CRITICAL ZEROS OF CLASS GROUP L-FUNCTIONS

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ABSTRACT OF THE DISSERTATION

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For an imaginary quadratic field, we define and study L-functions associated to the characters of the ideal class group of that field. After proving the main properties of these L-functions, we analyze them together as a family, with the goal of counting their zeros on the critical line. Our main result shows that any positive proportion of L-functions in this family has critical zeros of height bounded by an absolute constant times a factor that depends only slightly on the discriminant of the field. The main tools we make use of are the approximate functional equation and the equidistribution of Heegner points, which we use to obtain explicit formulas for the average of the square of the L-functions on the critical line.

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Dedication

To my parents Cléa and Benedito, and my brother Luis, for all the *saudade* we had to feel.

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Chapter 1

Introduction

1.1 Historical context

The study of zeros of L-functions starts with B. Riemann's 1859 memoir [10], entitled "On the number of prime numbers below a given quantity." In this article, Riemann showed a direct and powerful connection between the location of zeros of Euler's zeta function and the distribution of prime numbers. The strength of these ideas was shown by J. Hadamard and C. J. de la Valée Poussin, who proved the Prime Number Theorem by demonstrating that $\zeta(s)$ has no zeros on the line $\mathrm{Re}(s) = 1$. However, this is far from the expected location of the zeros of $\zeta(s)$, as the Riemann Hypothesis postulates that all zeros of nonnegative real part should lie on the so-called *critical line* $\mathrm{Re}(s) = 1/2$. Knowing the location of the zeros of $\zeta(s)$ so precisely would be a huge breakthrough in number theory, and would give incredible results about the distribution of prime numbers, among many other consequences.

Given the importance of the zeros of $\zeta(s)$, their location and distribution have been widely studied. Riemann showed in his original article that if

$$N(T) := \#\{\rho = \beta + i\gamma \,:\, \zeta(\rho) = 0,\ 0 < \gamma \leq T,\ \text{and } 0 \leq \beta \leq 1\}$$

is the number of nontrivial zeros of height up to T, then for $T \geq 2$ we have

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T),$$

but this does not give us much information about the real part of such zeros, or how many of them are on the critical line. Using a clever method for counting sign changes of the function $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$, G. H. Hardy and J. E. Littlewood were able to prove in [5] that the number of zeros of $\zeta(s)$ on the critical line of height up to T is at least growing linearly with T. Later, A. Selberg used in [11] these same ideas with the improvement of the mollification of $\zeta(s)$ to show that in fact a (very small) positive proportion of all of the zeros of $\zeta(s)$ lie on the critical line. A very different approach was used by N. Levinson in [9], which improved Selberg's result by showing that at least one third of zeros of $\zeta(s)$ have real part $1/2$, and there have been many improvements on the proportion of zeros on the critical line since. The interested reader can find modern proofs of these statements in [7] and [8].

With the development of the theory of L-functions, in particular starting with Dirichlet L-functions, the ideas of Riemann were greatly extended and generalized. As with the Riemann zeta function, the location of the zeros of many L-functions encodes valuable arithmetical information of all sorts and in many different contexts. For example, the zeros of Dirichlet L-functions $L(s, \chi)$ can be related to prime numbers in arithmetic progressions, and showing that $L(s, \chi)$ has no zeros on the line $\mathrm{Re}(s) = 1$ for all Dirichlet characters χ implies the Prime Number Theorem for arithmetic progressions. Moreover, for any civilized L-function, the knowledge that it has no zeros on the line $\mathrm{Re}(s) = 1$ gives a new verson of the Prime Number Theorem related to that L-function; see Section 5.6 of [8] for a precise formulation.

However, proving results as Hardy and Littlewood's or Selberg's count of critical zeros for most L-functions with good uniformity is a very difficult task. One way we can achieve reasonable results today is by considering not a single L-function, but instead a family of closely related L-functions, and taking an average over a family gives us better control over the coefficients and behavior of the L-functions.

It is in this context that we set out to study the critical zeros of the family of class group L-functions $L_K(s, \chi)$ associated to an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$ of discriminant $-D$. This is a very natural family of L-functions to study, coming from characters of the ideal class group of K. For fixed D, this family contains h elements, where h is the class number of the field K. Furthermore, since it is known that $h \ll \sqrt{D} \log D$ (with $h \ll \sqrt{D} \log \log D$ expected by the Grand Riemann Hypothesis), it is quite a small family relative to the size of the conductor D.

This family has been studied in the mathematical literature as well. W. Duke,

J. Friedlander and H. Iwaniec were able to use amplification methods with this family in order to obtain conditional subconvexity bounds in [3], as well as other interesting formulas for averages over the family. They rely on the equidistribution of Heegner points proved by Duke in [2] as one of the main tools to work with this family. Later, V. Blomer in [1] was able to use Duke, Friedlander and Iwaniec's ideas to apply mollification to the family and prove that a positive proportion of these class group L-functions does not vanish at the central point $s = 1/2$. We also mention the work of N. Templier which we will use heavily in this dissertation. Templier applied in [13] the equidistribution of Heegner points in a different way to get an explicit formula for the average of the squares of the class group L-functions on the critical line and also to improve on some of the results of [3].

Similarly, we will use the orthogonality of characters and the equidistribution of Heegner points to be able to successfully work with this family. We will follow Hardy and Littlewood's method of counting critical zeros by counting sign changes to show the following result, which gives zeros for any proportion $p < 1$ of class group L-functions. (Hence we get critical zeros for almost all class group L-functions in a weak sense, since we can choose p arbitrarily close to 1 but not $p = 1$.)

Theorem 1.1. *Let $p < 1$. Then at least ph of the h class group L-functions associated to the field $K = \mathbb{Q}(\sqrt{-D})$ have at least one critical zero of height up to $C[L(1, \chi_D) + 1]$, where $C = C(p)$ is a constant that depends only on p.*

In fact, we will prove a slightly more precise result, see Theorem 4.3. This will be the main result of this dissertation. The main achievement here is the uniformity with respect to D. If we are allowed to let the interval increase in length arbitrarily with D, then the result would be meaningless. However, we succeed in proving that given $p < 1$, there is one length of interval that contains zeros of ph of L-functions, and varies only sligthly with D, by a factor of $L(1, \chi_D) \ll \log D$. This factor $L(1, \chi_D)$ is essential in the sense that it comes from a fundamental difference in the families for varying D, according to the respective L-functions being more or less lacunary in a sense, and not from some weakness in our arguments.

The expected number of critical zeros in a large constant interval for a single class group L-function is in fact about $C \log D$ for some constant C, see Theorem 5.8 of [8]. However, to obtain this count would probably require the mollification of the class group L-functions (Selberg's method), which we are currently incapable of achieving. Note that Blomer succeeds in mollifying this family, but he was interested only in the values at the central point $s = 1/2$. Even then, the relatively small count of zeros that we obtain here is interesting and new because of its uniformity with respect to the conductor, and because of the small size of the family also with respect to the conductor.

1.2 The structure of this text

While our main goal in this dissertation is to count critical zeros, we start with developing the basic theory of class group L-functions in Chapter 2. In that chapter, we properly define the class group L-functions and prove some of their most important properties, such as the functional equation and their relation to Eisenstein series. Finally, we introduce the importance of Heegner points, state a version of the equidistribution theorem, and discuss the consequences of the equidistribution that we will use to count the critical zeros.

In Chapter 3, we develop the approximate functional equation (3.1) for the class group L-functions and prove some of its properties, only those that we will use in counting critical zeros. This equation will be one of the main tools we use because it is a way to use the orthogonality of class group characters in a very explicit manner.

Finally, we study the critical zeros of the class group L-functions in Chapter 4. We start by describing the method of Hardy and Littlewood, which involves estimating two integrals. Then in Section 4.2 we state the bounds that we will require of these integrals and the main result that we will prove. Our main result will be Theorem 4.3. In the last two sections of Chapter 4 we prove the bounds on the integrals, in fact proving sligthly stronger bounds than the ones we actually need for the proof of Theorem 4.3. In particular, we explain in Section 4.4 why the factor $L(1, \chi_D)$ that appears in the main result is really necessary in the method of Hardy and Littlewood.

The author would like to acknowledge once more the numerous mathematical suggestions and arguments that his advisor, H. Iwaniec, has offered. In particular, in the proof of Proposition 4.5.

Chapter 2

Class Group L-functions

2.1 Introduction

Throughout this text, we fix $K = \mathbb{Q}(\sqrt{-D})$ an imaginary quadratic field of discriminant $-D$. We denote the ring of integers of K by $\mathcal{O}_K$. Associated with this field we have the *Dedekind zeta-function* defined by

$$\zeta_K(s) := \sum_{\mathfrak{a}} \frac{1}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \left(1 - \frac{1}{N\mathfrak{p}^s}\right)^{-1}, \qquad \text{for } \operatorname{Re}(s) > 1,$$

where $\sum_{\mathfrak{a}}$ denotes a sum over all nonzero integral ideals $\mathfrak{a}$ of $\mathcal{O}_K$ and $\prod_{\mathfrak{p}}$ denotes a product over all prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$. Then if we let $L_\infty(s) := (\sqrt{D}/2\pi)^s\Gamma(s)$, we call $\Lambda_K(s) := L_\infty(s)\zeta_K(s)$ the *completed Dedekind zeta-function*. It satisfies the functional equation

$$\Lambda_K(s) = \Lambda_K(1-s),$$

which gives an analytic continuation of $\zeta_K(s)$ to $\mathbb{C} \setminus \{1\}$ with a simple pole at $s = 1$. In fact, we will see that the Dedekind zeta function of quadratic fields factors as $\zeta_K(s) = \zeta(s)L(s, \chi_D)$, where $\chi_D(n) = \left(\frac{-D}{n}\right)$ is the Kronecker symbol. Hence the residue of $\zeta_K(s)$ at $s = 1$ is $L(1, \chi_D)$.

Given two nonzero fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ of K, we say that these two ideals are *equivalent*, and write $\mathfrak{a} \sim \mathfrak{b}$, if there exists some principal fractional ideal (α) of K such that $\mathfrak{b} = (\alpha)\mathfrak{a}$. With this equivalence relation, the set of equivalence classes of nonzero fractional ideals forms a group, called the *ideal class group of* K, which we denote by $\mathcal{H}$. The *ideal class* of $\mathfrak{a}$ is the equivalence class of $\mathfrak{a}$ in $\mathcal{H}$ and will be denoted by $[\mathfrak{a}]$. The group $\mathcal{H}$ is a finite group, whose order is an important arithmetical constant that denote by h, called the *ideal class number of* K, or just *class number*. It is known that it satisfies (see [8] for Siegel's lower bound, and we will verify the upper bound in the next section)

$$D^{\frac{1}{2}-\varepsilon} \ll h \ll \sqrt{D}\log D. \tag{2.1}$$

Furthermore, assuming the Riemann Hypothesis for the Dirichlet L-function $L(s,\chi_D)$, one shows that in fact

$$\frac{\sqrt{D}}{\log\log D} \ll h \ll \sqrt{D}\log\log D.$$

The Dirichlet class number formula says that $L(1,\chi_D) = 2\pi h/w\sqrt{D}$, where w is the number of units in $\mathcal{O}_K$, see Corollary 2.8. This way, we have that

$$D^{-\varepsilon} \ll L(1,\chi_D) \ll \log D.$$

Under the Riemann Hypothesis for $L(s,\chi_D)$ we get

$$\frac{1}{\log\log D} \ll L(1,\chi_D) \ll \log\log D.$$

Let $\hat{\mathcal{H}}$ be the set of characters $\chi : \mathcal{H} \to \mathbb{C}^*$ on the group $\mathcal{H}$. Given a nonzero integral ideal $\mathfrak{a}$ of K, we naturally define $\chi(\mathfrak{a}) := \chi([\mathfrak{a}])$. This way, χ is completely multiplicative over integral ideals of K. We have h such characters, and the orthogonality relation

$$\frac{1}{h}\sum_{\chi\in\mathcal{H}} \chi(\mathfrak{a}) = \begin{cases} 1, & \text{if } \mathfrak{a} \text{ is a principal ideal,} \\ 0, & \text{otherwise.} \end{cases}$$

To each character $\chi \in \hat{\mathcal{H}}$ we can associate the *class group L-function* defined by

$$L_K(s,\chi) := \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1}, \qquad \text{for } \operatorname{Re}(s) > 1.$$

In fact, if $\chi = 1$ is the principal character, then $L_K(s,1) = \zeta_K(s)$. We let $\Lambda_K(s,\chi) := L_\infty(s)L_K(s,\chi)$ be the *completed class group L-function*, and it satisfies the functional equation

$$\Lambda_K(s,\chi) = \Lambda_K(1-s,\chi), \tag{2.2}$$

which we will prove in Section 2.4. This gives the analytic continuation for $L_K(s,\chi)$ to $\mathbb{C}$ for $\chi \neq 1$. This way $L_K(s,\chi)$ is an L-function of degree 2 and conductor D. Just like $\zeta_K(s)$, when the character χ is real, then the class group L-function $L_K(s,\chi)$ factors into Dirichlet L-functions, see [12] for a proof.

