarrow{1},\,0\right)}\partial_s^{\mathsf{k}-2}\left(\mathcal{Q}v - \mathcal{Q}\tilde{v}\right)$$

$$\frac{\partial_s^{\mathsf{k}-2}\mathsf{v} - \partial_s^{\mathsf{k}-2}\tilde{\mathsf{v}}}{s} \;\&\; \left\{s\left(\partial_s^{\mathsf{k}-1}\mathsf{v} - \partial_s^{\mathsf{k}-1}\tilde{\mathsf{v}}\right) - \left(\partial_s^{\mathsf{k}-2}\mathsf{v} - \partial_s^{\mathsf{k}-2}\tilde{\mathsf{v}}\right)\right\} \to 0 \quad \text{as } s \nearrow \infty \tag{2.75}$$

and also

$$\parallel s^{\mathsf{k}+1}\partial_s^{\mathsf{k}-2}\mathsf{v} \parallel_{L^\infty[R,\,\infty)} \leq C\left(n,\,\mathsf{k},\,\mathcal{C},\,\parallel f \parallel_{C^{\mathsf{k}}(\mathcal{K})},\,M\right) \tag{2.76}$$

$$\parallel s^{\mathsf{k}+1}\left(\partial_s^{\mathsf{k}-2}\mathsf{v} - \partial_s^{\mathsf{k}-2}\tilde{\mathsf{v}}\right) \parallel_{L^\infty[R,\,\infty)} \leq C\left(n,\,\mathsf{k},\,\mathcal{C},\,\parallel f \parallel_{C^{\mathsf{k}+1}(\mathcal{K})},\,M\right) \parallel v - \tilde{v} \parallel \tag{2.77}$$

By the gradient estimates in Lemma 2.2 (with $\gamma = k+1$), Lemma 2.3, (2.76) and (2.77), (2.72, 2.73) and (2.74, 2.75) imply that

$$\parallel s^{\mathsf{k}+2}\partial_s^{\mathsf{k}-1}\mathsf{v} \parallel_{L^\infty[R,\,\infty)} \leq \left\{8\frac{\partial_n f\left(\overrightarrow{1},\,0\right)}{1+\sigma^2} \parallel s^{\mathsf{k}+1}\partial_s^{\mathsf{k}-2}\left(\frac{f\left(\overrightarrow{1},\,0\right)}{\sigma s^3}\right) \parallel_{L^\infty[R,\,\infty)}\right. \tag{2.78}$$
$$+2\left(\mathsf{k}-2\right) \parallel s^{\mathsf{k}+1}\partial_s^{\mathsf{k}-2}\mathsf{v} \parallel_{L^\infty[R,\,\infty)} + 4\sum_{i=1}^{n-1}\frac{\partial_i f\left(\overrightarrow{1},\,0\right)}{\sigma^2} \parallel s^{\mathsf{k}+1}\partial_s^{\mathsf{k}-2}\left(\frac{v}{s^2}\right) \parallel_{L^\infty[R,\,\infty)}$$
$$\left.+4\frac{\parallel s^{\mathsf{k}+2}\partial_s^{\mathsf{k}-2}\mathcal{Q}v \parallel_{L^\infty[R,\,\infty)}}{R}\right\}$$
$$\leq C\left(n,\,\mathsf{k},\,\mathcal{C},\,\parallel f \parallel_{C^{\mathsf{k}}(\mathcal{K})},\,M\right)$$

$$\parallel s^{\mathsf{k}+1}\partial_s^{\mathsf{k}}\mathsf{v} \parallel_{L^\infty[R,\,\infty)} \leq C\left(n,\,\mathsf{k},\,\mathcal{C},\,\parallel f \parallel_{C^{\mathsf{k}}(\mathcal{K})},\,M\right) \tag{2.79}$$

$$\parallel s^{\mathsf{k}+2}\left(\partial_s^{\mathsf{k}-1}\mathsf{v} - \partial_s^{\mathsf{k}-1}\tilde{\mathsf{v}}\right) \parallel_{L^\infty[R,\,\infty)} \leq \left\{2\left(\mathsf{k}-2\right) \parallel s^{\mathsf{k}+1}\left(\partial_s^{\mathsf{k}-2}\mathsf{v} - \partial_s^{\mathsf{k}-2}\tilde{\mathsf{v}}\right) \parallel_{L^\infty[R,\,\infty)}\right. \tag{2.80}$$

$$+4\sum_{i=1}^{n-1}\frac{\partial_i f\left(\overrightarrow{1},0\right)}{\sigma^2}\parallel s^{\mathsf{k}+1}\partial_s^{\mathsf{k}-2}\left(\frac{v-\tilde{v}}{s^2}\right)\parallel_{L^\infty[R,\infty)}+4\frac{\parallel s^{\mathsf{k}+2}\left(\partial_s^{\mathsf{k}-2}\mathcal{Q}v-\partial_s^{\mathsf{k}-2}\mathcal{Q}\tilde{v}\right)\parallel_{L^\infty[R,\infty)}}{R}\Big\}$$

$$\leq C\left(n,\mathsf{k},\mathcal{C},\parallel f\parallel_{C^{\mathsf{k}+1}(\mathcal{K})},M\right)\parallel v-\tilde{v}\parallel$$

$$\parallel s^{\mathsf{k}+1}\left(\partial_s^{\mathsf{k}}\mathsf{v}-\partial_s^{\mathsf{k}}\tilde{\mathsf{v}}\right)\parallel_{L^\infty[R,\infty)}\leq C\left(n,\mathsf{k},\mathcal{C},\parallel f\parallel_{C^{\mathsf{k}+1}(\mathcal{K})},M\right)\parallel v-\tilde{v}\parallel \tag{2.81}$$

From (2.55), (2.68) to (2.76), (2.78) and (2.79), we see that $\mathcal{F}v(s)=\frac{f\left(\overrightarrow{1},0\right)}{\sigma s}+\mathsf{v}(s)\in\Im$ and it satisfies

$$\parallel\mathcal{F}v\parallel\leq\parallel\frac{f\left(\overrightarrow{1},0\right)}{\sigma s}\parallel+\parallel\mathsf{v}\parallel \tag{2.82}$$

$$\leq\left|\frac{f\left(\overrightarrow{1},0\right)}{\sigma}\right|\parallel\frac{1}{s}\parallel+\max\left\{\frac{1}{R^2}\sum_{j=0}^{\mathsf{k}-1}\parallel s^{j+3}\partial_s^j\mathsf{v}\parallel_{L^\infty[R,\infty)},\ \frac{1}{R}\parallel s^{\mathsf{k}+1}\partial_s^{\mathsf{k}}\mathsf{v}\parallel_{L^\infty[R,\infty)}\right\}$$

$$\leq\left|\frac{f\left(\overrightarrow{1},0\right)}{\sigma}\right|\parallel\frac{1}{s}\parallel+\frac{C\left(n,\mathsf{k},\mathcal{C},\parallel f\parallel_{C^{\mathsf{k}}(\mathcal{K})},M\right)}{R}$$

Besides, from (2.61), (2.70) to (2.77), (2.80) and (2.81), we have

$$\parallel\mathcal{F}v-\mathcal{F}\tilde{v}\parallel=\parallel\mathsf{v}-\tilde{\mathsf{v}}\parallel \tag{2.83}$$

$$\leq\max\left\{\frac{1}{R^2}\sum_{j=0}^{\mathsf{k}-1}\parallel s^{j+3}\left(\partial_s^j\mathsf{v}-\partial_s^j\tilde{\mathsf{v}}\right)\parallel_{L^\infty[R,\infty)},\ \frac{1}{R}\parallel s^{\mathsf{k}+1}\left(\partial_s^{\mathsf{k}}\mathsf{v}-\partial_s^{\mathsf{k}}\tilde{\mathsf{v}}\right)\parallel_{L^\infty[R,\infty)}\right\}$$

$$\leq\frac{C\left(n,\mathsf{k},\mathcal{C},\parallel f\parallel_{C^{\mathsf{k}+1}(\mathcal{K})},M\right)}{R}\parallel v-\tilde{v}\parallel$$

Now choose

$$M=\left|\frac{f\left(\overrightarrow{1},0\right)}{\sigma}\right|\parallel\frac{1}{s}\parallel+\frac{1}{2}$$

and take R even larger so that

$$\frac{C\left(n,\mathsf{k},\mathcal{C},\parallel f\parallel_{C^{\mathsf{k}+1}(\mathcal{K})},M\right)}{R}\leq\frac{1}{2}$$

Then we have $\mathcal{F}:\boldsymbol{B}\to\boldsymbol{B}$ is a contraction.

By the contraction mapping theorem, there is a unique fixed point u of $\mathcal{F}$ in $\boldsymbol{B}$. Moreover, let

$$\mathsf{u}(s)=\mathcal{F}u(s)-\frac{f\left(\overrightarrow{1},0\right)}{\sigma s}=u(s)-\frac{f\left(\overrightarrow{1},0\right)}{\sigma s}$$

then by (2.55), (2.68) to (2.76), (2.78) and (2.79), $\mathsf{u} \in C_0^{\mathsf{k}}[R,\, \infty)$ satisfies

$$\Big\{ \parallel s^3\, \mathsf{u} \parallel_{L^\infty[R,\, \infty)} + \parallel s^4 \partial_s \mathsf{u} \parallel_{L^\infty[R,\, \infty)} + \cdots + \parallel s^{\mathsf{k}+2} \partial_s^{\mathsf{k}-1} \mathsf{u} \parallel_{L^\infty[R,\, \infty)}$$

$$+ \parallel s^{\mathsf{k}+1} \partial_s^{\mathsf{k}} \mathsf{u} \parallel_{L^\infty[R,\, \infty)} \Big\} \leq C\left(n,\, \mathsf{k},\, \mathcal{C},\, \parallel f \parallel_{C^{\mathsf{k}}(\mathcal{K})}\right)$$

Then the conclusion follows immediately. $\square$

The following theorem is a direct result of Theorem 2.4:

Theorem 2.6. *There exist* $R = R\left(n,\, \mathsf{k},\, \mathcal{C},\, \mathcal{K},\, \parallel f \parallel_{C^{\mathsf{k}+1}(\mathcal{K})}\right) \geq 1$ *and* $\mathsf{u} \in C_0^{\mathsf{k}}[R,\, \infty)$ *such that*

$$\Sigma \equiv \left\{ \left(\left(\sigma s + \frac{f\left(\overrightarrow{1},\, 0\right)}{\sigma s} + \mathsf{u}\,(s) \right) \nu,\; s \right) \Big|\, \nu \in \mathbf{S}^{n-1},\, s \in [R,\, \infty) \right\}$$

is a rotationally symmetric F *self-shrinker which is* C^{k}*asymptotic to* $\mathcal{C}$ *at infinity. Besides, the corresponding self-similar solution to the* F *curvature flow is given by*

$$\Sigma_t = \sqrt{-t}\,\Sigma = \left\{ \left(\left(\sigma s\, - t\, \frac{f\left(\overrightarrow{1},\, 0\right)}{\sigma s} + \mathsf{u}_t\,(s) \right) \nu,\; s \right) \Big|\, \nu \in \mathbf{S}^{n-1},\, s \in [\sqrt{-t}R,\, \infty) \right\}$$

for $t \in [-1,\, 0)$*, where* $\mathsf{u}_t\,(s) = \sqrt{-t}\, \mathsf{u}\left(\frac{s}{\sqrt{-t}}\right)$ *and it satisfies*

$$\parallel s^3\, \mathsf{u}_t \parallel_{L^\infty[\sqrt{-t}R,\, \infty)} + \parallel s^4 \partial_s \mathsf{u}_t \parallel_{L^\infty[\sqrt{-t}R,\, \infty)} + \cdots + \parallel s^{k+2} \partial_s^{k-1} \mathsf{u}_t \parallel_{L^\infty[\sqrt{-t}R,\, \infty)}$$

$$\leq C\left(n,\, \mathsf{k},\, \mathcal{C},\, \parallel f \parallel_{C^{\mathsf{k}}(\mathcal{K})}\right)\, (-t)^2$$

$$\parallel s^{k+1} \partial_s^k \mathsf{u}_t \parallel_{L^\infty[\sqrt{-t}R,\, \infty)} \leq C\left(n,\, \mathsf{k},\, \mathcal{C},\, \parallel f \parallel_{C^{\mathsf{k}}(\mathcal{K})}\right)\, (-t)$$

for all $t \in [-1,\, 0)$.

Chapter 3

Analysis of Velázquez's solution to the mean curvature flow with a type II singularity

3.1 Introduction

J.J.L. Velázquez in [V] constructed a solution to the mean curvature flow which develops a type II singularity. Below is his result:

Theorem 3.1. *Let $n \geq 4$ be a positive integer. If $t_0 < 0$ and $|t_0| \ll 1$ (depending on n), then there is a $O(n) \times O(n)$ symmetric mean curvature flow $\{\Sigma_t\}_{t_0 \leq t < 0}$ so that*

1. *$\{\Sigma_t\}_{t_0 \leq t < 0}$ develops a type* II *singularity at O as $t \nearrow 0$ in the sense that there is $0 < \sigma = \sigma(n) < \frac{1}{2}$ (see (3.23)) so that the second fundamental form of Σ_t satisfies*

$$\limsup_{t \nearrow 0} \sup_{\Sigma_t \cap B\left(O; \sqrt{-t}\right)} (-t)^{\frac{1}{2}+\sigma} \left|A_{\Sigma_t}\right| > 0$$

2. *The type* I *rescaled hypersurfaces*

$$\left\{ \Pi_s = \frac{1}{\sqrt{-t}} \Sigma_t \Big|_{t=-e^{-s}} \right\}_{-\ln(-t_0) \leq s < \infty}$$

C^2-converge to Simons' cone $\mathcal{C}$ in any fixed annulus centered at O (i.e. $B(O; \mathrm{R}) \setminus B(O; \mathrm{r})$ with $0 < \mathrm{r} < \mathrm{R} < \infty$) as $s \nearrow \infty$.

3. *The type* II *rescaled hypersurfaces*

$$\left\{ \Gamma_\tau = \frac{1}{(-t)^{\frac{1}{2}+\sigma}} \Sigma_t \Big|_{t=-(2\sigma\tau)^{\frac{-1}{2\sigma}}} \right\}_{\frac{1}{2\sigma(-t_0)^{2\sigma}} \leq \tau < \infty}$$

locally C^0-converges to a minimal hypersurface $\mathcal{M}_k$ (see Section 3.2), which is tangent to Simons' cone $\mathcal{C}$ at infinity.

Velázquez's idea is to find a $O(n) \times O(n)$ symmetric solution to the "normalized mean curvature flow" $\{\Pi_s\}_{s_0 \leq s < \infty}$ which exists for a long time and converges (locally and away from O) to Simons' cone $\mathcal{C}$ as $s \nearrow \infty$. Note that the minimal cone $\mathcal{C}$ is a self-shrinker with a singularity at the origin and that this singularity of $\mathcal{C}$ forces the normalized mean curvature flow $\{\Pi_s\}_{s_0 \leq s < \infty}$ to develop a singularity at O as $s \nearrow \infty$. Consequently, the corresponding mean curvature flow $\{\Sigma_t\}_{t_0 \leq t < 0}$ develop a type II singularity at O in finite time (as $t \nearrow 0$). In addition, he used the comparison principle to show that the type II rescaled hypersurfaces convergers locally uniformly, in the C^0 sense, to a minimal hypersurface $\mathcal{M}_k$.

The motivation of studying Vel*á*zquez's solution comes from two natural questions. The first one is whether the minimal hypersurface $\mathcal{M}_k$ is the singularity model of the type II singularity at O? Note that the minimal hypersurface is stationary, which is a special case of the "translating mean curvature flow". Vel*á*zquez's result make us believe that this is true. However, we cannot be assured by his result since he only show that the type II rescaled hypersurfaces converges to $\mathcal{M}_k$ in the C^0 sense. Secondly, we would like to know whether the mean curvature of Vel*á*zquez's solution blows up as $t \nearrow 0$ or not. There is a long-lasting question in the study of mean curvature flow: "Does the mean curvature blow up at the first singular time?" The answer is positive under a variety of hypotheses. For instance, if the mean curvature flow is rotationally symmetric or its singularities belong to type I, then the mean curvature must blow up (see [K] and [LS]). People believe this is true in general for low-dimensional mean curvature flow, and it has been verified by Li and Wang (see [LW]) for the 2-dimensional case. However, people are skeptical about this for high-dimensional mean curvature flow, and they think Vel*á*zquez's solution might be a counterexample. Heuristically speaking, the type II rescaling of Vel*á*zquez's solution converges to a "minimal hypersurface", so it seems that there is a chance for the mean curvature of Vel*á*zquez's solution to stay bounded upto the first singular time.

Here we answer both of the above questions. More explicitly, we show the following:

Theorem 3.2. *Let* $\{\Sigma_t\}_{t_0 \leq t < 0}$ *be Velázquez's solution in Theorem 3.1 with* $n \geq 5$. *By choosing proper initial data outside a small ball centered at* O*, the origin is the only singularity of the solution at the first singular time* $t = 0$. *Moreover, the type* II *rescaled* *hypersurfaces* $\{\Gamma_\tau\}_{\frac{1}{2\sigma(-t_0)^{2\sigma}}\leq\tau<\infty}$ *converges locally smoothly to the minimal hypersurface* $\mathcal{M}_k$ *as* $\tau \nearrow \infty$. *It follows that the second fundamental form of* Σ_t *satisfies*

$$0 < \limsup_{t\nearrow 0} \sup_{\Sigma_t} (-t)^{\frac{1}{2}+\sigma} |A_{\Sigma_t}| < \infty$$

In addition, the mean curvature of Σ_t *blows up as* $t \nearrow 0$ *at a rate which smaller than that of the second fundamental form. More precisely, there hold*

$$\limsup_{t\nearrow 0} \sup_{\Sigma_t\cap B\left(O;C(n)(-t)^{\frac{1}{2}+\sigma}\right)} (-t)^{\frac{1}{2}-\sigma} |H_{\Sigma_t}| > 0$$

$$\limsup_{t\nearrow 0} \sup_{\Sigma_t} (-t)^{\frac{1}{2}+(1-2\varrho)\sigma} |H_{\Sigma_t}| < \infty$$

for some constant $0 < \varrho = \varrho(n) < 1$.

Proof. The smooth convergence of the type II rescaled hypersurfaces $\{\Gamma_\tau\}$ to $\mathcal{M}_k$ as $\tau \nearrow \infty$ and the fact that the origin is the only singularity of $\{\Sigma_t\}$ at $t = 0$ follow from Theorem 3.17 (see also Remark 3.18). The blow-up rates of the second fundamental form A_{Σ_t} and mean curvature H_{Σ_t} can be found in Proposition 3.19, Proposition 3.20, Proposition 3.21 and Proposition 3.22. □

To improve the convergence of the type II rescaled flow, all we need is to derive some smooth estimates (see Proposition 3.13 and Proposition 3.14). One of the key ingredients to achieve that is to use the curvature estimates in [EH]. As for the blow-up of the mean curvature, it follows from the smooth convergence of type II rescaled flow and L'Hôpital's rule. Moreover, by modifying Velázquez's estimates, we show that the blow-up rate of the mean curvature is smaller than that of the second fundamental form.

The chapter is organized as follows. In Section 3.2, we introduce the minimal hypersurface $\mathcal{M}_k$ found by Velázquez and then derive some smooth estimates for it. In Section 3.3, we specify the set up for constructing Velázquez's solution and define various regions and rescalings for analyzing the solution. In Section 3.4, we state the key a priori estimates (Proposition 3.13 and Proposition 3.14) and explain how to use them to construct Velázquez's solution (for the sake of completeness) and to see the behavior of the solution in different regions (see Theorem 3.17). In Section 3.5, we explain why the mean curvature blows up and why its blow-up rate is smaller than that of the second fundamental form. Lastly, in Section 3.6, Section 3.7 and Section 3.8 we prove Proposition 3.13 and Proposition 3.14 for completion of the argument.

3.2 Minimal hypersurfaces tangent to Simons' cone at infinity

Let

$$\mathcal{C} = \left\{ (r\nu,\, r\omega) \,\middle|\, r > 0;\, \nu,\, \omega \in \mathbb{S}^{n-1} \right\}$$

be Simons' cone, where $n \geq 4$ is a positive integer and $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$. It is shown in [V] that there is a smooth minimal hypersurface

$$\mathcal{M} = \left\{ \left(r\nu,\, \hat{\psi}\,(r)\,\omega \right) \middle|\, r \geq 0;\, \nu,\, \omega \in \mathbb{S}^{n-1} \right\}$$

in $\mathbb{R}^{2n}$ which is tangent to $\mathcal{C}$ at infinity, and that the function $\hat{\psi}\,(r)$ satisfies

$$\frac{\partial_{rr}^2 \hat{\psi}}{1 + \left(\partial_r \hat{\psi} \right)^2} + (n-1) \left(\frac{\partial_r \hat{\psi}}{r} - \frac{1}{\hat{\psi}} \right) = 0$$

and

$$\begin{cases} \partial_{rr}^2 \hat{\psi}\,(r) > 0 \\ \partial_r \hat{\psi}\,(0) = 0, \quad \lim_{r \nearrow \infty} \frac{\partial_r \hat{\psi}(r) - 1}{r^{\alpha-1}} = \alpha\, 2^{\frac{\alpha+1}{2}} \\ \hat{\psi}\,(r) > r, \quad \lim_{r \nearrow \infty} \frac{\hat{\psi}(r) - r}{r^{\alpha}} = 2^{\frac{\alpha+1}{2}} \end{cases}$$

where

$$\alpha = \frac{-\,(2n-3) + \sqrt{4n^2 - 20n + 17}}{2} \in [-2,\, -1)$$

is a root of the quadratic polynomial

$$\alpha\,(\alpha - 1) + 2\,(n-1)\,(\alpha + 1) = 0 \tag{3.1}$$

By symmetry, studying $\mathcal{M}$ is equivalent to analyzing the projected curves

$$\bar{\mathcal{M}} = \left\{ \left(r,\, \hat{\psi}\,(r) \right) \middle|\, r \geq 0 \right\}$$

$$\bar{\mathcal{C}} = \left\{ (r,\, r) \middle|\, r > 0 \right\} \tag{3.2}$$

Note that $\bar{\mathcal{M}}$ is a convex curve which lies above $\bar{\mathcal{C}}$ (i.e. $\hat{\psi}(r) > r$ for $r \geq 0$); moreover, $\bar{\mathcal{M}}$ intersects orthogonally with the vertical ray $\{(0, r) | \, r > 0\}$ (i.e. $\partial_r \hat{\psi}(0) = 0$) and is asymptotic to $\bar{\mathcal{C}}$ at infinity (i.e. $\hat{\psi}(r) = r + O(r^{\alpha})$ as $r \nearrow \infty$). Therefore, $\bar{\mathcal{M}}$ is a graph over $\bar{\mathcal{C}}$; more precisely,

$$\bar{\mathcal{M}} = \left\{ r\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) + \psi(r)\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \middle| \, r \geq \frac{\hat{\psi}(0)}{\sqrt{2}} \right\}$$

$$= \left\{ \left((r - \psi(r)) \frac{1}{\sqrt{2}}, \, (r + \psi(r)) \frac{1}{\sqrt{2}} \right) \middle| \, r \geq \frac{\hat{\psi}(0)}{\sqrt{2}} \right\}$$

Vel*á*zquez in [V] showed that the function $\psi(r)$ satisfies

$$\frac{\partial^2_{rr}\psi}{1 + (\partial_r \psi)^2} + 2(n-1)\frac{r\,\partial_r\psi + \psi}{r^2 - \psi^2} = 0$$

and

$$\begin{cases} \partial^2_{rr}\psi(r) > 0 \\ \\ \partial_r \psi\left(\frac{\hat{\psi}(0)}{\sqrt{2}}\right) = -1, \quad \lim_{r \nearrow \infty} \frac{\partial_r \psi(r)}{r^{\alpha-1}} = \alpha \\ \\ \psi\left(\frac{\hat{\psi}(0)}{\sqrt{2}}\right) = \frac{\hat{\psi}(0)}{\sqrt{2}}, \quad \lim_{r \nearrow \infty} \frac{\psi(r)}{r^{\alpha}} = 1 \end{cases}$$

More generally, for each $k > 0$, we can define

$$\mathcal{M}_k = k^{\frac{1}{1-\alpha}} \mathcal{M}$$

Then $\mathcal{M}_k$ is also a minimal hypersurface in $\mathbb{R}^{2n}$ which is tangent to $\mathcal{C}$ at infinity. Notice that

$$\mathcal{M}_k = \left\{ \left(r\,\nu, \, \hat{\psi}_k(r)\,\omega \right) \middle| \, r \geq 0; \, \nu, \omega \in \mathbb{S}^{n-1} \right\}$$

where

$$\hat{\psi}_k(r) = k^{\frac{1}{1-\alpha}} \, \hat{\psi}\left(k^{\frac{-1}{1-\alpha}} r \right) \tag{3.3}$$

By rescaling, we deduce that

$$\frac{\partial^2_{rr}\hat{\psi}_k}{1 + \left(\partial_r \hat{\psi}_k\right)^2} + (n-1)\left(\frac{\partial_r \hat{\psi}_k}{r} - \frac{1}{\hat{\psi}_k} \right) = 0 \tag{3.4}$$

$$\begin{cases} \partial^2_{rr}\hat{\psi}_k(r) > 0 \\ \\ \partial_r\hat{\psi}_k(0) = 0, \quad \lim_{r\nearrow\infty} \frac{\partial_r\hat{\psi}_k(r)-1}{r^{\alpha-1}} = k\alpha\, 2^{\frac{\alpha+1}{2}} \\ \\ \hat{\psi}_k(r) > r, \quad \lim_{r\nearrow\infty} \frac{\hat{\psi}_k(r)-r}{r^{\alpha}} = k\, 2^{\frac{\alpha+1}{2}} \end{cases}$$

Moreover, there holds a "monotonic" property of the rescaling family, i.e. $\hat{\psi}_{k_1}(r) < \hat{\psi}_{k_2}(r)$ whenever $0 < k_1 < k_2 < \infty$. To see that, let's first derive the following lemma.

Lemma 3.3. *The function $\hat{\psi}_k(r)$ satisfies*

$$\hat{\psi}_k(r) - r\,\partial_r\hat{\psi}_k(r) > 0 \tag{3.5}$$

for $r \geq 0$. In addition, there holds

$$\lim_{r\nearrow\infty} \frac{\hat{\psi}(r) - r\,\partial_r\hat{\psi}(r)}{r^{\alpha}} = (1-\alpha)\ 2^{\frac{\alpha+1}{2}} \tag{3.6}$$

Proof. Notice that

$$\partial_r\left(\hat{\psi}(r) - r\,\partial_r\hat{\psi}(r)\right) \ = \ -r\,\partial^2_{rr}\hat{\psi} \ < 0$$

which means the function $\hat{\psi}(r) - r\,\partial_r\hat{\psi}$ is decreasing. Furthermore, we have

$$\lim_{r\nearrow\infty} \frac{\hat{\psi}(r) - r\,\partial_r\hat{\psi}(r)}{r^{\alpha}} = \lim_{r\nearrow\infty}\left(\frac{\hat{\psi}(r)-r}{r^{\alpha}} + \frac{1-\partial_r\hat{\psi}(r)}{r^{\alpha-1}}\right) = (1-\alpha)\ 2^{\frac{\alpha+1}{2}}\ > 0$$

which implies

$$\hat{\psi}(r) - r\,\partial_r\hat{\psi}(r) > 0$$

for $r \gg 1$. The conclusions follow immediately. □

Now we show the monotonic property of the rescaling family.

Lemma 3.4. *There holds*

$$\partial_k\,\hat{\psi}_k > 0$$

In other words, $\hat{\psi}_k$ is monotonically increasing in k.

Proof. By definition, we have

$$\partial_k\,\hat{\psi}_k(z) = \partial_k\left(k^{\frac{1}{1-\alpha}}\ \hat{\psi}\left(k^{\frac{-1}{1-\alpha}}\,z\right)\right)$$

$$= \partial_k \, k^{\frac{1}{1-\alpha}} \, \left(\hat{\psi}(r) - r \, \partial_r \hat{\psi}(r) \right) \Big|_{r = k^{\frac{-1}{1-\alpha}} z} > 0$$

□

On the other hand, notice that the projected curve of $\mathcal{M}_k$ is also a graph over over $\bar{\mathcal{C}}$, i.e.

$$\bar{\mathcal{M}}_k = \left\{ \left(r, \, \hat{\psi}_k(r) \right) \Big| \; r \geq 0 \right\} \tag{3.7}$$

$$= \left\{ \left((r - \psi_k(r)) \frac{1}{\sqrt{2}}, \, (r + \psi_k(r)) \frac{1}{\sqrt{2}} \right) \Bigg| \; r \geq \frac{\hat{\psi}_k(0)}{\sqrt{2}} \right\}$$

where

$$\psi_k(r) = k^{\frac{1}{1-\alpha}} \; \psi\left(k^{\frac{-1}{1-\alpha}} r \right) \tag{3.8}$$

By rescaling, the function $\psi_k(r)$ satisfies

$$\frac{\partial^2_{rr} \psi_k}{1 + (\partial_r \psi_k)^2} + 2(n-1) \frac{r \, \partial_r \psi_k \, + \, \psi_k}{r^2 - \psi_k^2} = 0 \tag{3.9}$$

$$\begin{cases} \partial^2_{rr} \psi_k(r) > 0 \\ \partial_r \psi_k \left(\frac{\hat{\psi}_k(0)}{\sqrt{2}} \right) = -1, \quad \lim_{r \nearrow \infty} \frac{\partial_r \psi_k(r)}{r^{\alpha-1}} = k\alpha \\ \psi_k \left(\frac{\hat{\psi}_k(0)}{\sqrt{2}} \right) = \frac{\hat{\psi}_k(0)}{\sqrt{2}}, \quad \lim_{r \nearrow \infty} \frac{\psi_k(r)}{r^{\alpha}} = k \end{cases}$$

Note that $\psi_k(r) \searrow 0$ as $r \nearrow \infty$. Below we have the decay estimates for $\psi_k(r)$.

Lemma 3.5. *For any $m \in \mathbb{Z}_+$, there holds*

$$\left| \partial_r^m \psi_k(r) \right| \, \leq \, C(n, m) \, k r^{\alpha - m}$$

for $r \geq \frac{\hat{\psi}_k(0)}{\sqrt{2}}$.

Proof. By rescaling, it is sufficient to check for $k = 1$.

From

$$\lim_{r \to \infty} \frac{\psi(r)}{r^{\alpha}} \, = \, 1 \, = \, \lim_{r \to \infty} \frac{\partial_r \psi(r)}{\alpha r^{\alpha-1}}$$

we have

$$\max \left\{ \left| \frac{\psi(r)}{r} \right|, \, \left| \partial_r \psi(r) \right| \right\} \, \leq \, C(n) \, r^{\alpha - 1}$$

for $r \geq \frac{\hat{\psi}(0)}{\sqrt{2}}$. In particular, there is $R \gg 1$ (depending on n) so that

$$\max\left\{\left|\frac{\psi(r)}{r}\right|,\ |\partial_r \psi(r)|\right\} \leq \frac{1}{3}$$

for $r \geq R$. By (3.9), we have

$$\partial_{rr}^2 \psi(r) = -2(n-1)\left(1 + (\partial_r \psi(r))^2\right) \frac{r\,\partial_r \psi(r) + \psi(r)}{r^2 - \psi^2(r)}$$

It follows that

$$\left|\partial_{rr}^2 \psi(r)\right| \leq C(n)\, r^{\alpha-2}$$

for $r \geq R$. Continuing differentiating the equation of $\psi(r)$ and using induction yields

$$|\partial_r^m \psi(r)| \leq C(n, m)\, r^{\alpha-m}$$

for $r \geq R$, $m \in \mathbb{Z}_+$.

On the other hand, by the above choice of $R = R(n)$, we have

$$\sup_{\frac{\hat{\psi}(0)}{\sqrt{2}} \leq r \leq R} r^{m-\alpha} |\partial_r^m \psi(r)| \leq R^{m-\alpha} \sup_{\frac{\hat{\psi}(0)}{\sqrt{2}} \leq r \leq R} |\partial_r^m \psi(r)| \leq C(n, m)$$

for any $m \in \mathbb{Z}_+$. Therefore, we conclude that for any $m \in \mathbb{Z}_+$

$$|\partial_r^m \psi(r)| \leq C(n, m)\, r^{\alpha-m}$$

for $r \geq \frac{\hat{\psi}(0)}{\sqrt{2}}$. $\square$

As a corollary, we have the following decay estimates for the higher order derivatives of $\hat{\psi}_k(r)$.

Lemma 3.6. *For any $m \geq 2$, there holds*

$$\left|\partial_r^m \hat{\psi}_k(r)\right| \leq C(n, m)\, k r^{\alpha-m}$$

for $r \geq 0$.

Proof. By rescaling, it is sufficient to check for $k = 1$.

Let's first parametrize the projected curve $\bar{\mathcal{M}}$ by

$$\mathcal{Z} = \left((r - \psi_k(r))\frac{1}{\sqrt{2}},\ (r + \psi_k(r))\frac{1}{\sqrt{2}}\right)$$

In this parametrization, the normal curvature of $\bar{\mathcal{M}}$ is given by

$$A_{\bar{\mathcal{M}}} = \frac{\partial_{rr}^2 \psi(r)}{\left(1 + (\partial_r \psi(r))^2\right)^{\frac{3}{2}}}$$

Let $\nabla_{\bar{\mathcal{M}}}$ be the covariant derivative of $\bar{\mathcal{M}}$, i.e.

$$\nabla_{\bar{\mathcal{M}}} f = \frac{\partial_r f(r)}{\sqrt{1 + (\partial_r \psi(r))^2}} \qquad \text{for } f \in C^1\left(\bar{\mathcal{M}}\right)$$

By Lemma 3.5, there is $R \gg 1$ (depending on n) so that

$$\max\left\{\left|\frac{\psi(r)}{r}\right|, \, |\partial_r \psi(r)|\right\} \le \frac{1}{3}$$

and

$$|\mathcal{Z}|^m \left|\nabla_{\bar{\mathcal{M}}}^m A_{\bar{\mathcal{M}}}\right| \le C(n, m) \, |\mathcal{Z}|^{\alpha - 2} \tag{3.10}$$

for $r \ge R$, $m \in \mathbb{Z}_+$. Notice that

$$|\mathcal{Z}| = \sqrt{r^2 + \psi^2(r)}$$

is comparible with r for $r \ge R$.

Next, let's reparametrize $\bar{\mathcal{M}}$ by

$$\mathcal{Z} = \left(r, \, \hat{\psi}(r)\right) \tag{3.11}$$

In this parametrization, the normal curvature is given by

$$A_{\bar{\mathcal{M}}} = \frac{\partial_{rr}^2 \hat{\psi}(r)}{\left(1 + \left(\partial_r \hat{\psi}(r)\right)^2\right)^{\frac{3}{2}}} \tag{3.12}$$

and the covariant derivative is defined by

$$\nabla_{\bar{\mathcal{M}}} f = \frac{\partial_r f(r)}{\sqrt{1 + \left(\partial_r \hat{\psi}(r)\right)^2}} \qquad \text{for } f \in C^1\left(\bar{\mathcal{M}}\right) \tag{3.13}$$

Note also that by (3.4), we have

$$\begin{gathered} 0 \le \tfrac{\hat{\psi}(r)}{r} \le C(n) \\ 0 \le \partial_r \hat{\psi}(r) \le 1 \end{gathered} \tag{3.14}$$

for $r \geq R = R(n)$. Then by (3.10), (3.11), (3.12), (3.13) and (3.14), we infer that

$$\left|\partial_r^m \hat{\psi}(r)\right| \leq C(n, m)\, r^{\alpha - m}$$

for $r \geq 2R$, $m \geq 2$.

On the other hand, by the above choice of $R = R(n)$, there holds

$$\sup_{0 \leq r \leq 2R} r^{m-\alpha} \left|\partial_r^m \hat{\psi}(r)\right| \leq (2R)^{m-\alpha} \sup_{0 \leq r \leq 2R} \left|\partial_r^m \hat{\psi}(r)\right| \leq C(n, m)$$

for any $m \geq 2$. Consequently, we get

$$\left|\partial_r^m \hat{\psi}(r)\right| \leq C(n, m)\, r^{\alpha - m}$$

for $r \geq 0$, $m \geq 2$. □

Lastly, we conclude this section by estimating the difference between ψ_k and its asymptotic function appeared in (3.9).

Lemma 3.7. *The function $\psi_k(r)$ satisfies*

$$|\psi_k(r) - kr^{\alpha}| \leq C(n)\; k^3\, r^{3\alpha - 2}$$

$$\left|\partial_r \psi_k(r) - k\alpha r^{\alpha - 1}\right| \leq C(n)\; k^3\, r^{3\alpha - 3}$$

for $r \geq \frac{\hat{\psi}_k(0)}{\sqrt{2}}$.

Proof. Without loss of generality, we may assume $k = 1$.

First, let's rewrite the equation of $\psi(r)$ as

$$r\, \partial_{rr}^2 \psi = -2(n-1) \frac{1 + (\partial_r \psi)^2}{1 - \left(\frac{\psi}{r}\right)^2} \left(\partial_r \psi + \frac{\psi}{r}\right) \tag{3.15}$$

Let

$$P = \partial_r \psi(r), \qquad Q = \frac{\psi(r)}{r}$$

and

$$\mathfrak{h} = \ln(r)$$

Then from (3.15), we deduce

$$\begin{cases} \partial_{\mathfrak{h}} P = -2\,(n-1)\,\frac{1+P^2}{1-Q^2}\,(P+Q) \\ \\ \partial_{\mathfrak{h}} Q = P - Q \end{cases} \tag{3.16}$$

On the other hand, by (3.1), we can also deduce that

$$r\,\partial_{rr}^2 r^\alpha = -2\,(n-1)\left(\partial_r r^\alpha + \frac{r^\alpha}{r}\right)$$

Let

$$P_* = \partial_r r^\alpha = \alpha r^{\alpha-1}, \qquad Q_* = \frac{r^\alpha}{r} = r^{\alpha-1}$$

and

$$\mathfrak{h} = \ln(r)$$

Similarly, there holds

$$\begin{cases} \partial_{\mathfrak{h}} P_* = -2\,(n-1)\,(P_* + Q_*) \\ \\ \partial_{\mathfrak{h}} Q_* = P_* - Q_* \end{cases} \tag{3.17}$$

Now subtract (3.17) from (3.16) to get

$$\begin{cases} \partial_{\mathfrak{h}}\,(P-P_*) = -2\,(n-1)\,((P-P_*)+(Q-Q_*)) - 2\,(n-1)\,\frac{\left(P^2+Q^2\right)(P+Q)}{1-Q^2} \\ \\ \partial_{\mathfrak{h}}\,(Q-Q_*) = (P-P_*) - (Q-Q_*) \end{cases}$$

Note that by (3.9) we have

$$\lim_{r\to\infty} \frac{\psi(r) - r^\alpha}{r^\alpha} = 0 = \lim_{r\to\infty} \frac{\partial_r \psi(r) - \alpha r^{\alpha-1}}{r^{\alpha-1}}$$

which implies

$$\begin{cases} P - P_* = \partial_r \psi(r) - \alpha r^{\alpha-1} = o\left(r^{\alpha-1}\right) = o\left(e^{(\alpha-1)\mathfrak{h}}\right) \\ \\ Q - Q_* = \frac{\psi(r)}{r} - r^{\alpha-1} = o\left(r^{\alpha-1}\right) = o\left(e^{(\alpha-1)\mathfrak{h}}\right) \end{cases}$$

as $\mathfrak{h} \to \infty$. Now let

$$\Theta = \begin{pmatrix} P - P_* \\ \\ Q - Q_* \end{pmatrix}, \quad \boldsymbol{f}(\mathfrak{h}) = \begin{pmatrix} -2\,(n-1)\,\frac{\left(P^2+Q^2\right)(P+Q)}{1-Q^2} \\ \\ 0 \end{pmatrix}$$

and

$$\boldsymbol{L} = \begin{pmatrix} 2(n-1) & 2(n-1) \\ \\ -1 & 1 \end{pmatrix}$$

Then we have

$$\begin{cases} \partial_{\mathfrak{h}}\Theta + \boldsymbol{L}\Theta = \boldsymbol{f} \\ \\ \Theta(\mathfrak{h}) = o\left(e^{(\alpha-1)\mathfrak{h}}\right) \quad \text{as } \mathfrak{h} \to \infty \end{cases} \tag{3.18}$$

Notice that

$$\boldsymbol{L} = \begin{pmatrix} \alpha & \bar{\alpha} \\ \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\alpha+1 & 0 \\ \\ 0 & -\bar{\alpha}+1 \end{pmatrix} \begin{pmatrix} \alpha & \bar{\alpha} \\ \\ 1 & 1 \end{pmatrix}^{-1}$$

where

$$\bar{\alpha} = \frac{-(2n-3) - \sqrt{4n^2 - 20n + 17}}{2} < \alpha$$

and

$$|\boldsymbol{f}(\mathfrak{h})| \le C(n)\, e^{3(\alpha-1)\mathfrak{h}} \qquad \text{for } \mathfrak{h} \ge \ln\left(\frac{\hat{\psi}(0)}{\sqrt{2}}\right)$$

It follows that for any $R > \mathfrak{h} \ge \ln\left(\frac{\hat{\psi}(0)}{\sqrt{2}}\right)$,

$$\begin{aligned} |\Theta(\mathfrak{h})| &\le e^{(R-\mathfrak{h})(-\alpha+1)}\, |\Theta(R)| + \int_{\mathfrak{h}}^{R} e^{(\xi-\mathfrak{h})(-\alpha+1)}\, |\boldsymbol{f}(\xi)|\, d\xi \\ &\le \left(e^{(-\alpha+1)R}\, |\Theta(R)|\right) e^{(\alpha-1)\mathfrak{h}} + C(n)\, e^{3(\alpha-1)\mathfrak{h}} \end{aligned}$$

Note that

$$\Theta(R) = o\left(e^{(\alpha-1)R}\right)$$

as $R \to \infty$ by (3.18). Let $R \nearrow \infty$ to get

$$|\Theta(\mathfrak{h})| \le C(n)\, e^{3(\alpha-1)\mathfrak{h}} \qquad \text{for } \mathfrak{h} \ge \ln\left(\frac{\hat{\psi}(0)}{\sqrt{2}}\right)$$

which yields

$$\left|\partial_r \psi(r) - \alpha r^{\alpha-1}\right| + \left|\frac{\psi(r)}{r} - r^{\alpha-1}\right| \le C(n)\, r^{3(\alpha-1)} \quad \text{for } r \ge \frac{\hat{\psi}(0)}{\sqrt{2}}$$

$\square$

3.3 Admissible mean curvature flow

Let $n \geq 5$ be a positive integer and $\Lambda = \Lambda(n) \gg 1$, $0 < \rho \ll 1 \ll \beta$ (depending on n, Λ), $t_0 < 0$ with $|t_0| \ll 1$ (depending on n, Λ, ρ, β) be constants to be determined. Recall that an one-parameter family of smooth hypersurfaces $\{\Sigma_t\}_{t_0 \leq t \leq \mathring{t}}$ in $\mathbb{R}^{2n}$, where $\mathring{t} < 0$ is a constant, is called a mean curvature flow (MCF) provided that

$$\partial_t X_t \cdot N_{\Sigma_t} = H_{\Sigma_t} \tag{3.19}$$

where X_t is the position vector, N_{Σ_t} and H_{Σ_t} are the unit normal vector and mean curvature of Σ_t, respectively. We define the MCF $\{\Sigma_t\}_{t_0 \leq t \leq \mathring{t}}$ to be **admissible** if every time-sclice Σ_t is a complete, embedded and smooth hypersurface which satisfies

1. Σ_t is $O(n) \times O(n)$ symmetric and it can be parametrized as

$$\Sigma_t = \left\{ (x\,\nu,\ \hat{u}(x,\,t)\,\omega) \,\middle|\, x \geq 0;\ \nu,\ \omega \in \mathbb{S}^{n-1} \right\} \tag{3.20}$$

where $\hat{u}(x,\,t)$ is a smooth function which satisfies

$$\partial_t\,\hat{u} = \frac{\partial^2_{xx}\hat{u}}{1 + (\partial_x \hat{u})^2} + (n-1)\left(\frac{\partial_x \hat{u}}{x} - \frac{1}{\hat{u}}\right) \tag{3.21}$$

$$\hat{u}(0,\,t) > 0, \quad \partial_x \hat{u}(0,\,t) = 0$$

for $t_0 \leq t \leq \mathring{t}$. Note that the above condition means that the projected curve

$$\bar{\Sigma}_t = \{\,(x,\ \hat{u}(x,\,t))|\ x \geq 0\} \tag{3.22}$$

lives in the first quadrant and intersects orthogonally with the vertical ray $\{\,(0,\,x)|\ x > 0\}$.

2. The projected curve $\bar{\Sigma}_t$ is a graph over $\bar{\mathcal{C}}$ outside $B\left(O;\ \beta(-t)^{\frac{1}{2}+\sigma}\right)$, where

$$\sigma = -\frac{1}{2} + \frac{2}{1-\alpha} \in \left[\frac{1}{6},\ \frac{1}{2}\right) \tag{3.23}$$

Equivalently, this is saying that Σ_t is a normal graph over $\mathcal{C}$ outside $B\left(O;\ \beta(-t)^{\frac{1}{2}+\sigma}\right)$. In other words, we can reparametrize Σ_t by

$$X_t(x,\,\nu,\,\omega) = \left((x - u(x,\,t))\,\frac{\nu}{\sqrt{2}},\ (x + u(x,\,t))\,\frac{\omega}{\sqrt{2}}\right) \tag{3.24}$$

for $x \geq \beta(-t)^{\frac{1}{2}+\sigma}$, $\nu,\,\omega \in \mathbb{S}^{n-1}$, where $u(x,\,t)$ is a smooth function satisfying

$$\partial_t u = \frac{\partial^2_{xx} u}{1 + (\partial_x u)^2} + 2(n-1)\,\frac{x\,\partial_x u + u}{x^2 - u^2} \tag{3.25}$$

3. For the function $u(x,\,t)$, there holds

$$x^{i}\left|\partial_{x}^{i}u(x,\,t)\right|<\Lambda\left((-t)^{2}x^{\alpha}+x^{2\lambda_{2}+1}\right),\qquad i\in\{0,\,1,\,2\} \tag{3.26}$$

   for $\beta(-t)^{\frac{1}{2}+\sigma}\leq x\leq\rho$, $t_{0}\leq t\leq\mathring{t}$, where $\lambda_{2}=\frac{1}{2}(\alpha+3)$ is a constant (see Proposition 3.8).

In order to analyze an admissible MCF, below we divide the space into three (time-dependent) regions and do proper rescaling for small regions.

- The **outer region** – $\Sigma_{t}\setminus B\left(O;\,\sqrt{-t}\right)$
- The **intermediate region** – $\Sigma_{t}\cap\left(B\left(O;\,\sqrt{-t}\right)\setminus B\left(O;\,\beta(-t)^{\frac{1}{2}+\sigma}\right)\right)$: here we perform the "type I" rescaling

$$\Pi_{s}=\frac{1}{\sqrt{-t}}\Sigma_{t}\Big|_{t=-e^{-s}} \tag{3.27}$$

  By this rescaling, the intermediate region is then dilated to become

$$\Pi_{s}\cap\left(B(O;\,1)\setminus B\left(O;\,\beta e^{-\sigma s}\right)\right)$$

  for $s_{0}\leq s\leq\mathring{s}$, where $s_{0}=-\ln(-t_{0})$ and $\mathring{s}=-\ln\left(-\mathring{t}\right)$. Note that $s_{0}\gg1$ iff $|t_{0}|\ll1$.

- The **tip region** – $\Sigma_{t}\cap B\left(O;\,\beta(-t)^{\frac{1}{2}+\sigma}\right)$: here we perform the "type II" rescaling

$$\Gamma_{\tau}=\frac{1}{(-t)^{\frac{1}{2}+\sigma}}\Sigma_{t}\Big|_{t=-(2\sigma\tau)^{\frac{-1}{2\sigma}}} \tag{3.28}$$

  By this rescaling, the intermediate region is dilated to become

$$\Gamma_{\tau}\cap B(O;\,\beta)$$

  for $\tau_{0}\leq\tau\leq\mathring{\tau}$, where $\tau_{0}=\frac{1}{2\sigma(-t_{0})^{2\sigma}}$, $\mathring{\tau}=\frac{1}{2\sigma\left(-\mathring{t}\right)^{2\sigma}}$. Note that $\tau_{0}\gg1$ iff $|t_{0}|\ll1$.

In the outer region, we parametrize Σ_{t} by

$$X_{t}(x,\,\nu,\,\omega)=\left((x-u(x,\,t))\frac{\nu}{\sqrt{2}},\,(x+u(x,\,t))\frac{\omega}{\sqrt{2}}\right)$$

and study the function $u(x, t)$ via (3.25). In $B(O;\, \rho) \setminus B\left(O;\, \sqrt{-t}\right)$, Velázquez showed that by choosing suitable initial data (see Section 3.4), there holds

$$u(x, t) \sim x^{2\lambda_2+1}$$

However, the behavior outside $B(O;\, \rho)$ was not clear in [V]. Here we complete this part by providing smooth estimate for $\Sigma_t \setminus B(O;\, \rho)$.

