HOMOGENEOUS SOLUTIONS OF STATIONARY NAVIER-STOKES EQUATIONS WITH ISOLATED SINGULARITIES ON THE UNIT SPHERE.

by

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ABSTRACT OF THE DISSERTATION

Homogeneous solutions of stationary Navier-Stokes equations with isolated singularities on the unit sphere.

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We classify all $(-1)-$homogeneous axisymmetric no-swirl solutions of incompressible stationary Navier-Stokes equations in three dimension which are smooth on the unit sphere minus the south and north poles. We establish existence and nonexistence results of (-1)-homogeneous axisymmetric solutions with nonzero swirl emamnating from axisymmetric no-swirl solutions. We also establish asymptotic expansions for every (-1)-homogeneous axisymmetric solutions in a neighborhood of a singular point on the unit sphere. This dissertation is based on my papers [4], [5] and [6] joint with Li Li and Yanyan Li.

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Dedication

To My Parents.

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Chapter 1

Introduction

Consider (-1)-homogeneous solutions of incompressible stationary Navier-Stokes Equations (NSE) in $\mathbb{R}^3$:

$$\begin{cases} -\triangle u + u \cdot \nabla u + \nabla p = 0, \\ \text{div } u = 0. \end{cases} \tag{1.1}$$

The NSE is invariant under the scaling $u(x) \to \lambda u(\lambda x)$ and $p(x) \to \lambda^2 p(\lambda x)$, $\lambda > 0$.. It is natural to study (-1)-homogeneous solutions, namely, solutions which are invariant under this scaling.

In 1944, L.D. Landau discovered a 3-parameter family of explicit (-1)-homogeneous solutions of stationary NSE in $C^\infty(\mathbb{R}^3 \setminus \{0\})$. They are axisymmetric with no-swirl. He arrived at these solutions, now called *Landau solutions*, using the following ansatz: looking for solutions which are axisymmetric, no-swirl, and with two vanishing diagonal components of the t ensor of momentum flow density. Tian and Xin proved in [16] that all (-1)-homogeneous, axisymmetric nonzero solutions of the stationary NSE (1.1) in $C^2(\mathbb{R}^3 \setminus \{0\})$ are Landau solutions. Šverák established the following result in 2006:

Theorem A ([15]) *All (-1)-homogeneous nonzero solutions of (*1.1*) in* $C^2(\mathbb{R}^3 \setminus \{0\})$ *are Landau solutions.*

He also proved in the same paper that there is no nonzero (-1)-homogeneous solution of the stationary NSE in $C^2(\mathbb{R}^n \setminus \{0\})$ for $n \geq 4$. In dimension $n = 2$, he characterized all such solutions satisfying a zero flux condition.

Instead of studying (-1)-homogeneous solutions smooth on $\mathbb{S}^n$, we would like to analyze (-1)-homogeneous solutions in $\mathbb{R}^n$ with finite singularities on $\mathbb{S}^{n-1}$, as well as

(-1)-homogeneous solutions in half space $\mathbb{R}^n_+$ with finite singularities on $\mathbb{S}^n_+$ and zero velocity on $\partial\mathbb{R}^n_+$. In this thesis, we focus on axisymmetric solutions of the problem in $\mathbb{R}^3$ which have exactly one or two singularities on the unit sphere $\mathbb{S}^2$.

In polar coordinates (r, θ, ϕ), where r is the radial distance from the origin, θ is the angle between the radius vector and the positive x_3-axis, and ϕ is the meridian angle about the x_3-axis. A vector field u can be written as

$$u = u_r e_r + u_\theta e_\theta + u_\phi e_\phi, \tag{1.2}$$

where

$$e_r = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \quad e_\theta = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \quad e_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}.$$

A vector field u is called axisymmetric if u_r, u_θ and u_ϕ depend only on r and θ, and is called *no-swirl* if $u_\phi = 0$.

If u is (-1)-homogeneous and p is (-2)-homogeneous, system (1) is a system of partial differential equations of $u|_{\mathbb{S}^2}$ and $p|_{\mathbb{S}^2}$ on $\mathbb{S}^2$. For a (-1)-homogeneous and axisymmetric solution, (u, p) depends only on θ in polar coordinates, and the system on $\mathbb{S}^2$ takes the form

$$\begin{cases} \dfrac{d^2u_r}{d\theta^2} + (\cot\theta - u_\theta)\dfrac{du_r}{d\theta} + u_r^2 + u_\theta^2 + u_\phi^2 + 2p = 0; \\ \dfrac{d}{d\theta}(\dfrac{1}{2}u_\theta^2 - u_r + p) = \cot\theta u_\phi^2; \\ \dfrac{d}{d\theta}(\dfrac{du_\phi}{d\theta} + \cot\theta u_\phi) - u_\theta(\dfrac{du_\phi}{d\theta} + \cot\theta u_\phi) = 0; \\ u_r + \dfrac{du_\theta}{d\theta} + \cot\theta u_\theta = 0. \qquad \text{(divergence free condition)} \end{cases} \tag{1.3}$$

Since p is determined by u and its derivatives up to second order, in view of the first line of (1.3), we often say that u is a solution of (1.1) without mentioning p.

By the divergence free condition in (1.3), the radial component u_r of the velocity u is determined by u_θ and its first derivative.

Our first result classifies all (-1)-homogeneous axisymmetric no-swirl solutions u of (1.1) in $C^2(\mathbb{S}^2 \setminus \{S\})$, where S denotes the south pole of $\mathbb{S}^2$. In this case $u_\phi = 0$, and u_r, p can be determined by u_θ and its derivatives. So we only need to solve u_θ.

We introduce the following subsets of $\mathbb{R}^2$

$$\begin{aligned} J_1 &:= \{(\tau,\sigma) \mid \tau < 2, \sigma < \frac{1}{4}(4-\tau)\}, \\ J_2 &:= \{(\tau,\sigma) \mid \tau = 2, \sigma < \frac{1}{2}\}, \\ J_3 &:= \{(\tau,\sigma) \mid \tau \geq 2, \sigma = \frac{\tau}{4}\}, \end{aligned} \tag{1.4}$$

and $J := J_1 \cup J_2 \cup J_3$.

Theorem 1.0.1. *For every $(\tau,\sigma) \in J$, there exists a unique $u_\theta := (u_\theta)_{\tau,\sigma} \in C^\infty\left(\mathbb{S}^2 \setminus \{S\}\right)$ such that*

$$\lim_{\theta\to\pi^-} u_\theta \sin\theta = \tau, \qquad \lim_{\theta\to 0^+} \frac{u_\theta}{\sin\theta} = \sigma, \tag{1.5}$$

and the corresponding (u,p) satisfies (1.1) on $\mathbb{S}^2\setminus\{S\}$. Moreover, these are all axisymmetric no-swirl solutions in $C^2(\mathbb{S}^2 \setminus \{S\})$.

The solutions u_θ are explicitly given by, with $b := |1-\frac{\tau}{2}|$,

$$u_\theta = \begin{cases} \dfrac{1-\cos\theta}{\sin\theta}\left(1-b-\dfrac{2b(1-2\sigma-b)}{(1-2\sigma+b)(\frac{1+\cos\theta}{2})^{-b}+2\sigma-1+b}\right), & (\tau,\sigma)\in J_1; \\ \dfrac{1-\cos\theta}{\sin\theta}\left(1+\dfrac{2(1-2\sigma)}{(1-2\sigma)\ln\frac{1+\cos\theta}{2}-2}\right), & (\tau,\sigma)\in J_2; \\ \dfrac{(1+b)(1-\cos\theta)}{\sin\theta}, & (\tau,\sigma)\in J_3. \end{cases} \tag{1.6}$$

By saying (u,p) corresponding to u_θ we mean that on $\mathbb{S}^2$, u_r is determined by u_θ through the divergence free condition in (1.3), and p is determined by the first line in (1.3) using $u_\phi = 0$, and then u and p are extended respectively to (-1)-homogeneous and (-2)-homogeneous functions. Also, we often simply say that u satisfies (1.1) instead of saying that (u,p) satisfies (1.1).

We use $(u_{\tau,\sigma}, p_{\tau,\sigma})$ to denote the vector-valued function corresponding to $(u_\theta)_{\tau,\sigma}$.

In (1.6), $\{(u_\theta)_{\tau,\sigma} | \tau = 0, \sigma \in (-\infty,0)\cup(0,1)\}$ are Landau solutions. They can also be rewritten as

$$u_\theta = \frac{2\sin\theta}{\lambda+\cos\theta}, \quad |\lambda| > 1.$$

For an axisymmetric no-swirl solution $(u_{\tau,\sigma}, p_{\tau,\sigma})$ of (1.1) in $C^\infty(\mathbb{S}^2 \setminus \{S\})$, where $(u_\theta)_{\tau,\sigma}$ is given by Theorem 1.0.1, the linearized equation of (1.1) at $(u_{\tau,\sigma}, p_{\tau,\sigma})$ is

$$\begin{cases} -\triangle v + u_{\tau,\sigma}\cdot\nabla v + v\cdot\nabla u_{\tau,\sigma} + \nabla q = 0, \\ \text{div } v = 0. \end{cases} \tag{1.7}$$

Define

$$a_{\tau,\sigma}(\theta) = -\int_{\frac{\pi}{2}}^{\theta} (2\cot t + (u_\theta)_{\tau,\sigma})dt, \quad b_{\tau,\sigma}(\theta) = -\int_{\frac{\pi}{2}}^{\theta} (u_\theta)_{\tau,\sigma}dt,$$

and

$$v^1_{\tau,\sigma} = \begin{pmatrix} \frac{1}{\sin\theta}e^{-a_{\tau,\sigma}(\theta)}\frac{da_{\tau,\sigma}(\theta)}{d\theta} \\ \frac{1}{\sin\theta}e^{-a_{\tau,\sigma}(\theta)} \\ 0 \end{pmatrix}, \quad v^2_{\tau,\sigma} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sin\theta}\int_0^\theta e^{-b_{\tau,\sigma}(t)}\sin t dt \end{pmatrix}, \quad v^3_{\tau,\sigma} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sin\theta} \end{pmatrix}.$$

Then $\{v^1_{\tau,\sigma}, v^2_{\tau,\sigma}, v^3_{\tau,\sigma}\}$ are linearly independent solutions of (1.7), in polar coordinates, on $\mathbb{S}^2 \setminus \{S\}$.

$\{(u_\theta)_{\tau,\sigma} | (\tau,\sigma) \in J\}$ is a 2-parameter family of axisymmetric no-swirl solutions of (1.1) in $C^2(\mathbb{S}^2 \setminus \{S\})$. In the following theorem, we prove the existence of a curve of axisymmetric solutions with nonzero swirl in $C^2(\mathbb{S}^2 \setminus \{S\})$ emanating from $(u_\theta)_{\tau,\sigma}$ for each $(\tau,\sigma) \in J_1 \cup J_2 \cup \{J_3 \cap \{2 \le \tau < 3\}\}$. We also prove the nonexistence of such solutions for $(\tau,\sigma) \in J_3 \cap \{\tau > 3\}$.

Theorem 1.0.2. *Let K be a compact subset of one of the four sets J_1, J_2, $J_3 \cap \{2 < \tau < 3\}$ and $J_3 \cap \{\tau = 2\}$, then there exist $\delta = \delta(K) > 0$, and $(u,p) \in C^\infty(K \times (-\delta,\delta) \times (\mathbb{S}^2 \setminus \{S\}))$ such that for every $(\tau,\sigma,\beta) \in K \times (-\delta,\delta)$, $(u,p)(\tau,\sigma,\beta;\cdot) \in C^\infty(\mathbb{S}^2 \setminus \{S\})$ satisfies (1.1) in $\mathbb{R}^3 \setminus \{(0,0,x_3) | x_3 \le 0\}$, with nonzero swirl if $\beta \ne 0$, and $||\left(\sin\frac{\theta+\pi}{2}\right)(u(\tau,\sigma,\beta) - u_{\tau,\sigma})||_{L^\infty(\mathbb{S}^2\setminus\{S\})} \to 0$ as $\beta \to 0$. Moreover, $\frac{\partial}{\partial\beta}u(\tau,\sigma,\beta)|_{\beta=0} = v^2_{\tau,\sigma}$.*

On the other hand, for $(\tau,\sigma) \in J_3 \cap \{\tau > 3\}$, there does not exist any sequence of solutions $\{u^i\}$ of (1.1) in $C^\infty(\mathbb{S}^2 \setminus \{S\})$, with nonzero swirl, such that $||\left(\sin\frac{\theta+\pi}{2}\right)(u^i - u_{\tau,\sigma})||_{L^\infty(\mathbb{S}^2\setminus\{S\})} \to 0$ as $i \to \infty$.

In the above theorem, $(u,p) \in C^\infty(\mathbb{S}^2 \setminus \{S\})$ is understood to have been extended to $\mathbb{R}^3 \setminus \{(0,0,x_3 | x_3 \le 0)\}$ so that u is (-1)-homogeneous and p is (-2)-homogeneous. We use this convention throughout the paper unless otherwise stated.

Remark 1.0.1. *As far as we know, all previously known (-1)-homogeneous solutions* $u \in C^\infty(\mathbb{S}^2 \setminus \{S\}) \setminus C^\infty(\mathbb{S}^2)$ *of (*1.1*) satisfying* $\limsup_{y \to S} \mathrm{dist}(y,S)^N u(y) < \infty$ *for some* $N > 0$ *are axisymmetric with no swirl. The existence of such solutions with nonzero swirl are given by Theorem* 1.0.2*. A more detailed and stronger version of Theorem* 1.0.2*, including a uniqueness result, is given by Theorem* 3.2.1*, Theorem* 3.2.2 *and Theorem* 3.2.3 *in Chapter 3.*

In this thesis we work with new functions and a different variable:

$$x := \cos\theta, \quad U_r := u_r \sin\theta, \quad U_\theta := u_\theta \sin\theta, \quad U_\phi := u_\phi \sin\theta. \tag{1.8}$$

In particular, $x = 1$ and -1 correspond to the north and south poles N and S of $\mathbb{S}^2$ respectively, while $-1 < x < 1$ corresponds to $\mathbb{S}^2 \setminus \{S, N\}$. We will use " $'$ " to denote differentiation in x.

Our next two theorems are on the asymptotic behavior of a solution u in a punctured ball $B_\delta(S) \setminus \{S\}$ of $\mathbb{S}^2$, $\delta > 0$.

In the next two theorems, we will state that $U = (U_\theta, U_\phi)$ is a solution of (1.1), meaning that the u determined by U through (1.8) and (1.2), extended as a (-1)-homogeneous function, satisfies (1.1).

Theorem 1.0.3. *For* $\delta > 0$*, let* $U_\theta \in C^1(-1, -1+\delta]$*,* $U_\phi \in C^2(-1, -1+\delta]$*, and* $U = (U_\theta, U_\phi)$ *be an axisymmetric solution of (*1.1*). Then*

(i) $U_\theta(-1) := \lim_{x \to -1^+} U_\theta(x)$ *exists and is finite.*

(ii) $\lim_{x \to -1^+} (1+x) U_\theta'(x) = 0$*.*

(iii) If $U_\theta(-1) < 2$ *and* $U_\theta(-1) \neq 0$*, denote* $\alpha_0 = 1 - \frac{U_\theta(-1)}{2}$*, then there exist some constants* a_1, a_2 *such that for every* $\epsilon > 0$*,*

$$U_\theta(x) = U_\theta(-1) + a_1(1+x)^{\alpha_0} + a_2(1+x) + O((1+x)^{2\alpha_0 - \epsilon}) + O((1+x)^{2-\epsilon}).$$

If $U_\theta(-1) = 0$*, then there exist some constants* a_1, a_2 *such that for every* $\epsilon > 0$*,*

$$U_\theta(x) = a_1(1+x)\ln(1+x) + a_2(1+x) + O((1+x)^{2-\epsilon}).$$

If $U_\theta(-1) = 2$*, then, for every* $\epsilon > 0$*, either*

$$U_\theta(x) = 2 + \frac{4}{\ln(1+x)} + O((\ln(1+x))^{-2+\epsilon}),$$

or

$$U_\theta(x) = 2 + O((1+x)^{1-\epsilon}).$$

If $2 < U_\theta(-1) < 3$*, then there exist constants* a_1, a_2 *such that for every* $\epsilon > 0$*,*

$$U_\theta(x) = U_\theta(-1) + a_1(1+x)^{3-U_\theta(-1)} + a_2(1+x) + O((1+x)^{2(3-U_\theta(-1))-\epsilon}).$$

Recall that we denote $\alpha_0 = 1 - \frac{U_\theta(-1)}{2}$.

Theorem 1.0.4. *For* $\delta > 0$*, let* $U_\theta \in C^1(-1, -1+\delta)$*,* $U_\phi \in C^2(-1, -1+\delta)$*, and* $U = (U_\theta, U_\phi)$ *be an axisymmetric solution of (*1.1*). Then*

(i) If $U_\theta(-1) < 2$*, then* $U_\phi(-1)$ *exists and is finite, and there exist some constants* b_1, b_2, b_3 *such that*

$$U_\phi(x) = \begin{cases} U_\phi(-1) + b_1(1+x)^{\alpha_0} + b_2(1+x)^{2\alpha_0} + b_3(1+x)^{1+\alpha_0} \\ \quad + O((1+x)^{\alpha_0+2-\epsilon}) + O((1+x)^{3\alpha_0-\epsilon}), & \text{if } U_\theta(-1) \neq 0; \\ U_\phi(-1) + b_1(1+x) + b_2(1+x)^2\ln(1+x) + b_3(1+x)^2 \\ \quad + O((1+x)^{3-\epsilon}), & \text{if } U_\theta(-1) = 0. \end{cases}$$

(ii) If $2 < U_\theta(-1) < 3$*, then there exist some constants* b_1, b_2, b_3, b_4 *such that*

$$\begin{aligned} U_\phi(x) = & b_1(1+x)^{1-\frac{U_\theta(-1)}{2}} + b_2 + b_1b_3(1+x)^{4-\frac{3U_\theta(-1)}{2}} + b_1b_4(1+x)^{2-\frac{U_\theta(-1)}{2}} \\ & + b_1O((1+x)^{7-\frac{5U_\theta(-1)}{2}-\epsilon}). \end{aligned}$$

In particular, U_ϕ *is either a constant or an unbounded function in* $(-1, -1+\delta)$*.*

(iii) If $U_\theta(-1) \geq 3$*, then* U_ϕ *must be a constant in* $(-1, -1+\delta)$*.*

(iv) If $U_\theta(-1) = 2$*, then* $\eta := \lim_{x\to-1^+}(U_\theta - 2)\ln(1+x)$ *exists and is* 0 *or* 4*. If* $\eta = 0$*, then* U_ϕ *is either constant or unbounded, and there exist some constants* b_1, b_2 *such that*

$$U_\phi = b_1\ln(1+x) + b_2 + b_1O((1+x)^{1-\epsilon}).$$

If $\eta = 4$*, then* U_ϕ *is in* $L^\infty(-1, -1+\delta)$*, and there exists some constant* b *such that*

$$U_\phi = U_\phi(-1) + \frac{b}{\ln(1+x)} + O((\ln(1+x))^{-2+\epsilon}).$$

A consequence of Theorem 1.0.2 and Theorem 1.0.4 is

Corollary 1.0.1. *For every $\tau < 3$, there exists an axisymmetric solution (U_θ, U_ϕ) with nonzero swirl of (1.1) in $C^\infty(\mathbb{S}^2 \setminus \{S\})$ such that $U_\theta(-1) = \tau$. On the other hand, every axisymmetric solution (U_θ, U_ϕ) of (1.1) in $C^\infty(\mathbb{S}^2 \setminus \{S\})$ with $U_\theta(-1) \geq 3$ necessarily has zero swirl, i.e. $U_\phi \equiv 0$.*

Corollary 1.0.2. *There are similar results about the asymptotic behavior of solutions u in a punctured ball $B_\delta(N) \setminus \{N\}$ on $\mathbb{S}^2$, $\delta > 0$. We will state the results in detail in Chapter 4.*

Our next work focus on (-1)-homogeneous axisymmetric solutions which are smooth on $\mathbb{S}^2 \setminus \{S, N\}$, where S is the south pole and N is the north pole.

The first result on such solutions is a classification of (-1)-homogeneous axisymmetric no-swirl solutions which are smooth on $\mathbb{S}^2 \setminus \{S, N\}$.

For each $c_1 \geq -1, c_2 \geq -1$, define

$$\bar{c}_3(c_1, c_2) := -\frac{1}{2}\left(\sqrt{1+c_1} + \sqrt{1+c_2}\right)\left(\sqrt{1+c_1} + \sqrt{1+c_2} + 2\right). \tag{1.9}$$

Let $c := (c_1, c_2, c_3)$, we introduce the set

$$D := \{c \in \mathbb{R}^3 | c_1 \geq -1, c_2 \geq -1, c_3 \geq \bar{c}_3(c_1, c_2)\}, \tag{1.10}$$

and the following subsets D_i, $1 \leq i \leq 8$, of D as:

$$\begin{aligned}
&D_1 := \{c | c_1 > -1, c_2 > -1, c_3 > \bar{c}_3\}, \quad D_2 := \{c | c_1 = -1, c_2 > -1, c_3 > \bar{c}_3\},\\
&D_3 := \{c | c_1 > -1, c_2 = -1, c_3 > \bar{c}_3\}, \quad D_4 := \{c | c_1 = -1, c_2 = -1, c_3 > \bar{c}_3\},\\
&D_5 := \{c | c_1 > -1, c_2 > -1, c_3 = \bar{c}_3\}, \quad D_6 := \{c | c_1 = -1, c_2 > -1, c_3 = \bar{c}_3\},\\
&D_7 := \{c | c_1 > -1, c_2 = -1, c_3 = \bar{c}_3\}, \quad D_8 := \{c | c_1 = -1, c_2 = -1, c_3 = \bar{c}_3\}.
\end{aligned} \tag{1.11}$$

Then we have

Theorem 1.0.5. *There exist $\gamma^-, \gamma^+ \in C^\infty(D_k) \cap C^0(D)$, $1 \leq k \leq 8$, satisfying $\gamma^-(c) \leq \gamma^+(c)$ for all $c \in J$, where the equality holds if and only if $c \in \cup_{k=5}^8 D_k$, such that for each (c, γ) in the set*

$$E := \{(c, \gamma) \mid c_1 \geq -1, c_2 \geq -1, c_3 \geq \bar{c}_3(c_1, c_2), \gamma^-(c) \leq \gamma \leq \gamma^+(c)\},$$

there exists a unique C^1 solution $U_\theta^{c,\gamma}$ of (1.1) in $(-1,1)$ satisfying $U_\theta^{c,\gamma}(0)=\gamma$ and

$$(1-x^2)(U_\theta^{c,\gamma})' + 2xU_\theta^{c,\gamma} + \frac{1}{2}(U_\theta^{c,\gamma})^2 = c_1(1-x)+c_2(1+x)+c_3(1-x^2). \qquad (1.12)$$

Moreover, $\{U_\theta^{c,\gamma} | (c,\gamma)\in E\}$ are all (-1)-homogeneous axisymmetric no-swirl solutions of the Navier-Stokes equations (1.1) on $\mathbb{S}^2\setminus\{S,N\}$.

Remark 1.0.2. *The solution $U_\theta^{c,\gamma}$ is a continuous map from the set $E\setminus\{(c,\gamma)\in E | c\in\cup_{k=1}^3 D_k, \gamma=\gamma^-(c)$ or $\gamma=\gamma^+(c)\}$ to $C^0[-1,1]$. On the other hand, for every $c\in\cup_{k=1}^3 D_k$, we have either $\lim_{\gamma\to\gamma^+(c)} U_\theta^{c,\gamma}(-1)\neq U_\theta^{c,\gamma^+(c)}(-1)$ or $\lim_{\gamma\to\gamma^-(c)} U_\theta^{c,\gamma}(1)\neq U_\theta^{c,\gamma^-(c)}(1)$.*

Theorem 1.0.5 gives a classification of all axisymmetric, no-swirl solutions of the Navier-Stokes equations in $C^2(\mathbb{S}^2\setminus\{S,N\})$.

We denote by $(u^{c,\gamma},p^{c,\gamma})$ the axisymmetric no-swirl solution of (1.1) in $C^\infty(\mathbb{S}^2\setminus\{N,S\})$ obtained by $U_\theta^{c,\gamma}$. For each $(c,\gamma)\in E$, the linearized equation of (1.1) at $(u^{c,\gamma},p^{c,\gamma})$ is

$$\begin{cases} -\triangle v + u^{c,\gamma}\cdot\nabla v + v\cdot\nabla u^{c,\gamma} + \nabla q = 0, \\ \text{div } v = 0. \end{cases} \qquad (1.13)$$

Define

$$a_{c,\gamma}(\theta) = -\int_{\frac{\pi}{2}}^{\theta} \left(2\cot t + u_\theta^{c,\gamma}\right) dt, \quad b_{c,\gamma}(\theta) = -\int_{\frac{\pi}{2}}^{\theta} u_\theta^{c,\gamma} dt,$$

and

$$v_{c,\gamma}^1 = \begin{pmatrix} \frac{1}{\sin\theta} e^{-a_{c,\gamma}(\theta)} \frac{da_{c,\gamma}(\theta)}{d\theta} \\ \frac{1}{\sin\theta} e^{-a_{c,\gamma}(\theta)} \\ 0 \end{pmatrix}, \qquad v_{c,\gamma}^2 = \begin{pmatrix} \frac{-e^{-a_{c,\gamma}(\theta)}}{\sin\theta}\int_{\frac{\pi}{2}}^{\theta} e^{a_{c,\gamma}(t)}\sin t dt \frac{da_{c,\gamma}(\theta)}{d\theta} + 1 \\ \frac{-1}{\sin\theta} e^{-a_{c,\gamma}(\theta)}\int_{\frac{\pi}{2}}^{\theta} e^{a_{c,\gamma}(t)}\sin t dt \\ 0 \end{pmatrix},$$

$$v_{c,\gamma}^3 = \begin{pmatrix} 0 \\ 0 \\ \frac{-1}{\sin\theta}\int_{\frac{\pi}{2}}^{\theta} e^{-b_{c,\gamma}(t)}\sin t dt \end{pmatrix}, \qquad v_{c,\gamma}^4 = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sin\theta} \end{pmatrix}.$$

Then $\{v^i_{c,\gamma}, 1 \le i \le 4\}$ are linearly independent solutions of the linearized equation of (1.1) at $(u^{c,\gamma}, p^{c,\gamma})$, in spherical coordinates, on $\mathbb{S}^2 \setminus \{N, S\}$.

We introduce the following subsets of E:

For $1 \le k \le 4$, let

$$\begin{aligned} E_{k,1} &:= \{(c,\gamma) \in E | c \in D_k, c_1 < -\frac{3}{4}, \gamma = \gamma^+(c)\}, \\ E_{k,2} &:= \{(c,\gamma) \in E | c \in D_k, c_2 < -\frac{3}{4}, \gamma = \gamma^-(c)\}, \\ E_{k,3} &:= \{(c,\gamma) \in E | c \in D_k, \gamma^-(c) < \gamma < \gamma^+(c)\}, \end{aligned} \tag{1.14}$$

and for $5 \le k \le 8$, let $E_{k,l} := \{(c,\gamma) \in D_k \times \{\gamma^+(c)\} | c_1, c_2 < -\frac{3}{4}\}$.

Moreover, let

$$\hat{E} := \{(c,\gamma) \in E \setminus \cup_{1 \le k \le 8, 1 \le l \le 3} E_{k,l} | c_1 > -\frac{3}{4} \text{ or } c_2 > -\frac{3}{4}\}. \tag{1.15}$$

Let $B_\delta(0)$ be a ball in $\mathbb{R}^2$ with radius δ and centered at the origin.

Theorem 1.0.6. *Let K be a compact subset of one of the sets $E_{k,l}$, $1 \le k \le 8$, $1 \le l \le 3$. Then there exist $\delta = \delta(K) > 0$, and $(u,p) \in C^\infty(K \times B_\delta(0) \times (\mathbb{S}^2 \setminus \{N,S\}))$ such that for every $(c,\gamma,\beta) \in K \times B_\delta(0)$, $\beta = (\beta_3, \beta_4)$, $(u,p)(c,\gamma,\beta;\cdot) \in C^\infty(\mathbb{S}^2 \setminus \{N,S\})$ satisfies (1.1) in $\mathbb{R}^3 \setminus \{(0,0,x_3) | x_3 \in \mathbb{R}\}$, with nonzero swirl if $\beta \ne 0$, and $\| \sin\theta \, (u(c,\gamma,\beta) - u^{c,\gamma}) \|_{L^\infty(\mathbb{S}^2 \setminus \{N,S\})} \to 0$ as $\beta \to 0$. Moreover, $\frac{\partial}{\partial \beta_i} u(c,\gamma,\beta)|_{\beta=0} = v^i_{c,\gamma}$, $i = 3,4$.*

On the other hand, for $(c,\gamma) \in \hat{E}$, if there exists a sequence of solutions $\{u^i\}$ of (1.1) in $C^\infty(\mathbb{S}^2 \setminus \{N,S\})$, such that $\|\sin\theta(u^i - u^{c,\gamma})\|_{L^\infty(\mathbb{S}^2 \setminus \{N,S\})} \to 0$ as $i \to \infty$, then for sufficiently large i, $u^i = u^{c_i,\gamma_i} + \frac{C_i}{\sin\theta} \vec{e}_\phi$ for some constants c_i, γ_i, C_i. Moreover, $(c_i,\gamma_i) \to (c,\gamma)$ and $C_i \to 0$ as $i \to \infty$.

Landau interpreted the solutions he found (Landau solutions) as a jet discharged from a point. Experimentally, a pingpong ball can float and be stable in a jet of air (such as when we blow into a straw upwards). However, as pointed out by Šverák, the pressure in the center of the Landau jet is higher than the pressure nearby, and therefore the exact Landau jets solutions are unlikely to support a pingpong ball in a stable way. The real-life jets are turbulent and this plays an important role. The

Landau solutions could still be relevant when one thinks in terms of averaging, turbulent viscosity, Reynolds stress, etc. Still, the pressure profiles are of interest and in Chapter 6, we identify all axisymmetric no-swirl solutions in a neighborhood of the north pole of $\mathbb{S}^2$, which describe fluid jets with lower pressure in the center. It would be interesting to compare some of these solutions to real-life jets.

There have been some other papers on (-1)-homogeneous axisymmetric solutions of the stationary NSE (1.1), see [1], [9], [10], [11], [12], [13], [14], [17] and [18]. In the no-swirl case, the equations were converted to an equation of Riccati type in [13], see also [18] where various exact solutions on annulus regions of $\mathbb{S}^2$ were given. Some recent works also study (-1)-homogeneous solutions of the stationary NSE. Karch and Pilarczyk showed in [2] that Landau solutions are asymptotically stable under any L^2 perturbations. Luo and Shvydkoy studied in [7] classifications of homogeneous solutions to the stationary Euler equation with locally finite energy.

The organization of the paper is as follows. In Chapter 2, we reduce the NSE in the framework of spherical coordinates. We also give an alternative proof of the above mentioned result in [16] in the framework. In Section 3.1, we classify all (-1)-homogeneous axisymmetric no-swirl solutions of the stationary NSE (1.1) on $\mathbb{S}^2 \setminus \{S\}$. The existence part of Theorem 1.0.2 is established in Section 3.2. It is proved by using implicit function theorems in suitably chosen weighted norm Banach spaces. Three different sets of spaces are used according to which of the three parts of J, J_1, J_2 or $J_3 \cap \{2 \leq \tau < 3\}$, (τ, σ) belongs to. Asymptotic behavior of solutions in a punctured ball $B_\delta(S) \setminus \{S\}$ of $\mathbb{S}^2$ is studied in Chapter 4. Theorem 1.0.3, 1.0.4 and the nonexistence part, therefore the completion of Theorem 1.0.2, are established in this Chapter. We then study (-1)-homogeneous axisymmetric solutions on $\mathbb{S}^2 \setminus \{S, N\}$ in Chapter 5. We first classify all (-1)-homogeneous axisymmetric no-swirl solutions of the the stationary NSE (1.1) on $\mathbb{S}^2 \setminus \{S, N\}$ in Section 5.1, and prove Theorem 1.0.6 in Section 5.2 by implicit function theorems in suitably chosen weighted norm Banach spaces. Different spaces are used based on the behavior of solutions near the sigularities. Several results on first order ordinary differential equations used in Chapter 4 are given in Chapter 6.

Chapter 2, 3, 4 and 6 are from [4], Section 5.1 is from [5] and Section 5.2 is from [6].

Chapter 2

Reduction of equations

Our first attempt in proving Theorem 1.0.2 is to work with (u_θ, u_ϕ) and to find some spaces with appropriate weights on u_θ and u_ϕ together with their derivatives near the south pole S. However, we encounter difficulties of loss of derivatives when trying to apply implicit function theorems. As mentioned earlier, we work with new functions U_r, U_θ and U_ϕ, and a new variable x as defined in (1.8). Both formulations, with u and θ or with U and x are widely used in literature.

For any $-1 \le \delta_1 < \delta_2 \le 1$, system (1.3) in the range $\delta_1 < x < \delta_2$ can be reformulated into the following third order ODE system of U_θ, U_ϕ and p:

$$\begin{cases} -(1-x^2)U_\theta''' + 2xU_\theta'' - {U_\theta'}^2 - U_\theta U_\theta'' - \dfrac{U_\theta^2}{1-x^2} - \dfrac{U_\phi^2}{1-x^2} - 2p = 0, \\ (1-x^2)U_\theta'' - U_\theta U_\theta' - \dfrac{x}{1-x^2}U_\theta^2 - \dfrac{x}{1-x^2}U_\phi^2 - (1-x^2)p' = 0, \\ -(1-x^2)U_\phi'' - U_\theta U_\phi' = 0. \end{cases} \tag{2.1}$$

with the divergence free condition

$$U_r = U_\theta' \sin\theta. \tag{2.2}$$

Differentiating the first line of (2.1) in x, then subtracting $\frac{2}{1-x^2}$ times the second line, we have the following fourth order ODE system of U_θ and U_ϕ

$$\begin{cases} -(1-x^2)U_\theta'''' + 4xU_\theta''' - 3U_\theta' U_\theta'' - U_\theta U_\theta''' - \dfrac{2U_\phi U_\phi'}{1-x^2} = 0, \\ -(1-x^2)U_\phi'' - U_\theta U_\phi' = 0. \end{cases} \tag{2.3}$$

Since

$$-(1-x^2)U_\theta'''' + 4xU_\theta''' - 3U_\theta' U_\theta'' - U_\theta U_\theta''' = -\left((1-x^2)U_\theta' + 2xU_\theta + \frac{1}{2}U_\theta^2\right)''',$$

system (2.3) can be converted into

$$\begin{cases} (1-x^2)U_\theta' + 2xU_\theta + \dfrac{1}{2}U_\theta^2 + \displaystyle\int\int\int \dfrac{2U_\phi(s)U_\phi'(s)}{1-s^2}dsdtdl = c_1x^2 + c_2x + c_3, \\ (1-x^2)U_\phi'' + U_\theta U_\phi' = 0, \end{cases} \tag{2.4}$$

for some constants c_1, c_2, c_3. By (2.2), $U_r \in C((\delta_1,\delta_2),\mathbb{R})$ is well-defined if $U_\theta \in C^1((\delta_1,\delta_2),\mathbb{R})$, and $U_r = O(1)\sin\theta$ if U_θ' is bounded. The original Navier-Stokes system (1.1) is equivalent to (2.3) and (2.2).

If there exist some constants c_1, c_2, c_3 and $U_\theta \in C^1(\delta_1,\delta_2)$, $U_\phi \in C^2(\delta_1,\delta_2)$ such that (U_θ, U_ϕ) is a solution of (2.4) in (δ_1,δ_2), then the (-1)-homogeneous $u = (u_r, u_\theta, u_\phi)$ given in the corresponding domain on $\mathbb{S}^2$ by

$$u_r = U_\theta', \quad u_\theta = \frac{U_\theta}{\sin\theta}, \qquad u_\phi = \frac{U_\phi}{\sin\theta},$$

satisfies the stationary NSE (1.1). We will use $U = (U_\theta, U_\phi)$ to denote solutions of the stationary Navier-Stokes equations (1.1), with the meaning that u determined by U as above is a solution to (1.1).

With the above set up, we give an alternative proof of the following theorem:

Theorem B ([16]) *All (-1)-homogeneous nonzero axisymmetric solutions of (*1.1*) in* $C^2(\mathbb{R}^3\setminus\{0\})$ *are Landau solutions.*

Proof. Since the solution u is smooth in $\mathbb{R}^3\setminus\{0\}$, the components U_r, U_θ, U_ϕ and their derivatives are well-defined on $\mathbb{S}^2$. U_θ and U_ϕ vanish at $x=\pm 1$, $U_\theta = O(1)(1-x^2)$, U_θ', U_θ'' are bounded in $[-1,1]$.

From the second line of (2.4), we have

$$U_\phi' = ce^{-\int \frac{U_\theta}{1-s^2}ds},$$

for some constant c, so U_ϕ is monotone for $x \in [-1,1]$. Since $U_\phi(1) = U_\phi(-1) = 0$, we must have $U_\phi \equiv 0$, i.e. the solution does not have a swirling components.

Let x go to 1 in the first line of (2.4). Notice that $U_\theta = O(1)(1-x^2)$, and U_θ' is bounded, we obtain

$$c_1 + c_2 + c_3 = \lim_{x\to 1}\left((1-x^2)U_\theta' + 2xU_\theta + \frac{1}{2}U_\theta^2\right) = 0,$$

Differentiate the first line of (2.4) with respect to x, then send $x \to 1$, we have

$$2c_1 + c_2 = \lim_{x \to 1} \left((1 - x^2) U_\theta'' + 2U_\theta + U_\theta U_\theta' \right) = 0.$$

It follows that

$$c_1 x^2 + c_2 x + c_3 = c_1 (1 - x)^2.$$

Repeat the above analysis similarly as x goes to -1, we have

$$c_1 x^2 + c_2 x + c_3 = c_1 (1 + x)^2.$$

Therefore, we must have $c_1 = c_2 = c_3 = 0$, $U_\phi = 0$. It is now easy to see that u is a Landau solution, $u = \frac{2 \sin\theta}{\lambda + \cos\theta}$ with $|\lambda| > 1$. $\square$

Chapter 3

(-1)-homogeneous axisymmetric solutions on $\mathbb{S}^2 \setminus \{S\}$

3.1 Classification of axisymmetric no-swirl solutions on $\mathbb{S}^2 \setminus \{S\}$

In this section, we will prove Theorem 1.0.1, which classifies all (-1)-homogeneous axisymmetric no-swirl $C^\infty(\mathbb{S}^2 \setminus \{S\})$ solutions of (1.1). More generally, we study axisymmetric no-swirl solutions of (1.1) which are smooth in a neighborhood of the north pole.

By arguments used in Chapter 2, u is a solution of (1.1) in $\mathbb{S}^2 \setminus \{N, S\}$ if and only if U defined by (1.8) satisfies (2.4) in $(-1,1)$ for some constants c_1, c_2 and c_3. When the solution has no swirling component, (2.4) becomes

$$(1-x^2)U_\theta' + 2xU_\theta + \frac{1}{2}U_\theta^2 = c_1x^2 + c_2x + c_3. \tag{3.1}$$

Let u be a solution which is smooth in a neighborhood of the north pole, the proof of Theorem B in Chapter 2 actually shows that the polynomial on the right hand side of (3.1) must be $\mu(1-x)^2$ for some constant μ. Therefore, the NSE is

$$(1-x^2)U_\theta' + 2xU_\theta + \frac{1}{2}U_\theta^2 = \mu(1-x)^2. \tag{3.2}$$

Lemma 3.1.1. *Let $\mu, \gamma \in \mathbb{R}$ and $\delta \in [-1, 1)$, equation (3.2) has at most one solution $U_\theta \in C^1(\delta, 1)$ satisfying*

$$\lim_{x\to 1^-} U_\theta(x) = 0, \ \textit{and} \ \lim_{x\to 1^-} U_\theta'(x) = \gamma. \tag{3.3}$$

Proof. Let $U_\theta^{(i)}$ (i =1, 2) be two such solutions. Then $g_i(x) := (1-x^2)^{-1}U_\theta^{(i)}$ satisfies

$$g_i'(x) + \frac{1}{2}g_i^2(x) = \frac{\mu}{(1+x)^2}, \quad \delta < x < 1, \quad i = 1, 2.$$

Using (3.3) and the L'Hospital's rule,

$$\lim_{x\to 1^-} g_i(x) = -\frac{\gamma}{2}, \quad i = 1, 2.$$

So $g_i(x)$ can be extended as functions in $C^0(\delta,1]$, $g_1(1)=g_2(1)$, and g_1-g_2 satisfies $(g_1-g_2)'+\frac{1}{2}(g_1+g_2)(g_1-g_2)=0$ in $(\delta,1)$ with $(g_1-g_2)(1)=0$. It follows that $g_1\equiv g_2$ in $(\delta,1)$, so $U_\theta^{(1)}\equiv U_\theta^{(2)}$ in $(\delta,1)$. $\square$

Let $b:=\sqrt{|1+2\mu|}$, $\delta^*\in C(\mathbb{R}^2,[-1,1))$ be given by

$$\delta^*:=\delta^*(\mu,\gamma):=\begin{cases}-1, & \mu\ge-\frac{1}{2},\gamma\ge-(1+\sqrt{1+2\mu});\\ -1+2\left(\frac{\gamma+1-b}{\gamma+1+b}\right)^{-1/b}, & \mu>-\frac{1}{2},\gamma<-(1+\sqrt{1+2\mu});\\ -1+2e^{\frac{2}{\gamma+1}}, & \mu=-\frac{1}{2},\gamma<-1;\\ -1+2\exp\left(\frac{2}{b}(\arctan\frac{b}{\gamma+1}-\pi)\right), & \mu<-\frac{1}{2},\gamma>-1;\\ -1+2\exp\left(-\frac{\pi}{b}\right), & \mu<-\frac{1}{2},\gamma=-1;\\ -1+2\exp\left(\frac{2}{b}\arctan\frac{b}{\gamma+1}\right), & \mu<-\frac{1}{2},\gamma<-1.\end{cases} \tag{3.4}$$

Theorem 3.1.1 (Exact form of axisymmetric no-swirl solutions)**.** *For every $(\mu,\gamma)\in\mathbb{R}^2$, there exists a unique $U_\theta:=U_\theta(\mu,\gamma;\cdot)\in C^\infty(\delta^*,1)$ satisfying (3.2) in $(\delta^*,1)$ and*

$$\lim_{x\to1^-}U_\theta'(x)=\gamma. \tag{3.5}$$

The interval $(\delta^*,1)$ is the maximal interval of existence for U_θ, and in particular,

$$\lim_{x\to\delta^{*+}}|U_\theta(x)|=\infty,\ \text{ if } \delta^*>-1. \tag{3.6}$$

Moreover, U_θ is explicitly given by

$$U_\theta(x)=\begin{cases}(1-x)\left(1-b-\frac{2b(\gamma+1-b)}{(\gamma+1+b)(\frac{1+x}{2})^{-b}-\gamma-1+b}\right), & \mu>-\frac{1}{2},\\ (1-x)\left(1+\frac{2(\gamma+1)}{(\gamma+1)\ln\frac{1+x}{2}-2}\right), & \mu=-\frac{1}{2},\\ (1-x)\left(1+\frac{b(b\tan\frac{\beta(x)}{2}+\gamma+1)}{(\gamma+1)\tan\frac{\beta(x)}{2}-b}\right), & \mu<-\frac{1}{2},\end{cases} \tag{3.7}$$

where $b:=\sqrt{|1+2\mu|}$, and $\beta(x):=b\ln\frac{1+x}{2}$.

We will also use $U^{\mu,\gamma}$ to denote the axisymmetric no-swirl solution $(U_\theta(\mu,\gamma;\cdot),0)$ in the above theorem.

Let $u=u(\mu,\gamma)$ be the solution generated by $(U_\theta(\mu,\gamma),0)$, then $\{u(0,\gamma)\mid\gamma>-2,\gamma\neq0\}$ are Landau solutions. In particular, $U_\theta(x)=\frac{2(1-x^2)}{x+\lambda}$ with $|\lambda|>1$, and $\delta^*(0,\gamma)=-1$ for any $\gamma>-2$, $\gamma\neq0$.

It is easy to see that $U_\theta(\mu,\gamma)\neq U_\theta(\mu',\gamma')$ if $(\mu,\gamma)\neq(\mu',\gamma')$. Let I be defined by

$$I := \{(\mu,\gamma)\mid \mu\geq -\frac{1}{2}, \gamma\geq -1-\sqrt{1+2\mu}\},\quad I^c=\mathbb{R}^2\setminus I.$$

Then $\delta^*(\mu,\gamma)=-1$ if and only if $(\mu,\gamma)\in I$. Consequently, $u(\mu,\gamma)\in C^\infty(\mathbb{S}^2\setminus\{S\})\setminus C^1(\mathbb{S}^2)$ if and only if for all $(\mu,\gamma)\in I\setminus\{(0,\gamma)\mid\gamma>-2\}$. Also, it is not hard to see

$$\lim_{\gamma\to-\infty}\delta^*(\mu,\gamma)=1,\qquad \lim_{\mu\to-\infty}\delta^*(\mu,\gamma)=1.$$

$$\frac{\partial\delta^*(\mu,\gamma)}{\partial\mu}<0,\quad \frac{\partial\delta^*(\mu,\gamma)}{\partial\gamma}<0,\quad \text{for } (\mu,\gamma)\in I^c.$$

Proof of Theorem 3.1.1*:* For every $(\mu,\gamma)\in\mathbb{R}^2$, let a be a root of $\frac{1}{2}a^2-a=\mu$ (real or complex) then $h(x):=a(1-x)$ is a solution of (3.2). If U_θ is a solution of (3.2), denote $g:=U_\theta-h$, then g satisfies

$$(1-x^2)g'+2xg+hg+\frac{1}{2}g^2=0.$$

Multiplying both sides by the integrating factor $(1+x)^{a-1}(1-x)^{-1}$, we have

$$\tilde{g}'+\frac{1}{2}(1+x)^{-a}\tilde{g}^2=0,$$

where $\tilde{g}(x):=(1+x)^{a-1}(1-x)^{-1}g$. Solving this equation directly we have

$$\tilde{g}=\frac{2(1-a)}{(1+x)^{1-a}+c'},\quad \text{if } a\neq 1.$$

Then

$$U_\theta=a(1-x)+\frac{2(1-a)(1-x)(\frac{1+x}{2})^{1-a}}{(\frac{1+x}{2})^{1-a}+c} \tag{3.8}$$

where c is a (real or complex) constant.

Let $b:=\sqrt{|1+2\mu|}$. When $\mu>-\frac{1}{2}$, we can take $a=1+b$, $c=\frac{-\gamma-2+a}{\gamma+a}$. Then U_θ is the function in the first line of (3.7) which satisfies (3.5) and (3.2) in $(\delta^*,1)$ where δ^* is given in (3.4), and by Lemma 3.1.1 it is the only solution satisfying (3.2) and (3.5). Property (3.6) follows from standard ODE theory. U_θ are Landau solutions when $\mu=0$ and $\gamma>-2$, $\gamma\neq 0$,

$$U_\theta(x)=\frac{2(1-x^2)}{x+\lambda}$$

where $\lambda = -\frac{\gamma+4}{\gamma}$. It can be seen that when $\gamma > -2$, $\gamma \neq 0$, there is $|\lambda| > 1$.

When $\mu < -\frac{1}{2}$, we can take $a = 1 + ib$, $c = \frac{-\gamma-2+a}{\gamma+a}$. Then the real part of (3.8) can be rewritten as the function in the third line of (3.7), which satisfies the required properties.

When $\mu = -1/2$, we have $a = 1$ and

$$\tilde{g}' + \frac{1}{2}(1+x)^{-1}\tilde{g}^2 = 0$$

where $\tilde{g}(x) := (1-x)^{-1}g$. Thus,

$$U_\theta = (1-x) + \frac{1-x}{\frac{1}{2}\ln(1+x) + c}.$$

Choosing $c = \frac{-1}{\gamma+1} - \frac{1}{2}\ln 2$, then U_θ is the function in the second line of (3.7) which satisfies all the required properties. $\square$

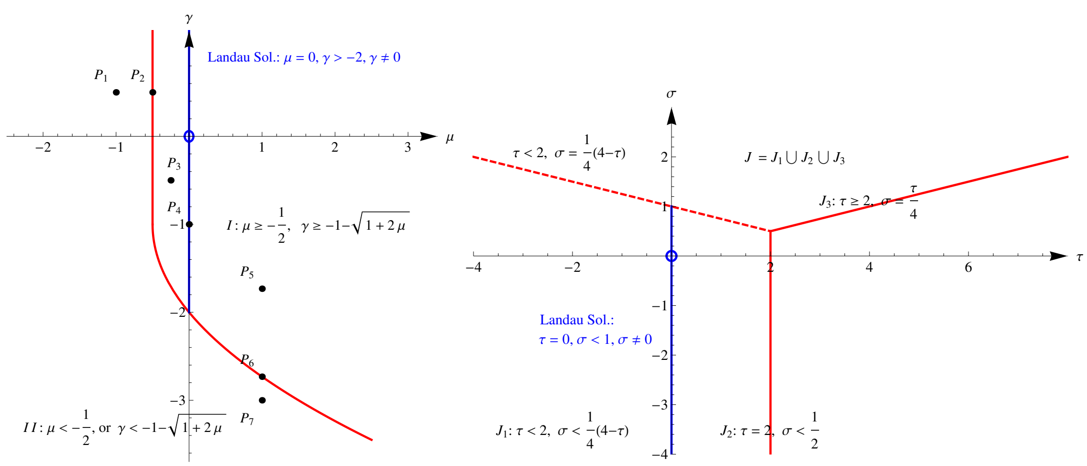

Figure 3.1: Dependence on parameters (μ, γ) or (τ, σ) of the maximal existence domains of the solutions to NSE

Figure 1 shows the dependence of the maximal existence domains on parameters (μ, γ) or (τ, σ). When the parameters $(\mu, \gamma) \in I$, or equivalently $(\tau, \sigma) \in J$, the solution is smooth on $\mathbb{S}^2 \setminus \{S\}$; When the parameters $(\mu, \gamma) \notin I$, or equivalently $(\tau, \sigma) \notin J$, the solution exists and is smooth in a neighborhood of the north pole $\{N\}$ but not on the entire $\mathbb{S}^2 \setminus \{S\}$. Some typical points are chosen in the (μ, γ) plane, (i.e. left part of Figure 1). The graph and stream lines at these points are presented in Section 6.2.

Here is an immediate consequence of Theorem 3.1.1:

Corollary 3.1.1. *Suppose U is an axisymmetric, no-swirl solution of Navier-Stokes equation and is smooth on $\mathbb{S}^2 \setminus \{S\}$, then $U_\theta(x)$ is given by a two-parameter-family (μ,γ) with $\mu \geq -\frac{1}{2}$, $\gamma \geq -1-\sqrt{1+2\mu}$:*

$$U_\theta(x) = \begin{cases} (1-x)\left(1-b-\frac{2b(\gamma+1-b)}{(\gamma+1+b)(\frac{1+x}{2})^{-b}-\gamma-1+b}\right), & \mu > -\frac{1}{2}, \gamma > -1-\sqrt{1+2\mu}, \\ (1-x)\left(1+\frac{2(\gamma+1)}{(\gamma+1)\ln\frac{1+x}{2}-2}\right) & \mu = -\frac{1}{2}, \gamma > -1, \\ (1+b)(1-x), & \mu \geq -\frac{1}{2}, \gamma = -1-\sqrt{1+2\mu}. \end{cases} \tag{3.9}$$

Since $U_\theta(x) = u_\theta \sin\theta$, $x=\cos\theta$ and (1.5), $\tau = \lim_{\theta\to\pi^-} u_\theta(x)\sin\theta = \lim_{x\to -1^+} U_\theta(x)$, $\gamma = \lim_{x\to 1} U_\theta'(x) = -2\lim_{\theta\to 0}\frac{u_\theta}{\sin\theta} = -2\sigma$, and $\mu = \lim_{x\to-1^+}\frac{1}{4}(\frac{1}{2}U_\theta^2(x) - 2U_\theta(x)) = \frac{1}{8}\tau^2 - \frac{1}{2}\tau$. The relation

$$\mu = \frac{1}{8}\tau^2 - \frac{1}{2}\tau, \quad \gamma = -2\sigma$$

gives a one-one correspondence between $\{u_{\tau,\sigma}|, (\tau,\sigma)\in J\}$ and $\{U^{\mu,\gamma}|(\mu,\gamma)\in I\}$ with $u_{\tau,\sigma}\sin\theta = U^{\mu,\gamma}$. Moreover, region J_1 corresponds to

$$I_1 := I^0 = \{(\mu,\gamma)\in I|, \mu > -\frac{1}{2}, \gamma > -1-\sqrt{1+2\mu}\},$$

boundary J_2 corresponds to

$$I_2 := \{(\mu,\gamma)\in I|\mu = -\frac{1}{2}, \gamma > -1\},$$

J_3 corresponds to

$$I_3 := \{(\mu,\gamma)\in I|\mu > -\frac{1}{2}, \gamma = -1-\sqrt{1+2\mu}\}.$$

Also, $J_3\cap\{2\leq\tau<3\}$ corresponds to $I_3\cap\{-\frac{1}{2}\leq\mu<-\frac{3}{8}\}$ and $J_3\cap\{\tau>3\}$ corresponds to $I_3\cap\{\mu>-\frac{3}{8}\}$.

Theorem 1.0.1 follows from the above corollary.

Remark 3.1.1. *From Theorem* 3.1.1 *and Corollary* 3.1.1 *we can see that $U_\theta(\mu,\gamma)$ exists on all $(-1,1]$ if and only if $(\mu,\gamma)\in I$, which is shown in the first graph of Figure 1, and the behavior of U_θ near the south pole is different when $(\mu,\gamma)\in I_1$, the interior of I, and when $(\mu,\gamma)\in\partial I$.*

When $(\mu,\gamma)\in I_1$, $\mu>-1/2$, we have for $-1<x<1$, $i,j\in\mathbb{Z}$

$$\begin{aligned}
&U_\theta^{\mu,\gamma}(x)=(1-x)(1-\sqrt{1+2\mu}+O(1)(1+x)^b),\\
&\partial_\mu^i U_\theta^{\mu,\gamma}(x)=(1-x)\left(-\frac{d^i}{d\mu^i}\sqrt{1+2\mu}+O(1)(1+x)^b\left|\ln\left(\frac{1+x}{2}\right)\right|^i\right),\quad i\geq 1,\\
&\left|\partial_\mu^i\partial_\gamma^j U_\theta^{\mu,\gamma}(x)\right|=O(1)(1-x)(1+x)^b\left|\ln\left(\frac{1+x}{2}\right)\right|^i,\quad i\geq 0, j\geq 1.
\end{aligned}\tag{3.10}$$

When $(\mu,\gamma)\in I_2$, we have

$$\begin{aligned}
&U_\theta^{\mu,\gamma}(x)=(1-x)\left(1+2\left(\ln\frac{1+x}{3}\right)^{-1}+O(1)\left(\ln\frac{1+x}{3}\right)^{-2}\right),\\
&\partial_\gamma^i U_\theta^{\mu,\gamma}(x)=O(1)(1-x)\left(\ln\frac{1+x}{3}\right)^{-2},\quad i\geq 1.
\end{aligned}\tag{3.11}$$

When $(\mu,\gamma)\in I_3$, $U_\theta(x)=(1+b)(1-x)$, which is a linear function, and

$$\begin{aligned}
&U_\theta^{\mu,\gamma}(x)=(1-x)(1+\sqrt{1+2\mu}),\\
&\partial_\mu^i U_\theta^{\mu,\gamma}(x)=\frac{\partial^i}{\partial\mu^i}\sqrt{1+2\mu}(1-x),\quad i\geq 1.
\end{aligned}\tag{3.12}$$

3.2 Existence of axisymmetric solutions with nonzero swirl on $\mathbb{S}^2\setminus\{S\}$

3.2.1 Framework of proofs

The set of all axisymmetric no swirl solutions of the NSE (1) in $C^\infty(\mathbb{S}^2\setminus\{S\})$ is classified in Section 3.1 as the two dimensional surface of solutions $\{U^{\mu,\gamma}=(U_\theta^{\mu,\gamma},0)\ \mid\ (\mu,\gamma)\in I\}$. In this chapter, we will use implicit function theorems in suitably chosen weighted normed spaces to prove the existence of a curve of axisymmetric solutions with nonzero swirl emanating from each $U^{\mu,\gamma}$ for $(\mu,\gamma)\in I\setminus(I_3\cap\{\mu\geq-\frac{3}{8}\})$.

Since $U_\theta(-1)$ affects the behavior of U_θ and U_ϕ near the singularity $x=-1$, we will need to use different function spaces according to the values of $U_\theta(-1)$. It is easy to check that $U^{\mu,\gamma}(-1)\in(-\infty,2)$ for $(\mu,\gamma)\in I_1$, $U^{\mu,\gamma}(-1)=2$ for $(\mu,\gamma)\in I_2$, $U^{\mu,\gamma}(-1)\in[2,3)$ for $(\mu,\gamma)\in I_3\cap\{-\frac{1}{2}\leq\mu<-\frac{3}{8}\}$. We will use three different sets of weighted normed spaces based on which of the three sets, I_1, I_2, and $I_3\cap\{-\frac{1}{2}\leq\mu<-\frac{3}{8}\}$, (μ,γ) belongs to.

On the other hand, $U^{\mu,\gamma}(-1)>3$ for $(\mu,\gamma)\in I_3\cap\{\mu>-\frac{3}{8}\}$. It will be proved in Chapter 4 that for every $(\mu,\gamma) \in I_3 \cap \{\mu > -\frac{3}{8}\}$, there exists no sequence of axisymmetric solution with nonzero swirl in $C^\infty(\mathbb{S}^2 \setminus \{S\})$ which converge to $U^{\mu,\gamma}$ in $L^\infty(-1,1)$.

For convenience, let us use $\bar{U}$ to denote axisymmetric no-swirl solutions of the stationary NSE.

In this section we denote $U = (U_\theta, U_\phi)$. The equations of axisymmetric solutions in $C^\infty(\mathbb{S}^2 \setminus \{S\})$ are of the form

$$\begin{cases} (1-x^2)U_\theta' + 2xU_\theta + \dfrac{1}{2}U_\theta^2 - \displaystyle\int_x^1 \int_l^1 \int_t^1 \frac{2U_\phi(s)U_\phi'(s)}{1-s^2} dsdtdl = \hat{\mu}(1-x)^2, \\ (1-x^2)U_\phi'' + U_\theta U_\phi' = 0, \end{cases} \tag{3.13}$$

where $\hat{\mu}$ is a constant.

We first introduce the implicit function theorem (IFT) which we use:

Theorem C ([8]) (Implicit Function Theorem) *Let* $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ *be Banach spaces and* f *a continuous mapping of an open set* $U \subset \mathbf{X} \times \mathbf{Y} \to \mathbf{Z}$. *Assume that* f *has a Fréchet derivative with respect to* x, $f_x(x,y)$ *which is continuous in* U. *Let* $(x_0, y_0) \in U$ *and* $f(x_0,y_0) = 0$. *If* $A = f_x(x_0,y_0)$ *is an isomorphism of* $\mathbf{X}$ *onto* $\mathbf{Z}$ *then*

(1) There is a ball $\{y : ||y - y_0|| < r\} = B_r(y_0)$ *and a unique continuous map* $u : B_r(y_0) \to \mathbf{X}$ *such that* $u(y_0) = x_0$ *and* $f(u(y),y) \equiv 0$.

(2) If f *is of class* C^1 *then* $u(y)$ *is of class* C^1 *and* $u_y(y) = -(f_x(u(y),y))^{-1} \circ f_y(u(y),y)$.

(3) $u_y(y)$ *belongs to* C^k *if* f *is in* C^k, $k > 1$.

We will work with $\tilde{U} := U - \bar{U}$, a calculation gives

$$(1-x^2)U_\theta' + 2xU_\theta + \frac{1}{2}U_\theta^2 - \mu(1-x)^2 = (1-x^2)\tilde{U}_\theta' + (2x + \bar{U}_\theta)\tilde{U}_\theta + \frac{1}{2}\tilde{U}_\theta^2,$$

where $\tilde{U}_\phi = U_\phi$. Denote

$$\psi[\tilde{U}_\phi](x) := \int_x^1 \int_l^1 \int_t^1 \frac{2\tilde{U}_\phi \tilde{U}_\phi'}{1-s^2} dsdtdl. \tag{3.14}$$

Define a map G on $(\mu,\gamma,\tilde U)$ by

$$G(\mu,\gamma,\tilde U)=\begin{pmatrix}(1-x^2)\tilde U_\theta'+(2x+\bar U_\theta)\tilde U_\theta+\frac12\tilde U_\theta^2-\psi[\tilde U_\phi](x)+\frac14\psi[\tilde U_\phi](-1)(1-x)^2\\(1-x^2)\tilde U_\phi''+(\tilde U_\theta+\bar U_\theta)\tilde U_\phi'\end{pmatrix}$$
$$=:\begin{pmatrix}\xi_\theta\\ \xi_\phi\end{pmatrix}. \tag{3.15}$$

If $(\mu,\gamma,\tilde U)$ satisfies $G(\mu,\gamma,\tilde U)=0$, then $U=\tilde U+\bar U$ gives a solution of (3.13)with $\hat\mu=\mu-\frac14\psi[\tilde U_\phi](-1)$ satisfying $U_\theta(-1)=\bar U_\theta(-1)$.

Let A and Q be maps of the form

$$A(\mu,\gamma,\tilde U)=\begin{pmatrix}A_\theta\\ A_\phi\end{pmatrix}:=\begin{pmatrix}(1-x^2)\tilde U_\theta'+(2x+\bar U_\theta)\tilde U_\theta\\(1-x^2)\tilde U_\phi''+\bar U_\theta\tilde U_\phi'\end{pmatrix}, \tag{3.16}$$

and

$$Q(\tilde U,\tilde V)=\begin{pmatrix}Q_\theta\\ Q_\phi\end{pmatrix}$$
$$:=\begin{pmatrix}\frac12\tilde U_\theta\tilde V_\theta-\int_x^1\int_l^1\int_t^1\frac{2\tilde U_\phi(s)\tilde V_\phi'(s)}{1-s^2}dsdtdl+\frac{(1-x)^2}{4}\int_{-1}^1\int_l^1\int_t^1\frac{2\tilde U_\phi(s)\tilde V_\phi'(s)}{1-s^2}dsdtdl\\ \tilde U_\theta\tilde V_\phi'\end{pmatrix}. \tag{3.17}$$

Then $G(\mu,\gamma,\tilde U)=A(\mu,\gamma,\tilde U)+Q(\tilde U,\tilde U)$.

By computation, the linearized operator of G with respect to $\tilde U$ at $(\mu,\gamma,\tilde U)$ is given by

$$L_{\tilde U}^{\mu,\gamma}\tilde V:=\begin{pmatrix}(1-x^2)\tilde V_\theta'+(2x+\bar U_\theta)\tilde V_\theta+\tilde U_\theta\tilde V_\theta-\Psi_{\tilde U_\phi}[\tilde V_\phi](x)+\frac14\Psi_{\tilde U_\phi}[\tilde V_\phi](-1)(1-x)^2\\(1-x^2)\tilde V_\phi''+(\tilde U_\theta+\bar U_\theta)\tilde V_\phi'+\tilde V_\theta\tilde U_\phi'\end{pmatrix} \tag{3.18}$$

where

$$\Psi_{\tilde U_\phi}[\tilde V_\phi](x):=\int_x^1\int_l^1\int_t^1\frac{2(\tilde U_\phi(s)\tilde V_\phi'(s)+\tilde V_\phi(s)\tilde U_\phi'(s))}{1-s^2}dsdtdl.$$

In particular, at $\tilde U=0$, the linearized operator of G with respect to $\tilde U$ is

$$L_0^{\mu,\gamma}\tilde V=\begin{pmatrix}(1-x^2)\tilde V_\theta'+(2x+\bar U_\theta)\tilde V_\theta\\(1-x^2)\tilde V_\phi''+\bar U_\theta\tilde V_\phi'\end{pmatrix}. \tag{3.19}$$

Let

$$a_{\mu,\gamma}(x) := \int_0^x \frac{2s + \bar{U}_\theta}{1 - s^2} ds, \quad b_{\mu,\gamma}(x) := \int_0^x \frac{\bar{U}_\theta}{1 - s^2} ds, \quad -1 < x < 1. \tag{3.20}$$

By Corollary 3.1.1, for all $(\mu, \gamma) \in I$, $\bar{U}_\theta$ is smooth in $(-1, 1]$ and $\bar{U}_\theta(x) = O(1 - x)$. So $a_{\mu,\gamma} \in C^\infty(-1, 1)$ and $b_{\mu,\gamma} \in C^\infty(-1, 1]$.

Note that this definition of $a(x)$ and $b(x)$ are consistent with the definition of $a(\theta)$ and $b(\theta)$ in Chapter 1, and $a(x) = -\ln(1 - x^2) + b(x)$. A calculation gives

$$a'_{\mu,\gamma}(x) = \frac{2x + \bar{U}_\theta(x)}{1 - x^2}, \quad a''_{\mu,\gamma}(x) = \frac{2 + \bar{U}'_\theta(x)}{1 - x^2} + \frac{4x^2 + 2x\bar{U}_\theta(x)}{(1 - x^2)^2}. \tag{3.21}$$

Consider the following system of ordinary differential equations in $(-1, 1)$:

$$\begin{cases} (1 - x^2)V'_\theta + 2xV_\theta + \bar{U}_\theta V_\theta = 0, \\ (1 - x^2)V''_\phi + \bar{U}_\theta V'_\phi = 0. \end{cases}$$

All solutions $V = (V_\theta, V_\phi) \in C^2((-1, 1), \mathbb{R}^2)$ are given by

$$V = c_1 V^1_{\mu,\gamma} + c_2 V^2_{\mu,\gamma} + c_3 V^3_{\mu,\gamma} \tag{3.22}$$

where $c_1, c_2, c_3 \in \mathbb{R}$, and

$$V^1_{\mu,\gamma} := \begin{pmatrix} e^{-a_{\mu,\gamma}(x)} \\ 0 \end{pmatrix}, \quad V^2_{\mu,\gamma} := \begin{pmatrix} 0 \\ \int_x^1 e^{-b_{\mu,\gamma}(t)} dt \end{pmatrix}, \quad V^3_{\mu,\gamma} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.23}$$

Next, by computation $(V^1_{\mu,\gamma})_\theta(0) = 1$, $(V^1_{\mu,\gamma})_\phi = 0$, $(V^2_{\mu,\gamma})_\theta = 0$ and

$$(V^2_{\mu,\gamma})_\phi(0) = \int_0^1 e^{-b_{\mu,\gamma}(t)} dt > 0$$

finite. Introduce the linear functionals l_1, l_2 acting on vector functions $V(x) = (V_\theta(x), V_\phi(x))$ by

$$l_1(V) := V_\theta(0), \quad l_2(V) := V_\phi(0). \tag{3.24}$$

It can be seen that for every $(\mu, \gamma) \in I$, the matrix $(l_i(V^j_{\mu,\gamma}))$ is a diagonal invertible matrix.

3.2.2 Existence of solutions with nonzero swirl near $U^{\mu,\gamma}$ when $(\mu,\gamma)\in I_1$

Let us first look at the problem near $U^{\mu,\gamma}$ when $(\mu,\gamma)\in I_1$. For some fixed $(\mu,\gamma)\in I_1$, write $\bar{U}=U^{\mu,\gamma}$, recall that in Corollary 3.1.1 we have

$$\bar{U}_\theta=(1-x)\left(1-b-\frac{2b(1+\gamma-b)}{(1+\gamma+b)(\frac{1+x}{2})^{-b}-\gamma-1+b}\right) \tag{3.25}$$

where $b=\sqrt{1+2\mu}$. It satisfies

$$(1-x^2)\bar{U}_\theta'+2x\bar{U}_\theta+\frac{1}{2}\bar{U}_\theta^2=\mu(1-x)^2.$$

Let us start from constructing the Banach spaces we use. Given a compact subset $K\subset I_1$, from the explicit formula of $U^{\mu,\gamma}$ in Section 3.1, $\bar{U}:=U^{\mu,\gamma}$ satisfies $\bar{U}_\theta(-1)<2$, so there exists an $\epsilon>0$, depending only on K, satisfying $\max\limits_{(\mu,\gamma)\in K}\dfrac{U_\theta^{\mu,\gamma}(-1)}{2}<\epsilon<1$ for all $(\mu,\gamma)\in K$. For this fixed ϵ, define

$$\begin{aligned}
\mathbf{M}_1=&\mathbf{M}_1(\epsilon)\\
:=&\Big\{\tilde{U}_\theta\in C([-1,1],\mathbb{R})\cap C^1((-1,1],\mathbb{R})\cap C^2((0,1),\mathbb{R})\mid \tilde{U}_\theta(1)=\tilde{U}_\theta(-1)=0,\\
&||(1+x)^{-1+\epsilon}\tilde{U}_\theta||_{L^\infty(-1,1)}<\infty, ||(1+x)^{\epsilon}\tilde{U}_\theta'||_{L^\infty(-1,1)}<\infty, ||\tilde{U}_\theta''||_{L^\infty(0,1)}<\infty\Big\},\\
\mathbf{M}_2=&\mathbf{M}_2(\epsilon)\\
:=&\Big\{\tilde{U}_\phi\in C^1((-1,1],\mathbb{R})\cap C^2((-1,1),\mathbb{R})\mid \tilde{U}_\phi(1)=0, ||\tilde{U}_\phi||_{L^\infty(-1,1)}<\infty,\\
&||(1+x)^{\varepsilon}\tilde{U}_\phi'||_{L^\infty(-1,1)}<\infty, ||(1+x)^{1+\varepsilon}\tilde{U}_\phi''||_{L^\infty(-1,1)}<\infty\Big\}
\end{aligned}$$

with the following norms accordingly:

$$\begin{aligned}
&||\tilde{U}_\theta||_{\mathbf{M}_1}:=||(1+x)^{-1+\epsilon}\tilde{U}_\theta||_{L^\infty(-1,1)}+||(1+x)^{\epsilon}\tilde{U}_\theta'||_{L^\infty(-1,1)}+||\tilde{U}_\theta''||_{L^\infty(0,1)},\\
&||\tilde{U}_\phi||_{\mathbf{M}_2}:=||\tilde{U}_\phi||_{L^\infty(-1,1)}+||(1+x)^{\varepsilon}\tilde{U}_\phi'||_{L^\infty(-1,1)}+||(1+x)^{1+\varepsilon}\tilde{U}_\phi''||_{L^\infty(-1,1)}.
\end{aligned}$$

Next, define the following function spaces:

$$\begin{aligned}
\mathbf{N}_1=\mathbf{N}_1(\epsilon):=&\big\{\xi_\theta\in C((-1,1],\mathbb{R})\cap C^1((0,1],\mathbb{R})\mid \xi_\theta(1)=\xi_\theta'(1)=\xi_\theta(-1)=0,\\
&||(1+x)^{-1+\epsilon}\xi_\theta||_{L^\infty(-1,1)}<\infty, ||\frac{\xi_\theta'}{1-x}||_{L^\infty(0,1)}<\infty\Big\},\\
\mathbf{N}_2=\mathbf{N}_2(\epsilon):=&\left\{\xi_\phi\in C((-1,1],\mathbb{R})\mid \xi_\phi(1)=0, ||\frac{(1+x)^{\varepsilon}\xi_\phi}{1-x}||_{L^\infty(-1,1)}<\infty\right\}
\end{aligned}$$

with the following norms accordingly:

$$||\xi_\theta||_{\mathbf{N}_1} := ||(1+x)^{-1+\epsilon}\xi_\theta||_{L^\infty(-1,1)} + ||\frac{\xi'_\theta}{1-x}||_{L^\infty(0,1)},$$
$$||\xi_\phi||_{\mathbf{N}_2} := ||\frac{(1+x)^\varepsilon \xi_\phi}{1-x}||_{L^\infty(-1,1)}.$$

Let $\mathbf{X} := \{\tilde{U} = (\tilde{U}_\theta, \tilde{U}_\phi) \mid \tilde{U}_\theta \in \mathbf{M}_1, \tilde{U}_\phi \in \mathbf{M}_2\}$ with the norm $||\tilde{U}||_{\mathbf{X}} := ||\tilde{U}_\theta||_{\mathbf{M}_1} + ||\tilde{U}_\phi||_{\mathbf{M}_2}$, and $\mathbf{Y} := \{\xi = (\xi_\theta, \xi_\phi) \mid \xi_\theta \in \mathbf{N}_1, \xi_\phi \in \mathbf{N}_2\}$ with the norm $||\xi||_{\mathbf{Y}} := ||\xi_\theta||_{\mathbf{N}_1} + ||\xi_\phi||_{\mathbf{N}_2}$. It is not difficult to verify that $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{X}$ and $\mathbf{Y}$ are Banach spaces.

Let $l_1, l_2 : \mathbf{X} \to \mathbb{R}$ be the bounded linear functionals defined by (3.24) for each $V \in \mathbf{X}$. Define

$$\mathbf{X}_1 := \ker l_1 \cap \ker l_2. \tag{3.26}$$

Theorem 3.2.1. *For every compact $K \subset I_1$, with $\max\{0, U_\theta^{\mu,\gamma}(-1)\} < 2\epsilon < 2$ for every $(\mu,\gamma) \in K$, there exist $\delta = \delta(K) > 0$, and $V \in C^\infty(K \times B_\delta(0), \mathbf{X}_1)$ satisfying $V(\mu,\gamma,0,0) = 0$ and $\frac{\partial V}{\partial \beta_i}|_{\beta=0} = 0$, $i = 1,2$, such that*

$$U = U^{\mu,\gamma} + \beta_1 V^1_{\mu,\gamma} + \beta_2 V^2_{\mu,\gamma} + V(\mu,\gamma,\beta_1,\beta_2) \tag{3.27}$$

satisfies equation (3.13) with $\hat{\mu} = \mu - \frac{1}{4}\psi[U_\phi](-1)$. Moreover, there exists some $\delta' = \delta'(K) > 0$, such that if $||U - U^{\mu,\gamma}||_{\mathbf{X}} < \delta'$, $(\mu,\gamma) \in K$, and U satisfies equation (3.13) with some constant $\hat{\mu}$, then (3.27) holds for some $|(\beta_1,\beta_2)| < \delta$.