Theorem 2.1 (Kronecker Factorization Formula). *If χ is a real class group character, then there exists a factorization $-D = D_1 D_2$ of $-D$ into fundamental discriminants D_1 and D_2 such that*

$$L_K(s,\chi) = L(s,\chi_{D_1})L(s,\chi_{D_2}).$$

2.2 Ideals of $\mathcal{O}_K$ and Heegner points

Here we recall some basic properties of ideals in the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$. Given $p \in \mathbb{Z}$ a prime number, we have that

$$\chi_D(p) := \left(\frac{-D}{p}\right) = \begin{cases} -1, & \text{if } p \text{ is inert, } p = \mathfrak{p} \\ 0, & \text{if } p \text{ is ramified, } p = \mathfrak{p}^2 \\ 1, & \text{if } p \text{ splits, } p = \mathfrak{p}\overline{\mathfrak{p}},\ \mathfrak{p} \neq \overline{\mathfrak{p}}. \end{cases}$$

From this we can prove the factorization $\zeta_K(s) = \zeta(s)L(s,\chi_D)$ by directly comparing the local factors of each L-function. In particular, we note that if $\lambda(n)$ denotes the number of ideals of norm n, then

$$\lambda(n) = \sum_{d|n} \chi_D(d).$$

Since every ideal class of K has an ideal representative of norm up to $\sqrt{D}$ by Minkowski's bound, it follows that

$$h \leq \sum_{n \leq \sqrt{D}} \lambda(n) \leq \sum_{n \leq \sqrt{D}} \tau(n) \ll \sqrt{D}\log D,$$

which checks the upper bound in (2.1).

For any integral ideal $\mathfrak{b}$, we can factor this ideal as $\mathfrak{b} = (l)\mathfrak{a}$, where $l \in \mathbb{Z}$ and $\mathfrak{a}$ has no rational integer factors. We call any ideal with this propery a *primitive ideal.* If $\mathfrak{a}$ is primitive, then we can explicitly write generators for $\mathfrak{a}$ as a two-dimensional lattice in the following way:

$$\mathfrak{a} = \mathbb{Z}a + \mathbb{Z}\frac{b + i\sqrt{D}}{2},$$

where $a = N\mathfrak{a}$, and b satisfies $b^2 + D \equiv 0 \pmod{4a}$. We also have a similar representation for the inverse ideal $\mathfrak{a}^{-1}$ as

$$\mathfrak{a}^{-1} = \mathbb{Z} + \mathbb{Z}\overline{z_{\mathfrak{a}}}, \qquad \text{where} \qquad z_{\mathfrak{a}} = \frac{b + i\sqrt{D}}{2a}. \tag{2.3}$$

These explicit representations make it very convenient to work with primitive ideals. The points in the upper half-plane $\mathbb{H}$ of the form $z_{\mathfrak{a}}$ are called *Heegner points*, and they are determined modulo 1.

The set of Heegner points is mapped to itself by the action of the modular group $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, and a primitive ideal $\mathfrak{a}$ such that its associated Heegner point $z_{\mathfrak{a}}$ belongs to the standard fundamental domain $\mathcal{F}$ of $\Gamma \setminus \mathbb{H}$ is called *reduced*. Every ideal class contains a unique primitive reduced ideal, which is the ideal with the smallest norm in its class. Therefore there are exactly h primitive reduced ideals, one representative for each ideal class, corresponding to h Heegner points in $\mathcal{F}$.

Interestingly enough, as the number of Heegner points in $\mathcal{F}$ increases as D increases, the Heegner points become equidistributed in $\mathcal{F}$, an important fact that was proved by W. Duke in [2]. N. Templier formulated a very neat and practical consequence of this equidistribution principle in Proposition 4.1 of [13], which we repeat here:

Theorem 2.2 (Templier)**.** *Let $A(z)$ be a C^∞ function on $\Gamma \setminus \mathbb{H}$ satisfying the following growth conditions for $z = x + iy \in \mathcal{F}$:*

$$A(z) \ll \max(\log y,\, 1), \tag{2.4}$$

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} A(z) \ll_{i,j} y^{A_{ij}} \tag{2.5}$$

for some $A_{ij} > 0$. Then

$$\frac{1}{h} \sum_{\mathfrak{a}}{}^{*} A(z_{\mathfrak{a}}) = \int_{\mathcal{F}} A(z) d\mu(z) + O(D^{-\delta}),$$

where $\sum_{\mathfrak{a}}^{*}$ denotes a sum over primitive reduced ideals, the integration measure is $d\mu(z) := (3/\pi) y^{-2} dx dy$, and the implied constant in the O-term is a linear combination of the implied constants in (2.4) *and* (2.5)*.*

2.3 Summing over lattice points

Given a primitive ideal $\mathfrak{a}$, we can use its underlying lattice structure with the Poisson summation formula to get a useful summation formula specific for these ideals.

Lemma 2.3. *Let $\mathfrak{a}$ be a primitive ideal of $\mathcal{O}_K$ with norm a, and $g : \mathbb{R}_{\geq 0} \to \mathbb{C}$ be a smooth function of Schwarz class. Then*

$$\sum_{\alpha \in \mathfrak{a}^{-1}} g\left(\frac{2a|\alpha|^2}{\sqrt{D}}\right) = \sum_{\alpha \in \mathfrak{a}^{-1}} h\left(\frac{2a|\alpha|^2}{\sqrt{D}}\right), \tag{2.6}$$

where $h(y)$ is given by

$$h(y) = \pi \int_0^\infty J_0(2\pi\sqrt{xy})g(x)dx$$

and $J_0(x)$ is the Bessel function.

Proof. Remembering the form of the elements of $\mathfrak{a}^{-1}$ given by (2.3), we let $z := z_{\mathfrak{a}} =: \beta + i\gamma$ and $f(x,y) = g(2a|x+yz|^2/\sqrt{D})$ so that we have

$$\sum_{\alpha \in \mathfrak{a}^{-1}} g\left(\frac{2a|\alpha|^2}{\sqrt{D}}\right) = \sum_{m,n \in \mathbb{Z}} g\left(\frac{2a|m+nz|^2}{\sqrt{D}}\right) = \sum_{m,n\in\mathbb{Z}} f(m,n) = \sum_{u,v\in\mathbb{Z}} \hat{f}(u,v), \tag{2.7}$$

where we used the Poisson summation formula. The Fourier transform of $f(x,y)$ is

$$\begin{aligned}
\hat{f}(u,v) &= \iint_{\mathbb{R}^2} g\left(\frac{2a|x+yz|^2}{\sqrt{D}}\right) e(-ux - vy)dxdy \\
&= \frac{\sqrt{D}}{2a\gamma} \iint_{\mathbb{R}^2} g(|x+iy|^2) e\left(-\frac{uD^{1/4}}{\sqrt{2a}}x - \frac{(v-\beta u)D^{1/4}}{\gamma\sqrt{2a}}y\right) dxdy \\
&= h\left(\left|\frac{uD^{1/4}}{\sqrt{2a}} + i\frac{(v-\beta u)D^{1/4}}{\gamma\sqrt{2a}}\right|^2\right) = h\left(\frac{\sqrt{D}|-v+uz|^2}{2a\gamma^2}\right) = h\left(\frac{\sqrt{D}|-v+uz|^2}{2D/4a}\right),
\end{aligned}$$

where we used Lemma 4.17 from [8] to calculate the integral of the radial function $(x,y) \mapsto g(|x+iy|^2)$. Inserting this into (2.7) we get the result. □

Now we can use (2.6) to count the number of points of norm up to X in $\mathfrak{a}^{-1}$. This will also provide us with a count of the number of ideals in an ideal class. The proof is exactly the same as the proof of Corollary 4.9 from [8], but we repeat it here to make sure we obtain the correct dependency on $\mathfrak{a}$ in the end.

Theorem 2.4. *Let $\mathfrak{a}$ be a primitive ideal of $\mathcal{O}_K$ of norm a. Then*

$$\#\{\alpha \in \mathfrak{a}^{-1} : |\alpha|^2 \leq X\} = \frac{2\pi a}{\sqrt{D}}X + O\left(\left(\frac{aX}{\sqrt{D}}\right)^{1/3}\right). \tag{2.8}$$

Proof. Let $r(k) := \#\{(m,n) \in \mathbb{Z}^2 : |m + nz_{\mathfrak{a}}|^2 = k\}$ so that

$$\#\{\alpha \in \mathfrak{a}^{-1} : |\alpha|^2 \leq X\} = \sum_{k \leq X} r(k),$$

and choose $g(x) = \min\{1, x, (X+Y-x)/Y\}$ for $0 \le x \le X+Y$, and $g(x) = 0$ otherwise. Then by using a change of variables in (2.6) we have

$$\sum_{k \le X} r(k) \le \sum_{\alpha \in \mathfrak{a}^{-1}} g(|\alpha|^2) = \frac{2a}{\sqrt{D}} h(0) + E, \tag{2.9}$$

where the error term E is given by

$$E := \frac{2a}{\sqrt{D}} \sum_{\alpha \in \mathfrak{a}^{-1} \setminus \{0\}} h\left(\frac{4a^2|\alpha|^2}{D}\right).$$

Now, the main term in the right-hand side of (2.9) is

$$h(0) = \pi \int_0^\infty g(u)du = \pi\left(X + \frac{Y-1}{2}\right). \tag{2.10}$$

For the error term, we recall (4.44) from [8] that gives the following bound on $h(y)$ for this particular choice of $g(x)$:

$$h(y) \ll y^{-3/4} X^{1/4} (1 + y/Z)^{-1/2},$$

where $Z = XY^{-2}$. Therefore we have

$$\begin{aligned} E &\ll \frac{2aX^{1/4}}{\sqrt{D}} \sum_{\alpha \in \mathfrak{a}^{-1} \setminus \{0\}} \left(\frac{4a^2|\alpha|^2}{D}\right)^{-3/4} \left(1 + \frac{4a^2|\alpha|^2}{DZ}\right)^{-1/2} \\ &\ll \frac{2aX^{1/4}}{\sqrt{D}Z^{3/4}} \iint_{\mathbb{R}^2} \left(\frac{4a^2|x+yz_{\mathfrak{a}}|^2}{DZ}\right)^{-3/4} \left(1 + \frac{4a^2|x+yz_{\mathfrak{a}}|^2}{DZ}\right)^{-1/2} dxdy, \end{aligned}$$

so making a substitution $x + y\frac{b}{2a} = r\cos\theta$, $y\frac{\sqrt{D}}{2a} = r\sin\theta$ we have

$$E \ll \frac{2aX^{1/4}}{\sqrt{D}Z^{3/4}} \int_0^\infty \left(\frac{4a^2r^2}{DZ}\right)^{-3/4} \left(1 + \frac{4a^2r^2}{DZ}\right)^{-1/2} \frac{2a}{\sqrt{D}} rdr \ll (X/Y)^{1/2}.$$

So from (2.9) and (2.10) we get

$$\sum_{k \le X} r(k) \le \frac{2\pi a}{\sqrt{D}} X + O\left(\frac{2aY}{\sqrt{D}} + \left(\frac{X}{Y}\right)^{1/2}\right).$$

