YAMABE PROBLEM ON COMPACT MANIFOLDS WITH BOUNDARY

by

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ABSTRACT OF THE DISSERTATION

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We proved the existence of conformal metric with nonzero constant scalar curvature and nonzero constant boundary mean curvature under some natural conditions. We also solved some remaining cases left open by J. Escobar [40]. Furthermore, we establish the compactness of minimizers which led to a partial affirmative answer to the Han-Li conjecture [50]. We also studied one types of Yamabe flow on compact manifolds with boundary, which has mean curvature equals to zero on the boundary. Convergence of flow is established under some conditions. In another work, We studied the classification of nonnegative solutions to polyharmonic functions with conformally invariant boundary conditions. We proved that nonnegative solutions of that elliptic equations have to be of the "*polynomials* plus *bubbles*" form. The presence of a polynomial part is a new phenomenon.

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Dedication

To My Parents.

To My Girlfriend Sai.

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Chapter 1

Introduction

1.1 Conformal metrics with constant scalar curvature and constant boundary mean curvature

It is well known that the Yamabe problem is to find a metric conformal to the background one on a closed compact manifold such that its scalar curvature is constant. This problem was solved by Yamabe, Trudinger, Aubin and Schoen. Analogous questions on manifolds with boundary are raised by many other researchers. Let us fix some notations before stating the questions. Suppose (M, g) is a compact manifold with boundary. R_g is the scalar curvature of M under metric g and h_g the mean curvature of the boundary ∂M. Let $[g]$ be the set of Riemannian metrics conformal to g. J. Escobar extended the Yamabe problem to manifolds with boundary in [38], [37] and [40]:

(a) Find $g \in [g_0]$ such that R_g is constant and $h_g = 0$ on the boundary. We call this **minimal boundary case**.

(b) Find $g \in [g_0]$ such that $R_g = 0$ and h_g is constant on the boundary. We call this **scalar-flat case**.

(c) Find $g \in [g_0]$ such that R_g is nonzero constant and h_g is nonzero constant on the boundary.

Case (a) and (b) are studied by many papers, for instance, in [4, 15, 26, 30, 38] for the minimal boundary case, in [2, 23, 37, 65, 66] for scalar flat case. Case (c) is a mixed version of the two previous cases, and therefore, shares the difficulties coming from both cases.

The problem is equivalent to finding a positive solution to the following PDE:

$$\begin{cases} -\dfrac{4(n-1)}{n-2}\Delta_{g_0}u + R_{g_0}u = c_1 u^{\frac{n+2}{n-2}}, & \text{in } M, \\ \dfrac{2}{n-2}\dfrac{\partial u}{\partial \nu_{g_0}} + h_{g_0}u = c_2 u^{\frac{n}{n-2}}, & \text{on } \partial M, \end{cases} \tag{1.1}$$

where $c_1, c_2 \in \mathbb{R}$, R_{g_0} is the scalar curvature, h_{g_0} is the mean curvature and ν_{g_0} is the outward unit normal on ∂M. Escobar initiated the investigation of this problem in [36, 40]. In the subsequent papers [49, 50], Z. C. Han and Y. Y. Li proposed the following (weak version) conjecture:

Conjecture 1.1.1 (Han-Li). *If $Y(M, \partial M) > 0$, given any positive constant c_1 and any $c_2 \in \mathbb{R}$, problem (1.1) is solvable.*

They proved that the conjecture is true when one of the following assumptions is fulfilled:

(a) $n \geq 5$ and ∂M admits at least one non-umbilic point (cf. [50]);

(b) $n \geq 3$ and (M, g_0) is locally conformally flat with umbilic boundary ∂M (cf. [49]).

Let us introduce some natural conformal invariants. The (generalized) Yamabe constant $Y(M, \partial M)$ is defined as

$$Y(M, \partial M) := \inf_{g \in [g_0]} \frac{\int_M R_g d\mu_g + 2(n-1)\int_{\partial M} h_g d\sigma_g}{(\int_M d\mu_g)^{\frac{n-2}{n}}}. \tag{1.2}$$

Similarly, we define (cf. [37])

$$Q(M, \partial M) := \inf_{g \in [g_0]} \frac{\int_M R_g d\mu_g + 2(n-1)\int_{\partial M} h_g d\sigma_g}{(\int_{\partial M} d\sigma_g)^{\frac{n-2}{n-1}}}.$$

It was first pointed out by Zhiren Jin (cf. [39]) that $Q(M, \partial M)$ could be $-\infty$, meanwhile $Y(M, \partial M) > -\infty$. If $Y(M, \partial M) > (=)0$, then there exists a conformal metric of g_0 with zero scalar curvature in M and positive (zero) mean curvature on ∂M.[1] Furthermore, $Y(M, \partial M) > 0$ if and only if $Q(M, \partial M) > 0$.

[1] From [38, Lemma 1.1], there exists $g_1 \in [g_0]$ such that $R_{g_1} > (=)0$ and $h_{g_1} = 0$. Let φ be a positive smooth minimizer of $\{\int_M (\frac{4(n-1)}{n-2}|\nabla\psi|^2_{g_1} + R_{g_1}\psi^2)d\mu_{g_1}; \psi \in H^1(M, g_1), \int_{\partial M}\psi^2 d\sigma_{g_1} = 1\}$, then $\varphi^{4/(n-2)}g_1$ is the desired conformal metric.

We remark that problem (1.1) is variational. The total scalar curvature plus total mean curvature functional is given by

$$E[u] = \int_M (\tfrac{4(n-1)}{n-2}|\nabla u|_{g_0}^2 + R_{g_0}u^2)d\mu_{g_0} + 2(n-1)\int_{\partial M} h_{g_0}u^2 d\sigma_{g_0}. \tag{1.3}$$

Given any $a, b > 0$, we define a conformal invariant on compact manifolds with boundary by

$$\begin{aligned} Y_{a,b}(M,\partial M) &= \inf_{g\in[g_0]} \frac{\int_M R_g d\mu_g + 2(n-1)\int_{\partial M} h_g d\sigma_g}{a\left(\int_M d\mu_g\right)^{\frac{n-2}{n}} + 2(n-1)b\left(\int_{\partial M} d\sigma_g\right)^{\frac{n-2}{n-1}}} \\ &= \inf_{0\not\equiv u\in H^1(M,g_0)} \mathcal{Q}_{a,b}[u], \end{aligned}$$

where

$$\mathcal{Q}_{a,b}[u] = \frac{E[u]}{a\left(\int_M |u|^{\frac{2n}{n-2}}d\mu_{g_0}\right)^{\frac{n-2}{n}} + 2(n-1)b\left(\int_{\partial M}|u|^{\frac{2(n-1)}{n-2}}d\sigma_{g_0}\right)^{\frac{n-2}{n-1}}}.$$

For $n \geq 3$, let $\mathbb{R}^n_+ = \{y = (y^1, \cdots, y^n) \in \mathbb{R}^n; y^n > 0\}$ denote the half space. The next theorem gives a criterion for the existence of a minimizer for $Y_{a,b}(M,\partial M)$, which is attained by subcritical approximations.

Theorem 1.1.2 ([28]). *Suppose $Y_{a,b}(M,\partial M) < Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1})$ for some given $a, b > 0$, then $Y_{a,b}(M,\partial M)$ can be achieved by a positive smooth minimizer.*

In [6, 7] Araujo also gave some characterization of critical points (including minimizers) of $E[u]$ under Escobar's non-homogeneous constraint (cf. [40]).

In order to apply Theorem 1.1.2 in the case of $Y(M,\partial M) > 0$, we need to construct a global test function $\bar{U}_{(x_0,\epsilon)}$ as a small perturbation of a bubble function W_ϵ with $x_0 \in \partial M$ and small $\epsilon > 0$, such that $\mathcal{Q}_{a,b}[\bar{U}_{(x_0,\epsilon)}] < Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1})$. We would like to mention some developments on the technique of constructing test functions in very closely related works. In dimension $n \geq 6$, Brendle [14] initiated this technique of constructing test functions through his study of the Yamabe flow. Subsequently Brendle and S. Chen [15] developed it to study the Yamabe problem with umbilic minimal boundary (i.e. $c_1 \in \mathbb{R}, c_2 = 0$). Not long after that S. Chen [23] adapted the same technique to scalar-flat and constant mean curvature problem with umbilic boundary (i.e. $c_1 = 0, c_2 \in \mathbb{R}$). One of the key ingredients in Almaraz [2] and Almaraz-L. Sun [4] is to extend such a technique to both the boundary ∂M has one non-umbilic point and the case of lower dimensions $3 \leq n \leq 5$. The correction term ψ in our test function (cf. (2.50)) origins from the linearization of scalar curvature and mean curvature at a round metric on a spherical cap, which has constant sectional curvature 4 (cf. Proposition 2.4.1).

We will use a notion of a *mass* associated to manifolds with boundary.

Definition 1.1.3. Let (N, g) be a Riemannian manifold with a boundary ∂N. We say that N is *asymptotically flat* with order $p > 0$, if there exist a compact set $N_0 \subset N$ and a diffeomorphism $F : N \backslash N_0 \to \mathbb{R}^n_+ \backslash \overline{B_1^+(0)}$ such that, in the coordinate chart defined by F (called *asymptotic coordinates* of N), there holds

$$|g_{ij}(y) - \delta_{ij}| + |y||\partial_k g_{ij}(y)| + |y|^2 |\partial^2_{kl} g_{ij}(y)| = O(|y|^{-p}), \quad \text{as} \quad |y| \to \infty,$$

where $i, j, k, l = 1, \cdots, n$, $B_1^+(0) = B_1(0) \cap \mathbb{R}^n_+$.

Provided that the following limit

$$m(g) := \lim_{R \to \infty} \left[\int_{\{y \in \mathbb{R}^n_+;\, |y| = R\}} \sum_{i,j=1}^{n} (g_{ij,j} - g_{jj,i}) \frac{y^i}{|y|} \, d\sigma + \int_{\{y \in \mathbb{R}^{n-1};\, |y| = R\}} \sum_{a=1}^{n-1} g_{na} \frac{y^a}{|y|} \, d\sigma \right]$$

exists, we call it the *mass* of (N, g). Moreover, $m(g)$ is a geometric invariant in the sense that it is independent of asymptotic coordinates. The definition of the mass $m(g)$ was first proposed by Marques. The following positive mass type conjecture was given in [2,3].

Conjecture 1.1.4 (**Positive mass with a non-compact boundary**)**.** If $R_g, h_g \geq 0$, then we have $m(g) \geq 0$ and the equality holds if and only if N is isometric to $\mathbb{R}^n_+$.

For $n \geq 3$, let $d = [(n-2)/2]$. As in [2], we define

$$\begin{aligned} \mathcal{Z} = \{x_0 \in \partial M; & \limsup_{x \to x_0} d_{g_0}(x, x_0)^{2-d} |W_{g_0}(x)|_{g_0} = 0 \text{ and} \\ & \limsup_{x \to x_0} d_{g_0}(x, x_0)^{1-d} |\mathring{\pi}_{g_0}(x)|_{g_0} = 0\}, \end{aligned}$$

where W_{g_0} denotes the Weyl tensor in M, π_{g_0} the second fundamental form and $\mathring{\pi}_{g_0}$ the trace-free second fundamental form on ∂M. Then $\mathcal{Z}$ only depends on the conformal structure of g_0, since W_{g_0} and $\mathring{\pi}_{g_0}$ are both pointwise conformal invariants. In particular, $\mathcal{Z} = \partial M$ when $n = 3$. Moreover, if the scalar curvature and the mean curvature are integrable on M and ∂M respectively and the decay order is $p > (n-2)/2$, the *mass* $m(g_0)$ is well-defined. This is the case for g_{x_0} when $x_0 \in \mathcal{Z}$.

For $x_0 \in \partial M$, let $g_{x_0} \in [g_0]$ be the metric induced by the conformal Fermi coordinates around x_0 (cf. [65]). Denote by G_{x_0} the Green's function of conformal Laplacian of g_{x_0} with pole at x_0, satisfying the boundary condition $\partial_{\nu_{g_{x_0}}} G_{x_0} - \frac{n-2}{2} h_{g_{x_0}} G_{x_0} = 0$ on $\partial M \setminus \{x_0\}$ (cf. (2.51)). Let $\bar{g}_{x_0} = G_{x_0}^{4/(n-2)} g_{x_0}$. Now we are ready to state our main result.

Theorem 1.1.5. *Let (M, g_0) be a smooth compact Riemannian manifold of dimension $n \geq 3$ with boundary. Suppose that M is not conformally equivalent to $\mathbb{R}^n_+$. If $Y(M, \partial M) > 0$, assume either $\partial M \backslash \mathcal{Z} \neq \emptyset$ or $m(\bar{g}_{x_0}) > 0$ for some $x_0 \in \mathcal{Z}$, then*

$$Y_{a,b}(M, \partial M) < Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1}).$$

We should point out that such assumptions on compact manifolds in Theorem 1.1.5 (or with some minor modifications) have been used in some closely related problems, for instance, Brendle [14] for the Yamabe flow in dimension $n \geq 6$, S. Chen [23] and Almaraz [2] for $c_1 = 0, c_2 \in \mathbb{R}$, Brendle-Chen [15] and Almaraz-L. Sun [4] for $c_1 \in \mathbb{R}, c_2 = 0$.

Recent advances in the above positive mass type theorem have played an important role in such conformal curvature problems (cf. [2, 3, 71] etc.). As a direct consequence of Theorem 1.1.5 and the positive mass type theorem proved in [3], we obtain

Theorem 1.1.6. *Let (M, g_0) be a smooth compact Riemannian manifold of dimension $n \geq 3$ with boudary. Suppose that M is not conformally equivalent to $\mathbb{R}^n_+$ and $Y(M, \partial M) > 0$. Assume that one of the following assumptions is satisfied:*

(i) $\partial M \setminus \mathcal{Z} \neq \emptyset$*;*

(ii) $3 \leq n \leq 7$ *or M is spin;*

(iii) $n \geq 8$ *and* (M, g_0) *is locally conformally flat with umbilic boundary* ∂M*.*

Then given any $a, b > 0$*, there exists at least one positive smooth minimizer* $u_{a,b}$ *for* $Y_{a,b}(M, \partial M)$*. Moreover, the conformal metric* $u_{a,b}^{4/(n-2)} g_0$*, modulo a positive constant multiple, has scalar curvature* 1 *and some positive constant boundary mean curvature.*

When $Y(M, \partial M) > 0$, Escobar proved in [40, Theorem 4.2] the existence of such conformal metrics in Theorem 1.1.6 under one of the following hypotheses:

(1) $3 \leq n \leq 5$;

(2) ∂M has at least a nonumbilic point;

(3) ∂M is umbilic and either M is locally conformally flat or the Weyl tensor does not identically vanish on ∂M.

Then we generalize the existence results to the cases including $n = 6, 7$ or M is spin. The remaining cases left by Escobar are the manifolds that are not locally conformally flat and ∂M is umbilic, and Weyl tensor vanishes identically on ∂M and $n \geq 6$. Thus our Theorem 1.1.6 also generalizes to this type of manifolds under the assumption $\partial M \backslash \mathcal{Z} \neq \emptyset$. We next prove the compactness of the minimizers for $Y_{a,b}(M, \partial M)$ when (a, b) varies in a compact set K of $\{(a, b); a \geq 0, b \geq 0\} \setminus \{(0, 0)\}$. We denote by $\mathcal{M}_{a,b}$ the set of positive smooth minimizers of $Y_{a,b}(M, \partial M)$ with the normalization (2.15).

Theorem 1.1.7. *Let* K *and* $\mathcal{M}_{a,b}$ *as defined above. Suppose* $Y_{a,b}(M, \partial M) < Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1})$ *for all* $(a, b) \in K$*, then there exists* $C = C(K, g_0)$ *such that*

$$C^{-1} \leq u_{a,b} \leq C, \quad \|u_{a,b}\|_{C^2(M)} \leq C, \quad \forall \ \ u_{a,b} \in \cup_{(a,b) \in K} \mathcal{M}_{a,b}.$$

It follows from Theorem 1.1.7 that in terms of normalized conformal metrics having scalar curvature 1, there exits a sequence of such conformal metrics such that their constant boundary mean curvatures go to $+\infty$. We refer the details to the end of Section 2.3. In contrast with our result, the constant mean curvature of such a conformal metric in [40, Theorem 4.2] only admits a small real number. Indeed, the smallness of $b \in \mathbb{R}$ in a conformal invariant $G_{a,b}(M)$ (see also Section 2.1) is very crucial in the proof of [40].

Remark 1.1.8. When $Y(M, \partial M) < 0$, as a direct consequence of [27, Theorem 1.1], there exists a conformal metric such that its scalar curvature equals -1 and its boundary mean curvature equals any negative real number.

1.2 Yamabe flow on compact manifolds with boundary

Let M^n be a closed manifold with dimension $n \geq 3$. In order to solve the Yamabe problem (see [85]), R. Hamilton introduced the Yamabe flow, which evolves Riemannian metrics on M according to the equation

$$\frac{\partial}{\partial t} g(t) = -(R_{g(t)} - \overline{R}_{g(t)}) g(t)\,,$$

where R_g denotes the scalar curvature of the metric g and $\overline{R}_g$ stands for the average

$$\left(\int_M dv_g\right)^{-1} \int_M R_g dv_g.$$

Here, dv_g is the volume form of (M, g). Although the Yamabe problem was solved using a different approach in [8, 74, 81], the Yamabe flow is a natural geometric deformation to metrics of constant scalar curvature. The convergence of the Yamabe flow on closed manifolds was studied in [31, 77, 87]. This question was solved in [13, 14] under the hypotheses of the positive mass theorem.

In this work, we study the convergence of the Yamabe flow on compact n-dimensional manifolds with boundary, when $n \geq 3$. For those manifolds, J. Escobar raised the question of existence of conformal metrics with constant scalar curvature which have the boundary as a minimal hypersurface. This problem was studied in [15, 39, 67]; see also [5, 49, 50].

We are interested in a formulation of the Yamabe flow for compact manifolds with minimal boundary proposed by S. Brendle in [12]. This flow evolves a conformal family of metrics $g(t)$, $t \geq 0$, according to the equations

$$\begin{cases} \dfrac{\partial}{\partial t} g(t) = -(R_{g(t)} - \overline{R}_{g(t)}) g(t)\,, & \text{in } M\,, \\ H_{g(t)} = 0\,, & \text{on } \partial M\,. \end{cases} \tag{1.4}$$

S. Brendle proved that

Theorem 1.2.1 ([12]). *Suppose that:*

(i) $Y(M,\partial M) \le 0$, *or*

(ii) $Y(M,\partial M) > 0$ *and* M *is locally conformally flat with umbilic boundary.*

where $Y(M,\partial M)$ *is defined in* (1.2)*. Then, for every initial metric* $g(0)$ *on* M *with minimal boundary, the flow* (1.4) *exists for all time* $t \ge 0$ *and converges to a constant scalar curvature metric with minimal boundary.*

Inspired by the ideas in [13, 14], we handle the remaining cases of this problem. Define

$$\mathcal{Z}_M = \{x_0 \in M\backslash\partial M\,;\ \limsup_{x\to x_0} d_{g_0}(x,x_0)^{2-d}|W_{g_0}(x)| = 0\}\,,$$

$$\mathcal{Z}_{\partial M} = \{x_0 \in \partial M\,;\ \limsup_{x\to x_0} d_{g_0}(x,x_0)^{2-d}|W_{g_0}(x)| = \limsup_{x\to x_0} d_{g_0}(x,x_0)^{1-d}|\mathring{\pi}_{g_0}(x)| = 0\}\,,$$

$$\text{and}\quad \mathcal{Z} = \mathcal{Z}_M \cup \mathcal{Z}_{\partial M}\,,$$

Our first result is the following:

Theorem 1.2.2. *Suppose that* (M^n, g_0) *is not conformally diffeomorphic to the hemisphere* S^n_+ *and satisfies* $Y(M,\partial M) > 0$*. If*

(a) $\mathcal{Z} = \emptyset$, *or*

(b) $n \le 7$, *or*

(c) M *is spin,*

then, for any initial metric $g(0)$ *on* M *with minimal boundary, the flow* (1.4) *exists for all time* $t \ge 0$ *and converges to a metric with constant scalar curvature and minimal boundary.*

Since the round sphere S^n minus a point is diffeomorphic to $\mathbb{R}^n$, which is spin, the following is an immediate consequence of Theorems 1.2.1 and 1.2.2:

Corollary 1.2.3. *If* $M \subset S^n$ *is a compact domain with smooth boundary, then the flow* (1.4)*, starting with any metric with minimal boundary, exists for all time* $t \ge 0$ *and converges to a metric with constant scalar curvature and minimal boundary.*

Condition (a) in Theorem 1.2.2 is particularly satisfied if the Weyl tensor and the trace-free second fundamental form are nonzero everywhere on $M \backslash \partial M$ and ∂M respectively. Conditions (b) and (c) allow us to make use of the positive mass theorem in [73, 75, 83] and its corresponding version for manifolds with a non-compact boundary in [3].

Before stating our main result, from which Theorem 1.2.2 follows, we need the Postive mass theorem on manifolds with boundary, see definition 1.1.3 and the theorem in [3]. The asymptotically flat manifolds used in this work are obtained as the generalized stereographic projections of the compact Riemannian manifold (M, g_0) with nonempty boundary. Those stereographic projections are performed around points $x_0 \in M$ by means of Green functions G_{x_0}, with singularity at x_0. After choosing a new background metric $g_{x_0} \in [g_0]$ with better coordinates expansion around x_0 (see Section 3.2), we consider the asymptotically flat manifold $(M \backslash \{x_0\}, \bar{g}_{x_0})$, where $\bar{g}_{x_0} = G_{x_0}^{\frac{4}{n-2}} g_{x_0}$ satisfies $R_{\bar{g}_{x_0}} \equiv 0$ and $H_{\bar{g}_{x_0}} \equiv 0$. If $x_0 \in \mathcal{Z}_{\partial M}$, according to Proposition 2.4.14 below, this manifold has asymptotic order $p > \frac{n-2}{2}$, so Conjecture 1.1.4 claims that $m(\bar{g}_{x_0}) > 0$ unless M is conformally equivalent to the unit hemisphere. If $x_0 \in \mathcal{Z}_M$, this manifold has asymptotic order $p > \frac{n-2}{2}$ (see [14, Proposition 19]), so the positive mass theorem prove by Schoen and Yau claims that $m_{ADM}(\bar{g}_{x_0}) > 0$.

Our main result, which implies Theorem 1.2.2, is the following:

Theorem 1.2.4. *Suppose that (M^n, g_0) is not conformally diffeomorphic to the unit hemisphere S^n_+ and satisfies $Y(M, \partial M) > 0$. Assume that $m_{ADM}(\bar{g}_{x_0}) > 0$ for all $x_0 \in \mathcal{Z}_M$ and $m(\bar{g}_{x_0}) > 0$ for all $x_0 \in \mathcal{Z}_{\partial M}$. Then, for any initial metric $g(0)$ with minimal boundary, the flow (1.4) exists for all $t \geq 0$ and converges to a constant scalar curvature metric with minimal boundary.*

The proof of Theorem 1.2.4 follows the arguments in [13]; see also [2]. An essential step is the construction of a family of test functions around each point $x_0 \in M$, whose energies are uniformly bounded by the Yamabe quotient $Y(S^n)$ if $x_0 \in M \backslash \partial M$, and by $Q(S^n_+)$ if $x_0 \in \partial M$. If $x_0 \in M \backslash \partial M$, the test functions used are essentially the ones introduced by S. Brendle in [14] for the case of closed manifolds. If $x_0 \in \partial M$, the functions used here were obtained in [15] in the case of umbilic boundary, where the authors address the existence of solutions to the Yamabe problem for manifolds with boundary. In this work, however, we estimate their energies without any assumption on the boundary.

An additional difficulty in controlling the energy of interior test functions by $Y(S^n)$ arises when their centers get close to the boundary (see Subsection 3.2.3). In this case, the techniques in [14] cannot be directly adapted because the standard (and symmetric) bubble in $\mathbb{R}^n$, which represents the sphere metric and is essential in the construction of the test functions, does not satisfy the Neumann boundary condition unless it is centered on $\partial\mathbb{R}^n_+$. However, here we are able to exploit the sign of this Neumann derivative, when centered in $\mathbb{R}^n_+\backslash\partial\mathbb{R}^n_+$, to obtain the necessary estimates.

1.3 Classification theorems for the polyharmonic equation

In the classical paper [16], Caffarelli-Gidas-Spruck established the asymptotic behavior for local positive solutions of the elliptic equation $-\Delta u = n(n-2)u^{\frac{n+2}{n-2}}$, $n \geq 3$, near an isolated singularity. Consequently, they proved that any positive entire solution of the equation has to be the form

$$\left(\frac{\lambda}{1+\lambda^2|x-x_0|^2}\right)^{\frac{n-2}{2}} \quad \text{for some } \lambda > 0,\ x_0 \in \mathbb{R}^n.$$

Particular interests of the above equation lie in its relation to the Yamabe problem (see Lee-Park [56]). Such Liouville type theorem has been extended to general conformally invariant nonlinear equations; see Lin [63] and Wei-Xu [82] for higher order semi-linear equations, Chen-Li-Ou [25], Li [58] and many others for integral equation, as well as Li-Li [57] for fully nonlinear second order elliptic equations.

Li-Zhu [62] and Ou [69] independently proved that any positive solution of

$$-\Delta u(x,t) = 0 \quad \text{in } \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty), \tag{1.5}$$

$$-\partial_t u(x,0) = (n-1)u^{\frac{n+1}{n-1}} \quad \text{on } \partial\mathbb{R}^{n+1}_+, \tag{1.6}$$

where $n \geq 2$, has to be the form

$$\left(\frac{\lambda}{\lambda^2|x-x_0|^2+(\lambda t+1)^2}\right)^{\frac{n-1}{2}}, \quad \lambda > 0, x_0 \in \mathbb{R}^n. \tag{1.7}$$

Throughout the thesis $\partial\mathbb{R}^{n+1}_+$ does not contain the infinity. See also Beckner [9] and Escobar [35] if u is an extremal of the sharp Sobolev trace inequality, and Li-Zhang [61], Jin-Li-Xiong [53] and references therein for related results. The isolated singularity problem has been studied recently by Caffarelli-Jin-Sire-Xiong [17], DelaTorre-González [33] and DelaTorre-del Pino-Gonzalez-Wei [32] as a special case. The nonlinear problem (1.5)-(1.6) arises from a boundary Yamabe problem or Riemann mapping problem of Escobar [37], sharp trace inequalities, nonlinear Neumann problems (see Cherrier [30]), and etc.

By the work Feffermann-Graham [41], Graham-Jenne-Mason-Sparling [47], and Graham-Zworski [48], there defines a class of conformally invariant operators on the conformal infinity of Poincaré-Einstein manifolds via scattering matrices. Such conformally invariant operators define fractional Q-curvatures. By the work Caffarelli-Silvestre [18], Chang-González [21], Yang [86] and Case-Chang [20], the boundary Yamabe problem mentioned above is the constant first order Q-curvature problem. If one considers the constant odd order Q-curvature problem on the conformal infinity of Poincaré ball or hyperbolic upper half space, it will lead to study positive solutions of nonlinear boundary value problem of polyharmonic equations

$$\Delta^m u(x,t) = 0 \quad \text{in } \mathbb{R}^{n+1}_+, \tag{1.8}$$

$$\partial_t \Delta^k u(x,0) = 0, \quad (-1)^m \partial_t \Delta^{m-1} u(x,0) = u^{\frac{n+(2m-1)}{n-(2m-1)}} \quad \text{on } \partial\mathbb{R}^{n+1}_+, \tag{1.9}$$

where $2 \le 2m < n+1$ is an integer, $k = 0, 1, \ldots, m-2$. One may view $(-1)^m \partial_t \Delta^{m-1}$ as $(-\partial_t)(-\Delta)^{m-1} \sim (-\Delta)^{\frac{1}{2}}(-\Delta)^{m-1}$. Hence, the above problem connects to

$$(-\Delta)^{\frac{2m-1}{2}} u = u^{\frac{n+(2m-1)}{n-(2m-1)}} \quad \text{in } \mathbb{R}^n. \tag{1.10}$$

However, we will see that (1.8)-(1.9) admits more solutions. Since we do not assume u to be a minimizer or belong to some Sobolev space of $\mathbb{R}^{n+1}_+$, there is no information of u near the infinity.

We are able to classify solutions of problem (1.8)-(1.9) and the subcritical cases.

Consider

$$\begin{cases} \Delta^m u(x,t) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_t \Delta^k u(x,0) = 0 & \text{on } \partial\mathbb{R}^{n+1}_+, \quad k = 0,1,\ldots,m-2, \\ (-1)^m \partial_t \Delta^{m-1} u(x,0) = u^p & \text{on } \partial\mathbb{R}^{n+1}_+, \end{cases} \tag{1.11}$$

where $2 \leq 2m < n+1$ is an integer and $1 < p \leq \frac{n+(2m-1)}{n-(2m-1)}$. We will show the nonnegative solutions of this problem are the composition of the following "bubbles" and some polynomials

$$U_{x_0,\lambda}(X) = c(n,m) \int_{\mathbb{R}^n} \frac{t^{2m-1}}{(|x-y|^2+t^2)^{\frac{n+2m-1}{2}}} \left(\frac{\lambda}{1+\lambda^2|y-x_0|^2} \right)^{\frac{n-2m+1}{2}} \mathrm{d}y \tag{1.12}$$

where $x_0 \in \mathbb{R}^n$ and $\lambda > 0$ and $c(n,m) > 0$ is some normalizing constant. The presence of polynomial part is a new phenomenon. More precisely

Theorem 1.3.1. Let $u \geq 0$ be a $C^{2m}(\mathbb{R}^{n+1}_+ \cup \partial\mathbb{R}^{n+1}_+)$ solution of (1.11). In case of that m is even, we additionally suppose that $u(x,t) = o((|x|^2+t^2)^{\frac{2m-1}{2}})$ as $x^2+t^2 \to \infty$. Then

(i) If $p = \frac{n+(2m-1)}{n-(2m-1)}$, we have

$$u(x,t) = U_{x_0,\lambda}(x,t) + \sum_{k=1}^{m-1} t^{2k} P_{2k}(x),$$

where $U_{x_0,\lambda}$ is defined in (1.12) for some $x_0 \in \mathbb{R}^n$ and $\lambda \geq 0$, and $P_{2k}(x)$ is a polynomial of degree $\leq 2m-2-2k$ satisfying $\liminf_{x\to\infty} P_{2k}(x) \geq 0$.

(ii) If $1 < p < \frac{n+(2m-1)}{n-(2m-1)}$, we have

$$u(x,t) = \sum_{k=1}^{m-1} t^{2k} P_{2k}(x),$$

where $P_{2k}(x) \geq 0$ is a polynomial of degree $\leq 2m-2-2k$.

Remark 1.3.2. For $m=1$, $U_{x_0,\lambda}$ defined (1.12) equals (1.7) up to a constant. For $m=2$, we have

$$U_{x_0,\lambda}(X) = C(n) \left(\frac{\lambda}{(1+\lambda t)^2 + \lambda^2|x-x_0|^2} \right)^{\frac{n-3}{2}} + C(n)(n-3)t \left(\frac{\lambda}{(1+\lambda t)^2 + \lambda^2|x-x_0|^2} \right)^{\frac{n-1}{2}}$$

with $C(n) = [2(n-3)(n^2-1)]^{\frac{n-3}{6}}$.

Remark 1.3.3. If m is even and the growth condition is removed, there is another class of solutions

$$H_a(x,t) = \frac{a}{(2m-1)!} t^{2m-1} + a^{\frac{1}{p}}, \quad a \geq 0. \tag{1.13}$$

We conjecture that for even m, all solutions have to be

$$\sum_{k=1}^{m-1} t^{2k} P_{2k}(x) + H_a(x,t) \quad \text{or} \quad \sum_{k=1}^{m-1} t^{2k} P_{2k}(x) + U_{x_0,\lambda}(x,t)$$

if $p = \frac{n+2m-1}{n-2m+1}$, while only the former expression can happen if $1 < p < \frac{n+2m-1}{n-2m+1}$.

By conformally transforming the upper half space to the unit ball, Theorem 1.3.1 implies that in the conformal class of the unit Euclidean ball there exist metrics with a single singular boundary point which have flat Q-curvature and constant boundary Q-curvature. See Section 4.5 of the paper for more details. When $m = 1$, there is no such metric which is singular on single boundary point because the polynomial part vanishes and the bubble is smooth at the infinity. Hence, boundary singular metrics have at least two singular points which is similar to the singular metrics on the unit sphere of constant scalar curvature; see Caffarelli-Gidas-Spruck [16] and Schoen [76]. Other possible applications of Theorem 1.3.1 would be seen in Jin-Li-Xiong [52], Li-Xiong [60] and references therein.

The proofs of Theorem 1.3.1 for $m = 1$ by Li-Zhu [62] or Ou [69] rely on the maximum principle in order to use the moving spheres/planes method. In contrast, for $m \geq 2$ the elliptic operators have nontrivial kernels and thus solutions of (1.11) could lose the maximum principle. To extract the kernels, we need to analyze the behavior of u near the infinity. Due to the conformal invariance of equations, the m-Kelvin transform u^* of u with respect to the unit sphere satisfies (1.8) and

$$\partial_t \Delta^k u^*(x,0) = 0, \quad (-1)^m \partial_t \Delta^{m-1} u^*(x,0) = |x|^{-\tau} u^*(x,0)^p \quad \text{in } \partial\mathbb{R}^{n+1}_+ \setminus \{0\},$$

where $k = 0, 1, \ldots, m-2$, and $\tau = [n + (2m-1)] - p[n - (2m-1)] \geq 0$. As Caffarelli-Gidas-Spruck [16], Lin [63] and Wei-Xu [82] did, one may wish to show $|x|^{-\tau} u^*(x,0)^p \in L^1$ near 0. However, since the linear equation itself would generate higher order singularities than the nonlinear term does, the methods of [16], [63] and [82] seem not to be applicable to $m \geq 2$. Even worse, this is wrong when m is even; see for instance the m-Kelvin transform of H_a, $a > 0$, in Remark 1.3.3. In fact, the method of [16] is by constructing test function which is of second order equation nature. And it is unclear how to adapt the ODE analysis procedure of [63] and [82] to our setting without information about the possible kernels. As the initial step, we prove that $u^*(x,0)$ belongs to L^1 (see Lemma 4.3.2), and then by a Poisson extension we are able to capture the singularity generated by the linear equation. A Liouville type theorem (see Proposition 4.2.1 and Theorem 4.2.2) for polyharmonic functions with a homogeneous boundary data plays an important role. Our method of proof of Theorem 4.2.2 is very flexible and can be easily adapted to polyharmonic functions with other homogeneous boundary data. Next, by subtracting the linear effect we prove $|x|^{-\tau}u^*(x,0)^p \in L^1$, where the growth condition is assumed if m is even. In this step a new method is developed. In particular, if p is less than the Serrin's exponent $\frac{n}{n-2m+1}$ we have to spend extra efforts. By a Neumann extension of $|x|^{-\tau}u^*(x,0)^p$ and making use of a boundary Bôcher theorem (see Corollary 4.2.4), we prove a crucial splitting result for u; see Proposition 4.4.1. It captures the polynomials $\sum_{k=1}^{m-1} t^{2k}P_{2k}(x)$ and implies the maximum principle for $v(x,t) := u(x,t) - \sum_{k=1}^{m-1} t^{2k}P_{2k}(x)$ which is completely controlled by the nonlinear effect. Since $v(x,0) = u(x,0)$, v satisfies a nonlinear integral equation. By Chen-Li-Ou [25], Li [58], or Dou-Zhu [34], $v(x,t)$ is then classified.

Our method of proof of Theorem 1.3.1 can be applied to constant fractional Q-curvature equation on the conformal infinity of hyperbolic upper half space, and can be applied to multiple nonlinear boundary conditions; see Chang-Qing [22], Branson-Gover [11] and Case [19] for the discussions of other conformally invariant boundary operators.

If $2m = n+1$, (1.9) will be replaced by

$$\partial_t \Delta^k u(x,0) = 0, \quad (-1)^m \partial_t \Delta^{m-1} u(x,0) = e^{(2m-1)u} \quad \text{on } \partial\mathbb{R}^{n+1}_+ \tag{1.14}$$

and u is not necessarily positive. When $m \geq 2$, in order to have a classification theorem one has to assume that (i) $\int_{\mathbb{R}^n} e^{(2m-1)u(x,0)} < \infty$, (ii) $|u(x,0)| = o(|x|^2)$ near the infinity, (iii) certain growth conditions on $u(x,t)$ near the infinity. See, for instance, Jin-Maalaoui-Martinazzi-Xiong [54] and references therein on why (i) and (ii) can not be dropped. Given (i), (ii) and (iii), one can prove a splitting theorem like Theorem 1.3.1 easily by the Bôcher theorem (see Corollary 4.2.4) and Xu [84]. We decide not to pursue it in this thesis.

Finally, we remark that there have been many papers devoted to Liouville theorems for nonnegative solutions of nonlinear polyharmonic equations with the homogeneous Dirichlet boundary condition or homogeneous Navier boundary condition; see Reichel-Weth [72], Lu-Wang-Zhu [64], Chen-Fang-Li [24] and references therein, where they proved that 0 is the unique solution.

Now, we describe briefly the content of each chapter: In Chapter 2, we will discuss the existence of conformal metrics with constant scalar curvature and constant mean curvature, which is a part of the paper [28]. In Chapter 3, we will discuss the Yamabe flow with minimal boundary condition in [4]. In Chapter 4, the classification of polyharmonic equation with conformally invariant boundary condition is considered as in [80].

Chapter 2

Conformal metrics with constant scalar curvature and constant mean curvature

In this chapter, we will explain the work included in [28].

2.1 Preliminaries

Let T_c be a negative real number, it follows from the classification theorem in [62] that all nonnegative solutions to the following PDE

$$\begin{cases} -\Delta v = n(n-2)v^{\frac{n+2}{n-2}}, & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial v}{\partial y^n} = (n-2)T_c v^{\frac{n}{n-2}}, & \text{on } \mathbb{R}^{n-1}. \end{cases} \tag{2.1}$$

must be either $v \equiv 0$ or $v(y) = W(y)$ (up to dilations and translations in variables $y^1, \cdots, y^{n-1}$), where

$$W(y) = \left(\frac{1}{1 + |y - T_c\mathbf{e}_n|^2}\right)^{\frac{n-2}{2}}$$

and $\mathbf{e}_n$ is the unit direction vector in n-th coordinate. In particular, we set

$$W_\epsilon(y) = \epsilon^{\frac{2-n}{2}} W(\epsilon^{-1}y) = \left(\frac{\epsilon}{\epsilon^2 + |y - T_c\epsilon\mathbf{e}_n|^2}\right)^{\frac{n-2}{2}}, \ \forall\, \epsilon > 0, \tag{2.2}$$

which satisfy (2.1) and are also the extremal functions of the associated Sobolev inequality induced by $Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1})$ (cf. [36, Theorem 3.3] or Lemma 2.2.5 below).

For each fixed $a, b > 0$, any positive minimizers of $Y_{a,b}(M, \partial M)$ satisfy

$$\begin{cases} -\dfrac{4(n-1)}{n-2}\Delta_{g_0} u + R_{g_0} u = \mu(M) a \left(\displaystyle\int_M u^{\frac{2n}{n-2}} d\mu_{g_0}\right)^{-\frac{2}{n}} u^{\frac{n+2}{n-2}} & \text{in } M, \\ \dfrac{2}{n-2}\dfrac{\partial u}{\partial \nu_{g_0}} + h_{g_0} u = \mu(M) b \left(\displaystyle\int_{\partial M} u^{\frac{2(n-1)}{n-2}} d\sigma_{g_0}\right)^{-\frac{1}{n-1}} u^{\frac{n}{n-2}} & \text{on } \partial M, \end{cases} \tag{2.3}$$

where $\mu(M) = Y_{a,b}(M, \partial M)$.

When $(M,g_0)=(\mathbb{R}^n_+,g_{\mathbb{R}^n})$, problem (2.3) is equivalent to the solvability of positive solutions to

$$\begin{cases}-\dfrac{4(n-1)}{n-2}\Delta u=\mu a\left(\displaystyle\int_{\mathbb{R}^n_+}u^{\frac{2n}{n-2}}dx\right)^{-\frac{2}{n}}u^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n_+,\\ -\dfrac{2}{n-2}\dfrac{\partial u}{\partial y^n}=\mu b\left(\displaystyle\int_{\mathbb{R}^{n-1}}u^{\frac{2(n-1)}{n-2}}d\sigma\right)^{-\frac{1}{n-1}}u^{\frac{n}{n-2}} & \text{on } \mathbb{R}^{n-1},\end{cases} \tag{2.4}$$

where $\mu=\mu(\mathbb{R}^n_+)=Y_{a,b}(\mathbb{R}^n_+,\mathbb{R}^{n-1})$. A simple but vital observation is that if u is a smooth positive solution to problem (2.4), so is c_*u for all $c_*\in\mathbb{R}_+$. Hence all positive solutions to problem (2.4) are in the form of, up to dilations and translations in variables $y^1,\cdots,y^{n-1}$,

$$c_*\left(\frac{1}{1+|y-T_c\mathbf{e}_n|^2}\right)^{\frac{n-2}{2}}$$

for all $c_*>0$ and some $T_c<0$ depending on n,a,b. We choose $c_*=1$ hereafter, namely, for this fixed $T_c<0$, the associated function

$$W(y)=\left(\frac{1}{1+|y-T_c\mathbf{e}_n|^2}\right)^{\frac{n-2}{2}}$$

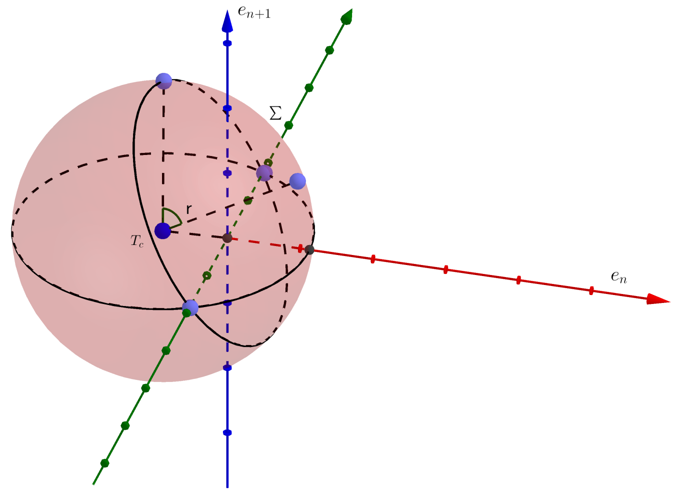
is a positive solution to both problems (2.1) and (2.4).

Denote by a mapping $\pi: S^n(T_c\mathbf{e}_n)\setminus\{T_c\mathbf{e}_n+\mathbf{e}_{n+1}\}\to\{\xi+T_c\mathbf{e}_n\in\mathbb{R}^{n+1};\xi^{n+1}=0\}\simeq\mathbb{R}^n$ the stereographic projection from the unit sphere $S^n(T_c\mathbf{e}_n)$ in $\mathbb{R}^{n+1}$ centered at $T_c\mathbf{e}_n$. Then for $y\in\mathbb{R}^n_+$, we set $\xi=\pi^{-1}(y)\in S^n$, namely (see also [50, (3.1) on page 831])

$$\begin{cases}\xi^a=\dfrac{2y^a}{1+|y-T_c\mathbf{e}_n|^2}, & \text{for } 1\le a\le n-1,\\ \xi^n=\dfrac{2(y^n-T_c)}{1+|y-T_c\mathbf{e}_n|^2}, & \\ \xi^{n+1}=\dfrac{|y-T_c\mathbf{e}_n|^2-1}{1+|y-T_c\mathbf{e}_n|^2}. & \end{cases}$$

Let Σ be a spherical cap (cf. Figure 2.1) equipped with a round metric $\frac14 g_{S^n}$, where g_{S^n} is the standard metric of the unit sphere $S^n(T_c\mathbf{e}_n)$. Then a direct computation shows

$$\frac14(\pi^{-1})^*(g_{S^n})=\left(\frac{1}{1+|y-T_c\mathbf{e}_n|^2}\right)^2 g_{\mathbb{R}^n}=W(y)^{\frac{4}{n-2}}g_{\mathbb{R}^n}.$$



Denote by ω_{n-1} the volume of the standard unit sphere in $\mathbb{R}^n$. Define

$$A=\int_{\mathbb{R}^n_+} W(y)^{\frac{2n}{n-2}}dy \quad \text{and} \quad B=\int_{\mathbb{R}^{n-1}} W(y)^{\frac{2(n-1)}{n-2}}d\sigma.$$

Notice that A,B only depend on n,T_c. Using (2.1) we get

$$\int_{\mathbb{R}^n_+}|\nabla W(y)|^2dy=n(n-2)A-(n-2)T_cB. \tag{2.5}$$

Recall that, from [36, Theorem 3.3] that $Y_{a,b}(\mathbb{R}^n_+,\mathbb{R}^{n-1})$ can be achieved by W with some T_c (up to dilations and translations in variables $y^1,\cdots,y^{n-1}$) modulo a positive constant multiple. Comparing (2.4) and (2.1), as well as the above comments, we have

$$\mu\frac{n-2}{4(n-1)}aA^{-\frac{2}{n}}=n(n-2), \quad \mu\frac{n-2}{2}bB^{-\frac{1}{n-1}}=-(n-2)T_c,$$

whence

$$-aA^{-\frac{2}{n}}T_c=2n(n-1)bB^{-\frac{1}{n-1}}.$$

Indeed we will establish that each pair of $a,b>0$ corresponds to a unique T_c satisfying the above identity.

Lemma 2.1.1. *Given any $a,b>0$, there exists a unique $T_c\in(-\infty,0)$ such that*

$$-aA^{-\frac{2}{n}}T_c=2n(n-1)bB^{-\frac{1}{n-1}}. \tag{2.6}$$

In particular, T_c is a continuous function of $(a,b)\in\mathbb{R}_+\times\mathbb{R}_+$. Moreover, for such a W satisfying (2.4) with the above unique T_c, there holds

$$Y_{a,b}(\mathbb{R}^n_+,\mathbb{R}^{n-1})=4n(n-1)a^{-1}A^{\frac{2}{n}}=-2T_cb^{-1}B^{\frac{1}{n-1}}.$$

Proof. Let $\cos r = \frac{-T_c}{\sqrt{1+T_c^2}}, r \in (0, \frac{\pi}{2})$, then A and B turn to

$$A(r) = \omega_{n-1} \int_0^r (\sin \tau)^{n-1} d\tau, \quad B(r) = \omega_{n-1} (\sin r)^{n-1}. \tag{2.7}$$

Then equation (2.6) is equivalent to finding some $r \in (0, \frac{\pi}{2})$ such that

$$f(r) := 2n(n-1)bA^{\frac{2}{n}} B^{-\frac{1}{n-1}} - a \cot r = 0.$$

First it is easy to verify that

$$\lim_{r \searrow 0} f(r) = -\infty \quad \text{and} \quad \lim_{r \nearrow \frac{\pi}{2}} f(r) = \mathit{constant} > 0.$$

Next we claim that $f(r)$ is increasing in $(0, \frac{\pi}{2})$. To see this, we have

$$\begin{aligned} &\frac{d}{dr} \log(B^{-\frac{1}{n-1}} A^{\frac{2}{n}}) = \frac{2}{n} \frac{A'}{A} - \frac{1}{n-1} \frac{B'}{B} \\ =& \frac{1}{\sin r \int_0^r (\sin \tau)^{n-1} d\tau} \left[\frac{2}{n} (\sin r)^n - \cos r \int_0^r (\sin \tau)^{n-1} d\tau \right]. \end{aligned}$$

Observe that

$$\cos r \int_0^r (\sin \tau)^{n-1} d\tau \leq \int_0^r (\sin \tau)^{n-1} \cos \tau d\tau = \frac{1}{n} (\sin r)^n.$$

This implies $(B^{-\frac{1}{n-1}} A^{\frac{2}{n}})(r)$ is increasing in $(0, \frac{\pi}{2})$, as well as is $f(r)$. Hence we conclude that there exists a unique $r \in (0, \frac{\pi}{2})$ such that $f(r) = 0$, namely there exists a unique $T_c < 0$ satisfying (2.6).

By [36, Theorem 3.3], (2.5) and (2.6), we get

$$\begin{aligned} Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1}) =& \frac{\frac{4(n-1)}{n-2} \int_{\mathbb{R}^n_+} |\nabla W|^2 dy}{aA^{\frac{n-2}{n}} + 2(n-1)bB^{\frac{n-2}{n-1}}} \\ =& \frac{4(n-1)}{n-2} \frac{n(n-2)A - (n-2)T_c B}{aA^{\frac{n-2}{n}} + 2(n-1)bB^{\frac{n-2}{n-1}}} \\ =& 4n(n-1)A^{\frac{2}{n}} a^{-1} = -2B^{\frac{1}{n-1}} T_c b^{-1}. \end{aligned} \tag{2.8}$$

In terms of the variable T_c, from (2.7) that $A(T_c)$ is increasing in $(-\infty, 0)$. One may regard T_c as a function of (a, b). Indeed one can show that $Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1})$ is continuous in $(a, b) \in \mathbb{R}_+ \times \mathbb{R}_+$ (see e.g. Proposition 2.3.1 below). From this and the third identity in (2.8), we get A is a continuous function of (a, b). Hence we conclude that T_c is a continuous function in (a, b). $\square$

From now on, we fix $T_c < 0$ as the unique one in Lemma 2.1.1 without otherwise stated. In [40], Escobar introduced a conformal invariant by $G_{a,b} = \inf\{E[u]; u \in C_{a,b}\}$, where $a > 0, b \in \mathbb{R}$ and

$$C_{a,b} = \left\{ u \in C^1(\bar{M}); a \int_M |u|^{\frac{2n}{n-2}} d\mu_{g_0} + b \int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} = 1 \right\}.$$

He established that $G_{a,b}(M) \le G_{a,b}(\mathbb{R}^n_+)$ holds for any compact Riemannian manifold with boundary. By similarly constructing a local test function as a perturbation of W_ϵ under the Fermi coordinates around a boundary point, one can mimick the proof of [40, Proposition 3.1] to show $Y_{a,b}(M, \partial M) \le Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1})$. Since it is more or less standard to the experts in this field, we omit the details here.

2.2 Existence of minimizers

The purpose of this section is to establish the following Theorem.

Theorem 2.2.1. *Suppose $Y_{a,b}(M, \partial M) < Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1})$ for some given $a, b > 0$, then $Y_{a,b}(M, \partial M)$ can be achieved by a positive smooth minimizer.*

We adopt the method of subcritical approximations to realize it. For $1 < q \le \frac{n+2}{n-2}$, we define

$$\mathcal{Q}^q_{a,b}[u] = \frac{E[u]}{a\left(\int_M |u|^{q+1} d\mu_{g_0}\right)^{\frac{2}{q+1}} + 2(n-1)b\left(\int_{\partial M} |u|^{\frac{q+3}{2}} d\sigma_{g_0}\right)^{\frac{4}{q+3}}}$$

for any $u \in H^1(M, g_0)$. Notice that $\mathcal{Q}^q_{a,b}[u]$ always has a lower bound when $Y(M, \partial M) \ge 0$, we set

$$\mu_q = \inf_{0 \not\equiv u \in H^1(M, g_0)} \mathcal{Q}^q_{a,b}[u].$$

For brevity, we use $\mu_{(n+2)/(n-2)} = Y_{a,b}(M, \partial M)$ and $\mathcal{Q}^{(n+2)/(n-2)}_{a,b}[u] = \mathcal{Q}_{a,b}[u]$.

Lemma 2.2.2. *Given $a, b > 0$, there holds $\limsup_{q \nearrow \frac{n+2}{n-2}} \mu_q \le Y_{a,b}(M, \partial M)$. Moreover, if $Y(M, \partial M) \ge 0$, there holds $\lim_{q \nearrow \frac{n+2}{n-2}} \mu_q = Y_{a,b}(M, \partial M)$.*

Proof. For any $\epsilon > 0$, there exists $\bar{u} > 0$ such that $\mathcal{Q}_{a,b}[\bar{u}] \le Y_{a,b}(M, \partial M) + \epsilon$. For each $\bar{u}$, there holds $\lim_{q \nearrow \frac{n+2}{n-2}} \mathcal{Q}^q_{a,b}[\bar{u}] = \mathcal{Q}_{a,b}[\bar{u}]$. Then we have

$$\limsup_{q \nearrow \frac{n+2}{n-2}} \mu_q \le \limsup_{q \nearrow \frac{n+2}{n-2}} \mathcal{Q}^q_{a,b}[\bar{u}] \le Y_{a,b}(M, \partial M) + \epsilon,$$

which gives the first assertion. If $Y(M,\partial M)\geq 0$, then $E[u]\geq 0$ for any $u\in H^1(M,g_0)$. Notice that

$$\mathcal{Q}_{a,b}[u]=\mathcal{Q}^q_{a,b}[u]\frac{a\left(\int_M|u|^{\frac{q+2}{2}}d\mu_{g_0}\right)^{\frac{2}{q+1}}+2(n-1)b\left(\int_{\partial M}|u|^{\frac{q+3}{2}}d\sigma_{g_0}\right)^{\frac{4}{q+3}}}{a\left(\int_M|u|^{\frac{2n}{n-2}}d\mu_{g_0}\right)^{\frac{n-2}{n}}+2(n-1)b\left(\int_{\partial M}|u|^{\frac{2(n-1)}{n-2}}d\sigma_{g_0}\right)^{\frac{n-2}{n-1}}}.$$

Hence the second assertion follows by Hölder's inequality and letting $q\nearrow\frac{n+2}{n-2}$. $\square$

Remark 2.2.3. We point out that there also holds $\lim_{q\nearrow\frac{n+2}{n-2}}\mu_q=Y_{a,b}(M,\partial M)$ when $Q(M,\partial M)$ is a negative real number (cf. [27, Remark 7.1]).

Again thanks to [27], it is enough to prove Theorem 2.2.1 when $Y(M,\partial M)\geq 0$.

Lemma 2.2.4. *Let (M,g_0) be a smooth compact Riemannian manifold of dimension $n\geq 3$. Let $2\leq p<\frac{2(n-1)}{n-2}$, then given any $\epsilon>0$, there exists $C=C(n,M,g_0)>0$ such that*

$$\left(\int_{\partial M}|\varphi|^p d\sigma_{g_0}\right)^{\frac{2}{p}}\leq\epsilon\int_M|\nabla\varphi|^2_{g_0}d\mu_{g_0}+\frac{C}{\epsilon}\int_M\varphi^2 d\mu_{g_0}$$

for any $\varphi\in H^1(M,g_0)$.

Proof. By negation, there exist some $\epsilon_0>0$ and $\{\varphi_j;j\in\mathbb{N}\}\subset H^1(M,g_0)$ such that

$$1=\left(\int_{\partial M}|\varphi_j|^p d\sigma_{g_0}\right)^{\frac{2}{p}}>\epsilon_0\int_M|\nabla\varphi_j|^2_{g_0}d\mu_{g_0}+\frac{j}{\epsilon_0}\int_M\varphi_j^2 d\mu_{g_0}.$$

From this, $\{\varphi_j\}$ is uniformly bounded in $H^1(M,g_0)$ and $\int_M\varphi_j^2 d\mu_{g_0}\to 0$ as $j\to\infty$. Then up to a subsequence, $\varphi_j\rightharpoonup\varphi$ weakly in $H^1(M,g_0)$, $\varphi_j\to\varphi$ strongly in $L^2(M,g_0)$ and $L^p(\partial M,g_0)$ as $j\to\infty$. Notice that $\varphi_j\to 0$ in $L^2(M,g_0)$ as $j\to\infty$. Thus we obtain $\varphi=0$ a.e. in M, which contradicts $\int_{\partial M}|\varphi|^p d\sigma_{g_0}=\lim_{j\to\infty}\int_{\partial M}|\varphi_j|^p d\sigma_{g_0}=1$. $\square$

Lemma 2.2.5. *Let (M,g_0) be a smooth compact Riemannian manifold of dimension $n\geq 3$ with boundary. Given $a,b>0$, then*

(i) Let $\varphi\in C_c^\infty(\overline{\mathbb{R}^n_+})$, there holds

$$\begin{aligned}&a\left(\int_{\mathbb{R}^n_+}|\varphi|^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{n}}+2(n-1)b\left(\int_{\mathbb{R}^{n-1}}|\varphi|^{\frac{2(n-1)}{n-2}}d\sigma\right)^{\frac{n-2}{n-1}}\\ \leq&\frac{1}{Y_{a,b}(\mathbb{R}^n_+,\mathbb{R}^{n-1})}\frac{4(n-1)}{n-2}\int_{\mathbb{R}^n_+}|\nabla\varphi|^2 dy,\end{aligned}$$

equality holds if and only if $\varphi(y) = W(y)$ up to dilations and translations in variables $y^1, \cdots, y^{n-1}$.

(ii) Suppose φ is a smooth function with compact support in a coordinate neighborhood $B_\rho(x_0) \cap \bar{M}$, then $\forall\ \epsilon > 0$ there exists ρ_0 such that $\rho \in (0, \rho_0)$,

$$\begin{aligned}&a\left(\int_M |\varphi|^{\frac{2n}{n-2}} d\mu_{g_0}\right)^{\frac{n-2}{n}} + 2(n-1)b\left(\int_{\partial M} |\varphi|^{\frac{2(n-1)}{n-2}} d\sigma_{g_0}\right)^{\frac{n-2}{n-1}}\\ \leq &\frac{1+\epsilon}{Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1})} \frac{4(n-1)}{n-2} \int_M |\nabla\varphi|^2_{g_0} d\mu_{g_0},\end{aligned}$$

where ρ_0 is independent of x_0.

(iii) Given $\epsilon > 0$, there exists $C(\epsilon)$ such that for every $\varphi \in H^1(M, g_0)$

$$\begin{aligned}&a\left(\int_M |\varphi|^{\frac{2n}{n-2}} d\mu_{g_0}\right)^{\frac{n-2}{n}} + 2(n-1)b\left(\int_{\partial M} |\varphi|^{\frac{2(n-1)}{n-2}} d\sigma_{g_0}\right)^{\frac{n-2}{n-1}}\\ \leq &\frac{1+\epsilon}{Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1})} \frac{4(n-1)}{n-2} \int_M |\nabla\varphi|^2_{g_0} d\mu_{g_0} + C(\epsilon)\int_M \varphi^2 d\mu_{g_0}.\end{aligned}$$

Proof. Assertion (i) is a direct consequence of [36, Theorem 3.3] and [62, Theorem 1.2]. Indeed (ii) and (iii) can be proved by a cut-and-paste argument.

(ii) Note that g_0 is Euclidean in $B_\rho(x_0) \cap \bar{M}$ up to order two under normal coordinates around $x_0 \in M$ or order one under the Fermi coordinates around $x_0 \in \partial M$. Then the inequality follows from (i) for every φ compactly supported in this coordinate chart.

(iii) Choose a finite covering of $\bar{M}$ by local coordinate charts, each of which satisfies the condition of part (ii). Through an argument of a partition of unity subordinate to this covering, the desired Sobolev inequality follows (e.g. [8]). □

Lemma 2.2.6. *For any $1 < q < \frac{n+2}{n-2}$, there exists a positive smooth minimizer u_q for μ_q.*

Proof. Let $\{u_i\} \subset H^1(M, g_0)$ be a minimizing sequence of nonnegative functions for μ_q with the normalization:

$$a\left(\int_M u_i^{q+1} d\mu_{g_0}\right)^{\frac{2}{q+1}} + 2(n-1)b\left(\int_{\partial M} u_i^{\frac{q+3}{2}} d\sigma_{g_0}\right)^{\frac{4}{q+3}} = 1, \forall\ i \in \mathbb{N}.$$

It is routine to show u_i is uniformly bounded in $H^1(M, g_0)$. Up to a subsequence, $u_i \rightharpoonup u_q$ in $H^1(M, g_0)$ and $u_i \to u_q$ in $L^{q+1}(M, g_0)$ and $L^{(q+3)/2}(\partial M, g_0)$ as $i \to \infty$.

Thus we obtain

$$a\left(\int_M u_q^{q+1}d\mu_{g_0}\right)^{\frac{2}{q+1}}+2(n-1)b\left(\int_{\partial M} u_q^{\frac{q+3}{2}}d\sigma_{g_0}\right)^{\frac{4}{q+3}}=1. \tag{2.9}$$

Then it follows from Lemma 2.2.4 and (2.9) that

$$\int_M u_q^{q+1}d\mu_{g_0}\geq C_0>0. \tag{2.10}$$

Next we claim that $\int_{\partial M} u_q^{(q+3)/2}d\sigma_{g_0}>0$. By contradiction, if $u_q=0$ a.e. on ∂M, namely $u_q\in H_0^1(M,g_0)$, then it yields

$$\mu_q=E[u_q]=\inf_{0\not\equiv v\in H_0^1(M,g_0)}\frac{E[v]}{a\left(\int_M |v|^{q+1}d\mu_{g_0}\right)^{\frac{2}{q+1}}}.$$

Thus the nonnegative minimizer $u_q\in H_0^1(M,g_0)$ weakly solves

$$\begin{cases}-\dfrac{4(n-1)}{n-2}\Delta_{g_0}v+R_{g_0}v=\mu_q a^{\frac{q+1}{2}}v^q & \text{in } M,\\ \dfrac{\partial v}{\partial\nu_{g_0}}=0 & \text{on } \partial M.\end{cases}$$

Hence a contradiction is reached by using Hopf boundary point lemma and (2.10).

Consequently u_q is a nonzero, nonnegative minimizer with normalization (2.9) for μ_q. Then $u_q\in H^1(M,g_0)$ weakly solves

$$\begin{cases}-\dfrac{4(n-1)}{n-2}\Delta_{g_0}u_q+R_{g_0}u_q=\mu_q a\left(\displaystyle\int_M u_q^{q+1}d\mu_{g_0}\right)^{\frac{1-q}{1+q}}u_q^q & \text{in } M,\\ \dfrac{2}{n-2}\dfrac{\partial u_q}{\partial\nu_{g_0}}+h_{g_0}u_q=\mu_q b\left(\displaystyle\int_{\partial M} u_q^{q+1}d\sigma_{g_0}\right)^{\frac{1-q}{q+3}}u_q^{\frac{q+1}{2}} & \text{on } \partial M.\end{cases} \tag{2.11}$$

Then the strong maximum principle gives $u_q>0$ in $\bar M$. Furthermore, a regularity theorem in [29] shows u_q is smooth in $\bar M$. $\square$

***Proof of Theorem* 2.2.1**. From Lemma 2.2.6 that for each $1<q<\frac{n+2}{n-2}$, there exists a positive minimizer $u_q\in H^1(M,g_0)$ with the normalization (2.9), which solves (2.11), namely for all $\psi\in H^1(M,g_0)$,

$$\begin{aligned}&\int_M\left(\frac{4(n-1)}{n-2}\langle\nabla u_q,\nabla\psi\rangle_{g_0}+R_{g_0}u_q\psi\right)d\mu_{g_0}+2(n-1)\int_{\partial M}h_{g_0}u_q\psi d\sigma_{g_0}\\ &-\mu_q\left[a\alpha_q\int_M u_q^q\psi d\mu_{g_0}+2(n-1)b\beta_q\int_{\partial M}u_q^{\frac{q+1}{2}}\psi d\sigma_{g_0}\right]=0,\end{aligned} \tag{2.12}$$

where $\alpha_q = \left(\int_M u_q^{q+1} d\mu_{g_0}\right)^{(1-q)/(1+q)}$, $\beta_q = \left(\int_{\partial M} u_q^{(q+3)/2} d\sigma_{g_0}\right)^{(1-q)/(q+3)}$. It follows from Lemma 2.2.2 and (2.9) that u_q is uniformly bounded in $H^1(M, g_0)$. Up to a subsequence, u_q weakly converges to some nonnegative function u in $H^1(M, g_0)$ as $q \nearrow \frac{n+2}{n-2}$, and u weakly solves (2.3). Meanwhile, by Lemma 2.2.2 we get $\mu_q \to Y_{a,b}(M, \partial M)$ as $q \nearrow \frac{n+2}{n-2}$.

From Lemma 2.2.5, for any $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that

$$\begin{aligned}
&a\left(\int_M u_q^{\frac{2n}{n-2}} d\mu_{g_0}\right)^{\frac{n-2}{n}} + 2(n-1)b\left(\int_{\partial M} u_q^{\frac{2(n-1)}{n-2}} d\sigma_{g_0}\right)^{\frac{n-2}{n-1}}\\
\leq&(Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1})^{-1} + \epsilon)\frac{4(n-1)}{n-2}\int_M |\nabla u_q|^2_{g_0} d\mu_{g_0} + C(\epsilon)\int_M u_q^2 d\mu_{g_0}.
\end{aligned}$$

By Hölder's inequality, we have

$$\left(\int_M d\mu_{g_0}\right)^{\frac{n-2}{n} - \frac{2}{q+1}} \left(\int_M u_q^{q+1} d\mu_{g_0}\right)^{\frac{2}{q+1}} \leq \left(\int_M u_q^{\frac{2n}{n-2}} d\mu_{g_0}\right)^{\frac{n-2}{n}},$$

$$\left(\int_{\partial M} d\sigma_{g_0}\right)^{\frac{n-2}{n-1} - \frac{4}{q+3}} \left(\int_{\partial M} u_q^{\frac{q+3}{2}} d\sigma_{g_0}\right)^{\frac{4}{q+3}} \leq \left(\int_{\partial M} u_q^{\frac{2(n-1)}{n-2}} d\sigma_{g_0}\right)^{\frac{n-2}{n-1}}.$$

By choosing q sufficiently close to $\frac{n+2}{n-2}$ and using the normalization (2.9), we get

$$\begin{aligned}
&1-\epsilon\\
\leq&(Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1})^{-1} + \epsilon)\frac{4(n-1)}{n-2}\int_M |\nabla u_q|^2_{g_0} d\mu_{g_0} + C(\epsilon)\int_M u_q^2 d\mu_{g_0}\\
=&(Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1})^{-1} + \epsilon)\left(\mu_q - \int_M R_{g_0} u_q^2 d\mu_{g_0} - 2(n-1)\int_{\partial M} h_{g_0} u_q^2 d\sigma_{g_0}\right)\\
&+ C(\epsilon)\int_M u_q^2 d\mu_{g_0}\\
\leq&(Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1})^{-1} + 2\epsilon)Y_{a,b}(M, \partial M) + C\int_M u_q^2 d\mu_{g_0},
\end{aligned}$$

where the last inequality follows from Lemmas 2.2.5 and 2.2.4. By choosing ϵ small enough and the assumption $Y_{a,b}(M, \partial M) < Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1})$, we get

$$\int_M u_q^2 d\mu_{g_0} \geq C_1 > 0,$$

where C_1 is independent of q. So α_q is uniformly bounded, then after passing to a subsequence we let $\bar{\alpha} = \lim_{q \nearrow \frac{n+2}{n-2}} \alpha_q > 0$. Meanwhile using $u_q \to u$ in $L^2(M, g_0)$ as $q \nearrow \frac{n+2}{n-2}$, we obtain

$$\int_M u^2 d\mu_{g_0} > 0. \tag{2.13}$$

Next we claim that with a constant C_2 independent of q, there holds

$$\int_{\partial M} u_q^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \geq C_2 > 0.$$

By negation, there exists a sequence $\{u_q\}$ such that

$$\lim_{q \nearrow \frac{n+2}{n-2}} \int_{\partial M} u_q^{\frac{q+3}{2}} d\sigma_{g_0} = 0,$$

then we obtain $\int_{\partial M} u^2 d\sigma_{g_0} = \lim_{q \nearrow \frac{n+2}{n-2}} \int_{\partial M} u_q^2 d\sigma_{g_0} = 0$, which implies $u = 0$ a. e. on ∂M. On the other hand, for any $\psi \in H^1(M, g_0)$, we get

$$\beta_q \left| \int_{\partial M} u_q^{\frac{q+1}{2}} \psi d\sigma_{g_0} \right| \leq \left(\int_{\partial M} u_q^{\frac{q+3}{2}} d\sigma_{g_0} \right)^{\frac{2}{q+3}} \|\psi\|_{L^{\frac{q+3}{2}}(M, g_0)} \to 0,$$

as $q \to \infty$. By letting $q \nearrow \frac{n+2}{n-2}$ in equation (2.12), u weakly solves

$$\begin{cases} -\dfrac{4(n-1)}{n-2} \Delta_{g_0} u + R_{g_0} u = a\bar{\alpha} Y_{a,b}(M, \partial M) u^{\frac{n+2}{n-2}} & \text{in } M, \\ \dfrac{\partial u}{\partial \nu_{g_0}} + \dfrac{n-2}{2} h_{g_0} u = 0 & \text{on } \partial M. \end{cases}$$

From (2.13), Hopf boundary point lemma gives $u > 0$ on ∂M. Hence we reach a contradiction.

