NEW METHODS FOR DESIGN OF FULL- AND REDUCED-ORDER OBSERVERS AND OBSERVER-BASED CONTROLLERS FOR SYSTEMS WITH SLOW AND FAST MODES

by

HEONJONG YOO

A dissertation submitted to the

Graduate School-New Brunswick

Rutgers, The State University of New Jersey

In partial fulfillment of the requirements

For the degree of

Doctor of Philosophy

Graduate Program in Electrical Engineering

Written under the direction of

Professor Zoran Gajić

And approved by

____

____

____

____

____

New Brunswick, New Jersey

May, 2017

ABSTRACT OF THE DISSERTATION

New Methods for Design of Full- and Reduced-Order Observers and Observer-Based Controllers for Systems with Slow and Fast Modes

by Heonjong Yoo

Dissertation Director:
Professor Zoran Gajić

This dissertation addresses the design of observer and observer-based controllers for singularly perturbed linear systems. To that end, we present an algorithm for the recursive solution of the singularly perturbed algebraic Sylvester equation. Due to the presence of a small singular perturbation parameter that indicates separation of the system variables into slow and fast, the corresponding algebraic Sylvester equation is numerically ill-conditioned. The observer driven controller design of singularly perturbed linear systems with the observer design done using the algebraic Sylvester equation is extremely ill-conditioned since the observer has to be much faster than the feedback system. The proposed method for the recursive reduced-order solution of the algebraic Sylvester equations removes ill-conditioning and iteratively obtains the solution in terms of four reduced-order numerically well-conditioned algebraic Sylvester equations corresponding to slow and fast variables. The convergence rate of the proposed algorithm is $O(\epsilon)$, where ϵ is a small positive singular perturbation parameter.

The new design technique for full-order Luenberger observers for systems with slow and fast modes is presented. The existing methods are able to design independent slow and fast observers with $O(\epsilon)$ accuracy only, where ϵ is a small positive singular perturbation parameter. In this dissertation, the design of independent slow and fast reduced-order observers was performed with the exact accuracy. The results obtained are extended

to the design of corresponding observer driven controllers. The design allows complete time-scale separation for both the observer and controller through the complete and exact decomposition into slow and fast time scale problems. This method reduces both off-line and on-line computations. The effectiveness of the new methods is demonstrated through both theoretical and simulation results.

The results obtained for the full-order observer of singularly perturbed linear systems are extended to design of reduced-order observers (using both the Sylvester equation and Luenberger observer formulations) and the design of corresponding controllers for singularly perturbed systems. In such design additional computational advantages are achieved due to the use of the reduced order observers. Several cases of reduced-order observer designs are considered depending on the measured state space variables: only all slow variables are measured, only all fast variables are measured, some combinations of the slow and fast variables are measured.

Acknowledgements

First and foremost I express my deepest and sincerest gratitude to my supervisor, Prof. Zoran Gajić, who has supported me throughout my dissertation with his guidance and patience. His wide knowledge, constructive comments, and logical way of thinking have been of great value for me. Without his guidance and persistent help this thesis would not have been possible.

I would like to thank my committee members, Prof. Quingze Zou, Prof. Haim Baruh, Prof. Jingang Yi, and Dr. Abraham Borno. I also would like to thank faculty in Electrical and Computer Engineering Department for their help to get strong backgrounds for my research.

My deepest gratitude goes to my family for their unflagging love and support. My wife and son as well as my family in Korea, have supported me for their love. This dissertation is simply impossible without them.

Lastly, I offer my regards and blessings to all of those who supported me in any respect during the completion of the dissertation.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

Traditionally, decomposing the original ill-conditioned singularly perturbed system into two subsystems resolves numerical ill-conditioning of the problem. The controller design may be then implemented at each subsystems level, and the results can be combined to produce a composite controller design for the original system. These reduced-order controller designs were presented with an $O(\epsilon)$, $O(\epsilon^2)$ accuracy [11] and exact the accuracy [12], [32]. It has been known that the observer for singularly perturbed systems can be designed with an $O(\epsilon)$ accuracy [29]. However, the $O(\epsilon^2)$ and higher accuracy designs for observers have not been presented so far. In this dissertation, we present the exact accuracy designs for the observer and the observer driven controller.

The book [16] summarizes guidance for the observer design in terms of the Sylvester algebraic equation. The aforementioned design, called the Sylvester approach here, applied to singularly perturbed systems, will suffer from ill-conditioning due to the presence of a small positive singular perturbation parameter ϵ. To overcome the problem, we propose the recursive reduced-order solution of the singularly perturbed algebraic Sylvester equation in Chapter 2. The proposed method was adopted from the corresponding method for solving the algebraic Lyapunov equation of singularly perturbed systems [3]. The method is extended to a Sylvester equation which is especially crucial to the observer design.

Chapter 3 is focused on the new design of an observer for singularly perturbed systems using the two-stage feedback design method [32]. The two-stage method is originally applied to the controller design of singularly perturbed linear systems. We extend this method to the observer design using the duality between the controller and the observer. In the last part of Chapter 3, we propose the design of observer driven controller for singularly perturbed systems putting the controller and observer design together.

We want to emphasize that the subject of Chapter 3 was studied in the master thesis of the author [51] in 2014. Here, we make the design more systematic fully based on the duality between the controller and observer designs for singularly perturbed linear systems. The results obtained in the course of this research have been submitted for journal publication. The main results of Chapter 2 have been submitted to Automatica [52], and the main results of Chapter 3 have been submitted to IEEE Transaction on Automatic Control [53]

In Chapter 4 and 5 we present an idea how to extend the results obtained in Chapter 2 and 3 to the design of reduced-order observers and corresponding controllers for singularly perturbed linear systems.

1.1 Introduction to Observer and Controller

Sometimes all state space variables are not available for measurements, or it is not practical to measure all of them, or it is too expensive to measure all state space variables. In order to be able to apply the state feedback control to a system, all of its state space variables must be available at all times. Thus, we face the problem of estimating system state space variables.

1.1.1 Full-Order Observer Design

Consider a linear time invariant system given as

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu_c(t),\ x_{t_0} = x_0 = \text{ unknown} \\ &y(t) = Cx(t) \end{aligned} \tag{1.1}$$

where $x(t) \in \Re^n, u(t) \in \Re^r, y(t) \in \Re^p$, and constant matrices A, B, C having appropriate dimensions. We may construct a full-order observer having the same matrices A, B, C that is

$$\begin{aligned} &\dot{\hat{x}}(t) = A\hat{x}(t) + Bu_c(t),\ \hat{x}_{t_0} = \hat{x}_0 \\ &\hat{y}(t) = C\hat{x}(t) \end{aligned} \tag{1.2}$$

Then, we compare the output $y(t)$ of the system (1.1) and the output $\hat{y}(t)$ of the full-order observer (1.2). These two outputs will be different since in the first case the system initial condition is unknown, and in the second case it has been chosen arbitrarily. The difference between these two outputs will generate an error signal

$$y(t) - \hat{y}(t) = Cx(t) - C\hat{x}(t) = Ce(t) \tag{1.3}$$

which can be used as the feedback signal to the full-order observer such that the estimation error $e(t)$ is reduced to zero. Considering the feedback signal (1.3), the observer structure is given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu_c(t) + K(y(t) - \hat{y}(t)) \tag{1.4}$$

Note that the observer has the same structure as the system plus the driving feedback term that contain information about the observation error. The observer is implemented on line as a dynamic system driven by the same input as the original system and the measurements coming from the original systems, that is

$$\begin{aligned} \dot{\hat{x}}(t) &= (A - KC)\hat{x}(t) + Bu_c(t) + Ky(t) \\ y(t) &= Cx(t), \;\; u_c(t) = F\hat{x}(t) \end{aligned} \tag{1.5}$$

This can be realized by proposing the system-observer structure as given in Figure. 1.1, [16] It is easy to derive an expression for dynamics of the observation error as

$$\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = (A - KC)e(t) \tag{1.6}$$

If the observer gain K is chosen such that the matrix $A - KC$ is asymptotically stable, then the error $e(t)$ can be reduced to zero at steady state. At this point, we need the following assumption.

Assumption 1.1.1. *The pair* (A, C) *is observable*

In practice, the observer eigenvalues should be chosen to be about $5 - 6$ times faster

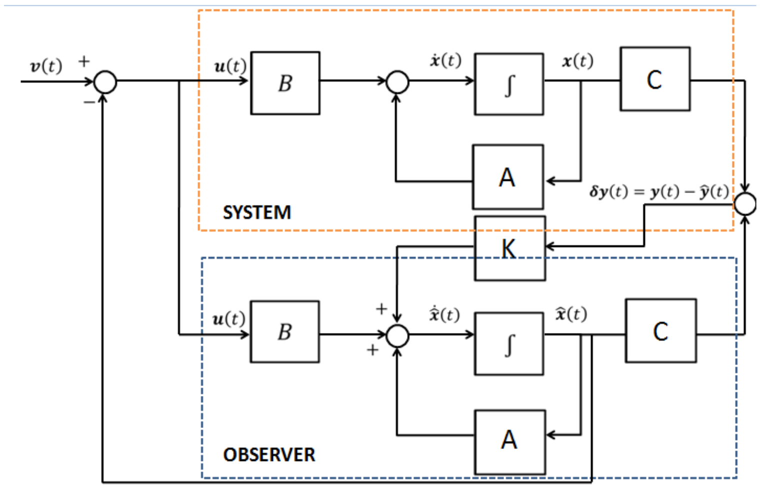

Figure 1.1: Full-order observer-based controller

than the system eigenvalues so that the minimal real part of the observer eigenvalues has to be $5-6$ times bigger than the maximal real part of system eigenvalues, that is

$$|\Re(\lambda_{min})|_{observer} > (5 \text{ or } 6) \times |\Re(\lambda_{max})|_{system} \tag{1.7}$$

1.1.2 Separation Principle

This section presents the fact that the observer-based controller preserves the closed-loop system eigenvalues. The system under state feedback control, that is $u(t) = -Fx(t)$ has the closed-loop form as

$$\dot{x}(t) = (A - BF)x(t) \tag{1.8}$$

so that the eigenvalues of the matrix $A - BF$ are the closed-loop system eigenvalues under state feedback. In the case of the observer-based controller, as given in Figure 1.1, the control input signal applied to the observer-based controller is given as

$$u_c(t) = -F\hat{x}(t) = -Fx(t) + Fe(t) \tag{1.9}$$

Substituting equation (1.9) in the full-order observer (1.5) and the system (1.8), we obtain the following augmented closed-loop matrix form

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & -BF \\ KC & A-KC-BF \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \tag{1.10}$$

At this point, we introduce the state transformation matrix given by

$$\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} = T_{aug} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \tag{1.11}$$

Since matrix T_{aug} is nonsingular, we can apply the similarity transformation to the closed-loop matrix form (1.10), which leads to

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A-BF & BF \\ 0 & A-KC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{1.12}$$

It is well known that the similarity transformation preserves the same eigenvalues as in the original system. The state matrix of the system (1.12) is upper block triangular and its eigenvalues are equal to the eigenvalues $\lambda(A-BF) \cup \lambda(A-KC)$, which indicates that the independent placement of observer and controller eigenvalues is possible.

1.2 An Observer for Singularly Perturbed Linear Systems

The singularly perturbed system (3.1) may be rewritten as

$$\begin{aligned} \dot{x}(t) &= \tilde{A}x(t) + \tilde{B}u(t),\ x(t_0) = x_0 \\ y(t) &= Cx(t) \end{aligned} \tag{1.13}$$

with

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\epsilon} & \frac{A_{22}}{\epsilon} \end{bmatrix},\ \tilde{B} = \begin{bmatrix} B_1 \\ \frac{B_2}{\epsilon} \end{bmatrix} \\ x(t) &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{1.14}$$

The corresponding full-order observer for the singularly perturbed system (1.13) is

$$\begin{aligned} \dot{\hat{x}}(t) &= (\tilde{A} - KC)\hat{x}(t) + \tilde{B}u(t) + Ky(t) \\ \hat{y}(t) &= C\hat{x}(t) \end{aligned} \tag{1.15}$$

where $\hat{x}(t)$ is an estimate of the state $x(t)$ in (1.13) and the state error is defined as

$$e(t) = \hat{x}(t) - x(t) \tag{1.16}$$

The role of the observer (3.17) is to reconstruct the state $x(t)$ of (1.13) in a uniformly asymptotic manner in the sense that

$$\lim_{t\to\infty} e(t) = 0 \tag{1.17}$$

The observability Assumption 1.1.1 is needed for (1.17) to hold

1.2.1 State Reconstruction for the Composite System

This section presents one of the results of a composite observer design based on the two slow and fast observers [29].

Lemma 1.2.1. *If the observer* (1.4) *is coupled to the system* (1.1) *with*

$$K = \begin{bmatrix} K_1 \\ \frac{1}{\epsilon}K_2 \end{bmatrix} \tag{1.18}$$

where

$$K_1 = \frac{1}{\epsilon^2}A_{12}A_{22}^{-1}K_2 + K_0[I - \frac{1}{\epsilon^2}C_2A_{22}^{-1}K_2] \tag{1.19}$$

and if $A_0 + K_0C_0$ and $A_{22} + K_2C_2$ are uniformly asymptotically stable, then the eigenvalues related to the error dynamics in the original coordinates satisfy

$$\begin{aligned} \lambda_i &= \lambda_i(A_0 + K_0C_0) + O(\epsilon),\ \ i = 1, ..., n_1 \\ \lambda_j &= \lambda_j(\frac{1}{\epsilon}A_{22} + \frac{1}{\epsilon}K_2C_2) + \frac{O(\epsilon)}{\epsilon},\ \ i = n_1 + j,\ \ j = 1, ..., n_2 \end{aligned} \tag{1.20}$$

1.3 Observer-based Controllers for Singularly Perturbed Systems

A dynamical feedback controllers for the singularly perturbed system (1.13) is given by

$$u_c(t) = F\hat{x}(t) \tag{1.21}$$

where $\hat{x}(t)$ is an estimate of the state $x(t)$ generated by the full-order observer (1.4). The overall closed-loop system for the original system (1.13) is given by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} \tilde{A} + \tilde{B}F & -\tilde{B}F \\ 0 & \tilde{A} - KC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{1.22}$$

It is required that the controller (3.19) be uniformly asymptotically stable in the sense that

$$\lim_{t\to\infty} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = 0 \tag{1.23}$$

Obviously, this may be achieved if and only if (1.13) is stabilizable by feedback (3.19) and the observer reconstruction error system

$$\dot{e}(t) = (\tilde{A} - KC)e(t),\ e(t_0) = \hat{x}(t_0) - x(t_0) \tag{1.24}$$

is asymptotically stable.

Observability Assumption 1.1.1 and the following assumption are needed for (3.19) to hold

Assumption 1.3.1. *The pair* $(\tilde{A}, B)$ *is controllable.*

1.3.1 A Composite Observer-based Controller

At this point, we need to introduce the observer driven controller proposed by [29].

Lemma 1.3.1. *If the observer and controller are coupled to the system* (1.13) *with*

$$F = \begin{bmatrix} F_1 & F_2 \end{bmatrix} \tag{1.25}$$

$$F_1 = [I + K_2 A_{22}^{-1} B_2] F_0 + F_2 A_{22}^{-1} A_{21} \tag{1.26}$$

$$K = \begin{bmatrix} K_1 \\ \frac{1}{\epsilon} K_2 \end{bmatrix} \tag{1.27}$$

$$K_1 = \frac{1}{\epsilon^2} A_{12} A_{22}^{-1} K_2 + K_0 [I - \frac{1}{\epsilon^2} C_2 A_{22}^{-1} K_2] \tag{1.28}$$

and if the slow subsystem and fast subsystem are each uniformly stabilizable by two observers and controllers, then there exists a positive ϵ^* *sufficiently small such that the original system* (1.13) *is uniformly completely stabilizable for any* $\epsilon \in (0, \epsilon^*]$.

This lemma indicates that the state and error dynamic can be reconstructed within $O(\epsilon)$ approximation. There are several papers for observers and observer driven controllers for singularly perturbed systems and *all of them did design with* $O(\epsilon)$ *accuracy* [29], [30], [31] and [34].

1.3.2 Design Procedure

The procedure for computing the feedback gain through the Lyapunov method is presented in [16]. Assume a controllable pair (A, B), where A is $\Re^{n\times n}$ and B is $\Re^{n\times m}$. Find a $\Re^{m\times n}$ real matrix F such that $(A - BF)$ has a set of desired eigenvalues that contains no eigenvalues of A.

Step 1. Select an $\Re^{n\times n}$ matrix $\Lambda_{desired}$ that has the desired set of eigenvalues. The form of $\Lambda_{desired}$ can be chosen arbitrarily. Often it is a diagonal matrix.

Step 2. Select an arbitrary $\Re^{m\times n}$ vector $\bar{F}$ such that $(\Lambda_{desired}, \bar{F})$ is observable.

Step 3. Solve the Sylvester equation $AP - P\Lambda_{desired} = B\bar{F}$ for the unique P.

Step 4. Compute the feedback gain $F = \bar{F}P^{-1}$ if the matrix P is invertible. If P is not invertible, go back to Step 2 and choose another $\bar{F}$.

1.4 Observer Eigenvalues Assignment

The corresponding Lyapunov method for obtaining the observer gain is to find the observer gain in the original coordinates. To find the observer gain, we need to transpose matrix $(A - KC)$. Consider the similarity transformation

$$P^{-1}(A^T - C^T K^T)P = \Lambda_{desired}^{obs} \tag{1.29}$$

where

$$\lambda(A^T - C^T K^T) = \lambda(\Lambda_{desired}^{obs}) = \lambda_{desired} \tag{1.30}$$

If (A, C) is observable, $\lambda(A - KC)$ can be arbitrarily located according to [16]. It is well known that the closed-loop eigenvalues of the observer should be located $5 - 6$ times farther to the left from the closed-loop system eigenvalues. Multiplying both side of (1.29) by P, (1.29) becomes the following Lyapunov equation

$$A^T P - P\Lambda_{desired}^{obs} = C^T \bar{K}^T \tag{1.31}$$

with

$$\bar{K}^T = K^T P \tag{1.32}$$

1.4.1 Design Procedure

For this section we introduce the procedure to compute the observer gain through the Lyapunov method. The following design procedure is presented in [16]. Consider the observable pair (A, C), where A is $\Re^{n\times n}$ and C is $\Re^{p\times n}$. Find a $\Re^{n\times p}$ real K such that $(A - KC)$ has any set of desired eigenvalues that contains no eigenvalues of A.

Step 1. Select an arbitrary matrix $\Lambda_{desired}^{obs}$ that has no common eigenvalues with those of A.

Step 2. Select an arbitrary $\Re^{p\times n}$ vector $\bar{K}^T$ such that $(\Lambda_{desired}^{obs}, \bar{K}^T)$ is observable.

Step 3. Solve for the unique P from the Sylvester equation $A^T P - P\Lambda_{desired}^{obs} = C^T \bar{K}^T$.

Step 4. Obtain the transposed observer gain from $K^T = \bar{K}^T P^{-1}$. If P is not invertible, go back to Step 2 and choose another $\bar{K}^T$.

1.5 Dissertation contributions

The Sylvester approach to the full-order observer design for singularly perturbed linear systems considered in Chapter 2. The aforementioned design, which can be applied to the normal linear system without numerical ill-conditional problem, can also be utilized to the singularly perturbed systems. To overcome the numerical ill-conditioning problem that comes from the perturbation parameter ϵ, we propose the recursive reduced-order solution of the singularly perturbed algebraic Sylvester equation. The proposed method was adopted from [3] where Lyapunov equation was considered. That method is extended to the Sylvester equation.

The two-stage feedback design approach is applied in the new design of an observer for singularly perturbed linear systems. The two-stage method is originally developed in [32] to the controller design of singularly perturbed linear systems with exact accuracy. We extend the two-stage method to the observer design using the duality between the controller and the observer. In the last part of Chapter 3, we propose the design of an observer driven controller for singularly perturbed linear systems by putting the observer and the corresponding controller designs together. Here, we want to emphasize that the proposed design method improves the accuracy from the previous observer design method for singularly perturbed systems : $O(\epsilon)$ design method [29], [30], [31] and [34].

We extend the results in Chapter 3 to the reduced-order observer design in Chapter 4 where the design algorithm is more complicated. To that end, five cases are considered. It should be emphasized that the proposed design produces the exact accuracy for the reduced-order observer designs for singularly perturbed systems.

The corresponding controller design based on the reduced-order observers is presented in Chapter 5. We have designed with very high accuracy the pure-slow and pure-fast reduced-order observer-based controllers for three out of five cases identified in Chapter 4.

To summarize the contribution of dissertation, we emphasize that an full-order and reduced-order observer and corresponding controller designs are implemented with an arbitrary high accuracy.

Chapter 2

Reduced-Order Algorithm for Eigenvalue Assignment of Singularly Perturbed Linear Systems

2.1 Introduction

The general algebraic Sylvester equation ([1], [2]) has many applications in engineering and sciences, including control systems [3],[4],[5],[16]. It is defined by

$$TA + MT + N = 0 \tag{2.1}$$

Its unique solution T exists under the assumption that matrices A and $-M$ has no eigenvalues in common [1], [2], [16].

Assumption 2.1.1. *Matrices A and $-M$ have no common eigenvalues.*

The classical method for numerical solution of (2.1) dates back to the reference [6]. However, it should be pointed out that solving the Sylvester algebraic equation numerically is not a simple task [7], [8]. Namely, it was stated in [7], [8] that the algorithm of [6] can not produce a highly accurate solution. In this paper, without loss of generality, we will consider the Sylvester equation encountered in the control system design of linear dynamic systems represented in state space formed by

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{2.2}$$

where $x(t) \in R^n$ are the state space variables, $u(t) \in R^m$ is the control input vector, $y(t) \in R^p$ is the vector of system measurements, and A, B, and C, $i, j = 1, 2$, are constant matrices of appropriate dimensions.

Forms of the Sylvester algebraic equations that appear in the observer and controller designs are respectively given by

$$
\begin{aligned}
&T_oA + MT_o - \bar{K}C = 0,\ \bar{K} = TK \\
&AT_c + T_cN - B\bar{F} = 0,\ \bar{F} = FT
\end{aligned}
\tag{2.3}
$$

where K stands for the observer feedback gain, and F is the system feedback gain. These Sylvester equations were extensively studied in a series of papers by [5], [7], [8]. It should be emphasized that in the observer driven controller design of singularly perturbed linear systems, [9], [10] due to the design requirement that the observer must be much faster that the system, the corresponding observer design algebraic Sylvester equation is extremely ill-conditioned.

2.2 Problem Statement

In this section, we study the Sylvester algebraic equation corresponding to singularly perturbed systems defined by ([11], [12])

$$
\begin{aligned}
&\dot{x}_1(t) = A_1x_1(t) + A_2x_2(t) + B_1u(t), x_1(t_0) = x_{10} \\
&\epsilon\dot{x}_2(t) = A_3x_1(t) + A_4x_2(t) + B_2u(t), x_2(t_0) = x_{20} \\
&y(t) = C_1x_1(t) + C_2x_2(t)
\end{aligned}
\tag{2.4}
$$

where $x_1(t) \in R^{n_1}, x_2(t) \in R^{n_2}, n_1 + n_2 = n$ are respectively slow and fast state variables and ϵ is a small positive singular perturbation parameter. The following is standard assumption used in theory of singular perturbation, [11].

Assumption 2.2.1. *The matrix A_4 is nonsingular.*

We study, without loss of generality, a variant of the observer design algebraic Sylvester equation (2.3) given by

$$
TA - A_{des}T = \bar{K}C,\ \bar{K} = TK,\ K = \begin{bmatrix} K_1 \\ \frac{1}{\epsilon}K_2 \end{bmatrix}
\tag{2.5}
$$

under the standard observer design assumptions [16].

Assumption 2.2.2. *The pair (A,C) is observable and the pair* $(A_{des}, \bar{K})$ *is controllable.*

The general existence condition given in Assumption 1, and specialized to (2.5) leads to the following assumption.

Assumption 2.2.3. $\lambda(A) \neq \lambda(A_{des})$

Having found an invertible solution of (2.5) then the observer gain is given by $K = T^{-1}\bar{K}$.

Note that Assumption 3 for single-input single-output systems is both sufficient and necessary condition for the existence of an invertible solution of (2.5). For multi-input multi-output systems it is only a necessary condition, [16] so that why a repetitive design algorithm has to be performed until an invertible solution T is obtained (see Section 5).

The system matrices defined in (2.4) and (2.5) are partitioned as

$$A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{\epsilon}A_3 & \frac{1}{\epsilon}A_4 \end{bmatrix}, \; A_{des} = \begin{bmatrix} A_s & 0 \\ 0 & \frac{1}{\epsilon}A_f \end{bmatrix}, \; C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \tag{2.6}$$

where A_{des} contains the desired observer closed-loop eigenvalues, that is, $\lambda(A_{des}) = \lambda(A - KC)$. We have found that the following scaling is appropriate for the solution matrix T

$$T = \begin{bmatrix} T_1 & \epsilon T_2 \\ \epsilon T_3 & \epsilon T_4 \end{bmatrix} \tag{2.7}$$

which is consistent with the structures of matrices defined in (2.5) and (2.6). Namely, the right-hand side of (2.5) is

$$\begin{aligned} \bar{K}C &= \begin{bmatrix} T_1 & \epsilon T_2 \\ \epsilon T_3 & \epsilon T_4 \end{bmatrix} \begin{bmatrix} K_1 \\ \frac{1}{\epsilon}K_2 \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} T_1K_1C_1 + T_1K_2C_1 & T_1K_1C_2 + T_2K_2C_2 \\ \epsilon T_3K_1C_1 + T_4K_2C_1 & \epsilon T_3K_1C_2 + T_4K_2C_2 \end{bmatrix} \\ &= - \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} = \begin{bmatrix} O(1) & O(1) \\ O(1) & O(1) \end{bmatrix} \end{aligned} \tag{2.8}$$

With the scaling chosen in (2.7), the left-hand side terms of (2.5), that is, TA and $A_{des}T$ are also both $O(1)$.

Due to the structure of matrices A and A_{des}, the singularly perturbed algebraic Sylvester equation defined in (2.5) is numerically ill-conditioned. To overcome numerical ill-conditioning, we propose in the next section a new recursive algorithm for solving (2.5) in terms of reduced-order well-defined algebraic Sylvester equations. The dual version of (2.5) used for the system controller design is given by

$$AT_c - T_c A_{des}^c = B\bar{F},\ \bar{F} = FT,\ B = \begin{bmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{bmatrix},\ A_{des}^c = \begin{bmatrix} A_{sc} & 0 \\ 0 & \frac{1}{\epsilon}A_{fc} \end{bmatrix} \tag{2.9}$$

It can be shown that the structure of T_c is given by

$$T_c = \begin{bmatrix} T_{1c} & T_{2c} \\ T_{3c} & \frac{1}{\epsilon}T_{4c} \end{bmatrix} \tag{2.10}$$

Algebraic Sylvester equation (2.9)-(2.10) will be solved numerically in terms of reduced-order numerically well-conditioned algebraic Sylvester equations under the standard controller design assumptions, [16].

Assumption 2.2.4. *The pair (A,B) is controllable and the pair* $(A_{des}^c, \bar{F})$ *is observable.*

Moreover, the existence of a unique solution of (2.9) requires the assumption dual to Assumption 4.

Assumption 2.2.5. $\lambda(A) \neq \lambda(A_{des}^c)$.

2.3 Parallel Algorithm for the Observer Design Sylvester Equation

The partitioned form of the Sylvester equation given in (2.5) subject to (2.6)-(2.8) is given by

$$\begin{aligned} &T_1A_1 + T_2A_3 - A_sT_1 + Q_1 = 0 \\ &T_1A_2 + T_2A_4 - \epsilon A_sT_2 + Q_2 = 0 \\ &\epsilon T_3A_1 + T_4A_3 - A_fT_3 + Q_3 = 0 \\ &\epsilon T_3A_2 + T_4A_4 - A_fT_4 + Q_4 = 0 \end{aligned} \tag{2.11}$$

Setting $\epsilon = 0$, the algebraic equations for zeroth-order approaximations of solutions $T_1^{(0)}, T_2^{(0)}, T_3^{(0)}, T_4^{(0)}$ are obtained as

$$T_1^{(0)}A_1 + T_2^{(0)}A_3 - A_sT_1^{(0)} + Q_1 = 0 \tag{2.12}$$

$$T_1^{(0)}A_2 + T_2^{(0)}A_4 + Q_2 = 0 \tag{2.13}$$

$$T_4^{(0)}A_3 - A_fT_3^{(0)} + Q_3 = 0 \tag{2.14}$$

$$T_4^{(0)}A_4 - A_fT_4^{(0)} + Q_4 = 0 \tag{2.15}$$

Unique solution $T_4^{(0)}$ can be obtained from the reduced-order Sylvester equation (2.15) under the following assumption.

Assumption 2.3.1. *Eigenvalues of A_4 and A_f have no eigenvalues in common.*

Since A_f, defined in (2.6), is chosen by the designer, this assumption is easily satisfied. From (2.13) and (2.14), we can obtain $T_3^{(0)}, T_2^{(0)}$ as

$$T_3^{(0)} = A_f^{-1}(T_4^{(0)}A_3 + Q_3) \tag{2.16}$$

$$T_2^{(0)} = -(Q_2 + T_1^{(0)}A_2)A_4^{-1} \tag{2.17}$$

Substituting (2.17) into (2.12) results in

$$T_1^{(0)}A_0 - A_sT_1^{(0)} + Q_0 = 0 \tag{2.18}$$

where

$$A_0 = A_1 - A_2A_4^{-1}A_3, \; Q_0 = Q_1 - Q_2A_4^{-1}A_3 \tag{2.19}$$

The unique solution $T_1^{(0)}$ of the reduced-order algebraic Sylvester equations (2.18) exists under the following assumption.

Assumption 2.3.2. *Matrices A_0 and A_s have no eigenvalues in common.*

Since A_s is chosen by the designer, this assumption is easily satisfied. Furthermore, $T_2^{(0)}$ and $T_3^{(0)}$ can be found from (2.16)-(2.17).

The solutions $T_1^{(0)}, T_2^{(0)}, T_3^{(0)}$, and $T_4^{(0)}$ are $O(\epsilon)$[1] close to the exact solutions, that is

$$\begin{aligned} T_1 &= T_1^{(0)} + \epsilon E_1 \\ T_2 &= T_2^{(0)} + \epsilon E_2 \\ T_3 &= T_3^{(0)} + \epsilon E_3 \\ T_4 &= T_4^{(0)} + \epsilon E_4 \end{aligned} \tag{2.20}$$

Subtracting (2.12)-(2.15) from (2.11) and using (2.20), we obtain the error equations (after some algebra) in the form

$$E_1 A_1 - A_s E_1 = -E_2 A_3 \tag{2.21}$$

$$E_2 A_4 - \epsilon A_s E_2 = -E_1 A_2 + A_s T_2^{(0)} \tag{2.22}$$

$$\epsilon E_3 A_1 - A_f E_3 = -T_3^{(0)} A_1 - E_4 A_3 \tag{2.23}$$

$$E_4 A_4 - A_f E_4 = -T_3^{(0)} A_2 - \epsilon E_3 A_2 \tag{2.24}$$

The error equations can be solved iteratively using the following fixed-point algorithm.

Algorithm I :

$$\begin{aligned} &E_1^{(i+1)} A_0 - A_s E_1^{(i+1)} = -A_s T_2^{(0)} A_4^{-1} A_3 - \epsilon A_s E_2^{(i)} A_4^{-1} A_3 \\ &- \epsilon A_s E_2^{(i+1)} + E_2^{(i+1)} A_4 = A_s T_2^{(0)} - E_1^{(i+1)} A_2 \\ &\epsilon E_3^{(i+1)} A_1 - A_f E_3^{(i+1)} = -T_3^{(0)} A_1 - E_4^{(i)} A_3 \\ &E_4^{(i+1)} A_4 - A_f E_4^{(i+1)} = -T_3^{(0)} A_2 - \epsilon E_3^{(i)} A_2 \end{aligned} \tag{2.25}$$

with starting points

$$\begin{aligned} &E_2^{(0)} = 0 \\ &E_3^{(0)} = A_f^{-1}(T_3^{(0)} A_1 + E_4^{(0)} A_3) \\ &E_4^{(0)} A_4 - A_f E_4^{(0)} + T_3^{(0)} A_2 = 0 \end{aligned} \tag{2.26}$$

---

[1] $O(\epsilon)$ is defined by $O(\epsilon) \leq c\epsilon$, where c is a bounded constant.

We first solve (2.22) as

$$E_2^{(i+1)} = [A_s T_2^{(0)} - E_1^{(i+1)} A_2 + \epsilon A_s E_2^{(i)}] A_4^{-1} \tag{2.27}$$

Substituting (2.27) in to (2.21) gives

$$E_1^{(i+1)} A_0 - A_s E_1^{(i+1)} = -A_s T_2^{(0)} A_4^{-1} A_3 - \epsilon A_s E_2^{(i)} A_4^{-1} A_3 \tag{2.28}$$

Equations (2.27) and (2.28) have very nice forms since the quantity E_2 in (2.28) is multiplied by a small parameter ϵ. Similarly, equations for E_3 and E_4 can be iteratively solved as

$$\epsilon E_3^{(i+1)} A_1 - A_f E_3^{(i+1)} = -T_3^{(0)} A_1 - E_4^{(i)} A_3 \tag{2.29}$$

$$E_4^{(i+1)} A_4 - A_f E_4^{(i+1)} = -T_3^{(0)} A_2 - \epsilon E_3^{(i)} A_2 \tag{2.30}$$

The following theorem presents the main feature of Algorithm I. Under assumptions 7 and 8, Algorithm I converges to the exact solution E with the rate of convergence of $O(\epsilon)$. The convergence is obtained for sufficiently small values of ϵ that makes the radius of convergence smaller than 1 in each iteration. Hence, after i iterations, the solution T is obtained with the accuracy of $O(\epsilon^i)$, that is

$$T_j^{(i)} = T_j^{(0)} + \epsilon E_j + O(\epsilon^i),\ j = 1, 2, 3, 4;\ i = 1, 2, ... \tag{2.31}$$

Proof of Theorem 1 :

For $i = 1$, (2.28) implies

$$E_1^{(1)} A_0 - A_s E_1^{(1)} = -A_s T_2^{(0)} A_4^{-1} A_3 - \epsilon A_s E_2^{(0)} A_4^{-1} A_3 = -A_s T_2^{(0)} A_4^{-1} A_3 \tag{2.32}$$

Note that $E_2^{(0)} = 0$. For $i = 2$, (2.28) produces

$$E_1^{(2)} A_0 - A_s E_1^{(2)} = -A_s T_2^{(0)} A_4^{-1} A_3 - \epsilon A_s E_2^{(1)} A_4^{-1} A_3 \tag{2.33}$$

Subtracting (2.32) from (2.33), we have

$$(E_1^{(2)} - E_1^{(1)})A_0 - A_s(E_1^{(2)} - E_1^{(1)}) = -\epsilon A_s E_2^{(1)} A_4^{-1} A_3 = O(\epsilon) \tag{2.34}$$

At this point, we conclude that

$$\|E_1^{(2)} - E_1^{(1)}\| = O(\epsilon) \tag{2.35}$$

In a similar way, we can write the relationship between $E_1^{(3)}$ and $E_1^{(2)}$ given as

$$(E_1^{(3)} - E_1^{(2)})A_0 - A_s(E_1^{(3)} - E_1^{(2)}) = -\epsilon A_s E_2^{(2)} A_4^{-1} A_3 = O(\epsilon) \tag{2.36}$$

which implies that

$$\|E_1^{(3)} - E_1^{(2)}\| = O(\epsilon) \tag{2.37}$$

Continuing the same procedure we obtain

$$\|E_1^{(i+1)} - E_1^{(i)}\| = O(\epsilon) \tag{2.38}$$

Now, we work with E_2 using (2.27). When $i = 0$, we have

$$E_2^{(1)} = [A_s T_2^{(0)} - E_1^{(1)} A_2 + \epsilon A_s E_2^{(0)}]A_4^{-1} \tag{2.39}$$

For $i = 1$

$$E_2^{(2)} = [A_s T_2^{(0)} - E_1^{(2)} A_2 + \epsilon A_s E_2^{(1)}]A_4^{-1} \tag{2.40}$$

Using the fact that $E_2^{(0)} = 0$, and the result established in (2.35), we get

$$\|E_2^{(2)} - E_2^{(1)}\| = O(\epsilon) \tag{2.41}$$

Considering (2.27) for $i = 2$ and using (2.41), we obtain

$$\|E_1^{(3)} - E_1^{(1)}\| = O(\epsilon^2) \tag{2.42}$$

If we keep repeating this process, we conclude that

$$\|E_1^{(i+1)} - E_1\| = O(\epsilon^{(i)}) \tag{2.43}$$

Similar procedures can be applied to (2.29) and (2.30), which produces

$$\|E_4^{(i+1)} - E_4^{(i)}\| = O(\epsilon) \tag{2.44}$$

and

$$\|E_3^{(i+1)} - E_3^{(i)}\| = O(\epsilon) \tag{2.45}$$

Results established in (2.38), (2.43)-(2.45), can be summarized in

$$\|E_j^{(i)} - E_j\| = O(\epsilon^i),\ j = 1, 2, 3, 4;\ i = 1, 2, ... \tag{2.46}$$

which completes the proof of the stated theorem.

2.4 Parallel Algorithm for the Controller Design Sylvester Equations

The controller design algebraic Sylvester equation defined in (2.9)-(2.10) can be partitioned as

$$\begin{aligned}
&A_1T_{1c} + A_2T_{3c} - T_{1c}A_{sc} + Q_{1c} = 0\\
&\epsilon A_1T_{2c} + A_2T_{4c} - T_{2c}A_{fc} + Q_{2c} = 0\\
&A_3T_{1c} + A_4T_{3c} - \epsilon T_{3c}A_{sc} + Q_{3c} = 0\\
&\epsilon A_3T_{2c} + A_4T_{4c} - T_{4c}A_{fc} + Q_{4c} = 0
\end{aligned} \tag{2.47}$$

where

$$Q_c = B\bar{F} = BFT_c = -\begin{bmatrix} Q_{1c} & \frac{1}{\epsilon}Q_{2c} \\ \frac{1}{\epsilon}Q_{3c} & \frac{1}{\epsilon^2}Q_{4c} \end{bmatrix} = \begin{bmatrix} O(1) & O(\frac{1}{\epsilon}) \\ O(\frac{1}{\epsilon}) & O(\frac{1}{\epsilon^2}) \end{bmatrix} \tag{2.48}$$

Setting $\epsilon = 0$ in (2.47), the zeroth-order approximations $T_{1c}^{(0)}, T_{2c}^{(0)}, T_{3c}^{(0)}$ and $T_{4c}^{(0)}$ can be obtained

$$A_1T_{1c}^{(0)} + A_2T_{3c}^{(0)} - T_{1c}^{(0)}A_{sc} + Q_{1c} = 0 \tag{2.49}$$

$$A_2T_{4c}^{(0)} - T_{4c}^{(0)}A_{fc} + Q_{4c} = 0 \tag{2.50}$$

$$A_3T_{1c}^{(0)} + A_4T_{3c}^{(0)} + Q_{3c} = 0 \tag{2.51}$$

$$A_4T_{4c}^{(0)} - T_{4c}^{(0)}A_{fc} + Q_{4c} = 0 \tag{2.52}$$

The unique solution $T_{4c}^{(0)}$ can be found from the reduced-order algebraic Sylvester equation (2.52) under the following assumption.

Assumption 2.4.1. *Matrices A_4 and A_{fc} have no eigenvalues in common.*

From (2.50)-(2.51), we can obtain $T_{2c}^{(0)}$ and $T_{3c}^{(0)}$ as

$$T_{2c}^{(0)} = (A_2T_{4c}^{(0)} + Q_{2c})A_{fc}^{-1} \tag{2.53}$$

$$T_{3c}^{(0)} = -A_4^{-1}(A_3T_{1c}^{(0)} + Q_{3c}) \tag{2.54}$$

Substituting (2.54) into (2.49) results in

$$A_0T_{1c}^{(0)} - T_{1c}^{(0)}A_{sc} + Q_{0c} = 0 \tag{2.55}$$

where

$$A_0 = A_1 - A_2A_4^{-1}A_3, Q_{0c} = Q_{1c} - A_2A_4^{-1}Q_{3c} \tag{2.56}$$

The solution $T_1^{(0)}$ can be obtained by solving the reduced-order algebraic Sylvester equation (2.55) under the assumption.

Assumption 2.4.2. *Matrices A_0 and A_{sc} have no eigenvalues in common.*

Since A_{sc} is chosen by the designer, this assumption is easily satisfied. Similarly, what we did in Section 3, we define the approximation errors as

$$\begin{aligned} T_{1c} &= T_{1c}^{(0)} + \epsilon E_{1c} \\ T_{2c} &= T_{2c}^{(0)} + \epsilon E_{2c} \\ T_{3c} &= T_{3c}^{(0)} + \epsilon E_{3c} \\ T_{4c} &= T_{4c}^{(0)} + \epsilon E_{4c} \end{aligned} \tag{2.57}$$

Subtracting (2.49)-(2.52) from (2.47) and using (2.52), we obtain the error equations (after some algebra) in the form

$$A_1 E_{1c} - E_{1c} A_{sc} = -A_2 E_{3c} \tag{2.58}$$

$$E_{2c} A_{fc} - \epsilon A_1 E_{2c} = -A_1 T_{2c}^{(0)} + A_2 E_{4c} \tag{2.59}$$

$$A_4 E_{3c} - \epsilon E_{3c} A_{sc} = -A_3 E_{1c} - T_{3c}^{(0)} A_{sc} \tag{2.60}$$

$$E_{4c} A_{fc} - A_4 E_{4c} = A_3 T_{2c}^{(0)} - \epsilon A_3 E_{2c} \tag{2.61}$$

These error equations can be solved using the fixed point algorithm, dual to Algorithm I, as follows

Algorithm: II :

$$\begin{aligned}
&A_0 E_{1c}^{(i+1)} - E_{1c}^{(i+1)} A_{sc} = -\epsilon A_2 A_4^{-1} E_{3c}^{(i)} A_{sc} - A_2 A_4^{-1} T_{3c}^{(0)} A_{sc} \\
&A_4 E_{3c}^{(i+1)} - \epsilon E_{3c}^{(i+1)} A_{sc} = -A_3 E_{1c}^{(i)} + T_{3c}^{(0)} A_{sc} \\
&E_{2c}^{(i+1)} A_{fc} - \epsilon A_1 E_{2c}^{(i+1)} = A_1 T_{2c}^{(0)} + A_2 E_{4c}^{(i)} \\
&E_{4c}^{(i+1)} A_{fc} - A_4 E_{4c}^{(i+1)} = A_3 T_{2c}^{(0)} + \epsilon A_3 E_{2c}^{(i)}
\end{aligned} \tag{2.62}$$

with starting points

$$\begin{aligned}
&E_{3c}^{(0)} = 0 \\
&E_{2c}^{(0)} = 0 \\
&A_0 E_{1c}^{(0)} - E_{1c}^{(0)} A_{sc} + A_2 A_4 T_{3c}^{(0)} A_{sc} = 0
\end{aligned} \tag{2.63}$$

The convergence proof of Algorithm II can be done via the dual arguments used in Algorithm I. Similarly, we can state the corresponding theorem dual to Theorem 1. Under Assumptions $5, 9$, and 10, Algorithm II converges for sufficiently small values of ϵ with the rate of $O(\epsilon)$ to the sought solution $T_j, j = 1, 2, 3, 4$, that is after i iterations, we have

$$T_{jc}^{(i)} = T_{jc}^{(0)} + \epsilon E_{jc} + O(\epsilon^i),\ \ j = 1, 2, 3, 4;\ \ i = 1, 2, 3, ... \tag{2.64}$$

The proof of Theorem 2 parallels the proof of Theorem 1, and hence it is omitted.

