# RISK-AVERSE OPTIMAL CONTROL OF DIFFUSION PROCESSES

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## ABSTRACT OF THE DISSERTATION

# Risk-Averse Optimal Control of Diffusion Processes

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This work analyzes an optimal control problem for which the performance is measured by a dynamic risk measure. While dynamic risk measures in discrete-time and the control problems associated are well understood, the continuous-time framework brings great challenges both in theory and practice. This study addresses modeling, numerical schemes and applications.

In the first part, we focus on the formulation of a risk-averse control problem. Specifically, we make use of a decoupled forward-backward system of stochastic differential equations to evaluate a fixed policy: the forward stochastic differential equation (SDE) characterizes the evolution of states, and the backward stochastic differential equation (BSDE) does the risk evaluation at any instant of time. Relying on the Markovian structure of the system, we obtain the corresponding dynamic programming equation via weak formulation and strong formulation; in the meanwhile, the risk-averse Hamilton-Jacobi-Bellman equation and its verification are derived under suitable assumptions.

In the second part, the main thrust is to find a convergent numerical method to solve the system in discrete-time setting. Specifically, we construct a piecewise-constant Markovian control to show its arbitrarily closeness to the optimal control. The results heavily relies on the regularity of the solution to generalized Hamilton-Jacobi-Bellman PDE.

In the third part, we propose a numerical method for risk evaluation defined by BSDE. Using dual representation of the risk measure, we converted risk valuation to a stochastic control problem, where the control is the *Radon-Nikodym derivative process*. The optimality conditions of such control problem enables us to use a piecewise-constant density (control) to arrive at a close approximation on a short interval. Then, the Bellman principle extends the approximation to any finite time horizon problem. Lastly, we give a financial application in risk management in conjunction with nested simulation.

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# Dedication

To my parents...

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# Chapter 1

# Introduction

## 1.1 Optimization Under Risk Aversion

### 1.1.1 Classiclal Risk Modeling

A great volume of literature in decision making under uncertainty is concerned with optimization of expected value, i.e.,

$$\min_{x\in\mathcal{X}} \mathbb{E}[Z_x] = \min_{x\in\mathcal{X}} \int_\Omega Z_x(\omega)\mathbb{P}(d\omega). \tag{1.1}$$

Here, $\mathcal{X}$ is the decision space and $Z$ is a random cost depending on decision $x \in \mathcal{X}$. Using expected value guiding decisions, however, is justified only if the *law of large numbers* can be invoked, because optimizing based on average requires observation of a large number of outcomes. In real life, we rarely enjoy these repetitions; usually, only a handful scenarios are available. This motivates the concept of *risk*, which can be defined as the existence of unlikely and undesirable outcomes. Namely, an event that incurs huge loss, but with low probability should drew special attention in many real world situations, such as, catastrophic risk management.

The first attemp to dealing with risk, to our knowledge, dates back to 1944, when von Neumann and Morgenstern [73] proposed *expected utility models* in economics. The resulting decision problems have the form:

$$\min_{x\in\mathcal{X}} \mathbb{E}[u(Z_x)] = \int_\Omega u(Z_x(\omega))d\mathbb{P}(\omega), \tag{1.2}$$

where $u : \mathbb{R} \mapsto \mathbb{R}$ is a non-decreasing dis-utility function. They derived the existence of a dis-utility function nonlinearly transforming the cost so that decision maker make a decision based on that. Such a model is characterized by a set of axioms, which were later criticized by economists. An alternative utility theory – dual utility – is built on the *distortion function* $w : [0,1] \mapsto \mathbb{R}$,

$$\min_{x\in\mathcal{X}} \int_0^1 F_{Z_x}^{-1}(p)dw(p),\ \ F_{Z_x}^{-1}(\cdot) \textit{ is the quantile function of } Z_x \tag{1.3}$$

The model [74, 62] attaches different weights to quantile functions aiming to amplify bad outcomes. The downside of the dis-utility theory is due to the difficulty to specify individuals', or a group of people's disutility functions in practice, which can lead to an undesired decision in the end.

In the field of operations research, similar ideas were employed. The optimization problem with chance constrained is postulated by [60, 52],

$$\min_{x\in\mathcal{X}} f(x) \ \textit{s.t.}\ \mathbb{P}(Z_x \le y) \ge 1-\alpha. \tag{1.4}$$

The interpretation can be, for instance, requiring the loss less than a certain benchmark $y$ to be very probable, i.e., at least $1-\alpha$. The model (1.4) can be generalized to individual constraints that are separable for each component of $Z_x$, or joint constraints enforcing the entire vector $Z_x$ to be below target with hight probability (particularly useful in coping with systemic risk). The recent advance of chance constraints optimization is so called *stochastic dominance constraints* by Ruszczyński and Dentcheva [18]. The stochastic dominance constraints consider the entire distribution of the outcome $Z_x$ rather than the probability of one event,

$$\min_{x\in\mathcal{X}} f(x) \ \textit{s.t.}\ Z_x \le_{SD} Y. \tag{1.5}$$

There are first order and second order of *stochastic orders*, both of which require that the cost $Z_x$ is less than random benchmark $Y$ in terms of distribution. It is interesting to point out that (1.5) is closely related to the models (1.2) and (1.3) in the sense that the dis-utility function is associated with the *Lagrangian multipliers* of the constraint present in (1.5).

In finance and engineering, researchers develop another approach to risk-averse optimization. Since expected value itself is not sufficient to incorporate risk, *variability* is taken into account as another object in the formulation. Here, the variability is considered as risk, which can be *variance*, *semi-deviation*, *deviation form quantile* and so on so forth. The most famous model of this kind is the *mean-variance model* of Markowitz [51], where the variability is measured by variance,

$$\min_{x,\in\mathcal{X}} \rho[Z_x] = \mathbb{E}[Z_x] + \kappa Var[Z_x],\ \ 0 \leq \kappa \leq \kappa_{max}. \tag{1.6}$$

The coeeficient $\kappa$ reflects one's risk aversion against variability; thus, by adjusting $\kappa$, we essentially place different levels of penalisation on the risk.

### 1.1.2 Modern Theory of Risk Measures

The modern theory of risk measures started in late 1990's by Artzner et al [2, 3]. Inspired by the capital adequacy rules of the Basel Accord, they proposed the notion of a *coherent risk measure*. The study was advanced by many authors, in particular, Föllmer, Schied [27, 28] and Frittelli, Rosazza Gianin [30] have extensive discussions on more general convex risk measure. They consider $X$ as a random variable representing payoff(or, gain), then the risk measure $\rho(\cdot)$ is simply a nonlinear mapping to quantify the uncertainty. A list of axioms are imposed to embody the evaluation, including *monotonicity*, *translation invariance*, *normalization*, *convexity* and *positive homogeneity*. Note, a convex risk measure is coherent if positive homogeinity is satisfied. The interpretation of monotonicity and normalization are trivial, and translation invariance can be understoood as the minimal capital reserve a company should hold in response to the the risk it exposes itself to, convexity arises due to the fact that diversification should not increase risk, and homogeneity postulates if an agent multiples his position by $\lambda$, his minimal capital reserves should be multipled by the same $\lambda$.

Another angle to view risk measure is to treat it as the equivalent amount of money one wants to pay to completely avoid the risk incurred by uncertainty, which is more close to the classical risk-averse models, e.g., utility theory, e.t.c.. In terms of the random loss, by following Ruszczyński and Shapiro [69, 68, 70], the risk measure is defined as

**Definition 1.1.1.** *(Axioms of Risk Measure) Given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{Z}$, a vector space of random variables on it, a functional $\rho : \mathcal{Z} \mapsto \mathbb{R}$ is a convex risk measure if it satisfies the following properties:*

- ***Convexity:*** *for all $Z, V \in \mathcal{Z}$ and $\lambda \in [0, 1]$,*

$$\rho(\lambda Z + (1 - \lambda)V) \leq \lambda\rho(Z) + (1 - \lambda)\rho(V).$$

- ***Monotonicity:*** *for all $Z, V \in \mathcal{Z}$, $Z \leq V$ almost surely implies $\rho(Z) \leq \rho(V)$*
- ***Translation Invariance:*** *for all $Z \in \mathcal{Z}$ and a constant $c \in \mathbb{R}$,*

$$\rho(Z + c) = \rho(Z) + c.$$

*If, additionally,* ***positive homogeneity*** *is imposed, i.e., for all $Z \in \mathcal{Z}$ and $\gamma \geq 0$,*

$$\rho(\gamma Z) = \gamma\rho(Z),$$

*it is then called* coherent risk measure.

**Remark 1.1.1.** *Here, the random variable on the vector space is understood as loss and $\mathcal{Z}$ is usually identified by $L^p(\Omega, \mathcal{F}, \mathbb{P})$ with $p \in [1, +\infty]$, while in [2, 3] only $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is considered.*

**Example 1.1.2.** *Let us give several well-known risk measures:*

1. Mean-semideviation (Ogryczak, Ruszczyński [53])

$$\rho(Z) ::= \mathbb{E}[Z] + \kappa\left(\mathbb{E}\left[\left((Z - \mathbb{E}[Z])_+\right)^p\right]\right)^{\frac{1}{p}}, \ \kappa \in [0, 1], \ p \geq 1.$$

2. Average Value at Risk (Rockafellar, Uryasev [64])

$$\rho(Z) := AV@R^+_\alpha(Z) := \frac{1}{\alpha}\int_0^\alpha F_Z^{-1}(1-\beta)d\beta = \min_\eta \in \mathbb{R}\left\{\eta + \frac{1}{\alpha}\mathbb{E}[(Z - \eta)_+]\right\}, \ \alpha \in (0, 1]$$

3. Entropic risk measure (Föllmer and Schied[28])

$$\rho(Z) := \frac{1}{\gamma}\ln\left(\mathbb{E}[e^{\gamma Z}]\right), \ \gamma > 0.$$

The appealing fact about risk measure is that they contain implicit protection against modeling under uncertainty. Mathematically, all convex risk measure have an alternative representation – *dual representation*, for cost depending on decision $x \in \mathcal{X}$,

$$\rho(Z_x) = \max_{\mu \in \mathcal{A}} \left(\mathbb{E}^\mu[Z_x] - \alpha^{min}(\mathbb{Q}^\mu)\right) \tag{1.7}$$

where the penalty function $\alpha^{min}$ maps probability measure into real value. Notice $\mathbb{Q}^\mu$ is a probability measure parameterized by $\mu$ that is absolutely continuous with respect to the original measure $\mathbb{P}$. The technical details can be found in, for example, [28], which mainly uses *separation theory* from functional analysis. In the coherent case, Ruszczyński and Shapiro in [70] give a more intuitive and concise proof. They used well-developed theory of convex analysis to obtain:

$$\rho(Z_x) = \max_{\mu \in \mathcal{A}} \mathbb{E}^{\mu}[Z_x]. \tag{1.8}$$

Clearly, as observed in (1.8), the coherent risk measure is the worst expected value over a certain set of probability measure.

Now, considering a risk averse optimization problem as before, but using $\rho(\cdot)$,

$$\min_{x \in \mathcal{X}} \rho(Z_x) = \min_{x \in \mathcal{X}} \max_{\mu \in \mathcal{A}} \mathbb{E}^{\mu}[Z_x]$$

By using dual representation, we found it can be interpreted as a game, i.e., min-max game, where one choose an action from an admissible set, and the opponent, or nature, selects a distribution $\mu \in \mathcal{A}$. Using game theory, it can be shown such optimization problems have an *equilibrium* under milde conditions,

$$\min_{x \in \mathcal{X}} \rho(Z_x) = \min_{x \in \mathcal{X}} \max_{\mu \in \mathcal{A}} \mathbb{E}^{\mu}[Z_x] = \max_{\mu \in \mathcal{A}} \min_{x \in \mathcal{X}} \mathbb{E}[Z_x]. \tag{1.9}$$

The introduction of risk measure opens a new field of stochastic programming, bringing both opportunities and challenges. The main difference is the nonlinearity of the objective function, which complicates the analysis of optimality conditions as well as developing corresponding efficient algorithms.

### 1.1.3 Risk-averse Optimization in Dynamic Setting – Discrete Time

In classical stochastic programming literature, it is very important to extend static models to a multi-stage setting. Namely, when data $\{Z_i\}_{i \in \{1,\ldots,T\}}$, modeled as a discrete-time continuous state space Markov chain (discrete time stochastic process), are revealed gradually over time, the decision sequence, $\{x_i\}_{i \in \{1,\ldots,T\}}$ should be adapted to the process. That is to say, the value of the decision process, chosen at stage $t$, only depends on the information generated by $Z_i$ up to time $t$, bu not the results of the future observations. This is requirement is also called *non-anticipativity*. The diagram below is a crystal clear illustration:

$$\textit{decision}(x_1) \sim \textit{observation}(Z_2) \sim \textit{decision}(x_2) \sim \cdots \textit{observation}(Z_T) \sim \textit{decision}(x_T).$$

Since the performance measure is conditional expectation (expectation is a special case of conditioning on the initial filtration), the *tower property* gives the nested structure. For such a mechanism, one can solve the problem by backward induction, or a dynamic programming equation. The former is more appropriate when the number of stages is relatively small but additional constraints can be imposed, the latter are powerful for large scale problem without additional constraints.

Adopting risk measure in multi-stage problem requires two fundamental elements. To motivate further discussion, let us suppose $(Z_1, ..., Z_T) \in \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_T$[1] is a random cost sequence from current time 1 to $T$, the objective will be to evaluate the risk associated with this sequence, i.e., discrete-time stochastic process. However, it is not sufficient to only have risk evaluation at the present time $t$ for the tail process $Z_t, ..., Z_T$; we should be able to evaluate the risk from tomorrow for the remaining sequence. In other words, in a dynamic setting, a collection of risk evaluators are required. As studied in [19], we call such evaluator – *conditional risk measures* and thus the family, $\{\rho_{t,s}(\cdot)\}_{t,s\in\{1,...,T\}}$ for $t \leq s$, *dynamic risk measure*. The evaluator $\rho_{t,s}(\cdot)$ is a mapping from $\mathcal{Z}_t \times \cdots \times \mathcal{Z}_s$ to $\mathcal{Z}_t$, that is, the risk evaluation at future time $t$ for tail cost $Z_t$ through $Z_s$ is uncertain now but becomes known at $t$, i.e.,

$$\rho_{1,T}(Z_1, Z_2, ..., Z_T) \in \mathcal{Z}_1 = \mathbb{R}$$

$$\rho_{2,T}(Z_2, Z_3, ..., Z_T) \in \mathcal{Z}_2$$

$$\rho_{3,T}(Z_3, Z_4, ..., Z_T) \in \mathcal{Z}_3$$

$$\cdots\cdots$$

Now, the question arises whether we can have arbitrary collection of risk evaluators $\rho_{t,s}(\cdot)$. The answer is no: the risk evaluators should have the properties of *time consistency*. If we have less risky tail process in the future, call it time $t$, and between now and future time $t$, nothing happens, then it should also look safe today.

**Definition 1.1.2.** *(Ruszczyński [70]) A dynamic risk measure $\{\rho_{t,T}\}_{t\in\{1,...,T\}}$ is* time consistent *if for all $\tau < \theta$, $Z_k = W_k$, $k = \tau, ..., \theta - 1$ and $\rho_{\theta,T}(Z_\theta, ..., Z_T) \leq \rho_{\theta,T}(W_\theta, ..., W_T)$ imply*

$$\rho_{\tau,T}(Z_\tau, ..., Z_T) \leq \rho_{\tau,T}(W_\tau, ..., W_T).$$

---

[1]Here, $\mathcal{Z}_t$, for $t \in \{1, ..., T\}$, are some vector space of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, e.g., $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

The big advantage of having the above definition is summarized in the following theorem:

**Theorem 1.1.3.** *(Ruszczyński [70]) Suppose a dynamic risk measure* $\{\rho_{t,T}\}_{t\in\{1,...,T\}}$ *is time-consistent, in addition,*

$$\rho_{t,T}(Z_t, ..., Z_T) = Z_t + \rho_{t,T}(0, Z_{t+1}, ..., Z_T)$$
$$\rho_{t,T}(0, ..., 0) = 0,$$

*then, for all t we have*

$$\rho_{t,T}(Z_t, ..., Z_T) = Z_t + \rho_t\Big(Z_{t+1} + \rho_{t+1}\Big(Z_{t+2} + \cdots + \rho_{T-1}(Z_T)\Big)\Big) \tag{1.10}$$

*where,*

$$\rho_t(Z_{t+1}) = \rho_{t,T}(0, Z_{t+1}, 0, ..., 0).$$

Very similar to the case of conditional expectation, the nested decomposition above connects all conditional risk evaluations; in addition to that, it gives an algorithm to evaluate the risk in a dynamic setting, namely, *backward induction*. Only one step risk evaluation is needed, for example, we can define one-step *mean-semideviation risk measure* as:

$$\rho_t(Z) := \mathbb{E}[Z\,|\,\mathcal{F}_t] + \kappa\Big(\mathbb{E}\Big[\Big((Z - \mathbb{E}[Z\,|\,\mathcal{F}_t])_+\Big)^p \,|\,\mathcal{F}_t\Big]\Big)^{\frac{1}{p}},\ \ \kappa \in [0, 1],\ \ p \geq 1.$$

for $Z \in \mathcal{Z}_{t+1}$.