This way we let $Y = (\sqrt{D}/2a)^{2/3} X^{1/3}$ to optimize the error term, and working in a similar fashion to obtain a lower bound we get the stated result. $\square$

We also note here the following useful upper bound, essentially obtained in passing in the proof of Theorem A1 of [3]:

Lemma 2.5. *Given a primitive reduced ideal $\mathfrak{a}$ of $\mathcal{O}_K$ of norm a, we have*

$$\#\{\alpha \in \mathfrak{a}^{-1} \setminus \mathbb{Z} : |\alpha|^2 \le X\} \le \frac{8a}{\sqrt{D}} X. \tag{2.11}$$

Proof. Let $\alpha \in \mathfrak{a}^{-1} \setminus \mathbb{Z}$ be such that $|\alpha|^2 \le X$. Then $\alpha = m + nz_{\mathfrak{a}}$ for some $m, n \in \mathbb{Z}$, and $n \ne 0$. Then

$$\left(m + n\frac{b}{2a}\right)^2 + \frac{n^2 D}{4a^2} = |\alpha|^2 \le X. \tag{2.12}$$

Therefore $n^2 \le 4a^2 X/D$, and so there are at most $2a\sqrt{X/D}$ choices for n. Since $a < \sqrt{D}$ by assumption, if also $2\sqrt{X} < 1$, then there are no possible choices for n and in this case we are done. For $2\sqrt{X} \ge 1$, it also follows from (2.12) that $\left(m + n\frac{b}{2a}\right)^2 \le X$, so given n there are at most $2\sqrt{X}+1$ choices for m. Therefore in total we have at most

$$2a\sqrt{\frac{X}{D}}(2\sqrt{X}+1) \le 2a\sqrt{\frac{X}{D}}4\sqrt{X} = \frac{8aX}{\sqrt{D}}$$

choices for the pair (m, n), which gives us the result. □

Lemma 2.6. *Let $f(y)$ be a function with continuous first derivative, $\mathfrak{a}$ a primitive ideal, and $a = N\mathfrak{a}$. Then*

$$\sum_{\substack{\alpha \in \mathfrak{a}^{-1} \\ A < |\alpha|^2 \le B}} f\left(|\alpha|^2\right) = \frac{2\pi a}{\sqrt{D}} \int_A^B f(t)dt + O\left(\frac{a^{1/3}}{D^{1/6}}\left[B^{1/3}|f(B)| + A^{1/3}|f(A)|\right]\right) + O\left(\frac{a^{1/3}}{D^{1/6}} \int_A^B t^{1/3}|f'(t)|dt\right).$$

Proof. This follows by partial summation with (2.8). □

2.4 The functional equation

With the above summation formulas, in particular with (2.6), we can prove the functional equation for the class group L-functions. The proof is pretty much the same as the proof of the functional equation for the Riemann zeta function with some small modifications. However, it is still instructive to see here an example of how we will work with elements in an ideal class in the specific case of imaginary quadratic fields. The first thing to notice is that we do not analyze the $L_K(s, \chi)$ directly. Instead, we fix an ideal class $A \in \mathcal{H}$ and define the L-function associated to this class by

$$\zeta_K(s,A) := \sum_{\mathfrak{a}\in A} \frac{1}{N\mathfrak{a}^s}, \qquad \text{for } \operatorname{Re}(s) > 1.$$

We will obtain an analytic continuation for $\zeta_K(s,A)$ first. This is enough since given an ideal character χ we have

$$L_K(s,\chi) = \sum_{A\in\mathcal{H}} \chi(A)\zeta_K(s,A). \tag{2.13}$$

The completed L-function of the ideal class is denoted by $\Lambda_K(s,A) := L_\infty(s)\zeta_K(s,A)$.

Fix s with $\operatorname{Re}(s) > 1$. We start with the following observation

$$\left(\frac{\sqrt{D}}{2\pi}\right)^s \Gamma(s) n^{-s} = \int_0^\infty e^{-y}\left(\frac{\sqrt{D}y}{2\pi}\right)^s \frac{dy}{y} = \int_0^\infty \exp\left(-y\frac{2\pi n}{\sqrt{D}}\right) y^s \frac{dy}{y}.$$

Letting $n = N\mathfrak{a}$ and adding over all ideals in A we get

$$\Lambda_K(s,A) = \int_0^\infty S(y,A) y^s \frac{dy}{y}, \qquad \text{where} \qquad S(y,A) := \sum_{\mathfrak{a}\in A} \exp\left(-y\frac{2\pi N\mathfrak{a}}{\sqrt{D}}\right). \tag{2.14}$$

Now we fix some primitive ideal $\mathfrak{a} \in A$ which will be our representative for this class. This way for any $\mathfrak{b} \in A$ there exists some $\alpha \in K$ such that $\mathfrak{b} = (\alpha)\mathfrak{a}$, and in particular $\alpha \in \mathfrak{a}^{-1} \setminus \{0\}$. On the other hand, given $\alpha \in \mathfrak{a}^{-1} \setminus \{0\}$ we can define $\mathfrak{b} := (\alpha)\mathfrak{a} \in A$. Hence we have the surjection

$$\begin{aligned} \mathfrak{a}^{-1} \setminus \{0\} &\to A \\ \alpha &\mapsto (\alpha)\mathfrak{a} \end{aligned} \tag{2.15}$$

which is w-to-1, where w is the number of units of K. Also note that writing $a = N\mathfrak{a}$ we have $N((\alpha)\mathfrak{a}) = |\alpha|^2 a$. Therefore we have

$$S(y,A) = \frac{1}{w} \sum_{\alpha\in\mathfrak{a}^{-1}} \exp\left(-y\frac{2\pi a|\alpha|^2}{\sqrt{D}}\right) - \frac{1}{w}, \tag{2.16}$$

the extra term $1/w$ corresponding to the zero element in $\mathfrak{a}^{-1}$. Now we use (2.6) with $g(x) = \exp(-\pi y x)$. We note that in this case we have

$$\begin{aligned} h\big(|u+iv|^2\big) &= \iint_{\mathbb{R}^2} g\big(|x_1+ix_2|^2\big) e(-ux_1 - vx_2) dx_1 dx_2 \\ &= \iint_{\mathbb{R}^2} \exp(-\pi y x_1^2)\exp(-\pi y x_2^2) e(-ux_1 - vx_2) dx_1 dx_2 \\ &= \frac{1}{y}\exp\big(-\pi(u/\sqrt{y})^2\big)\exp\big(-\pi(v/\sqrt{y})^2\big), \end{aligned}$$

where we recall that $f(x) = e^{-\pi x^2}$ is self-dual with respect to the Fourier transform. Therefore

$$h\big(|u+iv|^2\big) = \frac{1}{y}\exp\left(-\frac{\pi|u+iv|^2}{y}\right).$$

Hence by using (2.6) with (2.16) we have

$$\begin{aligned} S(y,A) &= \frac{1}{wy}\sum_{\alpha\in\mathfrak{a}^{-1}}\exp\left(-\frac{2\pi a|\alpha|^2}{y\sqrt{D}}\right) - \frac{1}{w} \\ &= \frac{1}{y}S\left(\frac{1}{y},A\right) + \frac{1}{wy} - \frac{1}{w}. \end{aligned} \tag{2.17}$$

Now we go back to (2.14). We have

$$\begin{aligned} \Lambda_K(s,A) &= \int_1^\infty S(y,A)y^s\frac{dy}{y} + \int_0^1 S(y,A)y^s\frac{dy}{y} \\ &= \int_1^\infty S(y,A)y^s\frac{dy}{y} + \int_1^\infty S\left(\frac{1}{y},A\right)y^{-s}\frac{dy}{y} \end{aligned}$$

Then using the functional equation (2.17) for $S(y,A)$ we get

$$\begin{aligned} \Lambda_K(s,A) &= \int_1^\infty S(y,A)(y^s+y^{1-s})\frac{dy}{y} + \frac{1}{w}\int_1^\infty (y-1)y^{-s}\frac{dy}{y} \\ &= \frac{1}{ws(s-1)} + \int_1^\infty S(y,A)(y^s+y^{1-s})\frac{dy}{y}, \end{aligned}$$

and since $S(y,A)$ is exponentially decreasing, the integral is actually absolutely convergent for all $s\in\mathbb{C}\setminus\{0,\,1\}$, so we have obtained an analytic continuation for $\zeta_K(s,A)$.

From here we can add ideal classes as in (2.13). If we let

$$S(y,\chi) := \sum_{\mathfrak{a}}\chi(\mathfrak{a})\exp\left(-\frac{2\pi N\mathfrak{a}}{\sqrt{D}}y\right),$$

then we get the following result.

Theorem 2.7 (Hecke)**.** *We have*

$$\Lambda_K(s,\chi) = \frac{\delta_\chi h}{ws(s-1)} + \int_1^\infty (y^{1-s}+y^s)S(y,\chi)\frac{dy}{y}, \tag{2.18}$$

where $\delta_\chi = 1$ *if* $\chi = 1$*, and* $\delta_\chi = 0$ *otherwise. In particular,* $\Lambda_K(s,\chi)$ *has an analytic continuation to* $\mathbb{C}$*, with a simple pole at* $s=1$ *only if* $\chi = 1$*, and satisfies the functional equation* (2.2)*.*

We can see the residue of $\zeta_K(s)$ from Hecke's formula, so in passing we have also proved the Dirichlet class number formula, remembering that $\zeta_K(s) = \zeta(s)L(s,\chi_D)$.

Corollary 2.8. *We have*

$$\operatorname*{Res}_{s=1} \zeta_K(s) = \frac{2\pi h}{w\sqrt{D}}.$$

2.5 Eisenstein series

There are very interesting connections between class group L-functions and Eisenstein series, which can be decisively exploited by using the equidistribution of Heegner points.

For $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ the modular group, we let

$$E(z,s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\operatorname{Im} \gamma z)^s = \frac{1}{2} \sum\sum_{(m,n)=1} \frac{y^s}{|m+nz|^{2s}} \qquad \text{for } z \in \mathbb{H} \text{ and } \operatorname{Re}(s) > 2.$$

It is known that $E(z,\cdot)$ has a functional equation with factor at infinity given by $\theta(s) := \pi^{-s}\Gamma(s)\zeta(2s)$, so that $E^*(z,s) := \theta(s)E(z,s)$ is the completed Eisenstein series, and it satisfies

$$E^*(z,s) = E^*(z,1-s).$$

This gives a meromorphic continuation of $E(z,s)$ to the complex plane in the s variable. The only pole of $E(z,\cdot)$ for $\operatorname{Re}(s) \geq 1/2$ is at $s=1$, which is a simple pole with residue $3/\pi$. See [6] for proofs about these facts regarding Eisenstein series.

The connection with the class group L-functions comes in the following way. Let A be an ideal class of K, and let $\mathfrak{a} \in A$ be a primitive ideal in that class. Then as we did in (2.15), we have a w-to-1 surjection from $\mathfrak{a}^{-1} \setminus \{0\}$ to A, so that we have

$$\zeta(s,A) = \frac{1}{w} \sum_{\alpha \in \mathfrak{a}^{-1} \setminus \{0\}} \frac{1}{a^s |\alpha|^{2s}} = \frac{1}{w} \sum\sum_{(m,n)\neq 0} \frac{1}{a^s |m+nz_{\mathfrak{a}}|^{2s}} = \frac{2}{w} \left(\frac{\sqrt{D}}{2}\right)^{-s} \zeta(2s) E(z_{\mathfrak{a}}, s),$$

where the $\zeta(2s)$ factor comes from the common divisors of the m and n in the sum. Therefore for any class group character χ we have

$$L_K(s,\chi) = \sum_{A \in \mathcal{H}} \chi(A)\zeta(s,A) = \frac{2}{w} \left(\frac{\sqrt{D}}{2}\right)^{-s} \zeta(2s) \sum_{\mathfrak{a}}{}^{*} \chi(\mathfrak{a}) E(z_{\mathfrak{a}}, s), \tag{2.19}$$

where $\sum_{\mathfrak{a}}^*$ represents a sum over the h reduced primitive ideals of $\mathcal{O}_K$. Moreover, we have the following very nice representation in terms of the completed L-functions

$$\Lambda_K(s,\chi) = \frac{2}{w} \sum_{\mathfrak{a}}{}^{*} \chi(\mathfrak{a}) E^*(z_{\mathfrak{a}}, s),$$

which also implies the functional equation (2.2).