In the intermediate region, we first do the type I rescaling and parametrize the rescaled hypersurface Π_s by

$$Y_s(y,\, \nu,\, \omega) = \left((y - v(y,\, s))\frac{\nu}{\sqrt{2}},\, (y + v(y,\, s))\frac{\omega}{\sqrt{2}}\right) \tag{3.29}$$

where

$$v(y,\, s) = \frac{1}{\sqrt{-t}}\, u\left(\sqrt{-t}\, y,\, t\right)\Big|_{t=-e^{-s}} \tag{3.30}$$

From (3.25), we derive

$$\partial_s v = \frac{\partial^2_{yy} v}{1 + (\partial_y v)^2} + 2(n-1)\frac{y\,\partial_y v + v}{y^2 - v^2} + \frac{1}{2}(-y\,\partial_y v + v) \tag{3.31}$$

Notice that (3.26) is equivalent to

$$y^i \left|\partial^i_y v(y,\, s)\right| < \Lambda e^{-\lambda_2 s}\left(y^\alpha + y^{2\lambda_2+1}\right), \qquad i \in \{0,\, 1,\, 2\} \tag{3.32}$$

for $\beta e^{-\sigma s} \le y \le \rho e^{\frac{s}{2}}$, $s_0 \le s \le \mathring{s}$. To study the function $v(y,\, s)$, Velázquez linearized (3.31) and showed that

$$v(y,\, s) \sim e^{-\lambda_2 s}\varphi_2(y)$$

by (3.32) and the choice of initial data (see Section 3.4), where λ_2 and $\varphi_2(y)$ are the first positive eigenvalue and eigenfunction of the linearized operator (see Proposition 3.8). More precisely, (3.31) can be rewritten as

$$\partial_s v = -\mathcal{L}v + \mathcal{Q}v \tag{3.33}$$

where

$$\begin{aligned}\mathcal{L}v &= -\left(\partial^2_{yy} v + 2(n-1)\frac{y\,\partial_y v + v}{y^2} + \frac{1}{2}(-y\,\partial_y v + v)\right) \\ &= -\left(y^{2(n-1)}e^{-\frac{y^2}{4}}\right)^{-1}\partial_y\left(y^{2(n-1)}e^{-\frac{y^2}{4}}\partial_y v\right) - \left(\frac{2(n-1)}{y^2} + \frac{1}{2}\right)v\end{aligned} \tag{3.34}$$

is the (negative) linearization of the RHS of (3.31), and

$$\mathcal{Q}v = -\frac{(\partial_y v)^2}{1+(\partial_y v)^2}\,\partial^2_{yy}v + 2(n-1)\,\frac{\left(\frac{v}{y}\right)^2}{1-\left(\frac{v}{y}\right)^2}\left(\frac{\partial_y v}{y}+\frac{v}{y^2}\right) \tag{3.35}$$

is the remaining (quadratic) parts. Velázquez showed that the linear differential operator $\mathcal{L}$ has the following properties (see [V]):

Proposition 3.8. *Define an inner product*

$$\langle \mathsf{v}_1,\, \mathsf{v}_2\rangle = \int_0^\infty \mathsf{v}_1(y)\,\mathsf{v}_2(y)\; y^{2(n-1)}e^{-\frac{y^2}{4}}\,dy$$

and the associated norm

$$\|\mathsf{v}\| \;=\; \sqrt{\langle \mathsf{v},\, \mathsf{v}\rangle}$$

Let $\mathbf{H}$ *be the Hilbert space formed by the completion of* $C_c^\infty(\mathbb{R}_+)$ *with respect to the following inner product:*

$$(\mathsf{v}_1,\, \mathsf{v}_2) \equiv \langle \partial_y\mathsf{v}_1,\, \partial_y\mathsf{v}_2\rangle + \langle \mathsf{v}_1,\, \mathsf{v}_2\rangle$$

Then we have

$$\left\|\frac{\mathsf{v}}{y}\right\|^2 \;\le\; \frac{4}{(2n-3)^2}\,\|\partial_y\mathsf{v}\|^2 \;+\; \frac{1}{2n-3}\,\|\mathsf{v}\|^2$$

and $\mathcal{L}$ *is a bounded linear operator in* $\mathbf{H}$*, which satisfies*

$$\langle \mathcal{L}\mathsf{v}_1,\, \mathsf{v}_2\rangle = \langle \partial_y\mathsf{v}_1,\, \partial_y\mathsf{v}_2\rangle - 2(n-1)\left\langle \frac{\mathsf{v}_1}{y},\, \frac{\mathsf{v}_2}{y}\right\rangle - \frac{1}{2}\langle \mathsf{v}_1,\, \mathsf{v}_2\rangle$$

$$\langle \mathcal{L}\mathsf{v},\, \mathsf{v}\rangle \;\ge\; \frac{4n^2-20n+17}{(2n-3)^2}\,\|\partial_y\mathsf{v}\|^2 \;-\; \frac{6n-7}{2(2n-3)}\,\|\mathsf{v}\|^2 \tag{3.36}$$

Note that $4n^2-20n+17\ge 1$ *if* $n\ge 4$.

Moreover, the eigenvalues and eigenfunctions of $\mathcal{L}$ *are given by*

$$\lambda_i = -\frac{1}{2}(1-\alpha) + i, \quad \textit{for}\;\; i = 0,\, 1,\, 2, \cdots \tag{3.37}$$

and

$$\varphi_i(y) = c_i\, y^\alpha\, M\left(-i,\, n+\alpha-\frac{1}{2};\, \frac{y^2}{4}\right)$$

respectively, where $c_i > 0$ *is the normalized constant so that*

$$\|\varphi_i\| = \sqrt{\langle \varphi_i, \, \varphi_i \rangle} = 1$$

and $M\,(\mathrm{a},\, \mathrm{b};\, \xi)$ *is the Kummer's function defined by*

$$M\,(\mathrm{a},\, \mathrm{b};\, \xi) = 1 + \sum_{j=1}^{\infty} \frac{\mathrm{a}\,(\mathrm{a}+1)\cdots(\mathrm{a}+j-1)}{\mathrm{b}\,(\mathrm{b}+1)\cdots(\mathrm{b}+j-1)} \frac{\xi^j}{j!}$$

and satisfying

$$\xi\, \partial^2_{\xi\xi} M\,(\mathrm{a},\, \mathrm{b};\, \xi) + (\mathrm{b} - \xi)\, \partial_\xi M\,(\mathrm{a},\, \mathrm{b};\, \xi) - \mathrm{a}\, M\,(\mathrm{a},\, \mathrm{b};\, \xi) = 0$$

In addition, the family of eigenfunctions $\{\varphi_i\}_{i=0,\,1,\,2,\cdots}$ *forms a complete orthonormal set in* $\mathbf{H}$*, and* λ_2 *is the first positive eigenvalue of* $\mathcal{L}$*, i.e.*

$$\lambda_0,\, \lambda_1 < 0, \qquad \lambda_2 > 0$$

Remark 3.9. The first three eigenfunctions of $\mathcal{L}$ are given by

$$\varphi_0\,(y) = c_0\, y^\alpha$$

$$\varphi_1\,(y) = c_1\, y^\alpha \left(1 + \varUpsilon_1 y^2\right)$$

$$\varphi_2\,(y) = c_2\, y^\alpha \left(1 + 2\varUpsilon_1 y^2 + \varUpsilon_2 y^4\right)$$

where

$$\varUpsilon_1 = \frac{-1}{4\left(n + \alpha - \frac{1}{2}\right)}, \qquad \varUpsilon_2 = \frac{1}{16\left(n + \alpha - \frac{1}{2}\right)\left(n + \alpha + \frac{1}{2}\right)}$$

Note that

$$\partial^2_{yy}\varphi_2\,(y) = c_2\, y^{\alpha-2} \left(\alpha\,(\alpha-1) + 2\varUpsilon_1\,(\alpha+2)\,(\alpha+1)\, y^2 + \varUpsilon_2\,(\alpha+4)\,(\alpha+3)\, y^4\right) > 0$$

for $y > 0$. In addition, for those constants, there hold

$$\alpha + 4 = 2\lambda_2 + 1$$

$$\sigma = \frac{\lambda_2}{1 - \alpha}$$

Furthermore, when $n \gg 1$, we have

$$\alpha \approx -1 - \frac{1}{n}, \qquad \sigma \approx \frac{1}{2} - \frac{1}{2n}$$

$$\lambda_0 \approx -1 - \frac{1}{2n}, \quad \lambda_1 \approx -\frac{1}{2n}, \quad \lambda_2 \approx 1 - \frac{1}{2n}$$

Lastly, in the tip region, we do the type II rescaling to get

$$\Gamma_\tau = \left\{ (z\,\nu,\ \hat{w}\,(z,\,\tau)\,\omega) \,\middle|\, z \geq 0;\ \nu,\,\omega \in \mathbb{S}^{n-1} \right\} \tag{3.38}$$

where

$$\hat{w}\,(z,\,\tau) = \frac{1}{(-t)^{\frac{1}{2}+\sigma}}\,\hat{u}\left((-t)^{\frac{1}{2}+\sigma}\,z,\ t \right)\Bigg|_{t=-(2\sigma\tau)^{\frac{-1}{2\sigma}}} \tag{3.39}$$

From (3.21) we derive

$$\partial_\tau \hat{w} = \frac{\partial^2_{zz}\hat{w}}{1 + (\partial_z \hat{w})^2} + (n-1)\left(\frac{\partial_z \hat{w}}{z} - \frac{1}{\hat{w}} \right) + \frac{\frac{1}{2}+\sigma}{2\sigma\tau}\,(-z\,\partial_z \hat{w} + \hat{w}) \tag{3.40}$$

$$\hat{w}\,(0,\,\tau) > 0, \quad \partial_z \hat{w}\,(0,\,\tau) = 0$$

for $\tau_0 \leq \tau \leq \mathring{\tau}$. Velázquez showed that by chooing suitable initial data (see Section 3.4), there holds

$$\hat{w}\,(z,\,\tau) \xrightarrow{C^0_{loc}} \hat{\psi}_k\,(z)$$

for some $k \approx 1$, where $\hat{\psi}_k$ is the function defined in Section 3.2. On the other hand, by the admissible condition and rescaling, we can regard the rescaled projected curve

$$\bar{\Gamma}_\tau = \{ (z,\ \hat{w}\,(z,\,\tau)) \,|\, z \geq 0 \} \tag{3.41}$$

as a graph over $\bar{\mathcal{C}}$ outside $B\,(O;\,\beta)$. In other words, Γ_τ can be reparametrized as a normal graph over $\mathcal{C}$ outside $B\,(O;\,\beta)$, say

$$Z_\tau\,(z,\,\nu,\,\omega) = \left((z - w\,(z,\,\tau))\,\frac{\nu}{\sqrt{2}},\ (z + w\,(z,\,\tau))\,\frac{\omega}{\sqrt{2}} \right) \tag{3.42}$$

for $z \geq \beta$, where

$$\begin{aligned} w\,(z,\,\tau) &= \frac{1}{(-t)^{\frac{1}{2}+\sigma}}\,u\left((-t)^{\frac{1}{2}+\sigma}\,z,\ t \right)\Bigg|_{t=-(2\sigma\tau)^{-\frac{1}{2\sigma}}} \\ &= e^{\sigma s}\ v\left(e^{-\sigma s} z,\ s\right)\Big|_{s=\frac{1}{2\sigma}\ln(2\sigma\tau)} \end{aligned} \tag{3.43}$$

From (3.25) we derive

$$\partial_\tau w = \frac{\partial^2_{zz} w}{1 + (\partial_z w)^2} + 2\,(n-1)\,\frac{z\,\partial_z w + w}{z^2 - w^2} + \frac{\frac{1}{2} + \sigma}{2\sigma\tau}\,(-z\,\partial_z w + w) \tag{3.44}$$

Notice that (3.26) is equivalent to

$$z^i \left|\partial^i_z w\,(z,\,\tau)\right| < \Lambda \left(z^\alpha + \frac{z^{2\lambda_2+1}}{(2\sigma\tau)^2} \right), \qquad i \in \{0,\,1,\,2\} \tag{3.45}$$

for $\beta \le z \le \rho\,(2\sigma\tau)^{\frac{1}{2}+\frac{1}{4\sigma}}$, $\tau_0 \le \tau \le \mathring{\tau}$.

3.4 Construction of Velázquez's solution

For readers' convenience and also for the sake of the completeness of the argument, in this section we show how Velázquez's solution is constructed. We basically follow Velázquez's idea in [V] and modify his proofs and estimates. Also, our setting is slightly different from that in [V] since we assume more condtions in order to get better results. The key step is Proposition 3.13 and Proposition 3.14. The main theorem in this section is Theorem 3.17.

The idea is as follows. At the initial time t_0, we would choose a bunch of "initial hypersurfaces" $\left\{\Sigma^{(a_0,\,a_1)}_{t_0}\right\}_{(a_0,\,a_1)}$ (as candidates) and move each of them by the mean curvature vector. We then manage to show that for each $\mathring{t} \in [t_0,\,0)$, there is an index $(a_0,\,a_1)$ for which the corresponding mean curvature flow $\left\{\Sigma^{(a_0,\,a_1)}_t\right\}_{t \ge t_0}$ exits and is admissible up to time $\mathring{t}$. In addition, we would establish uniform estimates for these solutions. Lastly, by the compactness theory, we then get a solution to the MCF which exists and is admissible for $t_0 \le t < 0$ and also admits those uniform estimates.

Let's start with choosing a proper family of **initial hypersurfaces**. Let

$$\left\{ \Sigma^{(a_0,\,a_1)}_{t_0} \,\middle|\, (a_0,\,a_1) \in \overline{B}^2\left(O;\, \beta^{2(\alpha-1)}\right) \right\}$$

be a continuous two-parameters family of complete, embedded and smooth hypersurfaces so that each element $\Sigma^{(a_0,\,a_1)}_{t_0}$ is **admissible at time** t_0 and satisfies

1. The funtion $v\,(y,\,s_0) = v^{(a_0,\,a_1)}\,(y,\,s_0)$ (defined in (3.29)) of the type I rescaled hypersurface

$$\Pi^{(a_0,\,a_1)}_{s_0} = \frac{1}{\sqrt{-t_0}}\Sigma^{(a_0,\,a_1)}_{t_0}$$

is given by

$$v\left(y,\, s_0\right) = e^{-\lambda_2 s_0}\left(\frac{1}{c_2}\varphi_2\left(y\right) + \frac{a_1}{c_1}\varphi_1\left(y\right) + \frac{a_0}{c_0}\varphi_0\left(y\right)\right) \tag{3.46}$$

$$= e^{-\lambda_2 s_0} y^{\alpha}\left(1 + a_1 + a_0 + \left(2 + a_1\right)\Upsilon_1 y^2 + \Upsilon_2 y^4\right)$$

for $\frac{1}{2}\beta e^{-\sigma s_0} \leq y \leq 2\rho e^{\frac{s_0}{2}}$ (see Proposition 3.8 and Remark 3.9).

2. The function $u(x,\, t_0) = u^{(a_0,\, a_1)}(x,\, t_0)$ (defined in (3.24)) of $\Sigma_{t_0}^{(a_0,\, a_1)}$ is chosen to be

$$u(x,\, t_0) \approx \frac{\Upsilon_2 x^{2\lambda_2 + 1}}{1 + x^4}$$

for $x \gtrsim \rho$ so that

$$\begin{cases} \left|u\left(x,\, t_0\right)\right| \leq \frac{1}{5}\, \min\left\{x,\, 1\right\} \\ \\ \left|\partial_x u\left(x,\, t_0\right)\right| \leq \frac{1}{5} \\ \\ \left|\partial_{xx}^2 u\left(x,\, t_0\right)\right| \leq C\left(n,\, \rho\right) \end{cases} \tag{3.47}$$

for $x \geq \frac{1}{6}\rho$.

3. The function $\hat{w}\left(\cdot,\, \tau_0\right) = \hat{w}^{(a_0,\, a_1)}\left(\cdot,\, \tau_0\right)$ (defined in (3.38)) of the type II rescaled hypersurface

$$\Gamma_{\tau_0}^{(a_0,\, a_1)} = \frac{1}{\left(-t_0\right)^{\frac{1}{2}+\sigma}} \Sigma_{t_0}^{(a_0,\, a_1)}$$

is chosen to be

$$\hat{w}\left(z,\, \tau_0\right) \approx \hat{\psi}_{1 + a_1 + a_0}\left(z\right)$$

for $0 \leq z \lesssim \beta$ so that

$$\begin{cases} \hat{\psi}_{1 - \beta^{\frac{3}{2}\alpha - \frac{5}{2}}}\left(z\right) \,<\, \hat{w}\left(z,\, \tau_0\right) \,<\, \hat{\psi}_{1 + \beta^{\frac{3}{2}\alpha - \frac{5}{2}}}\left(z\right) \\ \\ 0 \,=\, \partial_z \hat{w}\left(0,\, \tau_0\right) \,\leq\, \partial_z \hat{w}\left(z,\, \tau_0\right) \,<\, 1 \\ \\ 0 \,<\, \partial_{zz}^2 \hat{w}\left(z,\, \tau_0\right) \,\leq\, C\left(n\right) \end{cases} \tag{3.48}$$

for $0 \le z \le 5\beta$. Furthermore, if we reparametrize the projected curve $\bar{\Gamma}_{\tau_0}^{(a_0,\,a_1)}$ as a graph over $\bar{\mathcal{C}}$, the function $w^{(a_0,\,a_1)}(z,\,\tau_0) = w(z,\,\tau_0)$ (defined in (3.42)) satisfies

$$w(z,\,\tau_0) \approx \psi_{1+a_1+a_0}(z)$$

for $1 \lesssim z \lesssim \beta$ so that

$$\begin{cases} 0 \le w(z,\,\tau_0) \le C(n)\, z^{\alpha} \\ \\ |\partial_z w(z,\,\tau_0)| \le C(n)\, z^{\alpha-1} \\ \\ 0 < \partial_{zz}^2 w(z,\,\tau_0) \le C(n)\, z^{\alpha-2} \end{cases} \tag{3.49}$$

for $\frac{\hat{\psi}_2(0)}{\sqrt{2}} \le z \le 5\beta$,

The following remark shows that (3.46) fits in with the admissible condition and is compatible with (3.47).

Remark 3.10. By (3.30) and Remark 3.9, (3.46) is equivalent to

$$u(x,\,t_0) = (-t)^{\lambda_2+\frac{1}{2}}\left(\frac{1}{c_2}\varphi_2\left(\frac{x}{\sqrt{-t}}\right) + \frac{a_1}{c_1}\varphi_1\left(\frac{x}{\sqrt{-t}}\right) + \frac{a_0}{c_0}\varphi_0\left(\frac{x}{\sqrt{-t}}\right)\right)$$

$$= (1+a_1+a_0)(-t_0)^2 x^{\alpha} + (2+a_1)\Upsilon_1(-t_0)\, x^{\alpha+2} + \Upsilon_2 x^{2\lambda_2+1}$$

$$= x^{2\lambda_2+1}\left(\Upsilon_2 + (2+a_1)\Upsilon_1\left(\frac{-t_0}{x^2}\right) + (1+a_1+a_0)\left(\frac{-t_0}{x^2}\right)^2\right) \tag{3.50}$$

for $\frac{1}{2}\beta(-t_0)^{\frac{1}{2}+\sigma} \le x \le 2\rho$. In particular, there hold

$$x^i \left|\partial_x^i u(x,\,t)\right| \le C(n)\left((-t)^2 x^{\alpha} + x^{2\lambda_2+1}\right), \qquad i \in \{0,\,1,\,2\}$$

$$\left|\frac{u(x,\,t_0)}{x}\right| \le C(n)\left(\beta^{\alpha-1} + \rho^{2\lambda_2}\right) \tag{3.51}$$

for $\frac{1}{2}\beta(-t_0)^{\frac{1}{2}+\sigma} \le x \le 2\rho$. Thus, we may assume that

$$x^i \left|\partial_x^i u(x,\,t)\right| \le \frac{\Lambda}{3}\left((-t)^2 x^{\alpha} + x^{2\lambda_2+1}\right), \qquad i \in \{0,\,1,\,2\}$$

for $\beta\left(-t_0\right)^{\frac{1}{2}+\sigma} \leq x \leq \rho$, provided that $\Lambda \gg 1$ (depending on n). Also by (3.47), (3.48) and (3.51), we may assume that

$$\hat{u}\left(x,\, t_0\right) > 0$$

for $x \geq 0$, provided that $0 < \rho \ll 1 \ll \beta$ (depending on n). Furthermore, by (3.50) we have

$$u\left(x,\, t_0\right) = x^{2\lambda_2+1}\left(\Upsilon_2 + O\left(\frac{-t_0}{x^2}\right)\right)$$

for $\sqrt{-t_0} \lesssim x \leq 2\rho$, which is comparible with (3.47) provided that $0 < \rho \ll 1$ (depending on n) and $|t_0| \ll 1$ (depending on n, ρ).

The following remark shows that (3.46), (3.48) and (3.49) are compatible.

Remark 3.11. By (3.48), $\bar{\Gamma}_{\tau_0}^{(a_0,\, a_1)}$ (see (3.41)) is a convex curve which lies between $\bar{\mathcal{M}}_{1-\beta^{\frac{3}{2}\alpha-\frac{5}{2}}}$ and $\bar{\mathcal{M}}_{1+\beta^{\frac{3}{2}\alpha-\frac{5}{2}}}$ (see (3.7)) and intersects orthogonally with the vertical ray $\{(0,\, z)|\; z > 0\}$. Hence, if we reparametrize $\bar{\Gamma}_{\tau_0}^{(a_0,\, a_1)}$ as a graph over $\bar{\mathcal{C}}$, it follows that

$$\psi_{1-\beta^{\frac{3}{2}\alpha-\frac{5}{2}}}\left(z\right) \;<\; w\left(z,\, \tau_0\right) \;<\; \psi_{1+\beta^{\frac{3}{2}\alpha-\frac{5}{2}}}\left(z\right)$$

Then (3.49) is compatible with (3.48) in view of Lemma 3.5.

On the other hand, by (3.43) and Remark 3.9, (3.46) is equivalent to

$$\begin{aligned} w\left(z,\, \tau_0\right) &= \left(2\sigma\tau_0\right)^{\frac{\alpha}{2}}\left(\frac{1}{c_2}\varphi_2\left(\frac{z}{\sqrt{2\sigma\tau_0}}\right) + \sum_{i=0}^{1}\frac{a_i}{c_i}\varphi_i\left(\frac{z}{\sqrt{2\sigma\tau_0}}\right)\right) \\ &= z^{\alpha}\left(1 + a_1 + a_0 + \left(2 + a_1\right)\Upsilon_1\frac{z^2}{2\sigma\tau_0} + \Upsilon_2\left(\frac{z^2}{2\sigma\tau_0}\right)^2\right) \end{aligned} \tag{3.52}$$

for $\frac{1}{2}\beta \leq z \leq 2\rho\left(2\sigma\tau_0\right)^{\frac{1}{2}+\frac{1}{4\sigma}}$, which means

$$w\left(z,\, \tau_0\right) = \left(1 + a_1 + a_0 + O\left(\frac{z^2}{2\sigma\tau_0}\right)\right)z^{\alpha}$$

for $\frac{1}{2}\beta \leq z \leq \sqrt{2\sigma\tau_0}$. By Lemma 3.7, we then get

$$\begin{aligned} &\left|w\left(z,\, \tau_0\right) - \psi\left(z\right)\right| \;\leq\; \left|w\left(z,\, \tau_0\right) - z^{\alpha}\right| \;+\; \left|z^{\alpha} - \psi\left(z\right)\right| \\ \leq &\left(\left|a_0\right| + \left|a_1\right| + C\left(n\right)\left(\frac{z^2}{2\sigma\tau_0} + z^{2(\alpha-1)}\right)\right)z^{\alpha} \;\leq\; C\left(n\right)\beta^{2(\alpha-1)}z^{\alpha} \end{aligned}$$

for $\frac{1}{2}\beta \le z \le (2\sigma\tau_0)^{\frac{1}{3}}$, provided that $\beta \gg 1$ (depending on n) and $\tau_0 \gg 1$ (depending on n, β). Note also that Lemma 3.7 yields

$$\psi_{1\pm\beta^{\frac{3}{2}\alpha-\frac{5}{2}}}(z) - \psi(z) = \left(\pm\beta^{\frac{3}{2}\alpha-\frac{5}{2}} + O\left(z^{2(\alpha-1)}\right)\right) z^{\alpha}$$

in which we have

$$\frac{3}{2}\alpha - \frac{5}{2} > 2(\alpha-1)$$

Consequently, we get

$$\psi_{1-\beta^{\frac{3}{2}\alpha-\frac{5}{2}}}(z) < w(z,\tau_0) < \psi_{1+\beta^{\frac{3}{2}\alpha-\frac{5}{2}}}(z)$$

for $\frac{1}{2}\beta \le z \le (2\sigma\tau_0)^{\frac{1}{3}}$, provided that $\beta \gg 1$ (depending on n) and $\tau_0 \gg 1$ (depending on n, β).

Next, for each $(a_0, a_1) \in \overline{B}^2\left(O;\, \beta^{2(\alpha-1)}\right)$, by [EH] $\Sigma_{t_0}^{(a_0,\, a_1)}$ can be flowed by (3.19) for a short period of time. Let's denote the corresponding solution by $\left\{\Sigma_t^{(a_0,\, a_1)}\right\}$. Given $\mathring{t} \in [t_0, 0)$, let $\mathcal{O}_{\mathring{t}}$ be a set consisting of all $(a_0, a_1) \in B^2\left(O;\, \beta^{2(\alpha-1)}\right)$ for which

- The corresponding mean curvature flow $\left\{\Sigma_t^{(a_0,\, a_1)}\right\}$ exists for $t_0 \le t \le \mathring{t}$ and can be extended beyond time $\mathring{t}$.
- $\left\{\Sigma_t^{(a_0,\, a_1)}\right\}$ is **admissible** for $t_0 \le t \le \mathring{t}$.

Clearly,

$$\mathcal{O}_{t_0} = B^2\left(O;\, \beta^{2(\alpha-1)}\right)$$

and $\mathcal{O}_{\mathring{t}}$ is non-increasing in $\mathring{t}$.

Now let $\zeta(r)$ be a smooth, non-decreasing function so that

$$\zeta(r) = \begin{cases} 0, & \text{for } r \le 0 \\ \\ 1, & \text{for } r \ge 1 \end{cases} \tag{3.53}$$

For each $t \ge t_0$, we define a map $\Phi_t : \overline{\mathcal{O}}_t \to \mathbb{R}^2$ by

$$\Phi_t(a_0, a_1) = \left.\begin{pmatrix} \left\langle \zeta\left(e^{\sigma s}y - \beta\right)\zeta\left(\rho e^{\frac{s}{2}} - y\right) v(\cdot, s),\ c_0\,\varphi_0 \right\rangle \\ \\ \left\langle \zeta\left(e^{\sigma s}y - \beta\right)\zeta\left(\rho e^{\frac{s}{2}} - y\right) v(\cdot, s),\ c_1\,\varphi_1 \right\rangle \end{pmatrix}\right|_{s=-\ln(-t)} \tag{3.54}$$

where the inner product $\langle \cdot, \cdot \rangle$ is defined in Proposition 3.8 and $v(y, s) = v^{(a_0, a_1)}(y, s)$ is the function of $\Pi_s^{(a_0, a_1)}$ defined in (3.29) with $s = -\ln(-t)$. Note that the localized function

$$\tilde{v}(y, s) = \zeta(e^{\sigma s} y - \beta)\, \zeta\left(\rho e^{\frac{s}{2}} - y\right) v(y, s)$$

appeared in (3.54) is supported in $\left[\beta e^{-\sigma s}, \rho e^{\frac{s}{2}}\right]$ and would be studied carefully in Proposition 3.26. When $t = t_0$, we have the following lemma.

Lemma 3.12. *If $s_0 \gg 1$ (depending on n, ρ, β), there hold*

$$\left|\left\langle \zeta(e^{\sigma s_0} y - \beta)\, \zeta\left(\rho e^{\frac{s_0}{2}} - y\right) \varphi_i, \varphi_j \right\rangle - \delta_{ij}\right| \le C(n)\, e^{-2\left(n + \alpha - \frac{1}{2}\right)\sigma s_0}$$

$$\left\| \left(1 - \zeta(e^{\sigma s_0} y - \beta)\, \zeta\left(\rho e^{\frac{s_0}{2}} - y\right)\right) \varphi_i \right\| \le C(n)\, e^{-\left(n + \alpha - \frac{1}{2}\right)\sigma s_0}$$

for $i, j \in \{0, 1, 2\}$, where $s_0 = -\ln(-t_0)$ and φ_i is the i^{th} eigenfunction of $\mathcal{L}$ (see Proposition 3.8).

Proof. Notice that

$$\langle \varphi_i, \varphi_j \rangle = \delta_{ij}$$

and

$$\zeta(e^{\sigma s_0} y - \beta)\, \zeta\left(\rho e^{\frac{s_0}{2}} - y\right) \to 1 \qquad \text{as } s_0 \nearrow \infty$$

Then we compute

$$\begin{aligned}
&\left|\left\langle \zeta(e^{\sigma s_0} y - \beta)\, \zeta\left(\rho e^{\frac{s_0}{2}} - y\right) \varphi_i, \varphi_j \right\rangle - \delta_{ij}\right| \\
&= \left|\left\langle \left(1 - \zeta(e^{\sigma s_0} y - \beta)\, \zeta\left(\rho e^{\frac{s_0}{2}} - y\right)\right) \varphi_i, \varphi_j \right\rangle\right| \\
&\le \int_0^{(\beta+1)e^{-\sigma s_0}} |\varphi_i \varphi_j|\, y^{2(n-1)} e^{-\frac{y^2}{4}}\, dy + \int_{\rho e^{\frac{s_0}{2}} - 1}^{\infty} |\varphi_i \varphi_j|\, y^{2(n-1)} e^{-\frac{y^2}{4}}\, dy \\
&\le C(n) \left(\int_0^{(\beta+1)e^{-\sigma s_0}} y^{2\alpha} y^{2(n-1)} dy + \int_{\rho e^{\frac{s_0}{2}} - 1}^{\infty} y^{2\lambda_i + 2\lambda_j + 2} y^{2(n-1)} e^{-\frac{y^2}{4}}\, dy \right) \\
&\le C(n)\, e^{-2\left(n + \alpha - \frac{1}{2}\right)\sigma s_0}
\end{aligned}$$

It follows that

$$\left\| \left(1 - \zeta(e^{\sigma s_0} y - \beta)\, \zeta\left(\rho e^{\frac{s_0}{2}} - y\right)\right) \varphi_i \right\|^2$$

$$= \left\langle \left(1 - \zeta\left(e^{\sigma s_0} y - \beta\right) \zeta\left(\rho e^{\frac{s_0}{2}} - y\right)\right) \varphi_i,\, \left(1 - \zeta\left(e^{\sigma s_0} y - \beta\right) \zeta\left(\rho e^{\frac{s_0}{2}} - y\right)\right) \varphi_i \right\rangle$$

$$\leq \left\langle \left(1 - \zeta\left(e^{\sigma s_0} y - \beta\right) \zeta\left(\rho e^{\frac{s_0}{2}} - y\right)\right) \varphi_i,\, \varphi_i \right\rangle$$

$$\leq C(n)\, e^{-2\left(n+\alpha-\frac{1}{2}\right)\sigma s_0}$$

$\square$

By (3.46) and Lemma 3.12, the function Φ_{t_0} converges uniformly to the identity map in $\overline{B}^2\left(O;\, \beta^{2(\alpha-1)}\right)$ as $t_0 \nearrow 0$. Thus, if $|t_0| \ll 1$ (depending on n, β), we have

$$(0,\, 0) \notin \Phi_{t_0}\left(\partial \overline{B}^2\left(O;\, \beta^{2(\alpha-1)}\right)\right)$$

and

$$1 = \deg\left(\mathrm{Id},\, B^2\left(O;\, \beta^{2(\alpha-1)}\right),\, (0,\, 0)\right) = \deg\left(\Phi_{t_0},\, B^2\left(O;\, \beta^{2(\alpha-1)}\right),\, (0,\, 0)\right)$$

$$= \deg\left(\Phi_{t_0},\, \mathcal{O}_{t_0},\, (0,\, 0)\right) \tag{3.55}$$

In addition, notice that $\mathcal{O}_t$ is an open subset of $B^2\left(O;\, \beta^{2(\alpha-1)}\right)$ (by the continuous dependence on the initial data), and that Φ_t is continuous in the parameter t. Then we consider the following index set

$$\mathcal{I} = \left\{t \in [t_0,\, 0) \mid \deg\left(\Phi_t,\, \mathcal{O}_t,\, (0,\, 0)\right) = 1\right\}$$

Below are crucial a priori estimates of $\left\{\Sigma_t^{(a_0,\, a_1)}\right\}_{t_0 \leq t \leq t_1}$ for which

$$\Phi_{t_1}(a_0,\, a_1) = (0,\, 0)$$

We leave the proof in Section 3.6, Section 3.7 and Section 3.8.

Proposition 3.13. *Let $n \geq 5$ be a positive integer and choose $\varsigma = \varsigma(n) > 0$, $\vartheta = \vartheta(n) \in (0,\, 1)$ so that*

$$0 < \varsigma < \min\left\{\frac{n+\alpha-\frac{5}{2}}{1-\alpha},\, \frac{1}{\lambda_2}\right\} \tag{3.56}$$

$$\frac{-1-\alpha}{1-\alpha} < \vartheta < \min\left\{\frac{(1-\alpha)\varsigma}{n+\alpha+\frac{3}{2}},\, \frac{1-\alpha}{2-\alpha},\, \frac{1}{2\sigma}\right\} \tag{3.57}$$

Assume that $(a_0,\, a_1) \in \overline{\mathcal{O}}_{t_1}$ for which

$$\Phi_{t_1}(a_0,\, a_1) = (0,\, 0)$$

where $t_1 \in [t_0,\, 0)$ *is a constant. Suppose that*

$$(a_0,\, a_1) \in \overline{\mathcal{O}}_{\mathring{t}}$$

for some $\mathring{t} \in \left[t_1,\, e^{-1}t_1\right]$*. Then if* $\Lambda \gg 1$ *(depending on* n*),* $0 < \rho \ll 1 \ll \beta$ *(depending on* n*,* Λ*) and* $|t_0| \ll 1$ *(depending on* n*,* Λ*,* ρ*,* β*), we have the following estimates.*

1. *The function* $\hat{u}\,(x,\, t)$ *defined in (3.20) satisfies*

$$\partial^2_{xx}\hat{u}\,(x,\, t) \geq 0 \tag{3.58}$$

for $0 \leq x \leq \rho$*,* $t_0 \leq t \leq \mathring{t}$*.*

2. *The function* $u\,(x,\, t)$ *defined in (3.24) satisfies*

$$\begin{cases} |u\,(x,\, t)| \,\leq\, \frac{1}{3}\min\{x,\, 1\} \\ \\ |\partial_x u\,(x,\, t)| \,\leq\, \frac{1}{3} \\ \\ \left|\partial^2_{xx} u\,(x,\, t)\right| \,\leq\, C\,(n,\, \rho) \end{cases} \tag{3.59}$$

for $x \geq \frac{1}{3}\rho$*,* $t_0 \leq t \leq \mathring{t}$*, and*

$$x^i \left|\partial^i_x u\,(x,\, t)\right| \,\leq\, \frac{\Lambda}{2}\left((-t)^2\, x^{\alpha} + x^{2\lambda_2+1}\right), \qquad i \in \{0,\, 1,\, 2\} \tag{3.60}$$

for $\beta\,(-t)^{\frac{1}{2}+\sigma} \leq x \leq \rho$*,* $t_0 \leq t \leq \mathring{t}$*.*

3. *In the tip region, if we do the type* II *rescaling, the rescaled function* $\hat{w}\,(z,\, \tau)$ *defined in (3.39) satisfies*

$$\begin{cases} \hat{\psi}_{1-2\beta^{\alpha-3}}\,(z) \,<\, \hat{w}\,(z,\, \tau) \,<\, \hat{\psi}_{1+2\beta^{\alpha-3}}\,(z) \\ \\ 0 \,\leq\, \partial_z \hat{w}\,(z,\, \tau) \,\leq\, 1 + \beta^{\alpha-2} \\ \\ \left|\partial^2_{zz}\hat{w}\,(z,\, \tau)\right| \,\leq\, C\,(n) \end{cases} \tag{3.61}$$

for $0 \leq z \leq 3\beta$*,* $\tau_0 \leq \tau \leq \mathring{\tau}$*, where* $\mathring{\tau} = \frac{1}{2\sigma\left(-\mathring{t}\right)^{2\sigma}}$*.*

Furthermore, we have the following asymptotic formulas and smooth estimates for the solution in Proposition 3.13.

Proposition 3.14. *Under the hypothesis of Proposition 3.13, there is*

$$k \in \left(1 - C\left(n,\, \Lambda,\, \rho,\, \beta\right)\left(-t_0\right)^{\varsigma\lambda_2},\, 1 + C\left(n,\, \Lambda,\, \rho,\, \beta\right)\left(-t_0\right)^{\varsigma\lambda_2}\right)$$

so that for any given $0 < \delta \ll 1$, $m,\, l \in \mathbb{Z}_+$, the following smooth estimates hold.

1. In the **outer region**, *the function $u(x,\, t)$ of $\Sigma_t^{(a_0,\, a_1)}$ defined in (3.24) satisfies*

$$\left|\partial_x^m \partial_t^l u(x,\, t)\right| \leq C\left(n,\, \rho,\, \delta,\, m,\, l\right) \tag{3.62}$$

for $x \geq \frac{1}{2}\rho$, $t_0 + \delta^2 \leq t \leq \mathring{t}$, and

$$x^{m+2l}\left|\partial_x^m \partial_t^l\left(u\left(x,\, t\right) - \frac{k}{c_2}\left(-t\right)^{\lambda_2 + \frac{1}{2}}\varphi_2\left(\frac{x}{\sqrt{-t}}\right)\right)\right| \leq C\left(n,\, \Lambda,\, \delta,\, m,\, l\right)\rho^{4\lambda_2}\, x^{2\lambda_2+1} \tag{3.63}$$

for $(x,\, t)$ satisfying $\frac{1}{2}\sqrt{-t} \leq x \leq \frac{3}{4}\rho$, $t_0 + \delta^2 x^2 \leq t \leq \mathring{t}$. Note that

$$\frac{k}{c_2}\left(-t\right)^{\lambda_2+\frac{1}{2}}\varphi_2\left(\frac{x}{\sqrt{-t}}\right) = kx^{2\lambda_2+1}\left(\Upsilon_2 + 2\Upsilon_1\frac{-t}{x^2} + \left(\frac{-t}{x^2}\right)^2\right)$$

(see Proposition 3.8 and Remark 3.9).

2. In the **intermediate region**, *if we rescale the hypersurface by the type* I *rescaling (see (3.27)), then the function $v\left(y,\, s\right)$ of the rescaled hypersurface $\Pi_s^{(a_0,\, a_1)}$ defined in (3.29) satisfies*

$$y^{m+2l}\left|\partial_y^m \partial_s^l\left(v\left(y,\, s\right) - \frac{k}{c_2}e^{-\lambda_2 s}\varphi_2\left(y\right)\right)\right| \leq C\left(n,\, \Lambda,\, \delta,\, m,\, l\right)e^{-\varkappa s}e^{-\lambda_2 s}y^{\alpha+2} \tag{3.64}$$

for $(y,\, s)$ satisfying $e^{-\vartheta\sigma s} \leq y \leq 2$, $s_0 + \delta^2 y^2 \leq s \leq \mathring{s}$, and

$$y^{m+2l}\left|\partial_y^m \partial_s^l\left(v\left(y,\, s\right) - e^{-\sigma s}\,\psi_k\left(e^{\sigma s}y\right)\right)\right| \leq C\left(n,\, \Lambda,\, \delta,\, m,\, l\right)\beta^{\alpha-3}e^{-2\varrho\sigma\left(s-s_0\right)}e^{-\lambda_2 s}y^{\alpha} \tag{3.65}$$

for $(y,\, s)$ satisfying $\frac{3}{2}\beta e^{-\sigma s} \leq y \leq e^{-\vartheta\sigma s}$, $s_0 + \delta^2 y^2 \leq s \leq \mathring{s}$, where $\mathring{s} = -\ln\left(-\mathring{t}\right)$ and

$$\varkappa = \min\left\{\varsigma\lambda_2 - \vartheta\sigma\left(n + \alpha + \frac{3}{2}\right),\, \frac{\varsigma\lambda_2}{2},\, 2\left(\lambda_2 + \left(\alpha - 2\right)\vartheta\sigma\right)\right\} > 0 \tag{3.66}$$

$$\varrho = 1 - \frac{1}{2}\left(1-\alpha\right)\left(1-\vartheta\right) \in \left(0,\, \vartheta\right) \tag{3.67}$$

are constants. Note that

$$\frac{k}{c_2}e^{-\lambda_2 s}\varphi_2\left(y\right) = ke^{-\lambda_2 s}y^{\alpha}\left(1 + 2\Upsilon_1 y^2 + \Upsilon_2 y^4\right)$$

$$e^{-\sigma s}\,\psi_k\left(e^{\sigma s}y\right)=ke^{-\lambda_2 s}y^{\alpha}\left(1+O\left(\left(e^{\sigma s}y\right)^{-2(1-\alpha)}\right)\right)$$

(see Proposition 3.8 and (3.8)).

3. In the **tip region***, if we rescale the hypersurface by the type* II *rescaling (see (3.28)), then the function* $\hat{w}\,(z,\,\tau)$ *of the rescaled hypersurface* $\Gamma_\tau^{(a_0,\,a_1)}$ *defined in (3.38) satisfies*

$$\delta^{m+2l}\left|\partial_z^m\partial_\tau^l\left(\hat{w}\,(z,\,\tau)-\hat{\psi}_k\,(z)\right)\right|\le C\,(n,\,m,\,l)\,\beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho} \tag{3.68}$$

for $0\le z\le 2\beta$*,* $\tau_0+\delta^2\le\tau\le\mathring{\tau}$*, where* $\mathring{\tau}=\frac{1}{2\sigma\left(-\mathring{t}\right)^{2\sigma}}$*.*

Remark 3.15. By Proposition 3.13, Proposition 3.14 and [EH], we may infer that if $(a_0,\,a_1)\in\overline{\mathcal{O}}_{t_1}$ and

$$\Phi_{t_1}\,(a_0,\,a_1)=(0,\,0)$$

then $(a_0,\,a_1)\in\mathcal{O}_{e^{-1}t_1}$. In other words, $\Sigma_{t_0}^{(a_0,\,a_1)}$ is a "good" candidate of initial hypersurfaces to flow.

We then have the following corollary.

Corollary 3.16. *If* $|t_0|\ll 1$ *(depending on* n*), then we have* $\mathcal{I}=[t_0,\,0)$*.*

Proof. Notice that by (3.55) we have $t_0\in\mathcal{I}$. Then we would like to prove the corollary by induction.

Assume that $t_1\in\mathcal{I}$. The goal is to show that $t_2\in\mathcal{I}$ for any $t_2\in\left[t_1,\,e^{-1}t_1\right]$. By definition, there holds

$$\deg\left(\Phi_{t_1},\,\mathcal{O}_{t_1},\,(0,\,0)\right)=1$$

It follows that there is $(a_0,\,a_1)\in\mathcal{O}_{t_1}$ for which

$$\Phi_{t_1}\,(a_0,\,a_1)=(0,\,0)$$

By Remark 3.15, we then have $(a_0,\,a_1)\in\mathcal{O}_{t_2}$ and $(0,\,0)\notin\Phi_t\left(\partial\mathcal{O}_{t_2}\right)$ for all $t_1\le t\le t_2$. Consequently, $\mathcal{O}_{t_2}$ is non-empty and the degree of Φ_t at $(0,\,0)$ is well defined in $\mathcal{O}_{t_2}$ for each $t_1\le t\le t_2$. Since Φ_t is continuous in t, by the homotopy invariance of degree, there holds

$$\deg\left(\Phi_{t_2},\,\mathcal{O}_{t_2},\,(0,\,0)\right)=\deg\left(\Phi_{t_1},\,\mathcal{O}_{t_2},\,(0,\,0)\right)$$

In addition, by Remark 3.15, $(0,\,0) \notin \Phi_{t_1}(\mathcal{O}_{t_1} \setminus \mathcal{O}_{t_2})$, which, by the excision property of degree, implies that

$$\deg\left(\Phi_{t_1},\, \mathcal{O}_{t_2},\, (0,\,0)\right) = \deg\left(\Phi_{t_1},\, \mathcal{O}_{t_1},\, (0,\,0)\right) = 1$$

Therefore, we get $t_2 \in \mathcal{I}$. □

Now we are ready to prove the existence theorem of Velázquez's solution.

Theorem 3.17. *Let $n \geq 5$ be a positive integer. If $|t_0| \ll 1$ (depending on n), there is an* **admissible** *mean curvature flow $\{\Sigma_t\}_{t_0 \leq t < 0}$ (see Section 3.3) for which the the functions $\hat{u}(x,\,t)$ and $u(x,\,t)$ (defined in (3.20) and (3.24), respectively) satisfy (3.58) and (3.59). Besides, in the tip region, if we perform the type* II *rescaling, the rescaled function $\hat{w}(\cdot,\,\tau)$ (defined in (3.39)) satisfies (3.61).*

In addition, there is

$$k \in \left(1 - C(n)\,(-t_0)^{\varsigma\lambda_2},\, 1 + C(n)\,(-t_0)^{\varsigma\lambda_2}\right)$$

so that for any given $0 < \delta \ll 1$, $m,\, l \in \mathbb{Z}_+$, there hold

1. In the **outer region**, *the function $u(x,\,t)$ of Σ_t defined in (3.24) satisfies (3.62) and (3.63).*

2. In the **intermediate region**, *if we do the type* I *rescaling, the function $v(y,\,s)$ of the rescaled hypersurface Π_s defined in (3.29) satisfies (3.64) and (3.65).*

3. In the **tip region**, *if we do the type* II *rescaling, the function $\hat{w}(\cdot,\,\tau)$ of the rescaled hypersurface Γ_τ defined in (3.38) satisfies (3.68).*

Proof. Let $t_i > t_0$ be a sequence so that $t_i \nearrow 0$. By Corollary 3.16, there is $\left(a_0^i,\, a_1^i\right) \in \mathcal{O}_{t_i}$ for which

$$\Phi_{t_i}\left(a_0^i,\, a_1^i\right) = (0,\,0)$$

By the uniform estimates in Proposition 3.13 and Proposition 3.14, we may assume (by passing to a subsequence) that as $i \to \infty$,

$$k^{\left(a_0^i,\, a_1^i\right)} \to k$$

and the functions $\left\{\hat{u}^{\left(a_0^i,\,a_1^i\right)}(x,\,t)\right\}$ and $\left\{u^{\left(a_0^i,\,a_1^i\right)}(x,\,t)\right\}$ of $\Sigma_t^{\left(a_0^i,\,a_1^i\right)}$ (defined in (3.20) and (3.24)) converge locally smoothly to $\hat{u}(x,\,t)$ and $u(x,\,t)$, respectively. The conclusion follows immediately by passing the uniform estimates (in Proposition 3.13 and Proposition 3.14) to limit. $\square$

Remark 3.18. Let $\{\Sigma_t\}_{t_0 \le t < 0}$ be Velázquez's solution in Theorem 3.17. From (3.29), (3.30), (3.63) and (3.64), the type I rescaled hypersurfaces Π_s (see (3.27)) converges smoothly to $\mathcal{C}$ on any fixed annulus centered at O, i.e. for any $0 < r < R < \infty$,

$$\Pi_s \xrightarrow{C^\infty} \mathcal{C} \qquad \text{in } B(O;\,R) \setminus B(O;\,r)$$

as $s \nearrow \infty$. Likewise, from (3.38), (3.42), (3.43), (3.65) and (3.68), the type II rescaled hypersurfaces Γ_τ (see (3.28)) converges to $\mathcal{M}_k$ locally smoothly, i.e.

$$\Gamma_\tau \xrightarrow{C^\infty_{loc}} \mathcal{M}_k$$

In addition, by the admissible conditions, the projected curve $\bar{\Sigma}_t$ (see (3.22)) is a graph over $\bar{\mathcal{C}}$ outside $B\left(O;\,\beta(-t)^{\frac{1}{2}+\sigma}\right)$. By (3.58) and the admissible conditions, we know that inside $B\left(O;\,\beta(-t)^{\frac{1}{2}+\sigma}\right)$, $\bar{\Sigma}_t$ is a convex curve which intersects orthogonally with the vertical ray $\{(0,\,x)|\ x > 0\}$; moreover, if we zoom in at O by the type II rescaling, by (3.4) and (3.79), the rescaled curve $\bar{\Gamma}_\tau$ (see 3.41) lies above $\bar{\mathcal{C}}$ and tends to it for $z \nearrow \beta$. Therefore, $\bar{\Gamma}_\tau$ is a graph over $\bar{\mathcal{C}}$ inside $B(O;\,\beta)$, which in turn implies that $\bar{\Sigma}_t$ is also graph over $\bar{\mathcal{C}}$ inside $B\left(O;\,\beta(-t)^{\frac{1}{2}+\sigma}\right)$. Hence, we get

$$\bar{\Sigma}_t = \{(x,\,\hat{u}(x,\,t))|\ x \ge 0\}$$

$$= \left\{\left((x - u(x,\,t))\frac{1}{\sqrt{2}},\,(x + u(x,\,t))\frac{1}{\sqrt{2}}\right)\middle|\ x \ge \frac{\hat{u}(0,\,t)}{\sqrt{2}}\right\}$$

3.5 Type II singularity and blow-up of the mean curvature

In this section we explain why Velázquez's solution (see Theorem 3.17) develops a type II singularity at the origin and why its mean curvature blows up as $t \nearrow 0$. The lower bound for the blow-up rate of the second fundamental form is already shown in [V], while the upper bound (of the second fundamental form) and the blow-up of the mean curvature are new results.

To estimate the second fundamental form and mean curvature, we would use the asymptotic formulas in Theorem 3.17 to examine the solution in each region separately. Let's start with analyzing the outer region by (3.24), (3.59) and (3.60).

Proposition 3.19. *Let* $\{\Sigma_t\}_{t_0 \le t < 0}$ *be Velázquez's solution in Theorem 3.17. In the* **outer region**, *the second fundamental form of* Σ_t *is bounded by*

$$\sqrt{-t}\,|A_{\Sigma_t}| \le C(n)$$

for $\frac{1}{2}t_0 \le t < 0$.

Proof. In the outer region, we parametrize Σ_t by (3.24). The second fundamental form is then given by

$$A_{\Sigma_t} = \frac{1}{\sqrt{1+(\partial_x u)^2}} \begin{pmatrix} \frac{\partial_{xx}^2 u}{1+(\partial_x u)^2} & & \\ & \frac{1+\partial_x u}{x-u} I_{n-1} & \\ & & \frac{-1+\partial_x u}{x+u} I_{n-1} \end{pmatrix}$$

By (3.59) and (3.60), we have

$$\begin{cases} \max\left\{ \left| \frac{u(x,\,t)}{x} \right|,\ |\partial_x u(x,\,t)| \right\} \le \frac{1}{3} \\ \\ \left| \partial_{xx}^2 u(x,\,t) \right| \le C(n) \end{cases}$$

for $x \ge \sqrt{-t}$, $\frac{1}{2}t_0 \le t < 0$. The conclusion follows immediately. ☐

In the intermediate region, we first do the type I rescaling and study the rescaled hypersurface by (3.29), (3.30), (3.60), (3.64) and (3.65). Then we undo the rescaling to get the estimates for the solution.