To prove the theorem, we first study the properties of the Banach spaces $\mathbf{X}$ and $\mathbf{Y}$ we constructed.

With the fixed ϵ, we have

Lemma 3.2.1. *For every $\tilde{U} \in \mathbf{X}$, it satisfies the following*

$$|\tilde{U}_\phi(s)| \le ||\tilde{U}_\phi||_{\mathbf{M}_2}(1-s), \quad \forall\ -1 < s < 1, \tag{3.28}$$

$$|\tilde{U}_\theta(s)| \le ||\tilde{U}_\theta||_{\mathbf{M}_1}(1-s)(1+s)^{1-\epsilon}, \quad \forall\ -1 < s < 1. \tag{3.29}$$

Proof. For $s \in (0,1)$, there exists $y \in (s,1)$ such that

$$|\tilde{U}_\phi(s)| = |\tilde{U}_\phi(s) - \tilde{U}_\phi(1)| = |\tilde{U}'_\phi(y)|(1-s) \le (1-s)||\tilde{U}_\phi||_{\mathbf{M}_2},$$

while for $s \in (-1,0]$, $|\tilde{U}_\phi(s)| \le ||\tilde{U}_\phi||_{\mathbf{M}_2}$. So (3.28) is proved.

Now we prove (3.29). For $0 < s < 1$, $|(1+s)^{-1+\epsilon}\tilde{U}_\theta(s)| \le |\tilde{U}_\theta(s)| = |\tilde{U}_\theta(s)-\tilde{U}_\theta(1)| \le ||\tilde{U}'_\theta||_{L^\infty(0,1)}(1-s) \le ||\tilde{U}_\theta||_{\mathbf{M}_1}(1-s)$, and for $-1 < s \le 0$, $\left|(1+s)^{-1+\epsilon}(1-s)^{-1}\tilde{U}_\theta(s)\right| \le |(1+s)^{-1+\epsilon}\tilde{U}_\theta(s)| \le ||\tilde{U}_\theta||_{\mathbf{M}_1}$.

□

Lemma 3.2.2. *For every $\xi_\theta \in \mathbf{N}_1$,*

$$|\xi_\theta(x)| \le ||\xi_\theta||_{\mathbf{N}_1}(1+x)^{1-\epsilon}(1-x)^2, \quad \forall -1 < x \le 1. \tag{3.30}$$

Proof. If $\xi_\theta \in \mathbf{N}_1$, $\xi_\theta(1) = 0$. So for every $0 < x < 1$, there exists $y \in (x, 1)$ such that

$$|(1+x)^{-1+\epsilon}\xi_\theta(x)| \le |\xi_\theta(x)| = |\xi'_\theta(y)|(1-x) \le ||\xi_\theta||_{\mathbf{N}_1}(1-y)(1-x) \le ||\xi_\theta||_{\mathbf{N}_1}(1-x)^2.$$

For $-1 < x \le 0$, $|(1+x)^{-1+\epsilon}\xi_\theta(x)| \le ||\xi_\theta||_{\mathbf{N}_1} \le ||\xi_\theta||_{\mathbf{N}_1}(1-x)^2$. □

Near $\bar{U} = (\bar{U}_\theta, 0)$, we will prove the existence of a family of solutions $U(\mu,\gamma,\beta)$ in $\mathbf{X}$, $\beta = (\beta_1,\beta_2) \in \mathbb{R}^2$, which are (-1)-homogeneous, axisymmetric, with non-zero swirl when $\beta \ne 0$, and $U(\mu,\gamma,0) = \bar{U}$.

For $\tilde{U}_\phi \in \mathbf{M}_2$, let $\psi[\tilde{U}_\phi](x)$ be defined by (3.14). Define a map G on $K \times \mathbf{X}$ by (3.15) with $\bar{U}_\theta$ given by (3.25).

Proposition 3.2.1. *The map G is in $C^\infty(K\times\mathbf{X},\mathbf{Y})$ in the sense that G has continuous Fréchet derivatives of every order. Moreover, the Fréchet derivative of G with respect to $\tilde{U}$ at $(\mu,\gamma,\tilde{U}) \in \mathbf{X}$ is given by the linear bounded operator $L^{\mu,\gamma}_{\tilde{U}} : \mathbf{X} \to \mathbf{Y}$ defined as in (*3.18*).*

To prove Proposition 3.2.1, we first prove the following lemmas:

Lemma 3.2.3. *For every $(\mu,\gamma) \in K$, $A(\mu,\gamma,\cdot) : \mathbf{X} \to \mathbf{Y}$ defined by (*3.16*) is a well-defined bounded linear operator.*

Proof. In the following, C denotes a universal constant which may change from line to line. For convenience we denote $A = A(\mu,\gamma,\cdot)$ for some fixed $(\mu,\gamma) \in K$. We make use of the property of $\bar{U}_\theta$ that $\bar{U}_\theta(1) = 0$ and $\bar{U}_\theta \in C^2(-1,1] \cap L^\infty(-1,1)$.

A is clearly linear. For every $\tilde{U} \in \mathbf{X}$, we prove that $A\tilde{U}$ defined by (3.16) is in $\mathbf{Y}$ and there exists some constant C such that $||A\tilde{U}||_{\mathbf{Y}} \le C||\tilde{U}||_{\mathbf{X}}$ for all $\tilde{U} \in \mathbf{X}$.

By the fact that $\tilde{U}_\theta \in \mathbf{M}_1$ and (3.29), we have

$$|(1+x)^{-1+\epsilon} A_\theta| \le (1-x)|(1+x)^\epsilon \tilde{U}_\theta'| + (2+|\bar{U}_\theta|)(1+x)^{-1+\epsilon}|\tilde{U}_\theta| \le C(1-x)||\tilde{U}_\theta||_{\mathbf{M}_1}.$$

We also see from the above that $\lim_{x\to 1} A_\theta(x) = \lim_{x\to -1} A_\theta(x) = 0$. By computation $A_\theta' = (1-x^2)\tilde{U}_\theta'' + \bar{U}_\theta \tilde{U}_\theta' + (2+\bar{U}_\theta')\tilde{U}_\theta$. Then, by (3.29) and (3.25),

$$\frac{|A_\theta'|}{1-x} \le (1+x)|\tilde{U}_\theta''| + \frac{|\bar{U}_\theta|}{1-x}|\tilde{U}_\theta'| + (2+|\bar{U}_\theta'|)\frac{|\tilde{U}_\theta|}{1-x} \le C||\tilde{U}_\theta||_{\mathbf{M}_1},\ 0<x<1.$$

So we have $A_\theta \in \mathbf{N}_1$ and $||A_\theta||_{\mathbf{N}_1} \le C||\tilde{U}_\theta||_{\mathbf{M}_1}$.

Next, since $A_\phi = (1-x^2)\tilde{U}_\phi'' + \bar{U}_\theta \tilde{U}_\phi'$, by the fact that $\tilde{U}_\phi \in \mathbf{M}_2$ and (3.25) we have that

$$\left|\frac{(1+x)^\epsilon A_\phi}{1-x}\right| \le \frac{(1+x)^\epsilon}{1-x}(1-x^2)\frac{||\tilde{U}_\phi||_{\mathbf{M}_2}}{(1+x)^{1+\epsilon}} + \frac{(1+x)^\epsilon|\bar{U}_\theta|}{1-x} \cdot \frac{||\tilde{U}_\phi||_{\mathbf{M}_2}}{(1+x)^\epsilon} \le C||\tilde{U}_\phi||_{\mathbf{M}_2}.$$

We also see from the above that $\lim_{x\to 1} A_\phi(x) = 0$. So $A_\phi \in \mathbf{N}_1$, and $||A_\phi||_{\mathbf{N}_1} \le C||\tilde{U}_\phi||_{\mathbf{M}_2}$. We have proved that $A\tilde{U} \in \mathbf{Y}$ and $||A\tilde{U}||_{\mathbf{Y}} \le C||\tilde{U}||_{\mathbf{X}}$ for every $\tilde{U} \in \mathbf{X}$. The proof is finished. $\square$

Lemma 3.2.4. *The map $Q : \mathbf{X}\times\mathbf{X} \to \mathbf{Y}$ defined by (3.17) is a well-defined bounded bilinear operator.*

Proof. It is clear that Q is a bilinear operator. For every $\tilde{U}, \tilde{V} \in \mathbf{X}$, we will prove that $Q(\tilde{U}, \tilde{V})$ is in $\mathbf{Y}$ and there exists some constant C independent of $\tilde{U}$ and $\tilde{V}$ such that $||Q(\tilde{U}, \tilde{V})||_{\mathbf{Y}} \le C||\tilde{U}||_{\mathbf{X}}||\tilde{V}||_{\mathbf{X}}$.

For convenience we write

$$\psi(\tilde{U}, \tilde{V})(x) = \int_x^1 \int_l^1 \int_t^1 \frac{2\tilde{U}_\phi(s)\tilde{V}_\phi'(s)}{1-s^2} dsdtdl.$$

For $\tilde{U}, \tilde{V} \in \mathbf{X}$, we have, using (3.28) and the fact that $\tilde{U}_\phi, \tilde{V}_\phi \in \mathbf{M}_2$, that

$$\left|\frac{\tilde{U}_\phi(s)\tilde{V}_\phi'(s)}{1-s^2}\right| \le (1+s)^{-1-\epsilon}||\tilde{U}_\phi||_{\mathbf{M}_2}||\tilde{U}_\phi||_{\mathbf{M}_2}, \quad \forall -1<s<1. \tag{3.31}$$

It follows that $\psi(\tilde{U}, \tilde{V})(x)$ is well-defined and

$$|\psi(\tilde{U}, \tilde{V})(x)| \le C(\epsilon)(1-x)^3||\tilde{U}_\phi||_{\mathbf{M}_2}||\tilde{U}_\phi||_{\mathbf{M}_2}, \quad \forall -1<x<1. \tag{3.32}$$

Moreover, we have, in view of (3.31), that

$$\begin{aligned}
&|\psi(\tilde{U},\tilde{V})(x) - \frac{(1-x)^2}{4}\psi(\tilde{U},\tilde{V})(-1)| \\
&= \left|\psi(\tilde{U},\tilde{V})(x) - \psi(\tilde{U},\tilde{V})(-1) + \frac{(1+x)(3-x)}{4}\psi(\tilde{U},\tilde{V})(-1)\right| \\
&= \left|-\int_{-1}^{x}\int_{l}^{1}\int_{t}^{1}\frac{2\tilde{U}_\phi(s)\tilde{V}'_\phi(s)}{1-s^2}dsdtdl + \frac{(1+x)(3-x)}{4}\psi(\tilde{U},\tilde{V})(-1)\right| \\
&\le C(\epsilon)(1+x)||\tilde{U}_\phi||_{\mathbf{M_2}}||\tilde{V}_\phi||_{\mathbf{M_2}}, \quad \forall -1<x<1.
\end{aligned}$$

Thus, using (3.32), we have for any $x \in (-1,1)$

$$|\psi(\tilde{U},\tilde{V})(x) - \frac{(1-x)^2}{4}\psi(\tilde{U},\tilde{V})(-1)| \le C(\epsilon)(1+x)(1-x)^2||\tilde{U}_\phi||_{\mathbf{M_2}}||\tilde{V}_\phi||_{\mathbf{M_2}}. \quad (3.33)$$

So by (3.29) and (3.33),

$$\begin{aligned}
&|(1+x)^{-1+\epsilon}Q_\theta(x)| \\
&\le \frac{1}{2}|(1+x)^{-1+\epsilon}\tilde{U}_\theta(x)||\tilde{V}_\theta(x)| + (1+x)^{-1+\epsilon}\left|\psi(\tilde{U},\tilde{V})(x) - \frac{(1-x)^2}{4}\psi(\tilde{U},\tilde{V})(-1)\right| \\
&\le \frac{1}{2}(1-x)^2||\tilde{U}_\theta||_{\mathbf{M_1}}||\tilde{V}_\theta||_{\mathbf{M_1}} + C(\epsilon)(1+x)^\epsilon(1-x)^2||\tilde{U}_\phi||_{\mathbf{M_2}}||\tilde{V}_\phi||_{\mathbf{M_2}} \\
&\le C(\epsilon)(1-x)^2||\tilde{U}||_{\mathbf{X}}||\tilde{V}||_{\mathbf{X}}, \quad \forall -1<x<1.
\end{aligned}$$

From this we also have $\lim_{x\to 1} Q_\theta(x) = \lim_{x\to -1} Q_\theta(x) = 0$.

By computation,

$$Q'_\theta(x) = \frac{1}{2}\tilde{U}_\theta\tilde{V}'_\theta + \frac{1}{2}\tilde{U}'_\theta\tilde{V}_\theta + \int_x^1\int_t^1\frac{2\tilde{U}_\phi(s)\tilde{V}'_\phi(s)}{1-s^2}dsdt - \frac{1-x}{2}\psi(\tilde{U},\tilde{V})(-1), \text{ for } 0<x<1.$$

Using $\tilde{U} \in \mathbf{X}$, (3.28), (3.29) and (3.32), we see that for $0<x<1$,

$$\begin{aligned}
|Q'_\theta(x)| &\le \frac{1}{2}|\tilde{U}_\theta||\tilde{V}'_\theta| + \frac{1}{2}|\tilde{V}_\theta||\tilde{U}'_\theta| + \int_x^1\int_t^1\frac{2|\tilde{U}_\phi(s)||\tilde{V}'_\phi(s)|}{1-s^2}dsdt + \frac{|\psi(\tilde{U},\tilde{V})(-1)|}{2}(1-x) \\
&\le C(1-x)||\tilde{U}_\theta||_{\mathbf{M_1}}||\tilde{V}_\theta||_{\mathbf{M_1}} + 2||\tilde{U}_\phi||_{\mathbf{M_2}}||\tilde{V}_\phi||_{\mathbf{M_2}}\int_x^1\int_l^1(1+s)^{-\epsilon-1}dtdl \\
&+ C(\epsilon)(1-x)||\tilde{U}_\phi||_{\mathbf{M_2}}||\tilde{V}_\phi||_{\mathbf{M_2}} \\
&\le C(1-x)||\tilde{U}_\theta||_{\mathbf{M_1}}||\tilde{V}_\theta||_{\mathbf{M_1}} + C(\epsilon)(1-x)||\tilde{U}_\phi||_{\mathbf{M_2}}||\tilde{V}_\phi||_{\mathbf{M_2}} \\
&\le C(\epsilon)(1-x)||\tilde{U}||_{\mathbf{X}}||\tilde{V}||_{\mathbf{X}}.
\end{aligned}$$

So there is $Q_\theta \in \mathbf{N}_1$, and $||Q_\theta||_{\mathbf{N}_1} \le C(\epsilon)||\tilde{U}||_{\mathbf{X}}||\tilde{V}||_{\mathbf{X}}$.

Next, since $Q_\phi(x) = \tilde{U}_\theta(x)\tilde{V}'_\phi(x)$, for $-1<x<1$,

$$\left|\frac{(1+x)^\epsilon Q_\phi(x)}{1-x}\right| \le \frac{(1+x)^\epsilon}{1-x}|\tilde{U}_\theta(x)|\frac{||\tilde{V}_\phi||_{\mathbf{M_2}}}{(1+x)^\epsilon} \le 2||\tilde{U}_\theta||_{\mathbf{M_1}}||\tilde{V}_\phi||_{\mathbf{M_2}}.$$

We also see from the above that $\lim_{x\to 1} Q_\phi(x) = 0$. So $Q_\phi \in \mathbf{N}_2$, and

$$||Q_\phi||_{\mathbf{N_2}} \le ||\tilde{U}_\theta||_{\mathbf{M}_1} ||\tilde{V}_\phi||_{\mathbf{M_2}}.$$

Thus we have proved that $Q(\tilde{U},\tilde{V}) \in \mathbf{Y}$ and $||Q(\tilde{U},\tilde{V})||_{\mathbf{Y}} \le C||\tilde{U}||_{\mathbf{X}}||\tilde{V}||_{\mathbf{X}}$ for all $\tilde{U},\tilde{V} \in \mathbf{X}$. Lemma 3.2.4 is proved. □

Proof of Proposition 3.2.1*:* By definition, $G(\mu,\gamma,\tilde{U}) = A(\mu,\gamma,\tilde{U}) + Q(\tilde{U},\tilde{U})$ for $(\mu,\gamma,\tilde{U}) \in K \times \mathbf{X}$. Using standard theories in functional analysis, by Lemma 3.2.4 it is clear that Q is C^∞ on $I_1 \times \mathbf{X}$. By Lemma 3.2.3, $A(\mu,\gamma;\cdot) : \mathbf{X} \to \mathbf{Y}$ is C^∞ for each $(\mu,\gamma) \in I_1$. For all $i,j \ge 0$, $i+j \ne 0$, we have

$$\partial_\mu^i \partial_\gamma^j A(\mu,\gamma,\tilde{U}) = \partial_\mu^i \partial_\gamma^j U_\theta^{\mu,\gamma} \begin{pmatrix} \tilde{U}_\theta \\ \tilde{U}'_\phi \end{pmatrix}.$$

By (3.10), for each pair of integers (i,j) where $i,j \ge 0$, $i+j \ne 0$, there exists some constant $C = C(i,j,K)$, depending only on i,j,K, such that

$$|\partial_\mu^i \partial_\gamma^j U_\theta^{\mu,\gamma}(x)| \le C(i,j,K)(1-x), \quad -1 < x < 1. \tag{3.34}$$

From (3.25) we can also obtain

$$\left| \frac{d}{dx} \partial_\mu^i \partial_\gamma^j U_\theta^{\mu,\gamma}(x) \right| \le C(i,j,K), \quad 0 < x < 1.$$

Using the above estimates and the fact that $\tilde{U}_\theta \in \mathbf{M}_1$, we have

$$|(1+x)^{-1+\epsilon} \partial_\mu^i \partial_\gamma^j A_\theta(\mu,\gamma,\tilde{U})| \le C(i,j,K)(1-x)||\tilde{U}_\theta||_{\mathbf{M}_1}, \quad -1 < x < 1,$$

and

$$\begin{aligned} \left| \frac{d}{dx} \partial_\mu^i \partial_\gamma^j A_\theta(\mu,\gamma,\tilde{U}) \right| &\le \left| \frac{d}{dx} \partial_\mu^i \partial_\gamma^j U_\theta^{\mu,\gamma}(x) \right| |\tilde{U}_\theta(x)| + |\partial_\mu^i \partial_\gamma^j U_\theta^{\mu,\gamma}(x)| \left| \frac{d}{dx} \tilde{U}_\theta(x) \right| \\ &\le C(K)(1-x)||\tilde{U}_\theta||_{\mathbf{M}_1}, \quad 0 < x < 1. \end{aligned}$$

So $\partial_\mu^i \partial_\gamma^j A_\theta(\mu,\gamma,\tilde{U}) \in \mathbf{N}_1$, with $||\partial_\mu^i \partial_\gamma^j A_\theta(\mu,\gamma,\tilde{U})||_{\mathbf{N}_1} \le C(i,j,K)||\tilde{U}_\theta||_{\mathbf{M}_1}$ for all $(\mu,\gamma,\tilde{U}) \in K \times \mathbf{X}$.

Next, by (3.34) and the fact that $\tilde{U}_\phi \in \mathbf{M}_1$, we have

$$\frac{(1+x)^\epsilon}{1-x} |\partial_\mu^i \partial_\gamma^j A_\phi(\mu,\gamma,\tilde{U})(x)| = \frac{|\partial_\mu^i \partial_\gamma^j U_\theta^{\mu,\gamma}(x)|}{1-x} |(1+x)^\epsilon U'_\phi| \le C(i,j,K)||\tilde{U}_\phi||_{\mathbf{M_2}}.$$

So $\partial_\mu^i \partial_\gamma^j A_\phi(\mu,\gamma,\tilde U) \in \mathbf{N_2}$, with $||\partial_\mu^i \partial_\gamma^j A_\phi(\mu,\gamma,\tilde U)||_{\mathbf{N_2}} \le C(i,j,K)||\tilde U_\phi||_{\mathbf{M_2}}$ for all $(\mu,\gamma,\tilde U) \in K \times \mathbf{X}$. Thus $\partial_\mu^i \partial_\gamma^j A(\mu,\gamma,\tilde U) \in \mathbf{Y}$, with $||\partial_\mu^i \partial_\gamma^j A(\mu,\gamma,\tilde U)||_{\mathbf{Y}} \le C(i,j,K)||\tilde U||_{\mathbf{X}}$ for all $(\mu,\gamma,\tilde U) \in K \times \mathbf{X}$, $i,j \ge 0$, $i+j \ne 0$.

So for each $(\mu,\gamma) \in K$, $\partial_\mu^i \partial_\gamma^j A(\mu,\gamma;\cdot) : \mathbf{X} \to \mathbf{Y}$ is a bounded linear map with uniform bounded norm on K. Then by standard theories in functional analysis, $A : K \times \mathbf{X} \to \mathbf{Y}$ is C^∞. So G is a C^∞ map from $K \times \mathbf{X}$ to Y. By direct calculation we get its Fréchet derivative with respect to $\mathbf{X}$ is given by the linear bounded operator $L_{\tilde U}^{\mu,\gamma} : \mathbf{X} \to \mathbf{Y}$ defined as (3.18). The proof is finished. $\square$

By Proposition 3.2.1, $L_0^{\mu,\gamma} : \mathbf{X} \to \mathbf{Y}$, the Fréchet derivative of G with respect to $\tilde U$ at $\tilde U = 0$, is given by (3.19).

Let $a_{\mu,\gamma}(x), b_{\mu,\gamma}(x)$ be the functions defined by (3.20) with $\bar U_\theta$ given by (3.25). For $\xi = (\xi_\theta, \xi_\phi) \in \mathbf{Y}$, let the map $W^{\mu,\gamma}$ be defined as $W^{\mu,\gamma}(\xi) := (W_\theta^{\mu,\gamma}(\xi), W_\phi^{\mu,\gamma}(\xi))$, where

$$\begin{aligned} W_\theta^{\mu,\gamma}(\xi)(x) &:= e^{-a_{\mu,\gamma}(x)} \int_0^x e^{a_{\mu,\gamma}(s)} \frac{\xi_\theta(s)}{1-s^2} ds, \\ W_\phi^{\mu,\gamma}(\xi)(x) &:= \int_x^1 e^{-b_{\mu,\gamma}(t)} \int_t^1 e^{b_{\mu,\gamma}(s)} \frac{\xi_\phi(s)}{1-s^2} ds dt. \end{aligned} \tag{3.35}$$

A calculation gives

$$(W_\theta^{\mu,\gamma}(\xi))'(x) = -a'(x) W_\theta^{\mu,\gamma}(x) + \frac{\xi_\theta(x)}{1-x^2}. \tag{3.36}$$

Lemma 3.2.5. *For every $(\mu,\gamma) \in K$, $W^{\mu,\gamma} : \mathbf{Y} \to \mathbf{X}$ is continuous, and is a right inverse of $L_0^{\mu,\gamma}$.*

Proof. We make use of the property that $\bar U_\theta(1) = 0$, $\bar U_\theta \in C^2(-1,1] \cap C^0[-1,1]$ and $\bar U_\theta(-1) < 2\epsilon < 2$. For convenience let us write $W := W^{\mu,\gamma}(\xi)$ for $\xi \in \mathbf{Y}$, $a(x) = a_{\mu,\gamma}(x)$ and $b(x) = b_{\mu,\gamma}(x)$.

We first prove W is well-defined. Applying Lemma 3.2.2 in the expression of W_θ in (3.35),

$$|(1+x)^{-1+\epsilon} W_\theta(x)| \le (1+x)^{-1+\epsilon} ||\xi_\theta||_{\mathbf{N_1}} e^{-a(x)} \int_0^x e^{a(s)} (1-s)(1+s)^{-\epsilon} ds, \quad -1 < x < 1. \tag{3.37}$$

We make estimates first for $0 < x \le 1$ and then for $-1 < x \le 0$.

Case 1. $0 < x \le 1$.

Since $\bar{U}_\theta(x) = -\bar{U}'_\theta(1)(1-x) + O((1-x)^2)$,

$$b(x) = b(1) + \int_1^x \frac{\bar{U}_\theta}{1-s^2} ds = b(1) + \frac{1}{2}\bar{U}'_\theta(1)(1-x) + O(1)(1-x)^2, \quad 0 < x \le 1, \tag{3.38}$$

where $b(1) := \int_0^1 \frac{\bar{U}_\theta}{1-s^2} ds$ exists and is finite, we have

$$e^{b(x)} = e^{b(1)} \left[1 + \frac{1}{2}\bar{U}'_\theta(1)(1-x) + O(1)(1-x)^2 \right]. \tag{3.39}$$

Notice $a(x) = -\ln(1-x^2) + b(x)$, so

$$e^{a(x)} = \frac{e^{b(1)}}{2(1-x)} \left(1 + \frac{1}{2}\left(\bar{U}'_\theta(1) + 1\right)(1-x) + O(1)(1-x)^2 \right). \tag{3.40}$$

Then in (3.37), using the estimate of $a(x)$ and $e^{a(x)}$, it is not hard to see that there exists some positive constant C such that

$$e^{a(s)}(1-s)(1+s)^{-\epsilon} \le C, \quad e^{a(x)} \ge \frac{1}{C(1-x)}, \quad 0 < s < x < 1.$$

Thus

$$|W_\theta(x)| \le C||\xi_\theta||_{\mathbf{N}_1}(1-x), \quad 0 < x \le 1. \tag{3.41}$$

In particular, $W_\theta(1) = 0$.

By (3.21) and (3.36), for $0 < x < 1$,

$$|a'(x)| \le \frac{C}{(1-x)}, \quad |a''(x)| \le \frac{C}{(1-x)^2}, \tag{3.42}$$

$$|W'_\theta(x)| \le |a'(x)||W_\theta(x)| + \frac{|\xi_\theta(x)|}{1-x} \le C||\xi_\theta||_{\mathbf{N}_1}, \quad 0 < x \le 1,$$

where we have used (3.41), (3.42), the fact that $\xi \in \mathbf{N}_1$, and Lemma 3.2.2. Next,

$$\begin{aligned} W''_\theta(x) &= -a''(x)W_\theta - a'(x)W'_\theta(x) + \left(\frac{\xi_\theta(x)}{1-x^2}\right)' \\ &= ((a'(x))^2 - a''(x))W_\theta(x) - a'(x)\frac{\xi_\theta(x)}{1-x^2} + \frac{\xi'_\theta(x)}{1-x^2} + \frac{2x\xi_\theta(x)}{(1-x^2)^2}. \end{aligned}$$

Thus

$$|W''_\theta(x)| \le |(a'(x))^2 - a''(x)||W_\theta| + |a'(x)|\frac{|\xi_\theta|}{1-x^2} + \frac{|\xi'_\theta|}{(1-x)} + \frac{|\xi_\theta|}{(1-x)^2}.$$

By computation

$$(a'(x))^2 - a''(x) = \frac{\bar{U}_\theta^2 + 2x\bar{U}_\theta}{(1-x^2)^2} - \frac{2 + \bar{U}'_\theta}{1-x^2} = O\left(\frac{1}{1-x}\right).$$

It follows, using (3.41), (3.42) and Lemma 3.2.2, that

$$|W_\theta''(x)| \le C\left(\frac{|W_\theta(x)|}{1-x} + \frac{|\xi_\theta|}{(1-x)^2} + \frac{|\xi_\theta'|}{1-x}\right) \le C||\xi_\theta||_{\mathbf{N}_1}, \quad 0 < x < 1.$$

Case 2. $-1 < x \le 0$.

Recall that $\bar{U}_\theta(-1) < 2\epsilon < 2$. Moreover, $\bar{U}_\theta(x) = \bar{U}_\theta(-1) + O((1+x)^b)$ with $b = \sqrt{1+2\mu}$. Then we have , for $-1 < x \le 0$, that

$$b(x) = \frac{\bar{U}_\theta(-1)}{2}\ln(1+x) + O(1), \quad a(x) = (\frac{\bar{U}_\theta(-1)}{2} - 1)\ln(1+x) + O(1),$$
$$e^{a(x)} = (1+x)^{\frac{\bar{U}_\theta(-1)}{2}-1}e^{O(1)}, \quad e^{-a(x)} = (1+x)^{1-\frac{\bar{U}_\theta(-1)}{2}}e^{O(1)}.$$

So there exists some constant C such that

$$e^{a(s)}(1-s)(1+s)^{-\epsilon} \le C(1+s)^{\frac{\bar{U}_\theta(-1)}{2}-1-\epsilon}, \quad e^{-a(s)} \le C(1+s)^{1-\frac{\bar{U}_\theta(-1)}{2}}, \quad -1 < s \le 0.$$

Apply these estimates in (3.37), and use the fact that $\bar{U}_\theta(-1) < 2\epsilon$, we have

$$|(1+x)^{-1+\epsilon}W_\theta(x)| \le C\left|(1+x)^{-\frac{\bar{U}_\theta(-1)}{2}+\epsilon} - 1\right| ||\xi_\theta||_{\mathbf{N}_1} \le C||\xi_\theta||_{\mathbf{N}_1}, \quad -1 < x \le 0. \tag{3.43}$$

By (3.36), (3.21) and (3.43), we have, for $-1 < x \le 0$, that

$$|(1+x)^\epsilon W_\theta'(x)| \le |a'(x)(1+x)^\epsilon W_\theta(x)| + \frac{|\xi_\theta(x)|(1+x)^\epsilon}{1-x^2} \le C||\xi_\theta||_{\mathbf{N}_1}.$$

So we have shown that $W_\theta \in \mathbf{M}_1$, and $||W_\theta||_{\mathbf{M}_1} \le C||\xi_\theta||_{\mathbf{N}_1}$ for some constant C.

By the definition of $W_\phi(\xi)$ in (3.35) and the fact that $\xi_\phi \in \mathbf{N}_2$, we have, for every $-1 < x < 1$,

$$|W_\phi(x)| \le \int_x^1 e^{-b(t)}\int_t^1 e^{b(s)}\frac{|\xi_\phi(s)|}{1-s^2}dsdt \le ||\xi_\phi||_{\mathbf{N}_2}\int_x^1 e^{-b(t)}\int_t^1 e^{b(s)}(1+s)^{-1-\epsilon}dsdt.$$

Since $b(x) = \frac{\bar{U}_\theta(-1)}{2}\ln(1+x) + O(1)$ for all $-1 < x < 1$, there is some constant C such that

$$e^{b(s)} \le C(1+s)^{\frac{\bar{U}_\theta(-1)}{2}}, \quad e^{-b(t)} \le C(1+t)^{-\frac{\bar{U}_\theta(-1)}{2}}, \quad -1 < s, t \le 1. \tag{3.44}$$

So we have, using $\dfrac{\bar{U}_\theta(-1)}{2} < \epsilon < 1$, that for $-1 < x \le 1$,

$$\begin{aligned}|W_\phi(x)| &\le ||\xi_\phi||_{\mathbf{N}_2}\int_x^1 e^{-b(t)}\int_t^1 \frac{e^{b(s)}}{(1+s)^{1+\epsilon}}dsdt \\ &\le C||\xi_\phi||_{\mathbf{N}_2}\int_x^1 (1+t)^{-\frac{\bar{U}_\theta(-1)}{2}}\int_t^1 (1+s)^{\frac{\bar{U}_\theta(-1)}{2}-1-\epsilon}dsdt \\ &\le C(1-x)||\xi_\phi||_{\mathbf{N}_2}.\end{aligned}$$

In particular, $W_\phi(1) = 0$. By computation

$$W'_\phi(x) = -e^{-b(x)} \int_x^1 e^{b(s)} \frac{\xi_\phi(s)}{1-s^2} ds.$$

Thus, using (3.44) and the fact that $\xi_\phi \in \mathbf{N}_2$, we have for $-1 < x < 1$ that

$$\begin{aligned}
|(1+x)^\epsilon W'_\phi(x)| &\le ||\xi_\phi||_{\mathbf{N}_2} (1+x)^\epsilon e^{-b(x)} \int_x^1 e^{b(s)} (1+s)^{-1-\epsilon} ds \\
&\le C||\xi_\phi||_{\mathbf{N}_2} (1+x)^\epsilon (1+x)^{-\frac{\bar{U}_\theta(-1)}{2}} \int_x^1 (1+s)^{\frac{\bar{U}_\theta(-1)}{2}-1-\epsilon} ds \\
&\le C||\xi_\phi||_{\mathbf{N}_2}.
\end{aligned}$$

Similarly

$$W''_\phi(x) = b'(x) e^{-b(x)} \int_x^1 e^{b(s)} \frac{\xi_\phi(s)}{1-s^2} ds + \frac{\xi_\phi(x)}{1-x^2}.$$

Since $|b'(x)| = \left| \frac{\bar{U}_\theta(x)}{1-x^2} \right| \le \frac{C}{1+x}$ for all $-1 < x < 1$, using (3.44), that

$$|(1+x)^{1+\epsilon} W''_\phi(x)| \le C||\xi_\phi||_{\mathbf{N}_2}, \quad -1 < x < 1.$$

So $W_\phi \in \mathbf{M}_2$, and $||W_\phi||_{\mathbf{M}_2} \le C||\xi_\phi||_{\mathbf{N}_2}$ for some constant C.

Then $W^{\mu,\gamma}(\xi) \in \mathbf{X}$ for all $\xi \in \mathbf{Y}$, and $||W^{\mu,\gamma}(\xi)||_{\mathbf{X}} \le C||\xi||_{\mathbf{Y}}$ for some constant C. So $W^{\mu,\gamma} : \mathbf{Y} \to \mathbf{X}$ is well-defined and continuous. It can be checked directly that $W^{\mu,\gamma}$ is a right inverse of $L_0^{\mu,\gamma}$. □

Let $V^1_{\mu,\gamma}, V^2_{\mu,\gamma}, V^3_{\mu,\gamma}$ be vectors defined by (3.23), we have

Lemma 3.2.6. *$\{V^1_{\mu,\gamma}, V^2_{\mu,\gamma}\}$ is a basis of the kernel of $L_0^{\mu,\gamma} : \mathbf{X} \to \mathbf{Y}$.*

Proof. Let $V \in \mathbf{X}$, $L_0^{\mu,\gamma} V = 0$. We know that V is given by (3.22) for some $c_1, c_2, c_3 \in \mathbb{R}$. Since $\bar{U}_\theta(-1) < 2$, it is not hard to verify that $V^1_{\mu,\gamma}, V^2_{\mu,\gamma} \in \mathbf{X}$, and $V^3_{\mu,\gamma} \notin \mathbf{X}$. Since $V \in \mathbf{X}$, we must have $c_3 V^3_{\mu,\gamma} \in \mathbf{X}$, so $c_3 = 0$, and $V \in \text{span}\{V^1_{\mu,\gamma}, V^2_{\mu,\gamma}\}$. It is clear that $\{V^1_{\mu,\gamma}, V^2_{\mu,\gamma}\}$ is independent. So $\{V^1_{\mu,\gamma}, V^2_{\mu,\gamma}\}$ is a basis of the kernel. □

Corollary 3.2.1. *For any $\xi \in \mathbf{Y}$, all solutions of $L_0^{\mu,\gamma} V = \xi$, $V \in \mathbf{X}$, are given by*

$$V = W^{\mu,\gamma}(\xi) + c_1 V^1_{\mu,\gamma} + c_2 V^2_{\mu,\gamma}, \quad c_1, c_2 \in \mathbb{R}.$$

Namely,

$$V_\theta = W^{\mu,\gamma}_\theta(\xi) + c_1 e^{-a_{\mu,\gamma}(x)}, \quad V_\phi = W^{\mu,\gamma}_\phi(\xi) + c_2 \int_x^1 e^{-b_{\mu,\gamma}(t)} dt, \ c_1, c_2 \in \mathbb{R}.$$

Proof. By Lemma 3.2.5, $V - W^{\mu,\gamma}(\xi)$ is in the kernel of $L_0 : \mathbf{X} \to \mathbf{Y}$. The conclusion then follows from Lemma 3.2.6. $\square$

Let l_1, l_2 be the functionals on $\mathbf{X}$ defined by (3.24), and $\mathbf{X}_1$ be the subspace of $\mathbf{X}$ defined by (3.26). As shown in Section 3.2.1, the matrix $(l_i(V^j_{\mu,\gamma}))$ is a diagonal invertible matrix, for every $(\mu,\gamma) \in K$. So $\mathbf{X}_1(\mu,\gamma)$ is a closed subspace of $\mathbf{X}$, and

$$\mathbf{X} = \text{span}\{V^1_{\mu,\gamma}, V^2_{\mu,\gamma}\} \oplus \mathbf{X}_1(\mu,\gamma), \quad \forall (\mu,\gamma) \in K, \tag{3.45}$$

with the projection operator $P(\mu,\gamma) : \mathbf{X} \to \mathbf{X}_1$ given by

$$P(\mu,\gamma)V = V - l_1(V)V^1_{\mu,\gamma} - c(\mu,\gamma)l_2(V)V^2_{\mu,\gamma} \text{ for } V \in \mathbf{X}.$$

where $c(\mu,\gamma) = \left(\int_0^1 e^{-b_{\mu,\gamma}(t)}dt\right)^{-1} > 0$ for all $(\mu,\gamma) \in K$.

Lemma 3.2.7. *For each $(\mu,\gamma) \in K$, the operator $L_0^{\mu,\gamma} : \mathbf{X}_1 \to \mathbf{Y}$ is an isomorphism.*

Proof. By Corollary 3.2.1 and Lemma 3.2.6, $L_0^{\mu,\gamma} : \mathbf{X} \to \mathbf{Y}$ is surjective and $\ker L_0^{\mu,\gamma} =$ span $\{V^1_{\mu,\gamma}, V^2_{\mu,\gamma}\}$. The conclusion of the lemma then follows in view of the direct sum property (3.45). $\square$

Lemma 3.2.8. $V^1_{\mu,\gamma}, V^2_{\mu,\gamma} \in C^\infty(K, \mathbf{X})$.

Proof. We know $2\bar{\epsilon} := \max\{U_\theta^{\mu,\gamma}(-1) | (\mu,\gamma) \in K\} < 2\epsilon$. For convenience in this proof let us denote $a(x) = a_{\mu,\gamma}(x)$, $b(x) = b_{\mu,\gamma}(x)$ and $V^i = V^i_{\mu,\gamma}$, $i = 1, 2$.

By computation, using the explicit expression of $U_\theta^{\mu,\gamma}(x), a(x), a'(x), b(x), V_\theta^1(x)$ and $V_\phi^2(x)$ given by (3.25), (3.20), (3.21) and (3.23), and the estimate of $\partial_\mu^i \partial_\gamma^j U_\theta^{\mu,\gamma}$ in (3.10) for all $i, j \geq 0$. we have, for $(\mu,\gamma) \in K$, that

$$e^{-a(x)} = O(1)(1+x)^{1-\frac{U_\theta^{\mu,\gamma}(-1)}{2}}, \quad e^{-b(x)} = O(1)(1+x)^{-\frac{U_\theta^{\mu,\gamma}(-1)}{2}}, \quad -1 < x \leq 0.$$

So

$$\left|V_\theta^1(x)\right| = O(1)(1+x)^{1-\frac{U_\theta^{\mu,\gamma}(-1)}{2}} = O(1)(1+x)^{1-\bar{\epsilon}}, \quad V_\phi^2(x) = O(1), \quad -1 < x \leq 0,$$

and

$$\left|\frac{d}{dx}V_\theta^1(x)\right| = \left|e^{-a(x)}a'(x)\right| = O(1)(1+x)^{-\frac{U_\theta^{\mu,\gamma}(-1)}{2}} = O(1)(1+x)^{-\bar{\epsilon}}, \quad -1 < x \leq 0,$$

$$\left|\frac{d}{dx}V_\phi^2(x)\right| = e^{-b(x)} = O(1)(1+x)^{-\frac{U_\theta^{\mu,\gamma}(-1)}{2}} = O(1)(1+x)^{-\bar{\epsilon}}, \quad -1 < x \le 0.$$

Moreover,

$$\begin{aligned}
\frac{\partial^i}{\partial\mu^i}a(x) = \frac{\partial^i}{\partial\mu^i}b(x) &= \int_0^x \frac{1}{1-s^2}\frac{\partial^i}{\partial\mu^i}U^{\mu,\gamma}(s)ds \\
&= -\left(\frac{d^i}{d\mu^i}\sqrt{1+2\mu}\right)\ln(1+x) + O(1)\int_0^x (1+s)^{b-1}\left|\ln(1+s)\right|^i ds \\
&= -\left(\frac{d^i}{d\mu^i}\sqrt{1+2\mu}\right)\ln(1+x) + O(1)(1+x)^b\left|\ln(1+x)\right|^i,
\end{aligned}$$

where $|O(1)| \le C$ depending only on K and i. So we have

$$\left|\partial_\mu^i V_\theta^1(x)\right| = e^{-a(x)}O\left(\left|\ln(1+x)\right|^i\right) = O(1)(1+x)^{1-\bar{\epsilon}}\left|\ln(1+x)\right|^i, \quad -1 < x \le 0, i = 1,2,3...$$

Similarly,

$$\left|\partial_\gamma^j\partial_\mu^i V_\theta^1(x)\right| = e^{-a(x)}O\left((1+x)^b\left|\ln(1+x)\right|^i\right) = O(1)(1+x)^{1-\bar{\epsilon}}, \quad -1 < x \le 0, i = 1,2,3...$$

From the above we can see that for all $(\mu,\gamma) \in K$ and $i,j \ge 0$, there exists some constant $C = C(i,j,K)$, such that

$$\left|(1+x)^{-1+\epsilon}\partial_\gamma^j\partial_\mu^i V_\theta^1(x)\right| \le C, \quad \left|(1+x)^\epsilon\frac{d}{dx}\partial_\gamma^j\partial_\mu^i V_\theta^1(x)\right| \le C, \quad -1 < x \le 0.$$

We can also show that for $i,j \ge 0$,

$$\partial_\gamma^j\partial_\mu^i V_\theta^1(1) = 0,$$

and there exists some constant C such that

$$\left|\frac{d^l}{dx^l}\partial_\gamma^j\partial_\mu^i V_\theta^1(x)\right| \le C, \quad l = 0,1,2, \quad 0 \le x < 1.$$

The above imply that for all $i,j \ge 0$, $\partial_\gamma^j\partial_\mu^i V^1(x) \in \mathbf{X}$, and $V_\theta^1 \in C^\infty(K, M_1)$.

Similarly, we can show that $V_\phi^2 \in C^\infty(K, M_2)$. So $V^1, V^2 \in C^\infty(K, \mathbf{X})$. $\square$

Lemma 3.2.9. *There exists $C = C(K) > 0$ such that for all $(\mu,\gamma) \in K$, $(\beta_1,\beta_2) \in \mathbb{R}^2$, and $V \in \mathbf{X}_1$,*

$$||V||_{\mathbf{X}} + |(\beta_1,\beta_2)| \le C||\beta_1 V_{\mu,\gamma}^1 + \beta_2 V_{\mu,\gamma}^2 + V||_{\mathbf{X}}.$$

Proof. We prove the lemma by contradiction. Assume there exist a sequence $(\mu^i, \gamma^i) \in K$, and $(\beta_1^i, \beta_2^i) \in \mathbb{R}^2$, $V^i \in \mathbf{X}_1$, such that

$$||V^i||_{\mathbf{X}} + |(\beta_1^i, \beta_2^i)| \geq i||\beta_1^i V^1_{\mu^i,\gamma^i} + \beta_2^i V^2_{\mu^i,\gamma^i} + V^i||_{\mathbf{X}}. \tag{3.46}$$

Without loss of generality we can assume that

$$||V^i||_{\mathbf{X}} + |(\beta_1^i, \beta_2^i)| = 1.$$

Since K is compact, there exists a subsequence of (μ^i, γ^i), we still denote it as (μ^i, γ^i) and some $(\mu, \gamma) \in K$ such that $(\mu^i, \gamma^i) \to (\mu, \gamma) \in K$ as $i \to \infty$. Similarly, since $|(\beta_1^i, \beta_2^i)| \leq 1$, there exists some subsequence, still denote as (β_1^i, β_2^i), such that $(\beta_1^i, \beta_2^i) \to (\beta_1, \beta_2) \in \mathbb{R}^2$ as $i \to \infty$. By Lemma 3.2.8 we have

$$V^j_{\mu^i,\gamma^i} \to V^j_{\mu,\gamma}, \quad j = 1, 2.$$

By (3.46),

$$\beta_1^i V^1_{\mu^i,\gamma^i} + \beta_2^i V^2_{\mu^i,\gamma^i} + V^i \to 0.$$

This implies

$$V^i \to V := -(\beta_1 V^1_{\mu,\gamma} + \beta_2 V^2_{\mu,\gamma}).$$

On the other hand, $V^i \in \mathbf{X}_1$. Since $\mathbf{X}_1$ is a closed subspace of $\mathbf{X}$, we have $V \in \mathbf{X}_1$. Thus $V \in \mathbf{X}_1 \cap \operatorname{span}\{V^1_{\mu,\gamma}, V^2_{\mu,\gamma}\}$. So $V = 0$.

Since $V^1_{\mu,\gamma}, V^2_{\mu,\gamma}$ are independent for any $(\mu, \gamma) \in K$. We have $\beta_1 = \beta_2 = 0$. However, $||V^i||_{\mathbf{X}} + |(\beta_1^i, \beta_2^i)| = 1$ leads to $||V||_{\mathbf{X}} + |(\beta_1, \beta_2)| = 1$, contradiction. The lemma is proved. $\square$

Proof of Theorem 3.2.1*:* Define a map $F : K \times \mathbb{R}^2 \times \mathbf{X}_1 \to \mathbf{Y}$ by

$$F(\mu, \gamma, \beta_1, \beta_2, V) = G(\mu, \gamma, \beta_1 V^1_{\mu,\gamma} + \beta_2 V^2_{\mu,\gamma} + V).$$

By Proposition 3.2.1, G is a C^∞ map from $K \times \mathbf{X}$ to $\mathbf{Y}$. Let $\tilde{U} = \tilde{U}(\mu, \gamma, \beta_1, \beta_2, \bar{V}) = \beta_1 V^1_{\mu,\gamma} + \beta_2 V^2_{\mu,\gamma} + V$. Using Lemma 3.2.8, we have $\tilde{U} \in C^\infty(K \times \mathbb{R}^2 \times \mathbf{X}_1, \mathbf{X})$. So it concludes that $F \in C^\infty(K \times \mathbb{R}^2 \times \mathbf{X}_1, \mathbf{Y})$.

Next, by definition $F(\mu, \gamma, 0, 0, 0) = 0$ for all $(\mu, \gamma) \in K$. Fix some $(\bar{\mu}, \bar{\gamma}) \in K$, using Lemma 3.2.7, we have $F_V(\bar{\mu}, \bar{\gamma}, 0, 0, 0) = L_0^{\bar{\mu},\bar{\gamma}} : \mathbf{X}_1 \to \mathbf{Y}$ is an isomorphism.

Applying Theorem C, there exist some $\delta > 0$ and a unique $V \in C^\infty(B_\delta(\bar{\mu},\bar{\gamma}) \times B_\delta(0), \mathbf{X}_1)$, such that

$$F(\mu,\gamma,\beta_1,\beta_2,V(\mu,\gamma,\beta_1,\beta_2)) = 0, \quad \forall (\mu,\gamma) \in B_\delta(\bar{\mu},\bar{\gamma}), (\beta_1,\beta_2) \in B_\delta(0),$$

and

$$V(\bar{\mu},\bar{\gamma},0,0) = 0.$$

The uniqueness part of Theorem C holds in the sense that there exists some $0 < \bar{\delta} < \delta$, such that $B_{\bar{\delta}}(\bar{\mu},\bar{\gamma},0,0,0) \cap F^{-1}(0) \subset \{(\mu,\gamma,\beta_1,\beta_2,V(\mu,\gamma,\beta_1,\beta_2)) | (\mu,\gamma) \in B_\delta(\bar{\mu},\bar{\gamma}), \beta \in B_\delta(0)\}$.

Claim: there exists some $0 < \delta_1 < \frac{\bar{\delta}}{2}$, such that $V(\mu,\gamma,0,0) = 0$ for every $(\mu,\gamma) \in B_{\delta_1}(\bar{\mu},\bar{\gamma})$.

Proof of the claim: Since $V(\bar{\mu},\bar{\gamma},0,0) = 0$ and $V(\mu,\gamma,0,0)$ is continuous in (μ,γ), there exists some $0 < \delta_1 < \frac{\bar{\delta}}{2}$, such that for all $(\mu,\gamma) \in B_{\delta_1}(\bar{\mu},\bar{\gamma})$, $(\mu,\gamma,0,0,V(\mu,\gamma,0,0)) \in B_{\bar{\delta}(\bar{\mu},\bar{\gamma},0,0,0)}$. We know that for all $(\mu,\gamma) \in B_{\delta_1}(\bar{\mu},\bar{\gamma})$,

$$F(\mu,\gamma,0,0,0) = 0,$$

and

$$F(\mu,\gamma,0,0,V(\mu,\gamma,0,0)) = 0.$$

By the above mentioned uniqueness result, $V(\mu,\gamma,0,0) = 0$, for every $(\mu,\gamma) \in B_{\delta_1}(\bar{\mu},\bar{\gamma})$.

Now we have $V \in C^\infty(B_{\delta_1}(\bar{\mu},\bar{\gamma}) \times B_{\delta_1}(0), \mathbf{X}_1(\bar{\mu},\bar{\gamma}))$, and

$$F(\mu,\gamma,\beta_1,\beta_2,V(\mu,\gamma,\beta_1,\beta_2)) = 0, \quad \forall (\mu,\gamma) \in B_{\delta_1}(\bar{\mu},\bar{\gamma}), (\beta_1,\beta_2) \in B_{\delta_1}(0).$$

i.e.

$$G(\mu,\gamma,\beta_1 V^1_{\mu,\gamma} + \beta_2 V^2_{\mu,\gamma} + V(\mu,\gamma,\beta_1,\beta_2)) = 0, \quad \forall (\mu,\gamma) \in B_{\delta_1}(\bar{\mu},\bar{\gamma}), (\beta_1,\beta_2) \in B_{\delta_1}(0).$$

Take derivative of the above with respect to β_i at $(\mu,\gamma,0)$, $i = 1, 2$, we have

$$G_{\tilde{U}}(\mu,\gamma,0)(V^i_{\mu,\gamma} + \partial_{\beta_i} V(\mu,\gamma,0,0)) = 0.$$

Since $G_{\tilde{U}}(\mu,\gamma,0)V^i_{\mu,\gamma} = 0$ by Lemma 3.2.6, we have

$$G_{\tilde{U}}(\mu,\gamma,0)\partial_{\beta_i} V(\mu,\gamma,0,0) = 0.$$

But $\partial_{\beta_i} V(\mu,\gamma,0,0) \in C^\infty(\mathbf{X}_1)$, so

$$\partial_{\beta_i} V(\mu,\gamma,0,0) = 0, \quad i = 1,2.$$

Since K is compact, we can take δ_1 to be a universal constant for each $(\mu,\gamma) \in K$. So we have proved the existence of V in Theorem 3.2.1.

Next, let $(\mu,\gamma) \in B_{\delta_1}(\bar{\mu},\bar{\gamma})$. Let δ' be a small constant to be determined. For any U satisfies the equation (3.13) with $U - U^{\mu,\gamma} \in \mathbf{X}$, and $||U - U^{\mu,\gamma}||_{\mathbf{X}} \leq \delta'$ there exist some $\beta_1, \beta_2 \in \mathbb{R}$ and $V^* \in \mathbf{X}_1$ such that

$$U - U^{\mu,\gamma} = \beta_1 V^1_{\mu,\gamma} + \beta_2 V^2_{\mu,\gamma} + V^*.$$

Then by Lemma 3.2.9, there exists some constant $C > 0$ such that

$$\frac{1}{C}(|(\beta_1,\beta_2)| + ||V^*||_{\mathbf{X}}) \leq ||\beta_1 V^1_{\mu,\gamma} + \beta_2 V^2_{\mu,\gamma} + V^*||_{\mathbf{X}} \leq \delta'.$$

This gives $||V^*||_{\mathbf{X}} \leq C\delta'$.

Choose δ' small enough such that $C\delta' < \delta_1$. We have the uniqueness of V^*. So $V^* = V(\mu,\gamma,\beta_1,\beta_2)$ in (3.27). The theorem is proved. $\square$

3.2.3 Existence of solutions with nonzero swirl near $U^{\mu,\gamma}$ when $(\mu,\gamma) \in I_2$

Let us look at the problem near $U^{\mu,\gamma}$ when $\mu = -\frac{1}{2}$ and $\gamma > -1$. For such a fixed (μ,γ), write $\bar{U} = (\bar{U}_\theta, 0)$. Recall that in Corollary 3.1.1, we have

$$\bar{U}_\theta = (1-x)\left(1 + \frac{2(\gamma+1)}{(\gamma+1)\ln\frac{1+x}{2} - 2}\right). \tag{3.47}$$

It satisfies

$$(1-x^2)\bar{U}_\theta' + 2x\bar{U}_\theta + \frac{1}{2}\bar{U}_\theta^2 = -\frac{1}{2}(1-x)^2.$$

We will work with $\tilde{U} = U - \bar{U}$. Let $0 < \epsilon < \frac{1}{2}$, define

$$
\begin{aligned}
\mathbf{M}_1 := \Big\{ & \tilde{U}_\theta \in C([-1,1],\mathbf{R}) \cap C^1((-1,1],\mathbf{R}) \cap C^2((0,1),\mathbf{R}) \mid \\
& \tilde{U}_\theta(1) = \tilde{U}_\theta(-1) = 0, || \ln\left(\frac{1+x}{3}\right)\tilde{U}_\theta||_{L^\infty(-1,1)} < \infty, \\
& ||(1+x)\left(\ln\frac{1+x}{3}\right)^2 \tilde{U}'_\theta||_{L^\infty(-1,1)} < \infty, ||\tilde{U}''_\theta||_{L^\infty(0,1)} < \infty \Big\},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{M}_2 =& \mathbf{M}_2(\epsilon) \\
:=& \Big\{ \tilde{U}_\phi \in C^1((-1,1],\mathbf{R}) \cap C^2((-1,1),\mathbf{R}) \mid \tilde{U}_\phi(1) = 0, ||(1+x)^\epsilon \tilde{U}_\phi||_{L^\infty(-1,1)} < \infty, \\
& ||(1+x)^{1+\epsilon}\tilde{U}'_\phi||_{L^\infty(-1,1)} < \infty, ||(1+x)^{2+\epsilon}\tilde{U}''_\phi||_{L^\infty(-1,1)} < \infty \Big\}
\end{aligned}
$$

with the following norms accordingly:

$$
||\tilde{U}_\theta||_{\mathbf{M}_1} = || \ln\left(\frac{1+x}{3}\right)\tilde{U}_\theta||_{L^\infty(-1,1)} + || \left(\ln\frac{1+x}{3}\right)^2 (1+x)\tilde{U}'_\theta||_{L^\infty(-1,1)} + ||\tilde{U}''_\theta||_{L^\infty(0,1)},
$$

$$
||\tilde{U}_\phi||_{\mathbf{M}_2} := ||(1+x)^\epsilon \tilde{U}_\phi||_{L^\infty(-1,1)} + ||(1+x)^{1+\epsilon}\tilde{U}'_\phi||_{L^\infty(-1,1)} + ||(1+x)^{2+\epsilon}\tilde{U}''_\phi||_{L^\infty(-1,1)}.
$$

Next, define

$$
\begin{aligned}
\mathbf{N}_1 :=& \{\xi_\theta \in C([-1,1],\mathbf{R}) \cap C^1((0,1],\mathbf{R}) \mid \xi_\theta(1) = \xi_\theta(-1) = \xi'_\theta(1) = 0, \\
& || \left(\ln\frac{1+x}{3}\right)^2 \xi_\theta||_{L^\infty(-1,1)} < \infty, ||\frac{\xi'_\theta}{1-x}||_{L^\infty(0,1)} < \infty \Big\}, \\
\mathbf{N}_2 = \mathbf{N}_2(\epsilon) :=& \left\{ \xi_\phi \in C((-1,1],\mathbb{R}) \mid \xi_\phi(1) = 0, ||\frac{(1+x)^{1+\epsilon}\xi_\phi}{1-x}||_{L^\infty(-1,1)} < \infty \right\}
\end{aligned}
$$

with the following norms accordingly:

$$
\begin{aligned}
||\xi_\theta||_{\mathbf{N}_1} &:= || \left(\ln\frac{1+x}{3}\right)^2 \xi_\theta||_{L^\infty(-1,1)} + ||\frac{\xi'_\theta}{1-x}||_{L^\infty(0,1)}, \\
||\xi_\phi||_{\mathbf{N}_2} &:= ||\frac{(1+x)^{1+\varepsilon}\xi_\phi}{1-x}||_{L^\infty(-1,1)}.
\end{aligned}
$$

Let $\mathbf{X} := \{\tilde{U} = (\tilde{U}_\theta, \tilde{U}_\phi) \mid \tilde{U}_\theta \in \mathbf{M}_1, \tilde{U}_\phi \in \mathbf{M}_2\}$ with the norm $||\tilde{U}||_{\mathbf{X}} := ||\tilde{U}_\theta||_{\mathbf{M}_1} + ||\tilde{U}_\phi||_{\mathbf{M}_2}$, and $\mathbf{Y} := \{\xi = (\xi_\theta, \xi_\phi) \mid \xi_\theta \in \mathbf{N}_1, \xi_\phi \in \mathbf{N}_2\}$ with the norm $||\xi||_{\mathbf{Y}} := ||\xi_\theta||_{\mathbf{N}_1} + ||\xi_\phi||_{\mathbf{N}_2}$. It is not difficult to verify that $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{X}$ and $\mathbf{Y}$ are Banach spaces.

Let $l_1, l_2 : \mathbf{X} \to \mathbb{R}$ be the bounded linear functionals defined by (3.24) for each $V \in \mathbf{X}$. Define

$$
\mathbf{X}_1 := \ker l_1 \cap \ker l_2. \tag{3.48}
$$

Theorem 3.2.2. *For every compact subset K of $(-1,+\infty)$, there exists $\delta=\delta(K)>0$, and $V\in C^\infty(K\times B_\delta(0),\mathbf{X}_1)$ satisfying $V(\gamma,0,0)=0$ and $\frac{\partial V}{\partial\beta_i}|_{\beta=0}=0$, $i=1,2$, such that*

$$U=U^{-\frac{1}{2},\gamma}+\beta_1V^1_{-\frac{1}{2},\gamma}+\beta_2V^2_{-\frac{1}{2},\gamma}+V(\gamma,\beta_1,\beta_2) \tag{3.49}$$

satisfies equation (3.13) with $\hat{\mu}=-\frac{1}{2}-\frac{1}{4}\psi[U_\phi](-1)$. Moreover, there exists some $\delta'=\delta'(K)>0$, such that if $||U-U^{-\frac{1}{2},\gamma}||_{\mathbf{X}}<\delta'$, $\gamma\in K$, and U satisfies equation (3.13) with some constant $\hat{\mu}$, then (3.49) holds for some $|(\beta_1,\beta_2)|<\delta$.

To prove Theorem 3.2.2, we first study properties of the Banach spaces $\mathbf{X}$ and $\mathbf{Y}$.

With the fixed $\epsilon\in(0,1)$, we have

Lemma 3.2.10. *For every $\tilde{U}\in\mathbf{X}$, it satisfies*

$$|\tilde{U}_\phi(s)|\le(1-s)(1+s)^{-\epsilon}||\tilde{U}_\phi||_{\mathbf{M}_2},\quad\forall-1<s<1, \tag{3.50}$$

$$|\tilde{U}_\theta(s)|\le(\ln 3)\left(\ln\frac{2}{3}\right)^{-2}\left(\ln\frac{1+s}{3}\right)^{-1}(1-s)||\tilde{U}_\theta||_{\mathbf{M}_1},\quad\forall-1<s<1. \tag{3.51}$$

Proof. For $s\in(0,1)$, there exists $y\in(s,1)$ such that

$$|\tilde{U}_\phi(s)|=|\tilde{U}_\phi'(y)|(1-s)\le(1-s)||\tilde{U}_\phi||_{\mathbf{M}_2},$$

while for $s\in(-1,0]$, $|\tilde{U}_\phi(s)|\le(1+s)^{-\epsilon}||\tilde{U}_\phi||_{\mathbf{M}_2}\le(1-s)(1+s)^{-\epsilon}||\tilde{U}_\phi||_{\mathbf{M}_2}$. So (3.50) is proved.

Now we prove (3.51). For $0\le s<1$, by the fact that $\tilde{U}_\theta\in\mathbf{M}_1$, we have $|\tilde{U}_\theta'(s)|\le\left(\ln\frac{2}{3}\right)^{-2}||\tilde{U}_\theta||_{\mathbf{M}_1}$. So

$$\begin{aligned}\left|\left(\ln\frac{1+s}{3}\right)\tilde{U}_\theta(s)\right|&\le(\ln 3)|\tilde{U}_\theta(s)|=(\ln 3)|\tilde{U}_\theta(s)-\tilde{U}_\theta(1)|\le(\ln 3)||\tilde{U}_\theta'||_{L^\infty(0,1)}(1-s)\\&\le(\ln 3)\left(\ln\frac{2}{3}\right)^{-2}(1-s)||\tilde{U}_\theta||_{\mathbf{M}_1}.\end{aligned}$$

For $-1<s<0$, $\left|\left(\ln\frac{1+s}{3}\right)(1-s)^{-1}\tilde{U}_\theta(s)\right|\le\left|\left(\ln\frac{1+s}{3}\right)\tilde{U}_\theta(s)\right|\le||\tilde{U}_\theta||_{\mathbf{M}_1}$. So (3.51) is proved. □

Lemma 3.2.11. *For every $\xi_\theta\in\mathbf{N}_1$,*

$$|\xi_\theta(x)|\le(\ln 3)^2\left(\ln\frac{1+x}{3}\right)^{-2}(1-x)^2||\xi_\theta||_{\mathbf{N}_1},\quad-1<x<1.$$

Proof. If $\xi_\theta \in \mathbf{N}_1$, $\xi_\theta(1) = 0$. So for every $0 < x < 1$, there exists $y \in (x, 1)$ such that

$$\left|\left(\ln\frac{1+x}{3}\right)^2 \xi_\theta(x)\right| \le (\ln 3)^2|\xi_\theta(x)| = (\ln 3)^2|\xi_\theta'(y)(1-x)| \le (\ln 3)^2||\xi_\theta||_{\mathbf{N}_1}(1-y)(1-x)$$
$$\le (\ln 3)^2||\xi_\theta||_{\mathbf{N}_1}(1-x)^2.$$

For $-1 < x \le 0$, $\left|\left(\ln\frac{1+x}{3}\right)^2 \xi_\theta(x)\right| \le ||\xi_\theta||_{\mathbf{N}_1} \le ||\xi_\theta||_{\mathbf{N}_1}(1-x)^2$. □

Now let K be a compact subset of $(-1, +\infty)$. For $\tilde{U}_\phi \in \mathbf{M}_2$, let $\psi[\tilde{U}_\phi](x)$ be defined by (3.14). Then define a map G on $K \times \mathbf{X}$ such that for each $(\gamma, \tilde{U}) \in K \times \mathbf{X}$, $G(\gamma, \tilde{U}) = G(-\frac{1}{2}, \gamma, \tilde{U})$ given by (3.15) with $\bar{U}_\theta$ in (3.47). If $\tilde{U}$ satisfies $G(\gamma, \tilde{U}) = 0$, then $U = \tilde{U} + \bar{U}$ gives a solution of (3.13) with $\hat{\mu} = -\frac{1}{2} - \frac{1}{4}\psi[\tilde{U}_\phi](-1)$, satisfying $U_\theta(-1) = \bar{U}_\theta(-1) = 2$.

Proposition 3.2.2. *The map G is in $C^\infty(K \times \mathbf{X}, \mathbf{Y})$ in the sense that G has continuous Fréchet derivatives of every order. Moreover, the Fréchet derivative of G with respect to $\tilde{U}$ at $(\gamma, \tilde{U}) \in K \times \mathbf{X}$ is given by the linear operator $L_{\tilde{U}}^{-\frac{1}{2},\gamma} : \mathbf{X} \to \mathbf{Y}$ defined as in (*3.18*).*

To prove Proposition 3.2.2, we first prove the following lemmas:

Lemma 3.2.12. *For every $\gamma \in K$, the map $A(-\frac{1}{2}, \gamma, \cdot) : \mathbf{X} \to \mathbf{Y}$ defined by (*3.16*) is a bounded linear operator.*

Proof. For convenience we denote $A = A(-\frac{1}{2}, \gamma, \cdot)$. We make use of the properties of $\bar{U}_\theta$ that $\bar{U}_\theta(1) = 0$, $\bar{U}_\theta \in C^2(-1, 1] \cap L^\infty(-1, 1)$ and $\bar{U}_\theta - 2 = O(1)\dfrac{1}{\ln(1+x)}$.

A is clearly linear. For every $\tilde{U} \in \mathbf{X}$, we prove that $A\tilde{U}$ defined by (3.16) is in $\mathbf{Y}$ and there exists some constant C such that $||A\tilde{U}||_{\mathbf{Y}} \le C||\tilde{U}||_{\mathbf{X}}$ for all $\tilde{U} \in \mathbf{X}$.

By the fact that $\tilde{U}_\theta \in \mathbf{M}_1$ and (3.51), we have

$$\left|\left(\ln\frac{1+x}{3}\right)^2 A_\theta\right|$$
$$\le (1-x)\left|(1+x)\left(\ln\frac{1+x}{3}\right)^2 \tilde{U}_\theta'\right| + \left|(2x + \bar{U}_\theta)\ln\frac{1+x}{3}\right| \cdot \left|\tilde{U}_\theta \ln\frac{1+x}{3}\right|$$
$$\le C(1-x)||\tilde{U}_\theta||_{\mathbf{M}_1}.$$

From the above we also see that $\lim_{x\to 1} A_\theta(x) = \lim_{x\to -1} A_\theta(x) = 0$. By computation $A'_\theta = (1-x^2)\tilde{U}''_\theta + \bar{U}_\theta\tilde{U}'_\theta + (2+\bar{U}'_\theta)\tilde{U}_\theta$. Then by the fact that $\tilde{U}_\theta \in \mathbf{M}_1$ and (3.51), for $0 < x < 1$,

$$\frac{|A'_\theta(x)|}{1-x} \le (1+x)|\tilde{U}''_\theta| + |\tilde{U}'_\theta| + \frac{|\tilde{U}_\theta|}{1-x} \le C||\tilde{U}_\theta||_{\mathbf{M}_1}, \quad 0 < x < 1.$$

So $A_\theta \in \mathbf{N}_1$ and $||A_\theta||_{\mathbf{N}_1} \le C||\tilde{U}_\theta||_{\mathbf{M}_1}$.

Next, by the fact that $\tilde{U}_\phi \in \mathbf{M}_2$ and (3.47),

$$\frac{(1+x)^{1+\epsilon}}{1-x}|A_\phi| \le |(1+x)^{2+\epsilon}\tilde{U}''_\phi| + |(1+x)^{1+\epsilon}\frac{|\bar{U}_\theta|}{1-x}\tilde{U}'_\phi| \le C||\tilde{U}_\phi||_{\mathbf{M}_2}.$$

We also see from the above that $\lim_{x\to 1} A_\phi(x) = 0$. So $A_\phi \in \mathbf{N}_2$, and $||A_\phi||_{\mathbf{N}_2} \le C||\tilde{U}_\phi||_{\mathbf{M}_2}$. We have proved that $A\tilde{U} \in \mathbf{Y}$, and $||A\tilde{U}||_{\mathbf{Y}} \le C||\tilde{U}||_{\mathbf{X}}$ for every $\tilde{U} \in \mathbf{X}$. $\square$

Lemma 3.2.13. *The map $Q : \mathbf{X} \times \mathbf{X} \to \mathbf{Y}$ defined by (3.17) is a bounded bilinear operator.*

Proof. In the following, C denotes a universal constant which may change from line to line. It is clear that Q is a bilinear operator. For every $\tilde{U}, \tilde{V} \in \mathbf{X}$, we will prove that $Q(\tilde{U}, \tilde{V})$ is in $\mathbf{Y}$ and there exists some constant C independent of $\tilde{U}$ and $\tilde{V}$ such that $||Q(\tilde{U}, \tilde{V})||_{\mathbf{Y}} \le C||\tilde{U}||_{\mathbf{X}}||\tilde{V}||_{\mathbf{X}}$.

For convenience we write

$$\psi(\tilde{U}, \tilde{V})(x) = \int_x^1 \int_l^1 \int_t^1 \frac{2\tilde{U}_\phi(s)\tilde{V}'_\phi(s)}{1-s^2} ds dt dl.$$

For $\tilde{U}, \tilde{V} \in \mathbf{X}$, we have, using (3.50) in Lemma 3.2.10, that

$$\left|\frac{\tilde{U}_\phi(s)\tilde{V}'_\phi(s)}{1-s^2}\right| \le (1+s)^{-2-2\epsilon}||\tilde{U}_\phi||_{\mathbf{M}_2}||\tilde{V}_\phi||_{\mathbf{M}_2}. \tag{3.52}$$

It follows that $\psi(\tilde{U}, \tilde{V})(x)$ is well-defined and

$$|\psi(\tilde{U}, \tilde{V})(x)| \le C(\epsilon)(1-x)^3||\tilde{U}_\phi||_{\mathbf{M}_2}||\tilde{V}_\phi||_{\mathbf{M}_2}, \quad -1 < x < 1. \tag{3.53}$$

Moreover, we have, in view of (3.52), that

$$\begin{aligned}
&|\psi(\tilde U,\tilde V)(x) - \frac{(1-x)^2}{4}\psi(\tilde U,\tilde V)(-1)| \\
&= \left|\psi(\tilde U,\tilde V)(x) - \psi(\tilde U,\tilde V)(-1) + \frac{(1+x)(3-x)}{4}\psi(\tilde U,\tilde V)(-1)\right| \\
&= \left|-\int_{-1}^{x}\int_{l}^{1}\int_{t}^{1}\frac{2\tilde U_\phi(s)\tilde V_\phi'(s)}{1-s^2}dsdtdl + \frac{(1+x)(3-x)}{4}\psi(\tilde U,\tilde V)(-1)\right| \\
&\le C(1+x)^{1-2\epsilon}||\tilde U_\phi||_{\mathbf{M_2}}||\tilde V_\phi||_{\mathbf{M_2}}, \quad \forall -1<x\le 0.
\end{aligned} \tag{3.54}$$

Thus, using (3.53) and (3.54), we have

$$|\psi(\tilde U,\tilde V)(x) - \frac{(1-x)^2}{4}\psi(\tilde U,\tilde V)(-1)| \le C(\epsilon)(1+x)^{1-2\epsilon}(1-x)^2||\tilde U_\phi||_{\mathbf{M_2}}||\tilde V_\phi||_{\mathbf{M_2}}, \quad \forall -1<x<1. \tag{3.55}$$

So by (3.51), (3.55) and the fact that $\tilde U_\theta, \tilde V_\theta \in \mathbf{M_1}$, we have

$$\begin{aligned}
&|\left(\ln\frac{1+x}{3}\right)^2 Q_\theta(x)| \\
&\le \frac{1}{2}\left(\ln\frac{1+x}{3}\right)^2|\tilde U_\theta(x)||\tilde V_\theta(x)| + \left(\ln\frac{1+x}{3}\right)^2\left|\psi(\tilde U,\tilde V)(x) - \frac{(1-x)^2}{4}\psi(\tilde U,\tilde V)(-1)\right| \\
&\le C(1-x)^2||\tilde U_\theta||_{\mathbf{M_1}}||\tilde V_\theta||_{\mathbf{M_1}} + C\left(\ln\frac{1+x}{3}\right)^2(1+x)^{1-2\epsilon}(1-x)^2||\tilde U_\phi(s)||_{\mathbf{M_2}}||\tilde V_\phi||_{\mathbf{M_2}} \\
&\le C(1-x)^2||\tilde U||_{\mathbf{X}}||\tilde V||_{\mathbf{X}}, \quad \forall -1<x<1.
\end{aligned}$$

From this we also have $\lim_{x\to 1} Q_\theta(x) = \lim_{x\to -1} Q_\theta(x) = 0$.

A calculation gives

$$Q_\theta'(x) = \frac{1}{2}\tilde U_\theta\tilde V_\theta' + \frac{1}{2}\tilde U_\theta'\tilde V_\theta + \int_x^1\int_t^1\frac{2\tilde U_\phi(s)\tilde V_\phi'(s)}{1-s^2}dsdt - \frac{1-x}{2}\psi(\tilde U,\tilde V)(-1), \text{ for } 0<x<1.$$

Using $\tilde U \in \mathbf{X}$, (3.51), (3.50) and (3.52), we see that,

$$|Q_\theta'(x)| \le C(1-x)||\tilde U||_{\mathbf{X}}||\tilde V||_{\mathbf{X}}, \quad \forall 0<x<1.$$

So $Q_\theta \in \mathbf{N_1}$, and $||Q_\theta||_{\mathbf{N_1}} \le C||\tilde U||_{\mathbf{X}}||\tilde V||_{\mathbf{X}}$.