2.5 Observer And Controller Designs via the Sylvester Equations

In this section, we present the design of an observer and a controller using the Sylvester approach following the steps of Chen (2013). We will exploit two-time scale property so that the design is done in terms of reduced-order problems.

The goal is that $\lambda(A-KC) = \lambda^{observer}_{desired}$ are the desired observer eigenvalues. For the observer design, we first check that (A, C) is observable. The observer design procedure for the system defined in (2.2) has the following steps, Chen (2013).

Algorithm III: (Observer Design)

Step 1 : Choose $A_{des} = \begin{bmatrix} A_s & 0 \\ 0 & \frac{1}{\epsilon}A_f \end{bmatrix}$ such that $\lambda(A_{des}) \neq \lambda(A)$.

Step 2 : Guess $\bar{K} = \begin{bmatrix} \bar{K}_s \\ \frac{1}{\epsilon}\bar{K}_f \end{bmatrix}$ such that $(A_{des}, \bar{K})$ is controllable.

Step 3 : Solve $TA - A_{des}T = \bar{K}C$ using Algorithm I.

Step 4 : If T^{-1} does not exists, go back to Step 2 and guess another $\bar{K}$ and repeat the process until T^{-1} exists. If T^{-1} exists then $\hat{x}(t) = T^{-1}z(t)$ where the observer structure for $z(t)$ is given as, Chen (2013)

$$\begin{aligned} \dot{z}(t) &= A_{des}z(t) + TBu(t) + \bar{K}y(t) \\ \hat{x}(t) &= T^{-1}z(t) \end{aligned} \tag{2.65}$$

For the controller design, we first check that (A, B) is controllable. The state feedback controller for the system defined in (2.2) can be obtained using the following steps, [16]

Algorithm IV: (Controller Design)

Step 1 : Choose $A^c_{des} = \begin{bmatrix} A^c_s & 0 \\ 0 & \frac{1}{\epsilon}A^c_f \end{bmatrix}$ such that $\lambda(A^c_{des}) \neq \lambda(A)$

Step 2 : Guess $\bar{F} = \begin{bmatrix} \bar{F}_s & \bar{F}_f \end{bmatrix}$ such that $(A_{des}, \bar{F})$ is observable.

Step 3 : Solve $AT_c - TA^c_{des} = B\bar{F}$ using Algorithm II.

Step 4 : If T_c^{-1} exists then $F = \bar{F}T_c^{-1}$, otherwise go back to Step 2, guess another $\bar{F}$, and repeat the process.

The feedback system is given by

$$\dot{x}(t) = (A - BF)x(t), \quad \lambda(A - BF) = \lambda_{desired}^{system} \tag{2.66}$$

2.6 Simulation Results

Consider a $4^{th}-$ order system with the matrices A, B and C taken from [11].

$$A = \begin{bmatrix} 0 & 0.4000 & 0 & 0 \\ 0 & 0 & 0.3450 & 0 \\ 0 & -5.2400 & -4.6500 & 2.6200 \\ 0 & 0 & 0 & -10.0000 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The pair (A, C) is observable and we can proceed with the observer design algorithm. The designer decides to place observer eigenvalues at the desired location by choosing matrices A_s and A_f. In the following we will design a controller with the desired closed-loop eigenvalues placed at $\{-0.2, -0.3, -7, -8\}$. Note that (A, B) is controllable.

Observer Design Algorithm III :

We choose the observer eigenvalues such that it is roughly ten times faster than the closed-loop system. Consequently, we choose A_{des} as

$$A_{des} = \begin{bmatrix} A_s & 0 \\ 0 & \frac{1}{\epsilon} A_f \end{bmatrix} = \begin{bmatrix} -50 & 0 & 0 & 0 \\ 0 & -60 & 0 & 0 \\ 0 & 0 & -500 & 0 \\ 0 & 0 & 0 & -600 \end{bmatrix}$$

We choose $\bar{K}$ as

$$\bar{K} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 5 \\ 4 & 6 \end{bmatrix}$$

so that $(A_{des}, \bar{K})$ is controllable, as required in Step 2 of Algorithm III.

The matrix Q defined in (2.8) is given as

$$Q = -\bar{K}C = \begin{bmatrix} 1 & 0 & -3 & 0 \\ -2 & 0 & -4 & 0 \\ -3 & 0 & -5 & 0 \\ -4 & 0 & -6 & 0 \end{bmatrix}$$

The zeroth-order approximation $T_1^{(0)}, T_2^{(0)}, T_3^{(0)}$ and $T_4^{(0)}$ are obtained as

$$T_1^{(0)} = \begin{bmatrix} 0.02000000 & -0.06830400 \\ 0.03333333 & -0.07583907 \end{bmatrix},$$

$$T_2^{(0)} = \begin{bmatrix} -6.50229006 & -1.70360000 \\ -8.65841823 & -2.26850557 \end{bmatrix},$$

$$T_3^{(0)} = \begin{bmatrix} 0.06000000 & 0.00105783 \\ 0.06666666 & 0.00088015 \end{bmatrix},$$

$$T_4^{(0)} = \begin{bmatrix} 0.10093873 & -0.00053971 \\ 0.10078105 & -0.00044753 \end{bmatrix}$$

Performing iterations, we obtain the sixteen decimal digits accuracy after $i = 50$

$$E_1^{(50)} = \begin{bmatrix} 0 & 0.75071350 \\ 0 & 0.81924891 \end{bmatrix},$$

$$E_2^{(50)} = \begin{bmatrix} 71.63296737 & 16.60304059 \\ 93.80712798 & 22.30657343 \end{bmatrix},$$

$$E_3^{(50)} = 10^{-3} \times \begin{bmatrix} 0 & -0.48007370 \\ 0 & -0.44448673 \end{bmatrix},$$

$$E_4^{(50)} = 10^{-5} \times \begin{bmatrix} -0.70332400 & 0.00376063 \\ 0.48428406 & 0.00215054 \end{bmatrix}$$

The corresponding iterative solution $\hat{T}$ and the exact solution T (obtained by using the MATLAB function lyap to solve the full-order sylvester equation) are given by

$$\hat{T} = \begin{bmatrix} 0.02000000 & 0.00676734 & 0.06610066 & -0.00432960 \\ 0.03333333 & 0.00608581 & 0.07222945 & -0.00378482 \\ 0.00600000 & 0.00010100 & 0.01009380 & -0.00005400 \\ 0.00666666 & 0.00008357 & 0.01007805 & -0.00004475 \end{bmatrix} = T$$

The difference $\|T - \hat{T}\|$ is

$$\|T - \hat{T}\| = 1.677861779926784 \times 10^{-16}$$

The solution T is invertible in the first run of Algorithm III (see Section 5). The corresponding observer gain $K = (\hat{T})^{-1}\bar{K}$ is

$$K = 10^5 \times \begin{bmatrix} 0.01122902 & 0.01013856 \\ -0.78192851 & -0.92597450 \\ -0.00234100 & 0.00072447 \\ -1.20836830 & -1.39637444 \end{bmatrix}$$

Checking the corresponding observer closed loop eigenvalues, we have

$$\lambda(A - KC) = \begin{bmatrix} -49.9999999999540 \\ -60.0000000000346 \\ -499.9999999999835 \\ -599.9999999999814 \end{bmatrix}$$

which with the accuracy of $O(10^{-12})$ is close to the chosen desired eigenvalues of the matrix A_{des}. The observer's structure is given as in (2.65).

Algorithm IV : (Controller Design)

Similarly, we design a controller for the same system using the algorithm for solving the controller algebraic Sylvester equation from Section 4. We choose

$$A_{des}^c = \begin{bmatrix} A_s^c & 0 \\ 0 & \frac{1}{\epsilon} A_f^c \end{bmatrix} = \begin{bmatrix} -0.2 & 0 & 0 & 0 \\ 0 & -0.3 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

The zeroth-order approximations $T_{1c}^{(0)}, T_{2c}^{(0)}, T_{3c}^{(0)}$ and $T_{4c}^{(0)}$ are obtained as

$$T_{1c}^{(0)} = \begin{bmatrix} 2.05946684 & 2.91957364 \\ -1.02973342 & -2.18968023 \end{bmatrix},$$

$$T_{2c}^{(0)} = \begin{bmatrix} 0 & 0 \\ -0.18316110 & -0.16863805 \end{bmatrix},$$

$$T_{3c}^{(0)} = \begin{bmatrix} 0.59694691 & 1.90406976 \\ -1.00000000 & -1.00000000 \end{bmatrix},$$

$$T_{4c}^{(0)} = \begin{bmatrix} 0.37163120 & 0.39104477 \\ -0.33333333 & -0.50000000 \end{bmatrix}$$

Using the proposed algorithm, we obtain after $i = 50$

$$E_{1c}^{(50)} = \begin{bmatrix} -0.49558505 & -4.48463407 \\ 0.24779252 & 3.36347555 \end{bmatrix},$$

$$E_{2c}^{(50)} = \begin{bmatrix} 0.09430020 & 0.07899070 \\ 0.18135735 & 0.10656670 \end{bmatrix}$$

$$E_{3c}^{(50)} = \begin{bmatrix} -0.14364784 & -2.92476135 \\ -0.20408163 & -0.30927835 \end{bmatrix},$$

$$E_{4c}^{(50)} = \begin{bmatrix} -0.36797143 & -0.24711120 \\ 0 & 0 \end{bmatrix}$$

The iterative solution $\hat{T}^c$ and the exact solution T^c are given by

$$\hat{T}^c = \begin{bmatrix} 2.00990834 & 2.47111023 & 0.00943002 & 0.00789910 \\ -1.00495420 & -1.85333270 & -0.16502535 & -0.15798140 \\ 0.58258212 & 1.61159363 & 3.34834061 & 3.66333660 \\ -1.02040816 & -1.03092783 & -3.33333333 & -5.00000000 \end{bmatrix} = T^c$$

Their difference is

$$\|T^c - \hat{T}^c\| = 1.761132598848588 \times 10^{-14}$$

The solution T^c is invertible in the first run of Algorithm IV (see Section 5). The controller gain $F = \bar{F}(\hat{T}^c)^{-1}$ is given as

$$F = \begin{bmatrix} 0.92930633 & 1.02725633 & 0.43128625 & 0.08500000 \end{bmatrix}$$

Checking $\lambda(A - BF)$, we have

$$\lambda(A - BF) = \begin{bmatrix} -0.199999999999998 \\ -0.300000000000001 \\ -7.000000000000027 \\ -7.999999999999988 \end{bmatrix}$$

which have produced the desired eigenvalues with the accuracy of $O(10^{-13})$.

2.7 Conclusion

It has been shown that the numerically ill-conditioned Sylvester algebraic equation for singularly perturbed systems can be decomposed into four lower order well conditioned Sylvester equations. The recursive fixed-point type methods was utilized in order to obtain numerical solutions for such lower-order algebraic Sylvester equations. The corresponding observer and controller design algorithms for assignment of observer and controller closed-loop eigenvalues in terms of reduced-order slow and fast subproblems are presented. The main result of Chapter 2 have been submitted for publication in Automatica [52].

Chapter 3

New Designs of Linear Observers and Observer-Based Controllers for Singularly Perturbed Linear Systems

3.1 Introduction

Singularly perturbed systems have been studied in different set-ups by many researchers [11], [12], [19], [21], [25], [27] and [38]. Under ceratin conditions, a decoupling transformation was introduced, [17], such that a singularly perturbed linear system composed of two sub-systems can be internally decomposed into two reduced-order slow and fast sub-systems. After a system is internally decoupled into two subsystems, suitable control laws can be chosen for each subsystems. Traditionally, solutions of controller and observer designs for singularly perturbed linear systems were obtained with an $O(\epsilon)$ accuracy [18], [21], [24], [29], [30], [34], [35]. Observers nowadays play very important roles in all areas of science and engineering, see [22] and references there. There are several papers for observer driven controllers for singularly perturbed systems and all of them provide $O(\epsilon)$ accuracy only.

The paper is organized as follows. In Section 2, singularly perturbed systems are reviewed, including state transformation that decouples slow and fast modes, and observer and observer-driven controller for singularly perturbed systems. It was emphasized that the current design methods are with $O(\epsilon)$ accuracy. In Section 3, we review the results of [32],[33] in which exact eigenvalues assignment was implemented through a two-stage method. Sections 4 and 5 present new research results. The new two stage method for the observer design is presented in Section 4. The corresponding observer based controller design is presented in Section 5. In this paper, it will be shown how to design exactly (with arbitrary high accuracy) reduced-order slow and fast observers and corresponding controllers such that complete parallelism is introduced, which facilitates significant reduction in on- and off-line computational requirements and reduces the observer and controller signal processing time.

3.2 Review of Singularly Perturbed Linear Systems

Large-scale linear systems are encountered frequently in engineering [11], [12], [19], [21], [25],[27] and [38]. The crucial theme is how to reduce a large time scale system into a reduced form in which slow and fast dynamics is separated to enhance analysis, design, and simulation. Consider a linear time invariant singularly perturbed system, [11].

$$\begin{aligned}
&\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t),\ x_1(t_0) = x_1^0 & & \dot{x}(t) = Ax(t) + Bu(t)\\
&\dot{x}_2(t) = \frac{1}{\epsilon}A_{21}x_1(t) + \frac{1}{\epsilon}A_{22}x_2(t) + \frac{1}{\epsilon}B_2u(t),\ x_2(t_0) = x_2^0 & \Rightarrow\ & y(t) = Cx(t) + Du(t)\\
&y(t) = C_1x_1(t) + C_2x_2(t) + Du(t) & &
\end{aligned} \tag{3.1}$$

where ϵ is a small positive singular perturbation parameter that indicates seperation of state variables $x(t) \in \Re^n$ into slow $x_1(t) \in \Re^{n_1}$ and fast $x_2(t) \in \Re^{n_2}$, $n_1 + n_2 = n$ state variables. $u(t) \in \Re^m$ is the control input and $y(t) \in \Re^p$ the system measured output. ϵ usually represents small time constants, small masses, small resistances and capacitance, small moments of inertia. Often such systems have eigenvalues clustered into two groups, slow ones close to the imaginary axis and fast ones, far from the imaginary axis. The parameter ϵ can be taken as the ratio of the eigenvalue real parts,

$$\epsilon = \frac{\max_i\{\mathrm{Re}|\lambda_i^s(A)|\}}{\min_j\{\mathrm{Re}|\lambda_i^f(A)|\}} \tag{3.2}$$

Traditionally, the system (3.1) may be approximately decomposed into reduced slow system having n_1 slow modes and a fast subsystem having n_2 fast modes [11]. The reduced slow subsystem is obtained by setting $\epsilon = 0$ in (3.1), that is

$$\dot{x}_{1s}(t) = A_{11}x_{1s}(t) + A_{12}x_{2s}(t) + B_1u(t) \tag{3.3}$$

$$0 = A_{21}x_{1s}(t) + A_{22}x_{2s}(t) + B_2u(t) \tag{3.4}$$

$$y_s(t) = C_1 x_{1s}(t) + C_2 x_{2s}(t) + Du(t) \tag{3.5}$$

The standard assumption used for singularly perturbed linear system, [11], is

Assumption 3.2.1. *Matrix A_{22} is invertible.*

Under Assumption 1, $x_{2s}(t)$ can be obtained from (3.4) as

$$x_{2s}(t) = -A_{22}^{-1} A_{21} x_s(t) - A_{22}^{-1} B_2 u_s(t) \tag{3.6}$$

Substitution of (3.6) into (3.3) results in the approximate slow subsystem

$$\begin{aligned} \dot{x}_{1s}(t) &= A_0 x_{1s}(t) + B_0 u_s(t),\ x_{1s}(t_0) = x_1^0 \\ y_s(t) &= C_0 x_{1s}(t) + D_0 u_s(t) \end{aligned} \tag{3.7}$$

with

$$\begin{aligned} A_0 &= A_{11} - A_{12} A_{22}^{-1} A_{21},\ B_0 = B_1 - A_{12} A_{22}^{-1} B_2 \\ C_0 &= C_1 - C_2 A_{22}^{-1} A_{21},\ D_0 = -C_2 A_{22}^{-1} B_2 \end{aligned} \tag{3.8}$$

The fast subsystem for the approximate fast variable $x_f(t) = x_2(t) - x_{2s}(t)$ is defined as, [**?**].

$$\begin{aligned} \dot{x}_{2f}(\tau) &= A_{22} x_{2f}(\tau) + B_2 u_f(\tau),\ x_{2f}(t_0) = x_2^0 + A_{22}^{-1} A_{21} x_1^0 \\ y_f(\tau) &= C_2 x_{2f}(\tau) \end{aligned} \tag{3.9}$$

where $\tau = \frac{(t-t_0)}{\epsilon}$ represents the stretched time scale (fast time scale).

3.2.1 Chang Transformation

This is a similarity transformation defined by [15]

$$T_c = \begin{bmatrix} I_n & \epsilon H \\ -L & I_m - \epsilon L H \end{bmatrix},\ T_c^{-1} = \begin{bmatrix} I_n - \epsilon H L & -\epsilon H \\ L & I_m \end{bmatrix} \tag{3.10}$$

It is used to relate the original state variables $x_1(t)$ and $x_2(t)$ and $x_s(t)$ and $x_f(t)$- the pure slow and pure fast state variables.

$$\begin{bmatrix} x_s(t) \\ x_f(t) \end{bmatrix} = \begin{bmatrix} I_n - \epsilon HL & -\epsilon H \\ L & I_m \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = T_c^{-1} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{3.11}$$

The original state variables can be reconstructed from

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} I_n & \epsilon H \\ -L & I_m - \epsilon LH \end{bmatrix} \begin{bmatrix} x_s(t) \\ x_f(t) \end{bmatrix} = T_c \begin{bmatrix} x_s(t) \\ x_f(t) \end{bmatrix} \tag{3.12}$$

The matrices L and H satisfy the following algebraic equations

$$\begin{aligned} 0 &= \epsilon L(A_{11} - A_{12}L) + (A_{21} - A_{22}L) \\ 0 &= \epsilon(A_{11} - A_{12}L)H + A_{12} - H(A_{22} + \epsilon LA_{12}) \end{aligned} \tag{3.13}$$

The solutions for L and H can be obtained using either fixed-point iterations or Newton method, or eigenvector method [**?**]. Using the Chang transformation (3.10), the original system (3.1) becomes

$$\begin{aligned} \dot{x}_s(t) &= A_s x_s(t) + B_s u(t) \\ \epsilon \dot{x}_f(t) &= A_f x_f(t) + B_f u(t) \\ y(t) &= C_s x_s(t) + C_f x_f(t) + Du(t) \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} &A_s = A_{11} - A_{12}L,\ B_s = B_1(I_n - \epsilon HL) - \epsilon HB_2 \\ &A_f = A_{22} + \epsilon LA_{12},\ B_f = LB_1 + I_m B_2 \\ &C_s = C_1 - C_2 L,\ C_f = \epsilon C_1 H + C_2(I_m - \epsilon LH) \end{aligned} \tag{3.15}$$

It is known from [11], that the unique solutions of the L and H equations exist for sufficiently small ϵ under Assumption 1.

3.2.2 Linear Observers for Singularly Perturbed Systems

Consider a general linear time invariant system.

$$\begin{aligned} \dot{x}(t) &= Ax(t) \\ y(t) &= Cx(t) \end{aligned} \tag{3.16}$$

The design of linear observers for linear systems (3.16) requires the following assumption, [16].

Assumption 3.2.2. *The pair (A, C) is observable.*

The full-order observer for the singularly perturbed system (whose system and output variables are defined in (3.1)) is given by

$$\begin{aligned} &\begin{bmatrix} \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ \frac{1}{\epsilon}A_{21} & \frac{1}{\epsilon}A_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} K_1(t) \\ \frac{1}{\epsilon}K_2(t) \end{bmatrix} (y(t) - \hat{y}(t)) = A\hat{x}(t) + K(y(t) - \hat{y}(t)) \\ &= (A - KC)\hat{x}(t) + Ky(t) \\ &\hat{y}(t) = C_1\hat{x}_1(t) + C_2\hat{x}_2(t) = C\hat{x}(t) \end{aligned} \tag{3.17}$$

where $\hat{x}_1(t)$ and $\hat{x}_2(t)$ are estimates of the state variables $x_1(t)$ and $x_2(t)$. The state estimation (observation) error is defined as

$$e(t) = \hat{x}(t) - x(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) - \hat{x}_1(t) \\ x_2(t) - \hat{x}_2(t) \end{bmatrix} \tag{3.18}$$

The role of the observer (3.17) is to reconstruct the state $x(t)$ in an asymptotic manner in the sense that $\lim_{t\to\infty} e(t) = 0$. To achieve that goal, the observer gains K_1 and K_2 must be chosen to make the observer (3.17) asymptotically stable.

3.2.3 Observer-based Controllers for Singularly Perturbed Linear Systems

A observer-based feedback controller for the singularly perturbed system (3.1) is given by

$$u(t) = -F\hat{x}(t) \tag{3.19}$$

where $\hat{x}(t)$ is obtained from (3.17). The overall closed-loop system for the original system (3.1) and the observer (3.17) is given by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A - BF & BF \\ 0 & A - KC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{3.20}$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \frac{1}{\epsilon} A_{21} & \frac{1}{\epsilon} A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \frac{1}{\epsilon} B_2 \end{bmatrix}, \quad K = \begin{bmatrix} K_1 \\ \frac{1}{\epsilon} K_2 \end{bmatrix}, \quad F = \begin{bmatrix} F_1 & F_2 \end{bmatrix} \tag{3.21}$$

In many applications, it is required that the controller (3.19) produces asymptotic stabilization in the sense that

$$\lim_{t \to \infty} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = 0 \tag{3.22}$$

This may be achieved if and only if the system (3.1) is stabilizable by feedback (3.19) and the observer reconstruction error system is asymptotically stable

$$\dot{e}(t) = (A - KC)e(t), \ \ e(t_0) = \hat{x}(t_0) - x(t_0) \tag{3.23}$$

Assumption 2 produces the required condition for (3.23). For the system feedback stabilization, we need the next assumption, [16].

Assumption 3.2.3. *The pair (A, B) is stabilizable.*

3.3 Two-Stage Eigenvalue Assignment for Singularly Perturbed Linear Systems

This section reviews the exact assignment of both slow and fast eigenvalues via state feedback by following the two stage design of [32], [33]. This result will be used in Sections 4 and 5 for the exact design of pure-slow and pure-fast observers and corresponding observer driven controllers. The original singularly perturbed linear system given by (3.1) can be decoupled into the slow-fast form defined in (3.14) using the Chang transformation (3.10)-(3.13).

Suppose, we want to place n_1 slow open-loop eigenvalues and get n_1 new closed eigenvalues. We take the input for the slow subsystem as

$$u(t) = v(t) - F_s x_s(t) \tag{3.24}$$

Substituting (3.24) in (3.14), the following equations are obtained

$$\begin{aligned} \dot{x}_s(t) &= (A_s - B_s F_s) x_s(t) + B_s v(t) \\ \epsilon \dot{x}_f(t) &= A_f x_f(t) - B_f F_s x_s(t) + B_f v(t) \end{aligned} \tag{3.25}$$

The feedback gain F_s is chosen such to place slow eigenvalues at the desired locations, that is

$$\lambda(A_s - B_s F_s) = \lambda_s^{desired} \tag{3.26}$$

The eigenvalues of the slow subsystem can be arbitrarily located under the following assumption.

Assumption 3.3.1. *The pair (A_s, B_s) is controllable.*

The fast subsystem has now a term with the slow states $\frac{1}{\epsilon} B_f F_s x_s(t)$. The second transformation is needed to remove the slow term in $\dot{x}_f(t)$ as

$$x_{fnew}(t) = P x_s(t) + x_f(t) \tag{3.27}$$

where

$$\epsilon P(A_s - B_s F_s) - B_f F_s - A_f P = 0 \;\Rightarrow\; P^0 = A_f^{-1} B_f F_s + O(\epsilon) \tag{3.28}$$

Since $A_f = A_{22} + O(\epsilon)$, the algebraic Sylvester equation (3.28) has a unique solution for sufficiently small values of ϵ assuming that Assumption 1 is satisfied. The change of variables (3.27) leads to

$$\begin{aligned} &\dot{x}_{fnew}(t) = P\dot{x}_s(t) + \dot{x}_f(t) \\ &= [P(A_s - B_s F_s) - \frac{1}{\epsilon} B_f F_s - \frac{1}{\epsilon} A_f P] x_s(t) + \frac{1}{\epsilon} A_f x_{fnew}(t) + P B_s v(t) + \frac{1}{\epsilon} B_f v(t) \end{aligned} \tag{3.29}$$

Therefore, if the Sylvester equation (3.28) is satisfied, (3.29) becomes

$$\epsilon \dot{x}_{fnew}(t) = A_f x_{fnew}(t) + (B_f + \epsilon P B_s) v(t) = A_f x_{fnew}(t) + B_{fnew} v(t) \tag{3.30}$$

where $B_{fnew} = B_f + \epsilon P B_s$. The input $v(t)$ can be used to assign the fast subsystem eigenvalues independently of the slow subsystem. The input $v(t)$ can be taken as

$$v(t) = -F_{f2} x_{fnew}(t) \tag{3.31}$$

To locate the fast subsystem eigenvalues arbitrarily, we need the following assumption.

Assumption 3.3.2. *The pair (A_f, B_{fnew}) is controllable.*

This two stage method facilitates the independent slow and fast eigenvalue assignments. After obtaining the gains F_s and F_{f2}, we go back to the original coordinates to find the corresponding gains. The control input in the new coordinates is given by

$$u(t) = v(t) - F_s x_s(t) = -F_{f2} x_{fnew}(t) - F_s x_s(t) \tag{3.32}$$

Using the Chang transformation (3.10), the state feedback gains F_1 and F_2 in the original coordinates are obtained as follows [14], [32], [33].

$$u(t) = -Fx(t) = -\begin{bmatrix} F_s & F_{f2} \end{bmatrix} \begin{bmatrix} x_s(t) \\ x_{fnew}(t) \end{bmatrix} = -\begin{bmatrix} F_s + F_{f2} P & F_{f2} \end{bmatrix} T_c^{-1} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{3.33}$$

3.4 Two-Stage Observer Design for Singularly Perturbed Systems

The objective of this section is to present a new design technique for a full-order observer in slow and fast time scales. A full-order observer will be designed is Section 4.1 using the two-stage design. In Section 4.2, it is shown how to find the observer gain in the original coordinates. Observer design algorithm is formulated in Section 4.3. Numerical example is presented in Section 4.4.

3.4.1 Two-Stage Two-Time Scale Design of the Full-Order Observer

A full-order observer for the singularly perturbed system (3.1) is defined in (3.17). For simplicity of the design, we will assume that no control input is presented, that is $u(t) = 0$. When developing the two-stage method to a full-order observer we will start the duality between the controller and observer design. The duality says that $\dot{x}(t) = (A - BF)x(t)$ and $\dot{\hat{x}}(t) = (A - KC)\hat{x}(t) + Ky(t)$ can use the same procedure for the design of feedback matrices since transposing the observer feedback matrix, that is, $A^T - C^T K^T$ produces the dual form to the system feedback matrix $A - BF$.

Hence, it will be needed to transpose matrices A and KC of the full-order observer (3.17) and consider a hypothetical control system

$$\begin{aligned}
\dot{z}_1(t) &= A_{11}^T z_1(t) + \frac{1}{\epsilon} A_{21}^T z_2(t) + C_1^T \hat{u}(t) \\
\dot{z}_2(t) &= A_{12}^T z_1(t) + \frac{1}{\epsilon} A_{22}^T z_2(t) + C_2^T \hat{u}(t)
\end{aligned} \tag{3.34}$$

where states $z_1(t)$ and $z_2(t)$ are used for the purpose of design only. To transform (3.34) into an explicit singularly perturbed form we introduce $q_1(t) = z_1(t)$ and $q_2(t) = \frac{1}{\epsilon} z_2(t)$ which leads to

$$\begin{aligned}
\dot{q}_1(t) &= A_{11}^T q_1(t) + A_{21}^T q_2(t) + C_1^T \hat{u}(t) \\
\epsilon \dot{q}_2(t) &= A_{12}^T q_1(t) + A_{22}^T q_2(t) + \epsilon C_2^T \hat{u}(t)
\end{aligned} \tag{3.35}$$

The Chang transformation applied to (3.35) produces

$$\begin{aligned}
\dot{q}_s(t) &= A_{sq}^T q_s(t) + C_{sq}^T \hat{u}(t) \\
\epsilon \dot{q}_f(t) &= A_{fq}^T q_f(t) + \epsilon C_{fq}^T \hat{u}(t)
\end{aligned} \tag{3.36}$$

where

$$\begin{aligned} A_{sq}^T &= A_{11}^T - L^T A_{12}^T,\ A_{fq}^T = A_{22}^T + \epsilon A_{12}^T L^T \\ C_{sq}^T &= C_1^T - L^T C_2^T,\ C_{fq}^T = H^T C_1^T + \frac{1}{\epsilon}(I_m - \epsilon H^T L^T) C_2^T \end{aligned} \tag{3.37}$$

The Chang transformation needed for the proposed observer design relates the original state variables $q_1(t)$ and $q_2(t)$ and the slow and fast variables $q_s(t)$ and $q_f(t)$ as follows

$$\begin{bmatrix} q_s(t) \\ q_f(t) \end{bmatrix} = \begin{bmatrix} I_n & -\epsilon L^T \\ H^T & I_m - \epsilon H^T L^T \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = T_{cq}^T \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} \tag{3.38}$$

The state variables $q_1(t)$ and $q_2(t)$ can be reconstructed from the inverse transformation as

$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} I_n - \epsilon L^T H^T & \epsilon L^T \\ -H^T & I_m \end{bmatrix} \begin{bmatrix} q_s(t) \\ q_f(t) \end{bmatrix} = T_{cq}^{-T} \begin{bmatrix} q_s(t) \\ q_f(t) \end{bmatrix} \tag{3.39}$$

where L^T and H^T are the transposed solution of (3.13). The duality between the controller and observer designs requires that the controller gain, obtained from $\lambda(A - BF)$, is the same as the observer gain obtained from $\lambda(A^T - C^T K^T)$. The duality between the controller and observer designs means $A \to A^T, B \to C^T, F \to K^T$. From the sub-matrix point of view corresponding to slow and fast subsystems, the duality implies

$$\begin{aligned} &A_{11} \to A_{11}^T, A_{12} \to \frac{1}{\epsilon}A_{21}^T, \frac{1}{\epsilon}A_{21} \to A_{12}^T, A_{22} \to A_{22}^T, B_1 \to C_1^T, \frac{1}{\epsilon}B_2 \to C_2^T, F_1 \to K_1^T, \\ &F_2 \to \frac{1}{\epsilon}K_2^T, A_s \to A_{sq}^T, \frac{1}{\epsilon}A_f \to \frac{1}{\epsilon}A_{fq}^T, B_s \to C_{sq}^T, \frac{1}{\epsilon}B_f \to C_{fq}^T, F_s \to K_s^T, F_f \to \frac{1}{\epsilon}K_f^T \end{aligned} \tag{3.40}$$

The goal is to find the observer gain K using the two stage feedback design from Section 3.

We take $\hat{u}(t)$ for the slow subsystem as

$$\hat{u}(t) = -K_s^T q_s(t) + v(t) \tag{3.41}$$

Substituting (3.41) into (3.36), we have

$$\begin{aligned}\dot{q}_s(t) &= (A_{sq}^T - C_{sq}^T K_s^T)q_s(t) + C_{sq}^T v(t)\\ \epsilon\dot{q}_f(t) &= A_{fq}^T q_f(t) - \epsilon C_{fq}^T K_s^T q_s(t) + \epsilon C_{fq}^T v(t)\end{aligned} \tag{3.42}$$

At this point, it is possible to place the slow observer eigenvalues in the desired locations, that is

$$\lambda(A_{sq}^T - C_{sq}^T K_s^T) = \lambda(A_{sq} - K_s C_{sq}) = \lambda_s^{desired} \tag{3.43}$$

assuming that the following assumption is satisfied.

Assumption 3.4.1. *The pair* (A_{sq}, C_{sq}) *is observable.*

Now, the following change of coordinates is introduced

$$q_{fnew}(t) = P_o q_s(t) + q_f(t) \;\rightarrow\; q_f(t) = q_{fnew}(t) - P_o q_s(t) \tag{3.44}$$

where P_o satisfies the algebraic Sylvester equation

$$\epsilon P_o(A_{sq}^T - C_{sq}^T K_s^T) - \epsilon C_{fq}^T K_s^T - A_{fq}^T P_o = 0 \;\Rightarrow\; P_o = O(\epsilon) \tag{3.45}$$

The unique solution for P_o exist for sufficiently small values of ϵ under Assumption 1. By setting $\epsilon = 0$ in (3.45) we see that $A_{fq}^T P_o^{(0)} = 0 \Rightarrow P_o^{(0)} = 0$ and $P_o = O(\epsilon)$[1]. The change of variables in (3.44) results in

$$\begin{aligned}&\epsilon\dot{q}_{fnew}(t) = \epsilon P_o \dot{q}_s(t) + \epsilon\dot{q}_f(t)\\ &= [-A_{fq}^T P_o - \epsilon C_{fq}^T K_s^T + \epsilon P_o(A_{sq}^T - C_{sq}^T K_s^T)]q_s(t) + A_{fq}^T q_{fnew}(t) + \epsilon(C_{fq}^T + P_o C_{sq}^T)v(t)\end{aligned} \tag{3.46}$$

When the Sylvester equation (3.45) is satisfied, (3.46) becomes

$$\epsilon\dot{q}_{fnew}(t) = A_{fq}^T q_{fnew}(t) + \epsilon(C_{fq}^T + P_o C_{sq}^T)v(t) = A_{fq}^T q_{fnew}(t) + \epsilon C_{fnew}^T v(t) \tag{3.47}$$

where $C_{fnew}^T = C_{fq}^T + P_o C_{sq}^T$. The input $v(t)$ can be used to locate the fast subsystem

[1]An $O(\epsilon)$ is defined by $O(\epsilon) \leq k\epsilon$, where k is a bounded constant.

eigenvalues

$$v(t) = -K_{f2}^T q_{fnew}(t) \tag{3.48}$$

At this point, it is possible to locate the fast eigenvalues in the original coordinates at the desired location as

$$\lambda(A_{fq} - K_{f2}C_{fnew}) = \lambda_f^{desired} \tag{3.49}$$

If the following observability assumption is satisfied.

Assumption 3.4.2. *The pair (A_{fq}, C_{fnew}) is observable.*

Substituting (3.41) and (3.48) into (3.42) and (3.47), we obtain

$$\begin{bmatrix} \dot{q}_s(t) \\ \epsilon\dot{q}_{fnew}(t) \end{bmatrix} = \begin{bmatrix} (A_{sq} - K_sC_{sq})^T & -(K_{f2}C_{sq})^T \\ 0 & (A_{fq} - K_{f2}C_{fnew})^T \end{bmatrix} \begin{bmatrix} q_s(t) \\ q_{fnew}(t) \end{bmatrix} \tag{3.50}$$

The original coordinates $z_1(t), z_2(t)$ and $q_s(t), q_{fnew}(t)$ coordinates are related via

$$\begin{bmatrix} q_s(t) \\ q_{fnew}(t) \end{bmatrix} = T_2^T T_{cq}^T T_1^T \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \tag{3.51}$$

where

$$T_1^T = \begin{bmatrix} I_n & 0 \\ 0 & \frac{1}{\epsilon}I_m \end{bmatrix}, \; T_2^T = \begin{bmatrix} I_n & 0 \\ P_o & I_m \end{bmatrix} \tag{3.52}$$

with T_{cq}^T defined in (3.38). It is possible to reconstruct $z_1(t)$, $z_2(t)$ from $q_s(t)$, $q_{fnew}(t)$ via the inverse transformation

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = T_1^{-T} T_{cq}^{-T} T_2^{-T} \begin{bmatrix} q_s(t) \\ q_{fnew}(t) \end{bmatrix} = T_4^{-T} \begin{bmatrix} q_s(t) \\ q_{fnew}(t) \end{bmatrix} \tag{3.53}$$

From the above relation (3.53), we can construct the state transformation from $x_s(t)$, $x_{fnew}(t)$ to $x_1(t)$, $x_2(t)$ as follows

$$\begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} = T_4 \begin{bmatrix} \hat{x}_s(t) \\ \hat{x}_{fnew}(t) \end{bmatrix} \tag{3.54}$$

Applying above the state transformation (3.54) to (3.17), we get

$$
\begin{aligned}
T_4 \begin{bmatrix} \dot{\hat{x}}_s(t) \\ \dot{\hat{x}}_{fnew}(t) \end{bmatrix} &= (A - KC)T_4 \begin{bmatrix} \hat{x}_s(t) \\ \hat{x}_{fnew}(t) \end{bmatrix} + Ky(t) \\
\begin{bmatrix} \dot{\hat{x}}_s(t) \\ \dot{\hat{x}}_{fnew}(t) \end{bmatrix} &= T_4^{-1}(A - KC)T_4 \begin{bmatrix} \hat{x}_s(t) \\ \hat{x}_{fnew}(t) \end{bmatrix} + T_4^{-1}Ky(t)
\end{aligned}
\tag{3.55}
$$

Now we can present the observer configuration using the result in (3.50) and the duality between controller and observer designs

$$
\begin{bmatrix} \dot{\hat{x}}_s(t) \\ \epsilon\dot{\hat{x}}_{fnew}(t) \end{bmatrix} = \begin{bmatrix} A_{sq} - K_s C_{sq} & 0 \\ -\epsilon K_{f2} C_{sq} & A_{fq} - K_{f2} C_{fnew} \end{bmatrix} \begin{bmatrix} \hat{x}_s(t) \\ \hat{x}_{fnew}(t) \end{bmatrix} + \begin{bmatrix} K_s \\ \epsilon K_{f2} \end{bmatrix} y(t) \tag{3.56}
$$

where

$$
K_s = (I_n - \epsilon HL - \epsilon P_o^T L)K_1 - HK_2 - P_o^T K_2, \;\; K_{f2} = \epsilon L K_1 + \epsilon K_2 \tag{3.57}
$$

The block diagram for the sequential reduced-order observer is presented in Figure 3.1. The observer obtained in (3.56) has a sequential structure. The slow observer

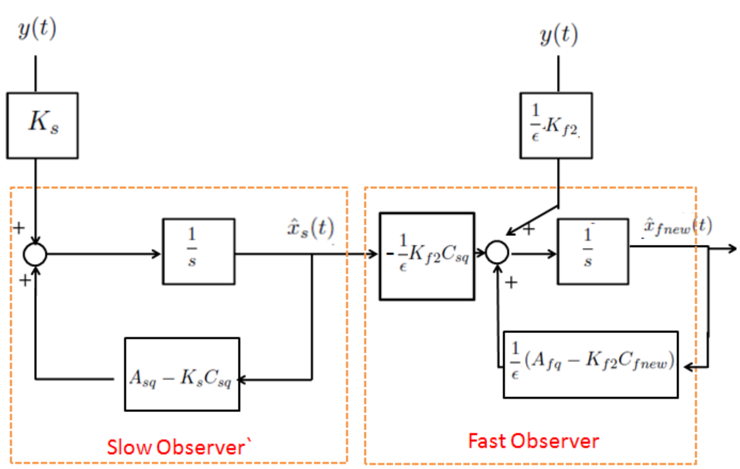

Figure 3.1: Sequential reduced-order slow and fast observers.

is independent of the fast and it is used to drive the fast observer. We can obtain a fully decoupled slow and fast observers working in parallel as follows. We change the coordinates once again given as

$$\hat{x}_{fnew2}(t) = P_{o2}\hat{x}_s(t) + \hat{x}_{fnew}(t) \;\rightarrow\; \hat{x}_{fnew}(t) = \hat{x}_{fnew2}(t) - P_{o2}\hat{x}_s(t) \tag{3.58}$$

where P_{o2} satisfies the algebraic Sylvester equation represented by

$$\epsilon P_{o2}(A_{sq} - K_s C_{sq}) - \epsilon K_{f2} C_{sq} - (A_{fq} - K_{f2} C_{fnew}) P_{o2} = 0 \;\Rightarrow\; P_{o2}^0 = O(\epsilon) \tag{3.59}$$

The linear algebraic equation (3.59) has a unique solution since $A_{fq} - K_{f2}C_{fnew}$ is an asymptotically stable fast subsystem feedback matrix. The change of variable (3.58) results in

$$\begin{aligned}
\epsilon\dot{\hat{x}}_{fnew2}(t) &= \epsilon P_{o2}\dot{\hat{x}}_s(t) + \epsilon\dot{\hat{x}}_{fnew}(t) \\
&= [\epsilon P_{o2}(A_{sq} - K_s C_{sq}) - \epsilon K_{f2} C_{sq} - (A_{fq} - K_{f2} C_{fnew}) P_{o2}]\hat{x}_s(t) \\
&+ (A_{fq} - K_{f2} C_{fnew})\hat{x}_{fnew2}(t) + K_{f3} y(t)
\end{aligned} \tag{3.60}$$

where

$$K_{f3} = \epsilon(P_{o2}K_s + K_{f2}) \tag{3.61}$$

Hence, if the second algebraic Sylvester equation (3.59) is satisfied, (3.60) becomes

$$\epsilon\dot{\hat{x}}_{fnew2}(t) = (A_{fq} - K_{f2}C_{fnew})\hat{x}_{fnew2}(t) + K_{f3}y(t) \tag{3.62}$$

At this point, we have the block-diagonalized form of the observer obtained as

$$\begin{aligned}
\dot{\hat{x}}_s(t) &= (A_{sq} - K_s C_{sq})\hat{x}_s(t) + K_s y(t) \\
\epsilon\dot{\hat{x}}_{fnew2}(t) &= (A_{fq} - K_{f2}C_{fnew})\hat{x}_{fnew2}(t) + K_{f3}y(t)
\end{aligned} \tag{3.63}$$

The original coordinates $\hat{x}_1(t)$, $\hat{x}_2(t)$ and the new coordinates $\hat{x}_s(t)$, $\hat{x}_{fnew2}(t)$ are related via

$$\begin{bmatrix} \hat{x}_s(t) \\ \hat{x}_{fnew2}(t) \end{bmatrix} = T_3 T_4^{-1} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} = T^{-1} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} \tag{3.64}$$

where

$$T_3 = \begin{bmatrix} I & 0 \\ P_{o2} & I \end{bmatrix} \tag{3.65}$$

Now, the original coordinates can be reconstructed via

$$\begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} = T_4 T_3^{-1} \begin{bmatrix} \hat{x}_s(t) \\ \hat{x}_{fnew2}(t) \end{bmatrix} = T \begin{bmatrix} \hat{x}_s(t) \\ \hat{x}_{fnew2}(t) \end{bmatrix} \tag{3.66}$$

In (3.63), we have the parallel slow and fast observer structure that is graphically represented in Figure 4.12. A summary of all matrices appearing in Figure 4.12 and equations used to obtain them is presented in Table 1.