Proposition 3.20. *Let* $\{\Sigma_t\}_{t_0 \le t < 0}$ *be Velázquez's solution in Theorem 3.17. In the* **intermediate region**, *the second fundamental form and the mean curvature of* Σ_t *are bounded by*

$$(-t)^{\frac{1}{2}+\sigma}\,|A_{\Sigma_t}| \le C(n)$$

$$(-t)^{\frac{1}{2}+(1-2\varrho)\sigma}\,|H_{\Sigma_t}|\,\leq\,C\,(n,\,t_0)$$

for $\frac{1}{2}t_0\leq t<0$*, where* $0<\sigma<\frac{1}{2}$ *and* $0<\varrho<1$ *are constants defined in (3.23) and (3.67), respectively.*

Proof. In the intermediate region, we rescale Velázquez's solution by

$$\Pi_s\,=\,\frac{1}{\sqrt{-t}}\Sigma_t\bigg|_{t=-e^{-s}}$$

which can be parametrized by (3.29). The second fundamental form and the mean curvature of Π_s are then given by

$$A_{\Pi_s}\,=\,\frac{1}{\sqrt{1+(\partial_y v)^2}}\begin{pmatrix}\frac{\partial^2_{yy}v}{1+(\partial_y v)^2} & & \\ & \frac{1+\partial_y v}{y-v}\,I_{n-1} & \\ & & \frac{-1+\partial_y v}{y+v}\,I_{n-1}\end{pmatrix}$$

$$\begin{aligned}H_{\Pi_s}\,&=\,\frac{1}{\sqrt{1+(\partial_y v)^2}}\left(\frac{\partial^2_{yy}v}{1+(\partial_y v)^2}+2\,(n-1)\,\frac{y\,\partial_y v+v}{y^2-v^2}\right)\\&=\,\frac{1}{\sqrt{1+(\partial_y v)^2}}\left(\partial_s v-\frac{1}{2}\,(-y\,\partial_y v+v)\right)\end{aligned}$$

By (3.30) and (3.60), we have

$$\begin{cases}\max\left\{\left|\frac{v(y,\,t)}{y}\right|,\,|\partial_y v\,(y,\,s)|\right\}\,\leq\,C\,(n)\,e^{-\lambda_2 s}y^{\alpha-1}\,\leq\,\frac{1}{3}\\[2ex] \left|\partial^2_{yy}v\,(y,\,s)\right|\,\leq\,C\,(n)\left(e^{-\lambda_2 s}y^{\alpha-1}\right)y^{-1}\,\leq\,C\,(n)\,e^{\sigma s}\end{cases}$$

for $\beta e^{-\sigma s}\leq y\leq 1$, $-\ln\left(-\frac{1}{2}t_0\right)\leq s<\infty$. Thus, we get

$$|A_{\Pi_s}|\,\leq\,C\,(n)\,e^{\sigma s}$$

in the intermediate region for $-\ln\left(-\frac{1}{2}t_0\right)\leq s<\infty$.

As for the mean curvature, notice that

$$v\,(y,\,s)\approx\begin{cases}\frac{k}{c_2}\,e^{-\lambda_2 s}\,\varphi_2\,(y) & \text{for }\;e^{-\vartheta\sigma s}\leq y\leq 1\\[2ex] e^{-\sigma s}\,\psi_k\,(e^{\sigma s}y) & \text{for }\;\beta e^{-\sigma s}\leq y\leq e^{-\vartheta\sigma s}\end{cases}$$

We then compute

$$\left(\partial_s + \frac{y}{2}\,\partial_y - \frac{1}{2}\right)\left(\frac{k}{c_2}e^{-\lambda_2 s}\varphi_2\left(y\right)\right)$$

$$= \left(\partial_s + \frac{y}{2}\,\partial_y - \frac{1}{2}\right)\left(ke^{-\lambda_2 s}y^{\alpha}\left(1 + 2\varUpsilon_1 y^2 + \varUpsilon_2 y^4\right)\right)$$

$$= -2ke^{-\lambda_2 s}y^{\alpha}\left(1 + \varUpsilon_1 y^2\right)$$

and

$$\left(\partial_s + \frac{y}{2}\,\partial_y - \frac{1}{2}\right)\left(e^{-\sigma s}\,\psi_k\left(e^{\sigma s}y\right)\right)$$

$$= -\left(\frac{1}{2} + \sigma\right)e^{-\sigma s}\left(\psi_k\left(z\right) - z\,\partial_z\psi_k\left(z\right)\right)\Big|_{z=e^{\sigma s}y}$$

$$= -\left(\frac{1}{2} + \sigma\right)e^{-\sigma s}\left(\left(1-\alpha\right)k\left(e^{\sigma s}y\right)^{\alpha} + O\left(\left(e^{\sigma s}y\right)^{3\alpha-2}\right)\right)$$

$$= -2ke^{-\lambda_2 s}y^{\alpha}\left(1 + O\left(\left(e^{\sigma s}y\right)^{-2(1-\alpha)}\right)\right)$$

It follows, by (3.64) and (3.65), that

$$\left|\left(\partial_s + \frac{y}{2}\,\partial_y - \frac{1}{2}\right)v\left(y,\, s\right)\right|$$

$$\leq \left|\left(\partial_s + \frac{y}{2}\,\partial_y - \frac{1}{2}\right)\left(\frac{k}{c_2}e^{-\lambda_2 s}\varphi_2\left(y\right)\right)\right| + C\left(n,\, t_0\right)e^{-\varkappa s}\left(e^{-\lambda_2 s}y^{\alpha}\right)$$

$$\leq \left|-2ke^{-\lambda_2 s}y^{\alpha}\left(1 + \varUpsilon_1 y^2\right)\right| + C\left(n,\, t_0\right)e^{-\varkappa s}\left(e^{-\lambda_2 s}y^{\alpha}\right)$$

$$\leq C\left(n,\, t_0\right)e^{-\lambda_2 s}y^{\alpha} \,\leq\, C\left(n,\, t_0\right)$$

for $e^{-\vartheta\sigma s} \leq y \leq 1$, $-\ln\left(-\frac{1}{2}t_0\right) \leq s < \infty$, and

$$\left|\left(\partial_s + \frac{y}{2}\,\partial_y - \frac{1}{2}\right)v\left(y,\, s\right)\right|$$

$$\leq \left|\left(\partial_s + \frac{y}{2}\,\partial_y - \frac{1}{2}\right)\left(e^{-\sigma s}\,\psi_k\left(e^{\sigma s}y\right)\right)\right| + C\left(n,\, t_0\right)e^{-2\varrho\sigma s}\left(e^{-\lambda_2 s}y^{\alpha-2}\right)$$

$$\leq \left|-2ke^{-\lambda_2 s}y^{\alpha}\left(1 + O\left(\left(e^{\sigma s}y\right)^{-2(1-\alpha)}\right)\right)\right| + C\left(n,\, t_0\right)\left(e^{-\lambda_2 s}y^{\alpha-1}\right)\left(e^{-2\varrho\sigma s}y^{-1}\right)$$

$$\leq C\left(n,\, t_0\right)e^{-\lambda_2 s}y^{\alpha-1}\left(y \,+\, e^{-2\varrho\sigma s}y^{-1}\right) \,\leq\, C\left(n,\, t_0\right)e^{(1-2\varrho)\sigma s}$$

for $\beta e^{-\sigma s} \leq y \leq e^{-\vartheta\sigma s}$, $-\ln\left(-\frac{1}{2}t_0\right) \leq s < \infty$. Consequently,

$$\left|H_{\Pi_s}\right| = \frac{\left|\partial_s v - \frac{1}{2}\left(-y\,\partial_y v + v\right)\right|}{\sqrt{1 + \left|\partial_y v\right|^2}} \,\leq\, C\left(n,\, t_0\right)e^{(1-2\varrho)\sigma s}$$

Lastly, by the relation

$$A_{\Pi_s}(y) = \sqrt{-t}\; A_{\Sigma_t}\left(\sqrt{-t}\, y\right)\Big|_{t=-e^{-s}}$$

$$H_{\Pi_s}(y) = \sqrt{-t}\; H_{\Sigma_t}\left(\sqrt{-t}\, y\right)\Big|_{t=-e^{-s}}$$

the conclusion follow easily. $\square$

In the tip region, we do the type II rescaling and study the rescaled hypersurface by (3.38), (3.61) and (3.68). Then we undo the rescaling to get estimates of the solution.

Proposition 3.21. *Let $\{\Sigma_t\}_{t_0\leq t<0}$ be Velázquez's solution in Theorem 3.17. In the* **tip region***, the second fundamental form and the mean curvature of Σ_t satisfy*

$$\frac{1}{C(n)} \leq (-t)^{\frac{1}{2}+\sigma}\left|A_{\Sigma_t}\right| \leq C(n)$$

$$(-t)^{\frac{1}{2}+(1-2\varrho)\sigma}\left|H_{\Sigma_t}\right| \leq C(n,\, t_0)$$

for $\frac{1}{2}t_0 \leq t < 0$, where $0<\sigma<\frac{1}{2}$ and $0<\varrho<1$ are constants defined in (3.23) and (3.67), respectively.

Proof. In the tip region, we first rescale Velázquez's solution by

$$\Gamma_\tau = \frac{1}{(-t)^{\frac{1}{2}+\sigma}}\Sigma_t\Bigg|_{t=-(2\sigma\tau)^{\frac{-1}{2\sigma}}}$$

which can be parametrized by (3.38). Then the second fundamental form and the mean curvature of Γ_τ are given by

$$A_{\Gamma_\tau} = \frac{1}{\sqrt{1+\left|\partial_z\hat{w}\right|^2}}\begin{pmatrix} \frac{\partial^2_{zz}\hat{w}}{1+\left|\partial_z\hat{w}\right|^2} & & \\ & \frac{\partial_z\hat{w}}{z}\, I_{n-1} & \\ & & \frac{-1}{\hat{w}}\, I_{n-1}\end{pmatrix}$$

$$H_{\Gamma_\tau} = \frac{1}{\sqrt{1+\left(\partial_z\hat{w}\right)^2}}\left(\frac{\partial^2_{zz}\hat{w}}{1+\left(\partial_z\hat{w}\right)^2} + (n-1)\left(\frac{\partial_z\hat{w}}{z} - \frac{1}{\hat{w}}\right)\right)$$

$$= \frac{1}{\sqrt{1+(\partial_z \hat{w})^2}} \left(\partial_\tau \hat{w} - \frac{\frac{1}{2}+\sigma}{2\sigma\tau} \left(-z\,\partial_z \hat{w} + \hat{w}\right) \right)$$

By (3.61), we have

$$\frac{1}{C(n)} \leq \hat{w}(z,\tau) \leq C(n)$$

$$|\partial_z \hat{w}(z,\tau)| + \left|\partial_{zz}^2 \hat{w}(z,\tau)\right| \leq C(n)$$

for $0 \leq z \leq \beta$, $\frac{1}{2\sigma}\left(-\frac{1}{2}t_0\right)^{-2\sigma} \leq \tau < \infty$. Thus, we get

$$\frac{1}{C(n)} \leq |A_{\Gamma_\tau}| \leq C(n)$$

As for the mean curvature, note, from (3.6), that

$$\left| \left(\partial_\tau + \frac{\frac{1}{2}+\sigma}{2\sigma\tau} z\,\partial_z - \frac{\frac{1}{2}+\sigma}{2\sigma\tau} \right) \hat{\psi}_k(z) \right| = \left| - \frac{\frac{1}{2}+\sigma}{2\sigma\tau} \left(\hat{\psi}_k(z) - z\,\partial_z \hat{\psi}_k(z) \right) \right| \leq \frac{C(n)}{2\sigma\tau}$$

By (3.68), we get

$$\begin{aligned} &\left| \left(\partial_\tau + \frac{\frac{1}{2}+\sigma}{2\sigma\tau} z\,\partial_z - \frac{\frac{1}{2}+\sigma}{2\sigma\tau} \right) \hat{w}(z,\tau) \right| \\ \leq &\left| \left(\partial_\tau + \frac{\frac{1}{2}+\sigma}{2\sigma\tau} z\,\partial_z - \frac{\frac{1}{2}+\sigma}{2\sigma\tau} \right) \hat{\psi}_k(z) \right| + C(n,t_0)(2\sigma\tau)^{-\varrho} \\ &\leq C(n,t_0)(2\sigma\tau)^{-\varrho} \end{aligned}$$

Thus,

$$|H_{\Gamma_\tau}| = \frac{\left| \partial_\tau \hat{w} - \frac{\frac{1}{2}+\sigma}{2\sigma\tau} \left(-z\,\partial_z \hat{w} + \hat{w}\right) \right|}{\sqrt{1+(\partial_z \hat{w})^2}} \leq C(n,t_0)(2\sigma\tau)^{-\varrho}$$

The conclusion follows by noting that

$$A_{\Gamma_\tau}(z) = (-t)^{\frac{1}{2}+\sigma} A_{\Sigma_t}\left((-t)^{\frac{1}{2}+\sigma} z \right) \Big|_{t=-(2\sigma\tau)^{\frac{-1}{2\sigma}}}$$

$$H_{\Gamma_\tau}(z) = (-t)^{\frac{1}{2}+\sigma} H_{\Sigma_t}\left((-t)^{\frac{1}{2}+\sigma} z \right) \Big|_{t=-(2\sigma\tau)^{\frac{-1}{2\sigma}}}$$

$\square$

Lastly, we would like to show that the mean curvature blows up in the tip region at a rate at least $\frac{1}{(-t)^{\frac{1}{2}-\sigma}}$ as $t \nearrow 0$.

Proposition 3.22. *Let* $\{\Sigma_t\}_{t_0\le t<0}$ *be Velázquez's solution in Theorem 3.17. Let* $H_{\Sigma_t}(x)$ *be the mean curvature of* Σ_t *at*

$$X_t(x,\,\nu,\,\omega)=(x\,\nu,\,\hat{u}(x,\,t)\,\omega)$$

(see (3.20)). Then for any $z\ge 0$*, there holds*

$$\limsup_{t\nearrow 0}\;(-t)^{\frac{1}{2}-\sigma}\left|H_{\Sigma_t}\left((-t)^{\frac{1}{2}+\sigma}\,z\right)\right|>0$$

Proof. Note that

$$H_{\Sigma_t}=\frac{1}{\sqrt{1+(\partial_x\hat{u})^2}}\left(\frac{\partial^2_{xx}\hat{u}}{1+(\partial_x\hat{u})^2}+(n-1)\left(\frac{\partial_x\hat{u}}{x}-\frac{1}{\hat{u}}\right)\right)$$
$$=\frac{\partial_t\hat{u}}{\sqrt{1+(\partial_x\hat{u})^2}}\tag{3.69}$$

We claim that for any $z\ge 0$, there holds

$$\limsup_{t\nearrow 0}\frac{\left|\partial_t\hat{u}\left((-t)^{\frac{1}{2}+\sigma}\,z,\,t\right)\right|}{(-t)^{-\frac{1}{2}+\sigma}}>0\tag{3.70}$$

The conclusion follows immediately from (3.61), (3.69) and (3.70).

To prove (3.70), we use a contradiction argument. Suppose that there is $z\ge 0$ so that

$$\limsup_{t\nearrow 0}\frac{\left|\partial_t\hat{u}\left((-t)^{\frac{1}{2}+\sigma}\,z,\,t\right)\right|}{(-t)^{-\frac{1}{2}+\sigma}}=0$$

then obviously,

$$\lim_{t\nearrow 0}\frac{\left|\partial_t\hat{u}\left((-t)^{\frac{1}{2}+\sigma}\,z,\,t\right)\right|}{(-t)^{-\frac{1}{2}+\sigma}}=0\tag{3.71}$$

Recall that by (3.68), we have

$$\frac{1}{(-t)^{\frac{1}{2}+\sigma}}\hat{u}\left((-t)^{\frac{1}{2}+\sigma}\,z,\,t\right)\;=\;\hat{w}\left(z,\,\frac{1}{2\sigma\,(-t)^{2\sigma}}\right)\;\to\;\hat{\psi}_k(z)\qquad\text{as}\quad t\nearrow 0$$

It follows, by L'Hôpital's rule, that

$$\hat{\psi}_k(z)=\lim_{t\nearrow 0}\frac{\hat{u}\left((-t)^{\frac{1}{2}+\sigma}\,z,\,t\right)}{(-t)^{\frac{1}{2}+\sigma}}=\lim_{t\nearrow 0}\left(\frac{\partial_t\hat{u}\left((-t)^{\frac{1}{2}+\sigma}\,z,\,t\right)}{-\left(\frac{1}{2}+\sigma\right)(-t)^{-\frac{1}{2}+\sigma}}+z\,\partial_z\hat{w}\left(z,\,\frac{1}{2\sigma\,(-t)^{2\sigma}}\right)\right)$$

Notice that the limit on the RHS exists because of (3.68) and (3.71), so L'Hôpital's rule is applicable here. Thus, we get

$$\lim_{t\nearrow 0}\frac{\partial_t\hat{u}\left((-t)^{\frac{1}{2}+\sigma}\,z,\,t\right)}{-\left(\frac{1}{2}+\sigma\right)(-t)^{-\frac{1}{2}+\sigma}}\;=\;\hat{\psi}_k(z)-z\,\partial_z\hat{\psi}_k(z)\;>0$$

by (3.5), which contradicts with (3.71). $\square$

3.6 C^0 estimates in Proposition 3.13 and Proposition 3.14

Starting from this section, we are devoted to prove Proposition 3.13 and Proposition 3.14. From now on, we focus on the estimate of the **admissible** MCF $\left\{\Sigma_t^{(a_0,\, a_1)}\right\}_{t_0 \leq t \leq \mathring{t}}$ for which

$$\Phi_{t_1}\left(a_0,\, a_1\right) = (0,\, 0) \tag{3.72}$$

where $t_0 \leq t_1 \leq \mathring{t} < 0$ are constants and $\mathring{t} \leq e^{-1} t_1$. In this section, we would show that if $0 < \rho \ll 1 \ll \beta$ (depending on $n,\, \Lambda$) and $|t_0| \ll 1$ (depending on $n,\, \Lambda,\, \rho,\, \beta$) , there holds

$$\sqrt{a_0^2 + a_1^2} \,\leq\, C\left(n,\, \Lambda,\, \rho,\, \beta\right)\left(-t_0\right)^{\varsigma\lambda_2} \tag{3.73}$$

where $\varsigma > 0$ is a constant defined in (3.56). Moreover, there is

$$k \in \left(1 - C\left(n,\, \Lambda,\, \rho,\, \beta\right)\left(-t_0\right)^{\varsigma\lambda_2},\; 1 + C\left(n,\, \Lambda,\, \rho,\, \beta\right)\left(-t_0\right)^{\varsigma\lambda_2}\right) \tag{3.74}$$

so that the following hold.

1. In the **outer region**, the function $u\left(x,\, t\right)$ of $\Sigma_t^{(a_0,\, a_1)}$ defined in (3.24) satisfies

$$\left|u(x,\, t) - u(x,\, t_0)\right| \,\leq\, C\left(n\right)\sqrt{t - t_0} \tag{3.75}$$

for $x \geq \frac{1}{5}\rho$, $t_0 \leq t \leq \mathring{t}$, and

$$\left|u\left(x,\, t\right) - \frac{k}{c_2}\left(-t\right)^{\lambda_2 + \frac{1}{2}}\varphi_2\left(\frac{x}{\sqrt{-t}}\right)\right| \,\leq\, C\left(n,\, \Lambda,\, \rho,\, \beta\right)\left(-t_0\right)^{\varkappa} x^{2\lambda_2 + 1} \tag{3.76}$$

for $\frac{1}{3}\sqrt{-t} \leq x \leq \rho$, $t_0 \leq t \leq \mathring{t}$, where $\varkappa > 0$ is a constant defined in (3.66). Note that

$$\frac{k}{c_2}\left(-t\right)^{\lambda_2 + \frac{1}{2}}\varphi_2\left(\frac{x}{\sqrt{-t}}\right) = k x^{2\lambda_2 + 1}\left(\Upsilon_2 + 2\Upsilon_1\left(\frac{-t}{x^2}\right) + \left(\frac{-t}{x^2}\right)^2\right)$$

2. In the **intermediateregion**, if we do the type I rescaling, the function $v\left(y,\, s\right)$ of the rescaled hypersurface $\Pi_s^{(a_0,\, a_1)}$ defined in (3.29) satisfies

$$\left|v\left(y,\, s\right) - \frac{k}{c_2}e^{-\lambda_2 s}\varphi_2\left(y\right)\right| \leq C\left(n,\, \Lambda,\, \rho,\, \beta\right)e^{-\varkappa s}\left(e^{-\lambda_2 s}y^{\alpha+2}\right) \tag{3.77}$$

for $\frac{1}{2}e^{-\vartheta\sigma s} \leq y \leq \sqrt{\varsigma\lambda_2 s}$, $s_0 \leq s \leq \mathring{s}$, and

$$\left|\left(v\left(y,\, s\right) \,-\, e^{-\sigma s}\,\psi_k\left(e^{\sigma s}y\right)\right)\right| \,\leq\, C\left(n\right)\beta^{\alpha-3}e^{-2\varrho\sigma\left(s - s_0\right)}\left(e^{-\lambda_2 s}y^{\alpha}\right) \tag{3.78}$$

for $\frac{4}{3}\beta e^{-\sigma s} \leq y \leq \frac{1}{2} e^{-\vartheta \sigma s}$, $s_0 \leq s \leq \mathring{s}$, where $\mathring{s} = -\ln\left(-\mathring{t}\right)$ and $0 < \varrho < \vartheta < 1$ are constants (see (3.57) and (3.67) for definition). Note that

$$\frac{k}{c_2} e^{-\lambda_2 s} \varphi_2\left(y\right) = k e^{-\lambda_2 s} y^{\alpha} \left(1 + 2\Upsilon_1 y^2 + \Upsilon_2 y^4\right)$$

$$e^{-\sigma s}\, \psi_k\left(e^{\sigma s} y\right) = k e^{-\lambda_2 s} y^{\alpha} \left(1 + O\left(\left(e^{\sigma s} y\right)^{-2(1-\alpha)}\right)\right)$$

3. In the **tip region**, if we do the type II rescaling, the function $\hat{w}\left(z, \tau\right)$ of the rescaled hypersurface $\Gamma_{\tau}^{(a_0,\, a_1)}$ defined in (3.38) satisfies

$$\hat{\psi}_{\left(1-\beta\alpha^{-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho}\right)k}\left(z\right) \;\leq\; \hat{w}\left(z, \tau\right) \;\leq\; \hat{\psi}_{\left(1+\beta\alpha^{-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho}\right)k}\left(z\right) \tag{3.79}$$

for $0 \leq z \leq \left(2\sigma\tau\right)^{\frac{1}{2}(1-\vartheta)}$, $\tau_0 \leq \tau \leq \mathring{\tau}$, where $\mathring{\tau} = \frac{1}{2\sigma\left(-\mathring{t}\right)^{2\sigma}}$.

To achieve that, we first establish (3.77) (see Proposition 3.26) by using the energy estimate and Sobolev inequality. Next, we use the comparison principle and the boundary values of (3.77) to show (3.76) (see Proposition 3.27) and (3.79) (see Proposition 3.28). Then we use (3.79) to deduce (3.78) by rescaling and analyzing the projected curves. Lastly, we use the gradient and curvature estimates in [EH] to prove (3.75) (see Proposition 3.29). The ideas of proving (3.76), (3.77) and (3.79) are due to Velázquez (see [V]). Here we improve his estimates to get better results.

Remark 3.23. By the above C^0 estimates, we deduce that

$$-2\left(\Upsilon_1^2 - \Upsilon_2\right) x^{2\lambda_2+1} \;\leq\; u\left(x, t\right) \;\leq\; 2\left(1 + 2\Upsilon_1 + \Upsilon_2\right) x^{2\lambda_2+1}$$

for $\sqrt{-t} \leq x \leq \rho$, $t_0 \leq t \leq \mathring{t}$, and

$$2\left(1 + 2\Upsilon_1 + \Upsilon_2\right) e^{-\lambda_2 s} y^{\alpha} \;\leq\; v\left(y, s\right) \;\leq\; 2 e^{-\lambda_2 s} y^{\alpha}$$

for $\frac{4}{3}\beta e^{-\sigma s} \leq y \leq 1$, $s_0 \leq s \leq \mathring{s}$, provided that $\beta \gg 1$ (depending on n) and $|t_0| \ll 1$ (depending on n, Λ, ρ, β). In Section 3.8, we would use these etstimates to choose the constant $\Lambda = \Lambda\left(n\right)$.

In order to prove (3.77), we need the following two lemmas. The first lemma is the energy estimates for solutions to a parabolic equation associated with the linear operator

$\mathcal{L}$ (see (3.34)). Recall that in Proposition 3.8, the eigenvalues of $\mathcal{L}$ satisfy $\lambda_i \geq \lambda_3 > 1$ for $i \geq 3$.

Lemma 3.24. *Let* $\mathbf{H}_*$ *be the closed subspace of* $\mathbf{H}$ *(see Proposition 3.8) spanned by eigenfunctions* $\{\varphi_i\}_{i\geq 3}$ *of* $\mathcal{L}$. *Given*

$$\mathsf{f}(\cdot,\, s) \in L^2\left([s_0,\, \mathring{s}]\,;\, L^2\left(\mathbb{R}_+,\, y^{2(n-1)}e^{-\frac{y^2}{4}}dy\right)\right)$$

and $\mathsf{h} \in \mathbf{H}_*$, *let* $\mathsf{v}(\cdot,\, s) \in C([s_0,\, \mathring{s}]\,;\, \mathbf{H}_*)$ *be the weak solution of*

$$\begin{cases} (\partial_s + \mathcal{L})\,\mathsf{v}(\cdot,\, s) = \mathsf{f}(\cdot,\, s) \quad \text{for } s_0 \leq s \leq \mathring{s} \\ \\ \mathsf{v}(\cdot,\, s_0) = \mathsf{h} \end{cases} \tag{3.80}$$

Then for any $0 < \delta < 1$, *there hold*

$$\begin{aligned} &\|\mathsf{v}(\cdot,\, s)\|^2 \\ &\leq e^{-2(1-\delta)\lambda_3(s-s_0)}\,\|\mathsf{v}(\cdot,\, s_0)\|^2 + \frac{1}{2\delta\lambda_3}\int_{s_0}^{s} e^{-2(1-\delta)\lambda_3(s-\xi)}\,\|\mathsf{f}(\cdot,\, \xi)\|^2\, d\xi \end{aligned}$$

and

$$\begin{aligned} &\langle \mathcal{L}\mathsf{v}(\cdot,\, s)\,,\, \mathsf{v}(\cdot,\, s)\rangle \\ &\leq e^{-2(1-\delta)\lambda_3(s-s_0)}\,\langle \mathcal{L}\mathsf{h},\, \mathsf{h}\rangle + \frac{1}{2\delta}\int_{s_0}^{s} e^{-2(1-\delta)\lambda_3(s-\xi)}\,\|\mathsf{f}(\cdot,\, \xi)\|^2\, d\xi \end{aligned}$$

for $s_0 \leq s \leq \mathring{s}$, *where the inner product* $\langle\cdot,\cdot\rangle$ *and the corresponding norm* $\|\cdot\|$ *are defined in Proposition 3.8.*

Proof. Let $\{\mathsf{v}_m\}_{m\geq 3}$ be the Galerkin's approximation of v. Namely,

$$\mathsf{v}_m(y,\, s) = \sum_{i=3}^{m}\left(e^{-\lambda_i(s-s_0)}\,\langle \mathsf{h},\, \varphi_i\rangle + \int_{s_0}^{s} e^{-\lambda_i(s-\xi)}\,\langle \mathsf{f}(\cdot,\, \xi)\,,\, \varphi_i\rangle\, d\xi\right)\varphi_i(y)$$

Then we have

$$\begin{cases} \partial_s\mathsf{v}_m(\cdot,\, s) + \mathcal{L}\mathsf{v}_m(\cdot,\, s) = \mathsf{f}_m(\cdot,\, s) \quad \text{for } s_0 \leq s \leq \mathring{s} \\ \\ \mathsf{v}_m(\cdot,\, s_0) = \sum_{i=3}^{m}\langle \mathsf{h},\, \varphi_i\rangle\,\varphi_i \;\to\; \mathsf{h} \quad \text{in } \mathbf{H}_* \end{cases}$$

where

$$\mathsf{f}_m(\cdot,\, s) = \sum_{i=3}^{m}\langle \mathsf{f}(\cdot,\, s)\,,\, \varphi_i\rangle\,\varphi_i \;\to\; \mathsf{f}(\cdot,\, s) \quad \text{in } L^2\left([s_0,\, \mathring{s}]\,;\, L^2\left(\mathbb{R}_+,\, y^{2(n-1)}e^{-\frac{y^2}{4}}dy\right)\right)$$

It follows that

$$\langle \partial_s \mathsf{v}_m(\cdot,\, s)\,,\, \mathsf{v}_m(\cdot,\, s)\rangle + \langle \mathcal{L}\mathsf{v}_m(\cdot,\, s)\,,\, \mathsf{v}_m(\cdot,\, s)\rangle = \langle \mathsf{f}_m(\cdot,\, s)\,,\, \mathsf{v}_m(\cdot,\, s)\rangle$$

which, by Cauchy-Schwarz inequality, yields

$$\begin{aligned} &\frac{1}{2}\,\partial_s \left\|\mathsf{v}_m(\cdot,\, s)\right\|^2 + \lambda_3 \left\|\mathsf{v}_m(\cdot,\, s)\right\|^2 \,\le\, \delta\lambda_3 \left\|\mathsf{v}_m(\cdot,\, s)\right\|^2 + \frac{1}{4\delta\lambda_3}\left\|\mathsf{f}_m(\cdot,\, s)\right\|^2 \\ \Leftrightarrow\quad &\partial_s \left\|\mathsf{v}_m(\cdot,\, s)\right\|^2 \,\le\, -2\,(1-\delta)\,\lambda_3 \left\|\mathsf{v}_m(\cdot,\, s)\right\|^2 + \frac{1}{2\delta\lambda_3}\left\|\mathsf{f}_m(\cdot,\, s)\right\|^2 \end{aligned}$$

for any $0 < \delta < 1$. Thus, by integrating the inquality with repect to s, we get

$$\begin{aligned} &\left\|\mathsf{v}_m(\cdot,\, s)\right\|^2 \\ &\le\, e^{-2(1-\delta)\lambda_3(s-s_0)}\left\|\mathsf{v}_m(\cdot,\, s_0)\right\|^2 + \frac{1}{2\delta\lambda_3}\int_{s_0}^{s} e^{-2(1-\delta)\lambda_3(s-\xi)}\left\|\mathsf{f}_m(\cdot,\, \xi)\right\|^2 d\xi \end{aligned} \tag{3.81}$$

for $s_0 \le s \le \mathring{s}$.

Similarly, we have

$$\langle \partial_s \mathsf{v}_m(\cdot,\, s)\,,\, \partial_s \mathsf{v}_m(\cdot,\, s)\rangle + \langle \mathcal{L}\mathsf{v}_m(\cdot,\, s)\,,\, \partial_s \mathsf{v}_m(\cdot,\, s)\rangle = \langle \mathsf{f}_m(\cdot,\, s)\,,\, \partial_s \mathsf{v}_m(\cdot,\, s)\rangle$$

Substitute $\partial_s \mathsf{v}_m(\cdot,\, s) = -\mathcal{L}\mathsf{v}_m(\cdot,\, s) + \mathsf{f}_m(\cdot,\, s)$ into the above equation to get

$$\frac{1}{2}\,\partial_s \langle \mathcal{L}\mathsf{v}_m(\cdot,\, s)\,,\, \mathsf{v}_m(\cdot,\, s)\rangle = -\langle \mathcal{L}\mathsf{v}_m(\cdot,\, s)\,,\, \mathcal{L}\mathsf{v}_m(\cdot,\, s)\rangle + \langle \mathcal{L}\mathsf{v}_m(\cdot,\, s)\,,\, \mathsf{f}_m(\cdot,\, s)\rangle$$

By Cauchy-Schwarz inequality, we get

$$\begin{aligned} &\partial_s \langle \mathcal{L}\mathsf{v}_m(\cdot,\, s)\,,\, \mathsf{v}_m(\cdot,\, s)\rangle \\ &\le -2\,(1-\delta)\,\langle \mathcal{L}\mathsf{v}_m(\cdot,\, s)\,,\, \mathcal{L}\mathsf{v}_m(\cdot,\, s)\rangle + \frac{1}{2\delta}\left\|\mathsf{f}_m(\cdot,\, s)\right\|^2 \\ &\le -2\,(1-\delta)\,\lambda_3\,\langle \mathcal{L}\mathsf{v}_m(\cdot,\, s)\,,\, \mathsf{v}_m(\cdot,\, s)\rangle + \frac{1}{2\delta}\left\|\mathsf{f}_m(\cdot,\, s)\right\|^2 \end{aligned}$$

for any $0 < \delta < 1$. Thus, we have

$$\begin{aligned} &\langle \mathcal{L}\mathsf{v}_m(\cdot,\, s)\,,\, \mathsf{v}_m(\cdot,\, s)\rangle \\ &\le e^{-2(1-\delta)\lambda_3(s-s_0)}\,\langle \mathcal{L}\mathsf{v}_m(\cdot,\, s_0)\,,\, \mathsf{v}_m(\cdot,\, s_0)\rangle \,+\, \frac{1}{2\delta}\int_{s_0}^{s} e^{-2(1-\delta)\lambda_3(s-\xi)}\left\|\mathsf{f}_m(\cdot,\, \xi)\right\|^2 d\xi \end{aligned} \tag{3.82}$$

for $s_0 \le s \le \mathring{s}$.

On the other hand, for any $m,\, l \geq 3$, there holds

$$\partial_s \left(\mathsf{v}_m\left(\cdot,\, s\right) - \mathsf{v}_l\left(\cdot,\, s\right)\right) + \mathcal{L}\left(\mathsf{v}_m\left(\cdot,\, s\right) - \mathsf{v}_l\left(\cdot,\, s\right)\right) = \mathsf{f}_m\left(\cdot,\, s\right) - \mathsf{f}_l\left(\cdot,\, s\right)$$

By the same arguments as above, for any $0 < \delta < 1$, we can deduce that

$$\left\|\mathsf{v}_m\left(\cdot,\, s\right) - \mathsf{v}_l\left(\cdot,\, s\right)\right\|^2 \tag{3.83}$$

$$\leq e^{-2(1-\delta)\lambda_3(s-s_0)} \left\|\mathsf{v}_m\left(\cdot,\, s_0\right) - \mathsf{v}_l\left(\cdot,\, s_0\right)\right\|^2 + \frac{1}{2\delta\lambda_3}\int_{s_0}^{s} e^{-2(1-\delta)\lambda_3(s-\xi)} \left\|\mathsf{f}_m\left(\cdot,\, \xi\right) - \mathsf{f}_l\left(\cdot,\, \xi\right)\right\|^2 d\xi$$

and

$$\left\langle \mathcal{L}\left(\mathsf{v}_m\left(\cdot,\, s\right) - \mathsf{v}_l\left(\cdot,\, s\right)\right),\, \left(\mathsf{v}_m\left(\cdot,\, s\right) - \mathsf{v}_l\left(\cdot,\, s\right)\right)\right\rangle \tag{3.84}$$

$$\leq e^{-2(1-\delta)\lambda_3(s-s_0)} \left\langle \mathcal{L}\left(\mathsf{v}_m\left(\cdot,\, s_0\right) - \mathsf{v}_l\left(\cdot,\, s_0\right)\right),\, \mathsf{v}_m\left(\cdot,\, s_0\right) - \mathsf{v}_l\left(\cdot,\, s_0\right)\right\rangle + \frac{1}{2\delta}\int_{s_0}^{s} e^{-2(1-\delta)\lambda_3(s-\xi)} \left\|\mathsf{f}_m\left(\cdot,\, \xi\right) - \mathsf{f}_l\left(\cdot,\, \xi\right)\right\|^2 d\xi$$

for $s_0 \leq s \leq \mathring{s}$. Therefore, by (3.36), (3.83), (3.84) and the uniqueness of weak solutions, we get

$$\mathsf{v}_m \to \mathsf{v} \quad \text{in } C\left(\left[s_0,\, \mathring{s}\right];\, \mathbf{H}_*\right)$$

The conclusion follows by passing (3.81) and (3.82) to limit. $\square$

The second lemma is a Sobolev type inequality for functions in $\mathbf{H}$, which is the Hilbert space defined in Proposition 3.8.

Lemma 3.25. *Functions in* $\mathbf{H}$ *are actually continuous, i.e.,* $\mathbf{H} \subset C\left(\mathbb{R}_+\right)$*. Moreover, for any* $\mathsf{v} \in \mathbf{H}$*, there holds*

$$\left|\mathsf{v}\left(y\right)\right| \leq C\left(n\right)\left(\frac{1}{y^{n-\frac{1}{2}}} + e^{\frac{(y+1)^2}{4}}\right)\left(\left\|\partial_y \mathsf{v}\right\| + \left\|\mathsf{v}\right\|\right)$$

for $y > 0$*.*

Proof. Let's first assume that $\mathsf{v} \in C^1(\mathbb{R}_+) \cap \mathbf{H}$.

For each $0 < y \leq 1$, by the fundamental theorem of calculus, we have

$$\mathsf{v}(y) = \mathsf{v}(z) + \int_z^y \partial_y \mathsf{v}(\xi)\, d\xi \qquad \forall\ \frac{y}{2} \leq z \leq y$$

which, by Hőlder's inequality, implies

$$|\mathsf{v}(y)|^2 \leq C\left(|\mathsf{v}(z)|^2 + y\int_{\frac{y}{2}}^y |\partial_y \mathsf{v}(\xi)|^2\, d\xi\right)$$

$$\leq C\,|\mathsf{v}(z)|^2 + C(n)\,\frac{y}{y^{2(n-1)}}\left(\int_{\frac{y}{2}}^y |\partial_y \mathsf{v}(\xi)|^2\, \xi^{2(n-1)} e^{-\frac{\xi^2}{4}}\, d\xi\right)$$

for $\frac{y}{2} \leq z \leq y$. Integrate the above inequality against $z^{2(n-1)} e^{-\frac{z^2}{4}}\, dz$ from $\frac{y}{2}$ to y to get

$$|\mathsf{v}(y)|^2 \left(\int_{\frac{y}{2}}^y z^{2(n-1)} e^{-\frac{z^2}{4}}\, dz\right) \leq C\int_{\frac{y}{2}}^y |\mathsf{v}(z)|^2\, z^{2(n-1)} e^{-\frac{z^2}{4}}\, dz$$

$$+\,C(n)\,\frac{1}{y^{2n-3}}\left(\int_{\frac{y}{2}}^y |\partial_y \mathsf{v}(\xi)|^2\, \xi^{2(n-1)} e^{-\frac{\xi^2}{4}}\, d\xi\right)\left(\int_{\frac{y}{2}}^y z^{2(n-1)} e^{-\frac{z^2}{4}}\, dz\right)$$

which implies

$$|\mathsf{v}(y)|^2 \leq C(n)\,\frac{1}{y^{2n-1}}\left(\int_{\frac{y}{2}}^y |\mathsf{v}(z)|^2\, z^{2(n-1)} e^{-\frac{z^2}{4}}\, dz\right)$$

$$+\,C(n)\,\frac{1}{y^{2n-3}}\left(\int_{\frac{y}{2}}^y |\partial_y \mathsf{v}(\xi)|^2\, \xi^{2(n-1)} e^{-\frac{\xi^2}{4}}\, d\xi\right)$$

That is,

$$|\mathsf{v}(y)| \leq C(n)\left(\frac{1}{y^{n-\frac{1}{2}}}\,\|\mathsf{v}\| + \frac{1}{y^{n-\frac{3}{2}}}\,\|\partial_y \mathsf{v}\|\right)$$

$$\leq C(n)\,\frac{1}{y^{n-\frac{1}{2}}}\left(\|\partial_y \mathsf{v}\| + \|\mathsf{v}\|\right)$$

for $0 < y \leq 1$.

Likewise, for each $y \geq 1$, by the fundamental theorem of calculus, we have

$$\mathsf{v}(y) = \mathsf{v}(z) - \int_y^z \partial_y \mathsf{v}(\xi)\, d\xi \qquad \forall\ y \leq z \leq y+1$$

which implies

$$|\mathsf{v}(y)|^2 \leq C\left(|\mathsf{v}(z)|^2 + \int_y^{y+1} |\partial_y \mathsf{v}(\xi)|^2\, d\xi\right)$$

$$\leq C\,|\mathsf{v}(z)|^2 + C\,y^{-2(n-1)} e^{\frac{(y+1)^2}{4}}\left(\int_y^{y+1} |\partial_y \mathsf{v}(\xi)|^2\, \xi^{2(n-1)} e^{-\frac{\xi^2}{4}}\, d\xi\right)$$

for $y \leq z \leq y+1$. Integrate both sides againt $z^{2(n-1)}e^{-\frac{z^2}{4}}\,dz$ from y to $y+1$ to get

$$|\mathsf{v}\,(y)|^2\left(\int_y^{y+1} z^{2(n-1)}e^{-\frac{z^2}{4}}dz\right) \leq C\int_y^{y+1}|\mathsf{v}\,(z)|^2\,z^{2(n-1)}e^{-\frac{z^2}{4}}dz$$

$$+\,C\,y^{-2(n-1)}e^{\frac{(y+1)^2}{4}}\left(\int_y^{y+1}|\partial_y\mathsf{v}\,(\xi)|^2\,\xi^{2(n-1)}e^{-\frac{\xi^2}{4}}d\xi\right)\left(\int_y^{y+1} z^{2(n-1)}e^{-\frac{z^2}{4}}dz\right)$$

which yields

$$\begin{aligned}|\mathsf{v}\,(y)|^2 &\leq C\,(n)\,y^{-2(n-1)}e^{\frac{(y+1)^2}{4}}\left(\|\mathsf{v}\|^2+\|\partial_y\mathsf{v}\|^2\right)\\ &\leq C\,(n)\,e^{\frac{(y+1)^2}{4}}\left(\|\partial_y\mathsf{v}\|+\|\mathsf{v}\|\right)\end{aligned}$$

for $y \geq 1$.

More generally, given a function $\mathsf{v}\in\mathbf{H}$, then choose a sequence $\{\mathsf{v}_i\}\subset C_c^1\,(\mathbb{R}_+)\cap\mathbf{H}$ so that

$$\mathsf{v}_i \xrightarrow{\mathbf{H}} \mathsf{v}$$

By the above arguments, we have

$$|\mathsf{v}_i\,(y)| \leq C\,(n)\left(\frac{1}{y^{n-\frac{1}{2}}}+e^{\frac{(y+1)^2}{4}}\right)\left(\|\partial_y\mathsf{v}_i\|+\|\mathsf{v}_i\|\right)$$

$$|\mathsf{v}_i\,(y)-\mathsf{v}_j\,(y)| \leq C\,(n)\left(\frac{1}{y^{n-\frac{1}{2}}}+e^{\frac{(y+1)^2}{4}}\right)\left(\|\partial_y\mathsf{v}_i-\partial_y\mathsf{v}_j\|+\|\mathsf{v}_i-\mathsf{v}_j\|\right)$$

for $y>0$. It follows, by the second inequality, that

$$\mathsf{v}_i \xrightarrow{C_{loc}} \mathsf{v}$$

Hence $\mathsf{v}\in C\,(\mathbb{R}_+)$. In addition, by passing the first inequality to limit, we get

$$|\mathsf{v}\,(y)| \leq C\,(n)\left(\frac{1}{y^{n-\frac{1}{2}}}+e^{\frac{(y+1)^2}{4}}\right)\left(\|\partial_y\mathsf{v}\|+\|\mathsf{v}\|\right)$$

for $y>0$. □

Now we are ready to prove (3.77). The idea is to linearize (3.31) and do Fourier expansion. The condition (3.72) allow us to control the evolution of components in negative eigenvalue functions. For the remainder terms, we can use the energy estimate and Sobolev inequality to get a L^∞ estimate.

Proposition 3.26. *If* $0 < \rho \ll 1 \ll \beta$ *(depending on* n, Λ*) and* $s_0 \gg 1$ *(depending on* n, Λ, ρ, β*), then (3.73) holds. Moreover, there is a constant* k *satisfying (3.74), for which the function* $v(y, s)$ *of the type* I *rescaled hypersurface* $\Pi_s^{(a_0, a_1)}$ *(see (3.31)) satisfies (3.77).*

Proof. Let

$$\widetilde{v}(y, s) = \zeta\left(e^{\sigma s} y - \beta\right) \zeta\left(\rho e^{\frac{s}{2}} - y\right) v(y, s)$$

then $\widetilde{v}(\cdot, s) \in C\left([s_0, \mathring{s}]; \mathbf{H}\right)$. From (3.33), we have

$$(\partial_s + \mathcal{L})\, v(\cdot, s) = \mathcal{Q}v(\cdot, s)$$

which implies

$$(\partial_s + \mathcal{L})\, \widetilde{v}(\cdot, s) = f(\cdot, s) \equiv f_{\mathrm{I}}(\cdot, s) + f_{\mathrm{II}}(\cdot, s) + f_{\mathrm{III}}(\cdot, s) \tag{3.85}$$

where

$$f_{\mathrm{I}}(y, s) = \zeta\left(e^{\sigma s} y - \beta\right) \zeta\left(\rho e^{\frac{s}{2}} - y\right) \mathcal{Q}v(y, s)$$

$$f_{\mathrm{II}}(y, s) = \zeta'\left(e^{\sigma s} y - \beta\right) e^{\sigma s}\left(-2\,\partial_y v(y, s) + \left(-\frac{2(n-1)}{y} + \left(\sigma + \frac{1}{2}\right) y\right) v(y, s)\right)$$
$$-\zeta''\left(e^{\sigma s} y - \beta\right) e^{2\sigma s}\, v(y, s)$$

$$f_{\mathrm{III}}(y, s) = \zeta'\left(\rho e^{\frac{s}{2}} - y\right)\left(\left(\frac{\rho}{2} e^{\frac{s}{2}} - \frac{y}{2} + \frac{2(n-1)}{y}\right) v(y, s) + 2\,\partial_y v(y, s)\right)$$
$$-\zeta''\left(\rho e^{\frac{s}{2}} - y\right) v(y, s)$$

We claim that

$$\| f(\cdot, s)\| \leq C(n, \Lambda, \rho, \beta)\, e^{-(1+\varsigma)\lambda_2 s} \tag{3.86}$$

for $s_0 \leq s \leq \mathring{s}$, provided that $0 < \rho \ll 1 \ll \beta$ (depending on n, Λ) and $s_0 \gg 1$ (depending on n, Λ, ρ, β), where the norm $\|\cdot\|$ is defined in Proposition 3.8. Notice that by (3.32), we have

$$\max\left\{\left|\frac{v(y, s)}{y}\right|, \left|\partial_y v(y, s)\right|\right\} \leq \Lambda e^{-\lambda_2 s}\left(y^{\alpha-1} + y^{2\lambda_2}\right) \lesssim \Lambda\left(\beta^{\alpha-1} + \rho^{2\lambda_2}\right)$$

for $\beta e^{-\sigma s} \leq y \leq \rho e^{\frac{s}{2}}$, so we have

$$\max \left\{ \left| \frac{v\,(y,\,s)}{y} \right|,\, \left| \partial_y v\,(y,\,s) \right| \right\} \,\leq\, \frac{1}{3}$$

for $\beta e^{-\sigma s} \,\leq\, y \,\leq\, \rho e^{\frac{s}{2}}$ provided that $0 < \rho \ll 1 \ll \beta$ (depending on $n,\, \Lambda$). To prove (3.86), we use (3.32) to get

$$\begin{aligned}
\left\| f_{\mathrm{I}} \right\| &= \left\| \zeta \left(e^{\sigma s} y - \beta \right) \zeta \left(\rho e^{\frac{s}{2}} - y \right) \, \mathcal{Q} v\,(y,\,s) \right\| \\
&\leq C\,(n)\,\Lambda^3 \left\| \left(e^{-\lambda_2 s} \left(y^{\alpha - 1} + y^{2\lambda_2} \right) \right)^2 e^{-\lambda_2 s} \left(y^{\alpha - 2} + y^{2\lambda_2 - 1} \right) \chi_{\left(\beta e^{-\sigma s},\, \rho e^{\frac{s}{2}} \right)} \right\| \\
&\leq C\,(n)\,\Lambda^3\, e^{-(1+\varsigma)\lambda_2 s} \left\| \left(e^{-\lambda_2 s} \left(y^{\alpha - 1} + y^{2\lambda_2} \right) \right)^{2-\varsigma} \left(y^{\alpha - 2 + \varsigma(\alpha - 1)} + y^{2\lambda_2 - 1 + 2\varsigma\lambda_2} \right) \chi_{\left(\beta e^{-\sigma s},\, \rho e^{\frac{s}{2}} \right)} \right\| \\
&\leq C\,(n)\,\Lambda^3\, e^{-(1+\varsigma)\lambda_2 s} \left\| \left(\beta^{\alpha - 1} + \rho^{2\lambda_2} \right)^{2-\varsigma} \left(y^{\alpha - 2 + \varsigma(\alpha - 1)} + y^{2\lambda_2 - 1 + 2\varsigma\lambda_2} \right) \chi_{\left(\beta e^{-\sigma s},\, \rho e^{\frac{s}{2}} \right)} \right\| \\
&\leq C\,(n)\,\Lambda^3\, e^{-(1+\varsigma)\lambda_2 s} \left(\int_0^\infty \left(y^{2(\alpha - 2 + \varsigma(\alpha - 1))} + y^{2(2\lambda_2 - 1 + 2\varsigma\lambda_2)} \right) y^{2(n-1)} e^{-\frac{y^2}{4}}\, dy \right)^{\frac{1}{2}} \\
&\leq C\,(n)\,\Lambda^3\, e^{-(1+\varsigma)\lambda_2 s}
\end{aligned}$$

since $\varsigma \leq \lambda_2^{-1} \leq 1$ and $2\,(\alpha - 2 + \varsigma\,(\alpha - 1)) + 2\,(n-1) > -1$;

$$\begin{aligned}
\left\| f_{\mathrm{II}} \right\| \,&\leq\, C\,(n)\,\Lambda \left\| e^{-\lambda_2 s} y^{\alpha - 2}\, \chi_{\left(\beta e^{-\sigma s},\, (\beta+1) e^{-\sigma s} \right)} \right\| \\
&\leq C\,(n)\,\Lambda\, e^{-\lambda_2 s} \left(\int_{\beta e^{-\sigma s}}^{(\beta+1)e^{-\sigma s}} y^{2(\alpha - 2)} y^{2(n-1)}\, dy \right)^{\frac{1}{2}} \\
&\leq C\,(n)\,\Lambda\, e^{-\lambda_2 s} \left(\beta e^{-\sigma s} \right)^{n + \alpha - \frac{5}{2}} \;\leq C\,(n)\,\Lambda\, \beta^{n + \alpha - \frac{5}{2}} e^{-(1+\varsigma)\lambda_2 s}
\end{aligned}$$

and

$$\begin{aligned}
\left\| f_{\mathrm{III}} \right\| \,&\leq\, C\,(n)\,\Lambda \left\| e^{-\lambda_2 s} y^{2\lambda_2 + 2} \chi_{\left(\rho e^{\frac{s}{2}} - 1,\, \rho e^{\frac{s}{2}} \right)} \right\| \\
&= C\,(n)\,\Lambda\, e^{-\lambda_2 s} \left(\int_{\rho e^{\frac{s}{2}} - 1}^{\rho e^{\frac{s}{2}}} y^{2(2\lambda_2 + 2)} y^{2(n-1)} e^{-\frac{y^2}{4}}\, dy \right)^{\frac{1}{2}} \\
&\leq\, C\,(n)\,\Lambda\, e^{-\lambda_2 s} e^{-s} \,\leq\, C\,(n)\,\Lambda\, e^{-(1+\varsigma)\lambda_2 s}
\end{aligned}$$

provided that $s_0 \gg 1$ (depending on $n,\, \rho$).