Next, since $Q_\phi(x) = \tilde U_\theta(x)\tilde V_\phi'(x)$, for $-1<x<1$,

$$\left|\frac{(1+x)^{1+\epsilon}Q_\phi}{1-x}\right| \le \frac{(1+x)^{1+\epsilon}}{1-x}|\tilde U_\theta(x)|\frac{||\tilde V_\phi||_{\mathbf{\tilde M_2}}}{(1+x)^{1+\epsilon}} \le C||\tilde U_\theta||_{\mathbf{\tilde M_1}}||\tilde V_\phi||_{\mathbf{M_2}}.$$

We also see from the above that $\lim_{x\to 1} Q_\phi(x) = 0$. So $Q_\phi \in \mathbf{N_2}$, and $||Q_\phi||_{\mathbf{N_2}} \le ||\tilde U_\theta||_{\mathbf{M_1}}||\tilde V_\phi||_{\mathbf{M_2}}$. Thus we have proved $Q(\tilde U,\tilde V) \in \mathbf{Y}$ and $||Q(\tilde U,\tilde V)||_{\mathbf{Y}} \le C||\tilde U||_{\mathbf{X}}||\tilde V||_{\mathbf{X}}$ for all $\tilde U, \tilde V \in \mathbf{X}$. The proof is finished. □

Proof of Proposition 3.2.2*:* By definition, $G(-\frac{1}{2},\gamma,\tilde{U}) = A(-\frac{1}{2},\gamma,\tilde{U}) + Q(\tilde{U},\tilde{U})$ for $(\gamma,\tilde{U}) \in K \times \mathbf{X}$. Using standard theories in functional analysis, by Lemma 3.2.13 it is clear that Q is C^∞ on $K \times \mathbf{X}$. By Lemma 3.2.12, $A(-\frac{1}{2},\gamma,\cdot) : \mathbf{X} \to \mathbf{Y}$ is C^∞ for each $\gamma \in K$. For all $i \geq 1$, we have

$$\partial_\gamma^i A(-\frac{1}{2},\gamma,\tilde{U}) = \partial_\gamma^i U_\theta^{-\frac{1}{2},\gamma} \begin{pmatrix} \tilde{U}_\theta \\ \tilde{U}'_\phi \end{pmatrix}.$$

By (3.11), for each integer $i \geq 1$, there exists some constant $C = C(i,K)$, depending only on i, K, such that

$$|\partial_\gamma^i U_\theta^{-\frac{1}{2},\gamma}(x)| \leq C(i,K)(1-x)\left(\ln\frac{1+x}{3}\right)^{-2}, \quad -1 < x < 1. \tag{3.56}$$

From (3.47) we also obtain

$$\left|\frac{d}{dx}\partial_\gamma^i U_\theta^{-\frac{1}{2},\gamma}(x)\right| \leq C(i,K), \quad 0 < x < 1.$$

Using the above estimates and the fact that $\tilde{U}_\theta \in \mathbf{M}_1$, we have

$$\left|\left(\ln\frac{1+x}{3}\right)^2 \partial_\gamma^i A_\theta(-\frac{1}{2},\gamma,\tilde{U})\right| \leq C(i,K)(1-x)||\tilde{U}_\theta||_{\mathbf{M}_1}, \quad -1 < x < 1,$$

and

$$\begin{aligned}\left|\frac{d}{dx}\partial_\gamma^i A_\theta(-\frac{1}{2},\gamma,\tilde{U})\right| &\leq \left|\frac{d}{dx}\partial_\gamma^i U_\theta^{-\frac{1}{2},\gamma}(x)\right| |\tilde{U}_\theta(x)| + |\partial_\gamma^i U_\theta^{-\frac{1}{2},\gamma}(x)| \left|\frac{d}{dx}\tilde{U}_\theta(x)\right| \\ &\leq C(i,K)(1-x)||\tilde{U}_\theta||_{\mathbf{M}_1}, \quad 0 < x < 1.\end{aligned}$$

So $\partial_\gamma^i A_\theta(-\frac{1}{2},\gamma,\tilde{U}) \in \mathbf{N}_1$, with $||\partial_\gamma^i A_\theta(-\frac{1}{2},\gamma,\tilde{U})||_{\mathbf{N}_1} \leq C(i,K)||\tilde{U}_\theta||_{\mathbf{M}_1}$ for all $(\gamma,\tilde{U}) \in K \times \mathbf{X}$.

Next, by (3.56) and the fact that $\tilde{U}_\phi \in \mathbf{M}_1$, we have

$$\frac{(1+x)^{1+\epsilon}}{1-x}|\partial_\gamma^i A_\phi(\mu,\gamma,\tilde{U})(x)| = \frac{|\partial_\gamma^i U_\theta^{-\frac{1}{2},\gamma}(x)|}{1-x}|(1+x)^{1+\epsilon}U'_\phi| \leq C(i,K)||\tilde{U}_\phi||_{\mathbf{M}_2}.$$

So $\partial_\gamma^i A_\phi(-\frac{1}{2},\gamma,\tilde{U}) \in \mathbf{N}_2$, with $||\partial_\gamma^i A_\phi(-\frac{1}{2},\gamma,\tilde{U})||_{\mathbf{N}_2} \leq C(i,K)||\tilde{U}_\phi||_{\mathbf{M}_2}$ for all $(\gamma,\tilde{U}) \in K \times \mathbf{X}$. Thus $\partial_\gamma^i A(-\frac{1}{2},\gamma,\tilde{U}) \in \mathbf{Y}$, with $||\partial_\gamma^i A(-\frac{1}{2},\gamma,\tilde{U})||_{\mathbf{Y}} \leq C(i,K)||\tilde{U}||_{\mathbf{X}}$ for all $(\gamma,\tilde{U}) \in K \times \mathbf{X}$, $i \geq 1$.

So for each $\gamma \in K$, $\partial_\gamma^i A(-\frac{1}{2},\gamma,\cdot) : \mathbf{X} \to \mathbf{Y}$ is a bounded linear map with uniform bounded norm on K. Then by standard theories in functional analysis, $A : K \times \mathbf{X} \to \mathbf{Y}$

is C^∞. So G is a C^∞ map from $K \times \mathbf{X}$ to $\mathbf{Y}$. By direct calculation we have that its Fréchet derivative with respect to $\mathbf{X}$ is given by the linear bounded operator $L_{\tilde{U}}^{-\frac{1}{2},\gamma} : \mathbf{X} \to \mathbf{Y}$ defined as (3.18). The proof is finished. $\square$

By Proposition 3.2.2, $L_0^{-\frac{1}{2},\gamma} : \mathbf{X} \to \mathbf{Y}$, the Fréchet derivative of G with respect to $\tilde{U}$ at $\tilde{U} = 0$, is given by (3.19).

Next, with $a_{-\frac{1}{2},\gamma}(x)$, $b_{-\frac{1}{2},\gamma}(x)$ defined by (3.20) with $\bar{U}_\theta$ given by (3.47), we define $W^{-\frac{1}{2},\gamma}(\xi)$ by (3.35) for $\xi \in \mathbf{Y}$. Then $a_{-\frac{1}{2},\gamma}(x)$ and $W^{-\frac{1}{2},\gamma}(\xi)(x)$ satisfy (3.21) and (3.36).

Lemma 3.2.14. *For every $\gamma \in K$, $W^{-\frac{1}{2},\gamma} : \mathbf{Y} \to \mathbf{X}$ is continuous and is a right inverse of $L_0^{-\frac{1}{2},\gamma}$.*

Proof. We make use of the properties that $U_\theta^{-\frac{1}{2},\gamma}(1) = 0$, $U_\theta^{-\frac{1}{2},\gamma} \in C^2(-1,1] \cap C^0[-1,1]$ and $\left|\left(\ln \frac{1+x}{3}\right)(\bar{U}_\theta(x) - 2)\right| \in L^\infty(-1,1)$. For convenience, we write $W := W^{-\frac{1}{2},\gamma}$, $a(x) = a_{-\frac{1}{2},\gamma}(x)$ and $b(x) = b_{-\frac{1}{2},\gamma}(x)$.

We first prove that W is well-defined, denote $W := W(\xi)$. Applying Lemma 3.2.11 in the expression of W_θ in (3.35), we have, for $-1 < x < 1$, that

$$\left|\left(\ln \frac{1+x}{3}\right) W_\theta(x)\right| \le C \left(\ln \frac{1+x}{3}\right) ||\xi_\theta||_{\mathbf{N}_1} e^{-a(x)} \int_0^x e^{a(s)}(1-s)(1+s)^{-1} \left(\ln \frac{1+s}{3}\right)^{-2} ds. \tag{3.57}$$

We make estimates first for $0 < x \le 1$ and then for $-1 < x \le 0$.

Case 1: $0 < x \le 1$.

By (3.47), $\bar{U}_\theta = -\bar{U}_\theta'(1)(1-x) + O((1-x)^2)$. Using similar arguments as in the proof of Lemma 3.2.5, $b(x)$ and $a(x)$ satisfy (3.38), (3.39) and (3.40). So there exists some positive constant C such that

$$e^{a(s)}(1-s)(1+s)^{-1} \left(\ln \frac{1+s}{3}\right)^{-2} \le C, \quad e^{a(x)} \ge \frac{1}{C(1-x)}, \quad 0 < s < x < 1.$$

Then using the above estimate in (3.57), we have that

$$|W_\theta(x)| \le C||\xi_\theta||_{\mathbf{N}_1}(1-x), \quad 0 < x \le 1. \tag{3.58}$$

In particular, $W_\theta(1) = 0$.

In (3.21), using $\bar{U}_\theta = -\bar{U}'_\theta(1)(1-x) + O((1-x)^2)$, we have

$$|a'(x)| \le \frac{C}{1-x}, \quad |a''(x)| \le \frac{C}{(1-x)^2}, \quad 0 < x < 1. \tag{3.59}$$

Then

$$|W'_\theta(x)| \le |a'(x)||W_\theta(x)| + \frac{|\xi_\theta(x)|}{1-x} \le C||\xi_\theta||_{\mathbf{N}_1}, \quad 0 < x < 1,$$

where we have used (3.58), (3.59), the fact that $\xi \in \mathbf{Y}$ and Lemma 3.2.11.

Next, A calculation gives

$$W''_\theta(x) = ((a'(x))^2 - a''(x))W_\theta(x) - a'(x)\frac{\xi_\theta(x)}{1-x^2} + \frac{\xi'_\theta(x)}{1-x^2} + \frac{2x\xi_\theta(x)}{(1-x^2)^2}.$$

So

$$|W''_\theta(x)| \le |(a'(x))^2 - a''(x)||W_\theta| + |a'(x)|\frac{|\xi_\theta|}{1-x^2} + \frac{|\xi'_\theta|}{(1-x)} + \frac{|\xi_\theta|}{(1-x)^2}.$$

By computation

$$(a'(x))^2 - a''(x) = \frac{\bar{U}_\theta^2 + 2x\bar{U}_\theta}{(1-x^2)^2} - \frac{2+\bar{U}'_\theta}{1-x^2} = O\left(\frac{1}{1-x}\right).$$

It follows, using (3.58), (3.59) and Lemma 3.2.11, that

$$|W''_\theta(x)| \le C\left(\frac{|W_\theta(x)|}{1-x} + \frac{|\xi_\theta|}{(1-x)^2} + \frac{|\xi'_\theta|}{1-x}\right) \le C||\xi_\theta||_{\mathbf{N}_1}, \quad 0 < x < 1.$$

Case 2: $-1 < x \le 0$.

In (3.47), since $\gamma > -1$, we have

$$\bar{U}_\theta(x) = 2 + \frac{4}{\ln\frac{1+x}{3}} + O\left(\left(\ln\frac{1+x}{3}\right)^{-2}\right). \tag{3.60}$$

Then we have, for $-1 < x \le 0$, that

$$b(x) = \ln\frac{1+x}{3} + 2\ln\left(-\ln\frac{1+x}{3}\right) + O(1), \quad a(x) = 2\ln\left(-\ln\frac{1+x}{3}\right) + O(1),$$

$$e^{a(x)} = \left(\ln\frac{1+x}{3}\right)^2 e^{O(1)}, \quad e^{-a(x)} = \left(\ln\frac{1+x}{3}\right)^{-2} e^{O(1)}.$$

So there exists some constant C such that for $-1 < x < s \le 0$

$$e^{a(s)}(1-s)(1+s)^{-1}\left(\ln\frac{1+s}{3}\right)^{-2} \le C(1+s)^{-1}, \quad e^{-a(x)} \le C\left(\ln\frac{1+x}{3}\right)^{-2}.$$

Apply these estimates in (3.57), we have

$$\left|\left(\ln\frac{1+x}{3}\right)W_\theta(x)\right| \le C||\xi_\theta||_{\mathbf{N}_1}\left(\ln\frac{1+x}{3}\right)^{-1}\int_0^x \frac{1}{1+s}ds \le C||\xi_\theta||_{\mathbf{N}_1}, \quad -1<x\le 0. \tag{3.61}$$

By (3.21) and (3.60), there exists some C such that

$$|a'(x)| \le \frac{C}{(1+x)\ln\frac{1+x}{3}}.$$

Then by (3.36), (3.61) and Lemma 3.2.11, we have, for $-1<x\le 0$, that

$$\begin{aligned}&\left|(1+x)\left(\ln\frac{1+x}{3}\right)^2 W_\theta'(x)\right|\\ \le&\left|a'(x)(1+x)\left(\ln\frac{1+x}{3}\right)^2 W_\theta(x)\right| + \left(\ln\frac{1+x}{3}\right)^2\frac{|\xi_\theta(x)|}{1-x} \le C||\xi_\theta||_{\mathbf{N}_1}.\end{aligned}$$

So we have shown that $W_\theta\in\mathbf{M}_1$, and $||W_\theta||_{\mathbf{M}_1}\le C||\xi_\theta||_{\mathbf{N}_1}$ for some constant C.

By the definition of $W_\phi(\xi)$ in (3.35) and the fact that $\xi_\phi\in\mathbf{N}_2$, we have, for every $-1<x<1$, that

$$|W_\phi(x)| \le \int_x^1 e^{-b(t)}\int_t^1 e^{b(s)}\frac{|\xi_\phi(s)|}{1-s^2}dsdt \le ||\xi_\phi||_{\mathbf{N}_2}\int_x^1 e^{-b(t)}\int_t^1 e^{b(s)}(1+s)^{-2-\epsilon}dsdt.$$

Since $b(x)=\ln\frac{1+x}{3}+2\ln\left(-\ln\frac{1+x}{3}\right)+O(1)$ for $-1<x<1$, there exists some constant C such that

$$e^{b(s)}\le C(1+s)\left(\ln\frac{1+s}{3}\right)^2, \quad e^{-b(t)}\le\frac{C}{(1+t)\left(\ln\frac{1+t}{3}\right)^2}, \quad -1<s,t\le 1. \tag{3.62}$$

So we have

$$\begin{aligned}(1+x)^\epsilon|W_\phi(x)| &\le C(1+x)^\epsilon||\xi_\phi||_{\mathbf{N}_2}\int_x^1(1+t)^{-1}\left(\ln\frac{1+t}{3}\right)^{-2}\int_t^1(1+s)^{-1-\epsilon}\left(\ln\frac{1+s}{3}\right)^2 dsdt\\ &\le C||\xi_\phi||_{\mathbf{N}_2}, \quad -1<x\le 1.\end{aligned}$$

For $0<x<1$, it can be seen from the above that $|W_\phi(x)|\le C||\xi_\phi||_{\mathbf{N}_2}(1-x)$. In particular, $W_\phi(1)=0$. By computation

$$W_\phi'(x) = -e^{-b(x)}\int_x^1 e^{b(s)}\frac{\xi_\phi(s)}{1-s^2}ds.$$

Using (3.62) and the fact that $\xi_\phi\in\mathbf{N}_2$, we have that for $-1<x<1$,

$$|(1+x)^{1+\epsilon}W_\phi'(x)| \le C||\xi_\phi||_{\mathbf{N}_2}(1+x)^\epsilon\left(\ln\frac{1+x}{3}\right)^{-2}\int_x^1(1+s)^{-1-\epsilon}\left(\ln\frac{1+s}{3}\right)^2 ds \le C||\xi_\phi||_{\mathbf{N}_2}.$$

Similarly,

$$W''_\phi(x) = b'(x)e^{-b(x)} \int_x^1 e^{b(s)} \frac{\xi_\phi(s)}{1-s^2} ds + \frac{\xi_\phi(x)}{1-x^2}.$$

By (3.60), $|b'(x)| = \frac{|\bar{U}_\theta(x)|}{1-x^2} = O((1+x)^{-1})$. Using (3.62), we have

$$|(1+x)^{2+\epsilon} W''_\phi(x)| \le C||\xi_\phi||_{\mathbf{N}_2}, \quad -1 < x < 1.$$

So $W_\phi \in \mathbf{M}_2$, and $||W_\phi||_{\mathbf{M}_2} \le C||\xi_\phi||_{\mathbf{N}_2}$ for some constant C.

Thus $W^{-\frac{1}{2},\gamma}(\xi) \in \mathbf{X}$ for all $\xi \in \mathbf{Y}$, and $||W^{-\frac{1}{2},\gamma}(\xi)||_{\mathbf{X}} \le C||\xi||_{\mathbf{Y}}$ for some constant C. So $W^{-\frac{1}{2},\gamma} : \mathbf{Y} \to \mathbf{X}$ is well-defined and continuous. It can be directly checked that W is a right inverse of $L_0^{-\frac{1}{2},\gamma}$. □

Let $V^1_{-\frac{1}{2},\gamma}, V^2_{-\frac{1}{2},\gamma}, V^3_{-\frac{1}{2},\gamma}$ be defined by (3.23) with related $a_{-\frac{1}{2},\gamma}(x)$ and $b_{-\frac{1}{2},\gamma}(x)$ in the current case, we have

Lemma 3.2.15. *$\{V^1_{-\frac{1}{2},\gamma}, V^2_{-\frac{1}{2},\gamma}\}$ is a basis of the kernel of $L_0^{-\frac{1}{2},\gamma} : \mathbf{X} \to \mathbf{Y}$.*

Proof. Let $V \in \mathbf{X}$, $L_0 V = 0$. It can be seen that V is given by $V = c_1 V^1_{-\frac{1}{2},\gamma} + c_2 V^2_{-\frac{1}{2},\gamma} + c_3 V^3_{-\frac{1}{2},\gamma}$ for some constants c_1, c_2, c_3. It is not hard to verify that $V^1_{-\frac{1}{2},\gamma}, V^2_{-\frac{1}{2},\gamma} \in \mathbf{X}$, and $V^3_{-\frac{1}{2},\gamma} \notin \mathbf{X}$. Since $V \in \mathbf{X}$, we must have $c_3 V^3 \in \mathbf{X}$, so $c_3 = 0$, and $V \in \text{span}\{V^1_{-\frac{1}{2},\gamma}, V^2_{-\frac{1}{2},\gamma}\}$. It is clear that $\{V^1_{-\frac{1}{2},\gamma}, V^2_{-\frac{1}{2},\gamma}\}$ is independent. So $\{V^1_{-\frac{1}{2},\gamma}, V^2_{-\frac{1}{2},\gamma}\}$ is a basis of the kernel. □

Corollary 3.2.2. *For any $\xi = (\xi_\theta, \xi_\phi) \in \mathbf{Y}$, all solutions of $L_0^{-\frac{1}{2},\gamma}(V) = \xi$, $V \in \mathbf{X}$, are given by*

$$V = W^{-\frac{1}{2},\gamma}(\xi) + c_1 V^1_{-\frac{1}{2},\gamma} + c_2 V^2_{-\frac{1}{2},\gamma}, \quad c_1, c_2 \in \mathbb{R}.$$

Namely,

$$V_\theta = W_\theta^{-\frac{1}{2},\gamma}(\xi) + c_1 e^{-a(x)}, \quad V_\phi = W_\phi^{-\frac{1}{2},\gamma}(\xi) + c_2 \int_x^1 e^{-b(t)} dt, \; c_1, c_2 \in \mathbb{R}.$$

Proof. By Lemma 3.2.14, $V - W^{-\frac{1}{2},\gamma}(\xi)$ is in the kernel of $L_0^{-\frac{1}{2},\gamma} : \mathbf{X} \to \mathbf{Y}$. The conclusion then follows from Lemma 3.2.15. □

Let l_1, l_2 be the functionals on $\mathbf{X}$ defined by (3.24), and $\mathbf{X}_1$ be the subspace of $\mathbf{X}$ defined by (3.48). As shown in Section 3.2.1, the matrix $(l_i(V^j_{-\frac{1}{2},\gamma}))$, $i, j = 1, 2$, is an invertible matrix, for every $\gamma \in K$. So $\mathbf{X}_1$ is a closed subspace of $\mathbf{X}$, and

$$\mathbf{X} = \mathrm{span}\{V^1_{-\frac{1}{2},\gamma}, V^2_{-\frac{1}{2},\gamma}\} \oplus \mathbf{X}_1, \quad \forall \gamma \in K, \tag{3.63}$$

with the projection operator $P(\gamma) : \mathbf{X} \to \mathbf{X}_1$ given by

$$P(\gamma)V = V - l_1(V)V^1_{-\frac{1}{2},\gamma} - c(\gamma)l_2(V)V^2_{-\frac{1}{2},\gamma} \text{ for } V \in \mathbf{X}.$$

where $c(\gamma) = \left(\int_0^1 e^{-b_{-\frac{1}{2},\gamma}(t)} dt\right)^{-1} > 0$ for all $\gamma \in K$.

Lemma 3.2.16. *The operator* $L_0^{-\frac{1}{2},\gamma} : \mathbf{X}_1 \to \mathbf{Y}$ *is an isomorphism.*

Proof. By Corollary 3.2.2 and Lemma 3.2.15, $L_0^{-\frac{1}{2},\gamma} : \mathbf{X} \to \mathbf{Y}$ is surjective and $\ker L_0 = \mathrm{span}\{V^1, V^2\}$. The conclusion of the lemma then follows in view of the property that $\mathbf{X} = \mathrm{span}\{V^1, V^2\} \oplus \mathbf{X}_1$. $\square$

Lemma 3.2.17. $V^1_{-\frac{1}{2},\gamma}, V^2_{-\frac{1}{2},\gamma} \in C^\infty((-1,\infty), \mathbf{X})$.

Proof. For convenience, in this proof we denote $a(x) = a_{-\frac{1}{2},\gamma}(x)$, $b(x) = b_{-\frac{1}{2},\gamma}(x)$ and $V^i = V^i_{-\frac{1}{2},\gamma}$, $i = 1, 2$.

By computation, using the explicit expression of $U_\theta^{-\frac{1}{2},\gamma}(x), a(x), a'(x), b(x), V^1_\theta(x)$ and $V^2_\phi(x)$ given by (3.47), (3.20), (3.21) and (3.23), and the estimates of $\partial^i_\gamma U^{-\frac{1}{2},\gamma}$ given by (3.11) for all $i \geq 0$, we have, for $\gamma \in K$, that

$$e^{-a(x)} = O(1)\left(\ln\frac{1+x}{3}\right)^{-2}, \quad e^{-b(x)} = O(1)\frac{1}{(1+x)\left(\ln\frac{1+x}{3}\right)^2}, \quad -1 < x \leq 0.$$

and

$$a'(x) = \frac{2x + U_\theta^{-\frac{1}{2},\gamma}(x)}{1-x^2} = O(1)\frac{1}{(1+x)\left(\ln\frac{1+x}{3}\right)}.$$

So

$$\left|V^1_\theta(x)\right| = O(1)\left(\ln\frac{1+x}{3}\right)^{-2}, \quad V^2_\phi(x) = O(1), \quad -1 < x \leq 0,$$

and

$$\left|\frac{d}{dx}V^1_\theta(x)\right| = \left|e^{-a(x)}a'(x)\right| = O(1)\frac{1}{(1+x)\left(\ln\frac{1+x}{3}\right)^3}, \quad -1 < x \leq 0,$$

$$\left|\frac{d}{dx}V^2_\phi(x)\right| = e^{-b(x)} = O(1)\frac{1}{(1+x)\left(\ln\frac{1+x}{3}\right)^2}, \quad -1 < x \leq 0.$$

Moreover,

$$\frac{\partial^i}{\partial\gamma^i}a(x) = \frac{\partial^i}{\partial\gamma^i}b(x) = \int_0^x \frac{1}{1-s^2}\frac{\partial^i}{\partial\gamma^i}U^{-\frac{1}{2},\gamma}(s)ds$$
$$= O(1)\int_0^x \frac{1}{(1+s)\left(\ln\frac{1+x}{3}\right)^2}ds = O(1),$$

and

$$\frac{\partial^i}{\partial\gamma^i}a'(x) = \frac{\partial^i}{\partial\gamma^i}b'(x) = \frac{1}{1-x^2}\frac{\partial^i}{\partial\gamma^i}U^{-\frac{1}{2},\gamma}(x) = O(1)\frac{1}{(1+x)\left(\ln\frac{1+x}{3}\right)^2},$$

where $|O(1)| \le C$ depending only on γ and i. So we have

$$\left|\partial_\gamma^i V_\theta^1(x)\right| = e^{-a(x)}O(1) = O(1)\left(\ln\frac{1+x}{3}\right)^{-2}, \quad -1 < x \le 0, i = 1,2,3...$$

From the above we can see that for all $\gamma > -1$ and $i \ge 0$, there exists some constant C, such that

$$\left|\left(\ln\frac{1+x}{3}\right)^2\partial_\gamma^i V_\theta^1(x)\right| \le C, \quad \left|(1+x)\left(\ln\frac{1+x}{3}\right)^2\frac{d}{dx}\partial_\gamma^i V_\theta^1(x)\right| \le C, \quad -1 < x \le 0.$$

We can also show that for $i \ge 0$,

$$\partial_\gamma^i V_\theta^1(1) = 0,$$

and there exists some constant C such that

$$\left|\frac{d^l}{dx^l}\partial_\gamma^i V_\theta^1(x)\right| \le C, \quad l = 0,1,2, \quad 0 \le x < 1.$$

The above imply that for all $i \ge 0$, $\partial_\gamma^i V^1(x) \in \mathbf{X}$, and $V_\theta^1 \in C^\infty((-1,+\infty),\mathbf{M}_1)$.

Similarly, we can show that $V_\phi^2 \in C^\infty((-1,+\infty),\mathbf{M}_2)$. So $V^1, V^2 \in C^\infty((-1,+\infty),\mathbf{X})$.

$\square$

Next, by similar arguments in the proof of Lemma 3.2.9, using Lemma 3.2.17, we have

Lemma 3.2.18. *There exists $C = C(K) > 0$ such that for all $\gamma \in K$, $(\beta_1,\beta_2) \in \mathbb{R}^2$, and $V \in \mathbf{X}_1$,*

$$||V||_{\mathbf{X}} + |(\beta_1,\beta_2)| \le C||\beta_1 V^1_{-\frac{1}{2},\gamma} + \beta_2 V^2_{-\frac{1}{2},\gamma} + V||_{\mathbf{X}}.$$

Proof of Theorem 3.2.2*:* Define a map $F : K \times \mathbb{R}^2 \times \mathbf{X}_1 \to \mathbf{Y}$ by

$$F(\gamma, \beta_1, \beta_2, V) = G(\gamma, \beta_1 V^1_{-\frac{1}{2},\gamma} + \beta_2 V^2_{-\frac{1}{2},\gamma} + V).$$

By Proposition 3.2.2, G is a C^∞ map from $K \times \mathbf{X}$ to $\mathbf{Y}$. Let $\tilde{U} = \tilde{U}(\gamma, \beta_1, \beta_2, V) = \beta_1 V^1_{-\frac{1}{2},\gamma} + \beta_2 V^2_{-\frac{1}{2},\gamma} + V$. Using Lemma 3.2.17, we have $\tilde{U} \in C^\infty(K \times \mathbb{R}^2 \times \mathbf{X}_1, \mathbf{X})$. So $F \in C^\infty(K \times \mathbb{R}^2 \times \mathbf{X}_1, \mathbf{Y})$.

Next, by definition $F(\gamma, 0, 0, 0) = 0$ for all $\gamma \in K$. Fix some $\bar{\gamma} \in K$, using Lemma 3.2.16, we have $F_V(\bar{\gamma}, 0, 0, 0) = L_0^{-\frac{1}{2},\bar{\gamma}} : \mathbf{X}_1 \to \mathbf{Y}$ is an isomorphism.

Applying Theorem C, there exist some $\delta > 0$ and a unique $V \in C^\infty(B_\delta(\bar{\gamma}) \times B_\delta(0), \mathbf{X}_1)$, such that

$$F(\gamma, \beta_1, \beta_2, V(\gamma, \beta_1, \beta_2)) = 0, \quad \forall \gamma \in B_\delta(\bar{\gamma}), (\beta_1, \beta_2) \in B_\delta(0),$$

and

$$V(\bar{\gamma}, 0, 0) = 0.$$

The uniqueness part of Theorem C holds in the sense that there exists some $0 < \bar{\delta} < \delta$, such that $B_{\bar{\delta}}(\bar{\gamma}, 0, 0, 0) \cap F^{-1}(0) \subset \{(\gamma, \beta_1, \beta_2, V(\gamma, \beta_1, \beta_2)) | (\gamma) \in B_\delta(\bar{\gamma}), \beta \in B_\delta(0)\}$.

Claim: there exists some $0 < \delta_1 < \frac{\bar{\delta}}{2}$, such that $V(\gamma, 0, 0) = 0$ for every $\gamma \in B_{\delta_1}(\bar{\gamma})$.

Proof of the claim: Since $V(\bar{\gamma}, 0, 0) = 0$ and $V(\gamma, 0, 0)$ is continuous in γ, there exists some $0 < \delta_1 < \frac{\bar{\delta}}{2}$, such that for all $\gamma \in B_{\delta_1}(\bar{\gamma})$, $(\gamma, 0, 0, V(\gamma, 0, 0)) \in B_{\bar{\delta}(\bar{\gamma},0,0,0)}$. We know that for all $\gamma \in B_{\delta_1}(\bar{\gamma})$,

$$F(\gamma, 0, 0, 0) = 0,$$

and

$$F(\gamma, 0, 0, V(\gamma, 0, 0)) = 0.$$

By the above mentioned uniqueness result, $V(\gamma, 0, 0) = 0$, for every $\gamma \in B_{\delta_1}(\bar{\gamma})$.

Now we have $V \in C^\infty(B_{\delta_1}(\bar{\gamma}) \times B_{\delta_1}(0), \mathbf{X}_1)$, and

$$F(\gamma, \beta_1, \beta_2, V(\gamma, \beta_1, \beta_2)) = 0, \quad \forall \gamma \in B_{\delta_1}(\bar{\gamma}), (\beta_1, \beta_2) \in B_{\delta_1}(0).$$

i.e.

$$G(\gamma, \beta_1 V^1_{-\frac{1}{2},\gamma} + \beta_2 V^2_{-\frac{1}{2},\gamma} + V(\gamma, \beta_1, \beta_2)) = 0, \quad \forall \gamma \in B_{\delta_1}(\bar{\gamma}), (\beta_1, \beta_2) \in B_{\delta_1}(0).$$

Take derivative of the above with respect to β_i at $(\gamma, 0)$, i=1,2, we have

$$G_{\tilde{U}}(\gamma,0)(V^i_{-\frac{1}{2},\gamma} + \partial_{\beta_i} V(\gamma,0,0)) = 0.$$

Since $G_{\tilde{U}}(\gamma,0)V^i_{-\frac{1}{2},\gamma} = 0$ by Lemma 3.2.15, we have

$$G_{\tilde{U}}(\gamma,0)\partial_{\beta_i} V(\gamma,0,0) = 0.$$

But $\partial_{\beta_i} V(\gamma,0,0) \in \mathbf{X}_1$, so

$$\partial_{\beta_i} V(\gamma,0,0) = 0, \quad i = 1,2.$$

Since K is compact, we can take δ_1 to be a universal constant for each $\gamma \in K$. So we have proved the existence of V in Theorem 3.2.2.

Next, let $\gamma \in B_{\delta_1}(\bar{\gamma})$. Let δ' be a small constant to be determined. For any U satisfying the equation (3.13) with $U - U^{-\frac{1}{2},\gamma} \in \mathbf{X}$, and $||U - U^{-\frac{1}{2},\gamma}||_{\mathbf{X}} \le \delta'$ there exist some $\beta_1, \beta_2 \in \mathbb{R}$ and $V^* \in \mathbf{X}_1$ such that

$$U - U^{-\frac{1}{2},\gamma} = \beta_1 V^1_{-\frac{1}{2},\gamma} + \beta_2 V^2_{-\frac{1}{2},\gamma} + V^*.$$

Then by Lemma 3.2.18, there exists some constant $C > 0$ such that

$$\frac{1}{C}(|(\beta_1,\beta_2)| + ||V^*||_{\mathbf{X}}) \le ||\beta_1 V^1_{-\frac{1}{2},\gamma} + \beta_2 V^2_{-\frac{1}{2},\gamma} + V^*||_{\mathbf{X}} \le \delta'.$$

This gives $||V^*||_{\mathbf{X}} \le C\delta'$.

Choose δ' small enough such that $C\delta' < \delta_1$. We have the uniqueness of V^*. So $V^* = V(\gamma,\beta_1,\beta_2)$ in (3.49). The theorem is proved. □

3.2.4 Existence of solutions with nonzero swirl near $U^{\mu,\gamma}$ when $(\mu,\gamma) \in I_3 \cap \{-\frac{1}{2} \le \mu < -\frac{3}{8}\}$

Next we look at the problem near $U^{\mu,\gamma}$ when $\mu \ge -\frac{1}{2}$ and $\gamma = -(1+\sqrt{1+2\mu})$. For such a fixed (μ,γ), write $\bar{U} = U^{\mu,\gamma}$. Recall that in Corollary 3.1.1 we have

$$\bar{U}_\theta = (1-x)(1+\sqrt{1+2\mu}). \tag{3.64}$$

It satisfies

$$(1-x^2)\bar{U}'_\theta + 2x\bar{U}_\theta + \frac{1}{2}\bar{U}^2_\theta = \mu(1-x)^2.$$

We will work with $\tilde{U} = U - \bar{U}$. Given a compact subset $K \in (-\frac{1}{2}, -\frac{3}{8})$ or $K = \{-\frac{1}{2}\}$, there exists an $\epsilon > 0$, depending only on K, satisfying $\max_{\mu \in K} \sqrt{1+2\mu} < \epsilon < \dfrac{1}{2}$. For this fixed ϵ, define

$$\begin{aligned}
\mathbf{M}_1 =& \mathbf{M}_1(\epsilon) \\
:=& \Big\{ \tilde{U}_\theta \in C([-1,1],\mathbb{R}) \cap C^1((-1,1],\mathbb{R}) \cap C^2((0,1),\mathbb{R}) \mid \tilde{U}_\theta(1) = \tilde{U}_\theta(-1) = 0, \\
& ||(1+x)^{-1+2\epsilon}\tilde{U}_\theta||_{L^\infty(-1,1)} < \infty, ||(1+x)^{2\epsilon}\tilde{U}_\theta'||_{L^\infty(-1,1)} < \infty, ||\tilde{U}_\theta''||_{L^\infty(0,1)} < \infty \Big\}, \\
\mathbf{M}_2 =& \mathbf{M}_2(\epsilon) \\
:=& \Big\{ \tilde{U}_\phi \in C^1((-1,1],\mathbf{R}) \cap C^2((-1,1),\mathbf{R}) \mid \tilde{U}_\phi(1) = 0, ||(1+x)^{\epsilon}\tilde{U}_\phi||_{L^\infty(-1,1)} < \infty, \\
& ||(1+x)^{1+\epsilon}\tilde{U}_\phi'||_{L^\infty(-1,1)} < \infty, ||(1+x)^{2+\epsilon}\tilde{U}_\phi''||_{L^\infty(-1,1)} < \infty \Big\}
\end{aligned}$$

with the following norms accordingly:

$$\begin{aligned}
||\tilde{U}_\theta||_{\mathbf{M}_1} &:= ||(1+x)^{-1+2\epsilon}\tilde{U}_\theta||_{L^\infty(-1,1)} + ||(1+x)^{2\epsilon}\tilde{U}_\theta'||_{L^\infty(-1,1)} + ||\tilde{U}_\theta''||_{L^\infty(0,1)}, \\
||\tilde{U}_\phi||_{\mathbf{M}_2} &:= ||(1+x)^{\epsilon}\tilde{U}_\phi||_{L^\infty(-1,1)} + ||(1+x)^{1+\epsilon}\tilde{U}_\phi'||_{L^\infty(-1,1)} + ||(1+x)^{2+\epsilon}\tilde{U}_\phi''||_{L^\infty(-1,1)}.
\end{aligned}$$

Next, define

$$\begin{aligned}
\mathbf{N}_1 = \mathbf{N}_1(\epsilon) &:= \big\{ \xi_\theta \in C((-1,1],\mathbb{R}) \cap C^1((0,1],\mathbb{R}) \mid \xi_\theta(1) = \xi_\theta'(1) = \xi_\theta(-1) = 0, \\
& \qquad ||(1+x)^{-1+2\epsilon}\xi_\theta||_{L^\infty(-1,1)} < \infty, ||\frac{\xi_\theta'}{1-x}||_{L^\infty(0,1)} < \infty \Big\}, \\
\mathbf{N}_2 = \mathbf{N}_2(\epsilon) &:= \Big\{ \xi_\phi \in C((-1,1],\mathbb{R}) \mid \xi_\phi(1) = 0, ||\frac{(1+x)^{1+\epsilon}\xi_\phi}{1-x}||_{L^\infty(-1,1)} < \infty \Big\}
\end{aligned}$$

with the following norms accordingly:

$$\begin{aligned}
||\xi_\theta||_{\mathbf{N}_1} &:= ||(1+x)^{-1+2\epsilon}\xi_\theta||_{L^\infty(-1,1)} + ||\frac{\xi_\theta'}{1-x}||_{L^\infty(0,1)}, \\
||\xi_\phi||_{\mathbf{N}_2} &:= ||\frac{(1+x)^{1+\varepsilon}\xi_\phi}{1-x}||_{L^\infty(-1,1)}.
\end{aligned}$$

Let $\mathbf{X} := \{\tilde{U} = (\tilde{U}_\theta, \tilde{U}_\phi) \mid \tilde{U}_\theta \in \mathbf{M}_1, \tilde{U}_\phi \in \mathbf{M}_2\}$ with the norm $||\tilde{U}||_{\mathbf{X}} := ||\tilde{U}_\theta||_{\mathbf{M}_1} + ||\tilde{U}_\phi||_{\mathbf{M}_2}$, and $\mathbf{Y} := \{\xi = (\xi_\theta, \xi_\phi) \mid \xi_\theta \in \mathbf{N}_1, \xi_\phi \in \mathbf{N}_2\}$ with the norm $||\xi||_{\mathbf{Y}} := ||\xi_\theta||_{\mathbf{N}_1} + ||\xi_\phi||_{\mathbf{N}_2}$. It is not difficult to verify that $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{X}$ and $\mathbf{Y}$ are Banach spaces.

Let $l_2 : \mathbf{X} \to \mathbb{R}$ be the bounded linear functional defined by (3.24) for each $V \in \mathbf{X}$. Define

$$\mathbf{X}_1 := \ker l_2. \tag{3.65}$$

Theorem 3.2.3. *For every compact subset K of $(-\frac{1}{2}, -\frac{3}{8})$ or $K = \{-\frac{1}{2}\}$, there exist $\delta = \delta(K) > 0$, and $V \in C^\infty(K \times B_\delta(0), \mathbf{X}_1)$ satisfying $V(\mu, 0) = 0$ and $\frac{\partial V}{\partial \beta}|_{\beta=0} = 0$, such that*

$$U = U^{\mu,-1-\sqrt{1+2\mu}} + \beta V^2_{\mu,-1-\sqrt{1+2\mu}} + V(\mu,\beta) \tag{3.66}$$

satisfies equation (3.13) with $\hat{\mu} = \mu - \dfrac{1}{4}\psi[U_\phi](-1)$. Moreover, there exists some $\delta' = \delta'(K) > 0$, such that if $||U - U^{\mu,-1-\sqrt{1+2\mu}}||_{\mathbf{X}} < \delta'$, $\mu \in K$, and U satisfies equation (3.13) with some constant $\hat{\mu}$, then (3.66) holds for some $|\beta| < \delta$.

To prove Theorem 3.2.3, we first study the properties of the Banach spaces $\mathbf{X}$ and $\mathbf{Y}$.

With the fixed ϵ, we have

Lemma 3.2.19. *For every $\tilde{U} \in \mathbf{X}$, it satisfies*

$$|\tilde{U}_\phi(s)| \le (1-s)(1+s)^{-\epsilon}||\tilde{U}_\phi||_{\mathbf{M}_2}, \quad \forall -1 < s < 1, \tag{3.67}$$

$$|\tilde{U}_\theta(s)| \le (1-s)(1+s)^{1-2\epsilon}||\tilde{U}_\theta||_{\mathbf{M}_1}, \quad \forall -1 < s < 1. \tag{3.68}$$

Lemma 3.2.20. *For every $\xi_\theta \in \mathbf{N}_1$,*

$$|\xi_\theta(s)| \le (1-s)^2(1+s)^{1-2\epsilon}||\xi_\theta||_{\mathbf{N}_1}, \quad \forall -1 < s < 1. \tag{3.69}$$

Now let K be a compact subset of $(-\frac{1}{2}, -\frac{3}{8})$ or $K = \{-\frac{1}{2}\}$. For $\tilde{U}_\phi \in \mathbf{M}_2$, let $\psi[\tilde{U}_\phi](x)$ be defined by (3.14). Then define a map G on $K \times \mathbf{X}$ such that for each $(\mu, \tilde{U}) \in K \times \mathbf{X}$, $G(\mu, \tilde{U}) = G(\mu, -1-\sqrt{1+2\mu}, \tilde{U})$ given by (3.15) with $\bar{U}_\theta$ in (3.64). If $\tilde{U}$ satisfies $G(\mu, \tilde{U}) = 0$, then $U = \tilde{U} + \bar{U}$ gives a solution of (3.13) with $\hat{\mu} = \mu - \frac{1}{4}\psi[\tilde{U}_\phi](-1)$, satisfying $U_\theta(-1) = \bar{U}_\theta(-1)$.

Proposition 3.2.3. *The map G is in $C^\infty(K \times \mathbf{X}, \mathbf{Y})$ in the sense that G has continuous Fréchet derivatives of every order. Moreover, the Fréchet derivative of G with respect to $\tilde{U}$ at $(\mu, \tilde{U}) \in K \times \mathbf{X}$ is given by the linear operator $L^\mu_{\tilde{U}} : \mathbf{X} \to \mathbf{Y}$ where $L^\mu := L^{\mu,-1-\sqrt{1+2\mu}}$ defined as in (3.18).*

To prove Proposition 3.2.3, we first have the following lemmas:

Lemma 3.2.21. *For every $\mu \in K$, the map $A(\mu, -1-\sqrt{1+2\mu}, \cdot) : \mathbf{X} \to \mathbf{Y}$ defined by (3.16) is a bounded linear operator.*

Proof. For convenience we denote $A = A(\mu, -1-\sqrt{1+2\mu}, \cdot)$. We make use of the properties of $\bar{U}_\theta$ that $\bar{U}_\theta(1) = 0$ and $\bar{U}_\theta \in C^2(-1, 1] \cap L^\infty(-1, 1)$.

A is clearly linear. For every $\tilde{U} \in \mathbf{X}$, we prove that $A\tilde{U}$ defined by (3.16) is in $\mathbf{Y}$ and there exists some constant C such that $||A\tilde{U}||_{\mathbf{Y}} \le C||\tilde{U}||_{\mathbf{X}}$ for all $\tilde{U} \in \mathbf{X}$.

By the fact that $\tilde{U}_\theta \in \mathbf{M}_1$ and (3.67), we have

$$\left|(1+x)^{-1+2\epsilon}A_\theta\right| \le (1-x)(1+x)^{2\epsilon}|\tilde{U}_\theta'| + (2+|\bar{U}_\theta|)(1+x)^{-1+2\epsilon}|\tilde{U}_\theta| \le C(1-x)||\tilde{U}_\theta||_{\mathbf{M}_1}.$$

We also see from the above that $\lim_{x\to 1} A_\theta(x) = \lim_{x\to -1} A_\theta(x) = 0$. By computation $A_\theta' = (1-x^2)\tilde{U}_\theta'' + \bar{U}_\theta\tilde{U}_\theta' + (2+\bar{U}_\theta')\tilde{U}_\theta$. Then by (3.64), (3.68) and the fact that $\tilde{U}_\theta \in \mathbf{M}_1$,

$$\frac{|A_\theta'(x)|}{1-x} \le C||\tilde{U}_\theta||_{\mathbf{M}_1}, \quad 0 < x < 1.$$

So $A_\theta \in \mathbf{N}_1$ and $||A_\theta||_{\mathbf{N}_1} \le C||\tilde{U}_\theta||_{\mathbf{M}_1}$.

Next, by the fact that $\tilde{U}_\phi \in \mathbf{M}_2$ and (3.64), with similar arguments in the proof of Lemma 3.2.12, we have

$$\frac{(1+x)^{1+\epsilon}}{1-x}|A_\phi| \le C||\tilde{U}_\phi||_{\mathbf{M}_2}, \quad -1 < x < 1.$$

In particular, $\lim_{x\to 1} A_\phi(x) = 0$. So $A_\phi \in \mathbf{N}_2$, and $||A_\phi||_{\mathbf{N}_2} \le C||\tilde{U}_\phi||_{\mathbf{M}_2}$. We have proved that $A\tilde{U} \in \mathbf{Y}$, and $||A\tilde{U}||_{\mathbf{Y}} \le C||\tilde{U}||_{\mathbf{X}}$ for every $\tilde{U} \in \mathbf{X}$. □

Lemma 3.2.22. *The map $Q : \mathbf{X} \times \mathbf{X} \to \mathbf{Y}$ defined by (3.17) is a bounded bilinear operator.*

Proof. It is clear that Q is a bilinear operator. For every $\tilde{U}, \tilde{V} \in \mathbf{X}$, we will prove that $Q(\tilde{U}, \tilde{V})$ is in $\mathbf{Y}$ and there exists some constant C independent of $\tilde{U}$ and $\tilde{V}$ such that $||Q(\tilde{U}, \tilde{V})||_{\mathbf{Y}} \le C||\tilde{U}||_{\mathbf{X}}||\tilde{V}||_{\mathbf{X}}$.

For convenience we write

$$\psi(\tilde{U}, \tilde{V})(x) = \int_x^1 \int_l^1 \int_t^1 \frac{2\tilde{U}_\phi(s)\tilde{V}_\phi'(s)}{1-s^2} dsdtdl.$$

By the same proof as that of Lemma 3.2.13, for $\tilde{U}_\phi, \tilde{V}_\phi \in \mathbf{M}_2$, we have for any $-1 < x < 1$

$$|\psi(\tilde{U},\tilde{V})(x) - \frac{(1-x)^2}{4}\psi(\tilde{U},\tilde{V})(-1)| \leq C(\epsilon)(1+x)^{1-2\epsilon}(1-x)^2||\tilde{U}_\phi||_{\mathbf{M}_2}||\tilde{V}_\phi||_{\mathbf{M}_2}. \quad (3.70)$$

So by (3.68), (3.70) and the fact that $\tilde{U}_\theta, \tilde{V}_\theta \in \mathbf{M}_1$, we have

$$\begin{aligned}
&|(1+x)^{-1+2\epsilon}Q_\theta(x)| \\
&\leq \frac{1}{2}(1+x)^{-1+2\epsilon}|\tilde{U}_\theta(x)||\tilde{V}_\theta(x)| + (1+x)^{-1+2\epsilon}|\psi(\tilde{U},\tilde{V})(x) - \frac{(1-x)^2}{4}\psi(\tilde{U},\tilde{V})(-1)| \\
&\leq C(1-x)^2||\tilde{U}_\theta||_{\mathbf{M}_1}||\tilde{V}_\theta||_{\mathbf{M}_1} + C(1-x)^2||\tilde{U}_\phi(s)||_{\mathbf{M}_2}||\tilde{V}_\phi||_{\mathbf{M}_2} \\
&\leq C(1-x)^2||\tilde{U}||_{\mathbf{X}}||\tilde{V}||_{\mathbf{X}}, \quad \forall -1 < x < 1.
\end{aligned}$$

Since $\epsilon < \frac{1}{2}$, from the above we also see that $\lim_{x\to 1} Q_\theta(x) = \lim_{x\to -1} Q_\theta(x) = 0$.

Using (3.67), (3.68) and the fact that $\tilde{U} \in \mathbf{X}$, with the same argument in the proof of Lemma 3.2.13, it can be shown that

$$|Q'_\theta(x)| \leq C(1-x)||\tilde{U}||_{\mathbf{X}}||\tilde{V}||_{\mathbf{X}}, \quad \forall 0 < x < 1.$$

So $Q_\theta \in \mathbf{N}_1$, and $||Q_\theta||_{\mathbf{N}_1} \leq C||\tilde{U}||_{\mathbf{X}}||\tilde{V}||_{\mathbf{X}}$.

Next, using (3.68) and similar proof of Lemma 3.2.13, we can prove

$$\left|\frac{(1+x)^{1+\epsilon}Q_\phi}{1-x}\right| \leq C||\tilde{U}_\theta||_{\tilde{\mathbf{M}}_1}||\tilde{V}_\phi||_{\tilde{\mathbf{M}}_2}, \quad -1 < x < 1,$$

and $\lim\limits_{x\to 1} Q_\phi(x) = 0$. So $Q_\phi \in \mathbf{N}_2$, and $||Q_\phi||_{\mathbf{N}_2} \leq ||\tilde{U}_\theta||_{\mathbf{M}_1}||\tilde{V}_\phi||_{\mathbf{M}_2}$. Thus we have proved $Q(\tilde{U},\tilde{V}) \in \mathbf{Y}$ and $||Q(\tilde{U},\tilde{V})||_{\mathbf{Y}} \leq C||\tilde{U}||_{\mathbf{X}}||\tilde{V}||_{\mathbf{X}}$ for all $\tilde{U},\tilde{V} \in \mathbf{X}$. The proof is finished. $\square$

Proof of Proposition 3.2.3*:* By definition, $G(\mu,\tilde{U}) = A(\mu, -1-\sqrt{1+2\mu}, \tilde{U}) + Q(\tilde{U},\tilde{U})$ for $(\mu,\tilde{U}) \in K \times \mathbf{X}$. Using standard theories in functional analysis, by Lemma 3.2.22 it is clear that Q is C^∞ on $K \times \mathbf{X}$. By Lemma 3.2.21, $A(\mu, -1-\sqrt{1+2\mu}, \cdot) : \mathbf{X} \to \mathbf{Y}$ is C^∞ for each $\mu \in K$. For all $i \geq 1$, we have

$$\partial^i_\mu A(\mu, -1-\sqrt{1+2\mu}, \tilde{U}) = \partial^i_\mu U_\theta^{\mu,-1-\sqrt{1+2\mu}} \begin{pmatrix} \tilde{U}_\theta \\ \tilde{U}'_\phi \end{pmatrix}.$$

By (3.12), for each integer $i \geq 1$, there exists some constant $C = C(i, K)$, depending only on i, K, such that

$$|\partial_\mu^i U_\theta^{\mu,-1-\sqrt{1+2\mu}}(x)| \leq C(i,K)(1-x), \quad -1 < x < 1. \tag{3.71}$$

From (3.64) we can also obtain

$$\left|\frac{d}{dx}\partial_\mu^i U_\theta^{\mu,-1-\sqrt{1+2\mu}}(x)\right| \leq C(i,K), \quad 0 < x < 1.$$

Using the above estimates and the fact that $\tilde{U}_\theta \in \mathbf{M_1}$, we have

$$\left|(1+x)^{-1+2\epsilon}\partial_\mu^i A_\theta(\mu, -1-\sqrt{1+2\mu}, \tilde{U})\right| \leq C(i,K)(1-x)||\tilde{U}_\theta||_{\mathbf{M_1}}, \quad -1 < x < 1,$$

and

$$\begin{aligned}
&\left|\frac{d}{dx}\partial_\mu^i A_\theta(\mu, -1-\sqrt{1+2\mu}, \tilde{U})\right| \\
\leq &\left|\frac{d}{dx}\partial_\mu^i U_\theta^{\mu,-1-\sqrt{1+2\mu}}(x)\right| |\tilde{U}_\theta(x)| + |\partial_\mu^i U_\theta^{\mu,-1-\sqrt{1+2\mu}}(x)| \left|\frac{d}{dx}\tilde{U}_\theta(x)\right| \\
\leq &C(i,K)(1-x)||\tilde{U}_\theta||_{\mathbf{M_1}}, \quad 0 < x < 1.
\end{aligned}$$

So $\partial_\mu^i A_\theta(\mu, -1-\sqrt{1+2\mu}, \tilde{U}) \in \mathbf{N_1}$, with $||\partial_\mu^i A_\theta(\mu, -1-\sqrt{1+2\mu}, \tilde{U})||_{\mathbf{N_1}} \leq C(i,K)||\tilde{U}_\theta||_{\mathbf{M_1}}$ for all $(\mu, \tilde{U}) \in K \times \mathbf{X}$.

Next, by (3.71) and the fact that $\tilde{U}_\phi \in \mathbf{M_1}$, we have

$$\frac{(1+x)^{1+\epsilon}}{1-x}|\partial_\mu^i A_\phi(\mu, -1-\sqrt{1+2\mu}, \tilde{U})| = \frac{|\partial_\mu^i U_\theta^{\mu,-1-\sqrt{1+2\mu}}|}{1-x}|(1+x)^{1+\epsilon}U_\phi'| \leq C(i,K)||\tilde{U}_\phi||_{\mathbf{M_2}}.$$

So $\partial_\mu^i A_\phi(\mu, -1-\sqrt{1+2\mu}, \tilde{U}) \in \mathbf{N_2}$, with

$$||\partial_\mu^i A_\phi(\mu, -1-\sqrt{1+2\mu}, \gamma, \tilde{U})||_{\mathbf{N_2}} \leq C(i,K)||\tilde{U}_\phi||_{\mathbf{M_2}}$$

for all $(\mu, \tilde{U}) \in K \times \mathbf{X}$. Thus $\partial_\mu^i A(\mu, -1-\sqrt{1+2\mu}, \tilde{U}) \in \mathbf{Y}$, with

$$||\partial_\mu^i A(\mu, -1-\sqrt{1+2\mu}, \tilde{U})||_{\mathbf{Y}} \leq C(i,K)||\tilde{U}||_{\mathbf{X}}$$

for all $(\mu, \tilde{U}) \in K \times \mathbf{X}$, $i \geq 1$.

So for each $\mu \in K$, $\partial_\mu^i A(\mu, -1-\sqrt{1+2\mu}, \cdot) : \mathbf{X} \to \mathbf{Y}$ is a bounded linear map with uniform bounded norm on K. Then by standard theories in functional analysis, $A : K \times \mathbf{X} \to \mathbf{Y}$ is C^∞. So G is a C^∞ map from $K \times \mathbf{X}$ to $\mathbf{Y}$. By direct calculation we get its Fréchet derivative with respect to $\mathbf{X}$ is given by the linear bounded operator $L_{\tilde{U}}^{\mu,-1-\sqrt{1+2\mu}} : \mathbf{X} \to \mathbf{Y}$ defined as (3.18). The proof is finished. $\square$

By Proposition 3.2.3, $L_0^{\mu} : \mathbf{X} \to \mathbf{Y}$, the Fréchet derivative of G at $\tilde{U} = 0$ is given by (3.19).

Next, let $a_\mu(x) = a_{\mu,-1-\sqrt{1+2\mu}}(x)$, $b_\mu(x) = b_{\mu,-1-\sqrt{1+2\mu}}(x)$ be the functions defined by (3.20) with $\bar{U}_\theta$ given by (3.64).

Since $\bar{U} = (1-x)(1+\sqrt{1+2\mu})$, we have

$$\begin{aligned} a_\mu(x) &= -\ln(1-x^2) + (1+\sqrt{1+2\mu})\ln(1+x), \\ b_\mu(x) &= (1+\sqrt{1+2\mu})\ln(1+x). \end{aligned} \tag{3.72}$$

For $\xi = (\xi_\theta, \xi_\phi) \in \mathbf{Y}$, by (3.69) and (3.72), we have

$$\int_{-1}^{1} e^{a_\mu(s)} \frac{|\xi_\theta(s)|}{1-s^2} ds \leq ||\xi_\theta||_{\mathbf{N}_1} \int_{-1}^{1} (1+s)^{\sqrt{1+2\mu}-2\epsilon} ds < \infty.$$

Let the map W^μ be defined as $W^\mu(\xi) := (W_\theta^\mu(\xi), W_\phi^\mu(\xi))$ by

$$\begin{aligned} W_\theta^\mu(\xi)(x) &= e^{-a_\mu(x)} \int_{-1}^{x} e^{a_\mu(s)} \frac{\xi_\theta(s)}{1-s^2} ds, \\ W_\phi^\mu(\xi)(x) &= \int_x^1 e^{-b_\mu(t)} \int_t^1 e^{b_\mu(s)} \frac{\xi_\phi(s)}{1-s^2} ds dt. \end{aligned} \tag{3.73}$$

Then W^μ satisfies (3.36).

Lemma 3.2.23. *$W^\mu : \mathbf{Y} \to \mathbf{X}$ is continuous and is a right inverse of L_0^μ.*

Proof. For convenience we write $W = W^\mu(\xi)$, $a(x) = a_\mu(x)$ and $b(x) = b_\mu(x)$.

We first prove that W is well-defined. For $\xi \in \mathbf{Y}$, denote $W := W(\xi)$. Applying Lemma 3.2.20 in the expression of W_θ in (3.73), we have

$$\left|(1+x)^{-1+2\epsilon} W_\theta(x)\right| \leq C(1+x)^{-1+2\epsilon} ||\xi_\theta||_{\mathbf{N}_1} e^{-a(x)} \int_{-1}^{x} e^{a(s)}(1-s)(1+s)^{-2\epsilon} ds, \quad -1 < x < 1. \tag{3.74}$$

Using (3.72), we have

$$e^{a(s)} = (1+s)^{\sqrt{1+2\mu}}(1-s)^{-1}, \quad e^{-a(x)} = (1+s)^{-\sqrt{1+2\mu}}(1-s), \quad -1 < s < x < 1.$$

Apply this in (3.74), it is not hard to see that

$$|W_\theta(x)| \leq C||\xi_\theta||_{\mathbf{N}_1}(1+x)^{1-2\epsilon}(1-x), \quad -1 < x \leq 1. \tag{3.75}$$

In particular $W_\theta(1) = 0$. Since $\epsilon < \frac{1}{2}$, $\lim_{x\to -1} W_\theta(x) = 0$,

By (3.72),

$$|a'(x)| \le \frac{C}{1-x^2}, \quad |a''(x)| \le \frac{C}{(1-x^2)^2}, \quad -1 < x < 1. \tag{3.76}$$

Using the above estimate of $|a'(x)|$, (3.69), (3.75) and (3.36), we have

$$|(1+x)^{2\epsilon} W_\theta'| \le (1+x)^{2\epsilon} |a'(x)||W_\theta(x)| + \frac{|\xi_\theta(x)|(1+x)^{2\epsilon}}{1-x^2} \le C||\xi_\theta||_{\mathbf{N}_1}, \quad -1 < x < 1.$$

Next, A calculation gives

$$W_\theta''(x) = ((a'(x))^2 - a''(x))W_\theta(x) - a'(x)\frac{\xi_\theta(x)}{1-x^2} + \frac{\xi_\theta'(x)}{1-x^2} + \frac{2x\xi_\theta(x)}{(1-x^2)^2}.$$

So

$$|W_\theta''(x)| \le |(a'(x))^2 - a''(x)||W_\theta| + |a'(x)|\frac{|\xi_\theta|}{1-x^2} + \frac{|\xi_\theta'|}{(1-x)} + \frac{|\xi_\theta|}{(1-x)^2}.$$

By (3.72), we have the estimate

$$(a'(x))^2 - a''(x) = O\left(\frac{1}{1-x}\right).$$

It follows, using (3.75), (3.76) and Lemma 3.2.20, that

$$|W_\theta''(x)| \le C\left(\frac{|W_\theta(x)|}{1-x} + \frac{|\xi_\theta|}{(1-x)^2} + \frac{|\xi_\theta'|}{1-x}\right) \le C||\xi_\theta||_{\mathbf{N}_1}, \quad 0 < x < 1.$$

So we have shown that $W_\theta \in \mathbf{M}_1$, and $||W_\theta||_{\mathbf{M}_1} \le C||\xi_\theta||_{\mathbf{N}_1}$ for some constant C.

By definition of $W_\phi(\xi)$ in (3.73) and the fact that $\xi_\phi \in \mathbf{N}_2$, we have, for every $-1 < x < 1$, that

$$|W_\phi(x)| \le ||\xi_\phi||_{\mathbf{N}_2} \int_x^1 e^{-b(t)} \int_t^1 e^{b(s)}(1+s)^{-2-\epsilon} ds dt.$$

Using (3.72), we have

$$e^{b(s)} = (1+s)^{1+\sqrt{1+2\mu}}, \quad e^{-b(t)} = (1+t)^{-1-\sqrt{1+2\mu}}, \quad -1 < s, t < 1. \tag{3.77}$$

So we have, using $\sqrt{1+2\mu} < \epsilon < \frac{1}{2}$,

$$\begin{aligned}|W_\phi(x)| &\le C||\xi_\phi||_{\mathbf{N}_2} \int_x^1 (1+t)^{-1-\sqrt{1+2\mu}} \int_t^1 (1+s)^{\sqrt{1+2\mu}-1-\epsilon} ds dt \\ &\le C||\xi_\phi||_{\mathbf{N}_2}(1+x)^{-\epsilon}, \quad -1 < x \le 1.\end{aligned}$$

For $0 < x < 1$, it can be seen from the above that $|W_\phi(x)| \leq C||\xi_\phi||_{\mathbf{N}_2}(1-x)$. In particular, $W_\phi(1) = 0$. By computation

$$W_\phi'(x) = -e^{-b(x)} \int_x^1 e^{b(s)} \frac{\xi_\phi(s)}{1-s^2} ds.$$

Using (3.77), $\epsilon > \sqrt{1+2\mu}$ and the fact that $\xi_\phi \in \mathbf{N}_2$, we have,

$$|(1+x)^{1+\epsilon} W_\phi'(x)| \leq C||\xi_\phi||_{\mathbf{N}_2} \quad -1 < x < 1.$$

Similarly,

$$W_\phi''(x) = b'(x) e^{-b(x)} \int_x^1 e^{b(s)} \frac{\xi_\phi(s)}{1-s^2} ds + \frac{\xi_\phi(x)}{1-x^2}.$$

By (3.72), $b'(x) = \frac{1+\sqrt{1+2\mu}}{1+x} = O((1+x)^{-1})$. Using (3.77), we have

$$|(1+x)^{2+\epsilon} W_\phi''(x)| \leq C||\xi_\phi||_{\mathbf{N}_2}, \quad -1 < x < 1.$$

So $W_\phi \in \mathbf{M}_2$, and $||W_\phi||_{\mathbf{M}_2} \leq C||\xi_\phi||_{\mathbf{N}_2}$ for some constant C.

Thus $W^\mu(\xi) \in \mathbf{X}$ for all $\xi \in \mathbf{Y}$, and $||W(\xi)||_{\mathbf{X}} \leq C||\xi||_{\mathbf{Y}}$ for some constant C. So $W^\mu : \mathbf{X} \to \mathbf{Y}$ is well-defined and continuous. It can be directly checked that W^μ is a right inverse of L_0^μ. $\square$

Let $V_\mu^i := V^i_{\mu,-1-\sqrt{1+2\mu}}$, $i = 1, 2, 3$, be defined by (3.23) with related $a_{\mu,-1-\sqrt{1+2\mu}} = a_\mu(x)$ and $b_{\mu,-1-\sqrt{1+2\mu}} = b_\mu(x)$ given by (3.72) , we have

Lemma 3.2.24. *$\{V_\mu^2\}$ is a basis of the kernel of $L_0^\mu : \mathbf{X} \to \mathbf{Y}$.*

Proof. By (3.72), it is not hard to verify that $V_\mu^2 \in \mathbf{X}$, and $V_\mu^1, V_\mu^3 \notin \mathbf{X}$. Then by similar proof as Lemma 3.2.6, we obtain the conclusion. $\square$

Corollary 3.2.3. *For any $\xi = (\xi_\theta, \xi_\phi) \in \mathbf{Y}$, all solutions of $L_0^\mu(V) = \xi$, $V \in \mathbf{X}$, are given by*

$$V = W^\mu(\xi) + cV_\mu^2, \quad c \in \mathbb{R}.$$

Namely,

$$V_\theta = W_\theta^\mu(\xi), \quad V_\phi = W_\phi^\mu(\xi) + c\int_x^1 e^{-b_\mu(t)} dt, \;\; c \in \mathbb{R}.$$

Proof. By Lemma 3.2.23, $V - W^\mu(\xi)$ is in the kernel of $L_0^\mu : \mathbf{X} \to \mathbf{Y}$. The conclusion then follows from Lemma 3.2.24. $\square$

Let l_2 be the functionals on $\mathbf{X}$ defined by (3.24), and $\mathbf{X}_1$ be the subspace of $\mathbf{X}$ defined by (3.65). As shown in Section 3.2.1, $l_2(V_\mu^2) > 0$ for every $\mu \in K$. So $\mathbf{X}_1$ is a closed subspace of $\mathbf{X}$, and

$$\mathbf{X} = \operatorname{span}\{V_\mu^2\} \oplus \mathbf{X}_1, \quad \forall \mu \in K,$$

with the projection operator $P(\mu) : \mathbf{X} \to \mathbf{X}_1$ given by

$$P(\mu)V = V - c(\mu) l_2(V) V_\mu^2 \text{ for } V \in \mathbf{X}.$$

where $c(\mu) = \left(\int_0^1 e^{-b_\mu(t)} dt \right)^{-1} > 0$ for all $\mu \in K$.

By Lemma 3.2.24 and Corollary 3.2.3, using similar proof as Lemma 3.2.7, we have

Lemma 3.2.25. *The operator $L_0^\mu : \mathbf{X}_1 \to \mathbf{Y}$ is an isomorphism.*

Lemma 3.2.26. *$V_\mu^2 \in C^\infty(K, \mathbf{X})$.*

Proof. For convenience, in this proof we denote $a(x) = a_\mu(x)$, $b(x) = b_\mu(x)$ and $V^2 = V_\mu^2$.

By computation, using the explicit expression of $U_\theta^{\mu,-1-\sqrt{1+2\mu}}(x), a(x), a'(x), b(x)$ and $V_\phi^2(x)$ given by (3.64), (3.72) and (3.23), and the estimates of $\partial_\mu^i U_\theta^{\mu,-1-\sqrt{1+2\mu}}$ in (3.12), we have, for $\mu \in (-\frac{1}{2}, -\frac{3}{8})$, that

$$e^{-b(x)} = (1+x)^{-1-\sqrt{1+2\mu}}, \quad -1 < x < 1.$$

So

$$V_\phi^2(x) = O(1)(1-x)(1+x)^{-\sqrt{1+2\mu}}, \quad \left| \frac{d}{dx} V_\phi^2(x) \right| = e^{-b(x)} = (1+x)^{-1-\sqrt{1+2\mu}},$$

$$\left| \frac{d^2}{dx^2} V_\phi^2(x) \right| = \left| b'(x) \right| e^{-b(x)} = O(1)(1+x)^{-2-\sqrt{1+2\mu}}, \quad -1 < x < 1.$$

Moreover,

$$\frac{\partial^i}{\partial \mu^i} b(x) = \frac{\partial^i}{\partial \mu^i} \sqrt{1+2\mu} \ln(1+x).$$

So we have, for $-1 < x < 1, i = 1, 2, 3...$, that

$$\left| \partial_\mu^i V_\phi^2(x) \right| = O(1)(1-x)(1+x)^{-\sqrt{1+2\mu}} (\ln(1+x))^i,$$

$$\left|\partial_\mu^i \frac{d}{dx} V_\phi^2(x)\right| = O(1)(1+x)^{-1-\sqrt{1+2\mu}}(\ln(1+x))^i,$$

$$\left|\partial_\mu^i \frac{d^2}{dx^2} V_\phi^2(x)\right| = O(1)(1+x)^{-2-\sqrt{1+2\mu}}(\ln(1+x))^i.$$

The above imply that for all $i \geq 0$, $\partial_\mu^i V^2(x) \in \mathbf{X}$, and $V_\phi^2 \in C^\infty(K, M_2)$. So $V^2 \in C^\infty(K, \mathbf{X})$. □

Next, by similar arguments in the proof of Lemma 3.2.9, using Lemma 3.2.26, we have

Lemma 3.2.27. *There exists $C = C(K) > 0$ such that for all $\mu \in K$, $\beta \in \mathbb{R}^2$, and $V \in \mathbf{X}_1$,*

$$||V||_{\mathbf{X}} + |\beta| \leq C||\beta V_\mu^2 + V||_{\mathbf{X}}.$$

Proof of Theorem 3.2.3*:* Define a map $F : K \times \mathbb{R} \times \mathbf{X}_1 \to \mathbf{Y}$ by

$$F(\mu, \beta, V) = G(\mu, \beta V_\mu^2 + V).$$

By Proposition 3.2.3, G is a C^∞ map from $K \times \mathbf{X}$ to $\mathbf{Y}$. Let $\tilde{U} = \tilde{U}(\mu, \beta, V) = \beta_2 V_\mu^2 + V$. Using Lemma 3.2.26, we have $\tilde{U} \in C^\infty(K \times \mathbb{R} \times \mathbf{X}_1, \mathbf{X})$. So it concludes that $F \in C^\infty(K \times \mathbb{R} \times \mathbf{X}_1, \mathbf{Y})$.

Next, by definition $F(\mu, 0, 0) = 0$ for all $\mu \in K$. Fix some $\bar{\mu} \in K$, using Lemma 3.2.25, we have $F_V(\bar{\mu}, 0, 0) = L_0^{\bar{\mu}} : \mathbf{X}_1 \to \mathbf{Y}$ is an isomorphism.

Applying Theorem C, there exist some $\delta > 0$ and a unique $V \in C^\infty(B_\delta(\bar{\mu}) \times B_\delta(0), \mathbf{X}_1)$, such that

$$F(\mu, \beta, V(\mu, \beta)) = 0, \quad \forall \mu \in B_\delta(\bar{\mu}), \beta \in B_\delta(0),$$

and

$$V(\bar{\mu}, 0) = 0.$$

The uniqueness part of Theorem C holds in the sense that there exists some $0 < \bar{\delta} < \delta$, such that $B_{\bar{\delta}}(\bar{\mu}, 0, 0) \cap F^{-1}(0) \subset \{(\mu, \beta, V(\mu, \beta)) | (\gamma) \in B_\delta(\bar{\mu}), \beta \in B_\delta(0)\}$.

Claim: there exists some $0 < \delta_1 < \frac{\bar{\delta}}{2}$, such that $V(\mu, 0) = 0$ for every $\mu \in B_{\delta_1}(\bar{\mu})$.

Proof of the claim: Since $V(\bar{\mu}, 0) = 0$ and $V(\mu, 0)$ is continuous in μ, there exists some $0 < \delta_1 < \frac{\bar{\delta}}{2}$, such that for all $\mu \in B_{\delta_1}(\bar{\mu})$, $(\mu, 0, V(\mu, 0)) \in B_{\bar{\delta}(\bar{\mu},0,0)}$. We know that for all $\mu \in B_{\delta_1}(\bar{\mu})$,

$$F(\mu, 0, 0) = 0,$$

and

$$F(\mu, 0, V(\mu, 0)) = 0.$$

By the above mentioned uniqueness result, $V(\mu, 0) = 0$, for every $\mu \in B_{\delta_1}(\bar{\mu})$.

Now we have $V \in C^\infty(B_{\delta_1}(\bar{\mu}) \times B_{\delta_1}(0), \mathbf{X}_1)$, and

$$F(\mu, \beta, V(\mu, \beta)) = 0, \quad \forall \mu \in B_{\delta_1}(\bar{\mu}), \beta \in B_{\delta_1}(0).$$

i.e.

$$G(\mu, \beta V_\mu^2 + V(\mu, \beta)) = 0, \quad \forall \mu \in B_{\delta_1}(\bar{\mu}), \beta \in B_{\delta_1}(0).$$

Take derivative of the above with respect to β at $(\mu, 0)$, we have

$$G_{\tilde{U}}(\mu, 0)(V_\mu^2 + \partial_\beta V(\mu, 0)) = 0.$$

Since $G_{\tilde{U}}(\mu, 0)V_\mu^2 = 0$ by Lemma 3.2.24, we have

$$G_{\tilde{U}}(\mu, 0)\partial_\beta V(\mu, 0) = 0.$$

But $\partial_{\beta_i} V(\mu, 0) \in C^\infty(\mathbf{X}_1)$, so

$$\partial_\beta V(\mu, 0) = 0.$$

Since K is compact, we can take δ_1 to be a universal constant for each $\mu \in K$. So we have proved the existence of V in Theorem 3.2.3.

Next, let $\mu \in B_{\delta_1}(\bar{\mu})$. Let δ' be a small constant to be determined. For any U satisfies the equation (3.13) with $U - U^{\mu,-1-\sqrt{1+2\mu}} \in \mathbf{X}$, and $||U - U^{\mu,-1-\sqrt{1+2\mu}}||_{\mathbf{X}} \leq \delta'$ there exist some $\beta \in \mathbb{R}$ and $V^* \in \mathbf{X}_1$ such that

$$U - U^{\mu,-1-\sqrt{1+2\mu}} = \beta V_\mu^2 + V^*.$$

Then by Lemma 3.2.27, there exists some constant $C > 0$ such that

$$\frac{1}{C}(|\beta| + ||V^*||_{\mathbf{X}}) \leq ||\beta V_\mu^2 + V^*||_{\mathbf{X}} \leq \delta'.$$

This gives $||V^*||_{\mathbf{X}} \le C\delta'$.

Choose δ' small enough such that $C\delta' < \delta_1$. We have the uniqueness of V^*. So $V^* = V(\mu, \beta)$ in (3.66). The theorem is proved. $\square$

Now with Theorem 3.2.1-3.2.3 we can give the

Proof of the existence part of Theorem 1.0.2*:* Recall the relation between the parameters (μ, γ) and (τ, σ)

$$\mu = \frac{1}{8}\tau^2 - \frac{1}{2}\tau, \quad \gamma = -2\sigma.$$

Let K be a compact subset of one of the four sets J_1, J_2, $J_3 \cap \{2 < \tau < 3\}$ and $J_3 \cap \{\tau = 2\}$, where J_1, J_2, J_3 are the sets defined by (1.10).

For $(\tau, \sigma) \in K \cap J_1$, let

$$u(\tau, \sigma, \beta) = \frac{1}{\sin\theta}\left(U^{\mu,\gamma} + \beta V^2_{\mu,\gamma} + V(\mu, \gamma, 0, \beta)\right), \quad \beta \in (-\delta, \delta),$$

where $\delta, V^2_{\mu,\gamma}$ and $V(\mu, \gamma, 0, \beta)$ are as in Theorem 3.2.1.