Consequently, after passing to a further subsequence, we let $0 < \bar{\beta} = \lim_{q \nearrow \frac{n+2}{n-2}} \beta_q$. Furthermore, Fatou's lemma gives

$$\bar{\alpha} \leq \left(\int_M u^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{-\frac{2}{n}}, \quad \bar{\beta} \leq \left(\int_{\partial M} u^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{-\frac{1}{n-1}}.$$

Letting $q \nearrow \frac{n+2}{n-2}$ in (2.12), we obtain

$$\begin{aligned} &\int_M \left(\frac{4(n-1)}{n-2} \langle \nabla u, \nabla \psi \rangle_{g_0} + R_{g_0} u \psi \right) d\mu_{g_0} + 2(n-1) \int_{\partial M} h_{g_0} u \psi d\sigma_{g_0} \\ &- Y_{a,b}(M, \partial M) \left[a\bar{\alpha} \int_M u^{\frac{n+2}{n-2}} \psi d\mu_{g_0} + 2(n-1) b \bar{\beta} \int_{\partial M} u^{\frac{n}{n-2}} \psi d\sigma_{g_0} \right] = 0, \end{aligned} \tag{2.14}$$

for all $\psi \in H^1(M, g_0)$. The strong maximum principle gives $u > 0$ in $\bar{M}$. Test (2.14) with u, it yields

$$\begin{aligned} &Y_{a,b}(M, \partial M) \\ \leq &\mathcal{Q}_{a,b}[u] = \frac{Y_{a,b}(M, \partial M) \left[a\bar{\alpha} \int_M u^{\frac{2n}{n-2}} d\mu_{g_0} + 2(n-1) b \bar{\beta} \int_{\partial M} u^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right]}{a \left(\int_M u^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{\frac{n-2}{n}} + 2(n-1) b \left(\int_{\partial M} u^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{\frac{n-2}{n-1}}} \\ \leq &Y_{a,b}(M, \partial M). \end{aligned}$$

From this, we conclude that

$$\bar{\alpha} = \left(\int_M u^{\frac{2n}{n-2}} d\mu_{g_0}\right)^{-\frac{2}{n}}, \quad \bar{\beta} = \left(\int_{\partial M} u^{\frac{2(n-1)}{n-2}} d\sigma_{g_0}\right)^{-\frac{1}{n-1}}$$

and $Y_{a,b}(M,\partial M) = \mathcal{Q}_{a,b}[u] = E[u]$. Then $u_q \to u$ in $H^1(M, g_0)$ as $q \nearrow \frac{n+2}{n-2}$ and u is a positive minimizer for $Y_{a,b}(M,\partial M)$ and weakly solves (2.3). The regularity of u can follow from a theorem by Cherrier [29]. $\square$

2.3 Compactness of minimizers for various (a,b)

For brevity, we denote by $u_{a,b}$ the positive smooth minimizer of $Y_{a,b}(M,\partial M)$ with the normalization

$$a\left(\int_M u_{a,b}^{\frac{2n}{n-2}} d\mu_{g_0}\right)^{\frac{n-2}{n}} + 2(n-1)b\left(\int_{\partial M} u_{a,b}^{\frac{2(n-1)}{n-2}} d\sigma_{g_0}\right)^{\frac{n-2}{n-1}} = 1. \tag{2.15}$$

Under the conformal change of $g = u_{a,b}^{4/(n-2)} g_0$, we have

$$R_g = aY_{a,b}(M,\partial M)\left(\int_M u_{a,b}^{\frac{2n}{n-2}} d\mu_{g_0}\right)^{-\frac{2}{n}}$$

and

$$h_g = bY_{a,b}(M,\partial M)\left(\int_{\partial M} u_{a,b}^{\frac{2(n-1)}{n-2}} d\sigma_{g_0}\right)^{-\frac{1}{n-1}}.$$

Modulo a positive constant multiple, we get $R_g = 1$ and

$$h_g = \frac{b}{\sqrt{a}}\sqrt{Y_{a,b}(M,\partial M)}\left(\int_M u_{a,b}^{\frac{2n}{n-2}} d\mu_{g_0}\right)^{\frac{1}{n}}\left(\int_{\partial M} u_{a,b}^{\frac{2(n-1)}{n-2}} d\sigma_{g_0}\right)^{-\frac{1}{n-1}}. \tag{2.16}$$

Let K be a compact set of $\{(a,b); a \geq 0, b \geq 0\} \setminus \{(0,0)\}$.

Proposition 2.3.1. *Assume $Y(M,\partial M) \geq 0$ and let $(a,b) \in K$, then $Y_{a,b}(M,\partial M)$ is non-increasing in a for any fixed b, as well as in b for any fixed a, and is continuous in K.*

Proof. The proof is in the spirit of that of [40, Proposition 3.2]. For simplicity, we only prove the assertions for a with fixed b, the others are similar. Notice that $Y(M,\partial M) \geq 0$, then $E[u] \geq 0$ for any $u \in H^1(M,g_0)$. For $0 \leq a_1 \leq a_2$, $Y_{a_1,b}(M,\partial M) \geq Y_{a_2,b}(M,\partial M)$ follows from

$$\mathcal{Q}_{a_1,b}[u] \geq \mathcal{Q}_{a_2,b}[u] \quad \text{for any} \quad u \in H^1(M,g_0).$$

Next we prove the continuity of $Y_{a,b}(M,\partial M)$ in K. Since $Y(M,\partial M)\geq 0$ we may assume the background metric g_0 satisfies $R_{g_0}=0$ in M and $h_{g_0}\geq 0$ on ∂M. Suppose $\{(a_m,b_m);m\in\mathbb{N}\}\subset K$ and $(a_m,b_m)\to(a,b)\in K$ as $m\to\infty$. We assume $a\geq 0, b>0$ for simplicity. On one hand, given any $\epsilon>0$, there exists a $u\in H^1(M,g_0)\setminus\{0\}$ such that $\mathcal{Q}_{a,b}[u]<Y_{a,b}(M,\partial M)+\epsilon$. For this fixed u, $\mathcal{Q}_{a_m,b_m}[u]\to\mathcal{Q}_{a,b}[u]$ as $m\to\infty$. Then

$$\varlimsup_{m\to\infty} Y_{a_m,b_m}(M,\partial M)\leq \lim_{m\to\infty}\mathcal{Q}_{a_m,b_m}[u]=\mathcal{Q}_{a,b}[u]<Y_{a,b}(M,\partial M)+\epsilon.$$

On the other hand, given any $\epsilon>0$, for each (a_m,b_m) there exists $u_m\in H^1(M,g_0)$ with

$$a_m\left(\int_M u_m^{\frac{2n}{n-2}}d\mu_{g_0}\right)^{\frac{n-2}{n}}+2(n-1)b_m\left(\int_{\partial M}u_m^{\frac{2(n-1)}{n-2}}d\sigma_{g_0}\right)^{\frac{n-2}{n-1}}=1$$

such that $E[u_m]<Y_{a_m,b_m}(M,\partial M)+\epsilon$.

Let $0\leq a_0=\inf_m a_m$ and $0<b_0=\inf_m b_m$. Then it follows from the monotonicity of $Y_{a,b}(M,\partial M)$ that

$$Y_{a_m,b_m}(M,\partial M)\leq Y_{a_m,b_0}(M,\partial M)\leq Y_{a_0,b_0}(M,\partial M).$$

From the above normalization of u_m, we get

$$\begin{aligned}&\frac{4(n-1)}{n-2}\int_M|\nabla u_m|_{g_0}^2d\mu_{g_0}=E[u_m]-2(n-1)\int_{\partial M}h_{g_0}u_m^2d\sigma_{g_0}\\ \leq&Y_{a_0,b_0}(M,\partial M)+\epsilon+C\int_{\partial M}u_m^2d\sigma_{g_0}\leq Y_{a_0,b_0}(M,\partial M)+C.\end{aligned}$$

This yields $\{u_m\}$ is uniformly bounded in $H^1(M,g_0)$. Thus for all sufficiently large m, we have

$$a\left(\int_M u_m^{\frac{2n}{n-2}}d\mu_{g_0}\right)^{\frac{n-2}{n}}+2(n-1)b\left(\int_{\partial M}u_m^{\frac{2(n-1)}{n-2}}d\sigma_{g_0}\right)^{\frac{n-2}{n-1}}>1-\epsilon.$$

Consequently, we obtain

$$Y_{a,b}(M,\partial M)\leq\mathcal{Q}_{a,b}[u_m]<\frac{E[u_m]}{1-\epsilon}<\frac{Y_{a_m,b_m}(M,\partial M)+\epsilon}{1-\epsilon}$$

for all sufficiently large m. □

Lemma 2.3.2. *Suppose $Y_{a,b}(M,\partial M)<Y_{a,b}(\mathbb{R}^n_+,\mathbb{R}^{n-1})$ for all $(a,b)\in K$. Let $u_{a,b}$ be any positive smooth minimizer for $Y_{a,b}(M,\partial M)$ satisfying the normalization (2.15), then there exists $C=C(K,g_0)>0$ such that*

$$\int_M u_{a,b}^2d\mu_{g_0}\geq C,\quad \int_{\partial M}u_{a,b}^{\frac{2(n-1)}{n-2}}d\sigma_{g_0}\geq C,\quad\forall (a,b)\in K.$$

Proof. For $a = 0$, the desired assertions are guaranteed by [37, Proposition 2.1]. So in the following we assume $a > 0$. Given any $\epsilon > 0$, by Lemma 2.2.5 it yields

$$
\begin{aligned}
&Y_{a,b}(M, \partial M)^{-1} E[u_{a,b}] \\
=&a \left(\int_M u^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{\frac{n-2}{n}} + 2(n-1)b \left(\int_{\partial M} u^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{\frac{n-2}{n-1}} \\
\leq&(Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1})^{-1} + \epsilon) \frac{4(n-1)}{n-2} \int_M |\nabla u_{a,b}|^2_{g_0} d\mu_{g_0} + C(\varepsilon) \int_M u^2_{a,b} d\mu_{g_0}.
\end{aligned}
$$

Since $Y(M, \partial M) \geq 0$, we choose an initial metric such that $R_{g_0} \geq 0$ and $h_{g_0} \geq 0$. From Proposition 2.3.1 that $Y_{a,b}(M, \partial M)$ is continuous in K, then there exists $k_0 > 0$ such that

$$
\min_K \{ Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1}) - Y_{a,b}(M, \partial M)) \} \geq k_0.
$$

By choosing ϵ sufficiently small, with a constant $C = C(k_0) > 0$ we obtain

$$
\int_M |\nabla u_{a,b}|^2_{g_0} d\mu_{g_0} \leq C \int_M u^2_{a,b} d\mu_{g_0}. \tag{2.17}
$$

First we claim that $\forall (a,b) \in K, \int_M u^2_{a,b} d\mu_{g_0} \geq \bar{C}_1(K, g_0) > 0$. Otherwise there exists a sequence of minimizers $u_m := u_{a_m,b_m}$ with $(a_m, b_m) \in K$ such that $\int_M u^2_m d\mu_{g_0} \to 0$, then (2.17) gives $\|u_m\|_{H^1(M,g_0)} \to 0$ as $m \to \infty$, which contradicts the normalization (2.15) of u_m.

Next we assert that $\forall (a,b) \in K, \int_{\partial M} u_{a,b}^{2(n-1)/(n-2)} d\sigma_{g_0} \geq C(K, g_0) > 0$. By negation, there exists a sequence of minimizers $u_m = u_{a_m,b_m}$ with $(a_m, b_m) \in K$ such that $\int_{\partial M} u_m^{2(n-1)/(n-2)} d\sigma_{g_0} \to 0$ as $m \to \infty$. Since K is compact, we may assume $(a_m, b_m) \to (a, b) \in K$ and $E[u_m] = Y_{a_m,b_m}(M, \partial M) \to Y_{a,b}(M, \partial M)$ by Proposition 2.3.1 as $m \to \infty$. Notice that $E[u_m] = Y_{a_m,b_m}(M, \partial M)$, it follows from Proposition 2.3.1 that u_m is uniformly bounded in $H^1(M, g_0)$. Up to a subsequence, there hold $u_m \rightharpoonup u$ weakly in $H^1(M, g_0)$ and

$$
\begin{aligned}
\int_M u^2 d\mu_{g_0} &= \lim_{m \to \infty} \int_M u^2_m d\mu_{g_0} \geq \bar{C}_1, \\
\int_{\partial M} u^2 d\sigma_{g_0} &= \lim_{m \to \infty} \int_{\partial M} u^2_m d\sigma_{g_0} = 0.
\end{aligned}
$$

This means $u \not\equiv 0$ and $u = 0$ a. e. on ∂M. On the other hand, u_m satisfies

$$
\begin{aligned}
&\int_M \left(\frac{4(n-1)}{n-2}\langle \nabla u_m, \nabla \psi\rangle_{g_0} + R_{g_0} u_m \psi\right) d\mu_{g_0} + 2(n-1)\int_{\partial M} h_{g_0} u_m \psi d\sigma_{g_0} \\
=& \left[2(n-1)b\left(\int_{\partial M} u_m^{\frac{2(n-1)}{n-2}} d\sigma_{g_0}\right)^{-\frac{1}{n-1}} \int_{\partial M} u_m^{\frac{n}{n-2}} \psi d\sigma_{g_0}\right. \\
&\left. + a\left(\int_M u_m^{\frac{2n}{n-2}} d\sigma_{g_0}\right)^{-\frac{2}{n}} \int_M u_m^{\frac{n+2}{n-2}} \psi d\sigma_{g_0}\right] Y_{a_m,b_m}(M, \partial M) \qquad (2.18)
\end{aligned}
$$

for all $\psi \in H^1(M, g_0)$. By Hölder's inequality and the normalization (2.15) for u_m, we have

$$
\left(\int_{\partial M} u_m^{\frac{2(n-1)}{n-2}} d\sigma_{g_0}\right)^{-\frac{1}{n-1}} \int_{\partial M} u_m^{\frac{n}{n-2}} \psi d\sigma_{g_0} \to 0
$$

and

$$
\int_M u_m^{\frac{2n}{n-2}} d\mu_{g_0} \to a^{\frac{n}{2-n}}, \quad \text{as} \quad m \to \infty.
$$

By letting $m \to \infty$ in (2.18), u weakly solves

$$
\begin{cases}
-\dfrac{4(n-1)}{n-2}\Delta_{g_0} u + R_{g_0} u = a^{\frac{n}{n-2}} Y_{a,b}(M, \partial M) u^{\frac{n+2}{n-2}} & \text{in } M, \\
\dfrac{2}{n-2}\dfrac{\partial u}{\partial \nu_{g_0}} + h_{g_0} u = 0 & \text{on } \partial M.
\end{cases}
$$

Then Hopf boundary point lemma gives $u > 0$ on ∂M. This yields a contradiction. $\square$

Based on these preparations, we are now in a position to establish the following compactness theorem

Theorem 2.3.3. *Let K and $\mathcal{M}_{a,b}$ as defined above. Suppose $Y_{a,b}(M, \partial M) < Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1})$ for all $(a, b) \in K$, then there exists $C = C(K, g_0)$ such that*

$$
C^{-1} \leq u_{a,b} \leq C, \quad \|u_{a,b}\|_{C^2(M)} \leq C, \quad \forall \ u_{a,b} \in \cup_{(a,b)\in K} \mathcal{M}_{a,b}.
$$

Proof. We only need proof the assertion for $Y(M, \partial M) \geq 0$ due to the same reason of [27].

First we claim that there exits $C = C(K, g_0)$ such that $u_{a,b} \leq C$ for any $(a, b) \in K$. By contradiction, suppose there exist sequences $\{(a_m, b_m); m \in \mathbb{N}\} \subset K$ and $\{p_m; m \in \mathbb{N}\} \subset \bar{M}$ such that

$$
r_m := u_{a_m,b_m}(p_m) = \max_{x \in \bar{M}} u_{a_m,b_m}(x) \to \infty \quad \text{as} \quad m \to \infty.
$$

For brevity, we set $u_m = u_{a_m,b_m}$. Since M is compact, we may assume $p_m \to p_0 \in \bar{M}$ as $m \to \infty$.

If $\lim_{m\to\infty} \mathrm{dist}_{g_0}(p_m, \partial M) r_m^{\frac{2}{n-2}} = \infty$, under normal coordinates around p_0, near p_0 there holds

$$(g_0)_{ij}(x) = \delta_{ij} + O(|x|^2).$$

Observe that

$$\frac{4(n-1)}{n-2}\frac{1}{\sqrt{\det g_0}}\partial_i(\sqrt{\det g_0}g_0^{ij}\partial_j u_m) - R_{g_0}u_m + \tilde{a}_m u_m^{\frac{n+2}{n-2}} = 0$$

in Ω_ρ, where

$$\tilde{a}_m = a_m\left(\int_M u_m^{\frac{2n}{n-2}}d\mu_{g_0}\right)^{-\frac{2}{n}} Y_{a_m,b_m}(M,\partial M).$$

Define $\rho_m = \rho r_m^{\frac{2}{n-2}}$ and

$$v_m(y) = r_m^{-1}u_m(\exp_{p_m}(y r_m^{-\frac{2}{n-2}})) \text{ for } y \in B_{\rho_m}(0) \subset \mathbb{R}^n.$$

Then $v_m(0) = 1$ and $0 < v_m(y) \le 1$ in $B_{\rho_m}(0)$. Let $g_m(y) = g_0(\exp_{p_m}(y r_m^{-\frac{2}{n-2}}))$, $f_m(y) = r_m^{-\frac{4}{n-2}} R_{g_0}(\exp_{p_m}(y r_m^{-\frac{2}{n-2}}))$. Then v_m satisfies

$$\frac{4(n-1)}{n-2}\frac{1}{\sqrt{\det g_m}}\frac{\partial}{\partial y^i}(\sqrt{\det g_m}g_m^{ij}\frac{\partial}{\partial y^j}v_m) - f_m v_m + \tilde{a}_m v_m^{\frac{n+2}{n-2}} = 0$$

in $B_{\rho_m}(0)$. As $m \to \infty$, there hold

$$(g_m)_{ij} \to \delta_{ij} \quad f_m \to 0 \text{ in } C^1(\hat{K}) \text{ for any compact set } \hat{K} \subset \mathbb{R}^n.$$

Since K is compact and from Lemma 2.3.2 that $\tilde{a}_m$ is uniformly bounded, up to a subsequence we get

$$(a_m, b_m) \to (a, b), \quad \tilde{a}_m \to \tilde{a}, \text{ as } m \to \infty.$$

From the $W^{2,p}$-estimate, $\|v_m\|_{C^\lambda(B_{r_m})}$ is uniformly bounded for any $\lambda \in (0,1)$. Applying Schauder interior estimates and the diagonal method to extract a subsequence from $\{v_m\}$, still denote as $\{v_m\}$, we obtain $v_m \to v$ in $C^{2,\lambda}(\hat{K})$, as $m \to \infty$. Moreover v satisfies

$$\frac{4(n-1)}{n-2}\Delta v + \tilde{a}v^{\frac{n-2}{n+2}} = 0 \text{ in } \mathbb{R}^n.$$

Notice that $v(0) = 1$ and $0 \le v \le 1$, the strong maximum principle gives $v > 0$. From Fatou's lemma, we have

$$\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} dx \le \liminf_{m\to\infty} \int_{B_{\rho_m}(0)} v_m^{\frac{2n}{n-2}} \sqrt{\det g_m} dx \le \liminf_{m\to\infty} \int_M u_m^{\frac{2n}{n-2}} d\mu_{g_0}. \tag{2.19}$$

Recall that

$$\tilde{a} = aY_{a,b}(M, \partial M) \lim_{m\to\infty} \left(\int_M u_m^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{-\frac{2}{n}}.$$

It is not hard to show that if $Y(M, \partial M) = 0$, then $Y_{a,b}(M, \partial M) = 0$ for any $(a, b) \in K$. If either $a = 0$ or $Y(M, \partial M) = 0$, then $\tilde{a} = 0$. Then the strong maximum principle gives $v \equiv 1$. Using similar arguments in Lemma 2.3.2, one can show that u_m is uniformly bounded in $H^1(M, g_0)$. From this and (2.19), we have

$$\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} dx \le C,$$

which contradicts $v \equiv 1$ in $\mathbb{R}^n$. If $Y(M, \partial M) > 0$ and $a > 0$, then $\tilde{a} > 0$. Observe that

$$\tilde{a} \int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} dx = \frac{4(n-1)}{n-2} \int_{\mathbb{R}^n} |\nabla v|^2 dx = 2^{\frac{2}{n}} a Y_{a,0}(\mathbb{R}^n_+, \mathbb{R}^{n-1}) \left(\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}. \tag{2.20}$$

Together with Proposition 2.3.1, (2.19) and (2.20) give

$$Y_{a,0}(\mathbb{R}^n_+, \mathbb{R}^{n-1}) \ge Y_{a,0}(M, \partial M) \ge Y_{a,b}(M, \partial M) \ge 2^{\frac{2}{n}} Y_{a,0}, (\mathbb{R}^n_+, \mathbb{R}^{n-1}),$$

which obviously yields a contradiction.

If $\lim_{m\to\infty} \mathrm{dist}_{g_0}(p_m, \partial M) r_m^{\frac{2}{n-2}} < \infty$. Let $X = (x^1, \cdots, x^{n-1})$ be the normal coordinates of $x \in \partial M$ around p_0 and $\nu(X) := \nu_{g_0}(X)$ be the unit outward normal at $x \in \partial M$. For small $t \ge 0$, $\exp_X(-t\nu(X)) : B^+_\rho(0) \to \Omega_\rho \subset M$ is a diffeomorphism, then $(x^1, \cdots, x^{n-1}, t)$ are called the Fermi coordinates around p_0. Without loss of generality, we assume $p_m \in \Omega_\rho$ and denote by $p_m = \exp_{X_m}(-t_m \nu(X_m))$.

Under these coordinates, we have

$$\begin{cases} \dfrac{4(n-1)}{n-2} \dfrac{1}{\sqrt{\det g_0}} \partial_i(\sqrt{\det g_0} g_0^{ij} \partial_j u_m) - R_{g_0} u_m + \tilde{a}_m u_m^{\frac{n+2}{n-2}} = 0 & \text{in } \Omega_\rho, \\ \dfrac{2}{n-2} \dfrac{\partial u_m}{\partial \nu_{g_0}} + h_{g_0} u_m = \tilde{b}_m u_m^{\frac{n}{n-2}} & \text{on } \partial\Omega_\rho \cap \partial M, \end{cases}$$

where

$$\tilde{a}_m = a_m \left(\int_M u_m^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{-\frac{2}{n}} Y_{a_m,b_m}(M,\partial M),$$
$$\tilde{b}_m = b_m \left(\int_{\partial M} u_m^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{-\frac{1}{n-1}} Y_{a_m,b_m}(M,\partial M).$$

Define $\rho_m = \rho r_m^{\frac{2}{n-2}}$ and

$$v_m(X,t) = r_m^{-1} u_m(\exp_{X_m}(-t_m r_m^{-\frac{2}{n-2}} \nu(X r_m^{-\frac{2}{n-2}}))) \quad \text{in} \quad B_{\rho_m}^+(0).$$

Then $v_m(0) = 1$ and $0 < v_m(X,t) \le 1$ in $B_{\rho_m}^+(0)$. We set

$$\begin{aligned} g_m(X,t) =& g_0\big(\exp_{X_m}(-t_m r_m^{-\frac{2}{n-2}} \nu(X r_m^{-\frac{2}{n-2}}))\big), \\ \tilde{f}_m(X,t) =& r_m^{-\frac{4}{n-2}} R_{g_0}\big(\exp_{X_m}(-t_m r_m^{-\frac{2}{n-2}} \nu(X r_m^{-\frac{2}{n-2}}))\big), \\ h_m(X) =& r_m^{-\frac{2}{n-2}} h_{g_0}(X r_m^{-\frac{2}{n-2}}). \end{aligned}$$

Thus v_m satisfies

$$\begin{cases} \dfrac{4(n-1)}{n-2} \dfrac{1}{\sqrt{\det g_m}} \partial_i(\sqrt{\det g_m} g_m^{ij} \partial_j v_m) - \tilde{f}_m v_m + \tilde{a}_m v_m^{\frac{n+2}{n-2}} = 0 & \text{in} \quad B_{\rho_m}^+, \\ -\dfrac{2}{n-2} \partial_t v_m + h_m v_m - \tilde{b}_m v_m^{\frac{n}{n-2}} = 0 & \text{on} \quad D_{\rho_m}. \end{cases}$$

Since $r_m \to \infty$ as $m \to \infty$, we have

$$(g_m)_{ij} \to \delta_{ij}, \quad \tilde{f}_m, h_m \to 0 \quad \text{in} \quad C^1(\tilde{K})$$

for any compact set $\tilde{K} \subset \overline{\mathbb{R}^n_+}$. Since K is compact and from Lemma 2.3.2 that $\tilde{a}_m, \tilde{b}_m$ are bounded, up to a subsequence we have

$$(a_m, b_m) \to (a,b), \quad \tilde{a}_m \to \tilde{a}, \quad \tilde{b}_m \to \tilde{b} \quad \text{as} \quad m \to \infty.$$

From $W^{2,p}$-estimate, $\|v_m\|_{C^\lambda(\overline{B_{\rho_m}^+})}$ is uniformly bounded for any $\lambda \in (0,1)$. Applying Schauder estimates and the diagonal method to extract a subsequence from $\{v_m\}$, still denote as $\{v_m\}$, we obtain $v_m \to v$ in $C^{2,\lambda}(\tilde{K})$, as $m \to \infty$. Moreover v satisfies

$$\begin{cases} \dfrac{4(n-1)}{n-2} \Delta v + \tilde{a} v^{\frac{n-2}{n+2}} = 0 & \text{in} \quad \mathbb{R}^n_+, \\ -\dfrac{\partial v}{\partial t} - \tilde{b} v^{\frac{n}{n-2}} = 0 & \text{on} \quad \mathbb{R}^{n-1}. \end{cases} \tag{2.21}$$

Notice that $v(0) = 1$ and $0 \le v \le 1$, the strong maximum principle gives $v > 0$. Recall that

$$\tilde{a} = aY_{a,b}(M, \partial M) \lim_{m\to\infty} \left(\int_M u_m^{\frac{2n}{n-2}} d\mu_{g_0}\right)^{-\frac{2}{n}},$$
$$\tilde{b} = bY_{a,b}(M, \partial M) \lim_{m\to\infty} \left(\int_{\partial M} u_m^{\frac{2(n-1)}{n-2}} d\sigma_{g_0}\right)^{-\frac{1}{n-1}}.$$

Fatou's lemma gives

$$\int_{\mathbb{R}^n_+} v^{\frac{2n}{n-2}} dx \le \liminf_{m\to\infty} \int_{B^+_{\rho m}} v_m^{\frac{2n}{n-2}} \sqrt{\det g_m} dx \le \liminf_{m\to\infty} \int_M u_m^{\frac{2n}{n-2}} d\mu_{g_0},$$
$$\int_{\mathbb{R}^{n-1}} v^{\frac{2(n-1)}{n-2}} d\sigma \le \liminf_{m\to\infty} \int_{D_{\rho m}} v_m^{\frac{2(n-1)}{n-2}} \sqrt{\det g_m} d\sigma \le \liminf_{m\to\infty} \int_{\partial M} u_m^{\frac{2(n-1)}{n-2}} d\sigma_{g_0}.$$

If $Y(M, \partial M) = 0$, then $\tilde{a} = \tilde{b} = 0$. Then the strong maximum principle gives $v \equiv 1$ in $\mathbb{R}^n_+$. As above, we also get $v \in L^{2n/(n-2)}(\mathbb{R}^n_+)$. Thus we reach a contradiction. If $Y(M, \partial M) > 0$, testing with v in problem (2.21), we get

$$\begin{aligned}
&\tilde{a} \int_{\mathbb{R}^n_+} v^{\frac{2n}{n-2}} dx + 2(n-1)\tilde{b} \int_{\mathbb{R}^{n-1}} v^{\frac{2(n-1)}{n-2}} d\sigma \\
=&\frac{4(n-1)}{n-2} \int_{\mathbb{R}^n_+} |\nabla v|^2 dx \\
=&\left[a \left(\int_{\mathbb{R}^n_+} v^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}} + 2(n-1)b \left(\int_{\mathbb{R}^{n-1}} v^{\frac{2(n-1)}{n-2}} d\sigma\right)^{\frac{n-2}{n-1}}\right] Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1}).
\end{aligned}$$

Since $a^2 + b^2 > 0$ and $Y(M, \partial M) > 0$ imply $\tilde{a}^2 + \tilde{b}^2 > 0$, we have

$$Y_{a,b}(M, \partial M) = \lim_{m\to\infty} Y_{a_m,b_m}(M, \partial M) \ge Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1}),$$

which contradicts the assumption $Y_{a,b}(M, \partial M) < Y_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1}), \ \forall\ (a, b) \in K$.

Finally based on the above upper bound, it follows from Lemma 2.3.2 and [2, Proposition A-4] that $\forall\ (a, b) \in K$, $u_{a,b}$ has a positive uniform lower bound. Then Schauder estimates give the C^2-estimate of $u_{a,b}$ in K. □

As a byproduct of Proposition 2.3.1, there hold

$$\lim_{a\to 0^+} Y_{a,b}(M, \partial M) = Y_{0,b}(M, \partial M), \quad \text{for any fixed } b > 0,$$
$$\lim_{b\to 0^+} Y_{a,b}(M, \partial M) = Y_{a,0}(M, \partial M), \quad \text{for any fixed } a > 0.$$

From these together with Theorem 2.3.3, when $Y(M,\partial M)>0$ expression (2.16) shows that the normalized conformal metric of scalar curvature 1 has positive constant mean curvature, which runs in a large set of $\mathbb{R}_+$.

2.4 Construction of test functions

In this section, we use the following notation: given any $\rho>0$, let

$$\begin{aligned} B_\rho^+(0)&=B_\rho(0)\cap\mathbb{R}^n_+; \qquad \partial^+B_\rho^+(0)=\partial B_\rho^+(0)\cap\mathbb{R}^n_+;\\ D_\rho(0)&=\partial B_\rho^+(0)\backslash\partial^+B_\rho^+(0). \end{aligned}$$

From now on, we assume $Y(M,\partial M)>0$. Recall that $d=[(n-2)/2]$ when $n\geq 3$. By the result of Marques [65], for each $x_0\in\partial M$ there exists a conformal metric $g_{x_0}=f_{x_0}^{4/(n-2)}g_0$ with $f_{x_0}(x_0)=1$. Suppose $\Psi_{x_0}:B^+_{2\rho}(0)\to M$ is the g_{x_0}-Fermi coordinates around x_0, set $x=\Psi_{x_0}(y)$ for $y\in B^+_{2\rho}(0)$. Under these coordinates, there hold $\det g_{x_0}=1+O(|y|^{2d+2})$, $(g_{x_0})_{ij}(0)=\delta_{ij}$ and $(g_{x_0})_{ni}(y)=\delta_{ni}$, for any $y\in B^+_{2\rho}(0)$ and $i,j=1,...,n$. Let $g_{x_0}=\exp(h)$, where exp denotes the matrix exponential, then the symmetric 2-tensor h has the following properties:

$$\begin{cases} tr\, h(y)=O(|y|^{2d+2})\,, & \text{for } y\in B^+_{2\rho}(0),\\ h_{ab}(0)=0\,, & \text{for } i,j=1,...,n\,,\\ h_{in}(y)=0\,, & \text{for } y\in B^+_{2\rho}(0),\ i=1,...,n\,,\\ \partial_a h_{bc}(0)=0\,, & \text{for } a,b,c=1,...,n-1\,,\\ \sum_{b=1}^{n-1}y^b h_{ab}(y)=0\,, & \text{for } y\in D_{2\rho}(0),\ a=1,...,n-1\,. \end{cases} \tag{2.22}$$

The last two properties follow from the fact that Fermi coordinates are normal on ∂M.

Convention. In the following, we let $a,b,c,\cdots$ range from 1 to $n-1$ and $i,j,k\cdots$ range from 1 to n. We adopt Einstein summation convention and simplify $B^+_\rho(0),\partial^+B^+_\rho(0)$, $D_\rho(0)$ by B^+_ρ, $\partial^+B^+_\rho$, D_ρ without otherwise stated.

Under these conformal Fermi coordinates, the mean curvature satisfies

$$\begin{aligned} h_{g_{x_0}}(x)=&-\frac{1}{2(n-1)}g^{ab}\partial_n g_{ab}(x)\\ =&-\frac{1}{2(n-1)}\partial_n(\log\det(g_{x_0}))(x)=O(|y|^{2d+1}). \end{aligned}\tag{2.23}$$

Let H_{ij} be the Taylor expansion of h_{ij} up to order d, namely

$$H_{ij} = \sum_{|\alpha|=1}^{d} \partial^\alpha h_{ij} y^\alpha,$$

where α is a multi-index and $\partial^\alpha h_{ij} = \partial^\alpha h_{ij}(0)$. Then H satisfies (2.22) except the first property replaced by $\mathrm{tr} H = 0$.

2.4.1 Linearization of scalar curvature and mean curvature

From (2.2) and (2.1), we get

$$W_\epsilon \partial_i \partial_j W_\epsilon - \frac{n}{n-2}\partial_i W_\epsilon \partial_j W_\epsilon = \frac{1}{n}\left(W_\epsilon \Delta W_\epsilon - \frac{n}{n-2}|\nabla W_\epsilon|^2\right)\delta_{ij} \quad \text{in} \quad \mathbb{R}^n_+. \tag{2.24}$$

Proposition 2.4.1. *Let V be a smooth vector field in $\overline{\mathbb{R}^n_+}$ satisfying $V_n = 0 = \partial_n V_a$ on $\mathbb{R}^{n-1}$, where $1 \leq a \leq n-1$. Let*

$$\psi = V_k \partial_k W_\epsilon + \frac{n-2}{2n} W_\epsilon \mathrm{div} V$$

and

$$S_{ij} = \partial_i V_j + \partial_j V_i - \frac{2}{n}\mathrm{div} V \delta_{ij}$$

be a conformal killing operator. Then we have

$$\Delta\psi + n(n+2) W_\epsilon^{\frac{4}{n-2}} \psi = \frac{n-2}{4(n-1)} W_\epsilon \partial_i \partial_j S_{ij} + \partial_i(\partial_j W_\epsilon S_{ij}) \quad \text{in} \quad \mathbb{R}^n_+ \tag{2.25}$$

and

$$\partial_n \psi - \frac{n}{n-2} W_\epsilon^{-1} \partial_n W_\epsilon \psi = \frac{1}{2}\partial_n W_\epsilon S_{nn} + \frac{n-2}{4(n-1)} W_\epsilon \partial_n S_{nn} \quad \text{on} \quad \mathbb{R}^{n-1}. \tag{2.26}$$

Proof. The linearized equations (2.25) and (2.26) for scalar curvature and mean curvature can be verified by direct computations in [14, Proposition 5] and [23, Proposition 5], respectively. Somewhat inspired by Brendle [14], we adopt a geometric proof of these linearized equations. It involves the first variation formulae for scalar curvature and mean curvature at a round metric of the spherical cap Σ.

Let $g_\Sigma = W_\epsilon^{4/(n-2)} g_{\mathbb{R}^n}$ be the standard spherical metric on Σ of constant sectional curvature 4, see also Section 2.1. We now consider a family of perturbed metrics of g_Σ:

$$W_\epsilon^{\frac{4}{n-2}} e^{tS} = \phi_t^*((W_\epsilon - t\psi)^{\frac{4}{n-2}} g_{\mathbb{R}^n}), \quad t \in \mathbb{R}, \tag{2.27}$$

where ϕ_t is one-parameter family of diffeomorphisms on S^n generated by V. Differentiating of (2.27) with respect to t and evaluating at $t=0$, we get

$$W_\epsilon^{\frac{4}{n-2}} S = \mathcal{L}_V(g_\Sigma) - \frac{4}{n-2}\psi W_\epsilon^{-1} g_\Sigma. \tag{2.28}$$

We remark that such a decomposition of symmetric 2-tensor is guaranteed by [10, Lemma 4.57]. Recall that the first variation of scalar curvature (cf. [10, Theorem 1.174 (e)]) is given by:

$$R'_g(h) = -h^{ik}R_{ik} + \nabla^i\nabla^k h_{ik} - \Delta_g \mathrm{tr}_g(h) \tag{2.29}$$

for any symmetric 2-tensor h, where ∇ indicates the covariant derivative of g.

On one hand, set $\tilde{g}_E = e^{tS}$, there holds

$$R_{W_\epsilon^{\frac{4}{n-2}}\tilde{g}_E} = W_\epsilon^{-\frac{n+2}{n-2}}\left(-\frac{4(n-1)}{n-2}\Delta_{\tilde{g}_E}W_\epsilon + R_{\tilde{g}_E}W_\epsilon\right).$$

Notice that $\det \tilde{g}_E = 1$ due to $\mathrm{tr}S = 0$, then

$$\frac{d}{dt}\Big|_{t=0}\Delta_{\tilde{g}_E}W_\epsilon = \frac{d}{dt}\Big|_{t=0}\partial_i(e^{-tS_{ij}}\partial_j W_\epsilon) = -\partial_i(S_{ij}\partial_j W_\epsilon)$$

and (2.29) gives

$$\frac{d}{dt}\Big|_{t=0}R_{\tilde{g}_E} = \partial_i\partial_j S_{ij}.$$

Thus we obtain

$$\begin{aligned} R'_{g_\Sigma}(W_\epsilon^{\frac{4}{n-2}}S) =& \frac{d}{dt}\Big|_{t=0}R_{W_\epsilon^{\frac{4}{n-2}}\tilde{g}_E} \\ =& W_\epsilon^{-\frac{n+2}{n-2}}\left(\frac{4(n-1)}{n-2}\partial_i(S_{ij}\partial_j W_\epsilon) + \partial_i\partial_j S_{ij}W_\epsilon\right). \end{aligned} \tag{2.30}$$

On the other hand, using (2.28) and (2.29), we have

$$R'_{g_\Sigma}(W_\epsilon^{\frac{4}{n-2}}S) = R'_{g_\Sigma}(\mathcal{L}_V(g_\Sigma)) - R'_{g_\Sigma}(\frac{4}{n-2}\psi W_\epsilon^{-1}g_\Sigma), \tag{2.31}$$

where $\mathcal{L}_V(g_\Sigma)$ denotes the Lie derivative of metric g_Σ along the vector field V. In particular, it is routine to verify that

$$R'_{g_\Sigma}(\mathcal{L}_V(g_\Sigma)) = 0. \tag{2.32}$$

It also follows from (2.29) that

$$
\begin{aligned}
&R'_{g_\Sigma}(\frac{4}{n-2}\psi W_\epsilon^{-1}g_\Sigma)\\
=&\frac{4}{n-2}\left[-4n(n-1)W_\epsilon^{-1}\psi+(1-n)\Delta_{g_\Sigma}(W_\epsilon^{-1}\psi)\right]\\
=&-\frac{4(n-1)}{n-2}\left[4nW_\epsilon^{-1}\psi+n(n-2)W_\epsilon^{-1}\psi+W_\epsilon^{-\frac{n+2}{n-2}}\Delta\psi\right]\\
=&-\frac{4(n-1)}{n-2}W_\epsilon^{-\frac{n+2}{n-2}}\left[n(n+2)W_\epsilon^{\frac{4}{n-2}}\psi+\Delta\psi\right].
\end{aligned}\tag{2.33}
$$

Putting (2.30)-(2.33) together, we obtain equation (2.25).

Next we need to show (2.26). Let ν_g be the unit outward normal on $\mathbb{R}^{n-1}$, then

$$\nu_g=-\frac{g^{ni}}{\sqrt{g^{nn}}}\partial_i$$

and

$$
\begin{aligned}
h_g=&-\frac{1}{n-1}g^{ab}\langle\nu_g,\nabla_{\partial_a}\partial_b\rangle=\frac{1}{n-1}g^{ab}g^{ni}(g^{nn})^{-\frac{1}{2}}g_{ij}\Gamma_{ab}^j\\
=&\frac{1}{n-1}g^{ab}\Gamma_{ab}^n(g^{nn})^{-\frac{1}{2}}.
\end{aligned}\tag{2.34}
$$

From conformal change formula of mean curvatures, we get

$$\frac{d}{dt}\Big|_{t=0}h_{W_\epsilon^{\frac{4}{n-2}}\tilde{g}_E}=\frac{2}{n-2}W_\epsilon^{-\frac{n}{n-2}}\frac{d}{dt}\Big|_{t=0}\left(\frac{\partial W_\epsilon}{\partial\nu_{\tilde{g}_E}}+\frac{n-2}{2}h_{\tilde{g}_E}W_\epsilon\right).\tag{2.35}$$

Observe that

$$\frac{\partial W_\epsilon}{\partial\nu_{\tilde{g}_E}}=-(\tilde{g}_E^{nn})^{-1/2}\tilde{g}_E^{ni}\partial_iW_\epsilon,$$

then

$$\frac{d}{dt}\Big|_{t=0}\frac{\partial W_\epsilon}{\partial\nu_{\tilde{g}_E}}=S_{ni}\partial_iW_\epsilon-\frac{1}{2}S_{nn}\partial_nW_\epsilon=\frac{1}{2}S_{nn}\partial_nW_\epsilon,\tag{2.36}$$

where the last identity follows from $S_{an}=0$ on $\mathbb{R}^{n-1}$ due to the assumption that $V_n=0=\partial_nV_a$ on $\mathbb{R}^{n-1}$. Recall that the Christoffel symbols of $\tilde{g}_E$ are given by

$$\tilde{\Gamma}_{ab}^n=\frac{1}{2}\tilde{g}_E^{ni}\left[\partial_b(\tilde{g}_E)_{ai}+\partial_a(\tilde{g}_E)_{ib}-\partial_i(\tilde{g}_E)_{ab}\right]$$

then

$$\frac{d}{dt}\Big|_{t=0}\tilde{\Gamma}_{ab}^n=-\frac{1}{2}\partial_nS_{ab},$$

due to $S_{an}=0$ on $\mathbb{R}^{n-1}$. From this and (2.34), we get

$$\frac{d}{dt}\Big|_{t=0}h_{\tilde{g}_E}=-\frac{1}{2(n-1)}\partial_nS_{aa}=\frac{1}{2(n-1)}\partial_nS_{nn},\tag{2.37}$$

where the last identity follows from $-S_{nn} = S_{aa}$ due to trS $= 0$. Plugging (2.36) and (2.37) into (2.35), we obtain

$$\frac{d}{dt}\Big|_{t=0} h_{W_\epsilon^{\frac{4}{n-2}}\tilde{g}_E} = \frac{2}{n-2} W_\epsilon^{-\frac{n}{n-2}} \left(\frac{1}{2}\partial_n W_\epsilon S_{nn} + \frac{n-2}{4(n-1)} W_\epsilon \partial_n S_{nn} \right). \tag{2.38}$$

On the other hand, using (2.28) we have

$$h'_{g_\Sigma}(W_\epsilon^{\frac{4}{n-2}} S) = h'_{g_\Sigma}(\mathcal{L}_V(g_\Sigma)) - h'_{g_\Sigma}(\frac{4}{n-2}\psi W_\epsilon^{-1} g_\Sigma) \quad \text{on} \quad \mathbb{R}^{n-1}. \tag{2.39}$$

First we assert that

$$h'_{g_\Sigma}(\mathcal{L}_V(g_\Sigma)) = 0 \qquad \text{on} \quad \mathbb{R}^{n-1}. \tag{2.40}$$

Next we compute

$$\begin{aligned} &\frac{d}{dt}\Big|_{t=0} h_{(W_\epsilon - t\psi)^{\frac{4}{n-2}} g_E} = -\frac{2}{n-2}\frac{d}{dt}\Big|_{t=0} \left[(W_\epsilon - t\psi)^{-\frac{n}{n-2}} \partial_n (W_\epsilon - t\psi) \right] \\ =&\frac{2}{n-2} W_\epsilon^{-\frac{n}{n-2}} \left(\partial_n \psi - \frac{n}{n-2} W_\epsilon^{-1} \partial_n W_\epsilon \psi \right). \end{aligned} \tag{2.41}$$

Therefore from (2.38)-(2.41), equation (2.26) follows.

It remains to show assertion (2.40). Define

$$\hat{S}_{ij} := \mathcal{L}_V(g_\Sigma)_{ij} = (V_k \partial_k W_\epsilon^{\frac{4}{n-2}})\delta_{ij} + W_\epsilon^{\frac{4}{n-2}}(\partial_i V_j + \partial_j V_i).$$

For brevity, we abuse $g = g_\Sigma$ for a while. Since $V_n = 0 = \partial_n V_a$ on $\mathbb{R}^n_+$, then $\hat{S}_{an} = 0$ on $\mathbb{R}^n_+$. Observe that

$$(\Gamma^n_{ab})' = \frac{1}{2} g^{ni}(\nabla_b \hat{S}_{ia} + \nabla_a \hat{S}_{ib} - \nabla_i \hat{S}_{ab}) = \frac{1}{2} W_\epsilon^{-\frac{4}{n-2}} (\nabla_b \hat{S}_{na} + \nabla_a \hat{S}_{nb} - \nabla_n \hat{S}_{ab}),$$

then

$$\begin{aligned} g^{ab}(\Gamma^n_{ab})' =& W_\epsilon^{-\frac{4}{n-2}} \left[g^{ab}\hat{S}_{na,b} - \frac{1}{2}\partial_n \mathrm{tr}_g(\hat{S}) \right] \\ =& W_\epsilon^{-\frac{4}{n-2}} \left[W_\epsilon^{-\frac{4}{n-2}} \hat{S}_{na,a} - \frac{1}{2}\partial_n (W_\epsilon^{-\frac{4}{n-2}} \hat{S}_{aa}) \right]. \end{aligned}$$

We compute

$$\begin{aligned} \hat{S}_{na,a} =& \partial_a \hat{S}_{na} - \Gamma^i_{na}\hat{S}_{ia} - \Gamma^i_{aa}\hat{S}_{ni} \\ =& -\Gamma^b_{na}\hat{S}_{ba} - \Gamma^n_{aa}\hat{S}_{nn} \\ =& -2T_c W_\epsilon^{\frac{2}{n-2}} [\hat{S}_{aa} - (n-1)\hat{S}_{nn}], \end{aligned}$$

where the last identity follows from

$$\begin{aligned}\Gamma^b_{na} =&\frac{1}{2}g^{bc}\partial_n g_{ca} = \frac{1}{2}W_\epsilon^{-\frac{4}{n-2}}\partial_n W_\epsilon^{\frac{4}{n-2}}\delta_{ab} = 2T_c W_\epsilon^{\frac{2}{n-2}}\delta_{ab},\\ \Gamma^n_{aa} =&-\frac{1}{2}g^{nn}\partial_n g_{aa} = -\frac{1}{2}W_\epsilon^{-\frac{4}{n-2}}\partial_n W_\epsilon^{\frac{4}{n-2}}\delta_{aa} = -2(n-1)T_c W_\epsilon^{\frac{2}{n-2}}\end{aligned}$$

in view of (2.1). From (2.34), we have

$$(n-1)(h_g)'(\hat{S}) = -\hat{S}^{ab}\pi_{ab} + \frac{n-1}{2}\frac{\hat{S}^{nn}}{g^{nn}}h_g + \frac{(\Gamma^n_{ab})'}{\sqrt{g^{nn}}}g^{ab}.$$

From (2.24) we get

$$\begin{aligned}&\partial_n\partial_a W_\epsilon^{\frac{4}{n-2}} = \frac{4}{n-2}\left[\frac{6-n}{n-2}W_\epsilon^{\frac{4}{n-2}-2}\partial_n W_\epsilon \partial_a W_\epsilon + \partial_n\partial_a W_\epsilon\right]\\ =&\frac{4}{n-2}\frac{6}{n-2}W_\epsilon^{\frac{4}{n-2}-2}\partial_n W_\epsilon\partial_a W_\epsilon = 6T_c W_\epsilon^{\frac{2}{n-2}}\partial_a W_\epsilon^{\frac{4}{n-2}},\end{aligned}$$

whence

$$\begin{aligned}&\partial_n\hat{S}_{aa} = \partial_n\left[(n-1)(V_k\partial_k W_\epsilon^{\frac{4}{n-2}}) + 2W_\epsilon^{\frac{4}{n-2}}\partial_a V_a\right]\\ =&(n-1)(V_a\partial_n\partial_a W_\epsilon^{\frac{4}{n-2}} + \partial_n V_n \partial_n W_\epsilon^{\frac{4}{n-2}}) + 2\partial_n W_\epsilon^{\frac{4}{n-2}}\partial_a V_a\\ =&(n-1)T_c W_\epsilon^{\frac{2}{n-2}}(6V_a\partial_a W_\epsilon^{\frac{4}{n-2}} + 4\partial_n V_n W_\epsilon^{\frac{4}{n-2}}) + 8T_c W_\epsilon^{\frac{6}{n-2}}\partial_a V_a.\end{aligned}$$

Then we have

$$\begin{aligned}&\partial_n(W_\epsilon^{-\frac{4}{n-2}}\hat{S}_{aa}) = W_\epsilon^{-\frac{4}{n-2}}\partial_n\hat{S}_{aa} + \partial_n W_\epsilon^{-\frac{4}{n-2}}\hat{S}_{aa}\\ =&(n-1)T_c W_\epsilon^{-\frac{2}{n-2}}(6V_a\partial_a W_\epsilon^{\frac{4}{n-2}} + 4\partial_n V_n W_\epsilon^{\frac{4}{n-2}})\\ &+ 8T_c W_\epsilon^{\frac{2}{n-2}}\partial_a V_a - 4T_c W_\epsilon^{-\frac{2}{n-2}}\hat{S}_{aa}.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}&-T_c^{-1}W_\epsilon^{\frac{6}{n-2}}g^{ab}(\Gamma^n_{ab})'\\ =&2[\hat{S}_{aa} - (n-1)\hat{S}_{nn}] + (n-1)(3V_a\partial_a W_\epsilon^{\frac{4}{n-2}} + 2\partial_n V_n W_\epsilon^{\frac{4}{n-2}})\\ &+ 4W_\epsilon^{\frac{4}{n-2}}\partial_a V_a - 2\hat{S}_{aa}\\ =&-2(n-1)\hat{S}_{nn} + 4W_\epsilon^{\frac{4}{n-2}}\partial_a V_a + (n-1)(3V_a\partial_a W_\epsilon^{\frac{4}{n-2}} + 2\partial_n V_n W_\epsilon^{\frac{4}{n-2}}).\end{aligned}$$

Putting these facts together and using $\pi_{ab} = -2T_c g_{ab}$, we conclude that

$$
\begin{aligned}
&(n-1)T_c^{-1}W_\epsilon^{\frac{4}{n-2}}(h_g)'(\hat{S})\\
=&2\hat{S}_{aa} - (n-1)\hat{S}_{nn} + T_c^{-1}W_\epsilon^{\frac{6}{n-2}}g^{ab}(\Gamma_{ab}^n)'\\
=&2\hat{S}_{aa} + (n-1)\hat{S}_{nn} - (n-1)(3V_a\partial_a W_\epsilon^{\frac{4}{n-2}} + 2\partial_n V_n W_\epsilon^{\frac{4}{n-2}}) - 4W_\epsilon^{\frac{4}{n-2}}\partial_a V_a\\
=&2\Big[(n-1)(V_a\partial_a W_\epsilon^{\frac{4}{n-2}}) + 2W_\epsilon^{\frac{4}{n-2}}\partial_a V_a\Big]\\
&+(n-1)\Big[(V_a\partial_a W_\epsilon^{\frac{4}{n-2}}) + 2W_\epsilon^{\frac{4}{n-2}}\partial_n V_n\Big]\\
&-(n-1)(3V_a\partial_a W_\epsilon^{\frac{4}{n-2}} + 2\partial_n V_n W_\epsilon^{\frac{4}{n-2}}) - 4W_\epsilon^{\frac{4}{n-2}}\partial_a V_a\\
=&0,
\end{aligned}
$$

which implies the desired assertion. □

2.4.2 Test functions and their energy estimates

Let $\chi(y) = \chi(|y|)$ be a smooth cut-off function in $\overline{\mathbb{R}^n_+}$ with $\chi = 1$ in B_1^+ and $\chi = 0$ in $\overline{\mathbb{R}^n_+}\backslash B_2^+$. For any $\rho > 0$, set $\chi_\rho(y) = \chi(|y|/\rho)$ for $y \in \mathbb{R}^n_+$. As in [15] and [23], given H_{ij} there exists a smooth vector field V in $\overline{\mathbb{R}^n_+}$ such that

$$
\begin{cases}
\sum\limits_{i=1}^n \partial_i\left[W_\epsilon^{\frac{2n}{n-2}}\Big(\chi_\rho H_{ij} - \partial_i V_j - \partial_j V_i + \frac{2}{n}(\mathrm{div} V)\delta_{ij}\Big)\right] = 0, & \text{in } \mathbb{R}^n_+,\\
\partial_n V_a = V_n = 0, & \text{on } \mathbb{R}^{n-1},
\end{cases}
\tag{2.42}
$$

where $1 \leq i, j \leq n, 1 \leq a \leq n-1$. Moreover, there holds

$$
|\partial^\beta V(y)| \leq C(n, |\beta|)\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|(\epsilon + |y|)^{|\alpha|+1-|\beta|}. \tag{2.43}
$$

We only sketch the proof of the construction of vector field V. Consider the spherical cap (Σ, g_S) as in Proposition 2.4.1 with $\epsilon = 1$. Define

$$
\mathscr{X} = \{V \in H^1(\Sigma, g_\Sigma); \langle V, \nu_{g_\Sigma}\rangle_{g_\Sigma} = 0 \text{ for a vector field } V \text{ on } \partial\Sigma\}
$$

and $\mathscr{H}$ the space of all trace-free symmetric two-tensors on Σ of class L^2. A conformal killing operator $\mathcal{D} : \mathscr{X} \to \mathscr{H}$ on Σ defined as

$$
\mathcal{D}_{g_\Sigma}V = \mathcal{L}_V(g_\Sigma) - \frac{2}{n}(\mathrm{div}_{g_\Sigma}V)g_\Sigma.
$$

Similarly as in the appendix of [15], we know that $\ker\mathcal{D}_{g_\Sigma}$ is finite dimensional. We define

$$\mathscr{X}_0 = \{V \in \mathscr{X}\,; \langle V, Z\rangle_{L^2(\Sigma, g_\Sigma)} = 0, \forall\, Z \in \ker\mathcal{D}_{g_\Sigma}\}.$$

Using a similar argument in [15, Proposition A.3], we assert that for any symmetric two-tensor $\tilde{h}$ with compact support in $\mathbb{R}^n_+$, there exists a unique vector field $V \in \mathscr{X}_0$ such that

$$\langle W^{\frac{4}{n-2}}\tilde{h} - \mathcal{D}_{g_\Sigma}V, \mathcal{D}_{g_\Sigma}Z\rangle_{L^2(\Sigma, g_\Sigma)} = 0 \ \text{ for all } \ Z \in \mathscr{X}.$$

Furthermore, with a dimensional constant C there holds

$$\|V\|^2_{L^2(\Sigma, g_\Sigma)} + \|\nabla V\|^2_{L^2(\Sigma, g_\Sigma)} \le C\|W^{\frac{4}{n-2}}\tilde{h}\|^2_{L^2(\Sigma, g_\Sigma)}.$$

Based on this estimate and using our W instead, we can construct the vector field V satisfying (2.42) and estimate (2.43) by mimicking the proofs of [23, Propositions 12-13].

As in Proposition 2.4.1, we define symmetric trace-free 2-tensors S and T in $\overline{\mathbb{R}^n_+}$ by

$$S_{ij} = \partial_i V_j + \partial_j V_i - \frac{2}{n}\mathrm{div}V\delta_{ij} \qquad \text{and} \qquad T = H - S\,. \tag{2.44}$$

It follows from (2.42) that T satisfies

$$W_\epsilon \partial_j T_{ij} + \frac{2n}{n-2}\partial_j W_\epsilon T_{ij} = 0\,, \quad \text{in } B^+_{2\rho}. \tag{2.45}$$

For $n \ge 3$, we define an auxiliary function $\psi = \psi_{\epsilon,\rho,H}$ by

$$\psi = \partial_i W_\epsilon V_i + \frac{n-2}{2n}W_\epsilon \mathrm{div}V. \tag{2.46}$$

When $n = 3$, then $d = 0$ and we choose $\psi = 0$. Using (2.43) and (2.2) of W_ϵ, in $B^+_{2\rho}$ we have

$$|\psi(y)| \le C(n, T_c)\epsilon^{\frac{n-2}{2}} \sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}|(\epsilon + |y|)^{|\alpha|+2-n}. \tag{2.47}$$

By the above construction of V and $H_{in} = 0$ in $B^+_{2\rho}$, we know

$$W_\epsilon \partial_i S_{ni} + \frac{2n}{n-2}\partial_i W_\epsilon S_{ni} = 0 \ \text{ in } \ B^+_{2\rho}$$

and $S_{na} = 0$ on $D_{2\rho}$. Thus we get

$$\partial_n S_{nn} = -\partial_a S_{na} - \frac{2n}{n-2} W_\epsilon^{-1} \partial_i W_\epsilon S_{ni} = -\frac{2n}{n-2} W_\epsilon^{-1} \partial_n W_\epsilon S_{nn} \quad \text{on} \quad D_{2\rho}.$$

Combining this and (2.26), we conclude that

$$\partial_n \psi - \frac{n}{n-2} W_\epsilon^{-1} \partial_n W_\epsilon \psi = -\frac{1}{2(n-1)} \partial_n W_\epsilon S_{nn} \qquad \text{on} \quad D_{2\rho}.$$

For future citation, we collect the linearized equations for scalar curvature and mean curvature in the following

Lemma 2.4.2. *The function ψ satisfies*

$$\begin{cases} \Delta\psi + n(n+2) W_\epsilon^{\frac{4}{n-2}} \psi = \dfrac{n-2}{4(n-1)} W_\epsilon \partial_i \partial_j S_{ij} + \partial_i(\partial_j W_\epsilon S_{ij}) & \text{in } B_{2\rho}^+, \\ \partial_n \psi - \dfrac{n}{n-2} W_\epsilon^{-1} \partial_n W_\epsilon \psi = -\dfrac{1}{2(n-1)} \partial_n W_\epsilon S_{nn} & \text{on } D_{2\rho}. \end{cases}$$

Similar to [23, Proposition 5], we collect and derive some properties associated to S and T.

Lemma 2.4.3. *(1) $S_{an} = 0 = T_{an}$, $0 \le a \le n-1$.*

(2) On $D_{2\rho}$, there hold

$$\begin{aligned} \partial_n S_{nn} &= -\frac{2n}{n-2} W_\epsilon^{-1} \partial_n W_\epsilon S_{nn}, \\ \partial_n S_{ab} &= -\frac{1}{n-1} \partial_n S_{nn} \delta_{ab}, \end{aligned}$$

where $1 \le a, b \le n-1$.

Based on Lemma 2.4.2, we rearrange [14, Propositions 5-6] as follows.

Proposition 2.4.4. *There holds*

$$\begin{aligned} &\frac{1}{4} Q_{ik,j} Q_{ik,j} - \frac{1}{2} Q_{ki,k} Q_{li,l} + 2 W_\epsilon^{\frac{2n}{n-2}} T_{ik} T_{ik} \\ =& \frac{1}{4} W_\epsilon^2 \partial_l H_{ik} \partial_l H_{ik} - \frac{2(n-1)}{n-2} \partial_k W_\epsilon \partial_l W_\epsilon H_{ik} H_{il} - 2 W_\epsilon \partial_k W_\epsilon H_{ik} \partial_l H_{il} \\ &- \frac{1}{2} W_\epsilon^2 \partial_k H_{ik} \partial_l H_{il} + \frac{8(n-1)}{n-2} \partial_i W_\epsilon \partial_k \psi H_{ik} - \frac{4(n-1)}{n-2} |\nabla\psi|^2 \\ &+ \frac{4(n-1)}{n-2} n(n+2) W_\epsilon^{\frac{4}{n-2}} \psi^2 - 2 W_\epsilon \psi \partial_i \partial_k H_{ik} + \mathrm{div}\xi, \end{aligned}$$

where

$$Q_{ij,k} = W_\epsilon \partial_k T_{ij} + \frac{2}{n-2}(\partial_l W_\epsilon T_{il}\delta_{jk} + \partial_l W_\epsilon T_{jl}\delta_{ik} - \partial_i W_\epsilon T_{jk} - \partial_j W_\epsilon T_{ik})$$

and the vector field ξ is given by

$$\begin{aligned}\xi_i =& 2W_\epsilon \psi \partial_k H_{ik} - 2W_\epsilon \partial_k \psi H_{ik} - 2\partial_k W_\epsilon \psi H_{ik} - \frac{1}{2} W_\epsilon^2 \partial_i S_{lk} H_{lk} \\ &+ W_\epsilon^2 \partial_l S_{kl} H_{ik} + 2W_\epsilon \partial_l W_\epsilon S_{kl} H_{ik} - W_\epsilon \psi \partial_k S_{ik} + W_\epsilon \partial_k \psi S_{ik} \\ &+ \partial_k W_\epsilon \psi S_{ik} + \frac{1}{4} W_\epsilon^2 \partial_i S_{lk} S_{lk} - \frac{1}{2} W_\epsilon^2 \partial_l S_{kl} S_{ik} - W_\epsilon \partial_l W_\epsilon S_{kl} S_{ik} \\ &- \frac{4(n-1)}{n-2} \partial_k W_\epsilon \psi S_{ik} + \frac{4(n-1)}{n-2} \psi \partial_i \psi - \frac{2}{n-2} W_\epsilon \partial_k W_\epsilon T_{lk} T_{il}. \end{aligned} \tag{2.48}$$

In particular, it yields

$$\xi_n = -\frac{n+2}{2(n-2)} W_\epsilon \partial_n W_\epsilon S_{nn}^2 + \frac{4n(n-1)}{(n-2)^2} W_\epsilon^{-1} \partial_n W_\epsilon \psi^2 \tag{2.49}$$

on $\mathbb{R}^{n-1}$.

Proposition 2.4.5. *There exists $\lambda^* = \lambda^*(n, T_c) > 0$ such that*

$$\begin{aligned}&\lambda^* \epsilon^{n-2} \sum_{i,j=1}^{n} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ij}|^2 \int_{B_\rho^+(0)} (\epsilon + |y|)^{2|\alpha|+2-2n} dy \\ \le& \frac{1}{4} \int_{B_\rho^+(0)} Q_{ij,k} Q_{ij,k} dy - \frac{n^2}{2(n-1)(n-2)} \int_{D_\rho(0)} \partial_n W_\epsilon W_\epsilon S_{nn}^2 d\sigma\end{aligned}$$

for all $2\epsilon \le \rho$.

Proof. Since only the unchanged sign condition of $\partial_n W_\epsilon$ on B_ρ^+ and Lemma 2.4.3 were required in [2, Lemma 3.4], we refer to similar arguments in [2, Proposition 3.5] for the details. □

Our test function is

$$\bar{U}_{(x_0,\epsilon)} = [\chi_\rho (W_\epsilon + \psi)] \circ \Psi_{x_0}^{-1} + (1 - \chi_\rho) \circ \Psi_{x_0}^{-1} \epsilon^{\frac{n-2}{2}} G, \tag{2.50}$$

where $G = G_{x_0}$ is the Green's function of the conformal Laplacian with pole at $x_0 \in \partial M$, coupled with a boundary condition, namely

$$\begin{cases} -\dfrac{4(n-1)}{n-2} \Delta_{g_{x_0}} G_{x_0} + R_{g_{x_0}} G_{x_0} = 0\,, & \text{in } M \backslash \{x_0\}\,, \\ \dfrac{2}{n-2} \dfrac{\partial G_{x_0}}{\partial \nu_{g_{x_0}}} + h_{g_{x_0}} G_{x_0} = 0\,, & \text{on } \partial M \backslash \{x_0\}\,. \end{cases} \tag{2.51}$$

We assume that G is normalized such that $\lim_{y\to 0} G(\Psi_{x_0}(y))|y|^{n-2} = 1$. Then G satisfies the following estimates near x_0, namely for sufficiently small $|y|$ (cf. [4, Proposition B-2]):

$$\begin{aligned}
&|G(\Psi_{x_0}(y)) - |y|^{2-n}| \\
\leq & C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}||y|^{|\alpha|+2-n} + \begin{cases} C|y|^{d+3-n}, & \text{if} \quad n \geq 5, \\ C(1 + |\log|y||), & \text{if} \quad n = 3, 4, \end{cases} \\
&|\nabla(G(\Psi_{x_0}(y)) - |y|^{2-n})| \leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}||y|^{|\alpha|+1-n} + C|y|^{d+2-n}.
\end{aligned} \tag{2.52}$$

Moreover, there holds

$$C(T_c, n)^{-1} \epsilon^{\frac{n-2}{2}} (\epsilon + |y|)^{2-n} \leq W_\epsilon(y) \leq C(T_c, n) \epsilon^{\frac{n-2}{2}} (\epsilon + |y|)^{2-n}.$$

We consider the flux integral as in [15, P.1006]

$$\begin{aligned}
\mathcal{I}(x_0, \rho) = & - \int_{\partial^+ B_\rho^+} |y|^{2-2n} (|y|^2 \partial_j h_{ij} - 2n y^j h_{ij}) \frac{y^i}{|y|} d\sigma \\
& + \frac{4(n-1)}{n-2} \int_{\partial^+ B_\rho^+} (|y|^{2-n} \partial_i G - G \partial_i |y|^{2-n}) \frac{y^i}{|y|} d\sigma
\end{aligned}$$

for $x_0 \in \partial M$ and all sufficiently small $\rho > 0$.

The following estimates on the expansion of scalar curvature can be found in [2, P. 2645], which follows from [14, Proposition 11] and [23, Proposition 3]. Keep in mind that the boundary is not necessarily umbilic here.