**Remark 1.1.4.** *A more recent study in refining definition of time consistency can be found in [23] where the comparison or riskiness is based on stochastic dominance. In that work, more generalized partial observable system is discussed when risk measure is the performance measure.*

Optimizing with dynamic risk measure can be accomplished with some effort. In particular, if structure is *Markovian* and the cost sequence is a function of the control sequence $\{x_i\}_{i\in\{1,...,T\}}$, then one can develop a corresponding dynamic programming equation. For a *Markov control* $u \in U(x)$ with $U$ being a *muiltifunction*,

$$\begin{cases} v_{T+1}(x) = c_{T+1}(x), & x \in \mathcal{X}, \\ v_t(x) = \min_{u\in U(x)}\ \{c_t(x, u) + \sigma_t(x, Q_t(x, u), v_{t+1})\}, & x \in \mathcal{X},\ \ t = T, ..., 1. \end{cases} \tag{1.11}$$

Here, $c$ is the cost function, the final stage depends only on the state while the running cost depends on both action and state, $\sigma_t(\cdot,\cdot,\cdot)$ is called *Markov risk mapping*, which plays the same role as conditional expectation in the classical control context. It takes current state, the *Markov transition kernel* $Q$ and value function of next time instance, evaluating the minimum risk of future cost. In [67], both *value iteration* and *policy iteration* are discussed, and the convergence of the methods are also guaranteed. If an additional constraint is imposed, backward induction can be used based on (1.10).

### 1.1.4 Risk-averse Optimization in Dynamic Setting – Continuous Time

In continuous-time setting, the stochastic optimization problem is known as *stochastic control*, which is concerned with the following problem:

$$V(0,x) = \inf_{u\in\mathcal{U}} \mathbb{E}\Big[ \int_0^T c(s, X_s, u_s)ds + \Phi(X_T) \Big], \tag{1.12}$$

where the process $X$ follows some continuous-time dynamics with initial state $X_0 = x$. Similar to the discrete-time case, the cost rate functional $c$ depends on time, state and control in general, while the final stage cost only depends on the final state. The set $\mathcal{U}$ is the set of admissible controls,which will be detailed in later chapters. As we can observe, the continuous-time stochastic control problem aims to find an optimal control process as well as the optimal value under such control. This is rather a challenging problem, because we shall not in general assume the optimal control of Markovian type. In fact, it turns out that in most cases, the optimal control is not a function of corresponding state. In addition, the dynamics of the system is also difficult to analyze. Let's give a brief discussion of the underlying dynamics of the optimal control problem above – *the stochastic differential equation (SDE)*.

One of the earliest works related to Brownian motion is by Bachelier [4] in his thesis, later on, Stratonovich and Itô made significant contributions the area and built the foundation (e.g., see [33, 34, 35]). The literature studying SDE is extensiv; among which one can refer to monographs Økesendal[54], Protter[61], Karatzas and Shreve [38], Rogers and Williams [66], Jacod and Shiryaev [36]. The stochastic differential equation can be viewed as a generalization of *ordinary differential equation(ODE)*, involving Itô integration. On complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ with $\mathbb{F}$ being the filtration generated by $d$-dimensional *Brownian motion*

$\{W_t\}_{t\geq 0}$, consider the SDE of the following form:

$$
(1.13)\qquad \begin{cases} dX_t = b(t,X,\omega)dt + \sigma(t,X,\omega)dW_t, \qquad t\in[0,T],\\ X_0 = \eta(\omega). \end{cases}
$$

Observe that the evolution of the state $X$ depends on the path generated and the randomness explicitly. This is a very general SDE with random coefficients, as *drift* functional $b$ and *diffusion* functional $\sigma$ are random themselves.

**Definition 1.1.3.** *Let the maps $b:[0,\infty)\times C([0,T]\times\mathbb{R}^n)\times\Omega\mapsto\mathbb{R}^n$ and $\sigma:[0,\infty)\times C([0,T]\times\mathbb{R}^n)\times\Omega\mapsto\mathbb{R}^{n\times m}$ be given on the probability space defined above. Let $\eta$ be a $\mathcal{F}_0$-measurable. An $\mathcal{F}_t$-adapted continuous process $\{X_t\}_{t\geq 0}$ is called a* solution *of* (1.13) *if*

$$
(1.14)\qquad X_0 = \eta,\ \ a.s.,
$$

$$
(1.15)\qquad \int_0^t \Big\{\big|b(s,X,\omega)\big| + \big|\sigma(s,X,\omega)\big|^2\Big\}ds < +\infty,\ \ t\geq 0,\ \ a.s.,
$$

$$
(1.16)\qquad X_t = \eta + \int_0^t b(s,X,\omega)ds + \int_0^t \sigma(s,X,\omega)dW_s,\ \ t\geq 0\ \ a.s..
$$

*If $\mathbb{P}(X_t = Y_t,\ 0\leq t<+\infty) = 1$, holds for any two solutions $X$ and $Y$ of* (1.13)*, we say that the solution is unique.*

**Assumption 1.1.5.** *For any $\omega\in\Omega$, the functions $b(\cdot,\cdot,\omega)$ and $\sigma(\cdot,\cdot,\omega)$ are progressively measurable w.r.t. the natural filtrtion on $C([0,T]\times\mathbb{R}^n)$ and for any $x\in C([0,T]\times\mathbb{R}^n)$, $b(\cdot,x,\cdot)$ and $\sigma(\cdot,x,\cdot)$ are both $\mathcal{F}_t$-adapted processes. Moreover, there exists an $L>0$ such that for all $t\in[0,\infty)$, $x.,y.\in C([0,T]\times\mathbb{R}^n)$, and $\omega\in\Omega$,*

$$
\begin{cases} |b(t,x.,\omega) - b(t,y.,\omega)| \leq L|x. - y.|_{C([0,T]\times\mathbb{R}^n)},\\ |\sigma(t,x.,\omega) - \sigma(t,y.,\omega)| \leq L|x. - y.|_{C([0,T]\times\mathbb{R}^n)},\\ |b(\cdot,0,\cdot)| + |\sigma(\cdot,0,\cdot)| \in \mathcal{M}^2[0,T],\ \ \forall T\geq 0. \end{cases}
$$

**Theorem 1.1.6.** *Let assumption 1.1.5 hold, then for any $\eta\in L^2(\Omega,\mathcal{F}_0,\mathbb{P})$,* (1.13) *admits a unique solution $X$ such that for any $T>0$,*

$$
\mathbb{E}[\max_{0\leq s\leq T}\big|X_s\big|^2] \leq K_T(1+\mathbb{E}[\big|\eta\big|^2])
$$

*and*

$$\mathbb{E}[|X_t - X_s|^2] \leq K_T(1 + \mathbb{E}[|\eta|^2]|t - s|^{\frac{l}{2}}, \ \forall s, t, \in [0, T].$$

*Moreover, if* $\hat{\eta} \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$ *is another random variable and* $\hat{X}$ *is the corresponding solution of* (1.13)*, then for any* $T > 0$*, there exists* $K_T > 0$ *such that*

$$\mathbb{E}[\max_{0 \leq s \leq T} |X_s - \hat{X}_s|] \leq K_T \mathbb{E}[|\eta - \hat{\eta}|]. \tag{1.17}$$

After establishing the well-posedness of the general SDE (1.13), let us restrict our attention to a specific parameterization. Since the controller should have impact on the dynamics of $X$, both drift and diffusion terms should depend on control. Meanwhile, to preserve the Markovian property, we assume that $b$ and $\sigma$ are only functions of the current state. As a consequence, a *controlled diffusion process* takes the forms,

$$\begin{cases} dX_t = b(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dW_t, \ t \in [0, T], \\ X_0 = \eta(\omega). \end{cases} \tag{1.18}$$

The randomness of the coefficients is absorbed into the control process so that the system is Markovian with respect to $(X, u)$.

The original control problem (1.21) can be treated in different ways, one can, as in the discrete-time case, derive dynamic programming equation,

$$V(t, x) = \inf_{u \in \mathcal{U}} \mathbb{E}\Big[\int_t^r c(s, X_s, u_s)ds + V(r, X_r)\Big], \ 0 \leq t \leq r \leq T, \tag{1.19}$$

and use Markov chain approximation [42] to solve the problem, or, obtain a *Hamilton-Jacobi-Bellman* equation,

$$\partial_t V(t, x) + \inf_{u \in U} \Big\{ c(t, x, u) + b(t, x, u)\partial_x V(t, x) + \frac{1}{2} tr(\sigma(t, x, u))\partial_{xx} V(t, x) \Big\} = 0 \tag{1.20}$$

$U$ is where the admissible control is valued. The analytical solution to such highly nonlinear partial differential equation is out of reach; numerical methods, such as finite difference, finite element, can be adopted to solve the problem numerically. Another approach is to work with the *stochastic maximum principle*. By defining the *Hamiltonian*, one can derive necessary conditions for optimal control $u^*$. Iterative methods can be designed to construct a piecewise constant control that is close to the original optimal control (see Bonnans [9]). For continuous time, the classical references are [77, 26, 1, 44, 5, 8]. A more generalized optimal control problem uses utility function as performance measure, references concerned with such formulations are [14, 24, 25].

As expectation is criticized for ignoring the shape of the distribution, we want to use nonlinear evaluation instead, that is,

$$\widetilde{V}(0,x) = \inf_{u \in \mathcal{U}} \rho\Big[ \int_0^T c(s, X_s, u_s) ds + \Phi(X_T) \Big], \tag{1.21}$$

for some continuous-time dynamic risk measure $\rho$, where $X$ satisfies (1.18). The operator $\rho$ has to evaluate the risk in a continuous-time dynamic setting. Before discussing the specific form of evaluation, we should, again, have *time consistency* enforced. Following Cheridito et al [12, 13], Barrieu, El Karoui [6], a dynamic risk measure $\{\rho_{s,T}\}_{0 \le s \le T}$ should also satisfy:

**Definition 1.1.4.** *(Time Consistency) For $Z, V \in L^p(\Omega, \mathcal{F}_T, \mathbb{P})$ and $s \le t \le T$, $\rho_{t,T}(Z) \le \rho_{t,T}(V)$ a.s. implies $\rho_{s,T}(Z) \le \rho_{s,T}(V)$ a.s..*

The intuition of time consistency is when the terminal random variable $Z$ is riskier than $V$ at time $t \ge s$ in the future, then at time $s$, $Z$ should still be considered riskier than $V$.

As for the specific computation regarding operator $\rho$, the continuous-time dynamic risk measure is representable by the backward stochastic differential equation (BSDE) under mild assumptions (see detailed discussion in the seminal paper by Coquet et al [15].) The history of backward stochastic differential equation (BSDE) goes back to 1980, when Bismut [8] studied the stochastic maximum principle. Peng and Pardoux [57, 58] extend linear BSDE to nonlinear BSDE, following which BSDE is extensively studied and becomes one of the most popular topics in applied probability, especially, for applications in mathematical finance. It is by nature a nonlinear evaluation of future outcomes. In the risk measure context, let us suppose $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, which stands for future loss at time $T$, the 1-dimensional BSDE takes the form,

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad s \in [t, T].$$

It can be shown that finding risk valuation of $\xi$ is equivalent to solving the above BSDE for $Y$, i.e., $\rho_{t,T}(\xi) := Y_t$. Furthermore, time consistency is automatic in this case,

$$\rho_{t,T}(\,\cdot\,) = \rho_{t,s}(\rho_{s,T}(\,\cdot\,)), \quad 0 \le s \le t \le T,$$

as discussed in Barrieu, El Karoui [6, 7], Peng [59] and Riedel citeRF. It is worth mentioning, by the convergence results from Briand et al[11], Stadje's work[72] presents an approach for the transition from convex risk measure in a discrete-time setting to their counterparts in continuous time. This allows obtaining continuous-time analogues of a collection of one-step convex risk measures.

In this thesis, we study the continuous-time optimal control problem with risk-aversion. As pointed out earlier, it requires an analysis of a backward stochastic differential equation that facilitates risk evaluation at any time. Such control problems are closely related to *forward-backward systems of stochastic differential equations (FBSDE)* (see, [75]). For controlled fully coupled FBDEs, Li and Wei[48] obtained the dynamic programming equation, and derived the corresponding *Hamilton-Jacobi-Bellman* equation. Maximum principle for forward-backward systems and corresponding games was derived in [55, 56], including models with Lévy processes. We presents a simplified yet significant model formulation, in addition, special approximation algorithm are developed.

## 1.2 Organization of Thesis

In chapter 2, we review the concept of nonlinear evaluations proposed by Peng [59], also his $\mathbb{F}_t$-consistent evaluation in a dynamic setting. Then, the connection to backward stochastic differential equation (BSDE) is established. We discuss general theory of BSDE and its properties. In particular, the features of the BSDE structure when interpreted as a dynamic risk measure will be emphasized and explored. In addition, the generalization of dual representation will be explained as a counterparty of dual representation in the static case (or the conditional case).

Chapter 3 plays the most significant role in terms of modeling. It starts by introducing decoupled forward - backward stochastic differential equation as the foundation of risk-averse modeling in a continuous-time setting. The forward stochastic differential equation (SDE) is controlled by functions satisfying admissibility, and backward stochastic differential equation (BSDE) measures the risk of accumulated cost and final cost. Both weak formulation and strong formulation will be discussed, as in [77]. The former requires switching from different probability space but being intuitive, the latter is more technical in terms of analysis. After the formulation step, risk-averse dynamic programming can be derived, which leads to a risk-averse Hamilton-Jacobi-Bellman equation.

In chapter 4, we propose a theoretical algorithm based on discretization of the risk-averse dynamic programming equation. The classical method based on the discussion in [41] works under expectation, however, it generates uncontrollable errors when non-linear evaluation is considered. We therefore use Borel ball technique to show the existence of a Markovian piece-wise constant $\epsilon$-optimal control, namely, we show the arbitrarily closeness of such policy to the original optimal control. Unfortunately, such constructions cannot lead to a convergence rate. This is an open questions for interested researchers and scholars, a possible solution is to use maximum principle for control of FBSDE system to design a penalty method. Such an algorithm is investigated by F. Bonnans in expectation, which should be possible to generalize.

The purpose of the final chapter – chapter 5 is to devise a numerical scheme for risk evaluation in a short-time interval. This is important if one wants to utilize the theoretical numerical schemes proposed in chapter 4. At each discretized interval, an efficient and ”accurate” risk evaluation should be available. This amounts to solving a FBSDE system on such interval. The key idea here is to convert risk evaluation into a optimal control problem, where the Radon Nikodym derivative process is the control. By exploring the stochastic maximum principle and taking advantage of regularity of a semi-linear PDE, we reduce a functional optimization to vector optimization with acceptable error. A financial application – risk management – is provided at the end, to illustrate the risk exposure of holding derivatives in the future. The numerical example and simulation are based on a tree structure and nested simulation.

# Chapter 2

# Dynamic Risk Measures in Continuous Time

## 2.1 $\mathbb{F}$-consistent Nonlinear Expectation

We establish a suitable framework and briefly review the concept of $\mathbb{F}$-consistent nonlinear expectations (for an extensive treatment, see [59]). Fix a finite horizon $[0,T]$ for $0 < T < \infty$, let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a probability space, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a filtration. A vector-valued stochastic process $\{X_t\}_{0\le t\le T}$ is said to be adapted to $\mathbb{F}$ if $X_t$ is an $\mathcal{F}_t$-measurable random variable for any $t \in [0,T]$.

Let us start by introducing the concept of a nonlinear expectation.