For $(\tau, \sigma) \in K \cap J_2$, let

$$u(\tau, \sigma, \beta) = \frac{1}{\sin\theta}\left(U^{-\frac{1}{2},\gamma} + \beta V^2_{-\frac{1}{2},\gamma} + V(\gamma, 0, \beta)\right), \quad \beta \in (-\delta, \delta),$$

where $\delta, V^2_{-\frac{1}{2},\gamma}$ and $V(\gamma, 0, \beta)$ are as in Theorem 3.2.2.

For $(\tau, \sigma) \in K \cap (J_3 \cap \{2 \le \tau < 3\})$, let

$$u(\tau, \sigma, \beta) = \frac{1}{\sin\theta}\left(U^{\mu,-1-\sqrt{1+2\mu}} + \beta V^2_{\mu,-1-\sqrt{1+2\mu}} + V(\mu, \beta)\right), \quad \beta \in (-\delta, \delta),$$

where $\delta, V^2_{\mu,-1-\sqrt{1+2\mu}}$ and $V(\mu, \beta)$ are as in Theorem 3.2.3.

With $u(\tau, \sigma, \beta)$ defined as the above, the existence part of Theorem 1.0.2 follows from Theorem 3.2.1-3.2.3.

3.3 Pingpong ball on top of a fountain

As mentioned in the introduction, the pressure of Landau solutions at the center of north pole is greater than the pressure nearby. In this section, we identify all (-1) homogeneous, axisymmetric, no-swirl solutions which describe outward jets with lower pressure in the center. We tend to believe that the pressure profiles are of interest and modification of these solutions is more likely to support a pingpong ball.

Set $\alpha := \gamma + 1$, consider below the exact form solutions in Theorem 3.1.1:

When $\mu > -\frac{1}{2}$, the solutions are expressed as

$$U_\theta(x) = (1-x)\left(1 - b - \frac{2b(\alpha-b)}{(\alpha+b)(\frac{1+x}{2})^{-b} - \alpha + b}\right)$$

where $b = \sqrt{1+2\mu}$. Then $u_r|_{x=1} = U'_\theta(1) = \gamma = \alpha - 1$. By L'Hospital's rule,

$$\lim_{x\to 1^-} \frac{U_\theta(x)}{1-x^2} = \lim_{x\to 1^-} \frac{U'_\theta(x)}{-2x} = -\frac{1}{2}U'_\theta(1).$$

From the second line of (2.1) with $U_\phi \equiv 0$, we have

$$\begin{aligned} \lim_{x\to 1^-} p' &= \lim_{x\to 1^-}\left(U''_\theta - \frac{1}{1-x^2}U_\theta U'_\theta - \frac{x}{(1-x^2)^2}U_\theta^2\right) = \frac{1}{2}(\alpha+b)(\alpha-b) + \frac{1}{2}U'^2_\theta - \frac{1}{4}U'^2_\theta \\ &= \frac{1}{2}(\alpha+b)(\alpha-b) + \frac{1}{4}(\alpha-1)^2 = \frac{3}{4}\alpha^2 - \frac{1}{2}\alpha + \frac{1}{4} - \frac{1}{2}b^2. \end{aligned}$$

Since $b = \sqrt{1+2\mu} > 0$, it can be proved that $u_r|_{x=1} = \alpha - 1 > 0$ and $\lim_{x\to 1^-} p'(x) = \frac{1}{2}(\alpha+b)(\alpha-b) + \frac{1}{4}(\alpha-1)^2 < 0$ if and only if $b > 1$, $1 < \alpha < \frac{1}{3} + \sqrt{\frac{2}{3}b^2 - \frac{2}{9}}$. Notice that $b > 1$, $1 < \alpha < \frac{1}{3} + \sqrt{\frac{2}{3}b^2 - \frac{2}{9}}$ implies

$$\mu > 0, \quad 0 < \gamma < \frac{2}{3}(\sqrt{1+3\mu} - 1).$$

Therefore, under the condition $\mu > 0$, $0 < \gamma < \frac{2}{3}(\sqrt{1+3\mu} - 1)$, we have $u_r \mid_{x=1} > 0$, $\frac{dp}{dx} \mid_{x=1} < 0$. The corresponding solutions describe fluid jets with lower pressure at north pole than nearby.

It remains to check the case when $u_r \mid_{x=1} > 0$, $\frac{dp}{dx} \mid_{x=1} = 0$. This condition implies

$$b > 1, \quad \alpha = \frac{1}{3} + \sqrt{\frac{2}{3}b^2 - \frac{2}{9}},$$

or equivalently,

$$\mu > 0, \quad \gamma = \frac{2}{3}\left(\sqrt{1+3\mu} - 1\right). \tag{3.78}$$

Notice that $\{(\mu,\gamma) \mid \mu > 0, \gamma = \frac{2}{3}\left(\sqrt{1+3\mu} - 1\right)\} \subset I$. We substitute (3.78) into the the first line of (3.7) in Theorem 3.1, then use the first line of (2.1) to derive the pressure p. Direct computation shows that

$$p(x) = C + f(b)(1-x)^2 + O(1)(1-x)^3,$$

where function

$$f(b)=\frac{1}{432}\left(54b^2-22-\sqrt{2(3b^2-1)}(15b^2+1)\right).$$

It can be checked that

$$f(1)=f'(1)=0;\qquad f'(b)<0,\forall b>1;\qquad f''(1)<0.$$

So $f(b)<0$ for all $b>1$. It means that when $p'\mid_{x=1}=0$, the pressure at the center of north pole is greater than the pressure nearby.

When $\mu=-\frac{1}{2}$, the solutions are expressed as

$$U_\theta(x)=(1-x)\left(1+\frac{2\alpha}{\alpha\ln\frac{1+x}{2}-2}\right),$$

and there is $\lim_{x\to1^-}u_r=\alpha-1$. Similarly, by L'Hospital's rule, we get

$$\lim_{x\to1^-}p'=\lim_{x\to1^-}\left(U_\theta''+\frac{1}{4}U_\theta'^2\right)=\frac{1}{2}\alpha^2+\frac{1}{4}(\alpha-1)^2.$$

It is not hard to see that $\lim_{x\to1^-}p'>0$ for any $\alpha\in\mathbb{R}$.

When $\mu<-\frac{1}{2}$, the solution can be exactly expressed as

$$U_\theta(x)=(1-x)\left(1+\frac{b(b\tan\frac{\beta(x)}{2}+\alpha)}{\alpha\tan\frac{\beta(x)}{2}-b}\right),$$

where $\beta(x)$ is determined by $\beta(x)=b\ln\frac{1+x}{2}$. There is $u_r|_{x=1}=\alpha-1$, and

$$\lim_{x\to1^-}p'=U_\theta''-\frac{1}{1-x^2}U_\theta U_\theta'-\frac{x}{(1-x^2)^2}U_\theta^2=\frac{\alpha^2}{2}+\frac{b^2}{2}+\frac{1}{4}(\alpha-1)^2.$$

It is not hard to see that $p_x'|_{x=1}>0$ for any $\alpha\in\mathbb{R}$.

According to the above computation, if $\mu\le0$, the fluid does not fit our pressure profile to support a pingpong ball. In particular, Landau solutions correspond to $\mu=0$, and they have greater pressure in the center.

Define the open set $I_p\subset I$ by

$$I_p:=\{(\mu,\gamma)\subset\mathbb{R}^2|\mu>0,0<\gamma<\frac{2}{3}(\sqrt{1+3\mu}-1)\}.$$

Theorem 3.3.1. *For any $(\mu,\gamma)\in I_p$, $u_r|_{x=1}>0$, $p'|_{x=1}<0$. For any $(\mu,\gamma)\in\mathbb{R}^2\setminus I_p$, either*

$$u_r|_{x=1}\le0,$$

or there exists $\delta > 0$ *such that*

$$p(x) < p(1), \text{ in } (1-\delta, 1).$$

Remark 3.3.1. *We have therefore identified all (-1) homogeneous, axisymmetric, no-swirl solutions of NSE, which describe outward jets with lower pressure in the center. They are* $\{u(\mu,\gamma) \mid (\mu,\gamma) \in I_p\}$.

In particular, those solutions which can not be extended to solutions in $C^\infty(\mathbb{S}^2 \setminus \{S\})$ *are not in this set. There are also many solutions in* $C^\infty(\mathbb{S}^2 \setminus \{S\})$*, including Landau solutions, not in this set.*

Chapter 4

Asymptotic behavior of solutions

In this Chapter we study the asymptotic behavior of (-1)-homogeneous axisymmetric solutions of (1.3) in a punctured ball around the north or south pole of $\mathbb{S}^2$. In particular we prove Theorem 1.0.3 and Theorem 1.0.4.

Recall that the Navier-Stokes equations for (-1)-homogeneous solutions have been converted to the system

$$\begin{cases} (1-x^2)U_\theta' + 2xU_\theta + \dfrac{1}{2}U_\theta^2 + \displaystyle\int_{x_0}^{x}\int_{x_0}^{l}\int_{x_0}^{t} \frac{2U_\phi(s)U_\phi'(s)}{1-s^2} ds dt dl = c_1x^2 + c_2x + c_3, \\ (1-x^2)U_\phi'' + U_\theta U_\phi' = 0. \end{cases} \tag{4.1}$$

where x_0 is some fixed number in $(-1, 1)$, c_1, c_2, c_3 are constants.

It follows from the second line of the above that

$$U_\phi'(x) = Ce^{-\int_{-1+\delta_1}^{x} \frac{U_\theta}{1-s^2} ds}. \tag{4.2}$$

Let $\delta > 0$ be a real number, H be a function of x, we consider the equation

$$(1-x^2)U_\theta'(x) + 2xU_\theta + \frac{1}{2}U_\theta^2 = H(x), \quad -1 < x \le -1 + \delta. \tag{4.3}$$

Define, with $x_0 = -1 + \delta$,

$$I(x) := \int_{x_0}^{x}\int_{x_0}^{l}\int_{x_0}^{t} \frac{2U_\phi(s)U_\phi'(s)}{1-s^2} ds dt dl. \tag{4.4}$$

We can write I as

$$\begin{aligned} I(x) &= \int_{x_0}^{x}\int_{s}^{x}\int_{t}^{x} \frac{2U_\phi(s)U_\phi'(s)}{1-s^2} dl dt ds = \int_{x_0}^{x} \frac{U_\phi(s)U_\phi'(s)(s-x)^2}{1-s^2} ds \\ &= -\frac{(x-x_0)^2}{2(1-x_0^2)} U_\phi^2(x_0) + I_1 \end{aligned} \tag{4.5}$$

where

$$I_1(x) = -\int_{x_0}^{x} \frac{U_\phi^2(s)(s-x)(1-sx)}{(1-s^2)^2} ds. \tag{4.6}$$

By computation

$$I_1'(x) = -\int_{x_0}^{x} \frac{U_\phi^2(s)(-s^2+2xs-1)}{(1-s^2)^2} ds < 0, \quad -1 < x < x_0. \tag{4.7}$$

Indeed, the first inequality in the above follows from $-s^2+2xs-1 \le -s^2+s^2+x^2-1 = x^2-1 < 0$, for all $-1 < x < s < x_0$.

Proof of (i) and (ii) of Theorem 1.0.3*:* We write the first equation of (4.1) as (4.3) with

$$H(x) = -I(x) + c_1x^2 + c_2x + c_3,$$

where $I(x)$ is defined in (4.4).

By (4.5) and (4.7), $H(x)$ is the sum of a bounded function and a monotonically increasing function in $(-1,-1+\delta]$. It follows that $H^+ \in L^\infty(-1,-1+\delta)$.

Let $g(x) := U_\theta(x)$, $a(x) := 1-x^2$ and $b(x) := 2x$. An application of Proposition 6.1.1 yields part (i) and (ii) of the theorem. □

For $H \in C[-1,-1+\delta]$, denote $\tau_1 = 2-\sqrt{4+2H(-1)}$, and $\tau_2 = 2+\sqrt{4+2H(-1)}$

Lemma 4.0.1. *For $\delta > 0$, $H \in C[-1,-1+\delta]$, let $U_\theta \in C^1(-1,-1+\delta]$ be a solution of (4.3) in $(-1,-1+\delta)$. Then*

$$U_\theta(-1) := \lim_{x\to -1^+} U_\theta(x) = \tau_1 \text{ or } \tau_2,$$

and $H(-1) = -2U_\theta(-1) + \frac{1}{2}U_\theta^2(-1) \ge -2$.

Proof. Let $g(x) := U_\theta(x)$, $a(x) := 1-x^2$ and $b(x) := 2x$. By Proposition 6.1.1, $U_\theta(-1) := \lim_{x\to-1^+} U_\theta(x)$ exists and is finite, and $\lim_{x\to-1^+}(1-x^2)U_\theta'(x) = 0$. Sending $x \to -1$ in (4.3) leads to

$$H(-1) = -2U_\theta(-1) + \frac{1}{2}U_\theta^2(-1) = \frac{1}{2}\left(U_\theta(-1)-2\right)^2 - 2.$$

Lemma 4.0.1 follows from the above. □

Now we are ready to give some further local asymptotic behavior of local solutions U of (4.1) as $x \to -1^+$. By part (i) of Theorem 1.0.3 we know that $\lim_{x\to -1^+} U_\theta(x) = U_\theta(-1)$ exists and is finite. Now let us prove part (iii) of Theorem 1.0.3.

Lemma 4.0.2. *For $\delta > 0$, $x_0 \in (-1, -1+\delta]$, let $U = (U_\theta, U_\phi)$ be a solution of system (4.1) in $(-1, -1+\delta)$, and $U_\theta \in C^1(-1, -1+\delta]$, $U_\phi \in C^2(-1,-1+\delta]$, with $U_\theta(-1) < 2$. Then if $U_\theta(-1) \neq 0$, there exist some constants a_1, a_2 and b_1, b_2, b_3, such that for any $\epsilon > 0$,*

$$\begin{aligned} U_\theta(x) =& U_\theta(-1) + a_1(1+x)^{\alpha_0} + a_2(1+x) + O((1+x)^{2\alpha_0-\epsilon}) + O((1+x)^{2-\epsilon}),\\ U_\phi(x) =& U_\phi(-1) + b_1(1+x)^{\alpha_0} + b_2(1+x)^{2\alpha_0} + b_3(1+x)^{1+\alpha_0}\\ &+ O((1+x)^{\alpha_0+2-\epsilon}) + O((1+x)^{3\alpha_0-\epsilon}) \end{aligned}$$

where $\alpha_0 = 1 - \frac{U_\theta(-1)}{2}$.

If $U_\theta(-1) = 0$, there exist some constants a_1, a_2 and b_1, b_2, b_3 such that for any $\epsilon > 0$,

$$U_\theta(x) = a_1(1+x)\ln(1+x) + a_2(1+x) + O((1+x)^{2-\epsilon}),$$

$$U_\phi(x) = U_\phi(-1) + b_1(1+x) + b_2(1+x)^2\ln(1+x) + b_3(1+x)^2 + O((1+x)^{3-\epsilon}).$$

Proof. Let $I(x)$ be defined by (4.4). The first equation of (4.1) can be written as

$$(1-x^2)U_\theta' + 2xU_\theta + \frac{1}{2}U_\theta^2 = \lambda + h(x),$$

where by Lemma 4.0.1, λ is a constant , $\lambda = -2U_\theta(-1) + \frac{1}{2}U_\theta^2(-1) = -\frac{\tau_1\tau_2}{2}$, and $h(x) = -I(x) + I(-1) + c_1(1+x) + c_2(1+x)^2$ for some constants c_1 and c_2.

Since $U_\theta(-1) < 2$, there exist $\delta_1, \epsilon > 0$ such that $\dfrac{U_\theta(x)}{1-x} \le \dfrac{U_\theta(-1)+\epsilon}{2} < 1$ for $-1 < x \le -1+\delta_1$.

For convenience denote $\tau_1 = U_\theta(-1)$ and let $\tau_2 = 4 - U_\theta(-1)$. It follows from (4.2) that for some constant C_1, $|U_\phi'| \le C_1(1+x)^{-\frac{\tau_1+\epsilon}{2}}$ and $|U_\phi(x)| \le C_1$ for $-1 < x < -1+\delta_1$. Then $I'''(x) = O((1+x)^{-1-\frac{\tau_1+\epsilon}{2}})$. Therefore both $I(-1)$ and $I'(-1)$ exist and are finite, and $I(x) = I(-1) + I'(-1)(1+x) + O((1+x)^2) + O((1+x)^{2-\frac{\tau_1}{2}-\epsilon})$. So $h(x) = (c_1 - I'(-1))(1+x) + O((1+x)^2) + O((1+x)^{2-\frac{\tau_1}{2}-\epsilon})$.

Rewriting the above equation as

$$(1-x^2)(U_\theta - \tau_1)' + \frac{1}{2}(U_\theta - \tau_1)(U_\theta - \tau_2) = \tilde{h}(x) := h(x) - 2(1+x)U_\theta.$$

Let $V := U_\theta - \tau_1$, $B := \frac{U_\theta - \tau_2}{2(1-x^2)}$, $H := \frac{\tilde{h}}{1-x^2}$. It can be checked that $B, H \in C(-1, -1+\delta]$, $H \in L^\infty(-1, -1+\delta)$ and $\lim_{x \to -1^+}(1+x)B(x) = \frac{\tau_1 - \tau_2}{4} = -\alpha_0 < 0$, and V, B, H satisfy

$$V'(x) + B(x)V(x) = H(x), \quad -1 < x < -1 + \delta.$$

So we can apply Lemma 6.1.7 with $\beta = \alpha_0$ and $b = 1$ to obtain $U_\theta - \tau_1 = O((1 + x)^{\min\{\alpha_0, 1\} - \epsilon})$ for any $\epsilon > 0$.

Next, use this estimate in (4.2), we have $U'_\phi = O(1)(1+x)^{-\frac{\tau_1}{2}}$. So $U_\phi = U_\phi(-1) + O(1)(1+x)^{1-\frac{\tau_1}{2}}$ and $I(x) = I(-1) + I'(-1)(1+x) + O((1+x)^{2-\frac{\tau_1}{2}-\epsilon})$ for any $\epsilon > 0$. Then by the estimate of $I(x)$ and U_θ, notice $\alpha_0 = 1 - \frac{\tau_1}{2}$, there is some constant d_1 such that $\tilde{h}(x) = d_1(1+x) + O((1+x)^{1+\min\{\alpha_0,1\}-\epsilon})$ for any $\epsilon > 0$. So $H = d_1 + O((1+x)^{\min\{\alpha_0,1\}-\epsilon})$. Moreover,

$$(1+x)B + \alpha_0 = O((1+x)^{\min\{\alpha_0,1\}-\epsilon}).$$

So we can apply Lemma 6.1.9 . If $\alpha_0 \neq 1$, there exist some constants a_1, a_2 such that

$$U_\theta - \tau_1 = a_1(1+x)^{\alpha_0} + a_2(1+x) + O((1+x)^{1+\min\{\alpha_0,1\}-\epsilon}) + O((1+x)^{\alpha_0+\min\{\alpha_0,1\}-\epsilon}).$$

Then by (4.2), we have estimate of U'_ϕ and $U_\phi(-1)$ exists and finite, and there exist some constants b_1, b_2, b_3 such that

$$\begin{aligned} U_\phi =& U_\phi(-1) + b_1(1+x)^{\alpha_0} + b_2(1+x)^{2\alpha_0} + b_3(1+x)^{1+\alpha_0} \\ &+ O((1+x)^{\alpha_0+1+\min\{\alpha_0,1\}-\epsilon}) + O((1+x)^{2\alpha_0+\min\{\alpha_0,1\}-\epsilon}) \end{aligned}$$

for any $\epsilon > 0$.

If $\alpha_0 = 1$, $U_\theta(-1) = 0$, there exist some constants a_1, a_2 such that

$$U_\theta = a_1(1+x)\ln(1+x) + a_2(1+x) + O((1+x)^{1+\min\{\alpha_0,1\}-\epsilon})$$

By (4.2), $U_\phi(-1)$ exists and there exist some constants b_1, b_2, b_3 such that

$$U_\phi = U_\phi(-1) + b_1(1+x) + b_2(1+x)^2\ln(1+x) + b_3(1+x)^2 + O((1+x)^{2+\min\{\alpha_0,1\}-\epsilon})$$

for any $\epsilon > 0$. $\square$

Lemma 4.0.3. *Let $U = (U_\theta, U_\phi)$ be a solution of system (4.1), and $U_\theta \in C^1(-1, -1+\delta]$, $U_\phi \in C^2(-1, -1+\delta]$, for some $\delta > 0$ and $x_0 \in (-1, -1+\delta]$, with $U_\theta(-1) = 2$. Then for some constants b_1 and b_2, and for any $\epsilon \in (0,1)$, either*

$$\begin{aligned} U_\theta &= 2 + \frac{4}{\ln(1+x)} + O((\ln(1+x))^{-2+\epsilon}), \\ U_\phi &= U_\phi(-1) + \frac{b_1}{\ln(1+x)} + O((\ln(1+x))^{-2+\epsilon}), \end{aligned} \tag{4.8}$$

or

$$\begin{aligned} U_\theta &= 2 + O((1+x)^{1-\epsilon}), \\ U_\phi &= b_1 \ln(1+x) + b_2 + b_1 O((1+x)^{1-\epsilon}). \end{aligned} \tag{4.9}$$

Proof. Let I be the triple integral defined by (4.4). The equation (4.3) can be written as

$$(1-x^2)(U_\theta - 2)' + \frac{1}{2}(U_\theta - 2)^2 = \tilde{h} := -I(x) + c_1 x^2 + c_2 x + c_3 + 2 - 2(1+x)U_\theta. \tag{4.10}$$

Since $U_\theta(-1) = 2$, for any $\epsilon > 0$,

$$|U_\phi'| \le C(1+x)^{-\frac{2+\epsilon}{2}},$$

and $|U_\phi| \le C(1+x)^{-\frac{\epsilon}{2}}$ for some constant $C > 0$. Thus $I(x) = I(-1) + O((1+x)^{1-\epsilon})$. So $\tilde{h} = O((1+x)^{1-\epsilon})$.

By (4.10), $g := (U_\theta - 2)\ln(1+x)$ satisfies

$$(1-x^2)\ln(1+x) g' - (1-x)g + \frac{1}{2}g^2 = \tilde{h}(x)(\ln(1+x))^2.$$

By Proposition 6.1.1 , $g \in L^\infty(-1, -1+\frac{\delta}{2})$, $\lim_{x\to -1^+} g(x)$ exists and is finite, $\lim_{x\to -1^+}(1-x^2)\ln(1+x)g' = 0$, and $-2g(1) + \frac{1}{2}g^2(1) = 0$. So $g(1) = 0$ or 4

Let us write

$$U_\theta(x) = 2 + \frac{\eta}{\ln(1+x)} + V.$$

We can see that $\eta = 0$ or 4, $V(-1) = 0$ and $V = o(\frac{1}{\ln(1+x)})$.

By (4.10), V satisfies

$$(1-x^2)V' + \frac{\eta}{\ln(1+x)}V + \frac{1}{2}V^2 = \hat{h},$$

where $\hat{h} := -I(x) + c_1x^2 + c_2x + c_3 - \frac{\frac{1}{2}\eta^2 - \eta(1-x)}{(\ln(1+x))^2} - 2(1+x)V - 4x - 2 - \frac{2\eta(1+x)}{\ln(1+x)}$. Since $\eta = 0$ or 4, $\hat{h} = O((1+x)^{1-\epsilon})$.

Let $B = \frac{\frac{1}{2}V + \frac{\eta}{\ln(1+x)}}{1-x^2}$, $H(x) = \frac{\hat{h}}{1-x^2}$. Then $B, H \in C(-1, -1+\delta]$ satisfy $H(x) = O((1+x)^{-\epsilon})$, $\lim_{x\to -1^+}(1+x)\ln(1+x)B = \frac{\eta}{2}$, $V = o(\frac{1}{\ln(1+x)})$. So we can apply Lemma 6.1.11 to conclude that $V = O((\ln(1+x))^{-2+\epsilon})$ if $\eta = 4$ and $V = O((1+x)^{1-\epsilon})$ if $\eta = 0$. We have established the estimates of U_θ in (4.8) and (4.9).

With estimates of U_θ in (4.8) and (4.9), we obtain from (4.2) the estimates of U_ϕ in (4.8) and (4.9). The lemma is proved. □

Remark 4.0.1. *This case does occur. For example, as given by Corollary* 3.1.1*, for all* $\gamma > -1$*,* $(U_\theta, U_\phi) = ((1-x)(1 + \frac{2(\gamma+1)}{(\gamma+1)\ln\frac{1+x}{2} - 2}), 0)$ *are smooth solutions on* $\mathbb{S}^2 \setminus \{S\}$*.*

Lemma 4.0.4. *Let* $U = (U_\theta, U_\phi)$ *be a solution of the system (*4.1*), and* $U_\theta \in C^1(-1, -1+\delta]$*,* $U_\phi \in C^2(-1, -1+\delta]$*, for some* $\delta > 0$ *and* $x_0 \in (-1, -1+\delta]$*. If* $2 < U_\theta(-1) < 3$*, there exist constants* a_1, a_2 *and* b_1, b_2, b_3, b_4 *such that for any* $\epsilon > 0$*,*

$$\begin{aligned} U_\theta(x) =& U_\theta(-1) + a_1(1+x)^{3-U_\theta(-1)} + a_2(1+x) + O((1+x)^{2(3-U_\theta(-1))-\epsilon}), \\ U_\phi(x) =& b_1(1+x)^{1-\frac{U_\theta(-1)}{2}} + b_2 + b_1b_3(1+x)^{4-\frac{3U_\theta(-1)}{2}} + b_1b_4(1+x)^{2-\frac{U_\theta(-1)}{2}} \\ &+ b_1O((1+x)^{7-\frac{5U_\theta(-1)}{2}-\epsilon}). \end{aligned} \tag{4.11}$$

Proof. Let $\tau_2 = U_\theta(-1)$, and $I(x)$ be the triple integral defined by (4.4). Using the fact $2 < U_\theta(-1) < 3$ and (4.2), for any $\epsilon > 0$, there exists some constant C_1 such that $|U_\phi'(x)| \le C_1(1+x)^{-\frac{\tau_2+\epsilon}{2}}$. Then by (4.4) we obtain that in the current situation $I(x) = I(-1) + O((1+x)^{3-\tau_2-\epsilon})$. So U_θ satisfies

$$(1-x^2)(U_\theta - \tau_2)' + \frac{1}{2}(U_\theta - \tau_1)(U_\theta - \tau_2) = \tilde{h} := -I(x) + I(-1) + c_1(1+x) + c_2(1+x)^2 - 2(1+x)U_\theta$$

where c_1, c_2 are constants. By the estimate of $I(x)$, $\tilde{h} = O((1+x)^{3-\tau_2-\epsilon})$. Let $V = U_\theta - \tau_2$, $B = \frac{U_\theta - \tau_1}{2(1-x^2)}$, $H = \frac{\tilde{h}}{1-x^2}$. Then $V \in C^1(-1, -1+\delta]$, $B, H \in C(-1, -1+\delta]$, satisfy $V' + BV = H$, and $H(x) = O((1+x)^{2-\tau_2-\epsilon})$, $\lim_{x\to -1^+}(1+x)B = \alpha_0 > 0$, and $\lim_{x\to -1^+} V(x)e^{\int_{-1+\delta}^x B(s)ds} = 0$. So we can apply Lemma 6.1.8 to obtain

$$U_\theta(x) - \tau_2 = O((1+x)^{3-\tau_2-\epsilon}).$$

With this estimate, we derive from (4.2) that $U_\phi' = C(1+x)^{-\frac{\tau_2}{2}}(1+O((1+x)^{3-\tau_2-\epsilon}))$. So $U_\phi = \frac{2}{2-\tau_2}C(1+x)^{1-\frac{\tau_2}{2}}(1+O((1+x)^{3-\tau_2-\epsilon}))$ and $I(x) = I(-1)+c_1'(1+x)^{3-\tau_2}+c_2'(1+x)+O((1+x)^{2(3-\tau_2)-\epsilon})$ for some constants c_1', c_2'. Let $\bar{b} = 3-\tau_2$. Then by the estimate of $I(x)$ and U_θ, there is some constant d_1 such that $\tilde{h}(x) = c_1'(1+x)^{\bar{b}}+d_1(1+x)+O((1+x)^{2\bar{b}-\epsilon})$. So $H = c_1'(1+x)^{\bar{b}-1} + d_1 + O((1+x)^{2\bar{b}-1-\epsilon})$. Moreover,

$$(1+x)B - \alpha_0 = O((1+x)^{\bar{b}-\epsilon}).$$

So we can apply Lemma 6.1.10 to obtain the first estimate of U_θ in (4.11). Then by (4.2), we have the estimate of U_ϕ in (4.11), using the first estimate in (4.11). $\square$

Part (iii) of Theorem 1.0.3 and part (i), (ii) and (iv) of Theorem 1.0.4 follow from Lemma 4.0.2-4.0.4. So Theorem 1.0.3 is proved. Next let us prove part (iii) of Theorem 1.0.4.

Lemma 4.0.5. *If $U = (U_\theta, U_\phi)$ is a solution of (4.1) and $U_\theta \in C^1(-1,-1+\delta)$, $0<\delta<2$, $U_\theta(-1) \geq 3$, then U_ϕ is a constant in $(-1,-1+\delta)$.*

Proof. We prove it by contradiction. Assume that U_ϕ is not a constant, then (4.2) holds for a nonzero constant C and we may assume that C is positive. Let $I(x)$ be given by (4.4) with $x_0 = -1+\delta$. Since U_θ and $(1-x^2)U_\theta'$ are bounded according to Theorem 1.0.3, $I(x)$ is bounded in view of (4.1). We divide the proof into two cases.

Case 1. $U_\theta(-1) > 3$.

If $U_\theta(-1) > 3$, there exist $a > 3$ such that $U_\theta(x) > a > 3$ for x close to -1. So by (4.2), there exists $c > 0$ such that $U_\phi' \geq c(1+x)^{-\frac{a}{2}}$ and $-U_\phi \geq c(1+x)^{-\frac{a}{2}+1}$ for x close to -1 . Then, using (4.4), we have $-I(x) \to +\infty$ as $x \to -1^+$, a contradiction.

Case 2. $U_\theta(-1) = 3$.

Since $U_\theta(-1) = 3$, we rewrite the first line of (4.1) as

$$(1-x^2)(U_\theta-3)' + \frac{1}{2}(U_\theta-1)(U_\theta-3) = \tilde{h}(x) := -2(1+x)U_\theta + Q(x) + I_1(-1) - I_1(x),$$

where I_1 is given by (4.6) , and $Q(x)$ is a quadratic polynomial with $Q(-1) = 0$.

By (4.7), $I_1(-1) - I_1(x) \geq 0$ in $(-1,-1+\delta)$. Thus, using the boundedness of U_θ and the fact that $Q(-1) = 0$, $\tilde{h}(x) \geq -C(1+x)$ in $(-1,-1+\delta)$ for some constant $C > 0$.

Let $V(x) = U_\theta(x) - 3$, $B(x) = \frac{U_\theta - 1}{2(1-x^2)}$ and $H(x) = \frac{\tilde{h}(x)}{1-x^2}$. Then (6.6), (6.7), (6.13) and (6.11) hold with $b = 1$, $\beta = -\frac{1}{2}$. By Lemma 6.1.8, see also Remark 6.1.2, we have, for some positive constant C, and for any $\epsilon > 0$, $U_\theta - 3 \geq -C(1+x)^{1-\epsilon}$ in $(-1, -1+\delta)$.

Next, in (4.2), apply the estimate of $U_\theta(x)$, in $(-1, -1+\delta)$ there is

$$U'_\phi(x) \geq ce^{-\frac{3}{2}\ln(1+x)} \geq c(1+x)^{-\frac{3}{2}}, \text{ for } x \text{ close to } -1.$$

Then $-U_\phi(x) \geq c(1+x)^{-\frac{1}{2}}$ for x close to -1.

$$-I'''(x) = -\frac{2U_\phi(x)U'_\phi(x)}{1-x^2} \geq C(1+x)^{-3}.$$

Thus $I \geq C|\ln(1+x)|$ is unbounded, contradiction. So U_ϕ is a constant. $\square$

Completion of the proof of Theorem 1.0.2*:* We have proved the existence part of the theorem in Chapter 3 for $(\tau, \sigma) \in J_1 \cup J_2 \cup (J_3 \cap \{2 \leq \tau < 3\})$. Now we prove the nonexistence part of the theorem.

For $(\tau, \sigma) \in J_3 \cap \{\tau > 3\}$, let $\{u^i\}$ be a sequence of solutions of (1.1) satisfying $|| \sin\frac{\theta+\pi}{2}(u^i - u_{\tau,\sigma})||_{L^\infty(\mathbb{S}^2 \setminus \{S\})} \to 0$ as $i \to \infty$. Let $U^i = \sin\theta u^i$ for all $i \in \mathbb{N}$. Recall that $U^{\mu,\gamma} = \sin\theta u_{\tau,\sigma}$ with $(\mu, \gamma) = (\frac{1}{8}\tau^2 - \frac{1}{2}\tau, -2\sigma)$. We have $||U^i_\theta - U^{\mu,\gamma}_\theta||_{L^\infty(-1,1]} \to 0$. By Theorem 1.0.3 part (a), $U^i(-1)$ must exists and is finite for every i. Since $U^{\mu,\gamma}(-1) > 3$, $U^i_\theta(-1) > 3$ for large i. Then by Theorem 1.0.4, U^i_ϕ must be constant for large i. Since $u^i \in C^\infty(\mathbb{S}^2 \setminus \{S\})$, $U^i_\phi(1) = 0$, so $U^i_\phi = 0$ for large i. The theorem is proved. $\square$

As stated in Chapter 1, we also have similar results for solutions in a punctured ball near the north pole. By making the transformation $\tilde{U}(x) = -U(-x)$ and applying Theorem 1.0.3 and Theorem 1.0.4 to $\tilde{U}$, we have the following results.

Theorem 4.0.1'. *For $\delta > 0$, let $U_\theta \in C^1(1-\delta, 1]$, $U_\phi \in C^2(1-\delta, 1]$, and $U = (U_\theta, U_\phi)$ be an axisymmetric solution of (*1.1*). Then*

(i) $U_\theta(1) := \lim_{x\to 1^-} U_\theta(x)$ exists and is finite.

(ii) $\lim_{x\to 1^-}(1-x)U'_\theta(x) = 0$.

(iii) If $U_\theta(1) > -2$ and $U_\theta(1) \neq 0$, denote $\alpha_0 = 1 + \frac{U_\theta(1)}{2}$, then there exist some constants a_1, a_2 such that for every $\epsilon > 0$,

$$U_\theta(x) = U_\theta(1) + a_1(1-x)^{\alpha_0} + a_2(1-x) + O((1-x)^{2\alpha_0-\epsilon}) + O((1-x)^{2-\epsilon}).$$

If $U_\theta(1) = 0$, then there exist some constants a_1, a_2 such that for every $\epsilon > 0$,

$$U_\theta(x) = a_1(1-x)\ln(1-x) + a_2(1+x) + O((1-x)^{2-\epsilon}).$$

If $U_\theta(1) = -2$, then, for every $\epsilon > 0$, either

$$U_\theta(x) = -2 - \frac{4}{\ln(1-x)} + O((\ln(1-x))^{-2+\epsilon}),$$

or

$$U_\theta(x) = -2 + O((1-x)^{1-\epsilon}).$$

If $-3 < U_\theta(1) < -2$, then there exist constants a_1, a_2 such that for every $\epsilon > 0$,

$$U_\theta(x) = U_\theta(1) + a_1(1-x)^{3+U_\theta(-1)} + a_2(1-x) + O((1-x)^{2(3+U_\theta(1))-\epsilon}).$$

Theorem 4.0.2'. *For $\delta > 0$, let $U_\theta \in C^1(1-\delta, 1)$, $U_\phi \in C^2(1-\delta, 1)$, and $U = (U_\theta, U_\phi)$ be an axisymmetric solution of (1.1). Then*

(i) If $U_\theta(1) > -2$, then $U_\phi(1)$ exists and is finite, and there exist some constants b_1, b_2, b_3 such that

$$U_\phi(x) = \begin{cases} U_\phi(1) + b_1(1-x)^{\alpha_0} + b_2(1-x)^{2\alpha_0} + b_3(1-x)^{1+\alpha_0} \\ \quad + O((1-x)^{\alpha_0+2-\epsilon}) + O((1-x)^{3\alpha_0-\epsilon}), & \text{if } U_\theta(1) \neq 0; \\ U_\phi(1) + b_1(1-x) + b_2(1-x)^2\ln(1-x) + b_3(1-x)^2 \\ \quad + O((1-x)^{3-\epsilon}), & \text{if } U_\theta(1) = 0. \end{cases}$$

(ii) If $-3 < U_\theta(1) < -2$, then there exist some constants b_1, b_2, b_3, b_4 such that

$$\begin{aligned} U_\phi(x) =& b_1(1-x)^{1+\frac{U_\theta(1)}{2}} + b_2 + b_1 b_3(1-x)^{4+\frac{3U_\theta(1)}{2}} + b_1 b_4(1-x)^{2+\frac{U_\theta(1)}{2}} \\ &+ b_1 O((1-x)^{7+\frac{5U_\theta(1)}{2}-\epsilon}). \end{aligned}$$

In particular, U_ϕ is either a constant or an unbounded function in $(1-\delta, 1)$.

(iii) If $U_\theta(1) \leq -3$, then U_ϕ must be a constant in $(1-\delta, 1)$.

(iv) If $U_\theta(1) = -2$, then $\eta := \lim_{x\to 1^-}(U_\theta + 2)\ln(1-x)$ exists and is 0 or -4. If $\eta = 0$, then U_ϕ is either constant or unbounded, and there exist some constants b_1, b_2 such that

$$U_\phi = b_1\ln(1-x) + b_2 + b_1 O((1-x)^{1-\epsilon}).$$

If $\eta = -4$, then U_ϕ is in $L^\infty(1-\delta, 1)$, and there exists some constant b such that

$$U_\phi = U_\phi(1) + \frac{b}{\ln(1-x)} + O((\ln(1-x))^{-2+\epsilon}).$$

Chapter 5

(-1)-homogeneous axisymmetric solutions on $\mathbb{S}^2 \setminus \{S, N\}$

5.1 Classification of axisymmetric no-swirl solutions on $\mathbb{S}^2 \setminus \{S, N\}$

In this section, we will prove Theorem 1.0.5, which classifies all (-1)-homogeneous axisymmetric no-swirl $C^\infty(\mathbb{S}^2 \setminus \{S, N\})$ solutions of (1.1).

By arguments used in Chapter 2, the NSE equations (1.1) of a (-1)-homogeneous axisymmetric no-swirl solution can be reduced to

$$(1-x^2)U_\theta' + 2xU_\theta + \frac{1}{2}U_\theta^2 = c_1(1-x) + c_2(1+x) + c_3(1-x^2), \tag{5.1}$$

for some constants c_1, c_2, c_3, where "′" denote differentiation in x.

Denote $c := (c_1, c_2, c_3)$, and

$$P_c(x) := c_1(1-x) + c_2(1+x) + c_3(1-x^2). \tag{5.2}$$

We will show that the existence of solutions of (5.1) in $C^1(-1,1)$ depends on the constants c_1, c_2 and c_3.

Define

$$\tau_1(c_1) := 2 - 2\sqrt{1+c_1}, \quad \tau_2(c_1) := 2 + 2\sqrt{1+c_1}, \tag{5.3}$$

$$\tau_1'(c_2) := -2 - 2\sqrt{1+c_2}, \quad \tau_2'(c_2) := -2 + 2\sqrt{1+c_2}, \tag{5.4}$$

Lemma 5.1.1. *Let $\delta > 0$, $U_\theta \in C^1(-1,-1+\delta)$ satisfy (5.1) with $c_1, c_2, c_3 \in \mathbb{R}$. Then $c_1 \geq -1$ and $U_\theta(-1) := \lim_{x\to -1^+} U_\theta(x)$ exists and is finite. Moreover,*

$$U_\theta(-1) = \tau_1(c_1) \quad or \quad \tau_2(c_1).$$

Proof. By Theorem 1.0.3, $\lim_{x\to-1^+} U_\theta(x)$ exists and is finite and

$$\lim_{x\to -1^+}(1+x)U_\theta'(x) = 0.$$

Sending x to -1 in (5.1) leads to

$$-2U_\theta(-1) + \frac{1}{2}U_\theta(-1)^2 = 2c_1.$$

Thus,

$$c_1 = \frac{1}{4}\left[U_\theta(-1) - 2\right]^2 - 1 \geq -1,$$

and $U_\theta(-1) = \tau_1(c_1)$ or $\tau_2(c_1)$. □

Lemma 5.1.1'. *Let $\delta > 0$, $U_\theta \in C^1(1-\delta, 1)$ satisfy (5.1) with $c_1, c_2, c_3 \in \mathbb{R}$. Then $c_2 \geq -1$ and $U_\theta(1) := \lim_{x\to 1^-} U_\theta(x)$ exists and is finite. Moreover,*

$$U_\theta(1) = \tau_1'(c_2) \quad \textit{or} \quad \tau_2'(c_2).$$

Proof. Consider $\tilde{U}_\theta(x) := -U_\theta(-x)$, and apply Lemma 5.1.1 to $\tilde{U}_\theta$. □

Recall that in Chapter 1 we defined

$$\bar{c}_3(c_1, c_2) := -\frac{1}{2}\left(\sqrt{1+c_1} + \sqrt{1+c_2}\right)\left(\sqrt{1+c_1} + \sqrt{1+c_2} + 2\right). \tag{5.5}$$

and the set

$$D := \{c \in \mathbb{R}^3 | c_1 \geq -1, c_2 \geq -1, c_3 \geq \bar{c}_3(c_1, c_2)\}. \tag{5.6}$$

Moreover, the subsets D_k, $1 \leq k \leq 8$ are defined as

$$\begin{aligned}
&D_1 := \{c | c_1 > -1, c_2 > -1, c_3 > \bar{c}_3\}, \quad D_2 := \{c | c_1 = -1, c_2 > -1, c_3 > \bar{c}_3\},\\
&D_3 := \{c | c_1 > -1, c_2 = -1, c_3 > \bar{c}_3\}, \quad D_4 := \{c | c_1 = -1, c_2 = -1, c_3 > \bar{c}_3\},\\
&D_5 := \{c | c_1 > -1, c_2 > -1, c_3 = \bar{c}_3\}, \quad D_6 := \{c | c_1 = -1, c_2 > -1, c_3 = \bar{c}_3\},\\
&D_7 := \{c | c_1 > -1, c_2 = -1, c_3 = \bar{c}_3\}, \quad D_8 := \{c | c_1 = -1, c_2 = -1, c_3 = \bar{c}_3\}.
\end{aligned} \tag{5.7}$$

Theorem 5.1.1. *There exist $\gamma^-, \gamma^+ \in C^\infty(D_k) \cap C^0(D)$, $1 \leq k \leq 8$, satisfying $\gamma^-(c) \leq \gamma^+(c)$ for all $c \in D$, where the equality holds if and only if $c \in \cup_{k=5}^8 D_k$, such that for each (c, γ) in the set*

$$E := \{(c, \gamma) \mid c_1 \geq -1, c_2 \geq -1, c_3 \geq \bar{c}_3(c_1, c_2), \gamma^-(c) \leq \gamma \leq \gamma^+(c)\},$$

there exists a unique C^1 solution $U_\theta^{c,\gamma}$ of (5.1) in $(-1, 1)$ satisfying $U_\theta^{c,\gamma}(0) = \gamma$. Moreover, these are all solutions of (5.1) in $C^1(-1, 1)$.

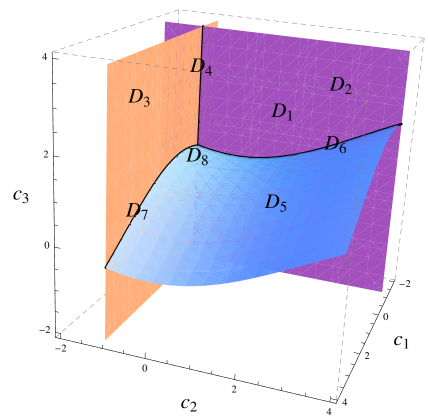

Figure 5.1: The parameter space D.

Though the axisymmetric no-swirl solution $\{U_\theta^{c,\gamma}\}$ do not have explicit formulas, they have the following nice properties.

Theorem 5.1.2. *Suppose* $(c,\gamma)\in E$, *then*

(i) If $c_3=\bar{c}_3(c_1,c_2)$, *then* $\gamma^-(c)=\gamma^+(c)$, *and equation* (5.1) *has a unique* $C^1(-1,1)$ *solution*

$$U_\theta^*(c_1,c_2)=\frac{\tau_2(c_1)}{2}(1-x)+\frac{\tau_1'(c_2)}{2}(1+x).$$

(ii) If $c_3>\bar{c}_3(c_1,c_2)$, *then* $\gamma^-(c)<\gamma^+(c)$, *and for any* $\gamma^-(c)\le\gamma<\gamma'\le\gamma^+(c)$, $U_\theta^{c,\gamma}<U_\theta^{c,\gamma'}$ *in* $(-1,1)$. *Moreover,*

$$\{(x,y)\mid -1<x<1, U_\theta^{c,\gamma^-(c)}(x)\le y\le U_\theta^{c,\gamma^+(c)}(x)\}=\bigcup_{\gamma\in[\gamma^-(c),\gamma^+(c)]}\{(x,U_\theta^{c,\gamma}(x)\mid -1<x<1\}$$

(iii) For any $(c,\gamma)\in I$, $U_\theta^{c,\gamma}(\pm1)$ *both exist and are finite. Moreover,*

$$U_\theta^{c,\gamma}(-1):=\begin{cases}\tau_2(c_1), & \text{when } \gamma=\gamma^+(c),\\ \tau_1(c_1), & \text{otherwise,}\end{cases}\qquad U_\theta^{c,\gamma}(1):=\begin{cases}\tau_1'(c_2), & \text{when } \gamma=\gamma^-(c),\\ \tau_2'(c_2), & \text{otherwise.}\end{cases}\tag{5.8}$$

Remark 5.1.1. *Theorem* 5.1.1 *gives a classification of all (-1)-homogeneous, axisymmetric, no swirl solutions of Navier-Stokes equations* (1.1) *in* $C^\infty(\mathbb{S}^2\setminus\{S,N\})$.

Remark 5.1.2. *Landau solutions correspond to* $c_1=0, c_2=0, c_3=0>\bar{c}_3(0,0)=-4, \gamma\in(-2,2)$, *and* $\gamma\neq0$.

Now let us define the following subsets of E: for $1 \le k \le 4$, let

$$F_{k,1} := \{(c,\gamma) \in E | c \in D_k, \gamma = \gamma^+(c)\},$$

$$F_{k,2} := \{(c,\gamma) \in E | c \in D_k, \gamma = \gamma^-(c)\},$$

$$F_{k,3} := \{(c,\gamma) \in E | c \in D_k, \gamma^-(c) < \gamma < \gamma^+(c)\}$$

and for $5 \le k \le 8$, let $F_{k,l} := D_k$, $l = 1,2,3$.

The solution $U_\theta^{c,\gamma}$ of equation (5.1) satisfying $U_\theta^{c,\gamma}(0) = \gamma$ has the following property:

Theorem 5.1.3. *Let K be a compact set contained in one of $F_{k,l}$, $1 \le k \le 8$, $l = 1,2,3$. Then the solution $U_\theta^{c,\gamma}$ of equation (5.1) is $C^\infty(K \times (-1,1))$. Moreover,*

(i) If $5 \le k \le 8$, or $k = 1$ and $l = 1,2,3$, or $(k,l) = (2,1)$ or $(3,2)$, then

$$|\partial_c^\alpha \partial_\gamma^j \bar{U}_\theta| \le C(m,K), \quad \textit{for any } 0 \le |\alpha| + j \le m, -1 < x < 1. \tag{5.9}$$

where $j = 0$ if $l = 1,2$, $\alpha_3 = 0$ if $5 \le k \le 8$; $\alpha_1 = 0$ if $k = 2,4$; and $\alpha_2 = 0$ if $k = 3,4$.

(ii) If $(k,l) = (2,2)$ or $(2,3)$ or $(4,2)$, then

$$\left(\ln \frac{1+x}{3}\right)^2 |\partial_c^\alpha \partial_\gamma^j \bar{U}_\theta| \le C(m,K), \quad \textit{for any } 1 \le |\alpha| + j \le m, \alpha_1 = 0, -1 < x < 1. \tag{5.10}$$

where $j = 0$ if $l = 2$, and $\alpha_2 = 0$ if $k = 4$.

(iii) If $(k,l) = (3,1), (3,3)$ or $(4,1)$, then

$$\left(\ln \frac{1-x}{3}\right)^2 |\partial_c^\alpha \partial_\gamma^j \bar{U}_\theta| \le C(m,K), \quad \textit{for any } 1 \le |\alpha| + j \le m, \alpha_2 = 0, -1 < x < 1. \tag{5.11}$$

where $j = 0$ if $l = 1$, and $\alpha_1 = 0$ if $k = 4$.

(iv) If $(k,l) = (4,3)$, then

$$\left(\ln \frac{1+x}{3}\right)^2 \left(\ln \frac{1-x}{3}\right)^2 |\partial_c^\alpha \partial_\gamma^j \bar{U}_\theta| \le C(m,K), \quad \textit{for any } 1 \le |\alpha| + j \le m, \alpha_1 = \alpha_2 = 0, -1 < x < 1. \tag{5.12}$$

Theorem 1.0.5 is a direct consequence of Theorem 5.1.1. Now let us turn to the proof of Theorem 5.1.1. In order to avoid the redundancy of the subscript in U_θ, we consider equation

$$(1-x^2)f' + 2xf + \frac{1}{2}f^2 = c_1(1-x) + c_2(1+x) + c_3(1-x^2). \tag{5.13}$$

instead of (5.1).

We first prove the following lemmas on local behaviors near $x = -1$ or $x = 1$. Roughly speaking, local existence of real analytic solutions is proved in Lemma 5.1.2 and Lemma 5.1.2'. Corollary 5.1.1 and Corollary 5.1.1' give the uniqueness of solution satisfying $f(-1) > 2$ or $f(1) < -2$. While Lemma 5.1.3 and Lemma 5.1.3' present some local comparison results.

Lemma 5.1.2. *Let $c_1 \geq -1$, $\tau = \tau_2(c_1)$ or $\tau = \tau_1(c_1) \notin \{0, -2, -4, -6, \cdots\}$. Then for every $c_2, c_3 \in \mathbb{R}$, there exist $\delta > 0$ depending only on an upper bound of $\sum_{i=1}^{3} |c_i|$ and a positive lower bound of $\inf_{k\in\mathbb{N}} |\tau + 2k|$, and a sequence $\{a_n\}_{n=1}^{\infty}$ depending only on c_1, c_2, c_3 and τ, such that*

$$\sum_{n=1}^{\infty} |a_n|\delta^n < \infty,$$

and

$$f(x) := \tau + \sum_{n=1}^{\infty} a_n(1+x)^n$$

is a real analytic solution of (5.13) in $(-1, -1+\delta)$. Moreover, f is the unique real analytic solution of (5.13) in $(-1, -1+\delta')$ satisfying $f(-1) = \tau$ for any $0 < \delta' < \delta$.

Proof. Let $s = 1 + x$. Rewrite

$$P(x) = 2c_1 + (-c_1 + c_2 + 2c_3)(1+x) - c_3(1+x)^2 =: \tilde{c}_1 + \tilde{c}_2 s + \tilde{c}_3 s^2.$$

Suppose $f = \tau + \sum_{n=1}^{\infty} a_n s^n$, then $f' = \sum_{n=1}^{\infty} n a_n s^{n-1}$. Plug them into (5.13),

$$\begin{aligned}
&\text{LHS}\\
&= s(2-s)\sum_{n=1}^{\infty} n a_n s^{n-1} + 2(s-1)(\tau + \sum_{n=1}^{\infty} a_n s^n) + \frac{1}{2}(\tau + \sum_{n=1}^{\infty} a_n s^n)^2\\
&= \frac{1}{2}\tau^2 - 2\tau + ((2+a_1)\tau)s + \sum_{n=2}^{\infty}[(2n-2+\tau)a_n + (3-n)a_{n-1} + \frac{1}{2}\sum_{k+l=n,k,l\geq 1} a_k a_l]s^n\\
&= \tilde{c}_1 + \tilde{c}_2 s + \tilde{c}_3 s^2 = \text{RHS}
\end{aligned}$$

Compare coefficients,

$$\begin{aligned}
&n = 0, \quad \tfrac{1}{2}\tau^2 - 2\tau = \tilde{c}_1, && \text{so } \tau = 2 \pm \sqrt{4 + 2\tilde{c}_1} = \tau_1(c_1) \text{ or } \tau_2(c_1),\\
&n = 1, \quad (a_1 + 2)\tau = \tilde{c}_2, && \text{so } a_1 = \frac{\tilde{c}_2}{\tau} - 2,\\
&n = 2, \quad (2+\tau)a_2 + a_1 + \tfrac{1}{2}a_1^2 = \tilde{c}_3, && \text{so } a_2 = \frac{1}{\tau+2}(\tilde{c}_3 - a_1 - \frac{1}{2}a_1^2).
\end{aligned}$$

For $n \geq 3$,

$$(2n - 2 + \tau)a_n + (3 - n)a_{n-1} + \frac{1}{2} \sum_{k+l=n,k,l\geq 1} a_k a_l = 0.$$

Since for any $n \geq 1$, $\tau \neq -2(n-1)$,

$$a_n = -\frac{1}{2n - 2 + \tau} \left(\frac{1}{2} \sum_{k+l=n,k,l\geq 1} a_k a_l + (3 - n)a_{n-1} \right), \tag{5.14}$$

it can be seen that a_n is determined by $a_1, ..., a_{n-1}$, thus determined by c_1, c_2, c_3 and τ.

Claim: there exists some $a > 0$ large, depending only on c_1, c_2, c_3 and τ, such that

$$|a_n| \leq a^n.$$

Proof: Choose $a > 1$ large such that for $1 \leq n \leq 100|\tau| + 100$, $|a_n| \leq a^n$.

Now for $n > 100|\tau| + 100$, suppose that for $1 \leq k \leq n-1$, $|a_k| \leq a^n$, then by induction and the recurrence formula (5.14),

$$|a_n| \leq \frac{2}{3(n-1)} \left| \frac{1}{2}(n-1)a^n + (n-3)a^{n-1} \right| \leq \left(\frac{1}{3} + \frac{2(n-3)}{3(n-1)a} \right) a^n \leq a^n.$$

The claim is proved.

So for $\delta < \frac{1}{a}$, $f = \tau + \sum_{n=1}^{\infty} a_n s^n$, with $s = 1 + x$, is a real analytic solution of (5.13) in $(-1, -1 + \delta)$. The uniqueness of f is clear from the proof above. □

Lemma 5.1.2'. *Let $c_2 \geq -1$, $\tau' = \tau_1'(c_2)$ or $\tau' = \tau_2'(c_2) \notin \{0, 2, 4, 6, \cdots\}$. Then for every $c_1, c_3 \in \mathbb{R}$, there exist $\delta > 0$, depending only on an upper bound of $\sum_{i=1}^{3} |c_i|$ and a positive lower bound of $\inf_{k\in\mathbb{N}} |\tau' - 2k|$, and a sequence $\{a_n\}_{n=1}^{\infty}$ depending only on c_1, c_2, c_3 and τ' such that*

$$\sum_{n=1}^{\infty} |a_n| \delta^n < \infty,$$

and

$$f(x) := \tau' + \sum_{n=1}^{\infty} a_n (1 - x)^n$$

is a real analytic solution of (5.13) in $(1 - \delta, 1)$. Moreover, f is the unique real analytic solution of (5.13) in $(1 - \delta', 1)$ satisfying $f(1) = \tau'$ for any $0 < \delta' < \delta$.

The following two lemmas give some local comparison results.

Lemma 5.1.3. *Suppose* $0<\delta<2$, $f_1, f_2 \in C^1(-1,-1+\delta]\cap C^0[-1,-1+\delta]$ *satisfy*

$$(1-x^2)f_1' + 2xf_1 + \frac{1}{2}f_1^2 \geq (1-x^2)f_2' + 2xf_2 + \frac{1}{2}f_2^2, \quad -1<x<-1+\delta.$$

Suppose also that one of the following two conditions holds.

(i) $f_1(-1) \geq f_2(-1) > 2$.

(ii) $f_1(-1) = f_2(-1) = 2$, *and*

$$\limsup_{x\to -1^+} \int_{-1+\delta}^{x} \frac{-2+f_1(s)}{1-s^2} ds < +\infty. \tag{5.15}$$

Then either

$$f_1 > f_2, \quad \textit{in } (-1,-1+\delta),$$

or there exists δ', $0<\delta'<\delta$ *such that*

$$f_1 \equiv f_2, \quad \textit{in } (-1,-1+\delta').$$

Proof. Let $g = f_1 - f_2$, then $g(-1) \geq 0$ and g satisfies

$$g' + b(x)g \geq \frac{1}{2(1-x^2)}g^2 \geq 0, \quad \text{for all } x \in (-1,-1+\delta). \tag{5.16}$$

where $b(x)$ is given by

$$b(x) = (1-x^2)^{-1}(2x+f_1). \tag{5.17}$$

Let

$$w(x) = e^{\int_{-1+\delta}^{x} b(s)ds} g(x).$$

Then w given satisfies, using (5.16), that

$$w'(x) \geq 0 \text{ in } (-1,-1+\delta). \tag{5.18}$$

Under condition either (i) or (ii), we have

$$\limsup_{x\to -1^+} \int_{-1+\delta}^{x} b(s)ds < +\infty.$$

Using this and the fact that $g(-1) \geq 0$, we have $\liminf_{x\to -1^+} w(x) \geq 0$. Therefore, using (5.18), we have either $w>0$ in $(-1,-1+\delta)$ or there exists a constant δ', $0<\delta'<\delta$ such that $w \equiv 0$ in $(-1,-1+\delta')$. The lemma is proved. □

Corollary 5.1.1. *For $c_1 > -1$, $c_2, c_3 \in \mathbb{R}$ and $0 < \delta < 2$, there exists at most one solution f of (5.13) in $C^1(-1, -1+\delta)$ satisfying*

$$\lim_{x \to -1^+} f(x) = \tau_2(c_1).$$

Proof. Since $\tau_2(c_1) > 2$ for $c_1 > -1$, the uniqueness follows from (i) of Lemma 5.1.3.

$\square$

Similarly, we have

Lemma 5.1.3'. *Suppose $0 < \delta < 2$, $f_1, f_2 \in C^1[1-\delta, 1) \cap C^0[1-\delta, 1]$ satisfy*

$$(1-x^2)f_1' + 2xf_1 + \frac{1}{2}f_1^2 \geq (1-x^2)f_2' + 2xf_2 + \frac{1}{2}f_2^2, \quad 1-\delta < x < 1.$$

Suppose also that one of the following two conditions holds.

(i) $f_1(1) \leq f_2(1) < -2$,

(ii) $f_1(1) = f_2(1) = -2$, and

$$\limsup_{x \to 1^-} \int_{1-\delta}^{x} \frac{2 + f_1(s)}{1-s^2} ds < +\infty.$$

Then either

$$f_1 < f_2, \quad \text{in } (1-\delta, 1),$$

or there exists δ', $0 < \delta' < \delta$ such that

$$f_1 \equiv f_2, \quad \text{in } (1-\delta', 1).$$

Corollary 5.1.1'. *For $c_2 > -1$, $c_1, c_3 \in \mathbb{R}$ and $0 < \delta < 2$, there exists at most one solution f of (5.13) in $C^1(1-\delta, 1)$ satisfying*

$$\lim_{x \to 1^-} f(x) = \tau_1'(c_2).$$

Now we are ready to analyze the global behavior of axisymmetric, no-swirl solutions of NSE in $(-1, 1)$. The behavior of solutions depends closely on parameters $c_1, c_2, c_3 \in \mathbb{R}$.

Lemma 5.1.4. *Suppose $c_1 \geq -1$, $c_2 \geq -1$, $c_3 = \bar{c}_3(c_1, c_2)$, then there exists a unique C^1 solution $f^*_{c_1,c_2}$ of (5.13) in $(-1, 1)$. Moreover,*

$$f^*_{c_1,c_2}(x) := ax + b$$

where

$$a = -\left(\sqrt{1+c_1} + \sqrt{1+c_2} + 2\right), \qquad b = \sqrt{1+c_1} - \sqrt{1+c_2},$$

and

$$f^*_{c_1,c_2}(-1) = \tau_2(c_1), \quad f^*_{c_1,c_2}(1) = \tau_1'(c_2).$$

Proof. A direct calculation shows that $f^* := f^*_{c_1,c_2}$ is a C^1 solution of (5.13) in $(-1, 1)$. It remains to prove the uniqueness.

Let f be a C^1 solution of (5.13) in $(-1, 1)$, $f \not\equiv f^*$. By Lemma 5.1.1 and Lemma 5.1.1', f can be extended as a function in $C^0[-1, 1]$, $f(-1) \in \{\tau_1(c_1), \tau_2(c_1)\}$, $f(1) \in \{\tau_1'(c_2), \tau_2'(c_2)\}$.

By Corollary 5.1.1 and (ii) of Lemma 5.1.3 with $f_1 = f^*$, we know that there exists a constant $0 < \delta_1 < \frac{1}{2}$ such that $f < f^*$ in $(-1, -1 + \delta_1)$. Similarly, by Corollary 5.1.1' and (ii) of Lemma 5.1.3' with $f_1 = f^*$, we know that there exists a constant $0 < \delta_2 < \frac{1}{2}$ such that $f > f^*$ in $(1 - \delta_2, 1)$.

Therefore, there exists a point $\bar{x} \in (-1+\delta_1, 1-\delta_2)$ such that $f(\bar{x}) = f^*(\bar{x})$. Standard uniqueness theory of ODE implies that $f \equiv f^*$ in $(-1, 1)$. This is a contradiction.

□

Lemma 5.1.5. *Suppose $c_1 \geq -1$, $c_2 \geq -1$, $c_3 < \bar{c}_3(c_1, c_2)$, then (5.13) has no solution in $C^1(-1, 1)$.*

Proof. If f is a C^1 solution of (5.13) in $(-1, 1)$. By Lemma 5.1.1 and Lemma 5.1.1', f can be extended as a function in $C^0[-1, 1]$, $f(-1) \in \{\tau_1(c_1), \tau_2(c_1)\}$, $f(1) \in \{\tau_1'(c_2), \tau_2'(c_2)\}$.

By Lemma 5.1.4, $f^* := f^*_{c_1,c_2}$ is the unique solution of (5.13) with $c_3 = \bar{c}_3(c_1, c_2)$. Since $c_3 < \bar{c}_3(c_1, c_2)$, $f \not\equiv f^*$ in any open interval in $(-1, 1)$. We first assume that $f(\bar{x}) > f^*(\bar{x})$ at some point $\bar{x} \in (-1, 1)$. Since $c_3 < \bar{c}_3(c_1, c_2)$ we have

$$(1 - x^2)f' + 2xf + \frac{1}{2}f^2 < (1 - x^2)f^{*\prime} + 2xf + \frac{1}{2}f^2, \quad -1 < x < 1. \tag{5.19}$$

Since $f(-1) \le f^*(-1)$, we have, in view of Lemma 5.1.3 with $f_1 = f^*$, $f_2 = f$, that there exists $\delta > 0$ such that $f < f^*$ in $(-1, -1+\delta)$.

Now with $f(\bar{x}) > f^*(\bar{x})$ and $f < f^*$ in $(-1, -1+\delta)$, there exist a point $\xi \in (-1+\delta, \bar{x})$ such that

$$f(\xi) = f^*(\xi), \qquad f'(\xi) \ge f^{*\prime}(\xi),$$

which contradicts inequality (5.19) at ξ.

Similar arguments will lead to a contradiction when $f(\bar{x}) < f^*(\bar{x})$ for some $\bar{x} \in (-1, 1)$ by showing $f > f^*$ near $x = 1$. The lemma is proved. □

Lemma 5.1.6. *Suppose $c_1 \ge -1$, $c_2 \ge -1$, $c_3 > \bar{c}_3(c_1, c_2)$. Let f_c^+ be the power series solution, obtained in Lemma 5.1.2 with $f_c^+(-1) = \tau_2(c_1)$, of (5.13) in $(-1, -1+\delta)$, then f_c^+ can be extended to be a solution of (5.13) in $(-1, 1)$, and $f_c^+(1) = \tau_2'(c_2)$.*

Let f_c^- be the power series solution, obtained in Lemma 5.1.2' with $f_c^-(1) = \tau_1'(c_2)$, of (5.13) in $(1-\delta, 1)$, then f_c^- can be extended to be a solution of (5.13) in $(-1, 1)$, and $f_c^-(-1) = \tau_1(c_1)$. Moreover, $f_c^- < f_c^+$ in $(-1, 1)$.

Proof. We only need to prove that $f^+ := f_c^+$ can be extended to be a solution of (5.13) in $(-1, 1)$ and $f^+(1) = \tau_2'(c_2)$, since similar arguments work for f_c^-.

Standard existence theory of ODE implies that f^+ can be extended to the maximal interval of existence, say $(-1, \xi)$, $\xi \in (-1+\delta, 1]$. Since $c_3 > \bar{c}_3(c_1, c_2)$, we have, with $f^* := f^*_{c_1, c_2}$,

$$(1-x^2)f^{+\prime} + 2xf^+ + \frac{1}{2}(f^+)^2 > (1-x^2)f^{*\prime} + 2xf^* + \frac{1}{2}f^{*2}, \quad -1 < x < \xi. \qquad (5.20)$$

Since $f^+(-1) = f^*(-1) = \tau_2(c_1) \ge 2$, by Lemma 5.1.3 with $f_1 = f^+$, $f_2 = f^*$ and the fact that f^+, f^* can not coincide in any open interval, we have $f^+ > f^*$ in $(-1, \xi)$.

If $\xi < 1$, since f^+ is bounded from below by f^*, there exists a sequence of points $\{x_i\}$ satisfying

$$x_1 < x_2 < x_3 < \cdots < \xi, \qquad \lim_{i\to\infty} x_i = \xi,$$

$$f^+(x_1) < f^+(x_2) < f^+(x_3) < \cdots, \qquad \lim_{i\to\infty} f^+(x_i) = +\infty.$$

Then, in each interval (x_i, x_{i+1}), we can find a point y_i such that

$$x_i < y_i < x_{i+1}, \quad f^+(y_i) \geq f^+(x_i), \quad {f^+}'(y_i) \geq 0.$$

Taking $x = y_i$ in equation (5.13), and sending i to infinity, we obtain a contradiction. So $\xi = 1$. By Lemma 5.1.1, $\lim_{x \to -1^+} f^+(x)$ exists and is finite.

We have extended f^+ to be a solution of (5.13) in $C^1(-1,1) \cap C^0[-1,1]$ and $f^+ > f^*$ in $(-1,1)$.

Similarly, f^- can be extended to $C^0[-1,1]$, and $f^- < f^* < f^+$ in $(-1,1)$.

By Lemma 5.1.1', $f^+(1) \in \{\tau_1'(c_2), \tau_2'(c_2)\}$. If $c_2 = -1$, $\tau_1'(c_2) = \tau_2'(c_2)$, so $f^+(1) = \tau_2'(c_2)$. If $c_2 > -1$, since $f^-(1) = \tau_1'(c_2)$ and $f^+ > f^-$, by Lemma 5.1.1', we have $f^+(1) = \tau_2'(c_2)$. Similarly, $f^-(-1) = \tau_1(c_1)$.

Lemma 5.1.6 is proved.

$\square$

By Lemma 5.1.4 and Lemma 5.1.6, for each $c \in D$, where D is defined by (5.6), $\gamma^+(c) := f_c^+(0)$, and $\gamma^-(c) := f_c^-(0)$ both exist and are finite. Denote H_i, $1 \leq i \leq 6$ to be the following subsets of D:

$$H_1 := \{c \in D | c_1 > -1, c_2 \geq -1, c_3 > \bar{c}_3\}, \quad H_2 := \{c \in D | c_1 = -1, c_2 \geq -1, c_3 > \bar{c}_3\},$$

and let

$$H_3 := \{c \in D | c_1 > -1, c_2 > -1, c_3 \geq \bar{c}_3\}, \quad H_4 := \{c \in D | c_1 = -1, c_2 > -1, c_3 \geq \bar{c}_3\},$$

$$H_5 := \{c \in D | c_1 > -1, c_2 = -1, c_3 \geq \bar{c}_3\}, \quad H_6 := \{c \in D | c_1 = -1, c_2 = -1, c_3 \geq \bar{c}_3\}$$

Lemma 5.1.7. *Let f_c^+ be the power series solution, obtained in Lemma 5.1.2 with $f_c^+(-1) = \tau_2(c_1)$. For any integer $m \geq 0$, and any compact subset K of one of H_i , $1 \leq i \leq 6$, there exists some $\delta > 0$, depending only on m and K, such that $f_c^+ \in C^m(K \times (-1, -1+\delta))$. Moreover, there exists some constant $C(m, K, \delta)$, depending only on m, K and δ, such that for any multi-index α satisfying $|\alpha| \leq m$,*

$$|\partial_c^\alpha f_c^+(x)| \leq C(m, K, \delta), \quad \forall x \in (-1, -1+\frac{\delta}{2}), c \in K.$$

Proof. Let $\alpha = (\alpha^1, \alpha^2, \alpha^3)$ denote a muti-index where $\alpha^i \geq 0$, $i = 1, 2, 3$. The partial derivative $\partial^\alpha = \partial_{c_1}^{\alpha_1} \partial_{c_2}^{\alpha_2} \partial_{c_3}^{\alpha_3}$ and the absolute value $|\alpha| = \sum_{i=1}^3 \alpha^i$.

By Lemma 5.1.2 and its proof, f_c^+ can be expressed as

$$f_c^+(x) = \tau + \sum_{n=1}^{\infty} a_n (1+x)^n, -1 < x < -1+\delta,$$

where

$$\tau = 2 + \sqrt{4+4c_1}, \tag{5.21}$$

$$a_1 = \frac{-c_1 + c_2 + 2c_3}{\tau} - 2, \quad a_2 = -\frac{1}{\tau+2}(c_3 + a_1 + \frac{1}{2}a_1^2) \tag{5.22}$$

and for $n \geq 3$,

$$a_n = -\frac{1}{2n-2+\tau}\left(\frac{1}{2}\sum_{k+l=n,k,l\geq 1} a_k a_l + (3-n)a_{n-1}\right). \tag{5.23}$$

By the above expressions and relations it can be seen that $\tau(c)$ and $a_n(c)$ are all C^∞ functions of c in J_1 or J_2. So to prove the lemma, we just need to show that there exists some $\delta > 0$, depending only on m and K, such that for any muti-index α satisfying $1 \leq |\alpha| \leq m$, the series

$$\frac{\partial^\alpha \tau}{\partial c^\alpha} + \sum_{n=1}^{\infty} \frac{\partial^\alpha a_n}{\partial c^\alpha}(1+x)^n \tag{5.24}$$

is absolutely convergent in $(-1, -1+\delta)$ uniformly for $c \in K$.

Case 1: $K \subset H_1$.

Let $C(m,K)$ be a constant depending only on m and K which may vary from line to line.

If K is a compact set in H_1, there exists some constant $\delta_1(K) > 0$, such that $4 + 4c_1 \geq \delta_1(K)$. Using this, (5.22), (5.23), and the fact that $\tau > 2$, we have

$$\left|\frac{\partial^\alpha \tau}{\partial c^\alpha}\right| \leq C(m,K), \quad \left|\frac{\partial^\alpha a_n}{\partial c^\alpha}\right| \leq C(m,K), \quad \forall 1 \leq n \leq 2, 0 \leq |\alpha| \leq m, c \in K. \tag{5.25}$$

Next, let $g_n(c) := \frac{1}{2n-2+\tau}$. By the above estimates and the fact that $\tau > 2$, we have

$$\left|\frac{\partial^\alpha}{\partial c^\alpha} g_n(c)\right| \leq \frac{C(m,K)}{n}, \quad \text{for all } 1 \leq |\alpha| \leq m, c \in K, \text{ and } n \geq 1. \tag{5.26}$$

To prove the series in (5.24) is convergent for all $1 \leq |\alpha| \leq m$ uniformly in K, we will show the following:

Claim: there exists some $a > 0$, depending only on m and K, such that

$$(P_n): \quad |\partial^\alpha a_n(c)| \leq a^{n(|\alpha|+1)}, \quad \text{for } 1 \leq |\alpha| \leq m, \text{ and } c \in K$$

holds for all $n \geq 1$.

Proof of the claim:

We prove this claim by induction on n. Let a be a constant to be determined in the proof.

By Lemma 5.1.2 and its proof, and estimate (5.25), there exists some constant $C(m, K)$, depending only on m and K, such that for any $a \geq C(m, K)$, $|a_n(c)| \leq a^n$ for all $c \in K$ and $n \geq 1$, and , (P_1) and (P_2) holds.

Now for $n \geq 3$, suppose that (P_k) holds for all $1 \leq k \leq n-1$.