Proposition 2.4.6. *The scalar curvature $R_{g_{x_0}}$ satisfies*

$$\begin{aligned}
&|R_{g_{x_0}} - \partial_i \partial_k H_{ik}| \leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}||y|^{|\alpha|-1} + C|y|^{d-1}, \\
&\left| R_{g_{x_0}} - \partial_i \partial_k h_{ik} + \partial_k (H_{ik} \partial_l H_{il}) - \frac{1}{2} \partial_k H_{ik} \partial_l H_{il} + \frac{1}{4} \partial_l H_{ik} \partial_l H_{ik} \right| \\
\leq & C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}|^2 |y|^{2|\alpha|-1} + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}||y|^{|\alpha|+d-1} + C|y|^{2d}
\end{aligned}$$

for $|y|$ sufficiently small.

In order to prove this theorem, we need to estimate the energy $E[\bar{U}_{(x_0,\epsilon)}]$. Notice that

$$\begin{aligned}E[\bar{U}_{(x_0,\epsilon)}]=&\int_M\left(\frac{4(n-1)}{n-2}|\nabla\bar{U}_{(x_0,\epsilon)}|^2_{g_{x_0}}+R_{g_{x_0}}\bar{U}^2_{(x_0,\epsilon)}\right)d\mu_{g_{x_0}}\\&+2(n-1)\int_{\partial M}h_{g_{x_0}}\bar{U}^2_{(x_0,\epsilon)}d\sigma_{g_{x_0}}.\end{aligned}$$

We will estimate $E[\bar{U}_{(x_0,\epsilon)}]$ in $\Psi_{x_0}(B^+_\rho)$ and $M\backslash\Psi_{x_0}(B^+_\rho)$ respectively.

Proposition 2.4.7. *With some sufficiently small $\rho_0>0$, there holds*

$$\begin{aligned}&\int_{B^+_\rho}\left[\frac{4(n-1)}{n-2}|\nabla(W_\epsilon+\psi)|^2_{g_{x_0}}+R_{g_{x_0}}(W_\epsilon+\psi)^2\right]dy\\&+2(n-1)\int_{D_\rho}h_{g_{x_0}}(W_\epsilon+\psi)^2d\sigma\\\leq&4n(n-1)\int_{B^+_\rho}W_\epsilon^{\frac{4}{n-2}}\left(W_\epsilon^2+\frac{n+2}{n-2}\psi^2\right)dy\\&+\int_{\partial^+B^+_\rho}\frac{4(n-1)}{n-2}\partial_iW_\epsilon W_\epsilon\frac{y^i}{|y|}d\sigma+\int_{\partial^+B^+_\rho}(W_\epsilon^2\partial_jh_{ij}-\partial_jW_\epsilon^2h_{ij})\frac{y^i}{|y|}d\sigma\\&-4(n-1)T_c\int_{D_\rho}W_\epsilon^{\frac{2}{n-2}}\left(W_\epsilon^2+2W_\epsilon\psi+\frac{n}{n-2}\psi^2-\frac{n-2}{8(n-1)^2}W_\epsilon^2S^2_{nn}\right)d\sigma\\&-\frac{1}{2}\lambda^*\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|^2\epsilon^{n-2}\int_{B^+_\rho}(\epsilon+|y|)^{2|\alpha|+2-2n}dy\\&+C\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|\epsilon^{n-2}\rho^{|\alpha|+2-n}+C\epsilon^{n-2}\rho^{2d+4-n}\end{aligned}$$

for $0<2\epsilon<\rho<\rho_0\leq 1$, where ρ_0 and C are some constants depending only on n,T_c,g_0.

Proof. Notice that $\bar{U}_{(x_0,\epsilon)}=W_\epsilon+\psi$ in B^+_ρ. First it follows from (2.23) and (2.46) that

$$\int_{D_\rho}h_{g_{x_0}}(W_\epsilon+\psi)^2d\sigma\leq C\int_{D_\rho}|y|^{2d+1}(W_\epsilon+\psi)^2d\sigma\leq C\epsilon^{n-2}\rho^{2d+2}. \tag{2.53}$$

Next we decompose

$$\begin{aligned}&\frac{4(n-1)}{n-2}|\nabla(W_\epsilon+\psi)|^2_{g_{x_0}}+R_{g_{x_0}}(W_\epsilon+\psi)^2\\=&\frac{4(n-1)}{n-2}|\nabla W_\epsilon|^2+\frac{4(n-1)}{n-2}n(n+2)W_\epsilon^{\frac{4}{n-2}}\psi^2+\sum_{i=1}^4J_i,\end{aligned}\tag{2.54}$$

where

$$
\begin{aligned}
J_1 =& \frac{8(n-1)}{n-2}\partial_i W_\epsilon \partial_i \psi - \frac{4(n-1)}{n-2}\partial_i W_\epsilon \partial_k W_\epsilon h_{ik} + W_\epsilon^2 \partial_i \partial_k h_{ik}, \\
& - W_\epsilon^2 \partial_k (H_{ik}\partial_l H_{il}) - 2W_\epsilon \partial_k W_\epsilon H_{ik}\partial_l H_{il}, \\
J_2 =& -\frac{1}{4}W_\epsilon^2 \partial_l H_{ik}\partial_l H_{ik} + \frac{2(n-1)}{n-2}\partial_k W_\epsilon \partial_l W_\epsilon H_{ik}H_{il} + 2W_\epsilon \partial_k W_\epsilon H_{ik}\partial_l H_{il} \\
& + \frac{1}{2}W_\epsilon^2 \partial_k H_{ik}\partial_l H_{il} + 2W_\epsilon \psi \partial_i \partial_k H_{ik} - \frac{8(n-1)}{n-2}\partial_i W_\epsilon \partial_k \psi H_{ik} \\
& + \frac{4(n-1)}{n-2}|\nabla \psi|^2 - \frac{4(n-1)}{n-2}n(n+2)W_\epsilon^{\frac{4}{n-2}}\psi^2, \\
J_3 =& \frac{4(n-1)}{n-2}(g_{x_0}^{ik} - \delta_{ik} + h_{ik} - \frac{1}{2}H_{il}H_{kl})\partial_i W_\epsilon \partial_k W_\epsilon \\
& + \left[R_{g_{x_0}} - \partial_i \partial_k h_{ik} + \partial_k (H_{ik}\partial_l H_{il}) - \frac{1}{2}\partial_k H_{ik}\partial_l H_{il} + \frac{1}{4}\partial_l H_{ik}\partial_l H_{ik}\right] W_\epsilon^2, \\
J_4 =& \frac{8(n-1)}{n-2}(g_{x_0}^{ik} - \delta_{ik} + H_{ik})\partial_i W_\epsilon \partial_k \psi + 2(R_{g_{x_0}} - \partial_i \partial_k H_{ik})W_\epsilon \psi \\
& + R_{g_{x_0}}\psi^2 + \frac{4(n-1)}{n-2}(g_{x_0}^{ik} - \delta_{ik})\partial_i \psi \partial_k \psi.
\end{aligned}
$$

We start with J_1. Rearrange J_1 as

$$
\begin{aligned}
J_1 =& \frac{8(n-1)}{n-2}\partial_i(\partial_i W_\epsilon \psi) - \frac{8(n-1)}{n-2}\psi \Delta W_\epsilon + \partial_i (W_\epsilon^2 \partial_k h_{ik}) - \partial_k (\partial_i W_\epsilon^2 h_{ik}) \\
& + 2\left(W_\epsilon \partial_i \partial_k W_\epsilon - \frac{n}{n-2}\partial_i W_\epsilon \partial_k W_\epsilon\right) h_{ik} - \partial_k (W_\epsilon^2 H_{ik}\partial_l H_{il}).
\end{aligned}
$$

Notice that W_ϵ satisfies

$$
\psi \Delta W_\epsilon = -\frac{(n-2)^2}{2}\partial_i (W_\epsilon^{\frac{2n}{n-2}} V_i).
$$

Thus using $V_n = 0$ on D_ρ, $H_{in} = h_{in} = 0, \operatorname{tr} h = O(|y|^{2d+2})$ in B_ρ^+ and (2.24), we have

$$
\begin{aligned}
& \int_{B_\rho^+} J_1 dy \\
=& -\frac{8(n-1)}{n-2}\int_{D_\rho} \partial_n W_\epsilon \psi d\sigma + \frac{8(n-1)}{n-2}\int_{\partial^+ B_\rho^+} \psi \partial_i W_\epsilon \frac{y^i}{|y|} d\sigma \\
& + 4(n-1)(n-2)\int_{\partial^+ B_\rho^+} W_\epsilon^{\frac{2n}{n-2}} V_i \frac{y^i}{|y|} d\sigma + \int_{\partial^+ B_\rho^+} (W_\epsilon^2 \partial_k h_{ik} - \partial_k W_\epsilon^2 h_{ik})\frac{y^i}{|y|} d\sigma \\
& + \int_{B_\rho^+} \frac{2}{n}(W_\epsilon \Delta W_\epsilon - \frac{n}{n-2}|\nabla W_\epsilon|^2)\mathrm{tr} h \, dy - \int_{\partial^+ B_\rho^+} W_\epsilon^2 H_{ik}\partial_l H_{il}\frac{y^k}{|y|} d\sigma.
\end{aligned}
$$

Using (2.43) and the expression (2.2) of W_ϵ, we estimate

$$\int_{\partial^+ B_\rho^+} \partial_i W_\epsilon \psi \frac{y^i}{|y|} d\sigma \leq C(n, T_c) \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}| \epsilon^{n-2} \rho^{|\alpha|+2-n},$$

$$\int_{\partial^+ B_\rho^+} W_\epsilon^{\frac{2n}{n-2}} V_i \frac{y^i}{|y|} d\sigma \leq C(n, T_c) \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}| \epsilon^{n} \rho^{|\alpha|-n},$$

$$\int_{B_\rho^+} (W_\epsilon \Delta W_\epsilon - \frac{n}{n-2} |\nabla W_\epsilon|^2) \mathrm{tr} h \, dy \leq C(n, T_c) \epsilon^{n-2} \rho^{2d+4-n}$$

and use $|\partial H_{ij}| \leq C$ to show

$$\int_{\partial^+ B_\rho^+} W_\epsilon^2 H_{ik} \partial_l H_{il} \frac{y^k}{|y|} d\sigma \leq C(n, T_c) \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}| \epsilon^{n-2} \rho^{|\alpha|+3-n}.$$

Hence combining the above estimates together, we obtain

$$\begin{aligned} \int_{B_\rho^+} J_1 dy \leq & - \int_{D_\rho} \frac{8(n-1)}{n-2} \partial_n W_\epsilon \psi d\sigma + \int_{\partial^+ B_\rho^+} (W_\epsilon^2 \partial_k h_{ik} - \partial_k W_\epsilon^2 h_{ik}) \frac{y^i}{|y|} d\sigma \\ & + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}| \epsilon^{n-2} \rho^{|\alpha|+2-n} + C \rho^{2d+4-n} \epsilon^{n-2}. \end{aligned} \tag{2.55}$$

For J_2, by Proposition 2.4.4 and (2.45) we have

$$J_2 = -\frac{1}{4} Q_{ik,l} Q_{ik,l} - 2 W_\epsilon^{\frac{2n}{n-2}} T_{ik} T_{ik} + \operatorname{div} \xi.$$

By (2.48) a direct computation yields

$$\int_{\partial^+ B_\rho^+} \xi_i \frac{y^i}{|y|} d\sigma \leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}|^2 \rho^{2|\alpha|+2-n} \epsilon^{n-2}.$$

From this and Proposition 2.4.5 we estimate

$$\begin{aligned} & \int_{B_\rho^+} J_2 dy \\ = & -\frac{1}{4} \int_{B_\rho^+} Q_{ik,l} Q_{ik,l} dy - \int_{B_\rho^+} 2 W_\epsilon^{\frac{2n}{n-2}} T_{ik} T_{ik} dy + \int_{\partial^+ B_\rho^+} \xi_i \frac{y^i}{|y|} d\sigma - \int_{D_\rho} \xi_n d\sigma \\ \leq & - \int_{D_\rho} \xi_n d\sigma - \frac{n^2}{2(n-1)(n-2)} \int_{D_\rho} \partial_n W_\epsilon W_\epsilon S_{nn}^2 d\sigma \\ & - \frac{1}{4} \lambda^* \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}|^2 \epsilon^{n-2} \int_{B_\rho^+} (\epsilon + |y|)^{2|\alpha|+2-2n} dy \\ & + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}|^2 \rho^{2|\alpha|+2-n} \epsilon^{n-2}. \end{aligned} \tag{2.56}$$

Observe that when $|y|$ is sufficiently small, there hold $|h| \le C|y|$ and

$$
\begin{aligned}
|g_{x_0}^{ik} - \delta_{ik}| \le& C|h|,\\
|g_{x_0}^{ik} - \delta_{ik} + H_{ik}| \le& C|h|^2 + O(|y|^{d+1}) \le C|h||y| + O(|y|^{d+1}),\\
|g_{x_0}^{ik} - \delta_{ik} + h_{ik} - \frac{1}{2}H_{il}H_{kl}| \le& C|h|^3 + O(|y|^{d+2}) \le C|h|^2|y| + O(|y|^{d+2}).
\end{aligned} \tag{2.57}
$$

By Proposition 2.4.6 and Young's inequality, we can bound J_3 and J_4 by

$$
\begin{aligned}
&J_3 + J_4\\
\le& C(n,T_c)\epsilon^{n-2}\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|^2(\epsilon+|y|)^{2|\alpha|+3-2n}\\
&+ C(n,T_c)\epsilon^{n-2}\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|(\epsilon+|y|)^{|\alpha|+d+3-2n}\\
&+ C(n,T_c)\epsilon^{n-2}(\epsilon+|y|)^{2d+4-2n}\\
\le& \frac{1}{2}\lambda^*\epsilon^{n-2}\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|^2(\epsilon+|y|)^{2|\alpha|+2-2n} + C\epsilon^{n-2}(\epsilon+|y|)^{2d+4-2n}.
\end{aligned} \tag{2.58}
$$

Consequently, combining the above (2.53), (2.55)-(2.58) and using the decomposition (2.54), we conclude that

$$
\begin{aligned}
&\int_{B_\rho^+}\left[\tfrac{4(n-1)}{n-2}|\nabla(W_\epsilon+\psi)|_{g_{x_0}}^2 + R_{g_{x_0}}(W_\epsilon+\psi)^2\right]dy + 2(n-1)\int_{D_\rho} h_{g_{x_0}}(W_\epsilon+\psi)^2 d\sigma\\
\le& \frac{4(n-1)}{n-2}\int_{B_\rho^+}\left[|\nabla W_\epsilon|^2 + n(n+2)W_\epsilon^{\frac{4}{n-2}}\psi^2\right]dy\\
&+ \int_{\partial^+ B_\rho^+}(W_\epsilon^2\partial_j h_{ij} - \partial_j W_\epsilon^2 h_{ij})\frac{y^i}{|y|}d\sigma - \int_{D_\rho}\frac{8(n-1)}{n-2}\partial_n W_\epsilon\psi d\sigma\\
&- \int_{D_\rho}\xi_n d\sigma - \frac{n^2}{2(n-1)(n-2)}\int_{D_\rho}\partial_n W_\epsilon W_\epsilon S_{nn}^2 d\sigma\\
&- \frac{1}{2}\lambda^*\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|^2\epsilon^{n-2}\int_{B_\rho^+}(\epsilon+|y|)^{2|\alpha|+2-2n}dy\\
&+ C\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|\epsilon^{n-2}\rho^{|\alpha|+2-n} + C\epsilon^{n-2}\rho^{2d+4-n}.
\end{aligned} \tag{2.59}
$$

Testing problem (2.1) with W_ϵ and integrating over B_ρ^+, via integration by parts we obtain

$$\begin{aligned}
&\frac{4(n-1)}{n-2}\int_{B_\rho^+}\left[|\nabla W_\epsilon|^2+n(n+2)W_\epsilon^{\frac{4}{n-2}}\psi^2\right]dy\\
=&4n(n-1)\int_{B_\rho^+}W_\epsilon^{\frac{4}{n-2}}\left(W_\epsilon^2+\frac{n+2}{n-2}\psi^2\right)dy\\
&+\frac{4(n-1)}{n-2}\int_{\partial^+B_\rho^+}W_\epsilon\partial_iW_\epsilon\frac{y^i}{|y|}d\sigma-4(n-1)T_c\int_{D_\rho}W_\epsilon^{\frac{2(n-1)}{n-2}}d\sigma.
\end{aligned}$$

Therefore, plugging this and (2.49) into (2.59) as well as again using (2.1), we obtain the desired assertion. □

Proposition 2.4.8. *There exists some sufficiently small ρ_0 such that*

$$\begin{aligned}
&4n(n-1)\int_{B_\rho^+}W_\epsilon^{\frac{4}{n-2}}\left(W_\epsilon^2+\frac{n+2}{n-2}\psi^2\right)dy\\
\le&aY_{a,b}(\mathbb{R}^n_+,\mathbb{R}^{n-1})\left(\int_{B_\rho^+}(W_\epsilon+\psi)^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{n}}+C\epsilon^n\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|\rho^{|\alpha|-n}\\
&+C\epsilon^n\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|^2\int_{B_\rho^+}(\epsilon+|y|)^{2|\alpha|+1-2n}dy
\end{aligned}$$

for all $0<2\epsilon\le\rho\le\rho_0$.

Proof. Notice that (2.47) gives $|\psi|\le C(\epsilon+|y|)W_\epsilon$ in $B_{2\rho}^+$. By Lemma 2.1.1, we get

$$aY_{a,b}(\mathbb{R}^n_+,\mathbb{R}^{n-1})=4n(n-1)\left(\int_{\mathbb{R}^n_+}W_\epsilon^{\frac{2n}{n-2}}dx\right)^{\frac{2}{n}}.$$

Together with the fact that $V_n=0$ on $D_{2\rho}$, the desired estimate can follow the same lines in [14, Propositions 14-15]. □

Proposition 2.4.9. *There exists some sufficiently small ρ_0 such that*

$$\begin{aligned}
&-4(n-1)T_c\int_{D_\rho}W_\epsilon^{\frac{2}{n-2}}\left(W_\epsilon^2+2W_\epsilon\psi+\frac{n}{n-2}\psi^2-\frac{n-2}{8(n-1)^2}W_\epsilon^2S_{nn}^2\right)d\sigma\\
\le&2(n-1)bY_{a,b}(\mathbb{R}^n_+,\mathbb{R}^{n-1})\left(\int_{D_\rho}(W_\epsilon+\psi)^{\frac{2(n-1)}{n-2}}d\sigma\right)^{\frac{n-2}{n-1}}\\
&+C\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|\rho^{|\alpha|+1-n}\epsilon^{n-1}\\
&+C\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|^2\epsilon^{n-1}\rho\int_{D_\rho}(\epsilon+|y|)^{2|\alpha|+2-2n}d\sigma
\end{aligned}$$

for all $0<2\epsilon\le\rho<\rho_0$.

Proof. Since (2.47) and (2.43) give $|\psi| \leq C(\epsilon+|y|)W_\epsilon$ and $|S_{nn}| \leq C(\epsilon+|y|)$ in $B^+_{2\rho}$, this assertion can follow the same lines in [23, Proposition 8] (see also [2, (3.23)]) by using

$$-2T_c\left(\int_{\mathbb{R}^{n-1}} W_\epsilon^{\frac{2(n-1)}{n-2}} d\sigma\right)^{\frac{1}{n-1}} = bY_{a,b}(\mathbb{R}^n_+, \mathbb{R}^{n-1})$$

in Lemma 2.1.1. □

For simplicity, we denote by $\Omega_\rho := \Psi_{x_0}(B^+_\rho)$ the coordinate ball of radius ρ under the Fermi coordinates around x_0.

Lemma 2.4.10. *If $0<\epsilon \ll \rho < \rho_0$ for some sufficiently small ρ_0, in $M \backslash \Omega_\rho$ there holds*

$$\begin{aligned}&|\bar{U}_{(x_0,\epsilon)} - \epsilon^{\frac{n-2}{2}} G|\\ \leq& C\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}|\rho^{|\alpha|+2-n}\epsilon^{\frac{n-2}{2}} + C\rho^{d+3-n}|\log\rho|\epsilon^{\frac{n-2}{2}} + C\rho^{1-n}\epsilon^{\frac{n}{2}}.\end{aligned}$$

Proof. For $x \in M\backslash\Omega_\rho$, let $y = \Psi_{x_0}^{-1}(x) \in \mathbb{R}^n_+ \setminus B^+_\rho$. In $M \setminus \Omega_\rho$, we have

$$\bar{U}_{(x_0,\epsilon)}(x) - \epsilon^{\frac{n-2}{2}}G(x) = \chi_\rho(y)\left[W_\epsilon(y) + \psi(y) - \epsilon^{\frac{n-2}{2}}G(\Psi_{x_0}(y))\right]. \tag{2.60}$$

Notice that

$$\begin{aligned}&W_\epsilon(y) - \epsilon^{\frac{n-2}{2}}|y|^{2-n}\\ =&\epsilon^{\frac{n-2}{2}}|y|^{2-n}\left[\left(1+\frac{(1+T_c^2)\epsilon^2}{|y|^2} - \frac{2y^nT_c\epsilon}{|y|^2}\right)^{\frac{2-n}{2}} - 1\right]\\ =&(n-2)y^n|y|^{-n}T_c\epsilon^{\frac{n}{2}} + O(\epsilon^{\frac{n+2}{2}}|y|^{-n}),\end{aligned}$$

then it yields

$$|W_\epsilon - \epsilon^{\frac{n-2}{2}}|y|^{2-n}| \leq C\epsilon^{\frac{n}{2}}\rho^{1-n} \quad \text{in} \quad B^+_{2\rho}\backslash B^+_\rho. \tag{2.61}$$

From this, (2.52) and (2.47), in $M\backslash\Omega_\rho$ we obtain

$$\begin{aligned}&|\bar{U}_{(x_0,\epsilon)} - \epsilon^{\frac{n-2}{2}} G|\\ \leq&|W_\epsilon - \epsilon^{\frac{n-2}{2}}|y|^{2-n}| + \epsilon^{\frac{n-2}{2}}|G-|y|^{2-n}| + |\psi|\\ \leq& C\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}|\rho^{|\alpha|+2-n}\epsilon^{\frac{n-2}{2}} + \underline{C\rho^{d+3-n}|\log\rho|\epsilon^{\frac{n-2}{2}}} + C\rho^{1-n}\epsilon^{\frac{n}{2}},\end{aligned}$$

[1]when $\epsilon \ll \rho < \rho_0$. $\square$

Lemma 2.4.11. *If $0 < \epsilon \ll \rho < \rho_0$ for some sufficiently small ρ_0, in $M\backslash\Omega_\rho$ there holds*

$$\begin{aligned}
&\rho^2 \left| \frac{4(n-1)}{n-2}\Delta_{g_{x_0}} \bar{U}_{(x_0,\epsilon)} - R_{g_{x_0}} \bar{U}_{(x_0,\epsilon)} \right| \\
\leq &C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}| \rho^{|\alpha|+2-n} \epsilon^{\frac{n-2}{2}} + C\rho^{d+3-n}\epsilon^{\frac{n-2}{2}} + C\rho^{1-n}\epsilon^{\frac{n}{2}}.
\end{aligned}$$

Proof. Since $\bar{U}_{(x_0,\epsilon)} = \epsilon^{\frac{n-2}{2}} G$ in $M\backslash\Omega_{2\rho}$, the estimate is trivial by the definition of G. Then it suffices to estimate the above inequality in $\Omega_{2\rho}\backslash\Omega_\rho$. To see this, by (2.60) we have

$$\begin{aligned}
&\Delta_{g_{x_0}} \bar{U}_{(x_0,\epsilon)} - \frac{n-2}{4(n-1)} R_{g_{x_0}} \bar{U}_{(x_0,\epsilon)} \\
=&(\Delta_{g_{x_0}}\chi_\rho)(W_\epsilon + \psi - \epsilon^{\frac{n-2}{2}}|y|^{2-n}) + 2\langle \nabla\chi_\rho, \nabla(W_\epsilon + \psi - \epsilon^{\frac{n-2}{2}}|y|^{2-n})\rangle_{g_{x_0}} \\
&- (\Delta_{g_{x_0}}\chi_\rho)\epsilon^{\frac{n-2}{2}}(G - |x|^{2-n}) - 2\epsilon^{\frac{n-2}{2}}\langle \nabla\chi_\rho, \nabla(G - |x|^{2-n})\rangle_{g_{x_0}} \\
&+ \chi_\rho \left[\Delta_{g_{x_0}}(W_\epsilon + \psi) - \frac{n-2}{4(n-1)} R_{g_{x_0}}(W_\epsilon + \psi) \right] \\
=&I_1 + I_2 + I_3,
\end{aligned}$$

where $I_i (i = 1, 2, 3)$ denotes the quantity in each corresponding line. By using (2.61) and $|\rho^2 \Delta_{g_{x_0}}\chi_\rho| + |\rho\nabla\chi_\rho|_{g_{x_0}} \leq C$, we get

$$\rho^2 I_1 \leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}| \rho^{|\alpha|+2-n} \epsilon^{\frac{n-2}{2}} + C\rho^{1-n}\epsilon^{\frac{n}{2}}.$$

Similarly (2.52) implies

$$\rho^2 I_2 \leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}| \rho^{|\alpha|+2-n} \epsilon^{\frac{n-2}{2}} + C\rho^{d+3-n}\epsilon^{\frac{n-2}{2}} + C\rho^{1-n}\epsilon^{\frac{n}{2}}.$$

For I_3, applying the property (2.47) of ψ and Proposition 2.4.6, we get

$$|R_{g_{x_0}}(W_\epsilon + \psi)| \leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}| \rho^{|\alpha|-n} \epsilon^{\frac{n-2}{2}} + C\rho^{d+1-n}\epsilon^{\frac{n-2}{2}}$$

---

[1]In view of (2.52), the underlined term can be precisely estimated by $C\rho^{d+3-n}\epsilon^{\frac{n-2}{2}}$ when $n \geq 5$ and $C|\log \rho|$ when $n = 3, 4$. Since this rough estimate goes through in the later part, we adopt it just for simplicity.

and

$$
\begin{aligned}
&|\Delta_{g_{x_0}}(W_\epsilon+\psi)| \\
\leq &|(\Delta_{g_{x_0}}-\Delta_{\mathbb{R}^n})(W_\epsilon+\psi)| + C\epsilon^{\frac{n+2}{2}}\rho^{-n-2} + C\epsilon^{\frac{n-2}{2}}\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|\rho^{|\alpha|-n} \\
\leq &C\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|\rho^{|\alpha|-n}\epsilon^{\frac{n-2}{2}} + C\rho^{d+1-n}\epsilon^{\frac{n-2}{2}} + C\rho^{-n-2}\epsilon^{\frac{n+2}{2}}
\end{aligned}
$$

Therefore

$$
\rho^2 I_3 \leq C\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|\rho^{|\alpha|+2-n}\epsilon^{\frac{n-2}{2}} + C\rho^{d+3-n}\epsilon^{\frac{n-2}{2}} + C\rho^{-n}\epsilon^{\frac{n+2}{2}}.
$$

Collecting all the above estimates on I_1-I_3, we get the desired assertion. □

We now arrive at the key Proposition 2.4.12.

Proposition 2.4.12. *If $0<\epsilon\ll\rho<\rho_0$ for some sufficiently small ρ_0, there holds*

$$
\begin{aligned}
&\int_M\left[\tfrac{4(n-1)}{n-2}|\nabla\bar{U}_{(x_0,\epsilon)}|^2_{g_{x_0}} + R_{g_{x_0}}\bar{U}^2_{(x_0,\epsilon)}\right]d\mu_{g_{x_0}} + 2(n-1)\int_{\partial M}h_{g_{x_0}}\bar{U}^2_{(x_0,\epsilon)}d\sigma_{g_{x_0}} \\
\leq &Y_{a,b}(\mathbb{R}^n,\mathbb{R}^{n-1})\left[a\left(\int_M\bar{U}_{(x_0,\epsilon)}^{\frac{2n}{n-2}}d\mu_{g_{x_0}}\right)^{\frac{n-2}{n}}\right. \\
&\qquad\qquad\qquad\qquad\left.+2(n-1)b\left(\int_{\partial M}\bar{U}_{(x_0,\epsilon)}^{\frac{2(n-1)}{n-2}}d\sigma_{g_{x_0}}\right)^{\frac{n-2}{n-1}}\right] \\
&-\epsilon^{n-2}\mathcal{I}(x_0,\rho) - \frac{1}{C}\eta_{\mathcal{Z}^c}(x_0)\lambda^*\epsilon^{n-2}\int_{B_\rho^+}|W_{g_0}(y)|^2_{g_0}(\epsilon+|y|)^{6-2n}dy \\
&-\frac{1}{C}\eta_{\mathcal{Z}^c}(x_0)\lambda^*\epsilon^{n-2}\int_{D_\rho}|\mathring{\pi}_{g_0}(y)|^2_{g_0}(\epsilon+|y|)^{5-2n}d\sigma + C\rho^{2d+4-n}|\log\rho|^2\epsilon^{n-2} \\
&+C\left(\frac{\epsilon}{\rho}\right)^{n-2}\frac{1}{\log(\rho/\epsilon)} + C^*\left(\frac{\epsilon}{\rho}\right)^{n-1},
\end{aligned}
$$

where $\eta_{\mathcal{Z}^c}$ is the characteristic function of $\mathcal{Z}^c=\partial M\setminus\mathcal{Z}$ defined on ∂M and C,C^* depend on n,g_0,T_c,ρ_0.

Proof. Observe that

$$
\begin{aligned}
&\int_{M\backslash\Omega_\rho}\left[\tfrac{4(n-1)}{n-2}|\nabla\bar{U}_{(x_0,\epsilon)}|^2_{g_{x_0}}+R_{g_{x_0}}\bar{U}^2_{(x_0,\epsilon)}\right]d\mu_{g_{x_0}}+2(n-1)\int_{\partial M\backslash\Omega_\rho}h_{g_{x_0}}\bar{U}^2_{(x_0,\epsilon)}d\sigma_{g_{x_0}}\\
=&\int_{M\backslash\Omega_\rho}\left(-\frac{4(n-1)}{n-2}\Delta_{g_{x_0}}\bar{U}_{(x_0,\epsilon)}+R_{g_{x_0}}\bar{U}^2_{(x_0,\epsilon)}\right)(\bar{U}_{(x_0,\epsilon)}-\epsilon^{\frac{n-2}{2}}G)d\mu_{g_{x_0}}\\
&+\frac{4(n-1)}{n-2}\int_{\partial(M\backslash\Omega_\rho)}\left[\frac{\partial\bar{U}_{(x_0,\epsilon)}}{\partial\nu_{g_{x_0}}}\bar{U}_{(x_0,\epsilon)}+\epsilon^{\frac{n-2}{2}}\left(\bar{U}_{(x_0,\epsilon)}\frac{\partial G}{\partial\nu_{g_{x_0}}}-G\frac{\partial\bar{U}_{(x_0,\epsilon)}}{\partial\nu_{g_{x_0}}}\right)\right]d\sigma_{g_{x_0}}\\
&+2(n-1)\int_{\partial M\backslash\Omega_\rho}h_{g_{x_0}}\bar{U}^2_{(x_0,\epsilon)}d\sigma_{g_{x_0}}\\
=&II_1+II_2+II_3,
\end{aligned}
$$

where $II_i(i=1,2,3)$ denotes the quantity in each corresponding line on the right hand side of the first identity. By Lemmas 2.4.10 and 2.4.11, we get

$$
\begin{aligned}
&\sup_{M\backslash\Omega_\rho}\left[|\bar{U}_{(x_0,\epsilon)}-\epsilon^{\frac{n-2}{2}}G|+\rho^2\left|\frac{4(n-1)}{n-2}\Delta_{g_{x_0}}\bar{U}_{(x_0,\epsilon)}-R_{g_{x_0}}\bar{U}_{(x_0,\epsilon)}\right|\right]\\
\le&C\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|\rho^{|\alpha|+2-n}\epsilon^{\frac{n-2}{2}}+C\rho^{d+3-n}|\log\rho|\epsilon^{\frac{n-2}{2}}+C\rho^{1-n}\epsilon^{\frac{n}{2}}.
\end{aligned}
$$

From this, one can estimate II_1 as

$$
\begin{aligned}
II_1=&\int_{\Omega_{2\rho}\backslash\bar{\Omega}_\rho}\left(-\tfrac{4(n-1)}{n-2}\Delta_{g_{x_0}}\bar{U}_{(x_0,\epsilon)}+R_{g_{x_0}}\bar{U}_{(x_0,\epsilon)}\right)(\bar{U}_{(x_0,\epsilon)}-\epsilon^{\frac{n-2}{2}}G)d\mu_{g_{x_0}}\\
\le&C\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|^2\rho^{2|\alpha|+2-n}\epsilon^{n-2}+C\rho^{2d+4-n}|\log\rho|^2\epsilon^{n-2}+C\rho^{-n}\epsilon^n.
\end{aligned}\tag{2.62}
$$

For II_2, we divide the integral into two parts $II_2=II_2^{(1)}+II_2^{(2)}$ according to $\partial(M\backslash\Omega_\rho)=(\partial M\backslash\Omega_\rho)\cup(\partial\Omega_\rho\backslash\partial M)$. Namely $II_2^{(1)}$ is the integral over $\partial M\backslash\Omega_\rho$ while $II_2^{(2)}$ is over $\partial\Omega_\rho\backslash\partial M$. Let us deal with $II_2^{(1)}$ first. In $\partial M\backslash\Omega_\rho$, by Lemma 2.4.2, (2.1), (2.51) and (2.23), we have

$$
\begin{aligned}
&\sup_{\partial M\cap(\Omega_{2\rho}\backslash\bar{\Omega}_\rho)}\left|\frac{\partial\bar{U}_{(x_0,\epsilon)}}{\partial\nu_{g_{x_0}}}\right|\\
\le&\sup_{\partial M\cap(\Omega_{2\rho}\backslash\bar{\Omega}_\rho)}\left[|\partial_n W_\epsilon+\partial_n\psi|+\epsilon^{\frac{n-2}{2}}\left|\frac{\partial G}{\partial\nu_{g_{x_0}}}\right|\right]\\
\le&C\epsilon^{\frac{n}{2}}\rho^{-n}+C\epsilon^{\frac{n}{2}}\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|\rho^{|\alpha|-n}+C\epsilon^{\frac{n-2}{2}}\rho^{2d+3-n}.
\end{aligned}
$$

Using (2.51), Lemma 2.4.10, (2.23) and $\bar{U}_{(x_0,\epsilon)} = \epsilon^{\frac{n-2}{2}} G$ in $M \setminus \Omega_{2\rho}$, we have

$$\begin{aligned}
&II_2^{(1)} + II_3 \\
=&\frac{4(n-1)}{n-2} \int_{\partial M \setminus \bar{\Omega}_\rho} \left[\frac{\partial \bar{U}_{(x_0,\epsilon)}}{\partial \nu_{g_{x_0}}} \bar{U}_{(x_0,\epsilon)} + \epsilon^{\frac{n-2}{2}} \left(\bar{U}_{(x_0,\epsilon)} \frac{\partial G}{\partial \nu_{g_{x_0}}} - G \frac{\partial \bar{U}_{(x_0,\epsilon)}}{\partial \nu_{g_{x_0}}} \right) \right] d\sigma_{g_{x_0}} \\
&+ II_3 \\
=&\frac{4(n-1)}{n-2} \int_{\partial M \cap (\Omega_{2\rho} \setminus \bar{\Omega}_\rho)} \frac{\partial \bar{U}_{(x_0,\epsilon)}}{\partial \nu_{g_{x_0}}} (\bar{U}_{(x_0,\epsilon)} - \epsilon^{\frac{n-2}{2}} G) d\sigma_{g_{x_0}} \\
&+ 2(n-1) \int_{\partial M \cap (\Omega_{2\rho} \setminus \bar{\Omega}_\rho)} h_{g_{x_0}} \bar{U}_{(x_0,\epsilon)} (\bar{U}_{(x_0,\epsilon)} - \epsilon^{\frac{n-2}{2}} G) d\sigma_{g_{x_0}} \\
\leq& C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}|^2 \rho^{2|\alpha|+1-n} \epsilon^{n-1} + |\partial^\alpha h_{ab}| \rho^{|\alpha|+1-n} \epsilon^{n-1} \\
&+ C\rho^{d+2-n} \epsilon^{n-1} + C\rho^{-n} \epsilon^n + C\rho^{2d+4-n} |\log \rho| \epsilon^{n-2}.
\end{aligned} \tag{2.63}$$

Next we start to estimate $II_2^{(2)}$ whose integral domain is $\partial \Omega_\rho \setminus \partial M$. It is not hard to verify that the outward unit normal $\nu_{g_{x_0}}$ on $\partial \Omega_\rho \setminus \partial M := \Psi_{x_0}(\partial^+ B_\rho^+)$ is given by

$$\nu_{g_{x_0}} = \frac{g_{x_0}^{ik} y^k}{\|y\|} \partial_{y^i} \quad \text{for} \quad y \in \partial^+ B_\rho^+,$$

where $\|y\|^2 := g_{x_0}^{kl} y^k y^l = \rho^2(1 + C|h|)$ on $\partial^+ B_\rho^+$. Note that $\bar{U}_{(x_0,\epsilon)} = W_\epsilon + \psi$ on $\partial^+ B_\rho^+$, by (2.57) we estimate

$$\begin{aligned}
&\int_{\partial \Omega_\rho \setminus \partial M} \frac{\partial \bar{U}_{(x_0,\epsilon)}}{\partial \nu_{g_{x_0}}} \bar{U}_{(x_0,\epsilon)} d\sigma_{g_{x_0}} \\
=& - \int_{\partial^+ B_\rho^+} g_{x_0}^{ij} \partial_i \bar{U}_{(x_0,\epsilon)} \frac{y^j}{\|y\|} \bar{U}_{(x_0,\epsilon)} d\sigma + O(\rho^{2d+4-n} \epsilon^{n-2}) \\
=& \int_{\partial^+ B_\rho^+} \left(-\partial_i \bar{U}_{(x_0,\epsilon)} + \partial_j \bar{U}_{(x_0,\epsilon)} h_{ij} \right) \frac{y^i}{|y|} (1 + C|h|) \bar{U}_{(x_0,\epsilon)} d\sigma \\
&+ C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}|^2 \rho^{2|\alpha|+2-n} \epsilon^{n-2} + O(\rho^{2d+4-n} \epsilon^{n-2}) \\
\leq& \int_{\partial^+ B_\rho^+} \left(-\partial_i W_\epsilon + \partial_j W_\epsilon h_{ij} \right) \frac{y^i}{|y|} W_\epsilon d\sigma + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}| \rho^{|\alpha|+2-n} \epsilon^{n-2} \\
&+ C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}|^2 \rho^{2|\alpha|+2-n} \epsilon^{n-2} + C\rho^{2d+4-n} \epsilon^{n-2}.
\end{aligned} \tag{2.64}$$

Similarly we have

$$
\begin{aligned}
&\epsilon^{\frac{n-2}{2}}\int_{\partial\Omega_\rho\setminus\partial M}\left(\bar{U}_{(x_0,\epsilon)}\frac{\partial G}{\partial\nu_{g_{x_0}}}-G\frac{\partial G}{\partial\nu_{g_{x_0}}}\right)d\sigma_{g_{x_0}}\\
\leq&-\epsilon^{\frac{n-2}{2}}\int_{\partial^+B_\rho^+}\left(\bar{U}_{(x_0,\epsilon)}\partial_iG-G\partial_i\bar{U}_{(x_0,\epsilon)}\right)\frac{y^i}{|y|}(1+C|h|)d\sigma\\
&+\epsilon^{\frac{n-2}{2}}\int_{\partial^+B_\rho^+}h_{ij}\frac{y^i}{|y|}(1+C|h|)\left(\bar{U}_{(x_0,\epsilon)}\partial_jG-G\partial_j\bar{U}_{(x_0,\epsilon)}\right)d\sigma\\
&+C\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|^2\rho^{2|\alpha|+2-n}\epsilon^{n-2}+C\rho^{2d+4-n}\epsilon^{n-2}\\
\leq&-\epsilon^{\frac{n-2}{2}}\int_{\partial^+B_\rho^+}\left(W_\epsilon\partial_iG-G\partial_iW_\epsilon\right)\frac{y^i}{|y|}d\sigma\\
&+\epsilon^{\frac{n-2}{2}}\int_{\partial^+B_\rho^+}h_{ij}\frac{y^i}{|y|}\left(W_\epsilon\partial_jG-G\partial_jW_\epsilon\right)d\sigma\\
&+C\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|\rho^{|\alpha|+2-n}\epsilon^{n-2}+C\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|^2\rho^{2|\alpha|+2-n}\epsilon^{n-2}\\
&+C\rho^{2d+4-n}\epsilon^{n-2}.
\end{aligned}\tag{2.65}
$$

From (2.52) and (2.61), on $\partial^+B_\rho^+$ we get

$$
\begin{aligned}
&\epsilon^{\frac{n-2}{2}}|\partial_iW_\epsilon G-\partial_iGW_\epsilon|\\
\leq&|\partial_iW_\epsilon(\epsilon^{\frac{n-2}{2}}G-W_\epsilon)|+|W_\epsilon\partial_i(\epsilon^{\frac{n-2}{2}}G-W_\epsilon)|\\
\leq&C\epsilon^{n-2}\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|\rho^{|\alpha|+3-2n}+C\epsilon^{n-2}\rho^{d+4-2n}|\log\rho|+C\epsilon^{n-1}\rho^{2-2n}
\end{aligned}
$$

and then

$$
\begin{aligned}
&\epsilon^{\frac{n-2}{2}}\int_{\partial^+B_\rho^+}h_{ij}\frac{y^i}{|y|}\left(W_\epsilon\partial_jG-G\partial_jW_\epsilon\right)d\sigma\\
\leq&C\epsilon^{n-2}\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|^2\rho^{2|\alpha|+2-n}+C\rho^{2d+4-n}|\log\rho|\epsilon^{n-2}+C\epsilon^{n-1}\rho^{2-n}.
\end{aligned}\tag{2.66}
$$

Hence plugging (2.66) into (2.65), we obtain

$$
\begin{aligned}
&\epsilon^{\frac{n-2}{2}}\int_{\partial\Omega_\rho\setminus\partial M}(\bar{U}_{(x_0,\epsilon)}\frac{\partial G}{\partial\nu_{g_{x_0}}}-G\frac{\partial G}{\partial\nu_{g_{x_0}}})d\sigma_{g_{x_0}}\\
\leq&-\epsilon^{\frac{n-2}{2}}\int_{\partial^+B_\rho^+}\left(W_\epsilon\partial_iG-G\partial_iW_\epsilon\right)\frac{y^i}{|y|}d\sigma+C\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|\rho^{|\alpha|+2-n}\epsilon^{n-2}\\
&+C\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|^2\rho^{2|\alpha|+2-n}\epsilon^{n-2}+C\rho^{2d+4-n}|\log\rho|\epsilon^{n-2}+C\epsilon^{n-1}\rho^{2-n}.
\end{aligned}\tag{2.67}
$$

Consequently combining (2.64) and (2.67), we can get

$$
\begin{aligned}
II_2^{(2)} \leq & -\tfrac{4(n-1)}{n-2} \int_{\partial^+ B_\rho^+} \left[\partial_i W_\epsilon W_\epsilon - \partial_j W_\epsilon W_\epsilon h_{ij} + \epsilon^{\frac{n-2}{2}} \left(W_\epsilon \partial_i G - G \partial_i W_\epsilon \right) \right] \frac{y^i}{|y|} d\sigma \\
& + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}| \rho^{|\alpha|+2-n} \epsilon^{n-2} + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}|^2 \rho^{2|\alpha|+2-n} \epsilon^{n-2} \\
& + C \rho^{2d+4-n} |\log \rho| \epsilon^{n-2} + C \epsilon^{n-1} \rho^{2-n}. \qquad (2.68)
\end{aligned}
$$

Therefore collecting the estimates (2.62) for II_1, (2.63) for $II_2^{(1)} + II_3$ and (2.68) for $II_2^{(2)}$ together, when $\epsilon \ll \rho < \rho_0$ we obtain

$$
\begin{aligned}
& \int_{M \setminus \Omega_\rho} \left[\tfrac{4(n-1)}{n-2} |\nabla \bar{U}_{(x_0,\epsilon)}|^2_{g_{x_0}} + R_{g_{x_0}} \bar{U}^2_{(x_0,\epsilon)} \right] d\mu_{g_{x_0}} + 2(n-1) \int_{\partial M \setminus \Omega_\rho} h_{g_{x_0}} \bar{U}^2_{(x_0,\epsilon)} d\sigma_{g_{x_0}} \\
\leq & \frac{4(n-1)}{n-2} \int_{\partial^+ B_\rho^+} \left[-\partial_i W_\epsilon W_\epsilon + \partial_j W_\epsilon W_\epsilon h_{ij} - \epsilon^{\frac{n-2}{2}} (W_\epsilon \partial_i G - G \partial_i W_\epsilon) \right] \frac{y^i}{|y|} d\sigma \\
& + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}| \rho^{|\alpha|+2-n} \epsilon^{n-2} + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |\partial^\alpha h_{ab}|^2 \rho^{2|\alpha|+2-n} \epsilon^{n-2} \\
& + C \rho^{2d+4-n} |\log \rho|^2 \epsilon^{n-2} + C \rho^{2-n} \epsilon^{n-1}. \qquad (2.69)
\end{aligned}
$$

Finally since $d\mu_{g_{x_0}} = \big(1 + O(|y|^{2d+2})\big) dy$ and $d\sigma_{g_{x_0}} = \big(1 + O(|y|^{2d+2})\big) d\sigma$ under the Fermi coordinates around $x_0 \in \partial M$, noticing that Propositions 2.4.7-2.4.9 and (2.69) give the estimates of energy $E[\bar{U}_{(x_0,\epsilon)}]$ in the interior of $B_\rho^+ = \Psi_{x_0}^{-1}(\Omega_\rho)$ and in the exterior of Ω_ρ respectively, we conclude that

$$
\begin{aligned}
&\int_M \left[\tfrac{4(n-1)}{n-2}|\nabla \bar{U}_{(x_0,\epsilon)}|^2_{g_{x_0}} + R_{g_{x_0}}\bar{U}^2_{(x_0,\epsilon)}\right] d\mu_{g_{x_0}} + 2(n-1)\int_{\partial M} h_{g_{x_0}}\bar{U}^2_{(x_0,\epsilon)} d\sigma_{g_{x_0}}\\
\le & Y_{a,b}(\mathbb{R}^n,\mathbb{R}^{n-1})\left[a\left(\int_M \bar{U}^{\frac{2n}{n-2}}_{(x_0,\epsilon)} d\mu_{g_{x_0}}\right)^{\frac{n-2}{n}}\right.\\
&\qquad\qquad\qquad\qquad\left. + 2(n-1)b\left(\int_{\partial M}\bar{U}^{\frac{2(n-1)}{n-2}}_{(x_0,\epsilon)} d\sigma_{g_{x_0}}\right)^{\frac{n-2}{n-1}}\right]\\
&+\int_{\partial^+ B^+_\rho}(W_\epsilon^2\partial_j h_{ij} + \frac{n}{n-2}\partial_j W_\epsilon^2 h_{ij})\frac{y^i}{|y|}d\sigma\\
&-\frac{4(n-1)}{n-2}\epsilon^{\frac{n-2}{2}}\int_{\partial^+ B^+_\rho}(W_\epsilon \partial_i G - G\partial_i W_\epsilon)\frac{y^i}{|y|}d\sigma\\
&-\frac{1}{4}\lambda^*\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|^2\epsilon^{n-2}\int_{B^+_\rho}(\epsilon+|y|)^{2|\alpha|+2-2n}dy\\
&+C\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|^2\epsilon^{n-1}\rho\int_{D_\rho}(\epsilon+|y|)^{2|\alpha|+2-2n}d\sigma\\
&+C\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|\rho^{|\alpha|+2-n}\epsilon^{n-2} + C\rho^{2d+4-n}|\log\rho|^2\epsilon^{n-2}\\
&+C\epsilon^{n-1}\rho^{2-n},
\end{aligned}
\tag{2.70}
$$

where we have used the following estimate:

$$
\begin{aligned}
&\epsilon^n\int_{B^+_\rho}(\epsilon+|y|)^{2|\alpha|+1-2n}dy\\
\le & C\epsilon^{n-1}\int_{B^+_\rho}(\epsilon+|y|)^{2|\alpha|+2-2n}dy \le \frac{\lambda^*}{4}\epsilon^{n-2}\int_{B^+_\rho}(\epsilon+|y|)^{2|\alpha|+2-2n}dy
\end{aligned}
$$

by choosing $\epsilon \ll \rho < \rho_0$. By (2.52) and the expression (2.2) of W_ϵ, we get

$$
\begin{aligned}
&\int_{\partial^+ B^+_\rho}(W_\epsilon^2\partial_j h_{ij} + \frac{n}{n-2}\partial_j W_\epsilon^2 h_{ij})\frac{y^i}{|y|}d\sigma\\
&-\frac{4(n-1)}{n-2}\epsilon^{\frac{n-2}{2}}\int_{\partial^+ B^+_\rho}(W_\epsilon \partial_i G - G\partial_i W_\epsilon)\frac{y^i}{|y|}d\sigma\\
\le & -\epsilon^{n-2}\mathcal{I}(x_0,\rho) + C\sum_{a,b=1}^{n-1}\sum_{|\alpha|=1}^{d}|\partial^\alpha h_{ab}|\rho^{|\alpha|+1-n}\epsilon^{n-1} + C\epsilon^{n-1}\rho^{1-n}.
\end{aligned}
\tag{2.71}
$$

Notice that

$$
|W_{g_0}(x)|_{g_0} = f_{x_0}^{\frac{4}{n-2}}|W_{g_{x_0}}(x)|_{g_{x_0}} \le C|\partial^2 h| + |\partial h| \quad \text{ in } \ M
$$

and

$$|\mathring{\pi}_{g_0}(x)|_{g_0} = f_{x_0}^{\frac{2}{n-2}} |W_{g_{x_0}}(x)|_{g_{x_0}} \leq C|\partial h| \qquad \text{on} \ \ \partial M.$$

By choosing ρ_0 small enough with all $\rho < \rho_0$, it is not hard to show that

$$C\epsilon^{n-1}\rho \int_{D_\rho} (\epsilon + |y|)^{2|\alpha|+2-2n} d\sigma \leq \frac{\lambda^*}{8}\epsilon^{n-2} \int_{B_\rho^+} (\epsilon + |y|)^{2|\alpha|+2-2n} dy.$$

Recall that we define by $\mathcal{Z}$ the set of all points $x_0 \in \partial M$ satisfying

$$\limsup_{x \to x_0} d_{g_0}(x, x_0)^{2-d} |W_{g_0}(x)|_{g_0} = \limsup_{x \to x_0} d_{g_0}(x, x_0)^{1-d} |\mathring{\pi}_{g_0}(x)|_{g_0} = 0.$$

From these estimates, (2.70) and (2.71), a similar argument in [2, Corollary 3.10] yields

$$\begin{aligned}
&\int_M \left[\tfrac{4(n-1)}{n-2}|\nabla \bar{U}_{(x_0,\epsilon)}|_{g_{x_0}}^2 + R_{g_{x_0}} \bar{U}_{(x_0,\epsilon)}^2\right] d\mu_{g_{x_0}} + 2(n-1)\int_{\partial M} h_{g_{x_0}} \bar{U}_{(x_0,\epsilon)}^2 d\sigma_{g_{x_0}} \\
\leq & Y_{a,b}(\mathbb{R}^n, \mathbb{R}^{n-1}) \left[a \left(\int_M \bar{U}_{(x_0,\epsilon)}^{\frac{2n}{n-2}} d\mu_{g_{x_0}} \right)^{\frac{n-2}{n}} \right. \\
& \qquad\qquad \left. + 2(n-1) b \left(\int_{\partial M} \bar{U}_{(x_0,\epsilon)}^{\frac{2(n-1)}{n-2}} d\sigma_{g_{x_0}} \right)^{\frac{n-2}{n-1}} \right] \\
& - \epsilon^{n-2}\mathcal{I}(x_0, \rho) - \frac{1}{C}\eta_{\mathcal{Z}^c}(x_0)\lambda^*\epsilon^{n-2}\int_{B_\rho^+} |W_{g_0}(y)|_{g_0}^2 (\epsilon + |y|)^{6-2n} dy \\
& - \frac{1}{C}\eta_{\mathcal{Z}^c}(x_0)\lambda^*\epsilon^{n-2}\int_{D_\rho} |\mathring{\pi}_{g_0}(y)|_{g_0}^2 (\epsilon + |y|)^{5-2n} d\sigma + C^*\rho^{2d+4-n}|\log\rho|^2\epsilon^{n-2} \\
& + C\left(\frac{\epsilon}{\rho}\right)^{n-2} \frac{1}{\log(\rho/\epsilon)} + C\left(\frac{\epsilon}{\rho}\right)^{n-1},
\end{aligned}$$

by recalling that $\eta_{\mathcal{Z}^c}$ is the characteristic function of $\mathcal{Z}^c = \partial M \setminus \mathcal{Z}$. □

Next we describe the continuity of $\mathcal{I}(x_0, \rho)$ over $\mathcal{Z}$ as in [2, Proposition 3.11] and some characterization of its limit as $\rho \to 0$ (cf. [15, Proposition 4.3]). We restate them here for convenience.

Proposition 2.4.13. *The functions $\mathcal{I}(x_0, \rho)$ converge to a continuous function $\mathcal{I}(x_0) : \mathcal{Z} \to \mathbb{R}$ uniformly for all $x_0 \in \mathcal{Z}$, as $\rho \to 0$.*

Proposition 2.4.14. *Let $x_0 \in \mathcal{Z}$ and consider inverted coordinates $\Phi : y \in \bar{M}\setminus\{x_0\} \mapsto z := y/|y|^2$, where $y = (y^1, \cdots, y^n)$ are Fermi coordinates centered at x_0. If we define the metric $\bar{g}_{x_0} = \Phi_*(G_{x_0}^{4/(n-2)} g_{x_0})$ on $\bar{M}\setminus\{x_0\}$, then the following statements hold:*

(i) $(\bar{M}\setminus\{x_0\}, \bar{g}_{x_0})$ is an asymptotically flat manifold with order $d+1 > \frac{n-2}{2}$ (in the sense of Definition 1.1.3), and satisfies $R_{\bar{g}_{x_0}} \equiv 0$ and $h_{\bar{g}_{x_0}} \equiv 0$.

(ii) We have

$$\mathcal{I}(x_0) = \lim_{R\to\infty}\left[\int_{\partial^+ B_R^+} \frac{z^i}{|z|}\partial_{z^j}\bar{g}_{x_0}(\partial_{z^i},\partial_{z^j})d\sigma - \int_{\partial^+ B_R^+} \frac{z^i}{|z|}\partial_{z^i}\bar{g}_{x_0}(\partial_{z^j},\partial_{z^j})d\sigma\right].$$

In particular, $\mathcal{I}(x_0)$ *is the mass* $m(\bar{g}_{x_0})$ *of* $(\bar{M}\backslash\{x_0\}, \bar{g}_{x_0})$.

Proof of Theorem 1.1.5. (i) When $\partial M \setminus \mathcal{Z} \neq \emptyset$, we choose $x_0 \in \partial M \setminus \mathcal{Z}$. Then the desired assertion follows from Proposition 2.4.13.

(ii) Assume that $\mathcal{I}(x_0) > 0$ for some $x_0 \in \mathcal{Z}$, it follows from Proposition 2.4.13 that

$$\mathcal{I}(x_0, \rho) > C^* \rho^{2d+4-n} |\log \rho|^2$$

for all $0 < \rho < \rho_0$, where ρ_0, C^* are the positive constants in Proposition 2.4.13. Based on the key estimate in Proposition 2.4.13, Theorem 1.1.5 follows the same lines of [2, Proposition 3.7]. $\square$

Chapter 3

Convergence of Yamabe flow on manifolds with minimal boundary

In this chapter, we are elaborating the work in [4].

3.1 Preliminary results and long-time existence

Notation. In this chapter, M^n will denote a compact manifold of dimension $n \geq 3$ with boundary ∂M, and g_0 will denote a background Riemannian metric on M. We will denote by $B_r(x)$ the metric ball in M of radius r with center $x \in M$.

For any Riemannian metric g on M, η_g will denote the inward unit normal vector to ∂M with respect to g and Δ_g the Laplace-Beltrami operator.

If $z_0 \in \mathbb{R}^n_+$, we set $B^+_r(z_0) = \{z \in \mathbb{R}^n_+\,;\ |z - z_0| < r\}$,

$$D_r(z_0) = B^+_r(z_0) \cap \partial\mathbb{R}^n_+\,, \quad \text{and} \quad \partial^+ B^+_r(z_0) = \partial B^+_r(z_0) \cap \mathbb{R}^n_+\,.$$

Finally, for any $z = (z_1, ..., z_n) \in \mathbb{R}^n$ we set $\bar{z} = (z_1, ..., z_{n-1}, 0) \in \partial\mathbb{R}^n_+ \cong \mathbb{R}^{n-1}$.

Convention. We assume that (M, g_0) satisfies $Y(M, \partial M) > 0$. According to [39, Lemma 1.1], we can also assume that $R_{g_0} > 0$ and $H_{g_0} \equiv 0$, after a conformal change of the metric. Multiplying g_0 by a positive constant, we can suppose that $\int_M dv_{g_0} = 1$. We will adopt the summation convention whenever confusion is not possible, and use indices $a, b, c, d = 1, ..., n$, and $i, j, k, l = 1, ..., n-1$.

If $g = u^{\frac{4}{n-2}} g_0$ for some positive smooth function u on M, we know that

$$\begin{cases} R_g = u^{-\frac{n+2}{n-2}} \left(-\dfrac{4(n-1)}{n-2} \Delta_{g_0} u + R_{g_0} u \right), & \text{in } M\,, \\ H_g = u^{-\frac{n}{n-2}} \left(-\dfrac{2(n-1)}{n-2} \dfrac{\partial}{\partial \eta_{g_0}} u + H_{g_0} u \right), & \text{on } \partial M\,, \end{cases} \tag{3.1}$$

and the operators $L_g = \frac{4(n-1)}{n-2}\Delta_g - R_g$ and $B_g = \frac{2(n-1)}{n-2}\frac{\partial}{\partial \eta_g} - H_g$ satisfy

$$L_{u^{\frac{4}{n-2}}g_0}(u^{-1}\zeta) = u^{-\frac{n+2}{n-2}} L_{g_0}\zeta, \tag{3.2}$$

$$B_{u^{\frac{4}{n-2}}g_0}(u^{-1}\zeta) = u^{-\frac{n}{n-2}} B_{g_0}\zeta\,, \tag{3.3}$$

for any smooth function ζ.

If $u(t) = u(\cdot, t)$ is a 1-parameter family of positive smooth functions on M and $g(t) = u(t)^{\frac{4}{n-2}} g_0$ with $H_{g_0} \equiv 0$, then (1.4) can be written as

$$\begin{cases} \dfrac{\partial}{\partial t} u(t) = -\dfrac{n-2}{4}(R_{g(t)} - \overline{R}_{g(t)})\, u(t), & \text{in } M, \\ \dfrac{\partial}{\partial \eta_{g_0}} u(t) = 0\,, & \text{on } \partial M, \end{cases} \tag{3.4}$$

The first equation of (3.4) can also be written as

$$\frac{\partial}{\partial t} u(t)^{\frac{n+2}{n-2}} = \frac{n+2}{4}\left(\frac{4(n-1)}{n-2}\Delta_{g_0} u - R_{g_0} u + \overline{R}_{g(t)} u^{\frac{n+2}{n-2}}\right).$$

Short-time existence of solutions to the equations (3.4) can be obtained by standard theory for quasilinear parabolic equations. Hence, the equations (3.4) have a solution $u(t)$ defined for all t in the maximal interval $[0, T_{max})$.

Taking $\partial/\partial\eta_{g_0}$ on both sides of the first equation of (3.4) and using the second one, one gets $\partial R_{g(t)}/\partial\eta_{g_0} = 0$ on ∂M. Hence the scalar curvature has evolution equations

$$\begin{cases} \dfrac{\partial}{\partial t} R_{g(t)} = (n-1)\Delta_{g(t)} R_{g(t)} + (R_{g(t)} - \overline{R}_{g(t)}) R_{g(t)}\,, & \text{in } M\,, \\ \dfrac{\partial}{\partial \eta_{g_{(t)}}} R_{g(t)} = 0\,, & \text{on } \partial M\,, \end{cases} \tag{3.5}$$

where the first equation comes from the well known first variation formula of scalar scalar curvature.

Observe that for all $t \geq 0$ we have

$$\frac{\partial}{\partial t} dv_{g(t)} = -\frac{n}{2}(R_{g(t)} - \overline{R}_{g(t)})\, dv_{g(t)} \tag{3.6}$$

and

$$\frac{\partial}{\partial t}\overline{R}_{g(t)} = -\frac{n-2}{2}\int_M (R_{g(t)} - \overline{R}_{g(t)})^2 dv_{g(t)}. \tag{3.7}$$

In particular, $\overline{R}_{g(t)}$ is decreasing and one can easily derive that (1.4) preserves the volume which we can normalize to

$$\int_M dv_{g(t)} = 1, \quad \text{for all } t \in [0, T_{max}).$$

So, $\overline{R}_{g(t)} \geq Y(M, \partial M) > 0$ for all $t \geq 0$.

Proposition 3.1.1. *We have $R_{g(t)} \geq \min\{\inf_M R_{g(0)}, 0\}$, for all $t \in [0, T_{max})$.*

Proof. Following (3.5), this is an application of maximum principle. □

Proposition 3.1.2. *For each $T \in (0, T_{max})$, there exist $C(T), c(T) > 0$ such that*

$$\sup_M u(t) \leq C(T) \quad \textit{and} \quad \inf_M u(t) \geq c(T), \quad \textit{for all}\, t \in [0, T]. \tag{3.8}$$

In particular, $T_{max} = \infty$.

Proof. Set $\sigma = 1 - \min\{\inf_M R_{g(0)}, 0\} = \max\{\sup_M(1 - R_{g(0)}), 1\}$. Then, by Proposition 3.1.1, $R_{g(t)} + \sigma \geq 1$ for all $t \in [0, T_{max})$. It follows from (3.4) and (3.7) that

$$\frac{\partial}{\partial t} \log u(t) = \frac{n-2}{4}(\overline{R}_{g(t)} - R_{g(t)}) \leq \frac{n-2}{4}(\overline{R}_{g(0)} + \sigma)$$

Then there exists $C(T) > 0$ such that $\sup_M u(t) \leq C(T)$ for all $t \in [0, T]$.

Defining $P = R_{g_0} + \sigma \left(\sup_{0 \leq t \leq T} \sup_M u(t)\right)^{\frac{4}{n-2}}$ we obtain

$$-\frac{4(n-1)}{n-2}\Delta_{g_0} u(t) + Pu(t) \geq -\frac{4(n-1)}{n-2}\Delta_{g_0} u(t) + R_{g_0} u(t) + \sigma u(t)^{\frac{n+2}{n-2}} = (R_{g(t)} + \sigma) u(t)^{\frac{n+2}{n-2}} \geq 0$$

for all $0 \leq t \leq T$. Then it follows from Proposition 3.A.4 in the Appendix that

$$\inf_M u(t) \left(\sup_M u(t)\right)^{\frac{n+2}{n-2}} \geq c(T) \int_M u(t)^{\frac{2n}{n-2}} dv_{g_0} = c(T),$$

by our normalization. This proves the second equation of (3.8).

Now we can follow [13, Proposition 2.6] to prove that if $0 < \alpha < \min\{4/n, 1\}$ then there is $\tilde{C}(T)$ such that

$$|u(x_1, t_1) - u(x_2, t_2)| \leq \tilde{C}(T)\big((t_1 - t_2)^{\alpha/2} + d_{g_0}(x_1, x_2)^\alpha\big)$$

for all $x_1, x_2 \in M$ and $t_1, t_2 \in [0, T]$ satisfying $0 < t_1 - t_2 < 1$. Then standard regularity theory for parabolic equations can be used to prove that all higher order derivatives of u are uniformly bounded on every fixed interval $[0, T]$. This implies the long-time existence of u. □

Set

$$\overline{R}_{\infty} = \lim_{t\to\infty} \overline{R}_{g(t)} > 0. \tag{3.9}$$

Because $\partial R_{g(t)}/\partial \eta_{g(t)} = 0$ holds on ∂M, we can follow the proof of Corollary 3.2 in [13] line by line, making use of (3.5), (3.6) and (3.7), to obtain

Corollary 3.1.3. *For any $1 < p < n/2 + 1$ we have*

$$\lim_{t\to\infty} \int_M |R_{g(t)} - \overline{R}_{\infty}|^p dv_{g(t)} = 0\,.$$

3.2 The test functions

In this section, we construct the test functions to be used in the blow-up analysis of Section 3.3. Those functions are perturbations of the symmetric functions U_ϵ (see (3.10) below), which represent the spherical metric on $\mathbb{R}^n$ and have maximum at the origin.

We will make use of the following coordinate systems:

Definition 3.2.1. Fix $x_0 \in \partial M$ and geodesic normal coordinates for ∂M centered at x_0. Let $(y_1, ..., y_{n-1})$ be the coordinates of $x \in \partial M$ and $\eta(x)$ be the inward unit vector normal to ∂M at x. For small $y_n \geq 0$, the point $\exp_x(y_n\eta(x)) \in M$ is said to have *Fermi coordinates* $(y_1, ..., y_n)$ (centered at x_0).

Definition 3.2.2. Let g be any (smooth) Riemannian metric on M. Consider $\bar{M}$ the double of M along its boundary and extend g to a (smooth) Riemannian metric $\bar{g}$ on $\bar{M}$. Fix $x_0 \in M$ and let $\bar{\psi}_{x_0} : B_r(0) \subset \mathbb{R}^n \to \bar{M}$ be normal coordinates (with respect to $\bar{g}$) centered at x_0. If $\tilde{B}_{x_0,r} = \bar{\psi}_{x_0}^{-1}(\bar{\psi}_{x_0}(B_r(0)) \cap M)$, we define the *extended normal coordinates* (centered at x_0)

$$\psi_{x_0} : \tilde{B}_{x_0,r} \subset \mathbb{R}^n \to M$$

as the restriction of $\bar{\psi}_{x_0}$ to $\tilde{B}_{x_0,r}$.

Observe that this definition depends on the metric $\tilde{g}$ chosen, but this is not a problem for our arguments in this section because we can fix the extension to $\tilde{M}$ of the background metric g_0.

Convention. We will refer to extended normal coordinates as *normal coordinates* for short.

Notation. We set $\tilde{D}_{x_0,r} = \tilde{B}_{x_0,r} \cap \psi_{x_0}^{-1}(\psi_{x_0}(\tilde{B}_{x_0,r}) \cap \partial M)$ and $\partial^+ \tilde{B}_{x_0,r} = \partial \tilde{B}_{x_0,r} \backslash \tilde{D}_{x_0,r} \subset \partial B_r(0)$.

Set $M_t = \{x \in M\, ;\ d_{g_0}(x, \partial M) \leq t\}$, which is defined for small $t > 0$. Let $\delta_0 > 0$ be a small constant to be chosen later. In the next subsections we will define three types of test functions:

- **Type A** test functions ($\bar{u}_{A;(x_0,\epsilon)}$): defined in Subsection 3.2.2 using Fermi coordinates centered at any $x_0 \in \partial M$ and with energy to be controlled by $Q(S^n_+)$.
- **Type B** test functions ($\bar{u}_{B;(x_0,\epsilon)}$): defined in Subsection 3.2.3 using normal coordinates centered at any $x_0 \in M_{2\delta_0} \backslash \partial M$ and with energy to be controlled by $Y(S^n)$.
- **Type C** test functions ($\bar{u}_{C;(x_0,\epsilon)}$): defined in Subsection 3.2.4 using normal coordinates centered at any $x_0 \in M \backslash M_{\delta_0}$ and with energy to be controlled by $Y(S^n)$.