Next, we would like to estimate the components of negative eigenvalue functions in the Fourier expansion of $\widetilde{v}(\cdot, s)$. For each $i \in \{0, 1\}$, by Proposition 3.8, (3.72) and (3.85), we have

$$\begin{cases} \partial_s \langle \widetilde{v}(\cdot, s), \varphi_i \rangle + \lambda_i \langle \widetilde{v}(\cdot, s), \varphi_i \rangle = \langle f(\cdot, s), \varphi_i \rangle \\ \\ \langle \widetilde{v}(\cdot, s_1), \varphi_i \rangle = 0 \end{cases}$$

Note that $\lambda_i = \lambda_2 - (2 - i) < 0$ and

$$\mathring{s} = -\ln\left(-\mathring{t}\right) \le -\ln\left(-e^{-1} t_1\right) = s_1 + 1$$

Therefore, for $s_1 \le s \le \mathring{s}$, we have

$$\begin{aligned} |\langle \widetilde{v}(\cdot, s), \varphi_i \rangle| &= \left| \int_{s_1}^{s} e^{-\lambda_i (s-\xi)} \langle f(\cdot, \xi), \varphi_i \rangle \, d\xi \right| \le \int_{s_1}^{s} e^{-(\lambda_2 - 2)(s-\xi)} \| f(\cdot, \xi) \| \, d\xi \\ &\le C(n, \Lambda, \rho, \beta) \, e^{-(\lambda_2 - 2)(s - s_1)} e^{-(1+\varsigma)\lambda_2 s_1} \\ &\le C(n, \Lambda, \rho, \beta) \, e^{-(1+\varsigma)\lambda_2 s} \end{aligned}$$

and for $s_0 \le s \le s_1$, we have

$$\begin{aligned} |\langle \widetilde{v}(\cdot, s), \varphi_i \rangle| &= \left| \int_{s}^{s_1} e^{\lambda_i (\xi - s)} \langle f(\cdot, \xi), \varphi_i \rangle \, d\xi \right| \le \int_{s}^{s_1} e^{(\lambda_2 - 1)(\xi - s)} \| f(\cdot, \xi) \| \, d\xi \\ &\le C(n, \Lambda, \rho, \beta) \, e^{-(1+\varsigma)\lambda_2 s} \end{aligned}$$

Thus, for $i \in \{0, 1\}$, there holds

$$|\langle \widetilde{v}(\cdot, s), \varphi_i \rangle| \le C(n, \Lambda, \rho, \beta) \, e^{-(1+\varsigma)\lambda_2 s} \tag{3.87}$$

for $s_0 \le s \le \mathring{s}$. In addition, for $i \in \{0, 1\}$, by Lemma 3.12 we have

$$\begin{aligned} &\left| \langle \widetilde{v}(\cdot, s_0), c_i \varphi_i \rangle - a_i e^{-\lambda_2 s_0} \right| \\ &= \left| \left\langle \zeta\left(e^{\sigma s_0} y - \beta\right) \zeta\left(\rho e^{\frac{s_0}{2}} - y\right) v(\cdot, s_0), c_i \varphi_i \right\rangle - a_i e^{-\lambda_2 s_0} \right| \\ &= e^{-\lambda_2 s_0} \left| \left\langle \zeta\left(e^{\sigma s_0} y - \beta\right) \zeta\left(\rho e^{\frac{s_0}{2}} - y\right) \left(\frac{1}{c_2} \varphi_2(y) + \frac{a_0}{c_0} \varphi_0(y) + \frac{a_1}{c_1} \varphi_1(y) \right), c_i \varphi_i \right\rangle - a_i \right| \\ &\le C(n, \Lambda, \rho, \beta) \, e^{-(1+2\varsigma)\lambda_2 s_0} \end{aligned}$$

which, together with (3.87), implies

$$|a_i| \le \left|e^{\lambda_2 s_0} \langle \widetilde{v}(\cdot,\, s_0)\,,\, c_i\, \varphi_i\rangle\right| + \left|e^{\lambda_2 s_0} \langle \widetilde{v}(\cdot,\, s_0)\,,\, c_i\, \varphi_i\rangle - a_i\right|$$

$$\le C\,(n,\, \Lambda,\, \rho,\, \beta)\, e^{-\varsigma\lambda_2 s_0}$$

We continue to estimate the components of the first positive eigenvalue functions in the Fourier expansion of $\widetilde{v}(\cdot,\, s)$. By Proposition 3.8, Lemma 3.12, (3.46) and (3.85), we have

$$\begin{cases} \partial_s \left(e^{\lambda_2 s} \langle \widetilde{v}(\cdot,\, s)\,,\, \varphi_2\rangle\right) = e^{\lambda_2 s} \langle f\,(\cdot,\, s)\,,\, \varphi_2\rangle \\ \\ \left|e^{\lambda_2 s_0} \langle \widetilde{v}(\cdot,\, s_0)\,,\, c_2 \varphi_2\rangle - 1\right| \le C\,(n)\, e^{-2\varsigma\lambda_2 s_0} \end{cases}$$

Now let

$$k = e^{\lambda_2 s_1} \langle \tilde{v}(\cdot,\, s_1)\,,\, c_2\, \varphi_2\rangle$$

then for $s_1 \le s \le \mathring{s}$, we have

$$\left|e^{\lambda_2 s} \langle \widetilde{v}(\cdot,\, s)\,,\, c_2 \varphi_2\rangle - k\right| = \left|e^{\lambda_2 s} \langle \widetilde{v}(\cdot,\, s)\,,\, \varphi_2\rangle - e^{\lambda_2 s_1} \langle \widetilde{v}(\cdot,\, s_1)\,,\, c_2\, \varphi_2\rangle\right|$$

$$= \left|\int_{s_1}^{s} e^{\lambda_2 \xi} \langle f\,(\cdot,\, \xi)\,,\, \varphi_2\rangle\, d\xi\right| \le \int_{s_1}^{s_1+1} e^{\lambda_2 \xi}\, \|f\,(\cdot,\, \xi)\|\, d\xi$$

$$\le C\,(n,\, \Lambda,\, \rho,\, \beta)\, e^{-\varsigma\lambda_2 s}$$

(since $\mathring{s} \le s_1 + 1$), and for $s_0 \le s \le s_1$ we have

$$\left|e^{\lambda_2 s} \langle \widetilde{v}(\cdot,\, s)\,,\, c_2\, \varphi_2\rangle - k\right| = \left|e^{\lambda_2 s} \langle \widetilde{v}(\cdot,\, s)\,,\, c_2\, \varphi_2\rangle - e^{\lambda_2 s_1} \langle \widetilde{v}(\cdot,\, s_1)\,,\, c_2\, \varphi_2\rangle\right|$$

$$= \left|\int_{s}^{s_1} e^{\lambda_2 \xi} \langle f\,(\cdot,\, \xi)\,,\, \varphi_2\rangle\, d\xi\right| \le \int_{s}^{s_1} e^{\lambda_2 \xi}\, \|f\,(\cdot,\, \xi)\|\, d\xi$$

$$\le C\,(n,\, \Lambda,\, \rho,\, \beta)\, e^{-\varsigma\lambda_2 s}$$

Thus, we get

$$|k - 1| \le \left|k - e^{\lambda_2 s_0} \langle \widetilde{v}(\cdot,\, s_0)\,,\, c_2\, \varphi_2\rangle\right| + \left|e^{\lambda_2 s_0} \langle \widetilde{v}(\cdot,\, s_0)\,,\, c_2\, \varphi_2\rangle - 1\right|$$

$$\le C\,(n,\, \Lambda,\, \rho,\, \beta)\, e^{-\varsigma\lambda_2 s}$$

and

$$\left|\langle \widetilde{v}(\cdot,\, s)\,,\, \varphi_2\rangle - \frac{k}{c_2} e^{-\lambda_2 s}\right| \le C\,(n,\, \Lambda,\, \rho,\, \beta)\, e^{-(1+\varsigma)\lambda_2 s} \tag{3.88}$$

for $s_0 \le s \le \mathring{s}$.

Now we would like to estimate the remaining parts in the Fourier expansion of $\widetilde{v}(\cdot,\, s)$. Let

$$\widetilde{v}_*(\cdot,\, s) = \widetilde{v}(\cdot,\, s) - \sum_{i=0}^{2} \langle \widetilde{v}(\cdot,\, s)\,,\, \varphi_i \rangle\, \varphi_i$$

then $\widetilde{v}_*(\cdot,\, s) \in C([s_0,\, s_1]\,;\, \mathbf{H}_*)$, where $\mathbf{H}_*$ is defined in Lemma 3.24. By Proposition 3.8 and (3.85), we have

$$(\partial_s + \mathcal{L})\,\widetilde{v}_*(\cdot,\, s) \;=\; f(\cdot,\, s) - \sum_{i=0}^{2} \langle f(\cdot,\, s)\,,\, \varphi_i \rangle\, \varphi_i \;\equiv\; f_*(\cdot,\, s)$$

Note that $\|f_*(\cdot,\, s)\| \;\le\; \|f(\cdot,\, s)\|$ and that $\lambda_3 \;=\; \lambda_2 + 1$. By Lemma 3.24, for any $0 < \delta < 1$, we have

$$\begin{aligned} &\|\widetilde{v}_*(\cdot,\, s)\|^2 \\ \le\; & e^{-2(1-\delta)(\lambda_2+1)(s-s_0)}\, \|\widetilde{v}_*(\cdot,\, s_0)\|^2 \;+\; \frac{1}{2\delta\lambda_3} \int_{s_0}^{s} e^{-2(1-\delta)(\lambda_2+1)(s-\xi)}\, \|f(\cdot,\, \xi)\|^2\, d\xi \end{aligned}$$

$$\begin{aligned} &\langle \mathcal{L}\widetilde{v}_*(\cdot,\, s)\,,\, \widetilde{v}_*(\cdot,\, s) \rangle \\ =\; & e^{-2(1-\delta)(\lambda_2+1)(s-s_0)}\, \langle \mathcal{L}\widetilde{v}_*(\cdot,\, s_0)\,,\, \widetilde{v}_*(\cdot,\, s_0) \rangle \;+\; \frac{1}{2\delta} \int_{s_0}^{s} e^{-2(1-\delta)(\lambda_2+1)(s-\xi)}\, \|f(\cdot,\, \xi)\|^2\, d\xi \end{aligned}$$

for $s_0 \le s \le \mathring{s}$. We claim that

$$\|\widetilde{v}_*(\cdot,\, s_0)\| \;+\; \|\mathcal{L}\widetilde{v}_*(\cdot,\, s_0)\| \le\; C(n,\, \Lambda,\, \rho,\, \beta)\, e^{-(1+\varsigma)\lambda_2 s_0} \tag{3.89}$$

Note that since $\varsigma < \lambda_2^{-1}$, there is $\delta \in (0,\, 1)$ so that $(1-\delta)(\lambda_2+1) > (1+\varsigma)\,\lambda_2$. Thus, we get

$$\|\widetilde{v}_*(\cdot,\, s)\|^2 + \langle \mathcal{L}\widetilde{v}_*(\cdot,\, s)\,,\, \widetilde{v}_*(\cdot,\, s) \rangle \;\le\; C(n,\, \Lambda,\, \rho,\, \beta)\, e^{-2(1+\varsigma)\lambda_2 s}$$

which, by (3.36), yields

$$\|\widetilde{v}_*(\cdot,\, s)\|^2 + \|\partial_y \widetilde{v}_*(\cdot,\, s)\|^2 \;\le C(n,\, \Lambda,\, \rho,\, \beta)\, e^{-2(1+\varsigma)\lambda_2 s}$$

By Lemma 3.25, we then get

$$\begin{aligned} |\widetilde{v}_*(y,\, s)| \;&\le\; C(n)\left(\|\partial_y \widetilde{v}_*(\cdot,\, s)\| \;+\; \|\widetilde{v}_*(\cdot,\, s)\|\right)\left(\frac{1}{y^{n-\frac{1}{2}}} + e^{\frac{(y+1)^2}{4}}\right) \\ &\le C(n,\, \Lambda,\, \rho,\, \beta)\, e^{-(1+\varsigma)\lambda_2 s}\left(\frac{1}{y^{n-\frac{1}{2}}} + e^{\frac{(y+1)^2}{4}}\right) \end{aligned} \tag{3.90}$$

for $s_0 \leq s \leq \mathring{s}$. To prove (3.89), we use Proposition 3.8, Lemma 3.12, (3.46) and previous computation for derving (3.87) and (3.88) to get

$$\|\widetilde{v}_*(\cdot,\, s_0)\| = \left\|\widetilde{v}(\cdot,\, s_0) - \sum_{i=0}^{2} \langle \widetilde{v}(\cdot,\, s_0)\,,\, \varphi_i\rangle\, \varphi_i\right\|$$

$$\leq \left\|\widetilde{v}(\cdot,\, s_0) - e^{-\lambda_2 s_0}\sum_{i=0}^{2}\frac{a_i}{c_i}\varphi_i\right\| + \left\|e^{-\lambda_2 s_0}\sum_{i=0}^{2}\frac{a_i}{c_i}\varphi_i - \sum_{i=0}^{2}\langle \widetilde{v}(\cdot,\, s_0)\,,\, \varphi_i\rangle\, \varphi_i\right\|$$

$$\leq e^{-\lambda_2 s_0}\left\|\left(1-\zeta\left(e^{\sigma s_0}y-\beta\right)\zeta\left(\rho e^{\frac{s_0}{2}}-y\right)\right)\sum_{i=0}^{2}\frac{a_i}{c_i}\varphi_i\right\| + \sum_{i=0}^{2}\frac{1}{c_i}\left|\langle \widetilde{v}(\cdot,\, s_0)\,,\, c_i\varphi_i\rangle - a_i e^{-\lambda_2 s_0}\right|$$

$$\leq\, C\left(n,\, \Lambda,\, \rho,\, \beta\right) e^{-(1+\varsigma)\lambda_2 s_0}$$

where $a_2 = 1$, and

$$\|\mathcal{L}\widetilde{v}_*(\cdot,\, s_0)\| = \left\|\mathcal{L}\left(\zeta\left(e^{\sigma s_0}y-\beta\right)\zeta\left(\rho e^{\frac{s_0}{2}}-y\right)v(\cdot,\, s_0)\right) - \sum_{i=0}^{2}\langle \widetilde{v}(\cdot,\, s)\,,\, \varphi_i\rangle\,\lambda_i\varphi_i\right\|$$

$$= \left\|\mathcal{L}\left(\zeta\left(e^{\sigma s_0}y-\beta\right)\zeta\left(\rho e^{\frac{s_0}{2}}-y\right)e^{-\lambda_2 s_0}\sum_{i=0}^{2}\frac{a_i}{c_i}\varphi_i\right) - \sum_{i=0}^{2}\langle \widetilde{v}(\cdot,\, s)\,,\, \varphi_i\rangle\,\lambda_i\varphi_i\right\|$$

$$\leq e^{-\lambda_2 s_0}\left\|\mathcal{L}\left(\zeta\left(e^{\sigma s_0}y-\beta\right)\zeta\left(\rho e^{\frac{s_0}{2}}-y\right)\sum_{i=0}^{2}\frac{a_i}{c_i}\varphi_i\right) - \sum_{i=0}^{2}\frac{a_i}{c_i}\lambda_i\varphi_i\right\|$$

$$+\left\|\sum_{i=0}^{2}\langle \widetilde{v}(\cdot,\, s)\,,\, \varphi_i\rangle\,\lambda_i\varphi_i - e^{-\lambda_2 s_0}\sum_{i=0}^{2}\frac{a_i}{c_i}\lambda_i\varphi_i\right\|$$

$$\leq \|h\| + \sum_{i=0}^{2}\frac{\lambda_i}{c_i}\left\|\langle \widetilde{v}(\cdot,\, s_0)\,,\, c_i\varphi_i\rangle - a_i e^{-\lambda_2 s_0}\right\|$$

where

$$h(y) = \zeta'\left(e^{\sigma s_0}y-\beta\right)e^{\sigma s_0}\left(-2\,\partial_y v(y,\, s_0) + \left(-\frac{2(n-1)}{y}+\frac{y}{2}\right)v(y,\, s_0)\right)$$

$$+\zeta'\left(\rho e^{\frac{s_0}{2}}-y\right)\left(\left(-\frac{y}{2}+\frac{2(n-1)}{y}\right)v(y,\, s_0)+2\,\partial_y v(y,\, s_0)\right)$$

$$-\zeta''\left(e^{\sigma s_0}y-\beta\right)e^{2\sigma s_0}v(y,\, s_0) - \zeta''\left(\rho e^{\frac{s_0}{2}}-y\right)v(y,\, s_0)$$

Note that by similar computation as for $f_{\mathrm{II}}(\cdot,\, s)$ and $f_{\mathrm{III}}(\cdot,\, s)$, we have

$$\|h\| \leq C\left(n,\, \Lambda,\, \rho,\, \beta\right) e^{-(1+\varsigma)\lambda_2 s_0}$$

Hence,

$$\|\mathcal{L}\widetilde{v}_*(\cdot,\, s_0)\| \leq\, C\left(n,\, \Lambda,\, \rho,\, \beta\right) e^{-(1+\varsigma)\lambda_2 s_0}$$

Lastly, combining (3.87), (3.88), and (3.90), we conclude

$$\left|\widetilde{v}\left(y,\,s\right)-\frac{k}{c_2}e^{-\lambda_2 s}\varphi_2\left(y\right)\right| = \left|\sum_{i=0}^{2}\left\langle\widetilde{v}\left(\cdot,\,s\right),\,\varphi_i\right\rangle\varphi_i\left(y\right)\,+\,\widetilde{v}_*\left(y,\,s\right)\,-\,\frac{k}{c_2}e^{-\lambda_2 s}\varphi_2\left(y\right)\right|$$

$$\leq \sum_{i=0}^{1}\left|\left\langle\widetilde{v}\left(\cdot,\,s\right),\,\varphi_i\right\rangle\varphi_i\left(y\right)\right|\,+\,\left|\left\langle\widetilde{v}\left(\cdot,\,s\right),\,\varphi_2\right\rangle\varphi_2\left(y\right)-\frac{k}{c_2}e^{-\lambda_2 s}\varphi_2\left(y\right)\right|\,+\,\left|\widetilde{v}_*\left(y,\,s\right)\right|$$

$$\leq C\left(n,\,\Lambda,\,\rho,\,\beta\right)e^{-(1+\varsigma)\lambda_2 s}\left(\frac{1}{y^{n-\frac{1}{2}}}+e^{\frac{(y+1)^2}{4}}\right)$$

for $s_0 \leq s \leq \mathring{s}$. As a result, for $\frac{1}{2}e^{-\vartheta\sigma s} \leq y \leq 1$, we have

$$\left|v\left(y,\,s\right)-\frac{k}{c_2}e^{-\lambda_2 s}\varphi_2\left(y\right)\right|\,\leq\,C\left(n,\,\Lambda,\,\rho,\,\beta\right)\left(\frac{e^{-\varsigma\lambda_2 s}}{y^{n+\alpha+\frac{3}{2}}}\right)e^{-\lambda_2 s}y^{\alpha+2}$$

$$\leq C\left(n,\,\Lambda,\,\rho,\,\beta\right)e^{-\left(\varsigma\lambda_2-\vartheta\sigma\left(n+\alpha+\frac{3}{2}\right)\right)s}e^{-\lambda_2 s}y^{\alpha+2}$$

and for $1 \leq y \leq \sqrt{\varsigma\lambda_2 s}$, we have

$$\left|v\left(y,\,s\right)-\frac{k}{c_2}e^{-\lambda_2 s}\varphi_2\left(y\right)\right|\,\leq\,C\left(n,\,\Lambda,\,\rho,\,\beta\right)\left(e^{-\varsigma\lambda_2 s}e^{\frac{(y+1)^2}{4}}\right)e^{-\lambda_2 s}y^{\alpha+2}$$

$$\leq C\left(n,\,\Lambda,\,\rho,\,\beta\right)e^{-\frac{\varsigma\lambda_2}{2}s}e^{-\lambda_2 s}y^{\alpha+2}$$

$\square$

As a corollary, by (3.30), Proposition 3.26 and Remark 3.9, we get

$$\left|u\left(x,\,t\right)-\frac{k}{c_2}\left(-t\right)^{\lambda_2+\frac{1}{2}}\varphi_2\left(\frac{x}{\sqrt{-t}}\right)\right|\,\leq\,C\left(n,\,\Lambda,\,\rho,\,\beta\right)\left(-t\right)^{\varkappa}\left(-t\right)x^{\alpha+2}$$

$$\leq C\left(n,\,\Lambda,\,\rho,\,\beta\right)\left(-t\right)^{\varkappa}x^{2\lambda_2+1} \tag{3.91}$$

for $\frac{1}{3}\sqrt{-t} \leq x \leq \sqrt{\varsigma\lambda_2\,t\,\ln\left(-t\right)}$, $t_0 \leq t \leq \mathring{t}$. Below we use (3.25), (3.50), (3.91) and the comparison principle to prove (3.76).

Proposition 3.27. *If* $0 < \rho \ll 1$ *(depending on* n, Λ*) and* $|t_0| \ll 1$ *(depending on* n, Λ, ρ*), there holds (3.76).*

Proof. First, by (3.26) we have

$$\max\left\{\left|\frac{u\left(x,\,t\right)}{x}\right|,\,\left|\partial_x u\left(x,\,t\right)\right|\right\}\,\leq\,\Lambda\left(\left(-t\right)^2x^{\alpha-1}+x^{2\lambda_2}\right)\,\leq\,\frac{1}{3}$$

for $\sqrt{\varsigma\lambda_2\, t\, \ln(-t)} \le x \le \rho$, $t_0 \le t \le \mathring{t}$, provided that $0 < \rho \ll 1$ (depending on n, Λ) and $|t_0| \ll 1$ (depending on n, Λ, ρ).

By (3.25), (3.26) and Remark 3.9, there holds

$$\begin{aligned}|\partial_t u(x,\, t)| \;&\le\; C(n)\left(\left|\partial^2_{xx} u(x,\, t)\right| + \left|\frac{\partial_x u(x,\, t)}{x}\right| + \left|\frac{u(x,\, t)}{x^2}\right|\right)\\ &\le\; C(n)\,\Lambda\left(x^{\alpha+2} + (-t)^2\, x^{\alpha-2}\right) \;\le\; C(n,\, \Lambda)\, x^{\alpha+2}\end{aligned}$$

for $\sqrt{\varsigma\lambda_2\, t\, \ln(-t)} \le x \le \rho$, $t_0 \le t \le \mathring{t}$. In addition, we have

$$\begin{aligned}\partial_t\left(k\,(-t)^{\lambda_2+\frac{1}{2}}\,\varphi_2\left(\frac{x}{\sqrt{-t}}\right)\right) &= k\;\partial_t\left(\Upsilon_2 x^{2\lambda_2+1} + 2\Upsilon_1(-t)\, x^{\alpha+2} + (-t)^2\, x^{\alpha}\right)\\ &= -2k\left(\Upsilon_1 x^{\alpha+2} + (-t)\, x^{\alpha}\right)\end{aligned}$$

Thus, we get

$$\left|\partial_t\left(u(x,\, t) - k\,(-t)^{\lambda_2+\frac{1}{2}}\,\varphi_2\left(\frac{x}{\sqrt{-t}}\right)\right)\right| \;\le\; C(n,\, \Lambda)\, x^{\alpha+2} \tag{3.92}$$

for $\sqrt{\varsigma\lambda_2\, t\, \ln(-t)} \le x \le \rho$, $t_0 \le t \le \mathring{t}$.

On the other hand, at time t_0, by (3.66), (3.73) and (3.74), there holds

$$\begin{aligned}&\left|u(x,\, t_0) - k\,(-t_0)^{\lambda_2+\frac{1}{2}}\,\varphi_2\left(\frac{x}{\sqrt{-t}}\right)\right|\\ &\le (-t_0)^{\lambda_2+\frac{1}{2}}\left(\frac{|k-1|}{c_0}\varphi_2\left(\frac{x}{\sqrt{-t}}\right) + \sum_{i=0}^{1}\frac{|a_i|}{c_i}\varphi_i\left(\frac{x}{\sqrt{-t}}\right)\right)\\ &\le C(n,\, \Lambda,\, \rho,\, \beta)\,(-t_0)^{\varkappa}\, x^{2\lambda_2+1}\end{aligned} \tag{3.93}$$

for $\sqrt{\varsigma\lambda_2\, t\, \ln(-t)} \le x \le \rho$. Moreover, by (3.91) we have

$$\left|u(x,\, t) - \frac{k}{c_2}(-t)^{\lambda_2+\frac{1}{2}}\,\varphi_2\left(\frac{x}{\sqrt{-t}}\right)\right| \;\le\; C(n,\, \Lambda,\, \rho,\, \beta)\,(-t_0)^{\varkappa}\, x^{2\lambda_2+1} \tag{3.94}$$

for $x = \sqrt{\varsigma\lambda_2\, t\, \ln(-t)}$, $t_0 \le t \le \mathring{t}$.

Combining (3.92), (3.93) and (3.94), we get

$$\begin{aligned}&\left|u(x,\, t) - k\,(-t)^{\lambda_2+\frac{1}{2}}\,\varphi_2\left(\frac{x}{\sqrt{-t}}\right)\right|\\ &\le C(n,\, \Lambda,\, \rho,\, \beta)\,(-t_0)^{\varkappa}\, x^{2\lambda_2+1} \,+\, C(n,\, \Lambda)\, x^{\alpha+2}\,(t-t_0)\\ &\le C(n,\, \Lambda,\, \rho,\, \beta)\,(-t_0)^{\varkappa}\, x^{2\lambda_2+1}\end{aligned}$$

for $\sqrt{\varsigma\lambda_2\, t\, \ln(-t)} \le x \le \rho$, $t_0 \le t \le \mathring{t}$. The conclusion follows by (3.91) and the above. $\square$

Next, by (3.30) and Proposition 3.26, we have

$$\left|w\left(z,\,\tau\right)-\frac{k}{c_2}\left(2\sigma\tau\right)^{\frac{\alpha}{2}}\varphi_2\left(\frac{z}{\sqrt{2\sigma\tau}}\right)\right|\,\leq\,C\left(n,\,\Lambda,\,\rho,\,\beta\right)\left(2\sigma\tau\right)^{-\frac{\varkappa}{2\sigma}}\,\frac{z^2}{2\sigma\tau}z^{\alpha}$$

for $\frac{1}{2}\left(2\sigma\tau\right)^{\frac{1}{2}(1-\vartheta)}\leq z\leq\sqrt{2\sigma\tau}$, $\tau_0\leq\tau\leq\mathring{\tau}$. Notice that

$$\frac{k}{c_2}\left(2\sigma\tau\right)^{\frac{\alpha}{2}}\varphi_2\left(\frac{z}{\sqrt{2\sigma\tau}}\right)=kz^{\alpha}\left(1+2\Upsilon_1\frac{z^2}{2\sigma\tau}+\Upsilon_2\left(\frac{z^2}{2\sigma\tau}\right)^2\right)$$

Hence we get

$$\left|w\left(z,\,\tau\right)-kz^{\alpha}\right|\,\leq\,C\left(n\right)\frac{z^2}{2\sigma\tau}z^{\alpha}$$

for $\frac{1}{2}\left(2\sigma\tau\right)^{\frac{1}{2}(1-\vartheta)}\leq z\leq\sqrt{2\sigma\tau}$, $\tau_0\leq\tau\leq\mathring{\tau}$, provided that $\tau_0\gg1$ (depending on n, Λ, ρ, β). On the other hand, by Lemma 3.7 and (3.74), we have

$$\left|\psi_k\left(z\right)-kz^{\alpha}\right|\,\leq\,C\left(n\right)k^3z^{3\alpha-2}\,\leq\,C\left(n\right)z^{3\alpha-2}$$

for $z\geq\frac{\hat{\psi}_2(0)}{\sqrt{2}}$, provided that $\tau_0\gg1$ (depending on n, Λ, ρ, β). Therefore, we get

$$\begin{aligned}\left|w\left(z,\,\tau\right)-\psi_k\left(z\right)\right|\,&\leq\,\left|w\left(z,\,\tau\right)-kz^{\alpha}\right|\,+\,\left|kz^{\alpha}-\psi_k\left(z\right)\right|\\&\leq C\left(n\right)\left(\frac{z^2}{2\sigma\tau}+z^{2(\alpha-1)}\right)z^{\alpha}\end{aligned}\tag{3.95}$$

for $\frac{1}{2}\left(2\sigma\tau\right)^{\frac{1}{2}(1-\vartheta)}\leq z\leq\sqrt{2\sigma\tau}$, $\tau_0\leq\tau\leq\mathring{\tau}$. Now consider the projected curves $\bar{\mathcal{M}}_k$ and $\bar{\Gamma}_{\tau}^{(a_0,\,a_1)}$ (see (3.7) and (3.41)), which can be viewed as graphes of $w\left(z,\,\tau\right)$ and $\psi_k\left(z\right)$ over $\bar{\mathcal{C}}$ (see (3.2)), respectively. Thus, (3.95) implies that

$$\left|\hat{w}\left(z,\,\tau\right)-\hat{\psi}_k\left(z\right)\right|\,\leq\,C\left(n\right)\left(\frac{z^2}{2\sigma\tau}+z^{2(\alpha-1)}\right)z^{\alpha}$$

for $\left(2\sigma\tau\right)^{\frac{1}{2}(1-\vartheta)}\leq z\leq\frac{1}{2}\sqrt{2\sigma\tau}$, $\tau_0\leq\tau\leq\mathring{\tau}$, provided that $\tau_0\gg1$ (depending on n, Λ, ρ, β). In particular, there holds

$$\left|\hat{w}\left(z,\,\tau\right)-\hat{\psi}_k\left(z\right)\right|\,\leq\,C\left(n\right)\left(2\sigma\tau\right)^{-\vartheta}z^{\alpha}\tag{3.96}$$

for $z=\left(2\sigma\tau\right)^{\frac{1}{2}(1-\vartheta)}$, $\tau_0\leq\tau\leq\mathring{\tau}$, since $0<\vartheta<\frac{1-\alpha}{2-\alpha}$ (see 3.57).

In addition, when $\tau=\tau_0$, by (3.52), (3.73) and (3.74), we have

$$\begin{aligned}&\left|w\left(z,\,\tau_0\right)-\psi_k\left(z\right)\right|\,\leq\,\left|w\left(z,\,\tau_0\right)-kz^{\alpha}\right|\,+\,\left|kz^{\alpha}-\psi_k\left(z\right)\right|\\&\leq\,\left(\left|k-1\right|\,+\,\left|a_0\right|\,+\,\left|a_1\right|\,+\,C\left(n\right)\left(\frac{z^2}{2\sigma\tau_0}+z^{2(\alpha-1)}\right)\right)z^{\alpha}\end{aligned}$$

$$\leq \left(C\left(n,\,\Lambda,\,\rho,\,\beta\right)\left(2\sigma\tau_0\right)^{-\frac{1-\alpha}{2}\varsigma}+C\left(n\right)\left(\left(2\sigma\tau_0\right)^{-\vartheta}+\beta^{2(\alpha-1)}\right)\right)z^{\alpha}$$

$$\leq C\left(n\right)\beta^{2(\alpha-1)}z^{\alpha}$$

for $\beta\leq z\leq 2\left(2\sigma\tau_0\right)^{\frac{1}{2}(1-\vartheta)}$, provided that $\tau_0\gg 1$ (depending on n, Λ, ρ, β). By reparametrizing $\bar{\Gamma}_{\tau_0}^{(a_0,\,a_1)}$ and $\bar{\mathcal{M}}_k$, we deduce that

$$\left|\hat{w}\left(z,\,\tau_0\right)-\hat{\psi}_k\left(z\right)\right|\leq C\left(n\right)\beta^{2(\alpha-1)}z^{\alpha} \tag{3.97}$$

for $\frac{3}{2}\beta\leq z\leq\left(2\sigma\tau_0\right)^{\frac{1}{2}(1-\vartheta)}$, provided that $\tau_0\gg 1$ (depending on n, Λ, ρ, β).

Below we use (3.40), (3.96), (3.97) and the comparison principle to prove (3.79). We follow Velázquez's idea of using the perturbation of $\hat{\psi}_k$ to construct barriers; moreover, we allow the perturbation to be time-dependent.

Proposition 3.28. *If $\beta\gg 1$ (depending on n) and $\tau_0\gg 1$ (depending on n, Λ, ρ, β), there holds (3.79). In particular, we have*

$$\left|\hat{w}\left(z,\,\tau\right)-\hat{\psi}_k\left(z\right)\right|\leq C\left(n\right)\beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho}z^{\alpha} \tag{3.98}$$

for $\beta\leq z\leq\left(2\sigma\tau\right)^{\frac{1}{2}(1-\vartheta)}$, $\tau_0\leq\tau\leq\mathring{\tau}$, and

$$\left|\hat{w}\left(z,\,\tau\right)-\hat{\psi}_k\left(z\right)\right|\leq C\left(n\right)\beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho} \tag{3.99}$$

for $0\leq z\leq 5\beta$, $\tau_0\leq\tau\leq\mathring{\tau}$.

Proof. Given functions $\lambda\left(\tau\right)$ and $\mu\left(\tau\right)$, we define the perturbation of $\hat{\psi}_k$ by

$$\hat{\psi}_k^{\lambda,\,\mu}\left(z,\,\tau\right)\equiv\hat{\psi}_{\lambda(\tau)\,k}\left(\frac{z}{\mu\left(\tau\right)}\right)=\lambda^{\frac{1}{1-\alpha}}\left(\tau\right)\,\hat{\psi}_k\left(\frac{z}{\lambda^{\frac{1}{1-\alpha}}\left(\tau\right)\,\mu\left(\tau\right)}\right)$$

(see also (3.3)). By (3.4), there holds

$$\partial_\tau\hat{\psi}_k^{\lambda,\,\mu}-\left(\frac{\partial_{zz}^2\hat{\psi}_k^{\lambda,\,\mu}}{1+\left(\partial_z\hat{\psi}_k^{\lambda,\,\mu}\right)^2}+\left(n-1\right)\left(\frac{\partial_z\hat{\psi}_k^{\lambda,\,\mu}}{z}-\frac{1}{\hat{\psi}_k^{\lambda,\,\mu}}\right)+\frac{\frac{1}{2}+\sigma}{2\sigma\tau}\left(-z\,\partial_z\hat{\psi}_k^{\lambda,\,\mu}+\hat{\psi}_k^{\lambda,\,\mu}\right)\right)$$

$$=\left(-\frac{\frac{1}{2}+\sigma}{2\sigma\tau}\lambda^{\frac{1}{1-\alpha}}+\frac{\lambda^{\frac{\alpha}{1-\alpha}}}{1-\alpha}\left(\partial_\tau\lambda\right)\right)\left(\hat{\psi}_k\left(r\right)-r\,\partial_r\hat{\psi}_k\left(r\right)\right)-\frac{\lambda^{\frac{1}{1-\alpha}}}{\mu}\left(\partial_\tau\mu\right)\left(r\,\partial_r\hat{\psi}_k\left(r\right)\right)\Bigg|_{r=\frac{z}{\lambda^{\frac{1}{1-\alpha}}\mu}}$$

$$+\frac{\mu^2-1}{\lambda^{\frac{1}{1-\alpha}}\mu^2}\left(\frac{\partial_{rr}^2\hat{\psi}_k(r)}{\left(1+\left(\partial_r\hat{\psi}_k(r)\right)^2\right)\left(1+\left(\frac{\partial_r\hat{\psi}_k(r)}{\mu}\right)^2\right)}+(n-1)\frac{\partial_r\hat{\psi}_k(r)}{r}\right)\Bigg|_{r=\frac{z}{\lambda^{\frac{1}{1-\alpha}}\mu}} \tag{3.100}$$

Notice that

$$\begin{cases} \partial_\lambda\left(\hat{\psi}_k^{\lambda,\mu}(z)\right) = \frac{\lambda^{\frac{\alpha}{1-\alpha}}}{1-\alpha}\left(\hat{\psi}_k(r) - r\,\partial_r\hat{\psi}_k(r)\right)\Big|_{r=\frac{z}{\lambda^{\frac{1}{1-\alpha}}\mu}} \\ \\ \partial_\mu\left(\hat{\psi}_k^{\lambda,\mu}(z)\right) = -\frac{\lambda^{\frac{1}{1-\alpha}}}{\mu}\left(r\,\partial_r\hat{\psi}_k(r)\right)\Big|_{r=\frac{z}{\lambda^{\frac{1}{1-\alpha}}\mu}} \end{cases} \tag{3.101}$$

Moreover, by (3.6), there holds

$$\lim_{r\nearrow\infty}\frac{\hat{\psi}_k(r) - r\,\partial_r\hat{\psi}_k}{r^\alpha} = k\lim_{r\nearrow\infty}\frac{\hat{\psi}(r) - r\,\partial_r\hat{\psi}}{r^\alpha} = k(1-\alpha)\,2^{\frac{\alpha+1}{2}}$$

which implies

$$\hat{\psi}_k(r) - r\,\partial_r\hat{\psi}_k = (1+o(1))(1-\alpha)\,2^{\frac{\alpha+1}{2}}r^\alpha \tag{3.102}$$

for $r\geq\beta$, if $\beta\gg 1$ (depending on n) and $\tau_0\gg 1$ (depending on n, Λ, ρ, β).

To get a lower barrier, we set

$$\hat{w}_-(z,\tau) = \hat{\psi}_k^{\lambda_-,\mu_-}(z,\tau)$$

with

$$\lambda_-(\tau) = 1-\beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho}, \qquad \mu_-(\tau)=1$$

where $\beta\gg 1$ (depending on n). Firstly, for the initial value, by Lemma 3.4 and (3.48), we have

$$\hat{w}_-(z,\tau_0) = \hat{\psi}_{\lambda_-(\tau_0)\,k}(z) = \hat{\psi}_{(1-\beta^{\alpha-3})(1+o(1))}(z) < \hat{w}(z,\tau_0) \tag{3.103}$$

for $0\leq z\leq\frac{3}{2}\beta$, provided that $\beta\gg 1$ (depending on n). Also, for each $\frac{3}{2}\beta\leq z\leq(2\sigma\tau_0)^{\frac{1}{2}(1-\vartheta)}$, by (3.101), (3.102), (3.97) and the mean value theorem, there is $\lambda_-(\tau_0)\leq\lambda_*\leq 1$ so that

$$\hat{w}_-(z,\tau_0) = \hat{\psi}_k(z) + (\lambda_-(\tau_0)-1)\,\partial_\lambda\left(\hat{\psi}_k^{\lambda,\mu}(z)\right)\Big|_{\lambda=\lambda_*,\,z=z_*\equiv\frac{z}{\lambda_*^{\frac{1}{1-\alpha}}}}$$

$$= \hat{\psi}_k(z) \;-\; \beta^{\alpha-3}\frac{\lambda_*^{\frac{\alpha}{1-\alpha}}}{1-\alpha}\left(\hat{\psi}_k(z_*) - z_*\,\partial_z\hat{\psi}_k(z_*)\right)$$

$$\le \hat{\psi}_k(z) \;-\; (1-o(1))\,\beta^{\alpha-3}2^{\frac{\alpha+1}{2}}z^{\alpha} \;<\; \hat{w}(z,\,\tau_0) \tag{3.104}$$

provided that $\beta \gg 1$ (depending on n). Secondly, for the boundary value, fix $\tau_0 \le \tau \le \mathring{\tau}$ and let $z = (2\sigma\tau)^{\frac{1}{2}(1-\vartheta)}$. By (3.96), (3.101), (3.102) and the mean value theorem, there is $\lambda_-(\tau_0) \le \lambda_* \le 1$ so that

$$\hat{w}_-(z,\,\tau_0) \;=\; \hat{\psi}_k(z) \;+\; (\lambda_-(\tau_0)-1)\;\partial_\lambda\left(\hat{\psi}_k^{\lambda,\,\mu}(z)\right)\Big|_{\lambda=\lambda_*,\;z=z_*\equiv\frac{z}{\lambda_*^{\frac{1}{1-\alpha}}}}$$

$$= \hat{\psi}_k(z) \;-\; \beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho}\frac{\lambda_*^{\frac{\alpha}{1-\alpha}}}{1-\alpha}\left(\hat{\psi}_k(z_*) - z_*\,\partial_z\hat{\psi}_k(z_*)\right)$$

$$\le \hat{\psi}_k(z) \;-(1-o(1))\,\beta^{\alpha-3}2^{\frac{\alpha+1}{2}}\left(\frac{\tau}{\tau_0}\right)^{-\varrho}z^{\alpha}$$

$$< \hat{\psi}_k(z) \;-C(n)\,(2\sigma\tau)^{-\vartheta}\,z^{\alpha} \;\le\; \hat{w}(z,\,\tau) \tag{3.105}$$

provided that $\tau_0 \gg 1$ (depending on n, β), since $0 < \varrho < \vartheta$. Thirdly, for the equation, by (3.100), there holds

$$\partial_\tau\hat{w}_- - \left(\frac{\partial^2_{zz}\hat{w}_-}{1+(\partial_z\hat{w}_-)^2} + (n-1)\left(\frac{\partial_z\hat{w}_-}{z} - \frac{1}{\hat{w}_-}\right) + \frac{\frac{1}{2}+\sigma}{2\sigma\tau}\left(-z\,\partial_z\hat{w}_- + \hat{w}_-\right)\right)$$

$$= \left(-\frac{\frac{1}{2}+\sigma}{2\sigma\tau}\lambda_-^{\frac{1}{1-\alpha}}(\tau) \;+\; \frac{\lambda_-^{\frac{\alpha}{1-\alpha}}(\tau)}{1-\alpha}\left(\partial_\tau\lambda_-(\tau)\right)\right)\left(\hat{\psi}_k - r\,\partial_r\hat{\psi}_k\right)\Bigg|_{r=\frac{z}{\lambda_-^{\frac{1}{1-\alpha}}(\tau)}}$$

$$= \frac{\lambda_-^{\frac{1}{1-\alpha}}(\tau)}{2\sigma\tau}\left(-\left(\frac{1}{2}+\sigma\right) + \frac{2\sigma\varrho\,\beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho}}{(1-\alpha)\,\lambda_-(\tau)}\right)\Bigg|_{r=\frac{z}{\lambda_-^{\frac{1}{1-\alpha}}(\tau)}} \le 0$$

for $0 \le z \le (2\sigma\tau)^{\frac{1}{2}(1-\vartheta)}$, $\tau_0 \le \tau \le \mathring{\tau}$, provided that $\beta \gg 1$ (depending on n). Then we subtract the above equation from (3.40) to get

$$\partial_\tau(\hat{w}-\hat{w}_-) - \left(\frac{1}{1+(\partial_z\hat{w})^2}\,\partial^2_{zz}(\hat{w}-\hat{w}_-) \;+\; \frac{n-1}{z}\,\partial_z(\hat{w}-\hat{w}_-)\right) \tag{3.106}$$

$$+\left(\frac{\partial^2_{zz}\hat{w}_-\left(\partial_z\hat{w}+\partial_z\hat{w}_-\right)}{\left(1+(\partial_z\hat{w})^2\right)\left(1+(\partial_z\hat{w}_-)^2\right)} \;+\; \frac{\frac{1}{2}+\sigma}{2\sigma\tau}z\right)\partial_z(\hat{w}-\hat{w}_-) - \left(\frac{n-1}{\hat{w}\,\hat{w}_-} \;+\; \frac{\frac{1}{2}+\sigma}{2\sigma\tau}\right)(\hat{w}-\hat{w}_-)$$

$$\ge\; 0$$

Now we are ready to show that $\hat{w}_{-}$ is a lower barrier. Let

$$\left(\hat{w}-\hat{w}_{-}\right)_{\min}(\tau)=\min_{0\leq z\leq(2\sigma\tau)^{\frac{1}{2}(1-\vartheta)}}\left(\hat{w}-\hat{w}_{-}\right)(z,\,\tau)$$

then by (3.103) and (3.104), we have

$$\left(\hat{w}-\hat{w}_{-}\right)_{\min}(\tau_{0})>0$$

We claim that

$$\left(\hat{w}-\hat{w}_{-}\right)_{\min}(\tau)\geq 0\qquad\forall\ \tau_{0}\leq\tau\leq\mathring{\tau}$$

Suppose the contrary, then there is $\tau_{0}<\tau_{1}^{*}\leq\mathring{\tau}$ so that

$$\left(\hat{w}-\hat{w}_{-}\right)_{\min}(\tau_{1}^{*})<0\tag{3.107}$$

Let $\tau_{0}^{*}\in[\tau_{0},\,\tau_{1}^{*})$ be the first time after which $\left(\hat{w}-\hat{w}_{-}\right)_{\min}$ stays negative all the way up to τ_{1}^{*}. By continuity, there holds

$$\left(\hat{w}-\hat{w}_{-}\right)_{\min}(\tau_{0}^{*})=0\tag{3.108}$$

On the other hand, by (3.105), the negative minimum of $\hat{w}-\hat{w}_{-}$ for each time-slice is achieved in $\left[0,\,(2\sigma\tau)^{\frac{1}{2}(1-\vartheta)}\right)$. Hence, applying the maximum principle to (3.106), we get

$$\partial_{\tau}\left(\hat{w}-\hat{w}_{-}\right)_{\min}-\left(\frac{n-1}{\hat{w}\,\hat{w}_{-}}+\frac{\frac{1}{2}+\sigma}{2\sigma\tau}\right)\left(\hat{w}-\hat{w}_{-}\right)_{\min}\geq 0$$

Notice that

$$\partial_{z}\hat{w}\,(0,\,\tau)\;=0=\;\partial_{z}\hat{w}_{-}\,(0,\,\tau)\qquad\forall\ \tau_{0}\leq\tau\leq\mathring{\tau}$$

So at $z=0$, by L'Hôpital's rule, the third term in (3.106) is interpreted as

$$\lim_{z\to 0}\frac{n-1}{z}\,\partial_{z}\left(\hat{w}-\hat{w}_{-}\right)(z,\,\tau)=(n-1)\;\partial_{zz}^{2}\left(\hat{w}-\hat{w}_{-}\right)(0,\,\tau)$$

It follows that

$$\partial_{\tau}\left(e^{-\int\frac{n-1}{\hat{w}\,\hat{w}_{-}}d\tau}\,\tau^{-\left(\frac{1}{2}+\frac{1}{4\sigma}\right)}\,\left(\hat{w}-\hat{w}_{-}\right)_{\min}(\tau)\right)\geq 0$$

which, together with (3.107), contradicts with (3.108).