Let $Q_n(c) := \sum_{k+l=n,k,l\geq 1} a_k a_l$. Then (5.23) can be written as

$$a_n = -\frac{1}{2} g_n Q_n + (n-3) g_n a_{n-1}.$$

So

$$\partial^\alpha a_n = -\frac{1}{2} \partial^\alpha (g_n Q_n) + (n-3) \partial^\alpha (g_n a_{n-1}). \tag{5.27}$$

Using (5.26), by computation we have

$$|\partial^\alpha (g_n Q_n)| \leq \frac{C(m, K)}{n} \max_{\alpha_1 \leq \alpha} |\partial^{\alpha - \alpha_1} Q_n|.$$

Let $a > a_0$, using the definition of $Q_n(c)$, by induction we have that,

$$\begin{aligned} |\partial^\alpha (g_n Q_n)| &\leq \frac{C(m, K)}{n} \sum_{k+l=n,k,l\geq 1} \max_{\alpha_2 \leq \alpha - \alpha_1} |\partial^{\alpha_2} a_k||\partial^{\alpha - \alpha_1 - \alpha_2} a_l| \\ &\leq C(m, K) \max_{\alpha_2 \leq \alpha - \alpha_1} \max_{k+l=n,k,l\geq 1} a^{k(|\alpha_2|+1)} a^{l(|\alpha - \alpha_1 - \alpha_2|+1)} \\ &\leq C(m, K) a^{n(|\alpha|+1) - |\alpha|} \end{aligned} \tag{5.28}$$

Similarly, by (5.26) and induction, we have

$$|\partial^\alpha (g_n a_{n-1})| \leq \frac{C(m, K)}{n} a^{(n-1)(|\alpha|+1)} \tag{5.29}$$

Plug (5.28) and (5.29) in (5.27), we have that for $|\alpha| \geq 1$,

$$|\partial^\alpha a_n| \leq C(m, K) a^{n(|\alpha|+1) - 1}.$$

Let $a > \max\{a_0, C(m,K), a_1\}$. We have

$$|\partial^\alpha a_n| \le a^{n(|\alpha|+1)}.$$

So the claim is true for all n. The lemma is proved for $K \subset H_1$.

Case 2: $K \subset H_2$, then $\tau = 2$ and $g_n(c)$ is a constant in K. By similar arguments as in Case 1, we have the same estimate for a_n and the proof is finished.

Case 3: $K \subset H_i$, $3 \le i \le 6$.

By the definition of $\bar{c}_3(c_1, c_2)$, we know $\bar{c}_3(c_1, c_2)$ is smooth in c_1, c_2 in each of $\{c_1 > -1, c_2 > -1\}$, $\{c_1 = -1, c_2 > -1\}$, $\{c_1 > -1, c_2 = -1\}$, and $\{c_1 = -1, c_2 = -1\}$ By similar arguments as in Case 1, we have the same estimate for τ and a_n and the proof is finished. □

Corollary 5.1.2. *For any $x_0 \in (-1,1)$, and any $K \subset H_i$, $1 \le i \le 6$, $f_c^+(x_0) \in C^\infty(K)$. Moreover, for any $\epsilon > 0$, $m \in \mathbb{N}$, there exists some constant $C(m,K,\epsilon)$ such that*

$$|\partial_c^\alpha f_c^+| \le C(m,K,\epsilon), \tag{5.30}$$

for any $0 \le |\alpha| \le m$, and $-1 < x < 1 - \epsilon$.

Proof. For any $x_0 \in (-1,1)$, there is some $\epsilon > 0$ such that $x_0 + \epsilon < 1$. By Lemma 5.1.7, there exist some $\delta > 0$, such that the power series solution $f_c^+ \in C^m(K \times (-1, -1+\delta))$ obtained in Lemma 5.1.2 with $f_c^+(-1) = \tau_2(c_1)$ or 2. So $f_c^+(-1 + \frac{\delta}{2}) \in C^m(K)$. Notice that $f_c^+(x)$ is the solution of the initial value problem

$$\begin{cases} (1-x^2)f' + 2xf + \dfrac{1}{2}f^2 = P_c(x) := c_1(1-x) + c_2(1+x) + c_3(1-x^2), \quad -1 < x < 1, \\ f(-1 + \dfrac{\delta}{2}) = f_c^+(-1 + \dfrac{\delta}{2}). \end{cases}$$

By stantdard ODE theories, $f_c^+(x_0)$ depends smoothly on $f_c^+(-1 + \frac{\delta}{2})$. So $f_c^+(x_0)$ is in $C^m(K)$ for all $m \in \mathbb{N}$.

Moreover, by Lemma 5.1.7, there exists some $\delta > 0$ and $C(m,K)$ such that $|\partial_c^\alpha f_c^+| \le C(m,K)$ for $-1 < x < -1 + \frac{\delta}{2}$. Then by standard ODE theory, for any $\epsilon > 0$ there exists some $C(m,K,\epsilon)$ such that (5.31) is true for all $-1 < x < -1 + \epsilon$.

□

Let

$$H_1' := \{c \in D | c_1 \geq -1, c_2 > -1, c_3 > \bar{c}_3\}, \quad H_2' := \{c \in D | c_1 \geq -1, c_2 = -1, c_3 > \bar{c}_3\}.$$

Similarly as Lemma 5.1.7 and Corollary 5.1.2 we have,

Lemma 5.1.7'. *Let f_c^- be the power series solution, obtained in Lemma 5.1.2' with $f_c^-(1) = \tau_1'(c_2)$. For any integer $m \geq 0$, and any compact set $K \subset H_1'$, H_2' or H_i, $3 \leq i \leq 6$, there exists some $\delta > 0$, depending only on m and K, such that $f_c^- \in C^m(K \times (1-\delta, 1))$. Moreover, there exists some constant $C(m, K, \delta)$, depending only on m, K and δ, such that for any multi-index α satisfying $|\alpha| \leq m$,*

$$|\partial_c^\alpha f_c^-(x)| \leq C(m, K, \delta), \quad \forall x \in (1 - \frac{\delta}{2}, 1), c \in K.$$

and

Corollary 5.1.2'. *For any $x_0 \in (-1, 1)$, and any $K \subset H_1'$, H_2' or H_i, $3 \leq i \leq 6$, $f_c^-(x_0) \in C^\infty(K)$. Moreover, for any $\epsilon > 0$, $m \in \mathbb{N}$, there exists some constant $C(m, K, \epsilon)$ such that*

$$|\partial_c^\alpha f_c^-| \leq C(m, K, \epsilon), \tag{5.31}$$

for any $0 \leq |\alpha| \leq m$, and $-1 + \epsilon < x < 1$.

Remark 5.1.3. *It can be actually seen that $H_1 = D_1 \cup D_3$, $H_2 = D_2 \cup D_4$, $H_1' = D_1 \cup D_2$, $H_2' = D_3 \cup D_4$, and $H_{i+2} = D_i \cup D_{i+4}$, $1 \leq i \leq 4$.*

Lemma 5.1.8. *Suppose $c_1 \geq -1$, $c_2 \geq -1$, $c_3 > \bar{c}_3(c_1, c_2)$. Let f_c^+, f_c^- be the unique C^1 solution of (5.13) in $(-1, 1)$ obtained in Lemma 5.1.6. Then any C^1 solution f of (5.13) in $(-1, 1)$ other than $f_c^\pm$ satisfies*

$$f_c^- < f < f_c^+, \quad \text{in } (-1, 1),$$

$$f(-1) = \tau_1(c_1), \quad f(1) = \tau_2'(c_2).$$

Proof. By Lemma 5.1.1 f can be extended to $C^0[-1, 1]$ with $f(-1) = \tau_1(c_1)$ or $\tau_2(c_1)$ $(\tau_1 \leq \tau_2)$.

We only need to prove $f < f_c^+$ in $(-1,1)$ and $f(-1) = \tau_1(c_1)$, since similar arguments imply that $f > f_c^-$ in $(-1,1)$ and $f(1) = \tau_2'(c_2)$.

From the standard uniqueness theory of ODE, we know that the graph of f and f_c^+ can not intersect in $(-1,1)$. So we either have $f < f_c^+$ in $(-1,1)$ or $f > f_c^+$ in $(-1,1)$.

If $f > f_c^+$ in $(-1,-1+\delta)$, then, by Lemma 5.1.1, $f(-1) = f_c^+(-1) = \tau_2(c_1) \geq 2$. Note that $f_1 := f_c^+$ satisfies (5.15), we can apply Lemma 5.1.3 with $f_1 = f_c^+$, $f_2 = f$ to obtain $f \leq f_c^+$, a contradiction. So $f < f_c^+$ in $(-1,1)$.

If $\tau_1(c_1) < \tau_2(c_1)$, the uniqueness result Corollary 5.1.1 implies that $f(-1) = \tau_1(c_1)$. If $\tau_1(c_1) = \tau_2(c_1)$, we again have $f(-1) = \tau_1(c_1)$. Lemma 5.1.8 is proved. $\square$

Suppose that $c_1 \geq -1$, $c_2 \geq -1$, $c_3 > \bar{c}_3(c_1,c_2)$. Let $f^{c,\gamma}$ be the unique local solution of (5.13) with $f^{c,\gamma}(0) = \gamma$. Denote

$$\gamma^-(c) := f_c^-(0), \quad \gamma^+(c) := f_c^+(0).$$

It is obvious that $f_c^- = f^{c,\gamma^-}$ and $f_c^+ = f^{c,\gamma^+}$. If $\gamma^-(c) < \gamma < \gamma^+(c)$, Lemma 5.1.8 implies that $f^{c,\gamma}$ can be extended to be a solution of (5.13) in $C^1(-1,1)$ satisfying $f^{c,\gamma}(-1) = \tau_1(c_1)$ and $f^{c,\gamma}(1) = \tau_2'(c_2)$.

Remark 5.1.4. *By Corollary* 5.1.2 *and Corollary* 5.1.2', *we know that* $\gamma^+(c)$ *is* C^∞ *in each* H_i, $1 \leq i \leq 6$, *and* $\gamma^-(c)$ *is* C^∞ *in each of* H_1', H_2', H_i, $3 \leq i \leq 6$. *But they are not* C^1 *at those points where* $c_1 = -1$ *or* $c_2 = -1$ *in* D.

Lemma 5.1.9. *Suppose* $c_1 \geq -1$, $c_2 \geq -1$, $c_3 > \bar{c}_3(c_1,c_2)$, *the graphs*

$$K_1(\gamma) := \{(x, f^{c,\gamma}(x)) \mid -1 < x < 1\}, \quad \gamma^-(c) < \gamma < \gamma^+(c).$$

foliate the set

$$K_2 := \{(x,y) \mid -1 < x < 1, f^{c,\gamma^-}(x) < y < f^{c,\gamma^+}(x)\}$$

in the sense that for any $\gamma, \gamma' \in \mathbb{R}$, $\gamma^-(c) \leq \gamma < \gamma' \leq \gamma^+(c)$, $f^{c,\gamma} < f^{c,\gamma'}$ *in* $(-1,1)$ *and* $K_2 = \bigcup_{\gamma^-(c)<\gamma<\gamma^+(c)} K_1(\gamma)$. *Moreover,* $f^{c,\gamma}$ *is a continuous function of* (c,γ,x) *in* $K \times [\gamma^-(c), \gamma^+(c)] \times (-1,1)$.

Proof. By standard uniqueness theories of ODE,

$$f^{c,\gamma^-} < f^{c,\gamma} < f^{c,\gamma'} < f^{c,\gamma^+} \quad \text{in } (-1,1), \qquad \gamma^-(c) < \gamma < \gamma' < \gamma^+(c).$$

It is obvious that $K_1(\gamma) \subseteq K_2$. On the other hand, let $(x_0, y_0) \in K_2$, so $-1 < x_0 < 1$ and $f^{c,\gamma^-}(x_0) < y_0 < f^{c,\gamma^+}(x_0)$. By standard existence and uniqueness theories of ODE,and Lemma 5.1.8, there exists a C^1 solution f of (5.13) in $(-1,1)$ satisfying $f(x_0) = y_0$ and $f^{c,\gamma^-} < f < f^{c,\gamma^+}$ in $(-1,1)$. In particular,

$$\gamma^-(c) = f^{c,\gamma^-}(0) < f(0) < f^{c,\gamma^+}(0) = \gamma^+(c),$$

$f = f^{c,\gamma}$ with $\gamma = f(0)$ and therefore $x_0, y_0 \in K_1(\gamma)$. We have proved that $K_2 = \bigcup_{\gamma^- < \gamma < \gamma^+} K_1(\gamma)$.

The continuity of $f^{c,\gamma}$ for (c, γ, x) in $K \times [\gamma^-(c), \gamma^+(c)] \times (-1,1)$ can be derived from (5.13), and the continuous dependence of ODE on its boundary conditions.

□

Theorem 5.1.1 and Theorem 5.1.2 follows from Lemma 5.1.4 - Lemma 5.1.9.

Proof of Theorem 5.1.3:

To prove Theorem 5.1.3, we make the following observations.

First, when $(c, \gamma) \in F_{k,l}$, $5 \le k \le 8$, $f^*_{c_1,c_2} = -\left(\sqrt{1+c_1} + \sqrt{1+c_2} + 2\right)x + (\sqrt{1+c_1} - \sqrt{1+c_2})$. So $f^*_{c_1,c_2}$ is smooth in $F_{k,l}$.

Next, by Corollary 5.1.2 and Corollary 5.1.2', we know that for $1 \le k \le 4$ and $l = 1, 2$, f_c^+ and f_c^- are smooth in $F_{k,l}$.

By standard ODE theory, since f satisfies (5.13), it is smooth in $F_{k,3}$ for each $1 \le k \le 4$.

So a solution f of the initial problem

$$\begin{cases} (1-x^2)f' + 2xf + \dfrac{1}{2}f^2 = P_c(x) := c_1(1-x) + c_2(1+x) + c_3(1-x^2), \quad -1 < x < 1, \\ f(-1+\dfrac{\delta}{2}) = f_c^+(-1+\dfrac{\delta}{2}). \end{cases}$$

is smooth with respect to (c, γ) in each $F_{k,l}$, $1 \le k \le 8$, $1 \le l \le 3$. It remains to prove the estimates (i)-(iv) in Theorem 5.1.3.

Notice that if $5 \leq k \leq 8$, the estimates in Theorem 5.1.3 (i) can be obtained by the expression of f^* directly. The remaining estimates are obtained from the following lemmas.

Lemma 5.1.10. *For any $\epsilon > 0$, $m \in \mathbb{N}$, and $K \in F_{1,3}$, $F_{1,2}$, $F_{3,2}$ and $F_{3,3}$, there exists some constant $C(m, K, \epsilon)$ such that*

$$|\partial_c^\alpha \partial_\gamma^j f| \leq C(m, K, \epsilon), \tag{5.32}$$

for any $1 \leq |\alpha| + j \leq m$, and $-1 < x < 1 - \epsilon$.

Proof. We prove the lemma by induction.

We use $C(m, K, \epsilon)$ and C to denote constants which may be different from line to line.

we know by (5.13) that

$$(1 - x^2)\left(\frac{\partial f}{\partial \gamma}\right)' + (2x + f)\left(\frac{\partial f}{\partial \gamma}\right) = 0,$$

and

$$(1 - x^2)\left(\frac{\partial f}{\partial c_i}\right)' + (2x + f)\left(\frac{\partial f}{\partial c_i}\right) = \partial_{c_i} P_c(x).$$

Denote

$$a(x) = a_{c,\gamma}(x) = \int_0^x \frac{2s + f}{1 - s^2} ds. \tag{5.33}$$

Then

$$\frac{\partial f}{\partial \gamma} = Ce^{-a(x)}, \tag{5.34}$$

and for $i = 1, 2, 3$,

$$\left(\frac{\partial f}{\partial c_i}\right) = Ce^{-a(x)} + e^{-a(x)} \int_0^x e^{a(s)} \frac{\partial_{c_i} P_c(s)}{1 - s^2} ds. \tag{5.35}$$

By the definition of $a(x)$, we have

$$e^{-a(x)} = O(1)(1 + x)^{1 - \frac{f(-1)}{2}}, \quad e^{a(x)} = O(1)(1 + x)^{\frac{f(-1)}{2} - 1}, \quad -1 < x < -1 + \epsilon.$$

Since when $(c, \gamma) \in K$, $f(-1) < 2$, there exists some $C(1, K, \epsilon)$, such that $e^{-a(x)} \leq C(1, K, \epsilon)$. Thus by (6.5) and (6.9) we have that for $-1 < x < 1 - \epsilon$,

$$|\partial_c^\alpha \partial_\gamma^j f| \leq C(1, K, \epsilon) \tag{5.36}$$

for all $|\alpha| + j = 1$.

Now for $m \geq 2$, suppose that $C(m_1, K, \epsilon)$ exist for all $1 \leq m_1 \leq m-1$, then

$$(1-x^2)(\partial_c^\alpha \partial_\gamma^j f)' + 2x\partial_c^\alpha \partial_\gamma^j f + \frac{1}{2}\partial_c^\alpha \partial_\gamma^j f^2 = \partial_c^\alpha \partial_\gamma^j P_c(x).$$

This leads to

$$(1-x^2)(\partial_c^\alpha \partial_\gamma^j f)' + (2x+f)\partial_c^\alpha \partial_\gamma^j f = h := -\frac{1}{2}\sum_{0<\alpha_1<\alpha, 0<j_1<j} \begin{pmatrix} \alpha \\ \alpha_1 \end{pmatrix} \begin{pmatrix} j \\ j_1 \end{pmatrix} \partial_c^{\alpha_1}\partial_\gamma^{j_1} f \partial_c^{\alpha-\alpha_1}\partial_\gamma^{j-j_1} f.$$

Then

$$\partial_c^\alpha \partial_\gamma^j f = Ce^{-a(x)} + e^{-a(x)}\int_{-1+\frac{\delta}{2}}^x e^{a(s)}\frac{h(s)}{1-s^2}ds.$$

By induction assumption, $h \in L^\infty(-1, 1-\epsilon)$ and there exists some $C(m, K, \epsilon)$ such that $|h|_{L^\infty(-1,1-\epsilon)} \leq C(m, K, \delta.\epsilon)$. So we have

$$|\partial_c^\alpha \partial_\gamma^j f|_{L^\infty(-1,1-\epsilon)} \leq C(m, K, \epsilon).$$

The proof is finished. $\square$

Similarly we have

Lemma 5.1.10'. *For any $\epsilon > 0$, $m \in \mathbb{N}$, and $K \in F_{1,3}$, $F_{2,3}$, $F_{1,1}$ or $F_{2,1}$, there exists some constant $C(m, K, \epsilon)$ such that*

$$|\partial_c^\alpha \partial_\gamma^j f| \leq C(m, K, \epsilon), \tag{5.37}$$

for any $1 \leq |\alpha| + j \leq m$, and $-1 + \epsilon < x < 1$.

Lemma 5.1.11. *For any $\epsilon > 0$, $m \in \mathbb{N}$, and $K \in F_{2,3}$, $F_{2,2}$, $F_{4,3}$ or $F_{4,2}$, there exists some constant $C(m, K, \epsilon)$ such that*

$$\left(\ln \frac{1+x}{3}\right)^2 |\partial_c^\alpha \partial_\gamma^j f| \leq C(m, K, \epsilon), \tag{5.38}$$

for any $\alpha = (0, \alpha_2, \alpha_3)$ and $j \geq 0$ satisfying $1 \leq |\alpha| + j \leq m$, and $-1 < x < 1 - \epsilon$.

Proof. We prove the lemma by induction.

Denote $C(m, K, \epsilon)$ and C to be constants which may vary from line to line. Similar as the proof of Lemma 5.1.10, we have (5.34) and (5.35) where $a(x)$ is defined by (5.33). Notice that in this case

$$f = 2 + \frac{4}{\ln(1+x)} + \frac{O(1)}{(\ln(1+x))^2}, \quad -1 < x < 1-\epsilon. \tag{5.39}$$

We have

$$e^{-a(x)} = O(1)(\ln(1+x))^{-2}, \qquad e^{a(x)} = O(1)(\ln(1+x))^2.$$

Notice in this case, $i = 2$ or 3 in (5.35), and $\partial_{c_i} P_c = O(1+x)$, so we have that for $-1 < x < 1-\epsilon$,

$$\left(\ln\frac{1+x}{3}\right)^2 |\partial_c^\alpha \partial_\gamma^j f| \le C(1, K, \epsilon) \tag{5.40}$$

for all $|\alpha| + j = 1$.

Now suppose that $C(m_1, K, \epsilon)$ exist for all $1 \le m_1 \le m-1$. Similar as the proof of the previous lemma we have

$$\partial_c^\alpha \partial_\gamma^j f = Ce^{-a(x)} + e^{-a(x)} \int_{-1+\frac{\delta}{2}}^{x} e^{a(s)} \frac{h(s)}{1-s^2} ds.$$

where

$$h := -\frac{1}{2} \sum_{0<\alpha_1<\alpha, 0<j_1<j} \begin{pmatrix} \alpha \\ \alpha_1 \end{pmatrix} \begin{pmatrix} j \\ j_1 \end{pmatrix} \partial_c^{\alpha_1} \partial_\gamma^{j_1} f \partial_c^{\alpha-\alpha_1} \partial_\gamma^{j-j_1} f.$$

Then by induction assumption $h \in L^\infty(-1, -1+\epsilon)$ and there is some $C(m, K, \epsilon)$ such that $\left(\ln\frac{1+x}{3}\right)^4 |h(x)| \le C(m, K, \epsilon)$ for all $-1 < x < 1-\epsilon$. Using this estimate we then have

$$\left(\ln\frac{1+x}{3}\right)^2 |\partial_c^\alpha \partial_\gamma^j f| \le C(m, K, \epsilon)$$

The lemma is proved. □

Similarly, we have

Lemma 5.1.11'. *For any $\epsilon > 0$, $m \in \mathbb{N}$, and $K \in F_{3,3}$, $F_{4,3}$, $F_{3,1}$ or $F_{4,1}$, there exists some constant $C(m, K, \epsilon)$ such that*

$$\left(\ln\frac{1-x}{3}\right)^2 |\partial_c^\alpha \partial_\gamma^j f| \le C(m, K, \epsilon), \tag{5.41}$$

for any $\alpha = (0, \alpha_2, \alpha_3)$ and $j \ge 0$ satisfying $1 \le |\alpha| + j \le m$, and $-1+\epsilon < x < 1$

Then by Corollary 5.1.2, Corollary 5.1.2', Lemma 5.1.10, 5.1.10', 5.1.11 and 5.1.11', Theorem 5.1.3 is proved.

From Theorem 5.1.3 we also have

Corollary 5.1.3. *Let K be a compact set contained in one of $F_{k,l}$, $1 \le k \le 8$, $l = 1, 2, 3$. Then for any positive integer m, there exists some constant $C(m, K)$, such that*

$$|\partial_c^\alpha \partial_\gamma^j \bar{U}_\theta'(x)| \le C(m,K), \quad |\partial_c^\alpha \partial_\gamma^j \bar{U}_\theta''(x)| \le C(m,K), \quad \forall 0 \le |\alpha| + j \le m, -\frac{1}{2} < x < \frac{1}{2}.$$

Proof. By equation (5.13),

$$\partial_c^\alpha \partial_\gamma^j \bar{U}_\theta' = \frac{1}{1-x^2} \partial_c^\alpha \partial_\gamma^j (P_c - 2x\bar{U}_\theta - \frac{1}{2}\bar{U}_\theta^2)$$

So by Theorem 5.1.3, there is some $C(m, K)$ such that for all $0 \le |\alpha| + j \le m$, and $-\frac{1}{2} < x < \frac{1}{2}$, we have

$$|\partial_c^\alpha \partial_\gamma^j \bar{U}_\theta'(x)| \le C(m,K).$$

Similarly we also have the estimate for $\partial_c^\alpha \partial_\gamma^j \bar{U}_\theta'(x)$ for $-\frac{1}{2} < x < \frac{1}{2}$. $\square$

5.2 Existence of axisymmetric solutions with nonzero swirl on $\mathbb{S}^2 \setminus \{S, N\}$

5.2.1 Framework of proofs

The set of all axisymmetric no swirl solutions of the NSE (1) in $C^\infty(\mathbb{S}^2 \setminus \{S, N\})$ is classified in Section 5.1 as the four parameter family $\{U^{c,\gamma} = (U_\theta^{c,\gamma}, 0) \mid (c, \gamma) \in E\}$. In this section, we will use Theorem C in Section 3.2 (Implicit Function Theorem) in suitably chosen weighted normed spaces to prove Theorem 1.0.6.

Denote $\bar{U} := U^{c,\gamma}$ for convenience. Recall that for each $(c, \gamma) \in E$, it satisfies

$$(1-x^2)\bar{U}_\theta' + 2x\bar{U}_\theta + \frac{1}{2}\bar{U}_\theta^2 = c_1(1-x) + c_2(1+x) + c_3(1-x^2), \tag{5.42}$$

and $\bar{U}_\theta(0) = \gamma$.

The equations of axisymmetric solutions in $C^\infty(\mathbb{S}^2 \setminus \{S, N\})$ are of the form

$$\begin{cases} (1-x^2)U_\theta' + 2xU_\theta + \frac{1}{2}U_\theta^2 + \int_0^x \int_0^l \int_0^t \frac{2U_\phi(s)U_\phi'(s)}{1-s^2} ds dt dl = P_{\hat{c}}(x) \\ (1-x^2)U_\phi'' + U_\theta U_\phi' = 0. \end{cases} \tag{5.43}$$

where

$$P_{\hat{c}} = \hat{c}_1(1-x) + \hat{c}_2(1+x) + \hat{c}_3(1-x^2),$$

and $\hat{c}_1, \hat{c}_2, \hat{c}_3$ are constants.

By Theorem 1.0.3 and Theorem 4.0.1', we know that if U is a solution of (5.43), then both $U_\theta(-1)$ and $U_\theta(1)$ exist and are finite. In particular, if $U_\theta(-1) = 2$, $\eta_1(U) := \lim_{x\to-1}(U_\theta - 2)\ln(1+x)$ exists and $\eta_1(U) = 0$ or 4. If $U_\theta(1) = -2$, $\eta_2(U) := \lim_{x\to1}(U_\theta + 2)\ln(1-x)$ exists and $\eta_2(U) = 0$ or -4. Moreover, the singular behaviors of U near the poles are affected by the values $U_\theta(-1)$, $U_\theta(1)$, $\eta_1(U)$ and $\eta_2(U)$.

So to prove the existence of axisymmetric solutions with nonzero swirl near $\bar{U}_\theta$ using IFT, we construct function spaces according to the values of $\bar{U}_\theta(-1)$, $\bar{U}_\theta(1)$, $\eta_1(\bar{U})$ and $\eta_2(\bar{U})$.

For convenience let us denote $\eta_1 = \eta_1(\bar{U})$ and $\eta_2 = \eta_2(\bar{U})$. Our proof of existence will be carried out in following separate cases:

Case 1: ($\bar{U}_\theta(-1) < 3$, $\bar{U}_\theta(-1) \neq 2$ or $\bar{U}_\theta(-1) = 2$ with $\eta_1 = 0$), while ($\bar{U}_\theta(1) > -3$, $\bar{U}_\theta(1) \neq -2$ or $\bar{U}_\theta(1) = -2$ with $\eta_2 = 0$).

Case 2: $\{\bar{U}_\theta(-1) = 2$ with $\eta_1 = 4$, while ($\bar{U}_\theta(1) > -3$, $\bar{U}_\theta(1) \neq -2$ or $\bar{U}_\theta(1) = -2$ with $\eta_2 = 0)\}$ OR $\{\bar{U}_\theta(1) = -2$ with $\eta_2 = -4$, while ($\bar{U}_\theta(-1) < 3$, $\bar{U}_\theta(-1) \neq 2$ or $\bar{U}_\theta(-1) = 2$ with $\eta_1 = 0)\}$.

Case 3: $\bar{U}_\theta(-1) = 2$ with $\eta_1 = 4$, and $\bar{U}_\theta(1) = -2$ with $\eta_2 = -4$.

Case 4: $\bar{U}_\theta(-1) \geq 3$ or $\bar{U}_\theta(1) \leq -3$.

Let $E_{k,l}$ be the sets defined by (1.11) and (1.14). By Lemma 5.1.2, if $c_1 = -1$, then $\eta_1(U^{c,\gamma^+}) = 0$. Then using Lemma 5.1.3, $\eta_1(U^{c,\gamma}) = 4$ for all $\gamma < \gamma^+$. Similarly, we have that if $c_2 = -1$, $\eta_2(U^{c,\gamma^-}) = 0$ and $\eta_2(U^{c,\gamma}) = -4$ for all $\gamma > \gamma^-$. Using this and Theorem 5.1.2 we have the following relations:

(i) $U^{c,\gamma}$ satisfies Case 1 if and only if $(c,\gamma) \in E_{k,l}$ with $(k,l) \in \mathscr{A}_1 := \{(k,l) \in \mathbb{Z}^2 | k = 1 \text{ or } 5 \leq k \leq 8, 1 \leq l \leq 3\} \cup \{(2,1),(3,2)\}$.

(ii) $U^{c,\gamma}$ satisfies Case 2 if and only if $(c,\gamma) \in E_{k,l}$ with $(k,l) \in \mathscr{A}_2 := \{(2,2),(2,3),(4,2)\}$

or $\mathscr{A}_3 := \{(3,1),(3,3),(4,1)\}$

(iii) $U^{c,\gamma}$ satisfies Case 3 if and only if $(c,\gamma) \in E_{k,l}$ with $(k,l) = (4,3)$.

In this section we denote $U = (U_\theta, U_\phi)$. We will work with $\tilde{U} := U - \bar{U}$, a calculation gives

$$(1-x^2)U_\theta' + 2xU_\theta + \frac{1}{2}U_\theta^2 - c_1(1-x) - c_2(1+x) - c_3(1-x^2) = (1-x^2)\tilde{U}_\theta' + (2x+\bar{U}_\theta)\tilde{U}_\theta + \frac{1}{2}\tilde{U}_\theta^2,$$

where $\tilde{U}_\phi = U_\phi$. Denote

$$\psi[\tilde{U}_\phi, \tilde{V}_\phi](x) := \int_0^x \int_0^l \int_0^t \frac{2\tilde{U}_\phi \tilde{V}_\phi'}{1-s^2} ds dt dl \tag{5.44}$$

and

$$\varphi_{c,\gamma}[\tilde{U}_\theta](x) := (1-x^2)\tilde{U}_\theta' + (2x+\bar{U}_\theta)\tilde{U}_\theta + \frac{1}{2}\tilde{U}_\theta^2. \tag{5.45}$$

For convenience write $\psi[\tilde{U}_\phi](x) := \psi[\tilde{U}_\phi, \tilde{U}_\phi](x)$. Define a map G on $(c,\gamma,\tilde{U})$ by

$$G(c,\gamma,\tilde{U}) = \begin{pmatrix} (1-x^2)\tilde{U}_\theta' + (2x+\bar{U}_\theta)\tilde{U}_\theta + \frac{1}{2}\tilde{U}_\theta^2 + \psi[\tilde{U}_\phi](x) - \tilde{P}(x) \\ (1-x^2)\tilde{U}_\phi'' + (\tilde{U}_\theta + \bar{U}_\theta)\tilde{U}_\phi' \end{pmatrix} \tag{5.46}$$

where

$$\tilde{P}(x) = \frac{1}{2}\psi[\tilde{U}_\phi](-1)(1-x) + \frac{1}{2}\psi[\tilde{U}_\phi](1)(1+x) - \frac{1}{2}(\varphi_{c,\gamma}[\tilde{U}_\theta])''(0)\cdot(1-x^2).$$

If $\tilde{U}$ satisfies $G(c,\gamma,\tilde{U}) = 0$, then $U = \tilde{U} + \bar{U}$ gives a solution of (5.43) with

$$\hat{c}_1 = c_1 + \frac{1}{2}\psi[\tilde{U}_\phi](-1), \quad \hat{c}_2 = c_2 + \frac{1}{2}\psi[\tilde{U}_\phi](1)$$

$$\hat{c}_3 = c_3 - \frac{1}{2}(\varphi_{c,\gamma}[\tilde{U}_\theta])''(0)$$

satisfying $U_\theta(-1) = \bar{U}_\theta(-1)$, $U_\theta(1) = \bar{U}_\theta(1)$.

Denote

$$l_{c,\gamma}[\tilde{U}_\theta](x) := (1-x^2)\tilde{U}_\theta'(x) + (2x+\bar{U}_\theta)\tilde{U}_\theta(x). \tag{5.47}$$

Let A and Q be maps of the form

$$A(c,\gamma,\tilde{U}) = \begin{pmatrix} A_\theta \\ A_\phi \end{pmatrix} := \begin{pmatrix} l_{c,\gamma}[\tilde{U}_\theta](x) + \frac{1}{2}(l_{c,\gamma}[\tilde{U}_\theta])''(0)\cdot(1-x^2) \\ (1-x^2)\tilde{U}_\phi'' + \bar{U}_\theta\tilde{U}_\phi' \end{pmatrix}, \tag{5.48}$$

and

$$Q(\tilde U,\tilde V) = \begin{pmatrix} Q_\theta \\ Q_\phi \end{pmatrix}$$

$$:= \begin{pmatrix} \frac{1}{2}\tilde U_\theta \tilde V_\theta + \psi[\tilde U_\phi, \tilde V_\phi](x) - \frac{1-x}{2}\psi[\tilde U_\phi, \tilde V_\phi](-1) - \frac{1+x}{2}\psi[\tilde U_\phi, \tilde V_\phi](1) + \frac{1}{4}(\tilde U_\theta \tilde V_\theta)''(0)(1-x^2) \\ \tilde U_\theta \tilde V_\phi' \end{pmatrix}. \tag{5.49}$$

Then $G(c,\gamma,\tilde U) = A(c,\gamma,\tilde U) + Q(\tilde U,\tilde U)$.

By computation, the linearized operator of G with respect to $\tilde U$ at $(c,\gamma,\tilde U)$ is given by

$$\begin{aligned} &L^{c,\gamma}_{\tilde U}\tilde V := A(c,\gamma,\tilde V) \\ &+ \begin{pmatrix} \tilde U_\theta \tilde V_\theta + \Psi_{\tilde U_\phi}[\tilde V_\phi](x) - \frac{1-x}{2}\Psi_{\tilde U_\phi}[\tilde V_\phi](-1) - \frac{1+x}{2}\Psi_{\tilde U_\phi}[\tilde V_\phi](1) + \frac{1}{2}(\tilde U_\theta \tilde V_\theta)''(0)(1-x^2) \\ \tilde U_\theta \tilde V_\phi' + \tilde V_\theta \tilde U_\phi' \end{pmatrix} \end{aligned} \tag{5.50}$$

where

$$\Psi_{\tilde U_\phi}[\tilde V_\phi](x) := \int_0^x \int_0^l \int_0^t \frac{2(\tilde U_\phi(s)\tilde V_\phi'(s) + \tilde V_\phi(s)\tilde U_\phi'(s))}{1-s^2} ds dt dl.$$

In particular, at $\tilde U = 0$, the linearized operator of G with respect to $\tilde U$ is

$$L^{c,\gamma}_0 \tilde V = A(c,\gamma,\tilde V) = \begin{pmatrix} l_{c,\gamma}[\tilde V_\theta](x) + \frac{1}{2}(l_{c,\gamma}[\tilde V_\theta])''(0)(1-x^2) \\ (1-x^2)\tilde V_\phi'' + \bar U_\theta \tilde V_\phi' \end{pmatrix}. \tag{5.51}$$

Let

$$a_{c,\gamma}(x) := \int_0^x \frac{2s + \bar U_\theta}{1-s^2} ds, \quad b_{c,\gamma}(x) := \int_0^x \frac{\bar U_\theta}{1-s^2} ds, \quad -1 < x < 1. \tag{5.52}$$

From the discussion in Section 5.1, for all $(c,\gamma) \in E$, $\bar U_\theta$ is smooth in $(-1,1)$. So $a_{c,\gamma}, b_{c,\gamma} \in C^\infty(-1,1)$.

By observation $a(x) = -\ln(1-x^2) + b(x)$. A calculation gives

$$a'_{c,\gamma}(x) = \frac{2x + \bar U_\theta(x)}{1-x^2}, \quad a''_{c,\gamma}(x) = \frac{2 + \bar U_\theta'(x)}{1-x^2} + \frac{4x^2 + 2x\bar U_\theta(x)}{(1-x^2)^2}. \tag{5.53}$$

Next, formally define the maps $W_\theta^{c,\gamma,i}$, $i = 1, 2a, 2b, 3$, on ξ_θ by

$$
\begin{aligned}
W_\theta^{c,\gamma,1}(\xi)(x) &:= e^{-a_{c,\gamma}(x)} \int_0^x e^{a_{c,\gamma}(s)} \frac{\xi_\theta(s)}{1-s^2} ds, \\
W_\theta^{c,\gamma,2a}(\xi)(x) &:= e^{-a_{c,\gamma}(x)} \int_{-1}^x e^{a_{c,\gamma}(s)} \frac{\xi_\theta(s)}{1-s^2} ds, \\
W_\theta^{c,\gamma,2b}(\xi)(x) &:= e^{-a_{c,\gamma}(x)} \int_{1}^x e^{a_{c,\gamma}(s)} \frac{\xi_\theta(s)}{1-s^2} ds, \\
W_\theta^{c,\gamma,3}(\xi)(x) &:= e^{-a_{c,\gamma}(x)} \int_{-1}^x e^{a_{c,\gamma}(s)} \left(\frac{\xi_\theta(s)}{1-s^2} - C_W^{c,\gamma}(\xi_\theta) \right) ds,
\end{aligned}
\tag{5.54}
$$

where

$$
C_W^{c,\gamma}(\xi_\theta) := \frac{1}{\int_{-1}^1 e^{a_{c,\gamma}(s)} ds} \int_{-1}^1 e^{a_{c,\gamma}(s)} \frac{\xi_\theta(s)}{1-s^2} ds.
$$

Define a map $W_\phi^{c,\gamma}$ on ξ_ϕ by

$$
W_\phi^{c,\gamma}(\xi)(x) := \int_0^x e^{-b_{c,\gamma}(t)} \int_0^t e^{b_{c,\gamma}(s)} \frac{\xi_\phi(s)}{1-s^2} ds dt. \tag{5.55}
$$

A calculation gives

$$
\begin{aligned}
(W_\theta^{c,\gamma,i}(\xi))'(x) &= -a'_{c,\gamma}(x) W_\theta^{c,\gamma,i}(x) + \frac{\xi_\theta(x)}{1-x^2}, \quad i = 1, 2a, 2b, \\
(W_\theta^{c,\gamma,3}(\xi))'(x) &= -a'_{c,\gamma}(x) W_\theta^{c,\gamma,3}(x) + \frac{\xi_\theta(x)}{1-x^2} - C_W^{c,\gamma}(\xi_\theta),
\end{aligned}
\tag{5.56}
$$

$$
(W_\phi^{c,\gamma}(\xi))'(x) = e^{-b_{c,\gamma}(x)} \int_0^x e^{b_{c,\gamma}(s)} \frac{\xi_\phi(s)}{1-s^2} ds, \tag{5.57}
$$

$$
(W_\phi^{c,\gamma}(\xi))''(x) = -b'_{c,\gamma}(x) (W_\phi^{c,\gamma}(\xi))'(x) + \frac{\xi_\phi(x)}{1-x^2}. \tag{5.58}
$$

We will prove in the following subsections that $W^{c,\gamma} = (W_\theta^{c,\gamma}, W_\phi^{c,\gamma})$ is, roughly speaking, a right inverse of $L_0^{c,\gamma}$.

Consider the following system of ordinary differential equations in $(-1, 1)$:

$$
\begin{cases}
(1-x^2)V_\theta' + (2x + \bar{U}_\theta)V_\theta + \frac{1}{2}(l_{c,\gamma}[\tilde{V}_\theta])''(0)(1-x^2) = 0, \\
(1-x^2)V_\phi'' + \bar{U}_\theta V_\phi' = 0.
\end{cases}
$$

All solutions $V = (V_\theta, V_\phi) \in C^1((-1,1), \mathbb{R}^2)$ are given by

$$
V = d_1 V_{c,\gamma}^1 + d_2 V_{c,\gamma}^2 + d_3 V_{c,\gamma}^3 + d_4 V_{c,\gamma}^4 \tag{5.59}
$$

where $d_1, d_2, d_3, d_4 \in \mathbb{R}$, and

$$V^1_{c,\gamma} := \begin{pmatrix} e^{-a_{c,\gamma}(x)} \\ 0 \end{pmatrix}, \quad V^2_{c,\gamma} := \begin{pmatrix} e^{-a_{c,\gamma}(x)} \int_0^x e^{a_{c,\gamma}(s)} ds \\ 0 \end{pmatrix},$$
$$V^3_{c,\gamma} := \begin{pmatrix} 0 \\ \int_0^x e^{-b_{c,\gamma}(t)} dt \end{pmatrix}, \quad V^4_{c,\gamma} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5.60}$$

Moreover, denote

$$V^{2a}_{c,\gamma} := \begin{pmatrix} e^{-a_{c,\gamma}(x)} \int_{-1}^x e^{a_{c,\gamma}(s)} ds \\ 0 \end{pmatrix}, \tag{5.61}$$

and

$$V^{2b}_{c,\gamma} := \begin{pmatrix} e^{-a_{c,\gamma}(x)} \int_{1}^x e^{a_{c,\gamma}(s)} ds \\ 0 \end{pmatrix}. \tag{5.62}$$

It can be seen that $V^{2a}_{c,\gamma} = V^2_{c,\gamma} + V^1_{c,\gamma} \int_{-1}^0 e^{a_{c,\gamma}(s)} ds$, and $V^{2b}_{c,\gamma} = V^2_{c,\gamma} - V^1_{c,\gamma} \int_0^1 e^{a_{c,\gamma}(s)} ds$. Next, introduce the linear functionals l_i, $1 \le i \le 4$ acting on vector-valued functions $V(x) = (V_\theta(x), V_\phi(x))$ by

$$l_1(V) := V_\theta(0), \quad l_2(V) := V'_\theta(0), \quad l_3(V) = V'_\phi(0), \quad l_4(V) = V_\phi(0). \tag{5.63}$$

By computation it can be checked that $(l_i(V^j_{c,\gamma}))$ is an invertible matrix, and it is also invertible if we replace $V^2_{c,\gamma}$ by $V^{2a}_{c,\gamma}$ or $V^{2b}_{c,\gamma}$ in this matrix.

5.2.2 Existence of axisymmetric, with swirl solutions around $U^{c,\gamma}$, when $(c, \gamma) \in E_{k,l}$ with $(k, l) \in \mathscr{A}_1$

Let us start from constructing the Banach spaces we use. Given a compact subset $K \subset E_{k,l}$ with $(k, l) \in \mathscr{A}_1$, we have $U^{c,\gamma}_\theta(-1) < 3$ and $U^{c,\gamma}_\theta(1) > -3$. So there exists an $0 < \epsilon < \frac{1}{2}$, depending only on K, satisfying that ,

$$\epsilon > \left(\frac{U^{c,\gamma}_\theta(-1)}{4}\right) \chi_{\{U^{c,\gamma}_\theta(-1)<2\}}(U^{c,\gamma}_\theta(-1)) + \left(\frac{U^{c,\gamma}_\theta(-1)}{2} - 1\right) \chi_{\{U^{c,\gamma}_\theta(-1)\ge 2\}}(U^{c,\gamma}_\theta(-1))$$

and

$$\epsilon > \left(-\frac{U^{c,\gamma}_\theta(1)}{4}\right) \chi_{\{U^{c,\gamma}_\theta(1)>-2\}}(U^{c,\gamma}_\theta(1)) + \left(-\frac{U^{c,\gamma}_\theta(1)}{2} - 1\right) \chi_{\{U^{c,\gamma}_\theta(1)\le -2\}}(U^{c,\gamma}_\theta(1))$$

for all $(c, \gamma) \in K$.

Denote $\bar{U}_\theta = U_\theta^{c,\gamma}$. Choose a fixed ϵ as above, define

$$
\begin{aligned}
\mathbf{M}_1 =& \mathbf{M}_1(\epsilon) := \{\tilde{U}_\theta \in C^3\left(-\frac{1}{2}, \frac{1}{2}\right) \cap C^1(-1,1) \cap C[-1,1] \mid \tilde{U}_\theta(-1) = \tilde{U}_\theta(1) = 0, \\
& \|(1-x^2)^{-1+2\epsilon}\tilde{U}_\theta\|_{L^\infty(-1,1)} < \infty, \|(1-x^2)^{2\epsilon}\tilde{U}_\theta'\|_{L^\infty(-1,1)} < \infty, \|\tilde{U}_\theta''\|_{L^\infty(-\frac{1}{2},\frac{1}{2})} < \infty, \\
& \|\tilde{U}_\theta'''\|_{L^\infty(-\frac{1}{2},\frac{1}{2})} < \infty\}, \\
\mathbf{M}_2 =& \mathbf{M}_2(\epsilon) := \{\tilde{U}_\phi \in C^2((-1,1),\mathbb{R}) \mid \|(1-x^2)^{\epsilon}\tilde{U}_\phi\|_{L^\infty(-1,1)} < \infty, \|(1-x^2)^{1+\epsilon}\tilde{U}_\phi'\|_{L^\infty(-1,1)} < \infty, \\
& \|(1-x^2)^{2+\epsilon}\tilde{U}_\phi''\|_{L^\infty(-1,1)} < \infty\}
\end{aligned}
$$

with the following norms accordingly

$$
\begin{aligned}
\|\tilde{U}_\theta\|_{\mathbf{M}_1} &= \|(1-x^2)^{-1+2\epsilon}\tilde{U}_\theta\|_{L^\infty(-1,1)} + \|(1-x^2)^{2\epsilon}\tilde{U}_\theta'\|_{L^\infty(-1,1)} + \|\tilde{U}_\theta''\|_{L^\infty(-\frac{1}{2},\frac{1}{2})} + \|\tilde{U}_\theta'''\|_{L^\infty(-\frac{1}{2},\frac{1}{2})}, \\
\|\tilde{U}_\phi\|_{\mathbf{M}_2} &= \|(1-x^2)^{\epsilon}\tilde{U}_\phi\|_{L^\infty(-1,1)} + \|(1-x^2)^{1+\epsilon}\tilde{U}_\phi'\|_{L^\infty(-1,1)} + \|(1-x^2)^{2+\epsilon}\tilde{U}_\phi''\|_{L^\infty(-1,1)}.
\end{aligned}
$$

Next define the following function spaces:

$$
\begin{aligned}
\mathbf{N}_1 = \mathbf{N}_1(\epsilon) := \{&\xi_\theta \in C^2(-\frac{1}{2}, \frac{1}{2}) \cap C((-1,1),\mathbb{R}) \mid \xi_\theta(-1) = \xi_\theta(1) = \xi_\theta''(0) = 0, \\
& \|(1-x^2)^{-1+2\epsilon}\xi_\theta\|_{L^\infty(-1,1)} < \infty, \|\xi_\theta'\|_{L^\infty(-\frac{1}{2},\frac{1}{2})} < \infty, \|\xi_\theta''\|_{L^\infty(-\frac{1}{2},\frac{1}{2})} < \infty\}, \\
\mathbf{N}_2 = \mathbf{N}_2(\epsilon) := \{&\xi_\phi \in C((-1,1),\mathbb{R}) \mid \|(1-x^2)^{1+\epsilon}\xi_\phi\|_{L^\infty(-1,1)} < \infty\},
\end{aligned}
$$

with the following norms accordingly:

$$
\begin{aligned}
&\|\xi_\theta\|_{\mathbf{N}_1} := \|(1-x^2)^{-1+2\epsilon}\xi_\theta\|_{L^\infty(-1,1)} + \|\xi_\theta'\|_{L^\infty(-\frac{1}{2},\frac{1}{2})} + \|\xi_\theta''\|_{L^\infty(-\frac{1}{2},\frac{1}{2})}, \\
&\|\xi_\phi\|_{\mathbf{N}_2} := \|(1-x^2)^{1+\epsilon}\xi_\phi\|_{L^\infty(-1,1)}.
\end{aligned}
$$

Then let $\mathbf{X} := \{\tilde{U} = (\tilde{U}_\theta, \tilde{U}_\phi) \mid \tilde{U}_\theta \in \mathbf{M}_1, \tilde{U}_\phi \in \mathbf{M}_2\}$ with norm $\|\tilde{U}\|_{\mathbf{X}} = \|\tilde{U}_\theta\|_{\mathbf{M}_1} + \|\tilde{U}_\phi\|_{\mathbf{M}_2}$, $\mathbf{Y} := \{\xi = (\xi_\theta, \xi_\phi) \mid \xi_\theta \in \mathbf{N}_1, \xi_\phi \in \mathbf{N}_2\}$, with the norm $\|\xi\|_{\mathbf{Y}} = \|\xi_\theta\|_{\mathbf{N}_1} + \|\xi_\phi\|_{\mathbf{N}_2}$. It can be proved that $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{X}$ and $\mathbf{Y}$ are Banach spaces.

Let $l_i : \mathbf{X} \to \mathbb{R}$, $1 \le i \le 4$, be the bounded linear functionals defined by (5.63) for each $V \in \mathbf{X}$. Define

$$
\mathbf{X}_1 := \ker l_1 \cap \ker l_2 \cap \ker l_3 \cap \ker l_4, \tag{5.64}
$$

It can be seen that $\mathbf{X}_1$ is independent of (c, γ).

Theorem 5.2.1. *For every compact subset $K \subset E_{1,3}$, there exist $\delta = \delta(K) > 0$, and $V \in C^\infty(K \times B_\delta(0), \mathbf{X}_1)$ satisfying $V(c,\gamma,0) = 0$ and $\frac{\partial V}{\partial \beta_i}\big|_{\beta=0} = 0$, $1 \le i \le 4$, $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$, such that*

$$
U = U^{c,\gamma} + \sum_{i=1}^{4} \beta_i V^i_{c,\gamma} + V(c,\gamma,\beta) \tag{5.65}
$$

satisfies equation (5.43*) with* $\hat{c}_1 = c_1 + \frac{1}{2}\psi[\tilde{U}_\phi](-1)$*,* $\hat{c}_2 = c_2 + \frac{1}{2}\psi[\tilde{U}_\phi](1)$*,* $\hat{c}_3 = c_3 - \frac{1}{2}(\varphi_{c,\gamma}[\tilde{U}_\theta])''(0)$*.*

Moreover, there exists some $\delta' = \delta'(K) > 0$*, such that if* $\|U - U^{c,\gamma}\|_{\mathbf{X}} < \delta'$*,* $(c,\gamma) \in K$*, and* U *satisfies equation (*5.43*) with some constant* $\hat{c}_1, \hat{c}_2, \hat{c}_3$*, then (*$5.65$*) holds for some* $|\beta| < \delta$*.*

Define

$$\mathbf{X}_2 := \ker l_1 \cap \ker l_3 \cap \ker l_4. \tag{5.66}$$

We have

Theorem 5.2.2. *For every compact subset* K *of* $E_{1,1}$ *or* $E_{2,1}$*, there exist* $\delta = \delta(K) > 0$*, and* $V \in C^\infty(K \times B_\delta(0), \mathbf{X}_2)$ *satisfying* $V(c,\gamma,0) = 0$ *and* $\dfrac{\partial V}{\partial \beta_i}\Big|_{\beta=0} = 0$*,* $i = 2,3,4$*,* $\beta = (\beta_2, \beta_3, \beta_4)$*, such that*

$$U = U^{c,\gamma} + \beta_2 V^{2a}_{c,\gamma} + \beta_3 V^3_{c,\gamma} + \beta_4 V^4_{c,\gamma} + V(c,\gamma,\beta) \tag{5.67}$$

satisfies equation (5.43*) with* $\hat{c}_1 = c_1 + \frac{1}{2}\psi[\tilde{U}_\phi](-1)$*,* $\hat{c}_2 = c_2 + \frac{1}{2}\psi[\tilde{U}_\phi](1)$*,* $\hat{c}_3 = c_3 - \frac{1}{2}(\varphi_{c,\gamma}[\tilde{U}_\theta])''(0)$*.*

Moreover, there exists some $\delta' = \delta'(K) > 0$*, such that if* $\|U - U^{c,\gamma}\|_{\mathbf{X}} < \delta'$*,* $(c,\gamma) \in K$*, and* U *satisfies equation (*5.43*) with some constant* $\hat{c}_1, \hat{c}_2, \hat{c}_3$*, then (*$5.67$*) holds for some* $|\beta| < \delta$*.*

Theorem 5.2.2'. *For every compact subset* K *of* $E_{1,2}$ *or* $E_{3,2}$*, there exist* $\delta = \delta(K) > 0$*, and* $V \in C^\infty(K \times B_\delta(0), \mathbf{X}_2)$ *satisfying* $V(c,\gamma,0) = 0$ *and* $\dfrac{\partial V}{\partial \beta_i}\Big|_{\beta=0} = 0$*,* $i = 2,3,4$*,* $\beta = (\beta_2, \beta_3, \beta_4)$*, such that*

$$U = U^{c,\gamma} + \beta_2 V^{2b}_{c,\gamma} + \beta_3 V^3_{c,\gamma} + \beta_4 V^4_{c,\gamma} + V(c,\gamma,\beta) \tag{5.68}$$

satisfies equation (5.43*) with* $\hat{c}_1 = c_1 + \frac{1}{2}\psi[\tilde{U}_\phi](-1)$*,* $\hat{c}_2 = c_2 + \frac{1}{2}\psi[\tilde{U}_\phi](1)$*,* $\hat{c}_3 = c_3 - \frac{1}{2}(\varphi_{c,\gamma}[\tilde{U}_\theta])''(0)$*.*

Moreover, there exists some $\delta' = \delta'(K) > 0$*, such that if* $\|U - U^{c,\gamma}\|_{\mathbf{X}} < \delta'$*,* $(c,\gamma) \in K$*, and* U *satisfies equation (*5.43*) with some constant* $\hat{c}_1, \hat{c}_2, \hat{c}_3$*, then (*$5.68$*) holds for some* $|\beta| < \delta$*.*

Define

$$\mathbf{X}_3 := \ker l_3 \cap \ker l_4. \tag{5.69}$$

We have

Theorem 5.2.3. *Let K be a compact set contained in one of $E_{k,l}$ with $5 \le k \le 8$ and $1 \le l \le 3$, there exist $\delta = \delta(K) > 0$, and $V \in C^\infty(K \times B_\delta(0), \mathbf{X}_3)$ satisfying $V(c,\gamma,0) = 0$ and $\frac{\partial V}{\partial \beta_i}\big|_{\beta=0} = 0$, $i = 3,4$, $\beta = (\beta_3, \beta_4)$, such that*

$$U = U^{c,\gamma} + \beta_3 V^3_{c,\gamma} + \beta_4 V^4_{c,\gamma} + V(c,\gamma,\beta) \tag{5.70}$$

satisfies equation (5.43) with $\hat{c}_1 = c_1 + \frac{1}{2}\psi[\tilde{U}_\phi](-1)$, $\hat{c}_2 = c_2 + \frac{1}{2}\psi[\tilde{U}_\phi](1)$, $\hat{c}_3 = c_3 - \frac{1}{2}(\varphi_{c,\gamma}[\tilde{U}_\theta])''(0)$.

Moreover, there exists some $\delta' = \delta'(K) > 0$, such that if $\|U - U^{c,\gamma}\|_{\mathbf{X}} < \delta'$, $(c,\gamma) \in K$, and U satisfies equation (5.43) with some constant $\hat{c}_1, \hat{c}_2, \hat{c}_3$, then (5.70) holds for some $|\beta| < \delta$.

For $\tilde{U}_\phi \in \mathbf{M}_2$, let $\psi[\tilde{U}_\phi](x)$ be defined by (5.44). Let K be a compact set contained in one of $E_{k,l}$ with $k = 1$ or $5 \le k \le 8$ or $(k,l) = (2,1)$ or $(3,2)$. Define a map $G = G(c,\gamma,\tilde{U})$ on $K \times \mathbf{X}$ by (5.46).

Proposition 5.2.1. *The map G is in $C^\infty(K \times \mathbf{X}, \mathbf{Y})$ in the sense that G has continuous Fréchet derivatives of every order. Moreover, the Fréchet derivative of G with respect to $\tilde{U}$ at $(c,\gamma,\tilde{U}) \in K \times \mathbf{X}$ is given by the linear bounded operator $L^{c,\gamma}_{\tilde{U}} : \mathbf{X} \to \mathbf{Y}$ defined as in (5.51).*

To prove Proposition 5.2.1, we first prove the following lemmas:

Lemma 5.2.1. *For every $(c,\gamma) \in K$, $A(c,\gamma,\cdot) : \mathbf{X} \to \mathbf{Y}$ defined by (5.48) is a well-defined bounded linear operator.*

Proof. In the following, C denotes a universal constant which may change from line to line. For convenience we denote $l = l_{c,\gamma}[\tilde{U}_\theta]$ defined by (5.47), and $A = A(c,\gamma,\cdot)$ for some fixed $(c,\gamma) \in K$. We make use of the property of $\bar{U}_\theta$ that $\bar{U}_\theta \in C^2(-1,1) \cap L^\infty(-1,1)$.

A is clearly linear. For every $\tilde{U} \in \mathbf{X}$, we prove that $A\tilde{U}$ defined by (5.48) is in $\mathbf{Y}$ and there exists some constant C such that $\|A\tilde{U}\|_{\mathbf{Y}} \leq C\|\tilde{U}\|_{\mathbf{X}}$ for all $\tilde{U} \in \mathbf{X}$.

By computation,

$$l'(x) = (1-x^2)\tilde{U}_\theta'' + \bar{U}_\theta \tilde{U}_\theta' + (2+\bar{U}_\theta')\tilde{U}_\theta$$

$$l''(x) = (1-x^2)\tilde{U}_\theta''' + (\bar{U}_\theta - 2x)\tilde{U}_\theta'' + 2(\bar{U}_\theta' + 1)\tilde{U}_\theta' + \bar{U}_\theta''\tilde{U}_\theta.$$

By the fact that $\tilde{U}_\theta \in \mathbf{M}_1$, we have,

$$|l''(0)| \leq |\tilde{U}_\theta'''(0)| + (|\bar{U}_\theta(0)|+2)|\tilde{U}_\theta''(0)| + 2(|\bar{U}_\theta'(0)|+1)|\tilde{U}_\theta'(0)| + |\bar{U}_\theta''(0)||\tilde{U}_\theta(0)| \leq C\|\tilde{U}_\theta\|_{\mathbf{M}_1}$$

For $-1 < x < 1$,

$$\begin{aligned}
|(1-x^2)^{-1+2\epsilon}A_\theta| &\leq |(1-x^2)^{-1+2\epsilon}l(x)| + \frac{1}{2}(l''(0))(1-x^2)^{2\epsilon} \\
&\leq |(1-x^2)^{2\epsilon}\tilde{U}_\theta'| + (2+|\bar{U}_\theta|)(1-x^2)^{-1+2\epsilon}|\tilde{U}_\theta| + \frac{1}{2}(1-x^2)^{2\epsilon}|l''(0)| \\
&\leq C\|\tilde{U}_\theta\|_{\mathbf{M}_1}.
\end{aligned}$$

For $-\frac{1}{2} < x < \frac{1}{2}$, we have

$$\begin{aligned}
|A_\theta'| &= |l'(x) - l''(0)x| \\
&\leq |\tilde{U}_\theta''| + |\bar{U}_\theta||\tilde{U}_\theta'| + (2+|\bar{U}_\theta'|)|\tilde{U}_\theta| + |l''(0)| \\
&\leq C\|\tilde{U}_\theta\|_{\mathbf{M}_1},
\end{aligned}$$

and

$$\begin{aligned}
|A_\theta''| &= |l''(x) - l''(0)| \\
&\leq |\tilde{U}_\theta'''| + (|\bar{U}_\theta|+2)|\tilde{U}_\theta''| + 2(|\bar{U}_\theta'|+1)|\tilde{U}_\theta'| + |\bar{U}_\theta''||\tilde{U}_\theta| + |l''(0)| \\
&\leq C\|\tilde{U}_\theta\|_{\mathbf{M}_1}
\end{aligned}$$

We also see from the above that $\lim_{x\to 1^-} A_\theta(x) = \lim_{x\to -1^+} A_\theta(x) = 0$. By computation $A_\theta''(0) = 0$. So we have $A_\theta \in \mathbf{N}_1$ and $\|A_\theta\|_{\mathbf{N}_1} \leq C\|\tilde{U}_\theta\|_{\mathbf{M}_1}$.

Next, since $A_\phi = (1-x^2)\tilde{U}_\phi'' + \bar{U}_\theta\tilde{U}_\phi'$, by the fact that $\tilde{U}_\phi \in \mathbf{M}_2$ we have that

$$\left|(1-x^2)^{1+\epsilon}A_\phi\right| \leq (1-x^2)^{2+\epsilon}|\tilde{U}_\phi''| + (1-x^2)^{1+\epsilon}|\bar{U}_\theta||\tilde{U}_\phi'| \leq C\|\tilde{U}_\phi\|_{\mathbf{M}_2}.$$

So $A_\phi \in \mathbf{N}_1$, and $\|A_\phi\|_{\mathbf{N}_1} \leq C\|\tilde{U}_\phi\|_{\mathbf{M}_2}$. We have proved that $A\tilde{U} \in \mathbf{Y}$ and $\|A\tilde{U}\|_{\mathbf{Y}} \leq C\|\tilde{U}\|_{\mathbf{X}}$ for every $\tilde{U} \in \mathbf{X}$. The proof is finished. $\square$

Lemma 5.2.2. *The map $Q : \mathbf{X} \times \mathbf{X} \to \mathbf{Y}$ defined by (5.49) is a well-defined bounded bilinear operator.*

Proof. In the following, C denotes a universal constant which may change from line to line. For convenience we denote $\psi = \psi[\tilde{U}_\phi, \tilde{V}_\phi]$ defined by (5.44).

It is clear that Q is a bilinear operator. For every $\tilde{U}, \tilde{V} \in \mathbf{X}$, we will prove that $Q(\tilde{U}, \tilde{V})$ is in $\mathbf{Y}$ and there exists some constant C independent of $\tilde{U}$ and $\tilde{V}$ such that $\|Q(\tilde{U}, \tilde{V})\|_{\mathbf{Y}} \le C\|\tilde{U}\|_{\mathbf{X}}\|\tilde{V}\|_{\mathbf{X}}$.

For $\tilde{U}, \tilde{V} \in \mathbf{X}$, we have, using the fact that $\tilde{U}_\phi, \tilde{V}_\phi \in \mathbf{M_2}$, that

$$\left|\frac{\tilde{U}_\phi(s)\tilde{V}'_\phi(s)}{1-s^2}\right| \le (1-s^2)^{-2-2\epsilon}\|\tilde{U}_\phi\|_{\mathbf{M_2}}\|\tilde{V}_\phi\|_{\mathbf{M_2}}, \quad \forall -1 < s < 1. \tag{5.71}$$

It follows that $\psi(\tilde{U}, \tilde{V})(x)$ is well-defined and

$$|\psi(-1)| \le C\|\tilde{U}_\phi\|_{\mathbf{M_2}}\|\tilde{V}_\phi\|_{\mathbf{M_2}}, \quad |\psi(1)| \le C\|\tilde{U}_\phi\|_{\mathbf{M_2}}\|\tilde{V}_\phi\|_{\mathbf{M_2}}. \tag{5.72}$$

Moreover,

$$\begin{aligned}
&\left|\psi(x) - \frac{1}{2}\psi(-1)(1-x) - \frac{1}{2}\psi(1)(1+x)\right| \\
&= \left|\frac{1}{2}\psi(x)(1-x) + \frac{1}{2}\psi(x)(1+x) - \frac{1}{2}\psi(-1)(1-x) - \frac{1}{2}\psi(1)(1+x)\right| \\
&\le \frac{1}{2}(1-x)|\psi(x) - \psi(-1)| + \frac{1}{2}(1+x)|\psi(x) - \psi(1)| \\
&= \frac{1}{2}(1-x)\left|\int_{-1}^{x}\int_0^l\int_0^t \frac{2\tilde{U}_\phi(s)\tilde{V}'_\phi(s)}{1-s^2}dsdtdl\right| + \frac{1}{2}(1+x)\left|\int_1^x\int_0^l\int_0^t \frac{2\tilde{U}_\phi(s)\tilde{V}'_\phi(s)}{1-s^2}dsdtdl\right| \\
&\le C(1-x)(1+x)^{1-2\epsilon}\|\tilde{U}_\phi\|_{\mathbf{M_2}}\|\tilde{V}_\phi\|_{\mathbf{M_2}} + C(1+x)(1-x)^{1-2\epsilon}\|\tilde{U}_\phi\|_{\mathbf{M_2}}\|\tilde{V}_\phi\|_{\mathbf{M_2}} \\
&\le C(1-x^2)^{1-2\epsilon}\|\tilde{U}_\phi\|_{\mathbf{M_2}}\|\tilde{V}_\phi\|_{\mathbf{M_2}}
\end{aligned} \tag{5.73}$$

By (5.71), we also have

$$|\psi'(x)| = \left|\int_0^x\int_0^t \frac{2\tilde{U}_\phi(s)\tilde{V}'_\phi(s)}{1-s^2}dsdt\right| \le C\|\tilde{U}_\phi\|_{\mathbf{M_2}}\|\tilde{V}_\phi\|_{\mathbf{M_2}}, \quad -\frac{1}{2} < x < \frac{1}{2} \tag{5.74}$$

and

$$|\psi''(x)| = \left|\int_0^x \frac{2\tilde{U}_\phi(s)\tilde{V}'_\phi(s)}{1-s^2}ds\right| \le C\|\tilde{U}_\phi\|_{\mathbf{M_2}}\|\tilde{V}_\phi\|_{\mathbf{M_2}}, \quad -\frac{1}{2} < x < \frac{1}{2} \tag{5.75}$$

Using the fact that $\tilde{U}_\theta, \tilde{V}_\theta \in \mathbf{M}_1$, we have

$$\begin{aligned} |(\tilde{U}_\theta \tilde{V}_\theta)''(0)| &\leq |\tilde{U}''_\theta(0)||\tilde{V}_\theta(0)| + 2|\tilde{U}'_\theta(0)||\tilde{V}'_\theta(0)| + |\tilde{U}_\theta(0)||\tilde{V}''_\theta(0)| \\ &\leq C\|\tilde{U}_\theta\|_{\mathbf{M}_1}\|\tilde{V}_\theta\|_{\mathbf{M}_1} \end{aligned} \tag{5.76}$$

So by (5.73), (5.76), and the fact that $\tilde{U}_\theta, \tilde{V}_\theta \in \mathbf{M}_1$, we have that for $-1 < x < 1$,

$$\begin{aligned} &|(1-x^2)^{-1+2\epsilon} Q_\theta(x)| \\ &\leq \frac{1}{2}|(1-x^2)^{-1+2\epsilon}\tilde{U}_\theta(x)||\tilde{V}_\theta(x)| + (1-x^2)^{-1+2\epsilon}\left|\psi(x) - \frac{1}{2}\psi(-1)(1-x) - \frac{1}{2}\psi(1)(1+x)\right| \\ &+ \frac{1}{4}(1-x^2)^{2\epsilon}|(\tilde{U}_\theta\tilde{V}_\theta)''(0)| \\ &\leq \frac{1}{2}\|\tilde{U}_\theta\|_{\mathbf{M}_1}\|\tilde{V}_\theta\|_{\mathbf{M}_1} + C\|\tilde{U}_\phi\|_{\mathbf{M}_2}\|\tilde{V}_\phi\|_{\mathbf{M}_2} + C(1-x^2)^{2\epsilon}\|\tilde{U}_\theta\|_{\mathbf{M}_1}\|\tilde{V}_\theta\|_{\mathbf{M}_1} \\ &\leq C\|\tilde{U}\|_{\mathbf{X}}\|\tilde{V}\|_{\mathbf{X}}. \end{aligned}$$

From this we also have $\lim_{x\to 1} Q_\theta(x) = \lim_{x\to -1} Q_\theta(x) = 0$.

By (5.74), (5.75), (5.76), and the fact that $\tilde{U}_\theta, \tilde{V}_\theta \in \mathbf{M}_1$, we have that for $-\frac{1}{2} < x < \frac{1}{2}$,

$$\begin{aligned} |Q'_\theta(x)| &= \left|\frac{1}{2}(\tilde{U}'_\theta\tilde{V}_\theta + \tilde{U}_\theta\tilde{V}'_\theta) + \psi'(x) + \frac{1}{2}(\psi(-1) - \psi(1)) - \frac{1}{2}(\tilde{U}_\theta\tilde{V}_\theta)''(0)x\right| \\ &\leq C\|\tilde{U}\|_{\mathbf{X}}\|\tilde{V}\|_{\mathbf{X}}, \end{aligned}$$

and

$$\begin{aligned} |Q''_\theta(x)| &= \left|\frac{1}{2}(\tilde{U}_\theta\tilde{V}_\theta)''(x) + \psi''(x) - \frac{1}{2}(\tilde{U}_\theta\tilde{V}_\theta)''(0)\right| \\ &= \left|\frac{1}{2}(\tilde{U}''_\theta\tilde{V}_\theta + 2\tilde{U}'_\theta\tilde{V}'_\theta + \tilde{U}_\theta\tilde{V}''_\theta) + \psi''(x) - \frac{1}{2}(\tilde{U}_\theta\tilde{V}_\theta)''(0)\right| \\ &\leq C\|\tilde{U}\|_{\mathbf{X}}\|\tilde{V}\|_{\mathbf{X}} \end{aligned}$$

From the above we also have $Q''_\theta(0) = 0$. So there is $Q_\theta \in \mathbf{N}_1$, and $\|Q_\theta\|_{\mathbf{N}_1} \leq C(\epsilon)\|\tilde{U}\|_{\mathbf{X}}\|\tilde{V}\|_{\mathbf{X}}$.

Next, since $Q_\phi(x) = \tilde{U}_\theta(x)\tilde{V}'_\phi(x)$, for $-1 < x < 1$,

$$\left|(1-x^2)^{1+\epsilon}Q_\phi(x)\right| \leq |\tilde{U}_\theta(x)|(1-x^2)^{1+\epsilon}|\tilde{V}'_\phi| \leq 2\|\tilde{U}_\theta\|_{\mathbf{M}_1}\|\tilde{V}_\phi\|_{\mathbf{M}_2}.$$

So $Q_\phi \in \mathbf{N}_2$, and $\|Q_\phi\|_{\mathbf{N}_2} \leq \|\tilde{U}_\theta\|_{\mathbf{M}_1}\|\tilde{V}_\phi\|_{\mathbf{M}_2}$. Thus we have proved that $Q(\tilde{U}, \tilde{V}) \in \mathbf{Y}$ and $\|Q(\tilde{U}, \tilde{V})\|_{\mathbf{Y}} \leq C\|\tilde{U}\|_{\mathbf{X}}\|\tilde{V}\|_{\mathbf{X}}$ for all $\tilde{U}, \tilde{V} \in \mathbf{X}$. Lemma 5.2.2 is proved. □

Proof of Proposition 5.2.1*:* By definition, $G(c,\gamma,\tilde U)=A(c,\gamma,\tilde U)+Q(\tilde U,\tilde U)$ for $(c,\gamma,\tilde U)\in K\times\mathbf{X}$. Using standard theories in functional analysis, by Lemma 5.2.2 it is clear that Q is C^∞ on $\mathbf{X}$. By Lemma 5.2.1, $A(c,\gamma;\cdot):\mathbf{X}\to\mathbf{Y}$ is continuous for each $(c,\gamma)\in K$. Let $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ be a multi-index where $\alpha_i\ge 0$, $i=1,2,3$, and $j\ge 0$. For all $|\alpha|+j\ge 1$, we have

$$\partial_c^\alpha\partial_\gamma^j A(c,\gamma,\tilde U)=\partial_c^\alpha\partial_\gamma^j U_\theta^{c,\gamma}\begin{pmatrix}\tilde U_\theta\\ \tilde U_\phi'\end{pmatrix}+\frac{1}{2}(\partial_c^\alpha\partial_\gamma^j U_\theta^{c,\gamma}\cdot\tilde U_\theta)''(0)\begin{pmatrix}1-x^2\\ 0\end{pmatrix}. \tag{5.77}$$

By Theorem 5.1.3 (i) and Corollary 5.1.3, we have

$$|(1-x^2)^{-1+2\epsilon}\partial_c^\alpha\partial_\gamma^j A_\theta(c,\gamma,\tilde U)|\le C(\alpha,j,K)\|\tilde U_\theta\|_{\mathbf{M}_1},\quad -1<x<1,$$

and for $-\frac{1}{2}<x<\frac{1}{2}$.

$$|\partial_c^\alpha\partial_\gamma^j A_\theta'(c,\gamma,\tilde U)|\le C(\alpha,j,K)\|\tilde U_\theta\|_{\mathbf{M}_1},\quad |\partial_c^\alpha\partial_\gamma^j A_\theta''(c,\gamma,\tilde U)|\le C(\alpha,j,K)\|\tilde U_\theta\|_{\mathbf{M}_1}.$$

The above estimates and (5.77) also implies that

$$\partial_c^\alpha\partial_\gamma^j A_\theta(c,\gamma,\tilde U)(-1)=\partial_c^\alpha\partial_\gamma^j A_\theta(c,\gamma,\tilde U)(1)=\partial_c^\alpha\partial_\gamma^j A_\theta(c,\gamma,\tilde U)''(0)=0.$$

So $\partial_c^\alpha\partial_\gamma^j A_\theta(c,\gamma,\tilde U)\in\mathbf{N}_1$, with $\|\partial_c^\alpha\partial_\gamma^j A_\theta(c,\gamma,\tilde U)\|_{\mathbf{N}_1}\le C(\alpha,j,K)\|\tilde U_\theta\|_{\mathbf{M}_1}$ for all $(c,\gamma,\tilde U)\in K\times\mathbf{X}$.

Next, by Theorem 5.1.3 (i) and the fact that $\tilde U_\phi\in\mathbf{M}_1$, we have

$$(1-x^2)^{1+\epsilon}|\partial_c^\alpha\partial_\gamma^j A_\phi(c,\gamma,\tilde U)(x)|=|\partial_c^\alpha\partial_\gamma^j U_\theta^{c,\gamma}(x)|\cdot|(1-x^2)^{1+\epsilon}\tilde U_\phi'|\le C(\alpha,j,K)\|\tilde U_\phi\|_{\mathbf{M}_2}. \tag{5.78}$$

So $\partial_c^\alpha\partial_\gamma^j A_\phi(c,\gamma,\tilde U)\in\mathbf{N}_2$ with $\|\partial_c^\alpha\partial_\gamma^j A_\phi(c,\gamma,\tilde U)\|_{\mathbf{N}_2}\le C(\alpha,j,K)\|\tilde U_\phi\|_{\mathbf{M}_2}$ for all $(c,\gamma,\tilde U)\in K\times\mathbf{X}$. Thus $\partial_c^\alpha\partial_\gamma^j A(c,\gamma,\tilde U)\in\mathbf{Y}$, with $\|\partial_c^\alpha\partial_\gamma^j A(c,\gamma,\tilde U)\|_{\mathbf{Y}}\le C(\alpha,j,K)\|\tilde U\|_{\mathbf{X}}$ for all $(c,\gamma,\tilde U)\in K\times\mathbf{X}$, $|\alpha|+j\ge 1$.