Table 3.1: Parameters of Slow and Fast subsystems

Slow data	Fast Data
$A_{sq} = A_{11} - A_{12}L$	$A_{fq} = A_{22} + \epsilon L A_{12}$
$C_{sq} = C_1 - C_2 L$	$C_{fnew} = \epsilon(C_1 - C_2 L)P_o^T + \epsilon C_1 H$
	$+C_2(I_m - \epsilon LH)$
$\lambda(A_{sq} - K_s C_{sq}) = \lambda_s^{desired} \Rightarrow K_s$	$\lambda(A_{fq} - K_{f2} C_{fnew}) = \lambda_f^{desired} \Rightarrow K_{f2}$
	$K_{f3} = \epsilon(P_{o2} K_s + K_{f2})$
$\epsilon P_o(A_{sq}^T - C_{sq}^T K_s^T) - \epsilon C_{fq}^T K_s^T - A_{fq}^T P_o$	$\epsilon P_{o2}(A_{sq} - K_s C_{sq}) - \epsilon K_{f2} C_{sq}$
$= 0$	$-(A_{fq} - K_{f2} C_{fnew}) P_{o2} = 0$

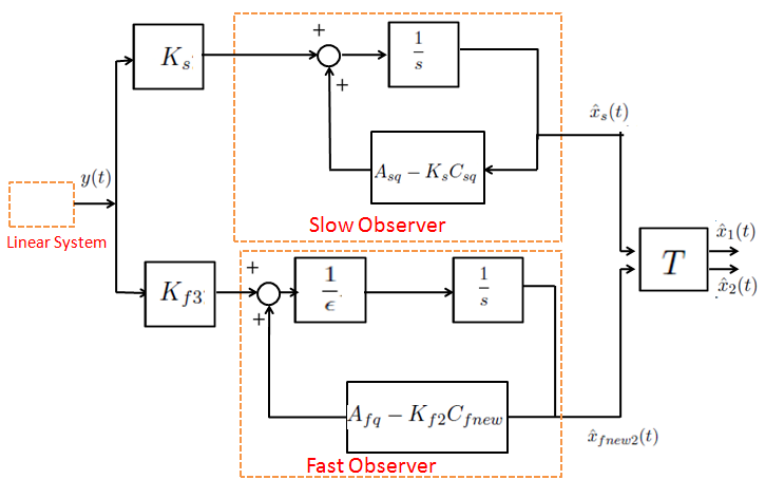

Figure 3.2: Slow-fast reduced-order parallel estimation (observation) with the reduced-order observers of dimensions $n_1 \times n_1$ and $n_2 \times n_2$, $n_1 + n_2 = n$, $n =$ order of the system.

3.4.2 Observation Error Equations

From (3.16)-(3.18), we have

$$\begin{bmatrix} \dot{e}_1(t) \\ \epsilon\dot{e}_2(t) \end{bmatrix} = (A - KC) \begin{bmatrix} \hat{e}_1(t) \\ \hat{e}_2(t) \end{bmatrix} \tag{3.67}$$

Using the state transformation (3.66), the original error coordinates $e_1(t), e_2(t)$ and the new error coordinates $e_s(t), e_{fnew2}(t)$ are related via

$$\begin{aligned} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} \\ &= T \begin{bmatrix} x_s(t) \\ x_{fnew2}(t) \end{bmatrix} - T \begin{bmatrix} \hat{x}_s(t) \\ \hat{x}_{fnew2}(t) \end{bmatrix} \\ &= T \begin{bmatrix} e_s(t) \\ e_{fnew2}(t) \end{bmatrix} \end{aligned} \tag{3.68}$$

From (3.67) and (3.68), we obtain

$$T^{-1} \begin{bmatrix} \dot{e}_1(t) \\ \epsilon\dot{e}_2(t) \end{bmatrix} = T^{-1}(A - KC)T \begin{bmatrix} \hat{e}_s(t) \\ \hat{e}_{fnew2}(t) \end{bmatrix} \tag{3.69}$$

which produces

$$\begin{aligned} \dot{e}_s(t) &= \hat{A}_s e_s(t) \\ \epsilon\dot{e}_{fnew2}(t) &= \hat{A}_f e_{fnew2}(t) \end{aligned} \tag{3.70}$$

$$\begin{aligned} \hat{A}_s &= A_{sq} - K_s C_{sq} \\ \hat{A}_f &= A_{fq} - K_{f2} C_{fnew} \end{aligned} \tag{3.71}$$

The asymptotic convergence of the error dynamic is generated since the eigenvalues satisfy

$$Re\lambda(_s) < 0, \;\; Re\lambda(_f) < 0 \tag{3.72}$$

3.4.3 Observer Gain in the Original Coordinates

We will show that the observer in the original coordinates is given by

$$K = (\begin{bmatrix} K_s^T + K_{f2}^T P_o & K_{f2}^T \end{bmatrix} T_{cq}^T T_1^T)^T = \begin{bmatrix} T_1 T_{cq}(K_s + P_o^T K_{f2}) \\ T_1 T_{cq} K_{f2} \end{bmatrix} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \tag{3.73}$$

where T_{cq} is the Chang transformation (3.38), P_o is the solution of the algebraic Sylvester equation (3.45). We previously set $K^T q(t) = v(t) - K_s^T q_s(t) = -K_s^T q_s(t) - K_{f2}^T q_{fnew}(t)$ in (3.41) and (3.48) , which implies

$$\begin{aligned} K^T q(t) &= \begin{bmatrix} K_s^T & K_{f2}^T \end{bmatrix} \begin{bmatrix} q_s(t) \\ q_{fnew}(t) \end{bmatrix} = \begin{bmatrix} K_s^T & K_{f2}^T \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 \\ P_o & I_{n_2} \end{bmatrix} \begin{bmatrix} q_s(t) \\ q_f(t) \end{bmatrix} \\ &= \begin{bmatrix} K_s^T + K_{f2}^T P_o & K_{f2}^T \end{bmatrix} T_{cq}^T \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} K_s^T + K_{f2}^T P_o & K_{f2}^T \end{bmatrix} T_{cq}^T T_1^T \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \end{aligned} \tag{3.74}$$

Hence $\begin{bmatrix} K_s^T + K_{f2}^T P_o & K_{f2}^T \end{bmatrix} T_{cq}^T T_1^T$ represents transpose of the observer gain matrix K in original coordinates. It is important to notify that the observer gain $K = f(K_s, K_{f2})$ can be obtained using computations with reduced order matrices K_s and K_{f2}.

3.4.4 Design Algorithm for Finding the Observer Gain

Given that the linear system (3.1) is observable, the following two-time scale design algorithm can be applied for the design of a full-order observer of singularly perturbed system.

Step 1. Transpose the matrices of the full-order observer from (3.17) and apply the change of variable to the hypothetical system defined in (3.35).

Step 2. Apply the Chang transformation (3.38) to (3.35) to get (3.36).

Step 3. Obtain the partitioned submatrices A_{sq}^T, $\frac{1}{\epsilon} A_{fq}^T$, C_{sq}^T and C_{fq}^T.

Step 4. Place the slow observer eigenvalues in the desired location and obtain the slow observer gain K_s^T using the eigenvalue assignment for $\lambda(A_{sq} - K_s C_{sq})$.

Step 5. Solve the reduced-order Sylvester algebraic equation (3.45) to get P_o.For the parallel observer structure, solve in addition for P_{o2} from (3.59).

Step 6. Place fast observer eigenvalues at the desired location using the eigenvalue assignment for $\lambda(A_{fq} - K_{f2}C_{fnew})$ and obtain K_{f2}.

Step 7. Find the observer gain in the original coordinates using (3.73) and check $\lambda(A - KC) = \lambda_s^{desired} \cup \lambda_f^{desired}$.

3.4.5 A Numerical Example

Consider a $4^{th}-$ order system with the system matrices A and C taken from [**?**]

$$A = \begin{bmatrix} 0 & 0.4000 & 0 & 0 \\ 0 & 0 & 0.3450 & 0 \\ 0 & -5.2400 & -4.6500 & 2.6200 \\ 0 & 0 & 0 & -10.0000 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}.$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Our goal is to design independently slow and fast reduced-order observers with the desired eigenvalues as $\lambda_s^{desired} = \{-50, -60\}$ and $\lambda_f^{desired} = \{-200, -300\}$. The observability matrix has full rank and therefore the pair (A, C) is observable.

According to Step 3 of Algorithm from Section 4.3, the following sub-matrices are obtained

$$A_{sq}^T = \begin{bmatrix} 0 & 0 \\ 0.4000 & -0.4282 \end{bmatrix}, \quad A_{fq}^T = \begin{bmatrix} -0.4221 & 0 \\ 0.2620 & -1.0000 \end{bmatrix},$$

$$C_{sq}^T = \begin{bmatrix} 1.0000 & 0 \\ 0 & -1.2412 \end{bmatrix}, \quad C_{fq}^T = \begin{bmatrix} 0.0086 & 1.1128 \\ 0.0032 & 0.0308 \end{bmatrix},$$

$$C_{fnew}^T = \begin{bmatrix} 0.0070 & -0.0756 \\ -0.0004 & 0.0563 \end{bmatrix}$$

Following Step 4, we place the slow eigenvalues in the original coordinates at $\{-50, -60\}$ via the slow feedback gain matrix

$$K_s^T = \begin{bmatrix} 50.0000 & 0 \\ -0.3223 & -47.9961 \end{bmatrix}$$

In Step 5 of the algorithm, we solve the Sylvester algebraic equations (3.45) and (3.59) and obtain matrices P_o and P_{o2} as

$$P_o = \begin{bmatrix} -0.0158 & 9.5761 \\ -0.0371 & -0.2052 \end{bmatrix}, \quad P_{o2} = \begin{bmatrix} 20.6930 & -0.2663 \\ 24.2598 & -2.9173 \end{bmatrix}$$

In Step 6 of the algorithm, we place the fast observer eigenvalues at the desired location $\{-200, -300\}$. The fast observer gain K_{f2}^T is given by

$$K_{f2}^T = \begin{bmatrix} 3103.957 & 6064.972 \\ 30.041 & 564.109 \end{bmatrix}, \quad K_{f3} = \begin{bmatrix} 413.861 & 3.615 \\ 727.796 & 69.631 \end{bmatrix}$$

In Step 7, using (3.73), matrix K is obtained as

$$K = \begin{bmatrix} 240.549 & -0.808 \\ 24146.674 & -43.824 \\ 1069.235 & 354.800 \\ 60649.720 & 5641.093 \end{bmatrix}$$

It can be checked that $\lambda(A - KC)$ in the original coordinate are given by

$$\lambda(A - KC) = \begin{bmatrix} -50.0000000000000 \\ -60.0000000000000 \\ -200.0000000000000 \\ -300.0000000000002 \end{bmatrix}$$

which is the same (with the accuracy of $O(10^{-14})$) as we placed the slow and fast eigenvalues using the two time scale decomposition designs. Figures 3.3 present the slow and fast observation errors. The observer initial conditions were chosen as $\hat{x}_s(0) = [1, 5]$ and $\hat{x}_{fnew2}(0) = [1, 3]$. In order to be able to run MATLAB simulink simulation we had to specify also the system initial conditions (these initial conditions are in general not known). We have chosen them as $\hat{x}_1(0) = [2, 2]$ and $\hat{x}_2(0) = [2, 2]$.

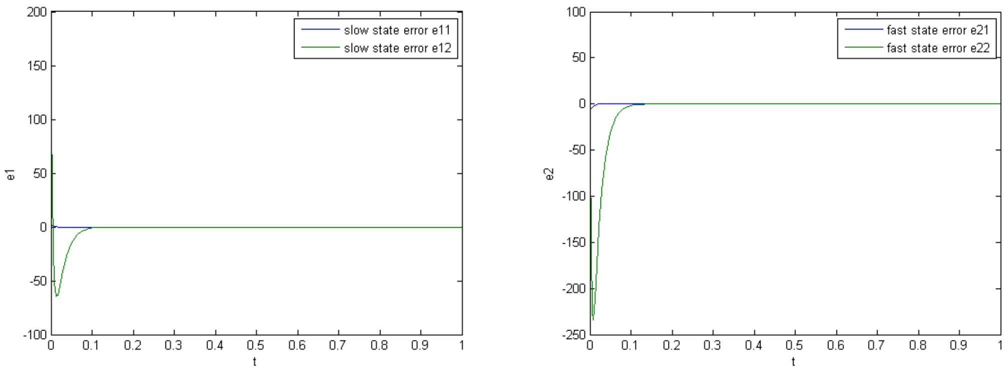

Figure 3.3: Convergence of the slow states $x_1(t) \in \Re^2$ and the fast states $x_2(t) \in \Re^2$

3.5 Slow and Fast Observer-based Controller Design for Singularly Perturbed Systems

In the previous section, we have accurately observed the states of the original system using independent the reduced-order slow and fast observers (3.63). In this section, we use these observers and consider the observer-based controller design for singularly perturbed linear systems. The observers are driven by the system measurements and control inputs with both observers implemented independently in the slow and fast time scales

$$\begin{aligned} &\dot{\hat{x}}_s(t) = (A_{sq} - K_s C_{sq})\hat{x}_s(t) + B_{s2}u(t) + K_s y(t) \\ &\epsilon \dot{\hat{x}}_{fnew2}(t) = (A_{fq} - K_{f2} C_{fnew})\hat{x}_{fnew2}(t) + B_{f2}u(t) \\ &+ K_{f3} y(t) \end{aligned} \tag{3.75}$$

where B_{s2}, B_{f2} are obtained from $T^{-1}B$ with T defined in (3.66), that is

$$\begin{aligned} &B_{s2} = (I_n - \epsilon HL)B_1 - \epsilon P_o^T L B_1 - H B_2 - P_o^T B_2, \\ &B_{f2} = \epsilon P_{o2}(I_n - \epsilon HL)B_1 - \epsilon^2 P_{o2} P_o^T L B_1 + \epsilon^2 L B_1 \\ &- \epsilon P_{o2} H B_2 - \epsilon P_{o2} P_o^T B_2 + \epsilon B_2 \end{aligned} \tag{3.76}$$

The control input in the $\hat{x}_s$-$\hat{x}_{fnew2}$ coordinates is given by

$$
\begin{aligned}
u(t) &= -F\hat{x}(t) = -\begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} \\
&= -\begin{bmatrix} F_1 & F_2 \end{bmatrix} T \begin{bmatrix} \hat{x}_s(t) \\ \hat{x}_{fnew2}(t) \end{bmatrix} \\
&= -\begin{bmatrix} F_{s2} & F_{f2} \end{bmatrix} \begin{bmatrix} \hat{x}_s(t) \\ \hat{x}_{fnew2}(t) \end{bmatrix} \\
&= -F_{s2}\hat{x}_s(t) - F_{f2}\hat{x}_{fnew2}(t)
\end{aligned}
\tag{3.77}
$$

$$
\begin{aligned}
F_{s2} &= F_1(I_n - P_o^T P_{o2} - HP_{o2}) - F_2L(I_n - P_o^T P_{o2}) \\
&- \frac{1}{\epsilon}F_2(I_m - \epsilon LH)P_{o2} \\
F_{f2} &= F_1(P_o^T + H) - F_2LP_o^T + \frac{1}{\epsilon}F_2(I_m - \epsilon LH)
\end{aligned}
\tag{3.78}
$$

The corresponding block diagram for the observer driven controller is presented in Figure 3.4. This block diagram clearly indicates full parallelism of the slow controller driven by the slow observer and the fast controller driven by the fast observer.

3.5.1 Numerical Example

Consider a $4^{th}-$ order system with the system matrices A, B and C defined in Section 4.C. The controllability matrix has full rank and therefore the pair (A, B) is controllable. The results obtained using MATLAB are given below. We locate the feedback system slow eigenvalues at $\lambda_{cs}^{desired} = (-2, -3)$ and the feedback system fast eigenvalues at $\lambda_{cf}^{desired} = (-7, -8)$, and the slow observer eigenvalues at $\lambda_{os}^{desired} = (-50, -60)$ and the fast observer eigenvalues at $\lambda_{of}^{desired} = (-200, -300)$, given in the previous numerical example. Following the design procedure of from Sections 4 and 5, the completely decoupled slow and fast observer in the x_s-x_{fnew2} coordinates, driven by the system measurements and control inputs, are

$$\dot{\hat{x}}_s(t) = \begin{bmatrix} -50.0000 & 0.0000 \\ 0.0000 & -60.0000 \end{bmatrix} \hat{x}_s(t)$$
$$+ \begin{bmatrix} 0.0046 \\ 0.4541 \end{bmatrix} u(t) + \begin{bmatrix} 50.0000 & -0.3223 \\ 0 & -47.9961 \end{bmatrix} y(t)$$

$$\dot{\hat{x}}_{fnew2}(t) = \begin{bmatrix} -200.0000 & 0.0000 \\ -0.0000 & -300.000 \end{bmatrix} \hat{x}_{fnew2}(t)$$
$$+ \begin{bmatrix} -0.0255 \\ -0.2129 \end{bmatrix} u(t) + \begin{bmatrix} 4138.6098 & 36.1545 \\ 7277.9664 & 696.3118 \end{bmatrix} y(t)$$

$$u(t) = -\begin{bmatrix} -6530.3242 & 91.6868 \end{bmatrix} \hat{x}_s(t)$$
$$- \begin{bmatrix} 332.8512 & -10.9008 \end{bmatrix} \hat{x}_{fnew2}(t)$$

The slow and fast controller gains F_{s2}, F_{f2} are obtained as

$$F_{s2} = \begin{bmatrix} -6530.3242 & 91.6868 \end{bmatrix},$$
$$F_{f2} = \begin{bmatrix} 332.8512 & -10.9008 \end{bmatrix}$$

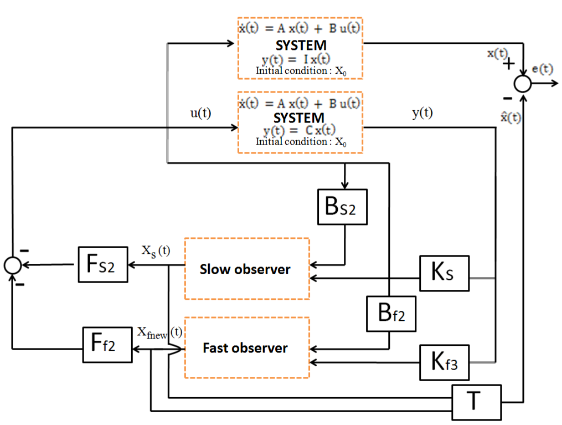

Figure 3.4: Complete parallelism and exact decomposition of the observer-based controller for singularly perturbed linear systems

3.6 Conclusions

We have designed with very high accuracy the pure-slow and pure-fast observer-based controllers. They are designed independently using the reduced-order slow and fast sub-matrices. The numerical ill-conditioning problem of the original system is removed. We have demonstrated that the full-order singularly perturbed system can be successfully controlled with the state feedback controllers designed on the subsystem levels. The two stage method is successfully implemented for both observer and controller designs. The main result of Chapter 3 have been submitted for publication in IEEE Transactions on Automatic Control [53]

Chapter 4

New Designs of Reduced-Order Observers for Singularly Perturbed Linear Systems

In Chapter 3, we have designed slow and fast full-order observers and observer-based controllers by placing eigenvalues using the two-stage feedback design for slow and fast subproblems. The numerically ill-conditioning problem is avoided using the two stage design method for singularly perturbed linear systems so that independent feedback controllers can be applied to each sub-system. We have demonstrated that the singularly perturbed system can be successfully controlled via the eigenvalue placement technique with the state feedback controllers and the full-order observers designed at the subsystem levels. The two stage method is successfully implemented for both the full-order observer and corresponding controller designs. In this chapter, we will consider the problem studied in Chapter 3, but using the reduced-order observers. The reduced-order observer for singularly perturbed systems have been studied only in a few papers [54]-[55], all of them producing accuracy of $O(\epsilon)$. The approach presented in this chapter will produce $O(\epsilon^k),\ k = 2, 3, ...$ accuracy, which for large k practically mean the exact accuracy.

4.1 Two-Stage Reduced-Order Observer Design for Singularly Perturbed Linear Systems

We extend the two-stage feedback design method to both the Sylvester equation based and the Luenberger reduced-order observers for systems that contain slow and fast modes [11], [16]. Consider a singularly perturbed linear system

$$\begin{aligned}\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t)\\ \epsilon\dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t)\end{aligned} \tag{4.1}$$

with the corresponding measurements

$$y(t) = Cx(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{4.2}$$

where $x_1(t) \in R^{n_1}, x_2(t) \in R^{n_2}, n_1 + n_2 = n, y(t) \in R^l$. We assume that the matrix C has full rank. The following assumption is standard in the theory of singularly perturbed linear systems [11].

Assumption 4.1.1. *Matrix A_{22} is nonsingular.*

The reduced-order observer for singularly perturbed linear systems will be considered for the following cases :

Case I. All slow variables are measured only, $\dim \mathrm{y(t)} = \mathrm{n}_1$,

$y(t) = x_1(t) \Rightarrow$ need only reduced order observer for $\epsilon \hat{x}_2(t)$

Case II. All fast variables are measured only, $\dim y(t) = n_2$,

$y(t) = x_2(t) \Rightarrow$ need only reduced order observer for $\epsilon \hat{x}_1(t)$

Case III. Only part of the slow variables vector is measured,

$l < n_1 < n \Rightarrow n - l > n_1 \Rightarrow y(t) = x_{11}(t),\ x_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix}$

$\Rightarrow$ need only reduced order observer for $\hat{x}_{12}(t), \epsilon \hat{x}_2(t)$

Case IV. Only a part of the fast variable vector is measured,

$l < n_2 < n \Rightarrow n - l > n_2 \Rightarrow y(t) = x_{22}(t),\ x_2(t) = \begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix}$

$\Rightarrow$ need only reduced order observer for $\hat{x}_1(t), \epsilon \hat{x}_{21}(t)$

Case V. Only parts of the slow and fast variables are measured,

$y(t) = C_1 x_1(t) + C_2 x_2(t),\ y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} x_{12}(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_{21}(t) \end{bmatrix}$

where $\mathrm{x}_{11}(t), x_{22}(t) \in R^l$.

Case I) indicates that only the slow variables are measured, so that the corresponding reduced-order observer has to estimate fast variables which are unmeasurable. Case II) says that only fast variables are measured which means the reduced-order observer must estimate the slow variables. These are simplistic situations in which the reduced-order observer has no singularly perturbed structure. However, there are general cases III) and IV) in which dimension of slow state n_1 and fast state n_2 are not the same as the dimension of measurement l.

4.2 Case I : All Slow Variables are Measured Only

Case I) says that the measurable states are the slow states $x_1(t)$ in the singularly perturbed linear system defined in (4.1), that is

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) \\ \epsilon\dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) \\ y(t) &= x_1(t) \end{aligned} \tag{4.3}$$

If we differentiate the output variable we will obtain

$$\dot{y}(t) = \dot{x}_1 = A_{11}x_1(t) + A_{12}x_2(t) \tag{4.4}$$

To construct an observer for $x_2(t)$, we use the knowledge that an observer has the same structure as the system plus the driving feedback term whose role is to reduce the observation error to zero. We define an observer for $x_2(t)$ as (See Appendix $A.2$)

$$\begin{aligned} \epsilon\dot{\hat{x}}_2(t) &= A_{21}x_1(t) + A_{22}\hat{x}_2(t) + K_{11}(\dot{y}(t) - \dot{\hat{y}}(t)) \\ \dot{\hat{y}}(t) &= \dot{x}_1 = A_{11}x_1(t) + A_{12}\hat{x}_2(t) \end{aligned} \tag{4.5}$$

The observation error dynamics can be obtained from $\dot{e}_2(t) = \dot{x}_2(t) - \dot{\hat{x}}_2(t)$ as

$$\epsilon\dot{e}_2(t) = (A_{22} - K_{11}A_{12})e_2(t) \tag{4.6}$$

To place the reduced-observer eigenvalues in the left half of the complex plane such that the observation error $e_2(t) \to 0$, we need the following assumption.

Assumption 4.2.1. *The pair* (A_{22}^T, A_{12}^T) *is controllable, which is equivalent to the pair* (A_{22}, A_{12}) *is observable.*

By applying the change of variables $\hat{x}_2(t) - \frac{1}{\epsilon}K_{11}y(t) = \hat{z}_2(t)$ in (4.5), in order to eliminate $\dot{y}(t)$, we obtain the fast subsystem observer given by

$$\epsilon\dot{\hat{z}}_2(t) = A_z\hat{z}_2(t) + K_z y(t) \tag{4.7}$$

where [39]

$$\begin{aligned} A_z &= A_{22} - K_{11}A_{12}, \\ K_z &= A_{21} - K_{11}A_{11} + \frac{1}{\epsilon}A_{22}K_{11} - \frac{1}{\epsilon}K_{11}A_{12}K_{11} \end{aligned} \tag{4.8}$$

The estimates of the original system state space variables are now obtained as

$$\hat{z}_2(t) + \frac{1}{\epsilon}K_{11}y(t) = \hat{x}_2(t) \tag{4.9}$$

The corresponding block diagram is presented in Figure 4.1.

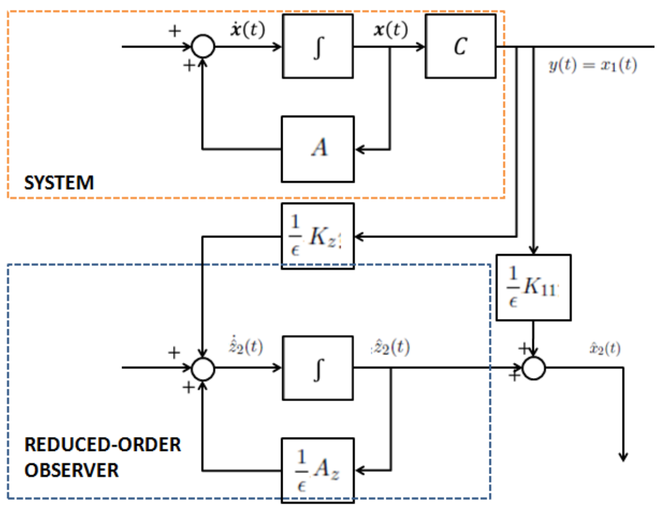

Figure 4.1: Case I : Reduced-order observer

4.2.1 Example 4.1

The system matrices are taken from [11] with modification of matrix C as

$$A = \begin{bmatrix} 0 & 0.4000 & 0 & 0 \\ 0 & 0 & 0.3450 & 0 \\ 0 & -5.2400 & -4.6500 & 2.6200 \\ 0 & 0 & 0 & -10.0000 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}. \; C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Our goal is to design the reduced-order observer with desired eigenvalues as $\lambda^{desired} = \{-1, -2\}$. The observability matrix has full rank and therefore the pair (A_{22}, A_{12}) is observable.

According to the Algorithm from Section 4.2, the following sub-matrices are obtained

$$A_z = \begin{bmatrix} -6.0000 & 0.2620 \\ -22.9008 & -1.0000 \end{bmatrix},\ K_{11} = \begin{bmatrix} 0 & 16.0434 \\ 0 & 66.3790 \end{bmatrix},\ K_z = \begin{bmatrix} 0 & -789.2196 \\ 0 & -4337.8692 \end{bmatrix}$$

In order to be able to run MATLAB Simulink simulation we had to specify also the system states initial conditions (these initial conditions are in general not known). We have chosen them as $x_1(0) = [2, 2]$ and $x_2(0) = [2, 2]$. From Appendix A.5, the initial condition for $\hat{x}_2(0)$ is given as

$$\hat{x}_2(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which results in

$$\hat{z}_2(0) = \hat{x}_2(0) - \frac{1}{\epsilon} K_{11} x_1(0)) = -\frac{1}{\epsilon} K_{11} x_1(0)) = \begin{bmatrix} -320.8695 \\ -1327.5804 \end{bmatrix}$$

so that we set $z_{21}(0) = [-320.8695]$ and $z_{22}(0) = [-1327.5804]$ in MATLAB simulation for the reduced-order observer. At this point, the initial condition for the error $e_2(0)$ is given as

$$e_2(0) = x_2(0) - \hat{x}_2(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

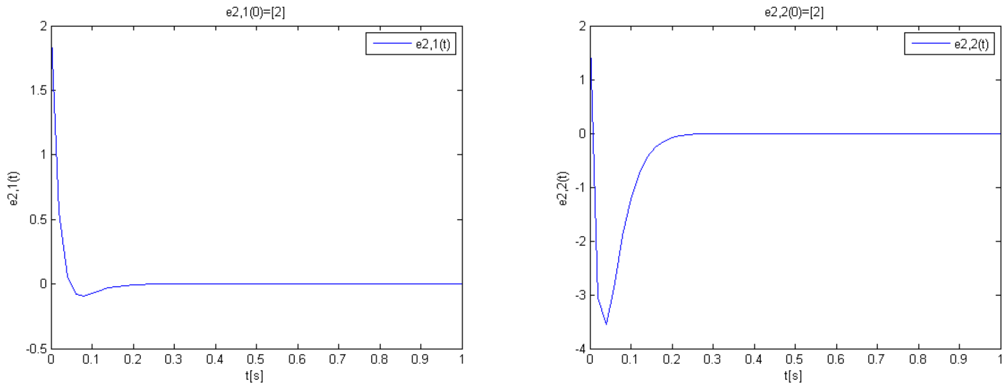

Figure 4.2: Case I : Convergence of the error state $e_2(t) = x_2(t) - \hat{x}_2(t)$

4.3 Case II : All Fast Variables are Measured Only

Case II) says that the measurable states are the fast states $x_2(t)$ in the singularly perturbed linear system defined in (3.1), that is

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) \\ \epsilon\dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) \\ y(t) &= x_2(t) \end{aligned} \tag{4.10}$$

If we differentiate the output variable we obtain

$$\dot{y}(t) = \dot{x}_2 = \frac{1}{\epsilon}A_{21}x_1(t) + \frac{1}{\epsilon}A_{22}x_2(t) \tag{4.11}$$

To construct an observer for $x_1(t)$, we use the knowledge that an observer has the same structure as the system plus the driving feedback term whose role is to reduce the observation error to zero given as (See Appendix A.3)

$$\begin{aligned} \dot{\hat{x}}_1(t) &= A_{11}\hat{x}_1(t) + A_{12}x_2(t) + K_{12}(\dot{y}(t) - \dot{\hat{y}}(t)) \\ \dot{\hat{y}}(t) &= \dot{x}_2(t) = \frac{1}{\epsilon}A_{21}\hat{x}_1(t) + \frac{1}{\epsilon}A_{22}x_2(t) \end{aligned} \tag{4.12}$$

The observation error dynamics can be obtained from $\dot{e}_1(t) = \dot{x}_1(t) - \dot{\hat{x}}_1(t)$ as

$$\dot{e}_1(t) = (A_{11} - \frac{1}{\epsilon}K_{12}A_{21})e_1(t) \tag{4.13}$$

To place the reduced-observer eigenvalues in the left half plane, we need the following assumption

Assumption 4.3.1. *The pair* $(A_{11}^T, \frac{1}{\epsilon}A_{21}^T)$ *is controllable, which is equivalent to the pair* $(A_{11}, \frac{1}{\epsilon}A_{21})$ *is observable.*

Using the fact that $\text{rank}(\alpha M) = \text{rank}(M), \alpha \neq 0$, it is easy to show that assumption 4.1.3 is equivalent to the following assumption

Assumption 4.3.2. *The pair* (A_{11}, A_{21}) *is observable.*

By introducing a change of variables $\hat{x}_1(t) - K_{12}y(t) = \hat{z}_1(t)$ (4.12) becomes

$$\dot{\hat{z}}_1(t) = A_z\hat{z}_1(t) + K_z y(t) \tag{4.14}$$

where [39]

$$\begin{aligned} A_z &= A_{11} - \frac{1}{\epsilon}K_{12}A_{21}, \\ K_z &= A_{12} + A_{11}K_{12} - \frac{1}{\epsilon}K_{12}A_{22} - \frac{1}{\epsilon}K_{12}A_{21}K_{12} \end{aligned} \tag{4.15}$$

Note that in this case the observer (4.14) is a fast observer given by

$$\epsilon\dot{\hat{z}}_1(t) = (\epsilon A_{11} - K_{12}A_{21})\hat{z}_1(t) + (\epsilon A_{12} - \epsilon A_{11}K_{12} - K_{12}A_{22} - K_{12}A_{21}K_{12})y(t) \tag{4.16}$$

The estimates of the original system state space variables are now obtained as

$$\hat{z}_1(t) + K_{12}y(t) = \hat{x}_1(t) \tag{4.17}$$

The corresponding observer is presented in Figure 4.3.

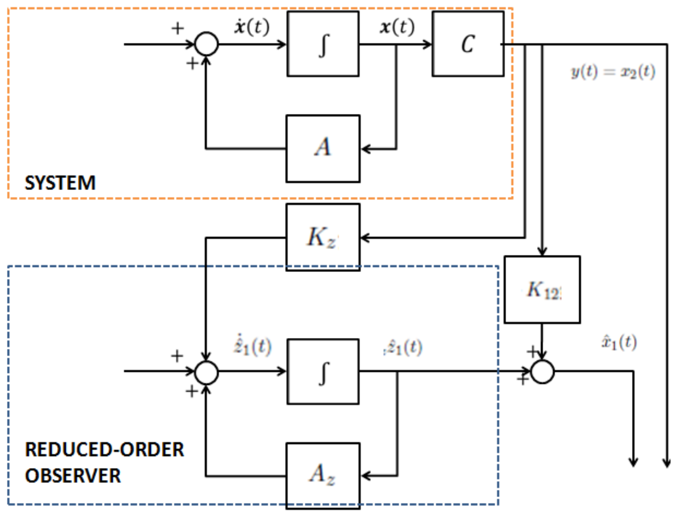

Figure 4.3: Case II : Reduced-order Observer

4.3.1 Example 4.2

The system matrices are given by

$$A = \begin{bmatrix} 0.2300 & 0.4000 & 0.5000 & 0.7000 \\ 0 & 0.2340 & 0.3460 & -1.0000 \\ 2.3400 & -5.2400 & -4.6500 & 2.6200 \\ 2.3600 & 5.6700 & 3.4500 & 1.2300 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}. \; C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Our goal is to design reduced-order observer with the desired eigenvalues, $\lambda^{desired} = \{-5, -7\}$. The observability matrix has full rank and therefore the pair (A_{11}, A_{21}) is observable.

According to the Algorithm from Section 4.3, the following sub-matrices are obtained

$$A_z = \begin{bmatrix} -5.0000 & 0.0000 \\ -0.0000 & -7.0000 \end{bmatrix}, \; K_{12} = \begin{bmatrix} 1.1199 & 1.1056 \\ -0.6659 & 0.6603 \end{bmatrix}, \; K_z = \begin{bmatrix} -3.7063 & -9.1223 \\ -0.3671 & -4.6898 \end{bmatrix}$$

In order to be able to run MATLAB Simulink simulation we had to specify also the system states initial conditions (these initial conditions are in general not known). We have chosen them as $x_1(0) = [2, 2]$ and $x_2(0) = [2, 2]$. From Appendix $A.5$, the initial condition for $\hat{x}_2(0)$ is given as

$$\hat{x}_1(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which results in

$$\hat{z}_1(0) = \hat{x}_1(0) - K_{12}x_2(0)) = -K_{12}x_2(0) = \begin{bmatrix} -4.4512 \\ 0.0113 \end{bmatrix}$$

so that we set $z_{11}(0) = [-4.4512]$ and $z_{12}(0) = [0.0113]$ in MATLAB simulation for the reduced-order observer The initial condition for the error $e_2(0)$ is given as

$$e_1(0) = x_1(0) - \hat{x}_1(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

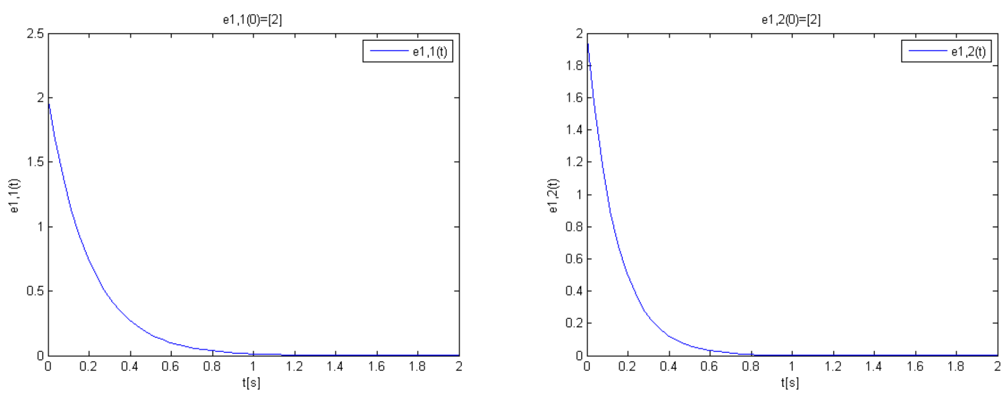

Figure 4.4: Case II : Convergence of the error state $e_1(t) = x_1(t) - \hat{x}_1(t)$

4.4 Case III : Only a Part of Slow Variables is Measured

Case III) says that the measurable states $x_{11}(t)$ are parts of the slow state $x_1(t)$ in the singularly perturbed linear system defined in (3.1), that is

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) \\ \epsilon\dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) \\ y(t) &= x_{11}(t) \end{aligned} \tag{4.18}$$

where

$$
\begin{aligned}
x_1(t) &= \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix}, \\
A_{11} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\ A_{12} = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \\
A_{21} &= \begin{bmatrix} a_{31} & a_{32} \end{bmatrix},\ A_{22} = \begin{bmatrix} a_{33} \end{bmatrix}
\end{aligned}
\tag{4.19}
$$

and $a_{11} \in R^{l\times l}, a_{12} \in R^{l\times(n_1-l)}, a_{13} \in R^{l\times n_2}, a_{21} \in R^{(n_1-l)\times l}, a_{22} \in R^{(n_1-l)\times(n_1-l)}, a_{23} \in R^{(n_1-l)\times n_2}, a_{31} \in R^{n_2\times l}, a_{32} \in R^{n_2\times(n_1-l)}, a_{33} \in R^{n_2\times n_2}, x_{11}(t) \in R^{l\times 1}$, $x_{12}(t) \in R^{(n_1-l)\times 1}, x_2(t) \in R^{n_2}, b_1 \in R^{l\times 1}, b_2 \in R^{(n-l)\times 1}$ and $y(t) \in R^{l\times 1}, p(t) \in R^{(n-l)\times 1}$. We assume that the slow states $x_1(t) \in R^{n_1}$, which might exceed the dimension of $y(t)$ in Case III, (4.18).

The system (4.18) with information (4.19) can be represented as

$$
\begin{aligned}
\dot{x}_{11}(t) &= A_{11}^r x_{11}(t) + A_{12}^r x_2^r(t) \\
\dot{x}_2^r(t) &= A_{21}^r x_{11}(t) + A_{22}^r x_2^r(t) \\
y(t) &= x_{11}(t)
\end{aligned}
\tag{4.20}
$$

where

$$
\begin{aligned}
x_2^r(t) &= \begin{bmatrix} x_{12}(t) \\ x_2(t) \end{bmatrix} \\
A_{11}^r &= \begin{bmatrix} a_{11} \end{bmatrix},\ A_{12}^r = \begin{bmatrix} a_{12} & a_{13} \end{bmatrix} \\
A_{21}^r &= \begin{bmatrix} a_{21} \\ \frac{1}{\epsilon}a_{31} \end{bmatrix},\ A_{22}^r = \begin{bmatrix} a_{22} & a_{23} \\ \frac{1}{\epsilon}a_{32} & \frac{1}{\epsilon}a_{33} \end{bmatrix}
\end{aligned}
\tag{4.21}
$$

At this point, the above redefined system can be used to design a reduced-order observer. To construct an observer for $x_2^r(t)$, we use the knowledge that an observer has the same structure as the system plus the driving feedback term whose role is to reduce the estimation error to zero. The reduced-order observer with the feedback information coming from $\dot{y}(t)$ is

$$
\dot{\hat{x}}_2^r(t) = A_{21}^r x_{11}(t) + A_{22}^r \hat{x}_2^r(t) + K_2(\dot{y}(t) - \dot{\hat{y}}(t)) \tag{4.22}
$$

If we differentiate the output variable $y(t)$, we obtain

$$\begin{aligned} \dot{y}(t) &= \dot{x}_{11}(t) = A_{11}^r x_{11}(t) + A_{12}^r x_2^r(t) \\ \dot{\hat{y}}(t) &= \dot{x}_{11}(t) = A_{11}^r x_{11}(t) + A_{12}^r \hat{x}_2^r(t) \end{aligned} \tag{4.23}$$

The error dynamic is governed by

$$\dot{e}_2(t) = \dot{x}_2^r(t) - \dot{\hat{x}}_2^r(t) = (A_{22}^r - K_2 A_{12}^r) e_2(t) \tag{4.24}$$

The following assumption is needed to make $e_2(t) \to 0$ at steady state.