Next, for the upper barrier, we set

$$\hat{w}_{+}\,(z,\,\tau)=\hat{\psi}_{k}^{\lambda_{+},\,\mu_{+}}\,(z,\,\tau)$$

with

$$\lambda_{+}(\tau)=1+\beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho}, \qquad \mu_{+}(\tau)=1+\delta\beta^{\alpha-3}(2\sigma\tau)^{-1+\varrho}\left(\frac{\tau}{\tau_0}\right)^{-\varrho}$$

where

$$\delta=\delta(n,\beta)=\frac{1}{4(1-\alpha)}\inf_{0\le r\le\frac{3}{2}\beta}\frac{\hat{\psi}_k(r)-r\,\partial_r\hat{\psi}_k(r)}{r\,\partial_r\hat{\psi}_k(r)}>0 \tag{3.109}$$

by (3.5). Note that by (see (3.4)),

$$0\le\partial_r\hat{\psi}_k(r)\le 1, \qquad \partial_{rr}^2\hat{\psi}_k(r)>0 \tag{3.110}$$

for all $r\ge 0$. Firstly, for the initial value, given $0\le z\le\frac{3}{2}\beta$, by Lemma 3.4, (3.48), (3.101), (3.102) and the mean value theorem, there are

$$1+\frac{1}{2}\beta^{\alpha-3}\le\lambda_*\le\lambda_{+}(\tau_0), \qquad 1\le\mu_*\le\mu_{+}(\tau_0)$$

so that

$$\begin{aligned}
\hat{w}_{+}(z,\tau_0)\;&=\;\hat{\psi}_k^{1+\frac{1}{2}\beta^{\alpha-3},\,1}(z,\tau_0)\\
&+\left(\lambda_{+}(\tau_0)-\left(1+\frac{1}{2}\beta^{\alpha-3}\right)\right)\partial_\lambda\left(\hat{\psi}_k^{\lambda,\,\mu}(z)\right)\Big|_{\lambda=\lambda_*,\;\mu=\mu_*,\;z=z_*\equiv\frac{z}{\lambda_*^{\frac{1}{1-\alpha}}\mu_*}}\\
&+(\mu_{+}(\tau_0)-1)\,\partial_\mu\left(\hat{\psi}_k^{\lambda,\,\mu}(z)\right)\Big|_{\lambda=\lambda_*,\;\mu=\mu_*,\;z=z_*\equiv\frac{z}{\lambda_*^{\frac{1}{1-\alpha}}\mu_*}}\\
&=\;\hat{\psi}_k^{1+\frac{1}{2}\beta^{\alpha-3},\,1}(z,\tau_0)\;+\;\frac{\beta^{\alpha-3}\lambda_*^{\frac{\alpha}{1-\alpha}}}{2(1-\alpha)}\left(\hat{\psi}_k(z_*)-z_*\,\partial_z\hat{\psi}_k(z_*)\right)\\
&\quad-\delta\beta^{\alpha-3}(2\sigma\tau_0)^{-1+\varrho}\,\frac{\lambda_*^{\frac{1}{1-\alpha}}}{\mu_*}\left(z_*\,\partial_z\hat{\psi}_k(z_*)\right)\\
&\ge\;\hat{\psi}_k^{1+\frac{1}{2}\beta^{\alpha-3},\,1}(z,\tau_0)\;+\;\frac{\beta^{\alpha-3}\lambda_*^{\frac{\alpha}{1-\alpha}}}{2(1-\alpha)}\left(1-\frac{\lambda_*}{2\mu_*}(2\sigma\tau_0)^{-1+\varrho}\right)\left(\hat{\psi}_k(z_*)-z_*\,\partial_z\hat{\psi}_k(z_*)\right)\\
&\ge\;\hat{\psi}_k^{1+\frac{1}{2}\beta^{\alpha-3},\,1}(z,\tau_0)\;=\;\hat{\psi}_{\left(1+\frac{1}{2}\beta^{\alpha-3}\right)k}(z,\tau_0)\;=\;\hat{\psi}_{\left(1+\frac{1}{2}\beta^{\alpha-3}\right)(1+o(1))}(z,\tau_0)\\
&>\;w(z,\tau_0)
\end{aligned} \tag{3.111}$$

provided that $\beta\gg 1$ (depending on n). Also, for each $\frac{3}{2}\beta\le z\le(2\sigma\tau_0)^{\frac{1}{2}(1-\vartheta)}$, by (3.67), (3.97), (3.101), (3.102), (3.109), (3.110) and the mean value theorem, there are

$$1\le\lambda_*\le\lambda_{+}(\tau_0), \qquad 1\le\mu_*\le\mu_{+}(\tau_0)$$

so that

$$\hat{w}_+ (z,\, \tau_0) \;=\; \hat{\psi}_k (z,\, \tau_0) \;+\; (\lambda_+ (\tau_0) - 1)\; \partial_\lambda \left(\hat{\psi}_k^{\lambda,\, \mu} (z) \right) \Big|_{\lambda=\lambda_*,\, \mu=\mu_*,\, z=z_* \equiv \frac{z}{\lambda_*^{\frac{1}{1-\alpha}} \mu_*}}$$

$$+\, (\mu_+ (\tau_0) - 1)\; \partial_\mu \left(\hat{\psi}_k^{\lambda,\, \mu} (z) \right) \Big|_{\lambda=\lambda_*,\, \mu=\mu_*,\, z=z_* \equiv \frac{z}{\lambda_*^{\frac{1}{1-\alpha}} \mu_*}}$$

$$= \hat{\psi}_k (z,\, \tau_0) + \frac{\beta^{\alpha-3} \lambda_*^{\frac{\alpha}{1-\alpha}}}{1-\alpha} \left(\hat{\psi}_k (z_*) - z_*\, \partial_z \hat{\psi}_k (z_*) \right) - \delta \beta^{\alpha-3} (2\sigma\tau_0)^{-1+\varrho}\, \frac{\lambda_*^{\frac{1}{1-\alpha}}}{\mu_*} \left(z_*\, \partial_z \hat{\psi}_k (z_*) \right)$$

$$\geq\; \hat{\psi}_k (z,\, \tau_0) \;+\; (1 + o(1))\, \beta^{\alpha-3} \mu_*^{-\alpha} 2^{\frac{\alpha+1}{2}} z^\alpha \;-\; \frac{\delta \beta^{\alpha-3}}{\mu_*^2} (2\sigma\tau_0)^{-\frac{1}{2}(1-\vartheta)(1-\alpha)}\, z$$

$$=\; \hat{\psi}_k (z,\, \tau_0) \;+\; \frac{1}{2} (1 + o(1))\, \beta^{\alpha-3} \mu_*^{-\alpha} 2^{\frac{\alpha+1}{2}} z^\alpha$$

$$+\, \frac{1}{2} \beta^{\alpha-3} z^\alpha \left((1 + o(1))\, 2^{\frac{\alpha+1}{2}} \mu_*^{-\alpha} \;-\; \frac{2\delta}{\mu_*^2} \left(\frac{z}{(2\sigma\tau_0)^{\frac{1}{2}(1-\vartheta)}} \right)^{1-\alpha} \right)$$

$$\geq\; \hat{\psi}_k (z,\, \tau_0) \;+\; \frac{1}{2} (1 + o(1))\, \beta^{\alpha-3} \mu_*^{-\alpha} 2^{\frac{\alpha+1}{2}} z^\alpha \;>\; w(z,\, \tau_0) \tag{3.112}$$

provided that $\beta \gg 1$ (depending on n), since $z \leq (2\sigma\tau_0)^{\frac{1}{2}(1-\vartheta)}$. Secondly, for the boundary value, fix $\tau_0 \leq \tau \leq \mathring{\tau}$ and let $z = (2\sigma\tau)^{\frac{1}{2}(1-\vartheta)}$, by (3.67), (3.96), (3.101), (3.102), (3.109), (3.110) and the mean value theorem, there are

$$1 \leq \lambda_* \leq \lambda_+ (\tau)\,, \qquad 1 \leq \mu_* \leq \mu_+ (\tau)$$

so that

$$\hat{w}_+ (z,\, \tau) \;=\; \hat{\psi}_k (z,\, \tau) \;+\; (\lambda_+ (\tau) - 1)\; \partial_\lambda \left(\hat{\psi}_k^{\lambda,\, \mu} (z) \right) \Big|_{\lambda=\lambda_*,\, \mu=\mu_*,\, z=z_* \equiv \frac{z}{\lambda_*^{\frac{1}{1-\alpha}} \mu_*}}$$

$$+\, (\mu_+ (\tau) - 1)\; \partial_\mu \left(\hat{\psi}_k^{\lambda,\, \mu} (z) \right) \Big|_{\lambda=\lambda_*,\, \mu=\mu_*,\, z=z_* \equiv \frac{z}{\lambda_*^{\frac{1}{1-\alpha}} \mu_*}}$$

$$=\; \hat{\psi}_k (z) \;+\; \beta^{\alpha-3} \left(\frac{\tau}{\tau_0} \right)^{-\varrho} \frac{\lambda_*^{\frac{\alpha}{1-\alpha}}}{1-\alpha} \left(\hat{\psi}_k (z_*) - z_*\, \partial_z \hat{\psi}_k (z_*) \right)$$

$$-\delta \beta^{\alpha-3} (2\sigma\tau)^{-1+\varrho} \left(\frac{\tau}{\tau_0} \right)^{-\varrho} \frac{\lambda_*^{\frac{1}{1-\alpha}}}{\mu_*} \left(z_*\, \partial_z \hat{\psi}_k (z_*) \right)$$

$$\geq\; \hat{\psi}_k (z) \;+\; (1 + o(1))\, \beta^{\alpha-3} \mu_*^{-\alpha} 2^{\frac{\alpha+1}{2}} \left(\frac{\tau}{\tau_0} \right)^{-\varrho} z^\alpha \;-\; \frac{\delta \beta^{\alpha-3}}{\mu_*^2} \left(\frac{\tau}{\tau_0} \right)^{-\varrho} (2\sigma\tau)^{-\frac{1}{2}(1-\vartheta)(1-\alpha)}\, z$$

$$\geq\; \hat{\psi}_k (z) \;+\; \frac{1}{2} (1 + o(1))\, \beta^{\alpha-3} \mu_*^{-\alpha} 2^{\frac{\alpha+1}{2}} \left(\frac{\tau}{\tau_0} \right)^{-\varrho} z^\alpha$$

$$+\frac{1}{2}\beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho}z^{\alpha}\left(\left(1+o\left(1\right)\right)2^{\frac{\alpha+1}{2}}\mu_*^{-\alpha}-\frac{2\delta}{\mu_*^2}\left(\frac{z}{\left(2\sigma\tau\right)^{\frac{1}{2}\left(1-\vartheta\right)}}\right)^{1-\alpha}\right)$$

$$>\hat{\psi}_k\left(z\right)-C\left(n\right)\left(2\sigma\tau\right)^{-\vartheta}z^{\alpha}\geq\hat{w}\left(z,\tau\right)\tag{3.113}$$

provided that $\tau_0 \gg 1$ (depending on n, β), since $z = (2\sigma\tau_0)^{\frac{1}{2}(1-\vartheta)}$ and $0 < \varrho < \vartheta$. Thirdly, by (3.100) and (3.110), there holds

$$\partial_\tau\hat{w}_+ -\left(\frac{\partial^2_{zz}\hat{w}_+}{1+\left(\partial_z\hat{w}_+\right)^2}+\left(n-1\right)\left(\frac{\partial_z\hat{w}_+}{z}-\frac{1}{\hat{w}_+}\right)+\frac{\frac{1}{2}+\sigma}{2\sigma\tau}\left(-z\,\partial_z\hat{w}_+ +\hat{w}_+\right)\right)$$

$$=\frac{\mu_+^2-1}{\lambda_+^{\frac{1}{1-\alpha}}\mu_+^2}\left(\frac{\partial^2_{rr}\hat{\psi}_k\left(r\right)}{\left(1+\left(\partial_r\hat{\psi}_k\left(r\right)\right)^2\right)\left(1+\left(\frac{\partial_r\hat{\psi}_k(r)}{\mu_+}\right)^2\right)}+\left(n-1\right)\frac{\partial_r\hat{\psi}_k\left(r\right)}{r}\right)\Bigg|_{r=\frac{z}{\lambda_+^{\frac{1}{1-\alpha}}\mu_+}}$$

$$+\left(-\frac{\frac{1}{2}+\sigma}{2\sigma\tau}\lambda_+^{\frac{1}{1-\alpha}}+\frac{\lambda_+^{\frac{\alpha}{1-\alpha}}}{1-\alpha}\left(\partial_\tau\lambda_+\right)\right)\left(\hat{\psi}_k\left(r\right)-r\,\partial_r\hat{\psi}_k\left(r\right)\right)\Bigg|_{r=\frac{z}{\lambda_+^{\frac{1}{1-\alpha}}\mu_+}}$$

$$-\frac{\lambda_+^{\frac{1}{1-\alpha}}}{\mu_+}\left(\partial_\tau\mu_+\right)\left(r\,\partial_r\hat{\psi}_k\left(r\right)\right)\Bigg|_{r=\frac{z}{\lambda_+^{\frac{1}{1-\alpha}}\mu_+}}$$

$$\geq 2\left(1-O\left(\beta^{\alpha-3}\right)\right)\delta\beta^{\alpha-3}\left(2\sigma\tau_0\right)^{\varrho}\left(2\sigma\tau\right)^{-1}\left(\frac{1}{4}\partial^2_{rr}\hat{\psi}_k\left(r\right)+\left(n-1\right)\frac{\partial_r\hat{\psi}_k\left(r\right)}{r}\right)\Bigg|_{r=\frac{z}{\lambda_+^{\frac{1}{1-\alpha}}\mu_+}}$$

$$-\left(1+O\left(\beta^{\alpha-3}\right)\right)\left(\frac{1}{2}+\sigma+\frac{2\sigma\varrho\beta^{\alpha-3}}{1-\alpha}\left(\frac{\tau}{\tau_0}\right)^{-\varrho}\right)\left(2\sigma\tau\right)^{-1}\left(\hat{\psi}_k\left(r\right)-r\,\partial_r\hat{\psi}_k\left(r\right)\right)\Bigg|_{r=\frac{z}{\lambda_+^{\frac{1}{1-\alpha}}\mu_+}}$$

$$\geq 0$$

provided that $\tau_0 \gg 1$ (depending on n, Λ, β), since

$$\frac{\partial_r\hat{\psi}_k\left(r\right)}{r}=\left(1+o\left(1\right)\right)r^{-1}>\left(1+o\left(1\right)\right)k\left(1-\alpha\right)2^{\frac{\alpha+1}{2}}r^{\alpha}=\hat{\psi}_k\left(r\right)-r\,\partial_r\hat{\psi}_k\left(r\right)$$

for $r \gg 1$ (noting that $\alpha < -1$). Then we subtract the equation of $\hat{w}_+(z,\tau)$ by (3.40) to get

$$\partial_\tau\left(\hat{w}_+-\hat{w}\right)-\left(\frac{1}{1+\left(\partial_z\hat{w}\right)^2}\partial^2_{zz}\left(\hat{w}_+-\hat{w}\right)+\frac{n-1}{z}\partial_z\left(\hat{w}_+-\hat{w}\right)\right)\tag{3.114}$$

$$+\left(\frac{\partial_{zz}^2\hat{w}_+\left(\partial_z\hat{w}_+ + \partial_z\hat{w}\right)}{\left(1+\left(\partial_z\hat{w}_+\right)^2\right)\left(1+\left(\partial_z\hat{w}\right)^2\right)} + \frac{\frac{1}{2}+\sigma}{2\sigma\tau}z\right)\partial_z\left(\hat{w}_+ - \hat{w}\right) - \left(\frac{n-1}{\hat{w}_+\,\hat{w}} + \frac{\frac{1}{2}+\sigma}{2\sigma\tau}\right)\left(\hat{w}_+ - \hat{w}\right)$$

$$\geq 0$$

To show that $\hat{w}_+$ is an upper barrier, let

$$\left(\hat{w}_+ - \hat{w}\right)_{\min}(\tau) = \min_{0\leq z\leq (2\sigma\tau)^{\frac{1}{2}(1-\vartheta)}}\left(\hat{w}_+ - \hat{w}\right)(z,\,\tau)$$

Note that by (3.111) and (3.112), we have

$$\left(\hat{w}_+ - \hat{w}\right)_{\min}(\tau_0) > 0$$

We claim that

$$\left(\hat{w}_+ - \hat{w}\right)_{\min}(\tau) \geq 0 \qquad \text{for } \tau_0 \leq \tau \leq \mathring{\tau}$$

Suppose the contrary, then there is $\tau_0 < \tau_1^* \leq \mathring{\tau}$ so that

$$\left(\hat{w}_+ - \hat{w}\right)_{\min}(\tau_1^*) < 0 \tag{3.115}$$

Let $\tau_0^* \in [\tau_0,\,\tau_1^*)$ be the first time after which $\left(\hat{w}^+ - \hat{w}\right)_{\min}$ is negative all the way up to τ_1^*, then by the continuity, we must have

$$\left(\hat{w}_+ - \hat{w}\right)_{\min}(\tau_0^*) = 0 \tag{3.116}$$

On the other hand, by (3.113), the minimum of $\hat{w}^+ - \hat{w}$ for each time-slice is achieved in $\left[0,\,(2\sigma\tau)^{\frac{1}{2}(1-\vartheta)}\right)$. Applying the maximum principle to (3.114), we get

$$\partial_\tau\left(\hat{w}_+ - \hat{w}\right)_{\min} - \left(\frac{n-1}{\hat{w}_+\,\hat{w}} + \frac{\frac{1}{2}+\sigma}{2\sigma\tau}\right)\left(\hat{w}_+ - \hat{w}\right)_{\min} \geq 0$$

Note that at $z = 0$, we always have

$$\partial_z\hat{w}(0,\,\tau) = 0 = \partial_z\hat{w}_+(0,\,\tau) \qquad \forall\ \tau_0 \leq \tau \leq \mathring{\tau}$$

so L'Hôpital's rule implies

$$\lim_{z\to 0}\frac{n-1}{z}\partial_z\left(\hat{w}_+ - \hat{w}\right)(z,\,\tau) = \frac{n-1}{z}\partial_z^2\left(\hat{w}_+ - \hat{w}\right)(0,\,\tau)$$

It follows that

$$\partial_\tau\left(e^{-\int\frac{n-1}{\hat{w}_+\,\hat{w}}d\tau}\tau^{-\frac{\frac{1}{2}+\sigma}{2\sigma}}\left(\hat{w}_+ - \hat{w}\right)_{\min}\right) \geq 0$$

which, together with (3.115), contraditcts with (3.116).

Lastly, by (3.101) and $\mu_+(\tau) \geq 1$, we have

$$\hat{w}_+(z,\,\tau) = \hat{\psi}_k^{\lambda_+,\,\mu_+}(z,\,\tau) \;\leq\; \hat{\psi}_k^{\lambda_+,\,1}(z,\,\tau) = \hat{\psi}_{\lambda_+(\tau)k}(z)$$

Thus, we get

$$\hat{\psi}_{\lambda_-(\tau)k}(z) = \hat{w}_-(z,\,\tau) \;\leq\; \hat{w}(z,\,\tau) \;\leq\; \hat{w}_+(z,\,\tau) \leq \hat{\psi}_{\lambda_+(\tau)k}(z)$$

For (3.98), given $\tau_0 \leq \tau \leq \mathring{\tau}$, $\beta \leq z \leq (2\sigma\tau)^{\frac{1}{2}(1-\vartheta)}$, by (3.101), (3.102) and the mean value theorem, there is $1 \leq \lambda_* \leq \lambda_+(\tau)$ so that

$$\begin{aligned}\hat{\psi}_k^{\lambda_+,\,1}(z,\,\tau) \;&=\; \hat{\psi}_k(z,\,\tau) \;+\; (\lambda_+(\tau)-1)\,\partial_\lambda\left(\hat{\psi}_k^{\lambda,\,\mu}(z)\right)\Big|_{\lambda=\lambda_*,\;z=z_*\equiv\frac{z}{\lambda_*^{\frac{1}{1-\alpha}}\mu_*}}\\
&=\; \hat{\psi}_k(z,\,\tau) \;+\; \beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho}\frac{\lambda_*^{\frac{\alpha}{1-\alpha}}}{1-\alpha}\left(\hat{\psi}_k(z_*) - z_*\,\partial_z\hat{\psi}_k(z_*)\right)\\
&\leq\; \hat{\psi}_k(z,\,\tau) \;+\; (1+o(1))\,2^{\frac{\alpha+1}{2}}\beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho} z^\alpha\end{aligned}$$

Similarly,

$$\hat{\psi}_k^{\lambda_-,\,1}(z,\,\tau) \;\geq\; \hat{\psi}_k(z,\,\tau) \;-\; (1+o(1))\,2^{\frac{\alpha+1}{2}}\beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho} z^\alpha$$

As for (3.99), given $\tau_0 \leq \tau \leq \mathring{\tau}$, $0 \leq z \leq 5\beta$, by (3.101), (3.102) and the mean value theorem, there is $1 \leq \lambda_* \leq \lambda_+(\tau)$ so that

$$\begin{aligned}\hat{\psi}_k^{\lambda_+,\,1}(z,\,\tau) \;&=\; \hat{\psi}_k(z,\,\tau) \;+\; (\lambda_+(\tau)-1)\,\partial_\lambda\left(\hat{\psi}_k^{\lambda,\,\mu}(z)\right)\Big|_{\lambda=\lambda_*,\;z=z_*\equiv\frac{z}{\lambda_*^{\frac{1}{1-\alpha}}\mu_*}}\\
&=\; \hat{\psi}_k(z,\,\tau) \;+\; \beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho}\frac{\lambda_*^{\frac{\alpha}{1-\alpha}}}{1-\alpha}\left(\hat{\psi}_k(z_*) - z_*\,\partial_z\hat{\psi}_k(z_*)\right)\\
&\leq\; \hat{\psi}_k(z,\,\tau) \;+\; \frac{\beta^{\alpha-3}\mathfrak{C}}{1-\alpha}\left(\frac{\tau}{\tau_0}\right)^{-\varrho}\end{aligned}$$

where

$$\mathfrak{C} = \sup_{r\geq 0}\left(\hat{\psi}_k(r) - r\,\partial_r\hat{\psi}_k(r)\right) \;\leq\; C(n)$$

(by (3.102)). Similarly,

$$\hat{\psi}_k^{\lambda_-,\,1}(z,\,\tau) \;\geq\; \hat{\psi}_k(z,\,\tau) \;-\; \frac{\beta^{\alpha-3}\mathfrak{C}}{1-\alpha}\left(\frac{\tau}{\tau_0}\right)^{-\varrho}$$

□

As a corollary, if we regard the projected curves $\bar{\Gamma}_{\tau}^{(a_0,\,a_1)}$ and $\bar{\mathcal{M}}_k$ as graphes over $\bar{\mathcal{C}}$, (3.98) implies

$$|w\,(z,\,\tau) - \psi_k\,(z)| \,\leq\, C\,(n)\,\beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho} z^{\alpha} \tag{3.117}$$

for $\frac{4}{3}\beta \leq z \leq \frac{1}{2}\,(2\sigma\tau)^{\frac{1}{2}(1-\vartheta)}$, $\tau_0 \leq \tau \leq \mathring{\tau}$. Then (3.78) follows immediately by (3.43).

Lastly, we prove (3.75) by using the gradient and curvature estimates in [EH].

Proposition 3.29. *If* $0 < \rho \ll 1$ *(depending on* n*,* Λ*) and* $|t_0| \ll \rho^2$ *(depending on* n*), there holds (3.75). Moreover, we have*

$$\begin{cases} |\partial_x u(x,\,t)| \,\lesssim\, 1 \\ \\ \left|\partial^2_{xx} u(x,\,t)\right| \,\leq\, \frac{C(n)}{\sqrt{t-t_0}} \end{cases} \tag{3.118}$$

for $x \geq \frac{1}{5}\rho$*,* $t_0 \leq t \leq \mathring{t}$*.*

Proof. For ease of notation, we denote $\Sigma_t^{(a_0,\,a_1)}$ by Σ_t. Let's first parametrize Σ_{t_0} by (3.24), i.e.

$$X_{t_0}\,(x,\,\nu,\,\omega) = \left((x - u\,(x,\,t_0))\,\frac{\nu}{\sqrt{2}},\,(x + u\,(x,\,t_0))\,\frac{\omega}{\sqrt{2}}\right)$$

for $x \geq \frac{1}{6}\rho$, $\nu,\,\omega \in \mathbb{S}^{n-1}$. Then the (upward) unit normal vector of Σ_{t_0} at X_{t_0} is given by

$$N_{\Sigma_{t_0}}\,(X_{t_0}) = \left(\left(\frac{1 + \partial_x u\,(x,\,t_0)}{\sqrt{1 + (\partial_x u\,(x,\,t_0))^2}}\right)\frac{-\nu}{\sqrt{2}},\,\left(\frac{1 - \partial_x u\,(x,\,t_0)}{\sqrt{1 + (\partial_x u\,(x,\,t_0))^2}}\right)\frac{\omega}{\sqrt{2}}\right)$$

Note that by (3.47) we have

$$\max\left\{\left|\frac{u\,(x,\,t_0)}{x}\right|,\,|\partial_x u\,(x,\,t_0)|\right\} \leq \frac{1}{3}$$

for $x \geq \frac{1}{6}\rho$.

Now fix $x_* \geq \frac{1}{5}\rho$ and let

$$\nu_* = \omega_* = \left(\overbrace{0,\cdots,0}^{\text{(n-1) copies}},\,1\right)$$

$$\mathbf{e} = \left(\frac{-1}{\sqrt{2}}\nu_*,\,\frac{1}{\sqrt{2}}\omega_*\right)$$

$$X_* = X_{t_0}(x_*,\, \nu_*,\, \omega_*) = \left((x_* - u(x_*,\, t_0)) \frac{\nu_*}{\sqrt{2}},\, (x_* + u(x_*,\, t_0)) \frac{\omega_*}{\sqrt{2}} \right)$$

Notice that

$$\begin{aligned} |X_{t_0} - X_*|^2 &\geq \frac{1}{2}(x - u(x,\, t_0))^2 \left(1 - (\nu \cdot \nu_*)^2\right) + \frac{1}{2}(x + u(x,\, t_0))^2 \left(1 - (\omega \cdot \omega_*)^2\right) \\ &\geq \frac{x^2}{2}\left(1 - \left|\frac{u(x,\, t_0)}{x}\right|\right)^2 \max\left\{1 - (\nu \cdot \nu_*)^2,\, 1 - (\omega \cdot \omega_*)^2\right\} \\ &\geq \frac{\rho^2}{9} \max\left\{1 - (\nu \cdot \nu_*)^2,\, 1 - (\omega \cdot \omega_*)^2\right\} \end{aligned}$$

Thus, for $X_{t_0} \in \Sigma_{t_0} \cap B\left(X_*;\, \frac{1}{30}\rho\right)$, there holds

$$\min\{\nu \cdot \nu_*,\, \omega \cdot \omega_*\} \geq \frac{\sqrt{91}}{10}$$

which implies

$$\begin{aligned} \left(N_{\Sigma_{t_0}}(X_{t_0}) \cdot \mathbf{e}\right)^{-1} &= \frac{2\sqrt{1 + (\partial_x u(x,\, t_0))^2}}{(1 + \partial_x u(x,\, t_0))(\nu \cdot \nu_*) + (1 - \partial_x u(x,\, t_0))(\omega \cdot \omega_*)} \\ &\leq \frac{\sqrt{10}}{\nu \cdot \nu_* + \omega \cdot \omega_*} \leq \frac{10\sqrt{10}}{2\sqrt{91}} \end{aligned} \tag{3.119}$$

By the gradient estimates in [EH], we then get

$$(N_{\Sigma_t}(X_t) \cdot \mathbf{e})^{-1} \leq \left(1 - \frac{|X_t - X_*|^2 + 2n(t - t_0)}{\left(\frac{1}{30}\rho\right)^2}\right)^{-1} \sup_{\Sigma_{t_0} \cap B\left(X_*;\, \frac{1}{30}\rho\right)} \left(N_{\Sigma_{t_0}} \cdot \mathbf{e}\right)^{-1}$$

for $X_t \in \Sigma_t \cap B\left(X_*;\, \sqrt{\left(\frac{1}{30}\rho\right)^2 - 2n(t - t_0)}\right)$, where $N_{\Sigma_t}(X_t)$ is the unit normal vector of Σ_t at X_t. Consequently,

$$(N(X_t) \cdot \mathbf{e})^{-1} \leq \left(1 - \left(\frac{30}{31}\right)^2\right) \frac{10\sqrt{10}}{2\sqrt{91}} \tag{3.120}$$

for $X_t \in \Sigma_t \cap B\left(X_*;\, \sqrt{\left(\frac{1}{31}\rho\right)^2 - 2n(t - t_0)}\right)$. It follows, by the curvature estimates in [EH], that

$$|A_{\Sigma_t}(X_t)| \leq C(n)\left(\frac{1}{\sqrt{t - t_0}} + \frac{1}{\rho}\right)$$

for $X_t \in \Sigma_t \cap B\left(X_*;\, \sqrt{\left(\frac{1}{32}\rho\right)^2 - 2n(t - t_0)}\right)$, where $A_{\Sigma_t}(X_t)$ is the second fundamental form of Σ_t at X_t. Thus, by choosing $|t_0| \ll \rho^2$ (depending on n), we may assume that

$$\sqrt{\left(\frac{1}{32}\rho\right)^2 - 2n(t - t_0)} \geq \frac{1}{33}\rho$$

for all $t_0 \le t \le \mathring{t}$, and

$$|A_{\Sigma_t}(X_t)| \le \frac{C(n)}{\sqrt{t - t_0}} \tag{3.121}$$

for $X_t \in \Sigma_t \cap B\left(X_*;\, \frac{\rho}{33}\right)$, $t_0 \le t \le \mathring{t}$.

Next, consider the "normal parametrization" for the MCF $\{\Sigma_t\}_{t_0 \le t \le \mathring{t}}$, i.e. let $X_t(x,\, \nu,\, \omega) = X(x,\, \nu,\, \omega;\, t)$ so that

$$\begin{cases} \partial_t X(x,\, \nu,\, \omega;\, t) = H_{\Sigma_t}(X(x,\, \nu,\, \omega;\, t))\, N_{\Sigma_t}(X(x,\, \nu,\, \omega;\, t)) \\ \\ X(x,\, \nu,\, \omega;\, t_0) = X_{t_0}(x,\, \nu,\, \omega) \end{cases}$$

For each $x \ge \rho$, ν, $\omega \in \mathbb{S}^{n-1}$, let $t_{(x,\nu,\omega)} \in \left(t_0,\, \mathring{t}\right]$ be the maximal time so that

$$X_t(x,\, \nu,\, \omega) \in \Sigma_t \cap B\left(X_{t_0}(x,\, \nu,\, \omega)\,;\, \frac{1}{33}\rho\right)$$

for all $t_0 \le t \le t_{(x,\nu,\omega)}$. Then we have

$$|\partial_t X_t(x,\, \nu,\, \omega)| = |H_{\Sigma_t}(X_t(x,\, \nu,\, \omega))| \le \frac{C(n)}{\sqrt{t - t_0}}$$

and hence

$$|X_t(x,\, \nu,\, \omega) - X_{t_0}(x,\, \nu,\, \omega)| \le C(n)\sqrt{t - t_0} \tag{3.122}$$

for all $t_0 \le t \le t_{(x,\nu,\omega)}$. Thus, if $|t_0| \ll 1$ (depending on n), we may assume that $t_{(x,\nu,\omega)} = \mathring{t}$ and

$$d_H\left(\Sigma_t \setminus B\left(O;\, \frac{1}{5}\rho\right),\, \Sigma_{t_0} \setminus B\left(O;\, \frac{1}{5}\rho\right)\right) \le C(n)\sqrt{t - t_0} \tag{3.123}$$

for all $t_0 \le t \le \mathring{t}$, where d_H is the Hausdorff distance. It follows that

$$|u(x,\, t) - u(x,\, t_0)| \le C(n)\sqrt{t - t_0}$$

for $x \ge \frac{1}{5}\rho$, $t_0 \le t \le \mathring{t}$.

Furthermore, by taking $x = x_*$, $\nu = \nu_*$, $\omega = \omega_*$ in (3.119) and replace t_0 by t, one could get

$$\left(N_{\Sigma_t}(X_t(x_*,\, \nu_*,\, \omega_*)) \cdot \mathbf{e}\right)^{-1} = \sqrt{1 + (\partial_x u(x_*,\, t))^2}$$

So by (3.120) and (3.122) , we have

$$|\partial_x u(x_*,\, t)| \lesssim 1 \tag{3.124}$$

for $t_0 \leq t \leq \mathring{t}$ (and any $x_* \geq \frac{1}{5}\rho$). For the second derivative, notice that

$$\frac{\left|\partial^2_{xx} u\left(x_*,\, t\right)\right|}{\left(1+\left(\partial_x u\left(x_*,\, t\right)\right)^2\right)^{\frac{3}{2}}} \leq \left|A_{\Sigma_t}\left(X_t\left(x_*,\, \nu_*,\, \omega_*\right)\right)\right|$$

By (3.121), (3.122) and (3.124), we conclude

$$\left|\partial^2_{xx} u(x_*,\, t)\right| \leq \frac{C\left(n\right)}{\sqrt{t-t_0}}$$

for $t_0 \leq t \leq \mathring{t}$ (and any $x_* \geq \frac{1}{5}\rho$). ☐

3.7 Smooth estimates in Proposition 3.13 and Proposition 3.14

This section is a continuation of Section 3.6. For ease of notation, from now on, let's denote $\Sigma_t^{(a_0,\, a_1)}$ by Σ_t, $\Gamma_\tau^{(a_0,\, a_1)}$ by Γ_τ and $\Pi_s^{(a_0,\, a_1)}$ by Π_s. Here we would like to show that if $0 < \rho \ll 1 \ll \beta$ (depending on n, Λ) and $|t_0| \ll 1$ (depending on n, Λ, ρ, β) , then

- In the **outer region**, the function $u\left(x,\, t\right)$ of $\Sigma_t^{(a_0,\, a_1)}$ defined in (3.24) satisfies (3.59).
- In the **tip region**, if we do the type II rescaling, the function $\hat{w}\left(z,\, \tau\right)$ of the rescaled hypersurface $\Gamma_\tau^{(a_0,\, a_1)}$ defined in (3.38) satisfies satisfies (3.61).

Moreover, for any $0 < \delta \ll 1$, $m,\, l \in \mathbb{Z}_+$, there hold the following higher order derivatives estimates.

1. In the **outer region**, the function $u\left(x,\, t\right)$ of $\Sigma_t^{(a_0,\, a_1)}$ defined in (3.24) satisfies (3.62) and (3.63) (see Proposition 3.33 and Proposition 3.34).
2. In the **intermediate region**, if we do the type I rescaling, the function $v\left(y,\, s\right)$ of the rescaled hypersurface $\Pi_s^{(a_0,\, a_1)}$ defined in (3.29) satisfies (3.64) and (3.65) (see Proposition 3.35).
3. In the **tip region**, if we do the type II rescaling, the function $\hat{w}\left(z,\, \tau\right)$ of the rescaled hypersurface $\Gamma_\tau^{(a_0,\, a_1)}$ defined in (3.38) satisfies (3.68) (see Proposition 3.41).

We establish (3.59) and (3.61) by using the maximum principle and curvature estimates in [EH]. Then we use Krylov-Safonov estimates and Schauder estimates, together with (3.26) (which is equivalent to (3.32) and (3.45)), (3.59) and (3.61), to derive (3.62), (3.63), (3.64), (3.65) and (3.68).

Let's start with proving (3.59). The C^0 estimats has already been shown in Proposition 3.27 and Proposition 3.29, in which we also get the first and second derivative bounds for $u(x,t)$ (see (3.118)). In the next lemma, we improve the first derivative bound in Proposition 3.29 by using the maximum principle, which turns out to be useful when we derive an improved second derivative estimate in Lemma 3.32.

Lemma 3.30. *If $0<\rho\ll 1$ (depending on n,Λ) and $|t_0|\ll 1$ (depending on n,ρ), there holds*

$$\sup_{x\geq\frac{1}{4}\rho}|\partial_x u(x,t)|\leq\sup_{x\geq\frac{1}{5}\rho}|\partial_x u(x,t_0)|+C(n,\rho)\sqrt{t-t_0}$$

for $t_0\leq t\leq\mathring{t}$.

Proof. First, differentiate (3.25) with respect to x to get

$$\partial_t(\partial_x u)-\frac{1}{1+(\partial_x u)^2}\partial^2_{xx}(\partial_x u)-\big(a(x,t)\,\partial^2_{xx}u+b(x,t)\big)\,\partial_x(\partial_x u)=f(x,t)$$

where

$$a(x,t)=\frac{-2\,\partial_x u(x,t)}{\left(1+(\partial_x u(x,t))^2\right)^2}$$

$$b(x,t)=\frac{2(n-1)}{x\left(1-\left(\frac{u(x,t)}{x}\right)^2\right)}$$

$$f(x,t)=\frac{-4(n-1)\left(\frac{u(x,t)}{x}\right)\left(1-(\partial_x u(x,t))^2\right)}{x^2\left(1-\left(\frac{u(x,t)}{x}\right)^2\right)^2}$$

For each $R\geq 2$, let $\eta(x)$ be a smooth function so that

$$\chi_{\left(\frac{1}{4}\rho,\,R-1\right)}\leq\eta\leq\chi_{\left(\frac{1}{5}\rho,\,R\right)}$$

$$|\partial_x\eta(x)|+\left|\partial^2_{xx}\eta(x)\right|\leq C(\rho)\tag{3.125}$$

It follows that

$$\partial_t (\eta\, \partial_x u) - \frac{1}{1+(\partial_x u)^2}\, \partial^2_{xx} (\eta\, \partial_x u) - \big(a(x,\, t)\, \partial^2_{xx} u + b(x,\, t) \big)\, \partial_x (\eta\, \partial_x u)$$

$$= -\left(\frac{\partial^2_{xx}\eta}{1+(\partial_x u)^2} + \big(a(x,\, t)\, \partial^2_{xx} u + b(x,\, t) \big)\, \partial_x \eta \right) (\partial_x u) \tag{3.126}$$

$$+ \eta(x)\, f(x,\, t) - \frac{2}{1+(\partial_x u)^2}\, \partial_x \eta\, \left(\partial^2_{xx} u \right)$$

Now let

$$(\eta\, \partial_x u)_{\max}(t) = \max_x \left(\eta(x)\, \partial_x u(x,\, t) \right)$$

By (3.26), (3.47) and (3.118), if $0 < \rho \ll 1$ (depending on n, Λ), $|t_0| \ll 1$ (depending on n, ρ), we may assume that

$$\begin{cases} \left| \frac{u(x,t)}{x} \right| \le \frac{1}{3} \\ \\ |\partial_x u(x,\, t)| \lesssim 1 \\ \\ \left| \partial^2_{xx} u(x,\, t) \right| \le \frac{C(n,\,\rho)}{\sqrt{t-t_0}} \end{cases} \tag{3.127}$$

for $x \ge \frac{1}{5}\rho$, $t_0 \le t \le \mathring{t}$. Thus, by (3.125) and (3.127), applying the maximum principle to (3.126) yields

$$\partial_t (\eta\, \partial_x u)_{\max} \le \frac{C(n,\, \rho)}{\sqrt{t-t_0}}$$

which implies

$$(\eta\, \partial_x u)_{\max}(t) \le (\eta\, \partial_x u)_{\max}(t_0) + C(n,\, \rho)\sqrt{t-t_0}$$

Likewise, if we define

$$(\eta\, \partial_x u)_{\min}(t) = \min_x \left(\eta(x)\, \partial_x u(x,\, t) \right)$$

by the same argument, we get

$$(\eta\, \partial_x u)_{\min}(t) \ge (\eta\, \partial_x u)_{\min}(t_0) - C(n,\, \rho)\sqrt{t-t_0}$$

□

Before moving on to the second derivative estimate, we derive the following lemma, which is about some properties of the cut-off functions to be used.

Lemma 3.31. *Let $\eta(r)$ be a smooth, non-increasing function so that*

$$\chi_{(-\infty,\,0)} \leq \eta \leq \chi_{(-\infty,\,1)}$$

and $\eta(r)$ vanishes at $r=1$ to infinite order. Then

$$\sup_r \frac{(\partial_r \eta(r))^2}{\eta(r)} < \infty$$

for $r \leq 1$.

Proof. By L'Hôpital's rule, we have

$$\lim_{r\nearrow 1} \frac{(\partial_r \eta(r))^2}{\eta(r)} = 2 \lim_{r\nearrow 1} \partial^2_{rr}\eta(r) = 0$$

Also, for $r \leq 0$ or $r > 1$, there holds

$$\frac{(\partial_r \eta(r))^2}{\eta(r)} = 0$$

Thus, the conclusion follows easily. □

Below is an improved estimate for the second derivative of $u(s,\,t)$ in the outer region. Note that the proof requres $|\partial_x u(x,\,t)| < \frac{1}{\sqrt{3}}$, which is guqranteed by (3.47) and Lemma 3.30.

Lemma 3.32. *If $0 < \rho \ll 1$ (depending on $n,\,\Lambda$) and $|t_0| \ll 1$ (depending on $n,\,\rho$), there holds*

$$\sup_{x \geq \frac{1}{3}\rho} \left|\partial^2_{xx} u(x,\,t)\right| \leq \sup_{x \geq \frac{1}{4}\rho} \left|\partial^2_{xx} u(x,\,t_0)\right| + C(n,\,\rho)$$

for $t_0 \leq t \leq \mathring{t}$.

Proof. Differentiating (3.25) with respect to x twice yields

$$\partial_t\left(\partial^2_{xx}u\right) - \frac{1}{1+(\partial_x u)^2}\,\partial^2_{xx}\left(\partial^2_{xx}u\right) - \left(\frac{-6\,\partial_x u}{\left(1+(\partial_x u)^2\right)^2}\left(\partial^2_{xx}u\right) + \frac{2\,(n-1)}{x\left(1-\left(\frac{u}{x}\right)^2\right)}\right)\partial_x\left(\partial^2_{xx}u\right)$$

$$= -\frac{2\left(1-3\left(\partial_x u\right)^2\right)}{\left(1+\left(\partial_x u\right)^2\right)^3}\left(\partial_{xx}^2 u\right)^3 - \frac{2\left(n-1\right)\left(1+\left(\frac{u}{x}\right)^2 - 6\left(\frac{u}{x}\right)\partial_x u\right)}{x^2\left(1-\left(\frac{u}{x}\right)^2\right)^2}\left(\partial_{xx}^2 u\right)$$

$$-\frac{4\left(n-1\right)\left(1-\left(\partial_x u\right)^2\right)}{x^3\left(1-\left(\frac{u}{x}\right)^2\right)^3}\left(\left(1+3\left(\frac{u}{x}\right)^2\right)\left(\partial_x u\right) - \left(3+\left(\frac{u}{x}\right)^2\right)\left(\frac{u}{x}\right)\right)$$

For each $R \geq 2$, let $\eta\left(x\right)$ be a smooth function so that

$$\chi_{\left(\frac{1}{3}\rho,\, R-1\right)} \leq \eta \leq \chi_{\left(\frac{1}{4}\rho,\, R\right)}$$

and $\eta\left(x\right)$ is increasing in $\left[\frac{1}{4}\rho,\ \frac{1}{3}\rho\right]$ and decreasing on $[R-1,\ R]$; moreove, $\eta\left(x\right)$ vanishes at $x = \frac{1}{4}\rho$ and $x = R$ to infinite order. Notice that by Lemma 3.31, we may assume

$$\frac{\left(\partial_x \eta\left(x\right)\right)^2}{\eta\left(x\right)} + \left|\partial_x \eta\left(x\right)\right| + \left|\partial_{xx}^2 \eta\left(x\right)\right| \leq C\left(\rho\right) \tag{3.128}$$

It follows that

$$\partial_t\left(\eta\,\partial_{xx}^2 u\right) - \frac{1}{1+\left(\partial_x u\right)^2}\,\partial_{xx}^2\left(\eta\,\partial_{xx}^2 u\right) - \left(\frac{-6\,\partial_x u}{\left(1+\left(\partial_x u\right)^2\right)^2}\left(\partial_{xx}^2 u\right) + \frac{2\left(n-1\right)}{x\left(1-\left(\frac{u}{x}\right)^2\right)}\right)\partial_x\left(\eta\,\partial_{xx}^2 u\right)$$

$$= -\frac{2\left(1-3\left(\partial_x u\right)^2\right)}{\left(1+\left(\partial_x u\right)^2\right)^3}\eta\left(\partial_{xx}^2 u\right)^3 - \frac{2\left(n-1\right)\eta\left(x\right)\left(1+\left(\frac{u}{x}\right)^2 - 6\left(\frac{u}{x}\right)\partial_x u\right)}{x^2\left(1-\left(\frac{u}{x}\right)^2\right)^2}\left(\partial_{xx}^2 u\right)$$

$$-\eta\left(x\right)\frac{4\left(n-1\right)\left(1-\left(\partial_x u\right)^2\right)}{x^3\left(1-\left(\frac{u}{x}\right)^2\right)^3}\left(\left(1+3\left(\frac{u}{x}\right)^2\right)\left(\partial_x u\right) - \left(3+\left(\frac{u}{x}\right)^2\right)\left(\frac{u}{x}\right)\right)$$

$$+\left(-\frac{\partial_{xx}^2 \eta}{1+\left(\partial_x u\right)^2} - \partial_x \eta\left(x\right)\left(\frac{-6\,\partial_x u}{\left(1+\left(\partial_x u\right)^2\right)^2}\left(\partial_{xx}^2 u\right) + \frac{2\left(n-1\right)}{x\left(1-\left(\frac{u}{x}\right)^2\right)}\right)\right)\left(\partial_{xx}^2 u\right)$$

$$-\frac{2}{1+\left(\partial_x u\right)^2}\,\partial_x \eta\,\partial_x\left(\partial_{xx}^2 u\right)$$

Note that we can rewrite the last term on the RHS of the above equation as

$$-\frac{2}{1+\left(\partial_x u\right)^2}\,\partial_x \eta\,\partial_x\left(\partial_{xx}^2 u\right) = -\frac{2}{1+\left(\partial_x u\right)^2}\frac{\partial_x \eta}{\eta}\left(\partial_x\left(\eta\,\partial_{xx}^2 u\right) - \left(\partial_x \eta\right)\left(\partial_{xx}^2 u\right)\right)$$

So the equation of $\eta\,\partial_{xx}^2 u$ can be rewritten as

$$\partial_t\left(\eta\,\partial_{xx}^2 u\right) - \frac{1}{1+\left(\partial_x u\right)^2}\,\partial_{xx}^2\left(\eta\,\partial_{xx}^2 u\right) \tag{3.129}$$

$$-\left(\frac{-6\,\partial_x u}{\left(1+(\partial_x u)^2\right)^2}\left(\partial^2_{xx}u\right)+\frac{2\,(n-1)}{x\left(1-\left(\frac{u}{x}\right)^2\right)}-\frac{2}{1+(\partial_x u)^2}\left(\frac{\partial_x\eta}{\eta}\right)\right)\partial_x\left(\eta\,\partial^2_{xx}u\right)$$

$$=-a\,(x,\,t)\,\eta\left(\partial^2_{xx}u\right)^3+b\,(x,\,t)\left(\partial^2_{xx}u\right)^2+c\,(x,\,t)\left(\partial^2_{xx}u\right)+\eta\,(x)\,f\,(x,\,t)$$

where

$$a\,(x,\,t)=\frac{2\left(1-3\,(\partial_x u)^2\right)}{\left(1+(\partial_x u)^2\right)^3}$$

$$b\,(x,\,t)=\frac{6\,\partial_x\eta\,\partial_x u}{\left(1+(\partial_x u)^2\right)^2}$$

$$c\,(x,\,t)=-\frac{2\,(n-1)\,\eta\,(x)\left(1+\left(\frac{u}{x}\right)^2-6\left(\frac{u}{x}\right)\partial_x u\right)}{x^2\left(1-\left(\frac{u}{x}\right)^2\right)^2}$$

$$-\frac{\partial^2_{xx}\eta}{1+(\partial_x u)^2}-\frac{2\,(n-1)\,\partial_x\eta}{x\left(1-\left(\frac{u}{x}\right)^2\right)}+\frac{2}{1+(\partial_x u)^2}\frac{(\partial_x\eta)^2}{\eta}$$

$$f\,(x,\,t)=-\frac{4\,(n-1)\left(1-(\partial_x u)^2\right)}{x^3\left(1-\left(\frac{u}{x}\right)^2\right)^3}\left(\left(1+3\left(\frac{u}{x}\right)^2\right)(\partial_x u)-\left(3+\left(\frac{u}{x}\right)^2\right)\left(\frac{u}{x}\right)\right)$$

By (3.26), (3.47), (3.118) and Lemma 3.30, if $0<\rho\ll 1$ (depending on n, Λ) and $|t_0|\ll 1$ (depending on n, ρ), we have

$$\max\left\{\left|\frac{u\,(x,\,t)}{x}\right|,\,|\partial_x u\,(x,\,t)|\right\}\le\frac{1}{3}$$

for $x\ge\frac{1}{4}\rho$, $t_0\le t\le\mathring{t}$, which, together with (3.128), implies

$$\begin{cases}\frac{972}{1000}\le a\,(x,\,t)\le 2\\[2ex] |b\,(x,\,t)|+|c\,(x,\,t)|+|f\,(x,\,t)|\le C\,(n,\,\rho)\end{cases}\tag{3.130}$$

for $x\ge\frac{1}{4}\rho$, $t_0\le t\le\mathring{t}$. Now let

$$M=\max_{\frac{1}{4}\rho\le x\le R,\;t_0\le t\le\mathring{t}}\eta\,(x)\,\partial^2_{xx}u\,(x,\,t)$$

If

$$M\le\max_{\frac{1}{4}\rho\le x\le R}\left(\eta\,(x)\,\partial^2_{xx}u\,(x,\,t_0)\right)_+$$

then we are done; otherwise, we have

$$M > \max_{\frac{1}{4}\rho \le x \le R} \left(\eta(x)\, \partial_{xx}^2 u(x,\, t_0)\right)_+$$

In the later case, let $(x_*,\, t_*)$ be a maximum point of $\eta\, \partial_{xx}^2 u$ in the spacetime, i.e.

$$\eta(x_*)\, \partial_{xx}^2 u(x_*,\, t_*) = M$$

then we have $\frac{1}{4}\rho < x_* < R,\ t_0 < t \le \mathring{t}$. Applying the maximum pricinple to (3.129) yields

$$\begin{aligned} 0 \le & -a(x_*,\, t_*)\, \eta(x_*) \left(\partial_{xx}^2 u(x_*,\, t_*)\right)^3 + b\left((x_*,\, t_*)\right) \left(\partial_{xx}^2 u(x_*,\, t_*)\right)^2 \\ & + c(x_*,\, t_*) \left(\partial_{xx}^2 u(x_*,\, t_*)\right) + \eta(x_*)\, f(x_*,\, t_*) \\ = & \frac{1}{\eta^2(x_*)} \left(-a(x_*,\, t_*)\, M^3 + b(x_*,\, t_*)\, M^2 + \eta(x_*)\, c(x_*,\, t_*)\, M + \eta^3(x_*)\, f(x_*,\, t_*)\right) \end{aligned}$$

It follows, by Young's inequality and (3.130), that

$$M^3 \le \frac{8}{3} \left(\frac{|b(x_*,\, t_*)|}{a(x_*,\, t_*)}\right)^3 + \frac{4\sqrt{2}}{3} \left(\frac{|c(x_*,\, t_*)|}{a(x_*,\, t_*)}\right)^{\frac{3}{2}} + \frac{|f(x_*,\, t_*)|}{a(x_*,\, t_*)} \le C(n,\, \rho)$$

Therefore, in either case, we have

$$\max_{\frac{1}{4}\rho \le x \le R,\, t_0 \le t \le \mathring{t}} \eta(x)\, \partial_{xx}^2 u(x,\, t) \le \max_{x \ge \frac{1}{4}\rho} \left(\eta(x)\, \partial_{xx}^2 u(x,\, t_0)\right)_+ + C(n,\, \rho)$$

Likewise, by the same argument, one could show that

$$\min_{\frac{\rho}{4} \le x \le R,\, t_0 \le t \le \mathring{t}} \eta(x)\, \partial_{xx}^2 u(x,\, t) \ge -\min_{x \ge \frac{\rho}{4}} \left(\eta(x)\, \partial_{xx}^2 u(x,\, t_0)\right) - C(n,\, \rho)$$

$\square$

In the next proposition, we apply the standard regularity theory for parabolic equations to (3.25), together with (3.59), to derive (3.62).

Proposition 3.33. *There holds (3.59).*

Proof. Given $0 < \delta \ll 1$, let's fix $x_* \ge \frac{1}{2}\rho,\ t_0 + \delta^2 \le t_* \le \mathring{t}$. By (3.59) and Krylov-Safonov Hölder estimates (applying to (3.25)), there is

$$\gamma = \gamma(n,\, \rho) \in (0,\, 1)$$

so that

$$[u]_{\gamma;\,Q\left(x_*,\,t_*;\,\frac{\delta}{2}\right)} \leq C\,(n,\,\rho,\,\delta)\,\|u\|_{L^\infty(Q(x_*,\,t_*;\,\delta))} \leq C\,(n,\,\rho,\,\delta) \tag{3.131}$$

Next, differentiate (3.25) with respect to x to get

$$\partial_t\,(\partial_x u) - \frac{1}{1+(\partial_x u)^2}\,\partial^2_{xx}\,(\partial_x u)$$
$$-\left(\frac{-2\,\partial_x u\,\partial^2_{xx}u}{\left(1+(\partial_x u)^2\right)^2} + \frac{2\,(n-1)}{x\left(1-\left(\frac{u}{x}\right)^2\right)}\right)\partial_x\,(\partial_x u) - \left(\frac{4\,(n-1)\left(\frac{u}{x}\right)\,\partial_x u}{x^2\left(1-\left(\frac{u}{x}\right)^2\right)^2}\right)(\partial_x u)$$
$$= \frac{-4\,(n-1)\left(\frac{u}{x}\right)}{x^2\left(1-\left(\frac{u}{x}\right)^2\right)^2}$$

Then by (3.59) and Krylov-Safonov Hölder estimates (applying to the above equation of $\partial_x u$), we may assume that for the same exponent γ, there holds

$$[\partial_x u]_{\gamma;\,Q\left(x_*,\,t_*;\,\frac{\delta}{2}\right)} \leq C\,(n,\,\rho,\,\delta)\left(\|\partial_x u\|_{L^\infty(Q(x_*,\,t_*;\,\delta))} + \left\|\frac{u}{x}\right\|_{L^\infty(Q(x_*,\,t_*;\,\delta))}\right)$$
$$\leq C\,(n,\,\rho,\,\delta) \tag{3.132}$$

It follows, by (3.59), (3.131), (3.132) and Schauder $C^{2,\gamma}$ estimates (applying to (3.25)), that

$$\left[\partial^2_{xx}u\right]_{\gamma;\,Q\left(x_*,\,t_*;\,\frac{\delta}{3}\right)} \leq C\,(n,\,\rho,\,\delta)\,\|u\|_{L^\infty\left(Q\left(x_*,\,t_*;\,\frac{\delta}{2}\right)\right)} \leq C\,(n,\,\rho,\,\delta)$$

By the bootstrap argument, one could show that for any $m \in \mathbb{Z}_+$, there holds

$$\|\partial_x^m u\|_{L^\infty\left(Q\left(x_*,\,t_*;\,\frac{\delta}{m+1}\right)\right)} + [\partial_x^m u]_{\gamma;\,Q\left(x_*,\,t_*;\,\frac{\delta}{m+1}\right)} \leq C\,(n,\,\rho,\,\delta,\,m) \tag{3.133}$$

Moreover, by (3.25) and (3.133), we immediately get

$$\|\partial_x^m \partial_t u\|_{L^\infty\left(Q\left(x_*,\,t_*;\,\frac{\delta}{m+3}\right)\right)} + [\partial_x^m \partial_t u]_{\gamma;\,Q\left(x_*,\,t_*;\,\frac{\delta}{m+3}\right)} \leq C\,(n,\,\rho,\,\delta,\,m)$$

for any $m \in \mathbb{Z}_+$. Differentiating (3.25) with respect to t and using the above estimates gives

$$\left\|\partial_x^m \partial_t^2 u\right\|_{L^\infty\left(Q\left(x_*,\,t_*;\,\frac{\delta}{m+5}\right)\right)} + \left[\partial_x^m \partial_t^2 u\right]_{\gamma;\,Q\left(x_*,\,t_*;\,\frac{\delta}{m+5}\right)} \leq C\,(n,\,\rho,\,\delta,\,m)$$

for any $m \in \mathbb{Z}_+$. Continuing this process and using induction yields

$$\left\|\partial_x^m \partial_t^l u\right\|_{L^\infty\left(Q\left(x_*,\,t_*;\,\frac{\delta}{m+2l+1}\right)\right)} + \left[\partial_x^m \partial_t^l u\right]_{\gamma;\,Q\left(x_*,\,t_*;\,\frac{\delta}{m+2l+1}\right)} \leq C\,(n,\,\rho,\,\delta,\,m,\,l)$$

for any $m,\,l \in \mathbb{Z}_+$. $\square$

In the following proposition, we prove (3.63) by using (3.25), (3.26), (3.76), (3.91) and the regularity theory for parabolic equations.