So for each $(c,\gamma)\in K$, $\partial_c^\alpha\partial_\gamma^j A(c,\gamma;\cdot):\mathbf{X}\to\mathbf{Y}$ is a bounded linear map with uniform bounded norm on K. Then by standard theories in functional analysis, $A:K\times\mathbf{X}\to\mathbf{Y}$ is C^∞. So G is a C^∞ map from $K\times\mathbf{X}$ to Y. By direct calculation we get its Fréchet derivative with respect to $\mathbf{X}$ is given by the linear bounded operator $L_{\tilde U}^{c,\gamma}:\mathbf{X}\to\mathbf{Y}$ defined as (5.50). The proof is finished. □

Let $a_{c,\gamma}(x), b_{c,\gamma}(x)$ be the functions defined by (5.52). For convenience we denote $a(x) = a_{c,\gamma}(x)$ and $b(x) = b_{c,\gamma}(x)$. We have

Lemma 5.2.3. *For $(c,\gamma) \in E_{k,l}$ with $(k,l) \in \mathscr{A}_1$, there exists some constant $C > 0$, depending only on (c,γ), such that for any $-1 < x < 1$,*

$$e^{b(x)} \le C(1+x)^{\frac{U_\theta^{c,\gamma}(-1)}{2}}(1-x)^{-\frac{U_\theta^{c,\gamma}(1)}{2}}, \quad e^{-b(x)} \le C(1+x)^{-\frac{U_\theta^{c,\gamma}(-1)}{2}}(1-x)^{\frac{U_\theta^{c,\gamma}(1)}{2}}, \tag{5.79}$$

and

$$e^{a(x)} \le C(1+x)^{\frac{U_\theta^{c,\gamma}(-1)}{2}-1}(1-x)^{-1-\frac{U_\theta^{c,\gamma}(1)}{2}}, \quad e^{-a(x)} \le C(1+x)^{1-\frac{U_\theta^{c,\gamma}(-1)}{2}}(1-x)^{1+\frac{U_\theta^{c,\gamma}(1)}{2}}. \tag{5.80}$$

Proof. Denote $\bar{U}_\theta := U_\theta^{c,\gamma}$, let

$$\alpha_0 = \min\left\{1, \left(1 - \frac{\bar{U}_\theta(-1)}{2}\right)\chi_{\{\bar{U}_\theta(-1)<2\}} + \chi_{\{\bar{U}_\theta(-1)\ge 2\}}, \left(1 + \frac{\bar{U}_\theta(1)}{2}\right)\}\chi_{\{\bar{U}_\theta(1)>-2\}} + \chi_{\{\bar{U}_\theta(1)\le -2\}}\right\}.$$

Since $(c,\gamma) \in E_{k,l}$ with $(k,l) \in \mathscr{A}_1$, we have $\bar{U}_\theta(-1) < 3$ and $\bar{U}_\theta(-1) \ne 2$, or $\bar{U}_\theta(-1) = 2$ with $\eta_1 = 0$, and $\bar{U}_\theta(1) > -3$ and $\bar{U}_\theta(1) \ne -2$, or $\bar{U}_\theta(1) = -2$ with $\eta_2 = 0$. According to Theorem 1.0.3 and Theorem 4.0.1', we then have

$$\bar{U}_\theta = \bar{U}_\theta(-1) + O((1+x)^{\alpha_0}) = \bar{U}_\theta(1) + O((1-x)^{\alpha_0}), \quad -1 < x < 1.$$

Thus, by definition of $a(x)$ and $b(x)$ in (5.52), for $-1 < x < 1$, we have

$$\begin{aligned} b(x) &= \frac{\bar{U}_\theta(-1)}{2}\ln(1+x) - \frac{\bar{U}_\theta(1)}{2}\ln(1-x) + O(1) \\ a(x) &= \left(\frac{\bar{U}_\theta(-1)}{2} - 1\right)\ln(1+x) - \left(\frac{\bar{U}_\theta(1)}{2} + 1\right)\ln(1-x) + O(1). \end{aligned} \tag{5.81}$$

The lemma then follows from the above estimates. $\square$

For $\xi = (\xi_\theta, \xi_\phi) \in \mathbf{Y}$, let the map $W^{c,\gamma}$ be defined as

$$W^{c,\gamma}(\xi) := (W_\theta^{c,\gamma}(\xi), W_\phi^{c,\gamma}(\xi)),$$

where

$$W_\theta^{c,\gamma}(\xi) = \begin{cases} W_\theta^{c,\gamma,1}(\xi) & \text{if } (c,\gamma) \in E_{1,3} \\ W_\theta^{c,\gamma,2a}(\xi) & \text{if } (c,\gamma) \in E_{1,1} \text{ or } E_{2,1} \\ W_\theta^{c,\gamma,2b}(\xi) & \text{if } (c,\gamma) \in E_{1,2} \text{ or } E_{3,2} \\ W_\theta^{c,\gamma,3}(\xi) & \text{if } (c,\gamma) \in E_{k,l} \text{ for } 5 \le k \le 8, \text{ and } 1 \le l \le 3. \end{cases}$$

$W_\theta^{c,\gamma,i}$, $i = 1, 2a, 2b, 3$ are defined by (5.54), and $W_\phi^{c,\gamma}(\xi)$ is defined by (5.55).

Lemma 5.2.4. *For every* $(c,\gamma) \in K$, $W^{c,\gamma} : \mathbf{Y} \to \mathbf{X}$ *is continuous, and is a right inverse of* $L_0^{c,\gamma}$.

Proof. In the following, C denotes a universal constant which may change from line to line. We make use of the property that $\bar{U}_\theta \in C^2(-1,1) \cap L^\infty(-1,1)$ and the range of ϵ. For convenience let us write $W := W^{c,\gamma}(\xi)$ and $W_\theta^i := W_\theta^{c,\gamma,i}(\xi)$ for $\xi \in \mathbf{Y}$.

By Lemma 5.2.3, we have the estimates (5.79) and (5.80).

We first prove W_θ is well-defined.

Claim. There exists $C > 0$, such that

$$|(1-x^2)^{-1+2\epsilon} W_\theta(x)| \le C\|\xi_\theta\|_{\mathbf{N}_1} \tag{5.82}$$

Proof. We prove the claim for each W^i, $i = 1, 2a, 2b, 3$.

Case 1. $(c,\gamma) \in E_{1,3}$, then $\bar{U}_\theta(-1) < 2$ and $\bar{U}_\theta(1) > -2$.

Using the fact that $\xi_\theta \in \mathbf{N}_1$, in the expression of $W_\theta = W_\theta^1$ in (5.54),

$$|(1-x^2)^{-1+2\epsilon} W_\theta^1(x)| \le (1-x^2)^{-1+2\epsilon} \|\xi_\theta\|_{\mathbf{N}_1} e^{-a(x)} \int_0^x e^{a(s)} (1-s^2)^{-2\epsilon} ds, \quad -1 < x < 1.$$

Applying (5.80) in Lemma 5.2.3, using the fact that $4\epsilon > \max\{\bar{U}_\theta(-1), -\bar{U}_\theta(1)\}$, we have

$$\begin{aligned}
&|(1-x^2)^{-1+2\epsilon} W_\theta^1(x)| \\
&\le \|\xi_\theta\|_{\mathbf{N}_1} (1+x)^{-\frac{\bar{U}_\theta(-1)}{2}+2\epsilon} (1-x)^{\frac{\bar{U}_\theta(1)}{2}+2\epsilon} \int_0^x (1+s)^{\frac{\bar{U}_\theta(-1)}{2}-1-2\epsilon} (1-s)^{-1-\frac{\bar{U}_\theta(1)}{2}-2\epsilon} ds \\
&\le C\|\xi_\theta\|_{\mathbf{N}_1} \left(1 + (1+x)^{-\frac{\bar{U}_\theta(-1)}{2}+2\epsilon}\right) \left(1 + (1-x)^{\frac{\bar{U}_\theta(1)}{2}+2\epsilon}\right) \\
&\le C\|\xi_\theta\|_{\mathbf{N}_1}
\end{aligned} \tag{5.83}$$

Case 2. $(c,\gamma) \in E_{1,1}$ or $E_{2,1}$, then $2 < \bar{U}_\theta(-1) < 3$ or $\bar{U}_\theta(-1) = 2$ with $\eta_1 = 0$, and $\bar{U}_\theta(1) > -2$

Using the fact that $\xi_\theta \in \mathbf{N}_1$, and (5.80) we first have

$$\int_{-1}^0 e^{a(s)} \frac{|\xi_\theta(s)|}{1-s^2} ds \le C\|\xi_\theta\|_{\mathbf{N}_1} \int_{-1}^0 (1+s)^{\frac{\bar{U}_\theta(-1)}{2}-1-2\epsilon} ds \le C\|\xi_\theta\|_{\mathbf{N}_1}$$

So the definition of W_θ^{2a} makes sense.

In the expression of $W_\theta = W_\theta^{2a}$ in (5.54),

$$|(1-x^2)^{-1+2\epsilon}W_\theta^{2a}(x)| \le (1-x^2)^{-1+2\epsilon}\|\xi_\theta\|_{\mathbf{N}_1}e^{-a(x)}\int_{-1}^{x}e^{a(s)}(1-s^2)^{-2\epsilon}ds, \quad -1<x<1.$$

Applying (5.80) in the above, using $\frac{\bar U_\theta(1)}{4} < \epsilon < \frac{1}{2}$ and $\bar U_\theta(-1) > 2$, we have

$$\begin{aligned}
&|(1-x^2)^{-1+2\epsilon}W_\theta^{2a}(x)| \\
&\le \|\xi_\theta\|_{\mathbf{N}_1}(1+x)^{-\frac{\bar U_\theta(-1)}{2}+2\epsilon}(1-x)^{\frac{\bar U_\theta(1)}{2}+2\epsilon}\int_{-1}^{x}(1+s)^{\frac{\bar U_\theta(-1)}{2}-1-2\epsilon}(1-s)^{-1-\frac{\bar U_\theta(1)}{2}-2\epsilon}ds \\
&\le C\|\xi_\theta\|_{\mathbf{N}_1}
\end{aligned} \tag{5.84}$$

Case 3. $(c,\gamma) \in E_{1,2}$ or $E_{3,2}$. Similar as in Case 2, we can prove

$$|(1-x^2)^{-1+2\epsilon}W_\theta^{2b}(x)| \le C\|\xi_\theta\|_{\mathbf{N}_1} \tag{5.85}$$

Case 4. $(c,\gamma) \in E_{k,l}$ for $5 \le k \le 8$, and $1 \le l \le 3$, then $2 < \bar U_\theta(-1) < 3$ or $\bar U_\theta(-1) = 2$ with $\eta_1 = 0$, and $-3 < \bar U_\theta(1) < -2$ or $\bar U_\theta(1) = -2$.

For convenience write

$$C_W := C_W^{c,\gamma,\xi}(\xi_\theta) = \frac{1}{\int_{-1}^{1}e^{a_{c,\gamma}(s)}ds}\int_{-1}^{1}e^{a_{c,\gamma}(s)}\frac{\xi_\theta(s)}{1-s^2}ds.$$

Using the fact that $\xi_\theta \in \mathbf{N}_1$, and (5.80) we first have

$$\int_{-1}^{1}e^{a(s)}\frac{|\xi_\theta(s)|}{1-s^2}ds \le C\|\xi_\theta\|_{\mathbf{N}_1}\int_{-1}^{1}(1+s)^{\frac{\bar U_\theta(-1)}{2}-1-2\epsilon}(1-s)^{-1-\frac{\bar U_\theta(1)}{2}-2\epsilon}ds \le C\|\xi_\theta\|_{\mathbf{N}_1}$$

and

$$\int_{-1}^{1}e^{a(s)}ds \ge C\int_{-1}^{1}(1+s)^{\frac{\bar U_\theta(-1)}{2}-1}(1-s)^{-1-\frac{\bar U_\theta(1)}{2}}ds \ge C > 0$$

So C_W is finite, and

$$|C_W| \le C\|\xi_\theta\|_{\mathbf{N}_1}. \tag{5.86}$$

So the definition of W_θ^3 makes sense.

For $-1 < x < 0$, using (5.80), the fact that $\xi_\theta \in \mathbf{N}_1$, $0 < \epsilon < \frac{1}{2}$, and $\bar U_\theta(-1) \ge 2$, we have

$$\begin{aligned}
|(1-x^2)^{-1+2\epsilon}W_\theta^3(x)| &\le (1-x^2)^{-1+2\epsilon}\|\xi_\theta\|_{\mathbf{N}_1}e^{-a(x)}\int_{-1}^{x}e^{a(s)}\left((1-s^2)^{-2\epsilon}-C_W\right)ds, \\
&\le C\|\xi_\theta\|_{\mathbf{N}_1}(1+x)^{-\frac{\bar U_\theta(-1)}{2}+2\epsilon}\int_{-1}^{x}(1+s)^{\frac{\bar U_\theta(-1)}{2}-1-2\epsilon}ds \\
&\le C\|\xi_\theta\|_{\mathbf{N}_1}
\end{aligned}$$

For $0 \le x < 1$, by computation,

$$
\begin{aligned}
W_\theta^3(x) &= e^{-a_{c,\gamma}(x)} \int_{-1}^{x} e^{a_{c,\gamma}(s)} \left(\frac{\xi_\theta(s)}{1-s^2} - C_W \right) ds \\
&= e^{-a_{c,\gamma}(x)} \int_{-1}^{x} e^{a_{c,\gamma}(s)} \frac{\xi_\theta(s)}{1-s^2} ds - e^{-a(x)} \frac{\int_{-1}^{1} e^{a_{c,\gamma}(s)} \frac{\xi_\theta(s)}{1-s^2} ds}{\int_{-1}^{1} e^{a_{c,\gamma}(s)} ds} \left(\int_{-1}^{1} e^{a(s)} ds + \int_{1}^{x} e^{a(s)} ds \right) \\
&= e^{-a_{c,\gamma}(x)} \int_{1}^{x} e^{a_{c,\gamma}(s)} \frac{\xi_\theta(s)}{1-s^2} ds - C_W e^{-a_{c,\gamma}(x)} \int_{1}^{x} e^{a_{c,\gamma}(s)} ds.
\end{aligned}
$$

Then using (5.80), the fact that $\xi_\theta \in \mathbf{N}_1$, and $0 < \epsilon < \frac{1}{2}$, $\bar{U}_\theta(1) < -2$, we have

$$
\begin{aligned}
|(1-x^2)^{-1+2\epsilon} W_\theta^3(x)| &\le C \|\xi_\theta\|_{\mathbf{N}_1} (1-x)^{\frac{\bar{U}_\theta(1)}{2}+2\epsilon} \int_{1}^{x} (1-s)^{-1-\frac{\bar{U}_\theta(1)}{2}-2\epsilon} ds \\
&\quad + C \|\xi_\theta\|_{\mathbf{N}_1} (1-x)^{\frac{\bar{U}_\theta(1)}{2}+2\epsilon} \int_{1}^{x} (1-s)^{-1-\frac{\bar{U}_\theta(1)}{2}} ds \\
&\le C \|\xi_\theta\|_{\mathbf{N}_1}
\end{aligned}
$$

Thus for all $-1 < x < 1$,

$$
(1-x^2)^{-1+2\epsilon} |W_\theta^3(x)| \le C \|\xi_\theta\|_{\mathbf{N}_1} \tag{5.87}
$$

So (5.82) can be obtained from (5.83), (5.84) and (5.87). The claim is proved. From this we also have $\lim_{x\to -1^+} W_\theta(x) = \lim_{x \to 1^-} W_\theta(x) = 0$.

By the first line of (5.56), (5.53) and (5.82), we have that for $i = 1, 2a, 2b$,

$$
\begin{aligned}
&|(1-x^2)^{2\epsilon} (W_\theta^i)'| \\
&\le |(2 + |\bar{U}_\theta|)(1-x^2)^{-1+2\epsilon} W_\theta^i| + (1-x^2)^{-1+2\epsilon} |\xi_\theta(x)| + C \|\xi_\theta\|_{\mathbf{N}_1} \\
&\le C \|\xi_\theta\|_{\mathbf{N}_1}, \quad -1 < x < 1.
\end{aligned}
$$

By the second line of (5.56), (5.53), (5.82), and (5.86), we have

$$
\begin{aligned}
&|(1-x^2)^{2\epsilon} (W_\theta^3)'| \\
&\le |(2 + |\bar{U}_\theta|)(1-x^2)^{-1+2\epsilon} W_\theta^3| + (1-x^2)^{-1+2\epsilon} |\xi_\theta(x)| + C \|\xi_\theta\|_{\mathbf{N}_1} + |C_W|(1-x^2)^{2\epsilon} \\
&\le C \|\xi_\theta\|_{\mathbf{N}_1}, \quad -1 < x < 1.
\end{aligned}
$$

Thus,

$$
|(1-x^2)^{2\epsilon} W_\theta'| \le C \|\xi_\theta\|_{\mathbf{N}_1}, \quad -1 < x < 1. \tag{5.88}
$$

By (5.53), it can be seen that $|a''(x)|, |a'''(x)| \le C$ for $-\frac{1}{2} < x < \frac{1}{2}$. Then using this fact and (5.82) and (5.88), we have, for $-\frac{1}{2} < x < \frac{1}{2}$,

$$
|W''(x)| = \left| a''(x) W_\theta(x) + a'(x) W_\theta'(x) - \left(\frac{\xi_\theta}{1-x^2} \right)' \right| \le C \|\xi_\theta\|_{\mathbf{N}_1},
$$

and

$$|W'''(x)| = \left| a'''(x)W_\theta(x) + 2a''(x)W_\theta'(x) + a'(x)W_\theta''(x) - \left(\frac{\xi_\theta}{1-x^2}\right)'' \right| \le C\|\xi_\theta\|_{\mathbf{N}_1}$$

So we have shown that $W_\theta \in \mathbf{M}_1$, and $\|W_\theta\|_{\mathbf{M}_1} \le C\|\xi_\theta\|_{\mathbf{N}_1}$ for some constant C.

By the definition of $W_\phi(\xi)$ in (5.55) , using (5.79) and the fact that $\xi_\phi \in \mathbf{N}_2$, we have, for every $-1 < x < 1$,

$$\begin{aligned}
&(1-x^2)^\epsilon |W_\phi(x)| \le (1-x^2)^\epsilon \int_0^x e^{-b(t)} \int_0^t e^{b(s)} \frac{|\xi_\phi(s)|}{1-s^2} ds dt \\
&\le \|\xi_\phi\|_{\mathbf{N}_2} (1-x^2)^\epsilon \int_0^x e^{-b(t)} \int_0^t e^{b(s)} (1-s^2)^{-2-\epsilon} ds dt. \\
&\le C\|\xi_\phi\|_{\mathbf{N}_2} (1-x^2)^\epsilon \int_0^x (1+t)^{-\frac{\bar{U}_\theta(-1)}{2}} (1-t)^{\frac{\bar{U}_\theta(1)}{2}} \int_0^t (1+s)^{\frac{\bar{U}_\theta(-1)}{2}-2-\epsilon} (1-s)^{-\frac{\bar{U}_\theta(1)}{2}-2-\epsilon} ds dt \\
&\le C(1-x^2)^\epsilon \|\xi_\phi\|_{\mathbf{N}_2} \int_0^x (1+t)^{-1-\epsilon} (1-t)^{-1-\epsilon} dt \\
&\le C\|\xi_\phi\|_{\mathbf{N}_2}
\end{aligned}$$

Using (5.57), (5.79) and the fact that $\xi_\phi \in \mathbf{N}_2$, we have, for every $-1 < x < 1$,

$$\begin{aligned}
&|(1-x^2)^{1+\epsilon} W_\phi'(x)| \\
&\le \|\xi_\phi\|_{\mathbf{N}_2} (1+x)^{-\frac{\bar{U}_\theta(-1)}{2}+1+\epsilon} (1-x)^{\frac{\bar{U}_\theta(1)}{2}+1+\epsilon} \int_0^x (1+s)^{\frac{\bar{U}_\theta(-1)}{2}-2-\epsilon} (1-s)^{-\frac{\bar{U}_\theta(1)}{2}-2-\epsilon} ds \\
&\le C\|\xi_\phi\|_{\mathbf{N}_2}.
\end{aligned} \tag{5.89}$$

Similarly, since $|b'(x)| = \frac{|\bar{U}_\theta|}{1-x^2}$, using (5.58), (5.89) and the fact that $\xi_\phi \in \mathbf{N}_2$, we have

$$|(1-x^2)^{2+\epsilon} W_\phi''(x)| \le C(1-x^2)^{1+\epsilon}|W_\phi'| + (1-x^2)^{1+\epsilon}|\xi_\phi| \le |C\|\xi_\phi\|_{\mathbf{N}_2}. \tag{5.90}$$

Then $W(\xi) \in \mathbf{X}$ for all $\xi \in \mathbf{Y}$, and $\|W(\xi)\|_{\mathbf{X}} \le C\|\xi\|_{\mathbf{Y}}$ for some constant C. So $W : \mathbf{Y} \to \mathbf{X}$ is well-defined and continuous.

By definition of W^i, $i = 1, 2a, 2b$, we have $l[W_\theta^i](x) = \xi_\theta$. So $(l[W_\theta^i])''(0) = \xi_\theta''(0) = 0$, then $l[W_\theta^i](x) + \frac{1}{2}(l[W_\theta^i])''(0)(1-x^2) = \xi_\theta$.

By definition of W^3, we have $l[W_\theta^3](x) = \xi_\theta - C_W(1-x^2)$. So $(l[W_\theta^3])''(0) = \xi_\theta''(0) + 2C_W = 2C_W$, then $l[W_\theta^3](x) + \frac{1}{2}(l[W_\theta^3])''(0)(1-x^2) = \xi_\theta$. Thus $L_0 W(\xi) = \xi$, W is a right inverse of L_0. □

Let $V_{c,\gamma}^i$, $1 \le i \le 4$, $V_{c,\gamma}^{2a}$, $V_{c,\gamma}^{2b}$ be vectors defined by (5.60), (5.61), (5.62), we have

Lemma 5.2.5.

$$\ker L_0^{c,\gamma} = \begin{cases} \operatorname{span}\{V_{c,\gamma}^1, V_{c,\gamma}^2, V_{c,\gamma}^3, V_{c,\gamma}^4\} & \textit{if } (c,\gamma) \in E_{1,3}, \\ \operatorname{span}\{V_{c,\gamma}^{2a}, V_{c,\gamma}^3, V_{c,\gamma}^4\} & \textit{if } (c,\gamma) \in E_{1,1} \textit{ or } E_{2,1}, \\ \operatorname{span}\{V_{c,\gamma}^{2b}, V_{c,\gamma}^3, V_{c,\gamma}^4\} & \textit{if } (c,\gamma) \in E_{1,2} \textit{ or } E_{3,2}, \\ \operatorname{span}\{V_{c,\gamma}^3, V_{c,\gamma}^4\} & \textit{if } (c,\gamma) \in E_{k,l} \textit{ for } 5 \le k \le 8, \textit{ and } 1 \le l \le 3. \end{cases}$$

Proof. Let $V \in \mathbf{X}$ satisfy $L_0^{c,\gamma} V = 0$. We know that V is given by (5.59) for some $d_1, d_2, d_3, d_4 \in \mathbb{R}$.

For convenience we denote $a(x) = a_{c,\gamma}(x)$, $b(x) = b_{c,\gamma}(x)$ and $V^i = V_{c,\gamma}^i$, $i = 1, 2, 2a, 2b, 3$.

By Lemma 5.2.3, and the expressions of V^1, V^2 in (5.60), we have that

$$V_\theta^1(x) = e^{-a(x)} = O(1)(1+x)^{1-\frac{\bar{U}_\theta(-1)}{2}}(1-x)^{1+\frac{\bar{U}_\theta(1)}{2}}. \tag{5.91}$$

If $\bar{U}_\theta(-1) \neq 0$, for $-1 < x \le 0$,

$$V_\theta^2(x) = e^{-a(x)} \int_0^x e^{a(s)} ds = O(1)(1+x)\left((1+x)^{-\frac{\bar{U}_\theta(-1)}{2}} + 1\right).$$

If $\bar{U}_\theta(-1) = 0$,

$$V_\theta^2(x) = O(1)(1+x)\ln(1+x), \quad -1 < x \le 0$$

Similarly, for all $0 \le x < 1$, $V_\theta^2(x) = O(1)(1-x)\left((1-x)^{\frac{\bar{U}_\theta(1)}{2}} + 1\right)$ if $\bar{U}_\theta(1) \neq 0$, and $V_\theta^2(x) = O(1)(1-x)\ln(1-x)$ if $\bar{U}_\theta(1) = 0$. So we have

$$\begin{aligned} V_\theta^2(x) =& O(1)(1-x^2)\left((1+x)^{-\frac{\bar{U}_\theta(-1)}{2}} + 1 + \ln(1+x)\chi_{\{\bar{U}_\theta(-1)=0\}}\right) \\ &\cdot \left((1-x)^{\frac{\bar{U}_\theta(1)}{2}} + 1 + \ln(1-x)\chi_{\{\bar{U}_\theta(1)=0\}}\right) \end{aligned} \tag{5.92}$$

Then by (5.53), we also have

$$\left|\frac{d}{dx}V_\theta^1(x)\right| = \left|e^{-a(x)}a'(x)\right| = O(1)(1+x)^{-\frac{\bar{U}_\theta(-1)}{2}}(1-x)^{\frac{\bar{U}_\theta(1)}{2}} \tag{5.93}$$

$$\begin{aligned} &\left|\frac{d}{dx}V_\theta^2(x)\right| = \left|V_\theta^2(x)a'(x) + 1\right| \\ &= O(1)\left((1+x)^{-\frac{\bar{U}_\theta(-1)}{2}} + 1 + \ln(1+x)\chi_{\{\bar{U}_\theta(-1)=0\}}\right)\left((1-x)^{\frac{\bar{U}_\theta(1)}{2}} + 1 + \ln(1-x)\chi_{\{\bar{U}_\theta(1)=0\}}\right) \end{aligned} \tag{5.94}$$

If $\bar{U}_\theta(-1) > 2$ or $\bar{U}_\theta(-1) = 2$ with $\eta_1 = 0$, by (5.61) and Lemma 5.2.3, we have that

$$\left|V_\theta^{2a}(x)\right| = \left|e^{-a(x)}\int_{-1}^{x} e^{a(s)}ds\right| = O(1)(1-x^2)(1+(1-x)^{\frac{\bar{U}_\theta(1)}{2}} + \ln(1-x)\chi_{\{\bar{U}_\theta(1)=0\}}), \tag{5.95}$$

and

$$\left|\frac{d}{dx}V_\theta^{2a}(x)\right| = \left|V_\theta^{2a}(x)a'(x)+1\right| = O(1)(1+(1-x)^{\frac{\bar{U}_\theta(1)}{2}} + \ln(1-x)\chi_{\{\bar{U}_\theta(1)=0\}}). \tag{5.96}$$

Similarly, we have if $\bar{U}_\theta(1) < -2$ or $\bar{U}_\theta(1) = -2$ with $\eta_2 = 0$,

$$\begin{aligned} \left|V_\theta^{2b}(x)\right| &= O(1)(1-x^2)(1+(1+x)^{-\frac{\bar{U}_\theta(-1)}{2}} + \ln(1+x)\chi_{\{\bar{U}_\theta(-1)=0\}}), \\ \left|\frac{d}{dx}V_\theta^{2b}(x)\right| &= O(1)(1+(1+x)^{-\frac{\bar{U}_\theta(-1)}{2}} + \ln(1+x)\chi_{\{\bar{U}_\theta(-1)=0\}}). \end{aligned} \tag{5.97}$$

Next, by computation we have for $i = 1, 2, 2a, 2b$

$$\frac{d^2}{dx^2}V_\theta^i = (V_\theta^i)'a'(x) + V_\theta^i a''(x), \quad \frac{d^3}{dx^3}V_\theta^i = (V_\theta^i)''a'(x) + 2(V_\theta^i)'a'(x) + V_\theta^i a'''(x).$$

Using the definition of $a(x)$ in (5.52), there exists some constant C, depending on c, γ, such that

$$\left|\frac{d^2}{dx^2}V_\theta^i\right| \le C, \quad \left|\frac{d^3}{dx^3}V_\theta^i\right| \le C, \quad -\frac{1}{2} < x < \frac{1}{2}, \quad i = 1, 2, 2a, 2b. \tag{5.98}$$

Moreover, by Lemma 5.2.3, and the expressions of V^3 in (5.60), we have

$$\begin{aligned} V_\phi^3(x) = \int_0^x e^{-b(t)}dt = O(1)&\left((1+x)^{1-\frac{\bar{U}_\theta(-1)}{2}} + 1 + \ln(1+x)\chi_{\{\bar{U}_\theta(-1)=2\}}\right) \\ &\cdot \left((1-x)^{1+\frac{\bar{U}_\theta(1)}{2}} + 1 + \ln(1-x)\chi_{\{\bar{U}_\theta(1)=-2\}}\right). \end{aligned} \tag{5.99}$$

and

$$\begin{aligned} \left|\frac{d}{dx}V_\phi^3(x)\right| &= e^{-b(x)} = O(1)(1+x)^{-\frac{\bar{U}_\theta(-1)}{2}}(1-x)^{\frac{\bar{U}_\theta(1)}{2}}, \\ \left|\frac{d^2}{dx^2}V_\phi^3(x)\right| &= e^{-b(x)}|b'(x)| = O(1)(1+x)^{-1-\frac{\bar{U}_\theta(-1)}{2}}(1-x)^{-1+\frac{\bar{U}_\theta(1)}{2}}. \end{aligned} \tag{5.100}$$

When $(c,\gamma) \in E_{1,3}$, $\bar{U}(-1) < 2$ and $\bar{U}(1) > -2$, using the estimates (5.91)-(5.94), (5.98), (5.99), (5.100), and the definition of $V_{c,\gamma}^4$, it is not hard to verify that $V_{c,\gamma}^i \in \mathbf{X}$, $1 \le i \le 4$. It is clear that $\{V_{c,\gamma}i, 1 \le i \le 4\}$ are independent. So $\{V_{c,\gamma}i, 1 \le i \le 4\}$ is a basis of the kernel.

Similarly, when $(c,\gamma) \in E_{1,1}$ or $E_{2,1}$, it can be checked that $\text{span}\{V^1_{c,\gamma}, V^2_{c,\gamma}\} = \text{span}\{V^1_{c,\gamma}, V^{2a}_{c,\gamma}\}$, where $V^{2a}_{c,\gamma}$, given by (5.61), is a linear combination of $V^1_{c,\gamma}, V^2_{c,\gamma}$. So $L^{c,\gamma}_0 V = 0$ implies

$$V = d_1 V^1_{c,\gamma} + d^a_2 V^{2a}_{c,\gamma} + d_3 V^3_{c,\gamma} + d_4 V^4_{c,\gamma}.$$

It can be checked by the estimates (5.91), (5.93), (5.95), (5.96), (5.98), (5.99), (5.100)that in this case $V^{2a}_{c,\gamma}, V^3_{c,\gamma}, V^4_{c,\gamma} \in \mathbf{X}$, and $V^1_{c,\gamma} \notin \mathbf{X}$. So $d_1 = 0$.

When $(c,\gamma) \in E_{1,2}$ or $E_{3,2}$, similarly as the proof of the previous case, we have that

$$V = d^b_2 V^{2b}_{c,\gamma} + d_3 V^3_{c,\gamma} + d_4 V^4_{c,\gamma},$$

for some constants d_2, d_3, d_4.

When $(c,\gamma) \in E_{k,l}$ for $5 \le k \le 8$, and $1 \le l \le 3$, by (5.91)-(5.94), (5.98), (5.99), and (5.100), we have $V^3_{c,\gamma}, V^4_{c,\gamma} \in \mathbf{X}$, and $V^1_{c,\gamma}, V^2_{c,\gamma} \notin \mathbf{X}$. If there exists some $d_1, d_2 \in \mathbb{R}$, such that $\hat{V}_\theta := d_1 V^1_{c,\gamma} + d_2 V^2_{c,\gamma} \in \mathbf{X}$, then by the fact that $\hat{V}_\theta(-1) = \hat{V}_\theta(1) = 0$, we have

$$d_1 = -d_2 \int_0^1 e^{a_{c,\gamma}(s)} ds = -d_2 \int_0^{-1} e^{a_{c,\gamma}(s)} ds.$$

This means $d_2 \int_{-1}^1 e^{a_{c,\gamma}(s)} ds = 0$, thus $d_2 = 0$, and $d_1 = 0$.

So the lemma is proved. $\square$

Corollary 5.2.1. *For any $\xi \in \mathbf{Y}$, all solutions of $L^{c,\gamma}_0 V = \xi$, $V \in \mathbf{X}$, are given by*

$$V = W^{c,\gamma}(\xi) + \begin{cases} d_1 V^1_{c,\gamma} + d_2 V^2_{c,\gamma} + d_3 V^3_{c,\gamma} + d_4 V^4_{c,\gamma}, & \text{if } (c,\gamma) \in E_{1,3} \\ d_2 V^{2a}_{c,\gamma} + d_3 V^3_{c,\gamma} + d_4 V^4_{c,\gamma}, & \text{if } (c,\gamma) \in E_{1,1} \text{ or } E_{2,1} \\ d_2 V^{2b}_{c,\gamma} + d_3 V^3_{c,\gamma} + d_4 V^4_{c,\gamma}, & \text{if } (c,\gamma) \in E_{1,2} \text{ or } E_{3,2} \\ d_3 V^3_{c,\gamma} + d_4 V^4_{c,\gamma}, & \text{if } (c,\gamma) \in E_{k,l} \text{ for } 5 \le k \le 8, \text{ and } 1 \le l \le 3 \end{cases}$$

Let l_1, l_2, l_3, l_4 be the functionals on $\mathbf{X}$ defined by (5.63), and $\mathbf{X}_i$, $i = 1,2,3$ be the subspaces of $\mathbf{X}$ defined by (5.64), (5.66) and (5.69). As shown in Section 5.2.1, the matrix $(l_i(V^j_{c,\gamma}))$ is invertible, for every $(c,\gamma) \in K$. So $\mathbf{X}_i$ is a closed subspace of $\mathbf{X}$, and

$$\mathbf{X} = \begin{cases} \text{span}\{V^1_{c,\gamma}, V^2_{c,\gamma}, V^3_{c,\gamma}, V^4_{c,\gamma}\} \oplus \mathbf{X}_1, & (c,\gamma) \in E_{1,3}, \\ \text{span}\{V^{2a}_{c,\gamma}, V^3_{c,\gamma}, V^4_{c,\gamma}\} \oplus \mathbf{X}_2, & (c,\gamma) \in E_{1,1} \text{ or } E_{2,1}, \\ \text{span}\{V^{2b}_{c,\gamma}, V^3_{c,\gamma}, V^4_{c,\gamma}\} \oplus \mathbf{X}_2, & (c,\gamma) \in E_{1,2} \text{ or } E_{3,2}, \\ \text{span}\{V^3_{c,\gamma}, V^4_{c,\gamma}\} \oplus \mathbf{X}_3, & (c,\gamma) \in E_{k,l} \text{ for } 5 \le k \le 8, \text{ and } 1 \le l \le 3, \end{cases} \tag{5.101}$$

with the projection operator $P_i : \mathbf{X} \to \mathbf{X}_i$ for $i = 1, 2a, 2b, 3$ given by

$$\begin{aligned}
P_1 V &= V - l_1(V)V^1_{c,\gamma} - \left(l_2(V) - l_1(V)l_2(V^1_{c,\gamma})\right) V^2_{c,\gamma} - l_3(V)V^3_{c,\gamma} - l_4(V)V^4_{c,\gamma},\\
P_{2a} V &= V - \frac{l_1(V)}{l_1(V^{2a}_{c,\gamma})} V^{2a}_{c,\gamma} - l_3(V)V^3_{c,\gamma} - l_4(V)V^4_{c,\gamma},\\
P_{2b} V &= V - \frac{l_1(V)}{l_1(V^{2b}_{c,\gamma})} V^{2b}_{c,\gamma} - l_3(V)V^3_{c,\gamma} - l_4(V)V^4_{c,\gamma},\\
P_3 V &= V - l_3(V)V^3_{c,\gamma} - l_4(V)V^4_{c,\gamma},
\end{aligned} \tag{5.102}$$

for all $V \in \mathbf{X}$.

Lemma 5.2.6. *If $(c, \gamma) \in E_{1,3}$, the operator $L^{c,\gamma}_0 : \mathbf{X}_1 \to \mathbf{Y}$ is an isomorphism.*

If $(c, \gamma) \in E_{1,1}$, $E_{1,2}$, $E_{3,2}$ or $E_{2,1}$, the operator $L^{c,\gamma}_0 : \mathbf{X}_2 \to \mathbf{Y}$ is an isomorphism.

If $(c, \gamma) \in E_{k,l}$ for $5 \le k \le 8$, and $1 \le l \le 3$, the operator $L^{c,\gamma}_0 : \mathbf{X}_3 \to \mathbf{Y}$ is an isomorphism.

Proof. By Corollary 5.2.1 and Lemma 5.2.5, $L^{c,\gamma}_0 : \mathbf{X} \to \mathbf{Y}$ is surjective and $\ker L^{c,\gamma}_0$ is given by Lemma 5.2.5. The conclusion of the lemma then follows in view of the direct sum property (5.101). □

Lemma 5.2.7. *$V^1_{c,\gamma}, V^2_{c,\gamma} \in C^\infty(K, \mathbf{X})$ for compact $K \subset E_{1,3}$.*

$V^{2a}_{c,\gamma} \in C^\infty(K, \mathbf{X})$ for compact $K \subset E_{1,1}$ or $E_{2,1}$.

$V^{2b}_{c,\gamma} \in C^\infty(K, \mathbf{X})$for compact $K \subset E_{1,2}$ or $E_{3,2}$.

$V^3_{c,\gamma}, V^4_{c,\gamma} \in C^\infty(K, \mathbf{X})$ for compact $K \subset E_{k,l}$ with $(k, l) \in \mathscr{A}_1$.

Proof. It is clear that $V^4_{c,\gamma} \in C^\infty(K, \mathbf{X})$ for all compact set K described as in the lemma.

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a multi-index where $\alpha_i \ge 0$, $i = 1, 2, 3$, and $j \ge 0$.

For convenience we denote $a(x) = a_{c,\gamma}(x)$, $b(x) = b_{c,\gamma}(x)$ and $V^i = V^i_{c,\gamma}$, $i = 1, 2, 2a, 2b, 3$.

Using Theorem 5.1.3 part (i), we have that for all $|\alpha| + j \ge 0$ and $(c, \gamma) \in K$,

$$\partial^\alpha_c \partial^j_\gamma a(x) = \partial^\alpha_c \partial^j_\gamma b(x) = \int_0^x \frac{1}{1 - s^2} \partial^\alpha_c \partial^j_\gamma U^{c,\gamma}(s) ds = O(1)|\ln(1 - x^2)|. \tag{5.103}$$

(1) If $K \subset E_{1,3}$, we have $U^{c,\gamma}_\theta(-1) < 2$ and $U^{c,\gamma}_\theta(1) > -2$.

Choose $\bar\epsilon < \epsilon$ such that $2\bar\epsilon > \max\{0, \frac{1}{2}U^{c,\gamma}_\theta(-1), -\frac{1}{2}U^{c,\gamma}_\theta(1) \mid (c, \gamma) \in K\}$.

Using the expressions of V^1, V^2 in (5.60), Lemma 5.2.3, the estimates (5.91), (5.92), (5.103) and Theorem 5.1.3 (i), we have that for all $|\alpha|+j\ge 0$ and $(c,\gamma)\in K$,

$$\left|\partial_c^\alpha\partial_\gamma^j V_\theta^1(x)\right| = e^{-a(x)}O\left(\left|\ln(1-x^2)\right|^{|\alpha|+j}\right) = O(1)(1-x^2)^{1-2\bar\epsilon}\left|\ln(1-x^2)\right|^{|\alpha|+j},$$
$$\left|\partial_c^\alpha\partial_\gamma^j V_\theta^2(x)\right| = \left|e^{-a(x)}\int_0^x e^{a(s)}ds\right| O\left(\left|\ln(1-x^2)\right|^{|\alpha|+j}\right) = O(1)(1-x^2)^{1-2\bar\epsilon}\left|\ln(1-x^2)\right|^{|\alpha|+j},$$

and

$$\left|\frac{d}{dx}\partial_c^\alpha\partial_\gamma^j V_\theta^1(x)\right| = e^{-a(x)}|a'(x)|O\left(\left|\ln(1-x^2)\right|^{|\alpha|+j}\right) = O(1)(1-x^2)^{-2\bar\epsilon}\left|\ln(1-x^2)\right|^{|\alpha|+j},$$
$$\left|\frac{d}{dx}\partial_c^\alpha\partial_\gamma^j V_\theta^1(x)\right| = |V_\theta^2(x)a'(x)+1|O\left(\left|\ln(1-x^2)\right|^{|\alpha|+j}\right) = O(1)(1-x^2)^{-2\bar\epsilon}\left|\ln(1-x^2)\right|^{|\alpha|+j}$$

From the above we can see that for all $|\alpha|+j\ge 0$, there exists some constant $C = C(\alpha,j,K)$, such that$(c,\gamma)\in K$,

$$\left|(1-x^2)^{-1+2\epsilon}\partial_c^\alpha\partial_\gamma^j V_\theta^i(x)\right|\le C, \quad \left|(1-x^2)^{2\epsilon}\frac{d}{dx}\partial_c^\alpha\partial_\gamma^j V_\theta^i(x)\right|\le C, \quad i=1,2.$$

From the above we also have that for $|\alpha|+j\ge 0$,

$$\partial_c^\alpha\partial_\gamma^j V_\theta^i(1) = \partial_c^\alpha\partial_\gamma^j V_\theta^i(-1) = 0, \quad i=1,2.$$

Next,using the definition of $a(x)$ in (5.52) and Corollary 5.1.3, there exists some constant $C=C(K)$, such that

$$\left|\frac{d^2}{dx^2}\partial_c^\alpha\partial_\gamma^j V_\theta^i\right|\le C, \quad \left|\frac{d^3}{dx^3}\partial_c^\alpha\partial_\gamma^j V_\theta^i\right|\le C, \quad \text{for all } -\frac{1}{2}<x<\frac{1}{2}, \quad i=1,2.$$

The above imply that for all $|\alpha|+j\ge 0$, $\partial_c^\alpha\partial_\gamma^j V_\theta^i\in M_1$, $i=1,2$, so $V^1, V^2\in C^\infty(K,\mathbf{X})$.

(2) If $K\subset E_{1,1}$ or $E_{2,1}$, we have $2\le U_\theta^{c,\gamma}(-1)<3$ with $U^{c,\gamma}(1)>-2$.

Choose $\bar\epsilon<\epsilon$ satisfying $2\bar\epsilon>\max\{U_\theta^{c,\gamma}(-1)-2, -\frac{1}{2}U_\theta^{c,\gamma}(1)\mid (c,\gamma)\in K\}$.

In this case $\gamma=\gamma^+(c_1,c_2,c_3)$. Using the expressions of V^{2a} in (5.61), Lemma 5.2.3, the estimates (5.95), (5.96), (5.103) and Theorem 5.1.3 (i), we have that for all $|\alpha|\ge 0$,

$$\left|\partial_c^\alpha V_\theta^{2a}(x)\right| = O(1)(1+x)(1-x)^{1-2\bar\epsilon}\left|\ln(1-x^2)\right|^{|\alpha|}.$$

and

$$\left|\frac{d}{dx}\partial_c^\alpha V_\theta^{2a}(x)\right| = O(1)(1-x)^{-2\bar\epsilon}\left|\ln(1-x^2)\right|^{|\alpha|}.$$

From the above we can see that for any $|\alpha| \geq 0$, there exists some constant $C = C(\alpha, K)$, such that for all $(c, \gamma) \in K$

$$\left|(1-x^2)^{-1+2\epsilon}\partial_c^\alpha V_\theta^{2a}(x)\right| \leq C, \quad \left|(1-x^2)^{2\epsilon}\frac{d}{dx}\partial_c^\alpha V_\theta^{2a}(x)\right| \leq C.$$

We also have that for $|\alpha| \geq 0$,

$$\partial_c^\alpha V_\theta^{2a}(1) = \partial_c^\alpha V_\theta^{2a}(-1) = 0.$$

Similarly as part (1), we have

$$\left|\frac{d^2}{dx^2}\partial_c^\alpha \partial_\gamma^j V_\theta^{2a}\right| \leq C, \quad \left|\frac{d^3}{dx^3}\partial_c^\alpha \partial_\gamma^j V_\theta^{2a}\right| \leq C, \qquad \text{for all } -\frac{1}{2} < x < \frac{1}{2}.$$

The above imply that for all $|\alpha| \geq 0$, $\partial_c^\alpha V_\theta^{2a} \in \mathbf{M}_1$, so $V_\theta^{2a} \in C^\infty(K, \mathbf{M}_1)$.

(3) If $K \subset E_{1,2}$ or $E_{3,2}$, then by similar argument as part (2), we have that $V_\theta^{2b} \in C^\infty(K, M_1)$.

(4) Let K be a subset of $E_{k,l}$ with $k = 1$ or $5 \leq k \leq 8$ or $(k, l) = (2, 1), (3, 2)$. Using the expressions of $V3$ in (5.60), Lemma 5.2.3, the estimates (5.99), (5.100), (5.103) and Theorem 5.1.3 (i), we have that for all $|\alpha| + j \geq 0$,

$$\left|\partial_c^\alpha \partial_\gamma^j V_\phi^3\right| = O(1)\left((1+x)^{1-\frac{\bar{U}_\theta(-1)}{2}} + 1\right)\left((1-x)^{1+\frac{\bar{U}_\theta(1)}{2}} + 1\right)|\ln(1-x^2)|^{|\alpha|+j+1},$$

$$\left|\frac{d}{dx}\partial_c^\alpha \partial_\gamma^j V_\phi^3(x)\right| = O(1)(1+x)^{-\frac{\bar{U}_\theta(-1)}{2}}(1-x)^{\frac{\bar{U}_\theta(1)}{2}}|\ln(1-x^2)|^{|\alpha|+j},$$

$$\left|\frac{d^2}{dx^2}\partial_c^\alpha \partial_\gamma^j V_\phi^3(x)\right| = e^{-b(x)}|b'(x)| = O(1)(1+x)^{-1-\frac{\bar{U}_\theta(-1)}{2}}(1-x)^{-1+\frac{\bar{U}_\theta(1)}{2}}|\ln(1-x^2)|^{|\alpha|+j}.$$

Since $\epsilon > \max\{0, \frac{\bar{U}_\theta(-1)}{2} - 1, -1 - \frac{\bar{U}_\theta(1)}{2}\}$, there exists some $C = C(\alpha, j, K)$ such that for all $(c, \gamma) \in K$,

$$|(1-x^2)^\epsilon \partial_c^\alpha \partial_\gamma^j V_\phi^3| \leq C, \quad \left|(1-x^2)^{1+\epsilon}\frac{d}{dx}\partial_c^\alpha \partial_\gamma^j V_\phi^3\right| \leq C, \quad \left|(1-x^2)^{2+\epsilon}\frac{d^2}{dx^2}\partial_c^\alpha \partial_\gamma^j V_\phi^3\right| \leq C.$$

The above imply that for any $|\alpha| + j \geq 0$, $\partial_c^\alpha \partial_\gamma^j V_\phi^3 \in \mathbf{M}_2$, so $V^3 \in C^\infty(K, \mathbf{X})$. □

Lemma 5.2.8. *(i) If $K \subset\subset E_{1,3}$, then there exists $C = C(K) > 0$ such that for all $(c, \gamma) \in K$, $\beta := (\beta_1, \beta_2, \beta_3, \beta_4) \in \mathbb{R}^4$, and $V \in \mathbf{X}_1$,*

$$\|V\|_{\mathbf{X}} + |\beta| \leq C\|\sum_{i=1}^4 \beta_i V_{c,\gamma}^i + V\|_{\mathbf{X}}.$$

(ii) If $\in K \subset\subset E_{1,1}$ or $E_{2,1}$, then there exists $C = C(K) > 0$ such that for all $(c,\gamma) \in K$, $(\beta_2, \beta_3, \beta_4) \in \mathbb{R}^3$, and $V \in \mathbf{X}_2$,

$$\|V\|_{\mathbf{X}} + |(\beta_2, \beta_3, \beta_4)| \le C\|\beta_2 V^{2a}_{c,\gamma} + \beta_3 V^3_{c,\gamma} + \beta_4 V^4_{c,\gamma} + V\|_{\mathbf{X}}.$$

(iii) If $\in K \subset\subset E_{1,2}$ or $E_{3,2}$, then there exists $C = C(K) > 0$ such that for all $(c,\gamma) \in K$, $(\beta_2, \beta_3, \beta_4) \in \mathbb{R}^3$, and $V \in \mathbf{X}_2$,

$$\|V\|_{\mathbf{X}} + |(\beta_2, \beta_3, \beta_4)| \le C\|\beta_2 V^{2b}_{c,\gamma} + \beta_3 V^3_{c,\gamma} + \beta_4 V^4_{c,\gamma} + V\|_{\mathbf{X}}.$$

(iv)If $K \subset\subset E_{k,l}$ with $5 \le k \le 8$, $1 \le l \le 3$, then there exists $C = C(K) > 0$ such that for all $(c,\gamma) \in K$, $(\beta_3, \beta_4) \in \mathbb{R}^2$, and $V \in \mathbf{X}_3$,

$$\|V\|_{\mathbf{X}} + |(\beta_3, \beta_4)| \le C\|\beta_3 V^3_{c,\gamma} + \beta_4 V^4_{c,\gamma} + V\|_{\mathbf{X}}.$$

Proof. We only prove (i). Similar arguments yield (ii), (iii) and (iv).

We use contradiction argument. Assume there exist a sequence $(c^i, \gamma^i) \in K$, $\beta^i := (\beta^i_1, \beta^i_2, \beta^i_3, \beta^i_4) \in \mathbb{R}^4$, and $V^i \in \mathbf{X}_1$, such that

$$\|V^i\|_{\mathbf{X}} + |\beta^i| \ge i\|\sum_{j=1}^4 \beta^i_j V^j_{c^i,\gamma^i} + V^i\|_{\mathbf{X}}. \tag{5.104}$$

Without loss of generality we assume that

$$\|V^i\|_{\mathbf{X}} + |\beta^i| = 1.$$

Since $|\beta^i| \le 1$, there exists some subsequence, still denote as β^i, and some $\beta \in \mathbb{R}^4$, such that $\beta^i \to \beta$ as $i \to \infty$.

Since K is compact, there exist a subsequence of (c^i, γ^i), still denoted as (c^i, γ^i), and some $(c,\gamma) \in K$ such that $(c^i, \gamma^i) \to (c,\gamma) \in K$ as $i \to \infty$. Then by Lemma 5.2.7 we have

$$V^j_{c^i,\gamma^i} \to V^j_{c,\gamma}, \quad 1 \le j \le 4.$$

By (5.104),

$$\sum_{j=1}^4 \beta^i_j V^j_{c^i,\gamma^i} + V^i \to 0.$$

This implies

$$V^i \to V := -\sum_{j=1}^4 \beta^i_j V^j_{c^i,\gamma^i}.$$

On the other hand, $V^i \in \mathbf{X}_1$. Since $\mathbf{X}_1$ is a closed subspace of $\mathbf{X}$, we have $V \in \mathbf{X}_1$. Thus $V \in \mathbf{X}_1 \cap \operatorname{span}\{V^1_{c,\gamma}, V^2_{c,\gamma}, V^3_{c,\gamma}\}$. So $V = 0$. Since $V^1_{c,\gamma}, V^2_{c,\gamma}, V^3_{c,\gamma}$ are independent for any $(c,\gamma) \in K$. We have $\beta_1 = \beta_2 = \beta_3 = 0$. However, $\|V^i\|_{\mathbf{X}} + |(\beta^i_1, \beta^i_2, \beta^i_3)| = 1$ leads to $\|V\|_{\mathbf{X}} + |(\beta_1, \beta_2, \beta_3)| = 1$, contradiction. The lemma is proved. $\square$

Proof of Theorem 5.2.1*:* Define a map $F : K \times \mathbb{R}^4 \times \mathbf{X}_1 \to \mathbf{Y}$ by

$$F(c,\gamma,\beta,V) = G(c,\gamma,\sum_{i=1}^4 \beta_i V^i_{c,\gamma} + V).$$

By Proposition 5.2.1, G is a C^∞ map from $K \times \mathbf{X}$ to $\mathbf{Y}$. Let $\tilde{U} = \tilde{U}(c,\gamma,\beta,V) = \sum_{i=1}^4 \beta_i V^i_{c,\gamma} + V$. Using Lemma 5.2.7, we have $\tilde{U} \in C^\infty(K \times \mathbb{R}^4 \times \mathbf{X}_1, \mathbf{X})$. So it concludes that $F \in C^\infty(K \times \mathbb{R}^4 \times \mathbf{X}_1, \mathbf{Y})$.

Next, by definition $F(c,\gamma,0,0) = 0$ for all $(c,\gamma) \in K$. Fix some $(\bar{c},\bar{\gamma}) \in K$, using Lemma 5.2.6, we have $F_V(\bar{c},\bar{\gamma},0,0) = L_0^{\bar{c},\bar{\gamma}} : \mathbf{X}_1 \to \mathbf{Y}$ is an isomorphism.

Applying Theorem C, there exist some $\delta > 0$ depending only on K and a unique $V \in C^\infty(B_\delta(\bar{c},\bar{\gamma}) \times B_\delta(0), \mathbf{X}_1)$, such that

$$F(c,\gamma,\beta,V(c,\gamma,\beta)) = 0, \quad \forall (c,\gamma) \in B_\delta(\bar{c},\bar{\gamma}), \beta \in B_\delta(0),$$

and

$$V(\bar{c},\bar{\gamma},0) = 0.$$

The uniqueness part of Theorem C holds in the sense that there exists some $0 < \bar{\delta} < \delta$, such that $B_{\bar{\delta}}(\bar{c},\bar{\gamma},0,0) \cap F^{-1}(0) \subset \{(c,\gamma,\beta,V(c,\gamma,\beta)) | (c,\gamma) \in B_\delta(\bar{c},\bar{\gamma}), \beta \in B_\delta(0)\}$.

Claim: there exists some $0 < \delta_1 < \frac{\bar{\delta}}{2}$, such that $V(c,\gamma,0) = 0$ for all $(c,\gamma) \in B_{\delta_1}(\bar{c},\bar{\gamma})$.

Proof of the claim: Since $V(\bar{c},\bar{\gamma},0) = 0$ and $V(c,\gamma,0)$ is continuous in (c,γ), there exists some $0 < \delta_1 < \frac{\bar{\delta}}{2}$, such that for all $(c,\gamma) \in B_{\delta_1}(\bar{c},\bar{\gamma})$, $(c,\gamma,0,V(c,\gamma,0)) \in B_{\bar{\delta}(\bar{c},\bar{\gamma},0,0)}$. We know that for all $(c,\gamma) \in B_{\delta_1}(\bar{c},\bar{\gamma})$,

$$F(c,\gamma,0,0) = 0,$$

and

$$F(c,\gamma,0,V(c,\gamma,0)) = 0.$$

By the above mentioned uniqueness result, $V(c,\gamma,0)=0$, for every $(c,\gamma)\in B_{\delta_1}(\bar{c},\bar{\gamma})$.

Now we have $V\in C^\infty(B_{\delta_1}(\bar{c},\bar{\gamma})\times B_{\delta_1}(0),\mathbf{X}_1(\bar{c},\bar{\gamma}))$, and

$$F(c,\gamma,\beta,V(c,\gamma,\beta))=0,\quad \forall (c,\gamma)\in B_{\delta_1}(\bar{c},\bar{\gamma}),\beta\in B_{\delta_1}(0).$$

i.e. for any $(c,\gamma)\in B_{\delta_1}(\bar{c},\bar{\gamma})$, $\beta\in B_{\delta_1}(0)$

$$G(c,\gamma,\sum_{i=1}^4\beta_i V^i_{c,\gamma}+V(c,\gamma,\beta))=0.$$

Take derivative of the above with respect to β_i at $(c,\gamma,0)$, $1\le i\le 4$, we have

$$G_{\tilde{U}}(c,\gamma,0)(V^i_{c,\gamma}+\partial_{\beta_i}V(c,\gamma,0))=0.$$

Since $G_{\tilde{U}}(c,\gamma,0)V^i_{c,\gamma}=0$ by Lemma 5.2.5, we have

$$G_{\tilde{U}}(c,\gamma,0)\partial_{\beta_i}V(c,\gamma,0)=0.$$

But $\partial_{\beta_i}V(c,\gamma,0)\in C^\infty(\mathbf{X}_1)$, so

$$\partial_{\beta_i}V(c,\gamma,0)=0,\quad 1\le i\le 4.$$

Since K is compact, we can take δ_1 to be a universal constant for each $(c,\gamma)\in K$. So we have proved the existence of V in Theorem 5.2.1.

Next, let $(c,\gamma)\in B_{\delta_1}(\bar{c},\bar{\gamma})$. Let δ' be a small constant to be determined. For any U satisfies the equation (5.43) with $U-U^{c,\gamma}\in\mathbf{X}$, and $\|U-U^{c,\gamma}\|_{\mathbf{X}}\le\delta'$ there exist some $\beta\in\mathbb{R}^4$ and $V^*\in\mathbf{X}_1$ such that

$$U-U^{c,\gamma}=\sum_{i=1}^4\beta_i V^i_{c,\gamma}+V^*.$$

Then by Lemma 5.2.8, there exists some constant $C>0$ such that

$$\frac{1}{C}(|\beta|+\|V^*\|_{\mathbf{X}})\le\|\sum_{i=1}^4\beta_i V^i_{c,\gamma}+V^*\|_{\mathbf{X}}\le\delta'.$$

This gives $\|V^*\|_{\mathbf{X}}\le C\delta'$.

Choose δ' small enough such that $C\delta'<\delta_1$. We have the uniqueness of V^*. So $V^*=V(c,\gamma,\beta)$ in (5.65). The theorem is proved. □

Theorem 5.2.2, 5.2.2' and Theorem 5.2.3 can be proved by replacing $\mathbf{X}_1$ by $\mathbf{X}_2$, $\mathbf{X}_3$, replacing $\sum_{i=1}^4\beta_i V^i_{c,\gamma}$ by $\beta_2V^{2a}_{c,\gamma}+\beta_3V^3_{c,\gamma}+\beta_4V^4_{c,\gamma}$, $\beta_2V^{2b}_{c,\gamma}+\beta_3V^3_{c,\gamma}+\beta_4V^4_{c,\gamma}$ and $\beta_3V^3_{c,\gamma}+\beta_4V^4_{c,\gamma}$ respectively.

5.2.3 Existence of axisymmetric, with swirl solutions around $\bar{U}$, when $(c,\gamma)\in E_{k,l}$ with $(k,l)\in\mathscr{A}_2$ or $\mathscr{A}_3$

Denote $\bar{U}_\theta=U_\theta^{c,\gamma}$. If $(c,\gamma)\in E_{k,l}$ with $(k,l)\in\mathscr{A}_2$, then $\bar{U}_\theta(-1)=2$ with $\eta_1=4$, and $-3<\bar{U}_\theta(1)\neq -2$ or $\bar{U}_\theta(1)=-2$ with $\eta_2=0$. If $(c,\gamma)\in E_{k,l}$ with $(k,l)\in\mathscr{A}_3$, then $\bar{U}_\theta(1)=-2$ with $\eta_2=-4$ and $3>\bar{U}_\theta(-1)\neq 2$ or $\bar{U}_\theta(-1)=2$ with $\eta_1=0$.

We only need to concentrate on the case when $(k,l)\in\mathscr{A}_2$, since the results of other cases can be obtained from this case by the transformation $\hat{x}=-x$ and $\hat{U}(\hat{x})=-U(-x)$.

Let us start from constructing the Banach spaces we use. Given a compact subset $K\subset E_{k,l}$ with $(k,l)\in\mathscr{A}_2$, there exists an $0<\epsilon<\frac{1}{2}$, depending only on K, satisfying that,

$$\epsilon>\left(-\frac{\bar{U}_\theta(1)}{4}\right)\chi_{\{\bar{U}_\theta(1)>-2\}}(\bar{U}_\theta(1))+\left(-\frac{\bar{U}_\theta(1)}{2}-1\right)\chi_{\{\bar{U}_\theta(1)\le -2\}}(\bar{U}_\theta(1)),$$

for all $\bar{U}_\theta=U_\theta^{c,\gamma}$ with $(c,\gamma)\in\mathscr{A}_2$.

Define

$$\begin{aligned}
\mathbf{M}_1=&\mathbf{M}_1(\epsilon):=\{\tilde{U}_\theta\in C^3(-\frac{1}{2},\frac{1}{2})\cap C^1(-1,1)\cap C[-1,1]\mid \tilde{U}_\theta(-1)=\tilde{U}_\theta(1)=0,\\
&\|\ln\frac{1+x}{3}(1-x)^{-1+2\epsilon}\tilde{U}_\theta\|_{L^\infty(-1,1)}<\infty,\|(1+x)\left(\ln\frac{1+x}{3}\right)^2(1-x)^{2\epsilon}\tilde{U}_\theta'\|_{L^\infty(-1,1)}<\infty,\\
&||\tilde{U}_\theta''||_{L^\infty(-\frac{1}{2},\frac{1}{2})}<\infty,||\tilde{U}_\theta'''||_{L^\infty(-\frac{1}{2},\frac{1}{2})}<\infty\},\\
\mathbf{M}_2=&\mathbf{M}_2(\epsilon):=\{\tilde{U}_\phi\in C^2((-1,1),\mathbb{R})\mid \|(1-x^2)^\epsilon\tilde{U}_\phi\|_{L^\infty(-1,1)}<\infty,\\
&\|(1-x^2)^{1+\epsilon}\tilde{U}_\phi'\|_{L^\infty(-1,1)}<\infty,\|(1-x^2)^{2+\epsilon}\tilde{U}_\phi''\|_{L^\infty(-1,1)}<\infty\}
\end{aligned}$$

with the following norms accordingly

$$\begin{aligned}
\|\tilde{U}_\theta\|_{\mathbf{M}_1}=&\|\ln\frac{1+x}{3}(1-x)^{-1+2\epsilon}\tilde{U}_\theta\|_{L^\infty(-1,1)}+\|(1+x)\left(\ln\frac{1+x}{3}\right)^2(1-x)^{2\epsilon}\tilde{U}_\theta'\|_{L^\infty(-1,1)}\\
&+||\tilde{U}_\theta''||_{L^\infty(-\frac{1}{2},\frac{1}{2})}+||\tilde{U}_\theta'''||_{L^\infty(-\frac{1}{2},\frac{1}{2})},\\
\|\tilde{U}_\phi\|_{\mathbf{M}_2}=&\|(1-x^2)^\epsilon\tilde{U}_\phi\|_{L^\infty(-1,1)}+\|(1-x^2)^{1+\epsilon}\tilde{U}_\phi'\|_{L^\infty(-1,1)}+\|(1-x^2)^{2+\epsilon}\tilde{U}_\phi''\|_{L^\infty(-1,1)}.
\end{aligned}$$

Next define the following function spaces:

$$\mathbf{N}_1 = \mathbf{N}_1(\epsilon) := \{\xi_\theta \in C^2(-\frac{1}{2}, \frac{1}{2}) \cap C[-1,1] \mid \xi_\theta(-1) = \xi_\theta(1) = \xi_\theta''(0) = 0,$$
$$\| \left(\ln \frac{1+x}{3}\right)^2 (1-x)^{-1+2\epsilon}\xi_\theta\|_{L^\infty(-1,1)} < \infty, ||\xi_\theta'||_{L^\infty(-\frac{1}{2},\frac{1}{2})} < \infty, ||\xi_\theta''||_{L^\infty(-\frac{1}{2},\frac{1}{2})} < \infty\},$$
$$\mathbf{N}_2 = \mathbf{N}_2(\epsilon) := \{\xi_\phi \in C((-1,1),\mathbb{R}) \mid \|(1-x^2)^{1+\epsilon}\xi_\phi\|_{L^\infty(-1,1)} < \infty\},$$

with the following norms accordingly:

$$\|\xi_\theta\|_{\mathbf{N}_1} := \| \left(\ln \frac{1+x}{3}\right)^2 (1-x)^{-1+2\epsilon}\xi_\theta\|_{L^\infty(-1,1)} + ||\xi_\theta'||_{L^\infty(-\frac{1}{2},\frac{1}{2})} + ||\xi_\theta''||_{L^\infty(-\frac{1}{2},\frac{1}{2})},$$
$$\|\xi_\phi\|_{\mathbf{N}_2} := \|(1-x^2)^{1+\epsilon}\xi_\phi\|_{L^\infty(-1,1)}.$$

Then let $\mathbf{X} := \{\tilde{U} = (\tilde{U}_\theta, \tilde{U}_\phi) \mid \tilde{U}_\theta \in \mathbf{M}_1, \tilde{U}_\phi \in \mathbf{M}_2\}$ with norm $\|\tilde{U}\|_{\mathbf{X}} = \|\tilde{U}_\theta\|_{\mathbf{M}_1} + \|\tilde{U}_\phi\|_{\mathbf{M}_2}$, $\mathbf{Y} := \{\xi = (\xi_\theta, \xi_\phi) \mid \xi_\theta \in \mathbf{N}_1, \xi_\phi \in \mathbf{N}_2\}$, with the norm $\|\xi\|_{\mathbf{Y}} = \|\xi_\theta\|_{\mathbf{N}_1} + \|\xi_\phi\|_{\mathbf{N}_2}$. It can be proved that $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{X}$ and $\mathbf{Y}$ are Banach spaces.

Let $l_i : \mathbf{X} \to \mathbb{R}$, $1 \le i \le 4$, be the bounded linear functionals defined by (5.63) for each $V \in \mathbf{X}$. Define $\mathbf{X}_1 := \ker l_1 \cap \ker l_2 \cap \ker l_3 \cap \ker l_4$. It can be seen that $\mathbf{X}_1$ is independent of (c, γ).

Theorem 5.2.4. *For every compact subset $K \subset E_{2,3}$, there exist $\delta = \delta(K) > 0$, and $V \in C^\infty(K \times B_\delta(0), \mathbf{X}_1)$ satisfying $V(c,\gamma,0) = 0$ and $\left.\frac{\partial V}{\partial \beta_i}\right|_{\beta=0} = 0$, $1 \le i \le 4$, $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$, such that*

$$U = U^{c,\gamma} + \sum_{i=1}^{4} \beta_i V^i_{c,\gamma} + V(c,\gamma,\beta) \tag{5.105}$$

*satisfies equation (*5.43*) with $\hat{c}_1 = c_1 + \frac{1}{2}\psi[\tilde{U}_\phi](-1)$, $\hat{c}_2 = c_2 + \frac{1}{2}\psi[\tilde{U}_\phi](1)$, $\hat{c}_3 = c_3 - \frac{1}{2}(\varphi_{c,\gamma}[\tilde{U}_\theta])''(0)$.*

*Moreover, there exists some $\delta' = \delta'(K) > 0$, such that if $\|U - U^{c,\gamma}\|_{\mathbf{X}} < \delta'$, $(c,\gamma) \in K$, and U satisfies equation (*5.43*) with some constant $\hat{c}_1, \hat{c}_2, \hat{c}_3$, then (*5.105*) holds for some $|\beta| < \delta$.*

Define $\mathbf{X}_2 := \ker l_1 \cap \ker l_3 \cap \ker l_4$. Then $\mathbf{X}_2$ is independent of (c, γ).

Theorem 5.2.5. *For every compact subset K of $E_{2,2}$ or $E_{4,2}$, there exist $\delta = \delta(K) > 0$, and $V \in C^\infty(K \times B_\delta(0), \mathbf{X}_2)$ satisfying $V(c,\gamma,0) = 0$ and $\left.\dfrac{\partial V}{\partial \beta_i}\right|_{\beta=0} = 0$, $i = 2,3,4$, $\beta = (\beta_2, \beta_3, \beta_4)$, such that*

$$U = U^{c,\gamma} + \beta_2 V^{2b}_{c,\gamma} + \beta_3 V^3_{c,\gamma} + \beta_4 V^4_{c,\gamma} + V(c,\gamma,\beta) \tag{5.106}$$

satisfies equation (5.43) with $\hat{c}_1 = c_1 + \frac{1}{2}\psi[\tilde{U}_\phi](-1)$, $\hat{c}_2 = c_2 + \frac{1}{2}\psi[\tilde{U}_\phi](1)$, $\hat{c}_3 = c_3 - \frac{1}{2}(\varphi_{c,\gamma}[\tilde{U}_\theta])''(0)$.

Moreover, there exists some $\delta' = \delta'(K) > 0$, such that if $\|U - U^{c,\gamma}\|_{\mathbf{X}} < \delta'$, $(c,\gamma) \in K$, and U satisfies equation (5.43) with some constant $\hat{c}_1, \hat{c}_2, \hat{c}_3$, then (5.106) holds for some $|\beta| < \delta$.

For $\tilde{U}_\phi \in \mathbf{M}_2$, let $\psi[\tilde{U}_\phi](x)$ be defined by (5.44). Let K be a compact subset contained in either $E_{2,2}$, $E_{2,3}$ or $E_{4,2}$. Define a map $G = G(c,\gamma,\tilde{U})$ on $K \times \mathbf{X}$ by (5.46).

Proposition 5.2.2. *The map G is in $C^\infty(K \times \mathbf{X}, \mathbf{Y})$ in the sense that G has continuous Fréchet derivatives of every order. Moreover, the Fréchet derivative of G with respect to $\tilde{U}$ at $(c,\gamma,\tilde{U}) \in K \times \mathbf{X}$ is given by the linear bounded operator $L^{c,\gamma}_{\tilde{U}} : \mathbf{X} \to \mathbf{Y}$ defined as in (5.51).*

To prove Proposition 5.2.2, we first prove the following lemmas:

Lemma 5.2.9. *For every $(c,\gamma) \in K$, $A(c,\gamma,\cdot) : \mathbf{X} \to \mathbf{Y}$ defined by (5.48) is a well-defined bounded linear operator.*

Proof. In the following, C denotes a universal constant which may change from line to line. For convenience we denote $l = l_{c,\gamma}[\tilde{U}_\theta]$ defined by (5.47), and $A = A(c,\gamma,\cdot)$ for some fixed $(c,\gamma) \in K$. We make use of the property of $\bar{U}_\theta$ that $\bar{U}_\theta \in C^2(-1,1) \cap L^\infty(-1,1)$. Moreover, by Theorem 1.0.3

$$\bar{U}_\theta = 2 + 4\left(\ln\frac{1+x}{3}\right)^{-1} + O(1)\left(\ln\frac{1+x}{3}\right)^{-2+\epsilon'},$$

for any $\epsilon' > 0$. So there exists some constant $C > 0$ such that

$$|2x + \bar{U}_\theta|\left|\ln\frac{1+x}{3}\right| \le C, \quad -1 < x < 1. \tag{5.107}$$

A is clearly linear. For every $\tilde{U} \in \mathbf{X}$, we prove that $A\tilde{U}$ defined by (5.48) is in $\mathbf{Y}$ and there exists some constant C such that $\|A\tilde{U}\|_{\mathbf{Y}} \leq C\|\tilde{U}\|_{\mathbf{X}}$ for all $\tilde{U} \in \mathbf{X}$.

By computation,

$$l'(x) = (1-x^2)\tilde{U}_\theta'' + \bar{U}_\theta\tilde{U}_\theta' + (2+\bar{U}_\theta')\tilde{U}_\theta,$$

$$l''(x) = (1-x^2)\tilde{U}_\theta''' + (\bar{U}_\theta - 2x)\tilde{U}_\theta'' + 2(\bar{U}_\theta'+1)\tilde{U}_\theta' + \bar{U}_\theta''\tilde{U}_\theta.$$

By the fact that $\tilde{U}_\theta \in \mathbf{M}_1$, we have $|l''(0)| \leq C||\tilde{U}_\theta||_{\mathbf{M}_1}$. So for $-1<x<1$, using (5.107), we have

$$\begin{aligned}
&|\left(\ln\frac{1+x}{3}\right)^2(1-x)^{-1+2\epsilon}A_\theta| \\
&\leq |\left(\ln\frac{1+x}{3}\right)^2(1-x)^{-1+2\epsilon}l(x)| + \frac{1}{2}|l''(0)|\left(\ln\frac{1+x}{3}\right)^2(1+x)(1-x)^{2\epsilon} \\
&\leq |\left(\ln\frac{1+x}{3}\right)^2(1+x)(1-x)^{2\epsilon}\tilde{U}_\theta'| + \left|(2x+\bar{U}_\theta)\left(\ln\frac{1+x}{3}\right)\right|\cdot\left|\left(\ln\frac{1+x}{3}\right)\right|(1-x)^{-1+2\epsilon}|\tilde{U}_\theta| \\
&\quad + C||\tilde{U}_\theta||\left(\ln\frac{1+x}{3}\right)^2(1+x)(1-x)^{2\epsilon} \\
&\leq C\|\tilde{U}_\theta\|_{\mathbf{M}_1}
\end{aligned}$$

We also see from the above that $\lim_{x\to1} A_\theta(x) = \lim_{x\to-1} A_\theta(x) = 0$.

For $-\frac{1}{2}<x<\frac{1}{2}$,

$$\begin{aligned}
|A_\theta'| &= |l'(x) - l''(0)x| \\
&\leq |\tilde{U}_\theta''| + |\bar{U}_\theta||\tilde{U}_\theta'| + (2+|\bar{U}_\theta'|)|\tilde{U}_\theta| + |l''(0)| \\
&\leq C\|\tilde{U}_\theta\|_{\mathbf{M}_1},
\end{aligned}$$

and

$$\begin{aligned}
|A_\theta''| &= |l''(x) - l''(0)| \\
&\leq |\tilde{U}_\theta'''| + (|\bar{U}_\theta|+2)|\tilde{U}_\theta''| + 2(|\bar{U}_\theta'|+1)|\tilde{U}_\theta'| + |\bar{U}_\theta''||\tilde{U}_\theta| + |l''(0)| \\
&\leq C\|\tilde{U}_\theta\|_{\mathbf{M}_1}
\end{aligned}$$

By computation $A_\theta''(0) = l''(0) - \frac{1}{2}l''(0)\cdot 2 = 0$. So we have $A_\theta \in \mathbf{N}_1$ and $\|A_\theta\|_{\mathbf{N}_1} \leq C\|\tilde{U}_\theta\|_{\mathbf{M}_1}$.