2.6 Explicit formulas for averages

Starting with the relationship between Eisenstein series and the class group L-functions and using the equidistribution of Heegner points in $\Gamma\backslash\mathbb{H}$ as in Theorem 2.2, N. Templier was able to prove in [13] the following very explicit formula for the average of the square of the class group L-functions.

Theorem 2.9 (Templier)**.** *For* $\mathrm{Re}(s) = 1/2$ *we have*

$$\frac{1}{h}\sum_{\chi}|L_K(s,\chi)|^2 = \frac{L(1,\chi_D)}{\zeta(2)}\left[\mathcal{L}_D + \gamma - \log 2 - 2\frac{\zeta'(2)}{\zeta(2)} + 2\,\mathrm{Re}\,\frac{\xi'(2s)}{\xi(2s)}\right]|\zeta(2s)|^2$$
$$+\,\mathrm{Re}\,\frac{\Gamma(s)}{\Gamma(1-s)}\left(\frac{\sqrt{D}}{2\pi}\right)^{2s-1}\frac{\zeta(2s)^3}{\zeta(4s)}L(2s,\chi_D) + O(|s|^A D^{-\delta}), \quad (2.20)$$

where $A, \delta > 0$*, and the implied constant are absolute,* $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$ *is the completed Riemann zeta function, and* $\mathcal{L}_D$ *is given by*

$$\mathcal{L}_D := \frac{1}{2}\log D + \frac{L'(1,\chi_D)}{L(1,\chi_D)}.$$

The average in the left-hand side of (2.20) had already been studied by W. Duke, J. Friedlander, and H. Iwaniec in [3] with similar results, but Templier was able to obtain an improved error term and a completely explicit formula for the right-hand side.

The term $\mathcal{L}_D$ is the largest term in (2.20) with respect to D. In fact, as seen in Section 3 of [13], we have $\mathcal{L}_D \gg \log D$. Furthermore, Templier remarks that assuming the Riemann Hypothesis for $L(s,\chi_D)$ we expect that

$$\mathcal{L}_D = \frac{1}{2}\log D + O(\log\log D).$$

Following the calculations that Templier performed in his proof of Theorem 2.9 we see that we can also get the following explicit formula, which we need to use in the future. We give an idea of the proof and how it uses the equidistribution of Heegner points, but we defer the main calculations to [13].

Theorem 2.10. *For* $s_1 \neq s_2$ *on the critical line with* $s_1 \neq 1-s_2$ *and* $s_i \neq 1/2$*, we have*

$$\frac{1}{h}\sum_{\chi}\Lambda_K(s_1,\chi)\Lambda_K(s_2,\chi) = \frac{2}{w}\sum_{u_1\in\{s_1,1-s_1\}}\ \sum_{u_2\in\{s_2,1-s_2\}}\frac{\theta(u_1)\theta(u_2)}{\theta(u_1+u_2)}\Lambda_K(u_1+u_2)$$
$$+\,O(|s_1s_2|^A|\Gamma(s_1)\Gamma(s_2)|hD^{-\delta}), \quad (2.21)$$

where $A, \delta > 0$ and the implied constant are absolute.

Sketch of proof. We will actually study the average with the L-functions $L_K(s,\chi)$ instead of the completed L-functions. By (2.19) we have

$$\frac{w^2}{4}\left(\frac{\sqrt{D}}{2}\right)^{s_1+s_2}\frac{1}{h^2}\sum_{\chi} L_K(s_1,\chi)L_K(s_2,\chi) = \frac{1}{h}\sum_{\mathfrak{a}}{}^{*}\,\zeta(2s_1)\zeta(2s_2)E(z_{\mathfrak{a}},s_1)E(z_{\mathfrak{a}},s_2). \tag{2.22}$$

The problem here is that the function $z \mapsto E(z,s_1)E(z,s_2)$ that appears in the right-hand side does not satisfy the conditions of Theorem 2.2 for us to use the equidistribution of Heegner points. Templier solved this problem by regularizing this function in the proof of Proposition 5.1 of [13]. There, he shows that the function

$$z \mapsto \zeta(2s_1)\zeta(2s_2)E(z,s_1)E(z,s_2) - B(s_1,s_2,z)$$

satisfies the conditions of Theorem 2.2 with zero integral, implied constants depending on s_1 and s_2 only polynomially, and where

$$\begin{aligned} B(s_1,s_2,z) := {}& \zeta(2s_1)\zeta(2s_2)E(s_1+s_2,z) \\ &+ \pi^{2s_2-1}\frac{\Gamma(1-s_2)}{\Gamma(s_2)}\zeta(2s_1)\zeta(2-2s_2)E(1+s_1-s_2,z) \\ &+ \pi^{2s_1-1}\frac{\Gamma(1-s_1)}{\Gamma(s_1)}\zeta(2-2s_1)\zeta(2s_2)E(1+s_2-s_1,z) \\ &+ \pi^{2s_1+2s_2-2}\frac{\Gamma(1-s_1)\Gamma(1-s_2)}{\Gamma(s_1)\Gamma(s_2)}\zeta(2-2s_1)\zeta(2-2s_2)E(2-s_1-s_2,z). \end{aligned}$$

But then by Theorem 2.2, we have

$$\frac{1}{h}\sum_{\mathfrak{a}}{}^{*}\,\zeta(2s_1)\zeta(2s_2)E(z_{\mathfrak{a}},s_1)E(z_{\mathfrak{a}},s_2) = \frac{1}{h}\sum_{\mathfrak{a}}{}^{*}\,B(s_1,s_2,z_{\mathfrak{a}}) + O(|s_1s_2|^{A}D^{-\delta}).$$

Then we perform the sum on the right-hand side by using (2.19) with $\chi = 1$. Using this in (2.22) and multiplying the resulting equation by $\pi^{-s_1}\Gamma(s_1)\pi^{-s_2}\Gamma(s_2)$ to complete the L-functions we get the result. □

Chapter 3

The Approximate Functional Equation

3.1 Introduction

In the future we want to work with the class group L-function $L_K(s,\chi)$ on the critical line $\mathrm{Re}(s) = 1/2$. Therefore we need a nice expression for this L-function on the critical line so that we can easily, for instance, use the orthogonality of the characters χ in an explicit and meaningful way. This will be accomplished by extensive use of the approximate functional equation (3.1). The proof is the same as the proof of Theorem 5.3 from [8] but we repeat it here for completeness.

Theorem 3.1. *Let χ be a class group character, and let $G(u)$ be a holomorphic, bounded and even function on the strip $-4 < \mathrm{Re}(u) < 4$ such that $G(0) = 1$. Then for s in the strip $0 < \mathrm{Re}(s) < 1$ we have*

$$L_K(s,\chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} V_s\left(\frac{N\mathfrak{a}}{\sqrt{D}}\right) + \varepsilon(s) \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^{1-s}} V_{1-s}\left(\frac{N\mathfrak{a}}{\sqrt{D}}\right) + R_\chi(s), \tag{3.1}$$

where we let

$$V_s(y) := \frac{1}{2\pi i}\int_{(3)} (2\pi y)^{-u} \frac{G(u)}{u}\frac{\Gamma(s+u)}{\Gamma(s)}du$$

and we have

$$\varepsilon(s) := \frac{L_\infty(1-s)}{L_\infty(s)}, \qquad \text{and} \qquad R_1(s) := \frac{h}{wL_\infty(s)}\left[\frac{G(s)}{s} + \frac{G(1-s)}{1-s}\right],$$

and $R_\chi(s) = 0$ if $\chi \neq 1$.

Proof. Let

$$I_K(s,\chi) := \frac{1}{2\pi i}\int_{(3)} \Lambda_K(s+u,\chi)\frac{G(u)}{u}du.$$

Then $I_K(s,\chi)$ converges absolutely because $\Lambda_K(s,\chi)$ is exponentially decreasing on vertical lines (because of the gamma factor). Then moving the integral to the line

$\mathrm{Re}(u) = -3$ and using the functional equation (2.2) we get

$$I_K(s, \chi) = \Lambda_K(s, \chi) - L_\infty(s) R_\chi(s) - I_K(1 - s, \chi), \tag{3.2}$$

where the terms $\Lambda_K(s, \chi)$ and $L_\infty(s) R_\chi(s)$ come from the residues of the integrand at the point $u = 0$, and when $\chi = 1$ also at $u = -s$ and $u = 1 - s$. Note that (2.18) gives the residue of $\Lambda_K(s, 1)$ at $s = 1$ and $s = 0$.

Now, we also have

$$\begin{aligned} I_K(s, \chi) &= \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} \frac{1}{2\pi i} \int_{(3)} \left(\frac{\sqrt{D}}{2\pi} \right)^{s+u} \Gamma(s+u) N\mathfrak{a}^{-u} \frac{G(u)}{u} du \\ &= L_\infty(s) \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} V_s \left(\frac{N\mathfrak{a}}{\sqrt{D}} \right), \end{aligned}$$

and applying this to (3.2) we get the result. □

Our choice for $G(u)$ will be $G(u) = \exp(u^2)$, which has exponential decay over vertical lines and satisfies the required properties. The meaning of $V_s(y)$ in the sums is explained with the following result, which will be proved in Section 3.4.

Proposition 3.2. *Let $s = \frac{1}{2} + it$ with $2 \leq T \leq t \leq 2T$. Then for and $y > 0$ and $A > 0$ we have*

$$V_s(y) = 1 + O\big((y/T)^A + (y/T)^{1/4} T^{-1}\big) \tag{3.3}$$

and

$$V_s(y) \ll (y/T)^{-A}, \tag{3.4}$$

where the implied constants depend only on A.

This way, the value of $V_s(y)$ for $y \gg T$ is very small, so the sums in (3.1) are in practice limited to $N\mathfrak{a} \ll T\sqrt{D}$.

We will also need bound for certain integrals of $V_{\frac{1}{2}+it}(y)$ with respect to t. First we will smooth the integration. For $T \geq 2$, we let $\varphi(t)$ be a smooth function such that

$$\varphi(t) = \begin{cases} 1, & \text{if } t \in [T+1, 2T-1] \\ 0, & \text{if } t \notin (T, 2T), \end{cases}$$

and $0 \le \varphi(t) \le 1$ otherwise. This way we choose $\varphi(t)$ depending on T, but with the condition that $\int |\varphi^{(k)}(t)|dt$ does not depend on T for $k \ge 1$. The integrals we will need and their respective bounds are given in the following result.

Proposition 3.3. *Let $X \neq 1$, and $\sigma \neq 0$, $\sigma > -1/2$. Then for any $k \ge 1$ we have*

$$\int_T^{2T} \varphi(t) X^{it} V_{\frac{1}{2}+it}(y) dt \ll \frac{\delta_\sigma + y^{-\sigma} T^\sigma}{|\log X|^k}, \tag{3.5}$$

$$\int_T^{2T} \varphi(t) X^{it} \frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{1}{2}+it)} V_{\frac{1}{2}-it}(y) dt \ll \frac{1}{|\log X|^k} (\delta_\sigma + y^{-\sigma} T^{\sigma+1} \log^k T), \tag{3.6}$$

where $\delta_\sigma = 1$ if $\sigma < 0$, and $\delta_\sigma = 0$ if $\sigma > 0$. The implied constant depends on σ and k.

This way, the remainder of this chapter will be devoted to the proof of Propositions 3.2 and 3.3. To prove these bounds we will use repeated integration by parts and integrate derivatives of $\Gamma(s+u)/\Gamma(s)$, $s = \frac{1}{2} + it$, with respect to t. Therefore we first take some time to prove properties of quotients of Γ functions.

3.2 The polygamma functions

In the next section we will take several derivatives of quotients of $\Gamma(s)$, so naturally the polygamma functions $\psi^{(n)}(z)$ will appear. We take some time here to recall their definition and state a few of their properties.