Proposition 3.34. *If* $0 < \rho \ll 1$ *(depending on* n*,* Λ*) and* $|t_0| \ll 1$ *(depending on* n*,* Λ*,* ρ*), there holds (3.63).*

Proof. Notice that by (3.26), we have

$$\max\left\{\left|\frac{u(x,\,t)}{x}\right|,\,|\partial_x u(x,\,t)|\right\} \leq \frac{1}{3} \tag{3.134}$$

$$x^i\left|\partial_x^i u(x,\,t)\right| \leq \Lambda\left((-t)^2 x^\alpha + x^{2\lambda_2+1}\right) \leq C(n,\,\Lambda)\,x^{2\lambda_2+1} \qquad \forall\ i \in \{0,\,1,\,2\} \tag{3.135}$$

for $\frac{1}{3}\sqrt{-t} \leq x \leq \rho$, $t_0 \leq t \leq \mathring{t}$, provided that $0 < \rho \ll 1$ (depending on n, Λ) and $|t_0| \ll 1$ (depending on n, Λ, ρ).

Given $0 < \delta \ll 1$, let's fix $(x_*,\,t_*)$ so that

$$\frac{1}{2}\sqrt{-t_*} \leq x_* \leq \frac{3}{4}\rho, \qquad t_0 + \delta^2 x_*^2 \leq t_* \leq \mathring{t}$$

Define

$$h(r,\,\iota) = u\left(rx_*,\,t_* + \iota x_*^2\right)$$

for $\frac{2}{3} \leq r \leq \frac{4}{3}$, $-\delta^2 \leq \iota \leq 0$. From (3.25), there holds

$$\partial_\iota h - a(r,\,\iota)\,\partial_{rr}^2 h - b(r,\,\iota)\,\partial_r h - c(r,\,\iota)\,h = 0 \tag{3.136}$$

where

$$a(r,\,\iota) = \left.\frac{1}{1 + (\partial_x u(x,\,t))^2}\right|_{x=rx_*,\,t=t_*+\iota x_*^2}$$

$$b(r,\,\iota) = \frac{1}{r}\left.\left(\frac{2(n-1)}{1 - \left(\frac{u(x,t)}{x}\right)^2}\right)\right|_{x=rx_*,\,t=t_*+\iota x_*^2}$$

$$c(r,\,\iota) = \frac{1}{r^2}\left.\left(\frac{2(n-1)}{1 - \left(\frac{u(x,t)}{x}\right)^2}\right)\right|_{x=rx_*,\,t=t_*+\iota x_*^2}$$

By (3.134), (3.135) and Krylov-Safonov Hölder estimates, there is

$$\gamma = \gamma(n,\,\Lambda) \in (0,\,1)$$

so that

$$[h]_{\gamma;Q\left(1,0;\,\frac{\delta}{2}\right)} \le C\,(n,\,\delta)\,\|h\|_{L^\infty(Q(1,0;\,\delta))} \le C\,(n,\,\Lambda,\,\delta)\,x_*^{2\lambda_2+1}$$

In other words, we get

$$x_*^\gamma\,[u]_{\gamma;\,Q\left(x_*,\,t_*;\,\frac{\delta}{2}x_*\right)} \le C\,(n,\,\Lambda,\,\delta)\,x_*^{2\lambda_2+1} \tag{3.137}$$

Next, differentiate (3.25) with respect to x to get

$$\begin{aligned} &\partial_t\,(\partial_x u) - \frac{1}{1+(\partial_x u)^2}\,\partial^2_{xx}\,(\partial_x u) \\ -\frac{1}{x}&\left(\frac{-2\,\partial_x u\,\left(x\,\partial^2_{xx}u\right)}{\left(1+(\partial_x u)^2\right)^2} + \frac{2\,(n-1)}{1-\left(\frac{u}{x}\right)^2}\right)\partial_x\,(\partial_x u) - \frac{1}{x^2}\left(\frac{4\,(n-1)\left(\frac{u}{x}\right)\,\partial_x u}{\left(1-\left(\frac{u}{x}\right)^2\right)^2}\right)(\partial_x u) \\ &= \frac{1}{x^2}\left(\frac{-4\,(n-1)}{\left(1-\left(\frac{u}{x}\right)^2\right)^2}\left(\frac{u}{x}\right)\right) \end{aligned} \tag{3.138}$$

Define

$$\tilde h\,(r,\,\iota) = \partial_x u\left(rx_*,\,t_* + \iota x_*^2\right)$$

then we have

$$\partial_\iota \tilde h - \tilde a\,(r,\,\iota)\,\partial^2_{rr}\tilde h - \tilde b\,(r,\,\iota)\,\partial_r \tilde h - \tilde c\,(r,\,\iota)\,\tilde h = \tilde f\,(r,\,\iota) \tag{3.139}$$

where

$$\tilde a\,(r,\,\iota) = \left.\frac{1}{1+(\partial_x u\,(x,\,t))^2}\right|_{x=rx_*,\,t=t_*+\iota x_*^2}$$

$$\tilde b\,(r,\,\iota) = \frac{1}{r}\left.\left(\frac{-2\,\partial_x u\,(x,\,t)\left(x\,\partial^2_{xx}u\,(x,\,t)\right)}{\left(1+(\partial_x u\,(x,\,t))^2\right)^2} + \frac{2\,(n-1)}{1-\left(\frac{u(x,\,t)}{x}\right)^2}\right)\right|_{x=rx_*,\,t=t_*+\iota x_*^2}$$

$$\tilde c\,(r,\,\iota) = \frac{1}{r^2}\left.\left(\frac{4\,(n-1)\left(\frac{u(x,\,t)}{x}\right)\,\partial_x u\,(x,\,t)}{\left(1-\left(\frac{u(x,\,t)}{x}\right)^2\right)^2}\right)\right|_{x=rx_*,\,t=t_*+\iota x_*^2}$$

$$\tilde f\,(r,\,\iota) = \frac{1}{r^2}\left.\left(\frac{-4\,(n-1)}{\left(1-\left(\frac{u(x,\,t)}{x}\right)^2\right)^2}\left(\frac{u\,(x,\,t)}{x}\right)\right)\right|_{x=rx_*,\,t=t_*+\iota x_*^2}$$

By (3.134), (3.135) and Krylov-Safonov Hölder estimates, we may assume that for the same exponent γ, there holds

$$\left[\tilde{h}\right]_{\gamma;Q\left(1,0;\,\frac{\delta}{2}\right)} \leq C\left(n,\,\Lambda,\,\delta\right)\left(\left\|\tilde{h}\right\|_{L^{\infty}(Q(1,0;\,\delta))}+\left\|\tilde{f}\right\|_{L^{\infty}(Q(1,0;\,\delta))}\right)$$
$$\leq C\left(n,\,\Lambda,\,\delta\right)x_{*}^{2\lambda_2}$$

which implies

$$x_{*}^{\gamma}\left[\partial_x u\right]_{\gamma;\,Q\left(x_*,\,t_*;\,\frac{\delta}{2}x_*\right)} \leq C\left(n,\,\Lambda,\,\delta\right)x_{*}^{2\lambda_2} \tag{3.140}$$

Thus, by (3.134), (3.135), (3.137), (3.140), applying Schauder $C^{2,\gamma}$ estimates to (3.136) yields

$$\left[\partial_{rr}^{2}h\right]_{\gamma;Q\left(1,\,0;\,\frac{\delta}{3}\right)} \leq C\left(n,\,\Lambda,\,\delta\right)\|h\|_{L^{\infty}\left(Q\left(1,\,0;\,\frac{\delta}{2}\right)\right)} \leq C\left(n,\,\Lambda,\,\delta\right)x_{*}^{2\lambda_2+1}$$

which implies

$$x_{*}^{2+\gamma}\left[\partial_{xx}^{2}u\right]_{\gamma;\,Q\left(x_*,\,t_*;\,\frac{\delta}{3}x_*\right)} \leq C\left(n,\,\Lambda,\,\delta\right)x_{*}^{2\lambda_2+1} \tag{3.141}$$

By the bootstrap and rescaling argument, one could show that for any $m\in\mathbb{Z}_+$, there holds

$$x_{*}^{m}\left\|\partial_x^m u\right\|_{L^{\infty}\left(Q\left(x_*,\,t_*;\,\frac{\delta}{m+1}x_*\right)\right)} + x_{*}^{m+\gamma}\left[\partial_x^m u\right]_{\gamma;\,Q\left(x_*,\,t_*;\,\frac{\delta}{m+1}x_*\right)}$$
$$\leq C\left(n,\,\Lambda,\,\delta,\,m\right)x_{*}^{2\lambda_2+1} \tag{3.142}$$

It follows, by (3.25) and (3.142), that

$$x_{*}^{m+2}\left\|\partial_x^m\partial_t u\right\|_{L^{\infty}\left(Q\left(x_*,\,t_*;\,\frac{\delta}{m+3}x_*\right)\right)} + x_{*}^{m+2+\gamma}\left[\partial_x^m\partial_t u\right]_{\gamma;\,Q\left(x_*,\,t_*;\,\frac{\delta}{m+3}x_*\right)}$$
$$\leq C\left(n,\,\Lambda,\,\delta,\,m\right)x_{*}^{2\lambda_2+1}$$

for any $m\in\mathbb{Z}_+$. Then differentiate (3.25) with respect to t and use the above estimates to get

$$x_{*}^{m+4}\left\|\partial_x^m\partial_t^2 u\right\|_{L^{\infty}\left(Q\left(x_*,\,t_*;\,\frac{\delta}{m+5}x_*\right)\right)} + x_{*}^{m+4+\gamma}\left[\partial_x^m\partial_t^2 u\right]_{\gamma;\,Q\left(x_*,\,t_*;\,\frac{\delta}{m+5}x_*\right)}$$
$$\leq C\left(n,\,\Lambda,\,\delta,\,m\right)x_{*}^{2\lambda_2+1}$$

Continuing this process and using induction yields

$$x_{*}^{m+2l}\left\|\partial_x^m\partial_t^l u\right\|_{L^{\infty}\left(Q\left(x_*,\,t_*;\,\frac{\delta}{m+2l+1}x_*\right)\right)} + x_{*}^{m+2l+\gamma}\left[\partial_x^m\partial_t^l u\right]_{\gamma;\,Q\left(x_*,\,t_*;\,\frac{\delta}{m+2l+1}x_*\right)}$$

$$\leq C\left(n,\,\Lambda,\,\delta,\,m\right)x_*^{2\lambda_2+1} \tag{3.143}$$

for any $m,\, l\in\mathbb{Z}_+$.

On the other hand, by Proposition 3.8, there holds

$$\left(\partial_s+\mathcal{L}\right)\left(ke^{-\lambda_2 s}\varphi_2\left(y\right)\right)=0$$

By a rescaling argument, we get

$$\left(\partial_t-\partial_{xx}^2-\frac{2\left(n-1\right)}{x}\partial_x-\frac{2\left(n-1\right)}{x^2}\right)\left(k\left(-t\right)^{\lambda_2+\frac{1}{2}}\varphi_2\left(\frac{x}{\sqrt{-t}}\right)\right)=0 \tag{3.144}$$

In addition, by (3.25) we have

$$\left(\partial_t-\partial_{xx}^2-\frac{2\left(n-1\right)}{x}\partial_x-\frac{2\left(n-1\right)}{x^2}\right)u\left(x,\,t\right)\;=\;\frac{f\left(x,\,t\right)}{x^2} \tag{3.145}$$

where

$$f\left(x,\,t\right)=-\frac{\left(\partial_x u\right)^2}{1+\left(\partial_x u\right)^2}\left(x^2\,\partial_{xx}^2u\right)+\frac{2\left(n-1\right)\left(\frac{u}{x}\right)^2}{1-\left(\frac{u}{x}\right)^2}\left(x\,\partial_x u\right)+\frac{2\left(n-1\right)\left(\frac{u}{x}\right)^2}{1-\left(\frac{u}{x}\right)^2}u$$

Note that by (3.134) and (3.143) we have

$$x_*^{m+2l}\left\|\partial_x^m\partial_t^l f\left(x,\,t\right)\right\|_{L^\infty\left(Q\left(x_*,\,t_*;\,\frac{\delta}{m+2l+1}x_*\right)\right)}+\,x_*^{m+2l+\gamma}\left[\partial_x^m\partial_t^l f\left(x,\,t\right)\right]_{\gamma;\,Q\left(x_*,\,t_*;\,\frac{\delta}{m+2l+1}x_*\right)}$$
$$\leq C\left(n,\,\Lambda,\,\delta,\,m,\,l\right)x_*^{4\lambda_2}\,x_*^{2\lambda_2+1} \tag{3.146}$$

for any $m,\, l\in\mathbb{Z}_+$. Subtract (3.144) from (3.145) to get

$$\left(\partial_t-\partial_{xx}^2-\frac{2\left(n-1\right)}{x}\partial_x-\frac{2\left(n-1\right)}{x^2}\right)\left(u\left(x,\,t\right)-k\left(-t\right)^{\lambda_2+\frac{1}{2}}\varphi_2\left(\frac{x}{\sqrt{-t}}\right)\right)\;=\;\frac{f\left(x,\,t\right)}{x^2}$$

Then by the rescaling argument, together with (3.146) and Schauder estimates, we get

$$x_*^{m+2l}\left\|\partial_x^m\partial_t^l\left(u\left(x,\,t\right)-k\left(-t\right)^{\lambda_2+\frac{1}{2}}\varphi_2\left(\frac{x}{\sqrt{-t}}\right)\right)\right\|_{L^\infty\left(Q\left(x_*,\,t_*;\,\frac{\delta}{m+2l+2}x_*\right)\right)}$$
$$\leq C\left(n,\,\Lambda,\,\delta,\,m,\,l\right)\left\|u\left(x,\,t\right)-k\left(-t\right)^{\lambda_2+\frac{1}{2}}\varphi_2\left(\frac{x}{\sqrt{-t}}\right)\right\|_{L^\infty\left(Q\left(x_*,\,t_*;\,\frac{\delta}{m+2l+1}x_*\right)\right)}$$
$$+C\left(n,\,\Lambda,\,\delta,\,m,\,l\right)\sum_{i=0}^{m}\sum_{j=0}^{l}x_*^{i+2j}\left\|\partial_x^i\partial_t^j f\left(x,\,t\right)\right\|_{L^\infty\left(Q\left(x_*,\,t_*;\,\frac{\delta}{m+2l+1}x_*\right)\right)}$$
$$+C\left(n,\,\Lambda,\,\delta,\,m,\,l\right)\sum_{i=0}^{m}\sum_{j=0}^{l}x_*^{i+2j+\gamma}\left[\partial_x^i\partial_t^j f\left(x,\,t\right)\right]_{\gamma;\,Q\left(x_*,\,t_*;\,\frac{\delta}{m+2l+1}x_*\right)}$$
$$\leq C\left(n,\,\Lambda,\,\delta,\,m,\,l\right)\left(\left(-t_0\right)^{\varsigma\lambda_2}+x_*^{4\lambda_2}\right)x_*^{2\lambda_2+1}$$

for any $m,\, l\in\mathbb{Z}_+$. $\square$

Below we use (3.31), (3.32), (3.77), (3.78) and the regularity theory to show (3.64) and (3.65).

Proposition 3.35. *If $\beta \gg 1$ (depending on n, Λ), $s_0 \gg 1$ (depending on n, Λ, β), there hold (3.64) and (3.65).*

Proof. By (3.32), we have

$$y^i \left|\partial_y^i v\left(y,\, s\right)\right| \leq \Lambda e^{-\lambda_2 s}\left(y^\alpha + y^{2\lambda_2+1}\right) \leq C\left(n,\, \Lambda\right) e^{-\lambda_2 s} y^\alpha \tag{3.147}$$

for $\beta e^{-\sigma s} \leq y \leq 3$, $s_0 \leq s \leq \mathring{s}$. In particular, we may assume that

$$\max\left\{\left|\frac{v\left(y,\, s\right)}{y}\right|,\, \left|\partial_y v\left(y,\, s\right)\right|\right\} \leq C\left(n,\, \Lambda\right) e^{-\lambda_2 s} y^{\alpha-1} \leq \frac{1}{3}$$

for $\beta e^{-\sigma s} \leq y \leq 3$, $s_0 \leq s \leq \mathring{s}$, provided that $\beta \gg 1$ (depending on n, Λ).

Now given $0 < \delta \ll 1$ and fix $(y_*,\, s_*)$ so that

$$\frac{3}{2}\beta e^{-\sigma s_*} \leq y_* \leq 2, \qquad s_0 + \delta^2 y_*^2 \leq s_* \leq \mathring{s}$$

From (3.31), we have

$$\partial_s v \;-\; \frac{1}{1+\left(\partial_y v\right)^2}\,\partial_{yy}^2 v \;-\; \frac{1}{y}\left(\frac{2\left(n-1\right)}{1-\left(\frac{v}{y}\right)^2} - \frac{y^2}{2}\right)\partial_y v \;-\; \frac{1}{y^2}\left(\frac{2\left(n-1\right)}{1-\left(\frac{v}{y}\right)^2} + \frac{y^2}{2}\right) v \;=\; 0$$

By (3.147) and Krylov-Safonov Hölder estimates, there is

$$\gamma = \gamma\left(n,\, \Lambda\right) \in \left(0,\, 1\right)$$

so that

$$y_*^\gamma \left[v\right]_{\gamma;\, Q\left(y_*,\, s_*;\, \frac{\delta}{2} y_*\right)} \leq C\left(n,\, \delta\right) \left\|v\right\|_{L^\infty\left(Q\left(y_*,\, s_*;\, \delta y_*\right)\right)} \leq C\left(n,\, \Lambda,\, \delta\right) e^{-\lambda_2 s_*} y_*^\alpha \tag{3.148}$$

Differentiate (3.31) with respect to y to get

$$\partial_s\left(\partial_y v\right) - \frac{1}{1+\left(\partial_y v\right)^2}\,\partial_{yy}^2\left(\partial_y v\right)$$

$$-\frac{1}{y}\left(\frac{-2\left(\partial_y v\right)\left(y\,\partial_{yy}^2 v\right)}{\left(1+\left(\partial_y v\right)^2\right)^2} + \frac{2\left(n-1\right)}{1-\left(\frac{v}{y}\right)^2} - \frac{y^2}{2}\right)\partial_y\left(\partial_y v\right) \;-\; \frac{1}{y^2}\left(\frac{4\left(n-1\right)\left(\frac{v}{y}\right)\partial_y v}{\left(1-\left(\frac{v}{y}\right)^2\right)^2}\right)\left(\partial_y v\right)$$

$$= \frac{1}{y^2}\left(\frac{-4\,(n-1)\left(\frac{v}{y}\right)}{\left(1-\left(\frac{v}{y}\right)^2\right)^2}\right)$$

By (3.147) and Krylov-Safonov Hölder estimates, we may assume that for the same γ, there holds

$$y_*^{\gamma}\,[\partial_y v]_{\gamma;\,Q\left(y_*,\,s_*;\,\frac{\delta}{2}y_*\right)} \le C\,(n,\,\Lambda,\,\delta)\left(\|\partial_y v\|_{L^\infty(Q(y_*,\,s_*;\,\delta y_*))} + \left\|\frac{v}{y}\right\|_{L^\infty(Q(y_*,\,s_*;\,\delta y_*))}\right)$$
$$\le C\,(n,\,\Lambda,\,\delta)\,e^{-\lambda_2 s_*} y_*^{\alpha-1} \tag{3.149}$$

By (3.147), (3.148) and (3.149), applying Schauder $C^{2,\,\gamma}$ estimates to (3.31) yields

$$y_*^{2+\gamma}\left[\partial_{yy}^2 v\right]_{\gamma;\,Q\left(y_*,\,s_*;\,\frac{\delta}{3}y_*\right)} \le C\,(n,\,\Lambda,\,\delta)\,\|v\|_{L^\infty\left(Q\left(y_*,\,s_*;\,\frac{\delta}{2}y_*\right)\right)} \le C\,(n,\,\Lambda,\,\delta)\,e^{-\lambda_2 s_*} y_*^{\alpha} \tag{3.150}$$

Then by the bootstrap argument, one could show that

$$y_*^{m}\left\|\partial_y^m v\,(y,\,s)\right\|_{L^\infty\left(Q\left(y_*,\,s_*;\,\frac{\delta}{m+1}y_*\right)\right)} + y_*^{m+\gamma}\left[\partial_y^m v\,(y,\,s)\right]_{\gamma;\,Q\left(y_*,\,s_*;\,\frac{\delta}{m+1}y_*\right)}$$
$$\le C\,(n,\,\Lambda,\,\delta,\,m)\,e^{-\lambda_2 s_*} y_*^{\alpha} \tag{3.151}$$

for all $m \in \mathbb{Z}_+$. Furthermore, by (3.31) and (3.151), we get

$$y_*^{m+2}\left\|\partial_y^m \partial_s v\,(y,\,s)\right\|_{L^\infty\left(Q\left(y_*,\,s_*;\,\frac{\delta}{m+3}y_*\right)\right)} + y_*^{m+2+\gamma}\left[\partial_y^m \partial_s v\,(y,\,s)\right]_{\gamma;\,Q\left(y_*,\,s_*;\,\frac{\delta}{m+3}y_*\right)}$$
$$\le C\,(n,\,\Lambda,\,\delta,\,m)\,e^{-\lambda_2 s_*} y_*^{\alpha}$$

for all $m \ge 0$. Diffrentiating (3.31) with respect to s and using the above estimates gives

$$y_*^{m+4}\left\|\partial_y^m \partial_s^2 v\,(y,\,s)\right\|_{L^\infty\left(Q\left(y_*,\,s_*;\,\frac{\delta}{m+5}y_*\right)\right)} + y_*^{m+4+\gamma}\left[\partial_y^m v\,(y,\,s)\right]_{\gamma;\,Q\left(y_*,\,s_*;\,\frac{\delta}{m+5}y_*\right)}$$
$$\le C\,(n,\,\Lambda,\,\delta,\,m)\,e^{-\lambda_2 s_*} y_*^{\alpha}$$

Continuing this process and using induction yields

$$y_*^{m+2l}\left\|\partial_y^m \partial_s^l v\,(y,\,s)\right\|_{L^\infty\left(Q\left(y_*,\,s_*;\,\frac{\delta}{m+2l+1}y_*\right)\right)} + y_*^{m+2l+\gamma}\left[\partial_y^m \partial_s^l v\,(y,\,s)\right]_{\gamma;\,Q\left(y_*,\,s_*;\,\frac{\delta}{m+2l+1}y_*\right)}$$
$$\le C\,(n,\,\Lambda,\,\delta,\,m,\,l)\,e^{-\lambda_2 s_*} y_*^{\alpha} \tag{3.152}$$

for any $m,\,l \in \mathbb{Z}_+$.

If $e^{-\vartheta\sigma s_*} \le y_* \le 2$, recall that by Proposition 3.8, there holds

$$(\partial_s + \mathcal{L})\left(\frac{k}{c_2} e^{-\lambda_2 s}\varphi_2(y)\right) = 0$$

That is,

$$\left(\partial_s - \partial_{yy}^2 + \frac{1}{y}\left(2(n-1) - \frac{y^2}{2}\right)\partial_y - \frac{1}{y^2}\left(2(n-1) + \frac{y^2}{2}\right)\right)\left(\frac{k}{c_2} e^{-\lambda_2 s}\varphi_2(y)\right) = 0 \tag{3.153}$$

In addition, from (3.31) we have

$$\left(\partial_s - \partial_{yy}^2 + \frac{1}{y}\left(2(n-1) - \frac{y^2}{2}\right)\partial_y - \frac{1}{y^2}\left(2(n-1) + \frac{y^2}{2}\right)\right) v(y, s) = \frac{h(y, s)}{y^2} \tag{3.154}$$

where

$$h(y, s) = -\frac{(\partial_y v)^2}{1 + (\partial_y v)^2}\left(y^2\,\partial_{yy}^2 v\right) + \frac{2(n-1)\left(\frac{v}{y}\right)^2}{1 - \left(\frac{v}{y}\right)^2}\left(y\,\partial_y v\right) + \frac{2(n-1)\left(\frac{v}{y}\right)^2}{1 - \left(\frac{v}{y}\right)^2} v$$

Notice that by (3.152), the function $h(y, s)$ satisfies

$$\begin{aligned}
y_*^{m+2l}\left\|\partial_y^m \partial_s^l h(y, s)\right\|_{L^\infty\left(Q\left(y_*, s_*; \frac{\delta}{m+2l+1} y_*\right)\right)} &+ y_*^{m+2l+\gamma}\left[\partial_y^m \partial_s^l h(y, s)\right]_{\gamma; Q\left(y_*, s_*; \frac{\delta}{m+2l+1} y_*\right)} \\
&\le C(n, \Lambda, \delta, m, l)\left(e^{-\lambda_2 s_*} y_*^{\alpha-1}\right)^2\left(e^{-\lambda_2 s_*} y_*^{\alpha}\right) \\
&= C(n, \Lambda, \delta, m, l)\left(e^{-\lambda_2 s_*} y_*^{\alpha-2}\right)^2\left(e^{-\lambda_2 s_*} y_*^{\alpha+2}\right) \\
&= C(n, \Lambda, \delta, m, l)\, e^{-\varkappa s_*}\left(e^{-\lambda_2 s_*} y_*^{\alpha+2}\right)
\end{aligned} \tag{3.155}$$

for any $m, l \in \mathbb{Z}_+$. Then we substract (3.153) from (3.154) to get

$$\left(\partial_s - \partial_{yy}^2 + \frac{1}{y}\left(2(n-1) - \frac{y^2}{2}\right)\partial_y - \frac{1}{y^2}\left(2(n-1) + \frac{y^2}{2}\right)\right)\left(v - \frac{k}{c_2} e^{-\lambda_2 s}\varphi_2(y)\right) = \frac{h}{y^2}$$

By (3.155) and Schauder estimates, we get

$$\begin{aligned}
&y_*^{m+2l}\left\|\partial_y^m \partial_s^l\left(v(y, s) - \frac{k}{c_2} e^{-\lambda_2 s}\varphi_2(y)\right)\right\|_{L^\infty\left(Q\left(y_*, s_*; \frac{\delta}{m+2l+2} y_*\right)\right)} \\
&\le C(n, \Lambda, \delta, m, l)\left\|v(y, s) - \frac{k}{c_2} e^{-\lambda_2 s}\varphi_2(y)\right\|_{L^\infty\left(Q\left(y_*, s_*; \frac{\delta}{m+2l+1} y_*\right)\right)} \\
&+ C(n, \Lambda, \delta, m, l)\sum_{i=0}^{m}\sum_{j=0}^{l} y_*^{i+2j}\left\|\partial_y^i \partial_s^j h\right\|_{L^\infty\left(Q\left(y_*, s_*; \frac{\delta}{m+2l+1} y_*\right)\right)}
\end{aligned}$$

$$+C\left(n,\,\Lambda,\,\delta,\,m,\,l\right)\sum_{i=0}^{m}\sum_{j=0}^{l}y_*^{i+2j+\gamma}\left[\partial_y^i\partial_s^j h\right]_{\gamma;\,Q\left(y_*,\,s_*;\,\frac{\delta}{m+2l+1}y_*\right)}$$

$$\leq C\left(n,\,\Lambda,\,\delta,\,m,\,l\right)e^{-\varkappa s_*}\left(e^{-\lambda_2 s_*}y_*^{\alpha+2}\right)$$

for any $m,\,l\in\mathbb{Z}_+$.

If $\frac{3}{2}\beta e^{-\sigma s_*}\leq y_*\leq e^{-\vartheta\sigma s_*}$, notice that

$$\partial_\tau\psi_k\left(z\right)=0=\frac{1}{1+\left(\partial_z\psi_k\left(z\right)\right)^2}\,\partial_{zz}^2\psi_k\left(z\right)\,+\,2\left(n-1\right)\frac{z\,\partial_z\psi_k\left(z\right)+\psi_k\left(z\right)}{z^2-\psi_k^2\left(z\right)}$$

Let

$$\breve{v}\left(y,\,s\right)=e^{-\sigma s}\,\psi_k\left(e^{\sigma s}y\right)\tag{3.156}$$

then we have

$$\partial_s\breve{v}+\sigma\left(-y\,\partial_y\breve{v}+\breve{v}\right)=\frac{1}{1+\left(\partial_y\breve{v}\right)^2}\,\partial_{yy}^2\breve{v}\,+\,2\left(n-1\right)\frac{y\,\partial_y\breve{v}+\breve{v}}{y^2-\breve{v}^2}$$

Then we subtract the above equation from (3.31) to get

$$\partial_s\left(v-\breve{v}\right)-a\left(y,\,s\right)\partial_{yy}^2\left(v-\breve{v}\right)-\frac{1}{y}\,b\left(y,\,s\right)\partial_z\left(v-\breve{v}\right)-\frac{1}{y^2}\,c\left(y,\,s\right)\left(v-\breve{v}\right)=\frac{1}{y^2}\,f\left(y,\,s\right)\tag{3.157}$$

where

$$a\left(z,\,\tau\right)=\frac{1}{1+\left(\partial_y v\right)^2}$$

$$b\left(z,\,\tau\right)=\frac{-\left(y\,\partial_{yy}^2\breve{v}\right)\left(\partial_y v+\partial_y\breve{v}\right)}{\left(1+\left(\partial_y v\right)^2\right)\left(1+\left(\partial_y\breve{v}\right)^2\right)}+\frac{2\left(n-1\right)}{1-\left(\frac{v}{y}\right)^2}-\frac{y^2}{2}$$

$$c\left(z,\,\tau\right)=\frac{2\left(n-1\right)\left(\partial_y\breve{v}+\frac{\breve{v}}{y}\right)\left(\frac{v}{y}+\frac{\breve{v}}{y}\right)}{\left(1-\left(\frac{v}{y}\right)^2\right)\left(1-\left(\frac{\breve{v}}{y}\right)^2\right)}+\frac{2\left(n-1\right)}{1-\left(\frac{v}{y}\right)^2}+\frac{y^2}{2}$$

$$f\left(z,\,\tau\right)=\left(\frac{1}{2}+\sigma\right)y^2\left(-y\,\partial_y\breve{v}+\breve{v}\right)$$

Note that by Lemma 3.5 and (3.156), we have

$$y^m\left|\partial_y^m\breve{v}\left(y,\,s\right)\right|\leq C\left(n,\,m\right)e^{-\lambda_2 s}y^\alpha\tag{3.158}$$

for $y\geq\beta$, which yields

$$y_*^{m+2l}\left\|\partial_y^m\partial_s^l f\left(y,\,s\right)\right\|_{L^\infty\left(Q\left(y_*,\,s_*;\,\frac{\delta}{m+2l+1}y_*\right)\right)}+y_*^{m+2l+\gamma}\left[\partial_y^m\partial_s^l f\left(y,\,s\right)\right]_{\gamma;\,Q\left(y_*,\,s_*;\,\frac{\delta}{m+2l+1}y_*\right)}$$

$$\leq C\left(n,\,\delta,\,m,\,l\right)\left(e^{-\lambda_{2}s_{*}}y_{*}^{\alpha+2}\right)$$

$$\leq C\left(n,\,\delta,\,m,\,l\right)e^{-2\vartheta\sigma s_{*}}\left(e^{-\lambda_{2}s_{*}}y_{*}^{\alpha}\right) \tag{3.159}$$

since $\frac{3}{2}\beta e^{-\sigma s_{*}}\leq y_{*}\leq e^{-\vartheta\sigma s_{*}}$. Thus, by (3.152), (3.158), (3.159) and applying Schauder estimates to (3.157), we get

$$y_{*}^{m+2l}\left\|\partial_{y}^{m}\partial_{s}^{l}\left(v\left(y,\,s\right)-\breve{v}\left(y,\,s\right)\right)\right\|_{L^{\infty}\left(Q\left(y_{*},\,s_{*};\,\frac{\delta}{m+2l+2}y_{*}\right)\right)}$$

$$\leq C\left(n,\,\Lambda,\,\delta,\,m,\,l\right)\left\|v\left(y,\,s\right)-\breve{v}\left(y,\,s\right)\right\|_{L^{\infty}\left(Q\left(y_{*},\,s_{*};\,\frac{\delta}{m+2l+1}y_{*}\right)\right)}$$

$$+C\left(n,\,\Lambda,\,\delta,\,m,\,l\right)\sum_{i=0}^{m}\sum_{j=0}^{l}y_{*}^{i+2j}\left\|\partial_{y}^{i}\partial_{s}^{j}f\right\|_{L^{\infty}\left(Q\left(y_{*},\,s_{*};\,\frac{\delta}{m+2l+1}y_{*}\right)\right)}$$

$$+C\left(n,\,\Lambda,\,\delta,\,m,\,l\right)\sum_{i=0}^{m}\sum_{j=0}^{l}y_{*}^{i+2j+\gamma}\left[\partial_{y}^{i}\partial_{s}^{j}f\right]_{\gamma;\,Q\left(y_{*},\,s_{*};\,\frac{\delta}{m+2l+1}y_{*}\right)}$$

$$\leq C\left(n,\,\Lambda,\,\delta,\,m,\,l\right)\left(\beta^{\alpha-3}e^{-2\varrho\sigma\left(s_{*}-s_{0}\right)}e^{-\lambda_{2}s_{*}}y_{*}^{\alpha}\,+\,e^{-2\vartheta\sigma s_{*}}\left(e^{-\lambda_{2}s_{*}}y_{*}^{\alpha}\right)\right)$$

$$\leq C\left(n,\,\Lambda,\,\delta,\,m,\,l\right)\beta^{\alpha-3}e^{-2\varrho\sigma\left(s_{*}-s_{0}\right)}e^{-\lambda_{2}s_{*}}y_{*}^{\alpha}$$

provided that $s_{0}\gg1$ (depending on $n,\,\beta$). Notice that $0<\varrho<\vartheta$. $\square$

Next, we would like to prove (3.61). The C^{0} estimate is already shown in Proposition 3.28. Below we would prove the first and second derivatives estimates in Lemma 3.38 and Lemma 3.40, respectively. Before that, notice that by (3.45) we have

$$z^{i}\left|\partial_{z}^{i}w\left(z,\,\tau\right)\right|\,\leq\,\Lambda\left(z^{\alpha}+\frac{z^{2\lambda_{2}+1}}{\left(2\sigma\tau\right)^{2}}\right)\,\leq\,C\left(n,\,\Lambda\right)z^{\alpha},\qquad i\in\left\{0,\,1,\,2\right\} \tag{3.160}$$

for $\beta\leq z\leq\left(2\sigma\tau\right)^{\frac{1}{2}\left(1-\vartheta\right)}$, $\tau_{0}\leq\tau\leq\mathring{\tau}$; in particular, we have

$$\max\left\{\left|\frac{w\left(z,\,\tau\right)}{z}\right|,\,\left|\partial_{z}w\left(z,\,\tau\right)\right|\right\}\,\leq\,\frac{1}{3} \tag{3.161}$$

for $\beta\leq z\leq\left(2\sigma\tau\right)^{\frac{1}{2}\left(1-\vartheta\right)}$, $\tau_{0}\leq\tau\leq\mathring{\tau}$, provided that $\beta\gg1$ (depending on $n,\,\Lambda$). In the following lemma, we show how to transform the above estimates for $w\left(z,\,\tau\right)$ to $\hat{w}\left(z,\,\tau\right)$ via the projected curve $\bar{\Gamma}_{\tau}$ defined in (3.41). This lemma is useful since it provides the "boundary values" for estimating $\hat{w}\left(z,\,\tau\right)$ in the rescaled tip region.

Lemma 3.36. *If $\beta \gg 1$ (depending on n, Λ) and $\tau_0 \gg 1$ (depending on n, Λ, ρ, β), there hold*

$$\left|\partial_z \hat{w}\left(z,\, \tau\right) - 1\right| \,\leq\, C\left(n,\, \Lambda\right) z^{\alpha - 1} \tag{3.162}$$

$$\left|\partial_{zz}^2 \hat{w}\left(z,\, \tau\right)\right| \,\leq\, C\left(n,\, \Lambda\right) z^{\alpha - 2} \tag{3.163}$$

for $2\beta \leq z \leq \frac{1}{2}\left(2\sigma\tau\right)^{\frac{1}{2}(1-\vartheta)}$, $\tau_0 \leq \tau \leq \mathring{\tau}$.

Proof. Let's first parametrize the projected curve $\bar{\Gamma}_\tau$ by

$$Z_\tau = \left(\left(z - w\left(z,\, \tau\right)\right)\frac{1}{\sqrt{2}},\, \left(z + w\left(z,\, \tau\right)\right)\frac{1}{\sqrt{2}}\right)$$

In this parametrization, there hold

$$N_{\bar{\Gamma}_\tau} \cdot \mathbf{e} = \frac{-\partial_z w\left(z,\, \tau\right)}{\sqrt{1 + \left(\partial_z w\left(z,\, \tau\right)\right)^2}}$$

$$A_{\bar{\Gamma}_\tau} \,=\, \frac{\partial_{zz}^2 w\left(z,\, \tau\right)}{\left(1 + \left(\partial_z w\left(z,\, \tau\right)\right)^2\right)^{\frac{3}{2}}}$$

where $N_{\bar{\Gamma}_\tau}$ and $A_{\bar{\Gamma}_\tau}$ are the (upward) unit normal vector and normal curvature of $\bar{\Gamma}_\tau$ at Z_τ, respectively, and

$$\mathbf{e} = \left(\frac{1}{\sqrt{2}},\, \frac{1}{\sqrt{2}}\right)$$

By (3.160) and (3.161), we get

$$z \,\leq\, \left|Z_\tau\right| = \sqrt{z^2 + \left(w\left(z,\, \tau\right)\right)^2} \,\leq\, \sqrt{\frac{10}{9}}z$$

$$\left|N_{\bar{\Gamma}_\tau} \cdot \mathbf{e}\right| \leq C\left(n\right)\Lambda\left|Z_\tau\right|^{\alpha - 1} \tag{3.164}$$

$$\left|A_{\bar{\Gamma}_\tau}\right| \,\leq\, C\left(n\right)\Lambda\left|Z_\tau\right|^{\alpha - 2} \tag{3.165}$$

for $\beta \leq z \leq \left(2\sigma\tau\right)^{\frac{1}{2}(1-\vartheta)}$, $\tau_0 \leq \tau \leq \mathring{\tau}$.

Now we reparametrize $\bar{\Gamma}_\tau$ as

$$Z_\tau = \left(z,\, \hat{w}\left(z,\, \tau\right)\right)$$

In that case, we have

$$|Z_\tau| = \sqrt{z^2 + (\hat{w}(z,\,\tau))^2}$$

$$N_{\bar{\Gamma}_\tau} \cdot \mathbf{e} = \frac{1 - \partial_z \hat{w}(z,\,\tau)}{\sqrt{2\left(1 + (\partial_z \hat{w}(z,\,\tau))^2\right)}} \tag{3.166}$$

$$A_{\bar{\Gamma}_\tau} = \frac{\partial^2_{zz} \hat{w}(z,\,\tau)}{\left(1 + (\partial_z \hat{w}(z,\,\tau))^2\right)^{\frac{3}{2}}} \tag{3.167}$$

Note that by (3.3), (3.74) and (3.79), there holds

$$\frac{1}{C(n)} \le \frac{|Z_\tau|}{z} \le C(n) \tag{3.168}$$

for $2\beta \le z \le \frac{1}{2}(2\sigma\tau)^{\frac{1}{2}(1-\vartheta)}$, provided that $\beta \gg 1$ (depending on n) and $\tau_0 \gg 1$ (depending on n, Λ, ρ, β). Moreover, by (3.164) we may assume

$$\left|N_{\bar{\Gamma}_\tau} \cdot \mathbf{e}\right| \le \frac{1}{100\sqrt{2}}$$

for $2\beta \le z \le \frac{1}{2}(2\sigma\tau)^{\frac{1}{2}(1-\vartheta)}$. Since

$$\lim_{p\to\pm\infty} \frac{1-p}{\sqrt{2(1+p^2)}} = \mp\frac{1}{\sqrt{2}}$$

it follows, by (3.166), that

$$|\partial_z \hat{w}(z,\,\tau)| \le C \tag{3.169}$$

for $2\beta \le z \le \frac{1}{2}\sqrt{2\sigma\tau}$. The conclusion follows by (3.164), (3.165), (3.166), (3.167), (3.168) and (3.169). $\square$

Remark 3.37. Note that for the last lemma, when $\tau = \tau_0$, by (3.52) we have

$$N_{\bar{\Gamma}_{\tau_0}} \cdot \mathbf{e} = \frac{-\partial_z w(z,\,\tau_0)}{\sqrt{1 + (\partial_z w(z,\,\tau_0))^2}} > 0$$

for $\frac{1}{2}\beta \le z \le (2\sigma\tau)^{\frac{1}{2}(1-\vartheta)}$, $\tau_0 \le \tau \le \mathring{\tau}$. Consequently, by the same argument and (3.166), we can show that

$$0 \le 1 - \partial_z \hat{w}(z,\,\tau_0) \le C(n,\,\Lambda)\, z^{\alpha-1} \tag{3.170}$$

for $\frac{1}{2}\beta \le z \le (2\sigma\tau_0)^{\frac{1}{2}(1-\vartheta)}$.

Below we use (3.40), (3.48), (3.162) and the maximum principle to show the first derivative estimate in (3.61).

Lemma 3.38. *If* $\beta \gg 1$ *(depending on* n, Λ*), there holds*

$$0 \leq \partial_z \hat{w}(z, \tau) \leq 1 + \beta^{\alpha - 2} \tag{3.171}$$

for $0 \leq z \leq \beta^2$, $\tau_0 \leq \tau \leq \mathring{\tau}$.

Proof. By differentiating (3.40), we get

$$\partial_\tau (\partial_z \hat{w}) = \frac{1}{1 + (\partial_z \hat{w})^2} \partial^2_{zz} (\partial_z \hat{w}) \tag{3.172}$$

$$+ \left(\frac{n-1}{z} - \frac{2\, \partial_z \hat{w}\, \partial^2_{zz} \hat{w}}{\left(1 + (\partial_z \hat{w})^2\right)^2} - \frac{\frac{1}{2} + \sigma}{2\sigma\tau} z \right) \partial_z (\partial_z \hat{w}) + (n-1) \left(\frac{1}{\hat{w}^2} - \frac{1}{z^2} \right) (\partial_z \hat{w})$$

Notice that for the last term on the RHS of (3.172), by (3.4) and (3.79), there holds

$$\hat{w}(z, \tau) > z \quad \Leftrightarrow \quad \frac{1}{\hat{w}^2(z, \tau)} - \frac{1}{z^2} < 0 \tag{3.173}$$

for $0 \leq z \leq \beta^2$, $\tau_0 \leq \tau \leq \mathring{\tau}$.

Let

$$(\partial_z \hat{w})_{\min} (\tau) = \min_{0 \leq z \leq \beta^2} \partial_z \hat{w}(z, \tau)$$

Then $(\partial_z \hat{w})_{\min} (\tau_0) \geq 0$ by (3.48). We claim that

$$(\partial_z \hat{w})_{\min} (\tau) \geq 0 \tag{3.174}$$

for $\tau_0 \leq \tau \leq \mathring{\tau}$. To prove that, we use a contradiction argument. Suppose that there is $\tau_1^* > \tau_0$ so that

$$(\partial_z \hat{w})_{\min} (\tau_1^*) < 0$$

Let $\tau_0^* > \tau_0$ be the first time after which $(\partial_z \hat{w})_{\min}$ stays negative all the way up to τ_1^*. By continuity, we have

$$(\partial_z \hat{w})_{\min} (\tau_0^*) \geq 0$$

Note that by (3.40) and (3.162), the negative minimum of $\partial_z \hat{w}(z, \tau)$ for each time-slice must be attained in $\left(0, \beta^2\right)$, provided that $\beta \gg 1$ (depending on n, Λ). Applying the maximum principle to (3.172) (and noting (3.173)) yields

$$\partial_\tau (\partial_z \hat{w})_{\min} \geq (n-1) \left(\frac{1}{\hat{w}^2} - \frac{1}{z^2} \right) (\partial_z \hat{w})_{\min} \geq 0$$

for $\tau_0^* \le \tau < \tau_1^*$. It follows that

$$(\partial_z \hat{w})_{\min}(\tau_0^*) \le (\partial_z \hat{w})_{\min}(\tau_1^*) < 0$$

which is a contradiction.

Next, let

$$(\partial_z \hat{w})_{\max}(\tau) = \max_{0 \le z \le \beta^2} \partial_z \hat{w}(z, \tau)$$

Then

$$(\partial_z \hat{w})_{\max}(\tau_0) \le 1$$

by (3.48) and (3.170). We claim that

$$(\partial_z \hat{w})_{\max}(\tau) \le 1 + \beta^{\alpha-2}$$

for $\tau_0 \le \tau \le \mathring{\tau}$. Suppose the contrary, then there is $\tau_1^* > \tau_0$ so that

$$(\partial_z \hat{w})_{\max}(\tau_1^*) > 1 + \beta^{\alpha-2}$$

Let $\tau_0^* > \tau_0$ be the first time after which $(\partial_z \hat{w})_{\max}$ is greater than $1 + \beta^{\alpha-2}$ all the way up to τ_1^*. By continuity, we have

$$(\partial_z \hat{w})_{\max}(\tau_0^*) \le 1 + \beta^{\alpha-2}$$

Notice that by (3.162), there holds

$$\partial_z \hat{w}\left(\beta^2, \tau\right) \le 1 + C(n, \Lambda)\, \beta^{2(\alpha-1)} < 1 + \beta^{\alpha-2}$$

provided that $\beta \gg 1$(depending on n, Λ). Thus, the maximum of $\partial_z \hat{w}(z, \tau)$ for each time-slice which is greater than $1 + \beta^{\alpha-2}$ must be attained in $\left(0, \beta^2\right)$, provided that $\beta \gg 1$ (depending on n, Λ). Applying the maximum principle to (3.172) (and using (3.173) and (3.174)) yields

$$\partial_\tau (\partial_z \hat{w})_{\max} \le 0$$

for $\tau_0^* \le \tau < \tau_1^*$. It follows that

$$(\partial_z \hat{w})_{\max}(\tau_0^*) \ge (\partial_z \hat{w})_{\max}(\tau_1^*) > 1 + \beta^{\alpha-2}$$

which is a contradiction. □

Then we start to show the second derivative estimate in (3.61). Note that the second fundamental form of Γ_τ (in the parametrization of (3.38)) is given by

$$A_{\Gamma_\tau} = \frac{1}{\sqrt{1+|\partial_z \hat{w}|^2}} \begin{pmatrix} \frac{\partial^2_{zz}\hat{w}}{1+|\partial_z \hat{w}|^2} & & \\ & \frac{\partial_z \hat{w}}{z} I_{n-1} & \\ & & \frac{-1}{\hat{w}} I_{n-1} \end{pmatrix} \tag{3.175}$$

By (3.79) and (3.171), to estimate $\partial^2_{zz}\hat{w}(z,\tau)$ is equivalent to estimate A_{Γ_τ}. In the following lemma, we derive an evolution equation of A_{Γ_τ} and use that, together with (3.48), (3.163) and the maximum principle, to show that A_{Γ_τ} can be estimated for a short period of time.

Lemma 3.39. *If $\beta \gg 1$ (depending on n, Λ), then there is $\delta > 0$ (depending on n) so that the second fundamental form of Γ_τ satisfies*

$$\max_{\Gamma_\tau \cap B(O;3\beta)} |A_{\Gamma_\tau}| \le C(n)$$

for $\tau_0 \le \tau \le \min\{\tau_0+\delta, \mathring{\tau}\}$. In particular, there holds

$$\left|\partial^2_{zz}\hat{w}(z,\tau)\right| \le C(n)$$

for $0 \le z \le 3\beta$, $\tau_0 \le \tau \le \min\{\tau_0+\delta, \mathring{\tau}\}$.

Proof. By (3.48), (3.79), (3.162), (3.163) and (3.175), the second fundamental form of Γ_τ satisfies

$$\mathfrak{C} \equiv |A_{\Gamma_\tau}|^2_{\max}(\tau_0) + \max_{Z_\tau \in \Gamma_\tau, |Z_\tau|=3\beta} |A_{\Gamma_\tau}(Z_\tau)|^2 \le C(n) \tag{3.176}$$

provided that $\beta \gg 1$ (depending on n, Λ). By reparametrization of the flow, we may derive an evolution equation for A_{Γ_τ} as follows:

$$(\partial_\tau - \triangle_{\Gamma_\tau})|A_{\Gamma_\tau}|^2 = -2|\nabla_{\Gamma_\tau} A_{\Gamma_\tau}|^2 + 2|A_{\Gamma_\tau}|^4 - \frac{1+2\sigma}{2\sigma\tau}|A_{\Gamma_\tau}|^2 \tag{3.177}$$

Let

$$h(\tau) = \max_{\Gamma_\tau \cap B(O;3\beta)} |A_{\Gamma_\tau}|^2$$

If $h(\tau) \le \mathfrak{C}$ for $\tau_0 \le \tau \le \mathring{\tau}$, then we are done. Otherwise, there is $\tau_1^* > \tau_0$ so that

$$h(\tau_1^*) > \mathfrak{C}$$

Let $\tau_0^* > \tau_0$ be the first time after which h is greater than $\mathfrak{C}$ all the way up to τ_1^*. By continuity, we have

$$h\left(\tau_0^*\right) \le \mathfrak{C} \tag{3.178}$$

Note that the maximum for each time-slice must be attained in the interior of $\Gamma_\tau \cap B\left(O;\, 3\beta\right)$. By applying the maximum principle to (3.177), we get

$$\partial_\tau h\left(\tau\right) \le 2\, h^2\left(\tau\right)$$

for $\tau_0^* \le \tau \le \tau_1^*$, which implies

$$h\left(\tau_1^*\right) \le \frac{h\left(\tau_0^*\right)}{1 - 2\left(\tau_1^* - \tau_0^*\right) h\left(\tau_0^*\right)} \tag{3.179}$$

Thus, by (3.176), (3.178) and (3.179), there is $\delta = \delta\left(n\right)$ so that

$$h\left(\tau\right) \le 2\mathfrak{C}$$

for $\tau_0^* \le \tau \le \min\left\{\tau_0^* + \delta,\, \tau_1^*\right\}$. For this choice of $\delta > 0$, we claim that

$$h\left(\tau\right) \le 2\mathfrak{C}$$

for $\tau_0 \le \tau \le \min\left\{\tau_0 + \delta,\, \mathring{\tau}\right\}$; otherwise, we may get a contradiction by the above argument. Then the conclusion follows immediately by (3.79), (3.171) and (3.175). $\square$

In the following lemma, we use Ecker-Huisken interior estimate for MCF to estimate A_{Γ_τ} for $\tau_0 + \delta \le \tau \le \mathring{\tau}$. Combining with Lemma 3.39, we then get the second derivative estimate in (3.61).