We fix $P_0 = P_0(M, g_0) > 0$ small such that (extended) normal coordinates with center x_0 are defined in $\tilde{B}_{x_0,2P_0}$ for all $x_0 \in M \backslash \partial M$, and Fermi coordinates with center at x_0 are defined in $B^+_{2P_0}(0)$ for all $x_0 \in \partial M$.

Convention. In what follows, we will use the normalization $\overline{R}_\infty = 4n(n-1)$, without loss of generality.

3.2.1 The auxiliary function and some algebraic preliminaries

Firstly we fix some notations. If $\epsilon > 0$, we define

$$W_\epsilon(y) = \left(\frac{\epsilon}{\epsilon^2 + |y|^2} \right)^{\frac{n-2}{2}} \quad \text{for } \ y \in \mathbb{R}^n\,. \tag{3.10}$$

It is well known that W_ϵ satisfies

$$\begin{cases} \Delta W_\epsilon + n(n-2) W_\epsilon^{\frac{n+2}{n-2}} = 0\,, & \text{in } \mathbb{R}^n_+\,, \\ \partial_n W_\epsilon = 0\,, & \text{on } \partial\mathbb{R}^n_+\,, \end{cases} \tag{3.11}$$

and

$$4n(n-1)\left(\int_{\mathbb{R}^n_+} W_\epsilon(y)^{\frac{2n}{n-2}}dy\right)^{\frac{2}{n}} = Q(S^n_+)\,. \tag{3.12}$$

In this subsection, $\mathcal{H}$ will denote a symmetric trace-free 2-tensor on $\mathbb{R}^n_+$ with components $\mathcal{H}_{ab}$, $a,b=1,...,n$, satisfying

$$\begin{cases} \mathcal{H}_{ab}(0)=0\,, & \text{for } a,b=1,...,n\,,\\ \mathcal{H}_{an}(x)=0\,, & \text{for } x\in\mathbb{R}^n_+,\ a=1,...,n\,,\\ \partial_k\mathcal{H}_{ij}(0)=0\,, & \text{for } i,j,k=1,...,n-1\,,\\ \sum_{j=1}^{n-1}x_j\mathcal{H}_{ij}(x)=0\,, & \text{for } x\in\partial\mathbb{R}^n_+,\ i=1,...,n-1\,. \end{cases} \tag{3.13}$$

We will also assume that those components are of the form

$$\mathcal{H}_{ab}(x)=\sum_{1\leq|\alpha|\leq d}h_{ab,\alpha}x^\alpha \qquad \text{for } x\in\mathbb{R}^n_+\,, \tag{3.14}$$

where $d=\left[\frac{n-2}{2}\right]$ and each α stands for a multi-index. Obviously, the constants $h_{ab,\alpha}\in\mathbb{R}$ satisfy $h_{an,\alpha}=0$ for any α, and $h_{ab,\alpha}=0$ for any $\alpha\neq(0,...,0,1)$ with $|\alpha|=1$, where $a,b=1,...,n$.

Let $\chi:\mathbb{R}\to\mathbb{R}$ be a non-negative smooth function such that $\chi|_{[0,4/3]}\equiv 1$ and $\chi|_{[5/3,\infty)}\equiv 0$. If $\rho>0$, we define

$$\chi_\rho(x)=\chi\left(\frac{|x|}{\rho}\right) \quad \text{for } x\in\mathbb{R}^n\,. \tag{3.15}$$

Notice that $\partial_n\chi_\rho=0$ on $\partial\mathbb{R}^n_+$.

Let $V=V(\epsilon,\rho,\mathcal{H})$ be the smooth vector field on $\mathbb{R}^n_+$ obtained in [15, Theorem A.4], which satisfies

$$\begin{cases} \sum_{b=1}^n\partial_b\left\{W_\epsilon^{\frac{2n}{n-2}}(\chi_\rho\mathcal{H}_{ab}-\partial_aV_b-\partial_bV_a+\frac{2}{n}(\mathrm{div}V)\delta_{ab})\right\}=0\,, & \text{in } \mathbb{R}^n_+\,,\\ \partial_nV_i=V_n=0\,, & \text{on } \partial\mathbb{R}^n_+\,, \end{cases} \tag{3.16}$$

for $a=1,...,n$, and $i=1,...,n-1$, and

$$|\partial^\beta V(x)|\leq C(n,|\beta|)\sum_{i,j=1}^{n-1}\sum_{|\alpha|=1}^{d}|h_{ij,\alpha}|(\epsilon+|x|)^{|\alpha|+1-|\beta|} \tag{3.17}$$

for any multi-index β. Here $\delta_{ab}=1$ if $a=b$ and $\delta_{ab}=0$ if $a\neq b$.

We define symmetric trace-free 2-tensors S and T on $\mathbb{R}^n_+$ by

$$S_{ab} = \partial_a V_b + \partial_b V_a - \frac{2}{n}\partial_c V_c \delta_{ab} \qquad \text{and} \qquad T = \mathcal{H} - S\,. \tag{3.18}$$

(Recall that we are adopting the summation convention.) Observe that $T_{in} = S_{in} = 0$ on $\partial\mathbb{R}^n_+$ for $i = 1, ..., n-1$. It follows from (3.16) that T satisfies

$$W_\epsilon \partial_b T_{ab} + \frac{2n}{n-2}\partial_b W_\epsilon T_{ab} = 0\,, \quad \text{in } B^+_\rho(0)\,, \quad \text{for } a = 1, ..., n\,. \tag{3.19}$$

In particular,

$$\frac{n-2}{4(n-1)} W_\epsilon \partial_a \partial_b T_{ab} + \partial_a(\partial_b W_\epsilon T_{ab}) = 0\,, \quad \text{in } B^+_\rho(0)\,, \tag{3.20}$$

where we have used $W_\epsilon \partial_a \partial_b W_\epsilon - \frac{n}{n-2}\partial_a W_\epsilon \partial_b W_\epsilon = \frac{1}{n}(W_\epsilon \Delta W_\epsilon - \frac{n}{n-2}|dW_\epsilon|^2)\delta_{ab}$ in $\mathbb{R}^n_+$ for all $a, b = 1, ..., n$.

Next we define the auxiliary function $\phi = \phi_{\epsilon,\rho,\mathcal{H}}$ by

$$\phi = \partial_a W_\epsilon V_a + \frac{n-2}{2n} W_\epsilon \partial_a V_a\,. \tag{3.21}$$

By a direct computation, we have

$$\begin{cases} \Delta\phi + n(n+2)W_\epsilon^{\frac{4}{n-2}}\phi = \frac{n-2}{4(n-1)} W_\epsilon \partial_b \partial_a \mathcal{H}_{ab} + \partial_b(\partial_a W_\epsilon \mathcal{H}_{ab}), & \text{in } B^+_\rho(0)\,, \\ \partial_n \phi = 0, & \text{on } \partial\mathbb{R}^n_+\,. \end{cases} \tag{3.22}$$

By the estimate (3.17), ϕ satisfies

$$|\phi(x)| \leq C\epsilon^{\frac{n-2}{2}} \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^{d} |h_{ij,\alpha}|(\epsilon + |x|)^{|\alpha|+2-n} \tag{3.23}$$

and

$$\left|\Delta\phi(x) + n(n+2)W_\epsilon^{\frac{4}{n-2}}\phi(x)\right| \leq C\epsilon^{\frac{n-2}{2}} \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^{d} |h_{ij,\alpha}|(\epsilon + |x|)^{|\alpha|-n}\,, \tag{3.24}$$

for all $x \in \mathbb{R}^n_+$.

Observe that if $n = 3$ then $d = 0$, in which case $\mathcal{H} \equiv 0$ and $\phi \equiv 0$.

Convention. In the rest of Subsection 3.2.1 we will assume that $n \geq 4$.

We define algebraic Schouten tensor and algebraic Weyl tensor by

$$A_{ac} = \partial_c\partial_e\mathcal{H}_{ae} + \partial_a\partial_e\mathcal{H}_{ce} - \partial_e\partial_e\mathcal{H}_{ac} - \frac{1}{n-1}\partial_e\partial_f\mathcal{H}_{ef}\delta_{ac}$$

and

$$\begin{aligned} Z_{abcd} =& \partial_b\partial_d\mathcal{H}_{ac} - \partial_b\partial_c\mathcal{H}_{ad} + \partial_a\partial_c\mathcal{H}_{db} - \partial_a\partial_d\mathcal{H}_{bc} \\ &+ \frac{1}{n-2}\left(A_{ac}\delta_{bd} - A_{ad}\delta_{bc} + A_{bd}\delta_{ac} - A_{bc}\delta_{db}\right) . \end{aligned} \tag{3.25}$$

We also set

$$Q_{ab,c} = W_\epsilon\partial_c T_{ab} - \frac{2}{n-2}\partial_a W_\epsilon T_{bc} - \frac{2}{n-2}\partial_b W_\epsilon T_{ac} + \frac{2}{n-2}\partial_d W_\epsilon T_{ad}\delta_{bc} + \frac{2}{n-2}\partial_d W_\epsilon T_{bd}\delta_{ac} . \tag{3.26}$$

Lemma 3.2.3. *If the tensor $\mathcal{H}$ satisfies*

$$\begin{cases} Z_{abcd} = 0, & in\, \mathbb{R}^n_+ , \\ \partial_n\mathcal{H}_{ij} = 0, & on\, \partial\mathbb{R}^n_+ , \end{cases}$$

then $\mathcal{H} = 0$ in $\mathbb{R}^n_+$.

Proof. Observe that the hypothesis $\partial_n\mathcal{H}_{ij} = 0$ on $\partial\mathbb{R}^n_+$ implies that $h_{ij,\alpha} = 0$ for $\alpha = (0, ..., 0, 1)$. In this case, the expression (3.14) can be written as

$$\mathcal{H}_{ab}(x) = \sum_{|\alpha|=2}^{d} h_{ab,\alpha}x^\alpha .$$

Now the result is just Proposition 2.3 in [15]. □

Proposition 3.2.4. *Set $U_r = B_{r/4}(0, ..., 0, \frac{3r}{2}) \subset \mathbb{R}^n_+$. Then there exists $C = C(n) > 0$ such that*

$$\sum_{i,j=1}^{n-1}\sum_{|\alpha|=1}^{d}|h_{ij,\alpha}|^2 r^{2|\alpha|-4+n} \le C\int_{U_r} Z_{abcd}Z_{abcd} + Cr^{-1}\int_{D_{\frac{5r}{3}}(0)\backslash D_{\frac{4r}{3}}(0)} \partial_n\mathcal{H}_{ij}\partial_n\mathcal{H}_{ij} ,$$

for all $r > 0$.

Proof. If $r = 1$, observe that the square roots of both sides of the inequality are norms in $\mathcal{H}$, due to Lemma 3.2.3. The general case follows by scaling. □

Lemma 3.2.5. *There exists $C = C(n) > 0$ such that*

$$\epsilon^{n-2} r^{6-2n} \int_{U_r} Z_{abcd} Z_{abcd} \leq \frac{C}{\theta} \int_{B^+_{2r}(0)\setminus B^+_r(0)} Q_{ab,c} Q_{ab,c} + \theta \epsilon^{n-2} \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^{d} |h_{ij,\alpha}|^2 r^{2|\alpha|+2-n}$$

for all $0 < \theta < 1$ and all $r \geq \epsilon$.

Proof. This follows from the third formula in the proof of Proposition 2.5 in [15], by means of Young's inequality. Observe that, in our calculations, we are using the range $1 \leq |\alpha| \leq d$ in the summation formulas, instead of the range $2 \leq |\alpha| \leq d$ used in [15]. □

Lemma 3.2.6. *There exists $C = C(n) > 0$ such that*

$$\epsilon^{n-2} r^{5-2n} \int_{D_{\frac{5r}{3}}(0)\setminus D_{\frac{4r}{3}}(0)} \partial_n \mathcal{H}_{ij} \partial_n \mathcal{H}_{ij} \leq C\theta \epsilon^{n-2} \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^{d} |h_{ij,\alpha}|^2 r^{2|\alpha|+2-n} + \frac{C}{\theta} \int_{B^+_{2r}(0)\setminus B^+_r(0)} Q_{ij,n} Q_{ij,n} \tag{3.27}$$

for all $0 < \theta < 1$ and all $r \geq \epsilon$.

Proof. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a non-negative smooth function such that $\chi(t) = 1$ for $t \in [4/3, 5/3]$ and $\chi(t) = 0$ for $t \notin [1, 2]$. For $r > 0$ and $x \in \mathbb{R}^n_+$ we define $\chi_r(x) = \chi(|x|/r)$. Observe that $\partial_n S_{ij} = -\frac{1}{n-1} \partial_n S_{nn} \delta_{ij}$ on $\partial \mathbb{R}^n_+$. On the other hand, (3.20) gives $\partial_n S_{nn} = -\partial_n T_{nn} = 0$. Thus, $\partial_n S_{ij} = 0$ and $\partial_n \mathcal{H}_{ij} = \partial_n T_{ij} = W_\epsilon^{-1} Q_{ij,n}$ on $\partial \mathbb{R}^n_+$. Integration by parts gives

$$\begin{aligned} \int_{\partial \mathbb{R}^n_+} W_\epsilon^{\frac{2(n-1)}{n-2}} \partial_n \mathcal{H}_{ij} \partial_n \mathcal{H}_{ij} \chi_r &= \int_{\partial \mathbb{R}^n_+} W_\epsilon^{\frac{2}{n-2}} Q_{ij,n} Q_{ij,n} \chi_r = -\int_{\mathbb{R}^n_+} \partial_n \big(W_\epsilon^{\frac{2}{n-2}} Q_{ij,n} Q_{ij,n} \chi_r \big) \\ &= -\int_{\mathbb{R}^n_+} \partial_n (W_\epsilon^{\frac{2}{n-2}} Q_{ij,n} \chi_r) Q_{ij,n} - \int_{\mathbb{R}^n_+} W_\epsilon^{\frac{2}{n-2}} \partial_n Q_{ij,n} Q_{ij,n} \chi_r \,. \end{aligned} \tag{3.28}$$

By using Young's inequality, the result now follows from the inequalities

$$W_\epsilon^{\frac{2(n-1)}{n-2}} \partial_n \mathcal{H}_{ij} \partial_n \mathcal{H}_{ij} \chi_r \geq C^{-1} \epsilon^{n-1} r^{2-2n} \partial_n \mathcal{H}_{ij} \partial_n \mathcal{H}_{ij} \chi_r$$

and

$$|\partial_n (W_\epsilon^{\frac{2}{n-2}} Q_{ij,n} \chi_r)| + |W_\epsilon^{\frac{2}{n-2}} \partial_n Q_{ij,n} \chi_r| \leq C \epsilon^{\frac{n}{2}} \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^{d} |h_{ij,\alpha}| r^{|\alpha|-2-n} \,.$$

□

Proposition 3.2.7. *There exists $\lambda = \lambda(n) > 0$ such that*

$$\lambda\epsilon^{n-2}\sum_{i,j=1}^{n-1}\sum_{|\alpha|=1}^{d}|h_{ij,\alpha}|^2\int_{B_\rho^+(0)}(\epsilon+|x|)^{2|\alpha|+2-2n}dx \le \frac{1}{4}\int_{B_\rho^+(0)}Q_{ab,c}Q_{ab,c}dx$$

for all $\rho \ge 2\epsilon$.

Proof. This follows from Proposition 3.2.4, Lemma 3.2.5, and Lemma 3.2.6. □

3.2.2 Type A test functions ($\bar{u}_{A;(x_0,\epsilon)}$)

In this subsection we use the same test functions as in [15] but we need to do some changes when estimating their energy by $Q(S^n_+)$ because the boundary does not need to be umbilical in our case.

For $\rho \in (0, P_0/2]$, the Fermi coordinates centered at $x_0 \in \partial M$ define a smooth map $\psi_{x_0} : B^+_\rho(0) \subset \mathbb{R}^n_+ \to M$. We will sometimes omit the symbols ψ_{x_0} in order to simplify our notations, identifying $\psi_{x_0}(x) \in M$ with $x \in B^+_\rho(0)$. In those coordinates, we have the properties $g_{ab}(0) = \delta_{ab}$ and $g_{nb}(x) = \delta_{nb}$, for any $x \in B^+_\rho(0)$ and $a, b = 1, ..., n$. If we write $g = \exp(h)$, where exp denotes the matrix exponential, then the symmetric 2-tensor h satisfies the following properties:

$$\begin{cases} h_{ab}(0) = 0\,, & \text{for } a, b = 1, ..., n\,, \\ h_{an}(x) = 0\,, & \text{for } x \in B^+_\rho(0),\ a = 1, ..., n\,, \\ \partial_k h_{ij}(0) = 0\,, & \text{for } i, j, k = 1, ..., n-1\,, \\ \sum_{j=1}^{n-1} x_j h_{ij}(x) = 0\,, & \text{for } x \in D_\rho(0),\ i = 1, ..., n-1\,. \end{cases}$$

The last two properties follow from the fact that Fermi coordinates are normal on the boundary.

According to [65, Proposition 3.1], for each $x_0 \in \partial M$ we can find a conformal metric $g_{x_0} = f_{x_0}^{\frac{4}{n-2}} g_0$, with $f_{x_0}(x_0) = 1$, and Fermi coordinates centered at x_0 such that $\det(g_{x_0})(x) = 1 + O(|x|^{2d+2})$, where $d = \left[\frac{n-2}{2}\right]$. In particular, if we write $g_{x_0} = \exp(h_{x_0})$, we have $\mathrm{tr}(h_{x_0})(x) = O(|x|^{2d+2})$. Moreover, $H_{g_{x_0}}$, the trace of the second fundamental form of ∂M, satisfies

$$H_{g_{x_0}}(x) = -\frac{1}{2}g^{ij}\partial_n g_{ij}(x) = -\frac{1}{2}\partial_n(\log\det(g_{x_0}))(x) = O(|x|^{2d+1})\,. \tag{3.29}$$

Since M is compact, we can assume that $1/2 \leq f_{x_0} \leq 3/2$ for any $x_0 \in \partial M$, choosing P_0 smaller if necessary.

Notation. In order to simplify our notations, in the coordinates above, we will write g_{ab} and g^{ab} instead of $(g_{x_0})_{ab}$ and $(g_{x_0})^{ab}$ respectively, and h_{ab} instead of $(h_{x_0})_{ab}$.

In this subsection, we denote by

$$\mathcal{H}_{ab}(x) = \sum_{1 \leq |\alpha| \leq d} h_{ab,\alpha} x^\alpha$$

the Taylor expansion of order d associated with the function $h_{ab}(x)$. Thus, we have $h_{ab}(x) = \mathcal{H}_{ab}(x) + O(|x|^{d+1})$. Observe that $\mathcal{H}$ is a symmetric trace-free 2-tensor on $\mathbb{R}^n_+$, which satisfies the properties (3.13) and has the form (3.14). Then we can use the function $\phi = \phi_{\epsilon,\rho,\mathcal{H}}$ (see formula (3.21)) and the results obtained in Subsection 3.2.1.

Recall the definitions of W_ϵ in (3.10), χ_ρ in (3.15), and $\overline{R}_\infty$ in (3.9). Define

$$\begin{aligned} \bar{U}_{(x_0,\epsilon)}(x) = &\left(\frac{4n(n-1)}{\overline{R}_\infty}\right)^{\frac{n-2}{4}} \chi_\rho(\psi_{x_0}^{-1}(x))\big(W_\epsilon(\psi_{x_0}^{-1}(x)) + \phi(\psi_{x_0}^{-1}(x))\big) \\ &+ \left(\frac{4n(n-1)}{\overline{R}_\infty}\right)^{\frac{n-2}{4}} \epsilon^{\frac{n-2}{2}}\big(1-\chi_\rho(\psi_{x_0}^{-1}(x))\big) G_{x_0}(x)\,, \end{aligned} \tag{3.30}$$

if $x \in \psi_{x_0}(B^+_{2\rho}(0))$, and $\bar{U}_{(x_0,\epsilon)}(x) = G_{x_0}(x)$ otherwise. Here, G_{x_0} is the Green's function of the conformal Laplacian $L_{g_{x_0}} = \Delta_{g_{x_0}} - \frac{n-2}{4(n-1)} R_{g_{x_0}}$, with pole at $x_0 \in \partial M$, satisfying the boundary condition

$$\frac{\partial}{\partial \eta_{g_{x_0}}} G_{x_0} - \frac{n-2}{2(n-1)} H_{g_{x_0}} G_{x_0} = 0 \tag{3.31}$$

and the normalization $\lim_{|y|\to 0} |y|^{n-2} G_{x_0}(\psi_{x_0}(y)) = 1$. This function, obtained in Proposition 3.B.2, satisfies

$$|G_{x_0}(\psi_{x_0}(y)) - |y|^{2-n}| \leq C \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^{d} |h_{ij,\alpha}||y|^{|\alpha|+2-n} + \begin{cases} C|y|^{d+3-n}, & \text{if } n \geq 5, \\ C(1+\log|y|), & \text{if } n = 3, 4, \end{cases} \tag{3.32}$$

$$\left|\frac{\partial}{\partial y_b}(G_{x_0}(\psi_{x_0}(y)) - |y|^{2-n})\right| \leq C \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^{d} |h_{ij,\alpha}||y|^{|\alpha|+1-n} + C|y|^{d+2-n}\,,$$

for all $b = 1, ..., n$.

We define the test function

$$\bar{u}_{A;(x_0,\epsilon)} = f_{x_0}\bar{U}_{(x_0,\epsilon)}\,. \tag{3.33}$$

Observe that this function also depends on the radius ρ above, which will be fixed later in Section 3.3. Such constant will also be referred to as ρ_A in order to avoid confusion with test functions of the other subsections.

Our main result in this subsection is the following estimate for the energy of $\bar{u}_{A;(x_0,\epsilon)}$:

Proposition 3.2.8. *Under the hypotheses of Theorem 1.2.4, there exists $P_1 = P_1(M, g_0) > 0$ such that*

$$\begin{aligned}
&\frac{\int_M \left\{\frac{4(n-1)}{n-2}|d\bar{u}_{A;(x_0,\epsilon)}|^2_{g_0} + R_{g_0}\bar{u}^2_{A;(x_0,\epsilon)}\right\}dv_{g_0}}{\left(\int_M \bar{u}_{A;(x_0,\epsilon)}^{\frac{2n}{n-2}}dv_{g_0}\right)^{\frac{n-2}{n}}}\\
&\quad= \frac{\int_M \left\{\frac{4(n-1)}{n-2}|d\bar{U}_{(x_0,\epsilon)}|^2_{g_{x_0}} + R_{g_{x_0}}\bar{U}^2_{(x_0,\epsilon)}\right\}dv_{g_{x_0}} + \int_{\partial M} 2H_{g_{x_0}}\bar{U}^2_{(x_0,\epsilon)}d\sigma_{g_{x_0}}}{\left(\int_M \bar{U}_{(x_0,\epsilon)}^{\frac{2n}{n-2}}dv_{g_{x_0}}\right)^{\frac{n-2}{n}}}\\
&\quad\leq Q(S^n_+)
\end{aligned}$$

for all $x_0 \in \partial M$ and $0 < 2\epsilon < \rho_A < P_1$.

Let λ be the constant obtained in Proposition 3.2.7.

Proposition 3.2.9. *There exist $C, P_1 > 0$, depending only on (M, g_0), such that*

$$\begin{aligned}
&\int_{B^+_\rho(0)}\left\{\frac{4(n-1)}{n-2}|d(W_\epsilon+\phi)|^2_{g_{x_0}} + R_{g_{x_0}}(W_\epsilon+\phi)^2\right\}dx + \int_{D_\rho(0)} 2H_{g_{x_0}}(W_\epsilon+\phi)^2 d\sigma\\
&\quad\leq 4n(n-1)\int_{B^+_\rho(0)} W_\epsilon^{\frac{4}{n-2}}\left(W_\epsilon^2 + \frac{n+2}{n-2}\phi^2\right)dx \qquad (3.34)\\
&\qquad+ \int_{\partial^+ B^+_\rho(0)}\left\{\frac{4(n-1)}{n-2}W_\epsilon\partial_a W_\epsilon + W_\epsilon^2\partial_b h_{ab} - \partial_b W_\epsilon^2 h_{ab}\right\}\frac{x_a}{|x|}d\sigma_\rho\\
&\qquad- \frac{\lambda}{2}\sum_{i,j=1}^{n-1}\sum_{|\alpha|=1}^{d}|h_{ij,\alpha}|^2\epsilon^{n-2}\int_{B^+_\rho(0)}(\epsilon+|x|)^{2|\alpha|+2-2n}dx \qquad (3.35)\\
&+ \qquad C\sum_{i,j=1}^{n-1}\sum_{|\alpha|=1}^{d}|h_{ij,\alpha}|\epsilon^{n-2}\rho^{|\alpha|+2-n} + C\epsilon^{n-2}\rho^{2d+4-n}
\end{aligned}$$

for all $0 < 2\epsilon \leq \rho \leq P_1$.

Proof. Following the steps in [15, Proposition 3.6] we obtain

$$\begin{aligned}
&\int_{B^+_\rho(0)}\left\{\frac{4(n-1)}{n-2}|d(W_\epsilon+\phi)|^2_{g_{x_0}}+R_{g_{x_0}}(W_\epsilon+\phi)^2\right\}dx+\int_{D_\rho(0)}2H_{g_{x_0}}(W_\epsilon+\phi)^2d\sigma\\
&\quad\leq\int_{B^+_\rho(0)}\frac{4(n-1)}{n-2}|dW_\epsilon|^2dx+\int_{B^+_\rho(0)}\frac{4(n-1)}{n-2}n(n+2)W_\epsilon^{\frac{4}{n-2}}\phi^2dx\\
&\qquad+\int_{\partial^+B^+_\rho(0)}\left(W_\epsilon^2\partial_bh_{ab}-\partial_bW_\epsilon^2h_{ab}\right)\frac{x_a}{|x|}d\sigma_\rho-\frac{1}{4}\int_{B^+_\rho(0)}Q_{ab,c}Q_{ab,c}dx\\
&\qquad+\frac{\lambda}{2}\sum_{i,j=1}^{n-1}\sum_{|\alpha|=1}^{d}|h_{ij,\alpha}|^2\epsilon^{n-2}\int_{B^+_\rho(0)}(\epsilon+|x|)^{2|\alpha|+2-2n}dx\\
&\qquad+C\sum_{i,j=1}^{n-1}\sum_{|\alpha|=1}^{d}|h_{ij,\alpha}|\epsilon^{n-2}\rho^{|\alpha|+2-n}+C\epsilon^{n-2}\rho^{2d+4-n}\,.
\end{aligned}$$

The result follows by making use of Proposition 3.2.7 and

$$|dW_\epsilon|^2=\partial_a(W_\epsilon\partial_aW_\epsilon)-W_\epsilon\Delta W_\epsilon=\partial_a(W_\epsilon\partial_aW_\epsilon)+n(n-2)W_\epsilon^{\frac{2n}{n-2}}\,.$$

□

As in [15, p. 1006], we define the flux integral

$$\begin{aligned}
\mathcal{I}(x_0,\rho)=\frac{4(n-1)}{n-2}&\int_{\partial^+B^+_\rho(0)}(|x|^{2-n}\partial_aG_{x_0}-\partial_a|x|^{2-n}G_{x_0})\frac{x_a}{|x|}d\sigma_\rho\\
&-\int_{\partial^+B^+_\rho(0)}|x|^{2-2n}(|x|^2\partial_bh_{ab}-2nx_bh_{ab})\frac{x_a}{|x|}d\sigma_\rho\,,
\end{aligned}\tag{3.36}$$

for $\rho>0$ sufficiently small.

Proposition 3.2.10. *There exists $P_1=P_1(M,g_0)>0$ such that*

$$\begin{aligned}
&\int_M\left\{\frac{4(n-1)}{n-2}|d\bar{U}_{(x_0,\epsilon)}|^2_{g_{x_0}}+R_{g_{x_0}}\bar{U}^2_{(x_0,\epsilon)}\right\}dv_{g_{x_0}}+\int_{\partial M}2H_{g_{x_0}}\bar{U}^2_{(x_0,\epsilon)}d\sigma_{g_{x_0}}\\
&\quad\leq Q(S^n_+)\left\{\int_M\bar{U}_{(x_0,\epsilon)}^{\frac{2n}{n-2}}dv_{g_{x_0}}\right\}^{\frac{n-2}{n}}-\epsilon^{n-2}\mathcal{I}(x_0,\rho)\\
&\qquad-\frac{\lambda}{4}\sum_{i,j=1}^{n-1}\sum_{|\alpha|=1}^{d}|h_{ij,\alpha}|^2\epsilon^{n-2}\int_{B^+_\rho(0)}(\epsilon+|x|)^{2|\alpha|+2-2n}dx\\
&\qquad+C\sum_{i,j=1}^{n-1}\sum_{|\alpha|=1}^{d}|h_{ij,\alpha}|\epsilon^{n-2}\rho^{|\alpha|+2-n}+C\epsilon^{n-2}\rho^{2d+4-n}+C\epsilon^n\rho^{-n}
\end{aligned}$$

for all $0<2\epsilon\leq\rho\leq P_1$.

Proof. Once we have proved Proposition 3.2.9, our proof is analogous to the one in [15, Proposition 4.1]. A necessary step is the estimate

$$4n(n-1)\int_{B^+_\rho(0)} W_\epsilon^{\frac{4}{n-2}}\left(W_\epsilon^2+\frac{n+2}{n-2}\phi^2\right)dx \tag{3.37}$$

$$\leq Q(S^n_+)\left(\int_{B^+_\rho(0)}(W_\epsilon+\phi)^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{n}}+\sum_{i,j=1}^{n-1}\sum_{|\alpha|=1}^{d}|h_{ij,\alpha}|\rho^{|\alpha|-n}\epsilon^n \tag{3.38}$$

$$+C\sum_{i,j=1}^{n-1}\sum_{|\alpha|=1}^{d}|h_{ij,\alpha}|^2\epsilon^{n-1}\int_{B^+_\rho(0)}(\epsilon+|x|)^{2|\alpha|+2-2n}dx$$

for all $0<2\epsilon\leq\rho\leq P_1$ and P_1 sufficiently small; see the proof of Proposition 3.2.24 below. □

Corollary 3.2.11. *There exist P_1, θ, $C_0>0$, depending only on (M,g_0), such that*

$$\int_M\left\{\frac{4(n-1)}{n-2}|d\bar{U}_{(x_0,\epsilon)}|^2_{g_{x_0}}+R_{g_{x_0}}\bar{U}^2_{(x_0,\epsilon)}\right\}dv_{g_{x_0}}+\int_{\partial M}2H_{g_{x_0}}\bar{U}^2_{(x_0,\epsilon)}d\sigma_{g_{x_0}}$$

$$\leq Q(S^n_+)\left\{\int_M\bar{U}_{(x_0,\epsilon)}^{\frac{2n}{n-2}}dv_{g_{x_0}}\right\}^{\frac{n-2}{n}}-\epsilon^{n-2}\mathcal{I}(x_0,\rho)-\theta\epsilon^{n-2}\int_{B^+_\rho(0)}|W_{g_0}(x)|^2(\epsilon+|x|)^{6-2n}dx$$

$$-\theta\epsilon^{n-2}\int_{D_\rho(0)}|\pi_{g_0}(x)|^2(\epsilon+|x|)^{5-2n}d\sigma+C_0\epsilon^{n-2}\rho^{2d+4-n}+C_0\left(\frac{\epsilon}{\rho}\right)^{n-2}\frac{1}{\log(\rho/\epsilon)}$$

for all $0<2\epsilon\leq\rho\leq P_1$. Here, we denote by W_{g_0} the Weyl tensor of (M,g_0) and by π_{g_0} the trace-free 2nd fundamental form of ∂M.

Proof. Similar to [2, Corollary 3.10]. □

Recall that we denote by $\mathcal{Z}_{\partial M}$ the set of all points $x_0\in\partial M$ such that

$$\limsup_{x\to x_0} d_{g_0}(x,x_0)^{2-d}|W_{g_0}(x)|=\limsup_{x\to x_0} d_{g_0}(x,x_0)^{1-d}|\pi_{g_0}(x)|=0\,.$$

Proposition 3.2.12. *The functions $\mathcal{I}(x_0,\rho)$ converge uniformly to a continuous function $I:\mathcal{Z}_{\partial M}\to\mathbb{R}$ as $\rho\to 0$.*

Proof. As in [2, Proposition 3.11] we can prove that

$$\sup_{x_0\in\mathcal{Z}_{\partial M}}|\mathcal{I}(x_0,\rho)-\mathcal{I}(x_0,\tilde{\rho})|\leq\begin{cases}C\rho^{2d+4-n} & \text{if } n\geq 5,\\ C\rho^{2d+4-n}(\log\rho) & \text{if } n=3,4,\end{cases}$$

for all $0<\tilde{\rho}<\rho$. The result follows. □

The following proposition, which is [2, Proposition 3.12] [1], relates $\mathcal{I}(x_0)$ with the mass,

Proposition 3.2.13. *Let $x_0 \in \mathcal{Z}_{\partial M}$ and consider inverted coordinates $y = x/|x|^2$, where $x = (x_1, ..., x_n)$ are Fermi coordinates centered at x_0. If we define the metric $\bar{g} = G_{x_0}^{\frac{4}{n-2}} g_{x_0}$ on $M \backslash \{x_0\}$, then the following statements hold:*

(i) *$(M\backslash\{x_0\}, \bar{g})$ is an asymptotically flat manifold with order $p > \frac{n-2}{2}$ (in the sense of Definition 1.1.3), and satisfies $R_{\bar{g}} \equiv 0$ and $H_{\bar{g}} \equiv 0$.*

(ii) *We have*

$$\mathcal{I}(x_0) = \lim_{R\to\infty} \left\{ \int_{\partial^+ B_R^+(0)} \frac{y_a}{|y|} \frac{\partial \bar{g}}{\partial y_b} \left(\frac{\partial}{\partial y_a}, \frac{\partial}{\partial y_b} \right) d\sigma_R - \int_{\partial^+ B_R^+(0)} \frac{y_a}{|y|} \frac{\partial \bar{g}}{\partial y_a} \left(\frac{\partial}{\partial y_b}, \frac{\partial}{\partial y_b} \right) d\sigma_R \right\}.$$

In particular, $\mathcal{I}(x_0)$ is the mass $m(\bar{g})$ of $(M\backslash\{x_0\}, \bar{g})$.

Proof of Proposition 3.2.8. Once we have proved Corollary 3.2.11, and Propositions 3.2.12 an 3.2.13, this proof follows the same lines as [2, Proposition 3.7]. $\square$

We now prove some further results for later use.

Proposition 3.2.14. [2] *For $x \in M$ and $\epsilon < \rho$,*

$$\begin{aligned} &\left| \frac{4(n-1)}{n-2} \Delta_{g_{x_0}} \bar{U}_{(x_0,\epsilon)} - R_{g_{x_0}} \bar{U}_{(x_0,\epsilon)} + \overline{R}_\infty \bar{U}_{(x_0,\epsilon)}^{\frac{n+2}{n-2}} \right| (x) \\ &\qquad\qquad \leq C \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n-2}{2}} (\epsilon^2 + |x|^2)^{-\frac{1}{2}} 1_{B^+_{2\rho}(0)}(x) + C \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n+2}{2}} 1_{M \backslash B^+_\rho(0)}(x) \\ &\qquad\qquad\quad + C(\epsilon^{\frac{n+2}{2}} \rho^{-2-n} + \epsilon^{\frac{n-2}{2}} \rho^{1-n} (\log \rho)) 1_{B^+_{2\rho}(0) \backslash B^+_\rho(0)}(x). \end{aligned}$$

---

[1] In [2, Propositions 3.11 and 3.12] a $\log \rho$ must be included in the arguments for dimensions 3 and 4, when the Green function has log in its expansion; see (3.32).

[2] *The $(\epsilon^2 + |x|^2)^{-\frac{1}{2}}$ term in this proposition is necessary only in dimension 3, when $d = 0$ and so $\mathcal{H} = 0$. On the other hand, the $\log \rho$ term is necessary only in dimensions 3 and 4, because of (3.32). The same terms are also necessary in the first inequality of [2, Proposition 3.13], but this does not affect any other results in that paper because weaker estimates similar to the ones obtained in Subsection 3.2.5 are also enough to [2].*

Proof. Note that after scaling, we are assuming $\overline{R}_\infty = 4n(n-1)$. Then

$$\begin{aligned}
\Delta_{g_{x_0}} \bar{U}_{(x_0,\epsilon)} &- \frac{n-2}{4(n-1)} R_{g_{x_0}} \bar{U}_{(x_0,\epsilon)} + \frac{n-2}{4(n-1)} \overline{R}_\infty \bar{U}_{(x_0,\epsilon)}^{\frac{n+2}{n-2}} \\
&= (\Delta_{g_{x_0}} \chi_\rho)(W_\epsilon + \phi - \epsilon^{\frac{n-2}{2}} |x|^{2-n}) + 2\langle d\chi_\rho, d(W_\epsilon + \phi - \epsilon^{\frac{n-2}{2}} |x|^{2-n}) \rangle_{g_{x_0}} \\
&\quad - (\Delta_{g_{x_0}} \chi_\rho) \epsilon^{\frac{n-2}{2}} (G_{x_0} - |x|^{2-n}) - 2\epsilon^{\frac{n-2}{2}} \langle d\chi_\rho, d(G_{x_0} - |x|^{2-n}) \rangle_{g_{x_0}} \\
&\quad + \chi_\rho \left(\Delta_{g_{x_0}} (W_\epsilon + \phi) - \frac{n-2}{4(n-1)} R_{g_{x_0}} (W_\epsilon + \phi) + n(n-2)(W_\epsilon + \phi)^{\frac{n+2}{n-2}} \right) \\
&\quad + n(n-2) \left(\left(\chi_\rho (W_\epsilon + \phi) + (1-\chi_\rho) \epsilon^{\frac{n-2}{2}} G_{x_0} \right)^{\frac{n+2}{n-2}} - \chi_\rho (W_\epsilon + \phi)^{\frac{n+2}{n-2}} \right) \\
&= I_1 + I_2 + I_3 + I_4
\end{aligned}$$

where I_i, i=1,2,3,4, denote the corresponding row.

To estimate I_1, notice that for $|x| \geq \rho > \epsilon$ we have

$$\left| (\epsilon^2 + |x|^2)^{\frac{2-n}{2}} - |x|^{2-n} \right| \leq C\epsilon^2 |x|^{-n} \tag{3.39}$$

and, equivalently, $|W_\epsilon - \epsilon^{\frac{n-2}{2}} |x|^{2-n}| \leq C\epsilon^{\frac{n+2}{2}} |x|^{-n}$. Then I_1 can be estimated as

$$|I_1| \leq C(\epsilon^{\frac{n+2}{2}} \rho^{-2-n} + \epsilon^{\frac{n-2}{2}} \rho^{1-n}) 1_{B_{2\rho}^+(0) \setminus B_\rho^+(0)}.$$

Recall the properties (3.32) of G_{x_0}. Then $|I_2| \leq C\epsilon^{\frac{n-2}{2}} \rho^{1-n} (\log \rho) 1_{B_{2\rho}^+(0) \setminus B_\rho^+(0)}$. Using (3.22) we calculate

$$|I_3| \leq C \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n-2}{2}} (\epsilon^2 + |x|^2)^{-\frac{1}{2}} 1_{B_{2\rho}^+(0)}.$$

Some elementary calculation reveals

$$|I_4| \leq C \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n+2}{2}} 1_{M \setminus B_\rho^+(0)}.$$

Combining all the estimates above, we get the conclusion. □

Proposition 3.2.15. *For $x \in \partial M$,*

$$\left| \frac{2(n-1)}{n-2} \frac{\partial}{\partial \eta_{g_{x_0}}} \bar{U}_{(x_0,\epsilon)} - H_{g_{x_0}} \bar{U}_{(x_0,\epsilon)} \right| (x) \leq C\rho \left(\frac{\epsilon}{\epsilon^2 + |\bar{x}|^2} \right)^{\frac{n-2}{2}} 1_{D_{2\rho}(0)}(x).$$

Proof. Observe that

$$\begin{aligned}
\frac{\partial}{\partial \eta_{g_{x_0}}} \bar{U}_{(x_0,\epsilon)} - \frac{n-2}{2(n-1)} H_{g_{x_0}} \bar{U}_{(x_0,\epsilon)} =& \chi_\rho \frac{\partial}{\partial \eta_{g_{x_0}}} (W_\epsilon + \phi) + \frac{n-2}{2(n-1)} \chi_\rho H_{g_{x_0}} (W_\epsilon + \phi) \\
&+ (1-\chi_\rho) \epsilon^{\frac{n-2}{2}} \left(\frac{\partial}{\partial \eta_{g_{x_0}}} G_{x_0} - \frac{n-2}{2(n-1)} H_{g_{x_0}} G_{x_0} \right).
\end{aligned}$$

Recall that we were using Fermi coordinates, thus $\eta_{g_{x_0}} = \partial_n$. The first and third terms are zero by the equations (3.11) and (3.22) while the middle one can be bounded as

$$|\chi_\rho H_{g_{x_0}}(W_\epsilon + \phi)| \leq C\rho \left(\frac{\epsilon}{\epsilon + |\bar{x}|^2}\right)^{\frac{n-2}{2}} 1_{D_{2\rho}(0)}.$$

□

3.2.3 Type B test functions ($\bar{u}_{B;(x_0,\epsilon)}$)

In this case the test functions we use are essentially the same as in [14]. However, when trying to control their energy by $Y(S^n)$, due to the proximity to the boundary, the argument in that paper cannot be directly applied. We are able to overcome this difficulty by exploiting the sign of $\partial_n W_\epsilon(0)$ (see the definition in (3.10)). Since all the argument is local, we do not make use of the positive mass theorem in this subsection.

Fix $x_0 \in M_{2\delta_0}\backslash\partial M$ and let $\psi_{x_0} : \tilde{B}_{x_0,2\rho} \subset \mathbb{R}^n \to M$ be normal coordinates centered at x_0, where $0 < \rho \leq P_0$. We will sometimes omit the symbols ψ_{x_0} in order to simplify our notations, identifying $\psi_{x_0}(x) \in M$ with $x \in \tilde{B}_{x_0,2\rho}$. In those coordinates, we have the properties $g_{ab}(0) = \delta_{ab}$ and $\partial_c g_{ab}(0) = 0$, for $a, b, c = 1, ..., n$. If we write $g = \exp(h)$, where exp denotes the matrix exponential, then the symmetric 2-tensor h satisfies the following properties:

$$\begin{cases} h_{ab}(0) = 0\,, & \text{for } a, b = 1, ..., n\,, \\ \partial_c h_{ab}(0) = 0\,, & \text{for } a, b, c = 1, ..., n\,, \\ \sum_{b=1}^n x_b h_{ab}(x) = 0\,, & \text{for } x \in \tilde{B}_{x_0\rho},\ a = 1, ..., n\,. \end{cases}$$

According to [55], we can find a conformal metric $g_{x_0} = f_{x_0}^{\frac{4}{n-2}} g_0$, with $f_{x_0}(x_0) = 1$, such that $\det(g_{x_0})(x) = 1 + O(|x|^{2d+2})$ in normal coordinates centered at x_0, again written $\psi_{x_0} : \tilde{B}_{x_0,2\rho} \to M$ for simplicity. We can suppose that $1/2 \leq f_{x_0} \leq 3/2$.

Notation. In order to simplify our notations, in the coordinates above, we will write g_{ab} and g^{ab} instead of $(g_{x_0})_{ab}$ and $(g_{x_0})^{ab}$ respectively, h_{ab} instead of $(h_{x_0})_{ab}$, and η^a instead of $(\eta_{x_0})^a$. We denote by $\nu = \nu_{x_0}$ the unit normal vector to $\tilde{D}_{x_0,\rho}$ with respect to the Euclidean metric, pointing the same way as η_{g_0}, and $\eta_{g_{x_0}}$ and write $\nu = \nu^a\partial_a$ and $\eta = \eta^a\partial_a$.

Set $\delta = d_{g_{x_0}}(x_0, \partial M)$. If $\bar{x}_0 \in \partial M$ is chosen such that $d_{g_{x_0}}(x_0, \bar{x}_0) = \delta$ then we can assume $\psi_{x_0}(\bar{x}_0) = (-\delta, 0)$. Thus, $\eta_{g_{x_0}}(-\delta, 0) = \nu(-\delta, 0) = \partial_n$ and there exists $C_0 = C_0(M, g_0) > 2$ such that

$$|\eta^a(x) - \delta_{an}| \le C_0|\bar{x}|, \quad \text{and} \tag{3.40}$$

$$|\nu^a(x) - \delta_{an}| \le C_0|\bar{x}|, \quad \text{for all } x \in \tilde{D}_{x_0,2\rho}, \tag{3.41}$$

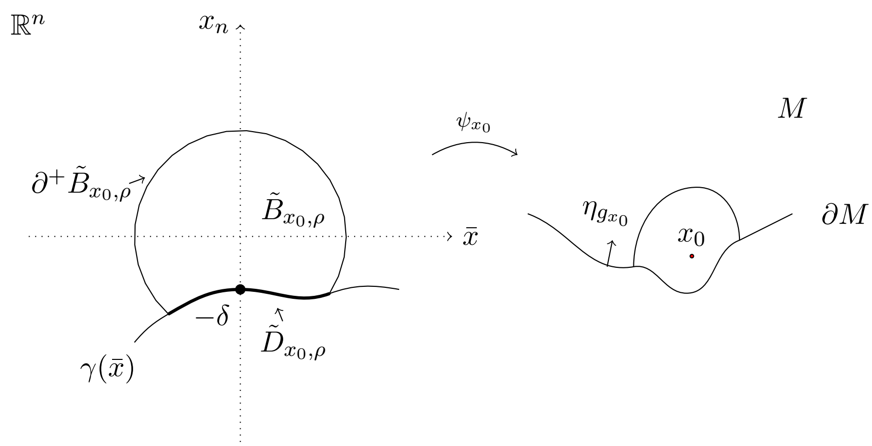
where $x = (x_1, \cdots, x_n) = (\bar{x}, x_n) \in \mathbb{R}^n$. We will also assume that $\tilde{D}_{x_0,2\rho}$ is the graph of a smooth function $\gamma = \gamma_{x_0}$ so that

$$\tilde{D}_{x_0,2\rho} = \{x = (\bar{x}, \gamma(\bar{x})) \mid |x| < 2\rho\}.$$

We can write $\gamma(\bar{x}) = -\delta + O(|\bar{x}|^2)$ and choose C_0 larger if necessary such that

$$|\gamma(\bar{x}) + \delta| \le C_0|\bar{x}|^2, \quad \text{for all } x \in \tilde{D}_{x_0,2\rho}. \tag{3.42}$$

See Figure 1.



In this subsection, we denote by

$$\mathcal{H}_{ab}(x) = \sum_{2 \le |\alpha| \le d} h_{ab,\alpha} x^\alpha$$

the Taylor expansion of order $d = \left[\frac{n-2}{2}\right]$ associated with the function $h_{ab}(x)$. Thus, $h_{ab}(x) = \mathcal{H}_{ab}(x) + O(|x|^{d+1})$. We define ϕ, S, T and $Q_{ab,c}$ as in Subsection 3.2.1 (see (3.21), (3.18) and (3.26)), except for the fact that, as in [14], the whole construction is done in $\mathbb{R}^n$ instead of $\mathbb{R}^n_+$. Then the first equation of (3.22) and the estimates (3.23) and (3.24) also hold, with $2 \le |\alpha| \le d$ replacing $1 \le |\alpha| \le d$.

Lemma 3.2.16. *There exists $\lambda = \lambda(n) > 0$ such that*

$$\lambda \epsilon^{n-2} \sum_{a,b=1}^{n} \sum_{|\alpha|=2}^{d} |h_{ab,\alpha}|^2 \int_{B_\rho(0)} (\epsilon + |x|)^{2|\alpha|+2-2n} dx \leq \frac{1}{4} \int_{B_\rho(0)} Q_{ab,c} Q_{ab,c}$$

for all $\rho \geq 2\epsilon$.

Proof. See [14, Corollary 10]. □

Recall the definitions of W_ϵ in (3.10), χ_ρ in (3.15), and $\overline{R}_\infty$ in (3.9). Set

$$\begin{aligned} \bar{U}_{(x_0,\epsilon)}(x) = \left(\frac{4n(n-1)}{\overline{R}_\infty}\right)^{\frac{n-2}{4}} \chi_\rho(\psi_{x_0}^{-1}(x)) \big(W_\epsilon(\psi_{x_0}^{-1}(x)) + \phi(\psi_{x_0}^{-1}(x))\big) \\ + \left(\frac{4n(n-1)}{\overline{R}_\infty}\right)^{\frac{n-2}{4}} \epsilon^{\frac{n-2}{2}} \big(1 - \chi_\rho(\psi_{x_0}^{-1}(x))\big) G_{x_0}(x)\,, \end{aligned}$$

if $x \in \psi_{x_0}(\tilde{B}_{x_0,2\rho})$, and $\bar{U}_{(x_0,\epsilon)}(x) = G_{x_0}(x)$ otherwise. Here, G_{x_0} is the Green's function of the conformal Laplacian $L_{g_{x_0}}$ with pole at $x_0 \in M \backslash \partial M$, satisfying the boundary condition (3.31) and the normalization $\lim_{|y| \to 0} |y|^{n-2} G_{x_0}(\psi_{x_0}(y)) = 1/2$. This function is obtained in Proposition 3.B.3 and satisfies, for some $C = C(M, g_0)$,

$$|G_{x_0}(\psi_{x_0}(y)) - |y|^{2-n}| \leq \begin{cases} C|y|^{3-n} + C\delta|y|^{1-n} & \text{if } n \geq 4, \\ C(1 + \log|y|) + C\delta|y|^{1-n} & \text{if } n = 3, \end{cases} \tag{3.43}$$

$$\left|\frac{\partial}{\partial y_b}(G_{x_0}(\psi_{x_0}(y)) - |y|^{2-n})\right| \leq C|y|^{2-n} + C\delta|y|^{-n}\,,$$

for all $b = 1, ..., n$ and $\psi_{x_0}(y) \in M_{\tilde{\delta}}$ for some small $\tilde{\delta} = \tilde{\delta}(M, g_0)$.

Define the test function

$$\bar{u}_{B;(x_0,\epsilon)} = f_{x_0} \bar{U}_{(x_0,\epsilon)}. \tag{3.44}$$

Observe that this function also depends on the radius ρ above, which will be fixed later in Section 3.3. Such constant will also be referred to as ρ_B in order to avoid confusion with test functions of the other subsections.

The main result of this subsection is the following:

Proposition 3.2.17. *Under the hypothesis of Theorem 1.2.4, there exist positive P_2 and C_B, depending only on (M, g_0), such that for any $\rho_B \leq P_2$ one can choose $\delta_0 < C_B \rho_B^2$*

satisfying

$$\frac{\int_M \left\{\frac{4(n-1)}{n-2}|d\bar{u}_{B;(x_0,\epsilon)}|^2_{g_0} + R_{g_0}\bar{u}^2_{B;(x_0,\epsilon)}\right\} dv_{g_0}}{\left(\int_M \bar{u}^{\frac{2n}{n-2}}_{B;(x_0,\epsilon)} dv_{g_0}\right)^{\frac{n-2}{n}}}$$

$$= \frac{\int_M \left\{\frac{4(n-1)}{n-2}|d\bar{U}_{(x_0,\epsilon)}|^2_{g_{x_0}} + R_{g_{x_0}}\bar{U}^2_{(x_0,\epsilon)}\right\} dv_{g_{x_0}} + \int_{\partial M} 2H_{g_{x_0}}\bar{U}^2_{(x_0,\epsilon)} d\sigma_{g_{x_0}}}{\left(\int_M \bar{U}^{\frac{2n}{n-2}}_{(x_0,\epsilon)} dv_{g_{x_0}}\right)^{\frac{n-2}{n}}}$$

$$\leq Y(S^n)$$

for all $x_0 \in M_{2\delta_0}\backslash\partial M$ *and* $0 < \epsilon < C_B^{-1} d_{g_0}(x_0, \partial M)$.

We will prove several lemmas before proceeding to the proof of Proposition 3.2.17.

Lemma 3.2.18. *If* $|\bar{x}| \leq 1/(2C_0)$, *then for* $\epsilon > 0$ *and* $0 < \delta < 1$ *we have*

$$\frac{1}{2C_0}(\epsilon^2 + |\bar{x}|^2 + \delta^2) < \epsilon^2 + |\bar{x}|^2 + \gamma(\bar{x})^2 < 2(\epsilon^2 + |\bar{x}|^2 + \delta^2). \tag{3.45}$$

Proof. First assume $\delta \geq C_0|\bar{x}|^2$. Since $|\gamma(\bar{x})| \geq \delta - C_0|\bar{x}|^2 \geq 0$, Cauchy's inequality implies

$$\gamma(\bar{x})^2 \geq \left(\delta - C_0|\bar{x}|^2\right)^2 \geq \delta^2 - 2C_0\delta|\bar{x}|^2 \geq \frac{1}{2}\delta^2 - 2C_0^2|\bar{x}|^4.$$

So,

$$\epsilon^2 + |\bar{x}|^2 + \gamma(\bar{x})^2 \geq \epsilon^2 + (1 - 2C_0^2|\bar{x}|^2)|\bar{x}|^2 + \frac{1}{2}\delta^2,$$

and our assumption $|\bar{x}|^2 \leq 1/(4C_0^2)$ gives

$$\epsilon^2 + |\bar{x}|^2 + \gamma(\bar{x})^2 \geq \epsilon^2 + \frac{1}{2}|\bar{x}|^2 + \frac{1}{2}\delta^2 > \frac{1}{2}(\epsilon^2 + |\bar{x}|^2 + \delta^2).$$

If $\delta < C_0|\bar{x}|^2$ we have

$$|\bar{x}|^2 + \gamma(\bar{x})^2 + \epsilon^2 > \frac{\delta^2}{2C_0} + \frac{|\bar{x}|^2}{2} + \epsilon^2 > \frac{1}{2C_0}(\delta^2 + |\bar{x}|^2 + \epsilon^2).$$

so the left part of (3.45) is proved.

As for the right part, notice that

$$\gamma(\bar{x})^2 \leq (\delta + C_0|\bar{x}|^2)^2 \leq 2\delta^2 + 2C_0^2|\bar{x}|^4.$$

Consequently,

$$\epsilon^2 + |\bar{x}|^2 + \gamma(\bar{x})^2 \leq \epsilon^2 + (1 + 2C_0^2|\bar{x}|^2)|\bar{x}|^2 + 2\delta^2 < 2(\epsilon^2 + |\bar{x}|^2 + \delta^2),$$

because our assumption on $|\bar{x}|$ implies $1 + 2C_0^2|\bar{x}|^2 \leq 2$. $\square$

Lemma 3.2.19. *If* $0 < \rho < 1/C_0$ *and* $0 < \delta \leq \rho/4$ *then*

$$\sqrt{|\bar{x}|^2 + \gamma(\bar{x}))^2} < \rho, \quad \textit{for all}\, |\bar{x}| \leq \rho/2.$$

Proof. From our assumption it is easy to get $\delta/\rho + C_0\rho/4 \leq 1/2$. Since

$$|\gamma(\bar{x})| \leq \delta + C_0|\bar{x}|^2 \leq \delta + C_0\rho^2/4,$$

we have

$$|\bar{x}|^2 + \gamma(\bar{x})^2 \leq \frac{\rho^2}{4} + \left(\delta + \frac{C_0\rho^2}{4}\right)^2 \leq \frac{\rho^2}{4} + \left(\frac{\rho}{2}\right)^2 = \frac{\rho^2}{2}.$$

$\square$

Lemma 3.2.20. *If* $0 < \rho \leq 1/C_0$ *and* $0 < \delta < 1$ *then*

$$\sqrt{|\bar{x}|^2 + \gamma(\bar{x})^2} > \delta/\sqrt{C_0}\,, \qquad \textit{for all}\ |\bar{x}| < \rho.$$

Proof. First assume $\delta \geq C_0|\bar{x}|^2$. Then $|\gamma(\bar{x})| \geq \delta - C_0|\bar{x}|^2 \geq 0$, which yields

$$\begin{aligned}\gamma(\bar{x})^2 &\geq (\delta - C_0|\bar{x}|^2)^2 = \delta^2 - 2\delta C_0|\bar{x}|^2 + C_0^2|\bar{x}|^4 \\ &\geq \delta^2 - \frac{\delta^2}{2} - 2C_0^2|\bar{x}|^4 + C_0^2|\bar{x}|^4 = \frac{\delta^2}{2} - C_0^2|\bar{x}|^4.\end{aligned}$$

Therefore, by the assumption $|\bar{x}| < \rho \leq 1/C_0$, we have

$$|\bar{x}|^2 + \gamma(\bar{x})^2 \geq (1 - C_0^2|\bar{x}|^2)|\bar{x}|^2 + \delta^2/2 \geq \delta^2/2 > \delta^2/C_0,$$

because $C_0 > 2$.

If $\delta < C_0|\bar{x}|^2$, since $0 < \delta < 1$, we have $\delta^2 < \delta < C_0|\bar{x}|^2$. Obviously

$$|\bar{x}|^2 + \gamma(\bar{x})^2 > \delta^2/C_0,$$

proving the result. $\square$

Lemma 3.2.21. *There exists $C = C(n)$ such that*

$$\int_{\{\bar{x}\in\mathbb{R}^{n-1}|\ |\bar{x}|\leq\rho\}} (\epsilon^2+|\bar{x}|^2+\delta^2)^{2-n}d\bar{x} \leq C\rho\delta^{2-n}, \qquad \textit{for } 0<\delta\leq\rho.$$

Proof. Just observe that

$$\begin{aligned}\int_{|\bar{x}|\leq\rho}(\epsilon^2+|\bar{x}|^2+\delta^2)^{2-n}d\bar{x} &\leq \int_{|\bar{x}|\leq\rho}(|\bar{x}|^2+\delta^2)^{2-n}d\bar{x}\\ &\leq \sqrt{2}\rho\int_{\mathbb{R}^{n-1}}(|\bar{x}|^2+\delta^2)^{\frac{3-2n}{2}}d\bar{x} = \sqrt{2}\rho\delta^{2-n}\int_{\mathbb{R}^{n-1}}(|\bar{y}|^2+1)^{\frac{3-2n}{2}}d\bar{y}.\end{aligned}$$

□

Lemma 3.2.22. *There exist $\tilde{c}, K, P_2 > 0$, depending only on (M, g_0), such that*

$$\frac{4(n-1)}{n-2}\int_{\tilde{D}_{x_0,\rho}} W_\epsilon\partial_\nu W_\epsilon d\sigma \geq \tilde{c}\epsilon^{n-2}\delta^{2-n}$$

when $0<\epsilon<\delta<K\rho$ and $\rho<P_2$.

Proof. Observe that $W_\epsilon\partial_a W_\epsilon = -(n-2)\epsilon^{n-2}(\epsilon^2+|x|^2)^{1-n}x_a$ and, on $\tilde{D}_{x_0,\rho}$,

$$W_\epsilon\partial_\nu W_\epsilon = W_\epsilon\nu^a\partial_a W_\epsilon = W_\epsilon\partial_n W_\epsilon + W_\epsilon(\nu^a-\delta_{an})\partial_a W_\epsilon.$$

Using (3.41) and Lemma 3.2.18, we have

$$\begin{aligned}|W_\epsilon(\nu^a-\delta_{an})\partial_a W_\epsilon|(x) &\leq (n-2)C\epsilon^{n-2}(\epsilon^2+|\bar{x}|^2+\gamma(\bar{x})^2)^{2-n}\\ &\leq (2C_0)^{n-2}(n-2)C\epsilon^{n-2}(\epsilon^2+|\bar{x}|^2+\delta^2)^{2-n}\end{aligned}$$

when $x=(\bar{x},\gamma(\bar{x}))\in\tilde{D}_{x_0,\rho}$ with $|\bar{x}|\leq(2C_0)^{-1}$. Hence if $\rho\leq(2C_0)^{-1}$ and $0<\delta\leq\rho$, then

$$\int_{\tilde{D}_{x_0,\rho}} W_\epsilon\partial_\nu W_\epsilon d\sigma \geq \int_{\tilde{D}_{x_0,\rho}} W_\epsilon\partial_n W_\epsilon d\sigma - C\rho\left(\frac{\epsilon}{\delta}\right)^{n-2},$$

where we used Lemma 3.2.21.

In order to estimate from below the r.h.s. of this last inequality, we see that

$$\begin{aligned}W_\epsilon\partial_n W_\epsilon(x) &= -(n-2)\epsilon^{n-2}(\epsilon^2+|\bar{x}|^2+\gamma(\bar{x})^2)^{1-n}\gamma(\bar{x})\\ &\geq (n-2)\epsilon^{n-2}(\epsilon^2+|\bar{x}|^2+\gamma(\bar{x})^2)^{1-n}(\delta-C_0|\bar{x}|^2)\\ &\geq (n-2)\epsilon^{n-2}\delta(\epsilon^2+|\bar{x}|^2+\gamma(\bar{x})^2)^{1-n}-(n-2)C_0\epsilon^{n-2}(\epsilon^2+|\bar{x}|^2+\gamma(\bar{x})^2)^{2-n}\\ &\geq (n-2)2^{1-n}\epsilon^{n-2}\delta(\epsilon^2+|\bar{x}|^2+\delta^2)^{1-n}-C\epsilon^{n-2}(\epsilon^2+|\bar{x}|^2+\delta^2)^{2-n}\end{aligned}$$

for $x = (\bar{x}, \gamma(\bar{x})) \in \tilde{D}_{x_0,\rho}$ with $|\bar{x}| \leq (2C_0)^{-1}$, where we used Lemma 3.2.18 in the last step.

Assume $0 < \rho < (2C_0)^{-1}$ and $0 < \delta \leq \rho/4$. According to Lemma 3.2.19,

$$\left\{(\bar{x}, \gamma(\bar{x})) \;\middle|\; |\bar{x}| \leq \rho/2\right\} \subset \tilde{D}_{x_0,\rho}.$$

Then

$$\begin{aligned}\int_{\tilde{D}_{x_0,\rho}} W_\epsilon \partial_n W_\epsilon d\sigma &\geq (n-2)2^{1-n}\epsilon^{n-2}\delta \int_{|\bar{x}|\leq\rho/2} (\epsilon^2 + |\bar{x}|^2 + \delta^2)^{1-n} d\bar{x} \\ &\quad - C\epsilon^{n-2} \int_{|\bar{x}|<\rho} (\epsilon^2 + |\bar{x}|^2 + \delta^2)^{2-n} d\bar{x} \\ &= I - II.\end{aligned}$$

Notice that

$$\begin{aligned}\delta \int_{|\bar{x}|\leq\rho/2} (\epsilon^2 + |\bar{x}|^2 + \delta^2)^{1-n} d\bar{x} &= \delta^{2-n} \int_{|\bar{y}|\leq\rho/2\delta} \left(\left(\frac{\epsilon}{\delta}\right)^2 + |\bar{y}|^2 + 1\right)^{1-n} d\bar{y} \\ &\geq 2^{1-n}\delta^{2-n} \int_{|\bar{y}|\leq\rho/2\delta} (|\bar{y}|^2 + 1)^{1-n} d\bar{y}\end{aligned}$$

for $0 < \epsilon < \delta$, because $(\epsilon/\delta)^2 + |\bar{y}|^2 + 1 < 2(|\bar{y}|^2 + 1)$.

Set $\alpha(n) = \int_{\mathbb{R}^{n-1}} (|\bar{y}|^2 + 1)^{1-n} d\bar{y}$ and observe that

$$\int_{|\bar{y}|\leq\rho/2\delta} (|\bar{y}|^2 + 1)^{1-n} d\bar{y} = \alpha(n) - \int_{|\bar{y}|>\rho/2\delta} (|\bar{y}|^2 + 1)^{1-n} d\bar{y} \geq \alpha(n) - C\left(\frac{\delta}{\rho}\right)^{n-1}.$$

Hence,

$$I \geq (n-2)2^{2-2n}\alpha(n)\left(\frac{\epsilon}{\delta}\right)^{n-2} - C\left(\frac{\delta}{\rho}\right)^{n-1}\left(\frac{\epsilon}{\delta}\right)^{n-2}.$$

On the other hand, $II \leq C\rho\,(\epsilon/\delta)^{n-2}$, by Lemma 3.2.21.

Putting things together, we obtain

$$\int_{\tilde{D}_{x_0,\rho}} W_\epsilon \partial_\nu W_\epsilon d\sigma \geq (n-2)2^{2-2n}\big(\alpha(n) - C(\delta/\rho)^{n-1} - C\rho\big)\,(\epsilon/\delta)^{n-2},$$

from which the result follows. □

Proposition 3.2.23. *There exists $P_2 = P_2(M, g_0) > 0$ such that if $0 < \delta \le \rho \le P_2$*

$$\begin{aligned}
&\int_{\tilde{B}_{x_0\rho}} \left\{ \frac{4(n-1)}{n-2} |d(W_\epsilon + \phi)|^2 + R_{g_{x_0}} (W_\epsilon + \phi)^2 \right\} dx \\
&\quad \le \frac{4(n-1)}{n-2} \int_{\tilde{B}_{x_0,\rho}} |dW_\epsilon|^2 dx + \int_{\tilde{B}_{x_0,\rho}} \frac{4(n-1)}{n-2} n(n+2) W_\epsilon^{\frac{4}{n-2}} \phi^2 dx \\
&\quad + \frac{\lambda}{2} \sum_{a,b=1}^{n} \sum_{|\alpha|=2}^{d} |h_{ab,\alpha}|^2 \epsilon^{n-2} \int_{\tilde{B}_{x_0,\rho}} (\epsilon + |x|)^{2|\alpha|+2-2n} dx \\
&\quad - \frac{1}{4} \int_{\tilde{B}_{x_0,\rho}} Q_{ab,c} Q_{ab,c}\, dx + C\rho \left(\frac{\epsilon}{\delta}\right)^{n-2} + C\rho \left(\frac{\epsilon}{\rho}\right)^{n-2}
\end{aligned}$$

for all $\epsilon \in (0, \rho/2]$. Here, λ is the constant obtained in Lemma 3.2.16.