Assumption 4.4.1. *The pair* (A_{22}^r, A_{12}^r) *is observable.*

The change of variable is required to remove $\dot{y}(t)$ term in (4.22) as

$$\hat{x}_2^r(t) - K_2 y(t) = \begin{bmatrix} \hat{x}_{12}(t) \\ \hat{x}_2(t) \end{bmatrix} - \begin{bmatrix} K_{21} \\ \frac{1}{\epsilon} K_{22}(t) \end{bmatrix} y(t) = \hat{z}_2^r(t) = \begin{bmatrix} \hat{z}_{12}(t) \\ \hat{z}_2(t) \end{bmatrix} \tag{4.25}$$

Applying this change of variables, (4.22) leads to

$$\dot{\hat{z}}_2^r(t) = A_z^r \hat{z}_2^r(t) + K_z^r y(t) \tag{4.26}$$

where

$$\begin{aligned} A_z^r &= A_{22}^r - K_2 A_{12}^r = \begin{bmatrix} a_{22} & a_{23} \\ \frac{1}{\epsilon} a_{32} & \frac{1}{\epsilon} a_{33} \end{bmatrix} - \begin{bmatrix} K_{21} \\ K_{22} \end{bmatrix} \begin{bmatrix} a_{12} & a_{13} \end{bmatrix}, \\ K_z^r &= A_{21}^r - K_2 A_{11}^r + A_{22}^r K_2 - K_2 A_{12}^r K_2, \\ &= \begin{bmatrix} a_{21} - K_{21} a_{11} + a_{22} K_{21} + \frac{1}{\epsilon} a_{23} K_{22} - K_{21}(a_{12} K_{21} + \frac{1}{\epsilon} a_{13} K_{22}) \\ \frac{1}{\epsilon} a_{31} - \frac{1}{\epsilon} K_{22} a_{11} + \frac{1}{\epsilon} a_{32} K_{21} + \frac{1}{\epsilon^2} a_{33} K_{22} - \frac{1}{\epsilon} K_{22}(a_{12} K_{21} + \frac{1}{\epsilon} a_{13} K_{22}) \end{bmatrix} = \begin{bmatrix} K_{21r} \\ K_{22r} \end{bmatrix} \end{aligned} \tag{4.27}$$

Since K_2 is determined by eigenvalue assignment in terms of two matrices A_{22}^r, A_{12}^r, we can apply the two stage method to overcome numerical ill-conditioning problem coming from the perturbation parameter presented in matrix A_{22}^r. Here, we are going to use the duality between the controller and the observer so that we will need to transpose matrices A_{22}^r and $K_2 A_{12}^r$ and consider hypothetical control system, that is

$$\begin{aligned} \dot{q}_{12}(t) &= a_{22}^T q_{12}(t) + \frac{1}{\epsilon} a_{32}^T q_2(t) + a_{12}^T \hat{u}(t) \\ \dot{q}_2(t) &= a_{23}^T q_{12}(t) + \frac{1}{\epsilon} a_{33}^T q_2(t) + a_{13}^T \hat{u}(t) \end{aligned} \tag{4.28}$$

where $\hat{u}(t) = -K_2^T q(t) = -\begin{bmatrix} K_{21}^T & K_{22}^T \end{bmatrix} \begin{bmatrix} q_{12}(t) \\ q_2(t) \end{bmatrix}$, and states $q_{12}(t), q_2(t)$ are used for the purpose of design only. Here, the goal is to find a reduced-order observer gain K_2 using the two-stage method. To transform (4.28) into an explicit singularly perturbed form we introduce $r_{12}(t) = q_{12}(t)$ and $r_2(t) = \frac{1}{\epsilon} q_2(t)$ which leads to

$$\begin{aligned} \dot{r}_{12}(t) &= a_{22}^T r_{12}(t) + a_{32}^T r_2(t) + a_{12}^T \hat{u}(t) \\ \epsilon \dot{r}_2(t) &= a_{23}^T r_{12}(t) + a_{33}^T r_2(t) + a_{13}^T \hat{u}(t) \end{aligned} \tag{4.29}$$

The Chang transformation applied to (4.29) produces

$$\begin{aligned} \dot{r}_s(t) &= A_{sr}^T r_s(t) + c_{sr}^T \hat{u}(t) \\ \epsilon \dot{r}_f(t) &= A_{fr}^T r_f(t) + c_{fr}^T \hat{u}(t) \end{aligned} \tag{4.30}$$

where

$$\begin{aligned} &A_{sr}^T = a_{22}^T - L_r^T a_{23}^T, \;\; A_{fr}^T = a_{33}^T + \epsilon a_{23}^T L_r^T \\ &C_{sr}^T = a_{12}^T - L_r^T a_{13}^T, \;\; C_{fr}^T = \epsilon H_r^T a_{12}^T + (I_{n_2} - \epsilon H_r^T L_r^T) a_{13}^T \end{aligned} \tag{4.31}$$

The goal is to find the observer gain K_2^T using the two stage feedback design. The Chang transformation needed for the proposed observer design relates the original state variables $r_{12}(t)$ and $r_2(t)$ and the slow and fast variables $r_s(t)$ and $r_f(t)$ as follows

$$\begin{bmatrix} r_s(t) \\ r_f(t) \end{bmatrix} = \begin{bmatrix} I_{(n_1 - l)} & -\epsilon L_r^T \\ H_r^T & I_{n_2} - \epsilon H_r^T L_r^T \end{bmatrix} \begin{bmatrix} r_{12}(t) \\ r_2(t) \end{bmatrix} = T_{cr}^T \begin{bmatrix} r_{12}(t) \\ r_2(t) \end{bmatrix} \tag{4.32}$$

The state variables $r_{12}(t)$ and $r_2(t)$ can be reconstructed from the inverse transformation as

$$\begin{bmatrix} r_{12}(t) \\ r_2(t) \end{bmatrix} = \begin{bmatrix} I_{(n_1 - l)} - \epsilon L_r^T H_r^T & \epsilon L_r^T \\ -H_r^T & I_{n_2} \end{bmatrix} \begin{bmatrix} r_s(t) \\ r_f(t) \end{bmatrix} = T_{cr}^{-T} \begin{bmatrix} r_s(t) \\ r_f(t) \end{bmatrix} \tag{4.33}$$

where L_r^T and H_r^T are the transposed solution given by

$$
\begin{aligned}
0 &= \epsilon(a_{22}^T - L_r^T a_{23}^T)L_r^T + (a_{32}^T - L_r^T a_{33}^T) \\
0 &= \epsilon H_r^T(a_{22}^T - L_r^T a_{23}^T) + a_{23}^T - (a_{33}^T + \epsilon a_{23}^T L_r^T)H_r^T
\end{aligned}
\tag{4.34}
$$

We take $\hat{u}(t)$ for the slow subsystem as

$$\hat{u}(t) = -K_{sr}^T r_s(t) + v(t) \tag{4.35}$$

Substituting (4.35) into (4.30), (4.30) becomes

$$
\begin{aligned}
\dot{r}_s(t) &= (A_{sr}^T - C_{sr}^T K_{sr}^T)r_s(t) + C_{sr}^T v(t) \\
\epsilon\dot{r}_f(t) &= A_{fr}^T r_f(t) - C_{fr}^T K_{sr}^T r_s(t) + C_{fr}^T v(t)
\end{aligned}
\tag{4.36}
$$

At this point, it is possible to place the slow observer eigenvalues in the desired locations, that is

$$\lambda(A_{sr}^T - C_{sr}^T K_{sr}^T) = \lambda(A_{sr} - K_{sr}C_{sr}) = \lambda_s^{desired} \tag{4.37}$$

assuming that the following assumption is satisfied.

Assumption 4.4.2. *The pair (A_{sr}, C_{sr}) is observable.*

Now, the following change of coordinates is introduced

$$r_{fnew}(t) = P_{or}r_s(t) + r_f(t) \;\rightarrow\; r_f(t) = r_{fnew}(t) - P_{or}r_s(t) \tag{4.38}$$

where P_{or} satisfies the algebraic Sylvester equation

$$\epsilon P_{or}(A_{sr}^T - C_{sr}^T K_{sr}^T) - C_{fr}^T K_{sr}^T - A_{fr}^T P_{or} = 0 \;\Rightarrow\; P_{or} = O(\epsilon) \tag{4.39}$$

The unique solution for P_{or} exist for sufficiently small values ϵ under Assumption 3.2.1.

The change of variables in (4.38) results in

$$\begin{aligned}\epsilon\dot{r}_{fnew}(t) &= \epsilon P_{or}\dot{r}_s(t) + \epsilon\dot{r}_f(t)\\ &= [-A_{fr}^T P_{or} - C_{fr}^T K_{sr}^T + \epsilon P_{or}(A_{sr}^T - C_{sr}^T K_{sr}^T)]r_s(t) + A_{fr}^T r_{fnew}(t)\\ &+ (C_{fr}^T + \epsilon P_{or} C_{sr}^T)v(t)\end{aligned} \tag{4.40}$$

When the Sylvester equation (4.39) is satisfied, (4.40) becomes

$$\epsilon\dot{r}_{fnew}(t) = A_{fr}^T r_{fnew}(t) + (C_{fr}^T + \epsilon P_{or} C_{sr}^T)v(t) = A_{fr}^T r_{fnew}(t) + C_{fnewr}^T v(t) \tag{4.41}$$

The input $v(t)$ can be used to locate the fast subsystem eigenvalues

$$v(t) = -K_{f2r}^T r_{fnew}(t) \tag{4.42}$$

At this point, it is possible to locate the fast eigenvalues in the original coordinates at the desired location (left half complex plane)

$$\lambda(A_{fr} - K_{f2r} C_{fnewr}) = \lambda_f^{desired} \tag{4.43}$$

if the following observability assumption is satisfied.

Assumption 4.4.3. *The pair* (A_{fr}, C_{fnewr}) *is observable.*

Substituting (4.35) and (4.42) into (4.30) and (4.41), we obtain

$$\begin{bmatrix}\dot{r}_s(t)\\ \epsilon\dot{r}_{fnew}(t)\end{bmatrix} = \begin{bmatrix}(A_{sr} - K_{sr}C_{sr})^T & -(K_{f2r}C_{sr})^T\\ 0 & (A_{fr} - K_{f2r}C_{fnewr})^T\end{bmatrix}\begin{bmatrix}r_s(t)\\ r_{fnew}(t)\end{bmatrix} \tag{4.44}$$

The original coordinates $\hat{q}_{12}(t), \hat{q}_2(t)$ and $r_s(t), r_{fnew}(t)$ coordinates are related via

$$\begin{bmatrix}r_s(t)\\ r_{fnew}(t)\end{bmatrix} = T_{2r}^T T_{cr}^T T_{1r}^T \begin{bmatrix}\hat{q}_{12}(t)\\ \hat{q}_2(t)\end{bmatrix} \tag{4.45}$$

where

$$T_{1r}^T = \begin{bmatrix} I_{(n_1-l)} & 0 \\ 0 & \frac{1}{\epsilon} I_{n_2} \end{bmatrix}, \; T_{2r}^T = \begin{bmatrix} I_{(n_1-l)} & 0 \\ P_{or} & I_{n_2} \end{bmatrix} \tag{4.46}$$

with T_{cr}^T defined in (4.32). It is possible to reconstruct $\hat{q}_{12}(t)$, $\hat{q}_2(t)$ from $r_s(t)$, $r_{fnew}(t)$ via the inverse transformation

$$\begin{bmatrix} \hat{q}_{12}(t) \\ \hat{q}_2(t) \end{bmatrix} = T_{1r}^{-T} T_{cr}^{-T} T_{2r}^{-T} \begin{bmatrix} r_s(t) \\ r_{fnew}(t) \end{bmatrix} = T_{4r}^{-T} \begin{bmatrix} r_s(t) \\ r_{fnew}(t) \end{bmatrix} \tag{4.47}$$

From the above relation (4.47), we can construct the state transformation from $z_s(t)$, $z_{fnew}(t)$ to $z_{12}(t)$, $z_2(t)$ as follows

$$\begin{bmatrix} \hat{z}_{12}(t) \\ \hat{z}_2(t) \end{bmatrix} = T_{4r} \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew}(t) \end{bmatrix} \tag{4.48}$$

Applying above the state transformation (4.48) to (4.26), we get

$$\begin{aligned} T_{4r} \begin{bmatrix} \dot{\hat{z}}_s(t) \\ \dot{\hat{z}}_{fnew}(t) \end{bmatrix} &= (A_{22}^r - K_2 A_{12}^r) T_{4r} \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew}(t) \end{bmatrix} + K_z^r y(t) \\ \begin{bmatrix} \dot{\hat{z}}_s(t) \\ \dot{\hat{z}}_{fnew}(t) \end{bmatrix} &= T_{4r}^{-1} (A_{22}^r - K_2 A_{12}^r) T_{4r} \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew}(t) \end{bmatrix} + T_{4r}^{-1} K_z^r y(t) \end{aligned} \tag{4.49}$$

Now we can present the observer configuration using the result in (4.44) and the duality between controller and observer designs

$$\begin{bmatrix} \dot{\hat{z}}_s(t) \\ \epsilon \dot{\hat{z}}_{fnew}(t) \end{bmatrix} = \begin{bmatrix} A_{sr} - K_{sr} C_{sr} & 0 \\ -\epsilon K_{f2r} C_{sr} & A_{fr} - K_{f2r} C_{fnewr} \end{bmatrix} \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew}(t) \end{bmatrix} + \begin{bmatrix} K_{sr}^* \\ K_{f2r}^* \end{bmatrix} y(t) \tag{4.50}$$

where K_{sr}^*, $\frac{1}{\epsilon} K_{f2r}^*$ can be obtained from $T_{4r}^{-1} K_z^r$. We can obtain a fully decoupled slow and fast reduced-order observers working in parallel as follows. We change the coordinates once again given as

$$\hat{z}_{fnew2}(t) = P_{o2r} \hat{z}_s(t) + \hat{z}_{fnew}(t) \;\rightarrow\; \hat{z}_{fnew}(t) = \hat{z}_{fnew2}(t) - P_{o2r} \hat{z}_s(t) \tag{4.51}$$

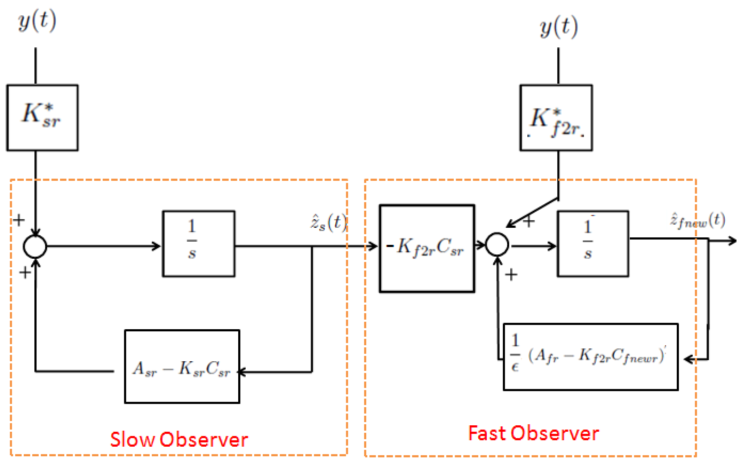

Figure 4.5: Case III : Sequential reduced-order slow and fast observers for the reduced-order observer

where P_{o2r} satisfies the algebraic Sylvester equation represented by

$$\epsilon P_{o2r}(A_{sr} - K_{sr}C_{sr}) - \epsilon K_{f2r}C_{sr} - (A_{fr} - K_{f2r}C_{fnewr})P_{o2r} = 0 \;\Rightarrow\; P^0_{o2r} = O(\epsilon) \quad (4.52)$$

The linear algebraic equation (4.52) has a unique solution since $A_{fr} - K_{f2r}C_{fnewr}$ is an asymptotically stable fast subsystem feedback matrix. The change of variable (4.51) results in

$$\begin{aligned}\epsilon\dot{\hat{z}}_{fnew2}(t) &= \epsilon P_{o2r}\dot{\hat{z}}_s(t) + \epsilon\dot{\hat{z}}_{fnew}(t)\\ &= [\epsilon P_{o2r}(A_{sr} - K_{sr}C_{sr}) - \epsilon K_{f2r}C_{sr} - (A_{fr} - K_{f2r}C_{fnewr})P_{o2r}]\hat{z}_s(t)\\ &+ (A_{fr} - K_{f2r}C_{fnewr})\hat{z}_{fnew2}(t) + K_{f3r}y(t)\end{aligned} \quad (4.53)$$

where

$$K_{f3r} = \epsilon(P_{o2r}K^*_{sr} + K^*_{f2r}) \quad (4.54)$$

Hence, if the second algebraic Sylvester equation (4.52) is satisfied, (4.53) becomes

$$\epsilon\dot{\hat{z}}_{fnew2}(t) = (A_{fr} - K_{f2r}C_{fnewr})\hat{z}_{fnew2}(t) + K_{f3r}y(t) \quad (4.55)$$

At this point, we have the block-diagonalized form of the observer obtained as

$$\begin{aligned}\dot{\hat{z}}_s(t) &= (A_{sr} - K_{sr}C_{sr})\hat{z}_s(t) + K^*_{sr}y(t)\\ \epsilon\dot{\hat{z}}_{fnew2}(t) &= (A_{fr} - K_{f2r}C_{fnewr})\hat{z}_{fnew2}(t) + K_{f3r}y(t)\end{aligned} \tag{4.56}$$

The original coordinates $\hat{z}_{12}(t)$, $\hat{z}_2(t)$ and the new coordinates $\hat{z}_s(t)$, $\hat{z}_{fnew2}(t)$ are related via

$$\begin{bmatrix}\hat{z}_s(t)\\ \hat{z}_{fnew2}(t)\end{bmatrix} = T_{3r}T_{4r}^{-1}\begin{bmatrix}\hat{z}_{12}(t)\\ \hat{z}_2(t)\end{bmatrix} = T_r^{-1}\begin{bmatrix}\hat{z}_{12}(t)\\ \hat{z}_2(t)\end{bmatrix} \tag{4.57}$$

where

$$T_{3r} = \begin{bmatrix} I & 0\\ P_{o2r} & I\end{bmatrix} \tag{4.58}$$

Now, the original coordinates can be reconstructed via

$$\begin{bmatrix}\hat{z}_{12}(t)\\ \hat{z}_2(t)\end{bmatrix} = T_{4r}T_{3r}^{-1}\begin{bmatrix}\hat{z}_s(t)\\ \hat{z}_{fnew2}(t)\end{bmatrix} = T_r\begin{bmatrix}\hat{z}_s(t)\\ \hat{z}_{fnew2}(t)\end{bmatrix} \tag{4.59}$$

At this point, the original state $\hat{x}_{12}(t)$ and $\hat{x}_2(t)$ can be reconstructed in terms of (4.25) and (4.59) given as

$$\begin{aligned}\begin{bmatrix}\hat{z}_{12}(t)\\ \hat{z}_2(t)\end{bmatrix} + \begin{bmatrix}K_{21}\\ \frac{1}{\epsilon}K_{22}\end{bmatrix}y(t) &= \begin{bmatrix}\hat{x}_{12}(t)\\ \hat{x}_2(t)\end{bmatrix}\\ \hat{z}^r_2(t) + K_2y(t) &= \hat{x}^r_2(t)\end{aligned} \tag{4.60}$$

4.4.1 Case III : Reduced-order Observation Error Equations

The error equation given in (4.24) is rewritten as

$$\begin{aligned}\dot{e}^r_2(t) = \dot{x}^r_2(t) - \dot{\hat{x}}^r_2(t) &= \begin{bmatrix}\dot{x}_{12}(t)\\ \dot{x}_2(t)\end{bmatrix} - \begin{bmatrix}\dot{\hat{x}}_{12}(t)\\ \dot{\hat{x}}_2(t)\end{bmatrix} = \begin{bmatrix}\dot{e}_{12}(t)\\ \dot{e}_2(t)\end{bmatrix}\\ &= (A^r_{22} - K_2A^r_{12})\begin{bmatrix}e_{12}(t)\\ e_2(t)\end{bmatrix}\end{aligned} \tag{4.61}$$

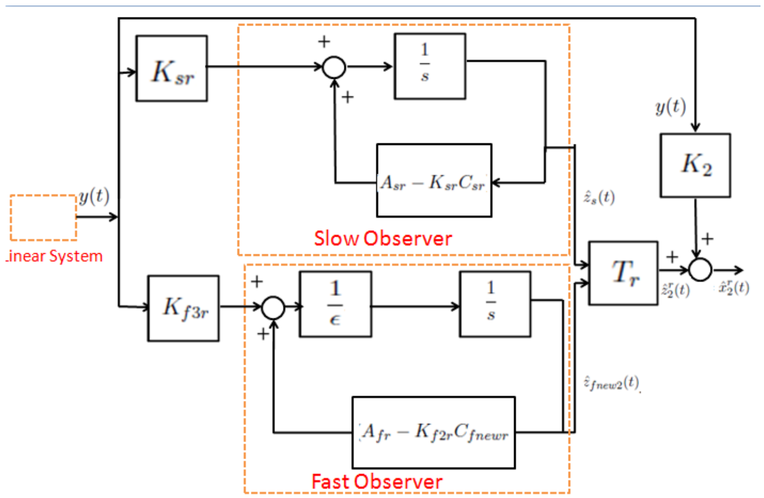

Figure 4.6: Case III : Slow-fast reduced-order parallel observation with the reduced-order observers of dimensions $(n_1 - l)$ and n_2, $(n_1 - l) + n_2 = n$, $(n - l)$ = order of unmeasurable states of the system.

Using state transformation defined in (4.59), the original error coordinates $e_{12}(t), e_2(t)$ and the new error coordinates $e^r_s(t), e^r_{fnew2}(t)$ are related via

$$\begin{aligned} \begin{bmatrix} e_{12}(t) \\ e_2(t) \end{bmatrix} &= \begin{bmatrix} x_{12}(t) \\ x_2(t) \end{bmatrix} - \begin{bmatrix} \hat{x}_{12}(t) \\ \hat{x}_2(t) \end{bmatrix} = T_r \begin{bmatrix} z_s(t) \\ z_{fnew2}(t) \end{bmatrix} - T_r \begin{bmatrix} \hat{z}_s(t) \\ \hat{x}_{fnew2}(t) \end{bmatrix} \\ &= T_r \begin{bmatrix} e^r_s(t) \\ e^r_{fnew2}(t) \end{bmatrix} \end{aligned} \tag{4.62}$$

Applying state transformation (4.62) into (4.61), (4.61) becomes

$$T^{-1} \begin{bmatrix} \dot{e}_{12}(t) \\ \epsilon\dot{e}_2(t) \end{bmatrix} = T_r^{-1}(A^r_{22} - K_2 A^r_{12}) T_r \begin{bmatrix} \hat{e}^r_s(t) \\ \hat{e}^r_{fnew2}(t) \end{bmatrix} \tag{4.63}$$

Analytical result for (4.63) is given as

$$\begin{aligned} \dot{e}^r_s(t) &= \hat{A}_{sr} e^r_s(t) \\ \epsilon\dot{e}^r_{fnew2}(t) &= \hat{A}_{fr} e^r_{fnew2}(t) \end{aligned} \tag{4.64}$$

where

$$\hat{A}_{sr} = A_{sr} - K_{sr}C_{sr}$$
$$\hat{A}_{fr} = A_{fr} - K_{f2r}C_{fnewr} \tag{4.65}$$

The convergence of the error dynamics will be obtained under the eigenvalues condition given as

$$Re\lambda(\hat{A}_{sr}) < 0, \ Re\lambda(\hat{A}_{fr}) < 0 \tag{4.66}$$

4.4.2 Case III : Reduced-order Observer Gain in the Original Coordinates

We will show that the observer in the original coordinates is given by

$$K_2 = (\begin{bmatrix} K_{sr}^T + K_{f2r}^T P_{or} & K_{f2r}^T \end{bmatrix} T_{cr}^T T_{1r}^T)^T = \begin{bmatrix} T_{1r}T_{cr}(K_{sr} + P_{or}^T K_{f2r}) \\ T_{1r}T_{cr}K_{f2r} \end{bmatrix} = \begin{bmatrix} K_{21} \\ K_{22} \end{bmatrix} \tag{4.67}$$

where T_{cr} is the Chang transformation (4.32), P_{or} is the solution of the algebraic Sylvester equation (4.39). We previously set $K_2^T r(t) = v(t) - K_{sr}^T r_s(t) = -K_{sr}^T r_s(t) - K_{f2r}^T r_{fnew}(t)$ in (4.35) and (4.42), which implies

$$\begin{aligned} K_2^T r(t) &= \begin{bmatrix} K_{sr}^T & K_{f2r}^T \end{bmatrix} \begin{bmatrix} r_s(t) \\ r_{fnew}(t) \end{bmatrix} = \begin{bmatrix} K_{sr}^T & K_{f2r}^T \end{bmatrix} \begin{bmatrix} I_{n_1-l} & 0 \\ P_{or} & I_{n_2} \end{bmatrix} \begin{bmatrix} r_s(t) \\ r_f(t) \end{bmatrix} \\ &= \begin{bmatrix} K_{sr}^T + K_{f2r}^T P_{or} & K_{f2r}^T \end{bmatrix} T_{cr}^T \begin{bmatrix} r_{12}(t) \\ r_2(t) \end{bmatrix} = \begin{bmatrix} K_{sr}^T + K_{f2r}^T P_{or} & K_{f2r}^T \end{bmatrix} T_{cr}^T T_{1r}^T \begin{bmatrix} q_{12}(t) \\ q_2(t) \end{bmatrix} \end{aligned} \tag{4.68}$$

Hence $\begin{bmatrix} K_{sr}^T + K_{f2r}^T P_{or} & K_{f2r}^T \end{bmatrix} T_{cr}^T T_{1r}^T$ represents transpose of the observer gain matrix K_2 in the original coordinates. It is important to notify that the observer gain $K_2 = f(K_{sr}, K_{f2r})$ can be obtained using computations with reduced-order matrices $K_{sr}, K_{sr,2}$. Using this fact, the observer gain matrix K_2 is given by (4.67).

4.4.3 Case III : Design Algorithm for Finding the reduced-order Observer Gain

Given that the linear system (A_{22}^r, A_{12}^r) is observable, the following two-time scale design algorithm can be applied for the design of a reduced-order observer for singularly perturbed linear system.

Step 1. Transpose the first part of matrices from (4.27) and apply the change of variable to the hypothetical system defined in (4.28).

Step 2. Apply the Chang transformation (4.33) to (4.29) to get (4.30).

Step 3. Obtain the partitioned submatrices $A_{sr}^T, \frac{1}{\epsilon}A_{fr}^T, C_{sr}^T$ and C_{fr}^T.

Step 4. Place the slow observer eigenvalues in the desired location and obtain the slow observer gain K_{sr}^T using the eigenvalue assignment for $\lambda(A_{sr} - K_{sr}C_{sr})$.

Step 5. Solve the reduced-order Sylvester algebraic equation (4.39) to get P_{or}.

Step 6. Place fast observer eigenvalues at the desired location using the eigenvalue assignment for $\frac{1}{\epsilon}\lambda(A_{fr} - K_{f2r}C_{fnewr})$ and obtain K_{f2r}.

Step 7. Find the reduced-order observer gain K_2 in the original coordinates using (4.67) and check $\lambda(A_{22}^r - K_2A_{12}^r) = \lambda_s^{desired} \cup \lambda_f^{desired}$.

4.4.4 Example 4.3

Consider a $4^{th}-$ order system with the system matrices A and C taken from [11]

$$A = \begin{bmatrix} 0 & 0.4000 & 0 & 0 \\ 0 & 0 & 0.3450 & 0 \\ 0 & -5.2400 & -4.6500 & 2.6200 \\ 0 & 0 & 0 & -10.0000 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}. \ C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

Our goal is to design independently slow and fast reduced-order observers with desired eigenvalues $\lambda_s^{desired} = \{-5\}$ and $\frac{1}{\epsilon}\lambda_f^{desired} = \{-20, -30\}$. The observability matrix has full rank and therefore the pair (A_{22}^r, A_{12}^r) is observable.

According to Steps 1 and 2 of the Algorithm from Section 4.4.3 in Casr III), the following sub-matrices are obtained

$$A_{sr}^T = \begin{bmatrix} -0.4282 \end{bmatrix}, \ \frac{1}{\epsilon} A_{fr}^T = \begin{bmatrix} -4.2218 & 0 \\ 2.6200 & -10.0000 \end{bmatrix},$$

$$C_{sr}^T = \begin{bmatrix} 0.4000 \end{bmatrix}, \ C_{fr}^T = \begin{bmatrix} -0.0364 \\ -0.0100 \end{bmatrix}, \ C_{fnewr}^T = \begin{bmatrix} 0.1773 \\ 0.0929 \end{bmatrix}$$

Following Step 4 in Section 4.4.3, we place the slow eigenvalues in the original coordinates at $\{-5\}$ via the slow feedback gain matrix

$$K_{sr}^T = \begin{bmatrix} 11.4294 \end{bmatrix}$$

In Step 3 of the algorithm, we solve the Sylvester algebraic equation and obtain matrix P_{or} as

$$P_{or} = \begin{bmatrix} 5.3426 \\ 2.5719 \end{bmatrix}, \ P_{o2r} = \begin{bmatrix} 0.1076 \\ 0.2295 \end{bmatrix}$$

In Step 4 of the algorithm, we place fast observer's eigenvalues at the desired location $\{-200, -300\}$. The fast observer gain K_{f2}^T is given by

$$K_{f2r}^T = \begin{bmatrix} 165.1035 & -276.5792 \end{bmatrix}, \ \frac{1}{\epsilon} K_{f3} = \begin{bmatrix} -808.9746 \\ 1391.2645 \end{bmatrix}$$

Step 5. Using (4.67), matrix K_2 is obtained as

$$K_2 = 10^3 \times \begin{bmatrix} 0.100874999999999 \\ 1.525831159420266 \\ -2.765792676180949 \end{bmatrix}$$

It can be checked that $\lambda(A_{22}^r - K_2 A_{12}^r)$ in the original coordinate are given by

$$\lambda(A_{22}^r - K_2 A_{12}^r) = \begin{bmatrix} -4.999999999999998 \\ -19.999999999999979 \\ -29.999999999999709 \end{bmatrix}$$

which is the same (with the accuracy of $O(10^{-14})$) as we placed the slow and fast eigenvalues using the two time scale decomposition designs. Figures 4.13 present the slow and fast observation errors. In order to be able to run MATLAB Simulink simulation we had to specify also the system states initial conditions (these initial conditions are in general not known). We have chosen them as $x_1(0) = [2, 2]$ and $x_2(0) = [2, 2]$. From Appendix $A.7$, the initial condition for $\hat{x}_2^r(0)$ is given as

$$\hat{x}_2^r(0) = \begin{bmatrix} x_{12}(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which results in

$$\hat{z}_2^r(0) = \hat{x}_2^r(0) - K_2 x_{11}(0) = -K_2 x_{11}(0) = \begin{bmatrix} -201.7499 \\ -3051.6623 \\ 5531.5853 \end{bmatrix}$$

Using (4.59), we obtain $\hat{z}_s(0), \hat{z}_{fnew2}(0)$ given as

$$\begin{bmatrix} \hat{z}_s(0) \\ \hat{z}_{fnew2}(0) \end{bmatrix} = T_r^{-1} \begin{bmatrix} \hat{z}_{12}(0) \\ \hat{z}_2(0) \end{bmatrix} = \begin{bmatrix} -22.8589 \\ -332.6667 \\ 547.9104 \end{bmatrix}$$

so that $z_s(0) = [-22.8589]$ and $z_{fnew2}(0) = [-332.6667, 547.9104]$ in MATLAB simulation for the reduced-order observer. At this point, the initial condition for the errors $e_{12}(0), e_2(0)$ are given as

$$\begin{bmatrix} e_{12}(0) \\ e_2(0) \end{bmatrix} = \begin{bmatrix} x_{12}(0) \\ x_2(0) \end{bmatrix} - \begin{bmatrix} \hat{x}_{12}(0) \\ \hat{x}_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

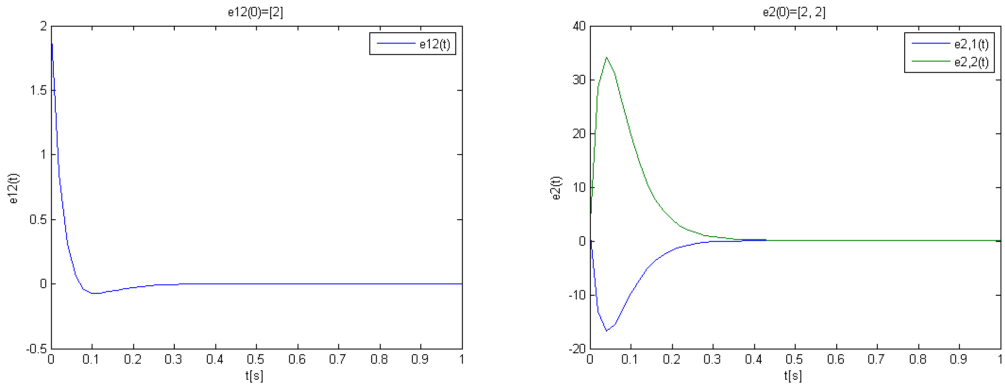

Figure 4.7: Case III : Convergence of the slow state observation error $e_{12}(t) = x_{12}(t) - \hat{x}_{12}(t)$ and the fast state observation error $e_2(t) = x_2(t) - \hat{x}_2(t)$ for the parallel structure from Fig. 4.6

4.5 Case IV: Only a Part of Fast Variables is Measured

Case IV says that the measurable states $x_{22}(t)$ are parts of fast states $x_2(t)$.

$$\begin{aligned}
\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) \\
\epsilon\dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) \\
y(t) &= I_l x_{22}(t)
\end{aligned} \tag{4.69}$$

where

$$\begin{aligned}
x_2(t) &= \begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix}, \\
A_{11} &= \begin{bmatrix} a_{11} \end{bmatrix}, \; A_{12} = \begin{bmatrix} a_{12} & a_{13} \end{bmatrix} \\
A_{21} &= \begin{bmatrix} a_{21} \\ a_{31} \end{bmatrix}, \; A_{22} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}
\end{aligned} \tag{4.70}$$

with $a_{33} \in R^{l\times l}, a_{32} \in R^{l\times(n_2-l)}, a_{31} \in R^{l\times n_1}, a_{23} \in R^{(n_2-l)\times l}, a_{22} \in R^{(n_2-l)\times(n_2-l)}$, $a_{21} \in R^{(n_2-l)\times n_1}, a_{13} \in R^{n_1\times l}, a_{12} \in R^{n_1\times(n_2-l)}, a_{11} \in R^{n_1\times n_1}, x_{22}(t) \in R^{l\times 1}$, $x_{21}(t) \in R^{n_2-l\times 1}, x_1(t) \in R^{n_1}$, $y(t) \in R^{l\times 1}$, and $p(t) \in R^{(n-l)\times 1}$.

We can also construct another form of a linear system to design the reduced-order observer

$$
\begin{aligned}
\dot{x}_1^r(t) &= A_{11}^r x_1^r(t) + A_{12}^r x_{22}(t) \\
\dot{x}_{22}(t) &= A_{21}^r x_1^r(t) + A_{22}^r x_{22}(t) \\
y(t) &= I_l x_{22}(t)
\end{aligned}
\tag{4.71}
$$

where

$$
\begin{aligned}
x_1^r(t) &= \begin{bmatrix} x_1(t) \\ x_{21}(t) \end{bmatrix} \\
A_{11}^r &= \begin{bmatrix} a_{11} & a_{12} \\ \frac{1}{\epsilon}a_{21} & \frac{1}{\epsilon}a_{22} \end{bmatrix}, \ A_{12}^r = \begin{bmatrix} a_{13} \\ \frac{1}{\epsilon}a_{23} \end{bmatrix} \\
A_{21}^r &= \begin{bmatrix} \frac{1}{\epsilon}a_{31} & \frac{1}{\epsilon}a_{32} \end{bmatrix}, \ A_{22}^r = \begin{bmatrix} \frac{1}{\epsilon}a_{33} \end{bmatrix}
\end{aligned}
\tag{4.72}
$$

At this point, the above redefined system (4.71) can be used to design the reduced-order observer. To construct an observer for $x_2^r(t)$, we use the knowledge that an observer has the same structure as the system plus the driving feedback term whose role is to reduce the estimation error to zero. The reduced-order observer with the feedback information coming from $\dot{y}(t)$ is

$$
\dot{\hat{x}}_1^r(t) = A_{11}^r \hat{x}_1^r(t) + A_{12}^r x_{22}(t) + K_3(\dot{y}(t) - \dot{\hat{y}}(t)) \tag{4.73}
$$

If we differentiate the output variable $y(t)$, we obtain

$$
\begin{aligned}
\dot{y}(t) &= \dot{x}_{22}(t) = A_{21}^r x_1^r(t) + A_{22}^r x_{22}(t) \\
\dot{\hat{y}}(t) &= \dot{x}_{22}(t) = A_{21}^r \hat{x}_1^r(t) + A_{22}^r x_{22}(t)
\end{aligned}
\tag{4.74}
$$

The error dynamic is governed by

$$
\dot{e}_1(t) = \dot{x}_1^r(t) - \dot{\hat{x}}_1^r(t) = (A_{11}^r - K_3 A_{21}^r)\dot{e}_1(t) \tag{4.75}
$$

The change of variable is required to remove $\dot{y}(t)$ and $\dot{\hat{y}}(t)$ terms in (4.74) given by

$$
\hat{x}_1^r(t) - K_3 y(t) = \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_{21}(t) \end{bmatrix} - \begin{bmatrix} K_{31} \\ K_{32} \end{bmatrix} y(t) = \hat{z}_1^r(t) = \begin{bmatrix} \hat{z}_1(t) \\ \hat{z}_{21}(t) \end{bmatrix} \tag{4.76}
$$

Applying the change of variable (4.76), (4.73) leads to

$$\dot{\hat{z}}_1^r(t) = A_{z,2}^r \hat{z}_1^r(t) + K_{z,2}^r y(t) \tag{4.77}$$

where

$$\begin{aligned}
A_{z,2}^r &= A_{11}^r - K_3 A_{21}^r = \begin{bmatrix} a_{11} & a_{12} \\ \frac{1}{\epsilon}a_{21} & \frac{1}{\epsilon}a_{22} \end{bmatrix} - \begin{bmatrix} K_{31} \\ K_{32} \end{bmatrix} \begin{bmatrix} \frac{1}{\epsilon}a_{31} & \frac{1}{\epsilon}a_{32} \end{bmatrix}, \\
K_{z,2}^r &= A_{12}^r - K_3 A_{22}^r + A_{11}^r K_3 - K_3 A_{21}^r K_3 \\
&= \begin{bmatrix} a_{21} - K_{21}a_{11} + a_{22}K_{21} + \frac{1}{\epsilon}a_{23}K_{22} - K_{21}(a_{12}K_{21} + \frac{1}{\epsilon}a_{13}K_{22}) \\ \frac{1}{\epsilon}a_{31} - \frac{1}{\epsilon}K_{22}a_{11} + \frac{1}{\epsilon}a_{32}K_{21} + \frac{1}{\epsilon^2}a_{33}K_{22} - \frac{1}{\epsilon}K_{22}(a_{12}K_{21} + \frac{1}{\epsilon}a_{13}K_{22}) \end{bmatrix} = \begin{bmatrix} K_{21r} \\ K_{22r} \end{bmatrix}
\end{aligned} \tag{4.78}$$

Since K_3 is determined by the eigenvalue assignment in terms of two matrices A_{11}^r, A_{21}^r, we can apply the two stage method to overcome numerical ill-conditioning problem coming from the singular perturbation parameter in matrix A_{11}^r.