Next, since $A_\phi = (1-x^2)\tilde{U}_\phi'' + \bar{U}_\theta\tilde{U}_\phi'$, by the fact that $\tilde{U}_\phi \in \mathbf{M}_2$ we have that

$$\left|(1-x^2)^{1+\epsilon}A_\phi\right| \leq (1-x^2)^{2+\epsilon}|\tilde{U}_\phi''| + (1-x^2)^{1+\epsilon}|\bar{U}_\theta||\tilde{U}_\phi'| \leq C\|\tilde{U}_\phi\|_{\mathbf{M}_2}.$$

So $A_\phi \in \mathbf{N}_1$, and $\|A_\phi\|_{\mathbf{N}_1} \le C\|\tilde{U}_\phi\|_{\mathbf{M}_2}$. We have proved that $A\tilde{U} \in \mathbf{Y}$ and $\|A\tilde{U}\|_{\mathbf{Y}} \le C\|\tilde{U}\|_{\mathbf{X}}$ for every $\tilde{U} \in \mathbf{X}$. The proof is finished. $\square$

Lemma 5.2.10. *The map $Q : \mathbf{X} \times \mathbf{X} \to \mathbf{Y}$ defined by (5.49) is a well-defined bounded bilinear operator.*

Proof. In the following, C denotes a universal constant which may change from line to line. For convenience we denote $\psi = \psi[\tilde{U}_\phi, \tilde{V}_\phi]$ defined by (5.44).

It is clear that Q is a bilinear operator. For every $\tilde{U}, \tilde{V} \in \mathbf{X}$, we will prove that $Q(\tilde{U}, \tilde{V})$ is in $\mathbf{Y}$ and there exists some constant C independent of $\tilde{U}$ and $\tilde{V}$ such that $\|Q(\tilde{U}, \tilde{V})\|_{\mathbf{Y}} \le C\|\tilde{U}\|_{\mathbf{X}}\|\tilde{V}\|_{\mathbf{X}}$.

For $\tilde{U}, \tilde{V} \in \mathbf{X}$, by the same arguments as in Lemma 5.2.2, there exists some constant $C > 0$ such that

$$|(\tilde{U}_\theta \tilde{V}_\theta)''(0)| \le C||\tilde{U}_\theta||_{\mathbf{M}_1}||\tilde{V}_\theta||_{\mathbf{M}_1},$$

and

$$\left|\psi(x) - \frac{1}{2}\psi(-1)(1-x) - \frac{1}{2}\psi(1)(1+x)\right| \le C(1-x^2)^{1-2\epsilon}||\tilde{U}_\phi||_{\mathbf{M}_2}||\tilde{V}_\phi||_{\mathbf{M}_2}, \quad -1 < x < 1.$$

So we have that for $-1 < x < 1$,

$$\begin{aligned}
&\left|\left(\ln\frac{1+x}{3}\right)^2 (1-x)^{-1+2\epsilon} Q_\theta(x)\right| \\
&\le \frac{1}{2}\left|\left(\ln\frac{1+x}{3}\right)(1-x)^{-1+2\epsilon}\tilde{U}_\theta(x)\right| \cdot \left|\left(\ln\frac{1+x}{3}\right)\tilde{V}_\theta(x)\right|, \\
&\quad + \left|\ln\frac{1+x}{3}\right|(1-x)^{-1+2\epsilon}\left|\psi(x) - \frac{1}{2}\psi(-1)(1-x) - \frac{1}{2}\psi(1)(1+x)\right|, \\
&\quad + \frac{1}{4}\left|\ln\frac{1+x}{3}\right|(1+x)(1-x)^{2\epsilon}|(\tilde{U}_\theta\tilde{V}_\theta)''(0)|, \\
&\le \frac{1}{2}\|\tilde{U}_\theta\|_{\mathbf{M}_1}\|\tilde{V}_\theta\|_{\mathbf{M}_1} + C\|\tilde{U}_\phi\|_{\mathbf{M}_2}\|\tilde{V}_\phi\|_{\mathbf{M}_2} + C(1-x^2)^{2\epsilon}\|\tilde{U}_\theta\|_{\mathbf{M}_1}\|\tilde{V}_\theta\|_{\mathbf{M}_1}, \\
&\le C\|\tilde{U}\|_{\mathbf{X}}\|\tilde{V}\|_{\mathbf{X}}.
\end{aligned}$$

From this we also have $\lim_{x\to 1} Q_\theta(x) = \lim_{x\to -1} Q_\theta(x) = 0$.

Similar as in Lemma 5.2.2, we have that for $-\frac{1}{2} < x < \frac{1}{2}$,

$$|Q_\theta'(x)| \le C\|\tilde{U}\|_{\mathbf{X}}\|\tilde{V}\|_{\mathbf{X}}, \quad |Q_\theta''(x)| \le C\|\tilde{U}\|_{\mathbf{X}}\|\tilde{V}\|_{\mathbf{X}}.$$

So there is $Q_\theta \in \mathbf{N}_1$, and $\|Q_\theta\|_{\mathbf{N}_1} \le C(\epsilon)\|\tilde{U}\|_{\mathbf{X}}\|\tilde{V}\|_{\mathbf{X}}$.

Next, since $Q_\phi(x) = \tilde{U}_\theta(x)\tilde{V}_\phi'(x)$, for $-1 < x < 1$,

$$\left|(1-x^2)^{1+\epsilon}Q_\phi(x)\right| \le |\tilde{U}_\theta(x)|(1-x^2)^{1+\epsilon}|\tilde{V}_\phi'| \le 2\|\tilde{U}_\theta\|_{\mathbf{M}_1}\|\tilde{V}_\phi\|_{\mathbf{M}_2}.$$

So $Q_\phi \in \mathbf{N}_2$, and $\|Q_\phi\|_{\mathbf{N}_2} \le \|\tilde{U}_\theta\|_{\mathbf{M}_1}\|\tilde{V}_\phi\|_{\mathbf{M}_2}$. Thus we have proved that $Q(\tilde{U},\tilde{V}) \in \mathbf{Y}$ and $\|Q(\tilde{U},\tilde{V})\|_{\mathbf{Y}} \le C\|\tilde{U}\|_{\mathbf{X}}\|\tilde{V}\|_{\mathbf{X}}$ for all $\tilde{U},\tilde{V} \in \mathbf{X}$. Lemma 5.2.10 is proved. $\square$

Proof of Proposition 5.2.2*:* By definition, $G(c,\gamma,\tilde{U}) = A(c,\gamma,\tilde{U}) + Q(\tilde{U},\tilde{U})$ for $(c,\gamma,\tilde{U}) \in K \times \mathbf{X}$. Using standard theories in functional analysis, by Lemma 5.2.10 it is clear that Q is C^∞ on $\mathbf{X}$. By Lemma 5.2.9, $A(c,\gamma;\cdot) : \mathbf{X} \to \mathbf{Y}$ is C^∞ for each $(c,\gamma) \in K$.

Let $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ be a multi-index where $\alpha_i \ge 0$, $i = 1,2,3$, and $j \ge 0$. For all $|\alpha| + j \ge 1$, we have

$$\partial_c^\alpha\partial_\gamma^j A(c,\gamma,\tilde{U}) = \partial_c^\alpha\partial_\gamma^j U_\theta^{c,\gamma}\begin{pmatrix}\tilde{U}_\theta\\ \tilde{U}_\phi'\end{pmatrix} + \frac{1}{2}(\partial_c^\alpha\partial_\gamma^j U_\theta^{c,\gamma}\cdot\tilde{U}_\theta)''(0)\begin{pmatrix}1-x^2\\ 0\end{pmatrix}. \tag{5.108}$$

By Theorem 5.1.3 (ii) and Corollary 5.1.3, we have

$$|\left(\ln\frac{1+x}{3}\right)^2(1-x)^{-1+2\epsilon}\partial_c^\alpha\partial_\gamma^j A_\theta(c,\gamma,\tilde{U})| \le C(\alpha,j,K)\|\tilde{U}_\theta\|_{\mathbf{M}_1}, \quad -1<x<1,$$

and for $-\frac{1}{2} < x < \frac{1}{2}$.

$$|\partial_c^\alpha\partial_\gamma^j A_\theta'(c,\gamma,\tilde{U})| \le C(\alpha,j,K)\|\tilde{U}_\theta\|_{\mathbf{M}_1}, \quad |\partial_c^\alpha\partial_\gamma^j A_\theta''(c,\gamma,\tilde{U})| \le C(\alpha,j,K)\|\tilde{U}_\theta\|_{\mathbf{M}_1}.$$

The above estimates and (5.108) also imply that

$$\partial_c^\alpha\partial_\gamma^j A_\theta(c,\gamma,\tilde{U})(-1) = \partial_c^\alpha\partial_\gamma^j A_\theta(c,\gamma,\tilde{U})(1) = \partial_c^\alpha\partial_\gamma^j A_\theta(c,\gamma,\tilde{U})''(0) = 0.$$

So $\partial_c^\alpha\partial_\gamma^j A_\theta(c,\gamma,\tilde{U}) \in \mathbf{N}_1$, with $\|\partial_c^\alpha\partial_\gamma^j A_\theta(c,\gamma,\tilde{U})\|_{\mathbf{N}_1} \le C(\alpha,j,K)\|\tilde{U}_\theta\|_{\mathbf{M}_1}$ for all $(c,\gamma,\tilde{U}) \in K \times \mathbf{X}$.

Next, by Theorem 5.1.3 (ii) and the fact that $\tilde{U}_\phi \in \mathbf{M}_1$, we have

$$(1-x^2)^{1+\epsilon}|\partial_c^\alpha\partial_\gamma^j A_\phi(c,\gamma,\tilde{U})(x)| = |\partial_c^\alpha\partial_\gamma^j U_\theta^{c,\gamma}(x)|\cdot|(1-x^2)^{1+\epsilon}U_\phi'| \le C(\alpha,j,K)\|\tilde{U}_\phi\|_{\mathbf{M}_2}. \tag{5.109}$$

So $\partial_c^\alpha \partial_\gamma^j A_\phi(c,\gamma,\tilde U)\in \mathbf{N}_2$ with $\|\partial_c^\alpha \partial_\gamma^j A_\phi(c,\gamma,\tilde U)\|_{\mathbf{N}_2}\le C(\alpha,j,K)\|\tilde U_\phi\|_{\mathbf{M}_2}$ for all $(c,\gamma,\tilde U)\in K\times \mathbf{X}$. Thus $\partial_c^\alpha \partial_\gamma^j A(c,\gamma,\tilde U)\in \mathbf{Y}$, with $\|\partial_c^\alpha \partial_\gamma^j A(c,\gamma,\tilde U)\|_{\mathbf{Y}}\le C(\alpha,j,K)\|\tilde U\|_{\mathbf{X}}$ for all $(c,\gamma,\tilde U)\in K\times\mathbf{X}$, $|\alpha|+j\ge 1$.

So for each $(c,\gamma)\in K$, $\partial_c^\alpha\partial_\gamma^j A(c,\gamma;\cdot):\mathbf{X}\to\mathbf{Y}$ is a bounded linear map with uniform bounded norm on K. Then by standard theories in functional analysis, $A: K\times\mathbf{X}\to\mathbf{Y}$ is C^∞. So G is a C^∞ map from $K\times\mathbf{X}$ to Y. By direct calculation we get its Fréchet derivative with respect to $\mathbf{X}$ is given by the linear bounded operator $L^{c,\gamma}_{\tilde U}:\mathbf{X}\to\mathbf{Y}$ defined as (5.50). The proof is finished. □

Let $a_{c,\gamma}(x), b_{c,\gamma}(x)$ be the functions defined by (5.52). For convenience we denote $a(x)=a_{c,\gamma}(x)$, $b(x)=b_{c,\gamma}(x)$, and $\bar U_\theta = U_\theta^{c,\gamma}$, we have

Lemma 5.2.11. *For $(c,\gamma)\in E_{k,l}$ with $(k,l)\in\mathscr{A}_2$, there exists some constant $C>0$, depending only on (c,γ), such that for any $-1<x<1$,*

$$e^{b(x)}\le C\left(\ln\frac{1+x}{3}\right)^2(1+x)(1-x)^{-\frac{\bar U_\theta(1)}{2}},\quad e^{-b(x)}\le C\left(\ln\frac{1+x}{3}\right)^{-2}(1+x)^{-1}(1-x)^{\frac{\bar U_\theta(1)}{2}}, \tag{5.110}$$

and

$$e^{a(x)}\le C\left(\ln\frac{1+x}{3}\right)^2(1-x)^{-1-\frac{\bar U_\theta(1)}{2}},\quad e^{-a(x)}\le C\left(\ln\frac{1+x}{3}\right)^{-2}(1-x)^{1+\frac{\bar U_\theta(1)}{2}}. \tag{5.111}$$

Proof. Let

$$\alpha_0=\min\left\{1,\left(1+\frac{\bar U_\theta(1)}{2}\right)\}\chi_{\{\bar U_\theta(1)>-2\}}+\chi_{\{\bar U_\theta(1)\le -2\}}\right\}$$

under the assumption of $\bar U_\theta$ in this case, by Theorem 1.0.3 we have that,

$$\bar U_\theta = 2+\frac{4}{\ln\frac{1+x}{3}}+O(1)\left(\ln\frac{1+x}{3}\right)^{-2+\epsilon'}=\bar U_\theta(1)+O((1-x)^{\alpha_0}),\quad -1<x<1.$$

for any $\epsilon'>0$.

Thus, by definition of $a(x)$ and $b(x)$ in (5.52), for $-1<x<1$, we have

$$\begin{aligned} b(x)&=\ln\left(\frac{1+x}{3}\right)+2\ln\left(-\ln\left(\frac{1+x}{3}\right)\right)-\frac{\bar U_\theta(1)}{2}\ln(1-x)+O(1),\\ a(x)&=2\ln\left(-\ln\left(\frac{1+x}{3}\right)\right)-\left(\frac{\bar U_\theta(1)}{2}+1\right)\ln(1-x)+O(1).\end{aligned} \tag{5.112}$$

The lemma then follows from the above etimates. □

For $\xi = (\xi_\theta, \xi_\phi) \in \mathbf{Y}$, let the map $W^{c,\gamma}$ be defined as

$$W^{c,\gamma}(\xi) := (W_\theta^{c,\gamma}(\xi), W_\phi^{c,\gamma}(\xi)),$$

where

$$W_\theta^{c,\gamma}(\xi) = \begin{cases} W_\theta^{c,\gamma,1}(\xi) & \text{if } (c,\gamma) \in E_{2,3}, \\ W_\theta^{c,\gamma,2b}(\xi) & \text{if } (c,\gamma) \in E_{2,2} \text{ or } E_{4,2}, \end{cases} \tag{5.113}$$

$W_\theta^{c,\gamma,i}$, $i = 1, 2b$ are defined by (5.54), and $W_\phi^{c,\gamma}(\xi)$ is defined by (5.55).

Lemma 5.2.12. *For every $(c,\gamma) \in K$, $W^{c,\gamma} : \mathbf{Y} \to \mathbf{X}$ is continuous, and is a right inverse of $L_0^{c,\gamma}$.*

Proof. In the following, C denotes a universal constant which may change from line to line. We make use of the property that $\bar{U}_\theta \in C^2(-1,1) \cap L^\infty(-1,1)$ and the range of ϵ. For convenience let us write $W := W^{c,\gamma}(\xi)$ and $W_\theta^i := W_\theta^{c,\gamma,i}(\xi)$ for $\xi \in \mathbf{Y}$.

By Lemma 5.2.11 we have the estimates (5.110) and (5.111).

We first prove $W_\theta : \mathbf{Y} \to \mathbf{X}$ is well-defined.

Claim. There exists $C > 0$, such that

$$\left| \left(\ln \frac{1+x}{3} \right) (1-x)^{-1+2\epsilon} W_\theta(x) \right| \le C \|\xi_\theta\|_{\mathbf{N}_1}. \tag{5.114}$$

Proof. We prove the claim for each W^i, $i = 1, 2b$.

Case 1. $(c,\gamma) \in E_{2,3}$, then $\bar{U}_\theta(-1) = 2$ with $\eta_1 = 4$ and $\bar{U}_\theta(1) > -2$.

In this case $W_\theta = W_\theta^1$.

Using the fact that $\xi_\theta \in \mathbf{N}_1$, in the expression of $W_\theta = W_\theta^1$ in (5.54), for any $-1 < x < 1$

$$\begin{aligned} &\left| \left(\ln \frac{1+x}{3} \right) (1-x)^{-1+2\epsilon} W_\theta^1(x) \right| \\ \le &\left| \ln \frac{1+x}{3} \right| (1-x)^{-1+2\epsilon} \|\xi_\theta\|_{\mathbf{N}_1} e^{-a(x)} \int_0^x \frac{e^{a(s)}}{1+s} \left(\ln \frac{1+s}{3} \right)^{-2} (1-s)^{-2\epsilon} ds. \end{aligned}$$

Applying (5.111) in the above, using the fact that $4\epsilon > -\bar{U}_\theta(1)$, we have

$$\begin{aligned}
&\left|\ln\left(\frac{1+x}{3}\right)(1-x)^{-1+2\epsilon}W_\theta^1(x)\right| \\
&\le \|\xi_\theta\|_{\mathbf{N}_1}\left|\ln\frac{1+x}{3}\right|^{-1}(1-x)^{\frac{\bar{U}_\theta(1)}{2}+2\epsilon}\int_0^x\frac{1}{1+s}(1-s)^{-1-\frac{\bar{U}_\theta(1)}{2}-2\epsilon}ds \\
&\le C\|\xi_\theta\|_{\mathbf{N}_1}\left(1+\left|\ln\frac{1+x}{3}\right|^{-1}\right)\left(1+(1-x)^{\frac{\bar{U}_\theta(1)}{2}+2\epsilon}\right) \\
&\le C\|\xi_\theta\|_{\mathbf{N}_1}.
\end{aligned} \tag{5.115}$$

Case 2. (c,γ) in $E_{2,2}$ or $E_{4,2}$, then $\bar{U}_\theta(-1)=2$ with $\eta_1=4$ and $-3<\bar{U}_\theta(1)<-2$ or $\bar{U}_\theta(1)=-2$ with $\eta_2=0$.

In this case $W_\theta=W_\theta^{2b}$.

Using the fact that $\xi_\theta\in\mathbf{N}_1$, and (5.111) we first have

$$\int_1^0 e^{a(s)}\frac{|\xi_\theta(s)|}{1-s^2}ds \le C\|\xi_\theta\|_{\mathbf{N}_1}\int_1^0(1-s)^{-\frac{\bar{U}_\theta(1)}{2}-1-2\epsilon}ds\le C\|\xi_\theta\|_{\mathbf{N}_1}.$$

So the definition of W_θ^{2b} makes sense.

In the expression of $W_\theta=W_\theta^{2b}$ in (5.54), we have for any $-1<x<1$ that

$$\begin{aligned}
&\left|\left(\ln\frac{1+x}{3}\right)(1-x)^{-1+2\epsilon}W_\theta^{2b}(x)\right| \\
\le &\left|\ln\frac{1+x}{3}\right|(1-x)^{-1+2\epsilon}\|\xi_\theta\|_{\mathbf{N}_1}e^{-a(x)}\int_1^x e^{a(s)}\left(\ln\frac{1+s}{3}\right)^{-2}\frac{1}{1+s}(1-s)^{-2\epsilon}ds.
\end{aligned}$$

Applying (5.111) in the above, using the fact that $\bar{U}_\theta(1)<-2$, and $\epsilon<\frac{1}{2}$, we have that

$$\begin{aligned}
&\left|\left(\ln\frac{1+x}{3}\right)(1-x)^{-1+2\epsilon}W_\theta^{2b}(x)\right| \\
&\le \|\xi_\theta\|_{\mathbf{N}_1}\left|\ln\frac{1+x}{3}\right|^{-1}(1-x)^{\frac{\bar{U}_\theta(1)}{2}+2\epsilon}\int_1^x\frac{1}{1+s}(1-s)^{-1-\frac{\bar{U}_\theta(1)}{2}-2\epsilon}ds, \\
&\le C\|\xi_\theta\|_{\mathbf{N}_1}\left(1+\left|\ln\frac{1+x}{3}\right|^{-1}\right), \\
&\le C\|\xi_\theta\|_{\mathbf{N}_1}.
\end{aligned} \tag{5.116}$$

So (5.114) can be obtained from (5.115) and (5.116). The claim is proved.

From the claim we also have that $\lim_{x\to-1}W_\theta(x)=\lim_{x\to1}W_\theta(x)=0$.

By (5.56), (5.53), (5.114), and the property that $\bar{U}_\theta=2+4\left(\ln\frac{1+x}{3}\right)^{-1}+O(1)\left(\ln\frac{1+x}{3}\right)^{-2+\epsilon'}$, we have that for $-1 < x < 1$,

$$\begin{aligned}
&\left|(1+x)\left(\ln\frac{1+x}{3}\right)^2(1-x)^{2\epsilon}W_\theta'\right| \\
&\le \left|(2x+\bar{U}_\theta)\left(\ln\frac{1+x}{3}\right)\right|\cdot\left|\left(\ln\frac{1+x}{3}\right)(1-x)^{-1+2\epsilon}W_\theta\right| + \left(\ln\frac{1+x}{3}\right)^2(1-x)^{-1+2\epsilon}|\xi_\theta(x)| \\
&\le C\|\xi_\theta\|_{\mathbf{N}_1}.
\end{aligned} \tag{5.117}$$

By (5.53), it can be seen that $|a''(x)|, |a'''(x)| \le C$ for $-\frac{1}{2} < x < \frac{1}{2}$. Then using this fact and (5.114) and (5.117), we have, for $-\frac{1}{2} < x < \frac{1}{2}$,

$$|W_\theta''(x)| = \left|a''(x)W_\theta(x) + a'(x)W_\theta'(x) + \left(\frac{\xi_\theta}{1-x^2}\right)'\right| \le C\|\xi_\theta\|_{\mathbf{N}_1},$$

and

$$|W_\theta'''(x)| = \left|a'''(x)W_\theta(x) + 2a''(x)W_\theta'(x) + a'(x)W_\theta''(x) + \left(\frac{\xi_\theta}{1-x^2}\right)''\right| \le C\|\xi_\theta\|_{\mathbf{N}_1}$$

So we have shown that $W_\theta \in \mathbf{M}_1$, and $\|W_\theta\|_{\mathbf{M}_1} \le C\|\xi_\theta\|_{\mathbf{N}_1}$ for some constant C.

By the definition of $W_\phi(\xi)$ in (5.55) , using (5.110) and the fact that $\xi_\phi \in \mathbf{N}_2$, we have, for every $-1 < x < 1$,

$$\begin{aligned}
&(1-x^2)^\epsilon|W_\phi(x)| \le (1-x^2)^\epsilon\int_0^x e^{-b(t)}\int_0^t e^{b(s)}\frac{|\xi_\phi(s)|}{1-s^2}dsdt \\
&\le \|\xi_\phi\|_{\mathbf{N}_2}(1-x^2)^\epsilon\int_0^x e^{-b(t)}\int_0^t e^{b(s)}(1-s^2)^{-2-\epsilon}dsdt. \\
&\le C\|\xi_\phi\|_{\mathbf{N}_2}(1-x^2)^\epsilon\int_0^x\left(\ln\frac{1+t}{3}\right)^{-2}(1+t)^{-1}(1-t)^{\frac{\bar{U}_\theta(1)}{2}} \\
&\quad\cdot\int_0^t\left(\ln\frac{1+s}{3}\right)^2(1+s)^{-1-\epsilon}(1-s)^{-\frac{\bar{U}_\theta(1)}{2}-2-\epsilon}dsdt \\
&\le C\|\xi_\phi\|_{\mathbf{N}_2}(1-x^2)^\epsilon\int_0^x(1-t^2)^{-1-\epsilon}dt \\
&\le C\|\xi_\phi\|_{\mathbf{N}_2}
\end{aligned}$$

Using (5.57), (5.110) and the fact that $\xi_\phi \in \mathbf{N}_2$, we have, for every $-1 < x < 1$,

$$\begin{aligned}
|(1-x^2)^{1+\epsilon}W_\phi'(x)| &\le \|\xi_\phi\|_{\mathbf{N}_2}\left(\ln\frac{1+x}{3}\right)^{-2}(1+x)^\epsilon(1-x)^{\frac{\bar{U}_\theta(1)}{2}+1+\epsilon} \\
&\quad\cdot\int_0^x\left(\ln\frac{1+s}{3}\right)^2(1+s)^{-1-\epsilon}(1-s)^{-\frac{\bar{U}_\theta(1)}{2}-2-\epsilon}ds \qquad (5.118) \\
&\le C\|\xi_\phi\|_{\mathbf{N}_2}
\end{aligned}$$

Similarly, since $|b'(x)| = \frac{|\bar{U}_\theta|}{1-x^2}$, using (5.58), (5.118) and the fact that $\xi_\phi \in \mathbf{N_2}$, we have

$$|(1-x^2)^{2+\epsilon} W''_\phi(x)| \le C(1-x^2)^{1+\epsilon}|W'_\phi| + (1-x^2)^{1+\epsilon}|\xi_\phi| \le |C\|\xi_\phi\|_{\mathbf{N_2}}.$$

Then $W(\xi) \in \mathbf{X}$ for all $\xi \in \mathbf{Y}$, and $\|W(\xi)\|_{\mathbf{X}} \le C\|\xi\|_{\mathbf{Y}}$ for some constant C. So $W : \mathbf{Y} \to \mathbf{X}$ is well-defined and continuous.

By definition of W^i, $i = 1, 2b$, we have $l[W^i_\theta](x) = \xi_\theta$. So $(l[W^i_\theta])''(0) = \xi''_\theta(0) = 0$, then $l[W^i_\theta](x) + \frac{1}{2}(l[W^i_\theta])''(0)(1-x^2) = \xi_\theta$. Thus $L_0 W(\xi) = \xi$, W is a right inverse of L_0. □

Let $V^i_{c,\gamma}$, $1 \le i \le 4$ and $V^{2b}_{c,\gamma}$ be vectors defined by (5.60) and (5.62), we have

Lemma 5.2.13.

$$\ker L_0^{c,\gamma} = \begin{cases} \mathrm{span}\{V^1_{c,\gamma}, V^2_{c,\gamma}, V^3_{c,\gamma}, V^4_{c,\gamma}\} & \textit{if } (c,\gamma) \in E_{2,3}, \\ \mathrm{span}\{V^{2b}_{c,\gamma}, V^3_{c,\gamma}, V^4_{c,\gamma}\} & \textit{if } (c,\gamma) \in E_{2,2} \textit{ or } E_{4,2}, \end{cases}$$

Proof. Let $V \in \mathbf{X}$, $L_0^{c,\gamma} V = 0$. We know that V is given by (5.59) for some $d_1, d_2, d_3, d_4 \in \mathbb{R}$. For convenience we denote $a(x) = a_{c,\gamma}(x)$, $b(x) = b_{c,\gamma}(x)$ and $V^i = V^i_{c,\gamma}$, $i = 1, 2, 2b, 3, 4$.

By Lemma 5.2.11, and the expressions of V^1, V^2 in (5.60), we have that

$$V^1_\theta(x) = e^{-a(x)} = O(1)\left(\ln\frac{1+x}{3}\right)^{-2}(1-x)^{1+\frac{\bar{U}_\theta(1)}{2}} \tag{5.119}$$

and

$$V^2_\theta(x) = e^{-a(x)}\int_0^x e^{a(s)}ds = O(1)\left(\ln\frac{1+x}{3}\right)^{-2}(1-x)\left((1-x)^{\frac{\bar{U}_\theta(1)}{2}} + 1 + \ln(1-x)\chi_{\{\bar{U}_\theta(1)=0\}}\right). \tag{5.120}$$

By (5.53), we also have

$$\left|\frac{d}{dx}V^1_\theta(x)\right| = \left|e^{-a(x)}a'(x)\right| = O(1)\left(\ln\frac{1+x}{3}\right)^{-2}(1+x)^{-1}(1-x)^{\frac{\bar{U}_\theta(1)}{2}} \tag{5.121}$$

$$\begin{aligned}\left|\frac{d}{dx}V^2_\theta(x)\right| &= \left|V^2_\theta(x)a'(x) + 1\right| \\ &= O(1)\left(\ln\frac{1+x}{3}\right)^{-2}(1+x)^{-1}\left((1-x)^{\frac{\bar{U}_\theta(1)}{2}} + 1 + \ln(1-x)\chi_{\{\bar{U}_\theta(1)=0\}}\right)\end{aligned} \tag{5.122}$$

If $\bar{U}_\theta(1) < -2$ or $\bar{U}_\theta(1) = -2$ with $\eta_2 = 0$,

$$\left|V_\theta^{2b}(x)\right| = O(1)\left(\ln\frac{1+x}{3}\right)^{-2}(1-x), \quad \left|\frac{d}{dx}V_\theta^{2b}(x)\right| = O(1)\left(\ln\frac{1+x}{3}\right)^{-2}(1+x)^{-1} \tag{5.123}$$

Next, by computation we have for $i = 1, 2, 2b$

$$\frac{d^2}{dx^2}V_\theta^i = (V_\theta^i)'a'(x) + V_\theta^i a''(x), \quad \frac{d^3}{dx^3}V_\theta^i = (V_\theta^i)''a'(x) + 2(V_\theta^i)'a'(x) + V_\theta^i a'''(x).$$

Using the definition of $a(x)$ in (5.52), there exists some constant C, depending on c, γ, such that

$$\left|\frac{d^2}{dx^2}V_\theta^i\right| \le C, \quad \left|\frac{d^3}{dx^3}V_\theta^i\right| \le C, \quad -\frac{1}{2} < x < \frac{1}{2}, \quad i = 1, 2, 2b. \tag{5.124}$$

Moreover, by Lemma 5.2.11, and the expressions of V^3 in (5.60), we have

$$V_\phi^3(x) = \int_0^x e^{-b(t)}dt = O(1)\left((1-x)^{1+\frac{\bar{U}_\theta(1)}{2}} + 1 + \ln(1-x)\chi_{\{\bar{U}_\theta(1)=-2\}}\right). \tag{5.125}$$

and

$$\begin{aligned}\left|\frac{d}{dx}V_\phi^3(x)\right| &= e^{-b(x)} = O(1)\left(\ln\frac{1+x}{3}\right)^{-2}(1+x)^{-1}(1-x)^{\frac{\bar{U}_\theta(1)}{2}}, \\ \left|\frac{d^2}{dx^2}V_\phi^3(x)\right| &= e^{-b(x)}|b'(x)| = O(1)\left(\ln\frac{1+x}{3}\right)^{-2}(1+x)^{-2}(1-x)^{-1+\frac{\bar{U}_\theta(1)}{2}}.\end{aligned} \tag{5.126}$$

When $(c, \gamma) \in E_{2,3}$, $\bar{U}(-1) = 2$ with $\eta_1 = 4$, and $\bar{U}(1) > -2$, using the estimates (5.119)–(5.122), (5.124)–(5.126), and the definition of $V_{c,\gamma}^4$, it is not hard to verify that $V_{c,\gamma}^i \in \mathbf{X}$, $1 \le i \le 4$. It is clear that $\{V_{c,\gamma}^i, 1 \le i \le 4\}$ are independent. So $\{V_{c,\gamma}i, 1 \le i \le 4\}$ is a basis of the kernel.

Similarly, when $(c, \gamma) \in E_{2,2}$ or $E_{4,2}$, it can be checked that $\text{span}\{V_{c,\gamma}^1, V_{c,\gamma}^2\} = \text{span}\{V_{c,\gamma}^1, V_{c,\gamma}^{2b}\}$, where $V_{c,\gamma}^{2b}$, given by (5.62), is a linear combination of $V_{c,\gamma}^1, V_{c,\gamma}^2$. So $L_0^{c,\gamma}V = 0$ implies

$$V = d_1V_{c,\gamma}^1 + d_2^aV_{c,\gamma}^{2b} + d_3V_{c,\gamma}^3 + d_4V_{c,\gamma}^4.$$

It can be checked by the estimates (5.119), (5.121), and (5.123)-(5.126) that in this case $V_{c,\gamma}^{2b}, V_{c,\gamma}^3, V_{c,\gamma}^4 \in \mathbf{X}$, and $V_{c,\gamma}^1 \notin \mathbf{X}$. So $d_1 = 0$.

The lemma is proved. $\square$

Corollary 5.2.2. *For any $\xi \in \mathbf{Y}$, all solutions of $L_0^{c,\gamma} V = \xi$, $V \in \mathbf{X}$, are given by*

$$V = W^{c,\gamma}(\xi) + \begin{cases} d_1 V^1_{c,\gamma} + d_2 V^2_{c,\gamma} + d_3 V^3_{c,\gamma} + d_4 V^4_{c,\gamma}, & \text{if } (c,\gamma) \in E_{2,3}, \\ d_2 V^{2b}_{c,\gamma} + d_3 V^3_{c,\gamma} + d_4 V^4_{c,\gamma}, & \text{if } (c,\gamma) \in E_{2,2} \text{ or } E_{4,2}. \end{cases}$$

Let l_i, $1 \le i \le 4$ be the functionals on $\mathbf{X}$ defined by (5.63) , and $\mathbf{X}_1 = \cap_{i=1}^4 \ker l_i$, $\mathbf{X}_2 = \ker l_1 \cap \ker l_3 \cap \ker l_4$. It can be checked that $\mathbf{X}_1$ and $\mathbf{X}_2$ are closed subspaces of $\mathbf{X}$, and

$$\mathbf{X} = \begin{cases} \operatorname{span}\{V^1_{c,\gamma}, V^2_{c,\gamma}, V^3_{c,\gamma}\} \oplus \mathbf{X}_1, & (c,\gamma) \in E_{2,3}, \\ \operatorname{span}\{V^{2b}_{c,\gamma}, V^3_{c,\gamma}\} \oplus \mathbf{X}_2, & (c,\gamma) \in E_{2,2} \text{ or } E_{4,2}, \end{cases} \tag{5.127}$$

with the projection operator $P_1 : \mathbf{X} \to \mathbf{X}_1$, $P_{2b} : \mathbf{X} \to \mathbf{X}_2$ given by (5.102).

Lemma 5.2.14. *If $(c,\gamma) \in E_{2,3}$, the operator $L_0^{c,\gamma} : \mathbf{X}_1 \to \mathbf{Y}$ is an isomorphism.*

If $(c,\gamma) \in E_{2,2}$ or $E_{4,2}$, the operator $L_0^{c,\gamma} : \mathbf{X}_2 \to \mathbf{Y}$ is an isomorphism.

Proof. By Corollary 5.2.2 and Lemma 5.2.13, $L_0^{c,\gamma} : \mathbf{X} \to \mathbf{Y}$ is surjective and $\ker L_0^{c,\gamma}$ is given by Lemma 5.2.13. The conclusion of the lemma then follows in view of the direct sum property (5.127). □

Lemma 5.2.15. *$V^1_{c,\gamma}, V^2_{c,\gamma} \in C^\infty(K, \mathbf{X})$ for compact $K \subset E_{2,3}$.*

$V^{2b}_{c,\gamma} \in C^\infty(K, \mathbf{X})$for compact $K \subset E_{2,2}$ or $E_{4,2}$.

$V^3_{c,\gamma}, V^4_{c,\gamma} \in C^\infty(K, \mathbf{X})$ for compact $K \subset E_{k,l}$ with $(k,l) \in \mathscr{A}_2$.

Proof. It is clear that $V^4_{c,\gamma} \in C^\infty(K, \mathbf{X})$ for all compact set K described as in the lemma.

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a multi-index where $\alpha_i \ge 0$, $i = 1, 2, 3$, and $j \ge 0$.

For convenience we denote $a(x) = a_{c,\gamma}(x)$, $b(x) = b_{c,\gamma}(x)$ and $V^i = V^i_{c,\gamma}$, $i = 1, 2, 2b, 3$.

Using Theorem 5.1.3 part (ii), we have that for all $|\alpha| + j \ge 1$ and $(c,\gamma) \in K$,

$$\partial_c^j \partial_\gamma^i a(x) = \partial_c^j \partial_\gamma^i b(x) = \int_0^x \frac{1}{1-s^2} \partial_c^j \partial_\gamma^i U^{c,\gamma}(s) ds = O(1)|\ln(1-x)|, \tag{5.128}$$

(1) If $K \subset E_{2,3}$, we have $U_\theta^{c,\gamma}(-1) = 2$ with $\eta_1 = 4$ and $U_\theta^{c,\gamma}(1) > -2$.

Choose $\bar{\epsilon} < \epsilon$ satisfying $2\bar{\epsilon} > \max\{0, -\frac{1}{2} U_\theta^{c,\gamma}(1) \mid (c,\gamma) \in K\}$.

By Lemma 5.2.13 we know that $V^1, V^2 \in \mathbf{X}$ in this case.

Using the expressions of V^1, V^2 in (5.60), Lemma 5.2.11, the estimates (5.119), (5.120), (5.128) and Theorem 5.1.3 (ii), we have that for all $|\alpha| + j \geq 1$ and $(c, \gamma) \in K$,

$$\left|\partial_c^\alpha \partial_\gamma^j V_\theta^1(x)\right| = e^{-a(x)} O\left(|\ln(1-x)|^{|\alpha|+j}\right) = O(1) \left(\ln \frac{1+x}{3}\right)^{-2} (1-x)^{1-2\bar{\epsilon}} |\ln(1-x)|^{|\alpha|+j},$$

$$\left|\partial_c^\alpha \partial_\gamma^j V_\theta^2(x)\right| = e^{-a(x)} \left|\int_0^x e^{a(s)} ds\right| O\left(|\ln(1-x)|^{|\alpha|+j}\right) = O(1) \left(\ln \frac{1+x}{3}\right)^{-2} (1-x)^{1-2\bar{\epsilon}} |\ln(1-x)|^{|\alpha|+j},$$

and

$$\left|\frac{d}{dx} \partial_c^\alpha \partial_\gamma^j V_\theta^1(x)\right| = e^{-a(x)} |a'(x)| O\left(|\ln(1-x)|^{|\alpha|+j}\right) = O(1) \left(\ln \frac{1+x}{3}\right)^{-2} \frac{(1-x)^{-2\bar{\epsilon}}}{1+x} |\ln(1-x)|^{|\alpha|+j},$$

$$\left|\frac{d}{dx} \partial_c^\alpha \partial_\gamma^j V_\theta^2(x)\right| = |V_\theta^2 a'(x) + 1| O\left(|\ln(1-x)|^{|\alpha|+j}\right) = O(1) \left(\ln \frac{1+x}{3}\right)^{-2} \frac{(1-x)^{-2\bar{\epsilon}}}{1+x} |\ln(1-x)|^{|\alpha|+j}.$$

From the above we can see that for all $|\alpha| + j \geq 1$, there exists some constant $C = C(\alpha, j, K)$, such that for $i = 1, 2$

$$\left|\left(\ln \frac{1+x}{3}\right) (1-x)^{-1+2\epsilon} \partial_c^\alpha \partial_\gamma^j V_\theta^i(x)\right| \leq C,$$

$$\left|\left(\ln \frac{1+x}{3}\right)^2 (1+x)(1-x)^{2\epsilon} \frac{d}{dx} \partial_c^\alpha \partial_\gamma^j V_\theta^i(x)\right| \leq C.$$

We also have that for $|\alpha| + j \geq 1$, $\partial_c^\alpha \partial_\gamma^j V_\theta^i(1) = \partial_c^\alpha \partial_\gamma^j V_\theta^i(-1) = 0$, $i = 1, 2$.

Next, using the definition of $a(x)$ in (5.52) and Corollary 5.1.3, there exists some constant $C = C(K)$, such that

$$\left|\frac{d^2}{dx^2} \partial_c^\alpha \partial_\gamma^j V_\theta^i\right| \leq C, \quad \left|\frac{d^3}{dx^3} \partial_c^\alpha \partial_\gamma^j V_\theta^i\right| \leq C, \quad \text{for all } -\frac{1}{2} < x < \frac{1}{2}, \quad i = 1, 2.$$

The above imply that for all $|\alpha| + j \geq 1$, $\partial_c^\alpha \partial_\gamma^j V_\theta^i \in M_1$, $i = 1, 2$, so $V^1, V^2 \in C^\infty(K, \mathbf{X})$.

(2) If $K \subset E_{2,2}$ or $E_{4,2}$, we have $U_\theta^{c,\gamma}(-1) = 2$ with $\eta_1 = 4$, and $U^{c,\gamma}(1) \in (-3, -2)$ or $U^{c,\gamma}(1) = -2$ with $\eta_2 = 0$.

By Lemma 5.2.13 we know that $V^{2b} \in \mathbf{X}$ in this case.

In this case $\gamma = \gamma^-(c_1, c_2, c_3)$. Using the expressions of V^{2b} in (5.62), Lemma 5.2.11, the estimates (5.123), (5.128) and Theorem 5.1.3 (ii), we have that for all $|\alpha| \geq 1$,

$$\left|\partial_c^\alpha V_\theta^{2b}(x)\right| = O(1) \left(\ln \frac{1+x}{3}\right)^{-2} (1-x) |\ln(1-x)|^{|\alpha|}.$$

and

$$\left|\partial_c^\alpha V_\theta^{2b}(x)\right| = O(1) \left(\ln \frac{1+x}{3}\right)^{-2} (1+x)^{-1} |\ln(1-x)|^{|\alpha|}.$$

From the above we see that for any $|\alpha| \geq 1$, there exists some constant $C = C(\alpha, K)$, such that for all $(c, \gamma) \in K$,

$$\left|\left(\ln \frac{1+x}{3}\right)(1-x)^{-1+2\epsilon}\partial_c^\alpha V_\theta^{2b}(x)\right| \leq C, \quad \left|\left(\ln \frac{1+x}{3}\right)^2 (1+x)(1-x)^{2\epsilon}\frac{d}{dx}\partial_c^\alpha V_\theta^{2b}(x)\right| \leq C.$$

We also have that for $|\alpha| \geq 1$, $\partial_c^\alpha V_\theta^{2b}(1) = \partial_c^\alpha V_\theta^{2b}(-1) = 0$.

Similarly as part (1), we have

$$\left|\frac{d^2}{dx^2}\partial_c^\alpha \partial_\gamma^j V_\theta^{2b}\right| \leq C, \quad \left|\frac{d^3}{dx^3}\partial_c^\alpha \partial_\gamma^j V_\theta^{2b}\right| \leq C, \quad \text{for all } -\frac{1}{2} < x < \frac{1}{2}.$$

The above imply that for all $|\alpha| \geq 1$, $\partial_c^\alpha V_\theta^{2b} \in \mathbf{M}_1$, so $V_\theta^{2b} \in C^\infty(K, \mathbf{M}_1)$.

(3) Let K be a subset of $E_{k,l}$ with $(k,l) \in \mathscr{A}_2$.

By Lemma 5.2.13 we know that $V^3 \in \mathbf{X}$.

Using the expressions of V^3 in (5.60), Lemma 5.2.11, the estimates (5.125), (5.126), (5.128) and Theorem 5.1.3 (ii), we have that for all $|\alpha| + j \geq 1$,

$$\begin{aligned}
\left|\partial_c^\alpha \partial_\gamma^j V_\phi^3\right| &= O(1)\left((1-x)^{1+\frac{\bar{U}_\theta(1)}{2}} + 1\right)|\ln(1-x)|^{|\alpha|+j+1}, \\
\left|\frac{d}{dx}\partial_c^\alpha \partial_\gamma^j V_\phi^3(x)\right| &= O(1)\left(\ln\frac{1+x}{3}\right)^{-2}(1+x)^{-1}(1-x)^{\frac{\bar{U}_\theta(1)}{2}}|\ln(1-x)|^{|\alpha|+j}, \\
\left|\frac{d^2}{dx^2}\partial_c^\alpha \partial_\gamma^j V_\phi^3(x)\right| &= O(1)\left(\ln\frac{1+x}{3}\right)^{-2}(1+x)^{-2}(1-x)^{-1+\frac{\bar{U}_\theta(1)}{2}}|\ln(1-x)|^{|\alpha|+j}.
\end{aligned}$$

Since $\epsilon > \max\{0, -1 - \frac{\bar{U}_\theta(1)}{2}\}$, there exists some $C = C(\alpha, j, K)$ such that for all $(c, \gamma) \in K$,

$$|(1-x^2)^\epsilon \partial_c^\alpha \partial_\gamma^j V_\phi^3| \leq C, \quad \left|(1-x^2)^{1+\epsilon}\frac{d}{dx}\partial_c^\alpha \partial_\gamma^j V_\phi^3\right| \leq C, \quad \left|(1-x^2)^{2+\epsilon}\frac{d^2}{dx^2}\partial_c^\alpha \partial_\gamma^j V_\phi^3\right| \leq C.$$

The above imply that for any $|\alpha| + j \geq 1$, $\partial_c^\alpha \partial_\gamma^j V_\phi^3 \in \mathbf{M}_2$, so $V^3 \in C^\infty(K, \mathbf{X})$. □

Similar arguments as in Lemma 5.2.8 imply the following lemma.

Lemma 5.2.16. *(i) If $K \subset\subset E_{2,3}$, then there exists $C = C(K) > 0$ such that for all $(c, \gamma) \in K$, $\beta := (\beta_1, \beta_2, \beta_3, \beta_4) \in \mathbb{R}^4$, and $V \in \mathbf{X}_1$,*

$$\|V\|_{\mathbf{X}} + |\beta| \leq C\|\sum_{i=1}^4 \beta_i V_{c,\gamma}^i + V\|_{\mathbf{X}}.$$

(ii) If $\in K \subset\subset E_{2,2}$ *or* $E_{4,2}$*, then there exists* $C = C(K) > 0$ *such that for all* $(c, \gamma) \in K$*,* $(\beta_2, \beta_3, \beta_4) \in \mathbb{R}^3$*, and* $V \in \mathbf{X}_{2b}$*,*

$$\|V\|_{\mathbf{X}} + |(\beta_2, \beta_3, \beta_4)| \leq C\|\beta_2 V_{c,\gamma}^{2b} + \beta_3 V_{c,\gamma}^3 + \beta_4 V_{c,\gamma}^4 + V\|_{\mathbf{X}}.$$

Proof of Theorem 5.2.4*:* Define a map $F : K \times \mathbb{R}^3 \times \mathbf{X}_1 \to \mathbf{Y}$ by

$$F(c, \gamma, \beta, V) = G(c, \gamma, \sum_{i=1}^{4} \beta_i V_{c,\gamma}^i + V).$$

By Proposition 5.2.2, G is a C^∞ map from $K \times \mathbf{X}$ to $\mathbf{Y}$. Let $\tilde{U} = \tilde{U}(c, \gamma, \beta, V) = \sum_{i=1}^4 \beta_i V_{c,\gamma}^i + V$. Using Lemma 5.2.15, we have $\tilde{U} \in C^\infty(K \times \mathbb{R}^4 \times \mathbf{X}_1, \mathbf{X})$. So it concludes that $F \in C^\infty(K \times \mathbb{R}^4 \times \mathbf{X}_1, \mathbf{Y})$.

Next, by definition $F(c, \gamma, 0, 0) = 0$ for all $(c, \gamma) \in K$. Fix some $(\bar{c}, \bar{\gamma}) \in K$, using Lemma 5.2.14, we have $F_V(\bar{c}, \bar{\gamma}, 0, 0) = L_0^{\bar{c},\bar{\gamma}} : \mathbf{X}_1 \to \mathbf{Y}$ is an isomorphism.

Applying Theorem C, there exist some $\delta > 0$ and a unique $V \in C^\infty(B_\delta(\bar{c}, \bar{\gamma}) \times B_\delta(0), \mathbf{X}_1)$, such that

$$F(c, \gamma, \beta, V(c, \gamma, \beta)) = 0, \quad \forall (c, \gamma) \in B_\delta(\bar{c}, \bar{\gamma}), \beta \in B_\delta(0),$$

and

$$V(\bar{c}, \bar{\gamma}, 0) = 0.$$

The uniqueness part of Theorem C holds in the sense that there exists some $0 < \bar{\delta} < \delta$, such that $B_{\bar{\delta}}(\bar{c}, \bar{\gamma}, 0, 0) \cap F^{-1}(0) \subset \{(c, \gamma, \beta, V(c, \gamma, \beta)) | (c, \gamma) \in B_\delta(\bar{c}, \bar{\gamma}), \beta \in B_\delta(0)\}$.

Claim: there exists some $0 < \delta_1 < \frac{\bar{\delta}}{2}$, such that $V(c, \gamma, 0) = 0$ for every $(c, \gamma) \in B_{\delta_1}(\bar{c}, \bar{\gamma})$.

Proof of the claim: Since $V(\bar{c}, \bar{\gamma}, 0) = 0$ and $V(c, \gamma, 0)$ is continuous in (c, γ), there exists some $0 < \delta_1 < \frac{\bar{\delta}}{2}$, such that for all $(c, \gamma) \in B_{\delta_1}(\bar{c}, \bar{\gamma})$, $(c, \gamma, 0, V(c, \gamma, 0)) \in B_{\bar{\delta}(\bar{c}, \bar{\gamma}, 0, 0)}$. We know that for all $(c, \gamma) \in B_{\delta_1}(\bar{c}, \bar{\gamma})$,

$$F(c, \gamma, 0, 0) = 0,$$

and

$$F(c, \gamma, 0, V(c, \gamma, 0)) = 0.$$

By the above mentioned uniqueness result, $V(c, \gamma, 0) = 0$, for every $(c, \gamma) \in B_{\delta_1}(\bar{c}, \bar{\gamma})$.

Now we have $V \in C^\infty(B_{\delta_1}(\bar{c},\bar{\gamma}) \times B_{\delta_1}(0), \mathbf{X}_1(\bar{c},\bar{\gamma}))$, and

$$F(c,\gamma,\beta,V(c,\gamma,\beta)) = 0, \quad \forall (c,\gamma) \in B_{\delta_1}(\bar{c},\bar{\gamma}), \beta \in B_{\delta_1}(0).$$

i.e. for any $(c,\gamma) \in B_{\delta_1}(\bar{c},\bar{\gamma})$, $\beta \in B_{\delta_1}(0)$

$$G(c,\gamma,\sum_{i=1}^{4} \beta_i V^i_{c,\gamma} + V(c,\gamma,\beta)) = 0.$$

Take derivative of the above with respect to β_i at $(c,\gamma,0)$, $1 \le i \le 4$, we have

$$G_{\tilde{U}}(c,\gamma,0)(V^i_{c,\gamma} + \partial_{\beta_i} V(c,\gamma,0)) = 0.$$

Since $G_{\tilde{U}}(c,\gamma,0)V^i_{c,\gamma} = 0$ by Lemma 5.2.13, we have

$$G_{\tilde{U}}(c,\gamma,0)\partial_{\beta_i} V(c,\gamma,0) = 0.$$

But $\partial_{\beta_i} V(c,\gamma,0) \in C^\infty(\mathbf{X}_1)$, so

$$\partial_{\beta_i} V(c,\gamma,0) = 0, \quad 1 \le i \le 4.$$

Since K is compact, we can take δ_1 to be a universal constant for each $(c,\gamma) \in K$. So we have proved the existence of V in Theorem 5.2.4.

Next, let $(c,\gamma) \in B_{\delta_1}(\bar{c},\bar{\gamma})$. Let δ' be a small constant to be determined. For any U satisfies the equation (5.43) with $U - U^{c,\gamma} \in \mathbf{X}$, and $\|U - U^{c,\gamma}\|_{\mathbf{X}} \le \delta'$ there exist some $\beta_1, \beta_2 \in \mathbb{R}$ and $V^* \in \mathbf{X}_1$ such that

$$U - U^{c,\gamma} = \sum_{i=1}^{4} \beta_i V^i_{c,\gamma} + V^*.$$

Then by Lemma 5.2.16, there exists some constant $C > 0$ such that

$$\frac{1}{C}(|(\beta_1,\beta_2,\beta_3)| + \|V^*\|_{\mathbf{X}}) \le \|\sum_{i=1}^{4} \beta_i V^i_{c,\gamma} + V^*\|_{\mathbf{X}} \le \delta'.$$

This gives $\|V^*\|_{\mathbf{X}} \le C\delta'$.

Choose δ' small enough such that $C\delta' < \delta_1$. We have the uniqueness of V^*. So $V^* = V(c,\gamma,\beta)$ in (5.105). The theorem is proved. $\square$

Theorem 5.2.5 can be proved by replacing $\mathbf{X}_1$ by $\mathbf{X}_2$, $\sum_{i=1}^{4} \beta_i V^i_{c,\gamma}$ by $\beta_2 V^{2b}_{c,\gamma} + \beta_3 V^3_{c,\gamma} + \beta_4 V^4_{c,\gamma}$ respectively.

The case $\bar{U}_\theta(1) = -2$ with $\eta_2 = -4$, ($\bar{U}_\theta(-1) < 3$, $\bar{U}_\theta(-1) \neq 2$ or $\bar{U}_\theta(-1) = 2$ with $\eta_1 = 0$) can conclude the following theorems and the theorems can be proved similarly.

Theorem 5.2.4'. *For every compact subset $K \subset E_{3,3}$, there exist $\delta = \delta(K) > 0$, and $V \in C^\infty(K \times B_\delta(0), \mathbf{X}_1)$ satisfying $V(c, \gamma, 0) = 0$ and $\left.\frac{\partial V}{\partial \beta_i}\right|_{\beta=0} = 0$, $1 \leq i \leq 4$, $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$, such that*

$$U = U^{c,\gamma} + \sum_{i=1}^{4} \beta_i V^i_{c,\gamma} + V(c, \gamma, \beta) \tag{5.129}$$

satisfies equation (5.43) with $\hat{c}_1 = c_1 + \frac{1}{2}\psi[\tilde{U}_\phi](-1)$, $\hat{c}_2 = c_2 + \frac{1}{2}\psi[\tilde{U}_\phi](1)$, $\hat{c}_3 = c_3 - \frac{1}{2}(\varphi_{c,\gamma}[\tilde{U}_\theta])''(0)$.

Moreover, there exists some $\delta' = \delta'(K) > 0$, such that if $\|U - U^{c,\gamma}\|_{\mathbf{X}} < \delta'$, $(c, \gamma) \in K$, and U satisfies equation (5.43) with some constant $\hat{c}_1, \hat{c}_2, \hat{c}_3$, then (5.129) holds for some $|\beta| < \delta$.

Recall that $V^{2a}_{c,\gamma}$ is defined by (5.61).

Theorem 5.2.5'. *For every compact subset K of $E_{3,1}$ or $E_{4,1}$, there exist $\delta = \delta(K) > 0$, and $V \in C^\infty(K \times B_\delta(0), \mathbf{X}_2)$ satisfying $V(c, \gamma, 0) = 0$ and $\left.\frac{\partial V}{\partial \beta_i}\right|_{\beta=0} = 0$, $i = 2, 3, 4$, $\beta = (\beta_2, \beta_3, \beta_4)$, such that*

$$U = U^{c,\gamma} + \beta_2 V^{2a}_{c,\gamma} + \beta_3 V^3_{c,\gamma} + \beta_4 V^4_{c,\gamma} + V(c, \gamma, \beta) \tag{5.130}$$

satisfies equation (5.43) with $\hat{c}_1 = c_1 + \frac{1}{2}\psi[\tilde{U}_\phi](-1)$, $\hat{c}_2 = c_2 + \frac{1}{2}\psi[\tilde{U}_\phi](1)$, $\hat{c}_3 = c_3 - \frac{1}{2}(\varphi_{c,\gamma}[\tilde{U}_\theta])''(0)$.

Moreover, there exists some $\delta' = \delta'(K) > 0$, such that if $\|U - U^{c,\gamma}\|_{\mathbf{X}} < \delta'$, $(c, \gamma) \in K$, and U satisfies equation (5.43) with some constant $\hat{c}_1, \hat{c}_2, \hat{c}_3$, then (5.130) holds for some $|\beta| < \delta$.

5.2.4 Existence of axisymmetric, with swirl solutions around $\bar{U}$, when $(c, \gamma) \in E_{4,3}$

If $(c, \gamma) \in E_{4,3}$, then $\bar{U}_\theta(-1) = 2$ with $\eta_1 = 4$ and $\bar{U}_\theta(1) = -2$ with $\eta_2 = -4$.

Let us start from constructing the Banach spaces we use. Choose $0 < \epsilon < \frac{1}{2}$, define

$$\mathbf{M}_1 = \mathbf{M}_1(\epsilon) := \Big\{ \tilde{U}_\theta \in C^3(-\frac{1}{2}, \frac{1}{2}) \cap C^1(-1,1) \cap C[-1,1] \mid \tilde{U}_\theta(1) = \tilde{U}_\theta(-1) = 0,$$

$$\| \ln \frac{1+x}{3} \ln \frac{1-x}{3} \tilde{U}_\theta \|_{L^\infty(-1,1)} < \infty, \|(1-x^2) \left(\ln \frac{1+x}{3}\right)^2 \left(\ln \frac{1-x}{3}\right)^2 \tilde{U}_\theta' \|_{L^\infty(-1,1)} < \infty,$$

$$||\tilde{U}_\theta''||_{L^\infty(-\frac{1}{2},\frac{1}{2})} < \infty, ||\tilde{U}_\theta'''||_{L^\infty(-\frac{1}{2},\frac{1}{2})} < \infty \Big\},$$

$$\mathbf{M}_2 = \mathbf{M}_2(\epsilon) := \Big\{ \tilde{U}_\phi \in C^2((-1,1),\mathbb{R}) \mid (1-x^2)^\epsilon \|\tilde{U}_\phi\|_{L^\infty(-1,1)} < \infty, \|(1-x^2)^{1+\varepsilon} \tilde{U}_\phi'\|_{L^\infty(-1,1)} < \infty,$$

$$\|(1-x^2)^{2+\varepsilon} \tilde{U}_\phi''\|_{L^\infty(-1,1)} < \infty \Big\}.$$

with the following norms accordingly

$$\|\tilde{U}_\theta\|_{\mathbf{M}_1} := \| \ln \frac{1+x}{3} \ln \frac{1-x}{3} \tilde{U}_\theta\|_{L^\infty(-1,1)} + \|(1-x^2) \left(\ln \frac{1+x}{3}\right)^2 \left(\ln \frac{1-x}{3}\right)^2 \tilde{U}_\theta'\|_{L^\infty(-1,1)}$$

$$+ ||\tilde{U}_\theta''||_{L^\infty(-\frac{1}{2},\frac{1}{2})} + ||\tilde{U}_\theta'''||_{L^\infty(-\frac{1}{2},\frac{1}{2})},$$

$$\|\tilde{U}_\phi\|_{\mathbf{M}_2} := (1-x^2)^\epsilon \|\tilde{U}_\phi\|_{L^\infty(-1,1)} + \|(1-x^2)^{1+\varepsilon} \tilde{U}_\phi'\|_{L^\infty(-1,1)} + \|(1-x^2)^{2+\varepsilon} \tilde{U}_\phi''\|_{L^\infty(-1,1)}.$$

Next, define the following function spaces:

$$\mathbf{N}_1 = \mathbf{N}_1(\epsilon) := \Big\{ \xi_\theta \in C^2(-\frac{1}{2}, \frac{1}{2}) \cap C[-1,1] \mid \xi_\theta(1) = \xi_\theta(-1) = \xi_\theta''(0) = 0,$$

$$\| \left(\ln \frac{1+x}{3}\right)^2 \left(\ln \frac{1-x}{3}\right)^2 \xi_\theta\|_{L^\infty(-1,1)} < \infty, ||\xi_\theta'||_{L^\infty(-\frac{1}{2},\frac{1}{2})} < \infty, ||\xi_\theta''||_{L^\infty(-\frac{1}{2},\frac{1}{2})} \Big\},$$

$$\mathbf{N}_2 = \mathbf{N}_2(\epsilon) := \big\{ \xi_\phi \in C((-1,1),\mathbb{R}) \mid \|(1-x^2)^{1+\varepsilon} \xi_\phi\|_{L^\infty(-1,1)} < \infty \big\}$$

with the following norms accordingly

$$\|\xi_\theta\|_{\mathbf{N}_1} = \| \left(\ln \frac{1+x}{3}\right)^2 \left(\ln \frac{1-x}{3}\right)^2 \xi_\theta\|_{L^\infty(-1,1)} + ||\xi_\theta'||_{L^\infty(-\frac{1}{2},\frac{1}{2})} + ||\xi_\theta''||_{L^\infty(-\frac{1}{2},\frac{1}{2})},$$

$$\|\xi_\phi\|_{\mathbf{N}_2} = \|(1-x^2)^{1+\epsilon} \xi_\phi\|_{L^\infty(-1,1)}.$$

Then let $\mathbf{X} := \{\tilde{U} = (\tilde{U}_\theta, \tilde{U}_\phi) \mid \tilde{U}_\theta \in \mathbf{M}_1, \tilde{U}_\phi \in \mathbf{M}_2\}$ with norm $\|\tilde{U}\|_{\mathbf{X}} = \|\tilde{U}_\theta\|_{\mathbf{M}_1} + \|\tilde{U}_\phi\|_{\mathbf{M}_2}$, $\mathbf{Y} := \{\xi = (\xi_\theta, \xi_\phi) \mid \xi_\theta \in \mathbf{N}_1, \xi_\phi \in \mathbf{N}_2\}$, with the norm $\|\xi\|_{\mathbf{Y}} = \|\xi_\theta\|_{\mathbf{N}_1} + \|\xi_\phi\|_{\mathbf{N}_2}$. It can be proved that $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{X}$ and $\mathbf{Y}$ are Banach spaces.

Let $l_i : \mathbf{X} \to \mathbb{R}$, $1 \le i \le 4$, be the bounded linear functionals defined by (5.63) for each $V \in \mathbf{X}$. Let $\mathbf{X}_1 := \cap_{i=1}^4 \ker l_i$. It can be seen that $\mathbf{X}_1$ is independent of (c, γ).

Theorem 5.2.6. *For every compact subset $K \subset E_{4,3}$, there exist $\delta = \delta(K) > 0$, and $V \in C^\infty(K \times B_\delta(0), \mathbf{X}_1)$ satisfying $V(c, \gamma, 0) = 0$ and $\left.\frac{\partial V}{\partial \beta_i}\right|_{\beta=0} = 0$, $1 \le i \le 4$,*

$\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$, such that

$$U = U^{c,\gamma} + \sum_{i=1}^{4} \beta_i V^i_{c,\gamma} + V(c, \gamma, \beta) \tag{5.131}$$

satisfies equation (5.43) with $\hat{c}_1 = c_1 + \frac{1}{2}\psi[\tilde{U}_\phi](-1)$, $\hat{c}_2 = c_2 + \frac{1}{2}\psi[\tilde{U}_\phi](1)$, $\hat{c}_3 = c_3 - \frac{1}{2}(\varphi_{c,\gamma}[\tilde{U}_\theta])''(0)$.

Moreover, there exists some $\delta' = \delta'(K) > 0$, such that if $\|U - U^{c,\gamma}\|_{\mathbf{X}} < \delta'$, $(c, \gamma) \in K$, and U satisfies equation (5.43) with some constant $\hat{c}_1, \hat{c}_2, \hat{c}_3$, then (5.131) holds for some $|\beta| < \delta$.

For $\tilde{U}_\phi \in \mathbf{M}_2$, let $\psi[\tilde{U}_\phi](x)$ be defined by (5.44).

Let K be a compact subset in $E_{4,3}$. Define a map $G = G(c, \gamma, \tilde{U})$ on $K \times \mathbf{X}$ by (5.46).

Proposition 5.2.3. *The map G is in $C^\infty(K \times \mathbf{X}, \mathbf{Y})$ in the sense that G has continuous Fréchet derivatives of every order. Moreover, the Fréchet derivative of G with respect to $\tilde{U}$ at $(c, \gamma, \tilde{U}) \in K \times \mathbf{X}$ is given by the linear bounded operator $L^{c,\gamma}_{\tilde{U}} : \mathbf{X} \to \mathbf{Y}$ defined as in (5.51).*

To prove Proposition 5.2.3, we first prove the following lemmas:

Lemma 5.2.17. *For every $(c, \gamma) \in K$, $A(c, \gamma, \cdot) : \mathbf{X} \to \mathbf{Y}$ defined by (5.48) is a well-defined bounded linear operator.*

Proof. In the following, C denotes a universal constant which may change from line to line. We denote $l = l_{c,\gamma}[\tilde{U}_\theta]$ defined by (5.47), and $A = A(c, \gamma, \cdot)$ for some fixed $(c, \gamma) \in K$. We make use of the property of $\bar{U}_\theta$ that $\bar{U}_\theta \in C^2(-1, 1) \cap L^\infty(-1, 1)$. Moreover, by Theorem 1.0.3 and Theorem 4.0.1', $\bar{U}_\theta = 2 + 4\left(\ln \frac{1+x}{3}\right)^{-1} + O(1)\left(\ln \frac{1+x}{3}\right)^{-2+\epsilon'} = -2 - 4\left(\ln \frac{1-x}{3}\right)^{-1} + O(1)\left(\ln \frac{1-x}{3}\right)^{-2+\epsilon'}$ for any $\epsilon' > 0$. So there exists some constant $C > 0$, such that

$$\left|(2x + \bar{U}_\theta)\left(\ln \frac{1+x}{3}\right)\left(\ln \frac{1-x}{3}\right)\right| \le C, \quad -1 < x < 1. \tag{5.132}$$

A is clearly linear. For every $\tilde{U} \in \mathbf{X}$, we prove that $A\tilde{U}$ defined by (5.48) is in $\mathbf{Y}$ and there exists some constant C such that $\|A\tilde{U}\|_{\mathbf{Y}} \le C\|\tilde{U}\|_{\mathbf{X}}$ for all $\tilde{U} \in \mathbf{X}$.

By computation,

$$l'(x) = (1-x^2)\tilde{U}_\theta'' + \bar{U}_\theta \tilde{U}_\theta' + (2+\bar{U}_\theta')\tilde{U}_\theta,$$

$$l''(x) = (1-x^2)\tilde{U}_\theta''' + (\bar{U}_\theta - 2x)\tilde{U}_\theta'' + 2(\bar{U}_\theta' + 1)\tilde{U}_\theta' + \bar{U}_\theta''\tilde{U}_\theta.$$

By the fact that $\tilde{U}_\theta \in \mathbf{M_1}$, we have that $|l''(0)| \leq C||\tilde{U}_\theta||_{\mathbf{M_1}}$. So for $-1 < x < 1$, we have

$$\begin{aligned}
&| \left(\ln \frac{1+x}{3}\right)^2 \left(\ln \frac{1-x}{3}\right)^2 A_\theta| \\
\leq &| \left(\ln \frac{1+x}{3}\right)^2 \left(\ln \frac{1-x}{3}\right)^2 (1-x^2)\tilde{U}_\theta'| + \left|(2x+\bar{U}_\theta) \left(\ln \frac{1+x}{3}\right)^2 \left(\ln \frac{1-x}{3}\right)^2\right| |\tilde{U}_\theta| \\
&+ \frac{1}{2} \left(\ln \frac{1+x}{3}\right)^2 \left(\ln \frac{1-x}{3}\right)^2 (1-x^2)|l''(0)| \\
\leq &C\|\tilde{U}_\theta\|_{\mathbf{M_1}}.
\end{aligned}$$

We also see from the above that $\lim_{x\to 1} A_\theta(x) = \lim_{x\to -1} A_\theta(x) = 0$.

For $-\frac{1}{2} < x < \frac{1}{2}$,

$$\begin{aligned}
|A_\theta'| &= |l'(x) - l''(0)x| \\
&\leq |\tilde{U}_\theta''| + |\bar{U}_\theta||\tilde{U}_\theta'| + (2 + |\bar{U}_\theta'|)|\tilde{U}_\theta| + |l''(0)| \\
&\leq C\|\tilde{U}_\theta\|_{\mathbf{M_1}},
\end{aligned}$$

and

$$\begin{aligned}
|A_\theta''| &= |l''(x) - l''(0)| \\
&\leq |\tilde{U}_\theta'''| + (|\bar{U}_\theta| + 2)|\tilde{U}_\theta''| + 2(|\bar{U}_\theta'| + 1)|\tilde{U}_\theta'| + |\bar{U}_\theta''||\tilde{U}_\theta| + |l''(0)| \\
&\leq C\|\tilde{U}_\theta\|_{\mathbf{M_1}}
\end{aligned}$$

By computation $A_\theta''(0) = l''(0) - \frac{1}{2}l''(0)\cdot 2 = 0$. So we have $A_\theta \in \mathbf{N_1}$ and $\|A_\theta\|_{\mathbf{N_1}} \leq C\|\tilde{U}_\theta\|_{\mathbf{M_1}}$.

Next, since $A_\phi = (1-x^2)\tilde{U}_\phi'' + \bar{U}_\theta\tilde{U}_\phi'$, by the fact that $\tilde{U}_\phi \in \mathbf{M_2}$ we have that

$$\left|(1-x^2)^{1+\epsilon}A_\phi\right| \leq (1-x^2)^{2+\epsilon}|\tilde{U}_\phi''| + (1-x^2)^{1+\epsilon}|\bar{U}_\theta||\tilde{U}_\phi'| \leq C\|\tilde{U}_\phi\|_{\mathbf{M_2}}.$$

So $A_\phi \in \mathbf{N_1}$, and $\|A_\phi\|_{\mathbf{N_1}} \leq C\|\tilde{U}_\phi\|_{\mathbf{M_2}}$. We have proved that $A\tilde{U} \in \mathbf{Y}$ and $\|A\tilde{U}\|_{\mathbf{Y}} \leq C\|\tilde{U}\|_{\mathbf{X}}$ for every $\tilde{U} \in \mathbf{X}$. The proof is finished.

$\square$

Lemma 5.2.18. *The map $Q : \mathbf{X} \times \mathbf{X} \to \mathbf{Y}$ defined by (5.49) is a well-defined bounded bilinear operator.*

Proof. In the following, C denotes a universal constant which may change from line to line. For convenience we denote $\psi = \psi[\tilde{U}_\phi, \tilde{V}_\phi]$ defined by (5.44).

It is clear that Q is a bilinear operator. For every $\tilde{U}, \tilde{V} \in \mathbf{X}$, we will prove that $Q(\tilde{U}, \tilde{V})$ is in $\mathbf{Y}$ and there exists some constant C independent of $\tilde{U}$ and $\tilde{V}$ such that $\|Q(\tilde{U}, \tilde{V})\|_{\mathbf{Y}} \le C\|\tilde{U}\|_{\mathbf{X}}\|\tilde{V}\|_{\mathbf{X}}$.

For $\tilde{U}, \tilde{V} \in \mathbf{X}$, we have, using the fact that $\tilde{U}_\phi, \tilde{V}_\phi \in \mathbf{M_2}$, that

$$\left|\frac{\tilde{U}_\phi(s)\tilde{V}_\phi'(s)}{1-s^2}\right| \le (1-s^2)^{-2-2\epsilon}\|\tilde{U}_\phi\|_{\mathbf{M_2}}\|\tilde{V}_\phi\|_{\mathbf{M_2}}, \quad \forall -1 < s < 1. \tag{5.133}$$

It follows that $\psi(\tilde{U}, \tilde{V})(x)$ is well-defined and

$$|\psi(-1)| \le C\|\tilde{U}_\phi\|_{\mathbf{M_2}}\|\tilde{V}_\phi\|_{\mathbf{M_2}}, \quad |\psi(1)| \le C\|\tilde{U}_\phi\|_{\mathbf{M_2}}\|\tilde{V}_\phi\|_{\mathbf{M_2}}. \tag{5.134}$$

Moreover,

$$\begin{aligned}
&\left|\psi(x) - \frac{1}{2}\psi(-1)(1-x) - \frac{1}{2}\psi(1)(1+x)\right| \\
&= \left|\frac{1}{2}\psi(x)(1-x) + \frac{1}{2}\psi(x)(1+x) - \frac{1}{2}\psi(-1)(1-x) - \frac{1}{2}\psi(1)(1+x)\right| \\
&\le \frac{1}{2}(1-x)|\psi(x)-\psi(-1)| + \frac{1}{2}(1+x)|\psi(x)-\psi(1)| \\
&= \frac{1}{2}(1-x)\left|\int_{-1}^{x}\int_0^l\int_0^t \frac{2\tilde{U}_\phi(s)\tilde{V}_\phi'(s)}{1-s^2}dsdtdl\right| + \frac{1}{2}(1+x)\left|\int_1^x\int_0^l\int_0^t \frac{2\tilde{U}_\phi(s)\tilde{V}_\phi'(s)}{1-s^2}dsdtdl\right| \\
&\le C(1-x)(1+x)^{1-2\epsilon}\|\tilde{U}_\phi\|_{\mathbf{M_2}}\|\tilde{V}_\phi\|_{\mathbf{M_2}} + C(1+x)(1-x)^{1-2\epsilon}\|\tilde{U}_\phi\|_{\mathbf{M_2}}\|\tilde{V}_\phi\|_{\mathbf{M_2}} \\
&\le C(1-x^2)^{1-2\epsilon}\|\tilde{U}_\phi\|_{\mathbf{M_2}}\|\tilde{V}_\phi\|_{\mathbf{M_2}}
\end{aligned} \tag{5.135}$$

By (5.133), we also have

$$|\psi'(x)| = \left|\int_0^x\int_0^t \frac{2\tilde{U}_\phi(s)\tilde{V}_\phi'(s)}{1-s^2}dsdt\right| \le C\|\tilde{U}_\phi\|_{\mathbf{M_2}}\|\tilde{V}_\phi\|_{\mathbf{M_2}}, \quad -\frac{1}{2} < x < \frac{1}{2} \tag{5.136}$$

and

$$|\psi''(x)| = \left|\int_0^x \frac{2\tilde{U}_\phi(s)\tilde{V}_\phi'(s)}{1-s^2}ds\right| \le C\|\tilde{U}_\phi\|_{\mathbf{M_2}}\|\tilde{V}_\phi\|_{\mathbf{M_2}}, \quad -\frac{1}{2} < x < \frac{1}{2} \tag{5.137}$$

Using the fact that $\tilde{U}_\theta, \tilde{V}_\theta \in \mathbf{M}_1$, we have

$$\begin{aligned} |(\tilde{U}_\theta \tilde{V}_\theta)''(0)| &\le |\tilde{U}_\theta''(0)||\tilde{V}_\theta(0)| + 2|\tilde{U}_\theta'(0)||\tilde{V}_\theta'(0)| + |\tilde{U}_\theta(0)||\tilde{V}_\theta''(0)| \\ &\le C\|\tilde{U}_\theta\|_{\mathbf{M}_1}\|\tilde{V}_\theta\|_{\mathbf{M}_1} \end{aligned} \tag{5.138}$$

So by (5.135), (5.138), and the fact that $\tilde{U}_\theta, \tilde{V}_\theta \in \mathbf{M}_1$, we have that for $-1 < x < 1$,

$$\begin{aligned} &|\left(\ln\frac{1+x}{3}\right)^2\left(\ln\frac{1-x}{3}\right)^2 Q_\theta(x)| \\ &\le \frac{1}{2}\left|\left(\ln\frac{1+x}{3}\right)\left(\ln\frac{1-x}{3}\right)\tilde{U}_\theta(x)\right| \cdot \left|\left(\ln\frac{1+x}{3}\right)\left(\ln\frac{1-x}{3}\right)\tilde{V}_\theta(x)\right| \\ &\quad + \left|\left(\ln\frac{1+x}{3}\right)^2 \cdot \left(\ln\frac{1-x}{3}\right)^2\right| \left|\psi(x) - \frac{1}{2}\psi(-1)(1-x) - \frac{1}{2}\psi(1)(1+x)\right| \\ &\quad + \frac{1}{4}\left|\left(\ln\frac{1+x}{3}\right)^2 \cdot \left(\ln\frac{1-x}{3}\right)^2\right|(1-x^2)|(\tilde{U}_\theta\tilde{V}_\theta)''(0)| \\ &\le C\|\tilde{U}\|_{\mathbf{X}}\|\tilde{V}\|_{\mathbf{X}} \end{aligned}$$

From this we also have $\lim_{x\to 1} Q_\theta(x) = \lim_{x\to -1} Q_\theta(x) = 0$. Similar as in Lemma 5.2.2, we have that for $-\frac{1}{2} < x < \frac{1}{2}$,

$$|Q_\theta'(x)| \le C\|\tilde{U}\|_{\mathbf{X}}\|\tilde{V}\|_{\mathbf{X}}, \quad |Q_\theta''(x)| \le C\|\tilde{U}\|_{\mathbf{X}}\|\tilde{V}\|_{\mathbf{X}}.$$

So there is $Q_\theta \in \mathbf{N}_1$, and $\|Q_\theta\|_{\mathbf{N}_1} \le C(\epsilon)\|\tilde{U}\|_{\mathbf{X}}\|\tilde{V}\|_{\mathbf{X}}$.