Proof. As in [15, Proposition 3.6], we can choose $0 < P_2 < 1$ such that

$$\begin{aligned}
&\int_{\tilde{B}_{x_0,\rho}} \left\{ \frac{4(n-1)}{n-2} |d(W_\epsilon + \phi)|^2 + R_{g_{x_0}} (W_\epsilon + \phi)^2 \right\} dx \\
&\le \frac{4(n-1)}{n-2} \int_{\tilde{B}_{x_0\rho}} |dW_\epsilon|^2 dx + \int_{\tilde{B}_{x_0,\rho}} \frac{4(n-1)}{n-2} n(n+2) W_\epsilon^{\frac{4}{n-2}} \phi^2 dx \\
&\quad + \int_{\partial^+ \tilde{B}_{x_0,\rho}} \left(W_\epsilon^2 \partial_b h_{ab} - \partial_b W_\epsilon^2 h_{ab} \right) \frac{x_a}{|x|} d\sigma_\rho - \frac{1}{4} \int_{\tilde{B}_{x_0,\rho}} Q_{ab,c} Q_{ab,c} dx \\
&\quad + \frac{\lambda}{2} \sum_{a,b=1}^{n} \sum_{|\alpha|=2}^{d} |h_{ab,\alpha}|^2 \epsilon^{n-2} \int_{\tilde{B}_{x_0,\rho}} (\epsilon + |x|)^{2|\alpha|+2-2n} dx \\
&\quad + C \sum_{a,b=1}^{n} \sum_{|\alpha|=2}^{d} |h_{ab,\alpha}| \epsilon^{n-2} \rho^{|\alpha|+2-n} + C\epsilon^{n-2} \rho^{2d+4-n} + \int_{\tilde{D}_{x_0,\rho}} \Psi d\sigma
\end{aligned}$$

holds for all $0 < 2\epsilon \le \rho \le P_2$, where

$$\begin{aligned}
\Psi = &- \frac{8(n-1)}{n-2} \left(\partial_a W_\epsilon \phi + \frac{(n-2)^2}{2} W_\epsilon^{\frac{2n}{n-2}} V_a \right) \nu^a - W_\epsilon^2 \partial_b h_{ab} \nu^a \\
&+ 2W_\epsilon (\partial_b W_\epsilon) h_{ab} \nu^a + W_\epsilon^2 \mathcal{H}_{ab} \partial_c \mathcal{H}_{ab} \nu^b - \nu^a \xi_a
\end{aligned}$$

comes from integration by parts. Here, ξ_a is a 1-tensor controlled by

$$|\xi_a(x)| \le C \sum_{a,b=1}^{n} \sum_{|\alpha|=2}^{d} |h_{ab,\alpha}|^2 \epsilon^{n-2} (\epsilon + |x|)^{3+2|\alpha|-2n}.$$

It is easy to estimate the following term on $\tilde{D}_{x_0,\rho}$

$$|W_\epsilon^{\frac{2n}{n-2}} V_a|(x) \le C\epsilon^n (\epsilon^2 + |\bar{x}|^2 + \gamma(\bar{x})^2)^{1-n} \le C\epsilon^{n-2} (\epsilon^2 + |\bar{x}|^2 + \gamma(\bar{x})^2)^{2-n}, \tag{3.46}$$

and all the other terms in Ψ can also be estimated by the r.h.s. of (3.46).

Choosing P_2 possibly smaller, from Lemmas 3.2.18 and 3.2.21 we get

$$\int_{\tilde{D}_{x_0,\rho}} \Psi d\sigma \leq C\left(\frac{\epsilon}{\delta}\right)^{n-2}\rho, \tag{3.47}$$

for $0<\delta\leq\rho$, from which the result follows. $\square$

Proposition 3.2.24. *There exist $P_2, C>0$, depending only on (M,g_0), such that*

$$\begin{aligned}\int_{\tilde{B}_{x_0,\rho}} &\left\{\frac{4(n-1)}{n-2}|d(W_\epsilon+\phi)|^2 + R_{g_{x_0}}(W_\epsilon+\phi)^2\right\}dx\\ &\leq Y(S^n)\left(\int_{\tilde{B}_{x_0,\rho}}(W_\epsilon+\phi)^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{n}} - (\tilde{c}-C\rho-C(\delta/\rho)^{n-2})\left(\frac{\epsilon}{\delta}\right)^{n-2}\\ &\quad -\frac{\lambda}{4}\sum_{a,b=1}^{n}\sum_{|\alpha|=2}^{d}|h_{ab,\alpha}|^2\epsilon^{n-2}\int_{\tilde{B}_{x_0,\rho}}(\epsilon+|x|)^{2|\alpha|+2-2n}dx\end{aligned}$$

for all $0<\rho\leq P_2$ and $0<\epsilon<\delta<K\rho$, where K and $\tilde{c}$ are the constants obtained in Lemma 3.2.22.

Proof. This result is a consequence of Proposition 3.2.23 and Lemma 3.2.16. Observe that

$$\begin{aligned}\frac{4(n-1)}{n-2}&\int_{\tilde{B}_{x_0,\rho}}|dW_\epsilon|^2dx + \int_{\tilde{B}_{x_0,\rho}}\frac{4(n-1)}{n-2}n(n+2)W_\epsilon^{\frac{4}{n-2}}\phi^2dx \\ &= \int_{\tilde{B}_{x_0,\rho}}\frac{4(n-1)}{n-2}\left(n(n-2)W_\epsilon^{\frac{2n}{n-2}}+n(n+2)W_\epsilon^{\frac{4}{n-2}}\phi^2\right)dx\\ &\quad -\int_{\tilde{D}_{x_0,\rho}}\frac{4(n-1)}{n-2}W_\epsilon\partial_\nu W_\epsilon d\sigma + \int_{\partial^+\tilde{B}_{x_0,\rho}}\frac{4(n-1)}{n-2}W_\epsilon\partial_a W_\epsilon\frac{x_a}{|x|}d\sigma\\ &\leq \int_{\tilde{B}_{x_0,\rho}}4n(n-1)W_\epsilon^{\frac{4}{n-2}}(W_\epsilon^2+\frac{n+2}{n-2}\phi^2)\,dx\\ &\quad -\int_{\tilde{D}_{x_0,\rho}}\frac{4(n-1)}{n-2}W_\epsilon\partial_\nu W_\epsilon d\sigma + C\left(\frac{\epsilon}{\rho}\right)^{n-2}.\end{aligned} \tag{3.48}$$

We shall handle the first two terms of the r.h.s. of (3.48) separately. As in [14, Proposition 14], we have

$$\left(W_\epsilon^2+\frac{n+2}{n-2}\phi^2\right)^{\frac{n}{n-2}} - (W_\epsilon+\phi)^{\frac{2n}{n-2}} + \frac{2n}{n-2}W_\epsilon^{\frac{n+2}{n-2}}\phi \leq C\sum_{a,b=1}^{n}\sum_{|\alpha|=2}^{d}|h_{ab,\alpha}|^2\epsilon^n(\epsilon+|x|)^{2|\alpha|+2-2n}$$

and

$$\begin{aligned}\int_{\tilde{B}_{x_0,\rho}}\frac{2n}{n-2}W_\epsilon^{\frac{n+2}{n-2}}\phi\,dx \geq \int_{\tilde{B}_{x_0,\rho}}\partial_a(W_\epsilon^{\frac{2n}{n-2}}V_a)\,dx &= \int_{\partial^+\tilde{B}_{x_0,\rho}}W_\epsilon^{\frac{2n}{n-2}}V_a\frac{x_a}{|x|}\,d\sigma - \int_{\tilde{D}_{x_0,\rho}}W_\epsilon^{\frac{2n}{n-2}}V_a\nu^a\,d\sigma\\ &\geq -C\rho^{1-n}\epsilon^n - C\rho\left(\frac{\epsilon}{\delta}\right)^{n-2}.\end{aligned}$$

Here, in the last step we estimated the integral on $\tilde{D}_{x_0,\rho}$ by (3.46) and Lemmas 3.2.18 and 3.2.21. So,

$$\int_{\tilde{B}_{x_0,\rho}} 4n(n-1)W_\epsilon^{\frac{4}{n-2}}(W_\epsilon^2+\frac{n+2}{n-2}\phi^2)\,dx \le Y(S^n)\left(\int_{\tilde{B}_{x_0,\rho}}(W_\epsilon^2+\frac{n+2}{n-2}\phi^2)^{\frac{n}{n-2}}dx\right)^{\frac{n-2}{n}} \tag{3.49}$$

$$\le Y(S^n)\left(\int_{\tilde{B}_{x_0,\rho}}(W_\epsilon+\phi)^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{n}}+C\rho\left(\frac{\epsilon}{\rho}\right)^n+C\rho\left(\frac{\epsilon}{\delta}\right)^{n-2}$$
$$+C\sum_{a,b=1}^{n}\sum_{|\alpha|=2}^{d}|h_{ab,\alpha}|^2\epsilon^n\int_{\tilde{B}_{x_0,\rho}}(\epsilon+|x|)^{2|\alpha|+2-2n}dx.$$

Recall that Lemma 3.2.22 says

$$-\int_{\tilde{D}_{x_0,\rho}}\frac{4(n-1)}{n-2}W_\epsilon\partial_\nu W_\epsilon d\sigma \le -\tilde{c}\left(\frac{\epsilon}{\delta}\right)^{n-2} \tag{3.50}$$

if $0<\epsilon<\delta<K\rho$ and $0<\rho<P_2$, for P_2 small enough.

Now it follows from Lemma 3.2.16 that

$$\lambda\epsilon^{n-2}\sum_{a,b=1}^{n}\sum_{|\alpha|=2}^{d}|h_{ab,\alpha}|^2\int_{\tilde{B}_{x_0,\rho}(0)}(\epsilon+|x|)^{2|\alpha|+2-2n}dx \le \frac{1}{4}\int_{B_\rho(0)}Q_{ab,c}Q_{ab,c}\,dx.$$

We claim that we can choose $P_2>0$ possibly smaller such that

$$\int_{B_\rho(0)\setminus\tilde{B}_{x_0,\rho}}Q_{ab,c}Q_{ab,c}\,dx \le C\rho^2\left(\frac{\epsilon}{\delta}\right)^{n-2}$$

for all $\rho<P_2$. In fact, from Lemma 3.2.20 we can choose P_2 small such that

$$B_\rho(0)\setminus\tilde{B}_{x_0,\rho}\subset B_\rho(0)\setminus B_{\delta/\sqrt{C_0}}(0)$$

for any $\rho<P_2$. Then using $Q_{ab,c}Q_{ab,c}\le C\epsilon^{n-2}(\epsilon+|x|)^{4-2n}$ we get

$$\int_{B_\rho(0)\setminus\tilde{B}_{x_0,\rho}}Q_{ab,c}Q_{ab,c}\,dx \le C\epsilon^{n-2}\int_{B_\rho(0)\setminus\tilde{B}_{x_0,\rho}}(\epsilon+|x|)^{4-2n}dx$$
$$\le C\epsilon^{n-2}\rho^2\int_{\mathbb{R}^n\setminus B_{\delta/\sqrt{C_0}}}(\epsilon+|x|)^{2-2n}dx \le C\epsilon^{n-2}\rho^2\delta^{2-n}.$$

In particular,

$$\lambda\epsilon^{n-2}\sum_{a,b=1}^{n}\sum_{|\alpha|=2}^{d}|h_{ab,\alpha}|^2\int_{\tilde{B}_{x_0,\rho}}(\epsilon+|x|)^{2|\alpha|+2-2n}dx \le \frac{1}{4}\int_{\tilde{B}_{x_0,\rho}}Q_{ab,c}Q_{ab,c}dx+C\rho^2\left(\frac{\epsilon}{\delta}\right)^{n-2}. \tag{3.51}$$

Now the result follows from Proposition 3.2.23 and estimates (3.48), (3.49), (3.50) and (3.51). □

Proposition 3.2.25. *There exist P_2 and K such that*

$$\int_M \left\{ \frac{4(n-1)}{n-2} |d\bar{U}_{(x_0,\epsilon)}|^2_{g_{x_0}} + R_{g_{x_0}} \bar{U}^2_{(x_0,\epsilon)} \right\} dv_{g_{x_0}} + \int_{\partial M} 2H_{g_{x_0}} \bar{U}^2_{(x_0,\epsilon)} d\sigma_{g_{x_0}}$$
$$\leq Y(S^n) \left(\int_M \bar{U}^{\frac{2n}{n-2}}_{(x_0,\epsilon)} dv_{g_{x_0}} \right)^{\frac{n-2}{n}} - \frac{\lambda}{4} \sum_{a,b=1}^{n} \sum_{|\alpha|=2}^{d} |h_{ab,\alpha}|^2 \epsilon^{n-2} \int_{\tilde{B}_{x_0,\rho}} (\epsilon + |x|)^{2|\alpha|+2-2n} dx - \frac{\tilde{c}}{2} \left(\frac{\epsilon}{\delta} \right)^{n-2}.$$

for all $0 < \epsilon < \delta < K\rho$ and $0 < \rho < P_2$.

Proof. We have

$$\int_{M \backslash \tilde{B}_{x_0,\rho}} \left\{ \frac{4(n-1)}{n-2} |d\bar{U}_{(x_0,\epsilon)}|^2_{g_{x_0}} + R_{g_{x_0}} \bar{U}^2_{(x_0,\epsilon)} \right\} dv_{g_{x_0}} + \int_{\partial M \backslash \tilde{D}_{x_0,\rho}} 2H_{g_{x_0}} \bar{U}^2_{(x_0,\epsilon)} d\sigma_{g_{x_0}} \leq C \left(\frac{\epsilon}{\rho} \right)^{n-2}.$$

As in the proof of Proposition 3.2.23,

$$\int_{\tilde{D}_{x_0,\rho}} 2H_{g_{x_0}} \bar{U}^2_{(x_0,\epsilon)} d\sigma_{g_{x_0}} \leq C\rho \left(\frac{\epsilon}{\delta} \right)^{n-2}.$$

The result now follows from Proposition 3.2.24 and the fact that $\det(g_{x_0})(x) = 1 + O(|x|^{2d+2})$. □

Proof of Proposition 3.2.17. Let P_2 and K be as in Proposition 3.2.25. Choose P_2 maybe smaller such that $P_2 < K$. Given $\rho_B \leq P_2$ choose $K' \leq \rho_B$ and $\delta'_0 \in (0, K'\rho_B)$. Observe that, in particular, one has $\delta'_0 < \rho_B^2$ and $\delta'_0 < K\rho_B$. By Proposition 3.2.25, the inequality we want to prove holds for all $0 < \epsilon < \delta < \delta'_0$ and $0 < \rho = \rho_B \leq P_2$, where $\delta = d_{g_{x_0}}(x_0, \partial M)$.

Now choose $C_B = C_B(M, g_0)$ such that $C_B^{-1}\delta \leq d_{g_0}(x_0, \partial M) \leq C_B\delta$, and take any $\delta_0 < C_B\delta'_0$. Then, because $\delta'_0 < \rho_B^2$, we have

$$\delta_0 < C_B\rho_B^2.$$

For any $\epsilon < C_B^{-1} d_{g_0}(x_0, \partial M)$ we have $\epsilon < C_B^{-1} d_{g_0}(x_0, \partial M) < \delta < \delta'_0$ and the inequality in Proposition 3.2.17 holds. □

We finally prove some results for later use.

Proposition 3.2.26. *For $x \in M$, $\epsilon < \rho$ and $\delta \leq C\rho^2$,*

$$\begin{aligned}
&\left|\frac{4(n-1)}{n-2}\Delta_{g_{x_0}}\bar{U}_{(x_0,\epsilon)} - R_{g_{x_0}}\bar{U}_{(x_0,\epsilon)} + \overline{R}_\infty \bar{U}_{(x_0,\epsilon)}^{\frac{n+2}{n-2}}\right|(x)\\
&\quad\leq C\rho^2\left(\frac{\epsilon}{\epsilon^2+|x|^2}\right)^{\frac{n-2}{2}} 1_{\tilde{B}_{x_0,\rho}}(x) + C\left(\frac{\epsilon}{\epsilon^2+|x|^2}\right)^{\frac{n+2}{2}} 1_{M\setminus\tilde{B}_{x_0,\rho}}(x)\\
&\quad\quad + C(\epsilon^{\frac{n+2}{2}}\rho^{-2-n} + \epsilon^{\frac{n-2}{2}}\rho^{1-n}(\log\rho))1_{\tilde{B}_{x_0,2\rho}\setminus\tilde{B}_{x_0,\rho}}(x).
\end{aligned}$$

Proof. The proof goes like that of Proposition 3.2.14 with I_1, I_2, I_3, I_4 being the same. Observing that we are using normal coordinates, we have

$$|I_3| \leq C\rho^2\left(\frac{\epsilon}{\epsilon^2+|x|^2}\right)^{\frac{n-2}{2}} 1_{\tilde{B}_{x_0,2\rho}}.$$

Using (3.43) we obtain $|I_2| \leq C\epsilon^{\frac{n-2}{2}}\rho^{1-n}(\log\rho)1_{\tilde{B}_{x_0,2\rho}\setminus\tilde{B}_{x_0,\rho}} + C\epsilon^{\frac{n-2}{2}}\delta\rho^{-1-n}1_{\tilde{B}_{x_0,2\rho}\setminus\tilde{B}_{x_0,\rho}}$, the $\log\rho$ being necessary only in dimension $n = 3$.

With the same estimate for I_1 and I_4 as in Proposition 3.2.14, we get the result. □

Proposition 3.2.27. *For $x \in \partial M$, $\epsilon < \rho$ and $\delta \leq C\rho^2$,*

$$\begin{aligned}
&\left|\frac{2(n-1)}{n-2}\frac{\partial}{\partial\eta_{g_{x_0}}}\bar{U}_{(x_0,\epsilon)} - H_{g_{x_0}}\bar{U}_{(x_0,\epsilon)}\right|(x)\\
&\qquad\leq C\frac{\delta}{\epsilon}\left(\frac{\epsilon}{\epsilon^2+|\bar{x}|^2}\right)^{\frac{n}{2}} 1_{\tilde{D}_{x_0,2\rho}}(x) + C\left(\frac{\epsilon}{\epsilon^2+|\bar{x}|^2}\right)^{\frac{n-2}{2}} 1_{\tilde{D}_{x_0,2\rho}}(x)\\
&\qquad\quad + C(\epsilon^{\frac{n+2}{2}}\rho^{-1-n} + \epsilon^{\frac{n-2}{2}}\rho^{2-n}(\log\rho))1_{\tilde{D}_{x_0,2\rho}\setminus\tilde{D}_{x_0,\rho}}(x).
\end{aligned}$$

Proof. Observe that, on ∂M,

$$\begin{aligned}
\frac{\partial\bar{U}_{(x_0,\epsilon)}}{\partial\eta_{g_{x_0}}} - \frac{n-2}{2(n-1)}H_{g_{x_0}}\bar{U}_{(x_0,\epsilon)} =& \frac{\partial\chi_\rho}{\partial\eta_{g_{x_0}}}(W_\epsilon + \phi - \epsilon^{\frac{n-2}{2}}|x|^{2-n}) + \frac{\partial\chi_\rho}{\partial\eta_{g_{x_0}}}\epsilon^{\frac{n-2}{2}}(|x|^{2-n} - G_{x_0})\\
&+ \chi_\rho\frac{\partial}{\partial\eta_{g_{x_0}}}(W_\epsilon+\phi) - \frac{n-2}{2(n-1)}\chi_\rho H_{g_{x_0}}(W_\epsilon+\phi)\\
&+ (1-\chi_\rho)\epsilon^{\frac{n-2}{2}}\left(\frac{\partial G_{x_0}}{\partial\eta_{g_{x_0}}} - \frac{n-2}{2(n-1)}H_{g_{x_0}}G_{x_0}\right),
\end{aligned}$$

where the last term is zero by the definition of G_{x_0}. Set

$$J_1 = \frac{\partial\chi_\rho}{\partial\eta_{g_{x_0}}}(W_\epsilon+\phi-\epsilon^{\frac{n-2}{2}}|x|^{2-n}), \qquad J_2 = \frac{\partial\chi_\rho}{\partial\eta_{g_{x_0}}}\epsilon^{\frac{n-2}{2}}(|x|^{2-n} - G_{x_0}),$$

$$J_3 = \chi_\rho\frac{\partial W_\epsilon}{\partial\eta_{g_{x_0}}}, \qquad J_4 = \chi_\rho\left(\frac{\partial\phi}{\partial\eta_{g_{x_0}}} - \frac{n-2}{2(n-1)}H_{g_{x_0}}(W_\epsilon+\phi)\right).$$

Recall (3.39) to bound

$$|J_1| \le \Big|\frac{\partial \chi_\rho}{\partial \eta_{g_{x_0}}}\Big|\big(\big|W_\epsilon - \epsilon^{\frac{n-2}{2}}|x|^{2-n}\big| + |\phi|\big) \le C(\epsilon^{\frac{n+2}{2}}\rho^{-1-n} + \epsilon^{\frac{n-2}{2}}\rho^{3-n})1_{\tilde{D}_{x_0,2\rho}\setminus\tilde{D}_{x_0,\rho}}.$$

For J_2, we can use the properties (3.43) of the Green function and the hypothesis $\delta \le C\rho^2$ to obtain

$$|J_2| \le \epsilon^{\frac{n-2}{2}}\Big|\frac{\partial \chi_\rho}{\partial \eta_{g_{x_0}}}\Big|\big||x|^{2-n} - G_{x_0}\big| \le C\epsilon^{\frac{n-2}{2}}\rho^{2-n}(\log\rho)1_{\tilde{D}_{x_0,2\rho}\setminus\tilde{D}_{x_0,\rho}}.$$

In order to estimate J_3, let us calculate $\partial W_\epsilon/\partial\eta_{g_{x_0}}$. Suppose $x = (\bar{x}, \gamma(\bar{x})) \in \tilde{D}_{x_0,\rho}$, then

$$\begin{aligned}\partial W_\epsilon/\partial\eta_{g_{x_0}}(x) &= -(n-2)\epsilon^{\frac{n-2}{2}}(\epsilon^2 + |x|^2)^{-\frac{n}{2}}x_a\eta^a(x)\\ &= -(n-2)\epsilon^{\frac{n-2}{2}}(\epsilon^2 + |\bar{x}|^2 + \gamma(\bar{x})^2)^{-\frac{n}{2}}(\gamma(\bar{x}) + (\eta^j(x) - \delta_{jn})x_j).\end{aligned}$$

Recall the properties (3.42) and (3.40) of γ and η. So,

$$\begin{aligned}\big|\partial W_\epsilon/\partial\eta_{g_{x_0}}\big|(x) &\le C\epsilon^{\frac{n-2}{2}}(\epsilon^2 + |\bar{x}|^2 + \gamma(\bar{x})^2)^{-\frac{n}{2}}(\delta + C|\bar{x}|^2)\\ &\le C\frac{\delta}{\epsilon}\left(\frac{\epsilon}{\epsilon^2 + |\bar{x}|^2}\right)^{\frac{n}{2}} + C\left(\frac{\epsilon}{\epsilon^2 + |\bar{x}|^2}\right)^{\frac{n-2}{2}}\end{aligned}$$

for $x \in \tilde{D}_{x_0,\rho}$. Consequently,

$$|J_3| \le C\frac{\delta}{\epsilon}\left(\frac{\epsilon}{\epsilon^2 + |\bar{x}|^2}\right)^{\frac{n}{2}}1_{\tilde{D}_{x_0,2\rho}} + C\left(\frac{\epsilon}{\epsilon^2 + |\bar{x}|^2}\right)^{\frac{n-2}{2}}1_{\tilde{D}_{x_0,2\rho}}.$$

Easily we can get

$$|J_4| \le C\chi_\rho\big(\Big|\frac{\partial\phi}{\partial\eta_{g_{x_0}}}\Big| + W_\epsilon + |\phi|\big) \le C\left(\frac{\epsilon}{\epsilon^2 + |\bar{x}|^2}\right)^{\frac{n-2}{2}}1_{\tilde{D}_{x_0,2\rho}}.$$

Combining all the results, we get the conclusion. □

Proposition 3.2.28. *For $x \in \partial M$, $\epsilon < \rho$ and $\delta \le C\rho^2$,*

$$\begin{aligned}&\Big(\frac{2(n-1)}{n-2}\frac{\partial}{\partial\eta_{g_{x_0}}}\bar{U}_{(x_0,\epsilon)} - H_{g_{x_0}}\bar{U}_{(x_0,\epsilon)}\Big)(x)\\ &\qquad \ge -C\left(\frac{\epsilon}{\epsilon^2 + |\bar{x}|^2}\right)^{\frac{n-2}{2}}1_{\tilde{D}_{x_0,2\rho}}(x) - C(\epsilon^{\frac{n+2}{2}}\rho^{-1-n} + \epsilon^{\frac{n-2}{2}}\rho^{2-n}(\log\rho))1_{\tilde{D}_{x_0,2\rho}\setminus\tilde{D}_{x_0,\rho}}(x).\end{aligned}$$

Proof. Observe that

$$\begin{aligned}\chi_\rho \frac{\partial}{\partial \eta_{g_{x_0}}} W_\epsilon &= \chi_\rho (n-2)\epsilon^{\frac{n-2}{2}}(\epsilon^2+|\bar{x}|^2+\gamma(\bar{x})^2)^{-\frac{n}{2}}(-\gamma(\bar{x})+(\delta_{jn}-\eta^j)x_j)\\ &\geq \chi_\rho (n-2)\epsilon^{\frac{n-2}{2}}(\epsilon^2+|\bar{x}|^2+\gamma(\bar{x})^2)^{-\frac{n}{2}}(\delta - C|\bar{x}|^2)\\ &\geq -C\epsilon^{\frac{n-2}{2}}(\epsilon^2+|\bar{x}|^2)^{\frac{2-n}{2}} 1_{\tilde{D}_{x_0,2\rho}},\end{aligned}$$

because $\delta > 0$. Now the result follows as in Proposition 3.2.27. □

3.2.4 Type C test functions ($\bar{u}_{C;(x_0,\epsilon)}$)

Our test functions in this case are the ones in [14], which are controlled by $Y(S^n)$ the same way as in that paper.

Recall that we assume that the background metric g_0 on M satisfies $H_{g_0} \equiv 0$ on ∂M. Fix $x_0 \in M \backslash M_{\delta_0}$ and let $\psi_{x_0} : B_{2\rho}(0) \subset \mathbb{R}^n \to B_{2\rho}(x_0) \subset M$ be normal coordinates centered at x_0, where ρ is small such that $0 < \rho \leq \delta_0/4$. As in Subsection 3.2.3, we choose a conformal metric $g_{x_0} = f_{x_0}^{\frac{4}{n-2}} g_0$ such that $\det(g_{x_0})(x) = 1 + O(|x|^{2d+2})$ in normal coordinates centered at x_0, still denoted by ψ_{x_0}. We assume $f_{x_0} \equiv 1$ in $M \backslash B_{2\rho}(x_0)$, which implies $H_{g_{x_0}} \equiv 0$ on ∂M.

Define ϕ as in Subsection 3.2.3 and set

$$\begin{aligned}\bar{U}_{(x_0,\epsilon)}(x) &= \left(\frac{4n(n-1)}{\overline{R}_\infty}\right)^{\frac{n-2}{4}} \chi_\rho(\psi_{x_0}^{-1}(x))\big(W_\epsilon(\psi_{x_0}^{-1}(x)) + \phi(\psi_{x_0}^{-1}(x))\big)\\ &\quad + \left(\frac{4n(n-1)}{\overline{R}_\infty}\right)^{\frac{n-2}{4}} \epsilon^{\frac{n-2}{2}}\big(1-\chi_\rho(\psi_{x_0}^{-1}(x))\big)G_{x_0}(x)\end{aligned} \tag{3.52}$$

if $x \in B_{2\rho}(x_0)$, and $\bar{U}_{(x_0,\epsilon)}(x) = G_{x_0}(x)$ otherwise. Here, G_{x_0} is the Green's function of the conformal Laplacian $L_{g_{x_0}} = \Delta_{g_{x_0}} - \frac{n-2}{4(n-1)}R_{g_{x_0}}$, with pole at $x_0 \in \partial M$, boundary condition (3.31) and the normalization $\lim_{|y|\to 0} |y|^{n-2} G_{x_0}(\psi_{x_0}(y)) = 1$. This function, obtained in Proposition 3.B.2, satisfies

$$|G_{x_0}(\psi_{x_0}(y)) - |y|^{2-n}| \leq C \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^{d} |h_{ij,\alpha}||y|^{|\alpha|+2-n} + \begin{cases} C|y|^{d+3-n}, & \text{if } n \geq 5,\\ C(1+\log|y|), & \text{if } n = 3,4, \end{cases} \tag{3.53}$$

$$\left|\frac{\partial}{\partial y_b}(G_{x_0}(\psi_{x_0}(y)) - |y|^{2-n})\right| \leq C \sum_{i,j=1}^{n-1} \sum_{|\alpha|=1}^{d} |h_{ij,\alpha}||y|^{|\alpha|+1-n} + C|y|^{d+2-n},$$

for some $C = C(M, g_0, \delta_0)$ for all $b = 1, ..., n$ and $x_0 \in M \backslash M_{\delta_0}$.

We define the test function

$$\bar{u}_{C;(x_0,\epsilon)} = f_{x_0} \bar{U}_{(x_0,\epsilon)}. \tag{3.54}$$

Observe that this function also depends on the radius ρ above, which will be fixed later in Section 3.3. Such constant will also be referred to as ρ_C in order to avoid confusion with test functions of the other subsections.

For later use we observe that $\frac{2(n-1)}{n-2} \frac{\partial}{\partial \eta_{g_0}} \bar{U}_{(x_0,\epsilon)} = B_{g_0} \bar{u}_{C;(x_0,\epsilon)} = B_{g_{x_0}} \bar{U}_{(x_0,\epsilon)} = 0$ on ∂M.

Our main result in this subsection is the following:

Proposition 3.2.29. *Under the hypothesis of Theorem 1.2.4, there exists $P_3 = P_3(M, g_0, \delta_0)$ such that*

$$\begin{aligned}
&\frac{\int_M \left\{ \frac{4(n-1)}{n-2} |d\bar{u}_{C;(x_0,\epsilon)}|^2_{g_0} + R_{g_0} \bar{u}^2_{C;(x_0,\epsilon)} \right\} dv_{g_0}}{\left(\int_M \bar{u}_{C;(x_0,\epsilon)}^{\frac{2n}{n-2}} dv_{g_0} \right)^{\frac{n-2}{n}}} \\
&\quad = \frac{\int_M \left\{ \frac{4(n-1)}{n-2} |d\bar{U}_{(x_0,\epsilon)}|^2_{g_{x_0}} + R_{g_{x_0}} \bar{U}^2_{(x_0,\epsilon)} \right\} dv_{g_{x_0}} + \int_{\partial M} 2H_{g_{x_0}} \bar{U}^2_{(x_0,\epsilon)} d\sigma_{g_{x_0}}}{\left(\int_M \bar{U}_{(x_0,\epsilon)}^{\frac{2n}{n-2}} dv_{g_{x_0}} \right)^{\frac{n-2}{n}}} \\
&\quad \leq Y(S^n)
\end{aligned}$$

for all $x_0 \in M \backslash M_{\delta_0}$ and $0 < 2\epsilon < \rho_C < P_3$.

Proof. Choose P_3 small such that for any $x_0 \in M \backslash M_{\delta_0}$ we have $d_{g_{x_0}}(x_0, \partial M) > 2P_3$. Choosing P_3 smaller if necessary (also depending on δ_0 because of the above estimates for G_{x_0}) the result is Corollary 3 and Proposition 19 in [14] with some obvious modifications, by making use of Theorem **??**. □

For later use we state the following result, which is proved as Proposition 3.2.26:

Proposition 3.2.30. *We can choose $P_3 = P_3(M, g_0, \delta_0)$ maybe smaller such that there* *is* $C = C(M, g_0)$ *such that*

$$\left|\frac{4(n-1)}{n-2}\Delta_{g_{x_0}}\bar{U}_{(x_0,\epsilon)} - R_{g_{x_0}}\bar{U}_{(x_0,\epsilon)} + \overline{R}_\infty \bar{U}_{(x_0,\epsilon)}^{\frac{n+2}{n-2}}\right|$$
$$\leq C\rho^2\left(\frac{\epsilon}{\epsilon^2+|x|^2}\right)^{\frac{n-2}{2}} 1_{B_{2\rho}(0)} + C\left(\frac{\epsilon}{\epsilon^2+|x|^2}\right)^{\frac{n+2}{2}} 1_{M\backslash B_\rho(0)}$$
$$+ C(\epsilon^{\frac{n+2}{2}}\rho^{-2-n} + \epsilon^{\frac{n-2}{2}}\rho^{3/4-n}(\log\rho))1_{B_{2\rho}(0)\backslash B_\rho(0)}$$

for all $x_0 \in M\backslash M_{\delta_0}$ *and* $\epsilon < \rho \leq P_3$.

Proof. As in Proposition 3.2.26, the proof follows the lines of Proposition 3.2.14, but the term I_2 is estimated by $|I_2| \leq C\epsilon^{\frac{n-2}{2}}\rho^{1-n}(\log\rho)$, where C depends on δ_0. Choose $P_3 < C^{-4}$. □

3.2.5 Further estimates

The results of this subsection are consequences of what was proved in Subsections 3.2.2, 3.2.3 and 3.2.4.

In this subsection, unless otherwise stated, if $x_0 \in \partial M$, $x_0 \in M_{\delta_0}\backslash\partial M$ or $x_0 \in M\backslash M_{2\delta_0}$, $\bar{u}_{(x_0,\epsilon)}$ will stand for $\bar{u}_{A;(x_0,\epsilon)}$, $\bar{u}_{B;(x_0,\epsilon)}$ or $\bar{u}_{C;(x_0,\epsilon)}$, respectively. If $x_0 \in M_{2\delta_0}\backslash M_{\delta_0}$, $\bar{u}_{(x_0,\epsilon)}$ will stand for $\bar{u}_{B;(x_0,\epsilon)}$ and $\bar{u}_{C;(x_0,\epsilon)}$, the results below holding for either. By the "radius" ρ of $\bar{u}_{(x_0,\epsilon)}$, we mean ρ_A, ρ_B or ρ_C, if $\bar{u}_{(x_0,\epsilon)} = \bar{u}_{A;(x_0,\epsilon)}$, $\bar{u}_{(x_0,\epsilon)} = \bar{u}_{B;(x_0,\epsilon)}$ or $\bar{u}_{(x_0,\epsilon)} = \bar{u}_{C;(x_0,\epsilon)}$, respectively.

We observe that whenever $\bar{u}_{(x_0,\epsilon)} = \bar{u}_{B;(x_0,\epsilon)}$ we have $d_{g_0}(x_0, \partial M) \leq \delta_0 \leq C\rho^2$, according to Proposition 3.2.17, because $x_0 \in M_{\delta_0}\backslash\partial M$ in this case. Hence, we can make use of Propositions 3.2.26, 3.2.27 and 3.2.28.

Corollary 3.2.31. *There exists* $C = C(M, g_0)$ *such that, for* $\epsilon < \rho$,

$$\left|\frac{4(n-1)}{n-2}\Delta_{g_0}\bar{u}_{(x_0,\epsilon)} - R_{g_0}\bar{u}_{(x_0,\epsilon)} + \overline{R}_\infty \bar{u}_{(x_0,\epsilon)}^{\frac{n+2}{n-2}}\right|$$
$$\leq C\rho^{-1/2}\left(\frac{\epsilon}{\epsilon^2+d_{g_0}(x,x_0)^2}\right)^{\frac{n-2}{2}}(\epsilon^2+d_{g_0}(x,x_0)^2)^{-\frac{1}{2}}1_{B_{4\rho}(x_0)}$$
$$+ C\left(\frac{\epsilon}{\epsilon^2+d_{g_0}(x,x_0)^2}\right)^{\frac{n+2}{2}} 1_{M\backslash B_{\rho/2}(x_0)}.$$

Proof. It is a consequence of Propositions 3.2.14, 3.2.26 and 3.2.30. □

Corollary 3.2.32. [3] *There exists $C = C(M, g_0)$ such that, if ρ is the radius of $\bar{u}_{(x_2,\epsilon_2)}$ and $\epsilon_2 < \rho$, we have*

$$\int_M \bar{u}_{(x_1,\epsilon_1)} \left| \frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_2,\epsilon_2)} - R_{g_0} \bar{u}_{(x_2,\epsilon_2)} + \overline{R}_\infty \bar{u}_{(x_2,\epsilon_2)}^{\frac{n+2}{n-2}} \right| dv_{g_0}$$
$$\leq C \left(\rho^{1/2} + \frac{\epsilon_2^2}{\rho^2} \right) \left(\frac{\epsilon_1 \epsilon_2}{\epsilon_2^2 + d_{g_0}(x_1,x_2)^2} \right)^{\frac{n-2}{2}}.$$

Proof. As in [13, Lemma B.5] we get

$$\int_{\{d_{g_0}(y,x_2) \geq \rho/2\}} \left(\frac{\epsilon_1}{\epsilon_1^2 + d_{g_0}(x_1,y)^2} \right)^{\frac{n-2}{2}} \left(\frac{\epsilon_2}{\epsilon_2^2 + d_{g_0}(x_2,y)^2} \right)^{\frac{n+2}{2}} dv_{g_0} \leq C \frac{\epsilon_2^2}{\rho^2} \left(\frac{\epsilon_1 \epsilon_2}{\epsilon_2^2 + d_{g_0}(x_1,x_2)^2} \right)^{\frac{n-2}{2}}. \tag{3.55}$$

We claim that

$$\int_{\{d_{g_0}(y,x_2) \leq 4\rho\}} \left(\frac{\epsilon_1}{\epsilon_1^2 + d_{g_0}(x_1,y)^2} \right)^{\frac{n-2}{2}} \left(\frac{\epsilon_2}{\epsilon_2^2 + d_{g_0}(x_2,y)^2} \right)^{\frac{n-2}{2}} (\epsilon_2^2 + d_{g_0}(x_2,y)^2)^{-\frac{1}{2}} dv_{g_0} \tag{3.56}$$
$$\leq C\rho \left(\frac{\epsilon_1 \epsilon_2}{\epsilon_2^2 + d_{g_0}(x_1,x_2)^2} \right)^{\frac{n-2}{2}}.$$

Set

$$A = \{2d_{g_0}(x_1,y) \leq \epsilon_2 + d_{12}\} \cap \{d_{g_0}(y,x_2) \leq 4\rho\}$$

and

$$B = \{2d_{g_0}(x_1,y) \geq \epsilon_2 + d_{12}\} \cap \{d_{g_0}(y,x_2) \leq 4\rho\}$$

where $d_{12} = d_{g_0}(x_1,x_2)$. Observe that on A we have

$$\epsilon_2 + d_{g_0}(y,x_2) \geq \epsilon_2 + d_{12} - d_{g_0}(y,x_1) \geq \frac{1}{2}(\epsilon_2 + d_{12}) \geq d_{g_0}(y,x_1)$$
$$\text{and} \quad d_{g_0}(y,x_1) \leq \frac{1}{2}(\epsilon_2 + d_{12}) \leq \epsilon_2 + d_{g_0}(y,x_2) \leq 5\rho.$$

Then

$$\int_A \left(\frac{\epsilon_1}{\epsilon_1^2 + d_{g_0}(x_1,y)^2} \right)^{\frac{n-2}{2}} \left(\frac{\epsilon_2}{\epsilon_2^2 + d_{g_0}(x_2,y)^2} \right)^{\frac{n-2}{2}} (\epsilon_2^2 + d_{g_0}(x_2,y)^2)^{-\frac{1}{2}} dv_{g_0} \tag{3.57}$$
$$\leq C \left(\frac{\epsilon_1 \epsilon_2}{\epsilon_2^2 + d_{12}^2} \right)^{\frac{n-2}{2}} \int_{\{d_{g_0}(y,x_1) \leq 5\rho\}} (\epsilon_1^2 + d_{g_0}(x_1,y)^2)^{\frac{2-n}{2}} d_{g_0}(x_1,y)^{-1} dv_{g_0}$$
$$\leq C \left(\frac{\epsilon_1 \epsilon_2}{\epsilon_2^2 + d_{12}^2} \right)^{\frac{n-2}{2}} \int_{\{d_{g_0}(y,x_1) \leq 5\rho\}} d_{g_0}(x_1,y)^{1-n} dv_{g_0}$$

---

[3] *For types A and B test functions in dimensions $n \geq 5$, the coefficient $\rho^{1/2}$ in this inequality can be improved to ρ. Indeed, ρ was worsen to $\rho^{1/2}$ due to the $\log \rho$ terms in Propositions 3.2.14 and 3.2.26, which are necessary only for $n = 3$ or 4, as observed in the footnote in Proposition 3.2.14.*

On the other hand,

$$\int_B \left(\frac{\epsilon_1}{\epsilon_1^2 + d_{g_0}(x_1,y)^2}\right)^{\frac{n-2}{2}} \left(\frac{\epsilon_2}{\epsilon_2^2 + d_{g_0}(x_2,y)^2}\right)^{\frac{n-2}{2}} (\epsilon_2^2 + d_{g_0}(x_2,y)^2)^{-\frac{1}{2}} dv_{g_0} \qquad (3.58)$$
$$\leq C\left(\frac{\epsilon_1\epsilon_2}{\epsilon_2^2 + d_{12}^2}\right)^{\frac{n-2}{2}} \int_{\{d_{g_0}(y,x_2)\leq 4\rho\}} d_{g_0}(x_2,y)^{1-n} dv_{g_0}.$$

The estimate (3.56) follows from (3.57) and (3.57) observing that the integrals on the right sides of those inequalities are bounded by $C\rho$.

The result now follows from (3.55), (3.56) and Corollary 3.2.31. □

Corollary 3.2.33. [4] *There exists $C = C(M, g_0)$ such that, if ρ is the radius of $\bar{u}_{(x_2,\epsilon_2)}$ and $\epsilon_2 < \rho$,*

$$\int_{\partial M} \bar{u}_{(x_1,\epsilon_1)} \frac{\partial}{\partial \eta_{g_0}} \bar{u}_{(x_2,\epsilon_2)} d\sigma_{g_0} \geq -C\left(\rho^{1/2} + \frac{\epsilon_2}{\rho}\right)\left(\frac{\epsilon_1\epsilon_2}{\epsilon_2^2 + d_{g_0}(x_1,x_2)^2}\right)^{\frac{n-2}{2}}.$$

Proof. Observe that the above integral vanishes when $\bar{u}_{(x_2,\epsilon_2)}$ is a type C test function. For types A and B test functions we estimate

$$\frac{\partial}{\partial \eta_{g_0}} \bar{u}_{(x_2,\epsilon_2)} \geq -C\rho^{-1/2}\left(\frac{\epsilon_2}{\epsilon_2^2 + d_{g_0}(x_2,y)^2}\right)^{\frac{n-2}{2}} 1_{\{d_{g_0}(y,x_2)\leq 4\rho\}\cap \partial M}$$
$$- C\left(\frac{\epsilon_2}{\epsilon_2^2 + d_{g_0}(x_2,y)^2}\right)^{\frac{n}{2}} 1_{\{d_{g_0}(y,x_2)\geq \rho/2\}\cap \partial M},$$

according to Propositions 3.2.15 and 3.2.28 and equation (3.3). As in [13, p.274-275] we can prove

$$\int_{\{d_{g_0}(y,x_2)\leq 4\rho\}\cap \partial M} \left(\frac{\epsilon_1}{\epsilon_1^2 + d_{g_0}(x_1,y)^2}\right)^{\frac{n-2}{2}} \left(\frac{\epsilon_2}{\epsilon_2^2 + d_{g_0}(x_2,y)^2}\right)^{\frac{n-2}{2}} d\sigma_{g_0} \leq C\rho\left(\frac{\epsilon_1\epsilon_2}{\epsilon_2^2 + d_{g_0}(x_1,x_2)^2}\right)^{\frac{n-2}{2}}$$

and

$$\int_{\{d_{g_0}(y,x_2)\geq \rho/2\}\cap \partial M} \left(\frac{\epsilon_1}{\epsilon_1^2 + d_{g_0}(x_1,y)^2}\right)^{\frac{n-2}{2}} \left(\frac{\epsilon_2}{\epsilon_2^2 + d_{g_0}(x_2,y)^2}\right)^{\frac{n}{2}} d\sigma_{g_0} \leq C\frac{\epsilon_2}{\rho}\left(\frac{\epsilon_1\epsilon_2}{\epsilon_2^2 + d_{g_0}(x_1,x_2)^2}\right)^{\frac{n-2}{2}}.$$

The result now follows. □

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[4] *Similarly to the footnote in Corollary 3.2.32, for types A and B test functions the coefficient $\rho^{1/2}$ can be improved to ρ if $n \geq 5$.*

Corollary 3.2.34. *For $\epsilon < \rho$ we have*

$$\left(\int_M \left|\frac{4(n-1)}{n-2}\Delta_{g_0}\bar{u}_{(x_0,\epsilon)} - R_{g_0}\bar{u}_{(x_0,\epsilon)} + \overline{R}_\infty \bar{u}_{(x_0,\epsilon)}^{\frac{n+2}{n-2}}\right|^{\frac{2n}{n+2}} dv_{g_0}\right)^{\frac{n+2}{2n}}$$

$$\leq C\left(\frac{\epsilon}{\rho}\right)^{\frac{n+2}{2}} + C\begin{cases} \epsilon\rho^{-1/2} & n \geq 5, \\ \epsilon\rho^{-1/2}\log(\rho/\epsilon) & n = 4, \\ \epsilon^{1/2} & n = 3. \end{cases}$$

Proof. The result follows easily from Corollary 3.2.31. □

Corollary 3.2.35. *If $\bar{u}_{(x_0,\epsilon)} = \bar{u}_{B;(x_0,\epsilon)}$ we have*

$$\left(\int_{\partial M}\left|\frac{2(n-1)}{n-2}\frac{\partial}{\partial\eta_{g_0}}\bar{u}_{(x_0,\epsilon)} - H_{g_0}\bar{u}_{(x_0,\epsilon)}\right|^{\frac{2(n-1)}{n}} d\sigma_{g_0}\right)^{\frac{n}{2(n-1)}}$$

$$\leq \begin{cases} C\left(\frac{\epsilon}{\delta}\right)^{\frac{n-2}{2}}\log\rho + \frac{\epsilon}{\rho} & n \geq 5, \\ C\left(\frac{\epsilon}{\delta}\right)\log\rho + \frac{\epsilon}{\rho}\log\frac{\rho}{\epsilon} & n = 4, \\ C\left(\frac{\epsilon}{\delta}\right)^{1/2}\log\rho + C\left(\frac{\epsilon}{\rho}\right)^{1/2} & n = 3, \end{cases}$$

for $\epsilon < \rho$, where $\delta = d_{g_0}(x_0, \partial M)$.

Proof. From Proposition 3.2.27, on ∂M we have

$$\left|\frac{2(n-1)}{n-2}\frac{\partial}{\partial\eta_{g_0}}\bar{u}_{(x_0,\epsilon)} - H_{g_0}\bar{u}_{(x_0,\epsilon)}\right| \leq C\frac{\delta}{\epsilon}\left(\frac{\epsilon}{\epsilon^2 + d_{g_0}(x,x_0)^2}\right)^{\frac{n}{2}} 1_{\{d_{g_0}(x,x_0)\leq 4\rho\}}$$
$$+ C\rho^{-1}\left(\frac{\epsilon}{\epsilon^2 + d_{g_0}(x,x_0)^2}\right)^{\frac{n-2}{2}} 1_{\{d_{g_0}(x,x_0)\leq 4\rho\}}.$$

Using $\delta \leq C\rho^2$, which in particular implies $\delta \leq C\rho$, the first term on the right side above is estimated by $C(\delta/\epsilon)^{(n-2)/2}(\epsilon + d_{g_0}(x,x_0))^{-n/2}1_{\{d_{g_0}(x,x_0)\leq 4\rho\}}$, and the result follows easily. □

3.3 Blow-up analysis

In this section, we carry out the blow-up analysis for sequences of solutions to the equations (3.4) that will be necessary for the proof of Theorem 1.2.4. Although the analysis goes along the lines of [13, Sections 4, 5 and 6], here we have to consider the possibility of both interior and boundary blow-up points, thus differing from the situation in [2, Section 4]. As we will see in Proposition 3.3.2 below, type A test functions are used to approximate solutions near boundary blow-up points. As for interior blow-up points, we make use of type B test functions if those points accumulate on the boundary, and type C ones otherwise.

Remark 3.3.1. Before proceeding to the blow-up analysis, we observe that one can choose ρ_A, ρ_B and ρ_C in Propositions 3.2.8, 3.2.17 and 3.2.29 in such a way that the inequalities of those propositions hold the three at the same time. To that end, choose δ_0 according to a small ρ_B in Proposition 3.2.17 and then ρ_C according to δ_0 in Proposition 3.2.29. Moreover, observe that given $C = C(M, g_0)$ one can always assume $\rho_A, \rho_B, \rho_C \leq C$. This last remark will be used in the proofs of Propositions 3.3.10 and 3.3.22 below.

Let $u(t)$, $t \geq 0$, be the solution of (3.4) obtained in Section 3.1, and let $\{t_\nu\}_{\nu=1}^{\infty}$ be any sequence satisfying $\lim_{\nu\to\infty} t_\nu = \infty$. We set $u_\nu = u(t_\nu)$ and $g_\nu = g(t_\nu) = u_\nu^{\frac{4}{n-2}} g_0$. Then

$$\int_M u_\nu^{\frac{2n}{n-2}} dv_{g_0} = \int_M dv_{g_\nu} = 1\,, \quad \text{for all } \nu\,.$$

It follows from Corollary 3.1.3 that

$$\int_M \left| \frac{4(n-1)}{n-2} \Delta_{g_0} u_\nu - R_{g_0} u_\nu + \overline{R}_\infty u_\nu^{\frac{n+2}{n-2}} \right|^{\frac{2n}{n+2}} dv_{g_0} = \int_M |R_{g_\nu} - \overline{R}_\infty|^{\frac{2n}{n+2}} dv_{g_\nu} \to 0$$

as $\nu \to \infty$.

The next proposition is an application of the decomposition result in [70], which plays the same role here as [79] did in [13, Proposition 4.1].

Proposition 3.3.2. *After passing to a subsequence, there exist an integer $m \geq 0$, a smooth function $u_\infty \geq 0$, and a sequence of m-tuplets $\{(x_{k,\nu}^*, \epsilon_{k,\nu}^*)_{1\leq k\leq m}\}_{\nu=1}^{\infty}$, such that:*

(i) The function u_∞ satisfies

$$\begin{cases} \frac{4(n-1)}{n-2} \Delta_{g_0} u_\infty - R_{g_0} u_\infty + \overline{R}_\infty u_\infty^{\frac{n+2}{n-2}} = 0\,, & \text{in } M\,, \\ \partial u_\infty / \partial \eta_{g_0} = 0\,, & \text{on } \partial M\,. \end{cases}$$

(ii) For all $i \neq j$,

$$\lim_{\nu\to\infty}\left\{\frac{\epsilon^*_{i,\nu}}{\epsilon^*_{j,\nu}}+\frac{\epsilon^*_{j,\nu}}{\epsilon^*_{i,\nu}}+\frac{d_{g_0}(x^*_{i,\nu},x^*_{j,\nu})^2}{\epsilon^*_{i,\nu}\epsilon^*_{j,\nu}}\right\}=\infty\,.$$

(iii) There are integers m_1, m_2, *with* $0 \le m_1 \le m_2 \le m$, *such that* $x^*_{k,\nu} \in \partial M$ *for* $1 \le k \le m_1$, $x^*_{k,\nu} \in M_{3\delta_0/2}\backslash\partial M$ *for* $m_1+1 \le k \le m_2$, $x^*_{k,\nu} \in M\backslash M_{3\delta_0/2}$ *for* $m_2+1 \le k \le m$, *and*

$$\lim_{\nu\to\infty} d_{g_0}(x^*_{k,\nu},\partial M)/\epsilon^*_{k,\nu}=\infty \qquad \text{if } k \ge m_1+1\,.$$

(iv) If

$$\bar u_{(x^*_{k,\nu},\epsilon^*_{k,\nu})}=\begin{cases}\bar u_{A;(x^*_{k,\nu},\epsilon^*_{k,\nu})} & \text{if}\, k\le m_1,\\[2mm] \bar u_{B;(x^*_{k,\nu},\epsilon^*_{k,\nu})} & \text{if}\, m_1+1\le k\le m_2,\\[2mm] \bar u_{C;(x^*_{k,\nu},\epsilon^*_{k,\nu})} & \text{if}\, k\ge m_2+1,\end{cases} \tag{3.59}$$

(see equations (3.33), (3.44) and (3.54)) then

$$\lim_{\nu\to\infty}\Big\|u_\nu-u_\infty-\sum_{k=1}^{m}\bar u_{(x^*_{k,\nu},\epsilon^*_{k,\nu})}\Big\|_{H^1(M)}=0\,.$$

Proof. By modifying the arguments in [70, Section 3] to the case of Riemannian manifolds, we can prove the existence of u_∞ and $\bar u_{(x^*_{k,\nu},\epsilon^*_{k,\nu})}$ satisfying (i) and (iv) except for, instead of using equations (3.59), the $\bar u_{(x^*_{k,\nu},\epsilon^*_{k,\nu})}$ are defined by

$$\bar u_{(x^*_{k,\nu},\epsilon^*_{k,\nu})}(x)=\left(\frac{4n(n-1)}{\overline{R}_\infty}\right)^{\frac{n-2}{4}}(\epsilon^*_{k,\nu})^{-\frac{n-2}{2}}\chi_\rho\big(\psi^{-1}_{x^*_{k,\nu}}(x)\big)\,u\big((\epsilon^*_{k,\nu})^{-1}\psi^{-1}_{x^*_{k,\nu}}(x)\big).$$

Here, $\psi_{x^*_{k,\nu}}$ are coordinates centered at $x^*_{k,\nu}$ and u satisfies

$$\Delta u+n(n-2)u^{\frac{n+2}{n-2}}=0 \quad \text{in } \mathbb{R}^n \tag{3.60}$$

if $\lim_{\nu\to\infty} d_{g_0}(x^*_{k,\nu},\partial M)/\epsilon^*_{k,\nu}=\infty$, and

$$\begin{cases}\Delta u+n(n-2)u^{\frac{n+2}{n-2}}=0 & \text{in } \{y=(y_1,...,y_n)\,|\,y_n\ge t\},\\[2mm] \frac{\partial}{\partial y_n}u=0 & \text{on } \{y=(y_1,...,y_{n-1},t)\},\end{cases} \tag{3.61}$$

for some $t\in\mathbb{R}$ if $d_{g_0}(x^*_{k,\nu},\partial M)/\epsilon^*_{k,\nu}$ is bounded.

Rearrange the indices and choose m_1 such that $k\ge m_1+1$ should (3.60) holds and $k\le m_1$ should (3.61) holds.

As in [68, Lemma 3.3], we can prove that $u \geq 0$ and also that (ii) holds. The classification results in [16, 62] (regularity was established in [29]) imply that $u(y) = W_\epsilon(y-z)$ (see (3.10)), for some $z = (z_1, ..., z_n) \in \mathbb{R}^n$ (with $z_n = t$ if $k \leq m_1$).

The points $x^*_{k,\nu}$ are now redefined as $\psi_{x^*_{k,\nu}}(z)$.[5] This establishes (iii).

For each pair $(x^*_{k,\nu}, \epsilon^*_{k,\nu})$, one can check that the difference between each function obtained above and the corresponding one defined by (3.59) converges to zero in $H^1(M)$. This proves (iv). □

Proposition 3.3.3. *If $u_\infty(x) = 0$ for some $x \in M$, then $u_\infty \equiv 0$.*

Proof. This is just a consequence of the maximum principle. □

Define the functionals

$$E(u) = \frac{\frac{4(n-1)}{n-2}\int_M |du|^2_{g_0} dv_{g_0} + \int_M R_{g_0} u^2 dv_{g_0}}{\left(\int_M u^{\frac{2n}{n-2}} dv_{g_0}\right)^{\frac{n-2}{n}}}$$

and

$$F(u) = \frac{\frac{4(n-1)}{n-2}\int_M |du|^2_{g_0} dv_{g_0} + \int_M R_{g_0} u^2 dv_{g_0}}{\int_M u^{\frac{2n}{n-2}} dv_{g_0}}\,.$$

Observe that $\overline{R}_\infty = F(u_\infty)$. Hence,

$$1 = \lim_{\nu\to\infty}\int_M u_\nu^{\frac{2n}{n-2}} dv_{g_0} = \lim_{\nu\to\infty}\left\{\int_M u_\infty^{\frac{2n}{n-2}} dv_{g_0} + \sum_{k=1}^m \int_M \bar{u}_{(x^*_{k,\nu},\epsilon^*_{k,\nu})}^{\frac{2n}{n-2}} dv_{g_0}\right\}.$$

The right side of this equation is $(E(u_\infty)/\overline{R}_\infty)^{n/2} + m_1(Q(S^n_+)/\overline{R}_\infty)^{n/2} + (m-m_1)(Y(S^n)/\overline{R}_\infty)^{n/2}$ if $u_\infty > 0$ and $m_1(Q(S^n_+)/\overline{R}_\infty)^{n/2} + (m-m_1)(Y(S^n)/\overline{R}_\infty)^{n/2}$ if $u_\infty \equiv 0$. Thus,

$$\overline{R}_\infty = \left(E(u_\infty)^{n/2} + m_1 Q(S^n_+)^{n/2} + (m-m_1)Y(S^n)^{n/2}\right)^{2/n} \quad \text{if } u_\infty > 0, \tag{3.62}$$
$$\text{and} \quad \overline{R}_\infty = \left(m_1 Q(S^n_+)^{n/2} + (m-m_1)Y(S^n)^{n/2}\right)^{2/n} \quad \text{if } u_\infty \equiv 0.$$

[5]To see that changing the centers $x^*_{j,\nu}$ as above does not change the limit in (ii), we consider, for fixed j, new centers $\bar{x}^*_{j,\nu}$ satisfying $d_{g_0}(x^*_{j,\nu}, \bar{x}^*_{j,\nu})/\epsilon^*_{j,\nu} \leq C$ (the term $\epsilon^*_{j,\nu}$ in the quotient comes from the rescaling). If the limit in (ii) holds with $\epsilon^*_{j,\nu}/\epsilon^*_{i,\nu} \to \infty$, that relation does not change after replacing the centers. So, let us assume $\epsilon^*_{j,\nu}/\epsilon^*_{i,\nu} \leq C$ without loss of generality. The triangle inequality gives

$$d_{g_0}(x^*_{i,\nu}, \bar{x}^*_{j,\nu})^2 \geq \left(d_{g_0}(x^*_{i,\nu}, x^*_{j,\nu}) - d_{g_0}(x^*_{j,\nu}, \bar{x}^*_{j,\nu})\right)^2 \geq \frac{1}{2} d_{g_0}(x^*_{i,\nu}, x^*_{j,\nu})^2 - C d_{g_0}(x^*_{j,\nu}, \bar{x}^*_{j,\nu})^2.$$

Hence,

$$\frac{d_{g_0}(x^*_{i,\nu}, \bar{x}^*_{j,\nu})^2}{\epsilon^*_{i,\nu}\epsilon^*_{j,\nu}} \geq \frac{1}{2}\frac{d_{g_0}(x^*_{i,\nu}, x^*_{j,\nu})^2}{\epsilon^*_{i,\nu}\epsilon^*_{j,\nu}} - C\frac{\epsilon^*_{j,\nu}}{\epsilon^*_{i,\nu}}\left(\frac{d_{g_0}(x^*_{j,\nu}, \bar{x}^*_{j,\nu})}{\epsilon^*_{j,\nu}}\right)^2 \geq \frac{1}{2}\frac{d_{g_0}(x^*_{i,\nu}, x^*_{j,\nu})^2}{\epsilon^*_{i,\nu}\epsilon^*_{j,\nu}} - C\,,$$

so that (ii) still holds with $\bar{x}^*_{j,\nu}$ replacing $x^*_{j,\nu}$.

3.3.1 The case $u_\infty \equiv 0$

We set

$$\mathcal{A}_\nu = \Big\{ (x_k, \epsilon_k, \alpha_k)_{k=1,\dots,m} \in (M \times \mathbb{R}_+ \times \mathbb{R}_+)^m \,, \text{ such that} \\ x_k \in \partial M \text{ if } k \le m_1 \,,\; x_k \in M\backslash\partial M \text{ if } k \ge m_1+1, \\ d_{g_0}(x_k, x^*_{k,\nu}) \le \epsilon^*_{k,\nu} \,,\; \frac{1}{2} \le \frac{\epsilon_k}{\epsilon^*_{k,\nu}} \le 2 \,,\; \frac{1}{2} \le \alpha_k \le 2 \Big\}. \tag{3.63}$$

For each ν, we can choose a triplet $(x_{k,\nu}, \epsilon_{k,\nu}, \alpha_{k,\nu})_{k=1,\dots,m} \in \mathcal{A}_\nu$ such that

$$\int_M \frac{4(n-1)}{n-2} \Big| d\big(u_\nu - \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}\big)\Big|^2_{g_0} dv_{g_0} + \int_M R_{g_0} \big(u_\nu - \sum_{k=1}^m \alpha_{k,\nu}\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}\big)^2 dv_{g_0} \\ \le \int_M \frac{4(n-1)}{n-2} \Big| d\big(u_\nu - \sum_{k=1}^m \alpha_k \bar{u}_{(x_k,\epsilon_k)}\big)\Big|^2_{g_0} dv_{g_0} + \int_M R_{g_0}\big(u_\nu - \sum_{k=1}^m \alpha_k \bar{u}_{(x_k,\epsilon_k)}\big)^2 dv_{g_0}$$

for all $(x_k, \epsilon_k, \alpha_k)_{k=1,\dots,m} \in \mathcal{A}_\nu$. Here, $\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})} = \bar{u}_{A;(x_{k,\nu},\epsilon_{k,\nu})}$ and $\bar{u}_{(x_k,\epsilon_k)} = \bar{u}_{A;(x_k,\epsilon_k)}$ if $k \le m_1$, $\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})} = \bar{u}_{B;(x_{k,\nu},\epsilon_{k,\nu})}$ and $\bar{u}_{(x_k,\epsilon_k)} = \bar{u}_{B;(x_k,\epsilon_k)}$ if $m_1 + 1 \le k \le m_2$, and $\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})} = \bar{u}_{C;(x_{k,\nu},\epsilon_{k,\nu})}$ and $\bar{u}_{(x_k,\epsilon_k)} = \bar{u}_{C;(x_k,\epsilon_k)}$ if $k \ge m_2 + 1$; see (3.33), (3.44) and (3.54).

Proposition 3.3.4. *If $k \ge m_1 + 1$, then $\lim_{\nu\to\infty} d_{g_0}(x_{k,\nu}, \partial M)/\epsilon_{k,\nu} = \infty$.*

Proof. It follows from the triangle inequality and (3.63) that

$$\frac{d_{g_0}(x_{k,\nu},\partial M)}{\epsilon_{k,\nu}} \ge \frac{d_{g_0}(x_{k,\nu},\partial M)}{2\epsilon^*_{k,\nu}} \ge \frac{d_{g_0}(x^*_{k,\nu},\partial M)}{2\epsilon^*_{k,\nu}} - \frac{1}{2}.$$

Now the right side goes to infinity as $\nu \to \infty$ by (iii) of Proposition 3.3.2. □

Proposition 3.3.5. *We have:*

(i) For all $i \ne j$,

$$\lim_{\nu\to\infty} \left\{ \frac{\epsilon_{i,\nu}}{\epsilon_{j,\nu}} + \frac{\epsilon_{j,\nu}}{\epsilon_{i,\nu}} + \frac{d_{g_0}(x_{i,\nu},x_{j,\nu})^2}{\epsilon_{i,\nu}\epsilon_{j,\nu}} \right\} = \infty\,.$$

(ii) We have

$$\lim_{\nu\to\infty} \Big\| u_\nu - \sum_{k=1}^m \alpha_{k,\nu}\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}\Big\|_{H^1(M)} = 0\,.$$

Proof. This is a simple consequence of Proposition 3.3.2 and the definition of $(x_{k,\nu}, \epsilon_{k,\nu}, \alpha_{k,\nu})$; see [13, Propostion 5.1] for details. □

Proposition 3.3.6. *We have*

$$d_{g_0}(x_{k,\nu},x^*_{k,\nu})\leq o(1)\epsilon^*_{k,\nu}\,,\quad \frac{\epsilon_{k,\nu}}{\epsilon^*_{k,\nu}}=1+o(1)\,,\quad \textit{and}\quad \alpha_{k,\nu}=1+o(1)\,,$$

for all $k=1,...,m$. In particular, $(x_{k,\nu},\epsilon_{k,\nu},\alpha_{k,\nu})_{k=1,...,m}$ is an interior point of $\mathcal{A}_\nu$ for ν sufficiently large.

Proof. It follows from Propositions 3.3.2 and 3.3.5 that

$$\begin{aligned}&\Big\|\sum_{k=1}^m\alpha_{k,\nu}\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}-\sum_{k=1}^m\bar{u}_{(x^*_{k,\nu},\epsilon^*_{k,\nu})}\Big\|_{H^1(M)}\\&\qquad\leq\Big\|u_\nu-\sum_{k=1}^m\bar{u}_{(x^*_{k,\nu},\epsilon^*_{k,\nu})}\Big\|_{H^1(M)}+\Big\|u_\nu-\sum_{k=1}^m\alpha_{k,\nu}\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}\Big\|_{H^1(M)}=o(1).\end{aligned}$$

Now the result follows. □

Notation. We write $u_\nu=v_\nu+w_\nu$, where

$$v_\nu=\sum_{k=1}^m\alpha_{k,\nu}\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}\quad\text{and}\quad w_\nu=u_\nu-\sum_{k=1}^m\alpha_{k,\nu}\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}\,. \tag{3.64}$$

Observe that by Proposition 3.3.5 we have

$$\int_M\frac{4(n-1)}{n-2}|dw_\nu|^2_{g_0}dv_{g_0}+\int_M R_{g_0}w_\nu^2dv_{g_0}=o(1)\,. \tag{3.65}$$

Set

$$C_\nu=\Big(\int_{\partial M}|\psi|^{\frac{2(n-1)}{n-2}}d\sigma_{g_0}\Big)^{\frac{n-2}{2(n-1)}}+\Big(\int_M|\psi|^{\frac{2n}{n-2}}dv_{g_0}\Big)^{\frac{n-2}{2n}}\,.$$

Proposition 3.3.7. *Fix $\rho\leq P_0$. Let $\psi_{k,\nu}:\Omega_{k,\nu}=B^+_\rho(0)\subset\mathbb{R}^n_+\to M$ be Fermi coordinates centered at $x_{k,\nu}$ if $1\leq k\leq m_1$, and let $\psi_{k,\nu}:\Omega_{k,\nu}=\tilde{B}_{x_{k,\nu},\rho}\subset R^n\to M$ be normal coordinates centered at $x_{k,\nu}$ if $m_1+1\leq k\leq m$ (see Definitions 3.2.1 and 3.2.2). We have:*

(i) $\Big|\displaystyle\int_M\bar{u}^{\frac{n+2}{n-2}}_{(x_{k,\nu},\epsilon_{k,\nu})}\,\psi\,dv_{g_0}\Big|\leq o(1)\,C_\nu\,.$

(ii) $\Big|\displaystyle\int_{\Omega_{k,\nu}}\bar{u}^{\frac{n+2}{n-2}}_{(x_{k,\nu},\epsilon_{k,\nu})}\frac{\epsilon^2_{k,\nu}-|\psi^{-1}_{k,\nu}(x)|^2}{\epsilon^2_{k,\nu}+|\psi^{-1}_{k,\nu}(x)|^2}\,\psi\,dv_{g_0}\Big|\leq o(1)\,C_\nu\,.$

(iii) $\Big|\displaystyle\int_{\Omega_{k,\nu}}\bar{u}^{\frac{n+2}{n-2}}_{(x_{k,\nu},\epsilon_{k,\nu})}\frac{\epsilon_{k,\nu}\psi^{-1}_{k,\nu}(x)}{\epsilon^2_{k,\nu}+|\psi^{-1}_{k,\nu}(x)|^2}\,\psi\,dv_{g_0}\Big|\leq o(1)\,C_\nu,\quad \textit{if}\; m_1+1\leq k\leq m,$

and $\Big|\displaystyle\int_{\Omega_{k,\nu}}\bar{u}^{\frac{n+2}{n-2}}_{(x_{k,\nu},\epsilon_{k,\nu})}\frac{\epsilon_{k,\nu}\overline{\psi^{-1}_{k,\nu}(x)}}{\epsilon^2_{k,\nu}+|\psi^{-1}_{k,\nu}(x)|^2}\,\psi\,dv_{g_0}\Big|\leq o(1)\,C_\nu,\quad \textit{if}\; k\leq m_1,$

where we are denoting $\bar{y}=(y_1,...,y_{n-1})$ for any $y=(y_1,...,y_n)\in\mathbb{R}^n$.