Lemma 3.40. *If $\beta \gg 1$ (depending on n, Λ), there holds*

$$\left|\partial_{zz}^2 \hat{w}\left(z,\, \tau\right)\right| \le C\left(n\right)$$

for $0 \le z \le 3\beta$, $\tau_0 \le \tau \le \mathring{\tau}$.

Proof. By Lemma 3.39, there is $\delta = \delta\left(n\right)$ so that

$$\left|\partial_{zz}^2 \hat{w}\left(z,\, \tau\right)\right| \le C\left(n\right)$$

for $0 \le z \le 3\beta$, $\tau_0 \le \tau \le \min\left\{\tau_0 + \delta,\, \mathring{\tau}\right\}$. Hence, to prove the lemma, we have to consider the case when $\mathring{\tau} - \tau_0 > \delta$.

Fix $\tau_0 + \delta \leq \tau_* \leq \mathring{\tau}$ and let

$$\Xi_\iota = (2\sigma\tau_*)^{\frac{1}{2}+\frac{1}{4\sigma}}\ \Sigma_{-(2\sigma\tau_*)^{\frac{-1}{2\sigma}}\left(1-\frac{\iota}{2\sigma\tau_*}\right)}$$

$$= \left\{ \left(r\nu,\ \hat{h}\left(r,\ \iota\right)\omega \right) \middle| r \geq 0,\ \nu \in \mathbb{S}^{n-1},\ \omega \in \mathbb{S}^{n-1} \right\}$$

where

$$\hat{h}\left(r,\ \iota\right) = (2\sigma\tau_*)^{\frac{1}{2}+\frac{1}{4\sigma}}\ \hat{u}\left(\frac{r}{(2\sigma\tau_*)^{\frac{1}{2}+\frac{1}{4\sigma}}},\ -\left(2\sigma\tau_*\right)^{\frac{-1}{2\sigma}}\left(1 - \frac{\iota}{2\sigma\tau_*}\right) \right)$$

Then $\{\Xi_\iota\}$ defines a MCF for $-\left(2\sigma\tau_*\right)\left(\left(\frac{\tau_*}{\tau_0}\right)^{\frac{1}{2\sigma}} - 1\right) \leq \iota \leq 0$. Note that

$$\Xi_0 = (2\sigma\tau_*)^{\frac{1}{2}+\frac{1}{4\sigma}}\ \Sigma_{-(2\sigma\tau_*)^{\frac{-1}{2\sigma}}} = \Gamma_{\tau_*}$$

and

$$\left(2\sigma\tau_*\right)\left(\left(\frac{\tau_*}{\tau_0}\right)^{\frac{1}{2\sigma}} - 1\right) \geq \frac{\delta}{2}$$

provided that $\tau_0 \gg 1$(depending on n). By (3.39), we may rewrite $\hat{h}\left(r,\ \iota\right)$ as

$$\hat{h}\left(r,\ \iota\right) = \left(1 - \frac{\iota}{2\sigma\tau_*}\right)^{\frac{1}{2}+\sigma}\ \hat{w}\left(\frac{r}{\left(1 - \frac{\iota}{2\sigma\tau_*}\right)^{\frac{1}{2}+\sigma}},\ \frac{\tau_*}{\left(1 - \frac{\iota}{2\sigma\tau_*}\right)^{2\sigma}} \right)$$

By (3.79) and (3.171), we have

$$\hat{h}\left(r,\ \iota\right) \geq \frac{\hat{\psi}\left(0\right)}{2} \tag{3.180}$$

$$\left|\partial_r \hat{h}\left(r,\ \iota\right)\right| = \left|\partial_z \hat{w}\left(\frac{r}{\left(1 - \frac{\iota}{2\sigma\tau_*}\right)^{\frac{1}{2}+\sigma}},\ \frac{\tau_*}{\left(1 - \frac{\iota}{2\sigma\tau_*}\right)^{2\sigma}} \right)\right| \leq \frac{4}{3} \tag{3.181}$$

for $0 \leq r \leq 4\beta$, $-\frac{\delta}{2} \leq \iota \leq 0$, provided that $\tau_0 \gg 1$ (depending on n). Note that the unit normal vector of Ξ_ι at $\mathcal{X}_\iota\left(r,\ \nu,\ \omega\right) = \left(r\nu,\ \hat{h}\left(r,\ \iota\right)\omega\right)$ is given by

$$N_{\Xi_\iota}\left(r,\ \nu,\ \omega\right) = \frac{\left(-\partial_r \hat{h}\left(r,\ \iota\right)\nu,\ \omega\right)}{\sqrt{1 + \left(\partial_r \hat{h}\left(r,\ \iota\right)\right)^2}}$$

which satisfies

$$\left(N_{\Xi_\iota}\left(r,\ \nu,\ \omega\right) \cdot \mathbf{e}\right)^{-1} = \frac{\sqrt{1 + \left(\partial_r \hat{h}\left(r,\ \iota\right)\right)^2}}{\left(\vec{0},\ \omega\right) \cdot \mathbf{e}} \tag{3.182}$$

where

$$\mathbf{e} = \Bigg(\overbrace{0, \cdots, 0}^{\text{(2n-1) copies}}, 1\Bigg), \qquad \vec{0} = \Bigg(\overbrace{0, \cdots, 0}^{n\text{ copies}}\Bigg)$$

Now fix $0 \le z_* \le 3\beta$ and let

$$\mathcal{X}_* = \left(z_* \nu_*,\, \hat{h}\,(z_*,\, 0)\ \omega_*\right) = \left(z_* \nu_*,\, \hat{w}\,(z_*,\, \tau_*)\ \omega_*\right)$$

where $\nu_* = \omega_* = \Bigg(\overbrace{0, \cdots, 0}^{\text{(n-1) copies}}, 1\Bigg)$, we claim that

$$\left(N_{\Xi_\iota}\,(r,\, \nu,\, \omega) \cdot \mathbf{e}\right)^{-1} \le \frac{5\sqrt{2}}{3} \tag{3.183}$$

for $\mathcal{X}_\iota \in \Xi_\iota \cap B^{2n}\left(\mathcal{X}_*,\, \frac{\hat{\psi}(0)}{2\sqrt{2}}\right)$, $-\frac{\delta}{2} \le \iota \le 0$. Then by the curvature estimate in [EH], the second fundamental form of Γ_{τ_*} at $\mathcal{X}_*$ satisfies

$$\left|A_{\Gamma_{\tau_*}}\,(\mathcal{X}_*)\right| \;=\; \left|A_{\Xi_0}\,(\mathcal{X}_*)\right| \;\le\; C\,(n) \left(\frac{2\sqrt{2}}{\hat{\psi}\,(0)} + \sqrt{\frac{2}{\delta}}\right) = C\,(n)$$

It follows that

$$\frac{\left|\partial^2_{zz} \hat{w}\,(z_*,\, \tau_*)\right|}{\left(1 + \left(\partial_z \hat{w}\,(z_*,\, \tau_*)\right)^2\right)^{\frac{3}{2}}} \;\le\; \left|A_{\Gamma_\tau}\,(\mathcal{X}_*)\right| \;\le\; C\,(n)$$

Now let's come back to (3.183). First notice that for each

$$\mathcal{X}_\iota\,(r,\, \nu,\, \omega) \;\in\; \Xi_\iota \cap B^{2n}\left(\mathcal{X}_*,\, \frac{\hat{\psi}\,(0)}{2\sqrt{2}}\right), \qquad -\frac{\delta}{2} \le \iota \le 0$$

there holds

$$\hat{h}\,(r,\, \iota)\,\sqrt{1 - \left(\left(\vec{0},\, \omega\right) \cdot \mathbf{e}\right)^2} \;\le\; \left|\mathcal{X}_\iota\,(r,\, \nu,\, \omega) - \mathcal{X}_*\right| \;\le\; \frac{\hat{\psi}\,(0)}{2\sqrt{2}}$$

which, together with (3.180), implies

$$\left(\vec{0},\, \omega\right) \cdot \mathbf{e} \;\ge\; \frac{1}{\sqrt{2}} \tag{3.184}$$

Then (3.183) follows by (3.181), (3.182) and (3.184). $\square$

Below we use (3.4), (3.40), (3.61), (3.79) and the standard regularity theory for parabolic equations to prove (3.68).

Proposition 3.41. *If $\beta \gg 1$ (depending on n) and $\tau_0 \gg 1$ (depending on n, β), there holds (3.68).*

Proof. Firstly, let $\hat{\boldsymbol{w}}(\boldsymbol{z},\tau)$ and $\hat{\boldsymbol{\psi}}_k(\boldsymbol{z})$ be radially symmetric functions so that

$$\hat{\boldsymbol{w}}(\boldsymbol{z},\tau) = \hat{w}(z,\tau)\Big|_{z=|\boldsymbol{z}|}, \qquad \hat{\boldsymbol{\psi}}_k(\boldsymbol{z}) = \hat{\psi}_k(z)\Big|_{z=|\boldsymbol{z}|}$$

where $\boldsymbol{z} = (\boldsymbol{z}_1, \cdots, \boldsymbol{z}_n)$. Note that

$$\partial_{\boldsymbol{z}_i}\hat{\boldsymbol{w}} = \partial_z \hat{w}\,\frac{\boldsymbol{z}_i}{|\boldsymbol{z}|}$$

$$\partial^2_{\boldsymbol{z}_i\boldsymbol{z}_j}\hat{\boldsymbol{w}} = \partial^2_{zz}\hat{w}\,\frac{\boldsymbol{z}_i\,\boldsymbol{z}_j}{|\boldsymbol{z}|^2} + \partial_z\hat{w}\,\frac{|\boldsymbol{z}|^2\,\delta_{ij} - \boldsymbol{z}_i\,\boldsymbol{z}_j}{|\boldsymbol{z}|^3}$$

Then by (3.99), (3.171) and (3.61), there hold

$$\begin{cases} \left|\hat{\boldsymbol{w}}(\boldsymbol{z},\tau) - \hat{\boldsymbol{\psi}}_k(\boldsymbol{z})\right| \le C(n)\,\beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho} \\[2em] |\nabla\hat{\boldsymbol{w}}(\boldsymbol{z},\tau)| \le 1 + \beta^{\alpha-2} \\[2em] \left|\nabla^2\hat{\boldsymbol{w}}(\boldsymbol{z},\tau)\right| \le C(n) \end{cases} \tag{3.185}$$

for $\boldsymbol{z} \in B(O;3\beta)$, $\tau_0 \le \tau \le \mathring{\tau}$, $m \in \mathbb{Z}_+$, where

$$\nabla = (\partial_{\boldsymbol{z}_1}, \cdots, \partial_{\boldsymbol{z}_n})$$

Also, by (3.4) and Lemma 3.6, we get

$$\left\|\nabla^m\hat{\boldsymbol{\psi}}_k\right\|_{L^\infty} \le C(n,m) \tag{3.186}$$

for all $m \ge 1$. In addition, from (3.4) and (3.40), we have

$$\partial_\tau\hat{w} = \frac{\sqrt{1+(\partial_z\hat{w})^2}}{z^{n-1}}\,\partial_z\left(\frac{z^{n-1}}{\sqrt{1+(\partial_z w)^2}}\,\partial_z\hat{w}\right) - \frac{n-1}{\hat{w}} + \frac{\frac{1}{2}+\sigma}{2\sigma\tau}\left(-z\,\partial_z\hat{w} + \hat{w}\right)$$

and

$$\partial_\tau\hat{\psi}_k = 0 = \frac{\sqrt{1+\left(\partial_z\hat{\psi}_k\right)^2}}{z^{n-1}}\,\partial_z\left(\frac{z^{n-1}}{\sqrt{1+\left(\partial_z\hat{\psi}_k\right)^2}}\,\partial_z\hat{\psi}_k\right) - \frac{n-1}{\hat{\psi}_k}$$

which yield

$$\partial_\tau \hat{\boldsymbol{w}} = \sqrt{1 + |\nabla \hat{\boldsymbol{w}}|^2}\ \nabla \cdot \frac{\nabla \hat{\boldsymbol{w}}}{\sqrt{1 + |\nabla \hat{\boldsymbol{w}}|^2}} - \frac{n-1}{\hat{\boldsymbol{w}}} + \frac{\frac{1}{2}+\sigma}{2\sigma\tau}\left(-\boldsymbol{z}\cdot\nabla\hat{\boldsymbol{w}} + \hat{\boldsymbol{w}}\right)$$

$$= \sum_{i,\,j=1}^{n}\left(\delta_{ij} - \frac{\partial_{\boldsymbol{z}_i}\hat{\boldsymbol{w}}\,\partial_{\boldsymbol{z}_j}\hat{\boldsymbol{w}}}{1+|\nabla\hat{\boldsymbol{w}}|^2}\right)\partial^2_{\boldsymbol{z}_i\boldsymbol{z}_j}\hat{\boldsymbol{w}} - \sum_{i=1}^{n}\left(\frac{\frac{1}{2}+\sigma}{2\sigma\tau}\boldsymbol{z}_i\right)\partial_{\boldsymbol{z}_i}\hat{\boldsymbol{w}} + \left(\frac{\frac{1}{2}+\sigma}{2\sigma\tau}\right)\hat{\boldsymbol{w}} - \frac{n-1}{\hat{\boldsymbol{w}}} \tag{3.187}$$

and

$$\partial_\tau\hat{\boldsymbol{\psi}}_k = 0 = \sqrt{1+\left|\nabla\hat{\boldsymbol{\psi}}_k\right|^2}\ \nabla\cdot\frac{\nabla\hat{\boldsymbol{\psi}}_k}{\sqrt{1+\left|\nabla\hat{\boldsymbol{\psi}}_k\right|^2}} - \frac{(n-1)}{\hat{\boldsymbol{\psi}}_k}$$

$$= \sum_{i,\,j=1}^{n}\left(\delta_{ij} - \frac{\partial_{\boldsymbol{z}_i}\hat{\boldsymbol{\psi}}_k\,\partial_{\boldsymbol{z}_j}\hat{\boldsymbol{\psi}}_k}{1+\left|\nabla\hat{\boldsymbol{\psi}}_k\right|^2}\right)\partial^2_{\boldsymbol{z}_i\boldsymbol{z}_j}\hat{\boldsymbol{\psi}}_k - \frac{n-1}{\hat{\boldsymbol{\psi}}_k} \tag{3.188}$$

Then we subtract (3.188) from (3.187) to get

$$\partial_\tau\left(\hat{\boldsymbol{w}} - \hat{\boldsymbol{\psi}}_k\right) - \sum_{i,\,j=1}^{n}\left(\delta_{ij} - \frac{\partial_{\boldsymbol{z}_i}\hat{\boldsymbol{w}}\,\partial_{\boldsymbol{z}_j}\hat{\boldsymbol{w}}}{1+|\nabla\hat{\boldsymbol{w}}|^2}\right)\partial^2_{\boldsymbol{z}_i\boldsymbol{z}_j}\left(\hat{\boldsymbol{w}} - \hat{\boldsymbol{\psi}}_k\right)$$

$$-\sum_{q=1}^{n}\left(\frac{\sum_{i,\,j=1}^{n}\partial_{\boldsymbol{z}_i}\hat{\boldsymbol{\psi}}_k\,\partial_{\boldsymbol{z}_j}\hat{\boldsymbol{\psi}}_k\,\partial^2_{\boldsymbol{z}_i\boldsymbol{z}_j}\hat{\boldsymbol{\psi}}_k\left(\partial_{\boldsymbol{z}_q}\hat{\boldsymbol{w}} + \partial_{\boldsymbol{z}_q}\hat{\boldsymbol{\psi}}_k\right)}{\left(1+|\nabla\hat{\boldsymbol{w}}|^2\right)\left(1+\left|\nabla\hat{\boldsymbol{\psi}}_k\right|^2\right)}\right)\partial_{\boldsymbol{z}_q}\left(\hat{\boldsymbol{w}} - \hat{\boldsymbol{\psi}}_k\right)$$

$$+\sum_{q=1}^{n}\left(\frac{\sum_{i=1}^{n}\partial^2_{\boldsymbol{z}_i\boldsymbol{z}_q}\hat{\boldsymbol{\psi}}_k\left(\partial_{\boldsymbol{z}_i}\hat{\boldsymbol{w}} + \partial_{\boldsymbol{z}_i}\hat{\boldsymbol{\psi}}_k\right)}{1+|\nabla\hat{\boldsymbol{w}}|^2} + \frac{\frac{1}{2}+\sigma}{2\sigma\tau}\boldsymbol{z}_q\right)\partial_{\boldsymbol{z}_q}\left(\hat{\boldsymbol{w}} - \hat{\boldsymbol{\psi}}_k\right)$$

$$-\left(\frac{n-1}{\hat{\boldsymbol{w}}\,\hat{\boldsymbol{\psi}}_k} + \frac{\frac{1}{2}+\sigma}{2\sigma\tau}\right)\left(\hat{\boldsymbol{w}} - \hat{\boldsymbol{\psi}}_k\right)$$

$$= \frac{\frac{1}{2}+\sigma}{2\sigma\tau}\left(-\boldsymbol{z}\cdot\nabla\hat{\boldsymbol{\psi}}_k(\boldsymbol{z}) + \hat{\boldsymbol{\psi}}_k(\boldsymbol{z})\right) \equiv \boldsymbol{f}(\boldsymbol{z},\,\tau) \tag{3.189}$$

Note that by (3.6), we have

$$\left|\nabla^m\partial_\tau^l\boldsymbol{f}(\boldsymbol{z},\,\tau)\right| \le C(n,\,m,\,l)\,\tau^{-1} \tag{3.190}$$

for $\boldsymbol{z}\in\mathbb{R}^n$, $\tau_0\le\tau\le\mathring{\tau}$, $m\ge 0$.

Now fix $0<\delta\ll 1$ and $\boldsymbol{z}_*\in B(O;\,2\beta)$, $\tau_0+\delta^2\le\tau_*\le\mathring{\tau}$. By (3.185), (3.186) and Krylov-Safonov Hölder estimate (applying to (3.189)), there is

$$\gamma = \gamma(n)\in(0,\,1)$$

so that

$$\delta^{\gamma}\left[\hat{\boldsymbol{w}}-\hat{\boldsymbol{\psi}}_k\right]_{\gamma;\,Q\left(\boldsymbol{z}_*,\,\tau_*;\,\frac{1}{2}\delta\right)} \tag{3.191}$$

$$\leq C\left(n\right)\left(\left\|\hat{\boldsymbol{w}}-\hat{\boldsymbol{\psi}}_k\right\|_{L^\infty\left(Q\left(\boldsymbol{z}_*,\,\tau_*;\,\delta\right)\right)}+\delta^2\left\|\boldsymbol{f}\right\|_{L^\infty\left(Q\left(\boldsymbol{z}_*,\,\tau_*;\,\delta\right)\right)}\right)\;\leq\;C\left(n\right)$$

provided that $\beta \gg 1$ (depending on n) and $\tau_0 \gg 1$ (depending on n, β). Next, for each $p \in \{1, \cdots, n\}$, differentiate (3.187) with respect to $\boldsymbol{z}_p$ to get

$$\partial_\tau\left(\partial_{\boldsymbol{z}_p}\hat{\boldsymbol{w}}\right)=\triangle\left(\partial_{\boldsymbol{z}_p}\hat{\boldsymbol{w}}\right)-\frac{\nabla^2\left(\partial_{\boldsymbol{z}_p}\hat{\boldsymbol{w}}\right)\left(\nabla\hat{\boldsymbol{w}},\,\nabla\hat{\boldsymbol{w}}\right)}{1+\left|\nabla\hat{\boldsymbol{w}}\right|^2}$$

$$+\frac{1}{2}\left\langle\left(\frac{\left\langle\nabla\ln\left(1+\left|\nabla\hat{\boldsymbol{w}}\right|^2\right),\,\nabla\hat{\boldsymbol{w}}\right\rangle}{1+\left|\nabla\hat{\boldsymbol{w}}\right|^2}\nabla\hat{\boldsymbol{w}}\;-\;\nabla\ln\left(1+\left|\nabla\hat{\boldsymbol{w}}\right|^2\right)\right),\;\;\nabla\left(\partial_{\boldsymbol{z}_p}\hat{\boldsymbol{w}}\right)\right\rangle$$

$$-\left\langle\frac{\frac{1}{2}+\sigma}{2\sigma\tau}\,\boldsymbol{z},\;\;\nabla\left(\partial_{\boldsymbol{z}_p}\hat{\boldsymbol{w}}\right)\right\rangle+\frac{n-1}{\hat{\boldsymbol{w}}^2}\left(\partial_{\boldsymbol{z}_p}\hat{\boldsymbol{w}}\right)$$

$$=\sum_{i,\,j=1}^{n}\left(\delta_{ij}-\frac{\partial_{\boldsymbol{z}_i}\hat{\boldsymbol{w}}\,\partial_{\boldsymbol{z}_j}\hat{\boldsymbol{w}}}{1+\left|\nabla\hat{\boldsymbol{w}}\right|^2}\right)\partial^2_{\boldsymbol{z}_i\boldsymbol{z}_j}\left(\partial_{\boldsymbol{z}_p}\hat{\boldsymbol{w}}\right)$$

$$+\sum_{q=1}^{n}\left(\frac{\sum_{i,\,j=1}^{n}\partial_{\boldsymbol{z}_i}\hat{\boldsymbol{w}}\,\partial_{\boldsymbol{z}_j}\hat{\boldsymbol{w}}\,\partial_{\boldsymbol{z}_q}\hat{\boldsymbol{w}}\,\partial^2_{\boldsymbol{z}_i\boldsymbol{z}_j}\hat{\boldsymbol{w}}\;-\;\sum_{i=1}^{n}\left(1+\left|\nabla\hat{\boldsymbol{w}}\right|^2\right)\partial_{\boldsymbol{z}_i}\hat{\boldsymbol{w}}\,\partial^2_{\boldsymbol{z}_i\boldsymbol{z}_q}\hat{\boldsymbol{w}}}{\left(1+\left|\nabla\hat{\boldsymbol{w}}\right|^2\right)^2}\right)\partial_{\boldsymbol{z}_q}\left(\partial_{\boldsymbol{z}_p}\hat{\boldsymbol{w}}\right)$$

$$-\sum_{q=1}^{n}\frac{\frac{1}{2}+\sigma}{2\sigma\tau}\,\boldsymbol{z}_q\,\partial_{\boldsymbol{z}_q}\left(\partial_{\boldsymbol{z}_p}\hat{\boldsymbol{w}}\right)\;+\;\frac{n-1}{\hat{\boldsymbol{w}}^2}\left(\partial_{\boldsymbol{z}_p}\hat{\boldsymbol{w}}\right)$$

Then by (3.185) and Krylov-Safonov Hölder estimates, we may assume that for the same exponent γ, there holds

$$\delta^{1+\gamma}\left[\nabla\hat{\boldsymbol{w}}\right]_{\gamma;\,Q\left(\boldsymbol{z}_*,\,\tau_*;\,\frac{1}{2}\delta\right)}\;\leq\;C\left(n\right)\delta\left\|\nabla\hat{\boldsymbol{w}}\right\|_{L^\infty\left(Q\left(\boldsymbol{z}_*,\,\tau_*;\,\delta\right)\right)}\;\leq\;C\left(n\right) \tag{3.192}$$

Therefore, by (3.185), (3.186), (3.191) and (3.192), we can apply Schauder $C^{2,\,\gamma}$ estimates to (3.189) to get

$$\delta\left\|\nabla\left(\hat{\boldsymbol{w}}-\hat{\boldsymbol{\psi}}_k\right)\right\|_{L^\infty\left(Q\left(\boldsymbol{z}_*,\,\tau_*;\,\frac{1}{3}\delta\right)\right)}\;+\;\delta^2\left\|\nabla^2\left(\hat{\boldsymbol{w}}-\hat{\boldsymbol{\psi}}_k\right)\right\|_{L^\infty\left(Q\left(\boldsymbol{z}_*,\,\tau_*;\,\frac{1}{3}\delta\right)\right)}$$

$$+\,\delta^{2+\gamma}\left[\nabla^2\left(\hat{\boldsymbol{w}}-\hat{\boldsymbol{\psi}}_k\right)\right]_{\gamma;\,Q\left(\boldsymbol{z}_*,\,\tau_*;\,\frac{1}{3}\delta\right)}$$

$$\leq C\left(n\right)\left(\left\|\hat{\boldsymbol{w}}-\hat{\boldsymbol{\psi}}_k\right\|_{L^\infty\left(Q\left(\boldsymbol{z}_*,\,\tau_*;\,\frac{1}{2}\delta\right)\right)}+\delta^2\left\|\boldsymbol{f}\right\|_{L^\infty\left(Q\left(\boldsymbol{z}_*,\,\tau_*;\,\frac{1}{2}\delta\right)\right)}+\delta^{2+\gamma}\left[\boldsymbol{f}\right]_{\gamma;\,Q\left(\boldsymbol{z}_*,\,\tau_*;\,\frac{1}{2}\delta\right)}\right)$$

$$\leq C(n)\left(\beta^{\alpha-3}\left(\frac{\tau_*}{\tau_0}\right)^{-\varrho}+\tau_*^{-1}\right)\leq C(n)\,\beta^{2(\alpha-1)}\left(\frac{\tau_*}{\tau_0}\right)^{-\varrho} \tag{3.193}$$

provided that $\tau_0 \gg 1$ (depending on n, β).

The conclusion follows by using the bootstrap argument on (3.189) and repeatedly differentiating equations with respect to τ. □

3.8 Determining the constant Λ

In this section, we would finish the proof of Proposition 3.13 and Proposition 3.14. What's left is to show (3.58) and choose $\Lambda = \Lambda(n) \gg 1$ so that (3.60) holds. To this end, it suffices to show that

1. In the **outer region**, the function $u(x, t)$ defined in (3.24) satisfies

$$x^i\left|\partial_x^i u(x,\,t)\right| \leq C(n)\,x^{2\lambda_2+1} \qquad \forall\ i\in\{0,\,1,\,2\} \tag{3.194}$$

$$\partial^2_{xx}u(x,\,t)\geq 0 \tag{3.195}$$

for $\sqrt{-t} \leq x \leq \rho$, $t_0 \leq t \leq \mathring{t}$;

2. In the **intermediate region**, if we perform the type I rescaling, the type I rescaled function $v(y, s)$ defined in (3.29) satisfies

$$y^i\left|\partial_y^i v(y,\,s)\right| \leq C(n)\,e^{-\lambda_2 s}y^{\alpha} \qquad \forall\ i\in\{0,\,1,\,2\} \tag{3.196}$$

$$\partial^2_{yy}v(y,\,s)\geq 0 \tag{3.197}$$

for $2\beta e^{-\sigma s} \leq y \leq 1$, $s_0 < s \leq \mathring{s}$;

3. Near the **tip region**, if we perform the type II rescaling, the type II rescaled function $w(z, \tau)$ defined in (3.42) satisfies

$$z^i\left|\partial_z^i w(z,\,\tau)\right| \leq C(n)\,z^{\alpha} \qquad \forall\ i\in\{0,\,1,\,2\} \tag{3.198}$$

for $\beta \leq z \leq 2\beta$, $\tau_0 \leq \tau \leq \mathring{\tau}$. In addition, the type II rescaled function $\hat{w}(z, \tau)$ defined in (3.38) satisfies

$$\partial^2_{zz}\hat{w}(z,\,\tau)\geq 0 \tag{3.199}$$

for $0 \leq z \leq 5\beta$, $\tau_0 \leq \tau \leq \mathring{\tau}$.

Note that (3.196) is equivalent to

$$x^i \left|\partial_x^i u\left(x,\, t\right)\right| \leq C\left(n\right)\left(-t\right)^2 x^{\alpha} \qquad \forall\ i \in \{0,\, 1,\, 2\}$$

for $2\beta\left(-t\right)^{\frac{1}{2}+\sigma} \leq x \leq \sqrt{-t}$, $t_0 \leq t \leq \mathring{t}$ (see (3.30) and (3.37)). Also, (3.198) is equivalent to

$$x^i \left|\partial_x^i u\left(x,\, t\right)\right| \leq C\left(n\right)\left(-t\right)^2 x^{\alpha} \qquad \forall\ i \in \{0,\, 1,\, 2\}$$

for $\beta\left(-t\right)^{\frac{1}{2}+\sigma} \leq x \leq 2\beta\left(-t\right)^{\frac{1}{2}+\sigma}$, $t_0 \leq t \leq \mathring{t}$ (see (3.23) and (3.43)). Moreover, by (3.195), (3.197), (3.199) and rescaling, we can show (3.58), i.e. the projected curve $\bar{\Sigma}_t$ is convex in $B\left(O;\, \rho\right)$ for $t_0 \leq t \leq \mathring{t}$.

Recall that in Remark 3.23, we already show the C^0 estimates in (3.194) and (3.196). As for the derivatives, notice that the smooth estimates in Proposition 3.13 does not imply (3.194), (3.196) and (3.198), since those estimates doest not extend to the initial time. Therefore, in this section we compensate that by showing how to estimate the quantities in (3.194), (3.196) and (3.198) from the initial time to some extent. The idea is to derive evolution equations for these quantities and use the following lemma (see Lemma 3.42), together with (3.46), (3.50) and (3.52), to show that they can be bounded in terms of n for a short period of time. Below is the lemma which we would use to prove the derivatives estimates in (3.194) and (3.196).

Lemma 3.42. *Suppose that $h\left(r,\, \iota\right)$ is a function which satisfies*

$$\partial_\iota h \,-\, a\left(r,\, \iota\right)\partial_{rr}^2 h \,-\, b\left(r,\, \iota\right)\partial_r h \,=\, f\left(r,\, \iota\right)$$

for $\frac{1}{2} \leq r \leq \frac{3}{2}$, $0 \leq \iota \leq \mathcal{T}$, with

$$a\left(r,\, \iota\right) > 0$$

$$\max\Big\{\left|a\left(r,\, \iota\right)\right|,\, \left|b\left(r,\, \iota\right)\right|\Big\} \,\leq\, M$$

for $\frac{1}{2} \leq r \leq \frac{3}{2}$, $0 \leq \iota \leq \mathcal{T}$, where $\mathcal{T}$, $M > 0$ are constants. Then there hold

$$h\left(r,\, \iota\right) \,\leq\, \max_{\frac{1}{2}\leq \mathrm{r} \leq \frac{3}{2}} h\left(\mathrm{r},\, 0\right) \,+\, C\left(M\right)\iota\left(\|h\|_{L^\infty\left(\left[\frac{1}{2},\, \frac{3}{2}\right]\times[0,\mathcal{T}]\right)} + \|f\|_{L^\infty\left(\left[\frac{1}{2},\, \frac{3}{2}\right]\times[0,\mathcal{T}]\right)}\right)$$

$$h\left(r,\, \iota\right) \,\geq\, \min_{\frac{1}{2}\leq \mathrm{r} \leq \frac{3}{2}} h\left(\mathrm{r},\, 0\right) \,-\, C\left(M\right)\iota\left(\|h\|_{L^\infty\left(\left[\frac{1}{2},\, \frac{3}{2}\right]\times[0,\mathcal{T}]\right)} + \|f\|_{L^\infty\left(\left[\frac{1}{2},\, \frac{3}{2}\right]\times[0,\mathcal{T}]\right)}\right)$$

for $\frac{3}{4} \leq r \leq \frac{5}{4}$, $0 \leq \iota \leq \mathcal{T}$.

Proof. Let $\eta(r)$ be a smooth function so that

$$\chi_{\left[\frac{3}{4},\,\frac{5}{4}\right]} \leq \eta \leq \chi_{\left[\frac{1}{2},\,\frac{3}{2}\right]}$$

and $\eta(r)$ vanishes at $\frac{1}{2}$ and $\frac{3}{2}$ to infinite order. Note that by Lemma 3.31, we may assume that

$$\frac{(\partial_r \eta(r))^2}{\eta(r)} + |\partial_r \eta(r)| + \left|\partial_{rr}^2 \eta(r)\right| \lesssim 1$$

It follows that

$$\partial_\iota(\eta h) - a(r,\,\iota)\,\partial_{rr}^2(\eta h) - b(r,\,\iota)\,\partial_r(\eta h) \tag{3.200}$$

$$= \eta\, f(r,\,\iota) - \left(a(r,\,\iota)\,\partial_{rr}^2\eta + b(r,\,\iota)\,\partial_r\eta\right)h - 2\,a(r,\,\iota)\;\partial_r\eta\,\partial_r h$$

For the last term on RHS of (3.200), if we evaluate it at any maximum point of $\eta(r)\,h(r,\,\iota)$ for each time-slice, either $\eta = 0$ and hence

$$\partial_r \eta = 0 \;\Rightarrow\; -2\,a(r,\,\iota)\;\partial_r\eta\,\partial_r h = 0 \tag{3.201}$$

or $0 < \eta \leq 1$, in which case we have

$$\partial_r(\eta h) = 0 \;\Leftrightarrow\; \eta\,\partial_r h + h\,\partial_r \eta = 0$$

which yields

$$-2\,a(r,\,\iota)\;\partial_r\eta\,\partial_r h = 2\,a(r,\,\iota)\;\frac{(\partial_r\eta)^2}{\eta}h \tag{3.202}$$

Now let

$$(\eta h)_{\max}(\iota) = \max_r\,(\eta(r)\,h(r,\,\iota))$$

By (3.201) and (3.202), if we apply the maximum principle to (3.200), we get

$$\partial_\iota(\eta h)_{\max} \leq C(M)\left(\|h\|_{L^\infty\left(\left[\frac{1}{2},\,\frac{3}{2}\right]\times[0,\,\mathcal{T}]\right)} + \|f\|_{L^\infty\left(\left[\frac{1}{2},\,\frac{3}{2}\right]\times[0,\,\mathcal{T}]\right)}\right)$$

which implies

$$(\eta h)_{\max}(\iota) \;\leq\; (\eta h)_{\max}(0) + C(M)\,\iota\left(\|h\|_{L^\infty\left(\left[\frac{1}{2},\,\frac{3}{2}\right]\times[0,\,\mathcal{T}]\right)} + \|f\|_{L^\infty\left(\left[\frac{1}{2},\,\frac{3}{2}\right]\times[0,\,\mathcal{T}]\right)}\right)$$

Similarly, if we define

$$(\eta h)_{\min}(\iota) = \min_r\,(\eta(r)\,h(r,\,\iota))$$

then we have

$$(\eta h)_{\min}(\iota) \geq (\eta h)_{\min}(0) - C(M)\,\iota\left(\|h\|_{L^\infty\left(\left[\frac{1}{2},\,\frac{3}{2}\right]\times[0,\mathcal{T}]\right)} + \|f\|_{L^\infty\left(\left[\frac{1}{2},\,\frac{3}{2}\right]\times[0,\mathcal{T}]\right)}\right)$$

□

To prove the derivatives estimates in (3.194), we divide the region into two parts: $\frac{3}{4}\rho \leq x \leq \rho$ and $\sqrt{-t} \leq x \leq \frac{3}{4}\rho$. In the following proposition, we show (3.194) for $\frac{3}{4}\rho \leq x \leq \rho$ by using (3.25), (3.50), (3.59) and Lemma 3.42.

Proposition 3.43. *If $|t_0| \ll 1$ (depending on n, Λ, ρ, β), then there hold*

$$\frac{1}{2}\Upsilon_2(2\lambda_2+1)\,x^{2\lambda_2} \leq \partial_x u(x,\,t) \leq \frac{3}{2}\Upsilon_2(2\lambda_2+1)\,x^{2\lambda_2} \tag{3.203}$$

$$\partial^2_{xx}u(x,\,t) \leq \frac{3}{2}\Upsilon_2(2\lambda_2+1)(2\lambda_2)\,x^{2\lambda_2-1} \tag{3.204}$$

$$\partial^2_{xx}u(x,\,t) \geq \frac{1}{2}\Upsilon_2(2\lambda_2+1)(2\lambda_2)\,x^{2\lambda_2-1} > 0 \tag{3.205}$$

for $\frac{3}{4}\rho \leq x \leq \frac{5}{4}\rho$, $t_0 \leq t \leq \mathring{t}$.

Proof. Let

$$h(r,\,\iota) = x^{-2\lambda_2}\partial_x u(x,\,t)\Big|_{x=r\rho,\,t=t_0+\iota\rho^2}$$

From (3.25), we derive

$$\partial_\iota h - a(r,\,\iota)\,\partial^2_{rr}h - b(r,\,\iota)\,\partial_r h = f(r,\,\iota)$$

where

$$a(r,\,\iota) = \frac{1}{1+(\partial_x u(x,\,t))^2}\Bigg|_{x=r\rho,\,t=t_0+\iota\rho^2}$$

$$b(r,\,\iota) = \frac{1}{r}\left(\frac{-2x\,(\partial_x u(x,\,t))\left(\partial^2_{xx}u(x,\,t)\right)}{\left(1+(\partial_x u(x,\,t))^2\right)^2} + \frac{2(n-1)}{1-\left(\frac{u(x,t)}{x}\right)^2}\right)\Bigg|_{x=r\rho,\,t=t_0+\iota\rho^2}$$

$$f(r,\,\iota) = \frac{\rho^{-2\lambda_2+1}}{r^{2\lambda_2+1}}\left(\left(\frac{2\lambda_2}{1+(\partial_x u(x,\,t))^2}\right)\left(\partial^2_{xx}u(x,\,t)\right)\right)\Bigg|_{x=r\rho,\,t=t_0+\iota\rho^2}$$

$$+\frac{\rho^{-2\lambda_2}}{r^{2\lambda_2+2}}\left(2\lambda_2\left(\frac{-2x\left(\partial_x u\left(x,\,t\right)\right)\left(\partial_{xx}^2 u\left(x,\,t\right)\right)}{\left(1+\left(\partial_x u\left(x,\,t\right)\right)^2\right)^2}+\frac{2\left(n-1\right)}{1-\left(\frac{u\left(x,\,t\right)}{x}\right)^2}\right)\right)\left(\partial_x u\left(x,\,t\right)\right)\bigg|_{x=r\rho,\,t=t_0+\iota\rho^2}$$

$$+\frac{\rho^{-2\lambda_2}}{r^{2\lambda_2+2}}\left(-\frac{2\lambda_2\left(2\lambda_2+1\right)}{1+\left(\partial_x u\left(x,\,t\right)\right)^2}\right)\left(\partial_x u\left(x,\,t\right)\right)\bigg|_{x=r\rho,\,t=t_0+\iota\rho^2}$$

$$+\frac{\rho^{-2\lambda_2-1}}{r^{2\lambda_2+3}}\left(\left(\frac{4\left(n-1\right)\left(\left(\partial_x u\left(x,\,t\right)\right)^2-1\right)}{\left(1-\left(\frac{u\left(x,\,t\right)}{x}\right)^2\right)^2}\right)\left(u\left(x,\,t\right)\right)\right)\bigg|_{x=r\rho,\,t=t_0+\iota\rho^2}$$

It follows, by (3.59) and Lemma 3.42, that

$$\min_{\frac{1}{2}\leq \mathrm{r}\leq\frac{3}{2}} h\left(\mathrm{r},\,0\right)\,-\,C\left(n,\,\rho\right)\iota\,\leq\,h\left(r,\,\iota\right)\,\leq\,\max_{\frac{1}{2}\leq \mathrm{r}\leq\frac{3}{2}} h\left(\mathrm{r},\,0\right)\,+\,C\left(n,\,\rho\right)\iota$$

for $\frac{3}{4}\leq r\leq\frac{5}{4}$. Undoing the change of variables, we get

$$x_*^{-2\lambda_2}\partial_x u\left(x_*,\,t\right)\,\leq\,\max_{\frac{1}{2}\rho\leq x\leq\frac{3}{2}\rho}\left(x^{-2\lambda_2}\partial_x u\left(x,\,t_0\right)\right)\,+\,C\left(n,\,\rho\right)\frac{t-t_0}{\rho^2}$$

$$x_*^{-2\lambda_2}\partial_x u\left(x_*,\,t\right)\,\geq\,\min_{\frac{1}{2}\rho\leq x\leq\frac{3}{2}\rho}\left(x^{-2\lambda_2}\partial_x u\left(x,\,t_0\right)\right)\,-\,C\left(n,\,\rho\right)\frac{t-t_0}{\rho^2}$$

for $\frac{3}{4}\rho\leq x_*\leq\frac{5}{4}\rho$, $t_0\leq t\leq\mathring{t}$. Therefore, if $|t_0|\ll 1$ (depending on n, Λ, ρ, β), then (3.203) follows immediately from the above, (3.50) and (3.73).

For the second derivative, note that we have the following evolution equation:

$$\partial_t\left(x^{-2\lambda_2+1}\partial_{xx}^2 u\right)-\frac{1}{1+\left(\partial_x u\right)^2}\,\partial_{xx}^2\left(x^{-2\lambda_2+1}\partial_{xx}^2 u\right)$$

$$-\frac{1}{x}\left(\frac{-6x\left(\partial_x u\right)\left(\partial_{xx}^2 u\right)}{\left(1+\left(\partial_x u\right)^2\right)^2}+\frac{2\left(n-1\right)}{1-\left(\frac{u}{x}\right)^2}+\frac{2\left(2\lambda_2-1\right)}{1+\left(\partial_x u\right)^2}\right)\partial_x\left(x^{-2\lambda_2+1}\partial_{xx}^2 u\right)$$

$$=\frac{1}{x^{2\lambda_2+1}}\left(\frac{-2x^2\left(\partial_{xx}^2 u\right)^2\left(1-3\left(\partial_x u\right)^2\right)}{\left(1+\left(\partial_x u\right)^2\right)^3}+\frac{12\left(n-1\right)\left(\frac{u}{x}\right)\partial_x u}{\left(1-\left(\frac{u}{x}\right)^2\right)^2}-\frac{2\left(n-1\right)\left(1+\left(\frac{u}{x}\right)^2\right)}{\left(1-\left(\frac{u}{x}\right)^2\right)^2}\right)\left(\partial_{xx}^2 u\right)$$

$$+\frac{2\lambda_2-1}{x^{2\lambda_2+1}}\left(\left(\frac{-6x\left(\partial_x u\right)\left(\partial_{xx}^2 u\right)}{\left(1+\left(\partial_x u\right)^2\right)^2}+\frac{2\left(n-1\right)}{1-\left(\frac{u}{x}\right)^2}\right)+\frac{2\lambda_2-2}{1+\left(\partial_x u\right)^2}\right)\left(\partial_{xx}^2 u\right)$$

$$+\frac{1}{x^{2\lambda_2+2}}\left(\frac{4\left(n-1\right)\left(\left(\partial_x u\right)^2-1\right)\left(1+3\left(\frac{u}{x}\right)^2\right)}{\left(1-\left(\frac{u}{x}\right)^2\right)^3}\right)\left(\partial_x u\right)$$

$$+\frac{1}{x^{2\lambda_2+3}}\left(\frac{4\,(n-1)\left(1-(\partial_x u)^2\right)\left(\left(\frac{u}{x}\right)^2+3\right)}{\left(1-\left(\frac{u}{x}\right)^2\right)^3}\right)(u)$$

By the same argument (as for the first derivative), we can show (3.204) and (3.205). $\square$

Now we show the derivatives estimates in (3.194) for $\sqrt{-t}\le x\le \frac{3}{4}\rho$ by using (3.25), (3.26), (3.50), (3.63) and Lemma 3.42.

Proposition 3.44. *If $0<\rho\ll 1$ (depending on n, Λ) and $|t_0|\ll 1$ (depending on n, Λ, ρ, β), then there hold*

$$2\left(\alpha+2\Upsilon_1\left(\alpha+2\right)+\Upsilon_2\left(2\lambda_2+1\right)\right)x^{2\lambda_2}\;\le\;\partial_x u\left(x,\,t\right)\;\le\;2\Upsilon_2\left(2\lambda_2+1\right)x^{2\lambda_2}\tag{3.206}$$

$$\partial^2_{xx}u\left(x,\,t\right)\;\le\;2\left(\alpha\left(\alpha-1\right)+2\Upsilon_1\left(\alpha+2\right)\left(\alpha+1\right)+\Upsilon_2\left(2\lambda_2+1\right)\left(2\lambda_2\right)\right)x^{2\lambda_2-1}\tag{3.207}$$

$$\partial^2_{xx}u\left(x,\,t\right)\;\ge\;\frac{1}{2}\Upsilon_2\left(2\lambda_2+1\right)\left(2\lambda_2\right)x^{2\lambda_2-1}\;>0\tag{3.208}$$

for $\sqrt{-t}\le x\le\frac{3}{4}\rho$, $t_0\le t\le \mathring{t}$.

Proof. First, fix $x_*\in\left[\frac{2}{3}\sqrt{-t_0},\,\frac{3}{4}\rho\right]$ and let

$$h\left(r,\,\iota\right)=\left.x^{-2\lambda_2}\partial_x u\left(x,\,t\right)\right|_{x=rx_*,\,t=t_0+\iota x_*^2}$$

From (3.25), we derive

$$\partial_\iota h-a\left(r,\,\iota\right)\partial^2_{rr}h-b\left(r,\,\iota\right)\partial_r h=f\left(r,\,\iota\right)$$

where

$$a\left(r,\,\iota\right)=\left.\frac{1}{1+\left(\partial_x u\left(x,\,t\right)\right)^2}\right|_{x=rx_*,\,t=t_0+\iota x_*^2}$$

$$b\left(r,\,\iota\right)=\frac{1}{r}\left.\left(\frac{-2\,\partial_x u\left(x,\,t\right)\left(x\,\partial^2_{xx}u\left(x,\,t\right)\right)}{\left(1+\left(\partial_x u\left(x,\,t\right)\right)^2\right)^2}+\frac{2\left(n-1\right)}{1-\left(\frac{u(x,\,t)}{x}\right)^2}\right)\right|_{x=rx_*,\,t=t_0+\iota x_*^2}$$

$$f\left(r,\,\iota\right)=\frac{1}{r^2}\left.\left(\left(\frac{2\lambda_2}{1+\left(\partial_x u\left(x,\,t\right)\right)^2}\right)\left(x^{-2\lambda_2+1}\partial^2_{xx}u\left(x,\,t\right)\right)\right)\right|_{x=rx_*,\,t=t_0+\iota x_*^2}$$

$$+\frac{1}{r^2}\left(2\lambda_2\left(\frac{-2\,\partial_x u(x,t)\left(x\,\partial_{xx}^2 u(x,t)\right)}{\left(1+(\partial_x u(x,t))^2\right)^2}+\frac{2(n-1)}{1-\left(\frac{u(x,t)}{x}\right)^2}\right)\right)\left(x^{-2\lambda_2}\partial_x u(x,t)\right)\Bigg|_{x=rx_*,\,t=t_0+\iota x_*^2}$$

$$+\frac{1}{r^2}\left(-\frac{2\lambda_2(2\lambda_2+1)}{1+(\partial_x u(x,t))^2}\right)\left(x^{-2\lambda_2}\partial_x u(x,t)\right)\Big|_{x=rx_*,\,t=t_0+\iota x_*^2}$$

$$+\frac{1}{r^2}\left(\left(\frac{4(n-1)\left((\partial_x u(x,t))^2-1\right)}{\left(1-\left(\frac{u(x,t)}{x}\right)^2\right)^2}\right)\left(x^{-2\lambda_2-1}u(x,t)\right)\right)\Bigg|_{x=rx_*,\,t=t_0+\iota x_*^2}$$

Notice that by (3.26) we have

$$\max\left\{\left|\frac{u(x,t)}{x}\right|,\,\left|\partial_x u(x,t)\right|,\,\left|x\,\partial_{xx}^2 u(x,t)\right|\right\}\le C(n,\Lambda)\,x^{2\lambda_2}\le\frac{1}{3}$$

$$x^{-2\lambda_2-1+i}\left|\partial_x^i u(x,t)\right|\le C(n,\Lambda),\qquad i\in\{0,1,2\}$$

for $\frac{1}{2}\sqrt{-t}\le x\le\rho,\ t_0\le t\le\mathring{t}$, provided that $0<\rho\ll 1$ (depending on n,Λ) . It follows, by Lemma 3.42, that

$$\min_{\frac{1}{2}\le \mathrm{r}\le\frac{3}{2}} h(\mathrm{r},0)-C(n,\Lambda)\,\iota\le h(r,\iota)\le\max_{\frac{1}{2}\le \mathrm{r}\le\frac{3}{2}} h(\mathrm{r},0)+C(n,\Lambda)\,\iota$$

which implies

$$x_*^{-2\lambda_2}\partial_x u(x_*,t)\le\max_{\frac{1}{2}\sqrt{-t_0}\le x\le\rho}\left(x^{-2\lambda_2}\partial_x u(x,t_0)\right)+C(n,\Lambda)\,\frac{t-t_0}{\rho^2}$$

$$x_*^{-2\lambda_2}\partial_x u(x_*,t)\ge\min_{\frac{1}{2}\sqrt{-t_0}\le x\le\rho}\left(x^{-2\lambda_2}\partial_x u(x,t_0)\right)-C(n,\Lambda)\,\frac{t-t_0}{\rho^2}$$

for $t_0\le t\le t_0+\delta^2x_*^2$. Thus, by (3.50) and (3.73), we can choose $0<\delta\ll 1$ (depending on n,Λ) so that

$$2\left(\alpha+2\varUpsilon_1(\alpha+2)+\varUpsilon_2(2\lambda_2+1)\right)x^{2\lambda_2}\le\partial_x u(x,t)\le 2\varUpsilon_2(2\lambda_2+1)\,x^{2\lambda_2}\tag{3.209}$$

for (x,t) satisfying $\sqrt{-t}\le x\le\frac{3}{4}\rho,\ t_0\le t\le t_0+\delta^2x^2$, provided that $|t_0|\ll 1$ (depending on n,Λ,ρ,β).