**Definition 2.1.1.** *For* $0 \le T < \infty$, *a* nonlinear expectation *is a functional* $\rho_{0,T}\colon L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \to \mathbb{R}$ *satisfying the strict monotonicity property:*

$$\text{if } \xi_1 \ge \xi_2 \text{ a.s., then } \rho_{0,T}[\,\xi_1\,] \ge \rho_{0,T}[\,\xi_2\,];$$

$$\text{if } \xi_1 \ge \xi_2 \text{ a.s., then } \rho_{0,T}[\,\xi_1\,] = \rho_{0,T}[\,\xi_2\,] \text{ if and only if } \xi_1 = \xi_2 \text{ a.s.};$$

*and the constant preservation property:*

$$\rho_{0,T}[\,c\mathbb{1}.\,] = c, \quad \forall\, c \in \mathbb{R}.$$

*where* $\mathbb{1}.$ *is the indicator function of an event.*

Based on that, the $\mathbb{F}$-consistent nonlinear expectation is defined as follows.

**Definition 2.1.2.** *For a filtered probability space* $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, *a nonlinear expectation* $\rho_{0,T}[\,\cdot\,]$ *is* $\mathbb{F}$-consistent *if for every* $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ *and every* $t \in [0,T]$ *a random variable* $\eta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ *exists such that*

$$\rho_{0,T}[\,\xi\mathbb{1}_A\,] = \rho_{0,T}[\,\eta\mathbb{1}_A\,] \quad \forall A \in \mathcal{F}_t.$$

The variable $\eta$ in Definition 2.1.2 is uniquely defined, we denote it by $\rho_{t,T}[\,\xi\,]$. Observe that such identification also defines conditional expectation. Therefore, we interpret $\rho_{t,T}[\,\xi\,]$ as a nonlinear conditional expectation of $\xi$ at time $t$. The nonlinear expectation defined above preserves essential properties of conditional expectation.

**Proposition 2.1.1.** *Let $\rho[\,\cdot\,]$ be as in definition 2.1.1, for each $t \in [0,T]$ and $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, there exists a $\rho_{t,T}[\,\xi\,] \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ satisfying definition 2.1.2, then $\{\rho_{t,T}[\,\xi\,]\}_{0\le t\le T}$ satisfies the following axioms:*

(i) ***Monotonicity:*** *$\rho_{t,T}[\,\xi\,] \ge \rho_{t,T}[\,\xi'\,]$ a.s., if $\xi \ge \xi'$ a.s.;*

(ii) ***Constant-preserving:*** *$\rho_{t,T}[\,\xi\,] = \xi$, if $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})$;*

(iii) ***Local Property:*** *for each $t$, $\rho_{t,T}[\,\mathbb{1}_A\xi\,] = \mathbb{1}_A\rho_{t,T}[\,\xi\,]$, where $A \in \mathcal{F}_t$;*

(iv) ***Time consistency:*** *$\rho_{s,t}[\,\rho_{t,T}[\,\xi\,]\,] = \rho_{s,T}[\,\xi\,]$, if $s \le t \le T$.*

It follows that $\mathbb{F}$-consistent nonlinear expectations are special cases of dynamic time-consistent measures of risk, enjoying a number of useful properties. They do not, however, have the properties of convexity, translation invariance, or positive homogeneity, unless additional assumptions are made. We shall return to this in later sections.

## 2.2 Connection of Dynamic Risk Measures to Backward Stochastic Differential Equations

The first introduction of BSDE was by Bismut [8], who analyzed optimality conditions of stochastic control problems; in that case, the BSDE is linear in the sense that the driver $g$ is linear in $y$ and $z$. The main development is due to Pardoux and Peng in their seminal papers [57, 58], where they generalized the maximum principle; a nonlinear BSDE is postulated, with a nonlinear driver, under certain regularity conditions. In the last decades, nonlinear BSDE became a popular in various areas of mathematical finance [37, 22, 16, 17, 21], such as derivative pricing, hedging strategies, recursive utility and dynamic risk measures.

These wide applications result from the nature of BSDE being a nonlinear evaluation of future randomness generated by Brownian motion, which connects nonlinear expectation discussed in the previous section. This motivates the discussion of BSDE below, let us recall that the stochastic differential equation (SDE) is a nonlinear extension of a stochastic integral, because of the presence of a drift term. In the meanwhile, the *martingale representation (in Brownian case)* states the following, assuming $Y_t$ is a martingale, then

$$dY_t = Z_t dW_t, \quad \textit{or equivalently,} \quad Y_t = \xi - \int_t^T Z_s dW_s, \tag{2.1}$$

with $\xi$ being the random outcome at time $T$. If we take conditional expectation on both sides, we obtain

$$Y_t = \mathbb{E}_t[\,\xi\,]. \tag{2.2}$$

This implies that if we find a solution $(Y, Z)$ to (2.1), the $Y$ part gives a linear evaluation – *conditional expectation*. Notice that the $Z$ process in (2.1) must be an adapted process; it makes the evaluation $Y_t$ non-trivial, although, as far as evaluation is concerned, we're most interested in the $Y$ part. In the same spirit as generalization of Itô integral, adding a "drift term" to martingale representation leads to a nonlinear evaluation,

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \tag{2.3}$$

a *Backward Stochastic Differential Equation (BSDE)*. The "drift term" appearing above is usually called the *driver* of BSDE, it has a special structure, depending on the solution $(Y, Z)$.

In the following, we formally introduce BSDE and its theory, since it is heavily used in our formulation and numerical approximation. After the detour, we finally build the formal connection between BSDE and nonlinear expectation in a continuous time setting.

### 2.2.1 Initial Set-up

We equip $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ with a $d$-dimensional Brownian filtration, i.e., $\mathcal{F}_t = \sigma\{(W_s; 0 \le s \le T) \cup \mathcal{N}\}$, where $\mathcal{N}$ is the collection of $P$-null sets in $\Omega$. In this paper we consider the following 1-dimensional BSDE:

$$-dY_t = g(t, Y_t, Z_t)\, dt - Z_t\, dW_t, \quad Y_T = \xi, \tag{2.4}$$

where the data is the pair $(\xi, g)$, called the *terminal condition* and the *generator* (or *driver*), respectively. Here, $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, and $g : [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is a measurable function (with respect to the product $\sigma$-algebra), which is *nonanticipative*, that is, $g(t, Y_t, Z_t)$ is $\mathcal{F}_t$-progressively measurable for all $t \in [0, T]$.

The solution of the BSDE is a pair of processes $(Y,Z) \in \mathcal{S}^2[0,T] \times \mathcal{H}^{2,d}[0,T]$ such that

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s, \quad t \in [t,T]. \tag{2.5}$$

Two sets of assumptions are available, under which the existence and uniqueness of the solution of (2.4) can be guaranteed.

**Assumption 2.2.1** (Peng and Pardoux[57]). (i) *$g$ is jointly Lipschitz in $(y,z)$, i.e., a constant $K > 0$ exists such that for all $t \in [0,T]$, all $y_1, y_2 \in \mathbb{R}$ and all $z_1, z_2 \in \mathbb{R}^d$ we have*

$$|g(t,y_1,z_1) - g(t,y_2,z_2)| \le K(|y_1 - y_2| + |z_1 - z_2|) \quad a.s.;$$

(ii) *the process $g(\cdot, 0, 0) \in \mathcal{H}^2[0,T]$.*

**Assumption 2.2.2** (Kobylanski [40]). (i) *for every $C > 0$, there exists a $K$ such that for all $y \in [-C, C]$*

$$\left|\frac{\partial g(t,y,z)}{\partial y}\right| \le K(1 + |z|^2) \quad a.s.,$$

*and*

$$\left|\frac{\partial g(t,y,z)}{\partial z}\right| \le K(1 + |z|) \quad a.s..$$

(ii) *For some $K > 0$, $|g(t,y,z)| \le K(1 + |y| + |z|^2)$ a.s;*

(iii) $\xi \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$.

Notice, under assumption (2.2.2), the solution $Y$ is not in $\mathcal{S}^{2,d}[0,T]$, but we only have

$$\mathbb{E}\Big[\sup_{0 \le t \le T} \big|Y_t\big|\Big] < \infty. \tag{2.6}$$

In this paper, we mainly focus our attention on assumption 2.2.1.

**Theorem 2.2.3.** *Let Assumption 2.2.1 be satisfied. Then for any given $\xi \in \mathcal{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, the BSDE (2.4) admits a unique adapted solution $(Y_\cdot, Z_\cdot) \in \mathcal{M}[0,T]$.*

The proof can be found in numerous references. A special equivalent norm[1] is introduced under which we have a Banach space setting, namely, for any $\beta \in \mathbb{R}$, define $\mathcal{M}_\beta[0,T]$ to be the Banach space,

$$\mathcal{M}_\beta[0,T] := \mathcal{H}^2[0,T] \times \mathcal{H}^2[0,T] \tag{2.7}$$

[1] All the norms $\|\cdot\|_{\mathcal{M}_\beta[0,T]}$ with different $\beta$ are equivalent.

equipped with the norm

$$\|(Y_\cdot, Z_\cdot)\|_{\mathcal{M}_\beta[0,T]} := \Big\{\mathbb{E}\Big[\sup_{t\in[0,T]} |Y_t|^2 e^{2\beta t}\Big] + \mathbb{E}\Big[\int_0^T |Z_s|^2 e^{2\beta t} dt\Big]\Big\}^{\frac{1}{2}} \tag{2.8}$$

The advantage of this equivalent norm is to obtain a contraction mapping $\mathcal{T} : \mathcal{M}_\beta[0,T] \mapsto \mathcal{M}_\beta[0,T]$ by $(y,z) \mapsto (Y,Z)$ via the following BSDE:

$$dY_t = g(t, y_t, z_t)dt + Z_t dW_t, \quad Y_T = \xi. \tag{2.9}$$

Specifically, the following inequality holds:

$$\|\mathcal{T}(y,z) - \mathcal{T}(\bar{y},\bar{z})\|_{\mathcal{M}_\beta[0,T]} \leq \frac{1}{2}\|(y,z) - (\bar{y} - \bar{z})\|_{\mathcal{M}_\beta[0,T]}, \quad \forall (y,z), (\bar{y},\bar{z}) \in \mathcal{M}_\beta[0,T]. \tag{2.10}$$

Then, we proceed similar to the proof of existence and uniqueness of SDE solution; we can claim that $\mathcal{T}$ has a unique fixed point, which is an adapted solution to (2.4).

### 2.2.2 Some Useful Results on BSDE

We recall essential properties of BSDE that will be used later for risk-averse modeling. The first results follow from the uniqueness of BSDE (2.4). Fix $t_0 \in [0,T]$, denote

$$\mathcal{F}_t^{t_0} := \sigma\{ (W_s - W_{t_0}; t_0 \leq s \leq t) \cup \mathcal{N}\}. \tag{2.11}$$

**Proposition 2.2.4.** *We assume that $g$ satisfies* Assumption *2.2.1; moreover, for a fixed $t_0 \in [0,T]$ and for each $(y,z) \in \mathbb{R} \times \mathbb{R}^d$, the process $g(\cdot, y, z)$ is $\mathcal{F}_t^{t_0}$-progressively measurable on the interval $[t_0,T]$ and $\xi \in L^2(\Omega, \mathcal{F}_T^{t_0}, \mathbb{P})$. Then the solution $(Y,Z)$ of BSDE* (2.4) *is also $\mathcal{F}_t^{t_0}$ - adapted on $[t_0,T]$. In particular, $Y_{t_0}$ and $Z_{t_0}$ are deterministic.*

*Proof.* Let $(Y'_\cdot, Z'_\cdot)$ be the solution of $\mathcal{F}_t^{t_0}$ -adapted solution, on the interval $[t_0,T]$ of the BSDE:

$$Y'_t = \xi + \int_t^T g(s, Y'_s, Z'_s)ds - \int_t^T Z'_s dW_s^0$$

where we denote $W_t^0 := W_t - W_{t_0}$ (observe that $(W_s^0)_{t_0\leq s\leq T}$ is an $\mathcal{F}_t^{t_0}$ - Brownian motion on $[t_0,T]$). But note $\mathcal{F}_t^{t_0} \subseteq \mathcal{F}_t$, $(Y'_t, Z'_t)_{t_0\leq s\leq T}$ is also $\mathcal{F}_t$ - adapted and $\int_t^T Z'_s dW_s = \int_t^T Z'_s dW_s^0$ for $t \in [t_0,T]$. Thus from the uniqueness result of BSDE, the solution $(Y_\cdot, Z_\cdot)$ of BSDE (2.4) coincides with $(Y', Z')$ on $[t_0,T]$. Thus $(Y_\cdot, Z_\cdot)$ is $\mathcal{F}_t^{t_0}$ - adapted. □

The significance of the above results will be evident when deriving a dynamic programming equation under both strong formulation and weak formulation.

As the data $(\xi, g)$ uniquely identify each BSDE, we state the *comparison theorem* which gives us more insight into the mechanism of BSDE. Consider BSDE below,

$$Y_t^1 = \xi_1 + \int_t^T g^1(s, Y_s^1, Z_s^1)\, ds - \int_t^T Z_s^1\, dW_s, \tag{2.12}$$

$$Y_t^2 = \xi_2 + \int_t^T g^2(s, Y_s^2, Z_s^2)\, ds - \int_t^T Z_s^2\, dW_s. \tag{2.13}$$

We have following theorems

**Theorem 2.2.5.** *Let the assumptions of Theorem 2.2.8 hold, and let* $(Y^1, Z^1)$ *and* $(Y^2, Z^2)$ *be the unique solutions of* (2.12) *and* (2.13)*, respectively, with* $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$*. If* $\xi_1 \geq \xi_2$ *a.s. and*

$$g^1(s, Y_s^1, Z_s^1) \geq g^2(s, Y_s^1, Z_s^1), \quad \forall\, s \in [0, T],\ a.s.,$$

*then* $Y_t^1 \geq Y_t^2$*, a.s., for all* $t \in [0, T]$.

*Proof.* Without loss of generality, let's set $t = 0$ and define

$$\hat{g}_s = g^1(s, Y_s^1, Z_s^1) - g^2(s, Y_s^2, Z_s^2),\ \hat{Y} = Y^1 - Y^2,\ \hat{Z} = Z^1 - Z^2,\ \hat{\xi} = \xi^1 - \xi^2. \tag{2.14}$$

By construction, $(\hat{Y}, \hat{Z})$ satisfies the following linear BSDE:

$$-d\hat{Y}_s = (a_s\hat{Y}_s + b_s\hat{Z}_s + \hat{g}_s)ds - \hat{Z}_s dW_s,\ \hat{Y}_T = \hat{\xi}, \tag{2.15}$$

where

$$a_s := \begin{cases} \frac{g^1(s,Y_s^1,Z_s^1) - g^1(s,Y_s^2,Z_s^1)}{Y_s^1 - Y_s^2}, & \text{if } Y_s^1 \neq Y_s^2, \\ 0, & \text{if } Y_s^1 = Y_s^2, \end{cases}$$

$$b_s := \begin{cases} \frac{g^1(s,Y_s^2,Z_s^1) - g^1(s,Y_s^2,Z_s^2)}{Z_s^1 - Z_s^2}, & \text{if } Z_s^1 \neq Z_s^2, \\ 0, & \text{if } Z_s^1 = Z_s^2. \end{cases}$$

Since $g^1$ satisfies Lipschitz condition, $|a_s| \leq C$ and $|b_s| \leq C$. We set

$$\Lambda_t := \exp\Big\{ \int_0^T b_s dW_s - \frac{1}{2}\int_0^T |b_s|^2 ds + \int_0^T a_s ds \Big\}. \tag{2.16}$$

Applying Itô's formula to $\Lambda_t \hat{Y}_t$ on the interval $[0,T]$ and taking expectation gives:

$$\hat{Y}_0 = \mathbb{E}\Big[\hat{Y}_T \Lambda_T + \int_0^T \Lambda_t \hat{g}_t dt\Big] \geq 0. \tag{2.17}$$

As desired, it follows that $Y_0^1 \geq Y_0^2$. □

**Proposition 2.2.6.** *For $L > 0$, the following estimate holds:*

(2.18)

$$\|Y^1. - Y^2.,Z^1. - Z^2.\|^2_{\mathcal{M}^2[0,T]} \leq K\{\mathbb{E}[|\xi^1 - \xi^2|^2] + \mathbb{E}[\int_0^T |g^1(s,Y_s^2,Z_s^2) - g^2(s,Y_s^1,Z_s^1)|^2 ds]\}.$$