Proof. It follows from the definition of $(x_{k,\nu}, \epsilon_{k,\nu}, \alpha_{k,\nu})$ that

$$\int_M \left(\frac{4(n-1)}{n-2} \langle d\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}, dw_\nu \rangle_{g_0} + R_{g_0} \bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})} w_\nu \right) dv_{g_0} = 0\,.$$

Integrating by parts, we obtain

$$\int_M \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})} - R_{g_0} \bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})} \right) w_\nu dv_{g_0} + \int_{\partial M} \frac{4(n-1)}{n-2} \frac{\partial}{\partial \eta_{g_0}} \bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})} w_\nu \, d\sigma_{g_0} = 0.$$

We claim that

$$\left\| \frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})} - R_{g_0} \bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})} + \overline{R}_\infty \bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}^{\frac{n+2}{n-2}} \right\|_{L^{\frac{2n}{n+2}}(M)} = o(1),$$

and

$$\left\| \frac{\partial}{\partial \eta_{g_0}} \bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})} \right\|_{L^{\frac{2(n-1)}{n}}(\partial M)} = o(1).$$

The first statement follows from Corollary 3.2.34. As for the second one, observe first that

$$\partial \bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})} / \partial \eta_{g_0} = 0$$

on ∂M if $\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})} = \bar{u}_{C;(x_{k,\nu},\epsilon_{k,\nu})}$. If $\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})} = \bar{u}_{A;(x_{k,\nu},\epsilon_{k,\nu})}$ this statement follows easily from Proposition 3.2.15 and (3.1), and if $\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})} = \bar{u}_{B;(x_{k,\nu},\epsilon_{k,\nu})}$ this is Corollary 3.2.35, also making use of Proposition 3.3.4.

This proves (i). The remaining statements follow similarly. □

Proposition 3.3.8. *There exists $c > 0$ such that*

$$\frac{n+2}{n-2} \overline{R}_\infty \int_M \sum_{k=1}^m \bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}^{\frac{4}{n-2}} \psi^2 \, dv_{g_0} \le (1-c) \left\{ \int_M \frac{4(n-1)}{n-2} |d\psi|_{g_0}^2 dv_{g_0} + \int_M R_{g_0} \psi^2 \, dv_{g_0} \right\}$$

for all ν sufficiently large.

Proof. Once we have proved Proposition 3.3.7, this proof is a contradiction argument similar to [13, Propostion 5.4] and [2, Proposition 4.6] and we will omit the details. Assume by contradiction that there is a sequence $\{\tilde{w}_\nu\}$ satisfying

$$\int_M \frac{4(n-1)}{n-2} |d\tilde{w}_\nu|_{g_0}^2 dv_{g_0} + \int_M R_{g_0} \tilde{w}_\nu^2 \, dv_{g_0} = 1$$

and

$$\lim_{\nu \to \infty} \frac{n+2}{n-2} \overline{R}_\infty \int_M \sum_{k=1}^m \bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}^{\frac{4}{n-2}} \tilde{w}_\nu^2 \, dv_{g_0} \ge 1\,.$$

After rescaling around $x_{k,\nu}$, the new sequence obtained converges (weakly in $H^1_{loc}(\mathbb{R}^n_+)$ if $k \le m_1$ and in $H^1_{loc}(\mathbb{R}^n)$ if $k \ge m_1 + 1$) to a certain $\hat{w}$. It turns out that one can choose $k \in \{1, ..., m\}$ in such way that $\hat{w}$ satisfies

$$\int_{\mathbb{R}^n_+} \left(\frac{1}{1+|y|^2} \right)^2 \hat{w}^2(y)\, dy > 0$$

and

$$\int_{\mathbb{R}^n_+} |d\hat{w}(y)|^2 dy \le n(n+2) \int_{\mathbb{R}^n_+} \left(\frac{1}{1+|y|^2} \right)^2 \hat{w}^2(y)\, dy$$

if $k \le m_1$, or the same two inequalities with $\mathbb{R}^n_+$ replaced by $\mathbb{R}^n$ if $k \ge m_1 + 1$.

On the other hand, if $k \le m_1$, due to Proposition 3.3.7, $\hat{w}$ satisfies

$$\int_{\mathbb{R}^n_+} \left(\frac{1}{1+|y|^2} \right)^{\frac{n+2}{2}} \hat{w}(y)\, dy = 0\,,$$

$$\int_{\mathbb{R}^n_+} \left(\frac{1}{1+|y|^2} \right)^{\frac{n+2}{2}} \frac{1-|y|^2}{1+|y|^2} \hat{w}(y)\, dy = 0\,,$$

$$\int_{\mathbb{R}^n_+} \left(\frac{1}{1+|y|^2} \right)^{\frac{n+2}{2}} \frac{y_j}{1+|y|^2} \hat{w}(y)\, dy = 0\,,$$

where $y = (y_1, ..., y_n)$, and $j = 1, ..., n-1$. By considering the corresponding equations on the round hemisphere we obtain a contradiction as in [2, Proposition 4.6]. If $k \ge m_1 + 1$, $\hat{w}$ satisfies the same last three equations (with $j = 1, ..., n$ for the last), but with $\mathbb{R}^n_+$ replaced by $\mathbb{R}^n$, and the same contradiction is reached by considering corresponding equations on the round sphere instead of the hemisphere. □

Corollary 3.3.9. *There exists $c > 0$ such that*

$$\frac{n+2}{n-2} \overline{R}_\infty \int_M v_\nu^{\frac{4}{n-2}} w_\nu^2\, dv_{g_0} \le (1-c) \left\{ \int_M \frac{4(n-1)}{n-2} |d\psi|^2_{g_0} dv_{g_0} + \int_M R_{g_0} \psi^2\, dv_{g_0} \right\}$$

for all ν sufficiently large.

Proof. By the definition of v_ν (equation (3.64)), we have

$$\lim_{\nu \to \infty} \int_M \left| v_\nu^{\frac{4}{n-2}} - \sum_{k=1}^m \bar{u}_{(x_{k,\nu}, \epsilon_{k,\nu})}^{\frac{4}{n-2}} \right|^{n/2} dv_{g_0} = 0\,.$$

Hence, the assertion follows from Proposition 3.3.8. □

Proposition 3.3.10. *For all ν sufficiently large, we have $E(v_\nu) \le \left(\sum_{k=1}^m E(\bar{u}_{(x_k, \epsilon_k)})^{n/2} \right)^{2/n}$.*

Proof. Choose a permutation $\sigma : \{1,...,m\}$ such that $\epsilon_{\sigma(i),\nu} \leq \epsilon_{\sigma(j),\nu}$ for all $i < j$. During this proof we will omit the symbol σ, writing $\epsilon_{i,\nu}$ instead of $\epsilon_{\sigma(i),\nu}$, so that $\epsilon_{i,\nu} \leq \epsilon_{j,\nu}$ for all $i < j$. After calculations similar to the ones in [13, Proposition 5.6] we obtain

$$\begin{aligned}
&E(v_\nu)\left(\int_M v_\nu^{\frac{2n}{n-2}} dv_{g_0}\right)^{\frac{n-2}{n}} \\
&\leq \left(\sum_{k=1}^m E(\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})})^{\frac{n}{2}}\right)^{\frac{2}{n}} \left(\int_M v_\nu^{\frac{2n}{n-2}} dv_{g_0}\right)^{\frac{n-2}{n}} - c\sum_{i<j}\left(\frac{\epsilon_{i,\nu}\epsilon_{j,\nu}}{\epsilon_{j,\nu}^2 + d_{g_0}(x_{i,\nu},x_{j,\nu})^2}\right)^{\frac{n-2}{2}} \\
&\quad - 2\int_M \sum_{i<j} \alpha_{i,\nu}\alpha_{j,\nu}\bar{u}_{(x_{i,\nu},\epsilon_{i,\nu})}\left(\frac{4(n-1)}{n-2}\Delta_{g_0}\bar{u}_{(x_{j,\nu},\epsilon_{j,\nu})} - R_{g_0}\bar{u}_{(x_{j,\nu},\epsilon_{j,\nu})} + \overline{R}_\infty \bar{u}_{(x_{j,\nu},\epsilon_{j,\nu})}^{\frac{n+2}{n-2}}\right) dv_{g_0} \\
&\quad - \frac{8(n-1)}{n-2}\int_{\partial M}\sum_{i<j}\alpha_{i,\nu}\alpha_{j,\nu}\bar{u}_{(x_{i,\nu},\epsilon_{i,\nu})}\frac{\partial \bar{u}_{(x_{j,\nu},\epsilon_{j,\nu})}}{\partial \eta_{g_0}} d\sigma_{g_0} \\
&\quad - 2\sum_{i<j}\alpha_{i,\nu}\alpha_{j,\nu}(F(\bar{u}_{(x_{j,\nu},\epsilon_{j,\nu})}) - \overline{R}_\infty)\int_M \bar{u}_{(x_{i,\nu},\epsilon_{i,\nu})}\bar{u}_{(x_{j,\nu},\epsilon_{j,\nu})}^{\frac{n+2}{n-2}} dv_{g_0}.
\end{aligned}$$

It is not difficult to see that $F(\bar{u}_{(x_{j,\nu},\epsilon_{j,\nu})}) = \overline{R}_\infty + o(1)$. This is more subtle in the case $\bar{u}_{(x_{j,\nu},\epsilon_{j,\nu})} = \bar{u}_{B;(x_{j,\nu},\epsilon_{j,\nu})}$, when we make use of Proposition 3.3.4 and Lemma 3.2.20. Then, because of [13, Lemma B.4], we have

$$|F(\bar{u}_{(x_{j,\nu},\epsilon_{j,\nu})}) - \overline{R}_\infty| \int_M \bar{u}_{(x_{i,\nu},\epsilon_{i,\nu})}\bar{u}_{(x_{j,\nu},\epsilon_{j,\nu})}^{\frac{n+2}{n-2}} dv_{g_0} \leq o(1)\left(\frac{\epsilon_{i,\nu}\epsilon_{j,\nu}}{\epsilon_{j,\nu}^2 + d_{g_0}(x_{i,\nu},x_{j,\nu})^2}\right)^{\frac{n-2}{2}}.$$

Then, using Corollaries 3.2.32 and 3.2.33,

$$\begin{aligned}
E(v_\nu)\left(\int_M v_\nu^{\frac{2n}{n-2}} dv_{g_0}\right)^{\frac{n-2}{n}} &\leq \left(\sum_{k=1}^m E(\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})})^{\frac{n}{2}}\right)^{\frac{2}{n}}\left(\int_M v_\nu^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\
&\quad - \sum_{i<j}(c - C\max\{\rho_A,\rho_B,\rho_C\}^{1/2} - o(1))\left(\frac{\epsilon_{i,\nu}\epsilon_{j,\nu}}{\epsilon_{j,\nu}^2 + d_{g_0}(x_{i,\nu},x_{j,\nu})^2}\right)^{\frac{n-2}{2}}.
\end{aligned}$$

Hence, the assertion follows by choosing ρ_A, ρ_B and ρ_C smaller if necessary (see Remark 3.3.1).

□

Corollary 3.3.11. *Under the hypothesis of Theorem 1.2.4, we have*

$$E(v_\nu) \leq \overline{R}_\infty, \qquad \textit{for all } \nu \textit{ sufficiently large.}$$

Proof. Using Propositions 3.2.8, 3.2.17 and 3.2.29, we obtain $E(\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}) \leq Q(S^n_+)$ for $k \leq m_1$, and $E(\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}) \leq Y(S^n)$ for $k \geq m_1+1$. Then the result follows from Proposition 3.3.10 and (3.62). □

3.3.2 The case $u_\infty > 0$

Proposition 3.3.12. *There exist sequences $\{\psi_a\}_{a\in\mathbb{N}} \subset C^\infty(M)$ and $\{\lambda_a\}_{a\in\mathbb{N}} \subset \mathbb{R}$, with $\lambda_a > 0$, satisfying:*

(i) For all $a \in \mathbb{N}$,

$$\begin{cases} \frac{4(n-1)}{n-2}\Delta_{g_0}\psi_a - R_{g_0}\psi_a + \lambda_a u_\infty^{\frac{4}{n-2}}\psi_a = 0\,, & \textit{in}\, M\,, \\ \frac{\partial}{\partial\eta_{g_0}}\psi_a = 0\,, & \textit{on}\, \partial M\,. \end{cases}$$

(ii) For all $a, b \in \mathbb{N}$,

$$\int_M \psi_a\psi_b u_\infty^{\frac{4}{n-2}} dv_{g_0} = \begin{cases} 1\,, & \textit{if}\, a = b\,, \\ 0\,, & \textit{if}\, a \neq b\,. \end{cases}$$

(iii) The span of $\{\psi_a\}_{a\in\mathbb{N}}$ is dense in $L^2(M)$.

(iv) We have $\lim_{a\to\infty}\lambda_a = \infty$.

Proof. Since we are assuming $R_{g_0} > 0$, for each $f \in L^2(M)$ we can define $T(f) = u$, where $u \in H^1(M)$ is the unique solution of

$$\begin{cases} \frac{4(n-1)}{n-2}\Delta_{g_0}u - R_{g_0}u = f u_\infty^{\frac{4}{n-2}}\,, & \text{in}\, M\,, \\ \frac{\partial}{\partial\eta_{g_0}}u = 0\,, & \text{on}\, \partial M\,. \end{cases}$$

Since $H^1(M)$ is compactly embedded in $L^2(M)$, the operator $T : L^2(M) \to L^2(M)$ is compact. Integrating by parts, we see that T is symmetric with respect to the inner product $(\psi_1, \psi_2) \mapsto \int_M \psi_1\psi_2 u_\infty^{\frac{4}{n-2}} dv_{g_0}$. Then the result follows from the spectral theorem for compact operators. □

Let $A \subset \mathbb{N}$ be a finite set such that $\lambda_a > \frac{n+2}{n-2}\overline{R}_\infty$ for all $a \notin A$, and define the projection

$$\Gamma(f) = \sum_{a\notin A}\left(\int_M \psi_a f dv_{g_0}\right)\psi_a u_\infty^{\frac{4}{n-2}} = f - \sum_{a\in A}\left(\int_M \psi_a f dv_{g_0}\right)\psi_a u_\infty^{\frac{4}{n-2}}\,.$$

Lemma 3.3.13. *There exists $\zeta > 0$ with the following significance: for all $z = (z_1, ..., z_a) \in \mathbb{R}^A$ with $|z| \leq \zeta$, there exists a smooth function $\bar{u}_z$ satisfying $\partial \bar{u}_z / \partial \eta_{g_0} = 0$ on ∂M,*

$$\int_M u_\infty^{\frac{4}{n-2}} (\bar{u}_z - u_\infty) \psi_a dv_{g_0} = z_a \quad \text{for all}\, a \in A\,, \tag{3.66}$$

and

$$\Gamma \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_z - R_{g_0} \bar{u}_z + \overline{R}_\infty \bar{u}_z^{\frac{n+2}{n-2}} \right) = 0\,. \tag{3.67}$$

Moreover, the mapping $z \mapsto \bar{u}_z$ is real analytic.

Proof. This is just an application of the implicit function theorem. □

Lemma 3.3.14. *There exists $0 < \gamma < 1$ such that*

$$E(\bar{u}_z) - E(u_\infty) \leq C \sup_{a \in A} \left| \int_M \psi_a \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_z - R_{g_0} \bar{u}_z + \overline{R}_\infty \bar{u}_z^{\frac{n+2}{n-2}} \right) dv_{g_0} \right|^{1+\gamma},$$

if $|z|$ is sufficiently small.

Proof. Observe that the function $z \mapsto E(\bar{u}_z)$ is real analytic. According to results of Lojasiewicz (see equation (2.4) in [78, p.538]), there exists $0 < \gamma < 1$ such that

$$|E(\bar{u}_z) - E(u_\infty)| \leq \sup_{a \in A} \left| \frac{\partial}{\partial z_a} E(\bar{u}_z) \right|^{1+\gamma},$$

if $|z|$ is sufficiently small. Now we can follow the lines in [13, Lemma 6.5] to obtain the result. □

We set

$$\begin{aligned} \mathcal{A}_\nu = \Big\{ &(z, (x_k, \epsilon_k, \alpha_k)_{k=1,\ldots,m}) \in \mathbb{R}^A \times (M \times \mathbb{R}_+ \times \mathbb{R}_+)^m \,, \text{ such that} \\ &x_k \in \partial M \text{ if } k \leq m_1 \,,\ x_k \in M \backslash \partial M \text{ if } k \geq m_1 + 1, \\ &|z| \leq \zeta,\ d_{g_0}(x_k, x_{k,\nu}^*) \leq \epsilon_{k,\nu}^* \,,\ \frac{1}{2} \leq \frac{\epsilon_k}{\epsilon_{k,\nu}^*} \leq 2 \,,\ \frac{1}{2} \leq \alpha_k \leq 2 \Big\}. \end{aligned}$$

For each ν, we can choose a pair $(z_\nu, (x_{k,\nu}, \epsilon_{k,\nu}, \alpha_{k,\nu})_{k=1,\ldots,m}) \in \mathcal{A}_\nu$ such that

$$\begin{aligned} &\int_M \frac{4(n-1)}{n-2} \Big| d(u_\nu - \bar{u}_{z_\nu} - \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \epsilon_{k,\nu})}) \Big|_{g_0}^2 dv_{g_0} + \int_M R_{g_0} \big(u_\nu - \bar{u}_{z_\nu} - \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \epsilon_{k,\nu})} \big)^2 dv_{g_0} \\ &\leq \int_M \frac{4(n-1)}{n-2} \Big| d(u_\nu - \bar{u}_z - \sum_{k=1}^m \alpha_k \bar{u}_{(x_k, \epsilon_k)}) \Big|_{g_0}^2 dv_{g_0} + \int_M R_{g_0} \big(u_\nu - \bar{u}_z - \sum_{k=1}^m \alpha_k \bar{u}_{(x_k, \epsilon_k)} \big)^2 dv_{g_0} \end{aligned}$$

for all $(z,(x_k,\epsilon_k,\alpha_k)_{k=1,...,m}) \in \mathcal{A}_\nu$. Here, $\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})} = \bar{u}_{A;(x_{k,\nu},\epsilon_{k,\nu})}$ and $\bar{u}_{(x_k,\epsilon_k)} = \bar{u}_{A;(x_k,\epsilon_k)}$ if $k \le m_1$, $\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})} = \bar{u}_{B;(x_{k,\nu},\epsilon_{k,\nu})}$ and $\bar{u}_{(x_k,\epsilon_k)} = \bar{u}_{B;(x_k,\epsilon_k)}$ if $m_1+1 \le k \le m_2$, and $\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})} = \bar{u}_{C;(x_{k,\nu},\epsilon_{k,\nu})}$ and $\bar{u}_{(x_k,\epsilon_k)} = \bar{u}_{C;(x_k,\epsilon_k)}$ if $k \ge m_2+1$; see (3.33), (3.44) and (3.54).

The proofs of the next three propositions are similar to Propositions 3.3.4, 3.3.5 and 3.3.6.

Proposition 3.3.15. *If $k \ge m_1+1$, then $\lim_{\nu\to\infty} d_{g_0}(x_{k,\nu},\partial M)/\epsilon_{k,\nu} = \infty$.*

Proposition 3.3.16. *We have:*

(i) For all $i \neq j$,

$$\lim_{\nu\to\infty}\left\{\frac{\epsilon_{i,\nu}}{\epsilon_{j,\nu}}+\frac{\epsilon_{j,\nu}}{\epsilon_{i,\nu}}+\frac{d_{g_0}(x_{i,\nu},x_{j,\nu})^2}{\epsilon_{i,\nu}\epsilon_{j,\nu}}\right\}=\infty\,.$$

(ii) We have

$$\lim_{\nu\to\infty}\left\|u_\nu-\bar{u}_{z_\nu}-\sum_{k=1}^{m}\alpha_{k,\nu}\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}\right\|_{H^1(M)}=0\,.$$

Proposition 3.3.17. *We have $|z_\nu| = o(1)$, and*

$$d_{g_0}(x_{k,\nu},x^*_{k,\nu}) \le o(1)\,\epsilon^*_{k,\nu}\,,\quad \frac{\epsilon_{k,\nu}}{\epsilon^*_{k,\nu}} = 1+o(1)\,,\quad \textit{and}\quad \alpha_{k,\nu}=1+o(1)\,,$$

for all $k = 1,...,m$. In particular, $(z_\nu,(x_{k,\nu},\epsilon_{k,\nu},\alpha_{k,\nu})_{k=1,...,m})$ is an interior point of $\mathcal{A}_\nu$ for ν sufficiently large.

Notation. We write $u_\nu = v_\nu + w_\nu$, where

$$v_\nu=\bar{u}_{z_\nu}+\sum_{k=1}^{m}\alpha_{k,\nu}\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}\quad\text{and}\quad w_\nu=u_\nu-\bar{u}_{z_\nu}-\sum_{k=1}^{m}\alpha_{k,\nu}\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}\,. \tag{3.68}$$

Observe that by Proposition 3.3.16 we have

$$\int_M\frac{4(n-1)}{n-2}|dw_\nu|^2_{g_0}dv_{g_0}+\int_M R_{g_0}w_\nu^2dv_{g_0}=o(1)\,. \tag{3.69}$$

Set

$$C_\nu=\left(\int_{\partial M}|\psi|^{\frac{2(n-1)}{n-2}}d\sigma_{g_0}\right)^{\frac{n-2}{2(n-1)}}+\left(\int_M|\psi|^{\frac{2n}{n-2}}dv_{g_0}\right)^{\frac{n-2}{2n}}\,,$$

Proposition 3.3.18. *Fix $\rho \leq P_0$. Let $\psi_{k,\nu} : \Omega_{k,\nu} = B^+_\rho(0) \subset \mathbb{R}^n_+ \to M$ be Fermi coordinates centered at $x_{k,\nu}$ if $1 \leq k \leq m_1$, and let $\psi_{k,\nu} : \Omega_{k,\nu} = \tilde{B}_{x_{k,\nu}\rho} \subset R^n \to M$ be normal coordinates centered at $x_{k,\nu}$ if $m_1+1 \leq k \leq m$ (see Definitions 3.2.1 and 3.2.2). We have:*

(i) $\left|\int_M u_\infty^{\frac{4}{n-2}}\psi_a\,\psi\,dv_{g_0}\right| \leq o(1)\int_M |w_\nu|dv_{g_0}, \quad for\, a\in A.$

(ii) $\left|\int_M \bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}^{\frac{n+2}{n-2}}\,\psi\,dv_{g_0}\right| \leq o(1)\,C_\nu\,.$

(iii) $\left|\int_{\Omega_{k,\nu}} \bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}^{\frac{n+2}{n-2}}\frac{\epsilon_{k,\nu}^2-|\psi_{k,\nu}^{-1}(x)|^2}{\epsilon_{k,\nu}^2+|\psi_{k,\nu}^{-1}(x)|^2}\,\psi\,dv_{g_0}\right| \leq o(1)\,C_\nu\,.$

(iv) $\left|\int_{\Omega_{k,\nu}} \bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}^{\frac{n+2}{n-2}}\frac{\epsilon_{k,\nu}\psi_{k,\nu}^{-1}(x)}{\epsilon_{k,\nu}^2+|\psi_{k,\nu}^{-1}(x)|^2}\,\psi\,dv_{g_0}\right| \leq o(1)\,C_\nu, \quad if\, m_1+1\leq k\leq m,$

$$and\ \left|\int_{\Omega_{k,\nu}} \bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}^{\frac{n+2}{n-2}}\frac{\epsilon_{k,\nu}\overline{\psi_{k,\nu}^{-1}(x)}}{\epsilon_{k,\nu}^2+|\psi_{k,\nu}^{-1}(x)|^2}\,\psi\,dv_{g_0}\right| \leq o(1)\,C_\nu, \quad if\, k\leq m_1,$$

where we are denoting $\bar{y}=(y_1,...,y_{n-1})$ for any $y=(y_1,...,y_n)\in\mathbb{R}^n$.

Proof. (i) Set $\tilde{\psi}_{a,z} = \partial\bar{u}_z/\partial z_a$. It follows from the identities (3.66) and (3.67) that $\tilde{\psi}_{a,0}=\psi_a$ for all $a\in A$. By the definition of $(z_\nu,(x_{k,\nu},\epsilon_{k,\nu},\alpha_{k,\nu})_{1\leq k\leq m})$, we have

$$\int_M \frac{4(n-1)}{n-2}\langle d\tilde{\psi}_{a,z_\nu},w_\nu\rangle_{g_0}dv_{g_0}+\int_M R_{g_0}\tilde{\psi}_{a,z_\nu}w_\nu\,dv_{g_0}=0\,.$$

Hence,

$$\begin{aligned}
&\lambda_a\int_M u_\infty^{\frac{4}{n-2}}\psi_a w_\nu\,dv_{g_0} \qquad (3.70)\\
&=-\int_M\left(\frac{4(n-1)}{n-2}\Delta_{g_0}\psi_a-R_{g_0}\psi_a\right)w_\nu\,dv_{g_0}\\
&=\int_M\left(\frac{4(n-1)}{n-2}\Delta_{g_0}(\tilde{\psi}_{a,z_\nu}-\psi_a)-R_{g_0}(\tilde{\psi}_{a,z_\nu}-\psi_a)\right)w_\nu\,dv_{g_0}+\int_{\partial M}\frac{\partial\tilde{\psi}_{a,z_\nu}}{\partial\eta_{g_0}}w_\nu d\sigma_{g_0}\,.
\end{aligned}$$

However, we know that $\partial\tilde{\psi}_{a,z_\nu}/\partial\eta_{g_0}=0$ on ∂M. Then, since $\lambda_a>0$ and $|z_\nu|\to 0$ as $\nu\to\infty$, we conclude that the assertion (i) follows.

The proofs of (ii), (iii), and (iv) are similar to Proposition 3.3.7. □

Proposition 3.3.19. *There exists $c>0$ such that*

$$\frac{n+2}{n-2}\overline{R}_\infty\int_M\left(u_\infty^{\frac{4}{n-2}}+\sum_{k=1}^m\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}^{\frac{4}{n-2}}\right)\psi^2\,dv_{g_0}\leq(1-c)\int_M\left(\frac{4(n-1)}{n-2}|d\psi|^2_{g_0}+R_{g_0}\psi^2\right)dv_{g_0}$$

for all ν sufficiently large.

Proof. As in Proposition 3.3.8, once Proposition 3.3.18 is established, this proof is a contradiction argument similar to [13, Proposition 6.8] and [2, Proposition 4.18]. □

Corollary 3.3.20. *There exists $c > 0$ such that*

$$\frac{n+2}{n-2}\overline{R}_\infty \int_M v_\nu^{\frac{4}{n-2}} w_\nu^2 \, dv_{g_0} \leq (1-c) \int_M \left(\frac{4(n-1)}{n-2} |d\psi|_{g_0}^2 + R_{g_0}\psi^2 \right) dv_{g_0}$$

for all ν sufficiently large.

Proof. By the definition of v_ν (see (3.68)), we have

$$\lim_{\nu\to\infty} \int_M \Big| v_\nu^{\frac{4}{n-2}} - u_\infty^{\frac{4}{n-2}} - \sum_{k=1}^m \bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}^{\frac{4}{n-2}} \Big|^{\frac{n}{2}} dv_{g_0} = 0\,.$$

Hence, the assertion follows from Proposition 3.3.19. □

The next two propositions are similar to Propositions 6.14 and 6.15 of [13] and we will just outline their proofs.

Proposition 3.3.21. *There exist $C > 0$ and $0 < \gamma < 1$ such that*

$$E(\bar{u}_{z_\nu}) - E(u_\infty) \leq C \left\{ \int_M u_\nu^{\frac{2n}{n-2}} |R_{g_\nu} - \overline{R}_\infty|^{\frac{2n}{n+2}} dv_{g_0} \right\}^{\frac{n+2}{2n}(1+\gamma)} + C \sum_{k=1}^m \epsilon_{k,\nu}^{\frac{n-2}{2}(1+\gamma)}$$

if ν is sufficiently large.

Proof. As in [13, Lemmas 6.11 and 6.12], because $\partial u_\nu/\partial\eta_{g_0} = \partial\bar{u}_{z_\nu}/\partial\eta_{g_0} = 0$ on ∂M, we can show that there exists $C > 0$ such that

$$\|u_\nu - \bar{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}^{\frac{n+2}{n-2}} \leq C \|u_\nu^{\frac{n+2}{n-2}}(R_{g_\nu} - \overline{R}_\infty)\|_{L^{\frac{2n}{n+2}}(M)}^{\frac{n+2}{n-2}} + C \sum_{k=1}^m \epsilon_{k,\nu}^{\frac{n-2}{2}} \tag{3.71}$$

and

$$\|u_\nu - \bar{u}_{z_\nu}\|_{L^1(M)} \leq C \|u_\nu^{\frac{n+2}{n-2}}(R_{g_\nu} - \overline{R}_\infty)\|_{L^{\frac{2n}{n+2}}(M)} + C \sum_{k=1}^m \epsilon_{k,\nu}^{\frac{n-2}{2}}\,, \tag{3.72}$$

for ν sufficiently large.

We will prove the estimate

$$\begin{aligned} &\sup_{a\in A} \left| \int_M \psi_a \left(\frac{4(n-1)}{n-2} \Delta_{g_0}\bar{u}_{z_\nu} - R_{g_0}\bar{u}_{z_\nu} + \overline{R}_\infty \bar{u}_{z_\nu}^{\frac{n+2}{n-2}} \right) dv_{g_0} \right| \\ &\quad \leq C \left\{ \int_M u_\nu^{\frac{2n}{n-2}} |R_{g_\nu} - \overline{R}_\infty|^{\frac{2n}{n+2}} dv_{g_0} \right\}^{\frac{n+2}{2n}} + C \sum_{k=1}^m \epsilon_{k,\nu}^{\frac{n-2}{2}} \end{aligned} \tag{3.73}$$

for ν is sufficiently large.

Integrating by parts, we obtain

$$\begin{aligned}&\int_M \psi_a \left(\frac{4(n-1)}{n-2}\Delta_{g_0}\bar{u}_{z_\nu} - R_{g_0}\bar{u}_{z_\nu} + \overline{R}_\infty \bar{u}_{z_\nu}^{\frac{n+2}{n-2}}\right) dv_{g_0} \\ &\quad = \int_M \psi_a \left(\frac{4(n-1)}{n-2}\Delta_{g_0}u_\nu - R_{g_0}u_\nu + \overline{R}_\infty u_\nu^{\frac{n+2}{n-2}}\right) dv_{g_0} \\ &\quad + \lambda_a \int_M u_\infty^{\frac{4}{n-2}}\psi_a(u_\nu - \bar{u}_{z_\nu})\, dv_{g_0} - \overline{R}_\infty \int_M \psi_a(u_\nu^{\frac{n+2}{n-2}} - \bar{u}_{z_\nu}^{\frac{n+2}{n-2}})\, dv_{g_0}\,.\end{aligned}$$

Using the fact that $\frac{4(n-1)}{n-2}\Delta_{g_0}u_\nu - R_{g_0}u_\nu + \overline{R}_\infty u_\nu^{\frac{n+2}{n-2}} = -(R_{g_\nu} - \overline{R}_\infty)u_\nu^{\frac{n+2}{n-2}}$ and the pointwise estimate

$$|u_\nu^{\frac{n+2}{n-2}} - \bar{u}_{z_\nu}^{\frac{n+2}{n-2}}| \leq C\bar{u}_{z_\nu}^{\frac{4}{n-2}}|u_\nu - \bar{u}_{z_\nu}| + C|u_\nu - \bar{u}_{z_\nu}|^{\frac{n+2}{n-2}}\,,$$

we obtain

$$\begin{aligned}&\sup_{a\in A}\left|\int_M \psi_a \left(\frac{4(n-1)}{n-2}\Delta_{g_0}\bar{u}_{z_\nu} - R_{g_0}\bar{u}_{z_\nu} + \overline{R}_\infty \bar{u}_{z_\nu}^{\frac{n+2}{n-2}}\right) dv_{g_0}\right| \\ &\quad \leq C\|u_\nu^{\frac{n+2}{n-2}}(R_{g_\nu} - \overline{R}_\infty)\|_{L^{\frac{2n}{n+2}}(M)} + C\|u_\nu - \bar{u}_{z_\nu}\|_{L^1(M)} + C\|u_\nu - \bar{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}^{\frac{n+2}{n-2}}\,.\end{aligned}$$

Then it follows from (3.71) and (3.72) that

$$\begin{aligned}&\sup_{a\in A}\left|\int_M \psi_a \left(\frac{4(n-1)}{n-2}\Delta_{g_0}\bar{u}_{z_\nu} - R_{g_0}\bar{u}_{z_\nu} + \overline{R}_\infty \bar{u}_{z_\nu}^{\frac{n+2}{n-2}}\right) dv_{g_0}\right| \\ &\quad \leq C\|u_\nu^{\frac{n+2}{n-2}}(R_{g_\nu} - \overline{R}_\infty)\|_{L^{\frac{2n}{n+2}}(M)}^{\frac{n+2}{n-2}} + C\|u_\nu^{\frac{n+2}{n-2}}(R_{g_\nu} - \overline{R}_\infty)\|_{L^{\frac{2n}{n+2}}(M)} + C\sum_{k=1}^m \epsilon_{k,\nu}^{\frac{n-2}{2}}\,.\end{aligned} \tag{3.74}$$

On the other hand, by Corollary 3.1.3 we can assume

$$\|u_\nu^{\frac{n+2}{n-2}}(R_{g_\nu} - \overline{R}_\infty)\|_{L^{\frac{2n}{n+2}}(M)} = \left(\int_M |R_{g_\nu} - \overline{R}_\infty|^{\frac{2n}{n+2}} dv_{g_\nu}\right)^{\frac{n+2}{2n}} < 1. \tag{3.75}$$

The estimate (3.73) now follows using the inequality (3.75) in (3.74). Proposition 3.3.21 is a consequence of Lemma 3.3.14 and the estimate (3.73). □

Proposition 3.3.22. *There exists $c > 0$ such that*

$$E(v_\nu) \leq \left(E(\bar{u}_{z_\nu})^{\frac{n}{2}} + \sum_{k=1}^m E(\bar{u}_{x_k,\epsilon_{k,\nu}})^{\frac{n}{2}}\right)^{\frac{2}{n}} - c\sum_{k=1}^m \epsilon_{k,\nu}^{\frac{n-2}{2}}$$

if ν is sufficiently large.

Proof. Choose a permutation $\sigma : \{1, ..., m\}$ such that $\epsilon_{\sigma(i),\nu} \leq \epsilon_{\sigma(j),\nu}$ for all $i < j$. During this proof we will omit the symbol σ, writing $\epsilon_{i,\nu}$ instead of $\epsilon_{\sigma(i),\nu}$, so that $\epsilon_{i,\nu} \leq \epsilon_{j,\nu}$ for all $i < j$. After calculations similar to the ones in [13, Proposition 6.15], we obtain

$$\begin{aligned}
&E(v_\nu)\left(\int_M v_\nu^{\frac{2n}{n-2}} dv_{g_0}\right)^{\frac{n-2}{n}} \\
&\quad\leq \left(E(\bar{u}_{z_\nu})^{\frac{n}{2}} + \sum_{k=1}^{m} E(\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})})^{\frac{n}{2}}\right)^{\frac{2}{n}} \left(\int_M v_\nu^{\frac{2n}{n-2}} dv_{g_0}\right)^{\frac{n-2}{n}} \\
&\qquad - \sum_{k=1}^{m} 2\alpha_{k,\nu} \int_M \left(\frac{4(n-1)}{n-2}\Delta_{g_0}\bar{u}_{z_\nu} - R_{g_0}\bar{u}_{z_\nu} + F(\bar{u}_{z_\nu})\bar{u}_{z_\nu}^{\frac{n+2}{n-2}}\right) \bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})} dv_{g_0} \\
&\qquad - \sum_{i<j} 2\alpha_{i,\nu}\alpha_{j,\nu} \int_M \frac{4(n-1)}{n-2} \frac{\partial \bar{u}_{(x_{j,\nu},\epsilon_{j,\nu})}}{\partial \eta_{g_0}} \bar{u}_{(x_{i,\nu},\epsilon_{i,\nu})} dv_{g_0} \\
&\qquad - \sum_{i<j} 2\alpha_{i,\nu}\alpha_{j,\nu} \int_M \left(\frac{4(n-1)}{n-2}\Delta_{g_0}\bar{u}_{(x_{j,\nu},\epsilon_{j,\nu})} - R_{g_0}\bar{u}_{(x_{j,\nu},\epsilon_{j,\nu})} + F(\bar{u}_{(x_{j,\nu},\epsilon_{j,\nu})})\bar{u}_{(x_{j,\nu},\epsilon_{j,\nu})}^{\frac{n+2}{n-2}}\right) \bar{u}_{(x_{i,\nu},\epsilon_{i,\nu})} dv_{g_0} \\
&\qquad - c\sum_{k=1}^{m} \epsilon_{k,\nu}^{\frac{n-2}{2}} - c\sum_{i<j} \left(\frac{\epsilon_{i,\nu}\epsilon_{j,\nu}}{\epsilon_{j,\nu}^2 + d_{g_0}(x_{i,\nu}, x_{j,\nu})^2}\right)^{\frac{n-2}{2}}.
\end{aligned}$$

Since $F(\bar{u}_{z_\nu}) \to F(u_\infty) = \overline{R}_\infty$ as $\nu \to \infty$, we have the estimate

$$\int_M \left|\frac{4(n-1)}{n-2}\Delta_{g_0}\bar{u}_{z_\nu} - R_{g_0}\bar{u}_{z_\nu} + F(\bar{u}_{z_\nu})\bar{u}_{z_\nu}^{\frac{n+2}{n-2}}\right| \bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})} dv_{g_0} \leq o(1)\epsilon_{k,\nu}^{\frac{n-2}{2}}.$$

Now the assertion follows as in the proof of Proposition 3.3.10. □

Corollary 3.3.23. *Under the hypothesis of Theorem 1.2.4, there exist $C > 0$ and $0 < \gamma < 1$ such that*

$$E(v_\nu) \leq \overline{R}_\infty + C\left(\int_M u_\nu^{\frac{2n}{n-2}} |R_{g_\nu} - \overline{R}_\infty|^{\frac{2n}{n+2}} dv_{g_0}\right)^{\frac{n+2}{2n}(1+\gamma)},$$

if ν is sufficiently large.

Proof. Using Propositions 3.2.8, 3.2.17 and 3.2.29, we obtain $E(\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}) \leq Q(S^n_+)$ for all $k = 1, ..., m_1$ and $E(\bar{u}_{(x_{k,\nu},\epsilon_{k,\nu})}) \leq Y(S^n)$ for all $k = m_1 + 1, ..., m$. Then the result follows from Propositions 3.3.21 and 3.3.22 and (3.62). □

3.4 Proof of the main theorem

As in Sections 3 and 7 of [13], the proof of Theorem 1.2.4 is carried out in several propositions, whose proofs will be only sketched in what follows.

Let $u(t)$, $t \geq 0$, be the solution of (3.4) obtained in Section 3.1. The next proposition, which is analogous to [13, Proposition 3.3], is a crucial step in the argument.

Proposition 3.4.1. *Let $\{t_\nu\}_{\nu=1}^{\infty}$ be a sequence such that $\lim_{\nu\to\infty} t_\nu = \infty$. Then we can choose $0 < \gamma < 1$ and $C > 0$ such that, after passing to a subsequence, we have*

$$\overline{R}_{g(t_\nu)} - \overline{R}_\infty \leq C \left\{ \int_M u(t_\nu)^{\frac{2n}{n-2}} |R_{g(t_\nu)} - \overline{R}_\infty|^{\frac{2n}{n+2}} dv_{g_0} \right\}^{\frac{n+2}{2n}(1+\gamma)}$$

for all ν.

Proof. It is a long computation using Corollaries 3.3.9, 3.3.11, 3.3.20 and 3.3.23; see [13, Section 7]. □

Proposition 3.4.2. *There exists $C > 0$ such that*

$$\int_0^\infty \left\{ \int_M u(t)^{\frac{2n}{n-2}} (R_{g(t)} - \overline{R}_{g(t)})^2 dv_{g_0} \right\}^{\frac{1}{2}} dt \leq C$$

for all $t \geq 0$.

Proof. A simple contradiction argument using Corollary 3.1.3 and Proposition 3.4.1 (see [13, Proposition 3.4]) shows that there exist $0 < \gamma < 1$ and $t_0 > 0$ such that

$$\overline{R}_{g(t)} - \overline{R}_\infty \leq \left\{ \int_M u(t)^{\frac{2n}{n-2}} |R_{g(t)} - \overline{R}_\infty|^{\frac{2n}{n+2}} dv_{g_0} \right\}^{\frac{n+2}{2n}(1+\gamma)}$$

for all $t \geq t_0$. Then it follows that

$$\overline{R}_{g(t)} - \overline{R}_\infty \leq C \left\{ \int_M u(t)^{\frac{2n}{n-2}} |R_{g(t)} - \overline{R}_{g(t)}|^{\frac{2n}{n+2}} dv_{g_0} \right\}^{\frac{n+2}{2n}(1+\gamma)} + C(\overline{R}_{g(t)} - \overline{R}_\infty)^{1+\gamma},$$

hence

$$\overline{R}_{g(t)} - \overline{R}_\infty \leq C \left\{ \int_M u(t)^{\frac{2n}{n-2}} |R_{g(t)} - \overline{R}_{g(t)}|^{\frac{2n}{n+2}} dv_{g_0} \right\}^{\frac{n+2}{2n}(1+\gamma)} \tag{3.76}$$

for $t > 0$ sufficiently large. By (3.7) and (3.76), there exists $c > 0$ such that

$$\begin{aligned} \frac{d}{dt}(\overline{R}_{g(t)} - \overline{R}_\infty) &= -\frac{n-2}{2} \int_M (R_{g(t)} - \overline{R}_{g(t)})^2 \, u(t)^{\frac{2n}{n-2}} dv_{g_0} \\ &\leq -\frac{n-2}{2} \left\{ \int_M |R_{g(t)} - \overline{R}_{g(t)}|^{\frac{2n}{n+2}} u(t)^{\frac{2n}{n-2}} dv_{g_0} \right\}^{\frac{n+2}{n}} \leq -c(\overline{R}_{g(t)} - \overline{R}_\infty)^{\frac{2}{1+\gamma}} \end{aligned}$$

for $t > 0$ sufficiently large. Hence, $\frac{d}{dt}(\overline{R}_{g(t)} - \overline{R}_\infty)^{-\frac{1-\gamma}{1+\gamma}} \geq c$, which implies

$$\overline{R}_{g(t)} - \overline{R}_\infty \leq Ct^{-\frac{1+\gamma}{1-\gamma}}, \quad \text{for } t > 0 \text{ sufficiently large.}$$

Then using Hölder's inequality and the equation (3.7) we obtain

$$\begin{aligned}\int_T^{2T}\left(\int_M (R_{g(t)} - \overline{R}_{g(t)})^2 u(t)^{\frac{2n}{n-2}} dv_{g_0}\right)^{\frac{1}{2}} dt &\leq \left(\int_T^{2T} dt\right)^{\frac{1}{2}}\left(\int_T^{2T}\int_M (R_{g(t)} - \overline{R}_{g(t)})^2 u(t)^{\frac{2n}{n-2}} dv_{g_0}\, dt\right)^{\frac{1}{2}} \\ &= \left\{\frac{2}{n-2}T(\overline{R}_{g(T)} - \overline{R}_{g(2T)})\right\}^{\frac{1}{2}} \leq CT^{-\frac{\gamma}{1-\gamma}}\end{aligned}$$

for T sufficiently large. This implies

$$\begin{aligned}&\int_0^\infty \left(\int_M (R_{g(t)} - \overline{R}_{g(t)})^2 u(t)^{\frac{2n}{n-2}} dv_{g_0}\right)^{\frac{1}{2}} dt \\ &\quad= \int_0^1 \left(\int_M (R_{g(t)} - \overline{R}_{g(t)})^2 u(t)^{\frac{2n}{n-2}} dv_{g_0}\right)^{\frac{1}{2}} dt + \sum_{k=0}^\infty \int_{2^k}^{2^{k+1}} \left(\int_M (R_{g(t)} - \overline{R}_{g(t)})^2 u(t)^{\frac{2n}{n-2}} dv_{g_0}\right)^{\frac{1}{2}} dt \\ &\quad\leq C\sum_{k=0}^\infty 2^{-\frac{\gamma}{1-\gamma}k} \leq C\,,\end{aligned}$$

which concludes the proof. $\square$

Proposition 3.4.3. *There exist $C, c > 0$ such that*

$$\sup_M u(t) \leq C \quad \textit{and} \quad \inf_M u(t) \geq c\,, \quad \textit{for all}\, t \geq 0\,. \tag{3.77}$$

Proof. We first claim that, given $\gamma_0 > 0$, there exists $r > 0$ such that

$$\int_{B_r(x)} u(t)^{\frac{2n}{n-2}} dv_{g_0} \leq \gamma_0, \quad \text{for all } t \geq 0,\ x \in M\,. \tag{3.78}$$

Indeed, we can make use of Proposition 3.4.2 as in [13, Proposition 3.6] to obtain the above inequality.

Fix $n/2 < q < p < (n+2)/2$. According to Corollary 3.1.3 there is $C_2 > 0$ such that

$$\int_M |R_{g(t)}|^p dv_{g(t)} \leq C_2\,, \quad \text{for all } t \geq 0\,.$$

Set $\gamma_0 = \gamma_1^{\frac{p}{p-q}} C_2^{-\frac{q}{p-q}}$, where γ_1 is the constant obtained in Proposition 3.A.3. By (3.78), there is $r > 0$ such that

$$\int_{B_r(x)} dv_{g(t)} \leq \gamma_0\,, \quad \text{for all } t \geq 0,\ x \in M\,.$$

Then

$$\int_{B_r(x)} |R_{g(t)}|^q dv_{g(t)} \le \left\{ \int_{B_r(x)} dv_{g(t)} \right\}^{\frac{p-q}{p}} \left\{ \int_{B_r(x)} |R_{g(t)}|^p dv_{g(t)} \right\}^{\frac{q}{p}} \le \gamma_1 \, .$$

Hence, the first assertion of (3.77) follows from Proposition 3.A.3. The second one follows exactly as in the proof of the second estimate of (3.8). □

Proof of Theorem 1.2.4. Once we have proved Proposition 3.4.3, it follows as in [13, p.229] that all higher order derivatives of u are uniformly bounded. The uniqueness of the asymptotic limit of $R_{g(t)}$ follows from Proposition 3.4.2. □

Appendix 3.A Some elliptic estimates

Let (M,g) be a complete Riemannian manifold with boundary ∂M and let η_g be its unit normal vector pointing inwards.

Definition 3.A.1. We say that $u \in H^1(M)$ is a *subsolution* (resp. *supersolution*) of

$$\begin{cases} \Delta_g u + Pu = f \, , & \text{in } M \, , \\ \partial u/\partial \eta_g + \bar{P}u = \bar{f} \, , & \text{on } \partial M \, . \end{cases} \tag{A-79}$$

if, for all $0 \le v \in C_c^1(M)$, the following quantity is nonpositive (resp. nonnegative)

$$\int_M (< du, dv >_g - Puv + fv) dv_g + \int_{\partial M} (-\bar{P}uv + \bar{f}v) d\sigma_g .$$

The next proposition is similar to [45, Theorems 8.17 and 8.18]; see also [49, Lemma A.1].

Proposition 3.A.2. *Let $q > n$, $s > n-1$ and $P \in L^{q/2}(M)$, $\bar{P} \in L^s(\partial M)$ with $||P||_{L^{q/2}}(M) + ||\bar{P}||_{L^s}(\partial M) \le \Lambda$.*

(a) For any $p > 1$, there exists $C = C(n,p,q,s,g,\Lambda)$ and $r_0 = r_0(M,g)$ such that

$$\sup_{B_r^+(x)} u \le Cr^{-\frac{n}{p}} ||u||_{L^p(B_{2r}^+(x))} + Cr^{2-\frac{2n}{q}} ||f||_{L^{q/2}(B_{4r}^+(x))} + Cr^{1-\frac{n-1}{s}} ||\bar{f}||_{L^s(D_{4r}(x))}$$

for any $x \in \partial M$, $r < r_0$ and $0 \le u \in H^1(M)$ subsolution of (A-79).

(b) If $1 \le p < \frac{n}{n-2}$, there exists $C = C(n,p,q,s,g,\Lambda)$ and $r_0 = r_0(M,g)$ such that

$$r^{-\frac{n}{p}} ||u||_{L^p(B_{2r}^+(x))} \le C \inf_{B_r^+(x)} u + Cr^{2-\frac{2n}{q}} ||f||_{L^{q/2}(B_{4r}^+(x))} + Cr^{1-\frac{n-1}{s}} ||\bar{f}||_{L^s(D_{4r}(x))}$$

for any $x \in \partial M$, $r < r_0$ and $0 \le u \in H^1(M)$ supersolution of (A-79).

Proof. After rescaling we can assume $r = 1$. Let $\beta \neq 0$, $k = ||f||_{L^{q/2}(B_4^+)} + ||\bar{f}||_{L^s(D_4)}$ and $0 \leq \chi \in C_c^1(B_4^+)$. We will assume that $k > 0$. The general case will follow by tending k to zero. Set $\bar{u} = u + k$.

If u is a subsolution, by definition we have

$$\int_M < du, d(\chi^2 \bar{u}^\beta) >_g dv_g \leq \int_M (Pu - f)\chi^2 \bar{u}^\beta dv_g + \int_{\partial M} (\bar{P}u - \bar{f})\chi^2 \bar{u}^\beta d\sigma_g,$$

and we have the opposite inequality in case u is a supersolution. Choosing $\beta > 0$ should u be a subsolution and $\beta < 0$ should u be a supersolution, in both cases we obtain

$$\begin{aligned}\int_M \chi^2 \bar{u}^{\beta-1} |d\bar{u}|_g^2 dv_g &\leq |\beta|^{-1} \int_M 2\chi \bar{u}^\beta |d\chi|_g |d\bar{u}_g|\, dv_g \\ &+ |\beta|^{-1} \int_M \chi^2 (|P| + k^{-1}|f|)\bar{u}^{\beta+1} dv_g + |\beta|^{-1} \int_{\partial M} \chi^2 (|\bar{P}| + k^{-1}|\bar{f}|)\bar{u}^{\beta+1} d\sigma_g\end{aligned} \tag{A-80}$$

by means of $< du, d(\chi^2 \bar{u}^\beta) >_g = 2\chi \bar{u}^\beta < d\chi, d\bar{u} >_g + \beta \chi^2 \bar{u}^{\beta-1} |d\bar{u}|_g^2$. Applying Young's inequality to the last term of (A-80) we arrive at

$$\begin{aligned}\int_M \chi^2 \bar{u}^{\beta-1} |d\bar{u}|_g^2 dv_g &\leq C|\beta|^{-2} \int_M |d\chi|_g^2 \bar{u}^{\beta+1}\, dv_g \\ &+ C|\beta|^{-1} \int_M \chi^2 (|P| + k^{-1}|f|)\bar{u}^{\beta+1} dv_g + C|\beta|^{-1} \int_{\partial M} \chi^2 (|\bar{P}| + k^{-1}|\bar{f}|)\bar{u}^{\beta+1} d\sigma_g.\end{aligned} \tag{A-81}$$

Set $h = |P| + k^{-1}|f|$, $\bar{h} = |\bar{P}| + k^{-1}|\bar{f}|$ and

$$w = \begin{cases} \bar{u}^{\frac{\beta+1}{2}} & \text{if } \beta \neq -1, \\ \log \bar{u} & \text{if } \beta = -1. \end{cases}$$

Then (A-81) can be rewritten as

$$\begin{aligned}\int_M \chi^2 |dw|_g^2 dv_g &\leq C\frac{(\beta+1)^2}{|\beta|^2} \int_M |d\chi|_g^2 w^2\, dv_g \\ &+ C\frac{(\beta+1)^2}{|\beta|} \int_M \chi^2 h w^2 dv_g + C\frac{(\beta+1)^2}{|\beta|} \int_{\partial M} \chi^2 \bar{h} w^2 d\sigma_g\end{aligned} \tag{A-82}$$

if $\beta \neq -1$ and

$$\int_M \chi^2 |dw|_g^2 dv_g \leq C \int_M |d\chi|_g^2\, dv_g + C \int_M \chi^2 h dv_g + C \int_{\partial M} \chi^2 \bar{h} d\sigma_g \tag{A-83}$$

if $\beta = -1$. It follows from $\chi^2 |dw|^2 \geq \frac{1}{2}|d(\chi w)|^2 - w^2 |d\chi|^2$ and Sobolev inequalities that

$$\int_M (\chi w)^{\frac{2n}{n-2}} dv_g - C \int_M |d\chi|^2 w^2 dv_g \leq C \int_M \chi^2 |dw|_g^2 dv_g \tag{A-84}$$

In order to handle the right hand side of (A-82) we use Hölder's and interpolation inequalities to get

$$\begin{aligned}\int_M \chi^2 h w^2 dv_g &\le \|h\|_{L^{q/2}(B_4^+)} \|\chi w\|^2_{L^{2q/(q-2)}(B_4^+)} \\ &\le \|h\|_{L^{q/2}(B_4^+)} (\epsilon^{1/2}\|\chi w\|_{L^{2n/(n-2)}(B_4^+)} + \epsilon^{-\mu_1/2}\|\chi w\|_{L^2(B_4^+)})^2 \\ &\le 2\|h\|_{L^{q/2}(B_4^+)} (\epsilon\|\chi w\|^2_{L^{2n/(n-2)}(B_4^+)} + \epsilon^{-\mu_1}\|\chi w\|^2_{L^2(B_4^+)})\end{aligned} \tag{A-85}$$

where $\mu_1 = n/(q-n)$, and

$$\begin{aligned}\int_{\partial M} \chi^2 \bar{h} w^2 d\sigma_g &\le \|\bar{h}\|_{L^s(D_4)} \|\chi w\|^2_{L^{2s/(s-1)}(D_4)} \\ &\le \|\bar{h}\|_{L^s(D_4)} (\epsilon^{1/2}\|\chi w\|_{L^{2(n-1)/(n-2)}(D_4)} + \epsilon^{-\mu_2/2}\|\chi w\|_{L^2(D_4)})^2 \\ &\le 2\|\bar{h}\|_{L^s(D_4)} (\epsilon\|\chi w\|^2_{L^{2(n-1)/(n-2)}(D_4)} + \epsilon^{-\mu_2}\|\chi w\|^2_{L^2(D_4)})\end{aligned} \tag{A-86}$$

where $\mu_2 = (n-1)/(s+1-n)$. It follows from the Sobolev embedding theorems that

$$\epsilon^{-\mu_2}\int_{D_4} (\chi w)^2 d\sigma_g \le \epsilon \int_{B_4^+} |d(\chi w)|^2 dv_g + \epsilon^{-2\mu_2-1}\int_{B_4^+} (\chi w)^2 dv_g$$

and

$$\Big(\int_{D_4} (\chi w)^{\frac{2(n-1)}{n-2}} d\sigma_g\Big)^{\frac{n-2}{n-1}} \le C\int_{B_4^+} |d(\chi w)|^2 dv_g.$$

Then the inequality (A-86) becomes

$$\int_{\partial M} \chi^2 \bar{h} w^2 d\sigma_g \le C\epsilon\|\bar{h}\|_{L^s(D_4)} \int_{B_4^+} |d(\chi w)|^2 dv_g + C\epsilon^{-2\mu_2-1}\|\bar{h}\|_{L^s(D_4)}\int_{B_4^+} (\chi w)^2 dv_g. \tag{A-87}$$

Choosing $\epsilon = c|\beta|(\beta+1)^{-2}\Gamma^{-1}$ with $c>0$ small, we can make use of the inequalities (A-84), (A-85), (A-86) and (A-87) in (A-82) to obtain

$$\Big(\int_{B_4^+} (\chi w)^{\frac{2n}{n-2}} dv_g\Big)^{\frac{n-2}{n}} \le C(1+|\gamma|)^{2\mu}\int_{B_4^+} (|d\chi|^2+\chi^2)w^2 dv_g. \tag{A-88}$$

Here, $\gamma = \beta+1$, $\mu = max\{\mu_1+1, 2\mu_2+2\}$, and C depends on Γ and is bounded when $|\beta|$ is bounded away from zero.

For any $1 \le r_a \le r_b \le 3$ we choose χ as a cut-off function satisfying $0\le\chi\le 1$, $|d\chi| \le 2/(r_b-r_a)$ and

$$\begin{cases}\chi \equiv 1 & \text{in } B_{r_a}^+, \\ \chi \equiv 0 & \text{in } B_4^+\backslash B_{r_b}^+.\end{cases}$$

Using this in (A-88) we obtain

$$\Big(\int_{B^+_{r_a}} \bar{u}^{\frac{\gamma n}{n-2}} dv_g\Big)^{\frac{n-2}{n}} \leq \frac{C(1+|\gamma|)^{2\mu}}{r_b - r_a}\int_{B^+_{r_b}} \bar{u}^{\gamma} dv_g. \tag{A-89}$$

If we set $\Phi(e,r) = \Big(\int_{B^+_r} \bar{u}^e dv_g\Big)^{1/e}$ and $\delta = n/(n-2)$, the estimate (A-89) becomes

$$\begin{cases} \Phi(\delta\gamma, r_a) \leq \left(\dfrac{C(1+|\gamma|)^{\mu}}{r_b - r_a}\right)^{\frac{2}{|\gamma|}} \Phi(\gamma, r_b) & \text{if } \gamma > 0, \\ \Phi(\gamma, r_b) \leq \left(\dfrac{C(1+|\gamma|)^{\mu}}{r_b - r_a}\right)^{\frac{2}{|\gamma|}} \Phi(\delta\gamma, r_a) & \text{if } \gamma < 0. \end{cases} \tag{A-90}$$

It is well known that $\lim_{e\to\infty}\Phi(e,r) = \sup_{B^+_r}\bar{u}$ and $\lim_{e\to-\infty}\Phi(e,r) = \inf_{B^+_r}\bar{u}$. The rest of the proof follows as in [45, p.197-198] by iterating the first inequality in (A-90) to prove (a), and by using (A-83) and iterating the second inequality in (A-90) to prove (b). $\square$

Once we have established Proposition 3.A.2(a), the proof of the next proposition is similar to [2, Proposition A.3].

Proposition 3.A.3. *Let (M^n, g_0) be a compact Riemannian manifold with boundary ∂M and with dimension $n \geq 3$. For each $q > n/2$ we can find positive constants $\gamma_1 = \gamma_1(M, g_0, q)$ and $C = C(M, g_0, q)$ with the following significance: if $g = u^{\frac{4}{n-2}} g_0$ is a conformal metric satisfying*

$$\int_M dv_g \leq 1 \quad \text{and} \quad \int_{B_r(x)} |R_g|^q \, dv_g \leq \gamma_1$$

for $x \in M$, then we have

$$u(x) \leq C r^{-\frac{n-2}{2}} \left(\int_{B_r(x)} dv_g\right)^{\frac{n-2}{2n}}.$$

Using Proposition 3.A.2(b) and interior Harnack estimates for elliptic linear equations (see [45, Theorem 8.18]), one can prove the next proposition by adapting the arguments in [13, Proposition A.2].

Proposition 3.A.4. *Let (M, g_0) be a Riemannian manifold with boundary ∂M, P a smooth function on M, and suppose u that satisfies*

$$\begin{cases} -\Delta_{g_0} u(t) + Pu \geq 0\,, & \text{in } M\,, \\ \dfrac{\partial}{\partial \eta_{g_0}} u = 0\,, & \text{on } \partial M\,. \end{cases}$$

Then there exists $C = C(P, g_0)$ such that

$$C \inf_M u \geq \int_M u dv_{g_0}.$$

In particular,

$$\int_M u^{\frac{2n}{n-2}} dv_{g_0} \leq C \inf_M u \left(\sup_M u \right)^{\frac{n+2}{n-2}}.$$

Appendix 3.B Construction of the Green function on manifolds with boundary

In this section, we prove the existence of the Green function used in this thesis and some of its properties. The construction performed here extends the one in [2, Proposition B-2]; see also [68, p.201] and [8, p.106].

Lemma 3.B.1. *Let (M, g) be a connected Riemannian manifold of dimension $n \geq 2$ and fix $x \in M$ and $\alpha \in \mathbb{R}$. Let $u : M \backslash \{x\} \to \mathbb{R}$ be a function satisfying*

$$|u(y)| \leq C_0 d_g(x,y)^\alpha \quad \textit{and} \quad |\nabla_g u(y)|_g \leq C_0 d_g(x,y)^{\alpha-1},$$

for any $y \in M$, with $x \neq y$. Then, for any $0 < \theta \leq 1$, there exists $C_1 = C_1(M, g, C_0, \alpha)$ such that

$$|u(y) - u(z)| \leq C_1 d_g(y,z)^\theta (d_g(x,y)^{\alpha-\theta} + d_g(x,z)^{\alpha-\theta})$$

for any $y, z \in M$, with $y \neq x \neq z$.

This is [2, Lemma B.1]. For the reader's convenience, we provide the proof here.

Proof. Let $y \neq x$ and $z \neq x$.

<u>1st case:</u> $d_g(y,z) \leq \frac{1}{2} d_g(x,y)$. Let $\gamma : [0,1] \to M$ be a smooth curve such that $\gamma(0) = y$, $\gamma(1) = z$, and $\int_0^1 |\gamma'(t)|_g dt \leq \frac{3}{2} d_g(y,z)$.

Claim. We have $\frac{1}{4} d_g(x,y) \leq d_g(\gamma(t), x) \leq \frac{7}{4} d_g(x,y)$.

Indeed, since $d_g(y, \gamma(t)) \leq \frac{3}{2} d_g(y,z) \leq \frac{3}{4} d_g(x,y)$, we have

$$d_g(x, \gamma(t)) \geq d_g(x,y) - d_g(\gamma(t), y) \geq d_g(x,y) - \frac{3}{4} d_g(x,y) = \frac{1}{4} d_g(x,y).$$

Moreover,

$$d_g(\gamma(t),x) \leq d_g(\gamma(t),y) + d_g(y,x) \leq \frac{3}{4}d_g(x,y) + d_g(x,y) = \frac{7}{4}d_g(x,y)\,.$$

This proves the claim.

Observe that $u(z) - u(y) = \int_0^1 g(\nabla_g u(\gamma(t)), \gamma'(t))\,dt$. Thus,

$$\begin{aligned}|u(y)-u(z)| &\leq \sup_{t\in[0,1]} |\nabla_g u(\gamma(t))|_g \int_0^1 |\gamma'(t)|_g dt \leq C \sup_{t\in[0,1]} d_g(\gamma(t),x)^{\alpha-1}\frac{3}{2}d_g(y,z) \\ &\leq C(\alpha)d_g(x,y)^{\alpha-1}d_g(y,z) \leq C(\alpha)d_g(x,y)^{\alpha-\theta}d_g(y,z)^{\theta}\,.\end{aligned}$$

<u>2nd case:</u> $d_g(y,z) > \frac{1}{2}d_g(x,y)$. In this case, we have

$$\begin{aligned}|u(y)-u(z)| &\leq |u(y)| + |u(z)| \leq Cd_g(y,x)^{\alpha} + Cd_g(z,x)^{\alpha} \\ &\leq Cd_g(y,x)^{\alpha-\theta}d_g(z,y)^{\theta} + Cd_g(z,x)^{\alpha-\theta}(d_g(x,y)+d_g(y,z))^{\theta} \\ &\leq Cd_g(y,z)^{\theta}(d_g(x,y)^{\alpha-\theta} + d_g(x,z)^{\alpha-\theta})\,.\end{aligned}$$

□

Let (M,g) be a compact Riemannian manifold with boundary ∂M, dimension $n \geq 3$, and positive Sobolev quotient $Q(M)$.

Notation. We denote by L_g the conformal Laplacian $\Delta_g - \frac{n-2}{4(n-1)}R_g$, and by B_g the boundary conformal operator $\frac{\partial}{\partial\eta_g} - \frac{n-2}{2(n-1)}H_g$, where η_g is the inward unit normal vector to ∂M.

Set $d(x) = d_g(x,\partial M)$ for $x \in M$, and $M_\rho = \{x \in M\,;\, d(x) < \rho\}$ for $\rho > 0$. Choose $\tilde\rho_0 = \tilde\rho_0(M,g) > 0$ small such that the function

$$\begin{aligned}M_{2\tilde\rho_0} &\to \partial M \\ x &\mapsto \bar{x}\end{aligned}$$

is well defined and smooth, where $\bar{x}$ is defined by $d_g(x,\bar{x}) = d_g(x,\partial M)$, and $\tilde\rho_0/4$ is smaller than the injectivity radius of M. Then, for any $0 < t < 2\tilde\rho_0$, the set $\partial_t M = \{x \in M\,;\, d(x) = t\}$ is a smooth embedded $(n-1)$-submanifold of M. For each $x \in M_{\tilde\rho_0}$, define the function

$$\begin{aligned}M_{2\tilde\rho_0} &\to \partial_{d(x)}M \\ y &\mapsto y_x\,,\end{aligned}$$

where y_x is defined by $d_g(y, y_x) = d_g(y, \partial_{d(x)}M)$.

For any $x \in M_{\rho_0}$ and $\rho_0 \in (0, \tilde{\rho}_0)$, we define the local coordinates $\psi_x(y) = (y_1, ..., y_n)$ on $M_{2\rho_0}$, where $y_n = d(y)$, and $(y_1, ..., y_{n-1})$ are normal coordinates of y_x, centered at x, with respect to the submanifold $\partial_{d(x)}M$. Then $(x, y) \mapsto \psi_x(y)$ is locally defined and smooth. Observe that $\psi_x(x) = (0, ..., 0, d(x))$ for any $x \in M_{\tilde{\rho}_0}$, and that ψ_x are Fermi coordinates for any $x \in \partial M$. Moreover, in those coordinates we have $g_{an} \equiv \delta_{an}$ and $g_{ab}(x) = \delta_{ab}$, for $a, b = 1, ..., n$, and the inward normal unit vector to ∂M is $d\psi_x^{-1}(\partial/\partial y_n)$. Choosing $\tilde{\rho}_0$ possibly smaller, we can assume that, for any $x \in M_{\tilde{\rho}_0}$, $\psi_x(y) = (y_1, ..., y_n)$ is defined for $0 \leq y_n < 2\tilde{\rho}_0$ and $|(y_1, ..., y_{n-1})| < \tilde{\rho}_0$.

Proposition 3.B.2. *Let $\rho_0 \in (0, \tilde{\rho}_0)$, $x_0 \in M$ and $d = \left[\frac{n-2}{2}\right]$. Suppose that one of the following conditions holds:*

(a) $x_0 \in \partial M$ and there exist $C = C(M, g)$ and N sufficiently large such that

$$H_g(y) \leq C d_g(x_0, y)^N \,, \quad \text{for all}\, y \in \partial M\,; \tag{B-91}$$

(b) $x_0 \in M_{\rho_0/2}$ and $H_g \equiv 0$ on ∂M;

(c) $x_0 \in M \backslash M_{2\rho_0}$.