On the other hand, by this choice of $\delta=\delta(n,\Lambda)$, (3.63) implies

$$\left|\partial_x\left(u(x,t)-\frac{k}{c_2}(-t)^{\lambda_2+\frac{1}{2}}\varphi_2\left(\frac{x}{\sqrt{-t}}\right)\right)\right|\le C(n,\Lambda)\,\rho^{4\lambda_2}x^{2\lambda_2}$$

for $(x,\, t)$ satisfying $\sqrt{-t} \le x \le \frac{3}{4}\rho$, $t_0 + \delta^2 x^2 \le t \le \mathring{t}$, where

$$\partial_x \left(\frac{k}{c_2} \left(-t\right)^{\lambda_2 + \frac{1}{2}} \varphi_2 \left(\frac{x}{\sqrt{-t}} \right) \right) = k x^{2\lambda_2} \left(\Upsilon_2 \left(2\lambda_2 + 1\right) + 2\Upsilon_1 \left(\alpha + 2\right) \left(\frac{-t}{x^2} \right) + \alpha \left(\frac{-t}{x^2} \right)^2 \right)$$

It follows, by (3.74), that

$$2 \left(\alpha + 2\Upsilon_1 \left(\alpha + 2\right) + \Upsilon_2 \left(2\lambda_2 + 1\right) \right) x^{2\lambda_2} \ \le\ \partial_x u \left(x,\, t\right) \ \le\ 2\Upsilon_2 \left(2\lambda_2 + 1\right) x^{2\lambda_2} \tag{3.210}$$

for $(x,\, t)$ satisfying $\sqrt{-t} \le x \le \frac{3}{4}\rho$, $t_0 \le t \le t_0 + \delta^2 x^2$, provided that $|t_0| \ll 1$ (depending on n, Λ, ρ, β). Then (3.206) follows immediately from (3.209) and (3.210).

As for the second derivatives, we have the evolution equation:

$$\begin{aligned}
&\partial_t \left(x^{-2\lambda_2 + 1} \partial^2_{xx} u \right) - \frac{1}{1 + \left(\partial_x u\right)^2} \partial^2_{xx} \left(x^{-2\lambda_2 + 1} \partial^2_{xx} u \right) \\
&- \frac{1}{x} \left(\frac{-6\, \partial_x u \left(x\, \partial^2_{xx} u \right)}{\left(1 + \left(\partial_x u\right)^2 \right)^2} + \frac{2\left(n-1\right)}{1 - \left(\frac{u}{x}\right)^2} + \frac{2\left(2\lambda_2 - 1\right)}{1 + \left(\partial_x u\right)^2} \right) \partial_x \left(x^{-2\lambda_2 + 1} \partial^2_{xx} u \right) \\
= &\frac{1}{x^2} \left(\frac{-2 \left(x\, \partial^2_{xx} u \right)^2 \left(1 - 3 \left(\partial_x u\right)^2 \right)}{\left(1 + \left(\partial_x u\right)^2 \right)^3} + \frac{12\left(n-1\right) \left(\frac{u}{x}\right) \partial_x u}{\left(1 - \left(\frac{u}{x}\right)^2 \right)^2} - \frac{2\left(n-1\right) \left(1 + \left(\frac{u}{x}\right)^2 \right)}{\left(1 - \left(\frac{u}{x}\right)^2 \right)^2} \right) \left(x^{-2\lambda_2 + 1} \partial^2_{xx} u \right) \\
+ &\frac{1}{x^2} \left(\left(2\lambda_2 - 1\right) \left(\frac{-6\, \partial_x u \left(x\, \partial^2_{xx} u \right)}{\left(1 + \left(\partial_x u\right)^2 \right)^2} + \frac{2\left(n-1\right)}{1 - \left(\frac{u}{x}\right)^2} \right) + \frac{\left(2\lambda_2 - 1\right)\left(2\lambda_2 - 2\right)}{1 + \left(\partial_x u\right)^2} \right) \left(x^{-2\lambda_2 + 1} \partial^2_{xx} u \right) \\
&+ \frac{1}{x^2} \left(\frac{4\left(n-1\right) \left(\left(\partial_x u\right)^2 - 1 \right) \left(1 + 3 \left(\frac{u}{x}\right)^2 \right)}{\left(1 - \left(\frac{u}{x}\right)^2 \right)^3} \right) \left(x^{-2\lambda_2} \partial_x u \right) \\
&+ \frac{1}{x^2} \left(\frac{4\left(n-1\right) \left(1 - \left(\partial_x u\right)^2 \right) \left(\left(\frac{u}{x}\right)^2 + 3 \right)}{\left(1 - \left(\frac{u}{x}\right)^2 \right)^3} \right) \left(x^{-2\lambda_2 - 1} u \right)
\end{aligned}$$

By a similar argument, we can deduce (3.207) and (3.208). □

In the following proposition, we prove (3.196) by using (3.31), (3.32), (3.46), (3.64), (3.65) and Lemma 3.42.

Proposition 3.45. *If $\beta \gg 1$ (depending on n, Λ) and $s_0 \gg 1$ (depending on n, Λ, ρ, β), then there hold*

$$2 \left(\alpha + 8\Upsilon_1 \left(\alpha + 2\right) + 16\Upsilon_2 \left(2\lambda_2 + 1\right) \right) e^{-\lambda_2 s} y^{\alpha - 1} \ \le\ \partial_y v \left(y,\, s\right) \ \le\ \frac{1}{2} \alpha e^{-\lambda_2 s} y^{\alpha - 1} \tag{3.211}$$

$$\partial_{yy}^2 v(y,\,s) \;\le\; 2\left(\alpha(\alpha-1)+8\varUpsilon_1(\alpha+2)(\alpha+1)+16\varUpsilon_2(2\lambda_2+1)(2\lambda_2)\right)\,e^{-\lambda_2 s}y^{\alpha-2} \tag{3.212}$$

$$\partial_{yy}^2 v(y,\,s) \;\ge\; \frac{1}{2}\left(\alpha(\alpha-1)\right)e^{-\lambda_2 s}y^{\alpha-2} > 0 \tag{3.213}$$

for $2\beta e^{-\sigma s} \le y \le 1,\; s_0 < s \le \mathring{s}.$

Proof. Firstly, for each $y_* \in \left[\frac{5}{3}\beta e^{-\sigma s_0},\, 1\right]$, let

$$h(r,\,\iota) = e^{\lambda_2 s}y^{-\alpha+1}\partial_y v(y,\,s)\Big|_{y=ry_*,\,s=s_0+\iota y_*^2}$$

From (3.31), we derive

$$\partial_\iota h - a(r,\,\iota)\,\partial_{rr}^2 h - b(r,\,\iota)\,\partial_r h = f(r,\,\iota)$$

where

$$a(r,\,\iota) = \left.\frac{1}{1+\left(\partial_y v(y,\,s)\right)^2}\right|_{y=ry_*,\,s=s_0+\iota y_*^2}$$

$$b(r,\,\iota) = \frac{1}{r}\left(\frac{-2\left(\partial_y v(y,\,s)\right)\left(y\,\partial_{yy}^2 v(y,\,s)\right)}{\left(1+(\partial_y v)^2\right)^2} + \frac{2(n-1)}{1-\left(\frac{v(y,\,s)}{y}\right)^2} - \frac{y^2}{2}\right)\Bigg|_{y=ry_*,\,s=s_0+\iota y_*^2}$$

$$\begin{aligned}
f(r,\,\iota) = {} & \frac{1}{r^2}\left(\frac{2(\alpha-1)}{1+\left(\partial_y v(y,\,s)\right)^2}\right)\left(e^{\lambda_2 s}y^{-\alpha+2}\partial_{yy}^2 v(y,\,s)\right)\Big|_{y=ry_*,\,s=s_0+\iota y_*^2} \\
& + \frac{\alpha-1}{r^2}\left(\frac{-2\left(\partial_y v(y,\,s)\right)\left(y\,\partial_{yy}^2 v(y,\,s)\right)}{\left(1+\left(\partial_y v(y,\,s)\right)^2\right)^2} + \frac{2(n-1)}{1-\left(\frac{v(y,\,s)}{y}\right)^2}\right)\left(e^{\lambda_2 s}y^{-\alpha+1}\partial_y v(y,\,s)\right)\Bigg|_{y=ry_*,\,s=s_0+\iota y_*^2} \\
& + \frac{1}{r^2}\left(\frac{-\alpha(\alpha-1)}{1+\left(\partial_y v(y,\,s)\right)^2} - \frac{\alpha-1}{2}y^2 + \lambda_2 y^2\right)\left(e^{\lambda_2 s}y^{-\alpha+1}\partial_y v(y,\,s)\right)\Big|_{y=ry_*,\,s=s_0+\iota y_*^2} \\
& + \frac{1}{r^2}\left(\frac{4(n-1)\left(\left(\partial_y v(y,\,s)\right)^2-1\right)}{\left(1-\left(\frac{v(y,\,s)}{y}\right)^2\right)^2}\right)\left(e^{\lambda_2 s}y^{-\alpha}v(y,\,s)\right)\Bigg|_{y=ry_*,\,s=s_0+\iota y_*^2}
\end{aligned}$$

Notice that by (3.32) we have

$$\max\left\{\left|\frac{v(y,\,s)}{y}\right|,\,|\partial_y v(y,\,s)|\,,\,\left|y\,\partial^2_{yy}v(y,\,s)\right|\right\}\,\le\,C(n,\,\Lambda)\,e^{-\lambda_2 s}y^{\alpha-1}\,\le\,\frac{1}{3}$$

$$e^{\lambda_2 s}y^{-\alpha+i}\left|\partial^i_y v(y,\,s)\right|\,\le\,C(n,\,\Lambda)\qquad\forall\ i\in\{0,\,1,\,2\}$$

for $\frac{3}{2}\beta e^{-\sigma s}\le y\le 2$, $s_0\le s\le\mathring{s}$, provided that $\beta\gg 1$ (depending on n, Λ). Then by Lemma 3.42 and (3.32), we get

$$\min_{\frac{1}{2}\le \mathrm{r}\le\frac{3}{2}} h(\mathrm{r},\,0)\,-\,C(n,\,\Lambda)\,\iota\,\le\,h(r,\,\iota)\,\le\,\max_{\frac{1}{2}\le \mathrm{r}\le\frac{3}{2}} h(\mathrm{r},\,0)\,+\,C(n,\,\Lambda)\,\iota$$

which implies

$$e^{\lambda_2 s}y_*^{-\alpha+1}\partial_y v(y_*,\,s)\,\le\,\max_{\beta e^{-\sigma s}\le y\le 2}\left(e^{\lambda_2 s_0}y^{-\alpha+1}\partial_y v(y,\,s_0)\right)\,+\,C(n,\,\Lambda)\,\frac{s-s_0}{y_*^2}$$

$$e^{\lambda_2 s}y_*^{-\alpha+1}\partial_y v(y_*,\,s)\,\ge\,\min_{\beta e^{-\sigma s}\le y\le 2}\left(e^{\lambda_2 s_0}y^{-\alpha+1}\partial_y v(y,\,s_0)\right)\,-\,C(n,\,\Lambda)\,\frac{s-s_0}{y_*^2}$$

for $s_0\le s\le s_0+\delta^2 y_*^2$. It follows, by (3.46) and (3.73), that we can choose $0<\delta\ll 1$ (depending on n, Λ) so that

$$2\left(\alpha+8\Upsilon_1(\alpha+2)+16\Upsilon_2(2\lambda_2+1)\right)e^{-\lambda_2 s}y^{\alpha-1}\,\le\,\partial_y v(y,\,s)\,\le\,\frac{1}{2}\alpha e^{-\lambda_2 s}y^{\alpha-1} \tag{3.214}$$

for $(y,\,s)$ satisfying $2\beta e^{-\sigma s}\le y\le 1$, $s_0\le s\le s_0+\delta^2 y^2$, provided that $s_0\gg 1$ (depending on n, Λ, ρ, β).

On the other hand, by the above choice of $\delta=\delta(n,\,\Lambda)$, (3.64) and (3.65) yield

$$\left|\partial_y\left(v(y,\,s)-\frac{k}{c_2}e^{-\lambda_2 s}\,\varphi_2(y)\right)\right|\,\le\,C(n,\,\Lambda)\,e^{-\varkappa s}\left(e^{-\lambda_2 s}y^{\alpha+1}\right)$$

for $(y,\,s)$ satisfying $e^{-\vartheta\sigma s}\le y\le 1$, $s_0+\delta^2 y^2\le s\le\mathring{s}$, and

$$\left|\partial_y\left(v(y,\,s)-e^{-\sigma s}\,\psi_k(e^{\sigma s}y)\right)\right|\,\le\,C(n,\,\Lambda)\,\beta^{\alpha-2}e^{-2\varrho\sigma(s-s_0)}\left(e^{-\lambda_2 s}y^{\alpha-1}\right)$$

for $(y,\,s)$ satisfying $2\beta e^{-\sigma s}\le y\le e^{-\vartheta\sigma s}$, $s_0+\delta^2 y^2\le s\le\mathring{s}$. Note that

$$\partial_y\left(\frac{k}{c_2}e^{-\lambda_2 s}\,\varphi_2(y)\right)=ke^{-\lambda_2 s}y^{\alpha-1}\left(\alpha+2\Upsilon_1(\alpha+2)\,y^2+\Upsilon_2(2\lambda_2+1)\,y^4\right)$$

$$\partial_y\left(e^{-\sigma s}\,\psi_k\left(e^{\sigma s}y\right)\right)=ke^{-\lambda_2 s}y^{\alpha-1}\left(\alpha+O\left(\left(e^{\sigma s}y\right)^{-2(1-\alpha)}\right)\right)$$

It follows, by (3.74), that

$$2\left(\alpha+8\varUpsilon_1\left(\alpha+2\right)+16\varUpsilon_2\left(2\lambda_2+1\right)\right)e^{-\lambda_2 s}y^{\alpha-1}\;\le\;\partial_y v\left(y,\,s\right)\;\le\;\frac{1}{2}\alpha e^{-\lambda_2 s}y^{\alpha-1} \tag{3.215}$$

for $(y,\,s)$ satisfying $2\beta e^{-\sigma s}\le y\le 1$, $s_0+\delta^2y^2\le s\le\mathring{s}$, provided that $\beta\gg 1$ (depending on $n,\,\Lambda$) and $s_0\gg 1$ (depending on $n,\,\Lambda$). Then (3.211) follows from (3.214) and (3.215).

As for the second derivative, we derive the following evolution equation:

$$\begin{aligned}
&\partial_s\left(e^{\lambda_2 s}y^{-\alpha+2}\partial^2_{yy}v\right)-\frac{1}{1+\left(\partial_y v\right)^2}\,\partial^2_{yy}\left(e^{\lambda_2 s}y^{-\alpha+2}\partial^2_{yy}v\right)\\
&-\frac{1}{y}\left(\frac{-6\left(\partial_y v\right)\left(y\,\partial^2_{yy}v\right)}{\left(1+\left(\partial_y v\right)^2\right)^2}+\frac{2\left(n-1\right)}{1-\left(\frac{v}{y}\right)^2}-\frac{y^2}{2}+\frac{2\left(\alpha-2\right)}{1+\left(\partial_y v\right)^2}\right)\partial_y\left(e^{\lambda_2 s}y^{-\alpha+2}\partial^2_{yy}v\right)\\
&=\frac{1}{y^2}\left(\frac{-2\left(y\,\partial^2_{yy}v\right)^2\left(1-3\left(\partial_y v\right)^2\right)}{\left(1+\left(\partial_y v\right)^2\right)^3}-\frac{y^2}{2}+\lambda_2 y^2\right)\left(e^{\lambda_2 s}y^{-\alpha+2}\partial^2_{yy}v\right)\\
&+\frac{2\left(n-1\right)}{y^2}\left(\frac{4\left(\frac{v}{y}\right)\partial_y v-1-\left(\frac{v}{y}\right)^2}{\left(1-\left(\frac{v}{y}\right)^2\right)^2}\right)\left(e^{\lambda_2 s}y^{-\alpha+2}\partial^2_{yy}v\right)\\
&+\frac{\alpha-2}{y^2}\left(\frac{-6\left(\partial_y v\right)\left(y\,\partial^2_{yy}v\right)}{\left(1+\left(\partial_y v\right)^2\right)^2}+\frac{2\left(n-1\right)}{1-\left(\frac{v}{y}\right)^2}-\frac{y^2}{2}+\frac{\alpha-3}{1+\left(\partial_y v\right)^2}\right)\left(e^{\lambda_2 s}y^{-\alpha+2}\partial^2_{yy}v\right)\\
&+\frac{1}{y^2}\left(\frac{4\left(n-1\right)\left(\frac{v}{y}\right)\left(y\,\partial^2_{yy}v\right)}{\left(1-\left(\frac{v}{y}\right)^2\right)^2}-\frac{4\left(n-1\right)\left(1-\left(\partial_y v\right)^2\right)\left(1-3\left(\frac{v}{y}\right)^2\right)}{\left(1-\left(\frac{v}{y}\right)^2\right)^3}\right)\left(e^{\lambda_2 s}y^{-\alpha+1}\partial_y v\right)\\
&+\frac{1}{y^2}\left(\frac{4\left(n-1\right)\left(1-\left(\partial_y v\right)^2\right)\left(3+\left(\frac{v}{y}\right)^2\right)}{\left(1-\left(\frac{v}{y}\right)^2\right)^3}\right)\left(e^{\lambda_2 s}y^{-\alpha}v\right)
\end{aligned}$$

Using the same argument as for the first derivative, (3.212) and (3.213) can be proved.

□

Note that by (3.43) and (3.196), we get

$$z^{i}\left|\partial_{z}^{i}w\left(z,\,\tau\right)\right|\leq C\left(n\right)z^{\alpha}\qquad\forall\ i\in\{0,\,1,\,2\}\tag{3.216}$$

for $2\beta\leq z\leq\sqrt{2\sigma\tau}$, $\tau_0<\tau\leq\mathring{\tau}$. Also, by (3.205), (3.208), (3.213) and rescaling, the projected curve $\bar{\Gamma}_\tau$ (see (3.41)) is convex in the corresponding rescaled region. More explicitly, we have

$$\partial_{zz}^{2}\hat{w}\left(z,\,\tau\right)\geq0\tag{3.217}$$

for $3\beta\leq z\leq\rho\left(2\sigma\tau\right)^{\frac{1}{2}+\frac{1}{4\sigma}}$, $\tau_0\leq\tau\leq\mathring{\tau}$. Below we prove (3.199) by using (3.4), (3.40), (3.79), (3.171) and (3.217).

Lemma 3.46. *If $\beta\gg1$ (depending on n, Λ) and $\tau_0\gg1$ (depending on n, Λ, ρ, β), there holds (3.199).*

Proof. From (3.40), we deduce that

$$\begin{aligned}\partial_{\tau}\left(\partial_{zz}^{2}\hat{w}\right)&=\frac{1}{1+\left(\partial_{z}\hat{w}\right)^{2}}\,\partial_{zz}^{2}\left(\partial_{zz}^{2}\hat{w}\right)\\&+\left(\frac{n-1}{z}-\frac{6\left(\partial_{z}\hat{w}\right)\left(\partial_{zz}^{2}\hat{w}\right)}{\left(1+\left(\partial_{z}\hat{w}\right)^{2}\right)^{2}}-\frac{\frac{1}{2}+\sigma}{2\sigma\tau}z\right)\partial_{z}\left(\partial_{zz}^{2}\hat{w}\right)-\frac{2-6\left(\partial_{z}\hat{w}\right)^{2}}{1+\left(\partial_{z}\hat{w}\right)^{2}}\left(\partial_{zz}^{2}\hat{w}\right)^{3}\\&+\left(\left(n-1\right)\left(\frac{1}{\hat{w}^{2}}-\frac{2}{z^{2}}\right)-\frac{\frac{1}{2}+\sigma}{2\sigma\tau}\right)\left(\partial_{zz}^{2}\hat{w}\right)+2\left(n-1\right)\left(\frac{1}{z^{3}}-\frac{\partial_{z}\hat{w}}{\hat{w}^{3}}\right)\partial_{z}\hat{w}\end{aligned}\tag{3.218}$$

Notice that the last term on the RHS is positive, i.e.

$$2\left(n-1\right)\left(\frac{1}{z^{3}}-\frac{\partial_{z}\hat{w}\left(z,\,\tau\right)}{\hat{w}^{3}\left(z,\,\tau\right)}\right)\partial_{z}\hat{w}\left(z,\,\tau\right)\,>0\tag{3.219}$$

for $0\leq z\leq5\beta$, $\tau_0\leq\tau\leq\mathring{\tau}$, since by (3.4), (3.74), (3.79) and (3.171), we have

$$\begin{aligned}\left(\frac{\hat{w}\left(z,\,\tau\right)}{z}\right)^{3}\geq\left(\frac{\hat{\psi}_{1-2\beta^{\alpha-3}}\left(z\right)}{z}\right)^{3}&\geq\left(1\,+\,2^{\frac{\alpha+1}{2}}\left(1-2\beta^{\alpha-3}\right)\left(5\beta\right)^{\alpha-1}\right)^{3}\\&>\,1+\beta^{\alpha-2}\,\geq\,\partial_{z}\hat{w}\left(z,\,\tau\right)\end{aligned}\tag{3.220}$$

for $0\leq z\leq5\beta$, $\tau_0\leq\tau\leq\mathring{\tau}$, provided that $\beta\gg1$ (depending on n, Λ) and $\tau_0\gg1$ (depending on n, Λ, ρ, β).

Now let

$$\left(\partial_{zz}^{2}\hat{w}\right)_{\min}\left(\tau\right)=\min_{0\leq z\leq5\beta}\partial_{zz}^{2}\hat{w}\left(z,\,\tau\right)$$

Note that by (3.48) we have

$$\left(\partial^2_{zz}\hat{w}\right)_{\min}(\tau_0) > 0$$

Now we would like to prove

$$\left(\partial^2_{zz}\hat{w}\right)_{\min}(\tau) \geq 0$$

for $\tau_0 \leq \tau \leq \mathring{\tau}$ by contradiction. Suppose that $\left(\partial^2_{zz}\hat{w}\right)_{\min}(\tau)$ fails to be non-negative for all $\tau_0 \leq \tau \leq \mathring{\tau}$, there must be $\tau_1^* > \tau_0$ so that

$$\left(\partial^2_{zz}\hat{w}\right)_{\min}(\tau_1^*) < 0$$

Let $\tau_0^* \geq \tau_0$ be the first time after which $\left(\partial^2_{zz}\hat{w}\right)_{\min}$ is negative all the way up to τ_1^*. By continuity, we have

$$\left(\partial^2_{zz}\hat{w}\right)_{\min}(\tau_0^*) \geq 0$$

On the other hand, by (3.171) and (3.217), there hold

$$\partial^2_{zz}\hat{w}(0,\,\tau) = \lim_{z\searrow 0}\frac{\partial_z\hat{w}(z,\,\tau)}{z} \geq 0$$

$$\partial^2_{zz}\hat{w}(5\beta,\,\tau) > 0$$

for $\tau_0 \leq \tau \leq \mathring{\tau}$. As a result, the negative minimum of $\partial^2_{zz}\hat{w}(z,\,\tau)$ for each time-slice must be achieved in $(0,\,5\beta)$. Then by the maximum principle (applying to (3.218)), (3.79), (3.219) and (3.220), we get

$$\partial_\tau\left(\partial^2_{zz}\hat{w}\right)_{\min} \geq \left(-\frac{2-6\left(\partial_z\hat{w}\right)^2}{1+\left(\partial_z\hat{w}\right)^2}\left(\partial^2_{zz}\hat{w}\right)^2_{\min} + \left((n-1)\left(\frac{1}{\hat{w}^2}-\frac{2}{z^2}\right)-\frac{\frac{1}{2}+\sigma}{2\sigma\tau}\right)\right)\left(\partial^2_{zz}\hat{w}\right)_{\min}$$

$$\geq \left(6\left(\partial_z\hat{w}\right)^2\left(\partial^2_{zz}\hat{w}\right)^2_{\min}\right)\left(\partial^2_{zz}\hat{w}\right)_{\min} \geq 6\left(1+\beta^{\alpha-2}\right)^2\left(\partial^2_{zz}\hat{w}\right)^3_{\min}$$

for $\tau_0^* < \tau \leq \tau_1^*$. It follows that $\left(\partial^2_{zz}\hat{w}\right)_{\min}(\tau_0^*) < 0$, which is a contradiction. □

Recall that by the admissible conditions (see Section 3.3), the projected curve $\bar{\Gamma}_\tau$ (see (3.41)) is a graph over $\bar{\mathcal{C}}$ outside $B(O;\,\beta)$. By (3.199) and also the admissible conditions, we also know that inside $B(O;\,\beta)$, $\bar{\Gamma}_\tau$ is a convex curve which intersects orthogonally with the vertical ray $\{(0,\,z)|\; z>0\}$, i.e. $\partial_z\hat{w}(0,\,\tau) = 0$. Furthermore, by

(3.4) and (3.79), $\bar{\Gamma}_\tau$ lies above $\bar{\mathcal{C}}$ and tends to it as $z \nearrow \beta$. Therefore, we conclude that $\bar{\Gamma}_\tau$ is "entirely" a graph over $\bar{\mathcal{C}}$ and

$$\bar{\Gamma}_\tau = \{(z,\, \hat{w}(z,\, \tau))|\ z \geq 0\} \tag{3.221}$$

$$= \left\{ \left((z - w(z,\, \tau)) \frac{1}{\sqrt{2}},\ (z + w(z,\, \tau)) \frac{1}{\sqrt{2}} \right) \middle|\ z \geq \frac{\hat{w}(0,\, \tau)}{\sqrt{2}} \right\}$$

Remark 3.47. For the admissible conditions in Section 3.3, we only require the function $w(z,\, \tau)$ (see (3.42)) is defined for $z \gtrsim \beta$. However, by the convexity (see (3.199)) and the above argument, we find the domain of definition for $w(z,\, \tau)$ is given by

$$\frac{\hat{w}(0,\, \tau)}{\sqrt{2}} \leq z < \infty$$

On the other hand, by (3.74) and (3.79), we may assume that inside $B(O;\, 5\beta)$, $\bar{\Gamma}_\tau$ is bounded between $\bar{\mathcal{M}}_{\frac{1}{2}}$ and $\bar{\mathcal{M}}_{\frac{3}{2}}$, provided that $\beta \gg 1$ (depending on n) and $\tau_0 \gg 1$ (depending on n, Λ, ρ, β). In particular, we have

$$\sup_{\tau_0 \leq \tau \leq \mathring{\tau}} \frac{\hat{w}(0,\, \tau)}{\sqrt{2}} < \frac{\hat{\psi}_2(0)}{\sqrt{2}}$$

which means $w(z,\, \tau)$ is defined for $z \geq \frac{\hat{\psi}_2(0)}{\sqrt{2}}$, $\tau_0 \leq \tau \leq \mathring{\tau}$. In addition, since $\bar{\Gamma}_\tau$ is a convex curve which lies below $\bar{\mathcal{M}}_{\frac{3}{2}}$ and tends to $\bar{\mathcal{C}}$, we deduce that

$$0 \leq w(z,\, \tau) \leq \psi_{\frac{3}{2}}(z) \leq \frac{\psi_{\frac{3}{2}}\left(\frac{\hat{\psi}_2(0)}{\sqrt{2}}\right)}{\frac{\hat{\psi}_2(0)}{\sqrt{2}}} z \tag{3.222}$$

for $\frac{\hat{\psi}_2(0)}{\sqrt{2}} \leq z \leq 5\beta$, $\tau_0 \leq \tau \leq \mathring{\tau}$. Note that the slope of the linear function on the RHS satisfies

$$0 < \frac{\psi_{\frac{3}{2}}\left(\frac{\hat{\psi}_2(0)}{\sqrt{2}}\right)}{\frac{\hat{\psi}_2(0)}{\sqrt{2}}} < \frac{\psi_2\left(\frac{\hat{\psi}_2(0)}{\sqrt{2}}\right)}{\frac{\hat{\psi}_2(0)}{\sqrt{2}}} = 1$$

Lastly, in order to prove (3.198), we need the following two lemmas, which provide smooth estimates of the function $w(z,\, \tau)$ in the rescaled tip region.

Lemma 3.48. *If $\beta \gg 1$ (depending on n, Λ) and $\tau_0 \gg 1$ (depending on n, Λ, ρ, β),* *there holds*

$$\begin{cases} \left|w\left(z,\,\tau\right)-\psi_{k}\left(z\right)\right|\,\leq\,C\left(n\right)\beta^{\alpha-3}\left(\frac{\tau}{\tau_{0}}\right)^{-\varrho} \\ \\ -1\,\leq\,\partial_{z}w\left(z,\,\tau\right)\,\leq\,\frac{1}{3} \\ \\ 0\,\leq\,\partial_{zz}^{2}w\left(z,\,\tau\right)\,\leq\,C\left(n\right) \end{cases} \tag{3.223}$$

for $\frac{\hat{\psi}_{2}(0)}{\sqrt{2}}\leq z\leq3\beta,\,\tau_{0}\leq\tau\leq\mathring{\tau}$.

Proof. By (3.79), inside $B\left(O;\,5\beta\right)$, the projected curve $\bar{\Gamma}_{\tau}$ is bounded between $\bar{\mathcal{M}}_{\left(1-\beta^{\alpha-3}\left(\frac{\tau}{\tau_{0}}\right)^{-\varrho}\right)k}$ and $\bar{\mathcal{M}}_{\left(1-\beta^{\alpha-3}\left(\frac{\tau}{\tau_{0}}\right)^{-\varrho}\right)k}$, which implies

$$\psi_{\left(1-\beta^{\alpha-3}\left(\frac{\tau}{\tau_{0}}\right)^{-\varrho}\right)k}\left(z\right)\,\leq\,w\left(z,\,\tau\right)\,\leq\,\psi_{\left(1+\beta^{\alpha-3}\left(\frac{\tau}{\tau_{0}}\right)^{-\varrho}\right)k}\left(z\right)$$

for $\frac{\hat{\psi}_{2}(0)}{\sqrt{2}}\leq z\leq3\beta,\,\tau_{0}\leq\tau\leq\mathring{\tau}$. Then by (3.9), (3.74) and using a similar argument as in the proof of Proposition 3.28, we can derive the C^{0} estimate of (3.223).

As for the first derivative, note that by (3.45), (3.199), (3.217) and the admissible conditions in Section 3.3, $\bar{\Gamma}_{\tau}$ is a convex curve which intersects orthogonally with the vertical ray $\left\{(0,\,z)|\;z>0\right\}$. Thus, we have

$$\partial_{zz}^{2}w\left(z,\,\tau\right)\geq0$$

$$\partial_{z}w\left(z,\,\tau\right)\,\geq\,\partial_{z}w\left(\frac{\hat{w}(0,\tau)}{\sqrt{2}},\,\tau\right)\,=-1 \tag{3.224}$$

$$\partial_{z}w\left(z,\,\tau\right)\,\leq\,\partial_{z}w\left(3\beta,\,\tau\right)\,\leq\,C\left(n,\,\Lambda\right)\beta^{\alpha-1}\leq\frac{1}{3}$$

for $\frac{\hat{\psi}_{2}(0)}{\sqrt{2}}\leq z\leq3\beta,\,\tau_{0}\leq\tau\leq\mathring{\tau}$, provided that $\beta\gg1$ (depending on $n,\,\Lambda$).

Lastly, for the second derivative, notice that by (3.61), the normal curvature of $\bar{\Gamma}_{\tau}$ (in terms of $\hat{w}\left(z,\,\tau\right)$) satisfies

$$\left|A_{\bar{\Gamma}_{\tau}}\right|\,=\,\frac{\left|\partial_{zz}^{2}\hat{w}\left(z,\,\tau\right)\right|}{\left(1+\left(\partial_{z}\hat{w}\left(z,\,\tau\right)\right)^{2}\right)^{\frac{3}{2}}}\,\leq\,C\left(n\right) \tag{3.225}$$

for $0\,\leq\,z\,\leq\,3\beta,\;\tau_{0}\,\leq\,\tau\,\leq\,\mathring{\tau}$. Now if we reparametrize $\bar{\Gamma}_{\tau}$ by means of $w\left(z,\,\tau\right)$, the normal curvature is then given by

$$A_{\bar{\Gamma}_\tau} = \frac{\partial^2_{zz} w(z,\tau)}{\left(1+(\partial_z w(z,\tau))^2\right)^{\frac{3}{2}}} \tag{3.226}$$

The second derivative estimate in (3.61) follows from (3.224), (3.225) and (3.226). □

The following lemma can be regarded as a counterpart of Proposition 3.40.

Lemma 3.49. *If $\beta \gg 1$ (depending on n, Λ) and $|\tau_0| \gg 1$ (depending on n, Λ, ρ, β), then for any $0<\delta\ll 1$, $m, l \in \mathbb{Z}_+$, there holds*

$$\delta^{m+2l}\left|\partial_z^m \partial_\tau^l \left(w(z,\tau) - \psi_k(z)\right)\right| \le C(n,m,l)\,\beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho} \tag{3.227}$$

for (z,τ) satisfying $\hat{\psi}_2(0) \le z \le 2\beta$, $\tau_0+\delta^2 \le \tau \le \mathring{\tau}$.

Proof. By mimicking the proof of Proposition 3.41 and using (3.9), (3.44), (3.222), (3.223) and Lemma (3.5), we can deduce (3.227). □

Below we show that the C^0estimate of (3.198) follows directly from the C^0 estimate of (3.223).

Proposition 3.50. *If $\beta \gg 1$ (depending on n) and $\tau_0 \gg 1$ (depending on n, Λ, ρ, β), there holds*

$$|w(z,\tau)| \le C(n)\, z^\alpha \tag{3.228}$$

for $2\hat{\psi}_2(0) \le z \le 2\beta$, $\tau_0 \le \tau \le \mathring{\tau}$.

Proof. By Lemma 3.5, (3.74) and (3.223), we have

$$\begin{aligned} z^{-\alpha}|w(z,\tau)| &\le z^{-\alpha}|\psi_k(z)| + z^{-\alpha}|w(z,\tau)-\psi_k(z)| \\ &\le z^{-\alpha}|\psi_k(z)| + (2\beta)^{-\alpha}|w(z,\tau)-\psi_k(z)| \\ &\le C(n)\left(1+\beta^{-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho}\right) \le C(n) \end{aligned}$$

for $2\hat{\psi}_2(0) \le z \le 2\beta$, $\tau_0 \le \tau \le \mathring{\tau}$, provided that $\beta \gg 1$ (depending on n). □

In the following proposition, we show the first derivative estimate of (3.198) by using the maximum principle and (3.227).

Proposition 3.51. *If $\beta \gg 1$ (depending on n) and $\tau_0 \gg 1$ (depending on n, Λ, ρ, β), there holds*

$$\left|\partial_z w\left(z,\,\tau\right)\right| \,\leq\, C\left(n\right) z^{\alpha-1} \tag{3.229}$$

for $2\hat{\psi}_2\left(0\right) \leq z \leq 2\beta$, $\tau_0 \leq \tau \leq \mathring{\tau}$.

Proof. From (3.44), we derive

$$\partial_\tau\left(z^{-\alpha+1}\partial_z w\right) - \frac{1}{1+\left(\partial_z w\right)^2}\,\partial_{zz}^2\left(z^{-\alpha+1}\partial_z w\right) \tag{3.230}$$

$$-\left(\frac{-2\,\partial_z w\,\partial_{zz}^2 w}{\left(1+\left(\partial_z w\right)^2\right)^2} + \frac{2\left(n-1\right)}{z\left(1-\left(\frac{w}{z}\right)^2\right)} - \left(\frac{1}{2}+\sigma\right)\frac{z}{2\sigma\tau}\right)\partial_z\left(z^{-\alpha+1}\partial_z w\right)$$

$$= z^{-\alpha}\left(\frac{2\left(\alpha-1\right)}{1+\left(\partial_z w\right)^2}\,\partial_{zz}^2 w - \frac{4\left(n-1\right)\left(1-\left(\partial_z w\right)^2\right)}{z^2\left(1-\left(\frac{w}{z}\right)^2\right)^2}\,w\right)$$

$$+\left(\alpha-1\right)z^{-\alpha}\left(\frac{-2\left(\partial_z w\right)\left(\partial_{zz}^2 w\right)}{\left(1+\left(\partial_z w\right)^2\right)^2} + \frac{2\left(n-1\right)}{z\left(1-\left(\frac{w}{z}\right)^2\right)} - \left(\frac{1}{2}+\sigma\right)\frac{z}{2\sigma\tau} - \frac{\alpha}{z\left(1+\left(\partial_z w\right)^2\right)}\right)\left(\partial_z w\right)$$

Let

$$M_{\text{boundary}} = \max_{\tau_0\leq\tau\leq\mathring{\tau}}\left\{z^{-\alpha+1}\partial_z w\left(z,\,\tau\right)\Big|_{z=2\hat{\psi}_2(0)},\; z^{-\alpha+1}\partial_z w\left(z,\,\tau\right)\Big|_{z=2\beta}\right\}$$

$$M_{\text{initial}} = \max_{2\hat{\psi}_2(0)\leq z\leq 2\beta} z^{-\alpha+1}\partial_z w\left(z,\,\tau_0\right)$$

Then by (3.216) and (3.223), we have

$$M_{\text{boundary}} \,\leq\, C\left(n\right)$$

By (3.49), we have

$$M_{\text{initial}} \leq C\left(n\right)$$

Let

$$h\left(\tau\right) = \max_{2\hat{\psi}_2(0)\leq z\leq 2\beta} z^{-\alpha+1}\partial_z w\left(z,\,\tau\right)$$

and

$$M = \max\left\{M_{\text{boundary}},\, M_{\text{initial}}\right\}$$

If $h\left(\tau\right) \leq M$ for $\tau_0 \leq \tau \leq \mathring{\tau}$, then we are done. Otherwise, there is $\tau_1^* > \tau_0$ for which

$$h\left(\tau_1^*\right) > M$$

Let τ_0^* be the first time after which h is greater than M all the way upto time τ_1^*. By continuity, we have

$$h\left(\tau_0^*\right) \le M$$

Applying the maximum principle to (3.230) (and using (3.222) and (3.223)) yields

$$\partial_\tau h \,\le\, C\left(n\right)\beta^{-\alpha}$$

which implies that

$$h\left(\tau\right) \,\le\, M + C\left(n\right)\beta^{-\alpha}\left(\tau - \tau_0^*\right)$$

for $\tau_0^* \le \tau \le \tau_1^*$. Now choose $0 < \varepsilon \ll 1$ (depending on n) so that

$$h\left(\tau\right) \,\le\, M + 1$$

for $\tau_0^* \le \tau \le \tau_0^* + \varepsilon\beta^\alpha$. By the above argument, we claim that

$$\max_{2\hat{\psi}_2(0)\le z\le 2\beta} z^{-\alpha+1}\partial_z w\left(z,\,\tau\right) \,\le\, M+1 \tag{3.231}$$

for $\tau_0 \le \tau \le \tau_0 + \varepsilon\beta^\alpha$; otherwise, we would get a contradiction by the above argument.

On the other hand, by (3.227) we have

$$\left(\varepsilon\beta^\alpha\right)^{\frac{1}{2}}\left|\partial_z\left(w\left(z,\,\tau\right) - \psi_k\left(z\right)\right)\right| \,\le\, C\left(n\right)\beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho}$$

for $\hat{\psi}_2\left(0\right) \le z \le 2\beta$, $\tau_0 + \varepsilon\beta^\alpha \le \tau \le \mathring{\tau}$. It follows, by (3.1), (3.74) and Lemma 3.5, that

$$\begin{aligned} z^{-\alpha+1}\partial_z w\left(z,\,\tau\right) \,&\le\, z^{-\alpha+1}\partial_z\psi_k\left(z\right) \,+\, C\left(n\right)\left(\varepsilon\beta^\alpha\right)^{-\frac{1}{2}}\beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho} z^{-\alpha+1} \\ &\le\, z^{-\alpha+1}\partial_z\psi_k\left(z\right) + C\left(n\right)\beta^{-2-\frac{\alpha}{2}} \,\le\, C\left(n\right) \end{aligned} \tag{3.232}$$

for $\hat{\psi}_2\left(0\right) \,\le\, z \,\le\, 2\beta$, $\tau_0 + \varepsilon\beta^\alpha \,\le\, \tau \,\le\, \mathring{\tau}$, provided that $\beta \gg 1$ (depending on n) and $\tau_0 \gg 1$ (depending on n, Λ, ρ, β). Note that $\varepsilon = \varepsilon\left(n\right)$.

Combining (3.231) with (3.232) yields

$$\partial_z w\left(z,\,\tau\right) \,\le\, C\left(n\right)z^{\alpha-1}$$

for $\hat{\psi}_2\left(0\right) \le z \le 2\beta$, $\tau_0 \le \tau \le \mathring{\tau}$. By a similar argument, we can show that

$$\partial_z w\left(z,\,\tau\right) \,\ge\, -C\left(n\right)z^{\alpha-1}$$

□

Next, given any constant p, from (3.44) we derive the following evolution equation in order to estimate the second derivative of (3.198).

$$\partial_\tau \left(z^{-p+2}\partial_{zz}^2 w\right) - \frac{1}{1+\left(\partial_z w\right)^2}\,\partial_{zz}^2 \left(z^{-p+2}\partial_{zz}^2 w\right) \tag{3.233}$$

$$-\left(\frac{-6\left(\partial_z w\right)\left(\partial_{zz}^2 w\right)}{\left(1+\left(\partial_z w\right)^2\right)^2}+\frac{2\left(n-1\right)}{z\left(1-\left(\frac{w}{z}\right)^2\right)}-\left(\frac{1}{2}+\sigma\right)\frac{z}{2\sigma\tau}+\frac{2\left(p-2\right)}{z\left(1+\left(\partial_z w\right)^2\right)}\right)\partial_z\left(z^{-p+2}\partial_{zz}^2 w\right)$$

$$=\left(\frac{-2\left(1-3\left(\partial_z w\right)^2\right)\left(\partial_{zz}^2 w\right)^2}{\left(1+\left(\partial_z w\right)^2\right)^3}+\frac{12\left(n-1\right)\left(\frac{w}{z}\right)\partial_z w}{z^2\left(1-\left(\frac{w}{z}\right)^2\right)^2}\right)\left(z^{-p+2}\partial_{zz}^2 w\right)$$

$$-\left(\frac{2\left(n-1\right)\left(1+\left(\frac{w}{z}\right)^2\right)}{z^2\left(1-\left(\frac{w}{z}\right)^2\right)^2}+2\left(n-1\right)\left(\frac{1}{2}+\sigma\right)\frac{1}{2\sigma\tau}\right)\left(z^{-p+2}\partial_{zz}^2 w\right)$$

$$+\left(p-2\right)\left(\frac{-6\left(\partial_z w\right)\left(\partial_{zz}^2 w\right)}{z\left(1+\left(\partial_z w\right)^2\right)^2}+\frac{2\left(n-1\right)}{z^2\left(1-\left(\frac{w}{z}\right)^2\right)}-\left(\frac{1}{2}+\sigma\right)\frac{1}{2\sigma\tau}+\frac{p-3}{z^2\left(1+\left(\partial_z w\right)^2\right)}\right)\left(z^{-p+2}\partial_{zz}^2 w\right)$$

$$+\frac{1}{z^2}\left(\frac{4\left(n-1\right)}{\left(1-\left(\frac{w}{z}\right)^2\right)^3}\left(\left(\partial_z w\right)^2+3\left(\frac{w}{z}\right)^2\left(\partial_z w\right)^2-1-3\left(\frac{w}{z}\right)^2\right)\right)\left(z^{-p+1}\partial_z w\right)$$

$$+\frac{1}{z^2}\left(\frac{4\left(n-1\right)}{\left(1-\left(\frac{w}{z}\right)^2\right)^3}\left(1-\left(\partial_z w\right)^2\right)\left(3+\left(\frac{w}{z}\right)^2\right)\right)\left(z^{-p} w\right)$$

The following lemma is essential for the derivation of the second derivative estimates in (3.198), and its proof is very similar to the one in the previous lemma

Lemma 3.52. *If $\tau_0 \gg 1$ (depending on n, Λ, ρ, β), there holds*

$$\left|z\,\partial_{zz}^2 w\left(z,\,\tau\right)\right| \le C\left(n\right)$$

for $2\hat{\psi}_2\left(0\right) \le z \le 2\beta$, $\tau_0 \le \tau \le \mathring{\tau}$.

Proof. Let

$$M_{\text{boundary}} = \max_{\tau_0\le\tau\le\mathring{\tau}}\left\{ z\,\partial_{zz}^2 w\left(z,\,\tau\right)\Big|_{z=2\hat{\psi}_2(0)},\; z\,\partial_{zz}^2 w\left(z,\,\tau\right)\Big|_{z=2\beta}\right\}$$

$$M_{\text{initial}} = \max_{2\hat{\psi}_2(0)\le z\le 2\beta} z\,\partial_{zz}^2 w\left(z,\,\tau_0\right)$$

By (3.49), (3.216) and (3.223), we have

$$M = \max\{M_{\text{boundary}},\, M_{\text{initial}}\} \le C(n)$$

Define

$$h(\tau) = \max_{2\hat{\psi}_2(0)\le z\le 2\beta} z\,\partial_{zz}^2 w(z,\tau)$$

If $h(\tau) \le M$ for $\tau_0 \le \tau \le \mathring{\tau}$, then we are done. Otherwise, there is $\tau_1^* > \tau_0$ for which

$$h(\tau_1^*) > M$$

Let τ_0^* be the first time after which h is greater than M all the way upto time τ_1^*. By continuity, we have

$$h(\tau_0^*) \le M$$

Applying the maximum principle to (3.233) with $p = 1$ (and using (3.222) and (3.223)) yields

$$\partial_\tau h(\tau) \le C(n)(h(\tau)+1)$$

which implies that

$$h(\tau) \le C(n)^{\tau-\tau_0^*}(M + C(n)) \le 2(M + C(n))$$

for $\tau_0^* \le \tau \le \tau_0^* + \varepsilon$, where $0 < \varepsilon = \varepsilon(n) \ll 1$. Thus, we claim that

$$\max_{2\hat{\psi}_2(0)\le z\le 2\beta} z\,\partial_{zz}^2 w(z,\tau) \le 2(M + C(n)) \tag{3.234}$$

for $\tau_0 \le \tau \le \tau_0 + \varepsilon$; otherwise, we would get a contradiction by the above argument.

On the other hand, by (3.227) we have

$$\varepsilon\left|\partial_{zz}^2(w(z,\tau) - \psi_k(z))\right| \le C(n)\,\beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho}$$

for $2\hat{\psi}_2(0) \le z \le 2\beta$, $\tau_0 + \varepsilon \le \tau \le \mathring{\tau}$, which, together with (3.1), (3.74) and Lemma 3.5, implies

$$z\,\partial_{zz}^2 w(z,\tau) \le z\,\partial_{zz}^2\psi_k(z) + C(n)\,\varepsilon^{-1}\beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho} z \le C(n) \tag{3.235}$$

for $2\hat{\psi}_2(0) \le z \le 2\beta$, $\tau_0 + \varepsilon \le \tau \le \mathring{\tau}$ (since $\varepsilon = \varepsilon(n)$).

By (3.234) and (3.235), we get

$$z\,\partial_{zz}^{2}w\left(z,\,\tau\right)\;\leq\;C\left(n\right)$$

for $2\hat{\psi}_2\left(0\right)\leq z\leq 2\beta$, $\tau_0\leq\tau\leq\mathring{\tau}$. Similarly, by a similar argument, we can show that

$$z\,\partial_{zz}^{2}w\left(z,\,\tau\right)\;\geq\;-C\left(n\right)$$

□

Now we are ready to show the second derivative estimate of (3.198) with the help of the previous lemma.

Proposition 3.53. *If $\tau_0\gg 1$ (depending on n), there holds*

$$\left|\partial_{zz}^{2}w\left(z,\,\tau\right)\right|\;\leq\;C\left(n\right)z^{\alpha-2}$$

for $2\hat{\psi}_2\left(0\right)\leq z\leq 2\beta$, $\tau_0\leq\tau\leq\mathring{\tau}$.

Proof. Let

$$M_{\text{boundary}}=\max_{\tau_0\leq\tau\leq\mathring{\tau}}\left\{z^{-\alpha+2}\partial_{zz}^{2}w\left(z,\,\tau\right)\Big|_{z=2\hat{\psi}_2(0)}\,,\;z^{-\alpha+2}\partial_{zz}^{2}w\left(z,\,\tau\right)\Big|_{z=2\beta}\right\}$$

$$M_{\text{initial}}=\max_{2\hat{\psi}_2(0)\leq z\leq 2\beta}\;z^{-\alpha+2}\partial_{zz}^{2}w\left(z,\,\tau_0\right)$$

By (3.49), (3.216) and (3.223), we have

$$M=\max\left\{M_{\text{boundary}},\,M_{\text{initial}}\right\}\leq C\left(n\right)$$

Define

$$h\left(\tau\right)=\max_{2\hat{\psi}_2(0)\leq z\leq 2\beta}\;z^{-\alpha+2}\partial_{zz}^{2}w\left(z,\,\tau\right)$$

If $h\left(\tau\right)\leq M$ for $\tau_0\leq\tau\leq\mathring{\tau}$, then we are done. Otherwise, there is $\tau_1^*>\tau_0$ for which

$$h\left(\tau_1^*\right)>M$$

Let τ_0^* be the first time after which h is greater than M all the way upto time τ_1^*. By continuity, we have

$$h\left(\tau_0^*\right)\leq M$$

By applying the maximum principle to (3.233) with $p=\alpha$ and using (3.222), (3.223), (3.228) and (3.229), we get

$$\partial_\tau h(\tau) \le C(n)(h(\tau)+1)$$

which implies that

$$h(\tau) \le C(n)^{\tau-\tau_0}(M+C(n)) \le 2(M+C(n))$$

for $\tau_0^* \le \tau \le \tau_0^*+\varepsilon$, where $0<\varepsilon=\varepsilon(n)\ll 1$. Thus, we infer that

$$\max_{2\hat{\psi}_2(0)\le z\le 2\beta} z^{-\alpha+2}\partial_{zz}^2 w(z,\tau) \le 2(M+C(n)) \tag{3.236}$$

for $\tau_0 \le \tau \le \tau_0+\varepsilon$, since otherwise, we would get a contradiction by the above argument.

On the other hand, by (3.227) we have

$$\varepsilon\left|\partial_{zz}^2(w(z,\tau)-\psi_k(z))\right| \le C(n)\beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho}$$

for $2\hat{\psi}_2(0) \le z \le 2\beta$, $\tau_0+\varepsilon \le \tau \le \mathring{\tau}$, which, together with (3.74) and Lemma 3.5, implies

$$\begin{aligned} z^{-\alpha+2}\partial_{zz}^2 w(z,\tau) &\le z^{-\alpha+2}\partial_{zz}^2\psi_k(z) + C(n)\beta^{\alpha-3}\left(\frac{\tau}{\tau_0}\right)^{-\varrho} z^{-\alpha+2} \\ &\le z^{-\alpha+2}\partial_{zz}^2\psi_k(z) + C(n)\beta^{-1} \le C(n) \end{aligned} \tag{3.237}$$

for $2\hat{\psi}_2(0) \le z \le 2\beta$, $\tau_0+\varepsilon \le \tau \le \mathring{\tau}$, provided that $\beta \gg 1$ (depending on n). Notice that $\varepsilon=\varepsilon(n)$.

Combining (3.236) with (3.237) yields

$$\partial_{zz}^2 w(z,\tau) \le C(n) z^{\alpha-2}$$

for $2\hat{\psi}_2(0) \le z \le 2\beta$, $\tau_0 \le \tau \le \mathring{\tau}$. Likewise, by a similar argument, we can show

$$\partial_{zz}^2 w(z,\tau) \ge -C(n) z^{\alpha-2}$$

for $2\hat{\psi}_2(0) \le z \le 2\beta$, $\tau_0 \le \tau \le \mathring{\tau}$. $\square$

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