*Proof.* Set $\hat{Y}, \hat{Z}, \hat{\xi}$ and $\hat{g}$ as in proof of theorem 2.2.5 and apply Itô's formula to $|\hat{Y}.|^2$, we obtain

$$\begin{aligned}
&|\hat{Y}_t|^2 + \int_t^T |\hat{Z}_s|^2 ds \\
=&|\hat{\xi}|^2 - 2\int_t^T \hat{Y}_s(g^1(s,Y_s^1,Z_s^1) - g^2(s,Y_s^2,Z_s^2))ds - 2\int_t^T \hat{Y}_s\hat{Z}_s dW_s \\
\leq&|\hat{\xi}|^2 + 2\int_t^T |\hat{Y}_s||\hat{g}_s| + K|\hat{Y}_s|(|\hat{Y}_s| + |\hat{Z}_s|)ds - 2\int_t^T \hat{Y}_s\hat{Z}_s dW_s \\
\leq&|\hat{\xi}|^2 + 2\int_t^T [(1+2K+2K^2)|\hat{Y}_s|^2 + \frac{1}{2}|\hat{Z}_s|^2 + |\hat{g}_s|^2]ds - 2\int_t^T \hat{Y}_s\hat{Z}_s dW_s
\end{aligned} \tag{2.19}$$

Taking expectation on both sides, for any $t \in [0,T]$,

(2.20)

$$\mathbb{E}\Big[|\hat{Y}_t|^2\Big] + \frac{1}{2}\mathbb{E}\Big[\int_t^T |\hat{Z}_s|^2 ds\Big] \leq \mathbb{E}\Big[|\hat{\xi}|^2\Big] + \mathbb{E}\Big[\int_t^T |\hat{g}_s|^2 ds\Big] + (1+2K+2K^2)\int_t^T \mathbb{E}\Big[|\hat{Y}_s|^2\Big]ds.$$

By *Gronwall's inequality*,

$$\mathbb{E}\Big[|\hat{Y}_t|^2\Big] + \frac{1}{2}\mathbb{E}\Big[\int_t^T |\hat{Z}_s|^2 ds\Big] \leq L\Big\{\mathbb{E}\Big[|\hat{\xi}|^2\Big] + \mathbb{E}\Big[\int_t^T |\hat{g}_s|^2 ds\Big]\Big\}. \tag{2.21}$$

On the other hand, by *Burkholder-Davis-Gundy's(BDG) inequality*, it follows

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]} |\hat{Y}_t|^2\Big] \leq L\Big\{\mathbb{E}\Big[|\hat{\xi}|^2\Big] + \mathbb{E}\Big[\int_0^T |\hat{g}_s|^2 ds\Big]\Big\} + 2\mathbb{E}\Big[\sup_{t\in[0,T]} \Big|\int_0^t \hat{Y}_s\hat{Z}_s dW_s\Big|\Big] \\
\leq& L\Big\{\mathbb{E}\Big[|\hat{\xi}|^2\Big] + \mathbb{E}\Big[\int_0^T |\hat{g}_s|^2 ds\Big]\Big\} + L\Big(\mathbb{E}\Big[\sup_{t\in[0,T]} |\hat{Y}_t|^2\Big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}\Big[\int_0^T |\hat{Z}_s|^2 ds\Big]\Big)^{\frac{1}{2}}
\end{aligned} \tag{2.22}$$

The result follows immediately. □

### 2.2.3 $g$-Expectation by BSDE

For this subsection, we closely follow [59] and refer the readers to the proofs there. The earlier discussion on BSDE indicates the nonlinearity of the evaluation is due to the driver $g$. Under assumption 2.2.1, we have:

**Definition 2.2.1.** *For each* $0 \le t \le T$ *and* $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$*, the* $g$-Expectation *at time* $t$ *is the operator* $\rho^g_{t,T} : L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \to L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ *defined as follows:*

$$\rho^g_{t,T}[\,\xi\,] := Y_t, \tag{2.23}$$

*where* $(Y_\cdot, Z_\cdot) \in \mathcal{S}^{2,d}[t,T] \times \mathcal{H}^2[t,T]$ *is the unique solution of* (2.4).

The solution $(Y_\cdot, Z_\cdot)$ is a functional of the driver, that is, by specifying $g$, we obtain a certain nonlinear evaluation. Let's make the following assumption:

**Assumption 2.2.7.** *Assume the driver* $g$ *satisfies:*

(i) $g(\cdot, 0, 0) \equiv 0$*;*

(ii) *$g$ is independent of $z$.*

The following theorem by Coquet Hu, Mémin and Peng [15] reveals the relationship between $g$-expectation and $\mathbb{F}$-consistent nonlinear expectation.

**Theorem 2.2.8.** *Let the driver $g$ satisfy assumption 2.2.1 and the condition (i) of assumption 2.2.7. Then the system of $g$-Expectation* $(\rho^g_{t,T})_{0\le t\le T}$ *defined in* (2.23) *is a system of* $\mathbb{F}$*-consistent conditional nonlinear expectations. Furthermore, we have*

$$\lim_{s\uparrow t} \rho^g_{s,t}[\,\xi\,] = \xi, \quad \forall\, \xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}),\ t \in [0,T].$$

Surprisingly, Coquet, Hu, Mémin, and Peng proved in [15] that every $\mathbb{F}$-consistent nonlinear expectation which is dominated by $\rho^{\mu,\nu}_{0,T}$ (a $g$-evaluation with $g = \mu|y| + \nu|z|$ with some $\nu, \mu > 0$) is in fact a $g$-evaluation for some $g$. The domination is understood as follows: $\rho_{0,T}[Y + \eta] - \rho_{0,T}[Y] \le \rho^{\mu,\nu}_{0,T}[\eta]$, for all $Y, \eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.

The importance of (ii) in assumption 2.2.7 is economically significant, as the following proposition implies:

**Proposition 2.2.9.** *Let g satisfy assumption of theorem 2.2.8, then,*

$$\rho^g_{t,T}[\xi+\eta] = \rho^g_{t,T}[\xi] + \eta, \tag{2.24}$$

*for* $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ *and* $\eta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})$*, if and only if g satisfies (ii) of assumption 2.2.7.*

**Remark 2.2.10.** *By imposing the independence assumption of g, g-Expectation preserves the* "take known out" *property, satisfied as well by the linear conditional expectation. It is also essential to the application of* Bellman's principle.

### 2.2.4 Representation of a Dynamic Risk Measure

From now on we shall use only $g$-evaluations as time-consistent dynamic measures of risk. To ensure desirable properties of the resulting measures of risk, we enforce additional conditions on the driver $g$.

**Assumption 2.2.11.** *The driver g satisfies assumption 2.2.7 and for almost all* $t \in [0,T]$ *the following conditions:*

(i) $g(t,\cdot)$ *is convex for all* $t \in [0,T]$*;*

(ii) $g(t,\cdot)$ *is positively homogeneous for all* $t \in [0,T]$.

Under these conditions, one can derive new properties of the evaluations $\rho^g_{t,T}[\cdot]$, $t \in [0,T]$, in addition to the general properties of $\mathbb{F}$-consistent nonlinear expectations stated in Proposition 2.1.1.

**Theorem 2.2.12.** *Suppose g satisfies Assumptions 2.2.1 and 2.2.7. Then the system of g-evaluations* $\rho^g_{t,r}$, $0 \le t \le r \le T$ *has the following properties:*

(i) ***Normalization****:* $\rho^g_{t,r}(0) = 0$*;*

(ii) ***Translation Property****: for all* $\xi \in L^2(\Omega, \mathcal{F}_r, \mathbb{P})$ *and* $\eta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})$*,*

$$\rho^g_{t,r}(\xi+\eta) = \rho^g_{t,r}(\xi) + \eta, \quad a.s.; \tag{2.25}$$

*If, additionally, condition (i) of of Assumption 2.2.11 is satisfied, then* $\rho^g_{t,r}$ *has the following property:*

(iii) ***Convexity****: for all* $\xi, \xi' \in L^2(\Omega, \mathcal{F}_r, \mathbb{P})$ *and all* $\lambda \in L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$ *such that* $0 \le \lambda \le 1$*,*

$$\rho^g_{t,r}(\lambda\xi + (1-\lambda)\xi') \le \lambda\rho^g_{t,r}(\xi) + (1-\lambda)\rho^g_{t,r}(\xi'), \quad a.s..$$

*Moreover, if g also satisfies condition (ii) of Assumption 2.2.11, then $\rho^g_{t,r}$ has also the following property:*

(iv) ***Positive Homogeneity**: for all $\xi \in L^2(\Omega, \mathcal{F}_r, \mathbb{P})$ and all $\beta \in L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$ such that $\beta \geq 0$, we have*

$$\rho^g_{t,r}(\beta\xi) = \beta\rho^g_{t,r}(\xi), \quad a.s..$$

It follows that under Assumptions 2.2.1 and 2.2.11, the $g$-evaluations $\rho^g_{t,r}$ are convex or coherent conditional measures of risk (depending on whether (ii) is assumed or not). Theorem 2.2.12 provides us a concrete object to work on in order to evaluate abstract $\mathbb{F}$-consistent nonlinear expectation, namely, a BSDE with a special driver.

*Proof.* The normalization property is trivial. The translation property is due to the equivalence between

(2.26)
$$Y_t = \xi + \int_t^T g(s, Z_s)ds - \int_t^T Z_s dW_s, \quad and \quad Y_t + \eta = \xi + \eta + \int_t^T g(s, Z_s)ds - \int_t^T Z_s dW_s.$$

A similar argument works for positive homogeneity, i.e.,

(2.27)
$$Y_t = \xi + \int_t^T g(s, Z_s)ds - \int_t^T Z_s dW_s, \quad and \quad \beta Y_t = \beta\xi + \int_t^T g(s, \beta Z_s)ds - \int_t^T \beta Z_s dW_s.$$

If the terminal value $\xi$ is multipled, then the solution of BSDE becomes $(\beta Y_t, \beta Z_t)$, where the $Y$-part stands for the risk evaluation. The convexity requires the analysis of the maximum solution of a BSDE. We call $(Y., Z.)$ a *supersolution* of BSDE (2.4) if, for all $s, t \in [0, T]$ with $s \leq t$, it holds:

$$Y_s \geq Y_t + \int_s^t g(r, Z_r)dr + \int_t^T Z_r dW_r \quad and \quad Y_T \geq \xi. \tag{2.28}$$

We also define

$$\Lambda(\xi, g) = \Big\{(Y, Z) \in S^2[0, T] \times H^{2,d}[0, T] : (2.28)\ holds.\Big\} \tag{2.29}$$

Then, by the dual representation introduced below[2],

$$\rho^g_{t,T}[\,\xi\,] = \text{ess inf}\{Y_t : (Y., Z.) \in \Lambda(\xi, g)\}. \tag{2.30}$$

[2]The supersolution can thought as the suprimum of a $\mu$-parameterized LBSDE.

Consider following two supersolutions, $(Y_\cdot, Z_\cdot,)$ and $(Y'_\cdot, Z'_\cdot)$, of the corresponding BSDEs, respectively,

$$\text{(2.31)}\qquad \begin{aligned} \lambda Y_t &\geq \lambda\xi + \int_t^T \lambda g(s, Z_s)ds - \int_t^T \lambda Z_S\, dW_s, \\ \lambda Y'_t &\geq (1-\lambda)\xi' + \int_t^T (1-\lambda) g(s, Z'_s)ds - \int_t^T (1-\lambda) Z'_s dW_s, \end{aligned}$$

By convexity of $g$, we obtain $\forall (Y_\cdot, Z_\cdot) \in \Lambda(\xi, g)$ and $(Y'_\cdot, Z'_\cdot) \in \Lambda(\xi', g)$,

$$\text{(2.32)}\qquad (\lambda Y_\cdot + (1-\lambda)Y'_\cdot, \lambda Z_\cdot + (1-\lambda)Z'_\cdot) \in \Lambda(\lambda\xi + (1-\lambda)\xi', g)$$

Thus,

$$\text{(2.33)}\qquad \lambda\Lambda(\xi, g) + (1-\lambda)\Lambda(\xi', g) \subset \Lambda(\lambda\xi + (1-\lambda)\xi', g).$$

In particular,

$$\text{(2.34)}\qquad \rho^g_{t,T}[\lambda\xi + (1-\lambda)\xi'] \leq \lambda\rho^g_{t,T}[\xi] + (1-\lambda)\rho^g_{t,T}[\xi'],$$

as desired. □

Recall that in the static(or, conditional) case, the convex risk measure admits a dual representation. Barrieu and El Karoui in [6, 7] discovered the continuous-time version dual representation as a generalization of *Girsanov's theorem*. The key part is again the driver $g$ of a BSDE. The *Legendre-Fenchel transformation* yields the conjugate function of $g$ as,

$$\text{(2.35)}\qquad G(t, \mu) := \sup_{z \in \mathbb{R}^d} \{\langle \mu, z\rangle - g(t, z)\}.$$

Since $g$ is continuous (actually Lipschitz continuous),

$$\text{(2.36)}\qquad g(t, z) = g^{**}(t, z) = \sup_{\mu \in \mathbb{R}^d} \Big\{\langle \mu, z\rangle - G(t, \mu)\Big\}.$$

In addition, for positively homogeneous $g$,

$$\text{(2.37)}\qquad g(t, z) = \sup_{\mu \in dom(G)} \langle \mu, z\rangle.$$

**Definition 2.2.2.** *Dynamic risk measure* $\rho^g_{t,T}[\cdot]$ *is said to have a* dual representation *if there exists a set* $\mathcal{A}$ *of admissible controls such that for t, T and* $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$*,*

$$\text{(2.38)}\qquad \rho^g_{t,T}[\xi] = \sup_{\mu \in \mathcal{A}} \mathbb{E}_t^{\mathbb{Q}^\mu}\Big[\xi_T - \int_t^T G(t, \mu_t)dt\Big]$$

*where* $\mathbb{Q}^\mu$ *is a probability measure absolutely continuous with respect to* $\mathbb{P}$*, with Radon Nikoydm derivative* $\mu$*.*

**Theorem 2.2.13.** *Given Assumptions 2.2.1 and 2.2.11, and with $G(\cdot,\cdot)$ being the polar process associated with $g(\cdot,\cdot)$, then we have*

(i) *For all $t \in [0,T]$, $g(t,z) = \sup_{\mu \in \mathrm{dom}(G)} \langle \mu, z\rangle - G(t,\mu)$ has an optimal progressively measurable solution $\mu^*_\cdot$ in the sub-differential of g at z, i.e., $\partial g(t,z)$;*

(ii) $\rho^g_{t,T}[\,\cdot\,]$ *has the following dual representation, for any $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$,*

$$\rho^g_{t,T}[\,\xi\,] = \sup_{\mu\in\mathcal{A}} \mathbb{E}^{\mathbb{Q}^\mu}_t\Big[\,\xi - \int_t^T g(s,\mu_s)ds\,\Big] = \mathbb{E}^{\mathbb{Q}^{\mu^*}}_t\Big[\,\xi - \int_t^T G(s,\mu^*)ds\Big] \tag{2.39}$$

*where $\mathcal{A}$ is the space of A-valued bounded adapted process with A being convex, closed and bounded, and*

$$d\Gamma^\mu_t = \mu_t \Gamma^\mu_t dW_t,\ \ \Gamma^\mu_0 = 1,\ \ t \in \mathcal{I}. \tag{2.40}$$

Theorem 2.2.13 implies that the solution of BSDE with convex driver can be interpreted as the maximum solution of a family of linear BSDEs (LBSDE). Barrieu and El Karoui [6] also worked out the case when $\mathcal{A}$ is the set of bounded mean oscillation martingales, but we will focus on the bounded density process only.

**Remark 2.2.14.** *It is now clear the supersolution discussed in theorem 2.2.12, the dynamic risk measure $\rho^g_{t,T}[\xi]$ is the maximum solution,*

$$\rho^g_{t,T}[\xi] = Y_t = \sup_{\mu\in\mathcal{A}} Y^\mu_t \tag{2.41}$$

*where*

$$Y^\mu_t = \xi + \int_t^T \sup_{\mu\in\mathcal{A}} \{\langle \mu_s, Z^\mu_s\rangle - G(s,\mu_s)\}ds - \int_t^T Z^\mu_s dW_s. \tag{2.42}$$

**Corollary 2.2.15.** *Given the same assumption of theorem 2.2.13, if $\rho_{t,T}[\cdot]$ is positive homogeneous dynamic risk measure, i.e., g is positive homogeneous, then*

(i) *For all $t \in [0,T]$, $g(t,z) = \sup_{\mu\in\mathrm{dom}(G)} \langle \mu, z\rangle$ has an optimal progressively measurable solution $\mu^*_\cdot$ in the sub-differential of g at* 0*, i.e., $\partial g(t,0)$;*

(ii) $\rho^g_{t,T}[\,\cdot\,]$ *has the following dual representation, for any $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$,*

$$\rho^g_{t,r}(\xi) = \sup_{\Gamma^\mu \in \mathcal{A}_{t,r}} \mathbb{E}^{\mathbb{Q}^\mu}[\,\xi\,] = \mathbb{E}[\,\Gamma^\mu \xi\,] \tag{2.43}$$

*where $\mathcal{A}_{t,r} = \partial \rho^g_{t,r}(0)$ is defined as follows:*

$$\mathcal{A}_{t,r} = \left\{ \exp\left( \int_t^r \mu_s\, dW_s - \frac{1}{2}\int_t^r |\mu_s|^2\, ds \right) : \mu \in \mathcal{H}^2[t,r],\ \mu_s \in \partial g(s,0),\ \ s \in [t,r] \right\}. \tag{2.44}$$

*where $\Gamma^\mu$ is an exponential martingale as in* (2.40).