Here, we are going to use the duality between the controller and the observer so that it will be needed to transpose matrices A_{22}^r and $K_2 A_{12}^r$ and consider a hypothetical control system, that is

$$\begin{aligned}
\dot{q}_1(t) &= a_{11}^T q_1(t) + \frac{1}{\epsilon} a_{21}^T q_{21}(t) + \frac{1}{\epsilon} a_{31}^T \hat{u}(t) \\
\dot{q}_{21}(t) &= a_{12}^T q_1(t) + \frac{1}{\epsilon} a_{22}^T q_{21}(t) + \frac{1}{\epsilon} a_{32}^T \hat{u}(t)
\end{aligned} \tag{4.79}$$

where $\hat{u}(t) = -K_3^T q(t) = -\begin{bmatrix} K_{31}^T & K_{32}^T \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_{21}(t) \end{bmatrix}$. States $q_1(t), q_{21}(t)$ are used for the purpose of design only. Here, the goal is to find a reduced-order observer gain K_3 using the two-stage method. To transform (4.79) into an explicit singularly perturbed form, we introduce $r_1(t) = q_1(t)$ and $r_{21}(t) = \frac{1}{\epsilon} q_{21}(t)$ which leads to

$$\begin{aligned}
\dot{r}_1(t) &= a_{11}^T r_1(t) + a_{21}^T r_{21}(t) + \frac{1}{\epsilon} a_{31}^T \hat{u}(t) \\
\epsilon \dot{r}_{21}(t) &= a_{12}^T r_1(t) + a_{22}^T r_{21}(t) + \frac{1}{\epsilon} a_{32}^T \hat{u}(t)
\end{aligned} \tag{4.80}$$

The Chang transformation applied to (4.80) produces

$$\begin{aligned}\dot{r}_{s,2}(t) &= A_{sr,2}^T r_{s,2}(t) + c_{sr,2}^T \hat{u}(t)\\ \epsilon\dot{r}_{f,2}(t) &= A_{fr,2}^T r_{f,2}(t) + c_{fr,2}^T \hat{u}(t)\end{aligned} \tag{4.81}$$

where

$$\begin{aligned}&A_{sr,2}^T = a_{11}^T - L_{r,2}^T a_{12}^T,\ A_{fr,2}^T = a_{22}^T + \epsilon a_{12}^T L_{r,2}^T\\ &C_{sr,2}^T = \frac{1}{\epsilon}a_{31}^T - \frac{1}{\epsilon}L_{r,2}^T a_{32}^T,\ c_{fr,2}^T = H_{r,2}^T a_{31}^T + \frac{1}{\epsilon}(I_{(n_2-l)} - \epsilon H_{r,2}^T L_{r,2}^T) a_{32}^T\end{aligned} \tag{4.82}$$

The goal is to find the observer gain K_3 using the two stage feedback design. The Chang transformation needed for the proposed observer design relates the original state variables $r_1(t)$ and $r_{21}(t)$ and the slow and fast variables $r_{s,2}(t)$ and $r_{f,2}(t)$ as follows

$$\begin{bmatrix} r_{s,2}(t)\\ r_{f,2}(t)\end{bmatrix} = \begin{bmatrix} I_{n_1} & -\epsilon L_{r,2}^T\\ H_{r,2}^T & I_{(n_2-l)} - \epsilon H_{r,2}^T L_{r,2}^T\end{bmatrix}\begin{bmatrix} r_1(t)\\ r_{21}(t)\end{bmatrix} = T_{cr,2}^T\begin{bmatrix} r_1(t)\\ r_{21}(t)\end{bmatrix} \tag{4.83}$$

The state variables $r_1(t)$ and $r_{21}(t)$ can be reconstructed from the inverse transformation as

$$\begin{bmatrix} r_1(t)\\ r_{21}(t)\end{bmatrix} = \begin{bmatrix} I_{n_1} - \epsilon L_{r,2}^T H_{r,2}^T & \epsilon L_{r,2}^T\\ -H_{r,2}^T & I_{(n_2-l)}\end{bmatrix}\begin{bmatrix} r_{s,2}(t)\\ r_{f,2}(t)\end{bmatrix} = T_{cr,2}^{-T}\begin{bmatrix} r_{s,2}(t)\\ r_{f,2}(t)\end{bmatrix} \tag{4.84}$$

where $L_{r,2}^T$ and $H_{r,2}^T$ are the solution given as

$$\begin{aligned}0 &= \epsilon(a_{22}^T - L_{r,2}^T a_{23}^T)L_{r,2}^T + (a_{32}^T - L_{r,2}^T a_{33}^T)\\ 0 &= \epsilon H_{r,2}^T(a_{22}^T - L_{r,2}^T a_{23}^T) + a_{23}^T - (a_{33}^T + \epsilon a_{23}^T L_{r,2}^T)H_{r,2}^T\end{aligned} \tag{4.85}$$

We take $\hat{u}(t)$ for the slow subsystem as

$$\hat{u}(t) = -K_{sr,2}^T r_{s,2}(t) + v(t) \tag{4.86}$$

Substituting (4.86) to (4.81), (4.81) becomes

$$\begin{aligned} \dot{r}_{s,2}(t) &= (A_{sr,2}^T - C_{sr,2}^T K_{sr,2}^T) r_{s,2}(t) + C_{sr,2}^T v(t) \\ \epsilon \dot{r}_{f,2}(t) &= A_{fr,2}^T r_{f,2}(t) - \epsilon C_{fr,2}^T K_{sr,2}^T r_{s,2}(t) + \epsilon C_{fr,2}^T v(t) \end{aligned} \tag{4.87}$$

At this point, it is possible to place the slow observer eigenvalues in the desired locations, that is

$$\lambda(A_{sr,2}^T - C_{sr,2}^T K_{sr,2}^T) = \lambda(A_{sr,2} - K_{sr,2} C_{sr,2}) = \lambda_s^{desired} \tag{4.88}$$

assuming that the following assumption is satisfied.

Assumption 4.5.1. *The pair* $(A_{sr,2}, C_{sr,2})$ *is observable.*

Now, the following change of coordinates is introduced

$$r_{fnew,2}(t) = P_{or,2} r_{s,2}(t) + r_{f,2}(t) \;\rightarrow\; r_{f,2}(t) = r_{fnew,2}(t) - P_{or,2} r_{s,2}(t) \tag{4.89}$$

where $P_{or,2}$ satisfies the algebraic Sylvester equation

$$\epsilon P_{or,2}(A_{sr,2}^T - C_{sr,2}^T K_{sr,2}^T) - C_{fr,2}^T K_{sr,2}^T - A_{fr,2}^T P_{or,2} = 0 \;\Rightarrow\; P_{or,2} = O(1) \tag{4.90}$$

The unique solution for $P_{or,2}$ exist for sufficiently small values of ϵ under Assumption 3.2.1. The change of variables in (4.89) results in

$$\begin{aligned} &\epsilon \dot{r}_{fnew,2}(t) = \epsilon P_{or,2} \dot{r}_{s,2}(t) + \epsilon \dot{r}_{f,2}(t) \\ &= [-A_{fr,2}^T P_{or,2} - C_{fr,2}^T K_{sr,2}^T + \epsilon P_{or,2}(A_{sr,2}^T - C_{sr,2}^T K_{sr,2}^T)] r_{s,2}(t) + A_{fr,2}^T r_{fnew,2}(t) \\ &+ (C_{fr,2}^T + \epsilon P_{or,2} C_{sr,2}^T) v(t) \end{aligned} \tag{4.91}$$

When the Sylvester equation (4.90) is satisfied, (4.91) becomes

$$\epsilon \dot{r}_{fnew,2}(t) = A_{fr,2}^T r_{fnew,2}(t) + (C_{fr,2}^T + \epsilon P_{or,2} C_{sr,2}^T) v(t) = A_{fr,2}^T r_{fnew,2}(t) + C_{fnewr,2}^T v(t) \tag{4.92}$$

The input $v(t)$ can be used to locate the fast subsystem eigenvalues

$$v(t) = -K_{f2r,2}^T r_{fnew,2}(t) \tag{4.93}$$

At this point, it is possible to locate the fast eigenvalues in the original coordinates at the desired location as

$$\lambda(A_{fr,2} - K_{f2r,2}C_{fnewr,2}) = \lambda_f^{desired} \tag{4.94}$$

If the following observability assumption is satisfied.

Assumption 4.5.2. *The pair $(A_{fr,2}, C_{fnewr,2})$ is observable.*

Substituting (4.86) and (4.93) into (4.87) and (4.92), we obtain

$$\begin{bmatrix} \dot{r}_{s,2}(t) \\ \epsilon\dot{r}_{fnew,2}(t) \end{bmatrix} = \begin{bmatrix} (A_{sr,2} - K_{sr,2}C_{sr,2})^T & -(K_{f2r,2}C_{sr,2})^T \\ 0 & (A_{fr,2} - K_{f2r,2}C_{fnewr,2})^T \end{bmatrix} \begin{bmatrix} r_{s,2}(t) \\ r_{fnew,2}(t) \end{bmatrix} \tag{4.95}$$

The original coordinates $\hat{q}_1(t), \hat{q}_{21}(t)$ and $r_{s,2}(t), r_{fnew,2}(t)$ coordinates are related via

$$\begin{bmatrix} r_{s,2}(t) \\ r_{fnew,2}(t) \end{bmatrix} = T_{2r,2}^T T_{cr,2}^T T_{1r,2}^T \begin{bmatrix} \hat{q}_1(t) \\ \hat{q}_{21}(t) \end{bmatrix} \tag{4.96}$$

where

$$T_{1r,2}^T = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \frac{1}{\epsilon}I_{(n_2-l)} \end{bmatrix}, \; T_{2r,2}^T = \begin{bmatrix} I_{n_1} & 0 \\ P_{or,2} & I_{(n_2-l)} \end{bmatrix} \tag{4.97}$$

with $T_{cr,2}^T$ defined in (4.83). It is possible to reconstruct $\hat{q}_1(t)$, $\hat{q}_{21}(t)$ from $r_{s,2}(t)$, $r_{fnew,2}(t)$ via the inverse transformation

$$\begin{bmatrix} \hat{q}_1(t) \\ \hat{q}_{21}(t) \end{bmatrix} = T_{1r,2}^{-T} T_{cr,2}^{-T} T_{2r,2}^{-T} \begin{bmatrix} r_{s,2}(t) \\ r_{fnew,2}(t) \end{bmatrix} = T_{4r,2}^{-T} \begin{bmatrix} r_{s,2}(t) \\ r_{fnew,2}(t) \end{bmatrix} \tag{4.98}$$

From the above relation (4.98), we can construct the state transformation from $z_{s,2}(t)$,

$z_{fnew,2}(t)$ to $q_1(t)$, $q_{21}(t)$ as follows

$$\begin{bmatrix} \hat{z}_1(t) \\ \hat{z}_{21}(t) \end{bmatrix} = T_{4r,2} \begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{fnew,2}(t) \end{bmatrix} \tag{4.99}$$

Applying the state transformation (4.99) to (4.77), we get

$$\begin{aligned} T_{4r,2} \begin{bmatrix} \dot{\hat{z}}_{s,2}(t) \\ \dot{\hat{z}}_{fnew,2}(t) \end{bmatrix} &= (A_{11}^r - K_3 A_{21}^r) T_{4r,2} \begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{fnew,2}(t) \end{bmatrix} + K_z^r y(t) \\ \begin{bmatrix} \dot{\hat{z}}_{s,2}(t) \\ \dot{\hat{z}}_{fnew,2}(t) \end{bmatrix} &= T_{4r,2}^{-1} (A_{11}^r - K_3 A_{21}^r) T_{4r,2} \begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{fnew,2}(t) \end{bmatrix} + T_{4r,2}^{-1} K_{z,2}^r y(t) \end{aligned} \tag{4.100}$$

Now we can present the observer configuration using the result in (4.95) and the duality between the controller and the observer designs

$$\begin{aligned} \begin{bmatrix} \dot{\hat{z}}_{s,2}(t) \\ \epsilon \dot{\hat{z}}_{fnew,2}(t) \end{bmatrix} &= \begin{bmatrix} A_{sr,2} - K_{sr,2} C_{sr,2} & 0 \\ -\epsilon K_{f2r,2} C_{sr,2} & A_{fr,2} - K_{f2r,2} C_{fnewr,2} \end{bmatrix} \begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{fnew,2}(t) \end{bmatrix} \\ &+ \begin{bmatrix} K_{sr,2}^* \\ K_{f2r,2}^* \end{bmatrix} y(t) \end{aligned} \tag{4.101}$$

where $K_{sr,2}^*$, $\frac{1}{\epsilon} K_{f2r,2}^*$ can be obtained from $T_{4r,2}^{-1} K_{z,2}^r{}^r$. We can obtain a fully decoupled

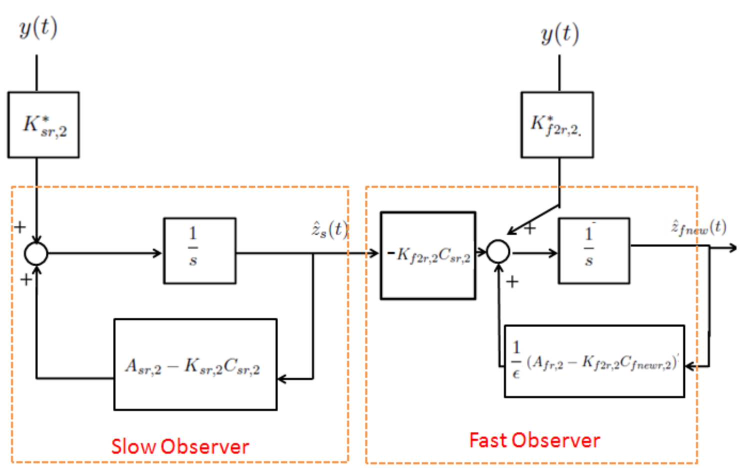

Figure 4.8: Case IV : Sequential reduced-order slow and fast observers for the reduced-order observer

slow and fast reduced-order observers working in parallel as follows. We change the coordinates once again given as

$$\hat{z}_{fnew2,2}(t) = P_{o2r,2}\hat{z}_{s,2}(t) + \hat{z}_{fnew,2}(t) \;\rightarrow\; \hat{z}_{fnew,2}(t) = \hat{z}_{fnew2,2}(t) - P_{o2r,2}\hat{z}_{s,2}(t) \quad (4.102)$$

where $P_{o2r,2}$ satisfies the algebraic Sylvester equation represented by

$$\begin{aligned} &\epsilon P_{o2r,2}(A_{sr,2} - K_{sr,2}C_{sr,2}) - \epsilon K_{f2r,2}C_{sr,2} - (A_{fr,2} - K_{f2r,2}C_{fnewr,2})P_{o2r,2} = 0 \\ &\Rightarrow\; P^0_{o2r,2} = O(1) \end{aligned} \quad (4.103)$$

The linear algebraic equation (4.103) has a unique solution since $A_{fr,2} - K_{f2r,2}C_{fnewr,2}$ is an asymptotically stable fast subsystem feedback matrix. The change of variable (4.102) results in

$$\begin{aligned} \epsilon\dot{\hat{z}}_{fnew2,2}(t) &= \epsilon P_{o2r,2}\dot{\hat{z}}_{s,2}(t) + \epsilon\dot{\hat{z}}_{fnew2,2}(t) \\ &= [\epsilon P_{o2r,2}(A_{sr,2} - K_{sr,2}C_{sr,2}) - \epsilon K_{f2r,2}C_{sr,2} \\ &- (A_{fr,2} - K_{f2r,2}C_{fnewr,2})P_{o2r,2}]\hat{z}_{s,2}(t) \\ &+ (A_{fr,2} - K_{f2r,2}C_{fnewr,2})\hat{z}_{fnew2,2}(t) + K_{f3r,2}y(t) \end{aligned} \quad (4.104)$$

where

$$K_{f3r,2} = \epsilon(P_{o2r,2}K_{sr,2} + K_{f2r,2}) \quad (4.105)$$

Hence, if the second algebraic Sylvester equation (4.103) is satisfied, (4.104) becomes

$$\epsilon\dot{\hat{z}}_{fnew2,2}(t) = (A_{fr,2} - K_{f2r,2}C_{fnewr,2})\hat{z}_{fnew2,2}(t) + K_{f3r,2}y(t) \quad (4.106)$$

At this point, we have the block-diagonalized form of the observer obtained as

$$\begin{aligned} \dot{\hat{z}}_{s,2}(t) &= (A_{sr,2} - K_{sr,2}C_{sr,2})\hat{z}_{s,2}(t) + K_{sr,2}y(t) \\ \epsilon\dot{\hat{z}}_{fnew2,2}(t) &= (A_{fr,2} - K_{f2r,2}C_{fnewr,2})\hat{z}_{fnew2,2}(t) + K_{f3r,2}y(t) \end{aligned} \quad (4.107)$$

The original coordinates $\hat{z}_1(t)$, $\hat{z}_{21}(t)$ and the new coordinates $\hat{z}_{s,2}(t)$, $\hat{z}_{fnew2,2}(t)$ are related via

$$\begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{fnew2,2}(t) \end{bmatrix} = T_{3r,2}T_{4r,2}^{-1} \begin{bmatrix} \hat{z}_1(t) \\ \hat{z}_{21}(t) \end{bmatrix} = T_{r,2}^{-1} \begin{bmatrix} \hat{z}_1(t) \\ \hat{z}_{21}(t) \end{bmatrix} \tag{4.108}$$

where

$$T_{3r,2} = \begin{bmatrix} I_{n_1} & 0 \\ P_{o2r,2} & I_{n_2-l} \end{bmatrix} \tag{4.109}$$

Now, the original coordinates can be reconstructed via

$$\begin{bmatrix} \hat{z}_1(t) \\ \hat{z}_{21}(t) \end{bmatrix} = T_{4r,2}T_{3r,2}^{-1} \begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{fnew2,2}(t) \end{bmatrix} = T_{r,2} \begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{fnew2,2}(t) \end{bmatrix} \tag{4.110}$$

At this point, the original state $\hat{x}_1(t)$ and $\hat{x}_{21}(t)$ can be reconstructed in terms of (4.76) and (4.110) given as

$$\begin{bmatrix} \hat{z}_1(t) \\ \hat{z}_{21}(t) \end{bmatrix} + \begin{bmatrix} K_{31} \\ K_{32} \end{bmatrix} y(t) = \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_{21}(t) \end{bmatrix} \tag{4.111}$$

$$\hat{z}_1^r + K_3 y(t) = \hat{x}_1^r(t)$$

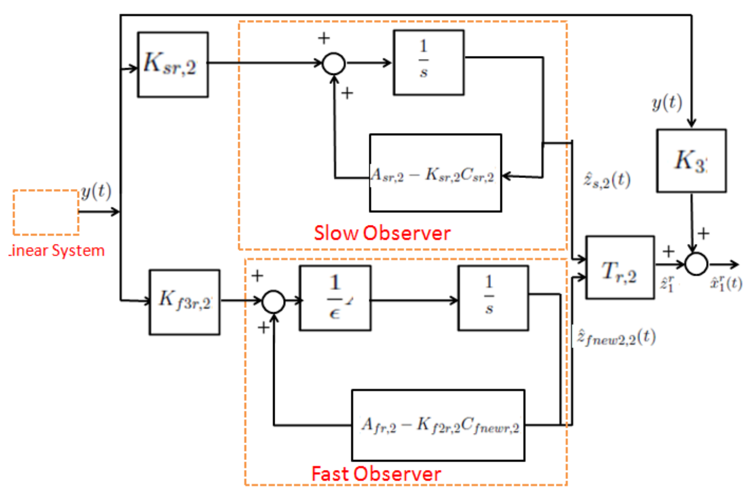

Figure 4.9: Case IV : Slow-fast reduced-order parallel observation with the reduced-order observers of dimensions n_1 and $(n_2 - l)$, $n_1 + (n_2 - l) = (n - l)$, $(n - l)$ = order of unmeasurable states of the system.

4.5.1 Case IV : Reduced-order Observation Error Equations

The error equation given in (4.75) is rewritten as

$$\dot{e}_1^r(t) = \dot{x}_1^r(t) - \dot{\hat{x}}_1^r(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_{21}(t) \end{bmatrix} - \begin{bmatrix} \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_{21}(t) \end{bmatrix} = \begin{bmatrix} \dot{e}_1(t) \\ \dot{e}_{21}(t) \end{bmatrix} = (A_{11}^r - K_3 A_{21}^r) \begin{bmatrix} e_1(t) \\ e_{21}(t) \end{bmatrix} \tag{4.112}$$

Using the state transformation defined in (4.110), the original error coordinates $e_1(t), e_{21}(t)$ and the new error coordinates $e_{s,2}^r(t), e_{fnew2,2}^r(t)$ are related via

$$\begin{aligned} \begin{bmatrix} e_1(t) \\ e_{21}(t) \end{bmatrix} &= \begin{bmatrix} x_1(t) \\ x_{21}(t) \end{bmatrix} - \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_{21}(t) \end{bmatrix} = T_{r,2} \begin{bmatrix} z_{s,2}(t) \\ z_{fnew2,2}(t) \end{bmatrix} - T_{r,2} \begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{x}_{fnew2,2}(t) \end{bmatrix} \\ &= T_{r,2} \begin{bmatrix} e_{s,2}^r(t) \\ e_{fnew2,2}^r(t) \end{bmatrix} \end{aligned} \tag{4.113}$$

Applying the state transformation (4.113), (4.112) becomes

$$T_{r,2}^{-1} \begin{bmatrix} \dot{e}_1(t) \\ \epsilon \dot{e}_{21}(t) \end{bmatrix} = T_{r,2}^{-1} (A_{11}^r - K_3 A_{21}^r) T_{r,2} \begin{bmatrix} \hat{e}_{s,2}^r(t) \\ \hat{e}_{fnew2,2}^r(t) \end{bmatrix} \tag{4.114}$$

Analytical result for (4.114) is given as

$$\begin{aligned} \dot{e}_{s,2}^r(t) &= \hat{A}_{sr,2} e_{s,2}^r(t) \\ \epsilon \dot{e}_{fnew2,2}^r(t) &= \hat{A}_{fr,2} e_{fnew2,2}^r(t) \end{aligned} \tag{4.115}$$

where

$$\begin{aligned} \hat{A}_{sr,2} &= A_{sr,2} - K_{sr,2} C_{sr,2} \\ \hat{A}_{fr,2} &= A_{fr,2} - K_{f2r,2} C_{fnewr,2} \end{aligned} \tag{4.116}$$

The convergence of the error dynamics will be obtained under the eigenvalues condition given as

$$Re\lambda(\hat{A}_{sr,2}) < 0, \;\; Re\lambda(\hat{A}_{fr,2}) < 0 \tag{4.117}$$

4.5.2 Case IV : Reduced-order Observer Gain in the Original Coordinates

We will show that the observer in the original coordinates is given by

$$K_3 = (\begin{bmatrix} K_{sr,2}^T + K_{f2r,2}^T P_{or,2} & K_{f2r,2}^T \end{bmatrix} T_{cr,2}^T T_{1r,2}^T)^T = \begin{bmatrix} T_{1r,2} T_{cr,2} (K_{sr,2} + P_{or,2}^T K_{f2r,2}) \\ T_{1r,2} T_{cr,2} K_{f2r,2} \end{bmatrix}$$
$$= \begin{bmatrix} K_{31} \\ K_{32} \end{bmatrix} \tag{4.118}$$

where $T_{cr,2}$ is the Chang transformation (4.83), $P_{or,2}$ is the solution of the algebraic Sylvester equation (4.103). We previously set $K_3^T r(t) = v(t) - K_{sr,2}^T r_{s,2}(t)$ $= -K_{sr,2}^T r_{s,2}(t) - K_{f2r,2}^T r_{fnew,2}(t)$ in (4.86) and (4.93) , which implies

$$K_3^T r(t) = \begin{bmatrix} K_{sr,2}^T & K_{f2r,2}^T \end{bmatrix} \begin{bmatrix} r_{s,2}(t) \\ r_{fnew,2}(t) \end{bmatrix} = \begin{bmatrix} K_{sr,2}^T & K_{f2r,2}^T \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 \\ P_{or,2} & I_{(n_2-l)} \end{bmatrix} \begin{bmatrix} r_{s,2}(t) \\ r_{f,2}(t) \end{bmatrix}$$
$$= \begin{bmatrix} K_{sr,2}^T + K_{f2r,2}^T P_{or,2} & K_{f2r,2}^T \end{bmatrix} T_{cr,2}^T \begin{bmatrix} r_1(t) \\ r_{21}(t) \end{bmatrix}$$
$$= \begin{bmatrix} K_{sr,2}^T + K_{f2r,2}^T P_{or,2} & K_{f2r,2}^T \end{bmatrix} T_{cr,2}^T T_{1r,2}^T \begin{bmatrix} q_1(t) \\ q_{21}(t) \end{bmatrix} \tag{4.119}$$

Hence $\begin{bmatrix} K_{sr,2}^T + K_{f2r,2}^T P_{or,2} & K_{f2r,2}^T \end{bmatrix} T_{cr,2}^T T_{1r,2}^T$ represents transpose of the observer gain matrix K_3 in the original coordinates. It is important to notice that the observer gain $K_3 = f(K_{sr,2}, K_{f2r,2})$ can be obtained using computations with reduced order matrices $K_{f2r}, K_{f2r,2}$. From this fact, the observer gain matrix K_3 is given by (4.118).

4.5.3 Case IV : Design Algorithm for Finding the Reduced-order Observer Gain

Given that the linear system (A_{11}^r, A_{21}^r) is observable, the following two-time scale design algorithm can be applied for the design of a reduced-order observer for a singularly perturbed system.

Step 1. Transpose matrices in (4.77) and apply the change of variable to the hypothetical system defined in (4.79).

Step 2. Apply the Chang transformation (4.83) to (4.80) to get (4.81).

Step 3. Obtain the partitioned sub-matrices A^T_{sr2}, $\frac{1}{\epsilon}A^T_{fr2}$, C^T_{sr2} and C^T_{fr2}.

Step 4. Place the slow observer eigenvalues in the desired location and obtain the slow observer gain K^T_{sr2} using the eigenvalue assignment for $\lambda(A_{sr2} - K_{sr2}C_{sr2})$.

Step 5. Solve the reduced-order Sylvester algebraic equation (4.90) to get P_{or2}.

Step 6. Place fast observer eigenvalues at the desired location using the eigenvalue assignment for $\frac{1}{\epsilon}\lambda(A_{fr2} - K_{f3r}C_{fnewr2})$ and obtain K_{f3r}.

Step 7. Find the reduced-order observer gain K_3 in the original coordinates using (4.67) and check $\lambda(A^r_{11} - K_3A^r_{21}) = \lambda^{desired}_s \cup \lambda^{desired}_f$.

4.5.4 Example 4.4

Consider a $4^{th}-$ order system with the system matrices A and C given as

$$A = \begin{bmatrix} 0 & 0 & 0 & -1.0000 \\ 0 & -0.5240 & -0.4650 & 0.2620 \\ -6.5400 & -5.7800 & -3.4500 & 0 \\ 0 & -4.0000 & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}. \ C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

Our goal is to design independently slow and fast reduced-order observers with desired $\lambda^{desired}_s = \{-1, -2\}$ and $\frac{1}{\epsilon}\lambda^{desired}_f = \{-10\}$. The observability matrix has full rank and therefore the pair (A^r_{22}, A^r_{12}) is observable.

According to Steps 1 and 2 of the Algorithm from Section 4.5.3 in Case IV, the following sub-matrices are obtained

$$A^T_{sr,2} = \begin{bmatrix} 0 & 0.7268 \\ 0 & 0.2103 \end{bmatrix}, \ \frac{1}{\epsilon}A^T_{fr,2} = \begin{bmatrix} -4.184 \end{bmatrix},$$

$$C^T_{sr,2} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}, \ C^T_{fr,2} = \begin{bmatrix} -0.4232 \end{bmatrix}, \ C^T_{fnewr,2} = \begin{bmatrix} -1.1190 \end{bmatrix}$$

Following Step 4 in Case IV, we place the slow eigenvalues in the original coordinates at $\{-50, -60\}$ via the slow feedback gain matrix

$$K_{sr,2}^T = \begin{bmatrix} -0.6880 & -0.8026 \end{bmatrix}$$

In Step 3 of the algorithm, we solve the Sylvester algebraic equation and obtain matrix P_{or} as

$$P_{or,2} = \begin{bmatrix} 1.8397 & 1.7393 \end{bmatrix}, \; P_{o2r,2} = \begin{bmatrix} -0.0210 & 0.2887 \end{bmatrix}$$

In Step 4 of the algorithm, we place fast observer's eigenvalues at the desired location $\{-28, -32\}$. The fast observer gain K_{f2}^T is given by

$$K_{f2r,2}^T = \begin{bmatrix} -0.5197 \end{bmatrix}, \; \frac{1}{\epsilon} K_{f3r,2} = \begin{bmatrix} 0.7441 \end{bmatrix}$$

Step 5. Using (4.67), matrix K_2 is obtained as

$$K_3 = 10^4 \times \begin{bmatrix} -1.6441 \\ -2.2565 \\ 0.9356 \end{bmatrix}$$

It can be checked that $\lambda(A_{22}^r - K_2 A_{12}^r)$ in the original coordinate are given by

$$\lambda(A_{11}^r - K_3 A_{21}^r) = \begin{bmatrix} -0.999999999996988 \\ -2.000000000003534 \\ -9.999999999999481 \end{bmatrix}$$

which is the same (with the accuracy of $O(10^{-14})$) as we placed the slow and fast eigenvalues using the two time scale decomposition designs. Figures 4.10 presents the slow and fast observation errors. In order to be able to run MATLAB Simulink simulation we had to specify also the system states initial conditions (these initial conditions are in general not known). We have chosen them as $x_1(0) = [2, 2]$ and $x_2(0) = [2, 2]$. From

Appendix $A.7$, the initial condition for $\hat{x}_1^r(0)$ is given as

$$\hat{x}_1^r(0) = \begin{bmatrix} x_1(0) \\ x_{21}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which results in

$$\hat{z}_1^r(0) = \hat{x}_1^r(0) - K_3 x_{22}(0) = -K_3 x_{22}(0) = \begin{bmatrix} 3.2883 \\ 4.5130 \\ -1.8712 \end{bmatrix}$$

Using (4.110), we obtain $\hat{z}_{s,2}(0), \hat{z}_{fnew2,2}(0)$ as

$$\begin{bmatrix} \hat{z}_{s,2}(0) \\ \hat{z}_{fnew2,2}(0) \end{bmatrix} = T_{r,2}^{-1} \begin{bmatrix} \hat{z}_1(0) \\ \hat{z}_{21}(0) \end{bmatrix} = \begin{bmatrix} 1.3759 \\ 1.6051 \\ 1.4741 \end{bmatrix}$$

so that $z_{s,2}(0) = [1.3759, 1.6051]$ and $z_{fnew2}(0) = [1.4741]$ in MATLAB simulation for the reduced-order observer. At this point, the initial condition for the errors $e_{12}(0), e_2(0)$ are given as

$$\begin{bmatrix} e_1(0) \\ e_{21}(0) \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_{21}(0) \end{bmatrix} - \begin{bmatrix} \hat{x}_1(0) \\ \hat{x}_{21}(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

4.6 Case V : Only a Part of Slow and Fast Variables are Measured

Case V) says that the measurable states $x_{11}(t), x_{21}(t)$ are parts of the slow state $x_1(t)$ and the fast state $x_2(t)$ in the singularly perturbed linear system defined in (3.1), that

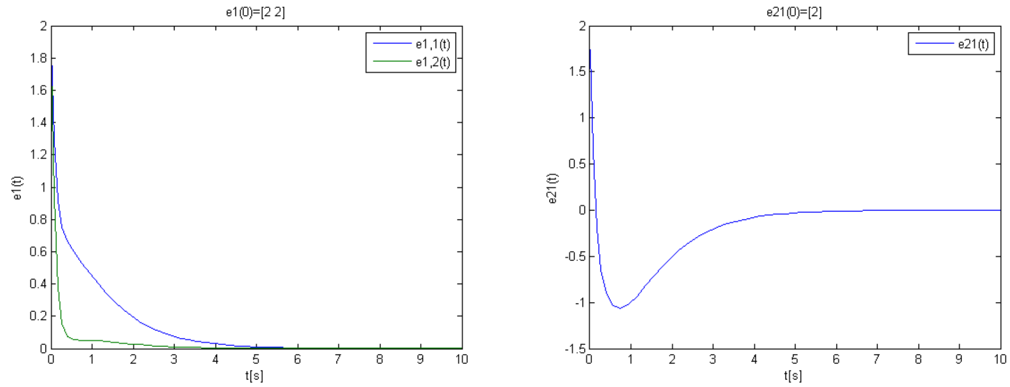

Figure 4.10: Case IV : Convergence of the slow state observation error $e_1(t) = x_1(t) - \hat{x}_1(t)$ and the fast state observation error $e_{21}(t) = x_{21}(t) - \hat{x}_{21}(t)$ for the parallel structure from Fig. 4.9

is

$$\begin{aligned}
\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) \\
\epsilon\dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) \\
y(t) &= \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \\ x_{21}(t) \\ x_{22}(t) \end{bmatrix} = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \end{bmatrix} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
\end{aligned} \tag{4.120}$$

where

$$\begin{aligned}
x_1(t) &= \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix}, x_2(t) = \begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix} \\
A_{11} &= \begin{bmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{bmatrix}, \ A_{12} = \begin{bmatrix} a_{13}^* & a_{14}^* \\ a_{23}^* & a_{24}^* \end{bmatrix} \\
A_{21} &= \begin{bmatrix} a_{31}^* & a_{32}^* \\ a_{41}^* & a_{42}^* \end{bmatrix}, \ A_{22} = \begin{bmatrix} a_{33}^* & a_{34}^* \\ a_{43}^* & a_{44}^* \end{bmatrix}
\end{aligned} \tag{4.121}$$

where $x_{11}(t) \in R^{l_1}, x_{12}(t) \in R^{(n_1-l_1)}, x_{21}(t) \in R^{l_2}, x_{22}(t) \in R^{(n_2-l_1)}$ and $a_{11} \in R^{l_1 \times l_1}$, $a_{12} \in R^{l_1 \times (n_1-l_1)}, a_{13} \in R^{l_1 \times l_2}, a_{14} \in R^{l_1 \times (n_2-l_2)}, a_{21} \in R^{(n_1-l_1) \times l_1}$, $a_{22} \in R^{(n_1-l_1) \times (n_1-l_1)}, a_{23} \in R^{(n_1-l_1) \times l_2}, a_{24} \in R^{(n_1-l_1) \times (n_2-l_2)}, a_{31} \in R^{l_2 \times l_1}$, $a_{32} \in R^{l_2 \times (n_1-l_1)}, a_{33} \in R^{l_2 \times l_2}, a_{34} \in R^{l_2 \times (n_2-l_2)}, a_{41} \in R^{(n_2-l_2) \times l_1}$, $a_{42} \in R^{(n_2-l_2) \times (n_1-l_1)}, a_{43} \in R^{(n_2-l_2) \times l_2}, a_{44} \in R^{(n_2-l_2) \times (n_2-l_2)}$,

$y(t) \in R^{(l_1+l_2)}$, and $p(t) \in R^{(n-l)\times 1}$.

We assume $x_{11}(t)$ and $x_{21}(t)$ are directly measured and present in $y(t)$.

The system (4.120) with information (4.121) can be redefined as

$$\begin{aligned}
\dot{x}_m(t) &= A_1^r x_m(t) + A_2^r x_u(t) \\
\dot{x}_u(t) &= A_3^r x_m(t) + A_4^r x_u(t) \\
y(t) &= \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \\ x_{21}(t) \\ x_{22}(t) \end{bmatrix} = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \end{bmatrix} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
\end{aligned} \tag{4.122}$$

where

$$\begin{aligned}
x_m(t) &= \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \end{bmatrix}, x_u(t) = \begin{bmatrix} x_{12}(t) \\ x_{22}(t) \end{bmatrix} \\
A_1^r &= \begin{bmatrix} a_{11}^* & a_{13}^* \\ \frac{1}{\epsilon}a_{31}^* & \frac{1}{\epsilon}a_{33}^* \end{bmatrix}, \ A_2^r = \begin{bmatrix} a_{12}^* & a_{14}^* \\ \frac{1}{\epsilon}a_{32}^* & \frac{1}{\epsilon}a_{34}^* \end{bmatrix} \\
A_3^r &= \begin{bmatrix} a_{21}^* & a_{23}^* \\ \frac{1}{\epsilon}a_{41}^* & \frac{1}{\epsilon}a_{43}^* \end{bmatrix}, \ A_4^r = \begin{bmatrix} a_{22}^* & a_{24}^* \\ \frac{1}{\epsilon}a_{42}^* & \frac{1}{\epsilon}a_{44}^* \end{bmatrix}
\end{aligned} \tag{4.123}$$

where $x_m(t)$ is the measurable states and $x_u(t)$ is the unmeasurable states. A_1^r, A_3^r are elements in (4.121) relevant to the measurable states, A_2^r, A_4^r are elements in (4.121) relevant to the unmeasurable states.

At this point, the above redefined system can be used to design a reduced-order observer. To construct an observer for $x_u(t)$, we use the knowledge that an observer has the same structure as the system plus the driving feedback term whose role is to reduced the estimation error to zero. The reduced-order observer with the feedback information coming from $\dot{y}(t)$ is

$$\dot{\hat{x}}_u(t) = A_3^r x_m(t) + A_4^r \hat{x}_u(t) + K_4(\dot{y}(t) - \dot{\hat{y}}(t)) \tag{4.124}$$

If we differentiate the output variable $y(t)$, we obtain

$$\begin{aligned} \dot{y}(t) &= \dot{x}_m(t) = A_1^r x_m(t) + A_2^r x_u(t) \\ \dot{\hat{y}}(t) &= \dot{x}_m(t) = A_1^r x_m(t) + A_2^r \hat{x}_u(t) \end{aligned} \tag{4.125}$$

The error dynamic is governed by

$$\dot{e}_u(t) = \dot{x}_u(t) - \dot{\hat{x}}_u(t) = (A_4^r - K_4 A_2^r) e_u(t) \tag{4.126}$$

The following assumption is needed.

Assumption 4.6.1. *The pair (A_4^r, A_2^r) is observable.*

The change of variable is required to remove $\dot{y}(t)$ terms in (4.124)

$$\hat{x}_u(t) - K_4 y(t) = \begin{bmatrix} \hat{x}_{12}(t) \\ \hat{x}_{22}(t) \end{bmatrix} - \begin{bmatrix} K_{41} \\ K_{42}(t) \end{bmatrix} y(t) = \hat{z}_u(t) = \begin{bmatrix} \hat{z}_{12}(t) \\ \hat{z}_{22}(t) \end{bmatrix} \tag{4.127}$$

Applying the change of variable (4.25) into (4.22), (4.22) leads to

$$\dot{\hat{z}}_u(t) = A_{z,3}^r \hat{z}_u(t) + K_{z,3}^r y(t) \tag{4.128}$$

where

$$\begin{aligned} A_{z,3}^r &= A_4^r - K_4 A_2^r = \begin{bmatrix} a_{22}^* & a_{24}^* \\ \frac{1}{\epsilon} a_{42}^* & \frac{1}{\epsilon} a_{44}^* \end{bmatrix} - \begin{bmatrix} K_{41} \\ K_{42} \end{bmatrix} \begin{bmatrix} a_{12}^* & a_{14}^* \\ \frac{1}{\epsilon} a_{32}^* & \frac{1}{\epsilon} a_{34}^* \end{bmatrix}, \\ K_{z,3}^r &= A_3^r - K_4 A_1^r + A_4^r K_4 - K_4 A_2^r K_4 \\ &= \begin{bmatrix} a_{21} - K_{21} a_{11} + a_{22} K_{21} + \frac{1}{\epsilon} a_{23} K_{22} - K_{21}(a_{12} K_{21} + \frac{1}{\epsilon} a_{13} K_{22}) \\ \frac{1}{\epsilon} a_{31} - \frac{1}{\epsilon} K_{22} a_{11} + \frac{1}{\epsilon} a_{32} K_{21} + \frac{1}{\epsilon^2} a_{33} K_{22} - \frac{1}{\epsilon} K_{22}(a_{12} K_{21} + \frac{1}{\epsilon} a_{13} K_{22}) \end{bmatrix} = \begin{bmatrix} K_{41r} \\ K_{42r} \end{bmatrix} \end{aligned} \tag{4.129}$$

Since K_4 is determined by eigenvalue assignment in terms of two matrices A_2^r, A_4^r, we can apply the two-stage method to overcome numerical ill-conditioning problem coming from the small singular perturbation parameter presented in matrix $A_{z,3}^r$. Here, we are going to use the duality between the controller and the observer so that it will be needed to transpose matrices A_4^r and $K_4 A_2^r$ and the considered hypothetical control system, that is

$$\begin{aligned}\dot{q}_{12}(t) &= {a_{22}^*}^T q_{12}(t) + \frac{1}{\epsilon}{a_{42}^*}^T q_{22}(t) + \begin{bmatrix}{a_{12}^*}^T & \frac{1}{\epsilon}{a_{32}^*}^T\end{bmatrix}\hat{u}(t)\\ \dot{q}_{22}(t) &= {a_{24}^*}^T q_{12}(t) + \frac{1}{\epsilon}{a_{44}^*}^T q_{22}(t) + \begin{bmatrix}{a_{14}^*}^T & \frac{1}{\epsilon}{a_{34}^*}^T\end{bmatrix}\hat{u}(t)\end{aligned} \tag{4.130}$$

where $\hat{u}(t) = -K_4^T q(t) = -\begin{bmatrix}K_{41}^T & K_{42}^T\end{bmatrix}\begin{bmatrix}q_{12}(t)\\ q_{22}(t)\end{bmatrix}$. States $q_{12}(t), q_{22}(t)$ are used for the purpose of design only. Here, the goal is to find a reduced-order observer gain K_4 using the two-stage method. To transform (4.130) into an explicit singularly perturbed form we introduce $r_{12}(t) = q_{12}(t)$ and $r_{22}(t) = \frac{1}{\epsilon}q_{22}(t)$ which leads to

$$\begin{aligned}\dot{r}_{12}(t) &= {a_{22}^*}^T r_{12}(t) + {a_{42}^*}^T r_{22}(t) + \begin{bmatrix}{a_{12}^*}^T & \frac{1}{\epsilon}{a_{32}^*}^T\end{bmatrix}\hat{u}(t)\\ \epsilon\dot{r}_{22}(t) &= {a_{24}^*}^T r_{12}(t) + {a_{44}^*}^T r_{22}(t) + \begin{bmatrix}{a_{14}^*}^T & \frac{1}{\epsilon}{a_{34}^*}^T\end{bmatrix}\hat{u}(t)\end{aligned} \tag{4.131}$$

The Chang transformation applied to (4.131) produces

$$\begin{aligned}\dot{r}_{s,3}(t) &= A_{sr,3}^T r_{s,3}(t) + C_{sr,3}^T \hat{u}(t)\\ \epsilon\dot{r}_{f,3}(t) &= A_{fr,3}^T r_{f,3}(t) + C_{fr,3}^T \hat{u}(t)\end{aligned} \tag{4.132}$$

where

$$C_{sr,3}^T = \begin{bmatrix}C_{sr,31}^T & C_{sr,32}^T\end{bmatrix},\ C_{fr,3}^T = \begin{bmatrix}C_{fr,31}^T & C_{fr,32}^T\end{bmatrix} \tag{4.133}$$

and

$$\begin{aligned}&A_{sr,3}^T = {a_{22}^*}^T - L_{r,3}^T {a_{24}^*}^T,\ A_{fr,3}^T = {a_{44}^*}^T + \epsilon {a_{24}^*}^T L_{r,3}^T\\ &C_{sr,31}^T = {a_{12}^*}^T - L_{r,3}^T {a_{14}^*}^T,\ C_{sr,32}^T = \frac{1}{\epsilon}{a_{32}^*}^T - L_{r,3}^T \frac{1}{\epsilon}{a_{34}^*}^T\\ &C_{fr,31}^T = \epsilon H_{r,3}^T {a_{12}^*}^T + (I_{(n_2-l_2)} - \epsilon H_{r,3}^T L_{r,3}^T){a_{14}^*}^T,\\ &C_{fr,32}^T = \epsilon H_{r,3}^T \frac{1}{\epsilon}{a_{32}^*}^T + (I_{(n_2-l_2)} - \epsilon H_{r,3}^T L_{r,3}^T)\frac{1}{\epsilon}{a_{34}^*}^T\end{aligned} \tag{4.134}$$

The goal is to find the observer gain K_4^T using the two stage feedback design. The Chang transformation needed for the proposed observer design relates the original state variables $r_{12}(t)$ and $r_{22}(t)$ and the slow and fast variables $r_{s,3}(t)$ and $r_{f,3}(t)$ as follows

$$\begin{bmatrix} r_{s,3}(t) \\ r_{f,3}(t) \end{bmatrix} = \begin{bmatrix} I_{(n_1-l_1)} & -\epsilon L_{r,3}^T \\ H_{r,3}^T & I_{(n_2-l_2)} - \epsilon H_{r,3}^T L_{r,3}^T \end{bmatrix} \begin{bmatrix} r_{12}(t) \\ r_{22}(t) \end{bmatrix} = T_{cr,3}^T \begin{bmatrix} r_{12}(t) \\ r_{22}(t) \end{bmatrix} \tag{4.135}$$

The state variables $r_{12}(t)$ and $r_{22}(t)$ can be reconstructed from the inverse transformation as

$$\begin{bmatrix} r_{12}(t) \\ r_{22}(t) \end{bmatrix} = \begin{bmatrix} I_{(n_1-l_1)} - \epsilon L_{r,3}^T H_{r,3}^T & \epsilon L_{r,3}^T \\ -H_{r,3}^T & I_{(n_2-l_2)} \end{bmatrix} \begin{bmatrix} r_{s,3}(t) \\ r_{f,3}(t) \end{bmatrix} = T_{cr,3}^{-T} \begin{bmatrix} r_{s,3}(t) \\ r_{f,3}(t) \end{bmatrix} \tag{4.136}$$

where $L_{r,3}^T$ and $H_{r,3}^T$ are the transposed solutions obtained from

$$\begin{aligned} 0 &= \epsilon({a_{22}^*}^T - L_{r,3}^T {a_{24}^*}^T) L_{r,3}^T + ({a_{42}^*}^T - L_{r,3}^T {a_{44}^*}^T) \\ 0 &= \epsilon H_{r,3}^T ({a_{22}^*}^T - L_{r,3}^T {a_{24}^*}^T) + {a_{24}^*}^T - ({a_{44}^*}^T + \epsilon {a_{24}^*}^T L_{r,3}^T) H_{r,3}^T \end{aligned} \tag{4.137}$$

We take $\hat{u}(t)$ for the slow subsystem as

$$\hat{u}(t) = -K_{sr,3}^T r_{s,3}(t) + v(t) \tag{4.138}$$

where

$$K_{sr,3}^T = \begin{bmatrix} K_{sr,31}^T \\ K_{sr,32}^T \end{bmatrix} \tag{4.139}$$

Substituting (4.138) to (4.133) produces

$$\begin{aligned} \dot{r}_{s,3}(t) &= (A_{sr,3}^T - C_{sr,3}^T K_{sr,3}^T) r_{s,3}(t) + C_{sr,3}^T v(t) \\ \epsilon \dot{r}_{f,3}(t) &= A_{fr,3}^T r_{f,3}(t) - C_{fr,3} K_{sr,3}^T r_{s,3}(t) + C_{fr,3} v(t) \end{aligned} \tag{4.140}$$

At this point, it is possible to place the slow observer eigenvalues in the desired locations, that is

$$\lambda(A_{sr,3}^T - C_{sr,3}^T K_{sr,3}^T) = \lambda(A_{sr,3} - K_{sr,3} C_{sr,3}) = \lambda_s^{desired} \tag{4.141}$$

assuming that the following assumption is satisfied.