Then there exists a positive $G_{x_0} \in C^\infty(M \backslash \{x_0\})$ satisfying

$$\begin{cases} L_g G_{x_0} = 0\,, & \text{in}\, M\backslash\{x_0\}\,, \\ B_g G_{x_0} = 0\,, & \text{on}\, \partial M\backslash\{x_0\}\,, \end{cases} \tag{B-92}$$

$$(n-2)\sigma_{n-1}\phi(x_0) = -\int_M G_{x_0}(y) L_g \phi(y) dv_g(y) - \int_{\partial M} G_{x_0}(y) B_g \phi(y) d\sigma_g(y) \tag{B-93}$$

for any $\phi \in C^2(M)$. Moreover, the following properties hold:

(P1) There exists $C = C(M, g)$ such that, for any $y \in M$ with $y \neq x_0$,

$$|G_{x_0}(y)| \leq C d_g(x_0, y)^{2-n} \quad \text{and} \quad |\nabla_g G_{x_0}(y)| \leq C d_g(x_0, y)^{1-n}\,.$$

(P2) If $x_0 \in \partial M$ consider Fermi coordinates $y = (y_1, ..., y_n)$ centered at that point. In those coordinates, write $g_{ab} = \exp(h_{ab})$, $a, b = 1, ..., n$, where

$$\left| h_{ab}(y) - \sum_{|\alpha|=1}^{d} h_{ab,\alpha} y^\alpha \right| \leq C(M, g) |y|^{d+1}, \tag{B-94}$$

where $h_{ab,\alpha} \in \mathbb{R}$ *and each* α *stands for a multi-index. Then there exists* $C = C(M, g, \rho_0)$ *such that* [6]

$$\left|G_{x_0}(y) - |y|^{2-n}\right| \leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |h_{ab,\alpha}| d_g(x_0, y)^{|\alpha|+2-n} + \begin{cases} C d_g(x_0, y)^{d+3-n} & \text{if } n \geq 5, \\ C(1 + |\log d_g(x_0, y)|) & \text{if } n = 3, 4, \end{cases} \tag{B-95}$$

$$\left|\nabla_g (G_{x_0}(y) - |y|^{2-n})\right| \leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^{d} |h_{ab,\alpha}| d_g(x_0, y)^{|\alpha|+1-n} + C d_g(x_0, y)^{d+2-n}.$$

(P3) If $x_0 \in M_{\rho_0/2}$ *consider the coordinate system* ψ_{x_0} *defined above. Then there exists* $C = C(M, g, \rho_0)$ *such that*

$$\left|G_{x_0}(y) - |(y_1, ..., y_{n-1}, y_n - d(x))|^{2-n} - |(y_1, ..., y_{n-1}, y_n + d(x))|^{2-n}\right| \leq C d_g(x_0, y)^{3-n},$$

$$\left|\nabla_g \big(G_{x_0}(y) - |(y_1, ..., y_{n-1}, y_n - d(x)))|^{2-n} - |(y_1, ..., y_{n-1}, y_n + d(x))|^{2-n}\big)\right| \leq C d_g(x_0, y)^{2-n},$$

if $n \geq 4$ *and*

$$\left|G_{x_0}(y) - |(y_1, ..., y_{n-1}, y_n - d(x))|^{2-n} - |(y_1, ..., y_{n-1}, y_n + d(x))|^{2-n}\right| \leq C(1 + |\log d_g(x_0, y)|)\,,$$

$$\left|\nabla_g \big(G_{x_0}(y) - |(y_1, ..., y_{n-1}, y_n - d(x))|^{2-n} - |(y_1, ..., y_{n-1}, y_n + d(x))|^{2-n}\big)\right| \leq C d_g(x_0, y)^{-1}\,,$$

if $n = 3$.

(P4) If $x_0 \in M \backslash M_{2\rho_0}$ *consider normal coordinates* $y = (y_1, ..., y_n)$ *centered at that point. As in (P2), write* $g_{ab} = \exp(h_{ab})$ *where* h_{ab} *satisfies (B-94). Then there exists* $C = C(M, g, \rho_0)$ *such that the estimates (B-95) hold. (Observe that in this case the sums range from* $|\alpha| = 2$ *to* d *instead of from* $\alpha = 1$ *to* d*.)*

Proof. Let $\chi : \mathbb{R}_+ \to [0, 1]$ be a smooth cutoff function satisfying $\chi(t) = 1$ for $t < \rho_0/2$, and $\chi(t) = 0$ for $t \geq \rho_0$. For each $x \in M_{\rho_0}$, set

$$K_1(x, y) = \chi(y_n/2)\chi(|(y_1, ..., y_{n-1})|) \cdot \left\{|(y_1, ..., y_{n-1}, y_n - d(x))|^{2-n} + |(y_1, ..., y_{n-1}, y_n + d(x))|^{2-n}\right\}\,,$$

[6] *The* log *term in dimensions* 3 *and* 4 *should also be included in [2, Proposition B-1]. However, that term does not affect the results in that paper as observed in the footnote in Proposition 3.2.14 above.*

where we are using the coordinates $\psi_x(y) = (y_1, ..., y_n)$. Observe that

$$\sum_{a=1}^{n} \frac{\partial^2}{\partial y_a^2} K_1(x,y) = 0\,, \quad \text{for } |(y_1, ..., y_{n-1})| < \rho_0/2\,,\ 0 \le y_n < \rho_0\,, \text{ and } x \neq y\,.$$

Moreover, $\partial K_1/\partial y_n(x,y) = 0$ if $y \in \partial M$ with $x \neq y$.

For each $x \in M\backslash M_{\rho_0/2}$, set

$$K_2(x,y) = \chi(4d_g(y,x))d_g(y,x)^{2-n}\,, \quad \text{if } 0 < d_g(y,x) < \rho_0/4\,,$$

and 0 otherwise. If we express $y \mapsto K_2(x,y)$ in normal coordinates $(y_1, ..., y_n)$ centered at x, we have $K_2(x,y) = \chi(4|(y_1, ..., y_n)|)|(y_1, ..., y_n)|^{2-n}$, and thus

$$\sum_{a=1}^{n} \frac{\partial^2}{\partial y_a^2} K_2(x,y) = 0\,, \quad \text{for } 0 < d_g(y,x) < \rho_0/8\,.$$

Define $K : M \times M\backslash D_M \to \mathbb{R}$ by the expression

$$K(x,y) = \chi(d(x))K_1(x,y) + (1 - \chi(d(x)))K_2(x,y)\,,$$

where $D_M = \{(x,x) \in M \times M\,;\, x \in M\}$. Thus, $K(x,y) = K_1(x,y)$ if $x \in M_{\rho_0/2}$, and $K(x,y) = K_2(x,y)$ if $x \in M\backslash M_{\rho_0}$. Observe that $\partial K/\partial \eta_{g,y}(x,y) = 0$ if $y \in \partial M$ with $y \neq x$.

Expressing $y \mapsto K_1(x,y)$ and $y \mapsto K_2(x,y)$ in their respective coordinate systems (as described above) one can check that there exists $C = C(M, g, \rho_0)$ such that

$$|L_{g,y}K(x,y)| \le C d_g(x,y)^{1-n}\,.$$

For any $\phi \in C^2(M)$ and $x \in M$, we have

$$(n-2)\sigma_{n-1}\phi(x) \tag{B-96}$$
$$= \int_M \Big(\Delta_{g,y}K(x,y)\phi(y) - K(x,y)\Delta_g\phi(y)\Big)dv_g(y) - \int_{\partial M} K(x,y)\frac{\partial}{\partial \eta_g}\phi(y)d\sigma_g(y)\,. \tag{B-97}$$

Indeed, this expression holds with $K_1(x,y)$ replacing $K(x,y)$ when $x \in M_{\rho_0}$, and with $K_2(x,y)$ replacing $K(x,y)$ when $x \in M\backslash M_{\rho_0/2}$. In particular, $\Delta_{distr,y}K(x,y) = \Delta_{g,y}K(x,y) - (n-2)\sigma_{n-1}\delta_x$.

We define $\Gamma_k : M \times M\backslash D_M \to \mathbb{R}$ inductively by setting

$$\Gamma_1(x,y) = L_{g,y}K(x,y)$$

and

$$\Gamma_{k+1}(x,y) = \int_M \Gamma_k(x,z)\Gamma_1(z,y)dv_g(z)\,.$$

According to [8, Proposition 4.12], which is a result due to Giraud ([46, p.50]), we have

$$|\Gamma_k(x,y)| \leq \begin{cases} Cd_g(x,y)^{k-n}\,, & \text{if } k<n\,,\\ C(1+|\log d_g(x,y)|)\,, & \text{if } k=n\,,\\ C\,, & \text{if } k>n\,, \end{cases} \tag{B-98}$$

for some $C=C(M,g,\rho_0)$. Moreover, Γ_k is continuous on $M\times M$ for $k>n$, and on $M\times M\backslash D_M$ for $k\leq n$.

If (a) or (b) holds we can refine the estimate (B-98) around the point x_0, using the expansion $g_{ab}=\exp(h_{ab})$. Since $K(x,y)=K_1(x,y)$ for $x\in M_{\rho_0/2}$ and $K(x,y)=K_2(x,y)$ for $x\in M\backslash M_{\rho_0}$, one can see that

$$|L_{g,y}K(x_0,y)| \leq C\sum_{a,b=1}^{n}\sum_{|\alpha|=1}^{d}|h_{ab,\alpha}|d_g(x_0,y)^{|\alpha|-n}+Cd_g(x_0,y)^{d+1-n},$$

for some $C=C(M,g,\rho_0)$, if (a) or (b) holds. Then Giraud's result implies

$$|\Gamma_k(x_0,y)| \leq C\sum_{a,b=1}^{n}\sum_{|\alpha|=1}^{d}|h_{ab,\alpha}|d_g(x_0,y)^{k-1+|\alpha|-n}+d_g(x_0,y)^{k+d-n}\,,\text{if} k<n-d\,. \tag{B-99}$$

Claim 1. Given $0<\theta<1$, there exists $C=C(M,g,\rho_0,\theta)$ such that

$$|\Gamma_{n+1}(x,y)-\Gamma_{n+1}(x,y')| \leq Cd_g(y,y')^{\theta}\,, \quad \text{for any } y\neq x\neq y'\,. \tag{B-100}$$

In particular, $\Gamma_{n+1}(x_0,\cdot)\in C^{0,\theta}(M)$.

Indeed, observe that $|\Gamma_1(x,y)-\Gamma_1(x,y')| \leq Cd_g(y,y')^{\theta}(d_g(x,y)^{1-\theta-n}+d_g(x,y')^{1-\theta-n})$, according to Lemma 3.B.1. So, Claim 1 follows from the estimates (B-98) and Giraud's result.

Set

$$F_k(x,y) = K(x,y)+\sum_{j=1}^{k}\int_M \Gamma_j(x,z)K(z,y)dv_g(z)\,.$$

Claim 2. For any $\phi\in C^2(M)$ and $x\in M$, and for all $k=1,2,...$, we have

$$\begin{aligned}\phi(x) = &-\int_M F_k(x,y)L_g\phi(y)dv_g(y)-\int_{\partial M}F_k(x,y)B_g\phi(y)d\sigma_g(y)\\ &+\int_M \Gamma_{k+1}(x,y)\phi(y)dv_g(y)-\int_{\partial M}\frac{n-2}{2(n-1)}H_g(y)F_k(x,y)\phi(y)d\sigma_g(y)\,.\end{aligned} \tag{B-101}$$

Claim 2 can be proved by induction on k.

Claim 3. For any $x \in M$ and $0 < \theta < 1$, the function $y \mapsto F_n(x, y)$ is in $C^{1,\theta}(M\backslash\{x\})$ and satisfies

$$|F_n(x,y)| \le C d_g(x,y)^{2-n}\,, \quad |\nabla_{g,y} F_n(x,y)|_g \le C d_g(x,y)^{1-n}\,, \tag{B-102}$$

and

$$\frac{|\nabla_{g,y} F_n(x,y) - \nabla_{g,y'} F_n(x,y')|_g}{d_g(y,y')^\theta} \le C d_g(x,y)^{1-\theta-n} + C d_g(x,y')^{1-\theta-n}\,, \tag{B-103}$$

for some $C = C(M, g, \rho_0)$. In particular, for any $x \in \partial M$, $y \mapsto \partial F_n/\partial\eta_{g,y}(x,y)$ defines a continuous function on $\partial M\backslash\{x\}$.

As a consequence of Claim 3, if $x_0 \in \partial M$ we can choose N large enough in the hypothesis (a) such that $y \mapsto H_g(y) F_n(x_0, y)$ is in $C^{1,\theta}(\partial M)$ for $0 < \theta < 1$ and satisfies

$$\|H_g(\cdot) F_n(x_0, \cdot)\|_{C^{1,\theta}(\partial M)} \le C(M, g, \rho_0, \theta)\,. \tag{B-104}$$

It is clear that (B-104) also holds if $x_0 \in M\backslash M_{\rho_0}$ with no assumptions on H_g, and that its left side vanishes under the hypothesis (b). In particular (B-104) holds should (a), (b) or (c) holds.

Let us prove Claim 3. Choose $y \neq x$ and a smooth curve y_t such that $y_0 = y$. Then, for any $r > 0$,

$$\frac{d}{dt}\int_{M\backslash B_r(y)} \Gamma_j(x,z) K(z, y_t) dv_g(z) = \int_{M\backslash B_r(y)} \Gamma_j(x,z) \frac{d}{dt} K(z, y_t) dv_g(z)$$

For any $r > 0$ such that $2r < d_g(x, y)$ and t small, we have

$$\begin{aligned} &\int_{B_r(y)} \Gamma_j(x,z) \Big| \frac{K(z,y_t) - K(z,y)}{t} \Big| dv_g(z) \\ &\qquad \le C \int_{B_r(y)} d_g(x,z)^{1-n} (d_g(z,y_t)^{1-n} + d_g(z,y)^{1-n}) dv_g(z) \\ &\qquad \le C 2^{n-1} d_g(x,y)^{1-n} \int_{B_r(y)} (d_g(z,y_t)^{1-n} + d_g(z,y)^{1-n}) dv_g(z) \end{aligned} \tag{B-105}$$

and the right-hand side goes to 0 as $r \to 0$. Here, $B_r(y)$ stands for the geodesic ball centered at y. Hence,

$$\frac{d}{dt}\int_M \Gamma_j(x,z) K(z, y_t) dv_g(z) = \int_M \Gamma_j(x,z) \frac{d}{dt} K(z, y_t) dv_g(z) \tag{B-106}$$

and the estimates in (B-102) follow from Giraud's result.

Now,

$$\begin{aligned}
&\frac{1}{d_g(y,y')^\theta}\Big|\int_M \Gamma_j(x,z)\frac{\partial}{\partial y_i}K(z,y)dv_g(z)-\int_M \Gamma_j(x,z)\frac{\partial}{\partial y_i}K(z,y')dv_g(z)\Big|\\
&\qquad\leq \int_M \Gamma_j(x,z)\Big|\frac{\frac{\partial}{\partial y_i}K(z,y)-\frac{\partial}{\partial y_i}K(z,y')}{d_g(y,y')^\theta}\Big|dv_g(z)\\
&\qquad\leq C\int_M d_g(x,z)^{1-n}(d_g(z,y)^{1-\theta-n}+d_g(z,y')^{1-\theta-n})dv_g(z)\\
&\qquad\leq C(d_g(x,y)^{2-\theta-n}+d_g(x,y')^{2-\theta-n})\,,
\end{aligned}$$

where we used Lemma 3.B.1 in the second inequality, and Giraud's result in the last one.

This proves Claim 3.

Using the hypothesis $Q(M)>0$, we define $u_{x_0}\in C^{2,\theta}(M)$ as the unique solution of

$$\begin{cases} L_g u_{x_0}(y)=-\Gamma_{n+1}(x_0,y)\,, & \text{in } M\,,\\ B_g u_{x_0}(y)=\frac{n-2}{2(n-1)}H_g(y)F_n(x_0,y)\,, & \text{on } \partial M\,. \end{cases} \tag{B-107}$$

It satisfies

$$\|u_{x_0}\|_{C^{2,\theta}(M)}\leq C\|u_{x_0}\|_{C^0(M)}+C\|\Gamma_{n+1}(x_0,\cdot)\|_{C^{0,\theta}(M)}+C\|H_g(\cdot)F_n(x_0,\cdot)\|_{C^{1,\theta}(\partial M)} \tag{B-108}$$

where $C=C(M,g,\rho_0,\theta)$ (see [45, Theorems 6.30 and 6.31].

Claim 4. There exists $C=C(M,g,\rho_0,\theta)$ such that $\|u_{x_0}\|_{C^{2,\theta}(M)}\leq C$.

Indeed, using (B-101) with $k=n$ and any $\phi\in C^2(M)$, one can see that

$$\sup_M|\phi|\leq C\sup_M|L_g\phi|+C\sup_{\partial M}|B_g\phi|+C\|\phi\|_{L^2(M)}+C\|\phi\|_{L^2(\partial M)}\,.$$

Since $Q(M)>0$, there exists $C=C(M,g)$ such that

$$\int_M\phi^2dv_g+\int_{\partial M}\phi^2d\sigma_g\leq C\int_M|L_g(\phi)\phi|dv_g+C\int_{\partial M}|B_g(\phi)\phi|d\sigma_g\,.$$

Thus, the Young's inequality implies

$$\int_M\phi^2dv_g+\int_{\partial M}\phi^2d\sigma_g\leq C\int_M L_g(\phi)^2dv_g+C\int_{\partial M}B_g(\phi)^2d\sigma_g\,.$$

Hence, $\|\phi\|_{C^0(M)} \le C\|L_g\phi\|_{C^0(M)} + C\|B_g\phi\|_{C^0(\partial M)}$. Setting $\phi = u_{x_0}$ and using the equations (B-107), we see that

$$\|u_{x_0}\|_{C^0(M)} \le C\|\Gamma_{n+1}(x_0,\cdot)\|_{C^0(M)} + C\|H_g(\cdot)F_n(x_0,\cdot)\|_{C^0(\partial M)}\,. \tag{B-109}$$

Claim 4 follows from the estimates (B-98), (B-100), (B-104), (B-108), and (B-109).

We define the function $G_{x_0} \in C^{1,\theta}(M\backslash\{x_0\})$ by

$$G_{x_0}(y) = K(x_0,y) + \sum_{k=1}^{n}\int_M \Gamma_i(x_0,z)K(z,y)dv_g(z) + u_{x_0}(y)\,.$$

One can check that the formula (B-93) holds.

Claim 5. We have $G_{x_0} \in C^\infty(M\backslash\{x_0\})$ and (B-92).

In order to prove Claim 5, we rewrite (B-96) as

$$\begin{aligned}\int_M K(x,y)L_g\phi(y)dv_g(y) &+ \int_{\partial M} K(x,y)B_g\phi(y)d\sigma_g(y) \\ &= \int_M L_{g,y}K(x,y)\phi(y)dv_g(y) - \phi(x) - \int_{\partial M}\frac{n-2}{2(n-1)}H_g(y)K(x,y)\phi(y)d\sigma_g(y)\,.\end{aligned} \tag{B-110}$$

Thus,

$$\begin{aligned}&\int_M\left\{\int_M \Gamma_j(x,z)K(z,y)dv_g(z)\right\}L_g\phi(y)dv_g(y) + \int_{\partial M}\left\{\int_M \Gamma_j(x,z)K(z,y)dv_g(z)\right\}B_g\phi(y)d\sigma_g(y)\\
&= \int_M \Gamma_j(x,z)\left\{\int_M K(z,y)L_g\phi(y)dv_g(y) + \int_{\partial M}K(z,y)B_g\phi(y)d\sigma_g(y)\right\}dv_g(z)\\
&= \int_M \Gamma_j(x,z)\int_M L_{g,y}K(z,y)\phi(y)dv_g(y)dv_g(z)\\
&\qquad - \int_M \Gamma_j(x,z)\left\{\int_{\partial M}\frac{n-2}{2(n-1)}H_g(y)K(z,y)\phi(y)d\sigma_g(y) + \phi(z)\right\}dv_g(z)\\
&= \int_M\left\{\int_M \Gamma_j(x,z)L_{g,y}K(z,y)dv_g(z) - \Gamma_j(x,y)\right\}\phi(y)dv_g(y)\\
&\qquad - \int_{\partial M}\left\{\int_M \Gamma_j(x,z)K(z,y)dv_g(z)\right\}\frac{n-2}{2(n-1)}H_g(y)\phi(y)d\sigma_g(y)\,,\end{aligned}$$

where we used (B-110) in the second equality. Hence, we proved that the equations

$$\begin{cases} L_{g,y}\int_M \Gamma_j(x,z)K(z,y)dv_g(z) = \Gamma_{j+1}(x,y) - \Gamma_j(x,y)\,, & \text{in } M\,,\\ B_{g,y}\int_M \Gamma_j(x,z)K(z,y)dv_g(z) = -\frac{n-2}{2(n-1)}H_g(y)\int_M \Gamma_j(x,z)K(z,y)dv_g(z)\,, & \text{on } \partial M\,,\end{cases}$$

hold in the sense of distributions. Then it is easy to check that the equations (B-92) hold in the sense of distributions. Since $G_{x_0} \in C^{1,\theta}(M\backslash\{x_0\})$, elliptic regularity arguments imply that $G_{x_0} \in C^\infty(M\backslash\{x_0\})$. This proves Claim 5.

The property (P1) follows from (B-102) and Claim 4. In order to prove (P2),(P3) and (P4), we use (B-98), (B-99), (B-106) and Claim 4.

Claim 6. The function G_{x_0} is positive on $M\backslash\{x_0\}$.

Let us prove Claim 6. Let

$$G_{x_0}^{-}=\begin{cases}-G_{x_0}\,, & \text{if}\, G_{x_0}<0\,,\\ 0\,, & \text{if}\, G_{x_0}\geq 0\,.\end{cases}$$

Since $G_{x_0}^{-}$ has support in $M\backslash\{x_0\}$, one has

$$\begin{aligned}0&=-\int_M G_{x_0}^{-}L_gG_{x_0}dv_g-\int_{\partial M}G_{x_0}^{-}B_gG_{x_0}d\sigma_g\\ &=\int_M\left(|\nabla_gG_{x_0}^{-}|_g^2+\frac{n-2}{4(n-1)}R_g(G_{x_0}^{-})^2\right)dv_g+\int_{\partial M}\frac{n-2}{2(n-1)}H_g(G_{x_0}^{-})^2d\sigma_g\,.\end{aligned}$$

By the hypothesis $Q(M)>0$, we have $G_{x_0}^{-}\equiv 0$ which implies $G_{x_0}\geq 0$.

We now change the metric by a conformal positive factor $u\in C^{\infty}(M)$ such that $\tilde{g}=u^{\frac{4}{n-2}}g$ satisfies $R_{\tilde{g}}>0$ in M and $H_{\tilde{g}}\equiv 0$ on ∂M (see [40]). Observing the conformal properties (3.2) and (3.3), we see that $\tilde{G}=u^{-1}G_{x_0}\geq 0$ satisfies $L_{\tilde{g}}\tilde{G}=0$ in $M\backslash\{x_0\}$ and $B_{\tilde{g}}\tilde{G}=0$ on $\partial M\backslash\{x_0\}$. Then the strong maximum principle implies $\tilde{G}>0$, proving Claim 6.

This finishes the proof of Proposition 3.B.2. □

Let (M,g_0) be a Riemannian manifold with $Q(M)>0$ and $H_{g_0}\equiv 0$. Let $g_{x_0}=f_{x_0}^{\frac{4}{n-2}}g_0$ be a conformal metric satisfying

$$|f_{x_0}(x)-1|\leq C(M,g_0)d_{g_0}(x,x_0).$$

Notation. For a Riemannian metric g we set $M_{t,g}=\{x\in M:d_g(x,\partial M)<t\}$ and $\partial_{t,g}M=\{x\in M:d_g(x,\partial M)=t\}$.

Proposition 3.B.3. *If ρ_0 is sufficiently small and $x_0\in M_{\rho_0,g_{x_0}}\backslash\partial M$, then there exists a positive $G_{x_0}\in C^{\infty}(M\backslash\{x_0\})$ satisfying*

$$\begin{cases}L_{g_{x_0}}G_{x_0}=0, & \textit{in}\, M\backslash\{x_0\},\\ B_{g_{x_0}}G_{x_0}=0, & \textit{on}\,\partial M,\end{cases}\tag{B-111}$$

and there exists $C = C(M, g_0, \rho_0)$ such that

$$|G_{x_0}(y) - |\phi_0(y)|^{2-n}| \leq \begin{cases} C|\phi_0(y)|^{3-n} + Cd_{g_{x_0}}(x_0, \partial M)|\phi_0(y)|^{1-n} & n \geq 4, \\ C(1 + |\log(|\phi_0(y)|)|) + Cd_{g_{x_0}}(x_0, \partial M)|\phi_0(y)|^{1-n} & n = 3, \end{cases} \tag{B-112}$$

$$|\nabla_{g_{x_0}}(G_{x_0}(y) - |\phi_0(y)|^{2-n})| \leq C|\phi_0(y)|^{1-n} + Cd_{g_{x_0}}(x_0, \partial M)|\phi_0(y)|^{-n}, \tag{B-113}$$

where $\phi_0(y) = (y_1, ..., y_n)$ are g_{x_0}- normal coordinates centered at x_0.

Proof. We will use the notation $d(x) = d_{g_0}(x, \partial M)$. Let us define the coordinate system $\psi_0(y) = (y_1, ..., y_n)$ on M_{ρ_0, g_0} where $(y_1, \cdots, y_{n-1})$ are normal coordinates of y_{x_0} on $\partial_{d(x_0), g_0}M$ centered at y_{x_0}, with respect to the metric induced by g_0, and $y_n = d(x) - d(x_0)$. Here, $y_{x_0} \in \partial_{d(x_0), g_0}M$ is such that $d_{g_0}(y, y_{x_0}) = d_{g_0}(y, \partial_{d(x_0), g_0}M)$. This differs from ψ_{x_0} defined above by a translation in the last coordinate.

According to Proposition 3.B.2 one can construct a function G_0, satisfying

$$\begin{cases} L_{g_0}G_0 = 0\,, & \text{in } M \backslash \{x_0\}\,, \\ B_{g_0}G_0 = 0\,, & \text{on } \partial M\,, \end{cases}$$

$$\Big|G_0(y) - \frac{1}{2}\big(|(y_1, ..., y_n)|^{2-n} + |(y_1, ..., y_{n-1}, y_n + 2d(x_0))|^{2-n}\big)\Big|$$
$$\leq \begin{cases} Cd_{g_0}(y, x_0)^{3-n} & n \geq 4, \\ C(1 + |\log d_{g_0}(y, x_0)|) & n = 3, \end{cases}$$

and

$$\Big|\nabla_{g_0}\big(G_0(y) - \frac{1}{2}\big(|(y_1, ..., y_n)|^{2-n} + |(y_1, ..., y_{n-1}, y_n + 2d(x_0))|^{2-n}\big)\big)\Big| \leq Cd_{g_0}(y, x_0)^{2-n}.$$

for some $C = C(M, g_0, \rho_0)$. Using $|(y_1, ..., y_{n-1}, y_n + 2d(x_0))| \geq |(y_1, ..., y_n)|$ and Lemma 3.B.1 we have

$$\big||(y_1, ..., y_n)|^{2-n} - |(y_1, ..., y_{n-1}, y_n + 2d(x_0))|^{2-n}\big| \leq Cd(x_0)|(y_1, ..., y_n)|^{1-n},$$
$$\big|\nabla|(y_1, ..., y_n)|^{2-n} - \nabla|(y_1, ..., y_{n-1}, y_n + 2d(x_0))|^{2-n}\big| \leq Cd(x_0)|(y_1, ..., y_n)|^{-n}.$$

Then

$$|G_0(y) - |\psi_0(y)|^{2-n}| \le \begin{cases} Cd_{g_0}(y,x_0)^{3-n} + Cd(x_0)d_{g_0}(y,x_0)^{1-n} & n \ge 4, \\ C(1 + |\log d_{g_0}(y,x_0)|) + Cd(x_0)d_{g_0}(y,x_0)^{1-n} & n = 3, \end{cases} \tag{B-114}$$

$$|\nabla_{g_0}(G_0(y) - |\psi_0(y)|^{2-n})| \le Cd_{g_0}(y,x_0)^{2-n} + Cd(x_0)d_{g_0}(y,x_0)^{-n}. \tag{B-115}$$

Now we change this to the conformal metric g_{x_0}. Let $\phi_0(y) = (y_1, ..., y_n)$ be g_{x_0}-conformal normal coordinates centered at x_0. By the definition of ϕ_0 and ψ_0 one can check that $\xi = \phi_0 \circ \psi_0^{-1}$ satisfies $\xi(0) = 0$ and $d\xi(0) = id_{\mathbb{R}^n}$. Since M is compact, one can find $C = C(M, g_0)$ uniform in x_0 such that

$$|\xi(y_1, ..., y_n) - (y_1, ..., y_n)| \le C|(y_1, ..., y_n)|^2. \tag{B-116}$$

The function $G_{x_0} = f_{x_0}^{-1} G_0$ satisfies (B-111), so we shall prove (B-112) and (B-113). Observe that

$$|G_{x_0}(y) - G_0(y)| \le Cd_{g_0}(y,x_0)|G_{x_0}(y)| \le Cd_{g_0}(y,x_0)^{3-n}. \tag{B-117}$$

Combining (B-114), (B-116) and (B-117), one gets (B-112) from the following steps:

$$\begin{aligned} &|G_{x_0}(y) - |\phi_0(y)|^{2-n}| \\ \le&|G_{x_0}(y) - G_0(y)| + |G_0(y) - |\psi_0(y)|^{2-n}| + ||\psi_0(y)|^{2-n} - |\xi \circ \psi_0(y)|^{2-n}| \\ \le&Cd_{g_0}(y,x_0)^{3-n} + Cd(x_0)d_{g_0}(y,x_0)^{1-n} + C|\psi_0(y)|^{3-n} \\ \le&Cd_{g_0}(y,x_0)^{3-n} + Cd_{g_{x_0}}(x_0, \partial M)(x_0)d_{g_0}(y,x_0)^{1-n} \end{aligned}$$

for $n \ge 4$ and with obvious modifications for $n = 3$. Similarly, using (B-115), (B-116) and (B-117), one gets (B-113). $\square$

Chapter 4

Classification theorems for solutions of higher order boundary conformally invariant problems

For all the work in this chapter, we will use the following notations as in [80].

Notations:

X	$(x,t)=(x^1,\dots,x^n,t)\subset\mathbb{R}^{n+1}$
$B_r(X)$	ball with radial r centered at X in $\mathbb{R}^{n+1}$ and $B_r=B_r(0)$
B_r^+	$B_r\cap\mathbb{R}^{n+1}_+$
$\partial^+B_r^+$	$\partial B_r^+\cap\mathbb{R}^{n+1}_+$
D_r	ball centered at the origin in $\mathbb{R}^n$, identifying $D_r=\partial B_r^+\backslash\overline{\partial^+B_r^+}$
$[f]_r$	$⨍_{\partial D_r} f\mathrm{d}\sigma$, the integral average of f over ∂D_r
χ_A	the characteristic function of the measurable set A in the Euclidean spaces

We will always assume $2m<n+1$ if it is not specified. We will use the Green identity and its variants repeatedly:

$$\begin{aligned}\int_{B_1^+}(u\Delta^m\phi-\phi\Delta^m u)\,dX=&\sum_{i=1}^m\int_{\partial^+B_1^+}\left[(\Delta^{i-1}u)\frac{\partial(\Delta^{m-i}\phi)}{\partial\nu}-(\Delta^{m-i}\phi)\frac{\partial(\Delta^{i-1}u)}{\partial\nu}\right]dS\\&-\sum_{i=1}^m\int_{D_1}\left[(\Delta^{i-1}u)\partial_t(\Delta^{m-i}\phi)-(\Delta^{m-i}\phi)\partial_t(\Delta^{i-1}u)\right]dx\end{aligned}$$

where ν is the outer unit normal of $\partial^+B_1^+$.

4.1 Preliminary

Let us recall that Δ^m is invariant under the m-Kelvin transformations

$$u_{X_0,\lambda}(X):=\left(\frac{\lambda}{|X-X_0|}\right)^{n-2m+1}u\left(X_0+\frac{\lambda^2(X-X_0)}{|X-X_0|^2}\right),$$

where $2m<n+1$, $X_0\in\mathbb{R}^{n+1}$ and $\lambda>0$. Namely, if $u\in C^{2m}(\mathbb{R}^{n+1})$ then there holds

$$\Delta^m u_{X_0,\lambda}(X)=\left(\frac{\lambda}{|X-X_0|}\right)^{n+2m-1}\Delta^m u\left(X_0+\frac{\lambda^2(X-X_0)}{|X-X_0|^2}\right)\quad\text{for }X\neq X_0.\quad(4.1)$$

There are various of boundary conditions for the polyharmonic equation, see Agmon-Douglis-Nirenberg [1] or Gazzola-Grunau-Sweers [43]. For the later use, we only consider two of them. One is like the Dirichlet condition and the other is a Neumann condition. We will be concerned with bounds of singular integrals involving the Poisson kernel and Neumann function, respectively. These bounds will play important roles in the proof of the main theorem.

4.1.1 Poisson kernel for a Dirichlet problem

Let us consider the boundary value problem

$$\begin{cases} \Delta^m v(x,t) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ v(x,0) = f(x) & \text{on } \partial\,\mathbb{R}^{n+1}_+, \\ \partial_t \Delta^k v(x,0) = 0 & \text{on } \partial\,\mathbb{R}^{n+1}_+, \end{cases} \tag{4.2}$$

where f is a smooth bounded function in $\mathbb{R}^n$, and $k = 0, \ldots, m-2$ (if $m = 1$, then we do not have this boundary condition). Let

$$\mathcal{P}_m(x,t) = \beta(n,m)\frac{t^{2m-1}}{(|x|^2+t^2)^{\frac{n+2m-1}{2}}},$$

where $\beta(n,m) = \pi^{-\frac{n}{2}}\Gamma(\frac{n+2m-1}{2})/\Gamma(m-\frac{1}{2})$ is the normalizing constant such that

$$\int_{\mathbb{R}^n} \mathcal{P}_m(x,1)\,\mathrm{d}x = 1.$$

Note that $\mathcal{P}_1$ is the standard upper space Poisson kernel for Laplace equation. Define

$$v(x,t) = \mathcal{P}_m * f(x,t) = \beta(n,m)\int_{\mathbb{R}^n}\frac{t^{2m-1}f(y)}{(|x-y|^2+t^2)^{\frac{n+2m-1}{2}}}\,\mathrm{d}y. \tag{4.3}$$

Lemma 4.1.1. If $f \in L^q(\mathbb{R}^n)$ for some $1 \le q \le \infty$, then v belongs to weak-$L^{\frac{n+1}{n}}(\mathbb{R}^{n+1}_+)$ if $q = 1$ and belongs to $L^{\frac{(n+1)q}{n}}(\mathbb{R}^{n+1}_+)$ if $q > 1$. Moreover,

$$|\{X : |\mathcal{P}_m * f(X)| > \lambda\}| \le c(n,m,1)\lambda^{-\frac{n+1}{n}}\|f\|^{\frac{n+1}{n}}_{L^1(\mathbb{R}^n)}, \quad \forall\, \lambda > 0,$$

and

$$||\mathcal{P}_m * f||_{L^{\frac{(n+1)q}{n}}(\mathbb{R}^{n+1}_+)} \le c(n,m,q)||f||_{L^q(\mathbb{R}^n)}, \quad \text{for } q > 1,$$

where $c(n,m,q) > 0$ is constants depending only n, m and q.

Proof. The proof by now is standard. When $m=1$, see Hang-Wang-Yan [51]. When $q=\infty$, it is easy to show. By the Marcinkiewicz interpolation theorem, it thus suffices to show the $q=1$ case. Without loss of generality, we may assume that $\|f\|_{L^1(\mathbb{R}^n)}=1$. First, note that for any $t>0$ there holds

$$|\mathcal{P}_m * f(x,t)| \leq \beta(n,m)t^{-n}.$$

In addition, for any number $a>0$,

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+\cap\{0<t<a\}} |\mathcal{P}_m * f(x,t)|\,\mathrm{d}x\mathrm{d}t\\ \leq&\int_{\mathbb{R}^n}|f(y)|\mathrm{d}y\int_0^a\int_{\mathbb{R}^n}\frac{\beta(n,m)t^{2m-1}}{(|x-y|^2+t^2)^{\frac{n+2m-1}{2}}}\,\mathrm{d}x\mathrm{d}t=a.\end{aligned}$$

It follows that for any $\lambda>0$

$$\begin{aligned}&|\{(x,t):|\mathcal{P}_m * f(x,t)|>\lambda\}|\\ =&\left|\{(x,t):0<t<\beta(n,m)^{\frac{1}{n}}\lambda^{-\frac{1}{n}},|\mathcal{P}_m * f(x,t)|>\lambda\}\right|\\ \leq&\frac{1}{\lambda}\int_{\mathbb{R}^{n+1}_+\cap\{0<t<\beta(n,m)^{\frac{1}{n}}\lambda^{-\frac{1}{n}}\}}|\mathcal{P}_m * f|\,\mathrm{d}x\mathrm{d}t\\ \leq&\beta(n,m)^{\frac{1}{n}}\lambda^{-\frac{n+1}{n}}.\end{aligned}$$

Therefore, we complete the proof. □

Lemma 4.1.2. Suppose that f is a smooth function in $L^q(\mathbb{R}^n)$ for some $q\geq 1$. Then v defined by (4.3) is smooth and satisfies (4.2).

Proof. The smoothness of $v(x,t)$ is easy and we omit the details. Note that $\mathcal{P}_m(x-y,t)$ is the Kelvin transform of $\beta(n,m)t^{2m-1}$ with respect to $X_0=(y,0)$ and $\lambda=1$. It follows that $\Delta^m_{x,t}\mathcal{P}_m(x-y,t)=\beta(n,m)|X-X_0|^{-(n+2m-1)}\Delta^m_{x,t}t^{2m-1}=0$ for any $x\in\mathbb{R}^n$ and $t>0$. Therefore, v satisfies the first equation of (4.2).

Next, let $\eta\geq 0$ be a cutoff function satisfying $\eta=1$ in $D_{1/2}$ and $\eta=0$ in $\mathbb{R}^n\setminus D_2$, and denote $\eta_{x_0}(x)=\eta(x-x_0)$ for any $x_0\in\mathbb{R}^n$. Let $v_1=\mathcal{P}_m*(f\eta_{x_0})$ and $v_2=\mathcal{P}_m*(f(1-\eta_{x_0}))$, then $v=v_1+v_2$. Clearly,

$$\lim_{(x,t)\to(x_0,0)} v_2(x,t)\to 0.$$

By the change of variables $x - y = tz$, we see that

$$v_1(x,t) = \beta(n,m) \int_{\mathbb{R}^n} \frac{(f\eta_{x_0})(x - tz)}{(|z|^2+1)^{\frac{n+2m-1}{2}}} \, \mathrm{d}z.$$

Sending $t \to 0$, by Lebesgue dominated convergence theorem we obtain

$$v_1(x,t) \to f(x_0) \quad \text{when } (x,t) \to (x_0, 0).$$

Hence, by the arbitrary choice of x_0, we verified the second line of (4.2).

Finally, for any $0 \le k \le m-2$, note that $\Delta^k t^{2m-1} = (2m-1)\cdots(2m-2k)t^{2m-1-2k}$ with $2m-1-2k \ge 2$. It follows that

$$\lim_{(x,t)\to(x_0,0)} \partial_t \Delta^k v_2(x,t) = 0.$$

Making use of $k \le m-2$ and Lebesgue dominated convergence theorem, we see that as $t \to 0$,

$$\begin{aligned}
&\partial_t \Delta^k v_1(x,t) \\
&= \beta(n,m) \int_{\mathbb{R}^n} \frac{\partial_t \sum_{j=0}^k C(j) \Delta_x^{k-j} \partial_t^{2j} (f\eta_{x_0})(x-tz)}{(|z|^2+1)^{\frac{n+2m-1}{2}}} \, \mathrm{d}z \\
&= \beta(n,m) \int_{\mathbb{R}^n} \frac{\sum_{j=0}^k C(j) \Delta_x^{k-j} \sum_{|\alpha|=2j+1} C(\alpha) D_x^\alpha (f\eta_{x_0})(x-tz) z_1^{\alpha_1} \cdots z_n^{\alpha_n}}{(|z|^2+1)^{\frac{n+2m-1}{2}}} \, \mathrm{d}z \\
&\to -\beta(n,m) \int_{\mathbb{R}^n} \frac{\sum_{j=0}^k C(j) \sum_{|\alpha|=2j+1} C(\alpha) \Delta_x^{k-j} D_x^\alpha (f\eta_{x_0})(x) z_1^{\alpha_1} \cdots z_n^{\alpha_n}}{(|z|^2+1)^{\frac{n+2m-1}{2}}} \, \mathrm{d}z = 0
\end{aligned}$$

where $C(j)$ and $C(\alpha)$ are some binomial constants and we used the oddness of the integrand in the last equality.

Therefore, we complete the proof. $\square$

Remark 4.1.3. If $f \in L^1(\mathbb{R}^n)$ is smooth in an open set $\Omega \in \mathbb{R}^n$ for instance. From the proof of Lemma 4.1.2, we see that v will satisfy boundary conditions of (4.2) on Ω pointwisely.

Next lemma shows the convolution with $\mathcal{P}_m$ commutes with $m-$Kelvin transformation.

Lemma 4.1.4. Suppose $f_{x_0,\lambda}(x) := |x|^{2m-1-n} f(x_0 + \frac{\lambda(x-x_0)}{|x-x_0|^2}) \in L^1(\mathbb{R}^n)$ for some $x_0 \in \mathbb{R}^n$ and $\lambda > 0$. Let $X_0 = (x_0, 0)$. Then $v_{X_0,\lambda} = \mathcal{P}_m * f_{x_0,\lambda}$.

Proof. We only verify the case $x_0 = 0$ and $\lambda = 1$, because the other situations are similar. Since $f_{0,1} \in L^1(\mathbb{R}^n)$, $\mathcal{P}_m * f_{0,1}$ is well-defined. By direct computations,

$$\begin{aligned}
v_{0,1}(X) &= |X|^{2m-1-n}\mathcal{P}_m * f\left(\frac{x}{|X|^2}, \frac{t}{|X|^2}\right) \\
&= \beta(n,m)|X|^{2m-1-n}\int_{\mathbb{R}^n} \frac{(t/|X|^2)^{2m-1}f(y)}{(|x/|X|^2-y|^2+(t/|X|^2)^2)^{\frac{n+2m-1}{2}}}\,\mathrm{d}y \\
&= \beta(n,m)|X|^{-2m+1-n}\int_{\mathbb{R}^n} \frac{t^{2m-1}f(y)}{(|x/|X|^2-y|^2+(t/|X|^2)^2)^{\frac{n+2m-1}{2}}}\,\mathrm{d}y \\
&= \beta(n,m)\int_{\mathbb{R}^n} \frac{t^{2m-1}|y|^{1-2m-n}f(y)}{(t^2+|y/|y|^2-x|^2)^{\frac{n+2m-1}{2}}}\,\mathrm{d}y \\
&= \beta(n,m)\int_{\mathbb{R}^n} \frac{t^{2m-1}|z|^{2m-1-n}f(\frac{z}{|z|^2})}{(t^2+|z-x|^2)^{\frac{n+2m-1}{2}}}\,\mathrm{d}z = \mathcal{P}_m * f_{0,1}(X),
\end{aligned}$$

where in the fourth step we used the elementary equality

$$|X|^2\left(\left(\frac{t}{|X|^2}\right)^2 + \left|\frac{x}{|X|^2}-y\right|^2\right) = |y|^2\left(t^2+\left|\frac{y}{|y|^2}-x\right|^2\right).$$

□

Remark 4.1.5. Actually the proof holds whenever $\mathcal{P}_m * f_{x_0,\lambda}$ is well defined, for example $f_{x_0,\lambda} \in L^1_{loc}(\mathbb{R}^n)$ and bounded at infinity.

Lemma 4.1.6. Let $v \in C^{2m}(\mathbb{R}^{n+1}_+ \cup \partial\mathbb{R}^{n+1}_+)$ be a solution of (4.2). Then for any $X_0 = (x_0, 0)$ and $\lambda > 0$, $v_{X_0,\lambda}$ satisfies (4.2) with f replaced by $f_{x_0,\lambda}$, except the the boundary point X_0.

Proof. It follows from direct computations. □

4.1.2 Neumann function for a Neumann problem

Now, we consider

$$\begin{cases}
\Delta^m v(x,t) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\
\partial_t \Delta^k v(x,0) = 0 & \text{on } \partial\,\mathbb{R}^{n+1}_+, \\
(-1)^m \partial_t \Delta^{m-1} v(x,0) = f(x) & \text{on } \partial\,\mathbb{R}^{n+1}_+,
\end{cases} \tag{4.4}$$

where f is a smooth function belonging to $L^q(\mathbb{R}^n)$ for some $q \geq 1$, and $k = 0, \ldots, m-2$.

Let

$$\mathcal{N}_m(x,t) = \gamma(n,m)\frac{1}{(|x|^2+t^2)^{\frac{n-2m+1}{2}}},$$

where $\gamma(n,m) = \pi^{\frac{n+1}{2}}\Gamma(\frac{n-2m+1}{2})/\Gamma(m)$. Define

$$v(x,t) := \mathcal{N}_m * f(x,t) = \gamma(n,m)\int_{\mathbb{R}^n} \frac{f(y)}{(t^2+|x-y|^2)^{\frac{n-2m+1}{2}}}\,\mathrm{d}y. \tag{4.5}$$

Lemma 4.1.7. If $f \in L^1(\mathbb{R}^n)$, then $v(x,t)$ belongs to weak$-L^{\frac{n+1}{n-2m+1}}(\mathbb{R}^{n+1}_+)$. Moreover,

$$|\{(x,t) : |v(x,t)| > \lambda\}| \le C(n,m)\lambda^{-\frac{n+1}{n-2m+1}}||f||_{L^1(\mathbb{R}^n)}^{\frac{n+1}{n-2m+1}} \quad \text{for every } \lambda > 0,$$

where $C(n,m) > 0$ is a constant depending only n and m.

Proof. The lemma was proved by Dou-Zhu [34] and we include a proof below for completeness and convenience of the readers.

After scaling, assume $\int_{\mathbb{R}^n} f(y)dy = 1$. Split v as

$$\begin{aligned} v(x,t) &= \gamma(n,m)\left(\int_{\mathbb{R}^n\cap\{|x-y|\le r\}} + \int_{\mathbb{R}^n\cap\{|x-y|>r\}}\right)\frac{f(y)}{(t^2+|x-y|^2)^{\frac{n-2m+1}{2}}}\,\mathrm{d}y \\ &=: v_1(x,t) + v_2(x,t), \end{aligned}$$

where r will be fixed later. By direct computations, we have

$$\begin{aligned} ||v_1||_{L^1(\mathbb{R}^{n+1}_+)} &= \gamma(n,m)\int_{\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^n\cap\{|x-y|\le r\}} \frac{|f(y)|}{(t^2+|x-y|^2)^{\frac{n-2m+1}{2}}}\mathrm{d}y\,\mathrm{d}X \\ &\le \gamma(n,m)\int_{\mathbb{R}^n}|f(y)|\,\mathrm{d}y\int_{\mathbb{R}^{n+1}_+\cap B_r}\frac{1}{|X|^{n-2m+1}}\,\mathrm{d}X \le C_1 r^{2m}, \end{aligned}$$

and

$$|v_2| \le C_2 r^{2m-n-1},$$

where C_1, C_2 are constants depending only n and m. Observing the inequality

$$|\{(x,t) : |v| \ge 2\lambda\}| \le |\{(x,t) : |v_1| \ge \lambda\}| + |\{(x,t) : |v_2| \ge \lambda\}|,$$

one can choose r as $C_2 r^{2m-n-1} = \lambda$, then $|\{(x,t) : |v_2| \ge \lambda\}| = 0$. Thus

$$\begin{aligned} |\{(x,t) : |v| \ge 2\lambda\}| &\le |\{(x,t) : |v_1| \ge \lambda\}| \le C\frac{1}{\lambda}||v_1||_{L^1(\mathbb{R}^{n+1}_+)} \\ &\le C\frac{r^{2m}}{\lambda} = C\lambda^{-\frac{n+1}{n-2m+1}}. \end{aligned}$$

By scaling, we complete the proof of the lemma. $\square$

We refer to Dou-Zhu [34] for strong type bounds for the convolution operator involving the Neumann function.

Lemma 4.1.8. Suppose that f is a smooth function belonging to $L^q(\mathbb{R}^n)$ for some $q \geq 1$. Then v defined by (4.5) is smooth and satisfies (4.4).

Proof. The smoothness and the first two lines of (4.4) are easy to show. For the last boundary condition, observe that

$$\Delta^k |X|^{2m-n-1} = (2m-n-1)\cdots(2m-n+1-2k)(2m-2)\cdots(2m-2k)|X|^{2m-n-1-2k}$$

for any $k \geq 1$. It follows that

$$\begin{aligned}\partial_t \Delta^{m-1} v(x,t) =& (2m-n-1)\cdots(1-n)(2m-2)\cdots 2\gamma(n,m)\int_{\mathbb{R}^n} \frac{tf(y)}{(|x-y|^2+t^2)^{\frac{n+1}{2}}}\mathrm{d}y \\ =& (-1)^m 2^{2m}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n-2m+1}{2})}\Gamma(m)\gamma(n,m)\int_{\mathbb{R}^n} \frac{tf(y)}{(|x-y|^2+t^2)^{\frac{n+1}{2}}}\mathrm{d}y\end{aligned}$$

therefore

$$\partial_t \Delta^{m-1} v(x,0) = (-1)^m f(x).$$

This verifies the last boundary condition.

Therefore, we complete the proof. □

Lemma 4.1.9. Suppose $f_{x_0,\lambda}(x) := |x|^{2m-1-n} f(x_0 + \frac{\lambda(x-x_0)}{|x-x_0|^2}) \in L^1(\mathbb{R}^n)$ for some $x_0 \in \mathbb{R}^n$ and $\lambda > 0$. Let $X_0 = (x_0, 0)$. Then $v_{X_0,\lambda} = \mathcal{N}_m * (|x|^{-2(2m-1)} f_{x_0,\lambda})$.

Proof. It is similar to the proof of Lemma 4.1.4, thus we omit the details. Same as the remark 4.1.5, the proof holds whenever $\mathcal{N}_m * (|x|^{-2(2m-1)} f_{x_0,\lambda})$ is well defined. □

Lemma 4.1.10. Let $v \in C^{2m}(\mathbb{R}^{n+1}_+ \cup \partial\mathbb{R}^{n+1}_+)$ be a solution of (4.4). Then for any $X_0 = (x_0, 0)$ and $\lambda > 0$, $v_{X_0,\lambda}$ satisfies (4.4) with $f(x)$ replaced by $|x|^{-2(2m-1)} f_{x_0,\lambda}(x)$, except the the boundary point X_0.

Proof. It follows from direct computations. □

4.2 Polyharmonic functions with homogeneous boundary data

4.2.1 Extensions of Liouville theorem

It is well-known that every nonnegative solution of

$$\begin{cases} \Delta u(x,t) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ u = 0 & \text{on } \partial\mathbb{R}^{n+1}_+, \end{cases}$$

has to equal at for some $a \geq 0$. A simple proof of this result is based on the boundary Harnack inequality. In this subsection, we extend this result to polyharmonic functions with homogeneous boundary conditions, for which we don't have a boundary Harnack inequality.

Proposition 4.2.1. Let $u \in C^{2m}(\mathbb{R}^{n+1}_+ \cup \partial\mathbb{R}^{n+1}_+)$ be a solution of

$$\begin{cases} \Delta^m u(x,t) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ u(x,0) = 0 & \text{on } \partial\,\mathbb{R}^{n+1}_+, \\ \partial_t \Delta^k u(x,0) = 0 & \text{on } \partial\,\mathbb{R}^{n+1}_+, \quad k = 0, 1, \cdots, m-2. \end{cases} \tag{4.6}$$

Suppose that $u^*(X) \in L^1(B_1^+)$, where $u^* := u_{0,1}$ is the $m-$Kelvin transform of u with respect to $X_0 = 0$ and $\lambda = 1$. Then

$$u(x,t) = \sum_{k=1}^{m-1} t^{2k} P_{2k}(x) + c_0 t^{2m-1}, \tag{4.7}$$

where $P_{2k}(x)$ are polynomials w. r. t. x of degree $\leq 2m-1-2k$.

In addition if we assume $u^*(X) \geq g(X)$ for some $g \in L^{\frac{n+1}{n}}(B_1^+)$, then $c_0 \geq 0$, and $\deg P_{2k} \leq 2m-2-2k$. In particular $P_{2(m-1)}$ must be a constant.

Proof. For any $r > 0$, let $v(X) = u^*(rX)$. Then $v(X)$ satisfies (4.6) pointwisely except the origin. By the standard estimates for solutions of linear elliptic PDEs, we have

$$\|v\|_{L^\infty(B^+_{5/4} \backslash B^+_{3/4})} \leq C(m,n) \|v\|_{L^1(B^+_{3/2} \backslash B^+_{1/2})}. \tag{4.8}$$

See [1] or Theorem 2.20 of [43] precisely. Notice that

$$\|v\|_{L^1(B^+_{3/2} \backslash B^+_{1/2})} = \frac{1}{r^{n+1}} \|u^*\|_{L^1(B^+_{3r/2} \backslash B^+_{r/2})} = o(r^{-(n+1)}) \quad \text{as } r \to 0.$$

Together with (4.8), the above inequality yields

$$|u^*(X)| = o(|X|^{-(n+1)}) \quad \text{as } |X| \to 0.$$

Since $u(X) = |X|^{2m-1-n} u^*\left(X/|X|^2\right)$, we obtain

$$|u(X)| = o(|X|^{2m}) \quad \text{as } |X| \to \infty. \tag{4.9}$$

For every $R > 0$, by the standard estimates for solutions of linear elliptic PDEs we obtain

$$\|\nabla^{2m} u\|_{L^\infty(B_R^+)} \le CR^{-2m}\|u\|_{L^\infty(B_{2R}^+)},$$

where $C > 0$ is independent of R. Sending $R \to \infty$ and making use of (4.9) we have

$$\nabla^{2m} u \equiv 0 \quad \text{in } \mathbb{R}_+^{n+1}.$$

It follows that u is a polynomial of degree at most $2m-1$. Sorting u by the degree of t, one can have

$$u(x,t) = \sum_{l=0}^{2m-2} t^l P_l(x) + c_0 t^{2m-1}$$

where $P_l(x)$ is a polynomial of x with degree $\le 2m-1-l$. The boundary conditions of u imply $P_l \equiv 0$ when $l \le 2m-2$ and is odd. Indeed, suppose the contrary and let $P_{l_0} \ne 0$ of the least odd order l_0. Set $k_0 = (l_0-1)/2 \le m-2$ which is an integer. Then

$$u(x,t) = \sum_{k=1}^{k_0} t^{2k} P_{2k}(x) + t^{l_0} P_{l_0}(x) + \sum_{l=l_0}^{2m-2} t^l P_l(x).$$

Applying $\partial_t \Delta^{k_0}$ to u, then $\partial_t \Delta^{k_0}(t^{2k} P_{2k}(x))(x,0) = 0$ and $\partial_t \Delta^{k_0}(t^l P_l(x))(x,0) = 0$ for any $l > l_0$. Since $\partial_t \Delta^{k_0} u(x,0) = 0$,

$$0 = \partial_t \Delta^{k_0}(t^{l_0} P_{l_0}(x))(x,0) = l_0! P_{l_0}(x).$$

Hence, we proved the claim. It follows that

$$u(x,t) = \sum_{k=1}^{m-1} t^{2k} P_{2k}(x) + c_0 t^{2m-1}.$$

If $u^* \ge g$ for some g as stated in the theorem. For any polynomial P with $\deg P < 2m-1-2k$, we have

$$(t^{2k}P(x))^* = |X|^{2m-1-n}\left(\frac{t}{|X|^2}\right)^{2k} P\left(\frac{x}{|X|^2}\right) = O(|X|^{1-n}) \text{ as } |X| \to 0 \tag{4.10}$$

which means $(t^{2k}P(x))^* \in L^{\frac{n+1}{n}}(B_1^+)$. Absorbing all these lower order terms of P_{2k} to g and collecting all the leading terms of each P_{2k} to be $\tilde{u}$, we have

$$\tilde{u}^* = \sum_{k=1}^{m-1} |X|^{2m-1-n} \left(\frac{t}{|X|^2}\right)^{2k} \tilde{P}_{2k}\left(\frac{x}{|X|^2}\right) + c_0|X|^{2m-1-n}\left(\frac{t}{|X|^2}\right)^{2m-1} \geq \tilde{g}$$

where $\tilde{P}_{2k}$ are homogeneous polynomial in x with degree equals to $2m-1-2k$ or $\tilde{P}_{2k} \equiv 0$. By the homogeneity,

$$\tilde{u}^* = |X|^{1-2m-n} t^{2m-1} \left(\sum_{k=1}^{m-1} \tilde{P}_{2k}\left(\frac{x}{t}\right) + c_0\right).$$

Note that $\tilde{P}_{2k}$ is a homogeneous polynomial of odd degree and thus $\tilde{P}_{2k}(-y) = -\tilde{P}_{2k}(y)$. Therefore if some $\tilde{P}_{2k}$ is not zero, then $\sum_{k=1}^{m-1} \tilde{P}_{2k}(y) + c_0$ will be negative on some open set $A \subset \mathbb{R}^n$ with measure $|A| = \infty$. This leads to $\tilde{u}^* < 0$ on set $A^+ = \{(x,t) \in B_1^+ | x/t \in A\}$ with $|A^+| > 0$. While on this set, $\tilde{u}^* \notin L^{\frac{n+1}{n}}$, which will violate the fact $\tilde{u} \geq \tilde{g}$ with $\tilde{g} \in L^{\frac{n+1}{n}}(B_1^+)$. Indeed, take a bounded subset E of A with $|E| > 0$, notice when $t_0 > 0$ small enough, we have $\{(tx,t) : x \in E, 0 < t < t_0\} \subset A^+$, then

$$\begin{aligned}
\int\int_{A^+} |\tilde{u}|^{\frac{n+1}{n}} \mathrm{d}x\mathrm{d}t \geq & \int_0^{t_0} \int_{tE} |\tilde{u}^*|^{\frac{n+1}{n}} \mathrm{d}x\mathrm{d}t \\
= & \int_0^{t_0} \int_{tE} \left[|X|^{1-2m-n} t^{2m-1} \left| \sum_{k=1}^{m} \tilde{P}_{2k}(x/t) + c_0 \right| \right]^{\frac{n+1}{n}} \mathrm{d}x\mathrm{d}t \\
= & \int_0^{t_0} t^{-1} \int_E \left[(|y|^2+1)^{\frac{1-2m-n}{2}} \left| \sum_{k=1}^{m} \tilde{P}_{2k}(y) + c_0 \right| \right]^{\frac{n+1}{n}} \mathrm{d}y\mathrm{d}t \\
> & c \int_0^{t_0} t^{-1} \mathrm{d}t = \infty \text{ for some } c > 0,
\end{aligned}$$

where we have changed variable $x = ty$. Therefore, $\tilde{P}_{2k} \equiv 0$ for $1 \leq k \leq m-1$ and $c_0 \geq 0$.

We complete the proof of the proposition. □

Theorem 4.2.2. Let $0 \leq u \in C^{2m}(\mathbb{R}_+^{n+1} \cup \partial\mathbb{R}_+^{n+1})$ be a solution of (4.6). Then

$$u(x,t) = \sum_{k=1}^{m-1} t^{2k} P_{2k}(x) + c_0 t^{2m-1}, \tag{4.11}$$

where $P_{2k}(x)$ are polynomials w. r. t. x of degree $\leq 2m-2-2k$, and $c_0 \geq 0$.

Proof. By Proposition 4.2.1, it suffices to show $u^* \in L^1(B_1^+)$. Note that u^* satisfies (4.6) except the origin. Define

$$\eta_\varepsilon(t) = \begin{cases} \frac{1}{2m!}(t-\varepsilon)^{2m} & \text{for } t \geq \varepsilon, \\ 0 & \text{for } t < \varepsilon. \end{cases}$$

Since u^* is smooth in on $\partial^+ B_1^+$ and $\eta(t) \in C^{2m-1,1}$, multiplying both sides of the polyharmonic equation of u^* and using Green's identity we have

$$\int_{B_1^+ \cap \{t > \varepsilon\}} u^*(X)\, dX \leq C,$$

where C is independent of ε. Sending $\varepsilon \to 0$ and using $u^* \geq 0$, by Lebesgue's monotone convergence theorem we have $u^* \in L^1(B_1^+)$.

Therefore, we complete the proof. □

4.2.2 Extensions of Bôcher theorem

In this subsection, we will give some extensions of the classical Bôcher theorem which says that every nonnegative harmonic function in the punctured unit ball is decomposed to the fundamental solution multiplied by a constant plus a harmonic function cross the origin. Let

$$\Phi(X) = c(m,n) \begin{cases} |X|^{2m-n-1} & \text{if } 2m < n+1, \\ \ln |X| & \text{if } 2m = n+1, \end{cases}$$

be the fundamental solution of $(-\Delta)^m$, where $c(m,n)$ is a normalization constant such that $(-\Delta)^m \Phi(X) = \delta_0$.

Theorem 4.2.3. Let $u \in C^{2m}(B_1 \backslash \{0\})$ be a solution of $(-\Delta)^{2m} u = 0$ in $B_1 \backslash \{0\} \subset \mathbb{R}^{n+1}$. Suppose $u \in L^1(B_1)$, then

$$u(X) = h(X) + \sum_{|\alpha| \leq 2m-1} c_\alpha D^\alpha \Phi(X) \quad \text{in } B_1,$$

where $\alpha = (\alpha_1, \ldots, \alpha_{n+1}) \in \mathbb{N}^{n+1}$ is multi-index, c_α are constants, and h is a smooth solution of $(-\Delta)^{2m} h = 0$ in B_1. If in addition assume $u \geq g$ for some g belonging to weak-$L^{\frac{n+1}{n-1}}(B_1)$, then $c_\alpha = 0$ for $|\alpha| = 2m-1$.

Proof. The first part of theorem was proved by Futamura-Kishi-Mizuta [42]. For the second part, noticing when $|\alpha| = 2m-1$, $D^\alpha\Phi(X)$ is homogeneous and has negative part comparable to $|X|^{-n}$, which does not belong to weak-$L^{\frac{n+1}{n-1}}(B_1)$. So $c_\alpha = 0$ for such α. $\square$

We refer to Futamura-Kishi-Mizuta [42], Ghergu-Moradifam-Taliaferro [44] and references therein for related works on Bôcher's theorem of higher order equations.

Corollary 4.2.4. Let $u \in C^{2m}(\bar{B}_1^+ \setminus \{0\})$ be a solution of

$$\begin{cases} (-\Delta)^m u = 0 & \text{in } B_1^+, \\ \partial_t u = \partial_t \Delta u = \cdots = \partial_t \Delta^{m-1} u = 0 & \text{on } D_1 \setminus \{0\}. \end{cases} \tag{4.12}$$

Suppose that $u \in L^1(B_1^+)$ and $u \geq g$ for some g belonging to weak-$L^{\frac{n+1}{n-1}}(B_1^+)$, then

$$u(X) = h(X) + \sum_{|\alpha| \leq 2m-2} c_\alpha D^\alpha \Phi(X),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) \in \mathbb{N}^{n+1}$ with α_{n+1} being even, and $h(X)$ satisfies

$$\begin{cases} (-\Delta)^m h = 0 & \text{in } B_1^+, \\ \partial_t h = \partial_t \Delta h = \cdots = \partial_t \Delta^{m-1} h = 0 & \text{on } D_1. \end{cases} \tag{4.13}$$

Proof. Let $u(x,t) = u(x,-t)$ and $g(x,t) = g(x,-t)$ for $t < 0$. We abuse the notation to denote these two new functions still as u and g, respectively. From the boundary condition and regularity theory for Poisson equation, we have $(-\Delta)^{m-1}u, (-\Delta)^{m-2}u, \ldots, u$ are smooth in $B_1 \setminus \{0\}$. Consequently, Theorem 4.2.3 implies the decomposition of u. The boundary condition actually implies we can only have $D^\alpha\Phi$ in the decomposition with α_{n+1} of $\alpha = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ is even, see the proof of the last statement of Proposition 4.2.1.

Therefore, we complete the proof. $\square$

Bôcher theorem for positive harmonic functions can be viewed as a stronger version of Liouville theorem. Indeed, *Corollary* 4.2.5. Let $u \in C^{2m}(\mathbb{R}^{n+1}_+ \cup \{\partial \mathbb{R}^{n+1}_+ \setminus \{0\}\})$ be a solution of

$$\begin{cases} (-\Delta)^m u = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_t u = \partial_t \Delta u = \cdots = \partial_t \Delta^{m-1} u = 0 & \text{on } \partial\mathbb{R}^{n+1}_+ \setminus \{0\}. \end{cases} \tag{4.14}$$

Suppose that $u \in L^1(B_1^+)$ and $u \geq g$ for some g belonging to weak-$L^{\frac{n+1}{n-1}}(B_1^+)$, and $\lim\limits_{|X|\to\infty} u(X) = 0$. Then

$$u(X) = \sum_{|\alpha| \leq 2m-2} c_\alpha D^\alpha \Phi(X) \quad \forall\ X \in \mathbb{R}^{n+1}_+,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) \in \mathbb{N}^{n+1}$ with α_{n+1} being even.

Proof. Applying Corollary 4.2.4 with B_1^+ replaced by consecutively large half balls, we have

$$u(X) = h(X) + \sum_{|\alpha| \leq 2m-2} c_\alpha D^\alpha \Phi(X) \quad \forall\ X \in \mathbb{R}^{n+1}_+,$$

with each α's α_{n+1} even. Since $|\alpha| \leq 2m-2$,

$$\lim_{|X|\to\infty} |h(X)| \leq \lim_{|X|\to\infty} |u(X)| + \lim_{|X|\to\infty} \left| \sum_{|\alpha| \leq 2m-2} c_\alpha D^\alpha \Phi(X) \right| = 0.$$

By the (4.13), extending h to lower half plane one can get a smooth polyharmonic function on $\mathbb{R}^{n+1}$ which is bounded and converges to 0 as $|X| \to \infty$. By the interior estimates for solutions of linear elliptic PDEs, one can easily obtain that $h \equiv 0$. Therefore, we complete the proof. $\square$

The method of proof of Proposition 4.2.1 can give a direct proof of Corollary 4.2.5. Corollary 4.2.4 is of independent interest and will be useful in study of local analysis of solutions of the nonlinear problem.