Similar to the static dual representation, if the evaluation is coherent, the polar process (penalty function) $G(\cdot,\cdot)$ equals to 0 in the its domain for any time $t \in [0, T]$. As a result, the dynamic risk measure is equivalent to choosing an optimal *Radon Nikodym process*, the exponential density, to maximize the expected value after change of measure. The following lemma for coherent risk measure is important.

**Lemma 2.2.16.** *A constant $C$ exists, such that for all $0 \le t \le r \le T$ and all $\Gamma^{\mu}_{t,r} \in \mathcal{A}_{t,r}$ we have*

$$\|\Gamma^{\mu}_{t,r} - 1\|^2 \le \frac{r-t}{T} e^{CT}. \tag{2.45}$$

*Proof.* It follows from the definition of $\mathcal{A}_{t,r}$ that $\Gamma_{t,r}$ is the solution of the SDE

$$d\Gamma^{\mu}_{t,s} = \mu_s \Gamma^{\mu}_{t,s}\, dW_s, \quad \mu_s \in \partial g(s,0), \quad s \in [t,r], \quad \Gamma^{\mu}_{t,t} = 1. \tag{2.46}$$

Using Itô isometry, we obtain the chain of relations

$$\|\Gamma^{\mu}_{t,r} - 1\|^2 = \int_t^r \|\mu_s \Gamma^{\mu}_{t,s}\|^2\, ds \le \int_t^r \|\mu_s\|^2 \|\Gamma^{\mu}_{t,s}\|^2\, ds \le \int_t^r \|\mu_s\|^2 (1 + \|\Gamma^{\mu}_{t,s} - 1\|^2)\, ds. \tag{2.47}$$

If $u$ is a uniform upper bound on the norm of the subgradients of $g(s,0)$ we deduce that $\|\Gamma^{\mu}_{t,r} - 1\|^2 \le \Delta_s$, $s \in [t,r]$, where $\Delta$ satisfies the ODE: $\frac{d\Delta_s}{ds} = u(1 + \Delta_s)$, with $\Delta_t = 0$. Consequently,

$$\|\Gamma^{\mu}_{t,r} - 1\|^2 \le \Delta_r = e^{u^2(r-t)} - 1. \tag{2.48}$$

The convexity of the exponential function yields the postulated bound. □

# Chapter 3

# Formulation of Risk-averse Control Problem

In this chapter, we introduce the underlying controlled diffusion process, together with backward stochastic differential equation (BSDEs); the resulting FBSDE system enables evaluation of risk in the system dynamically. After policy evaluation is addressed, we pass to dynamic programming equation with two approaches: weak formulation and strong formulation. Finally, we derive the risk-averse Hamilton-Jacobi-Bellman equation and discuss its verification.

## 3.1 Decoupled FBSDE system

As the foundation element of our analysis and modeling, we devote this section to an introduction to a decoupled forward backward stochastic differential equation system(FBSDEs).

### 3.1.1 Foundation of FBSDE

On the same probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, we consider a *stochastic differential equation* (SDE) with initial condition $(t, \eta)$, where $t \in [0, T)$ and $\eta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$:

$$\begin{cases} dX_s^{t,\eta} = b(s, X_s^{t,\eta})\, ds + \sigma(s, X_s^{t,\eta})\, dW_s, \quad s \in [t, T], \\ X_t^{t,\eta} = \eta, \end{cases} \tag{3.1}$$

where $\{W_s\}_{t\leq s\leq T}$ is a $d$-dimensional Brownian motion. Here, $b : [0, T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^{n\times d}$ are $\mathcal{F}_t$-adapted processes.

**Assumption 3.1.1.**

(i) $|b(\cdot, 0)| + |\sigma(\cdot, 0)| \in \mathcal{H}^2[0, T]$*;*

(ii) *$b$ and $\sigma$ are* Lipschitz *in x, i.e., a constant $C > 0$ exists such that for all $t \in [0, T]$ and $x_1, x_2 \in \mathbb{R}^n$ we have:*

$$|b(t, x_1) - b(t, x_2)| + |\sigma(t, x_1) - \sigma(t, x_2)| \leq C|x_1 - x_2| \quad a.s..$$

From the standard theory of stochastic differential equations with random coefficients, Assumption 5.3.1 implies the existence of solution and its uniqueness. Together with that, we also consider a BSDE with the terminal condition specified by a function of the final value of the solution of (3.1),

$$\begin{cases} -dY_s^{t,\eta} = f(s, X_s^{t,\eta}, Y_s^{t,\eta}, Z_s^{t,\eta})\, ds - Z_s^{t,\eta}\, dW_s, \quad s \in [t, T] \\ Y_T^{t,\eta} = \Phi(X_T^{t,\eta}), \end{cases} \tag{3.2}$$

where $f : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is an $\mathcal{F}_t$-adapted process, and the function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is bounded and Borel measurable. We call this system a *decoupled forward backward stochastic differential equation*. Its essential feature is that the solution of the BSDE does not interfere with the dynamics of the SDE.

**Assumption 3.1.2.**

(i) $|f(\cdot, 0, 0, 0)| \in \mathcal{H}^2[0, T]$ *and* $\Phi(\cdot) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ *;*

(ii) $f$ *is* jointly Lipschitz *in* $(x, y, z)$ *and* $\Phi$ *is* Lipschitz *in x, i.e., a constant* $C > 0$ *exists such that for all* $t \in [0, T]$ *and* $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$ *we have:*

$$|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| \le C(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|), \quad a.s.,$$

$$|\Phi(x_1) - \Phi(x_2)| \le C|x_1 - x_2|, \quad a.s..$$

The following proposition follows from the fact that $\Phi(\cdot)$ and $f(\cdot, \cdot, \cdot, \cdot)$ provide standard parameters for the BSDE (3.2), and from the discussion in Chapter 2.

**Proposition 3.1.3.** *If Assumptions 5.3.1 and 3.1.2 are satisfied, then the equation (3.2) has a unique solution* $(Y^{t,\eta}, Z^{t,\eta}) \in \mathcal{S}^2[t, T] \times \mathcal{H}^{2,d}[t, T]$.

The following two prior estimates of a decoupled FBSDE system are standard.

**Proposition 3.1.4.** *under assumption 3.1.1, for any* $t \in [0, T]$ *and* $\eta, \eta' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$*, there exists* $C_0(C)$ *such that*

$$\mathbb{E}_t\Big[\sup_{s\in[t,T]} \Big|X_s^{t,\eta} - X_s^{t,\eta'}\Big|\Big] \le C_0\Big|\eta - \eta'\Big|, \ a.s. \tag{3.3}$$

*and*

$$\mathbb{E}_t\Big[\sup_{s\in[t,T]} \Big|X_s^{t,\eta}\Big|^2\Big] \le C_0\Big|\eta\Big|^2, \ a.s.. \tag{3.4}$$

**Lemma 3.1.5.** *Suppose Assumptions 3.1.1 and 3.1.2 are satisfied, then for any $t < T$, and any different initial states $\eta$, $\eta' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$, we have following estimates:*

$$|Y_t^{t,\eta} - Y_t^{t,\eta'}| \le C_0|\eta - \eta'|^{\frac{1}{2}}, \ \ a.s. \tag{3.5}$$

*and*

$$|Y_t^{t,\eta}| \le C_0(1 + |\eta|), \ \ a.s.. \tag{3.6}$$

### 3.1.2 Markovian Properties of Decoupled FBSDEs

For any $x \in \mathbb{R}^n$, we define:

$$v(t, x) := Y_t^{t,x}, \tag{3.7}$$

then, according to (3.5) and (3.6), we have

$$\begin{aligned} &|v(t, x) - v(t, x')| \le C_0|x - x'|^{\frac{1}{2}}, \ \ a.s., \\ &|v(t, x)| \le C_0(1 + |x|) \ a.s.. \end{aligned} \tag{3.8}$$

In general case, $v$ is a random function, i.e., for any $x \in \mathbb{R}^n$, $v(\cdot, x)$ is a $\mathcal{F}_t$-adapted process (this is the case when $\Phi(x)$, $f(t, x, y, z)$, $b(t, x)$ and $\sigma(t, x)$ are random functions). The situation would simplify under the following assumption.

**Assumption 3.1.6.** *$\forall(t, x, y, z)$, $\Phi(x)$, $f(t, x, y, z)$, $b(t, x)$ and $\sigma(t, x)$ are deterministic functions.*

Under the above assumption, $v$ is a deterministic function with respect to $(t, x)$. As a special case, if $f$ only depends on $(t, x)$, then

$$v(t, x) = \mathbb{E}\Big[ \int_t^T f(s, X_s^{t,x})ds + \Phi(X_T^{t,x}) \Big]$$

which leads to Feynman-Kac equation, i.e., $v(t, x)$ satisfies a nonlinear partial differential equation.

We shall not assume Assumption 3.1.6 for control problems discussed later. Let us introduce the following definition and an important theorem stating that randomness of the value function results only from the randomness of the starting point, even if the coefficients are ”arbitrarily” random.

**Definition 3.1.1.** *For a* $t \in [0, T]$*, we call* $\{A_i\}_{i=1}^N \subset \mathcal{F}_t$ *a* partition *of measurable space* $(\Omega, \mathcal{F}_t)$ *if* $\bigcup_{i=1}^N A_i = \Omega$ *and*

$$A_i \in \mathcal{F}_t,\ i = 1, ..., N;\ A_i \cap A_j = \emptyset,\ \textit{for}\ i \neq j$$

*here* $N \in \mathbb{Z}^+$.

**Theorem 3.1.7.** *Under Assumption 5.3.1 and 3.1.2, for any* $\eta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}, \mathbb{R}^n)$*, we have*

$$v(t, \eta) = Y_t^{t,\eta} \tag{3.9}$$

*Proof.* We first consider the case of simple function,

$$\eta = \Sigma_{i=1}^N \mathbb{1}_{A_i} x_i \tag{3.10}$$

with $\{A_i\}_{i=1}^N$ being a partition of $(\Omega, \mathcal{F}_t)$, $x_i \in \mathbb{R}^n$, $i = 1, 2, ..., N$. For each $i$, denote

$$(X_s^i, Y_s^i, Z_s^i) \equiv (X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})|_{x=x_i} \tag{3.11}$$

where $X^i$, for all $i$'s, is the solution of SDEs

$$X_s^i = x_i + \int_t^s b(r, X_r^i)dr + \int_t^s \sigma(r, X_r^i)dW_r, \quad s \in [t, T], \tag{3.12}$$

$(Y^i, Z^i)$, for all $i$'s, is the solution of BSDEs

$$Y_s^i = \Phi(X_T^i) + \int_s^T f(r, X_r^i, Y_r^i, Z_r^i) - \int_s^T Z_r^i dW_r, \quad s \in [t, T]. \tag{3.13}$$

Multiplying by $\mathbb{1}_{A_i}$ on the both sides of SDE and BSDE and take summation over all $i$'s, using the fact that $\Sigma_i \psi(x_i)\mathbb{1}_{A_i} = \psi(\Sigma_i \mathbb{1}_{A_i} x_i)$ for Borel any measurable function $\psi(\cdot)$, we have

$$\Sigma_{i=1}^N \mathbb{1}_{A_i} X_s^i = x_i + \int_t^s b(r, \Sigma_{i=1}^N \mathbb{1}_{A_i} X_r^i)dr + \int_t^s \sigma(r, \Sigma_{i=1}^N \mathbb{1}_{A_i} X_r^i)dW_r \tag{3.14}$$

$$\begin{aligned} \Sigma_{i=1}^N \mathbb{1}_{A_i} Y_s^i =& \Phi(\Sigma_{i=1}^N \mathbb{1}_{A_i} X_T^i) + \\ & \int_s^T f(r, \Sigma_{i=1}^N \mathbb{1}_{A_i} X_r^i, \Sigma_{i=1}^N \mathbb{1}_{A_i} Y_r^i, \Sigma_{i=1}^N \mathbb{1}_{A_i} Z_r^i) - \int_s^T \Sigma_{i=1}^N \mathbb{1}_{A_i} Z_r^i dW_r \end{aligned} \tag{3.15}$$

By the uniqueness property of the solution of SDE and BSDE,

$$X_s^{t,\eta} = \Sigma_{i=1}^N X_s^i \mathbb{1}_{A_i} \tag{3.16}$$

and

$$(Y_s^{t,\eta}, Z_s^{t,\eta}) = (\Sigma_{i=1}^N \mathbb{1}_{A_i} Y_s^i, \Sigma_{i=1}^N \mathbb{1}_{A_i} Z_s^i) \tag{3.17}$$

Thus, from (3.9),

$$Y_t^{t,\eta} = \Sigma_{i=1}^N Y_t^i \mathbb{1}_{A_i} = \Sigma_{i=1}^N v(t,x_i)\mathbb{1}_{A_i} = v(t, \Sigma_{i=1}^N x_i \mathbb{1}_{A_i}) = v(t,\eta). \tag{3.18}$$

The proof is completed as $\eta$ is a simple function. For general $\eta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$, we can choose a sequence $\{\eta_i\}_{i\in\mathbb{Z}^+}$ that is converging to $\eta$ in $L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$. Then from the estimates (3.5), (3.6) and (3.8) we have

$$\mathbb{E}_t[|Y_t^{t,\eta_i} - Y_t^{t,\eta}|^2] \leq C_0 \mathbb{E}_t[|\eta_i - \eta|] \to 0 \tag{3.19}$$

and

$$\mathbb{E}_t[|v(t,\eta_i) - v(t,\eta)|^2] \leq C_0 \mathbb{E}_t[|\eta_i - \eta|] \to 0 \tag{3.20}$$

as desired. □

## 3.2 Formulation of a Risk-Averse Control Problem

Our objective is to evaluate the risk of cumulative cost generated by diffusion process in a continuous-time setting via FBSDE system. In the sequel, we will study both a strong formulation and a weak formulation of the risk-averse control problem.