Assumption 4.6.2. *The pair* $(A_{sr,3}, C_{sr,3})$ *is observable.*

Now, the following change of coordinates is introduced

$$r_{fnew,3}(t) = P_{or,3}r_{s,3}(t) + r_{f,3}(t) \ \rightarrow \ r_{f,3}(t) = r_{fnew,3}(t) - P_{or,3}r_{s,3}(t) \tag{4.142}$$

where $P_{or,3}$ satisfies the algebraic Sylvester equation

$$\begin{aligned} &\epsilon P_{or,3}(A^T_{sr,3} - C^T_{sr,3}K^T_{sr,3}) - C^T_{fr,3}K^T_{sr,3} - A^T_{fr,3}P_{or,3} = 0 \\ &\Rightarrow \ P_{or,3} = O(\epsilon) \end{aligned} \tag{4.143}$$

The unique solution for $P_{or,3}$ exist for sufficiently small values of ϵ under Assumption 3.2.1. The change of variables in (4.142) results in

$$\begin{aligned} &\epsilon\dot{r}_{fnew,3}(t) = \epsilon P_{or,3}\dot{r}_{s,3}(t) + \epsilon\dot{r}_{f,3}(t) \\ &= [-A^T_{fr,3}P_{or,3} - C^T_{fr,3}K^T_{sr,3} + \epsilon P_{or,3}(A^T_{sr,3} - C^T_{sr,3}K^T_{sr,3})]r_{s,3}(t) \\ &+ A^T_{fr,3}r_{fnew,3}(t) + (C^T_{fr,3} + \epsilon P_{or,3}C^T_{sr,3})v(t) \end{aligned} \tag{4.144}$$

When the Sylvester equation (4.143) is satisfied, (4.144) becomes

$$\begin{aligned} \epsilon\dot{r}_{fnew,3}(t) &= A^T_{fr,3}r_{fnew,3}(t) + (C^T_{fr,3} + \epsilon P_{or,3}C^T_{sr,3})v(t) \\ &= A^T_{fr,3}r_{fnew,3}(t) + C^T_{fnewr,3}v(t) \end{aligned} \tag{4.145}$$

The input $v(t)$ can be used to locate the fast subsystem eigenvalues

$$v(t) = -K^T_{f2r,3}r_{fnew,3}(t) \tag{4.146}$$

where

$$K^T_{f2r,3} = \begin{bmatrix} K^T_{f2r,31} \\ K^T_{f2r,32} \end{bmatrix} \tag{4.147}$$

At this point, it is possible to locate the fast eigenvalues in the original coordinates at the desired location as

$$\lambda(A_{fr,3} - K^T_{f2r,3}C_{fnewr,3}) = \lambda^{desired}_f \tag{4.148}$$

if the following observability assumption is satisfied.

Assumption 4.6.3. *The pair* $(A_{fr,3}, C_{fnewr,3})$ *is observable.*

Substituting (4.138) and (4.146) into (4.133) and (4.145), we obtain

$$\begin{bmatrix} \dot{r}_{s,3}(t) \\ \epsilon\dot{r}_{fnew,3}(t) \end{bmatrix} = \begin{bmatrix} (A_{sr,3} - K_{sr,3}C_{sr,3})^T & -(K_{f2r,3}C_{sr,3})^T \\ 0 & (A_{fr,3} - K_{f2r,3}C_{fnewr,3})^T \end{bmatrix} \begin{bmatrix} r_{s,3}(t) \\ r_{fnew,3}(t) \end{bmatrix} \tag{4.149}$$

The original coordinates $q_{12}(t), q_{22}(t)$ and $r_{s,3}(t), r_{fnew,3}(t)$ coordinates are related via

$$\begin{bmatrix} r_{s,3}(t) \\ r_{fnew,3}(t) \end{bmatrix} = T_{2r,3}^T T_{cr,3}^T T_{1r,3}^T \begin{bmatrix} \hat{q}_{12}(t) \\ \hat{q}_{22}(t) \end{bmatrix} \tag{4.150}$$

where

$$T_{1r,3}^T = \begin{bmatrix} I_{(n_1-l_1)} & 0 \\ 0 & \frac{1}{\epsilon}I_{(n_2-l_2)} \end{bmatrix}, \; T_{2r,3}^T = \begin{bmatrix} I_{(n_1-l_1)} & 0 \\ P_{or,3} & I_{(n_2-l_2)} \end{bmatrix} \tag{4.151}$$

with $T_{cr,3}^T$ defined in (4.135). It is possible to reconstruct $\hat{q}_{12}(t)$, $\hat{q}_{22}(t)$ from $r_{s,3}(t)$, $r_{fnew,3}(t)$ via the inverse transformation

$$\begin{bmatrix} \hat{q}_{12}(t) \\ \hat{q}_{22}(t) \end{bmatrix} = T_{1r,3}^{-T} T_{cr,3}^{-T} T_{2r,3}^{-T} \begin{bmatrix} r_{s,3}(t) \\ r_{fnew,3}(t) \end{bmatrix} = T_{4r,3}^{-T} \begin{bmatrix} r_{s,3}(t) \\ r_{fnew,3}(t) \end{bmatrix} \tag{4.152}$$

From the above relation (4.152), we can construct the state transformation from $z_{s,3}(t)$, $z_{fnew,3}(t)$ to $z_{12}(t)$, $z_{22}(t)$ as follows

$$\begin{bmatrix} \hat{z}_{12}(t) \\ \hat{z}_{22}(t) \end{bmatrix} = T_{4r,3} \begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew,3}(t) \end{bmatrix} \tag{4.153}$$

Applying the state transformation (4.153) to (4.128), we get

$$\begin{aligned} T_{4r,3} \begin{bmatrix} \dot{\hat{z}}_{s,3}(t) \\ \dot{\hat{z}}_{fnew,3}(t) \end{bmatrix} &= (A_4^r - K_4 A_2^r) T_{4r,3} \begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew,3}(t) \end{bmatrix} + K_{z,3}^r y(t) \\ \begin{bmatrix} \dot{\hat{z}}_{s,3}(t) \\ \dot{\hat{z}}_{fnew,3}(t) \end{bmatrix} &= T_{4r,3}^{-1}(A_4^r - K_4 A_2^r) T_{4r,3} \begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew,3}(t) \end{bmatrix} + T_{4r,3}^{-1} K_{z,3}^r y(t) \end{aligned} \tag{4.154}$$

Now we can present the observer configuration using the result in (4.149) and the duality between the controller and the observer designs

$$\begin{aligned}\begin{bmatrix}\dot{\hat{z}}_{s,3}(t)\\ \epsilon\dot{\hat{z}}_{fnew,3}(t)\end{bmatrix} &= \begin{bmatrix}A_{sr,3}-K_{sr,3}C_{sr,3} & 0\\ -\epsilon K_{f2r,3}C_{sr,3} & A_{fr,3}-K_{f2r,3}C_{fnewr,3}\end{bmatrix}\begin{bmatrix}\hat{z}_{s,3}(t)\\ \hat{z}_{fnew,3}(t)\end{bmatrix}\\ &+\begin{bmatrix}K^*_{sr,3}\\ \epsilon K^*_{f2r,3}\end{bmatrix}y(t)\end{aligned} \tag{4.155}$$

where $K^*_{sr,3}, K^*_{f2r,3}$ can be obtained from $T^{-1}_{4r,3}K^r_{z,3}$.

We can obtain a fully decoupled slow and fast reduced-order observers working in

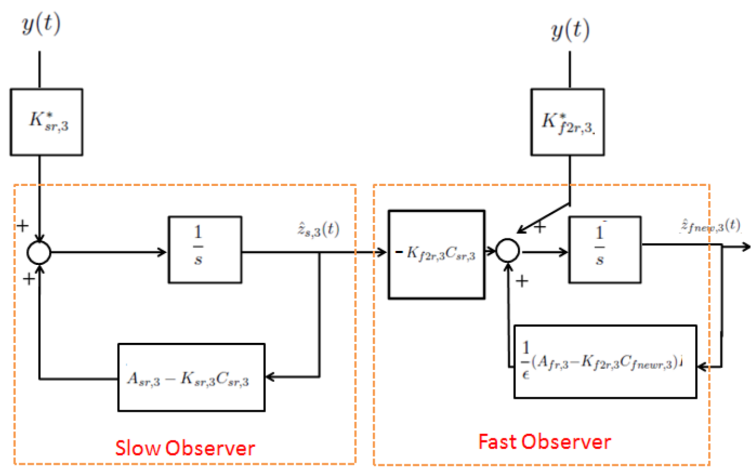

Figure 4.11: Case V : Sequential reduced-order slow and fast observers for the reduced-order observer

parallel as follows. We change the coordinates once again given as

$$\hat{z}_{fnew2,3}(t) = P_{o2r,3}\hat{z}_{s,3}(t)+\hat{z}_{fnew,3}(t) \ \rightarrow\ \hat{z}_{fnew,3}(t) = \hat{z}_{fnew2,3}(t)-P_{o2r,3}\hat{z}_{s,3}(t) \tag{4.156}$$

where $P_{o2r,3}$ satisfies the algebraic Sylvester equation represented by

$$\begin{aligned}&\epsilon P_{o2r,3}(A_{sr,3}-K_{sr,3}C_{sr,3})-\epsilon K_{f2r,3}C_{sr,3}-(A_{fr,3}-K_{f2r,3}C_{fnewr,3})P_{o2r,3}=0\\ &\Rightarrow\ P^0_{o2r,3}=O(\epsilon)\end{aligned} \tag{4.157}$$

The linear algebraic equation (4.157) has a unique solution since $A_{fr,3}-K_{f2r,3}C_{fnewr,3}$ is an asymptotically stable fast subsystem feedback matrix. The change of variable

(4.156) results in

$$\epsilon\dot{\hat{z}}_{fnew2,3}(t) = \epsilon P_{o2r,3}\dot{\hat{z}}_{s,3}(t) + \epsilon\dot{\hat{z}}_{fnew,3}(t)$$
$$= [\epsilon P_{o2r,3}(A_{sr,3} - K_{sr,3}C_{sr,3}) - \epsilon K_{f2r,3}C_{sr,3} - (A_{fr,3} - K_{f2r,3}C_{fnewr,3})P_{o2r,3}]\hat{z}_{s,3}(t)$$
$$+ (A_{fr,3} - K_{f2r,3}C_{fnewr,3})\hat{z}_{fnew2,3}(t) + K_{f3r,3}y(t) \tag{4.158}$$

where

$$K_{f3r,3} = \epsilon(P_{o2r,3}K^*_{sr,3} + K^*_{f2r,3}) \tag{4.159}$$

Hence, if the second algebraic Sylvester equation (4.157) is satisfied, (4.158) becomes

$$\epsilon\dot{\hat{z}}_{fnew2,3}(t) = (A_{fr,3} - K_{f2r,3}C_{fnewr,3})\hat{z}_{fnew2,3}(t) + K_{f3r,3}y(t) \tag{4.160}$$

At this point, we have the block-diagonalized form of the observer obtained as

$$\begin{aligned} \dot{\hat{z}}_{s,3}(t) &= (A_{sr,3} - K_{sr,3}C_{sr,3})\hat{z}_{s,3}(t) + K^*_{sr,3}y(t) \\ \epsilon\dot{\hat{z}}_{fnew2,3}(t) &= (A_{fr,3} - K_{f2r,3}C_{fnewr,3})\hat{z}_{fnew2,3}(t) + K_{f3r,3}y(t) \end{aligned} \tag{4.161}$$

The original coordinates $\hat{z}_{12}(t)$, $\hat{z}_2(t)$ and the new coordinates $\hat{z}_s(t)$, $\hat{z}_{fnew2}(t)$ are related via

$$\begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew2,3}(t) \end{bmatrix} = T_{3r,3}T_{4r,3}^{-1}\begin{bmatrix} \hat{z}_{12}(t) \\ \hat{z}_{22}(t) \end{bmatrix} = T_{r,3}^{-1}\begin{bmatrix} \hat{z}_{12}(t) \\ \hat{z}_{22}(t) \end{bmatrix} \tag{4.162}$$

where

$$T_{3r,3} = \begin{bmatrix} I & 0 \\ P_{o2r,3} & I \end{bmatrix} \tag{4.163}$$

Now, the original coordinates can be reconstructed via

$$\begin{bmatrix} \hat{z}_{12}(t) \\ \hat{z}_{22}(t) \end{bmatrix} = T_{4r,3}T_{3r,3}^{-1}\begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew2,3}(t) \end{bmatrix} = T_{r,3}\begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew2,3}(t) \end{bmatrix} \tag{4.164}$$

The original state $\hat{x}_{12}(t)$ and $\hat{x}_{22}(t)$ can be reconstructed in terms of (4.127) and (4.164) as

$$\begin{bmatrix} \hat{z}_{12}(t) \\ \hat{z}_{22}(t) \end{bmatrix} + \begin{bmatrix} K_{41} \\ K_{42} \end{bmatrix} y(t) = \begin{bmatrix} \hat{x}_{12}(t) \\ \hat{x}_{22}(t) \end{bmatrix} \tag{4.165}$$

$$\hat{z}_u(t) + K_4 y(t) = \hat{x}_u(t)$$

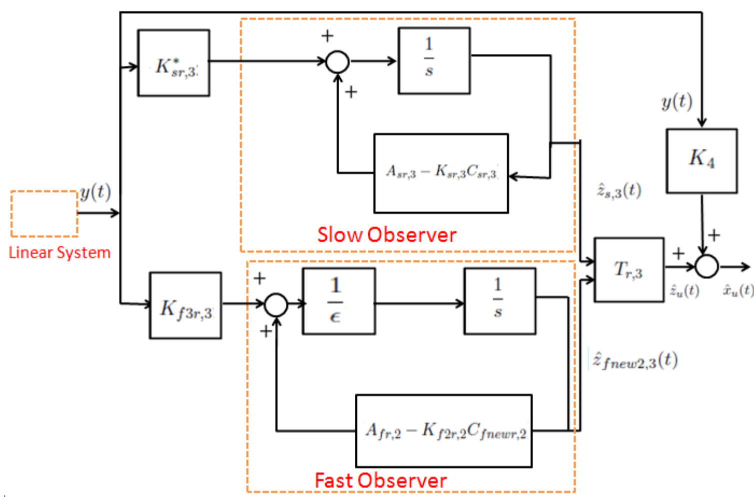

Figure 4.12: Case V : Slow-fast reduced-order parallel observation with the reduced-order observers of dimensions $(n_1 - l_1)$ and $(n_2 - l_2)$, $(n_1 - l_1) + (n_2 - l_2) = n - (l_1 + l_2)$, $(n - (l_1 + l_2))$ = order of unmeasurable states of the system.

4.6.1 Case V : Reduced-order Observation Error Equations

The error equation given in (4.126) is rewritten as

$$\begin{aligned} \dot{e}_u(t) = \dot{x}_u(t) - \dot{\hat{x}}_u(t) &= \begin{bmatrix} \dot{x}_{12}(t) \\ \dot{x}_{22}(t) \end{bmatrix} - \begin{bmatrix} \dot{\hat{x}}_{12}(t) \\ \dot{\hat{x}}_{22}(t) \end{bmatrix} = \begin{bmatrix} \dot{e}_{12}(t) \\ \dot{e}_{22}(t) \end{bmatrix} \\ &= (A_4^r - K_4 A_2^r) \begin{bmatrix} e_{12}(t) \\ e_{22}(t) \end{bmatrix} \end{aligned} \tag{4.166}$$

Using the state transformation defined in (4.164), the original error coordinates $e_{12}(t)$, $e_{22}(t)$ and the new error coordinates $e^r_{s,3}(t), e^r_{fnew2,3}(t)$ are related via

$$\begin{aligned}\begin{bmatrix} e_{12}(t) \\ e_{22}(t) \end{bmatrix} &= \begin{bmatrix} x_{12}(t) \\ x_{22}(t) \end{bmatrix} - \begin{bmatrix} \hat{x}_{12}(t) \\ \hat{x}_{22}(t) \end{bmatrix} = T_{r,3} \begin{bmatrix} z_{s,3}(t) \\ z_{fnew2,3}(t) \end{bmatrix} - T_{r,3} \begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew2,3}(t) \end{bmatrix} \\ &= T_{r,3} \begin{bmatrix} e^r_{s,3}(t) \\ e^r_{fnew2,3}(t) \end{bmatrix}\end{aligned} \tag{4.167}$$

Applying the state transformation (4.167) to (4.166) produces

$$T^{-1}_{r,3} \begin{bmatrix} \dot{e}_{12}(t) \\ \epsilon \dot{e}_{22}(t) \end{bmatrix} = T^{-1}_{r,3}(A^r_4 - K_4 A^r_2) T_{r,3} \begin{bmatrix} \hat{e}^r_{s,3}(t) \\ \hat{e}^r_{fnew2,3}(t) \end{bmatrix} \tag{4.168}$$

Analytical result for (4.168) is given as

$$\begin{aligned}\dot{e}^r_{s,3}(t) &= \hat{A}_{sr,3} e^r_{s,3}(t) \\ \epsilon \dot{e}^r_{fnew2,3}(t) &= \hat{A}_{fr,3} e^r_{fnew2,3}(t)\end{aligned} \tag{4.169}$$

where

$$\begin{aligned}\hat{A}_{sr,3} &= A_{sr,3} - K_{sr,3} C_{sr,3} \\ \hat{A}_{fr,3} &= A_{fr,3} - K_{f2r,3} C_{fnewr,3}\end{aligned} \tag{4.170}$$

The convergence of the error dynamic will be obtained under the eigenvalues condition given as

$$Re\lambda(\hat{A}_{sr,3}) < 0, \;\; Re\lambda(\hat{A}_{fr,3}) < 0 \tag{4.171}$$

4.6.2 Case V : Reduced-order Observer Gain in the Original Coordinates

We will show that the observer in the original coordinates is given by

$$K_4 = (\begin{bmatrix} K_{sr,3}^T + K_{f2r,3}^T P_{or,3} & K_{f2r,3}^T \end{bmatrix} T_{cr,3}^T T_{1r,3}^T)^T = \begin{bmatrix} T_{1r,3} T_{cr,3}(K_{sr,3} + P_{or,3}^T K_{f2r,3}) \\ T_{1r,3} T_{cr,3} K_{f2r,3} \end{bmatrix}$$
$$= \begin{bmatrix} K_{41} \\ K_{42} \end{bmatrix} \tag{4.172}$$

where

$$K_{sr,3}^T = \begin{bmatrix} K_{sr,31}^T \\ K_{sr,32}^T \end{bmatrix}, K_{f2r,3}^T = \begin{bmatrix} K_{f2r,31}^T \\ K_{f2r,32}^T \end{bmatrix} \tag{4.173}$$

where $T_{cr,3}$ is the Chang transformation (4.135). $P_{or,3}$ is the solution of the algebraic Sylvester equation (4.143). We previously set $K_4^T r(t) = v(t) - K_{sr,3}^T r_{s,3}(t) = -K_{sr,3}^T r_{s,3}(t) - K_{f2r,3}^T r_{fnew,3}(t)$ in (4.138) and (4.146), which implies

$$K_4^T r(t) = \begin{bmatrix} K_{sr,3}^T & K_{f2r,3}^T \end{bmatrix} \begin{bmatrix} r_{s,3}(t) \\ r_{fnew,3}(t) \end{bmatrix} = \begin{bmatrix} K_{sr,3}^T & K_{f2r,3}^T \end{bmatrix} \begin{bmatrix} I_{(n_1-l_1)} & 0 \\ P_{or,3} & I_{(n_2-l_2)} \end{bmatrix} \begin{bmatrix} r_{s,3}(t) \\ r_{f,3}(t) \end{bmatrix}$$
$$= \begin{bmatrix} K_{sr,3}^T + K_{f2r,3}^T P_{or,3} & K_{f2r,3}^T \end{bmatrix} T_{cr,3}^T \begin{bmatrix} r_{12}(t) \\ r_{22}(t) \end{bmatrix}$$
$$= \begin{bmatrix} K_{sr,3}^T + K_{f2r,3}^T P_{or,3} & K_{f2r,3}^T \end{bmatrix} T_{cr,3}^T T_{1r,3}^T \begin{bmatrix} q_{12}(t) \\ q_{22}(t) \end{bmatrix} \tag{4.174}$$

Hence $\begin{bmatrix} K_{sr,3}^T + K_{f2r,3}^T P_{or,3} & K_{f2r,3}^T \end{bmatrix} T_{cr,3}^T T_{1r,3}^T$ represents transpose of the observer gain matrix K_4 in the original coordinates. It is important to note that the observer gain $K_4 = f(K_{sr,3}, K_{f2r,3})$ can be obtained using computations with reduced-order matrices $K_{sr,3}, K_{f2r,3}$. From this fact, the observer gain matrix K_4 are given by (4.172).

4.6.3 Case V : Design Algorithm for Finding the Reduced-order Observer Gain

Given that the linear system (A_4^r, A_2^r) is observable, the following two-time scale design algorithm can be applied for the design of a reduced-order observer for singularly perturbed linear system.

Step 1. Transpose the first part of matrices from (4.129) and apply the change of variable to the hypothetical system defined in (4.130).

Step 2. Apply the Chang transformation (4.135) to (4.131) to get (4.133).

Step 3. Obtain the partitioned submatrices $A_{sr,3}^T$, $\frac{1}{\epsilon}A_{fr,3}^T$, $C_{sr,3}^T$ and $C_{fr,3}^T$.

Step 4. Place the slow observer eigenvalues in the desired location and obtain the slow observer gain $K_{sr,3}^T$ using the eigenvalue assignment for $\lambda(A_{sr,3} - K_{sr,3}C_{sr,3})$.

Step 5. Solve the reduced-order Sylvester algebraic equation (4.143) to get $P_{or,3}$.

Step 6. Place fast observer eigenvalues at the desired location using the eigenvalue assignment for $\frac{1}{\epsilon}\lambda(A_{fr,3} - K_{f2r,3}C_{fnewr,3})$ and obtain $K_{f2r,3}$.

Step 7. Find the reduced-order observer gain K_4 in the original coordinates using (4.172) and check $\lambda((A_4^r - K_4A_2^r)) = \lambda_s^{desired} \cup \lambda_f^{desired}$.

4.6.4 Example 4.5

Consider a $4^{th}-$ order system with the system matrices A and C given as

$$A = \begin{bmatrix} 0 & 0.4000 & 0 & 0 \\ 0 & 0 & 0.3450 & 0 \\ 0 & -52.4000 & -46.5000 & 26.2000 \\ 0 & 0 & 0 & -100.0000 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}. \; C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Our goal is to design independently slow and fast reduced-order observers with desired $\lambda_s^{desired} = \{-1\}$ and $\frac{1}{\epsilon}\lambda_f^{desired} = \{-200\}$. The observability matrix has full rank and therefore the pair (A_4^r, A_2^r) is observable.

According to Steps 1 and 2 of the Algorithm from Section 4.63 in Case V), the following sub-matrices are obtained

$$A_{sr}^T = \begin{bmatrix}0\end{bmatrix}, \ \frac{1}{\epsilon}A_{fr}^T = \begin{bmatrix}-100\end{bmatrix},$$
$$C_{sr}^T = \begin{bmatrix}0.4000 & -52.4000\end{bmatrix}, \ C_{fr}^T = \begin{bmatrix}0 & 26.2000\end{bmatrix}, \ C_{fnewr}^T = \begin{bmatrix}-0.0020 & 26.4646\end{bmatrix}$$

Following Step 4 from Section 4.63 in Case V), we place the slow eigenvalues in the original coordinates at -1 via the slow feedback gain matrix

$$K_{sr}^T = \begin{bmatrix}0.000145670667754 \\ -0.019082857475819\end{bmatrix}$$

The Step 3 of the algorithm solves the Sylvester algebraic equation and obtains matrix P_{or}, P_{o2r} as

$$P_{or} = \begin{bmatrix}-0.505021076632777\end{bmatrix}, \ P_{o2r} = \begin{bmatrix}-0.009949760280849\end{bmatrix}$$

In Step 4 of the algorithm, we place fast observer's eigenvalues at the desired location $\{-200, -300\}$. The fast observer gain K_{f2}^T is given by

$$K_{f2r}^T = \begin{bmatrix}-0.000002884282566 \\ 0.037786281339235\end{bmatrix}, \ \frac{1}{\epsilon}K_{f3} = \begin{bmatrix}0.000008667341580 & -0.058327718759739\end{bmatrix}$$

Step 5. Using (4.67), matrix K_2 is obtained as

$$K_4 = \begin{bmatrix}0.000147127291241 & -0.038165725959708 \\ -0.000288428256597 & 3.778628133923465\end{bmatrix}$$

It can be checked that $\lambda(A_4^r - K_4A_2^r)$ in the original coordinates

$$\lambda(A_4^r - K_4A_2^r) = \begin{bmatrix}-1.0000000000000 \\ -200.0000000000000\end{bmatrix}$$

which is the same (with the accuracy of $O(10^{-14})$) as we had placed the slow and fast eigenvalues using the two time scale decomposition designs. Figures 4.13 present the slow and fast observation errors.

In order to be able to run MATLAB Simulink simulation we had to specify also the system states initial conditions (these initial conditions are in general not known). We have chosen them as $x_1(0) = [2, 2]$ and $x_2(0) = [2, 2]$. From Appendix A.9, the initial condition for $\hat{x}_u(0)$ is given as

$$\hat{x}_u(0) = \begin{bmatrix} \hat{x}_{12}(0) \\ \hat{x}_{22}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which results in

$$\hat{z}_u(0) = \hat{x}_m(0) - K_4 y(0) = -K_4 \hat{x}_m(0) = \begin{bmatrix} 0.076037197336934 \\ -7.556679411333735 \end{bmatrix}$$

Using (4.162), we obtain $\hat{z}_{s,3}(0), \hat{z}_{fnew2,3}(0)$ as

$$\begin{bmatrix} \hat{z}_{s,3}(0) \\ \hat{z}_{fnew2,3}(0) \end{bmatrix} = T_{r,3}^{-1} \begin{bmatrix} \hat{z}_{12}(0) \\ \hat{z}_{22}(0) \end{bmatrix} = \begin{bmatrix} 0.037874373616129 \\ -0.075943635051605 \end{bmatrix}$$

so that $z_s(0) = [0.037874373616129]$ and $z_{fnew2}(0) = [-0.075943635051605]$ in MATLAB simulation for the reduced-order observer. At this point, the initial condition for error $e_{12}(0), e_{22}(0)$ are given as

$$\begin{bmatrix} e_{12}(0) \\ e_{22}(0) \end{bmatrix} = x_u(t) - \hat{x}_u(t) = \begin{bmatrix} x_{12}(0) \\ x_{22}(0) \end{bmatrix} - \begin{bmatrix} \hat{x}_{12}(0) \\ \hat{x}_{22}(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

4.7 Conclusions

We have designed the reduced-order observers for singularly perturbed linear systems. There are five cases for the design based on the status of measured states : only all slow variables are measured (Case I), only all fast variables are measured (Case II), some combinations of slow and fast variables are measured (Case III - Case V). In Case I)

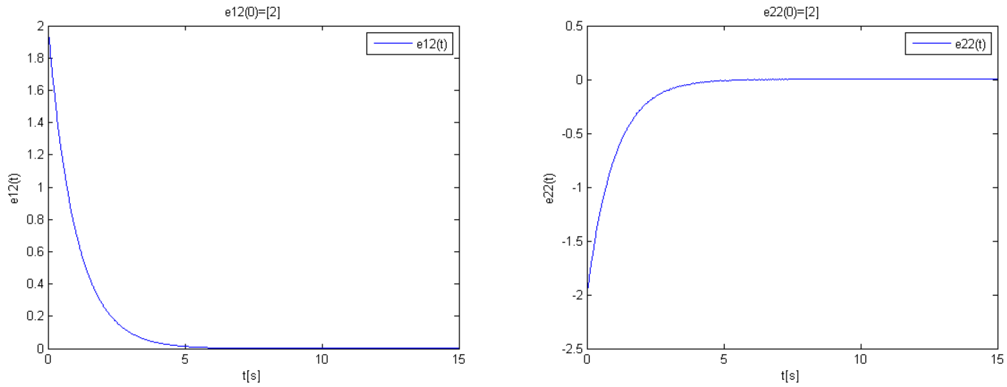

Figure 4.13: Case V : Convergence of the slow state observation error $e_{12}(t) = x_{12}(t) - \hat{x}_{12}(t)$ and the fast state observation error $e_{22}(t) = x_{22}(t) - \hat{x}_{22}(t)$ for the parallel structure from Fig. 4.12

and Case II) the reduced-order observer doesn't have the singularly perturbed structure, since the dimension of measurement l matches with the dimension of slow states n_1 and fast states n_2. That means if measurable states are slow states $x_1(t)$, corresponding reduced-order observer design can be applied to the fast states $x_2(t)$. Considering only fast states, there is no slow and fast decomposition for the reduced-order observer. Similarly, if measurable states are fast states $x_2(t)$, corresponding reduced-order observer design is implemented for the slow states $x_1(t)$. Considering only slow states, there is no slow and fast composition in which numerical ill-conditioning problem is not encountered. However, in Case III) to Case V) the condition that the dimension of measurement l is much smaller than the dimension of slow states n_1 and fast states n_2 makes the reduced-order observer to contain singular a perturbation parameter. The aforementioned condition causes numerical ill-conditioning for the eigenvalue assignment in the reduced-order observer design. To overcome the numerical ill-conditioning problem, we use the two-stage method presented in Chapter 3 for the eigenvalue assignment, which facilitate the reduced-order observer design.

Chapter 5

New Designs of Reduced-Order Observer-Based Controllers for Singularly Perturbed Linear Systems

In the previous chapter, we have designed reduced-order observers for singularly perturbed linear systems. For Cases I) and II), the corresponding reduced-order observers are not respectively pure fast and pure slow. For Cases III)-V), we have observed the states of the original system using both reduced-order slow and fast observers (4.56),(4.107),(5.34). For Cases III)-V), the two-stage method have been used to overcome the numerical-ill conditioning problem. In this chapter, we use these observers and consider the observer-based controller designs for singularly perturbed linear systems. The observers are driven by the system measurements and control inputs with observers implemented independently in the slow and fast time scales.

5.1 Case I: Controller Design when All Slow Variables are Measured Only

Consider a linear time invariant singularly perturbed control system, [11]

$$
\begin{aligned}
\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \\
\dot{x}_2(t) &= \frac{1}{\epsilon}A_{21}x_1(t) + \frac{1}{\epsilon}A_{22}x_2(t) + \frac{1}{\epsilon}B_2u(t) \\
y(t) &= x_1(t)
\end{aligned}
\tag{5.1}
$$

where ϵ is a small positive singular perturbation parameter that indicates seperation of state variables $x(t) \in \Re^n$ into slow $x_1(t) \in \Re^{n_1}$ and fast $x_2(t) \in \Re^{n_2}$, $n_1 + n_2 = n$. $u(t) \in \Re^m$ is the control input and $y(t) \in \Re^p$ the system measured output.

The reduced-order observer for the system defined in (5.1) was derived in Chapter 4, Section 4.2, given by

$$\epsilon\dot{\hat{z}}_2(t) = A_z\hat{z}_2(t) + B_z u(t) + K_z y(t) \tag{5.2}$$

where [39]

$$\begin{aligned} A_z &= A_{22} - K_{11}A_{12}, \\ B_z &= B_2 - K_{11}B_1 \\ K_z &= A_{21} - K_{11}A_{11} + \frac{1}{\epsilon}A_{22}K_{11} - \frac{1}{\epsilon}K_{11}A_{12}K_{11} \end{aligned} \tag{5.3}$$

The state estimation of the fast variables is obtained from

$$\hat{x}_2(t) = \hat{z}_2(t) + \frac{1}{\epsilon}K_{11}y(t) \tag{5.4}$$

so that

$$\hat{x}(t) = \begin{bmatrix} x_1(t) \\ \hat{x}_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ \hat{x}_2(t) \end{bmatrix} \tag{5.5}$$

The matrix K_{11} is chosen to stabilize the reduced order observer (5.2), that is

$$\lambda(A_{22} - K_{11}A_{12}) = \lambda(A_{22}^T - A_{12}^T K_{11}^T) = \lambda_{robs}^{desired} \tag{5.6}$$

In the following, the Chang transformation matrix, [15], will be needed

$$T_c = \begin{bmatrix} I_n & \epsilon H \\ -L & I_m - \epsilon LH \end{bmatrix},\ T_c^{-1} = \begin{bmatrix} I_n - \epsilon HL & -\epsilon H \\ L & I_m \end{bmatrix} \tag{5.7}$$

where matrices L and H satisfy the algebraic equations

$$\begin{aligned} 0 &= \epsilon L(A_{11} - A_{12}L) + (A_{21} - A_{22}L) \\ 0 &= \epsilon(A_{11} - A_{12}L)H + A_{12} - H(A_{22} + \epsilon LA_{12}) \end{aligned} \tag{5.8}$$

The solutions for L and H can be obtained using either the fixed-point iterations or Newton method or eigenvector method [12].

Using the separation principle, the observer based controller design via the two stage design was considered in Chapter 3, as

$$u(t) = -F\hat{x}(t) = -\begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ \hat{x}_2(t) \end{bmatrix} = -\begin{bmatrix} F_s + F_{f2}P & F_{f2} \end{bmatrix} T_c^{-1} \begin{bmatrix} y(t) \\ \hat{x}_2(t) \end{bmatrix} \tag{5.9}$$

The feedback gain F_s is chosen such to place slow eigenvalues at the desired locations, that is

$$\lambda(A_s - B_sF_s) = \lambda_s^{desired} \tag{5.10}$$

The matrix P is obtained from the Sylvester algebraic equation

$$\epsilon P(A_s - B_sF_s) - B_fF_s - A_fP = 0 \;\Rightarrow\; P^{(0)} = A_f^{-1}B_fF_s \tag{5.11}$$

where

$$\begin{aligned} A_s &= A_{11} - A_{12}L,\ B_s = B_1(I_n - \epsilon HL) - \epsilon HB_2 \\ A_f &= A_{22} + \epsilon LA_{12},\ B_f = \epsilon LB_1 + B_2 \end{aligned} \tag{5.12}$$

The feedback gain F_{f2} is chosen such to place the fast eigenvalues at the desired locations, that is

$$\lambda(A_f - (B_f + \epsilon PB_s)F_{f2}) = \lambda(A_f - B_{fnew}F_{f2}) = \lambda_f^{desired} \tag{5.13}$$

Based on information from (5.2), (5.6), (5.9) and Figure 4.1, we present in Figure 5.1 the block diagram for the reduced-order observer-based controller when only all state variables are perfectly measured. In (5.10) and (5.13) we have chosen the feedback gains for the eigenvalue assignment problem. However, any F_1 and F_2 can be used to control the system and provide corresponding design requirements.

5.1.1 Case I: Numerical Example

Consider a $4^{th}-$ order system with the system matrices A, B, and C defined in Example 4.1 in Sections 4.2.1. The controllability matrix has full rank and therefore the pair (A, B) is controllable. The results obtained using MATLAB are given below. We locate the feedback system slow eigenvalues at $\lambda_{cs}^{desired} = (-2, -3)$ and the feedback

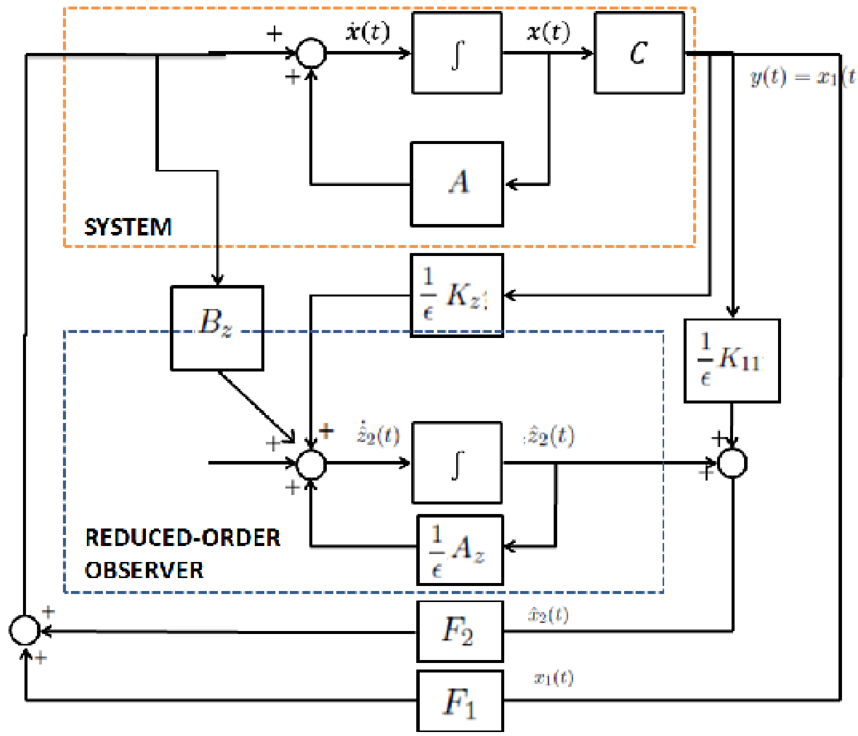

Figure 5.1: Case I: Reduced-order observer based controller design for a singularly perturbed linear systems

system fast eigenvalues at $\lambda_{cf}^{desired} = (-7, -8)$, and the reduced-order observer eigenvalues at $\lambda_{robs}^{desired} = (-50, -60)$, given in the previous numerical example. Following the design procedure from Example 4.1 in Sections 4.2.1 and 5.1, the observer matrices A_z,K_z, K_{11}, F_1, F_2 are given as

$$A_z = \begin{bmatrix} -108.9999 & 0.2620 \\ -11034.3511 & -1.0000 \end{bmatrix}, \; K_{11} = \begin{bmatrix} 0 & 314.5942 \\ 0 & 31983.6265 \end{bmatrix},$$
$$K_z = \begin{bmatrix} 0 & -259111.1037 \\ 0 & -35033265.2948 \end{bmatrix},$$

and the feedback gains are obtained as

$$F_1 = \begin{bmatrix} 92.9306 & 37.8637 \end{bmatrix}, \; F_2 = \begin{bmatrix} 2.4356 & 0.5349 \end{bmatrix}$$

5.2 Case II : Controller Design when All Fast Variables are Measured Only

Consider a linear time invariant singularly perturbed control system, [11]

$$\begin{aligned}\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t)\\ \dot{x}_2(t) &= \frac{1}{\epsilon}A_{21}x_1(t) + \frac{1}{\epsilon}A_{22}x_2(t) + \frac{1}{\epsilon}B_2u(t)\\ y(t) &= x_2(t)\end{aligned} \tag{5.14}$$

where ϵ is a small positive singular perturbation parameter that indicates separation of state variables $x(t) \in \Re^n$ into slow $x_1(t) \in \Re^{n_1}$ and fast $x_2(t) \in \Re^{n_2}$, $n_1 + n_2 = n$. $u(t) \in \Re^m$ is the control input and $y(t) \in \Re^p$ the system measured output.
The reduced-order observer for the system (5.14) is defined in Chapter 4, Section 4.3, and given by

$$\dot{\hat{z}}_1(t) = A_z\hat{z}_1(t) + B_zu(t) + K_zy(t) \tag{5.15}$$

where [39]

$$\begin{aligned}A_z &= A_{11} - \frac{1}{\epsilon}K_{12}A_{21},\\ B_z &= B_2 - \frac{1}{\epsilon}K_{12}B_1\\ K_z &= A_{12} + A_{11}K_{12} - \frac{1}{\epsilon}K_{12}A_{22} - \frac{1}{\epsilon}K_{12}A_{21}K_{12}\end{aligned} \tag{5.16}$$

The state estimation is obtained from

$$\hat{x}_1(t) = \hat{z}_1(t) + K_{12}y(t) \tag{5.17}$$

so that

$$\hat{x}(t) = \begin{bmatrix}\hat{x}_1(t)\\ x_2(t)\end{bmatrix} = \begin{bmatrix}\hat{x}_1(t)\\ y(t)\end{bmatrix} \tag{5.18}$$

The matrix K_{12} is chosen to stabilize the reduced-order observer (5.15), that is

$$\lambda(A_{11} - \frac{1}{\epsilon}K_{12}A_{21}) = \lambda(A_{11}^T - \frac{1}{\epsilon}A_{21}^TK_{12}^T) = \lambda_{robs}^{desired} \tag{5.19}$$

Additional matrices needed in this design can be obtained from (5.8),(5.11)-(5.12). Using the separation principle, the observer based controller can be designed via the two-stage design considered in Chapter 3.

$$u(t) = -Fx(t) = -\begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ x_2(t) \end{bmatrix} = -\begin{bmatrix} F_s + F_{f2}P & F_{f2} \end{bmatrix} T_c^{-1} \begin{bmatrix} \hat{x}_1(t) \\ y(t) \end{bmatrix} \tag{5.20}$$

The feedback gain F_s is chosen such to place slow eigenvalues at the desired locations, that is

$$\lambda(A_s - B_sF_s) = \lambda_s^{desired} \tag{5.21}$$

The feedback gain F_{f2} is chosen such to place fast eigenvalues at the desired locations, that is

$$\lambda(A_f - (B_f + \epsilon PB_s)F_{f2}) = \lambda(A_f - B_{fnew}F_{f2}) = \lambda_f^{desired} \tag{5.22}$$

In Figure 5.2, the block diagram for the reduced-order observer-based controller when only all fast variables are perfectly measured, is presented.