The *polygamma function of order n* is defined by

$$\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(z) \qquad \text{for} \qquad n \ge 0.$$

This way $\psi^{(n)}(z)$ is a meromorphic function with poles of order $n+1$ at the nonpositive integers. For the polygamma function of order 0 we have

$$\psi^{(0)}(z) \ll \log |z| \qquad \text{for } \operatorname{Re}(z) > 0,$$

and we also have the integral representation

$$\psi^{(0)}(z) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}} \right) dt \qquad \text{for} \qquad \operatorname{Re}(z) > 0,$$

see Section 8.36 of [4]. Therefore taking derivatives we have for $n \ge 1$ and $\operatorname{Re}(z) > 0$ that

$$\psi^{(n)}(z) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-zt}}{1-e^{-t}} dt.$$

From here doing a substitution $u := |z|t$ we see that

$$\psi^{(n)}(z) \ll \frac{1}{|z|^n}, \qquad \text{for} \qquad n \geq 1 \text{ and } \operatorname{Re}(z) > 0.$$

These bounds are good enough for the applications that we have in mind in the next section.

3.3 Estimates of quotients of $\Gamma(s)$ and derivatives

Here we write $s = \frac{1}{2} + it$, and think of u as a variable with $|u|$ much smaller than $|s|$. This is all we need because in the definition of $V_s(y)$, the function $G(u)$ is exponentially decreasing over vertical lines, so in effect u in the contour integral is very small, while $T \leq t \leq 2T$, so $|s|$ is very large. So let us write $u = \sigma + i\tau$ with $\sigma > -1/2$, and assume that $|u| \leq |s|^{1/2}$.

Lemma 3.4. *For s and u as above, we have*

$$\frac{\Gamma(s+u)}{\Gamma(s)} = (it)^u \left[1 + O\left(\frac{|u|^4+1}{|s|}\right)\right].$$

Proof. We start with Stirling's formula to get

$$\begin{aligned}
\frac{\Gamma(s+u)}{\Gamma(s)} &= \frac{(s+u)^{-1/2}}{s^{-1/2}} \left(\frac{s+u}{e}\right)^{s+u} \left(\frac{e}{s}\right)^s \frac{1+O(|s+u|^{-1})}{1+O(|s|^{-1})} \\
&= \left(1+\frac{u}{s}\right)^{s+u-\frac{1}{2}} \left(\frac{s}{e}\right)^u \left[1+O\left(\frac{|u|^2+1}{|s|}\right)\right] \\
&= \left(\frac{s}{e}\right)^u \exp\left[\left(s+u-\frac{1}{2}\right)\log\left(1+\frac{u}{s}\right)\right] \cdot \left[1+O\left(\frac{|u|^2+1}{|s|}\right)\right].
\end{aligned}$$

Now, using the Taylor expansion of $\log(1+z)$ we have

$$\left(s+u-\frac{1}{2}\right)\log\left(1+\frac{u}{s}\right) = \left(s+u-\frac{1}{2}\right)\left[\frac{u}{s} - \frac{u^2}{2s^2} + O\left(\frac{|u|^3}{|s|^3}\right)\right] = u + O\left(\frac{|u|^4+1}{|s|}\right).$$

Then since $e^{O(\varepsilon)} = 1 + O(\varepsilon)$ we get

$$\frac{\Gamma(s+u)}{\Gamma(s)} = s^u \cdot \left[1+O\left(\frac{|u|^4+1}{|s|}\right)\right]$$

and from this we get the desired result, since

$$\frac{s^u}{(it)^u} = \left(1+\frac{1}{i2t}\right)^u = \exp\left[u\log\left(1+\frac{1}{i2t}\right)\right] = \exp\left[O\left(\frac{|u|}{|s|}\right)\right] = 1+O\left(\frac{|u|}{|s|}\right). \qquad \square$$

Lemma 3.5. *For $k \geq 1$, and s and u as above, we have*

$$\frac{d^k}{dt^k}\left(\frac{\Gamma(s+u)}{\Gamma(s)}\right) \ll |s|^{\sigma-k}(|u|^{k+4}+1),$$

with the implied constant depending on k.

Proof. Note that

$$\frac{d}{dt}\left(\frac{\Gamma(s+u)}{\Gamma(s)}\right) = i\big[\psi^{(0)}(s+u)-\psi^{(0)}(s)\big]\frac{\Gamma(s+u)}{\Gamma(s)}, \tag{3.7}$$

Then taking more derivatives we have

$$\frac{d^k}{dt^k}\left(\frac{\Gamma(s+u)}{\Gamma(s)}\right) = \frac{\Gamma(s+u)}{\Gamma(s)}\Big\{i\big[\psi^{(k-1)}(s+u)-\psi^{(k-1)}(s)\big]+\cdots+i^k\big[\psi^{(0)}(s+u)-\psi^{(0)}(s)\big]^k\Big\}. \tag{3.8}$$

Then for $m \geq 0$ we have

$$\psi^{(m)}(s+u)-\psi^{(m)}(s) \ll |u|\psi^{(m+1)}(s) \ll \frac{|u|}{|s|^{m+1}}.$$

Using this and Lemma 3.4 in (3.8) we get the result. □

Lemma 3.6. *For $T \leq t \leq 2T$ and $k \geq 0$ we have*

$$\frac{d^k}{dt^k}\left(\frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{1}{2}+it)}\right) \ll \log^k T,$$

with the implied constant depending on k.

Proof. In the same spirit as in the proof of Lemma 3.5, we take the derivative

$$\frac{d}{dt}\left(\frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{1}{2}+it)}\right) = (-i)\left[\psi^{(0)}\left(\frac{1}{2}+it\right)+\psi^{(0)}\left(\frac{1}{2}-it\right)\right]\frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{1}{2}+it)},$$

so that taking more derivatives we see that

$$\begin{aligned}\frac{d^k}{dt^k}\left(\frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{1}{2}+it)}\right) = \frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{1}{2}+it)}\Bigg\{&(-i)\left[\psi^{(k-1)}\left(\frac{1}{2}+it\right)+\psi^{(k-1)}\left(\frac{1}{2}-it\right)\right]\\ &+\cdots+(-i)^k\left[\psi^{(0)}\left(\frac{1}{2}+it\right)+\psi^{(0)}\left(\frac{1}{2}-it\right)\right]^k\Bigg\}.\end{aligned}$$

So to conclude we note that if $s=\frac{1}{2}+it$ we have $|\Gamma(1-s)/\Gamma(s)|=1$, and $\psi^{(0)}(s) \ll \log T$. □

We will also need at least one simple bound for u and s without constraints on u and s.

Lemma 3.7. *Let $s = \frac{1}{2} + it$ with $T \le t \le 2T$, let $\mathrm{Re}(u) = \sigma > -1/2$, and let $k \ge 0$. Then we have*

$$\frac{d^k}{dt^k}\left(\frac{\Gamma(s+u)}{\Gamma(s)}\right) \ll T^\sigma \log^k T \exp\left(\frac{\pi}{2}|u|\right)\log^k|u|,$$

the implied constant depending on k and σ.

Proof. Using Stirling's formula we have

$$\frac{\Gamma(s+u)}{\Gamma(s)} \ll |s+u|^\sigma \exp\left(\frac{\pi}{2}(|s|-|s+u|)\right) \ll (|s|+3)^\sigma \exp\left(\frac{\pi}{2}|u|\right).$$

Using this and the bounds for the polygamma functions, this result follows from (3.8).

□

3.4 Proof of Proposition 3.2

Let us prove (3.3) first. We move the line of integration in the definition of $V_s(y)$ to the line $\mathrm{Re}(u) = -1/4$, going over the pole at $u = 0$ to get

$$\begin{aligned} V_s(y) &= 1 + \frac{1}{2\pi i}\int_{(-1/4)} (2\pi y)^{-u}\frac{G(u)}{u}\frac{\Gamma(s+u)}{\Gamma(s)}du \\ &= 1 + \frac{1}{2\pi i}\int_{\frac{-1}{4}-iU}^{\frac{-1}{4}+iU} (2\pi y)^{-u}\frac{G(u)}{u}\frac{\Gamma(s+u)}{\Gamma(s)}du + O(e^{-U^2+\frac{\pi}{2}U}y^{1/4}t^{-1/4}), \end{aligned}$$

where we used Lemma 3.7 to truncate the integral, and the exponential decay in the error term comes from the exponential decay of $G(u)$. We choose U to satisfy $e^{-U^2+\frac{\pi}{2}U} \ll T^{-1}$, but such that U is still small in comparison to T, say $U \le T^{1/2}$. Now, we use Lemma 3.4 to get

$$V_s(y) - 1 = \frac{1}{2\pi i}\int_{\frac{-1}{4}-iU}^{\frac{-1}{4}+iU}\left(\frac{it}{2\pi y}\right)^u \frac{G(u)}{u}du + O(T^{-1}y^{1/4}T^{-1/4}).$$

Finally, moving the integral to the line $[-A-iU, -A+iU]$ we get the required bound for $V_s(y)-1$. Note that the integrals over the horizontal lines $[-A \pm iU, -\frac{1}{4} \pm iU]$ are negligible since over these lines we have $G(u) \ll \exp(-U^2)$.

The bound (3.4) can be obtained in a similar fashion by moving the integration in the definition of $V_s(y)$ to the line $\mathrm{Re}(u) = A$ instead. □

3.5 Proof of Proposition 3.3

Let $s = \frac{1}{2} + it$ with $T \le t \le 2T$. Note that we can move the integral in the definition of $V_s(y)$ to the line $\mathrm{Re}(u) = \sigma$ and we get

$$V_s(y) = \delta_\sigma + \frac{1}{2\pi i}\int_{(\sigma)} (2\pi y)^{-u}\frac{\Gamma(s+u)}{\Gamma(s)}\frac{G(u)}{u}du,$$

where δ_σ comes from taking the residue at the pole $u = 0$ when $\sigma < 0$. Let $l \ge 0$. We truncate this contour integral using Lemma 3.7 to get

$$\frac{d^l}{dt^l}\big[V_s(y) - \delta_\sigma\big] = \frac{1}{2\pi i}\int_{\sigma - iU}^{\sigma + iU} (2\pi y)^{-u}\frac{G(u)}{u}\frac{d^l}{dt^l}\left(\frac{\Gamma(s+u)}{\Gamma(s)}\right)du$$
$$+ O(y^{-\sigma}e^{-U^2+\pi U}T^\sigma \log^l T).$$

We choose U such that $e^{-U^2+\pi U} \le T^{-l-1}$ and $U \le T^{1/2}$ so that by Lemma 3.5 we get

$$\frac{d^l}{dt^l}\big[V_s(y) - \delta_\sigma\big] \ll y^{-\sigma}T^{\sigma - l}. \tag{3.9}$$

Therefore

$$\frac{d^k}{dt^k}\big[\varphi(t)V_s(y)\big] = \sum_{l=0}^{k}\binom{k}{l}\varphi^{(k-l)}(t)\frac{d^l}{dt^l}V_s(y) \ll |\varphi^{(k)}(t)|\delta_\sigma + \sum_{l=0}^{k}|\varphi^{(k-l)}(t)|y^{-\sigma}T^{\sigma-l}.$$

Then finally, by repeated integration by parts we have

$$\int_T^{2T}\varphi(t)X^{it}V_s(y)dt = \frac{1}{(i\log X)^k}\int_T^{2T}X^{it}\frac{d^k}{dt^k}\big[\varphi(t)V_s(y)\big]dt$$
$$\ll \frac{1}{|\log X|^k}\left(\delta_\sigma + y^{-\sigma}\sum_{l=0}^{k-1}T^{\sigma-l} + y^{-\sigma}T^{\sigma-k+1}\right) \ll \frac{\delta_\sigma + y^{-\sigma}T^\sigma}{|\log X|^k}.$$

This finishes the proof of (3.5).