Next, since $Q_\phi(x) = \tilde{U}_\theta(x)\tilde{V}_\phi'(x)$, for $-1 < x < 1$,

$$\left|(1-x^2)^{1+\epsilon}Q_\phi(x)\right| \le |\tilde{U}_\theta(x)|(1-x^2)^{1+\epsilon}|\tilde{V}_\phi'| \le 2\|\tilde{U}_\theta\|_{\mathbf{M}_1}\|\tilde{V}_\phi\|_{\mathbf{M}_2}.$$

So $Q_\phi \in \mathbf{N}_2$, and $\|Q_\phi\|_{\mathbf{N}_2} \le \|\tilde{U}_\theta\|_{\mathbf{M}_1}\|\tilde{V}_\phi\|_{\mathbf{M}_2}$. Thus we have proved that $Q(\tilde{U}, \tilde{V}) \in \mathbf{Y}$ and $\|Q(\tilde{U}, \tilde{V})\|_{\mathbf{Y}} \le C\|\tilde{U}\|_{\mathbf{X}}\|\tilde{V}\|_{\mathbf{X}}$ for all $\tilde{U}, \tilde{V} \in \mathbf{X}$. Lemma 5.2.18 is proved. □

Proof of Proposition 5.2.3*:* By definition, $G(c, \gamma, \tilde{U}) = A(c, \gamma, \tilde{U}) + Q(\tilde{U}, \tilde{U})$ for $(c, \gamma, \tilde{U}) \in K \times \mathbf{X}$. Using standard theories in functional analysis, by Lemma 5.2.18 it is clear that Q is C^∞ on $\mathbf{X}$. By Lemma 5.2.17, $A(c, \gamma; \cdot) : \mathbf{X} \to \mathbf{Y}$ is C^∞ for each $(c, \gamma) \in K$.

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a multi-index where $\alpha_i \ge 0$, $i = 1, 2, 3$, and $j \ge 0$. For all $|\alpha| + j \ge 1$, we have

$$\partial_c^\alpha \partial_\gamma^j A(c, \gamma, \tilde{U}) = \partial_c^\alpha \partial_\gamma^j U_\theta^{c,\gamma} \begin{pmatrix} \tilde{U}_\theta \\ \tilde{U}_\phi' \end{pmatrix} + \frac{1}{2}(\partial_c^\alpha \partial_\gamma^j U_\theta^{c,\gamma} \cdot \tilde{U}_\theta)''(0) \begin{pmatrix} 1 - x^2 \\ 0 \end{pmatrix}. \tag{5.139}$$

By Theorem 5.1.3 (iv) and Corollary 5.1.3, we have

$$\left|\ln\frac{1+x}{3}\right|^2\left|\ln\frac{1-x}{3}\right|^2\left|\partial_c^\alpha\partial_\gamma^j A_\theta(c,\gamma,\tilde U)\right|\le C(\alpha,j,K)\|\tilde U_\theta\|_{\mathbf{M}_1},\quad -1<x<1,$$

and for $-\frac{1}{2}<x<\frac{1}{2}$.

$$|\partial_c^\alpha\partial_\gamma^j A_\theta'(c,\gamma,\tilde U)|\le C(\alpha,j,K)\|\tilde U_\theta\|_{\mathbf{M}_1},\quad |\partial_c^\alpha\partial_\gamma^j A_\theta''(c,\gamma,\tilde U)|\le C(\alpha,j,K)\|\tilde U_\theta\|_{\mathbf{M}_1}.$$

The above estimates and (5.139) also imply that

$$\partial_c^\alpha\partial_\gamma^j A_\theta(c,\gamma,\tilde U)(-1)=\partial_c^\alpha\partial_\gamma^j A_\theta(c,\gamma,\tilde U)(1)=\partial_c^\alpha\partial_\gamma^j A_\theta(c,\gamma,\tilde U)''(0)=0.$$

So $\partial_c^\alpha\partial_\gamma^j A_\theta(c,\gamma,\tilde U)\in\mathbf{N}_1$, with $\|\partial_c^\alpha\partial_\gamma^j A_\theta(c,\gamma,\tilde U)\|_{\mathbf{N}_1}\le C(\alpha,j,K)\|\tilde U_\theta\|_{\mathbf{M}_1}$ for all $(c,\gamma,\tilde U)\in K\times\mathbf{X}$.

Next, by Theorem 5.1.3 (iv) and the fact that $\tilde U_\phi\in\mathbf{M}_1$, we have

$$(1-x^2)^{1+\epsilon}|\partial_c^\alpha\partial_\gamma^j A_\phi(c,\gamma,\tilde U)(x)|=|\partial_c^\alpha\partial_\gamma^j U_\theta^{c,\gamma}(x)|\cdot|(1-x^2)^{1+\epsilon}\tilde U_\phi'|\le C(\alpha,j,K)\|\tilde U_\phi\|_{\mathbf{M}_2}. \tag{5.140}$$

So $\partial_c^\alpha\partial_\gamma^j A_\phi(c,\gamma,\tilde U)\in\mathbf{N}_2$ with $\|\partial_c^\alpha\partial_\gamma^j A_\phi(c,\gamma,\tilde U)\|_{\mathbf{N}_2}\le C(\alpha,j,K)\|\tilde U_\phi\|_{\mathbf{M}_2}$ for all $(c,\gamma,\tilde U)\in K\times\mathbf{X}$. Thus $\partial_c^\alpha\partial_\gamma^j A(c,\gamma,\tilde U)\in\mathbf{Y}$, with $\|\partial_c^\alpha\partial_\gamma^j A(c,\gamma,\tilde U)\|_{\mathbf{Y}}\le C(\alpha,j,K)\|\tilde U\|_{\mathbf{X}}$ for all $(c,\gamma,\tilde U)\in K\times\mathbf{X}$, $|\alpha|+j\ge 0$.

So for each $(c,\gamma)\in K$, $\partial_c^\alpha\partial_\gamma^j A(c,\gamma;\cdot):\mathbf{X}\to\mathbf{Y}$ is a bounded linear map with uniform bounded norm on K. Then by standard theories in functional analysis, $A:K\times\mathbf{X}\to\mathbf{Y}$ is C^∞. So G is a C^∞ map from $K\times\mathbf{X}$ to Y. By direct calculation we get its Fréchet derivative with respect to $\mathbf{X}$ is given by the linear bounded operator $L_{\tilde U}^{c,\gamma}:\mathbf{X}\to\mathbf{Y}$ defined as (5.50). The proof is finished. $\square$

Let $a_{c,\gamma}(x), b_{c,\gamma}(x)$ be the functions defined by (5.52). For convenience we denote $a(x)=a_{c,\gamma}(x)$, $b(x)=b_{c,\gamma}(x)$, and $\bar U_\theta=U_\theta^{c,\gamma}$, we have

Lemma 5.2.19. *For $(c,\gamma)\in E_{4,3}$, there exists some constant $C>0$, depending only on (c,γ), such that for any $-1<x<1$,*

$$\begin{aligned} e^{b(x)}&\le C\left(\ln\frac{1+x}{3}\right)^2\left(\ln\frac{1-x}{3}\right)^2(1-x^2),\\ e^{-b(x)}&\le C\left(\ln\frac{1+x}{3}\right)^{-2}\left(\ln\frac{1-x}{3}\right)^{-2}(1-x^2)^{-1}, \end{aligned} \tag{5.141}$$

and

$$e^{a(x)} \le C \left|\ln \frac{1+x}{3}\right|^2 \left|\ln \frac{1-x}{3}\right|^2, \quad e^{-a(x)} \le C \left|\ln \frac{1+x}{3}\right|^{-2} \left|\ln \frac{1-x}{3}\right|^{-2}. \qquad (5.142)$$

Proof. Under the assumption of $\bar{U}_\theta$ in this case, we have for some small $\epsilon' > 0$

$$\bar{U}_\theta = 2 + \frac{4}{\ln \frac{1+x}{3}} + O(1)\left(\ln \frac{1+x}{3}\right)^{-2+\epsilon'}, \quad -1 < x < 0.$$

$$\bar{U}_\theta = -2 - \frac{4}{\ln \frac{1-x}{3}} + \left(\ln \frac{1-x}{3}\right)^{-2+\epsilon'}, \quad 0 < x < 1.$$

Thus, by definition of $a(x)$ and $b(x)$ in (5.52), for $-1 < x < 1$, we have

$$b(x) = \ln(1+x) + 2\ln(-\ln(1+x)) + \ln(1-x) + 2\ln(-\ln(1-x)) + O(1),$$

$$a(x) = 2\ln(-\ln(1+x)) + 2\ln(-\ln(1-x)) + O(1).$$

The Lemma follows from the above estimates. $\square$

For $\xi = (\xi_\theta, \xi_\phi) \in \mathbf{Y}$, let the map $W^{c,\gamma}$ be defined as

$$W^{c,\gamma}(\xi) := (W_\theta^{c,\gamma,1}(\xi), W_\phi^{c,\gamma}(\xi)),$$

where $W_\theta^{c,\gamma,1}$ and $W_\phi^{c,\gamma}(\xi)$ are defined by (5.54) and (5.55).

Lemma 5.2.20. *For every $(c,\gamma) \in K$, $W^{c,\gamma} : \mathbf{Y} \to \mathbf{X}$ is continuous, and is a right inverse of $L_0^{c,\gamma}$.*

Proof. In the following, C denotes a universal constant which may change from line to line.

We make use of the property that $\bar{U}_\theta \in C^2(-1,1) \cap L^\infty(-1,1)$ and the fact that $0 < \epsilon < 1$.

For convenience let us write $W := W^{c,\gamma}(\xi)$ and $W_\theta := W_\theta^{c,\gamma,1}(\xi)$ for $\xi \in \mathbf{Y}$.

By Lemma 5.2.19 we have the estimates (5.141) and (5.142).

We first prove $W_\theta : \mathbf{Y} \to \mathbf{X}$ is well-defined. Using Lemma 5.2.19 and the fact that $\xi_\theta \in \mathbf{N}_1$, we have, by the expression of $W_\theta = W_\theta^1$ in (5.54), for any $-1 < x < 1$ that

$$\begin{aligned} &\left|\ln \frac{1+x}{3} \ln \frac{1-x}{3} W_\theta^1(x)\right| \\ &\le \left|\ln \frac{1+x}{3} \ln \frac{1-x}{3}\right| \|\xi_\theta\|_{\mathbf{N}_1} e^{-a(x)} \int_0^x e^{a(s)} \left|\ln \frac{1+s}{3} \ln \frac{1+s}{3}\right|^{-2} ds \le C\|\xi_\theta\|_{\mathbf{N}_1}. \end{aligned} \qquad (5.143)$$

From the above we also have that $\lim_{x\to -1} W_\theta(x) = \lim_{x\to 1} W_\theta(x) = 0$.

By (5.56), (5.53), (5.143), and the property that $\bar{U}_\theta = 2 + O(1)\left(\ln \frac{1+x}{3}\right)^{-1} = -2 + O(1)\left(\ln \frac{1-x}{3}\right)^{-1}$, we have that for $-1 < x < 1$

$$\begin{aligned}
&\left|(1-x^2)\left(\ln \frac{1+x}{3}\right)^2 \left(\ln \frac{1-x}{3}\right)^2 W_\theta'\right| \\
\le &\left|(2x+\bar{U}_\theta)\left(\ln \frac{1+x}{3}\right)^2 \left(\ln \frac{1-x}{3}\right)^2 W_\theta\right| + \left(\ln \frac{1+x}{3}\right)^2 \left(\ln \frac{1-x}{3}\right)^2 |\xi_\theta(x)| \\
\le &\, C\|\xi_\theta\|_{\mathbf{N}_1}.
\end{aligned}$$

By (5.53), it can be seen that $|a''(x)|, |a'''(x)| \le C$ for $-\frac{1}{2} < x < \frac{1}{2}$. Then using this fact and (5.114) and (5.117), we have, for $-\frac{1}{2} < x < \frac{1}{2}$,

$$|W_\theta''(x)| = \left|a''(x)W_\theta(x) + a'(x)W_\theta'(x) + \left(\frac{\xi_\theta}{1-x^2}\right)'\right| \le C\|\xi_\theta\|_{\mathbf{N}_1},$$

and

$$|W_\theta'''(x)| = \left|a'''(x)W_\theta(x) + 2a''(x)W_\theta'(x) + a'(x)W_\theta''(x) + \left(\frac{\xi_\theta}{1-x^2}\right)''\right| \le C\|\xi_\theta\|_{\mathbf{N}_1}$$

So we have shown that $W_\theta \in \mathbf{M}_1$, and $\|W_\theta\|_{\mathbf{M}_1} \le C\|\xi_\theta\|_{\mathbf{N}_1}$ for some constant C. By the definition of $W_\phi(\xi)$ in (5.55) , (5.57), Lemma 5.2.19 and the fact that $\xi_\phi \in \mathbf{N}_2$, we have, for every $-1 < x < 1$,

$$\begin{aligned}
&(1-x^2)^\epsilon |W_\phi(x)| \le \int_0^x e^{-b(t)} \int_0^t e^{b(s)} \frac{|\xi_\phi(s)|}{1-s^2} ds dt \\
&\le C(1-x^2)^\epsilon \|\xi_\phi\|_{\mathbf{N}_2} \int_0^x \left(\ln \frac{1+t}{3}\right)^{-2} \left(\ln \frac{1-t}{3}\right)^{-2} (1-t^2)^{-1} \\
&\cdot \int_0^t \left(\ln \frac{1+s}{3}\right)^2 \left(\ln \frac{1-s}{3}\right)^2 (1-s^2)^{-1-\epsilon} ds dt \\
&\le C\|\xi_\phi\|_{\mathbf{N}_2},
\end{aligned}$$

and

$$|(1-x^2)^{1+\epsilon} W_\phi'(x)| \le (1-x^2)^{1+\epsilon} e^{-b(x)} \int_0^x e^{b(s)} \frac{|\xi_\phi(s)|}{1-s^2} ds \le C\|\xi_\phi\|_{\mathbf{N}_2}. \tag{5.144}$$

Similarly, since $|b'(x)| = \frac{|\bar{U}_\theta|}{1-x^2}$, using (5.58), (5.144) and the fact that $\xi_\phi \in \mathbf{N}_2$, we have

$$|(1-x^2)^{2+\epsilon} W_\phi''(x)| \le C\|\xi_\phi\|_{\mathbf{N}_2}.$$

Therefore $W(\xi) \in \mathbf{X}$ for all $\xi \in \mathbf{Y}$, and $\|W(\xi)\|_{\mathbf{X}} \le C\|\xi\|_{\mathbf{Y}}$ for some constant C. So $W : \mathbf{Y} \to \mathbf{X}$ is well-defined and continuous.

By definition of W, we have $l[W_\theta](x) = \xi_\theta$. So $(l[W_\theta])''(0) = \xi_\theta''(0) = 0$, $l[W_\theta](x) + \frac{1}{2}(l[W_\theta])''(0)(1-x^2) = \xi_\theta$. Thus $L_0W(\xi) = \xi$, W is a right inverse of L_0.

□

Let $V^i_{c,\gamma}$, $1 \le i \le 4$, be vectors defined by (5.60), we have

Lemma 5.2.21.

$$\ker L_0^{c,\gamma} = \operatorname{span}\{V^1_{c,\gamma}, V^2_{c,\gamma}, V^3_{c,\gamma}, V^4_{c,\gamma}\}.$$

Proof. Let $V \in \mathbf{X}$, $L_0^{c,\gamma}V = 0$. We know that V is given by (5.59) for some $d_1, d_2, d_3, d_4 \in \mathbb{R}$.

By Lemma 5.2.19, and the expressions of V^1, V^2 in (5.60), we have that

$$V^1_\theta(x) = e^{-a(x)} = O(1)\left|\ln\frac{1+x}{3}\right|^{-2}\left|\ln\frac{1-x}{3}\right|^{-2} \tag{5.145}$$

and

$$V^2_\theta(x) = e^{-a(x)}\int_0^x e^{a(s)}ds = O(1)\left|\ln\frac{1+x}{3}\right|^{-2}\left|\ln\frac{1+x}{3}\right|^{-2}. \tag{5.146}$$

By (5.53), we also have

$$\left|\frac{d}{dx}V^1_\theta(x)\right| = \left|e^{-a(x)}a'(x)\right| = O(1)\left|\ln\frac{1+x}{3}\right|^{-2}\left|\ln\frac{1-x}{3}\right|^{-2}(1-x^2)^{-1}, \tag{5.147}$$

$$\left|\frac{d}{dx}V^2_\theta(x)\right| = \left|V^2_\theta(x)a'(x)+1\right| = O(1)\left|\ln\frac{1+x}{3}\right|^{-2}\left|\ln\frac{1-x}{3}\right|^{-2}(1-x^2)^{-1}. \tag{5.148}$$

Next, by computation we have for $i = 1, 2$,

$$\frac{d^2}{dx^2}V^i_\theta = (V^i_\theta)'a'(x) + V^i_\theta a''(x), \quad \frac{d^3}{dx^3}V^i_\theta = (V^i_\theta)''a'(x) + 2(V^i_\theta)'a'(x) + V^i_\theta a'''(x).$$

Using the definition of $a(x)$ in (5.52), there exists some constant C, depending on c, γ, such that

$$\left|\frac{d^2}{dx^2}V^i_\theta\right| \le C, \quad \left|\frac{d^3}{dx^3}V^i_\theta\right| \le C, \quad -\frac{1}{2} < x < \frac{1}{2}, i = 1, 2. \tag{5.149}$$

Moreover, by Lemma 5.2.19, and the expressions of V^3 in (5.60), we have

$$V^3_\phi(x) = \int_0^x \left|\ln\frac{1+t}{3}\ln\frac{1-t}{3}\right|^{-2}(1-t^2)^{-1}dt = O(1)\left|\ln\frac{1+x}{3}\right|^{-1}\left|\ln\frac{1-x}{3}\right|^{-1} \tag{5.150}$$

and

$$\begin{aligned}
\left|\frac{d}{dx}V_\phi^3(x)\right| &= e^{-b(x)} = O(1)\left|\ln\frac{1+x}{3}\ln\frac{1-x}{3}\right|^{-2}(1-x^2)^{-1},\\
\left|\frac{d^2}{dx^2}V_\phi^3(x)\right| &= e^{-b(x)}|b'(x)| = O(1)\left|\ln\frac{1+x}{3}\ln\frac{1-x}{3}\right|^{-2}(1-x^2)^{-2}.
\end{aligned} \tag{5.151}$$

Using the above estimates and the definition of $V^4_{c,\gamma}$, it is not hard to verify that $V^i_{c,\gamma} \in \mathbf{X}$, $1 \le i \le 4$. It is clear that $\{V^i_{c,\gamma}, 1 \le i \le 4\}$ are independent. So $\{V_{c,\gamma}i, 1 \le i \le 4\}$ is a basis of the kernel. □

Corollary 5.2.3. *For any $\xi \in \mathbf{Y}$, all solutions of $L_0^{c,\gamma}V = \xi$, $V \in \mathbf{X}$, are given by*

$$V = W^{c,\gamma}(\xi) + d_1V^1_{c,\gamma} + d_2V^2_{c,\gamma} + d_3V^3_{c,\gamma} + d_4V^4_{c,\gamma}.$$

Let l_i, $1 \le i \le 4$, be the functionals on $\mathbf{X}$ defined by (5.63), and $\mathbf{X}_1 = \cap \ker l_i$. As shown in Section 5.2.1, the matrix $(l_i(V^j_{c,\gamma}))$ is invertible, for every $(c,\gamma) \in K$. So $\mathbf{X}_i$ is a closed subspace of $\mathbf{X}$, and

$$\mathbf{X} = \mathrm{span}\{V^1_{c,\gamma}, V^2_{c,\gamma}, V^3_{c,\gamma}, V^4_{c,\gamma}\} \oplus \mathbf{X}_1, \tag{5.152}$$

with the projection operator $P_1 : \mathbf{X} \to \mathbf{X}_1$ given by (5.102).

Lemma 5.2.22. *The operator $L_0^{c,\gamma} : \mathbf{X}_1 \to \mathbf{Y}$ is an isomorphism.*

Proof. By Corollary 5.2.3 and Lemma 5.2.21, $L_0^{c,\gamma} : \mathbf{X} \to \mathbf{Y}$ is surjective and $\ker L_0^{c,\gamma}$ is given by Lemma 5.2.21. The conclusion of the lemma then follows in view of the direct sum property (5.152). □

Lemma 5.2.23. $V^i_{c,\gamma} \in C^\infty(K, \mathbf{X})$ *for all* $1 \le i \le 4$ *and* (c,γ) *in compact subset* K *of* $E_{4,3}$.

Proof. It is clear that $V^4_{c,\gamma} \in C^\infty(K, \mathbf{X})$ for all compact set K in $E_{4,3}$.

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a multi-index where $\alpha_i \ge 0$, $i = 1, 2, 3$, and $j \ge 0$.

For convenience we denote $a(x) = a_{c,\gamma}(x)$, $b(x) = b_{c,\gamma}(x)$ and $V^i = V^i_{c,\gamma}$, $i = 1, 2, 3$.

By Lemma 5.2.21 we know that $V^i \in \mathbf{X}$, $1 \le i \le 4$.

Using Theorem 5.1.3 part (iv), we have that for all $|\alpha| + j \ge 1$ and $(c,\gamma) \in K$,

$$\partial_c^j\partial_\gamma^i a(x) = \partial_c^j\partial_\gamma^i b(x) = \int_0^x \frac{1}{1-s^2}\partial_c^j\partial_\gamma^i U^{c,\gamma}(s)ds = O(1), \tag{5.153}$$

Using the expression of V^i, $1 \le i \le 4$ in (5.60), Lemma 5.2.19, (5.145), (5.146), (5.153) and theorem 5.1.3 (iv), we have that for all $|\alpha| + j \ge 1$, and $(c, \gamma) \in K$,

$$\left|\partial_c^\alpha \partial_\gamma^j V_\theta^1(x)\right| = e^{-a(x)} O(1) = O(1) \left|\ln \frac{1+x}{3} \ln \frac{1-x}{3}\right|^{-2},$$

$$\left|\partial_c^\alpha \partial_\gamma^j V_\theta^2(x)\right| = e^{-a(x)} \left|\int_0^x e^{a(s)} ds\right| O(1) = O(1) \left|\ln \frac{1+x}{3} \ln \frac{1-x}{3}\right|^{-2},$$

and

$$\left|\frac{d}{dx} \partial_c^\alpha \partial_\gamma^j V_\theta^1(x)\right| = e^{-a(x)} |a'(x)| O(1) = O(1) \left|\ln \frac{1+x}{3}\right|^{-2} \left|\ln \frac{1-x}{3}\right|^{-2} (1-x^2)^{-1},$$

$$\left|\frac{d}{dx} \partial_c^\alpha \partial_\gamma^j V_\theta^2(x)\right| = |V_\theta^2 a'(x) + 1| O(1) = O(1) \left|\ln \frac{1+x}{3}\right|^{-2} \left|\ln \frac{1-x}{3}\right|^{-2} (1-x^2)^{-1}.$$

From the above we can see that for all $|\alpha| + j \ge 1$, there exists some constant $C = C(\alpha, j, K)$, such that for $i = 1, 2$,

$$\left|\ln \frac{1+x}{3} \ln \frac{1-x}{3} \partial_c^\alpha \partial_\gamma^j V_\theta^i(x)\right| \le C,$$

$$\left|\ln \frac{1+x}{3} \ln \frac{1-x}{3}\right|^2 (1-x^2) \left|\frac{d}{dx} \partial_c^\alpha \partial_\gamma^j V_\theta^i(x)\right| \le C.$$

We also have that for $|\alpha| + j \ge 1$, $\partial_c^\alpha \partial_\gamma^j V_\theta^i(1) = \partial_c^\alpha \partial_\gamma^j V_\theta^i(-1) = 0$, $i = 1, 2$.

Next, using the definition of $a(x)$ in (5.52) and Corollary 5.1.3, there exists some constant $C = C(K)$, such that

$$\left|\frac{d^2}{dx^2} \partial_c^\alpha \partial_\gamma^j V_\theta^i\right| \le C, \quad \left|\frac{d^3}{dx^3} \partial_c^\alpha \partial_\gamma^j V_\theta^i\right| \le C, \qquad \text{for all } -\frac{1}{2} < x < \frac{1}{2}, \quad i = 1, 2.$$

The above imply that for all $|\alpha| + j \ge 1$, $\partial_c^\alpha \partial_\gamma^j V_\theta^i \in M_1$, $i = 1, 2$, so $V^1, V^2 \in C^\infty(K, \mathbf{X})$.

Using the expressions of V^3 in (5.60), Lemma 5.2.19, the estimates (5.150), (5.151), (5.153) and Theorem 5.1.3 (iv), we have that for all $|\alpha| + j \ge 1$,

$$\left|\partial_c^\alpha \partial_\gamma^j V_\phi^3\right| = O(1), \quad \left|\frac{d}{dx} \partial_c^\alpha \partial_\gamma^j V_\phi^3(x)\right| = O(1) \left|\ln \frac{1+x}{3} \ln \frac{1-x}{3}\right|^{-2} (1-x^2)^{-1},$$

$$\left|\frac{d^2}{dx^2} \partial_c^\alpha \partial_\gamma^j V_\phi^3(x)\right| = O(1) \left|\ln \frac{1+x}{3} \ln \frac{1-x}{3}\right|^{-2} (1-x^2)^{-2}.$$

Since $\epsilon > 0$, there exists some $C = C(\alpha, j, K)$ such that for all $(c, \gamma) \in K$,

$$|(1-x^2)^\epsilon \partial_c^\alpha \partial_\gamma^j V_\phi^3| \le C, \quad \left|(1-x^2)^{1+\epsilon} \frac{d}{dx} \partial_c^\alpha \partial_\gamma^j V_\phi^3\right| \le C, \quad \left|(1-x^2)^{2+\epsilon} \frac{d^2}{dx^2} \partial_c^\alpha \partial_\gamma^j V_\phi^3\right| \le C.$$

The above imply that for any $|\alpha| + j \geq 1$, $\partial_c^\alpha \partial_\gamma^j V_\phi^3 \in \mathbf{M}_2$, so $V^3 \in C^\infty(K, \mathbf{X})$.

$\square$

Similar arguments as in Lemma 5.2.8 imply the following lemma.

Lemma 5.2.24. *There exists $C = C(K) > 0$ such that for all $(c, \gamma) \in K \subset\subset E_{4,3}$, $\beta = (\beta_1, \beta_2, \beta_3, \beta_4) \in \mathbb{R}^4$, and $V \in \mathbf{X}_1$,*

$$\|V\|_{\mathbf{X}} + |\beta| \leq C \|\sum_{i=1}^{4} \beta_i V_{c,\gamma}^i + V\|_{\mathbf{X}}.$$

Proof of Theorem 5.2.6*:* Define a map $F : K \times \mathbb{R}^3 \times \mathbf{X}_1 \to \mathbf{Y}$ by

$$F(c, \gamma, \beta, V) = G(c, \gamma, \beta_1 V_{c,\gamma}^1 + \beta_2 V_{c,\gamma}^2 + \beta_3 V_{c,\gamma}^3 + V).$$

By Proposition 5.2.3, G is a C^∞ map from $K \times \mathbf{X}$ to $\mathbf{Y}$. Let $\tilde{U} = \tilde{U}(c, \gamma, \beta, V) = \beta_1 V_{c,\gamma}^1 + \beta_2 V_{c,\gamma}^2 + \beta_3 V_{c,\gamma}^3 + V$. Using Lemma 5.2.23, we have $\tilde{U} \in C^\infty(K \times \mathbb{R}^3 \times \mathbf{X}_1, \mathbf{X})$. So it concludes that $F \in C^\infty(K \times \mathbb{R}^3 \times \mathbf{X}_1, \mathbf{Y})$.

Next, by definition $F(c, \gamma, 0, 0) = 0$ for all $(c, \gamma) \in K$. Fix some $(\bar{c}, \bar{\gamma}) \in K$, using Lemma 5.2.22, we have $F_V(\bar{c}, \bar{\gamma}, 0, 0) = L_0^{\bar{c},\bar{\gamma}} : \mathbf{X}_1 \to \mathbf{Y}$ is an isomorphism.

Applying Theorem C, there exist some $\delta > 0$ and a unique $V \in C^\infty(B_\delta(\bar{c}, \bar{\gamma}) \times B_\delta(0), \mathbf{X}_1)$, such that

$$F(c, \gamma, \beta, V(c, \gamma, \beta)) = 0, \quad \forall (c, \gamma) \in B_\delta(\bar{c}, \bar{\gamma}), \beta \in B_\delta(0),$$

and

$$V(\bar{c}, \bar{\gamma}, 0) = 0.$$

The uniqueness part of Theorem C holds in the sense that there exists some $0 < \bar{\delta} < \delta$, such that $B_{\bar{\delta}}(\bar{c}, \bar{\gamma}, 0, 0) \cap F^{-1}(0) \subset \{(c, \gamma, \beta, V(c, \gamma, \beta)) | (c, \gamma) \in B_\delta(\bar{c}, \bar{\gamma}), \beta \in B_\delta(0)\}$.

Claim: there exists some $0 < \delta_1 < \frac{\bar{\delta}}{2}$, such that $V(c, \gamma, 0) = 0$ for every $(c, \gamma) \in B_{\delta_1}(\bar{c}, \bar{\gamma})$.

Proof of the claim: Since $V(\bar{c}, \bar{\gamma}, 0) = 0$ and $V(c, \gamma, 0)$ is continuous in (c, γ), there exists some $0 < \delta_1 < \frac{\bar{\delta}}{2}$, such that for all $(c, \gamma) \in B_{\delta_1}(\bar{c}, \bar{\gamma})$, $(c, \gamma, 0, V(c, \gamma, 0)) \in B_{\bar{\delta}(\bar{c}, \bar{\gamma}, 0, 0)}$. We know that for all $(c, \gamma) \in B_{\delta_1}(\bar{c}, \bar{\gamma})$,

$$F(c, \gamma, 0, 0) = 0,$$

and

$$F(c,\gamma,0,V(c,\gamma,0)) = 0.$$

By the above mentioned uniqueness result, $V(c,\gamma,0) = 0$, for every $(c,\gamma) \in B_{\delta_1}(\bar{c},\bar{\gamma})$.

Now we have $V \in C^\infty(B_{\delta_1}(\bar{c},\bar{\gamma}) \times B_{\delta_1}(0), \mathbf{X}_1(\bar{c},\bar{\gamma}))$, and

$$F(c,\gamma,\beta,V(c,\gamma,\beta)) = 0, \quad \forall (c,\gamma) \in B_{\delta_1}(\bar{c},\bar{\gamma}), \beta \in B_{\delta_1}(0).$$

i.e. for any $(c,\gamma) \in B_{\delta_1}(\bar{c},\bar{\gamma})$, $\beta \in B_{\delta_1}(0)$

$$G(c,\gamma,\beta_1 V^1_{c,\gamma} + \beta_2 V^2_{c,\gamma} + \beta_3 V^3_{c,\gamma} + V(c,\gamma,\beta)) = 0.$$

Take derivative of the above with respect to β_i at $(c,\gamma,0)$, $i = 1,2,3$, we have

$$G_{\tilde{U}}(c,\gamma,0)(V^i_{c,\gamma} + \partial_{\beta_i} V(c,\gamma,0)) = 0.$$

Since $G_{\tilde{U}}(c,\gamma,0)V^i_{c,\gamma} = 0$ by Lemma 5.2.21, we have

$$G_{\tilde{U}}(c,\gamma,0)\partial_{\beta_i} V(c,\gamma,0) = 0.$$

But $\partial_{\beta_i} V(c,\gamma,0) \in C^\infty(\mathbf{X}_1)$, so

$$\partial_{\beta_i} V(c,\gamma,0) = 0, \quad i = 1,2,3.$$

Since K is compact, we can take δ_1 to be a universal constant for each $(c,\gamma) \in K$. So we have proved the existence of V in Theorem 5.2.6.

Next, let $(c,\gamma) \in B_{\delta_1}(\bar{c},\bar{\gamma})$. Let δ' be a small constant to be determined. For any U satisfies the equation (5.43) with $U - U^{c,\gamma} \in \mathbf{X}$, and $\|U - U^{c,\gamma}\|_{\mathbf{X}} \le \delta'$ there exist some $\beta_1, \beta_2 \in \mathbb{R}$ and $V^* \in \mathbf{X}_1$ such that

$$U - U^{c,\gamma} = \beta_1 V^1_{c,\gamma} + \beta_2 V^2_{c,\gamma} + \beta_3 V^3_{c,\gamma} + V^*.$$

Then by Lemma 5.2.24, there exists some constant $C > 0$ such that

$$\frac{1}{C}(|(\beta_1,\beta_2,\beta_3)| + \|V^*\|_{\mathbf{X}}) \le \|\beta_1 V^1_{c,\gamma} + \beta_2 V^2_{c,\gamma} + \beta_3 V^3_{c,\gamma} + V^*\|_{\mathbf{X}} \le \delta'.$$

This gives $\|V^*\|_{\mathbf{X}} \le C\delta'$.

Choose δ' small enough such that $C\delta' < \delta_1$. We have the uniqueness of V^*. So $V^* = V(c,\gamma,\beta)$ in (5.131). Theorem 5.2.6 is proved. $\square$

Now with Theorem 5.2.1- 5.2.6 we can give the

Proof of Theorem 1.0.6*:* Let K be a compact subset of one of the four sets $E_{k,l}$, $1 \le k \le 8$ and $1 \le l \le 3$, where $I_{k,l}$ are the sets defined by (1.14).

For $(c,\gamma) \in K \cap E_{k,l}$ with $1 \le k \le 4$ and $l = 3$, let

$$(u_\theta(c,\gamma,\beta), u_\phi(c,\gamma,\beta)) = \frac{1}{\sin\theta}\left(U^{c,\gamma} + \beta_3 V^3_{c,\gamma} + \beta_4 V^4_{c,\gamma} + V(c,\gamma,0,0,\beta_3,\beta_4)\right), \quad \beta = (\beta_3,\beta_4) \in B_\delta,$$

where $\delta, V^3_{c,\gamma}, V^4_{c,\gamma}$ and $V(c,\gamma,0,0,\beta_3,\beta_4)$ are as in Theorem 5.2.1, Theorem 5.2.4, Theorem 5.2.4' and Theorem 5.2.6.

For $(c,\gamma) \in K \cap E_{k,l}$ with $1 \le k \le 4$ and $l = 1,2$, let

$$(u_\theta(c,\gamma,\beta), u_\phi(c,\gamma,\beta)) = \frac{1}{\sin\theta}\left(U^{c,\gamma} + \beta_3 V^3_{c,\gamma} + \beta_4 V^4_{c,\gamma} + V(c,\gamma,0,\beta_3,\beta_4)\right), \quad \beta = (\beta_3,\beta_4) \in B_\delta,$$

where $\delta, V^3_{c,\gamma}, V^4_{c,\gamma}$ and $V(c,\gamma,0,\beta_1,\beta_2)$ are as in Theorem 5.2.2, Theorem 5.2.2', Theorem 5.2.5 and Theorem 5.2.5'.

For $(c,\gamma) \in K \cap E_{k,l}$ with $5 \le k \le 8$ and $1 \le l \le 3$, let

$$(u_\theta(c,\gamma,\beta), u_\phi(c,\gamma,\beta)) = \frac{1}{\sin\theta}\left(U^{c,\gamma} + \beta_3 V^3_{c,\gamma} + \beta_4 V^4_{c,\gamma} + V(c,\gamma,\beta_3,\beta_4)\right), \quad \beta = (\beta_3,\beta_4) \in B_\delta,$$

where $\delta, V^3_{c,\gamma}, V^4_{c,\gamma}$ and $V(c,\gamma,\beta_3,\beta_4)$ are as in Theorem 5.2.2.

With $(u_\theta(c,\gamma,\beta), u_\phi(c,\gamma,\beta))$ defined as the above, the first part of Theorem 1.0.6 follows from Theorem 5.2.1- 5.2.6.

For the second part of Theorem 1.0.6, recall $U^{c,\gamma} = \sin\theta u^{c,\gamma}$. It is not hard to check that if $(c,\gamma) \in \hat{I}$, then $U^{c,\gamma}_\theta(-1) > 3$ or $U^{c,\gamma}_\theta(1) < -3$. let $\{u^i\}$ be a sequence of solutions of (1.1) satisfying $||\sin\theta(u^i - u^{c,\gamma})||_{L^\infty(\mathbb{S}^2\setminus\{S,N\})} \to 0$ as $i \to \infty$. Let $U^i = \sin\theta u^i$ for all $i \in \mathbb{N}$. We have $||U^i_\theta - U^{\mu,\gamma}_\theta||_{L^\infty(-1,1)} \to 0$. By Theorem 1.0.3 and Theorem 4.0.1', $U^i(\pm 1)$ must exists and is finite for every i. If $U^{c,\gamma}(-1) > 3$, $U^i_\theta(-1) > 3$ for large i. If $U^{c,\gamma}(1) < -3$, $U^i_\theta(1) < -3$ for large i. Then by Theorem 1.0.4 and Theorem 4.0.2', U^i_ϕ must be constants for large i. the theorem is then proved. $\square$

Chapter 6

Appendix

6.1 Asymptotic behavior of certain type of ODE

In Chapter 4, we have analyzed several equations of the following form:

Let $\delta > 0$ and $g \in C^1(-1, -1+\delta]$ be a solution of

$$a(x)g'(x) + b(x)g(x) + \frac{1}{2}g^2(x) = H(x), \quad -1 < x < -1+\delta. \tag{6.1}$$

We require $a(x), b(x) \in C(-1, -1+\delta]$ and $a(x)$ satisfy:

either (i) $a(x) > 0$ for every $x \in (-1, -1+\delta]$, and $\lim_{x \to -1^+} \int_x^{-1+\delta} \frac{1}{a(x)} = +\infty$,

or (ii) $a(x) < 0$ for every $x \in (-1, -1+\delta]$, and $\lim_{x \to -1^+} \int_x^{-1+\delta} \frac{1}{a(x)} = -\infty$.

Introduce $H^+(x) = \max\{H(x), 0\}$ and $H^-(x) = \max\{-H(x), 0\}$, so $H(x) = H^+(x) - H^-(x)$. This is for $b^+(x), b^-(x)$ as well.

Proposition 6.1.1. *For $\delta > 0$, let $H, a, b \in C(-1, -1+\delta]$ with $b, H^+ \in L^\infty(-1, -1+\delta)$ and $a(x)$ satisfies (i) or (ii) above. Suppose that $g \in C^1(-1, -1+\delta]$ is a solution of (6.1). Then $g \in L^\infty(-1, -1+\delta)$. If in addition, $\lim_{x \to -1^+} H(x)$ is assumed to exist, either finite or infinite, and $\lim_{x \to -1^+} b(x)$ exists and is finite, then $\lim_{x \to -1^+} g(x)$ exists and is finite, $\lim_{x \to -1^+} a(x)g'(x) = 0$.*

Lemma 6.1.1. *For $\delta > 0$, let $H, a, b \in C(-1, -1+\delta]$ with $a(x) > 0$ for $x \in (-1, -1+\delta)$. Suppose that $g \in C^1(-1, -1+\delta]$ is a solution of (6.1). Then*

$$g(x) \geq -A_1 := -\max\{4||b^+||_{L^\infty(-1,-1+\delta)}, \sqrt{8||H^+||_{L^\infty(-1,-1+\delta)}}, -g(-1+\delta)\}, \forall x \in (-1, -1+\delta).$$

Proof. If $A_1 = \infty$, done. So we assume $A_1 < \infty$. If $g(x) < -A_1$ for some $x \in (-1, -1+\delta)$, we have

$$a(x)g'(x) = H(x) - \frac{1}{2}g^2(x) - b(x)g(x) \leq H(x) - \frac{1}{4}g^2(x) \leq -\frac{1}{8}g^2(x) < 0.$$

Thus $g'(x) < 0$. This implies, given $g(-1+\delta) \geq -A_1$, that $g \geq -A_1$ on $(-1, -1+\delta)$. $\square$

Lemma 6.1.2. *In addition to the assumption of Lemma* 6.1.1*, we assume that*

$$\lim_{x\to -1^+} \int_x^{-1+\delta} \frac{1}{a(x)} = +\infty.$$

Then

$$g(x) \leq A_2 := \max\{4||b^-||_{L^\infty(-1,-1+\delta)}, \sqrt{8||H^+||_{L^\infty(-1,-1+\delta)}}\}, \quad \forall x \in (-1, -1+\delta).$$

Proof. If $g(\bar{x}) > A_2$ for some $\bar{x} \in (-1, -1+\delta)$, we have

$$a(\bar{x})g'(\bar{x}) = H(\bar{x}) - \frac{1}{2}g^2(\bar{x}) - b(\bar{x})g(\bar{x}) \leq H(\bar{x}) - \frac{1}{4}g^2(\bar{x}) \leq -\frac{1}{8}g^2(\bar{x}) < 0.$$

Thus $g'(\bar{x}) < 0$, and therefore for some $\epsilon > 0$, $g(x) > g(\bar{x}) > A_2$ for $\bar{x} - \epsilon < x < \bar{x}$. It follows that $g(x) > A_2$ for all $x \in (-1, \bar{x})$. Thus as shown above, $a(x)g'(x) < -\frac{1}{8}g^2(x)$ for all $-1 < x < \bar{x}$. It follows that $(g^{-1})'(x) \geq \frac{1}{8a(x)}$ and

$$\frac{1}{8}\int_x^{\bar{x}} \frac{ds}{a(s)} \leq g^{-1}(\bar{x}) - g^{-1}(x) \leq g^{-1}(\bar{x}) \leq \frac{1}{A_2}, \quad \forall -1 < x < \bar{x}.$$

This violates $\int_x^{-1+\delta} \frac{ds}{a(s)} = \infty$, a contradiction. $\square$

Lemma 6.1.3. *For $\delta > 0$, let $H, a, b \in C(-1, -1+\delta]$ with $a(x) < 0$ for $x \in (-1, -1+\delta)$. Suppose that $g \in C^1(-1, -1+\delta)$ is a solution of* (6.1)*. Then*

$$g(x) \leq \hat{A}_1 := \max\{4||b^-||_{L^\infty(-1,-1+\delta)}, \sqrt{8||H^+||_{L^\infty(-1,-1+\delta)}}, g(-1+\delta)\}, \quad \forall x \in (-1, -1+\delta).$$

Proof. Rewriting (6.1) as

$$(-a)(-g)' + (-b)(-g) + \frac{1}{2}(-g)^2 = H. \tag{6.2}$$

The conclusion follows from Lemma 6.1.1 with a, b and g there replaced by $-a, -b$ and $-g$. $\square$

Lemma 6.1.4. *In addition to the assumption of Lemma* 6.1.3*, we assume that*

$$\lim_{x\to -1^+} \int_x^{-1+\delta} \frac{1}{a(x)} = -\infty.$$

Then

$$g(x) \geq -\hat{A}_2 := -\max\{4||b^+||_{L^\infty(-1,-1+\delta)}, \sqrt{8||H^+||_{L^\infty(-1,-1+\delta)}}\}, \quad \forall x \in (-1, -1+\delta).$$

Proof. This follows from Lemma 6.1.2 as the way Lemma 6.1.3 being deduced from Lemma 6.1.1. □

Lemma 6.1.5. *For $\delta > 0$, let $b \in C^0(-1, -1+\delta] \cap L^\infty(-1, -1+\delta)$, $H \in C^0(-1, -1+\delta]$, and let $a \in C^0(-1, -1+\delta]$ be either positive or negative in the interval and satisfies $\lim_{x \to -1^+} \left| \int_x^{-1+\delta} \frac{ds}{a(s)} \right| = \infty$. Assume that $g \in C^1(-1, -1+\delta]$ is a solution of (6.1). Then*

$$\lambda := \sup_{-1 < x \le -1+\delta} \left(H(x) + \frac{1}{2}(b(x))^2 \right) \ge 0.$$

Proof. We only need to treat the case that $a(x) > 0$ since the other case can be converted to this case by rewriting (6.1) as (6.2). We prove it by contradiction. If not, then

$$a(x)g'(x) = H(x) - \frac{1}{2}b(x)^2 - \frac{1}{2}(g(x) + b(x))^2 \le \lambda < 0, \quad \forall -1 < x < -1+\delta.$$

It follows that

$$g(-1+\delta) - g(x) = \int_x^{-1+\delta} g'(s)ds \le \lambda \int_x^{-1+\delta} \frac{ds}{a(x)} \to -\infty \text{ as } x \to -1^+.$$

This implies

$$\lim_{x \to -1^+} g(x) = +\infty. \tag{6.3}$$

On the other hand, λ being negative implies that $H^+ \in L^\infty(-1, -1+\delta)$. An application of Lemma 6.1.2 gives that $g^+ \in L^\infty(-1, -1+\delta)$, violating (6.3). □

Lemma 6.1.6. *For $\delta > 0$, let $b \in C^0[-1, -1+\delta]$ and $H \in C^0(-1, -1+\delta]$ such that $\lim_{x \to -1^+} H(x)$ exists, is either finite or infinite, and let $a \in C^0(-1, -1+\delta]$ be either positive or negative in the interval. Assume that $g \in C^1(-1, -1+\delta]$ is a solution of (6.1). Then $\lim_{x \to -1^+} g(x)$ exists and $b(-1)g(-1) + \frac{1}{2}g(-1)^2 = H(-1)$.*

If in addition, $\lim_{x \to -1^+} \left| \int_x^{-1+\delta} \frac{ds}{a(s)} \right| = \infty$, then $\lim_{x \to -1^+} g(x)$ is finite if and only if $\lim_{x \to -1^+} H(x)$ is finite, and in this case $\lim_{x \to -1^+} a(x)g'(x) = 0$.

Proof. As before, we will only prove it when $a > 0$, since the $a < 0$ case follows after rewriting (6.1) as (6.2). We prove it by contradiction.

Assume that $\lim_{x \to -1^+} g(x)$ does not exist, then there exist $-\infty < \alpha_1 < \alpha_2 < \infty$ and two sequences $\{x_i\}$ and $\{y_i\}$ such that $x_1 > y_1 > x_2 > y_2 > \cdots > -1$, $\lim_{i \to \infty} x_i = \lim_{i \to \infty} y_i = -1$,

$g(x_i) = \alpha_1$ and $g(y_i) = \alpha_2$. Then for any $\alpha \in (\alpha_1, \alpha_2)$, there exists a $x_i > z_i > y_i$ such that $g(z_i) = \alpha$ and $g(z) < \alpha, \forall x_i \geq z > z_i$. Clearly $\lim_{i\to\infty} z_i = 1$ and $g'(z_i) \leq 0$. This leads to, in view of (6.1), $b(z_i)g(z_i) + \frac{1}{2}g^2(z_i) \geq H(z_i)$. Sending $i \to \infty$, we have $b(-1)\alpha + \frac{1}{2}\alpha^2 \geq \lim_{x\to-1^+} H(x)$.

Similarly, we can find $y_i > \hat{z}_i > x_{i+1}$ satisfying $g(\hat{z}_i) = \alpha$ and $g'(\hat{z}_i) \geq 0$, which leads to $b(-1)\alpha + \frac{1}{2}\alpha^2 \leq \lim_{x\to-1^+} H(x)$. So for any $\alpha \in (\alpha_1, \alpha_2)$, $b(-1)\alpha + \frac{1}{2}\alpha^2 = \lim_{x\to-1^+} H(x)$. Contradiction. We have proved that $\lim_{x\to-1^+} g(x)$ exists, either finite or infinite.

If $\lim_{x\to-1^+} H(x)$ is finite, then, in view of Lemma 6.1.1 and Lemma 6.1.2, $\lim_{x\to-1^+} g(x)$ is finite.

If $\lim_{x\to-1^+} H(x)$ is infinite, then, in view of Lemma 6.1.5, $\lim_{x\to-1^+} H(x) = +\infty$. We will show by contradiction that $\lim_{x\to-1^+} g$ is infinite. Suppose that the limit is finite, then $a(x)g'(x) = H(x) - b(x)g(x) - \frac{1}{2}g^2(x) \to +\infty$ as $x \to -1^+$. It follows that there exists $0 < \epsilon < \delta$, such that $g'(x) \geq \dfrac{1}{a(x)}$, for $-1 < x < -1 + \epsilon$. It follows that

$$g(-1+\epsilon) - g(x) \geq \int_x^{-1+\epsilon} \frac{ds}{a(s)} \to \infty \text{ as } x \to -1^+,$$

a contradiction to the finiteness of $\lim_{x\to-1^+} g(x)$.

We have proved that $\lim_{x\to-1^+} g(x)$ is finite if and only if $\lim_{x\to-1^+} H(x)$ is finite.

If $\lim_{x\to-1^+} g$ is finite, we see by sending x to -1^+ in (6.1) that $\lim_{x\to-1^+} a(x)g' = \mu$ for some $\mu \in \mathbb{R}$. Since g is bounded, $\mu = 0$. Indeed, if $\mu \neq 0$, we would have

$$\frac{2g'(x)}{\mu} \geq \frac{1}{a(x)}$$

for x close to -1, and an argument above would lead to a contradiction to the boundedness of g. $\square$

Proposition 6.1.1 follows from Lemma 6.1.1-6.1.4 and Lemma 6.1.6.

Next, we study asymptotic behavior of solution $V \in C^1(-1, -1+\delta]$ of

$$V' + BV = H \qquad \text{in } (-1, -1+\delta) \tag{6.4}$$

under various hypothesis on B and H.

Let $w := \int_{-1+\delta}^{x} B(s)ds$, then V can be expressed as

$$V(x) = V(x_0)e^{w(x_0)-w(x)} + e^{-w(x)} \int_{x_0}^{x} e^{w(s)} H(s)ds, \tag{6.5}$$

for every $x_0 \in (-1, -1+\delta]$.

Lemma 6.1.7. *For $\delta > 0$, $0 \le b \le 1$ and $\beta \ge 0$, let $B, H \in C(-1, -1+\delta]$ satisfy*

$$\inf_{-1<x\le -1+\delta} (1+x)^{1-b} H(x) > -\infty, \tag{6.6}$$

and

$$\lim_{x\to -1^+} (1+x)B(x) = -\beta. \tag{6.7}$$

Assume that $V \in C^1(-1, -1+\delta]$ and satisfies (6.4). Then for every $\epsilon > 0$, there exists some constant C, such that

$$V(x) \le C(1+x)^{\min\{b,\beta\}-\epsilon}, \quad \textit{for all } -1 < x \le -1+\delta. \tag{6.8}$$

Proof. By (6.7)

$$w(x) = (-\beta + o(1)) \ln(1+x), \tag{6.9}$$

where $o(1)$ denotes some quantity which tends to 0 as $x \to -1^+$.

Since $V \in C^1(-1, -1+\delta]$ is a solution of (6.4), (6.5) holds for every $x_0 \in (-1, -1+\delta]$. It follows from (6.9), (6.6) and (6.5) , with $x_0 = -1+\delta$, that

$$\begin{aligned} V(x) &\le (1+x)^{\beta+o(1)} + (1+x)^{\beta+o(1)} \int_{x}^{-1+\delta} (1+s)^{-\beta+b-1+o(1)} ds \\ &\le (1+x)^{\beta+o(1)} + (1+x)^{b+o(1)} \le C(1+x)^{\min\{b,\beta\}-\epsilon}. \end{aligned}$$

□

Remark 6.1.1. *In Lemma 6.1.7, if we replace (6.6) by*

$$\sup_{-1<x\le -1+\delta} (1+x)^{1-b} |H(x)| < \infty, \tag{6.10}$$

then we have, instead of (6.8), for any $\epsilon > 0$,

$$|V(x)| \le C(1+x)^{\min\{b,\beta\}-\epsilon}, \quad \textit{for all } -1 < x \le -1+\delta$$

instead of (6.8)

Lemma 6.1.8. *For $\delta > 0$, $0 < b \le 1$ and $\beta < 0$, let $B, H \in C(-1, -1+\delta]$ satisfy (6.6) and (6.7). Assume that $V \in C^1(-1, -1+\delta]$ and satisfies (6.4) and*

$$\limsup_{x \to -1^+} V(x) e^{\int_{-1+\delta}^{x} B(s)ds} \ge 0. \tag{6.11}$$

Then for every $\epsilon > 0$, there exists some constant C, such that

$$-V(x) \le C(1+x)^{b-\epsilon}, \text{for all } -1 < x \le -1+\delta. \tag{6.12}$$

Proof. Estimate (6.9) still holds by the assumption of B. For all $-1 < x_0 < x$, we obtain from (6.5) and (6.6) that

$$V(x) \ge V(x_0) e^{w(x_0) - w(x)} - C e^{-w(x)} \int_{x_0}^{x} e^{w(s)} (1+s)^{b-1} ds.$$

Sending $x_0 \to -1$ along a subsequence in (6.5), we have, in view of (6.11)

$$-V(x) \le C e^{-w(x)} \int_{-1}^{x} e^{w(s)} (1+s)^{b-1} ds.$$

By (6.9), for every $\epsilon > 0$, there exists some constant C, such that

$$-V(x) \le (1+x)^{\beta + o(1)} \int_{-1}^{x} (1+s)^{-\beta + b - 1 + o(1)} ds \le C(1+x)^{b-\epsilon}.$$

$\square$

Remark 6.1.2. *In Lemma 6.1.8, if we replace (6.6) and (6.11) respectively by (6.10) and*

$$\lim_{x \to -1^+} V(x) e^{\int_{-1+\delta}^{x} B(s)ds} = 0, \tag{6.13}$$

then we have, instead of (6.12), that for any $\epsilon > 0$,

$$|V(x)| \le C(1+x)^{b-\epsilon}, \quad \text{for all } -1 < x \le -1+\delta.$$

Lemma 6.1.9. *For $\delta, \beta, c_1, c_2 > 0$, let $B \in C(-1, -1+\delta]$ and $H \in C[-1, -1+\delta]$ satisfy*

$$H(x) = H(-1) + O((1+x)^{c_1}), \quad -1 < x \le -1+\delta, \tag{6.14}$$

and

$$(1+x)B(x) + \beta = O((1+x)^{c_2}). \tag{6.15}$$

Assume that $V \in C^1(-1,-1+\delta]$ *and satisfies* (6.4)*. Then there exists some constant* a_1*, such that for every* $0<\alpha<\min\{c_2+\beta, c_2+1, c_1+1\}$,

$$V(x) = a_1(1+x)^\beta + \begin{cases} \frac{H(-1)}{1-\beta}(1+x) \ \text{if}\ \beta \neq 1 \\ H(-1)(1+x)\ln(1+x) \ \text{if}\ \beta = 1 \end{cases} + O((1+x)^\alpha), \quad -1<x\le -1+\delta.$$

Proof. Since V is a solution of (6.4), (6.5) holds. By (6.15), we have, for some $a_3 \in \mathbb{R}$,

$$w(x) = -\beta\ln(1+x) + a_3 + O((1+x)^{c_2}). \tag{6.16}$$

We derive from (6.5), using (6.14) and the above that for some constant $a_1 \in \mathbb{R}$,

$$\begin{aligned} V(x) &= V(x_0)e^{w(x_0)-w(x)} + e^{-w(x)}\int_{x_0}^x e^{w(s)}H(s)ds \\ &= V(x_0)e^{w(x_0)}e^{-a_3}(1+x)^\beta(1+O((1+x)^{c_2})) \\ &+ (1+x)^\beta(1+O((1+x)^{c_2}))\int_{x_0}^x (1+s)^{-\beta}(H(-1)+O((1+s)^{\min\{c_1,c_2\}}))ds, \end{aligned}$$

from which we conclude the proof. □

Lemma 6.1.10. *For* $\delta, c_1, c_2 > 0$*,* $\beta < 0$ *,* $0<b<1$ *and* $\gamma_1,\gamma_2 \in \mathbb{R}$*, let* $B, H \in C(-1,-1+\delta]$ *satisfy* (6.15) *and*

$$H(x) = \gamma_1(1+x)^{b-1} + \gamma_2 + O((1+x)^{b-1+c_1}), \quad -1<x\le -1+\delta. \tag{6.17}$$

Assume that $V \in C^1(-1,-1+\delta]$ *and satisfies* (6.4) *and* $V(x) = o((1+x)^\beta)$*. Then*

$$V(x) = \frac{\gamma_1}{b-\beta}(1+x)^b + \frac{\gamma_2}{1-\beta}(1+x) + O((1+x)^{b+\min\{c_1,c_2\}}).$$

Proof. Expression (6.5) still holds. By (6.15), we have (6.16) for some $a_3 \in \mathbb{R}$. Since $V(x) = o((1+x)^\beta)$, we obtain, by sending x_0 to -1 in (6.5) similar to the arguments in the proof of Lemma 6.1.8, that

$$V(x) = e^{-w(x)}\int_{-1}^x e^{w(s)}H(s)ds.$$

We derive from the above using (6.16) and (6.17) that

$$\begin{aligned} V(x) &= (1+x)^\beta(1+O((1+x)^{c_2}))\int_{-1}^x (1+s)^{-\beta}(\gamma_1(1+x)^{b-1}+\gamma_2+O((1+s)^{b-1+\min\{c_1,c_2\}}))ds \\ &= \frac{\gamma_1}{b-\beta}(1+x)^b + \frac{\gamma_2}{1-\beta}(1+x) + O((1+x)^{b+\min\{c_1,c_2\}}). \end{aligned}$$

□

Lemma 6.1.11. *For $\delta > 0$, $0 < b \leq 1$ and $\beta \leq 0$, let $B, H \in C(-1, -1+\delta]$ satisfy (6.6) and*

$$\lim_{x\to -1^+} [(1+x)\ln(1+x)]B(x) = -\beta. \tag{6.18}$$

Assume that $V \in C^1(-1, -1+\delta]$ satisfies (6.4). When $\beta = 0$, we also assume (6.11). Then for every $\epsilon > 0$, there exists some constant C, such that for all $-1 < x \leq -1+\delta$,

$$\begin{cases} V(x) \leq C(\ln(1+x))^{\beta+\epsilon} & \text{if } \beta < 0, \\ V(x) \geq -C(1+x)^b |\ln(1+x)|^\epsilon & \text{if } \beta = 0. \end{cases} \tag{6.19}$$

Proof. By (6.18) ,

$$w(x) = (-\beta + o(1))\ln(-\ln(1+x)).$$

Expression (6.5) still holds for all $x_0 \in (-1, -1+\delta]$. If $\beta < 0$, take $x_0 = -1+\delta$,

$$\begin{aligned} V(x) &= V(x_0)e^{w(x_0)-w(x)} + e^{-w(x)} \int_{x_0}^x e^{w(s)} H(s) ds \\ &\leq |\ln(1+x)|^{\beta+o(1)} + |\ln(1+x)|^{\beta+o(1)} \int_{x_0}^x (\ln(1+s))^{-\beta+o(1)} (1+s)^{b-1+o(1)} ds \\ &\leq |\ln(1+x)|^{\beta+o(1)}. \end{aligned}$$

If $\beta = 0$, $w = o(1)\ln(-\ln(1+x))$. By (6.11), similar as in the proof of Lemma 6.1.8, sending x_0 to -1 along a subsequence in (6.5) gives

$$\begin{aligned} V(x) &\geq -Ce^{-w(x)} \int_{-1}^x e^{w(s)} (1+x)^{b-1} ds \geq -C(|\ln(1+x)|)^{o(1)} \int_{x_0}^x (|\ln(1+s)|)^{o(1)} (1+s)^{b-1} ds \\ &\geq -C(1+x)^b |\ln(1+x)|^\epsilon. \end{aligned}$$

□

Remark 6.1.3. *If in Lemma 6.1.11, we replace (6.6) and (6.11) by (6.10) and (6.13) respectively, then we have, instead of (6.19), that for any $\epsilon > 0$,*

$$|V(x)| \leq \begin{cases} C(\ln(1+x))^{\beta+\epsilon} & \text{if } \beta < 0, \\ C(1+x)^b |\ln(1+x)|^\epsilon & \text{if } \beta = 0. \end{cases}$$

6.2 Figures

For a given axisymmetric vector fields (u_r, u_θ), the stream lines can be represented in the cross section plane $x_1 = 0$. The shape of stream lines, along with the graph of

$(u_r,\, u_\theta)$, depends on parameters (μ, γ). In this section, we choose some typical points on the (μ, γ) plane, whose positions are shown in the left part of Figure 1. At each parameter point, we present the graph of u_r, u_θ, and the corresponding stream lines. In stead of presenting a full classification of all possible shapes of the stream lines, we prefer to emphasize that four border lines play important roles to determine the shape of stream lines.

1) The line $l_1 : \gamma = 0$ separates the stream lines which are upward and downward along positive x_3 axis near the north pole.

2) The line $l_2 : \mu = 0$, $(\gamma > -2)$ separates the stream lines which are inward and outward to negative x_3 axis near the south pole.

3) The line $l_3 : \gamma = -1 + \sqrt{1 + 2\mu}$, $(-\frac{1}{2} < \mu < 0)$ separates the stream lines which are upward and downward along negative x_3 axis near the south pole.

4) The line $l_4 : \mu = -\frac{3}{8}$ separates the stream lines by the amplitude of u_r and u_θ. Namely, on the left of l_4, u_r dominates, thus the stream line near south pole is vertical. While on the right of l_4, u_θ dominates, thus the stream line near south pole is horizontal.

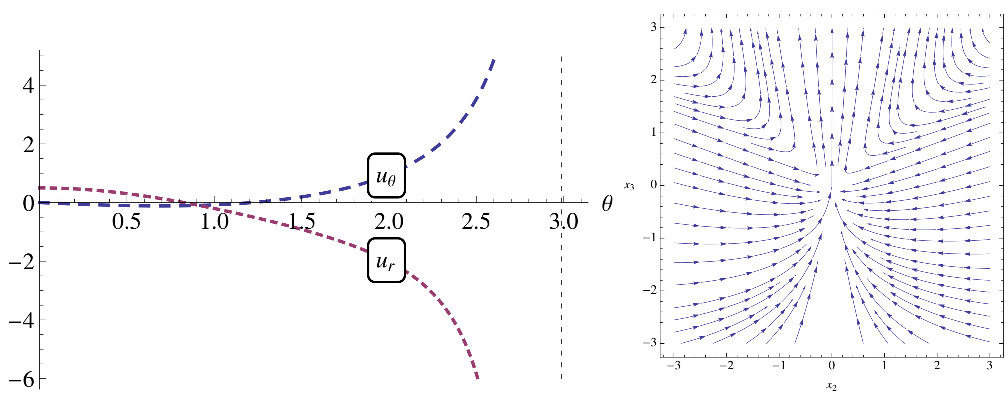

Figure 6.1: The graphs of u_θ, u_r and stream lines for P_1: $\mu = -1$, $\gamma = \frac{1}{2}$.

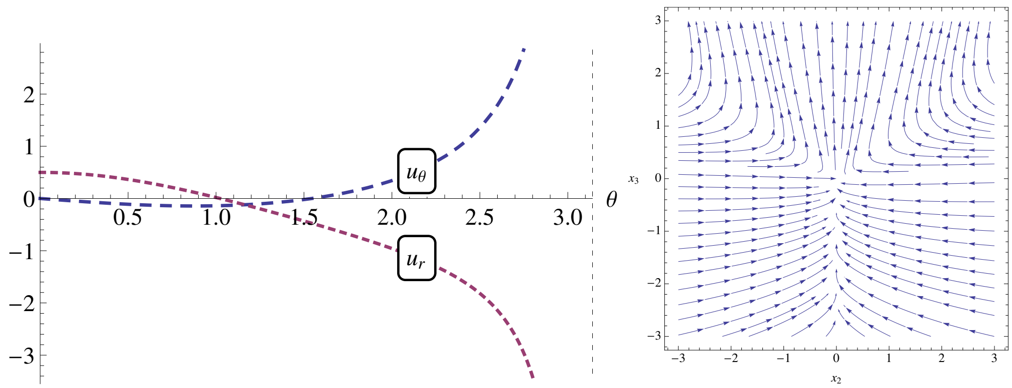

Figure 6.2: The graphs of u_θ, u_r and stream lines for P_2: $\mu = -\frac{1}{2}$, $\gamma = \frac{1}{2}$.

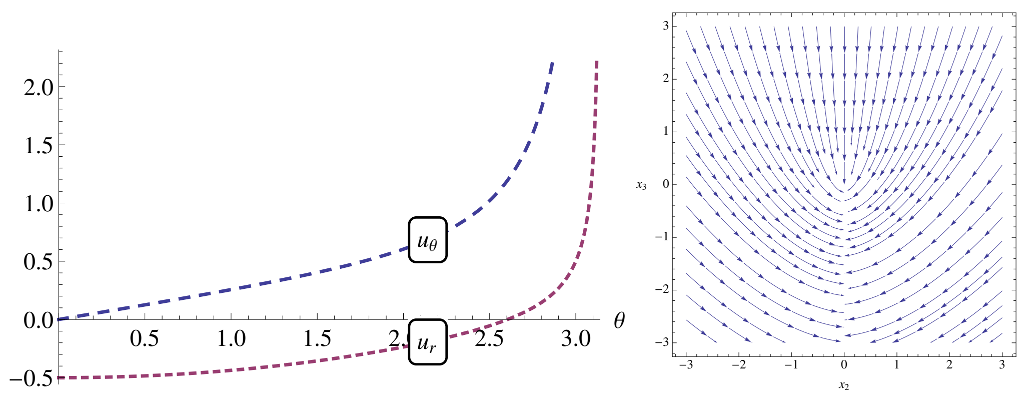

Figure 6.3: The graphs of u_θ, u_r and stream lines for P_3: $\mu = -\frac{1}{4}$, $\gamma = -\frac{1}{2}$.

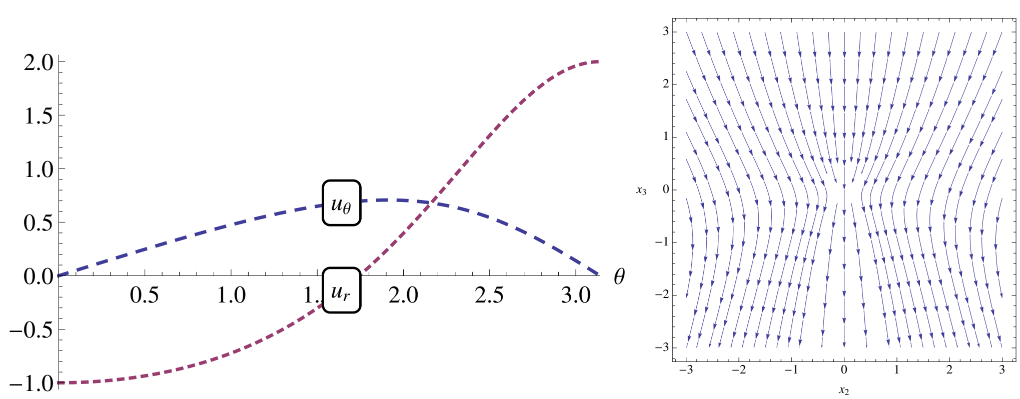

Figure 6.4: The graphs of u_θ, u_r and stream lines for P_4: $\mu = 0$, $\gamma = -1$.

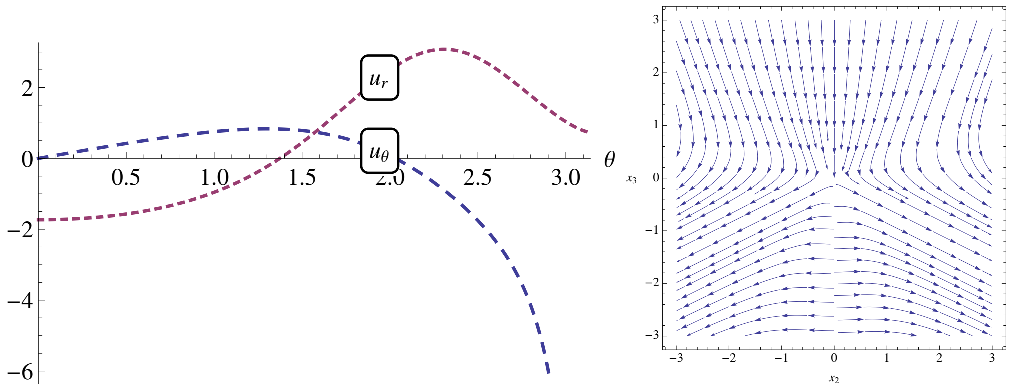

Figure 6.5: The graphs of u_θ, u_r and stream lines for P_5: $\mu = 1$, $\gamma = -\sqrt{3}$.

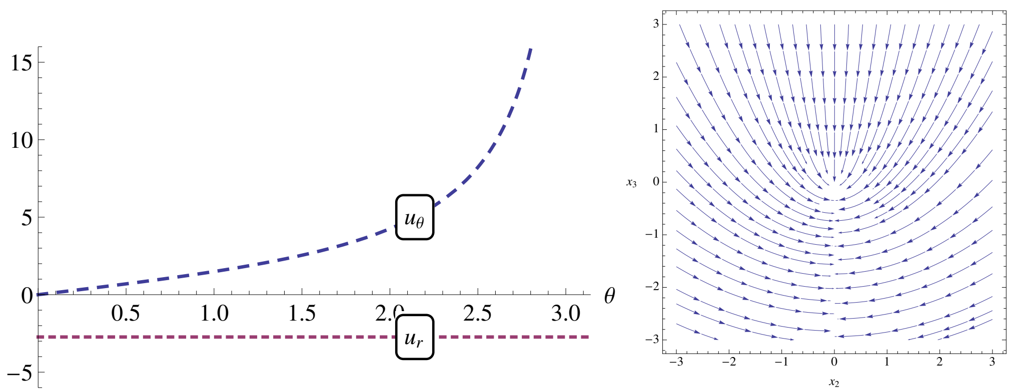

Figure 6.6: The graphs of u_θ, u_r and stream lines for P_6: $\mu = 1$, $\gamma = -1 - \sqrt{3}$.

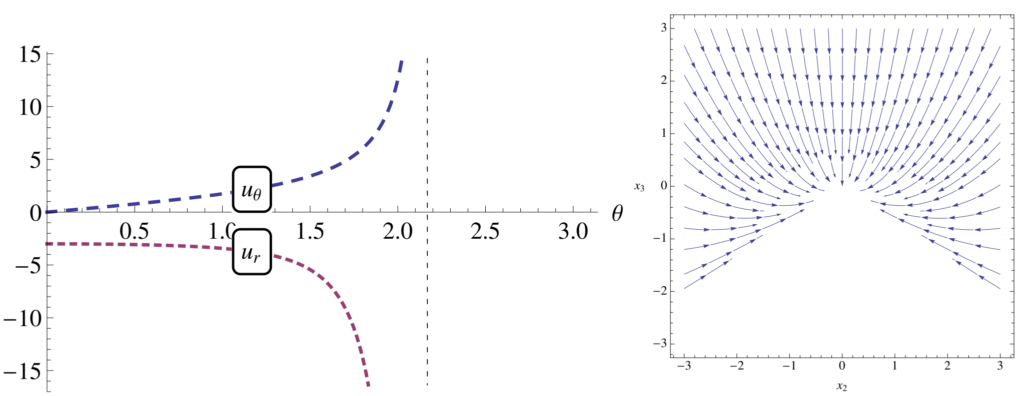

Figure 6.7: The graphs of u_θ, u_r and stream lines for P_7: $\mu = 1$, $\gamma = -3$.

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