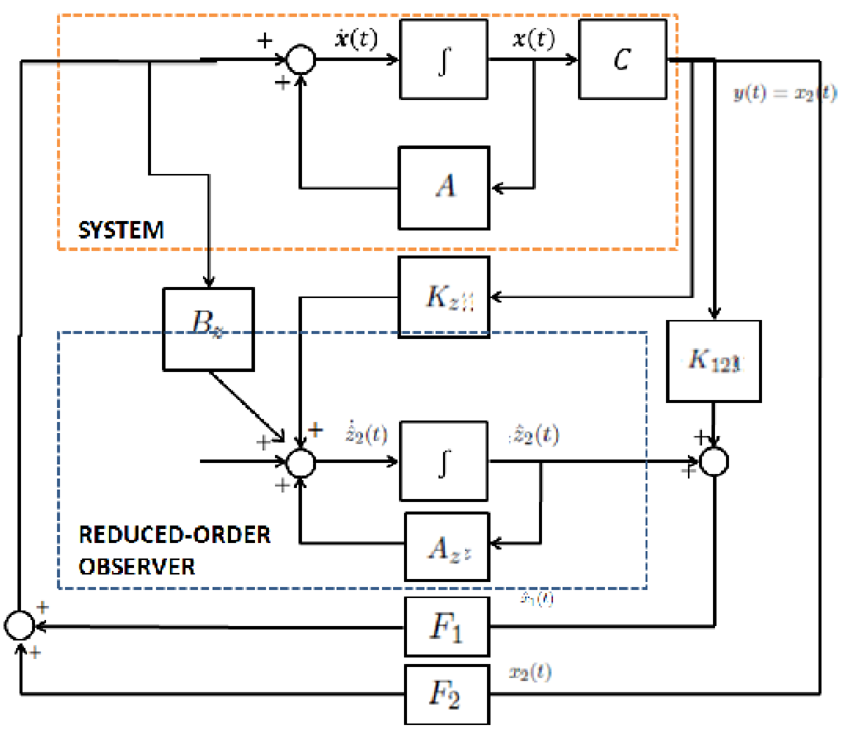

Figure 5.2: Case II: Slow and fast observer-based controller design for a singularly perturbed linear system

5.2.1 Case II: Numerical Example

Consider a $4^{th}-$ order system with the system matrices A, B, and C defined from Example 4.1 in Sections 4.2.1. The controllability matrix has full rank and therefore the pair (A, B) is controllable. We locate the feedback system slow eigenvalues at $\lambda_{cs}^{desired} = (-2, -3)$ and the feedback system fast eigenvalues at $\lambda_{cf}^{desired} = (-7, -8)$, and the reduced-order observer eigenvalues at $\lambda_{robs}^{desired} = (-50, -70)$, given in the previous numerical example. Following the design procedure of from Example 4.1 and in Sections 4.2.1 and 5.2, the observer matrices A_z,K_z, K_{12}, F_1, F_2 are given as

$$A_z = \begin{bmatrix} -49.9999 & 0.0000 \\ -0.0000 & -70.0000 \end{bmatrix}, \ K_{12} = \begin{bmatrix} 11.0734 & 10.3042 \\ -6.4660 & 6.4112 \end{bmatrix},$$
$$K_z = \begin{bmatrix} -537.2325 & -556.1992 \\ 400.7840 & -440.7330 \end{bmatrix},$$

and the feedback gains are

$$F_1 = \begin{bmatrix} 92.9306 & 37.8637 \end{bmatrix}, \ F_2 = \begin{bmatrix} 2.4356 & 0.5349 \end{bmatrix}$$

5.3 Case III: Controller Design when Only a Part of the Slow Variables is Measured

In Case III), the measurable states $x_{11}(t)$ are parts of the slow state vector $x_1(t)$ in the singularly perturbed linear system defined in (3.1), as

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) \\ \epsilon\dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) \\ y(t) &= x_{11}(t) \end{aligned} \tag{5.23}$$

where

$$x_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ x_{12}(t) \end{bmatrix}, \tag{5.24}$$

Use the following partitioning

$$A_{11} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \ A_{12} = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$
$$A_{21} = \begin{bmatrix} a_{31} & a_{32} \end{bmatrix}, \ A_{22} = \begin{bmatrix} a_{33} \end{bmatrix} \tag{5.25}$$

where $a_{11} \in R^{l\times l}, a_{12} \in R^{l\times(n_1-l)}, a_{13} \in R^{l\times n_2}, a_{21} \in R^{(n_1-l)\times l}, a_{22} \in R^{(n_1-l)\times(n_1-l)}$, $a_{23} \in R^{(n_1-l)\times n_2}, a_{31} \in R^{n_2\times l}, a_{32} \in R^{n_2\times(n_1-l)}, a_{33} \in R^{n_2\times n_2}$,
$x_{11}(t) \in R^{l\times 1}, x_{12}(t) \in R^{(n_1-l)\times 1}$ $, x_2(t) \in R^{n_2}, b_1 \in R^{l\times 1}, b_2 \in R^{(n-l)\times 1}$
and $y(t) \in R^{l\times 1}, p(t) \in R^{(n-l)\times 1}$.

The system (5.23) with (5.24)-(5.25) can be represented as

$$\begin{aligned} \dot{x}_{11}(t) &= A^r_{11}x_{11}(t) + A^r_{12}x^r_2(t) \\ \dot{x}^r_2(t) &= A^r_{21}x_{11}(t) + A^r_{22}x^r_2(t) \\ y(t) &= x_{11}(t) \end{aligned} \tag{5.26}$$

where

$$x^r_2(t) = \begin{bmatrix} x_{12}(t) \\ x_2(t) \end{bmatrix}$$
$$A^r_{11} = \begin{bmatrix} a_{11} \end{bmatrix}, \ A^r_{12} = \begin{bmatrix} a_{12} & a_{13} \end{bmatrix} \tag{5.27}$$
$$A^r_{21} = \begin{bmatrix} a_{21} \\ \frac{1}{\epsilon}a_{31} \end{bmatrix}, \ A^r_{22} = \begin{bmatrix} a_{22} & a_{23} \\ \frac{1}{\epsilon}a_{32} & \frac{1}{\epsilon}a_{33} \end{bmatrix}$$

The observer gains are obtained from

$$A^r_z = A^r_{22} - K_2A^r_{12} = \begin{bmatrix} a_{22} & a_{23} \\ \frac{1}{\epsilon}a_{32} & \frac{1}{\epsilon}a_{33} \end{bmatrix} - \begin{bmatrix} K_{21} \\ K_{22} \end{bmatrix} \begin{bmatrix} a_{12} & a_{13} \end{bmatrix},$$
$$K^r_z = A^r_{21} - K_2A^r_{11} + A^r_{22}K_2 - K_2A^r_{12}K_2,$$
$$= \begin{bmatrix} a_{21} - K_{21}a_{11} + a_{22}K_{21} + \frac{1}{\epsilon}a_{23}K_{22} - K_{21}(a_{12}K_{21} + \frac{1}{\epsilon}a_{13}K_{22}) \\ \frac{1}{\epsilon}a_{31} - \frac{1}{\epsilon}K_{22}a_{11} + \frac{1}{\epsilon}a_{32}K_{21} + \frac{1}{\epsilon^2}a_{33}K_{22} - \frac{1}{\epsilon}K_{22}(a_{12}K_{21} + \frac{1}{\epsilon}a_{13}K_{22}) \end{bmatrix} = \begin{bmatrix} K_{21r} \\ K_{22r} \end{bmatrix} \tag{5.28}$$

The reduced-order observer configuration obtained in Section 4.4 is given by

$$\begin{bmatrix} \dot{\hat{z}}_s(t) \\ \epsilon\dot{\hat{z}}_{fnew}(t) \end{bmatrix} = \begin{bmatrix} A_{sr} - K_{sr}C_{sr} & 0 \\ -\epsilon K_{f2r}C_{sr} & A_{fr} - K_{f2r}C_{fnewr} \end{bmatrix} \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew}(t) \end{bmatrix} + \begin{bmatrix} K^*_{sr} \\ K^*_{f2r} \end{bmatrix} y(t) \quad (5.29)$$

where K^*_{sr}, K^*_{f2r} are obtained from $T^{-1}_{4r}K^r_z$, with T_{4r} defined by

$$\begin{bmatrix} \hat{z}_{12}(t) \\ \hat{z}_2(t) \end{bmatrix} = T_{1r}T_{cr}T_{2r} = T_{4r} \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew}(t) \end{bmatrix} \quad (5.30)$$

where

$$T_{1r} = \begin{bmatrix} I_{(n_1-l)} & 0 \\ 0 & \frac{1}{\epsilon}I_{n_2} \end{bmatrix}, \; T_{2r} = \begin{bmatrix} I_{(n_1-l)} & P^T_{or} \\ 0 & I_{n_2} \end{bmatrix} \quad (5.31)$$

The Chang transformation is given by

$$T^T_{cr} = \begin{bmatrix} I_{(n_1-l)} & -\epsilon L^T_r \\ H^T_r & I_{n_2} - \epsilon H^T_r L^T_r \end{bmatrix} \quad (5.32)$$

where L^T_r and H^T_r are the transposed solution, that is

$$\begin{aligned} 0 &= \epsilon(a^T_{22} - L^T_r a^T_{23})L^T_r + (a^T_{32} - L^T_r a^T_{33}) \\ 0 &= \epsilon H^T_r(a^T_{22} - L^T_r a^T_{23}) + a^T_{23} - (a^T_{33} + \epsilon a^T_{23}L^T_r)H^T_r \end{aligned} \quad (5.33)$$

with a_{ij} matrices defined in (5.27)

The reduced-order observer (5.29) has a sequential structure. It can be block diagonalized and used as a parallel structure as follows

$$\begin{aligned} \dot{\hat{z}}_s(t) &= (A_{sr} - K_{sr}C_{sr})\hat{z}_s(t) + K^*_{sr}y(t) \\ \epsilon\dot{\hat{z}}_{fnew2}(t) &= (A_{fr} - K_{f2r}C_{fnewr})\hat{z}_{fnew2}(t) + K_{f3r}y(t) \end{aligned} \quad (5.34)$$

where

$$\begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew2}(t) \end{bmatrix} = \begin{bmatrix} I_{(n_1-l)} & 0 \\ P_{o2r} & I_{n_2} \end{bmatrix} \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew}(t) \end{bmatrix} = T_{3r} \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew}(t) \end{bmatrix} \quad (5.35)$$

The original coordinates $\hat{z}_{12}(t)$, $\hat{z}_2(t)$ and the coordinates $\hat{z}_s(t)$, $\hat{z}_{fnew2}(t)$ are related via

$$\begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew2}(t) \end{bmatrix} = T_{3r}T_{4r}^{-1} \begin{bmatrix} \hat{z}_{12}(t) \\ \hat{z}_2(t) \end{bmatrix} = T_r^{-1} \begin{bmatrix} \hat{z}_{12}(t) \\ \hat{z}_2(t) \end{bmatrix} \tag{5.36}$$

where P_{o2r} satisfies the algebraic Sylvester equation represented by

$$\epsilon P_{o2r}(A_{sr} - K_{sr}C_{sr}) - \epsilon K_{f2r}C_{sr} - (A_{fr} - K_{f2r}C_{fnewr})P_{o2r} = 0 \;\Rightarrow\; P_{o2r}^0 = O(\epsilon) \tag{5.37}$$

In the previous chapter, we have observed the original system state using independent reduced-order slow and fast observers (5.34). In this section, we use these observers and consider the observer-based controller design for singularly perturbed linear systems. The observer is driven by the system measurements and control inputs, that is

$$\begin{bmatrix} \dot{\hat{z}}_s(t) \\ \epsilon\dot{\hat{z}}_{fnew2}(t) \end{bmatrix} = \begin{bmatrix} A_{sr} - K_{sr}C_{sr} & 0 \\ 0 & A_{fr} - K_{f2r}C_{fnewr} \end{bmatrix} \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew2}(t) \end{bmatrix} + \begin{bmatrix} B_{sr2} \\ B_{f2r} \end{bmatrix} u(t)$$
$$+ \begin{bmatrix} K_{sr}^* \\ K_{f3r} \end{bmatrix} y(t) \tag{5.38}$$

Thus, these two observers (5.38) can be implemented independently in the slow and fast time scales

$$\begin{aligned} \dot{\hat{z}}_s(t) &= (A_{sr} - K_{sr}C_{sr})\hat{z}_s(t) + B_{sr2}u(t) + K_{sr}^*y(t) \\ \epsilon\dot{\hat{z}}_{fnew2}(t) &= (A_{fq} - K_{f2}C_{fnew})\hat{z}_{fnew2}(t) + B_{f2r}u(t) + K_{f3}y(t) \end{aligned} \tag{5.39}$$

where B_{sr2}, B_{f2r} can be obtained from $T_r^{-1}B$ as

$$\begin{aligned} &B_{sr2} = (I_{(n_1-l)} - \epsilon H_rL_r)B_{z1} - \epsilon P_{or}^TL_rB_{z1} - H_rB_{z2} - P_{or}^TB_{z2}, \\ &B_{f2r} = \epsilon P_{o2r}(I_{(n_1-l)} - \epsilon H_rL_r)B_{z1} - \epsilon^2P_{o2r}P_{or}^TL_rB_{z1} + \epsilon^2L_rB_{z1} - \epsilon P_{o2r}H_rB_{z2} \\ &- \epsilon P_{o2r}P_{or}^TB_{z2} + \epsilon B_{z2} \end{aligned} \tag{5.40}$$

The control input in the $\hat{z}_s$-$\hat{z}_{fnew2}$ coordinates is given by

$$\begin{aligned}
u(t) &= -F_r\hat{x}(t) = -\begin{bmatrix} F_{12} & F_2 \end{bmatrix}\begin{bmatrix} \hat{x}_{12}(t) \\ \hat{x}_2(t) \end{bmatrix} \\
&= -(\begin{bmatrix} F_{12} & F_2 \end{bmatrix}\begin{bmatrix} K_{21} \\ K_{22} \end{bmatrix} y(t) + \begin{bmatrix} F_{12} & F_2 \end{bmatrix} T_r \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew2}(t) \end{bmatrix}) \\
&= -(\begin{bmatrix} F_{12} & F_2 \end{bmatrix}\begin{bmatrix} K_{21} \\ K_{22} \end{bmatrix} y(t) + \begin{bmatrix} F_{sr2} & F_{fr2} \end{bmatrix} \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew2}(t) \end{bmatrix}) \\
&= -F_r K_2 y(t) - \begin{bmatrix} F_{sr2} & F_{fr2} \end{bmatrix} \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew2}(t) \end{bmatrix}
\end{aligned} \tag{5.41}$$

with

$$\begin{aligned}
F_{sr2} &= F_{12}(I_{(n_1-l)} - P_{or}^T P_{o2r} - H_r P_{o2r}) - F_2 L(I_n - P_{or}^T P_{o2r}) - \frac{1}{\epsilon} F_2 (I_{n_2} - \epsilon L_r H_r) P_{o2r} \\
F_{fr2} &= F_{12}(P_{or}^T + H_r) - F_2 L_r P_{or}^T + \frac{1}{\epsilon} F_2 (I_{n_2} - \epsilon L_r H_r)
\end{aligned} \tag{5.42}$$

Here, F_r is taken from (3.33), and $F_{12} \in R^{1\times(n_1-l)}$, $F_2 \in R^{1\times(n_2)}$. The corresponding block diagram for the observer driven controller is presented in Figure 5.3. This block diagram clearly indicates full parallelism of the slow controller driven by the slow observer and the fast controller driven by the fast observer.

The remaining matrices obtained in (5.29) are given by

$$\begin{aligned}
&A_{sr}^T = a_{22}^T - L^T a_{23}^T,\ A_{fr}^T = a_{33}^T + \epsilon a_{23}^T L^T \\
&C_{sr}^T = a_{12}^T - L^T a_{13}^T,\ C_{fr}^T = \epsilon H^T a_{12}^T + (I_{n_2} - \epsilon H^T L^T) a_{13}^T \\
&C_{fnewr} = C_{fr} + \epsilon C_{sr} P_{or}^T
\end{aligned} \tag{5.43}$$

K_{sr}, K_{f2r}, P_{or} can be obtained from the formulas in Section 4.4, that is

$$\begin{aligned}
&\lambda(A_{sr}^T - C_{sr}^T K_{sr}^T) = \lambda(A_{sr} - K_{sr} C_{sr}) = \lambda_s^{desired} \\
&\lambda(A_{fr} - K_{f2r} C_{fnewr}) = \lambda_f^{desired} \\
&\epsilon P_{or}(A_{sr}^T - C_{sr}^T K_{sr}^T) - C_{fr}^T K_{sr}^T - A_{fr}^T P_{or} = 0 \ \Rightarrow\ P_{or} = O(\epsilon)
\end{aligned} \tag{5.44}$$

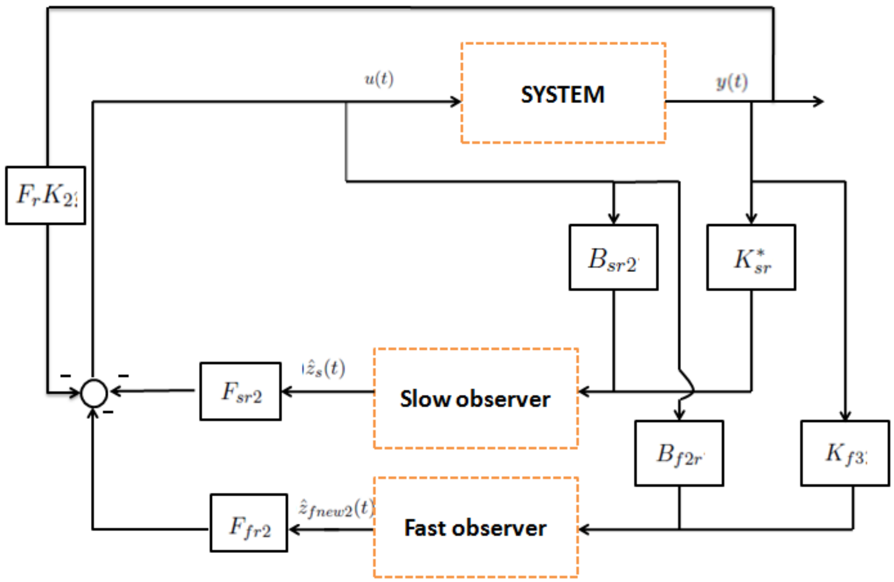

Figure 5.3: Case III: Slow and fast observer-based controller design for a singularly perturbed linear systems with the system feedback gains obtained in (5.41)

5.3.1 Case III : Numerical Example

Consider a $4^{th}-$ order system with the system matrices A, B, and C defined in Section 4.4. The controllability matrix has full rank and therefore the pair (A, B) is controllable. We locate the feedback system slow eigenvalues at $\lambda_{cs}^{desired} = (-2, -3)$ and the feedback system fast eigenvalues at $\lambda_{cf}^{desired} = (-7, -8)$, and the slow observer eigenvalues at $\lambda_{os}^{desired} = -50$ and the fast observer eigenvalues at $\lambda_{of}^{desired} = (-200, -300)$, given in the previous numerical example. Following the design procedure of from Sections 5.3, the completely decoupled slow and fast observer in the z_s-z_{fnew2} coordinates, driven by the system measurements and control inputs, are

$$\begin{aligned} \dot{\hat{z}}_s(t) &= \begin{bmatrix} -50.0000 \end{bmatrix} \hat{z}_s(t) \\ &+ \begin{bmatrix} -2.323033338614758 \end{bmatrix} u(t) + \begin{bmatrix} -6196.4741 \end{bmatrix} y(t) \end{aligned}$$

$$\dot{\hat{z}}_{fnew2}(t) = \begin{bmatrix} -1693.6295 & 113.2762 \\ -18376.0229 & 1193.6295 \end{bmatrix} \hat{z}_{fnew2}(t)$$

$$+ \begin{bmatrix} -0.2499 \\ 0.4666 \end{bmatrix} u(t) + \begin{bmatrix} -2859901.4120 \\ -33059040.2923 \end{bmatrix} y(t)$$

$$u(t) = -\begin{bmatrix} 1761.6414 \end{bmatrix} \hat{z}_s(t)$$

$$-\begin{bmatrix} 26.9824 & 5.1780 \end{bmatrix} \hat{z}_{fnew2}(t)$$

The slow and fast controller gains F_{sr2}, F_{fr2} are obtained as

$$F_{sr2} = \begin{bmatrix} 1761.6414 \end{bmatrix},$$

$$F_{fr2} = \begin{bmatrix} 26.9824 & 5.1780 \end{bmatrix}$$

5.4 Case IV: Controller Design when Only a Part of Fast Variables is Measured

In Case IV), the measurable states $x_{21}(t)$ are parts of the slow state vector $x_2(t)$ in the singularly perturbed linear system defined in (3.1), as

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) \\ \epsilon\dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) \\ y(t) &= I_l x_{22}(t) \end{aligned} \tag{5.45}$$

where

$$x_2(t) = \begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix} = \begin{bmatrix} x_{21}(t) \\ y(t) \end{bmatrix}, \tag{5.46}$$

using the following partitioning

$$\begin{aligned} A_{11} &= \begin{bmatrix} a_{11} \end{bmatrix}, \ A_{12} = \begin{bmatrix} a_{12} & a_{13} \end{bmatrix} \\ A_{21} &= \begin{bmatrix} a_{21} \\ a_{31} \end{bmatrix}, \ A_{22} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \end{aligned} \tag{5.47}$$

where $a_{33} \in R^{l\times l}, a_{32} \in R^{l\times(n_2-l)}, a_{31} \in R^{l\times n_1}, a_{23} \in R^{(n_2-l)\times l}, a_{22} \in R^{(n_2-l)\times(n_2-l)}$, $a_{21} \in R^{(n_2-l)\times n_1}, a_{13} \in R^{n_1\times l}, a_{12} \in R^{n_1\times(n_2-l)}, a_{11} \in R^{n_1\times n_1}, x_{22}(t) \in R^{l\times 1}$, $x_{21}(t) \in R^{n_2-l\times 1}, x_1(t) \in R^{n_1}$, $y(t) \in R^{l\times 1}$, and $p(t) \in R^{(n-l)\times 1}$.

The system (5.45) with (5.46)-(5.47) can be represented as

$$\begin{aligned} \dot{x}_1^r(t) &= A_{11}^r x_1^r(t) + A_{12}^r x_{22}(t) \\ \dot{x}_{22}(t) &= A_{21}^r x_1^r(t) + A_{22}^r x_{22}(t) \\ y(t) &= I_l x_{22}(t) \end{aligned} \tag{5.48}$$

where

$$\begin{aligned} x_1^r(t) &= \begin{bmatrix} x_1(t) \\ x_{21}(t) \end{bmatrix} \\ A_{11}^r &= \begin{bmatrix} a_{11} & a_{12} \\ \frac{1}{\epsilon}a_{21} & \frac{1}{\epsilon}a_{22} \end{bmatrix},\ A_{12}^r = \begin{bmatrix} a_{13} \\ \frac{1}{\epsilon}a_{23} \end{bmatrix} \\ A_{21}^r &= \begin{bmatrix} \frac{1}{\epsilon}a_{31} & \frac{1}{\epsilon}a_{32} \end{bmatrix},\ A_{22}^r = \begin{bmatrix} \frac{1}{\epsilon}a_{33} \end{bmatrix} \end{aligned} \tag{5.49}$$

The reduced-order observer configuration is obtained in Section 4.5 is given by

$$\begin{aligned} \begin{bmatrix} \dot{\hat{z}}_{s,2}(t) \\ \epsilon\dot{\hat{z}}_{fnew,2}(t) \end{bmatrix} &= \begin{bmatrix} A_{sr,2} - K_{sr,2}C_{sr,2} & 0 \\ -\epsilon K_{f2r,2}C_{sr,2} & A_{fr,2} - K_{f2r,2}C_{fnewr,2} \end{bmatrix} \begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{fnew,2}(t) \end{bmatrix} \\ &+ \begin{bmatrix} K_{sr,2}^* \\ K_{f2r,2}^* \end{bmatrix} y(t) \end{aligned} \tag{5.50}$$

where $K_{sr,2}^*, K_{f2r,2}^*$ are determined by $T_{4r,2}^{-1}K_{z,2}^r$, with $T_{4r,2}(t)$ defined by

$$\begin{bmatrix} \hat{z}_1(t) \\ \hat{z}_{21}(t) \end{bmatrix} = T_{1r,2}T_{cr,2}T_{2r,2} \begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{fnew,2}(t) \end{bmatrix} = T_{4r,2} \begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{fnew,2}(t) \end{bmatrix} \tag{5.51}$$

where

$$T_{1r,2}^T = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \frac{1}{\epsilon}I_{(n_2-l)} \end{bmatrix},\ T_{2r,2}^T = \begin{bmatrix} I_{n_1} & 0 \\ P_{or,2} & I_{(n_2-l)} \end{bmatrix} \tag{5.52}$$

The observer gains are obtained from

$$A^r_{z,2} = A^r_{11} - K_3 A^r_{21} = \begin{bmatrix} a_{11} & a_{12} \\ \frac{1}{\epsilon}a_{21} & \frac{1}{\epsilon}a_{22} \end{bmatrix} - \begin{bmatrix} K_{31} \\ K_{32} \end{bmatrix} \begin{bmatrix} \frac{1}{\epsilon}a_{31} & \frac{1}{\epsilon}a_{32} \end{bmatrix},$$

$$K^r_{z,2} = A^r_{12} - K_3 A^r_{22} + A^r_{11} K_3 - K_3 A^r_{21} K_3$$

$$= \begin{bmatrix} a_{21} - K_{21}a_{11} + a_{22}K_{21} + \frac{1}{\epsilon}a_{23}K_{22} - K_{21}(a_{12}K_{21} + \frac{1}{\epsilon}a_{13}K_{22}) \\ \frac{1}{\epsilon}a_{31} - \frac{1}{\epsilon}K_{22}a_{11} + \frac{1}{\epsilon}a_{32}K_{21} + \frac{1}{\epsilon^2}a_{33}K_{22} - \frac{1}{\epsilon}K_{22}(a_{12}K_{21} + \frac{1}{\epsilon}a_{13}K_{22}) \end{bmatrix} = \begin{bmatrix} K_{21r} \\ K_{22r} \end{bmatrix} \tag{5.53}$$

The Chang transformation needed for the reduced-order observer design is given as

$$T^T_{cr,2} = \begin{bmatrix} I_{n_1} & -\epsilon L^T_{r,2} \\ H^T_{r,2} & I_{(n_2-l)} - \epsilon H^T_{r,2} L^T_{r,2} \end{bmatrix} \tag{5.54}$$

where $L^T_{r,2}$ and $H^T_{r,2}$ are the transposed solution, that is

$$\begin{aligned} 0 &= \epsilon(a^T_{22} - L^T_{r,2}a^T_{23})L^T_{r,2} + (a^T_{32} - L^T_{r,2}a^T_{33}) \\ 0 &= \epsilon H^T_{r,2}(a^T_{22} - L^T_{r,2}a^T_{23}) + a^T_{23} - (a^T_{33} + \epsilon a^T_{23}L^T_{r,2})H^T_{r,2} \end{aligned} \tag{5.55}$$

with a_{ij} matrices defined in (5.49)

The reduced-order observer (5.50) has a sequential structure. It can be block diagonalized and used as a parallel structure as

$$\begin{aligned} \dot{\hat{z}}_{s,2}(t) &= (A_{sr,2} - K_{sr,2}C_{sr,2})\hat{z}_{s,2}(t) + K_{sr,2}y(t) \\ \epsilon\dot{\hat{z}}_{fnew2,2}(t) &= (A_{fr,2} - K_{f2r,2}C_{fnewr,2})\hat{z}_{fnew2,2}(t) + K_{f3r,2}y(t) \end{aligned} \tag{5.56}$$

where

$$\begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{fnew2,2}(t) \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 \\ P_{o2r,2} & I_{n_2-l} \end{bmatrix} \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew}(t) \end{bmatrix} = T_{3r,2} \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew}(t) \end{bmatrix} \tag{5.57}$$

The original coordinates $\hat{z}_1(t)$, $\hat{z}_{21}(t)$ and the coordinates $\hat{z}_{s,2}(t)$, $\hat{z}_{fnew2,2}(t)$ are related via

$$\begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{fnew2,2}(t) \end{bmatrix} = T_{3r,2}T^{-1}_{4r,2} \begin{bmatrix} \hat{z}_1(t) \\ \hat{z}_{21}(t) \end{bmatrix} = T^{-1}_{r,2} \begin{bmatrix} \hat{z}_1(t) \\ \hat{z}_{21}(t) \end{bmatrix} \tag{5.58}$$

where $P_{o2r,2}$ satisfies the algebraic Sylvester equation represented by

$$\begin{aligned}
&\epsilon P_{o2r,2}(A_{sr,2} - K_{sr,2}C_{sr,2}) - \epsilon K_{f2r,2}C_{sr,2} - (A_{fr,2} - K_{f2r,2}C_{fnewr,2})P_{o2r,2} = 0 \\
&\Rightarrow \; P^{0}_{o2r,2} = O(1)
\end{aligned} \tag{5.59}$$

In Section 4.5, we have observed the original system state using independent reduced-order slow and fast observers (5.56). In this section, we use these observers and consider the observer-based controller design for singularly perturbed linear systems. The observer is driven by the system measurements and control inputs, that is

$$\begin{aligned}
&\begin{bmatrix} \dot{\hat{z}}_{s,2}(t) \\ \epsilon\dot{\hat{z}}_{fnew2,2}(t) \end{bmatrix} = \begin{bmatrix} A_{sr,2} - K_{sr,2}C_{sr,2} & 0 \\ 0 & A_{fr,2} - K_{f2r,2}C_{fnewr,2} \end{bmatrix} \begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{fnew2,2}(t) \end{bmatrix} \\
&+ \begin{bmatrix} B_{s2r,2} \\ B_{f2r,2} \end{bmatrix} u(t) + \begin{bmatrix} K^{*}_{sr,2} \\ K_{f3r,2} \end{bmatrix} y(t)
\end{aligned} \tag{5.60}$$

These two observers (5.60) can be implemented independently in the slow and fast time scales

$$\begin{aligned}
&\dot{\hat{z}}_{s,2}(t) = (A_{sr,2} - K_{sr,2}C_{sr,2})\hat{z}_{s,2}(t) + B_{s2r,2}u(t) + K^{*}_{sr,2}y(t) \\
&\epsilon\dot{\hat{z}}_{fnew2,2}(t) = (A_{fr,2} - K_{f2r,2}C_{fnewr,2})\hat{z}_{fnew2,2}(t) + B_{f2r,2}u(t) + K_{f3r,2}y(t)
\end{aligned} \tag{5.61}$$

where $B_{sr2,2}, B_{f2r,2}$ can be obtained from $T^{-1}_{r,2}B$ as

$$\begin{aligned}
&B_{s2r,2} = (I_{n_1} - \epsilon H_{r,2}L_{r,2})B_{z1} - \epsilon P^{T}_{or,2}L_{r,2}B_{z1} - H_{r,2}B_{z2} - P^{T}_{or,2}B_{z2}, \\
&B_{f2r,2} = \epsilon P_{o2r,2}(I_{n_1} - \epsilon H_{r,2}L_{r,2})B_{z1} - \epsilon^{2}P_{o2r,2}P^{T}_{or,2}L_{r,2}B_{z1} + \epsilon^{2}L_{r,2}B_{z1} \\
&- \epsilon P_{o2r,2}H_{r,2}B_{z2} - \epsilon P_{o2r,2}P^{T}_{or,2}B_{z2} + \epsilon B_{z2}
\end{aligned} \tag{5.62}$$

The control input in the $\hat{z}_{s,2}$-$\hat{z}_{fnew2,2}$ coordinates is given by

$$\begin{aligned}
u(t) &= -F_r\hat{x}(t) = -\begin{bmatrix} F_1 & F_{21} \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_{21}(t) \end{bmatrix} \\
&= -(\begin{bmatrix} F_1 & F_{21} \end{bmatrix} \begin{bmatrix} K_{31} \\ K_{32} \end{bmatrix} y(t) + \begin{bmatrix} F_1 & F_{21} \end{bmatrix} T_{r,2} \begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{fnew2,2}(t) \end{bmatrix}) \\
&= -(\begin{bmatrix} F_1 & F_{21} \end{bmatrix} \begin{bmatrix} K_{31} \\ K_{32} \end{bmatrix} y(t) + \begin{bmatrix} F_{sr2,2} & F_{fr2,2} \end{bmatrix} \begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{fnew2,2}(t) \end{bmatrix}) \\
&= -F_{r,2}K_3 y(t) - \begin{bmatrix} F_{sr2,2} & F_{fr2,2} \end{bmatrix} \begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{fnew2,2}(t) \end{bmatrix}
\end{aligned} \tag{5.63}$$

with

$$\begin{aligned}
&F_{sr2,2} = F_1(I_{n_1} - P_{or,2}^T P_{o2r,2} - H_{r,2}P_{o2r,2}) - F_{21}L(I_{(n_2-l)} - P_{or,2}^T P_{o2r,2}) \\
&- \frac{1}{\epsilon}F_2(I_{(n_2-l)} - \epsilon L_{r,2}H_{r,2})P_{o2r,2} \\
&F_{fr2,2} = F_1(P_{or,2}^T + H_{r,2}) - F_{21}L_{r,2}P_{or,2}^T + \frac{1}{\epsilon}F_{21}(I_{(n_2-l)} - \epsilon L_{r,2}H_{r,2})
\end{aligned} \tag{5.64}$$

Here, F_r is taken from (3.33), and $F_1 \in R^{1\times n_1}$, $F_{21} \in R^{1\times(n_2-l)}$. The corresponding block diagram for the observer driven controller is presented in Figure 5.4. This block diagram clearly indicates full parallelism of the slow controller driven by the slow observer and the fast controller driven by the fast observer.

The remaining matrices introduced in (5.50) are given by

$$\begin{aligned}
&A_{sr,2}^T = a_{11}^T - L_{r,2}^T a_{12}^T,\ A_{fr,2}^T = a_{22}^T + \epsilon a_{12}^T L_{r,2}^T \\
&C_{sr,2}^T = \frac{1}{\epsilon}a_{31}^T - \frac{1}{\epsilon}L_{r,2}^T a_{32}^T,\ c_{fr,2}^T = H_{r,2}^T a_{31}^T + \frac{1}{\epsilon}(I_{(n_2-l)} - \epsilon H_{r,2}^T L_{r,2}^T)a_{32}^T, \\
&C_{fnewr,2} = C_{fr,2} + \epsilon C_{sr,2}P_{or,2}^T
\end{aligned} \tag{5.65}$$

$K_{sr,2}, K_{f2r,2}, P_{or,2}$ can be obtained from the formulas in Section 4.5, that is

$$\begin{aligned}
&\lambda(A_{sr,2}^T - C_{sr,2}^T K_{sr,2}^T) = \lambda(A_{sr,2} - K_{sr,2}C_{sr,2}) = \lambda_s^{desired} \\
&\lambda(A_{fr,2} - K_{f2r,2}C_{fnewr,2}) = \lambda_f^{desired} \\
&\epsilon P_{or,2}(A_{sr,2}^T - C_{sr,2}^T K_{sr,2}^T) - C_{fr,2}^T K_{sr,2}^T - A_{fr,2}^T P_{or,2} = 0 \;\Rightarrow\; P_{or,2} = O(1)
\end{aligned} \tag{5.66}$$

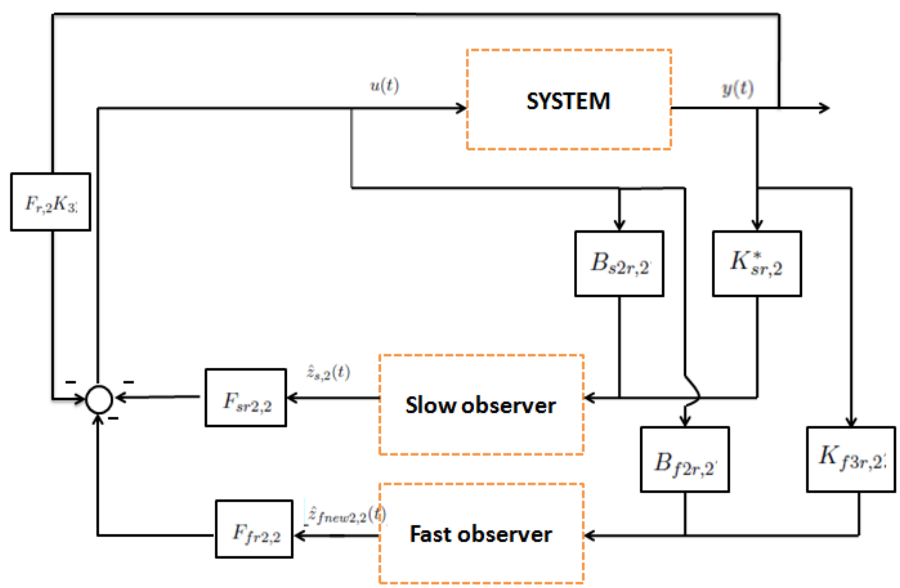

Figure 5.4: Case IV: Slow and fast observer-based controller design for a singularly perturbed linear systems with the system feedback gains obtained in (5.63)

5.4.1 Case IV : Numerical Example

Consider a $4^{th}-$ order system with the system matrices A, B, and C defined in Section 4.5. The controllability matrix has full rank and therefore the pair (A, B) is controllable. We locate the feedback system slow eigenvalues at $\lambda_{cs}^{desired} = (-2, -3)$ and the feedback system fast eigenvalues at $\lambda_{cf}^{desired} = (-7, -8)$, and the slow observer eigenvalues at $\lambda_{os}^{desired} = (-50, -60)$ and the fast observer eigenvalues at $\lambda_{of}^{desired} = -300$, given in the previous numerical example. Following the design procedure from Sections 5.4, the completely decoupled slow and fast observer in the z_s-z_{fnew2} coordinates, driven by the system measurements and control inputs, are

$$\dot{\hat{z}}_{s,2}(t) = \begin{bmatrix} -0.0000 & -4127.7367 \\ 0.7267 & -110.0000 \end{bmatrix} \hat{z}_{s,2}(t)$$
$$+ \begin{bmatrix} 4.7409 \\ -3.9951 \end{bmatrix} u(t) + \begin{bmatrix} 113729.6251 \\ 2281.0457 \end{bmatrix} y(t)$$

$$\dot{\hat{z}}_{fnew2,2}(t) = \begin{bmatrix} -300.0000 \end{bmatrix} \hat{z}_{fnew2,2}(t)$$
$$+ \begin{bmatrix} -779.5149 \end{bmatrix} u(t) + \begin{bmatrix} 437703.1601 \end{bmatrix} y(t)$$

$$u(t) = - \begin{bmatrix} -6530.3242 & 91.6868 \end{bmatrix} \hat{z}_s(t)$$
$$- \begin{bmatrix} 332.8512 & -10.9008 \end{bmatrix} \hat{z}_{fnew2}(t)$$

The slow and fast controller gains $F_{sr2,2}, F_{fr2,2}$ are obtained as

$$F_{sr2,2} = \begin{bmatrix} -6530.3242 & 91.6868 \end{bmatrix},$$
$$F_{fr2,2} = \begin{bmatrix} 332.8512 & -10.9008 \end{bmatrix}$$

5.5 Case V : Controller Design when Only a Part of Slow and Fast Variables are Measured

In Case V), the measurable states $x_{11}(t), x_{21}(t)$ are parts of the slow state vector $x_1(t)$ and the fast state $x_2(t)$ in the singularly perturbed linear system defined in (3.1), as

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) \\ \epsilon\dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) \\ y(t) &= \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \\ x_{21}(t) \\ x_{22}(t) \end{bmatrix} = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \end{bmatrix} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \end{aligned} \tag{5.67}$$

where

$$x_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix}, x_2(t) = \begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix} = \begin{bmatrix} y_1(t) \\ x_{12}(t) \end{bmatrix}, x_2(t) = \begin{bmatrix} x_{21}(t) \\ y_2(t) \end{bmatrix} \tag{5.68}$$

Using the following partitioning

$$\begin{aligned} A_{11} &= \begin{bmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{bmatrix}, \; A_{12} = \begin{bmatrix} a_{13}^* & a_{14}^* \\ a_{23}^* & a_{24}^* \end{bmatrix} \\ A_{21} &= \begin{bmatrix} a_{31}^* & a_{32}^* \\ a_{41}^* & a_{42}^* \end{bmatrix}, \; A_{22} = \begin{bmatrix} a_{33}^* & a_{34}^* \\ a_{43}^* & a_{44}^* \end{bmatrix} \end{aligned} \tag{5.69}$$

where $x_{11}(t) \in R^{l_1}, x_{12}(t) \in R^{(n_1-l_1)}, x_{21}(t) \in R^{l_2}, x_{22}(t) \in R^{(n_2-l_1)}$ and $a_{11} \in R^{l_1 \times l_1}$, $a_{12} \in R^{l_1 \times (n_1-l_1)}, a_{13} \in R^{l_1 \times l_2}, a_{14} \in R^{l_1 \times (n_2-l_2)}, a_{21} \in R^{(n_1-l_1) \times l_1}$, $a_{22} \in R^{(n_1-l_1) \times (n_1-l_1)}, a_{23} \in R^{(n_1-l_1) \times l_2}, a_{24} \in R^{(n_1-l_1) \times (n_2-l_2)}, a_{31} \in R^{l_2 \times l_1}$, $a_{32} \in R^{l_2 \times (n_1-l_1)}, a_{33} \in R^{l_2 \times l_2}, a_{34} \in R^{l_2 \times (n_2-l_2)}, a_{41} \in R^{(n_2-l_2) \times l_1}$, $a_{42} \in R^{(n_2-l_2) \times (n_1-l_1)}, a_{43} \in R^{(n_2-l_2) \times l_2}, a_{44} \in R^{(n_2-l_2) \times (n_2-l_2)}$, $y(t) \in R^{(l_1+l_2)}$, and $p(t) \in R^{(n-l) \times 1}$.