### 3.2.1 Strong Formulation

Let's first introduce the class of control processes:

**Definition 3.2.1.** *A stochastic process $u(\cdot)$ is called an* admissible control *if $u(\cdot)$ is taken from the set:*

$$\mathcal{U} := \{\, u : [0,T] \times \Omega \mapsto U \mid u(\cdot) \textit{ is } (\mathcal{F}_t)_{t\geq 0}\textit{-adapted} \,\} \tag{3.21}$$

*where $U \in \mathbb{R}^m$ is a compact set for some $m \in \mathbb{N}$.*

The forward SDE with random coefficients introduced previously can be controlled by composing drift function $b$ and diffusion function $\sigma$ with control process. In other words, the randomness of the coefficients is due to the control function alone, which leads to a *controlled* *diffusion process*. For any $u(\cdot) \in \mathcal{U}$, initial value $\zeta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$, consider

$$\begin{cases} dX_s^{t,\zeta;u} = b(s, X_s^{t,\zeta;u}, u_s)\, ds + \sigma(s, X_s^{t,\zeta;u}, u_s)\, dW_s, \quad s \in [t, T], \\ X_t^{t,\zeta;u} = \zeta, \end{cases} \tag{3.22}$$

with Borel measurable functions $b : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d}$. We also introduce the *cost rate* function which is a measurable map $c : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$, and the *final stage cost* is given by a measurable function $\Phi : \mathbb{R}^n \to \mathbb{R}$. Therefore, the random cost accumulated on the interval $[t, T]$ for any $t \in [0, T]$ is

$$\xi_{t,T}(u, \zeta) := \int_t^T c(s, X_s^{t,\zeta;u}, u_s)\, ds + \Phi(X_T^{t,\zeta;u}), \quad \text{a.s..} \tag{3.23}$$

**Assumption 3.2.1.** *A constant $C > 0$ exists such that, for any $s \in [t, T]$ and $(x_1, u_1), (x_2, u_2) \in \mathbb{R}^n \times U$, measurable functions $b$, $\sigma$, $c$, and $\Phi$ satisfy the following conditions:*

$$\begin{aligned} |b(s, x_1, u_1) - b(s, x_2, u_2)| + |\sigma(s, x_1, u_1) - \sigma(s, x_2, u_2)| + |c(s, x_1, u_1) - c(s, x_2, u_2)| \\ \le C(|x_1 - x_2| + |u_1 - u_2|), \end{aligned} \tag{3.24}$$

$$|b(s, x_1, u_1)| + |\sigma(s, x_1, u_1)| + |c(s, x_1, u_1)| + |\Phi(x_1)| \le C(1 + |x_1| + |u_1|). \tag{3.25}$$

Under *Assumption* 3.2.1, the controlled diffusion process (3.22) has a strong solution and cost functional has desired regularity. Now, we can define the *control value function*:

$$V^u(t, x) := \rho_{t,T}^g[\, \xi_{t,T}(u, x)\, ] = Y_t^{t,x;u}, \;\; a.s.. \tag{3.26}$$

By theorem 3.1.7 (proved in Peng [59]), we have, for any $\zeta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$,

$$V^u(t, \zeta) = Y_t^{t,\zeta;u}. \tag{3.27}$$

Using Definition 2.2.1, we can express the control value function as follows:

$$\begin{aligned} V^u(t, \zeta) &= \xi_{t,T}(u, \zeta) + \int_t^T g(s, Z_s^{t,\zeta;u})\, ds - \int_t^T Z_s^{t,\zeta;u}\, dW_s \\ &= \Phi(X_T^{t,\zeta;u}) + \int_t^T [c(s, X_s^{t,\zeta;u}, u_s) + g(s, Z_s^{t,\zeta;u})]\, ds - \int_t^T Z_s^{t,\zeta;u}\, dW_s. \end{aligned}$$

Equivalently, we need to evaluate the following BSDE:

$$\begin{cases} -dY_s^{t,\zeta;u} = [c(s, X_s^{t,\zeta;u}, u_s) + g(s, Z_s^{t,\zeta;u})]\, ds - Z_s^{t,\zeta;u}\, dW_s, \quad s \in [t, T] \\ Y_T^{t,\zeta;u} = \Phi(X_T^{t,\zeta;u}). \end{cases} \tag{3.28}$$

with Borel measurable function $g : [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}$. Suppose Assumptions 3.2.1, 2.2.1, and 2.2.11 are satisfied, then for every $(t,\zeta) \in [0,T] \times L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$, (3.28) has a unique solution $(Y^{t,\zeta;u}, Z^{t,\zeta;u}) \in \mathcal{S}^2[t,T] \times \mathcal{H}^{2,d}[t,T]$.

The policy evaluation amounts to solving (3.28), that is, for any fixed policy $u(\cdot)$, the risk of cumulative cost and final cost can be represented as $Y_t^{t,\zeta;u}$, the solution of (3.28) above. The following estimates is based on Proposition 2.2.6:

**Proposition 3.2.2.** *Under assumption 3.2.1, for some constant* $K > 0$ *and* $u, u' \in \mathcal{U}$*,*

(i) $|Y_t^{t,\zeta;u} - Y_t^{t,\zeta';u}| \leq C|\zeta - \zeta'|, \quad a.s.;$

(ii) $|Y_t^{t,\zeta;u}| \leq C(1 + |\zeta|), \quad a.s.;$

(iii) $|Y_t^{t,\zeta;u} - Y_t^{t,\zeta;u'}| \leq C\mathbb{E}_t\Big[ \int_t^T \big|u(s) - u'(s)\big|^2 ds \Big], \quad a.s..$

We can now define the *value function* of the risk-averse control problem: given $(t,x) \in [0,T) \times \mathbb{R}^n$,

$$V(t,x) := \inf_{u(\cdot)\in\mathcal{U}} V^u(t,x). \tag{3.29}$$

**Proposition 3.2.3.** *Under Assumption 3.2.1, the value function* $V(t,x)$ *is a deterministic function.*

*Proof.* The key step of this proof is to construct a control sequence $\{u^i(\cdot)\}$ satisfying

$$\lim_{i\to\infty} V^u(t,x) = V(t,x) \tag{3.30}$$

Set,

$$u_s^i = \Sigma_{j=1}^{N_i} u_s^{ij} \mathbb{1}_{A_{ij}}$$

where $u^{ij}(\cdot)$ are $\mathcal{U}$-valued $\mathcal{F}_s^t$-adapted process, and for each $i$, $\{A_{ij}\}_{j=1}^{N_i}$, is a partition of $(\Omega, \mathcal{F}_t)$, use exactly what was used to prove theorem 3.1.7, we obtain

$$\Sigma_{j=1}^{N_i} \mathbb{1}_{A_{ij}} V^{u^{ij}}(t,x) = V^{\Sigma_{j=1}^{N_i} \nu_s^{ij} \mathbb{1}_{A_{ij} u^{ij}}}(t,x) = V^u(t,x)$$

Notice that $u^{i1}(\cdot)$ is $(\mathcal{F}_s^t)$-adapted, $V^{u^{ij}}(t,x)$, where $j = 1,2,...,N_i$ are deterministic. Thus, without loss of generality, we can assume that

$$V^{u^{i1}}(t,x) \geq V^{u^{ij}}(t,x), \quad \forall i = 2,...,N_i$$

from which we get immediately $\lim_i V^{u^{i1}}(t,x) = V(t,x)$, thus $V(t,x)$ is deterministic. ◻

We also have estimates of the value function, which follow easily from Proposition 3.2.2:

**Proposition 3.2.4.** *For any $t \in [0,T]$ and $x, x' \in \mathbb{R}^n$, there exists a constant $C > 0$ such that,*

(i) $|V(t,x) - V(t,x')| \leq C|x - x'|$;

(ii) $|V(t,x)| \leq C(1 + |x|)$.

The inequalities below are decisive for the dynamic programming equation:

**Lemma 3.2.5.** *Fix a $t \in [0,T]$ and $\zeta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$, for any $u(\cdot) \in \mathcal{U}$, we have*

$$V(t,\zeta) \leq Y_t^{t,\zeta;u}, \tag{3.31}$$

*Conversely, for any $\epsilon > 0$, there exists a admissible control $u(\cdot) \in \mathcal{U}$, such that*

$$V(t,\zeta) + \epsilon \geq Y_t^{t,\zeta;u}. \tag{3.32}$$

*Proof.* In proposition 3.2.4, we already established the continuity for $V$ with respect to $x$ and $Y_t^{t,\zeta;u}$ with respect to $(\zeta, u)$, then it is only needed to discuss the situation that $\zeta$ and $u(\cdot)$ are simple functions, i.e.,

$$\zeta = \Sigma_{i=1}^{N} \mathbb{1}_{A_i} x_i \tag{3.33}$$

$$u(\cdot) = \Sigma_{i=1}^{N} u_s^i \mathbb{1}_{A_i} u^i(\cdot) \tag{3.34}$$

where for $i = 1, 2, ..., N$, $x_i \in \mathbb{R}^n$ is $\mathcal{F}_s^t$-adapted, $\{A_i\}_{i=1}^N$ is a partition of $(\Omega, \mathcal{F}_t)$. This enables us to use the same technique as before to get:

$$Y_t^{t,\zeta;u} = \Sigma_{i=1}^{N} \mathbb{1}_{A_i} Y_t^{t,x^i;u^i} \geq \Sigma_{i=1}^{N} \mathbb{1}_{A_i} v(t, x_i) = v(t, \Sigma_{t=1}^{N} \mathbb{1}_{A_i} x_i) = V(t,\zeta) \tag{3.35}$$

then (3.31) is proved. Similarly, we can prove (3.32), define $\eta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$,

$$\eta = \Sigma_{i=1}^{\infty} x^i \mathbb{1}_{A_i} \tag{3.36}$$

such that

$$|\eta - \zeta| \leq \left(\frac{\epsilon}{3C}\right)^{\frac{1}{\alpha}} \tag{3.37}$$

where $\{A_i\}_{i=1}^{\infty}$ is a partition of $(\Omega, \mathcal{F}_t)$, $x_i \in \mathbb{R}^n (i = 1, 2, ...)$. Thus, for any $u(\cdot) \in \mathcal{U}$,

$$\begin{gathered} |Y_t^{t,\eta;u} - Y_t^{t,\eta;u}| \leq \frac{\epsilon}{3} \\ |u(t,\zeta) - u(t,\eta)| \leq \frac{\epsilon}{3} \end{gathered} \tag{3.38}$$

then for each $x_i$, we choose $\mathcal{F}_s^t$-adapted control $u^i(\cdot)$, such that

$$V(t,x_i) + \frac{\epsilon}{3} \geq Y_t^{t,x_i;u^i} \tag{3.39}$$

define

$$u(\cdot) := \Sigma_{i=1}^{\infty} u^i \mathbb{1}_{A_i} \tag{3.40}$$

from (3.38), we have

$$\begin{aligned} Y_t^{t,\zeta;u} &\leq |Y_t^{t,\eta;u} - Y_t^{t,\zeta;u}| + Y_t^{t,\eta;u} \leq \frac{\epsilon}{3} + \Sigma_{i=1}^{\infty} Y_t^{t,x_i;u^i} \mathbb{1}_{A_i} \\ &\leq \frac{\epsilon}{3} + \Sigma_{i=1}^{\infty}\Big(V(t,x_i) + \frac{\epsilon}{3}\Big)\mathbb{1}_{A_i} = \frac{2\epsilon}{3} + \Sigma_{i=1}^{\infty} V(t,x_i)\mathbb{1}_{A_i} \\ &= \frac{2\epsilon}{3} + V(t,\eta) \leq \epsilon + V(t,\zeta) \end{aligned} \tag{3.41}$$

thus we have (3.32). □

**Theorem 3.2.6.** *Suppose Assumption 3.2.1, 2.2.1 and 2.2.11 are satisfied. Then, for any* $(t,x) \in [0,T) \times \mathbb{R}^n$,

$$V(t,x) = \inf_{u(\cdot)\in\mathcal{U}} \rho_{t,r}^g\Big[\int_t^r c(s, X_s^{t,x;u}, u_s)\,ds + V(r, X_r^{t,x;u})\Big], \quad 0 \leq t \leq r \leq T, \quad x \in \mathbb{R}^n. \tag{3.42}$$

*Proof.* We have

$$\begin{aligned} V(t,x) &= \inf_{u(\cdot)\in\mathcal{U}} \rho_{t,T}^g\Big[\int_t^T c(s, X_s^{t,x;u}, u_s) ds + \Phi(X_T^{t,x;u})\Big] \\ &= \inf_{u(\cdot)\in\mathcal{U}} \rho_{t,r}^g\Big[\rho_{r,T}^g\Big[\int_t^T c(s, X_s^{t,x;u}, u_s)\,ds + \Phi(X_T^{t,x;u})\Big]\Big] \\ &= \inf_{u(\cdot)\in\mathcal{U}} \rho_{t,r}^g\Big[\int_t^r c(s, X_s^{t,x;u}, u_s)\,ds + \rho_{r,T}^g\Big[\int_r^T c(s, X_s^{t,x;u}, u_s)\,ds + \Phi(X_T^{t,x;u})\Big]\Big] \\ &= \inf_{u(\cdot)\in\mathcal{U}} \rho_{t,r}^g\Big[\int_t^r c(s, X_s^{t,x;u}, u_s)\,ds + Y_r^{r,X_r^{t,x;u};u}\Big] \\ &\geq \inf_{u(\cdot)\in\mathcal{U}} \rho_{t,r}^g\Big[\int_t^r c(s, X_s^{t,x;u}, u_s)\,ds + V(r, X_r^{t,x;u})\Big] \end{aligned}$$

where time consistency, translation invariant, comparison theorem.

On the other hand, for any $\epsilon > 0$, there exists an admissible control $\bar{u}(\cdot) \in \mathcal{U}$ such that

$$V(r, X_r^{t,x;u}) \geq Y_r^{r,X_r^{t,x;u};\bar{u}} - \epsilon$$

Again, by comparison theorem,

$$
\begin{aligned}
V(t,x) &\le \inf_{u(\cdot)\in\mathcal{U}} \rho^g_{t,r}\Big[\int_t^r c(s, X_s^{t,x;u}, u_s)\,ds + Y_r^{r,X_r^{t,x;u};\bar{u}}\Big] \\
&\le \inf_{u(\cdot)\in\mathcal{U}} \rho^g_{t,r}\Big[\int_t^r c(s, X_s^{t,x;u}, u_s)\,ds + V(r, X_r^{t,x;u})\Big] + \varepsilon
\end{aligned}
$$

Since $\epsilon$ can be arbitrarily small, dynamic programming equation (3.42) follows. □

### 3.2.2 Weak Formulation

We now discuss the weak formulation. For any initial condition $(t,x) \in [0,T) \times \mathbb{R}^n$, the state evolution is driven by the following equation:

$$
(3.43)\qquad dX_s^{t,x;u} = b(s, X_s^{t,x;u}, u_s)\,ds + \sigma(s, X_s^{t,x;u}, u_s)\,dW_s, \quad s \in [t,T], \quad X_t^{t,x;u} = x.
$$

Let us define the *control value function* $V^u$ as follows:

$$
(3.44)\qquad V^u(t,x) = \rho^g_{t,T}[\xi_{t,T}(u)],
$$

where, similarly to (3.23), $\xi_{t,T}(u)$ is the cost accumulated in the interval $[t,T]$:

$$
(3.45)\qquad \xi_{t,T}(u) = \int_t^T c(s, X_s^{t,x;u}, u_s)\,ds + \Phi(X_T^{t,x;u}).
$$

Using Definition 2.2.1, we can express the control value function as follows:

$$
\begin{aligned}
V^u(t,x) &= \xi_{t,T}(u) + \int_t^T g(s, Z_s^{t,x;u})\,ds - \int_t^T Z_s^{t,x;u}\,dW_s \\
&= \Phi(X_T^{t,x;u}) + \int_t^T [c(s, X_s^{t,x;u}, U_s) + g(s, Z_s^{t,x;u})]\,ds - \int_t^T Z_s^{t,x;u}\,dW_s.
\end{aligned}
$$

Equivalently, we need to evaluate the following BSDE on $[t,T]$:

$$
(3.46)\qquad -dY_s^{t,x;u} = [c(s, X_s^{t,x;u}, u_s) + g(s, Z_s^{t,x;u})]\,ds - Z_s^{t,x;u}\,dW_s, \quad Y_T^{t,x;u} = \Phi(X_T^{t,x;u}).
$$

Let us now define the *admissible control system* (as defined in [77], p.177):

**Definition 3.2.2.** $\mathcal{U}^w[t,T]$ *is called an* admissible control system *if it satisfies the following conditions:*

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ *is a complete probability space;*

(ii) $\{W(s)\}_{s\geq t}$ *is an d-dimensional standard Brownian motion defined on* $(\Omega, \mathcal{F}, \mathbb{P})$ *over* $[t, T]$ *and* $\mathbb{F}^t = (\mathcal{F}_s)_{s\in[t,T]}$, *where* $\mathcal{F}_s^t = \sigma\{(W_s; t \leq s \leq T) \cup \mathcal{N}\}$ *and* $\mathcal{N}$ *is the collection of all P-null sets in* $\mathcal{F}$*;*

(iii) $u : [t, T] \times \Omega \to U$ *is an* $\mathcal{F}_s^t$*-adapted process on* $(\Omega, \mathcal{F}, \mathbb{P})$ *with* $\mathbb{E}\Big[\int_t^T |u_s|^2\, ds\Big] < +\infty$*;*

(iv) *For any* $x \in \mathbb{R}^n$ *the system* (3.43)–(3.46) *admits a unique solution* $(X, Y, Z)$ *on* $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}^t)$.