The proof of (3.6) is very similar. We have by (3.9) and Lemma 3.6 that

$$\frac{d^l}{dt^l}\left\{\frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{1}{2}+it)}[V_{1-s}(y) - \delta_\sigma]\right\} = \sum_{j=0}^{l}\binom{l}{j}\frac{d^j}{dt^j}\left[\frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{1}{2}+it)}\right]\cdot\frac{d^{l-j}}{dt^{l-j}}[V_{1-s}(y) - \delta_\sigma]$$
$$\ll y^{-\sigma}T^\sigma\log^l T$$

so that

$$\frac{d^k}{dt^k}\left[\varphi(t)\frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{1}{2}+it)}V_{1-s}(y)\right] \ll |\varphi^{(k)}(t)|\delta_\sigma + \sum_{l=0}^{k}|\varphi^{(k-l)}(t)|y^{-\sigma}T^\sigma\log^l T.$$

Hence integrating by parts we get

$$\int_T^{2T} X^{it}\varphi(t)\frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{1}{2}+it)}V_{\frac{1}{2}-it}(y)dt \ll \frac{1}{|\log X|^k}\left(\delta_\sigma + y^{-\sigma}T^{\sigma+1}\log^k T\right)$$

which finishes the proof of (3.6). □

Chapter 4

Critical Zeros

4.1 Introduction

Now we move to the problem of counting critical zeros of class group L-functions. We proceed by adapting the method that Hardy and Littlewood used in [5] to show that if $N_0(T)$ is the number of critical zeros of the Riemann zeta function, then $N_0(T) \gg T$, see Chapter 24 of [8].

Let χ be a class group character of $K = \mathbb{Q}(\sqrt{-D})$, and define

$$f_\chi(t) := \frac{L_\infty(\frac{1}{2}+it)}{|L_\infty(\frac{1}{2}+it)|} L_K\left(\frac{1}{2}+it, \chi\right) = \frac{\Lambda_K(\frac{1}{2}+it, \chi)}{|L_\infty(\frac{1}{2}+it)|}.$$

Because of the functional equation, the completed L-function $\Lambda_K(s,\chi)$ is real-valued on the critical line, so the function $f_\chi(t)$ is real-valued. We replace the goal of counting zeros of $L_K(s,\chi)$ with counting the sign changes of $f_\chi(t)$. For each sign change of $f_\chi(t)$ in a given interval, there must be at least one zero of $f_\chi(t)$ in that interval, and therefore, a zero of $L_K(s,\chi)$ on the critical line. On the other hand, we can determine if $f_\chi(t)$ changes sign on a given interval by comparing the integral of $f_\chi(t)$ with the integral of $|f_\chi(t)|$ on that interval. If these two integrals have different absolute values, then there must be a sign change of $f_\chi(t)$ in that interval.

However, we do not suceed by analyzing one $f_\chi(t)$ in isolation. Instead, we consider an average over all the $f_\chi(t)$ for varying characters χ, over a fixed field K. This averaging will allow us to use more tools such as the orthogonality of characters and the equidistribution of Heegner points to get better control of the integrals we need to estimate.

We also keep in mind that our goal is to obtain a result that is as uniform as possible over D. This way we want to study an interval of length relatively fixed relative to D, and D will be much larger. In essence, we think of D as going to infinity while the interval remains constant.

Fix $T > 2$. The interval over which we will integrate $f_\chi(t)$ and its absolute value will be $[T, 2T]$. Furthermore, we add a smoothing to the integration by choosing $\varphi(t)$ to be a smooth function such that

$$\varphi(t) = \begin{cases} 1, & \text{if } t \in [T+1, 2T-1] \\ \\ 0, & \text{if } t \notin (T, 2T), \end{cases}$$

and $0 \leq \varphi(t) \leq 1$ otherwise. So we choose $\varphi(t)$ depending on T, but such that $\int |\varphi^{(k)}(t)| dt$ does not depend on T for any $k \geq 1$. Then we let

$$I(\chi) := \int_T^{2T} \varphi(t) f_\chi(t) dt, \qquad \text{and} \qquad J(\chi) := \int_T^{2T} \varphi(t) |f_\chi(t)| dt$$

be the two integrals we will compare to detect the sign changes of $f_\chi(t)$. In fact, if we can show that $|I(\chi)| < J(\chi)$, then $f_\chi(t)$ must have a sign change between T and $2T$. But again, we do not analyze $I(\chi)$ and $J(\chi)$ individually.

4.2 The main result

The estimations that we will obtain for the integrals $J(\chi)$ and $I(\chi)$ will be the following.

Lemma 4.1. *Let $B > 0$. For any class group character χ we have*

$$J(\chi) \geq T - K(\chi),$$

where $K(\chi) \geq 0$ satisfies

$$\frac{1}{h} \sum_\chi K(\chi) \ll 1 \qquad \textit{for} \qquad T \leq \log^B D,$$

and the implied constant depends only on B.

Lemma 4.2. *Let $A, B > 0$. Then for $T \leq \log^B D$ we have*

$$\frac{1}{h} \sum_\chi |I(\chi)|^2 \ll L(1, \chi_D) T + \log^{-A} D,$$

with the implied constant depending only on A and B.

We will prove these two lemmas in the following sections with slightly more precise results, see Propositions 4.4 and 4.5. From them, we conclude the following Theorem, which is the main result of this text.

Theorem 4.3. *Let $0 < p < 1$ and $B > 0$. Assume that $T \leq \log^B D$ and that*

$$T \geq C\left[L(1, \chi_D) + 1\right],$$

where $C = C(p, B)$ is a constant that depends only on p and B. Then of the h class group L-functions $L_K(s, \chi)$, at least ph of them has a critical zero of imaginary part between T and $2T$.

Proof. Let $\varepsilon := 1 - p > 0$ and let $N \leq h$ be the number of L-functions $L_K(s, \chi)$ that do *not* have a critical zero of imaginary part between T and $2T$. Assuming that $C[L(1, \chi_D) + 1] \leq T \leq \log^B D$, we want to prove that $N/h \leq \varepsilon$ for large enough C. For a class group character χ, let

$$\delta(\chi) := \begin{cases} 1, & \text{if } |I(\chi)| = J(\chi), \\ 0, & \text{otherwise.} \end{cases}$$

This way, $\sum_\chi \delta(\chi) = N$. Also, if $\delta(\chi) = 1$ we have $T - K(\chi) \leq J(\chi) = |I(\chi)|$, so no matter the value of $\delta(\chi)$ we have

$$\delta(\chi)T \leq |I(\chi)| + K(\chi)$$

for any character χ. So on average over χ we have

$$T \cdot \frac{1}{h}\sum_\chi \delta(\chi) \leq \frac{1}{h}\sum_\chi \left[|I(\chi)| + K(\chi)\right] \leq \left(\frac{1}{h}\sum_\chi |I(\chi)|^2\right)^{1/2} + \frac{1}{h}\sum_\chi K(\chi)$$

Therefore by Lemmas 4.1 and 4.2 we have

$$N/h \ll \left(T^{-1}L(1, \chi_D) + T^{-2}\right)^{1/2} + T^{-1} \ll C^{-1/2}. \qquad \square$$

4.3 Estimating the $J(\chi)$

We start on the way to the proof of Lemma 4.1 by estimating $J(\chi)$ from below. We have

$$\begin{aligned} J(\chi) &= \int_T^{2T} \varphi(t) \left| L_K\left(\frac{1}{2}+it,\chi\right)\right| dt \geq \left| \int_T^{2T} \varphi(t) L_K\left(\frac{1}{2}+it,\chi\right) dt \right| \\ &\geq T - \left| \int_T^{2T} \varphi(t)\left[L_K\left(\frac{1}{2}+it,\chi\right)-1\right] dt\right| =: T - K(\chi), \end{aligned}$$

so now we need an upper bound for $K(\chi)$. We use (3.1) to write

$$K(\chi) \leq K_0 + K_1(\chi) + K_2(\chi) + \left| \int_T^{2T} \varphi(t) R_\chi\left(\frac{1}{2}+it\right) dt\right|, \tag{4.1}$$

where

$$\begin{aligned} K_0 &:= \left| \int_T^{2T} \left[\varphi(t) V_{\frac{1}{2}+it}\left(\frac{1}{\sqrt{D}}\right) - 1\right] dt \right|, \\ K_1(\chi) &:= \left| \int_T^{2T} \varphi(t) \sum_{N\mathfrak{a}>1} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^{\frac{1}{2}+it}} V_{\frac{1}{2}+it}\left(\frac{N\mathfrak{a}}{\sqrt{D}}\right) dt \right|, \quad \text{and} \\ K_2(\chi) &:= \left| \int_T^{2T} \varphi(t) \varepsilon\left(\frac{1}{2}+it\right) \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^{\frac{1}{2}-it}} V_{\frac{1}{2}-it}\left(\frac{N\mathfrak{a}}{\sqrt{D}}\right) dt \right|. \end{aligned}$$

Now we estimate each term separately, but $K_1(\chi)$ and $K_2(\chi)$ only on average over all class group characters χ.

By (3.9) with $\sigma = -\varepsilon < 0$ we have

$$\begin{aligned} K_0 &\ll 1 + \left| \int_T^{2T} \varphi(t) \left[V_{\frac{1}{2}+it}\left(\frac{1}{\sqrt{D}}\right) - 1\right] dt \right| \ll 1 + T\left(\frac{1}{\sqrt{D}}\right)^{\varepsilon} T^{-\varepsilon} \\ &\ll 1 + \frac{T}{T^{\varepsilon} D^{\varepsilon/2}}, \end{aligned}$$

and the integral on $R_\chi(\frac{1}{2}+it)$ when $\chi = 1$ is exponentially decreasing in T since $G(s)$ is exponentially decreasing over the line $\mathrm{Re}(s) = 1/2$.

Now we estimate $K_1(\chi)$ and $K_2(\chi)$.

Proposition 4.4. *Assume that $T \leq e^{\sqrt{\log D}}$. Then for any $k \geq 0$ we have*

$$\frac{1}{h}\sum_{\chi} K_1(\chi)^2 \ll 1 + \frac{T}{\log^k D}, \qquad \text{and} \qquad \frac{1}{h}\sum_{\chi} K_2(\chi)^2 \ll \frac{T^3}{\log^k D}, \tag{4.2}$$

and this implies that

$$\frac{1}{h}\sum_{\chi} K(\chi) \ll 1 + \frac{T^{3/2}}{\log^k D}, \tag{4.3}$$

with the implied constants depending only on k.

Proof. Note that (4.3) follows from (4.2) by using the Cauchy-Schwarz inequality to the average of (4.1) over characters χ, so now we only need to prove (4.2).

Let us prove the bound concerning $K_1(\chi)$. We write

$$K_1(\chi) = \left| \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^{1/2}} W_1(N\mathfrak{a}) \right|,$$

where $W_1(1) := 0$ and for $n > 1$ we let

$$W_1(n) := \int_T^{2T} \varphi(t) n^{-it} V_{\frac{1}{2}+it}\left(\frac{n}{\sqrt{D}}\right) dt.$$

Then on average over χ we use the orthogonality of the class group characters to get

$$\begin{aligned} \frac{1}{h}\sum_{\chi} K_1(\chi)^2 = \sum_{\mathfrak{a}\sim\mathfrak{b}} \frac{W_1(N\mathfrak{a})\overline{W_1(N\mathfrak{b})}}{N\mathfrak{a}^{1/2}N\mathfrak{b}^{1/2}} &= \sum_{\mathfrak{a}}{}^{*} \sum_{\mathfrak{b}_1\sim\mathfrak{a}} \sum_{\mathfrak{b}_2\sim\mathfrak{a}} \frac{W_1(N\mathfrak{b}_1)\overline{W_1(N\mathfrak{b}_2)}}{N\mathfrak{b}_1^{1/2}N\mathfrak{b}_2^{1/2}} \\ &= \frac{1}{w^2} \sum_{\mathfrak{a}}{}^{*} \left| \sum_{\alpha\in\mathfrak{a}^{-1}\setminus\{0\}} \frac{W_1(a|\alpha|^2)}{\sqrt{a}|\alpha|} \right|^2, \end{aligned} \tag{4.4}$$

where we used the w-to-1 surjection (2.15) from $\mathfrak{a}^{-1} \setminus \{0\}$ to the ideal class $[\mathfrak{a}]$. Remember that $\sum_{\mathfrak{a}}^{*}$ denotes a sum over the h primitive reduced ideals, and $a = N\mathfrak{a}$. So now we estimate the inner sum of (4.4).