The system (5.67) with (5.68)-(5.69) can be represented as

$$\begin{aligned} \dot{x}_m(t) &= A_1^r x_m(t) + A_2^r x_u(t) \\ \dot{x}_u(t) &= A_3^r x_m(t) + A_4^r x_u(t) \\ y(t) &= \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \\ x_{21}(t) \\ x_{22}(t) \end{bmatrix} = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \end{bmatrix} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \end{aligned} \tag{5.70}$$

where

$$x_m(t) = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \end{bmatrix}, x_u(t) = \begin{bmatrix} x_{12}(t) \\ x_{22}(t) \end{bmatrix}$$

$$A_1^r = \begin{bmatrix} a_{11}^* & a_{13}^* \\ \frac{1}{\epsilon}a_{31}^* & \frac{1}{\epsilon}a_{33}^* \end{bmatrix}, \ A_2^r = \begin{bmatrix} a_{12}^* & a_{14}^* \\ \frac{1}{\epsilon}a_{32}^* & \frac{1}{\epsilon}a_{34}^* \end{bmatrix} \tag{5.71}$$

$$A_3^r = \begin{bmatrix} a_{21}^* & a_{23}^* \\ \frac{1}{\epsilon}a_{41}^* & \frac{1}{\epsilon}a_{43}^* \end{bmatrix}, \ A_4^r = \begin{bmatrix} a_{22}^* & a_{24}^* \\ \frac{1}{\epsilon}a_{42}^* & \frac{1}{\epsilon}a_{44}^* \end{bmatrix}$$

$x_m(t)$ are the measurable states and $x_u(t)$ are the unmeasurable states. A_1^r, A_3^r are elements in (5.68) relevant to the measurable states, A_2^r, A_4^r are elements in (5.68) relevant to the unmeasurable states.

The reduced-order observer configuration obtained in Section 4.6 is given by

$$\begin{bmatrix} \dot{\hat{z}}_{s,3}(t) \\ \epsilon\dot{\hat{z}}_{fnew,3}(t) \end{bmatrix} = \begin{bmatrix} A_{sr,3} - K_{sr,3}C_{sr,3} & 0 \\ -\epsilon K_{f2r,3}C_{sr,3} & A_{fr,3} - K_{f2r,3}C_{fnewr,3} \end{bmatrix} \begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew,3}(t) \end{bmatrix} + \begin{bmatrix} K_{sr,3}^* \\ \epsilon K_{f2r,3}^* \end{bmatrix} y(t) \tag{5.72}$$

where $K_{sr,3}^*, K_{f2r,3}^*$ are determined by $T_{4r,3}^{-1}K_{z,3}^r$, with $T_{4r,3}(t)$ defined by

$$\begin{bmatrix} \hat{z}_{12}(t) \\ \hat{z}_{22}(t) \end{bmatrix} = T_{1r,3}T_{cr,3}T_{2r,3} \begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew,3}(t) \end{bmatrix} = T_{4r,3} \begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew,3}(t) \end{bmatrix} \tag{5.73}$$

where

$$T_{1r,3}^T = \begin{bmatrix} I_{(n_1-l_1)} & 0 \\ 0 & \frac{1}{\epsilon}I_{(n_2-l_2)} \end{bmatrix}, \ T_{2r,3}^T = \begin{bmatrix} I_{(n_1-l_1)} & 0 \\ P_{or,3} & I_{(n_2-l_2)} \end{bmatrix} \tag{5.74}$$

The Chang transformation is given by

$$T_{cr,3}^T = \begin{bmatrix} I_{(n_1-l_1)} & -\epsilon L_{r,3}^T \\ H_{r,3}^T & I_{(n_2-l_2)} - \epsilon H_{r,3}^T L_{r,3}^T \end{bmatrix} \tag{5.75}$$

where $L_{r,3}^T$ and $H_{r,3}^T$ are the transposed solution, that is

$$\begin{aligned}
0 &= \epsilon({a_{22}^*}^T - L_{r,3}^T {a_{24}^*}^T) L_{r,3}^T + ({a_{42}^*}^T - L_{r,3}^T {a_{44}^*}^T) \\
0 &= \epsilon H_{r,3}^T ({a_{22}^*}^T - L_{r,3}^T {a_{24}^*}^T) + {a_{24}^*}^T - ({a_{44}^*}^T + \epsilon {a_{24}^*}^T L_{r,3}^T) H_{r,3}^T
\end{aligned} \tag{5.76}$$

with a_{ij}^* matrices defined in (5.71)

The observer gains are obtained from

$$\begin{aligned}
&A_{z,3}^r = A_4^r - K_4 A_2^r = \begin{bmatrix} a_{22}^* & a_{24}^* \\ \frac{1}{\epsilon} a_{42}^* & \frac{1}{\epsilon} a_{44}^* \end{bmatrix} - \begin{bmatrix} K_{41} \\ K_{42} \end{bmatrix} \begin{bmatrix} a_{12}^* & a_{14}^* \\ \frac{1}{\epsilon} a_{32}^* & \frac{1}{\epsilon} a_{34}^* \end{bmatrix}, \\
&K_{z,3}^r = A_3^r - K_4 A_1^r + A_4^r K_4 - K_4 A_2^r K_4 \\
&= \begin{bmatrix} a_{21} - K_{21} a_{11} + a_{22} K_{21} + \frac{1}{\epsilon} a_{23} K_{22} - K_{21}(a_{12} K_{21} + \frac{1}{\epsilon} a_{13} K_{22}) \\ \frac{1}{\epsilon} a_{31} - \frac{1}{\epsilon} K_{22} a_{11} + \frac{1}{\epsilon} a_{32} K_{21} + \frac{1}{\epsilon^2} a_{33} K_{22} - \frac{1}{\epsilon} K_{22}(a_{12} K_{21} + \frac{1}{\epsilon} a_{13} K_{22}) \end{bmatrix} = \begin{bmatrix} K_{41r} \\ K_{42r} \end{bmatrix}
\end{aligned} \tag{5.77}$$

The reduced-order observer (5.72) has a sequential structure. It can be block diagonalized and used as a parallel structure as

$$\begin{aligned}
\dot{\hat{z}}_{s,3}(t) &= (A_{sr,3} - K_{sr,3} C_{sr,3}) \hat{z}_{s,3}(t) + K_{sr,3}^* y(t) \\
\epsilon \dot{\hat{z}}_{fnew2,3}(t) &= (A_{fr,3} - K_{f2r,3} C_{fnewr,3}) \hat{z}_{fnew2,3}(t) + K_{f3r,3} y(t)
\end{aligned} \tag{5.78}$$

where

$$\begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew2,3}(t) \end{bmatrix} = \begin{bmatrix} I_{(n_1-l_1)} & 0 \\ P_{o2r,3} & I_{(n_2-l_2)} \end{bmatrix} \begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew,3}(t) \end{bmatrix} = T_{3r,3} \begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew,3}(t) \end{bmatrix} \tag{5.79}$$

The original coordinates $\hat{z}_{12}(t)$, $\hat{z}_{22}(t)$ and the coordinates $\hat{z}_{s,3}(t)$, $\hat{z}_{fnew2,3}(t)$ are related via

$$\begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew2,3}(t) \end{bmatrix} = T_{3r,3} T_{4r,3}^{-1} \begin{bmatrix} \hat{z}_{12}(t) \\ \hat{z}_{22}(t) \end{bmatrix} = T_{r,3}^{-1} \begin{bmatrix} \hat{z}_{12}(t) \\ \hat{z}_{22}(t) \end{bmatrix} \tag{5.80}$$

where $P_{o2r,3}$ satisfies the algebraic Sylvester equation represented by

$$\begin{aligned}
&\epsilon P_{o2r,3}(A_{sr,3} - K_{sr,3} C_{sr,3}) - \epsilon K_{f2r,3} C_{sr,3} - (A_{fr,3} - K_{f2r,3} C_{fnewr,3}) P_{o2r,3} = 0 \\
&\Rightarrow \; P_{o2r,3}^0 = O(\epsilon)
\end{aligned} \tag{5.81}$$

In the previous chapter, we have observed the original system state using independent reduced-order slow and fast observers (5.78). In this section, we use these observers and consider the observer-based controller design for singularly perturbed linear systems. The observer is driven by the system measurements and control inputs, that is

$$\begin{bmatrix} \dot{\hat{z}}_{s,3}(t) \\ \epsilon\dot{\hat{z}}_{fnew2,3}(t) \end{bmatrix} = \begin{bmatrix} A_{sr,3} - K_{sr,3}C_{sr,3} & 0 \\ 0 & A_{fr,3} - K_{f2r,3}C_{fnewr,3} \end{bmatrix} \begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew2,3}(t) \end{bmatrix} + \begin{bmatrix} B_{sr2,3} \\ B_{f2r,3} \end{bmatrix} u(t) + \begin{bmatrix} K^*_{sr,3} \\ K_{f3r,3} \end{bmatrix} y(t) \tag{5.82}$$

Thus, these two observers (5.82) can be implemented independently in the slow and fast time scales

$$\begin{aligned} \dot{\hat{z}}_{s,3}(t) &= (A_{sr,3} - K_{sr,3}C_{sr})\hat{z}_{s,3}(t) + B_{sr2,3}u(t) + K^*_{sr,3}y(t) \\ \epsilon\dot{\hat{z}}_{fnew2,3}(t) &= (A_{fr,3} - K_{fr2,3}C_{fnewr,3})\hat{z}_{fnew2,3}(t) + B_{f2r,3}u(t) + K_{f3r,3}y(t) \end{aligned} \tag{5.83}$$

where $B_{sr2,3}, B_{f2r,3}$ can be obtained from $T^{-1}_{r,3}B$ as

$$\begin{aligned} &B_{sr2,3} = (I_{(n_1-l_1)} - \epsilon H_{r,3}L_{r,3})B_{z1} - \epsilon P^T_{or,3}L_{r,3}B_{z1} - H_{r,3}B_{z2} - P^T_{or,3}B_{z2}, \\ &B_{f2r,3} = \epsilon P_{o2r,3}(I_{(n_1-l_1)} - \epsilon H_{r,3}L_{r,3})B_{z1} - \epsilon^2 P_{o2r,3}P^T_{or,3}L_{r,3}B_{z1} + \epsilon^2 L_{r,3}B_{z1} \\ &- \epsilon P_{o2r,3}H_{r,3}B_{z2} - \epsilon P_{o2r,3}P^T_{or,3}B_{z2} + \epsilon B_{z2} \end{aligned} \tag{5.84}$$

The control input in the $\hat{z}_{s,3}$-$\hat{z}_{fnew2,3}$ coordinates is given by

$$\begin{aligned} u(t) = -F_{r,3}\hat{x}(t) &= -\begin{bmatrix} F_{12} & F_{21} \end{bmatrix} \begin{bmatrix} \hat{x}_{12}(t) \\ \hat{x}_{21}(t) \end{bmatrix} \\ &= -(\begin{bmatrix} F_{12} & F_{21} \end{bmatrix} \begin{bmatrix} K_{41} \\ K_{42} \end{bmatrix} y(t) + \begin{bmatrix} F_{12} & F_{21} \end{bmatrix} T_r \begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew2,3}(t) \end{bmatrix}) \\ &= -(\begin{bmatrix} F_{12} & F_{21} \end{bmatrix} \begin{bmatrix} K_{41} \\ K_{42} \end{bmatrix} y(t) + \begin{bmatrix} F_{sr2,3} & F_{fr2,3} \end{bmatrix} \begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew2,3}(t) \end{bmatrix}) \\ &= -F_{r,3}K_4 y(t) - \begin{bmatrix} F_{sr2,3} & F_{fr2,3} \end{bmatrix} \begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew2,3}(t) \end{bmatrix} \end{aligned} \tag{5.85}$$

with

$$
\begin{aligned}
&F_{sr2,3} = F_{12}(I_{(n_1-l_1)} - P_{or,3}^T P_{o2r,3} - H_{r,3}P_{o2r,3}) - F_{21}L(I_{(n_2-l_2)} - P_{or,3}^T P_{o2r,3}) \\
&- \frac{1}{\epsilon}F_{21}(I_{(n_2-l_2)} - \epsilon L_{r,3}H_{r,3})P_{o2r,3} \\
&F_{fr2,3} = F_{12}(P_{or,3}^T + H_{r,3}) - F_{21}L_{r,3}P_{or,3}^T + \frac{1}{\epsilon}F_{21}(I_{(n_2-l_2)} - \epsilon L_{r,3}H_{r,3})
\end{aligned}
\tag{5.86}
$$

Here, F_r is taken from (3.33), and $F_{12} \in R^{1\times(n_1-l_1)},\ F_{21} \in R^{1\times(n_2-l_2)}$. The corresponding block diagram for the observer driven controller is presented in Figure 5.5. This block diagram clearly indicates full parallelism of the slow controller driven by the slow observer and the fast controller driven by the fast observer.

The remaining matrices introduced in (5.72) are given by

$$
\begin{aligned}
&A_{sr,3}^T = {a_{22}^*}^T - L_{r,3}^T {a_{24}^*}^T,\ A_{fr,3}^T = {a_{44}^*}^T + \epsilon {a_{24}^*}^T L_{r,3}^T, \\
&C_{sr,31}^T = {a_{12}^*}^T - L_{r,3}^T {a_{14}^*}^T,\ C_{sr,32}^T = \frac{1}{\epsilon}{a_{32}^*}^T - L_{r,3}^T \frac{1}{\epsilon}{a_{34}^*}^T, \\
&C_{fr,31}^T = \epsilon H_{r,3}^T {a_{12}^*}^T + (I_{(n_2-l_2)} - \epsilon H_{r,3}^T L_{r,3}^T){a_{14}^*}^T, \\
&C_{fr,32}^T = \epsilon H_{r,3}^T \frac{1}{\epsilon}{a_{32}^*}^T + (I_{(n_2-l_2)} - \epsilon H_{r,3}^T L_{r,3}^T)\frac{1}{\epsilon}{a_{34}^*}^T, \\
&C_{sr,3}^T = \begin{bmatrix} C_{sr,31}^T & C_{sr,32}^T \end{bmatrix},\ C_{fr,3}^T = \begin{bmatrix} C_{fr,31}^T & C_{fr,32}^T \end{bmatrix} \\
&C_{fnewr,3} = C_{fr,3} + \epsilon C_{sr,3}P_{or,3}^T
\end{aligned}
\tag{5.87}
$$

$K_{sr,3}, K_{f2r,3}, P_{or,3}$ can be obtained from the formula in Section 4.6, that is

$$
\begin{aligned}
&\lambda(A_{sr,3}^T - C_{sr,3}^T K_{sr,3}^T) = \lambda(A_{sr,3} - K_{sr,3}C_{sr,3}) = \lambda_s^{desired} \\
&\lambda(A_{fr,3} - K_{f2r,3}^T C_{fnewr,3}) = \lambda_f^{desired} \\
&\epsilon P_{or,3}(A_{sr,3}^T - C_{sr,3}^T K_{sr,3}^T) - C_{fr,3}^T K_{sr,3}^T - A_{fr,3}^T P_{or,3} = 0 \\
&\Rightarrow\ P_{or,3} = O(\epsilon)
\end{aligned}
\tag{5.88}
$$

5.5.1 Case V : Numerical Example

Consider a $4^{th}-$ order system with the system matrices A, B, and C defined in Section 4.C. The controllability matrix has full rank and therefore the pair (A, B) is controllable. We locate the feedback system slow eigenvalues at $\lambda_{cs}^{desired} = (-2, -3)$ and the feedback

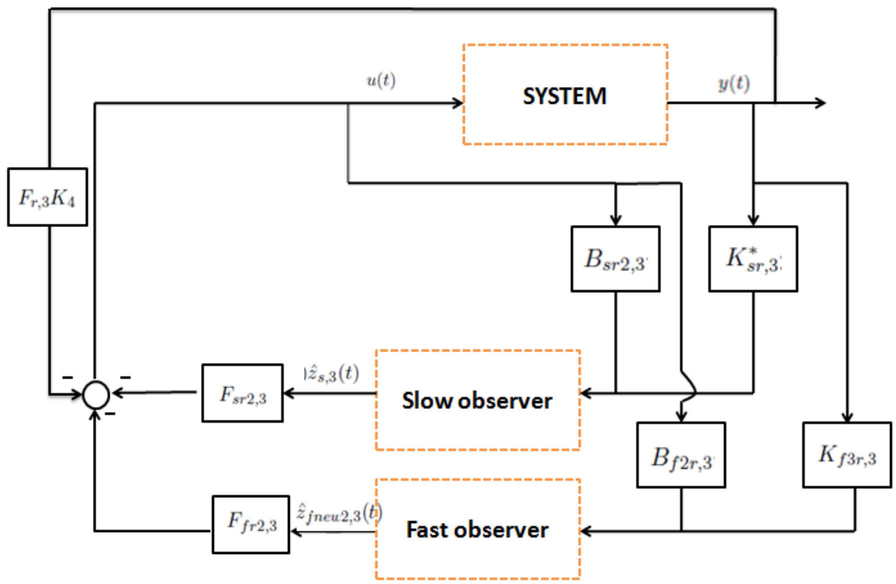

Figure 5.5: Case V: Slow and fast observer-based controller design for a singularly perturbed linear systems with the system feedback gains obtained in (5.85)

system fast eigenvalues at $\lambda_{cf}^{desired} = (-7, -8)$, and the slow observer eigenvalues at $\lambda_{os}^{desired} = -50$ and the fast observer eigenvalues at $\lambda_{of}^{desired} = (-200, -300)$, given in the previous numerical example. Following the design procedure of from Sections 5.5, the completely decoupled slow and fast observers in the $z_{s,3}$-$z_{fnew2,3}$ coordinates, driven by the system measurements and control inputs, are

$$\begin{aligned}\dot{\hat{z}}_{s,3}(t) &= \begin{bmatrix}-50.0000\end{bmatrix}\hat{z}_s(t)\\ &+\begin{bmatrix}-2.323033338614758\end{bmatrix}u(t)+\begin{bmatrix}-6196.4741\end{bmatrix}y(t)\end{aligned}$$

$$\begin{aligned}\dot{\hat{z}}_{fnew2,3}(t) &= \begin{bmatrix}-1693.6295 & 113.2762\\ -18376.0229 & 1193.6295\end{bmatrix}\hat{z}_{fnew2}(t)\\ &+\begin{bmatrix}-0.2499\\ 0.4666\end{bmatrix}u(t)+\begin{bmatrix}-2859901.4120\\ -33059040.2923\end{bmatrix}y(t)\end{aligned}$$

$$\begin{aligned}u(t) &= -\begin{bmatrix}-6530.3242 & 91.6868\end{bmatrix}\hat{z}_s(t)\\ &-\begin{bmatrix}332.8512 & -10.9008\end{bmatrix}\hat{z}_{fnew2}(t)\end{aligned}$$

The slow and fast controller gains $F_{sr2,3}, F_{fr2,3}$ are obtained as

$$F_{sr2,3} = \begin{bmatrix} -6530.3242 & 91.6868 \end{bmatrix},$$
$$F_{fr2,3} = \begin{bmatrix} 332.8512 & -10.9008 \end{bmatrix}$$

5.6 Conclusions

We have designed with high accuracy reduced-order observer-based controllers for singularly perturbed linear systems in Chapter 5. The numerical ill-conditioning problem of the original system is removed. We have demonstrated that the full-order singularly perturbed system can be successfully controlled with the state feedback reduced-order controllers designed on the subsystem levels. The two stage method is successfully implemented for both observer and controller designs from Case III) to Case V).

Chapter 6

Conclusions and Future work

6.1 Conclusions

We have designed with very high accuracy the pure-slow and pure-fast observer-based controllers. They are designed independently using the reduced-order slow and fast sub- system matrices. The numerical ill-conditioning problem of the original system is removed. We have demonstrated that the full-order singularly perturbed linear system can be successfully controlled with the state feedback controllers designed on the sub-system levels. The two stage method is successfully implemented for both observer and controller designs. Furthermore, we extend the two stage method to the reduced-order observer design and apply it to observer-based controller design. We consider several cases: Case I to Case V for the reduced-order observer design of singularly perturbed linear systems in order to account for different measurement situations.

6.2 Future Work

In the future, more realistic models of singularly perturbed linear system could be and should be developed since we did not consider noise in the state space model. In that case, we plan to extend this approach to design of the Kalman filter and Kalman filter based controllers for singularly perturbed linear systems [40]-[53]. Corresponding controllers may be designed in the future using multiple time scales [39]. The study of the corresponding discrete-time problems is also an interesting area for future research. Extensions to multi-time scale systems are interesting future research topics. In addition, studying the sensitivity of presented algorithms should be addressed in the future.

Appendix A

Proof

A.1 Rank Condition in Section 4.3

If the pair (A, C) is observable, we form matrix given as

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix}$$

The rank condition after scalar multiplication is unchanged

$$\text{rank}(\alpha A) = \text{rank}(A)$$

In Section 4.2.1, the pair $(A_{11}, \frac{1}{\epsilon}A_{21})$ is observable, which implies

$$\text{rank} \begin{bmatrix} \frac{1}{\epsilon}A_{21} \\ \frac{1}{\epsilon}A_{21}A_{11} \\ \frac{1}{\epsilon}A_{21}A_{11}^2 \\ . \\ . \\ . \\ \frac{1}{\epsilon}A_{21}A_{11}^{n-1} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{21} \\ A_{21}A_{11} \\ A_{21}A_{11}^2 \\ . \\ . \\ . \\ A_{21}A_{11}^{n-1} \end{bmatrix}$$

A.2 Reduced-Order Observer Design for Section 4.2

The unmeasured portion of the system is

$$\epsilon \dot{x}_2(t) = (A_{21}x_1(t)) + A_{22}x_2(t) \tag{A.1}$$

The term within the parentheses is a known quantity. Because there are n_1 measured states, the number of unmeasured states is $n - n_1$, so that we will build an observer of order $n - n_1$ to estimate these states. The observer structure is given by the following procedure (this is the same procedure used for the full-order observer): copy the system equation, replace unknown quantities by their estimates, and add a correction term multiplied by the observer gain. The correction term is the difference between the plant output and the observer output, producing

$$\epsilon \dot{x}_2(t) = A_{22}x_2(t) + (A_{21}x_1(t)) + K_{11}(\text{correction term}) \tag{A.2}$$

The correction term in the full-order observer case was $(y(t) - C\hat{x}(t))$. In the present case, it is

$$y(t) - \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = y(t) - x_1(t) = 0 \tag{A.3}$$

Hence, using the output will not provide any useful information. However, if the output is available, we can assume that their derivatives are also available. Now, we observe that the derivative of the plant output is equal to the measured portion of the system, i.e.,

$$\dot{y}(t) = \dot{x}_1(t) = A_{12}x_2(t) + A_{11}x_1(t) \rightarrow \dot{y}(t) - A_{11}x_1(t) = A_{12}x_2(t) \tag{A.4}$$

where we have collected the measured quantities on the left-hand side. We can use the known quantities on the left as a substitute for the plant output, and the right hand side as the observer output. Substituting this in the observer equation, we obtain

$$\epsilon \dot{x}_2(t) = A_{22}x_2(t) + (A_{21}x_1(t)) + K_{11}(\dot{y}(t) - A_{11}x_1(t) - A_{12}x_2(t)) \tag{A.5}$$

which corresponds to (4.5).

A.3 Reduced-Order Observer Design for Section 4.3

The unmeasured portion of the system is

$$\dot{x}_1(t) = (A_{12}x_2(t)) + A_{11}x_1(t) \tag{A.6}$$

The term within the parentheses is a known quantity. Because there are n_2 measured states, the number of unmeasured states is $n - n_2$, so that we will build an observer of order $n - n_2$ to estimate these states. The observer structure is given by the following procedure (this is the same procedure used for the full-order observer): copy the system equation, replace unknown quantities by their estimates, and add a correction term multiplied by the observer gin. The correction term is the difference between the plant output and the observer output, which produces

$$\dot{x}_1(t) = A_{11}x_1(t) + (A_{12}x_2(t)) + K_{12}(\text{correction term}) \tag{A.7}$$

The correction term in the full-order observer case was $(y(t) - C\hat{x}(t))$. In the present case, it is

$$y(t) - \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = y(t) - x_2(t) = 0 \tag{A.8}$$

Hence, using the output will not provide any useful information. However, if the output is available, we can assume that their derivatives are also available. Now, we observe that the derivative of the plant output is equal to the measured portion of the system, i.e.,

$$\dot{y}(t) = \dot{x}_2(t) = \frac{1}{\epsilon}A_{21}x_1(t) + \frac{1}{\epsilon}A_{22}x_2(t) \rightarrow \dot{y}(t) - \frac{1}{\epsilon}A_{22}x_2(t) = \frac{1}{\epsilon}A_{21}x_1(t) \tag{A.9}$$

where we have collected the measured quantities on the left-hand side. We can use the known quantities on the left as a substitute for plant output, and the right hand side as the observer output. Substituting this in the observer equation, we obtain

$$\dot{x}_1(t) = A_{11}x_1(t) + (A_{12}x_2(t)) + K_{12}(\dot{y}(t) - \frac{1}{\epsilon}A_{21}x_1(t) - \frac{1}{\epsilon}A_{22}x_2(t)) \tag{A.10}$$

which corresponds to (4.12).

A.4 Least Square Solution for the Full-Order Observer

From measurements, we have $n-$ unknown components of $\hat{x}(t)$, but $l-$ equations given as

$$\begin{aligned} y(t) &= C\hat{x}(t) \\ \text{at } \text{ t=0}, \ y(0) &= C\hat{x}(0) \end{aligned} \tag{A.11}$$

Multiplying by C^T on both side of the second equation in (A.11), we obtain

$$\begin{aligned} C^T C\hat{x}(0) &= C^T y(0) \\ \hat{x}(0) &= (C^T C)^{-1} C^T y(0) \end{aligned} \tag{A.12}$$

which gives the least square solution for $\hat{x}(0)$.

A.5 Case I : Least Square Solution for the Reduced-Order Observer in Section 4.2.1

From (4.3) at $t = 0$, the measurements are given as

$$y(0) = C_{\mathrm{I}}x(0) = x_1(0) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{A.13}$$

We can obtain the least square solution for $\hat{x}_1(0), \hat{x}_2(0)$ given as

$$\begin{bmatrix} \hat{x}_1(0) \\ \hat{x}_2(0) \end{bmatrix} = (C_{\mathrm{I}}^T C_{\mathrm{I}})^{\sharp} C_{\mathrm{I}}^T y(0) == (C_{\mathrm{I}}^T C_{\mathrm{I}})^{\sharp} C_{\mathrm{I}}^T x_1(0) \tag{A.14}$$

where $\sharp$ is a generalized inverse. We used the Penrose inverse since pinv exists in MATLAB.

A.6 Case II : Least Square Solution for the Reduced-Order Observer in Section 4.3.1

From (4.10) at $t = 0$, the measurements are given as

$$y(0) = C_{\mathrm{II}}x(0) = x_2(0) = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{A.15}$$

We can obtain the least square solution for $\hat{x}_1(0), \hat{x}_2(0)$ given as

$$\begin{bmatrix} \hat{x}_1(0) \\ \hat{x}_2(0) \end{bmatrix} = (C_{\mathrm{II}}^T C_{\mathrm{II}})^\sharp C_{\mathrm{II}}^T y(0) == (C_{\mathrm{II}}^T \times C_{\mathrm{II}})^\sharp C_{\mathrm{II}}^T x_2(0) \tag{A.16}$$

A.7 Case III : Least Square Solution for the Reduced-Order Observer in Section 4.4.4

From (4.18) at $t = 0$, the measurements are given as

$$y(0) = C_{\mathrm{III}}x(0) = x_{11}(0) = \begin{bmatrix} I & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11}(0) \\ x_{12}(0) \\ x_2(0) \end{bmatrix} \tag{A.17}$$

We can obtain the least square solution for $\hat{x}_{11}(0), \hat{x}_{12}(0), \hat{x}_2(0)$ given as

$$\begin{bmatrix} \hat{x}_{11}(0) \\ \hat{x}_{12}(0) \\ \hat{x}_2(0) \end{bmatrix} = (C_{\mathrm{III}}^T C_{\mathrm{III}})^\sharp C_{\mathrm{III}}^T y(0) == (C_{\mathrm{III}}^T C_{\mathrm{III}})^\sharp C_{\mathrm{III}}^T x_{11}(0) \tag{A.18}$$

A.8 Case IV : Least Square Solution for the Reduced-Order Observer in Section

From (4.69) at $t = 0$, the measurements are given as

$$y(0) = C_{\mathrm{IV}}x(0) = x_{22}(0) = \begin{bmatrix} 0 & 0 & I \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_{21}(0) \\ x_{22}(0) \end{bmatrix} \tag{A.19}$$

We can obtain the least square solution for $\hat{x}_1(0), \hat{x}_{21}(0), \hat{x}_{22}(0)$ given as

$$\begin{bmatrix} \hat{x}_1(0) \\ \hat{x}_{21}(0) \\ \hat{x}_{22}(0) \end{bmatrix} = (C_{\mathrm{IV}}^T C_{\mathrm{IV}})^\sharp C_{\mathrm{IV}}^T y(0) == (C_{\mathrm{IV}}^T C_{\mathrm{IV}})^\sharp C_{\mathrm{IV}}^T x_{22}(0) \tag{A.20}$$

A.9 Case V : Least Square Solution for the Reduced-Order Observer in Section 4.6.4

From (4.120) at $t = 0$, the measurements are given as

$$y(0) = C_{\mathrm{V}}x(0) = x_{11}(0) + x_{22}(0) = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} x_{11}(0) \\ x_{12}(0) \\ x_{21}(0) \\ x_{22}(0) \end{bmatrix} \tag{A.21}$$

We can obtain the least square solution for $\hat{x}_{11}(0), \hat{x}_{12}(0), \hat{x}_{21}(0), \hat{x}_{22}(0)$ given as

$$\begin{bmatrix} \hat{x}_{11}(0) \\ \hat{x}_{12}(0) \\ \hat{x}_{21}(0) \\ \hat{x}_{22}(0) \end{bmatrix} = (C_{\mathrm{V}}^T C_{\mathrm{V}})^\sharp C_{\mathrm{V}}^T y(0) == (C_{\mathrm{V}}^T C_{\mathrm{V}})^\sharp C_{\mathrm{V}}^T (x_{11}(0) + x_{22}(0)) \tag{A.22}$$

Bibliography

[1] P. Lancaster, "Explicit solutions of linear matrix equations," *SIAM Review*, vol. 12, pp. 544-566, 1970.

[2] P. Lancaster and M. Tismenetsky, *Theory of Matrices,* New York: Academic Press, 1985.

[3] Z. Gajic and M. Qureshi, *The Lyapunov Matrix Equation in System Stability and Control.* New York: Dover, 2008.

[4] D. Bini, B. Iannazzo, and B. Meini, *Numerical Solution for Algebraic Riccati Equation,* Philadelphia: SIAM, 2012.

[5] C. C. Tsui, "Observer design - A survey," *Int. J. Automat. and Comput.*, vol. 12, pp. 50-61, 2015.

[6] R. H. Bartels and G. W. Stewart, "Solution of the matrix equation AX+XB=C,"*Commun. Ass. Comput. Mach.*, vol. 15, pp. 820-826, 1972.

[7] Y-H. Gao and Z-Z. Bai, "On inexact Newton methods based on doubling iteration scheme for non-symmetric algebraic Riccati equations," *Numerical Linear Algebra and Applications*, vol. 18, pp. 325-341, 2011.

[8] J. Xue, S. Xu, and R-C. Li, "Accurate Solutions of M-Matrix Sylvester equations," *Numerische Mathematik*, vol. 120, pp. 638-670, 2012.

[9] C. C. Tsui, " A complete analytical solution to the equation $TA - FT = LC$ and its applications," *IEEE Trans. Automat. Contr.*, vol. 32, pp. 742-744 1987.

[10] C. C. Tsui, "On the solution to matrix equation $TA - FT = LC$ and its applications," *SIAM J. on Matr. Anal.*, vol. 14, pp. 33-44, 1993.

[11] P. Kokotovic, H. Khalil, and J. O'Reilly, *Singular Perturbation Methods in Control: Analysis and Design.* Orlando, FL: Academic, 1986.

[12] Z. Gajic and M. Lim, *Optimal Control of Singularly Perturbed Linear Systems with Applications : High Accuracy Techniques.* New York: Marcel Dekker, 2001.

[13] Ali, J., M. Hoang., N. Hussain, and D. Dochain. " Review and classification of recent observers applied in chemical process systems," *Computers and Chemical Engineering* , 76, 27-41, 2015.

[14] Anderson, L.R., "Pole placement and order reduction in two-time-scale control systems through Riccati iteration," *Mathematical Modeling*, 3, 93-101, 1982.

[15] Chang, K., "Singular perturbations of a general boundary value problem," *SIAM Journal of Mathematical Analysis,* 3, 520-526, 1972.

[16] Chen., C-T. *Linear System Theory and Design,* New York, NY: Oxford University Press, 1999.

[17] C., Kuehn. *Multiple Time Scale Dynamics,* Springer, 2015.

[18] Demetriou, M.A., Kazantzis, N., "Natural observer design for singularly perturbed vector second-order systems," *Transactions of the ASME Journal of Dynamic Systems, Measurement, and Control*, 127, 648-655, 2005.

[19] Dimitriev, M., Kurina, G., "Singular perturbations in control systems," *Automation and Remote Control,* 67, 1-43, 2006.

[20] Gajić, Z., M.T, Lim., "A new filtering method for linear singularly perturbed systems", IEEE *Transactions on Automatic Control*, 39, 1952-1955, 1994.

[21] Hsiao, FH., Hwang, JD., Pan, ST., "Stabilization of discrete singularly perturbed systems under composite observer-based control," *Transactions of the ASME Journal of Dynamic Systems, Measurement, and Control* , 123, 132-139, 2001.

[22] J., Ali, M., Hoang, N., Hussain, and D. Dochain., "Review and classification of recent observers applied in chemical process systems," *Computers and Chemical Engineering* , 76, 27-41, 2015.

[23] Kando, H., Aoyama, T., Iwazumi, T., "Observer design of singularly perturbed discrete systems via time-scale decomposition approach," *International Journal of Systems Science,* 21, 815-883, 1990.

[24] K-J, Lin, "Composite observer-based feedback design for singularly perturbed systems via LMI approach," *Proceedings of SICE Annual Conference;* 3056-3061, 2010.

[25] Liu, H.P., Sun, F.C., He, K.Z., "Survey of singularly perturbed control systems: theory and applications," *Control Theory and Applications,* 20, 1-7, 2003.

[26] Li, J-H., Li, T-H., "On the composite and reduced observer-based control of discrete two-time-scale systems." *Journal of the Franklin Institute*, 332, 47-66, 1995.

[27] Naidu, D., Calise, A., "Singular Perturbation and time scales in guidance and control of aerospace systems: A survey," *AIAA Journal of Guidance, Control and Dynamics,* 24, 1057-1078, 2001.

[28] Oloomi, H., Sawan, M. E., "The observer-based controller design of discrete-time singularly perturbed systems," IEEE *Transactions on Automatic Control*, 32, 246-248, 1987.

[29] O'Reilly, J., "Full-order observers for a class of singularly perturbed linear time-varying systems," *International Journal of Control,* 30, 745-756, 1979.

[30] O'Reilly, J., "Dynamical feedback control for a class of singularly perturbed linear systems using a full-order observer," *International Journal of Control*, 31, 1-10, 1980.

[31] O'Reilly, J., "Two time-scale feedback stabilization of linear time-varying singularly perturbed systems," *Automatica*, 308, 465-474, 1979.

[32] Phillips, R., "A two-stage design of linear feedback controls," IEEE *Transactions on Automatic Control,* 25, 1220-1223, 1980.

[33] Phillips, R., "The equivalence of time-scale decomposition techniques used in the analysis and design of linear systems," *International Journal of Control,* 37, 1239-1257, 1983.

[34] Porter, B., "Singular perturbation methods in the design of observers and stabilising feedback controllers for multivariable linear systems," *Electronics Letters,* IET, 10, 494-495, 1974.

[35] S., Javid., "Stabilization of time-varying singularly perturbed systems by observer-based slow-state feedback," IEEE *Transactions on Automatic Control*, 27, 702-704, 1982.

[36] Shouse, K. R., Taylor., "Discrete-time observers for singularly perturbed continuous-time systems," *IEEE Transactions on Automatic Control*, 40, 224-235, 1995.

[37] Sinha, A., *Linear Systems : Optimal and Robust Control,* Boca Raton, FL : CRC Press, 2007.

[38] Zhang, Y., Naidu, D.S., Cai, C., and Zou, Y., “Singular perturbation and time scales in control theories and applications : An Overview 2002-2012," *International Journal of Information and Systems Science,* 9, 1-36, 2014.

[39] Raymond T. Stefani et al., *Design of Feedback Control Systems,* Harcourt Brace Jovanovich College Publishers, 1994.

[40] Z. Gajic and X. Shen, “Parallel reduced-order controllers for stochastic linear singularly perturbed discrete systems," *IEEE Transaction On Automatic Control,* 1, 1-36, 1991.

[41] T. Grodt and Z. Gajic, “The Recursive reduced-order numerical solution of the singularly perturbed matrix differential riccati equation," *IEEE Transaction On Automatic Control,* 43, 751-754, 1988.

[42] W. Miranker, *Numerical Methods for Stiff Equations*, Amsterdam, The Netherlands: Reidel, 1981.

[46] R. Yackel and P. Kokotovic, “A boundary layer method for the matrix Riccati equation," *IEEE Transaction On Automatic Control,* 17, 17-24, 1973.

[44] Z. Gajic and M. T. Lim, “High Accuracy Techniques for Singularly Perturbed optimal Control and Filtering Problems," *International Conference on Continuous, Discrete and Impulsive Systems,* Canada, 2001.

[45] H. Mukaidani, “Near-optimal Kalman filters for multiparameter singularly perturbed linear systems," *IEEE Transaction On Circuits and Systems,* 50, 717-721, 2003.

[46] N. Prljaca and Z. Gajic, “A method for optimal control and filtering of multitimescale linear singularly-perturbed stochastic systems," *Automatica,* 44, 2149-2156, 2007.

[47] Z. Gajic and H. K. Khalil, “Multimodel strategies under random disturbances and imperfect partial observations," *Automatica,* 22, 121-125, 1986.

[51] C. Coumarbatch and Z. Gajic, “Parallel optimal Kalman filtering for stochastic systems in multimodeling form," *Transactions of ASME Journal of Dynamic Systems Measurement and Control,* 122, 542-550, 2000.

[52] A. J. Laub, “A Schur method for solving algebraic Riccati equations," *IEEE Transaction on Automatic Control,* 24, 913-921, 1979.

[53] D. L. Kleinman, “On the iterative technique for Riccati equation computations," *IEEE Transaction on Automatic Control,* 13, 114-115, 1968.

[51] H. Yoo, *Design of Observers for Systems with Slow and Fast modes*, *M.S Thesis, Rutgers University,* 2014.

[52] H. Yoo and Zoran Gajic, “Recursive reduced-order eigenvalue assignment for singularly perturbed algebraic Sylvester equation approach," submitted to *Automatica,* oct-2016.

[53] H. Yoo and Z. Gajic, “New designs of linear observers and observer-based controllers for singularly perturbed linear systems," submitted to *IEEE Transactions on Automatic Control,* oct-2016.

[54] MINH T. TRAN, MAHMOUD E. SAWAN, “Low-order observer for discrete-time systems with slow and fast modes," *International journal of Systems Science,* 15, 1283-1288, 1984.

[55] K. R. Shouse and D. G. Taylor, “Reduced-Order Discrete-Time Recursive Observer for Linear Continuous-Time Singularly Perturbed Systems," *American Control Conference,* 2128-2129, 1993.