For notation simplicity, we write $u \in \mathcal{U}^{\mathrm{w}}[t, T]$ for $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}^t, W(\cdot), u(\cdot)) \in \mathcal{U}^{\mathrm{w}}[t, T]$. Now, we define the *optimal value function* $V : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ as follows:

$$V(t, x) = \inf_{u\in\mathcal{U}^{\mathrm{w}}[t,T]} V^u(t, x). \tag{3.47}$$

The weak formulation of a risk-averse control problem is the following:

**Problem W**: Given $(t, x) \in [0, T) \times \mathbb{R}^n$, find $u^* \in \mathcal{U}^{\mathrm{w}}[t, T]$ such that

$$V^{u^*}(t, x) = \inf_{u(\cdot)\in\mathcal{U}^{\mathrm{w}}[t,T]} V^u(t, x). \tag{3.48}$$

**Proposition 3.2.7.** *Suppose Assumptions 3.2.1, 2.2.1, and conditions (i) - (ii) of Assumption 2.2.11 are satisfied. Then for every* $(t, x) \in [0, T] \times \mathbb{R}^n$ *the system* (3.43)–(3.46) *has a unique solution* $(X^{t,x;u}, Y^{t,x;u}, Z^{t,x;u}) \in \mathcal{S}^{2,n}[t, T] \times \mathcal{S}^2[t, T] \times \mathcal{H}^{2,d}[t, T]$. *Furthermore* $V^u(t, x)$ *is deterministic.*

*Proof.* Consider the backward equation:

$$-d\bar{Y}_s^{t,x;u} = g(s, Z_s^{t,x;u})\, ds - Z_s^{t,x;u}\, dW_s, \quad s \in [t, T], \tag{3.49}$$

where

$$\bar{Y}_T^{t,x;u} = \int_t^T c(\tau, X_\tau^{t,x;u}, s_\tau)\, d\tau + \Phi(X_T^{t,x;u}).$$

By Proposition 3.1.3, equation (3.49) has a unique solution $(\bar{Y}^{t,x;u}, Z^{t,x;u})$. Then

$$Y_s^{t,x;u} = \bar{Y}_s^{t,x;u} - \int_t^s c(\tau, X_\tau^{t,x;u}, u_\tau)\, d\tau, \quad t \leq s \leq T,$$

satisfies (3.46) with the same process $Z^{t,x;u}$. Moreover, $Y_t^{t,x;u}$ is deterministic, which proves our last claim. □

Let's present a technical result that follows from *Proposition* 2.2.4.

**Proposition 3.2.8.** *Let* $(t,x) \in [0,T] \times \mathbb{R}^n$ *and* $u(\cdot) \in \mathcal{U}^w[t,T]$. *Then, for any* $r \in [t,T]$ *and* $\mathcal{F}_r^t$ *- measurable random variable* $\eta$,

$$V^u(r,\eta) = \rho_{r,T}\Big[\int_r^T c(s, X_s^{r,\eta;u}, u_s)\,ds + \Phi(X_T^{r,\eta;u})\Big], \quad \mathbb{P}\text{-}a.s.$$

*Proof.* Observe that $\eta$ is deterministic under the new probability space $(\Omega, \mathcal{F}, \mathbb{P}(\cdot|\mathcal{F}_r^t), \mathbb{F}^t)$. For any $s \geq r$, a progressively measurable process $\psi$ exists, such that

$$u_s(\omega) = \psi(\omega, W_{\cdot\wedge s}(\omega)) = \psi(s, \bar{W}_{\cdot\wedge s}(\omega) + W_r(\omega)),$$

where $\widetilde{W}_s = W_s - W_r$ is a standard Brownian motion. Then $u_s$ is adapted to $\mathcal{F}_s^r$ for $s \geq r$, and thus

$$(\Omega, \mathcal{F}, \mathbb{P}(\cdot|\mathcal{F}_r^t)(\omega'), \mathbb{F}^r, \widetilde{W}(\cdot), u(\cdot)) \in \mathcal{U}^{\mathrm{w}}[r,T].$$

where $\omega' \in \Omega'$ such that $\Omega' \in \mathcal{F}$ with $\mathbb{P}(\Omega') = 1$. Working under probability space $(\Omega, \mathcal{F}, \mathbb{P}(\cdot|\mathcal{F}_r^t)(\omega'))$, by Proposition 2.2.4, we obtain our result. □

Let us now derive the *dynamic programming equation* for the risk-averse control problem in weak setting.

**Theorem 3.2.9.** *Suppose Assumption 3.2.1, 2.2.1 and conditions (i)–(ii) of* Assumption *2.2.11 are satisfied. Then for any* $(t,x) \in [0,T) \times \mathbb{R}^n$,

$$V(t,x) = \inf_{u(\cdot)\in\mathcal{U}^w[t,r]} \rho_{t,r}^g\Big[\int_t^r c(s, X_s^{t,x;u}, u_s)\,ds + V(r, X_r^{t,x;u})\Big], \quad 0 \leq t \leq r \leq T, \quad x \in \mathbb{R}^n. \tag{3.50}$$

*Proof.* For any $\varepsilon > 0$, there exists $\tilde{u}(\cdot) \in \mathcal{U}^{\mathrm{w}}[t,T]$ such that $V(t,x) + \varepsilon \geq V^{\tilde{u}}(t,x)$. By using the definition of $V^{\tilde{u}}(t,x)$ and applying the time-consistency property (Theorem 2.2.8) and the translation property (Theorem 2.2.12), we obtain

$$\begin{aligned}
V(t,x) + \varepsilon \geq V^{\tilde{u}}(t,x) &= \rho_{t,r}^g\Big[\rho_{r,T}^g\Big[\int_t^T c(s, X_s^{t,x;\tilde{u}}, \tilde{u}_s)\,ds + \Phi(X_T^{t,x;\tilde{u}})\Big]\Big] \\
&= \rho_{t,r}^g\Big[\int_t^r c(s, X_s^{t,x;\tilde{u}}, \tilde{u}_s)\,ds + \rho_{r,T}^g\Big[\int_r^T c(s, X_s^{t,x;\tilde{u}}, \tilde{u}_s)\,ds + \Phi(X_T^{t,x;\tilde{u}})\Big]\Big] \\
&= \rho_{t,r}^g\Big[\int_t^r c(s, X_s^{t,x;\tilde{u}}, \tilde{u}_s)\,ds + \rho_{r,T}^g\Big[\int_r^T c(s, X_s^{r,x(r);\tilde{u}}, \tilde{u}_s)\,ds + \Phi(X_T^{r,x(r);\tilde{u}})\Big]\Big].
\end{aligned}$$

By virtue of Proposition 3.2.8,

$$\begin{aligned} V(t,x)+\varepsilon &\geq \rho^g_{t,r}\Big[\int_t^r c(s,X_s^{t,x;\tilde{u}},\tilde{u}_s)\,ds + V^{\tilde{u}}(r,x_r^{t,x;\tilde{u}})\Big] \\ &\geq \rho^g_{t,r}\Big[\int_t^r c(s,X_s^{t,x;\tilde{u}},\tilde{u}_s)\,ds + V(r,x_r^{t,x;\tilde{u}})\Big] \\ &\geq \inf_{u(\cdot)\in\mathcal{U}^{\mathrm{w}}[t,r]} \rho^g_{t,r}\Big[\int_t^r c(s,X_s^{t,x;u},u_s)\,ds + V(r,X_r^{t,x;u})\Big]. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have established the inequality "$\geq$" in (3.50).

To prove the reverse inequality, let $\varepsilon > 0$ be fixed, and let $\bar{u} \in \mathcal{U}^{\mathrm{w}}[t,r]$ be an $\varepsilon$-optimal solution of the problem on the right hand side of (3.50). Thus

$$\inf_{u(\cdot)\in\mathcal{U}^{\mathrm{w}}[t,r]} \rho^g_{t,r}\Big[\int_t^r c(s,X_s^{t,x;u},u_s)\,ds + V(r,X_r^{t,x;u})\Big] + \varepsilon \geq \rho^g_{t,r}\Big[\int_t^r c(s,X_s^{t,x;\bar{u}},\bar{u}_s)\,ds + V(r,X_r^{t,x;\bar{u}})\Big].$$

For every $y \in \mathbb{R}^n$, let $\tilde{u}(y) \in \mathcal{U}^{\mathrm{w}}[r,T]$ be such that $V(r,y)+\varepsilon \geq V^{\tilde{u}(y)}(r,y)$. Owing to the measurable selection theorem in Sect. 5.2 of [71], without loss of generality we may assume that the function $y \mapsto \tilde{u}(y)$ is Borel measurable. Now we can construct a control function

$$u_s^0 = \begin{cases} \bar{u}_s & s \in [t,r), \\ \tilde{u}_s(X_r^{t,x;\bar{u}}) & s \in [r,T]. \end{cases}$$

By construction, $u^0 \in \mathcal{U}^{\mathrm{w}}[t,T]$. Using the monotonicity, translation, and time-consistency properties, we obtain

$$\begin{aligned} \rho^g_{t,r}\Big[\int_t^r c(s,X_s^{t,x;\bar{u}},\bar{u}_s)\,ds + V(r,X_r^{t,x;\bar{u}})\Big] &\geq \rho^g_{t,r}\Big[\int_t^r c(s,X_s^{t,x;\bar{u}},\bar{u}_s)\,ds + V^{\tilde{u}_s(X_r^{t,x;\bar{u}})}(r,X_r^{t,x;\bar{u}}) - \varepsilon\Big] \\ &= \rho^g_{t,T}\Big[\int_t^T c(s,X_s^{t,x;u^0},u_s^0)\,ds + \Phi(X_T^{t,x;u^0})\Big] - \varepsilon \\ &= V^{u^0}(t,x) - \varepsilon. \end{aligned}$$

Combining the last three inequalities, we obtain

$$\inf_{u(\cdot)\in\mathcal{U}^{\mathrm{w}}[t,r]} \rho^g_{t,r}\Big[\int_t^r c(s,X_s^{t,x;u},u_s)\,ds + V(r,X_r^{t,x;u})\Big] + \varepsilon \geq V^{u^0}(t,x) - \varepsilon \geq V(t,x) - \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we proved the inequality "$\leq$" in (3.50). □

## 3.3 Risk-averse Hamilton-Jacobi-Bellman Equation

So far, we have derived a risk-averse dynamic programming equation in both formulations: (3.42) and (3.50), respectively. Now our goal is to work out a non-linear partial differential equation, similar to the Hamilton-Jacobi-Bellman equation in the classical control case. In this section, we will not distinguish between strong and weak formulation. With a little abuse of notation, we denote the admissible control set as $\mathcal{U}$.

For $\alpha \in U$ we define the *Laplacian operator* $\mathcal{L}^\alpha$ as follows: for $w \in C_{\mathrm{b}}^{1,2}([0,T]\times\mathbb{R}^n)$,

$$[\mathcal{L}^\alpha w](t,x) = \partial_t w(t,x) + \sum_{i,j=1}^{n} \frac{1}{2}(\sigma(t,x,\alpha)\sigma(t,x,\alpha)^\top)_{ij}\partial_{x_i x_j} w(t,x) + \sum_{i=1}^{n} b_i(t,x,\alpha)\partial_{x_i} w(t,x).$$

On the space $C_{\mathrm{b}}^{1,2}([0,T]\times\mathbb{R}^n)$, we consider the following equation

$$\min_{\alpha\in U}\Big\{c(t,x,\alpha) + [\mathcal{L}^\alpha v](t,x) + g\big(t,[\mathcal{D}_x v\cdot\sigma^\alpha](t,x)\big)\Big\} = 0, \tag{3.51}$$

with the boundary condition

$$v(T,x) = \Phi(T,x), \quad x\in\mathbb{R}^n. \tag{3.52}$$

We call (4.12)–(4.13) the *risk-averse Hamilton–Jacobi–Bellman equation* associated with the controlled system (3.22) and the risk functional (3.23). It is a generalization of the classical Hamilton–Jacobi–Bellman Equation with the extra term $g(\cdot,\cdot)$ responsible for risk aversion. In the special case, when $g \equiv 0$, we obtain the standard equation.

Let us recall the notion of a *viscosity solution* of such an equation.

**Definition 3.3.1.** *A function $v : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ satisfying* (4.13) *is called a* viscosity solution *of the equation* (4.12)–(4.13)*, if the following two conditions are satisfied:*

(i) *$v$ is a viscosity subsolution: for every $w \in C_b^{1,2}([t,T]\times\mathbb{R}^n)$ such that $w \geq v$ on $[0,T]\times\mathbb{R}^n$, and $\min_{(t,x)}[w(t,x) - v(t,x)] = 0$, and for every $(\bar{t},\bar{x}) \in [0,T)\times\mathbb{R}^n$ such that $w(\bar{t},\bar{x}) = v(\bar{t},\bar{x})$, we have*

$$\min_{\alpha\in U}\Big\{c(\bar{t},\bar{x},\alpha) + [\mathcal{L}^\alpha w](\bar{t},\bar{x}) + g\big(\bar{t},\mathcal{D}_x w(\bar{t},\bar{x})\cdot\sigma(\bar{t},\bar{x},\alpha)\big)\Big\} \geq 0;$$

(ii) *$v$ is a viscosity supersolution: for every $w \in C_b^{1,2}([t,T]\times\mathbb{R}^n)$ such that $w \leq v$ on $[0,T]\times\mathbb{R}^n$, and $\min_{(t,x)}[v(t,x) - w(t,x)] = 0$, and for every $(\bar{t},\bar{x}) \in [0,T)\times\mathbb{R}^n$ such that $w(\bar{t},\bar{x}) = v(\bar{t},\bar{x})$, we have*

$$\min_{\alpha\in U}\Big\{c(\bar{t},\bar{x},\alpha) + [\mathcal{L}^\alpha w](\bar{t},\bar{x}) + g\big(\bar{t},\mathcal{D}_x w(\bar{t},\bar{x})\cdot\sigma(\bar{t},\bar{x},\alpha)\big)\Big\} \leq 0.$$

**Theorem 3.3.1.** *Suppose Assumption 3.2.1, Assumption 2.2.1 and 2.2.11 are satisfied, in addition, the functions $b$, $\sigma$ are bounded in $x$. Then the value function $V(\cdot,\cdot)$ is a viscosity solution of the equation* (4.12)–(4.13).

*Proof.* Let $w \in C_b^{1,2}([t,T]\times\mathbb{R}^n)$ be such that $w \geq V$ on $[0,T]\times\mathbb{R}^n$, and $\min_{(t,x)}[w(t,x)-V(t,x)] = 0$. Consider a point $(\bar{t},\bar{x}) \in [0,T)\times\mathbb{R}^n$ such that $w(\bar{t},\bar{x}) = V(\bar{t},\bar{x})$. Let us take $\Delta t > 0$, and consider a constant control $u_s = \alpha$ for $s \in [\bar{t},\bar{t}+\Delta t]$. According to (3.42)(or (3.50)), we have

$$\begin{aligned} w(\bar{t},\bar{x}) = V(\bar{t},\bar{x}) &\leq \rho^g_{\bar{t},\bar{t}+\Delta t}\left(\int_{\bar{t}}^{\bar{t}+\Delta t} c(s,X_s^{\bar{t},\bar{x};\alpha},\alpha)\,ds + V(\bar{t}+\Delta t, X_{\bar{t}+\Delta t}^{\bar{t},\bar{x};\alpha})\right) \\ &\leq \rho^g_{\bar{t},\bar{t}+\Delta t}\left(\int_{\bar{t}}^{\bar{t}+\Delta t} c(s,X_s^{\bar{t},\bar{x};\alpha},\alpha)\,ds + w(\bar{t}+\Delta t, X_{\bar{t}+\Delta t}^{\bar{t},\bar{x};\alpha})\right). \end{aligned} \tag{3.53}$$

Using the translation property of $\rho_{\bar{t},\bar{t}+\Delta t}$, we obtain the inequality:

$$\rho^g_{\bar{t},\bar{t}+\Delta t}\left(\int_{\bar{t}}^{\bar{t}+\Delta t} c(s,X_s^{\bar{t},\bar{x};\alpha},\alpha)\,ds + w(\bar{t}+\Delta t, X_{\bar{t}+\Delta t}^{\bar{t},\bar{x};\alpha}) - w(\bar{t},\bar{x})\right) \geq 0.$$

Since $w \in C_b^{1,2}([t,T]\times\mathbb{R}^n)$, we can evaluate the difference $w(\bar{t}+\Delta t, X_{\bar{t}+\Delta t}^{\bar{t},\bar{x};\alpha}) - w(\bar{t},\bar{x})$ by Itô formula between $\bar{t}$ and $\bar{t}+\Delta t$:

$$w(\bar{t}+\Delta t, X_{\bar{t}+\Delta t}^{\bar{t},\bar{x};\alpha}) - w(\bar{t},\bar{x}) = \int_{\bar{t}}^{\bar{t}+\Delta t} [\mathcal{L}^\alpha w](s,X_s^{\bar{t},\bar{x};\alpha})\,ds + \int_{\bar{t}}^{\bar{t}+\Delta t} [\mathcal{D}_x w\cdot\sigma^\alpha](s,X_s^{\bar{t},\bar{x};\alpha})\,dW_s.$$