4.3 Isolated singularity for nonlinear boundary data

Now let us go back to the nonlinear boundary problems we want to study. Suppose $0 \leq u \in C^{2m}(\mathbb{R}^{n+1}_+ \cup \partial\mathbb{R}^{n+1}_+)$ be a solution of (1.11) with $1 < p \leq \frac{n+(2m-1)}{n-(2m-1)}$. Then, by

Lemma 4.1.10, $u^* = u_{0,1}$ satisfies

$$\begin{cases} \Delta^m u^*(x,t) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_t u^* = \partial_t \Delta u^* = \cdots = \partial_t \Delta^{m-2} u^*(x,0) = 0, & \text{on } \partial\mathbb{R}^{n+1}_+ \setminus \{0\}, \\ (-1)^m \partial_t \Delta^{m-1} u^*(x,0) = |x|^{-\tau} u^{*p} & \text{on } \partial\mathbb{R}^{n+1}_+ \setminus \{0\}, \end{cases} \tag{4.15}$$

where $\tau = [n + (2m-1)] - p[n - (2m-1)] \geq 0$. The goal of this section is to show:

Proposition 4.3.1. Let u^* be as above. If either one of the two items holds

(1) m is odd;

(2) m is even and $u(X) = o(|X|^{2m-1})$ as $|X| \to \infty$,

then

$$\int_{D_1} |x|^{-\tau} u^*(x,0)^p \mathrm{d}x < \infty. \tag{4.16}$$

Let us start from basic properties of u^*.

Lemma 4.3.2. Let u^* be a nonnegative solution of (4.15). Then

(i) $u^* \in L^1(B_1^+)$,

(ii) $\int_{D_1} |x|^{2m-\tau} u^*(x,0)^p dx < \infty$,

(iii) If $p > 1$, then $\int_{D_1} u^*(x,0)^s dx < \infty$ for some $s > 1$.

Proof. (i) $u^* \in L^1(B_1^+)$ was shown in the proof of Theorem 4.2.2.

(ii) Let $r = |X|$, and construct a smooth radial function ξ_ε such that $\Delta^m \xi_\varepsilon(r) = \chi_{\{r>\varepsilon\}}(r)$ for given $\varepsilon > 0$, and $\xi_\varepsilon = 0$ in $B_{\varepsilon/2}$. It is easy to show $\xi_\varepsilon \to \frac{1}{C(m,n)} r^{2m}$ in C^0, where $C(m,n) = \Delta^m r^{2m} > 0$. Since ξ_ε is radially symmetric, then $\partial_t \Delta^k \xi_\varepsilon(x,0) = 0$ for any $k \geq 0$. Noticing that $\partial_t \Delta^k u^*$ vanishes for $k = 0, \ldots, m-2$ and using ξ_ε as a test function in Green's identity, we obtain

$$\int_{D_1} \xi_\varepsilon(|x|)|x|^{-\tau} u^*(x,0)^p \, \mathrm{d}x \leq (-1)^m \int_{B_1^+} u^*(X) \chi_{\{r>\varepsilon\}}(|X|) \mathrm{d}X + C.$$

By item (i) and sending $\varepsilon \to 0$, then $|x|^{2m-\tau} u^*(x,0)^p \in L^1(D_1)$.

(iii) By the definition of τ, it is easy to check

$$p > \frac{2m-\tau}{n} + 1.$$

Choosing b such that

$$\max\left\{\frac{2m-\tau}{n} + 1, 1\right\} < b < p,$$

then from Hölder's inequality

$$\int_{D_1} u^*(x,0)^{\frac{p}{b}} \mathrm{d}x \le \left(\int_{D_1} |x|^{2m-\tau} u^*(x,0)^p \mathrm{d}x\right)^{\frac{1}{b}} \left(\int_{D_1} |x|^{-\frac{2m-\tau}{b-1}} \mathrm{d}x\right)^{1-\frac{1}{b}}.$$

Noticing $(2m-\tau)/(b-1) < n$, it yields $u^*(x,0) \in L^s(D_1)$ for $s = p/b > 1$.

Therefore, the lemma is proved. □

Since $u^*(x,0) \in L^1(D_1)$ and $u^*(x,0) \in L^\infty(\mathbb{R}^n \setminus D_1)$, then

$$v^* := \mathcal{P}_m * u^* \tag{4.17}$$

is well-defined.

Proposition 4.3.3. Let v^* be in (4.17). Then we have $v^* \in L^{\frac{(n+1)}{n}}(B_1^+)$ and

$$\begin{cases} \Delta^m v^*(x,t) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_t v^* = \partial_t \Delta v^* = \cdots = \partial_t \Delta^{m-2} v^*(x,0) = 0 & \text{on } \partial\,\mathbb{R}^{n+1}_+ \setminus \{0\}, \\ (-1)^m \partial_t \Delta^{m-1} v^*(x,0) = |x|^{-\tau} v^{*p} - c_0(-1)^m |x|^{-(2m-1+n)} & \text{on } \partial\,\mathbb{R}^{n+1}_+ \setminus \{0\}, \end{cases} \tag{4.18}$$

where $c_0 \ge 0$ is a constant.

Proof. Decompose $u^*(x,0) = u_1^*(x,0) + u_2^*(x,0)$ for $x \in \mathbb{R}^n$, where $u_1^*(x,0) = u^*(x,0)\chi_{D_1}(x)$ and χ_{D_1} is the characteristic function of D_1. Then $v^* = v_1^* + v_2^*$ with v_1^* and v_2^* are given by the corresponding Poisson type convolutions of $u_1^*(x,0)$ and $u_2^*(x,0)$ as in (4.17), respectively.

Since $u^* \in L^s(D_1)$ for some $s > 1$ by Lemma 4.3.2, we have $v_1^* \in L^{\frac{(n+1)s}{n}}(\mathbb{R}^{n+1}_+)$ by Lemma 4.1.1. On the other hand, since $u^*(x,0) = O(|x|^{2m-1-n})$ as $x \to \infty$, then

$u_2^*(x,0) \in L^q(\mathbb{R}^n)$ for any $q > \frac{n}{n+1-2m}$. Using Lemma 4.1.1 again yields $v_2^* \in L^{\bar{q}}(\mathbb{R}_+^{n+1})$ for any $\bar{q} > \frac{n+1}{n+1-2m}$. Restricting v_1^* and v_2^* in B_1^+ and notice that

$$\min\left\{\frac{(n+1)s}{n}, \frac{n+1}{n+1-2m}\right\} > \frac{n+1}{n},$$

we proved $v^* \in L^{\frac{n+1}{n}}(B_1^+)$.

By Lemma 4.1.6, v^* satisfies the first two lines of (4.18). Let $v = (v^*)_{0,1}$. By Lemma 4.1.4 and the remark after it, $v(x,0) = u(x,0)$ on $\mathbb{R}^n$. Define $w = u - v$, which satisfies (4.6) in Proposition 4.2.1. $w^* \geq -v^*$ will satisfy the assumption of Proposition 4.2.1, therefore we conclude

$$w(x,t) = \sum_{k=1}^{m-1} t^{2k} P_{2k}(x) + c_0 t^{2m-1}, \tag{4.19}$$

where $c_0 \geq 0$, $P_{2k}(x)$ are polynomials w. r. t. x of degree $\leq 2m-2-2k$. Therefore,

$$\partial_t \Delta^{m-1} v^* = \partial_t \Delta^{m-1} u^* - \partial_t \Delta^{m-1} w^*,$$

Since

$$\begin{aligned}\partial_t \Delta^{m-1} w^*(x,0) &= c_0 \partial_t \Delta^{m-1}(|X|^{1-2m-n} t^{2m-1})(x,0) \\ &= c_0 (2m-1)! |x|^{-(2m-1+n)},\end{aligned}$$

the proposition follows immediately. $\square$

Naively one may wish $c_0 = 0$, then u^* and v^* share the same equations. However, as we said in the introduction, there are special cases, for example when m is even, u^* will be the $m-$Kelvin transformation of $H_a(x,t)$ in (1.13), but $v^* \equiv a^{1/p}|X|^{2m-1-n}$, so $c_0 \neq 0$. On the other hand, we will prove that under the assumptions in Proposition 4.3.1, we have $c_0 = 0$. To that end, we need to analyze the symmetrization of the solutions. When applied to radially symmetric functions in $\mathbb{R}^{n+1}$ the Laplace operator Δ is expressed as

$$L = \frac{d^2}{dr^2} + \frac{n}{r}\frac{d}{dr}.$$

Lemma 4.3.4. Suppose that $w \in C^{2m}(\mathbb{R}^{n+1}_+ \cup \{\partial\,\mathbb{R}^{n+1}_+ \setminus \{0\}\})$ satisfies

$$\begin{cases} \Delta^m w(x,t) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_t w = \cdots = \partial_t \Delta^{m-2} w(x,0) = 0 & \text{on } \partial\mathbb{R}^{n+1}_+ \setminus \{0\}, \\ (-1)^m \partial_t \Delta^{m-1} w(x,0) = f(x) & \text{on } \partial\mathbb{R}^{n+1}_+ \setminus \{0\}. \end{cases}$$

Then

$$L^m \bar{w}(r) = (-1)^m \frac{\psi_{n-1}}{\psi_n} r^{-1} [f]_r, \tag{4.20}$$

where $\bar{w}(r) = \fint_{\partial^+ B_r^+} w(x,t)\,\mathrm{d}S_{x,t}$ and $[f]_r = \fint_{\partial D_r} f(x)\,\mathrm{d}\sigma$ and ψ_n, ψ_{n-1} are the volume constants.

Proof. By the definition of $\bar{w}$, taking derivatives leads to

$$r^n \bar{w}'(r) = \frac{1}{\psi_n} \int_{\partial^+ B_r^+} \frac{\partial w}{\partial \nu}\,\mathrm{d}S = -\frac{1}{\psi_n} \int_{B_1^+ \setminus B_r^+} \Delta w\,\mathrm{d}X + \frac{1}{\psi_n} \int_{\partial^+ B_1^+} \frac{\partial w}{\partial \nu}\,\mathrm{d}S,$$

where $r \in (0,1)$, ν is the outer unit normal of the boundary and we used $\partial_t w(x,0) = 0$. It follows that

$$L\bar{w} = \fint_{\partial^+ B_r^+} \Delta w\,\mathrm{d}S.$$

Using $\partial_t \Delta^k w(x,0) = 0$ for $k = 1, \ldots, m-2$ and repeating this process, we have

$$L^{m-1} \bar{w} = \fint_{\partial^+ B_r^+} \Delta^{m-1} w\,\mathrm{d}S. \tag{4.21}$$

By Green's identity, we have for any $0 < r < 1$

$$\begin{aligned} &\int_{\partial^+ B_1^+} \frac{\partial \Delta^{m-1} w}{\partial \nu}\,\mathrm{d}S - \int_{\partial^+ B_r^+} \frac{\partial \Delta^{m-1} w}{\partial \nu}\,\mathrm{d}S - \int_{D_1 \setminus D_r} \partial_t \Delta^{m-1} w\,\mathrm{d}x \\ = &\int_{B_1^+ \setminus B_r^+} \Delta^m w = 0. \end{aligned}$$

Taking derivative in r, we have

$$\frac{d}{dr} \int_{\partial^+ B_r^+} \frac{\partial \Delta^{m-1} w}{\partial \nu}\,\mathrm{d}S = (-1)^m \psi_{n-1} r^{n-1} [f]_r. \tag{4.22}$$

Since

$$\frac{d}{dr} \fint_{\partial^+ B_r^+} \Delta^{m-1} w\,\mathrm{d}S = \fint_{\partial^+ B_r^+} \frac{\partial \Delta^{m-1} w}{\partial \nu}\,\mathrm{d}S, \tag{4.23}$$

then (4.21) implies

$$
\begin{aligned}
L^m \bar{w}(r) =& \frac{1}{r^n}\frac{d}{dr}\left(r^n \frac{d}{dr} \fint_{\partial^+ B_r^+} \Delta^{m-1} w \, \mathrm{d}S\right) = \frac{1}{r^n}\frac{d}{dr}\left(r^n \fint_{\partial^+ B_r^+} \frac{\partial \Delta^{m-1} w}{\partial \nu} \, \mathrm{d}S\right) \\
=& \frac{1}{w_n r^n}\frac{d}{dr}\int_{\partial^+ B_r^+} \frac{\partial \Delta^{m-1} w}{\partial \nu} \, \mathrm{d}S = (-1)^m \frac{w_{n-1}}{w_n} r^{-1} [f]_r.
\end{aligned}
$$

Therefore, we complete the proof. □

Notice that u^* satisfies (4.15) and v^* satisfies (4.18). It follows from the above lemma that:

Corollary 4.3.5.

$$
L^m \bar{u^*}(r) = (-1)^m \frac{\psi_{n-1}}{\psi_n} r^{-\tau-1} [u^{*p}]_r, \tag{4.24}
$$

$$
L^m \bar{v^*}(r) = (-1)^m \frac{\psi_{n-1}}{\psi_n} \{r^{-\tau-1}[v^{*p}]_r - c_0(-1)^m r^{-n-2m}\}. \tag{4.25}
$$

Lemma 4.3.6. Under the assumptions in Proposition 4.3.1, we have $c_0 = 0$.

Proof. By the ODE of $\bar{u^*}$, one can integrate $2m$ times to get

$$
\bar{u^*}(r) = a\Phi(r) + \sum_{k=2}^{m} \left\{ b_k r^{2(m-k)-n+1} + c_k r^{2(m-k)} \right\} + \frac{(-1)^m \psi_{n-1}}{\psi_n} F(r), \tag{4.26}
$$

where a, b_k, c_k are constants depending only on C^{2m} norm of u^* near $\partial^+ B_1^+$, and

$$
F(r) = \int_r^1 r_{2m-1}^{-n} \int_{r_{2m-1}}^1 r_{2m-2}^{n} \int \cdots \int_{r_4}^1 r_3^n \int_{r_3}^1 r_2^{-n} \int_{r_2}^1 r_1^n r_1^{-\tau-1} [u^*(x,0)^p]_{r_1} \, \mathrm{d}r_1 \cdots \mathrm{d}r_{2m-1} \mathrm{d}r.
$$

If m is odd, (4.26) gives $\bar{u^*}(r) \le C b_m r^{-n+1}$ for small r. Similarly, $\bar{v^*}(r) \le C b_m r^{-n+1}$ for small r. Since u^* and v^* are positive, u^*, v^* and $w^* := u^* - v^*$ must belong to weak- $L^{\frac{n+1}{n-1}}(B_1^+)$. By (4.19), $c_0 |X|^{-(n+2m-1)} t^{2m-1}$ has to belong weak-$L^{\frac{n+1}{n-1}}(B_1^+)$, which forces $c_0 = 0$.

On the other hand, if m is even and $u(X) = o(|X|^{2m-1})$, by (4.19) and the fact that $w = u - v \le u$ we immediately have $c_0 = 0$.

In conclusion, we complete the proof. □

Next two lemmas can boost the regularity of u^* by iteration.

Lemma 4.3.7. Under the assumptions of Proposition 4.3.1. If $u^*(x,0) \in L^s(D_1)$ for some $s > 1$, then

$$\int_{D_1} |x|^{q-\tau} u^*(x,0)^p \mathrm{d}x < \infty$$

for any $q > 2m - n - 1 + \frac{n}{s}$.

Proof. By the proof of Proposition 4.3.3, $v^* \in L^{\tilde{s}}(B_1^+)$, where

$$\tilde{s} = \min\left\{\frac{(n+1)s}{n}, \frac{n+1}{n-2m+1}\right\}.$$

Fix any $q > 2m - n - 1 + \frac{n}{s}$. Choose $0 \le \eta(r) \in C^\infty(0,\infty)$ such that $\eta(r) = 0$ when $r < 1/2$ and $\eta(r) = 1$ when $r > 1$ and define

$$\phi_\varepsilon(X) = \eta\left(\frac{|X|}{\varepsilon}\right)|X|^q.$$

Multiplying v^* by ϕ_ε and using Green's identity over B_1^+, we have

$$\int_{B_1} v^* \Delta^m \phi_\varepsilon \mathrm{d}X = \int_{D_1} \eta\left(\frac{|x|}{\varepsilon}\right)|x|^{q-\tau} u^*(x,0)^p \,\mathrm{d}x + C.$$

Sending $\varepsilon \to 0$, the first term of RHS will converge to the integral we want to bound, while the LHS will be uniformly bounded. Indeed, by Hölder's inequality and the radial symmetry of ϕ_ε,

$$\begin{aligned}
\int_{B_1^+} |v^* \Delta^m \phi_\varepsilon| \mathrm{d}X &\le C \sum_{k=0}^{2m} \int_{B_1^+} v^* |X|^{q-k} \left|\frac{\mathrm{d}^{2m-k}}{\mathrm{d}r^{2m-k}} \eta\left(\frac{r}{\varepsilon}\right)\right| \mathrm{d}X \\
&\le C \int_{B_1^+} v^* |X|^{q-2m} \mathrm{d}X + C\varepsilon^{q-2m} \int_{B_\varepsilon^+} v^* \mathrm{d}X \\
&\le C(n,q)\|v^*\|_{L^{\tilde{s}}(B_1^+)} + C(n,q)\|v^*\|_{L^{\tilde{s}}(B_1^+)} \varepsilon^{q-2m+(n+1)(1-1/\tilde{s})} \\
&\le C,
\end{aligned}$$

where we used the assumption on q to give $q - 2m + (n+1)(1-1/\tilde{s}) > n/s - (n+1)/\tilde{s} \ge 0$.

Therefore, we complete the proof. □

Lemma 4.3.8. Assume the assumptions in Proposition 4.3.1. Then for any $1 < p \le \frac{n+2m-1}{n-2m+1}$ we have

$$\int_{D_1} |x|^{q-\tau} u^*(x,0)^p \mathrm{d}x < \infty \quad \text{and} \quad u^*(x,0) \in L^p(D_1)$$

where $q > 2m - n - 1 + \frac{n}{p}$. In particular, if $p > \frac{n}{n-2m+1}$, q can achieve 0 thus (4.16) holds.

Proof. Let us call $u^*(x,0)$ has $(q,s)-$property if

$$\int_{D_1} |x|^{q'-\tau} u^*(x,0)^p \,\mathrm{d}x < \infty \quad \forall\, q' > q \text{ and } u^*(x,0) \in L^{s'}(D_1) \quad \forall\, s' < s.$$

From Lemma 4.3.2 item (ii), $u^*(x,0)$ has $(q_0,s_0)-$property with $q_0 = 2m$, $s_0 = \frac{np}{n+(2m-\tau)^+} = \frac{np}{n+(q_0-\tau)^+} > 1$, where $a^+ = \max\{a,0\}$ for any constant a. From Lemma 4.3.7, we have

$$\int_{D_1} |x|^{q-\tau} u^*(x,0)^p \,\mathrm{d}x < \infty \quad \forall\, q > q_1 = 2m - n - 1 + \frac{n}{s_0}.$$

From this, one can repeat the proof of Lemma 4.3.2 item (ii) to see

$$u^*(x,0) \in L^s(D_1) \quad \forall\, s < s_1 = \frac{np}{n+(q_1-\tau)^+}.$$

Therefore $u^*(x,0)$ has $(q_1,s_1)-$property. Moreover, it is easy to see $q_1 < q_0$ and $s_1 > s_0$. By iterating all the above steps, we have $u^*(x,0)$ has $(q_k,s_k)-$property,

$$q_k = 2m - n - 1 + \frac{n}{s_{k-1}} \quad \text{and} \quad s_k = \frac{np}{n+(q_k-\tau)^+}. \tag{4.27}$$

Moreover $q_0 > q_1 \geq \cdots \geq q_k$ and $s_0 < s_1 \leq \cdots \leq s_k$.

Claim: There exist some k finite such that $q_k \leq \tau$ and $s_k = p$.

Suppose not, then we will have an infinite many $q_k > \tau$ which are non-increasing. Suppose $\lim_{k\to\infty} q_k = a \geq \tau$, consequently (4.27) implies

$$a = 2m - n - 1 + \frac{a-\tau+n}{p} = \frac{a}{p} + \frac{1-2m}{p} < a,$$

which is a contradiction. The claim is proved.

Thus after some finite steps, we will have $s_k = p$ and $q_k = 2m - n - 1 + \frac{n}{p} = \frac{\tau-2m+1}{p} < \tau$ for some k finite. Namely, $u^*(x,0) \in L^p(D_1)$ and

$$\int_{D_1} |x|^{q_k-\tau} u^*(x,0)^p \,\mathrm{d}x < \infty.$$

In particular if $p > \frac{n}{n-2m+1}$, then $q_k = 2m - n - 1 + \frac{n}{p} < 0$ and

$$\int_{D_1} |x|^{-\tau} u^*(x,0)^p \,\mathrm{d}x < \infty.$$

We complete the proof. $\square$

In order to prove Proposition 4.3.1 in the remaining range $1 < p \leq \frac{n}{n-2m+1}$, we need to investigate the singularity of u^* near origin more precisely. The following three lemmas are devoted to that. Let us build a bridge between the boundary integral and inner integral of v^*.

Lemma 4.3.9. Let v^* be defined by (4.17) and $\varepsilon \in [0,1)$. Then for any $r_0 > 0$ there exists a constant $C > 0$, depending only on m, n, ε, r_0, $\|u^*(\cdot,0)\|_{L^\infty(\mathbb{R}^n \setminus D_{2r_0})}$ and $\|u^*(\cdot,0)\|_{L^1(D_{r_0})}$, such that

$$\int_r^{2r} \rho^{-\varepsilon}\bar{v}^*(\rho)\,\mathrm{d}\rho \leq Cr^{1-n-\varepsilon}\int_{D_{r/2}} u^*(y,0)\,\mathrm{d}y + C\int_{r/2}^{r_0} \rho^{-\varepsilon}[u^*]_\rho\,\mathrm{d}\rho + Cr^{1-\varepsilon} \tag{4.28}$$

for any $r \in (0, r_0/4)$.

Proof. For any $r < r_0/4$, suppose $\rho \in [r, r_0]$, then we have

$$\begin{aligned}
⨍_{\partial^+ B_\rho^+} v^*(x,t)\,\mathrm{d}S =& ⨍_{\partial^+ B_\rho^+}\int_{0<|y|<r/2} \mathcal{P}_m(x-y,t)u^*(y,0)\mathrm{d}y\mathrm{d}S \\
&+ ⨍_{\partial^+ B_\rho^+}\int_{r/2<|y|<2r_0} \mathcal{P}_m(x-y,t)u^*(y,0)\,\mathrm{d}y\mathrm{d}S \\
&+ ⨍_{\partial^+ B_\rho^+}\int_{2r_0<|y|} \mathcal{P}_m(x-y,t)u^*(y,0)\,\mathrm{d}y\mathrm{d}S \\
:=& I_1 + I_2 + I_3.
\end{aligned}$$

By direct computations,

$$\begin{aligned}
I_1 =& \int_{0<|y|<r/2} u^*(y,0)\mathrm{d}y ⨍_{\partial^+ B_\rho^+} \mathcal{P}_m(x-y,t)\,\mathrm{d}S \\
\leq& C\rho^{-n}\int_{D_{r/2}} u^*(y,0)\,\mathrm{d}y \\
I_2 \leq& \int_{r/2<|y|<2r_0} u^*(y,0) ⨍_{\partial^+ B_\rho^+} \frac{1}{|X-Y|^n}\,\mathrm{d}S\mathrm{d}y \\
I_3 \leq& C.
\end{aligned}$$

where $X = (x,t)$, $Y = (y,0)$, and $C > 0$ depends only on m, n, r_0, $\|u^*(\cdot,0)\|_{L^\infty(\mathbb{R}^n \setminus D_{2r_0})}$ and $\|u^*(\cdot,0)\|_{L^1(D_{r_0})}$. It follows that

$$\begin{aligned}
&⨍_{\partial^+ B_\rho^+} v^*(x,t)\,\mathrm{d}S \\
\leq& C\rho^{-n}\int_{D_{r/2}} u^*(y,0)\,\mathrm{d}y + \int_{r/2<|y|<2r_0} u^*(y,0) ⨍_{\partial^+ B_\rho^+} \frac{1}{|X-Y|^n}\,\mathrm{d}S\mathrm{d}y + C
\end{aligned}$$

Multiplying both sides of the above inequality by $\rho^{-\varepsilon}$ and integrating from r to $2r$, we obtain

$$
\begin{aligned}
&\int_r^{2r} \rho^{-\varepsilon}\bar{v}^*(\rho)\,\mathrm{d}\rho \\
\leq& Cr^{1-n-\varepsilon}\int_{D_{r/2}} u^*(y,0)\,\mathrm{d}y + C\int_{r/2<|y|<2r_0} u^*(y,0)|y|^{1-n-\varepsilon}\,\mathrm{d}y + Cr^{1-\varepsilon} \\
=& Cr^{1-n-\varepsilon}\int_{D_{r/2}} u^*(y,0)\,\mathrm{d}y + C\int_{r/2}^{2r_0} \rho^{-\varepsilon}[u^*]_\rho\,\mathrm{d}\rho + Cr^{1-\varepsilon},
\end{aligned}
$$

where we used the inequality

$$\int_{\mathbb{R}^{n+1}} \frac{1}{|X-Y|^n}|X|^{-n-\varepsilon}\,\mathrm{d}X \leq C(n,\varepsilon)|Y|^{1-n-\varepsilon}$$

with taking $Y = (y,0)$. □

Lemma 4.3.10. Assume the assumptions of Proposition 4.3.1, then

$$\int_{D_r} u^*(x,0)\mathrm{d}x \leq C(\hat{p})r^{\hat{p}}.$$

where $\hat{p} > 2m - 1 + (n - n/p)/p$.

Proof. From Lemma 4.3.8 we have

$$\int_{D_r} |x|^{q-\tau}u^*(x,0)^p\mathrm{d}x = C(q) < \infty, \tag{4.29}$$

where $q > q' = 2m - n - 1 + \frac{n}{p}$. From Hölder's inequality, we have

$$\int_{D_r} u^*(x,0)\mathrm{d}x \leq \left(\int_{D_r} |x|^{q-\tau}u^*(x,0)^p\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{D_r} |x|^{-\frac{q-\tau}{p-1}}\mathrm{d}x\right)^{1-\frac{1}{p}}. \tag{4.30}$$

By the definition of τ, one can verify

$$(n - \frac{q-\tau}{p-1})(1-\frac{1}{p}) < (n - \frac{q'-\tau}{p-1})(1-\frac{1}{p}) = 2m - 1 + \frac{n-n/p}{p}.$$

It follows that

$$\left(\int_{D_r} |x|^{-\frac{q-\tau}{p-1}}\mathrm{d}x\right)^{1-\frac{1}{p}} \leq C(q)r^{(n-\frac{q-\tau}{p-1})(1-1/p)}. \tag{4.31}$$

Combining (4.29), (4.30) and (4.31) together, the lemma follows immediately. □

Lemma 4.3.11. Under the assumptions of Proposition 4.3.1 and $1 < p \le \frac{n}{n-2m+1}$. It is **impossible** to find small constants $r_0 > 0$ and $a_0 > 0$ such that

$$\bar{v^*}(r) \ge a_0 r^{2m-1-n} \quad \forall\, r \in (0, r_0). \tag{4.32}$$

Proof. Suppose the contrary that there exist $r_0 > 0$ and $a_0 > 0$ such that (4.32) holds. Clearly, we can take r_0 being sufficiently small. By Lemma 4.3.9 and Lemma 4.3.10, if r_0 is sufficiently small, we have for $r \in (0, r_0)$

$$r^{2m-n-\varepsilon} \le C(\varepsilon) r^{2m-n+(n-n/p)/p-\frac{1}{2}\varepsilon} + C\int_{r/2}^{r_0} \rho^{-\varepsilon}[u^*]_\rho \mathrm{d}\,\rho + C r^{1-\varepsilon}$$

where $\varepsilon \in (0,1)$. Taking ε sufficiently small and fix it, it follows that for r_0 sufficiently small and $r \in (0, r_0)$ there holds

$$\int_r^{r_0} \rho^{-\varepsilon}[u^*]_\rho \,\mathrm{d}\rho \ge \frac{1}{C} r^{2m-n-\varepsilon}. \tag{4.33}$$

By Hölder's inequality,

$$\begin{aligned}
\int_r^{r_0} \rho^{-\varepsilon}[u^*]_\rho \,\mathrm{d}\rho &\le C\left(\int_r^{r_0} \rho^{-\varepsilon p}[(u^*)^p]_\rho \,\mathrm{d}\rho\right)^{1/p} \\
&\le C\left(\int_r^{r_0} \rho^{q-\tau+n-1}[(u^*)^p]_\rho \,\mathrm{d}\rho\right)^{1/p} r^{-\frac{(q-\tau+n-1+\varepsilon p)^+}{p}} \\
&= C\left(\int_{D_{r_0}} |x|^{q-\tau} u^*(x,0)^p \,\mathrm{d}x\right)^{1/p} r^{-\frac{(q-\tau+n-1+\varepsilon p)^+}{p}} \\
&\le C r^{-\frac{(q-\tau+n-1+\varepsilon p)^+}{p}},
\end{aligned}$$

where we used Lemma 4.3.8 in the last inequality. Together with (4.33), the above inequality yields

$$r^{-\frac{(q-\tau+n-1+\varepsilon p)^+}{p}} \ge \frac{1}{C} r^{2m-n-\varepsilon} \quad \forall\, r \in (0, r_0). \tag{4.34}$$

Since $1 < p \le \frac{n}{n-2m+1}$, we have

$$q - \tau + n - 1 = \frac{n}{p} + p(n-2m+1) - n - 1 \le n - 2m,$$

and thus

$$-\frac{(q-\tau+n-1+\varepsilon p)^+}{p} > 2m - n - \varepsilon,$$

which makes (4.34) impossible.

Therefore, we complete the proof. $\square$

Lemma 4.3.12 (Dichotomy lemma). Suppose $\xi(r) \in C^{2m}(0,\infty)$, if there exists $c_1, \tilde{c}_1, r_1, \tilde{r}_1 > 0$

$$L^{m-1}\xi(r) \geq c_1 r^{1-n} \text{ for any } r \in (0, r_1) \quad \text{ or } \quad L^{m-1}\xi(r) \leq -\tilde{c}_1 r^{1-n} \text{ for any } r \in (0, \tilde{r}_1)$$

then there exists $c_m, \tilde{c}_m, r_m, \tilde{r}_m > 0$ such that either

$$\xi(r) \geq c_m r^{2m-1-n} \text{ for any } r \in (0, r_m) \quad \text{ or } \quad \xi(r) \leq -\tilde{c}_m r^{2m-1-n} \text{ for any } r \in (0, \tilde{r}_m).$$

Proof. We will prove it by induction. Define $\xi_k = L^k \xi$, for $k = 0, 1, \cdots, m-1$.

(i) Suppose $\xi_{m-1}(r) = r^{-n}(r^n \xi'_{m-2})' \leq -\tilde{c}_1 r^{1-n}$, which implies $r^n \xi'_{m-2}$ is decreasing. There are two cases:

Case 1: $\liminf\limits_{r\to 0} r^n \xi'_{m-2} \leq 0$. Then we have

$$r^n \xi'_{m-2}(r) \leq -\frac{\tilde{c}_1}{2} r^2, \quad 0 < r < \tilde{r}_1$$

which yields

$$\xi_{m-2}(\tilde{r}_1) - \xi_{m-2}(r) = \int_r^{\tilde{r}_1} \xi'_{m-2} \, \mathrm{d}\rho \leq -\frac{\tilde{c}_1}{2} \int_r^{\tilde{r}_1} \rho^{2-n} \, \mathrm{d}\rho = \frac{\tilde{c}_1}{2(n-3)} \rho^{3-n} \Big|_r^{\tilde{r}_1}. \tag{4.35}$$

Therefore, there exist $c_2, r_2 > 0$ such that

$$\xi_{m-2}(r) \geq c_2 r^{3-n}, \qquad \text{for } 0 < r < r_2 < \tilde{r}_1. \tag{4.36}$$

Case 2: There exists $\hat{r} > 0$ and $\hat{c} > 0$ such that

$$r^n \xi'_{m-2}(r) \geq \hat{c}, \quad \text{for } 0 < r < \hat{r} < \tilde{r}_1.$$

Arguing as (4.35), there exist $\tilde{c}_2, \tilde{r}_2 > 0$ such that

$$\xi_{m-2}(r) \leq -\tilde{c}_2 r^{1-n} \leq -\tilde{c}_2 r^{3-n}, \quad \text{for } 0 < r < \tilde{r}_2 < \hat{r}. \tag{4.37}$$

(ii) Suppose $\xi_{m-1} \geq c_1 r^{1-n}$ happens, which implies $r^n \xi'_{m-2}$ is increasing as r goes large. There are two cases:

Case 1: $\liminf\limits_{r\to 0} r^n \xi'_{m-2} \geq 0$, then we have

$$r^n \xi'_{m-2}(r) \geq \frac{c_1}{2} r^2, \quad 0 < r < r_1$$

which yields

$$\xi_{m-2}(r_1) - \xi_{m-2}(r) = \int_r^{r_1} \xi'_{m-2}(\rho)\,\mathrm{d}\rho \geq \frac{c_1}{2}\int_r^{r_1} \rho^{2-n}\,\mathrm{d}\rho.$$

Therefore, there exist $\tilde{c}_2, \tilde{r}_2 > 0$,

$$\xi_{m-2}(r) \leq -\tilde{c}_2 r^{3-n} \quad \text{for } 0 < r < \tilde{r}_2 < \hat{r}. \tag{4.38}$$

Case 2: There exist $\hat{c}, \hat{r} > 0$ such that

$$r^n \xi'_{m-2}(r) \leq -\hat{c} \quad \text{for } 0 < r < \hat{r} < r_1.$$

Arguing as before there exist $c_2, r_2 > 0$ such that

$$\xi_{m-2} \geq c_2 r^{1-n} \geq c_2 r^{3-n} \quad \text{for } 0 < r < r_2 < \hat{r}.$$

For both (i) and (ii), we reached the same conclusion

$$\xi_{m-2} \geq c_2 r^{3-n} \text{ for } r \in (0, r_2) \quad \text{or} \quad \xi_{m-2} \leq -\tilde{c}_2 r^{3-n} \text{ for } r \in (0, \tilde{r}_2).$$

Repeating this procedure, we obtain

$$\xi_k \geq c_{m-k} r^{2(m-k)-1-n} \text{ for } r \in (0, r_k) \quad \text{or} \quad \xi_k \leq -\tilde{c}_{m-k} r^{2(m-k)-1-n} \text{ for } r \in (0, \tilde{r}_k),$$

when $0 \leq k \leq m-1$. Taking $k = 0$, we complete the proof the lemma. □

Proof of Proposition 4.3.1. Since $p \geq \frac{n}{n-2m+1}$ was proved in Lemma 4.3.8, now we assume $1 < p \leq \frac{n}{n-2m+1}$. Suppose contrary that (4.16) is not true, then it necessarily has

$$\int_{D_1 \setminus D_r} |x|^{-\tau} u^*(x,0)^p\,\mathrm{d}x = \int_{D_1 \setminus D_r} |x|^{-\tau} v^*(x,0)^p\,\mathrm{d}x \to \infty \quad \text{as } r \to 0. \tag{4.39}$$

Make use of the equation of v^* and Green's identity, we have

$$\begin{aligned}\int_{\partial^+ B_1^+} \frac{\partial \Delta^{m-1} v^*}{\partial \nu}\,\mathrm{d}S - \int_{\partial^+ B_r^+} \frac{\partial \Delta^{m-1} v^*}{\partial \nu}\,\mathrm{d}S &= \int_{D_1 \setminus D_r} \partial_t \Delta^{m-1} v^*(x,0)\,\mathrm{d}x \\ &= \int_{D_1 \setminus D_r} (-1)^m |x|^{-\tau} v^*(x,0)^p \mathrm{d}x.\end{aligned}$$

If m is odd, by (4.39) there exists $r_0 > 0$ such that for all $0 < r < r_0$,

$$⨍_{\partial^+ B_r^+} \frac{\partial \Delta^{m-1} v^*}{\partial \nu}\,\mathrm{d}S \geq \frac{r^{-n}}{2} \int_{D_{1/2} \setminus D_r} |x|^{-\tau} v^*(x,0)^p\,\mathrm{d}x.$$

It follows that

$$\fint_{\partial^+ B_{r_0}^+} \Delta^{m-1} v^* \, \mathrm{d}S - \fint_{\partial^+ B_r^+} \Delta^{m-1} v^* \, \mathrm{d}S \geq \frac{1}{2} \int_r^{r_0} \lambda^{-n} \int_{D_{1/2} \setminus D_\lambda} |x|^{-\tau} v^*(x,0)^p \, \mathrm{d}x \mathrm{d}\lambda,$$

which together with (4.39) yield

$$\fint_{\partial^+ B_r^+} \Delta^{m-1} v^* \, \mathrm{d}S \leq -r^{1-n} \tag{4.40}$$

for all $0 < r < r_1 < r_0$, where r_1 is some fixed constant. Since (4.21) is also true for v^*, then we have $L^{m-1}\bar{v^*}(r) \leq -r^{1-n}$.

If m is even, by (4.39) there exists $r_0 > 0$ such that for all $0 < r < r_0$,

$$\fint_{\partial^+ B_r^+} \frac{\partial \Delta^{m-1} v^*}{\partial \nu} \, \mathrm{d}S \leq -\frac{r^{-n}}{2} \int_{D_{1/2} \setminus D_r} |x|^{-\tau} v^*(x,0)^p \, \mathrm{d}x.$$

It follows that

$$\fint_{\partial^+ B_{r_0}^+} \Delta^{m-1} v^* \, \mathrm{d}S - \fint_{\partial^+ B_r^+} \Delta^{m-1} v^* \, \mathrm{d}S \leq -\frac{1}{2} \int_r^{r_0} \lambda^{-n} \int_{D_{1/2} \setminus D_\lambda} |x|^{-\tau} v^*(x,0)^p \, \mathrm{d}x \mathrm{d}\lambda,$$

which together with (4.39) yield

$$\fint_{\partial^+ B_r^+} \Delta^{m-1} v^* \, \mathrm{d}S \geq r^{-n+1} \tag{4.41}$$

for all $0 < r < r_1 < r_0$, where r_1 is some fixed constant. For the same reason above, we have $L^{m-1}\bar{v^*}(r) \geq r^{1-n}$.

For each case, from Lemma 4.3.12 we obtain

$$\bar{v^*}(r) \geq c_2 r^{2m-1-n} \quad \text{or} \quad \bar{v^*}(r) \leq -c_2 r^{2m-1-n} \tag{4.42}$$

provided r is sufficiently small. The later case can not happen because of the positivity of v^*. The former case can not happen either because of Lemma 4.3.11.

Therefore, we complete the proof of Proposition 4.3.1.

□

4.4 Proof of main theorem

Proposition 4.4.1. Under the assumptions in Proposition 4.3.1, we have

$$u^*(x,t) = \sum_{k=1}^{m-1} |X|^{2m-n-1} \left(\frac{t}{|X|^2} \right)^{2k} P_{2k} \left(\frac{x}{|X|^2} \right) + \gamma(n,m) \int_{\mathbb{R}^n} \frac{|y|^{-\tau} u^*(y,0)^p}{(t^2 + |x-y|^2)^{\frac{n-2m+1}{2}}} \, \mathrm{d}y$$

where P_{2k} is a polynomial of x with degree $\leq 2m - 2 - 2k$.

Proof. Define

$$V(x,t) := \gamma(n,m)\int_{\mathbb{R}^n} \frac{|y|^{-\tau}u^*(y,0)^p}{(t^2+|x-y|^2)^{\frac{n-2m+1}{2}}}\,\mathrm{d}y. \tag{4.43}$$

In view of (4.16) and $|y|^{-\tau}u^*(y,0)^p = O(|y|^{-(n+2m-1)})$ as $y\to\infty$, V is well defined. Set $W := u^* - V$. By Lemma 4.1.8, W satisfies

$$\begin{cases} \Delta^m W = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_t W = \partial_t\Delta W = \cdots = \partial_t \Delta^{m-1}W = 0 & \text{on } \partial\mathbb{R}^{n+1}_+\backslash\{0\}. \end{cases}$$

Since $u^*\geq 0$, then $W\geq -V$. By Lemma 4.1.7 we obtain V is in weak$-L^{\frac{n+1}{n-2m+1}}(\mathbb{R}^{n+1}_+)$. It follows from Corollary 4.2.5 that

$$W(X) = \sum_{|\alpha|\leq 2m-2} c_\alpha D^\alpha \Phi(X), \tag{4.44}$$

where c_α are constants and the $(n+1)$-th component of each α is even. By the definition of $\Phi(X)$ and $2m<n+1$, $D^\alpha\Phi$ can be rewritten as

$$D^\alpha\Phi(X) = \sum_{\beta\leq\alpha} c_\beta X^\beta |X|^{2m-n-1-2|\beta|} = \sum_{\beta\leq\alpha} c_\beta \left(\frac{X}{|X|^2}\right)^\beta |X|^{2m-n-1}.$$

where $\beta\leq\alpha$ means $\beta_i\leq\alpha_i$ for all $1\leq i\leq n$. Grouping and reordering the terms according to the degree of t in (4.44) yield

$$W = \sum_{k=0}^{m-1} |X|^{2m-n-1}\left(\frac{t}{|X|^2}\right)^{2k} P_{2k}\left(\frac{x}{|X|^2}\right). \tag{4.45}$$

where P_{2k} is a polynomial on x with degree $\leq 2m-2-2k$. Then the proposition follows from:

Claim: $P_0(x)\equiv 0$.

Let $l_0 = \deg P_0 \geq 0$. Collect all the terms of degree l_0 in P_0 to be a homogeneous polynomial $\tilde{P}_0$.

If there is a nonempty open cone $\mathcal{S}\subset\mathbb{R}^n$ with 0 as the vertex such that $\tilde{P}_0(\frac{x}{|x|^2}) > c > 0$ on $\mathcal{S}\cap D_{r_0}$ for some constant c, then we can find $r_0>0$ small enough such that

$$|P_0(x/|x|^2) - \tilde{P}_0(x/|x|^2)| < \frac{1}{2}\tilde{P}_0(x/|x|^2) \quad \text{in } \mathcal{S}\cap D_{r_0}. \tag{4.46}$$

Therefore

$$u^*(x,0) = W(x,0)+V(x,0) \geq W(x,0) \geq \frac{1}{2}|x|^{2m-n-1}\tilde{P}_0\left(\frac{x}{|x|^2}\right) \quad \text{in } \mathcal{S}\cap D_{r_0}.$$

It leads to

$$\begin{aligned}\int_{D_1} |y|^{-\tau} u^*(y,0)^p \,\mathrm{d}y \geq & \int_{\mathcal{S}\cap D_{r_0}} |y|^{-\tau+p(2m-n-1)} \tilde{P}_0\left(\frac{y}{|y|^2}\right)^p \mathrm{d}y \\ \geq & c^p \int_{\mathcal{S}\cap D_{r_0}} |y|^{-n-2m+1} = \infty\end{aligned}$$

which contradicts to Proposition 4.3.1. By the homogeneity of $\tilde{P}_0$, we conclude $\tilde{P}_0(\frac{x}{|x|^2}) \leq 0$.

Suppose that $P_0(\frac{x}{|x|^2}) \leq 0$ but not identical to 0. Without loss of generality, one may assume $\inf_{|x|=1} \tilde{P}_0(x) = -1$ and denote cone $E := \{x \in \mathbb{R}^n : \tilde{P}_0(\frac{x}{|x|^2}) < -\frac{1}{2}|x|^{-l_0}\}$. For the same fake, we can find $r_0 > 0$ small enough such that

$$|P_0(x/|x|^2) - \tilde{P}_0(x/|x|^2)| < \frac{1}{2}\tilde{P}_0(x/|x|^2) \quad \text{in } E \cap D_{r_0}. \tag{4.47}$$

Moreover, there exists $\varepsilon_0 > 0$ such that

$$|\{D_r \cap E\}| \geq \varepsilon_0 r^n \quad \forall\ 0 < r < 1.$$

For some $\lambda > 0$ to be chosen later, let $\rho = (4\lambda)^{-1/(n+l_0+1-2m)}$. On $D_\rho \cap E$, there holds

$$|x|^{2m-n-1} P_0\left(\frac{x}{|x|^2}\right) \leq \frac{1}{2}|x|^{2m-n-1}\tilde{P}_0\left(\frac{x}{|x|^2}\right) \leq -\frac{1}{4}|x|^{2m-n-1-l_0} < -\lambda.$$

Therefore by noticing $W(x,0) = |x|^{2m-n-1}P_0(x/|x|^2)$, we have

$$|\{x \in \mathbb{R}^n : W(x,0) < -\lambda\}| \geq |\{D_\rho \cap E\}| \geq \varepsilon_0 (4\lambda)^{-\frac{n}{n-2m+1+l_0}}. \tag{4.48}$$

Decompose $V(x,0)$ as

$$V(x,0) = \int_{|y|\leq\delta} \frac{|y|^{-\tau}u^*(y,0)^p}{|x-y|^{n-2m+1}} \,\mathrm{d}y + \int_{|y|>\delta} \frac{|y|^{-\tau}u^*(y,0)^p}{|x-y|^{n-2m+1}} \,\mathrm{d}y := V_1(x) + V_2(x),$$

where $\delta > 0$ to be fixed. For any $\varepsilon > 0$, choose $\delta > 0$ such that $\int_{D_\delta} |y|^{-\tau}u^*(y,0)^p < \varepsilon$. From the weak type estimate of Riesz potential,

$$\left|\{x : V_1(x,0) > \frac{1}{2}\lambda\}\right| \leq C(m,n)(\varepsilon\lambda^{-1})^{\frac{n}{n-2m+1}}, \tag{4.49}$$

where $C(m,n) > 0$ depends only on m and n. Since $|y|^{-\tau}u^*(y,0)^p$ is smooth and bounded outside D_δ, V_2 is bounded. It follows that for $\lambda \geq 100\|V_2\|_{L^\infty} + 1$,

$$\left|\{x : V_2(x,0) > \frac{1}{2}\lambda\}\right| \leq C(\varepsilon\lambda^{-1})^{\frac{n}{n-2m+1}}, \tag{4.50}$$

where C is independent of ε. Combining (4.49) and (4.50), we can choose ε even small such that

$$\left|\{x : V(x,0) > \frac{1}{2}\lambda\}\right| \le \varepsilon_0 10^{-\frac{n}{n-2m+1+l_0}} \lambda^{-\frac{n}{n-2m+1}} \tag{4.51}$$

for all $\lambda > 100\|V_2\|_{L^\infty} + 1$. Note that

$$\{x : W(x,0) < -\lambda, |V(x,0)| < \lambda/2\} \subset \{x : u^*(x,0) < 0\} = \emptyset.$$

It follows from (4.48) and (4.51) that for sufficiently large λ,

$$\begin{aligned}
0 &= |\{x : W(x,0) < -\lambda, |V(x,0)| \le \lambda/2\}| \\
&\ge |\{x : W(x,0) < -\lambda\}| - |\{x : |V(x,0)| > \lambda/2\}| \\
&\ge \varepsilon_0 (4\lambda)^{-\frac{n}{n-2m+1+l_0}} - \varepsilon_0 10^{-\frac{n}{n-2m+1+l_0}} \lambda^{-\frac{n}{n-2m+1}} \\
&> 0.
\end{aligned}$$

We obtain a contradiction again. Hence, $\tilde{P}_0(\frac{x}{|x|^2}) = 0$ and thus the claim is proved.

Therefore, we complete the proof of Proposition 4.4.1.

□

Proof of Theorem 1.3.1. Let V be defined in (4.43). By Proposition 4.4.1, $V(x,0) = u^*(x,0)$ and $V^*(x,0) := V_{0,1}(x,0)$ is smooth in $\mathbb{R}^n$. It follows from (4.43) that

$$V(x,t) = \gamma(n,m) \int_{\mathbb{R}^n} \frac{|y|^{-\tau} V(y,0)^p}{(t^2 + |x-y|^2)^{\frac{n-2m+1}{2}}} \, dy,$$

from lemma 4.1.9, it is equivalent to

$$V^*(x,t) = \gamma(n,m) \int_{\mathbb{R}^n} \frac{V^*(y,0)^p}{(t^2 + |x-y|^2)^{\frac{n-2m+1}{2}}} \, dy. \tag{4.52}$$

on the condition that the right hand side integral converges. This is justified through

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus D_1} \frac{V^*(y,0)^p}{(t^2 + |x-y|^2)^{\frac{n-2m+1}{2}}} \, dy &\le C \int_{\mathbb{R}^n \setminus D_1} u(y,0)^p |y|^{-(n-2m+1)} \, dy \\
&= C \int_{D_1} |x|^{-\tau} u^*(x,0)^p \, dx < \infty.
\end{aligned}$$

Sending $t \to 0$ in (4.52), we see that

$$V^*(x,0) = \gamma(n,m) \int_{\mathbb{R}^n} \frac{V^*(y,0)^p}{|x-y|^{n-2m+1}} \, dy.$$

Since $V^*(x,0)$ is smooth in $\mathbb{R}^n$, it follows from Chen-Li-Ou [25] and Li [58] that

$$V^*(x,0)=0 \quad \text{if } p<\frac{n+2m-1}{n-2m+1},$$

and

$$V^*(x,0)=c_0(n,m)\left(\frac{\lambda}{1+\lambda^2|x-x_0|^2}\right)^{\frac{n-2m+1}{2}} \qquad \text{for some } \lambda\geq 0, x_0\in\mathbb{R}^n,$$

where $c_0(n,m)>0$ is a constant depending only on n,m, if $p=\frac{n+2m-1}{n-2m+1}$. One may also apply the moving planes or spheres method to (4.52) directly to prove the classification result; see Dou-Zhu [34]. By Proposition 4.4.1, Theorem 1.3.1 follows immediately.

□

4.5 An application to conformal geometry

Given Theorem 1.3.1, we construct metrics which is singular on single boundary point of the unit ball below. Define the map $F:\mathbb{R}^{n+1}_+\to B_1$ by

$$F(x,t)=\left(\frac{2x}{|x|^2+(t+1)^2},\frac{|X|^2-1}{|x|^2+(t+1)^2}\right).$$

Observe that $F(x,0)\to\mathbb{S}^n$,

$$F(x,0)=\left(\frac{2x}{|x|^2+1},\frac{|x|^2-1}{|x|^2+1}\right)$$

is the inverse of the stereographic projection. Let

$$v(F(X))=|J_F|^{-\frac{n-2m+1}{2}}u(X)$$

wherer $|J_F|$ is the Jacobian determinant of F.

Proposition 4.5.1. Assume the assumptions in Theorem 1.3.1. Suppose that $u>0$ in $\mathbb{R}^{n+1}_+\cup\partial\mathbb{R}^{n+1}_+$ and $p=\frac{n+2m-1}{n-(2m-1)}$. Let v be defined as above and $g=v^{\frac{4}{n-(2m-1)}}\mathrm{d}X^2$ in B_1 be a conformal metric of the flat metric. Then the $2m$-th order Q-curvature of g in B_1 is zero and the boundary $(2m-1)$-th order Q-curvature is constant on $\partial B_1\setminus\{(0,1)\}$.

If the polynomial part of in the conclusion of the Theorem 1.3.1 is nontrivial, then v blows up near the boundary point $(0,1)$.

Proof. By the conformal invariance, it is easy to check that $\Delta^m v = 0$, see Li-Mastrolia-Monticelli [59]. It follows that the $2m$-th order Q-curvature of g in B_1 is zero.

By the proof of Proposition 4.4.1 we see that

$$u(x,0) = \gamma(n,m) \int_{\mathbb{R}^n} \frac{u(y,0)^{\frac{n+2m-1}{n-(2m-1)}}}{|x-y|^{n-(2m-1)}} \,\mathrm{d}y.$$

It follows that

$$v(X) = \gamma(n,m) \int_{\partial B_1} \frac{v(Y,0)^{\frac{n+2m-1}{n-(2m-1)}}}{|X-Y|^{n-(2m-1)}} \,\mathrm{d}S_Y$$

and thus the $(2m-1)$-th order Q-curvature is a constant, see Jin-Li-Xiong [52] for more details.

If the polynomial part of in the conclusion of the Theorem 1.3.1 is nontrivial, by the definition of v it is easy to see v blows up near the boundary point $(0,1)$. □

Remark 4.5.2. Note that the scalar curvature metric g in Proposition 4.5.1 could be negative.

If $m = 2$, we have explicit equations of v, see Chang-Qing [22], Branson-Gover [11] and Case [19]:

$$\begin{cases} \Delta^2 v = 0 & \text{in } B_1(0,1), \\ \mathbf{B}_1^3 v = 0 & \text{on } \partial B_1(0,1) \setminus \{(0,1)\}, \\ \mathbf{B}_3^3 v = v^{\frac{n+3}{n-3}} & \text{on } \partial B_1(0,1) \setminus \{(0,1)\}, \end{cases} \tag{4.53}$$

where

$$\mathbf{B}_1^3 v = \frac{\partial v}{\partial \nu} + \frac{n-3}{2} v,$$

$$\mathbf{B}_3^3 v = -\frac{\partial \Delta v}{\partial \nu} - \frac{n-3}{2}\frac{\partial^2 v}{\partial \nu^2} - \frac{3n-5}{2}\Delta_{\mathbb{S}^n} v + \frac{3n^2-7n+6}{4}\frac{\partial v}{\partial \nu} + \frac{n^2-n+2}{4}\frac{n-3}{2} v.$$

Therefore, the metric g has flat 4-th order Q-curvature, flat mean curvature and constant 3-th order Q-curvature on the boundary. By Theorem 1.3.1, solutions of (4.53) satisfying

$$v(F(X)) = o(|X|^n)$$

are classified. If $m \geq 3$, the analogues of (4.53) can be found in Branson-Gover [11] but are more complicated. Similarly, Theorem 1.3.1 can be applied to them.

References

[1] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I., Comm. Pure Appl. Math. 12 (1959) 623–727.

[2] S. Almaraz, Convergence of scalar-flat metrics on manifolds with boundary under a yamabe-type flow, Journal of Differential Equations 259 (2015) 2626–2694.

[3] S. Almaraz, E. Barbosa, L. de Lima, A positive mass theorem for asymptotically flat manifolds with a non-compact boundary. To appear in Comm. Anal. Geom.

[4] S. Almaraz, L. Sun, Convergence of the yamabe flow on manifolds with minimal boundary, arXiv:1604.06789 (2016).

[5] A. Ambrosetti, Y. Li, A. Malchiodi, On the yamabe problem and the scalar curvature problem under boundary condtions, Math. Ann 322 (4) (2002) 667–699.

[6] H. Araujo, Critical points of the total scalar curvature plus total mean curvature functional, Indiana Univ Math. J. 52 (1) (2003) 85–107.

[7] H. Araujo, Existence and compactness of minimizers of the yamabe problem on manifolds with boundary, Comm. Anal. Geom. 12 (3) (2004) 487–510.

[8] T. Aubin, Some nonlinear problems in Riemannian geometry, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.

[9] W. Beckner, Sharp sobolev inequalities on the sphere and the Moser–Trudinger inequality, Annals of Mathematics 138 (1) (1993) 213–242.

[10] A. L. Besse, Einstein manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 10. Springer-Verlag, Berlin, 1987.

[11] T. Branson, A. R. Gover, Conformally invariant non-local operators, Pacific J. Math 201 (1) (2001) 19–60.

[12] S. Brendle, A generalization of the yamabe flow for manifolds with boundary, Asian J. Math. 6 (2002) 625–644.

[13] S. Brendle, Convergence of the yamabe flow for arbitrary initial energy, J. Differential Geo. 69 (2005) 217–278.

[14] S. Brendle, Convergence of the yamabe flow in dimension 6 and higher, Invent. Math. 170 (3) (2007) 541–576.

[15] S. Brendle, S. Chen, An existence theorem for the yamabe problem on manifolds with boundary, J. Eur. Math. Soc.(JEMS) 16 (5) (2014) 991–1016.

[16] L. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical sobolev growth, Comm. Pure Appl. Math. 42 (1989) 271–297.

[17] L. Caffarelli, T. Jin, Y. Sire, J. Xiong, Local analysis of solutions of fractional semilinear elliptic equations with isolated singularities, Archive for Rational Mechanics and Analysis 213 (1) (2014) 245–268.

[18] L. Caffarelli, L. Silvestre, An extension problem related to the fractional laplacian, Communications in Partial Differential Equations 32 (8) (2007) 1245–1260.

[19] J. S. Case, Boundary operators associated to the paneitz operator, Preprint.

[20] J. S. Case, S.-Y. A. Chang, On fractional $GJMS$ operators, Communications on Pure and Applied Mathematics 69 (6) (2016) 1017–1061.

[21] S.-Y. A. Chang, M. del Mar González, Fractional laplacian in conformal geometry, Advances in Mathematics 226 (2) (2011) 1410–1432.

[22] S.-Y. A. Chang, J. Qing, The zeta functional determinants on manifolds with boundary, Journal of Functional Analysis 147 (2) (1997) 327–362.

[23] S. Chen, Conformal deformation to scalar flat metrics with constant mean curvature on the boundary in higher dimensions, arXiv:0912.1302v2.

[24] W. Chen, Y. Fang, C. Li, Super poly-harmonic property of solutions for navier boundary problems on a half space, Journal of Functional Analysis 265 (8) (2013) 1522–1555.

[25] W. Chen, C. Li, B. Ou, Classification of solutions for an integral equation, Communications on Pure and Applied Mathematics 59 (3) (2006) 330–343.

[26] X. Chen, P. Ho, Conformal curvature flows on compact manifold of negative yamabe constant, to appear Indiana Univ. Math. J.

[27] X. Chen, P. Ho, L. Sun, Prescribed scalar curvature plus mean curvature flows in compact manifolds with boundary of negative conformal invariant, preprint arXiv:1605.01719v2 (2016).

[28] X. Chen, L. Sun, Existence of conformal metrics with constant scalar curvature and constant boundary mean curvature on compact manifolds, Preprint.

[29] P. Cherrier, Problèmes de neumann non linéaires sur les variétés riemannienes, J. Funct. Anal 57 (1984) 154–206.

[30] P. Cherrier, Problèmes de neumann non linéaires sur les variétés riemanniennes, Journal of Functional Analysis 57 (2) (1984) 154–206.

[31] B. Chow, The yamabe flow on locally conformally flat manifolds with positive ricci curvature, Comm. Pure Appl. Math 45 (1992) 1003–1014.

[32] A. DelaTorre, M. del Pino, M. González, J. Wei, Delaunay-type singular solutions for the fractional yamabe problem, Preprint.

[33] A. DelaTorre, M. González, Isolated singularities for a semilinear equation for the fractional laplacian arising in conformal geometry, Preprint.

[34] J. Dou, M. Zhu, Sharp Hardy Littlewood Sobolev inequality on the upper half space, International Mathematics Research Notices.

[35] J. Escobar, Sharp constant in a sobolev trace inequality, Indiana Univ. Math. J. 37 (1988) 687–698.

[36] J. Escobar, Uniqueness theorems on conformal deformation of metrics, Sobolev inequalities, and an eigenvalue estimate, Comm. Pure Appl. Math 43 (1990) 857–883.

[37] J. Escobar, Conformal deformation of a riemannian metric to a scalar flat metric with constant mean curvature on the boundary, Ann. of Math 136 (1) (1992) 1–50.

[38] J. Escobar, The yamabe problem on manifolds with boundary, J. Differential Geom 35 (1) (1992) 21–84.

[39] J. Escobar, Addendum:"conformal deformation of a riemannian metric to a scalar flat metric with constant mean curvature on the boundary" [ann. of math. (2) 136 (1992), no. 1, 1−50; mr1173925 (93e:53046)], Ann. of Math 2 (3) (1994) 749–750.

[40] J. Escobar, Conformal deformation of a riemannian metric to a constant scalar curvature metric with constant mean curvature on the boundary, Indiana Univ. Math. J 45 (4) (1996) 917–943.

[41] C. Fefferman, C. R. Graham, Conformal invariants, The mathematical heritage of Elie Cartan, Numero Hors Serie, 95116, Asterisque, 1985.

[42] T. Futamura, K. Kishi, Y. Mizuta, A generalization of Bôcher's theorem for polyharmonic functions, Hiroshima Math. J. 31 (1) (2001) 59–70.

[43] F. Gazzola, H.-C. Grunau, G. Sweers, Polyharmonic Boundary Value Problems, Lectures notes in Mathematics no. 1991, AMS, 2010.

[44] M. Ghergu, A. Moradifam, S. D. Taliaferro, Isolated singularities of polyharmonic inequalities, Journal of Functional Analysis 261 (3) (2011) 660–680.

[45] D. Gilbarg, N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag Berlin Heidelberg, Springer-Verlag, Berlin, 2001.

[46] G. Giraud, Sur le problème de Dirichlet généralisé (deuxième mémoire), Ann. Sci. École Norm. Sup. (3) 46 (1929) 131–245.

[47] C. R. Graham, R. Jenne, L. J. Mason, G. A. J. Sparling, Conformally invariant powers of the laplacian, I: Existence, Journal of the London Mathematical Society s2-46 (3) (1992) 557–565.

[48] R. C. Graham, M. Zworski, Scattering matrix in conformal geometry, Inventiones mathematicae 152 (1) (2003) 89–118.

[49] Z. C. Han, Y. Y. Li, The yamabe problem on manifolds with boundary: existence and compactness results, Duke Math. J. 99 (3) (1999) 489–542.

[50] Z. C. Han, Y. Y. Li, The existence of conformal metrics with constant scalar curvature and constant boundary mean curvature, Comm. Anal. Geom. 8 (4) (2000) 809–869.

[51] F. Hang, X. Wang, X. Yan, An integral equation in conformal geometry, Annales de l'Institut Henri Poincare (C) Non Linear Analysis 26 (1) (2009) 1–21.

[52] T. Jin, Y. Y. Li, J. Xiong, The nirenberg problem and its generalizations: A unified approach, Preprint.

[53] T. Jin, Y. Y. Li, J. Xiong, On a fractional nirenberg problem, part I: blow up analysis and compactness of solutions, J. Eur. Math. Soc. (JEMS) 16 (2014) 1111–1171.

[54] T. Jin, A. Maalaoui, L. Martinazzi, J. Xiong, Existence and asymptotics for solutions of a non-local Q-curvature equation in dimension three, Calculus of Variations and Partial Differential Equations 52 (3) (2015) 469–488.

[55] J. Lee, T. Parker, The yamabe problem, Bull. Amer. Math. Soc. (N.S.) 17 (1) (1987) 37–91.

[56] J. M. Lee, T. H. Parker, The yamabe problem, Bull. Amer. Math. Soc. (N.S.) 17 (1) (1987) 37–91.

[57] A. Li, Y. Y. Li, On some conformally invariant fully nonlinear equations, Communications on Pure and Applied Mathematics 56 (10) (2003) 1416–1464.

[58] Y. Y. Li, Remark on some conformally invariant integral equations: the method of moving spheres, J. Eur. Math. Soc. (JEMS) 6 (2) (2004) 153–180.

[59] Y. Y. Li, P. Mastrolia, D. D. Monticelli, On conformally invariant equations on -ii. exponential invariance, Nonlinear Analysis: Theory, Methods & Applications 75 (13) (2012) 5194–5211.

[60] Y. Y. Li, J. Xiong, Compactness of conformal metrics with constant Q-curvature. I., Preprint.

[61] Y. Y. Li, L. Zhang, Liouville type theorems and Harnack type inequalities for semilinear elliptic equations, J. Anal. Math 90 (2003) 27–87.

[62] Y. Y. Li, M. Zhu, Uniqueness theorems through the method of moving spheres, Duke Math. J. 80 (2) (1995) 383–417.

[63] C. S. Lin, A classification of solutions of a conformally invariant fourth order equation in $\mathbb{R}^n$", Commentarii Mathematici Helvetici 73 (2) (1998) 206–231.

[64] G. Lu, P. Wang, J. Zhu, Liouville-type theorems and decay estimates for solutions to higher order elliptic equations, Annales de l'Institut Henri Poincare (C) Non Linear Analysis 29 (5) (2012) 653–665.

[65] F. Marques, Existence results for the yamabe problem on manifolds with boundary, Indiana Univ. Math. J. 54 (6) (2005) 1599–1620.

[66] F. Marques, Conformal deformations to scalar-flat metrics with constant mean curvature on the boundary, Comm. Anal. Geom. 15 (2) (2007) 381–405.

[67] M. Mayer, C. Ndiaye, Proof of the remaining cases of the boundary yamabe problem, preprint (2015).

[68] E. H. Olivier Druet, F. Robert, Blow-up Theory for Elliptic PDEs in Riemannian Geometry (MN-45), Princeton University Press, 2004.

[69] B. Ou, Positive harmonic functions on the upper half space satisfying a nonlinear boundary condition, Differential Integral Equations 9 (5) (1996) 1157–1164.

[70] D. Pierotti, S. Terracini, On a neumann problem with critical exponent and critical nonlinearity on the boundary, Communications in Partial Differential Equations 20 (7-8) (1995) 1155–1187.

[71] S. Raulot, Green functions for the dirac operator under local boundary conditions and applications, Ann. Global Anal. Geom. 39 (4) (2011) 337–359.

[72] W. Reichel, T. Weth, Existence of solutions to nonlinear, subcritical higher order elliptic dirichlet problems, Journal of Differential Equations 248 (7) (2010) 1866–1878.

[73] R. Schoen, Variational theory for the total scalar curvature functional for Riemannian metrics and related topics. Comm. Pure Appl. Math. 39(5) (1986), 661–693., Topics in Calculus of Variations, Lecture Notes in Mathematics v. 1365, Springer-Verlag, New York pp. 120–154, 1989.

[74] R. Schoen, Conformal deformation of a riemannian metric to constant scalar curvature, J. Differential Geom 20 (1994) 479–495.

[75] R. Schoen, S. Yau, On the proof of the positive mass conjecture in general relativity, Comm. Math. Phys 65 (1979) 45–76.

[76] R. M. Schoen, The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation, Communications on Pure and Applied Mathematics 41 (3) (1988) 317–392.

[77] H. Schwetlick, M. Struwe, Convergence of the yamabe flow for large energies, J. Reine Angew. Math 562 (2003) 59–100.

[78] L. Simon, Asymptotics for a class of non-linear evolution equations with applications to geometric problems, Ann. of Math 118 (1983) 525–571.

[79] M. Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities, Math. Z 187 (1984) 511–517.

[80] L. Sun, J. Xiong, Classification theorems for solutions of higher order boundary conformally invariant problems, I, Journal of Functional Analysis.

[81] N. Trudinger, Remarks concerning the conformal deformation of a riemannian structure on compact manifolds, Ann. Scuola Norm. Sup. Pisa Cl. Sci 22 (3) (1968) 265–274.

[82] J. Wei, X. Xu, Classification of solutions of higher order conformally invariant equations, Mathematische Annalen 313 (2) (1999) 207–228.

[83] E. Witten, A new proof of the positive energy theorem, Comm. Math. Phys 80 (1981) 381–402.

[84] X. Xu, Uniqueness and non-existence theorems for conformally invariant equations, Journal of Functional Analysis 222 (1) (2005) 1–28.

[85] H. Yamabe, On a deformation of riemannian structures on compact manifolds, Osaka Math. J 12 (1960) 21–37.

[86] R. Yang, On higher order extensions for the fractional laplacian, Preprint.

[87] R. Ye, Global existence and convergence of the yamabe flow, J. Differential Geom 39 (1994) 35–50.

Publication List

1. Liming Sun, Jingang Xiong, Classification theorems for solutions of higher order boundary conformally invariant problems, I. Journal of Functional Analysis, to appear.

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3. Xuezhang Chen, Pak Tung Ho, Liming Sun, Prescribed scalar curvature plus mean curvature flows in compact manifolds with boundary of negative conformal invariant. Submitted. arXiv:1604.06789

4. Sérgio Almaraz, Liming Sun, Convergence of the Yamabe flow on manifolds with minimal boundary. Submitted. arXiv:1604.06789