By (3.5) with $\sigma = -\varepsilon < 0$ and $\sigma = C > 0$ we have

$$W_1(n) \ll \frac{1}{\log^{2k} n}\left[1 + \left(\frac{n}{T\sqrt{D}}\right)^{\varepsilon}\right], \qquad \text{and} \tag{4.5}$$

$$W_1(n) \ll \frac{1}{\log^{2k} n}\left(\frac{T\sqrt{D}}{n}\right)^{C}. \tag{4.6}$$

We note that (4.5) is better for n small, say $n \le L := T\sqrt{D}$, and (4.6) is better for n large, say $n > L$. This way we split the inner sum of (4.4) into two separate parts.

By (4.6) and Lemma 2.6 we get

$$\sum_{\substack{\alpha\in\mathfrak{a}^{-1} \\ a|\alpha|^2>L}} \frac{W_1(a|\alpha|^2)}{\sqrt{a}|\alpha|} \ll (T\sqrt{D})^C \sum_{|\alpha|^2>L/a} \frac{1}{a^{C+\frac{1}{2}}|\alpha|^{2C+1}\log^k(a|\alpha|^2)} \ll \frac{\sqrt{T}}{D^{1/4}\log^k D}.$$

Now using (4.5) and (2.11) with simple partial summation we have

$$\sum_{\substack{\alpha\in\mathfrak{a}^{-1}\setminus\mathbb{Z} \\ 1<a|\alpha|^2\le L}} \frac{W_1(a|\alpha|^2)}{\sqrt{a}|\alpha|} \ll \sum_{\substack{\alpha\in\mathfrak{a}^{-1}\setminus\mathbb{Z} \\ 1<a|\alpha|^2\le L}} \frac{1}{\sqrt{a}|\alpha|\log^k(a|\alpha|^2)} \ll \frac{\sqrt{T}}{D^{1/4}\log^k D},$$

while for $\alpha = l \in \mathbb{Z}$ we have

$$\sum_{\substack{l\in\mathbb{Z}\\ 1<al^2\le L}} \frac{W_1(al^2)}{\sqrt{a}|l|} \ll \sum_{\substack{l\in\mathbb{Z}\\ 1<al^2\le L}} \frac{1}{\sqrt{a}|l|\log^{2k}(al^2)} \ll \frac{1}{\sqrt{a}\log^k a}.$$

Hence combining these bounds we obtain

$$\sum_{\alpha\in\mathfrak{a}^{-1}\setminus\{0\}} \frac{W_1(a|\alpha|^2)}{\sqrt{a}|\alpha|} \ll \frac{1}{\sqrt{a}\log^k a} + \frac{\sqrt{T}}{D^{1/4}\log^k D}.$$

Finally, applying this to (4.4) while noticing that

$$\sum_{\mathfrak{a}}{}^{*} \left(\frac{1}{\sqrt{a}\log^k a}\right)^2 \ll \sum_n \frac{\tau(n)}{n\log^{2k} n} \ll 1,$$

we get

$$\frac{1}{h}\sum_\chi |K_1(\chi)|^2 \ll 1 + \frac{hT}{\sqrt{D}\log^{2k} D} \ll 1 + \frac{T}{\log^k D},$$

where we recall the bounds (2.1) for the class number.

The proof of the upper bound for $K_2(\chi)$ follows very similar ideas, but we use (3.6) instead of (3.5). We write

$$K_2(\chi) = \left|\sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^{1/2}} W_2(N\mathfrak{a})\right|,$$

where for $n \ge 1$ we define

$$\begin{aligned} W_2(n) &:= \int_T^{T+\Delta} \varphi(t) n^{it} \varepsilon\left(\frac{1}{2}+it\right) V_{\frac{1}{2}-it}\left(\frac{n}{\sqrt{D}}\right) dt \\ &= \int_T^{T+\Delta} \varphi(t) \left(\frac{4\pi^2 n}{D}\right)^{it} \frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{1}{2}+it)} V_{\frac{1}{2}-it}\left(\frac{n}{\sqrt{D}}\right) dt. \end{aligned}$$

Then we use the orthogonality of characters to get

$$\begin{aligned} \frac{1}{h}\sum_\chi K_2(\chi)^2 = \sum_{\mathfrak{a}\sim\mathfrak{b}} \frac{W_2(N\mathfrak{a})\overline{W_2(N\mathfrak{b})}}{N\mathfrak{a}^{1/2}N\mathfrak{b}^{1/2}} &= \sum_{\mathfrak{a}}{}^{*} \sum_{\mathfrak{b}_1\sim\mathfrak{a}} \sum_{\mathfrak{b}_2\sim\mathfrak{a}} \frac{W_2(N\mathfrak{b}_1)\overline{W_2(N\mathfrak{b}_2)}}{N\mathfrak{b}_1^{1/2}N\mathfrak{b}_2^{1/2}} \\ &= \frac{1}{w^2}\sum_{\mathfrak{a}}{}^{*} \left|\sum_{\alpha\in\mathfrak{a}^{-1}\setminus\{0\}} \frac{W_2(a|\alpha|^2)}{\sqrt{a}|\alpha|}\right|^2. \end{aligned} \tag{4.7}$$

So now our goal is to estimate the inner sum in (4.7). If we can prove that

$$\sum_{\alpha\in\mathfrak{a}^{-1}\setminus\{0\}} \frac{W_2(a|\alpha|^2)}{\sqrt{a}|\alpha|} \ll \frac{T^{3/2}}{\log^k D}\left(\frac{1}{\sqrt{a}\log^k a} + \frac{1}{D^{1/4}}\right), \tag{4.8}$$

then we are done as above.

Then by (3.6) for $\sigma = -\varepsilon < 0$ we have

$$W_2(n) \ll \frac{1}{|\log(4\pi^2 n/D)|^{2k}} \left[1 + T\log^{2k} T \left(\frac{n}{T\sqrt{D}}\right)^{\varepsilon}\right], \tag{4.9}$$

which we use when $n \le L := T\sqrt{D}$. Also by (3.4) and (3.6) for $\sigma = C$ large we have

$$W_2(n) \ll T\left(\frac{T\sqrt{D}}{n}\right)^C \tag{4.10}$$

$$W_2(n) \ll \frac{T\log^{2k} T}{|\log(4\pi^2 n/D)|^{2k}} \left(\frac{T\sqrt{D}}{n}\right)^C. \tag{4.11}$$

We will use (4.10) instead of (4.11) when $n > D/4\pi^2 T$ to avoid n for which $\log(4\pi^2 n/D)$ is very small. This way we need to estimate the inner sum in (4.7) a bit more carefully by separating the sum into three intervals. Using (4.9) we have

$$\sum_{\substack{\alpha\in\mathfrak{a}^{-1}\setminus\mathbb{Z}\\ 0<a|\alpha|^2\le L}} \frac{W_2(a|\alpha|^2)}{|\alpha|} \ll \sum_{\substack{\alpha\in\mathfrak{a}^{-1}\setminus\mathbb{Z}\\ 0<a|\alpha|^2\le L}} \frac{T\log^{2k} T}{|\alpha|\log^{2k}(D/4\pi^2 a|\alpha|^2)} \ll \frac{T\log^{2k} T}{\log^{2k} D} \sum_{\substack{\alpha\in\mathfrak{a}^{-1}\setminus\mathbb{Z}\\ 0<a|\alpha|^2\le L}} \frac{1}{|\alpha|},$$

where we just used the fact that, in the sum, $D/4\pi^2 a|\alpha|^2 \ge \sqrt{D}/T \gg D^{1/4}$. Then by partial summation and (2.11) we have

$$\sum_{\substack{\alpha\in\mathfrak{a}^{-1}\setminus\mathbb{Z}\\ 0<a|\alpha|^2\le L}} \frac{W_2(a|\alpha|^2)}{|\alpha|} \ll \frac{T\log^{2k} T}{\log^{2k} D}\frac{a}{\sqrt{D}}\sqrt{\frac{L}{a}} \ll \frac{\sqrt{a}T^{3/2}}{D^{1/4}\log^k D},$$

where we also used the assumption that $T \le e^{\sqrt{\log D}}$. This is exactly compatible with our goal (4.8). For $\alpha = l \in \mathbb{Z}$ we have

$$\sum_{\substack{l\in\mathbb{Z}\\ 0<al^2\le L}} \frac{W_2(al^2)}{\sqrt{a}l} \ll \sum_{\substack{l\in\mathbb{Z}\\ 0<al^2\le L}} \frac{T\log^{2k+2} T}{\sqrt{a}|l|\cdot|\log(4\pi^2 al^2/D)|^{2k+2}} \ll \frac{T\log^{2k+3} T}{\sqrt{a}\log^k a\log^k D},$$

which is also good enough for (4.8).

Now we use (4.11) to get

$$\sum_{\substack{\alpha\in\mathfrak{a}^{-1}\\ L<a|\alpha|^2\le\frac{D}{4\pi^2 T}}} \frac{W_2(a|\alpha|^2)}{|\alpha|} \ll T\log^{2k} T\frac{T^C D^{C/2}}{a^C} \sum_{L<a|\alpha|^2\le\frac{D}{4\pi^2 T}} \frac{1}{|\alpha|^{2C+1}\log^{2k}(D/4\pi^2 a|\alpha|^2)}.$$

Now we apply Lemma 2.6, noting that for T sufficiently large the integrand is strictly decreasing, so we obtain

$$\sum_{\substack{\alpha\in\mathfrak{a}^{-1}\\ L<a|\alpha|^2\le\frac{D}{4\pi^2T}}} \frac{W_2(a|\alpha|^2)}{|\alpha|}$$
$$\ll \log^{2k} T \frac{T^{C+1}D^{C/2}}{a^C}\left[\frac{a}{\sqrt{D}}\left(\frac{L}{a}\right)^{-C+\frac{1}{2}}\frac{1}{\log^{2k} D} + \left(\frac{a}{\sqrt{D}}\frac{L}{a}\right)^{1/3}\left(\frac{L}{a}\right)^{-C-\frac{1}{2}}\frac{1}{\log^{2k} D}\right]$$
$$\ll \frac{\sqrt{a}T^{3/2}}{D^{1/4}\log^k D}.$$

Finally, by (4.10) and Lemma 2.6 we have

$$\sum_{\substack{\alpha\in\mathfrak{a}^{-1}\\ a|\alpha|^2>\frac{D}{4\pi^2T}}} \frac{W_2(a|\alpha|^2)}{|\alpha|} \ll T\frac{T^C D^{C/2}}{a^C}\sum_{\substack{\alpha\in\mathfrak{a}^{-1}\\ a|\alpha|^2>\frac{D}{4\pi^2T}}}\frac{1}{|\alpha|^{2C+1}} \ll T\frac{\sqrt{a}T^{2C-\frac{1}{2}}}{D^{C/2}}$$

so using the assumption $T \ll e^{\sqrt{\log D}}$ this finishes the proof of (4.8). □

4.4 Estimating the $I(\chi)$

The hardest part of the proof of Theorem 4.3 and its required Lemmas is estimating the integral $I(\chi)$. While the previous section shows that we can estimate $J(\chi)$ and $K(\chi)$ in pretty elementary ways, we cannot do so for $I(\chi)$. Here we need to use the equidistribution of Heegner points and the connection between the class group L-functions and Eisenstein series by means of the formula (2.21). However, we obtain the following asymptotic formula for the average of the $|I(\chi)|^2$.

Proposition 4.5. *Let $A, B > 0$. Then for $T \le \log^B D$ we have*

$$\frac{1}{h}\sum_{\chi}|I(\chi)|^2 = \frac{12}{\pi}L(1,\chi_D)\int_T^{2T}\varphi(t)^2|\zeta(1+i2t)|^2dt + O(\log^{-A} D), \tag{4.12}$$

with the implied constant depending only on A and B.

This Proposition immediately gives Lemma 4.2 since

$$\lim_{T\to\infty}\frac{1}{T}\int_1^T |\zeta(1+it)|^2dt = \zeta(2),$$

see Theorem 7.2 from [14].

Note that (4.12) shows that the $L(1,\chi_D)$ term in Theorem 4.3 really cannot be improved within the method of Hardy and Littlewood. It is not the case that we failed to obtain good bounds for $I(\chi)$ to remove the $L(1,\chi_D)$ factor, but instead that $L(1,\chi_D)$ is an essential factor that appears in this method.

Proof. We have

$$I := \frac{1}{h}\sum_{\chi}|I(\chi)|^2 = \int_T^{2T}\int_T^{2T}\varphi(t_1)\varphi(t_2)\frac{1}{h}\sum_{\chi}\frac{\Lambda_K(s_1,\chi)\Lambda_K(s_2,\chi)}{|L_\infty(s_1)L_\infty(s_2)|}dt_1dt_2,$$

where we wrote $s_1 := \frac{1}{2}+it_1$ and $s_2 := \frac{1}{2}+it_2$ for compactness. From here we wish to use (2.21), and to do so we need to remove the diagonal t_1 close to t_2. We remove the integral over the (t_1,t_2) such that $|t_1-t_2|\le\varepsilon$, introducing an error of $O(\varepsilon DT^3)$ by using the convexity bound. Then for the double integral with $|t_1-t_2|>\varepsilon$ we use (2.21) to get

$$I = 2\operatorname{Re}U_1 + 2\operatorname{Re}U_2 + O(\varepsilon DT^3 + D^{-\delta}), \tag{4.13}$$

where

$$U_1 := \iint_{|t_1-t_2|>\varepsilon}\varphi(t_1)\varphi(t_2)Q^{it_1+it_2}\gamma(s_1)\gamma(s_2)\zeta(2s_1)\zeta(2s_2)Z(s_1+s_2)dt_1dt_2,$$

and U_2 is by definition

$$\iint_{|t_1-t_2|>\varepsilon}\varphi(t_1)\varphi(t_2)Q^{it_1-it_2}\gamma(s_1)\gamma(1-s_2)\zeta(2s_1)\zeta(2-2s_2)Z(s_1+1-s_2)dt_1dt_2,$$

with $Q := \sqrt{D}/2\pi$, $\gamma(s) := (\Gamma(s)/\Gamma(1-s))^{1/2} = \Gamma(s)/|\Gamma(s)|$, and $Z(s) := \zeta_K(s)/\zeta(2s)$. We note that with this definition $\gamma(s)$ is holomorphic in the strip $0<\operatorname{Re}(s)<1$. We evaluate the two integrals U_1 and U_2 separately, while U_2 is the hardest one because of the pole of $Z(s)$ at $s=1$.

So let us evaluate U_2. By a change of variables $t := t_1$ and $v := t_2 - t_1$ we have

$$U_2 = \int_T^{2T}\varphi(t)\gamma(s)\zeta(2s)\overline{V(t)}dt, \tag{4.14}$$

where $s := \frac{1}{2}+it$ and

$$V(t) := \int_{|v|>\varepsilon}\varphi(t+v)Q^{iv}\gamma(s+iv)\zeta(2s+i2v)Z(1+iv)dv.$$

The idea from here is to write $V(t)$ as a line integral in the complex plane. However, $\varphi(t)$ is not an analytic function, so instead we represent it by its Fourier transform. To help with the convergence we actually use the Fourier transform of the function $\varphi(t)/\eta(t)$, where $\eta(t) := t(1+t^8)^{-1}$. This way the Fourier transform is

$$\psi(x) := \int_{\mathbb{R}} \frac{\varphi(t)}{\eta(t)} e(xt)dt, \qquad \text{so that} \qquad \varphi(t) = \eta(t) \int_{\mathbb{R}} \psi(x)e(-xt)dx \tag{4.15}$$

by Fourier inversion. We note that $\psi(x)$ is rapidly decreasing as $|x| \to \infty$. By repeated partial integration we have

$$\psi(x) \ll T^8(1+|x|)^{-A} \tag{4.16}$$

for any $A > 0$, the implied constant depending only on A. Using (4.15) in the definition of $V(t)$ and reversing the order of integration we get

$$V(t) = \int_{\mathbb{R}} \psi(x)e(-xt)V(x,t)dx,$$

where

$$V(x,t) := \int_{|v|>\varepsilon} \eta(t+v)(Qe^{-2\pi x})^{iv}\gamma(s+iv)\zeta(2s+i2v)Z(1+iv)dv.$$

Note that the absolute convergence of the integral $V(x,t)$ is guaranteed by our choice of $\eta(t)$. Now we split the integration in $V(t)$ into two parts. Let $X > 0$ to be chosen later, and write $V(t) = V_0(t) + V_\infty(t)$, in which

$$V_0(t) := \int_{|x|<X} \psi(x)e(-xt)V(x,t)dx, \quad \text{and} \quad V_\infty(t) := \int_{|x|\geq X} \psi(x)e(-xt)V(x,t)dx.$$

Let us estimate $V_\infty(t)$ first. We write $\Phi(v) := \eta(t+v)(Qe^{-2\pi x})^{iv}\gamma(s+iv)\zeta(2s+i2v)$ so that

$$V(x,t) = \int_{|v|>\varepsilon} \Phi(v)Z(1+iv)dv,$$

and we want to keep v and $-v$ together to be able to cancel the pole of $Z(s)$ at $s = 1$. To this end, we write

$$V(x,t) = \int_{\varepsilon}^{\infty} \Phi(v)\left[Z(1+iv) + Z(1-iv)\right]dv - \int_{\varepsilon}^{\infty} \left[\Phi(v) - \Phi(-v)\right]Z(1-iv)dv. \tag{4.17}$$

Now, using a simple bound for $\zeta(s)$ on the line $\mathrm{Re}(s) = 1$, say $\zeta(1 + it) \ll \log|t|$ for $|t| > 2$, and canceling the pole of $\zeta(2s + i2v)$ with the zero of $\eta(t + v)$ at $v = -t$, one can verify that

$$\Phi(v) \ll \left[1 + (v + t)^6\right]^{-1}, \qquad \text{and}$$
$$\Phi(v) - \Phi(-v) \ll |v|\left[1 + (v + t)^6\right]^{-1}\big(|x| + \log D\big),$$

while for the function $Z(s)$ we have

$$Z(1 + iv) \ll \frac{v^2 + 1}{|v|}\log D, \qquad \text{and}$$
$$Z(1 + iv) + Z(1 - iv) \ll (|v| + 1)\log D,$$

the $\log D$ terms coming from the bounding $L(s, \chi_D)$ on the line $\mathrm{Re}(s) = 1$. Using the above bounds in (4.17) and then using the substitution $v' := v + t$ with $T \le t \le 2T$ we get

$$V(x, t) \ll T^2\big(|x| + \log D\big)\log D.$$

Going back to the definition of $V_\infty(t)$ using this bound and (4.16) we get

$$V_\infty(t) \ll X^{-A}T^{10}\log^2 D. \tag{4.18}$$

Now we will estimate $V(x, t)$ again but for $|x| < X$, and much more precisely so that we can evaluate the integral $V_0(t)$. Going back to the definition of $V(x, t)$, we write the integral as a line integral over the line $\mathrm{Re}(z) = 0$ with $|z| > \varepsilon$ so that

$$V(x, t) = -\int_{\substack{Re(z)=0,\\ |z|>\varepsilon}} \eta\left(s + z - \frac{1}{2}\right)(Qe^{-2\pi x})^z\gamma(s + z)\zeta(2s + 2z)Z(1 + z)dz \tag{4.19}$$

orientated from $-i\infty$ to $+i\infty$. We will complete this line integral with a semicircle around the origin to avoid the pole, and move the contour to the line $\mathrm{Re}(z) = -\delta < 0$, $\varepsilon < \delta$. Now, for $\mathrm{Re}(z) = -\delta$ we have the bound $L(1 + z, \chi_D) \ll |z|D^{3\delta/8}\log D$, which follows from the convexity principle and D. Burgess' subconvexity bound for $L(s, \chi_D)$ on the critical line, see the end of Section 5.2 of [8]. Therefore, we have

$$\int_{(-\delta)} \eta\left(s + z - \frac{1}{2}\right)(Qe^{-2\pi x})^z\gamma(s + z)\zeta(2s + 2z)Z(1 + z)dz \ll T^2D^{-\delta/8}e^{2\pi x\delta}\log D,$$

where we also use Stirling's formula to bound $\gamma(s+z)$ off the critical line. Therefore it follows from (4.19) that

$$V(x,t) = -\int_{C(\varepsilon)} \eta\left(s+z-\frac{1}{2}\right)(Qe^{-2\pi x})^z\gamma(s+z)\zeta(2s+2z)Z(1+z)dz \\ + O(T^2D^{-\delta/8}e^{2\pi x\delta}\log D),$$

where $C(\varepsilon) := \{z \in \mathbb{C} : |z| = 1 \text{ and } \operatorname{Re}(z) \le 0\}$ and it is traversed counterclockwise around the origin. Now, write

$$Z(1+z) =: \frac{R}{z} + L(z), \qquad \text{where} \qquad R := \operatorname*{Res}_{z=0} Z(1+z) = \frac{6}{\pi^2}L(1,\chi_D).$$

The remaining holomorphic function $L(z)$ together with the rest of the integrand is bounded by $O(e^{2\pi x\varepsilon}DT)$, so that

$$V(x,t) = -R\int_{C(\varepsilon)} \eta\left(s+z-\frac{1}{2}\right)(Qe^{-2\pi x})^z\gamma(s+z)\zeta(2s+2z)\frac{dz}{z} \\ + O(\varepsilon e^{2\pi x\varepsilon}DT) + O(T^2D^{-\delta/8}e^{2\pi x\delta}\log D).$$

Finally, since ε is very small we can approximate z by 0 in every term of the integrand except the polar term $1/z$. The error term obtained by doing this is bounded by the derivatives of the function, and so it is $O(\varepsilon x e^{2\pi x\varepsilon}DT)$ after integration. Therefore we have

$$\begin{aligned} V(x,t) &= -R\eta\left(s-\frac{1}{2}\right)\gamma(s)\zeta(2s)\int_{C(\varepsilon)}\frac{dz}{z} \\ &\qquad + O\big(\varepsilon(|x|+1)e^{2\pi x\varepsilon}DT + T^2D^{-\delta/8}e^{2\pi x\delta}\log D\big) \\ &= -R\eta\left(s-\frac{1}{2}\right)\gamma(s)\zeta(2s)\pi i + O\big(\varepsilon X e^{2\pi X\varepsilon}DT + T^2D^{-\delta/8}e^{2\pi X\delta}\log D\big) \end{aligned}$$

for $|x| < X$, which is the case for estimating $V_0(t)$. Choosing $X := \log D/20\pi$ we get

$$V(x,t) = \pi R\eta(t)\gamma(s)\zeta(2s) + O(\varepsilon D^2T + T^2D^{-\nu})$$

for $|x| < X$ and for some $\nu > 0$. We insert this back into the definition of $V_0(t)$ to get

$$\begin{aligned} V_0(t) &= \pi R\eta(t)\gamma(s)\zeta(2s)\int_{|x|<X}\psi(x)e(-xt)dx + O(\varepsilon XD^2T^9 + XT^{10}D^{-\nu}) \\ &= \pi R\varphi(t)\gamma(s)\zeta(2s) + O(\varepsilon T^9D^2\log D + T^2D^{-\nu}\log D + X^{-A}T^{10}\log D), \end{aligned} \tag{4.20}$$

where we used (4.16) to complete the integral over x and get (4.15).

We add (4.18) and (4.20) and use our assumption that $T \le \log^B D$ to get

$$V(t) = \pi R\varphi(t)\gamma(s)\zeta(2s) + O(\varepsilon T^9 D^2 \log D + \log^{-C} D).$$

Putting this back into (4.14) we get

$$U_2 = \frac{6}{\pi} L(1, \chi_D) \int_T^{2T} \varphi(t)^2 |\zeta(2s)|^2 dt + O(\varepsilon T^{10} D^2 \log D + T \log^{-C} D).$$

So finally we get the result from (4.13) after letting $\varepsilon \to 0$ and noticing that U_1 can be estimated in the same way, but only contributes to the error term since we do not need to avoid a pole in integrating U_1 as we did above for U_2. $\square$

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