For simplicity of presentation, we write $\sigma^\alpha(s,x)$ for $\sigma(s,x,\alpha)$ and $c^\alpha(s,x)$ for $c(s,x,\alpha)$. Substitution into the previous inequality yields:

$$\rho^g_{\bar{t},\bar{t}+\Delta t}\left(\int_{\bar{t}}^{\bar{t}+\Delta t} [c^\alpha + \mathcal{L}^\alpha w](s,X_s^{\bar{t},\bar{x};\alpha})\,ds + \int_{\bar{t}}^{\bar{t}+\Delta t} [\mathcal{D}_x w\cdot\sigma^\alpha](s,X_s^{\bar{t},\bar{x};\alpha})\,dW_s\right) \geq 0. \tag{3.54}$$

If $w \in C_b^{1,2}([t,T]\times\mathbb{R}^n)$, then the argument of the risk measure is well-defined. The evaluation of the risk measure amounts to solving the following backward stochastic differential equation:

$$\begin{aligned} Y_{\bar{t}} = \int_{\bar{t}}^{\bar{t}+\Delta t} [c^\alpha + \mathcal{L}^\alpha w](s,X_s^{\bar{t},\bar{x};\alpha})\,ds &+ \int_{\bar{t}}^{\bar{t}+\Delta t} [\mathcal{D}_x w\cdot\sigma^\alpha](s,X_s^{\bar{t},\bar{x};\alpha})\,dW_s \\ &+ \int_{\bar{t}}^{\bar{t}+\Delta t} g(s,Z_s)\,ds - \int_{\bar{t}}^{\bar{t}+\Delta t} Z_s\,dW_s. \end{aligned}$$

By Proposition 3.1.3, the equation has a unique solution:

$$\begin{aligned} Z_s &= [\mathcal{D}_x w\cdot\sigma^\alpha](s,X_s^{\bar{t},\bar{x};\alpha}), \quad \bar{t}\leq s\leq \bar{t}+\Delta t, \\ Y_{\bar{t}} &= \int_{\bar{t}}^{\bar{t}+\Delta t} \left\{[c^\alpha + \mathcal{L}^\alpha w](s,X_s^{\bar{t},\bar{x};\alpha}) + g\big(s,[\mathcal{D}_x w\cdot\sigma^\alpha](s,X_s^{\bar{t},\bar{x};\alpha})\big)\right\}ds. \end{aligned}$$

Substitution into (3.54) yields the inequality:

$$\int_{\bar{t}}^{\bar{t}+\Delta t} \left\{[c^\alpha + \mathcal{L}^\alpha w](s,X_s^{\bar{t},\bar{x};\alpha}) + g\big(s,[\mathcal{D}_x w\cdot\sigma^\alpha](s,X_s^{\bar{t},\bar{x};\alpha})\big)\right\}ds \geq 0. \tag{3.55}$$

Dividing by $\Delta t$ and letting $\Delta t \downarrow 0$, we obtain

$$[c^{\alpha} + \mathcal{L}^{\alpha} w](\bar{t}, \bar{x}) + g\big(\bar{t}, [\mathcal{D}_x w \cdot \sigma^{\alpha}](\bar{t}, \bar{x})\big) \geq 0.$$

Since $\alpha \in U$ was arbitrary, we conclude that

$$\min_{\alpha \in U} \Big\{ c(\bar{t}, \bar{x}, \alpha) + \mathcal{L}^{\alpha} w(\bar{t}, \bar{x}) + g\big(\bar{t}, [\mathcal{D}_x w \cdot \sigma^{\alpha}](\bar{t}, \bar{x})\big) \Big\} \geq 0. \tag{3.56}$$

Consequently, $V$ is a viscosity subsolution of (4.12)–(4.13) (satisfies condition (i) of Definition 3.3.1). The minimum in (3.56) is attained, owing to the compactness of $U$ and the continuity of the function in braces with respect to $\alpha$.

Now, let $w \in C_{\mathrm{b}}^{1,2}([t, T] \times \mathbb{R}^n)$ be such that $w \leq V$ on $[0, T] \times \mathbb{R}^n$, and $\min_{(t,x)}[V(t, x) - w(t, x)] = 0$. Consider a point $(\bar{t}, \bar{x}) \in [0, T) \times \mathbb{R}^n$ such that $w(\bar{t}, \bar{x}) = V(\bar{t}, \bar{x})$. Let $\tilde{u}(\cdot) \in \mathcal{U}$ be an $\varepsilon\Delta t$-optimal control in (3.42)(or, (3.50)) on the interval $[\bar{t}, \bar{t} + \Delta t]$. Proceeding exactly as in the derivation of (3.55), we obtain the inequality:

$$\int_{\bar{t}}^{\bar{t}+\Delta t} \Big\{ [c^{\tilde{u}_s} + \mathcal{L}^{\tilde{u}_s} w](s, X_s^{\bar{t},\bar{x};\tilde{u}}) + g\big(s, [\mathcal{D}_x w \cdot \sigma^{\tilde{u}_s}](s, X_s^{\bar{t},\bar{x};\tilde{u}})\big) \Big\}\, ds \leq \varepsilon \Delta t.$$

Therefore, we also have

$$\int_{\bar{t}}^{\bar{t}+\Delta t} \min_{\alpha \in U} \Big\{ [c^{\alpha} + \mathcal{L}^{\alpha} w](s, X_s^{\bar{t},\bar{x};\tilde{u}}) + g\big(s, [\mathcal{D}_x w \cdot \sigma^{\alpha}](s, X_s^{\bar{t},\bar{x};\tilde{u}})\big) \Big\}\, ds \leq \varepsilon \Delta t. \tag{3.57}$$

The function $(s, x, \alpha) \mapsto [c^{\alpha} + \mathcal{L}^{\alpha} w](s, x) + g\big(s, [\mathcal{D}_x w \cdot \sigma^{\alpha}](s, x)\big)$ is continuous, the set $U$ is compact, and the solution $s \mapsto X_s^{\bar{t},\bar{x};\tilde{u}}$ is continuous. Consequently, the function under the integral in (3.57) is continuous. Therefore, dividing both sides of (3.57) by $\Delta t$ and letting $\Delta t \downarrow 0$, we obtain the following inequality:

$$\min_{\alpha \in U} \Big\{ c(\bar{t}, \bar{x}, \alpha) + \mathcal{L}^{\alpha} w(\bar{t}, \bar{x}) + g\big(\bar{t}, [\mathcal{D}_x w \cdot \sigma^{\alpha}](\bar{t}, \bar{x})\big) \Big\} \leq \varepsilon. \tag{3.58}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $V$ is a viscosity supersolution of (4.12)–(4.13) (satisfies condition (ii) of Definition 3.3.1). □

It is clear that if $V \in C_{\mathrm{b}}^{1,2}([t, T] \times \mathbb{R}^n)$ then it satisfies (4.12)–(4.13). We can also prove the converse relation (*verification theorem*).

**Theorem 3.3.2.** *Suppose the assumptions of Theorem 3.3.1 are satisfied and let $K \in C_b^{1,2}([t, T] \times \mathbb{R}^n)$ satisfy (4.12)–(4.13). Then $K(t, x) \leq V^u(t, x)$ for any control $u(\cdot) \in \mathcal{U}$ and all $(t, x) \in$*

$[0,T] \times \mathbb{R}^n$. *Furthermore, if a control process* $u^* \in \mathcal{U}$ *exists, satisfying for almost all* $(s, \Omega) \in [0,T] \times \Omega$, *together with the corresponding trajectory* $X_s^{0,x;u^*}$, *the relation*

$$u_s^* \in \arg\min_{\alpha \in U} \Big\{ c(s, X_s^{0,x;u^*}, \alpha) + \mathcal{L}^\alpha K(s, X_s^{0,x;u^*}) + g\big(t, [\mathcal{D}_x K \cdot \sigma^\alpha](t, X_s^{0,x;u^*})\big) \Big\}, \tag{3.59}$$

*then* $K(t,x) = V(t,x) = V^{u^*}(t,x)$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^n$.

*Proof.* Let us fix $(t,x) \in [0,T) \times \mathbb{R}^n$, a control function $u \in \mathcal{U}$, and consider the process $K(s, X_s^{t,x;u})$ for $s \in [t,T]$. We can evaluate the difference $K(T, X_T^{t,x;u}) - K(t,x)$ by Itô formula between $t$ and $T$:

$$K(T, X_T^{t,x;u}) - K(t,x) = \int_t^T [\mathcal{L}^{u_s} K](s, X_s^{t,x;u})\, ds + \int_t^T [\mathcal{D}_x K \cdot \sigma^{u_s}](s, X_s^{t,x;u})\, dW_s.$$

Equation (4.12) for $K$ yields the inequality:

$$[\mathcal{L}^{u_s} K](s, X_s^{t,x;u}) + c(t, X_s^{t,x;u}, u_s) + g\big(t, [\mathcal{D}_x K \cdot \sigma^{u_s}](t, X_s^{t,x;u})\big) \geq 0.$$

Combining the last two relations we conclude that

$$\begin{aligned} K(t,x) &\leq \Phi(T, X_T^{t,x;u}) \\ &+ \int_t^T \big[ c(t, X_s^{t,x;u}, u_s) + g\big(s, [\mathcal{D}_x K \cdot \sigma^{u_s}](s, X_s^{t,x;u})\big)\big]\, ds - \int_t^T [\mathcal{D}_x K \cdot \sigma^{u_s}](s, X_s^{t,x;u})\, dW_s. \end{aligned} \tag{3.60}$$

Define $Z_s^{t,x;u} = [\mathcal{D}_x K \cdot \sigma^{u_s}](s, X_s^{t,x;u})$, for $s \in [t,T]$. It follows that $K(t,x) \leq Y_t^{t,x;u}$, where $(Y^{t,x;u}, Z^{t,x;u})$ is the solution of the backward equation (3.28). Consequently, $K(t,x) \leq V^u(t,x)$.

If a control process $u^*$ satisfying (3.59) exists, then inequality (3.60) becomes an equation for $u = u^*$. Moreover, $u^*$ is an element of $\mathcal{U}$, because the process $X_s^{t,x;u^*}$ is progressively measurable. Thus, $K(t,x) = V^{u^*}(t,x) = \min_{u \in \mathcal{U}} V^u(t,x) = V(t,x)$. □

# Chapter 4

# Approximation by Piecewise Constant Control

In the previous chapter, we analyzed the optimal control problem under risk aversion,

$$\min_{u(\cdot)\in\mathcal{U}} \rho^g_{0,T}\Big[\int_0^T c(s,X_s,u_s)ds + \Phi(X_T)\Big]$$
$$s.t.\ dX_s = b(s,X_s,u_s)ds + \sigma(s,X_s,u_s)dW_s,\ \ X_0 = x,\ \ s\in[0,T]$$

The corresponding dynamic programming equation and Hamilton-Jacobi-Bellman equation are derived (see (3.42) and (4.12)). Now, we are in a position to solve the problem of finding the optimal value function and the optimal control. There are two main approximation approaches to the numerical solution of the problem : one is to attack the partial differential equation using some numerical scheme, such as a finite difference method or finite element method; the other is to use the probabilistic approach, namely, to discretize the dynamic programming equation to get a control problem on discrete-time and continuous state Markov chain. The recent references for these considerations are [29, 43, 9, 41]. We will focus on the latter one.

## 4.1 Collapse of Approximation by Regularization (Mollification)

An interesting and remarkable method to transit to the discrete time setting in expectation case is N. Krylov's approximation by perturbing the coefficients and adopting integral regularization (see [41]). He constructed a family of perturbed systems with two types of perturbations: of the initial time and the initial state. For such a family, he integrated the value functions of a piecewise constant control with respect to the said initial time and state values. This yields regularized functions for which Itô calculus can be applied, then by using Hamilton-Jacobi-Bellman equation, he established an error bound of order $h^{\frac{1}{6}}$, between the optimal values of the original system and a system with piecewise constant controls with time step of size $h^2$.

By mimicking his scheme, we can define the corresponding value function $V_h : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ as follows:

$$V_h(t,x) = \inf_{u(\cdot)\in\mathcal{U}_h^t} V^u(t,x). \tag{4.1}$$

where $\mathcal{U}_h$ is collection of piecewise constant control based on discretization $h^2$. Also, we perturb both the forward process and the backward process, i.e., let $B = \{(\tau,\zeta) \in \mathbb{R}\times\mathbb{R}^n : \tau \in (-1,0),\ |\zeta| < 1\}$,

$$d\widetilde{X}_s = b(s+\varepsilon^2\tau_i, \widetilde{X}_s + \varepsilon\zeta_i, \alpha_i)\, ds + \sigma(s+\varepsilon^2\tau_i, \widetilde{X}_s + \varepsilon\zeta_i, \alpha_i)\, d\widetilde{W}_s, \tag{4.2}$$

$$d\widetilde{Y}_s = [c(s+\varepsilon^2\tau_i, \widetilde{X}_s + \varepsilon\zeta_i, \alpha_i) + g(s+\varepsilon^2\tau_i, \tilde{Z}_s)]\, ds - \tilde{Z}_s\, d\widetilde{W}_s, \tag{4.3}$$

$$s \in [t_i, t_{i+1}), \quad i = 0,1,\ldots,k,$$

with a fixed $\varepsilon > 0$, with the initial condition $\widetilde{X}_t = x$, and with the final condition $\widetilde{Y}_T = \Phi(\widetilde{X}_T)$. Similarly, the value function corresponding to (4.2)-(4.3) can be defined as $\widetilde{V}_{h,\epsilon}(t_i, x_i)$, which satisfies the following dynamic programming equation:

$$\begin{aligned}\widetilde{V}_{h,\varepsilon}(t,x) = \inf_{\alpha_i\in U}\inf_{\beta_i\in B}\rho^g_{t+\varepsilon^2\tau, t+h^2+\varepsilon^2\tau}\Big[\int_{t+\varepsilon^2\tau}^{t+h^2+\varepsilon^2\tau} c(s, X_s^{t+\varepsilon^2\tau, x+\varepsilon\zeta;\alpha_i}, \alpha_i)\, ds \\ + \widetilde{V}_{h,\varepsilon}(t+h^2, X_{t+h^2+\varepsilon^2\tau}^{t+\varepsilon^2\tau, x+\varepsilon\zeta;\alpha_i} - \varepsilon\zeta)\Big].\end{aligned} \tag{4.4}$$

The following estimates follow from [41] and standard estimates in decoupled FBSDE system,

(i) For $t \in [0,T]$ and $x \in \mathbb{R}^d$, we have

$$\left|\widetilde{V}_{h,\varepsilon}(t,x) - V_h(t,x)\right| \le Ne^{NT}\varepsilon, \tag{4.5}$$

(ii) For $t, r \in [0,T]$, and $x, y \in \mathbb{R}^n$, we have

$$\left|\widetilde{V}_{h,\varepsilon}(t,x) - \widetilde{V}_{h,\varepsilon}(r,y)\right| \le Ne^{NT}(|x-y| + |t-r|^{\frac{1}{2}}). \tag{4.6}$$

It implies the closeness of $V_h$ to $\widetilde{V}_{h,\epsilon}$ and shows regularity of $\widetilde{V}_{h,\epsilon}$ as expected.

The integral regularization technique brings in a smooth value function based on the following mollification: take a non-negative function $\varphi \in C^\infty(B)$ with $\int_B \varphi(\tau,\zeta)\, d\tau\, d\zeta = 1$, for $\varepsilon > 0$, re-scale the mollifier as $\varphi_\varepsilon(\tau,\zeta) = \varepsilon^{-n-2}\varphi(\tau/\varepsilon^2, \zeta/\varepsilon)$, and define the regularized value function by convolution

$$\widehat{V}_{h,\varepsilon}(t,x) = [\widetilde{V}_{h,\varepsilon} \star \varphi_\varepsilon](t,x) = \int_B \widetilde{V}_{h,\varepsilon}(t-\varepsilon^2\tau, x-\varepsilon\zeta)\varphi(\tau,\zeta)\, d\tau\, d\zeta,$$

where $t \in [0, T-\varepsilon^2]$ and $x \in \mathbb{R}^n$. It can be shown that $\hat{V}_{h,\epsilon}$ is bounded in terms of semi-norm $\|\cdot\|_{2,1}$, more importantly, the mollification yields negligible error. Namely, for $\varepsilon \geq h$,

$$\left\|\widehat{V}_{h,\varepsilon}\right\|_{2,1} \leq Ne^{NT}\varepsilon^{-2}, \quad \left|\widehat{V}_{h,\varepsilon} - \widetilde{V}_{h,\varepsilon}\right|_0 \leq Ne^{NT}\varepsilon. \tag{4.7}$$

We discovered an interesting result for smooth value function of risk-averse control problem that connects dynamic programming equation and Hamilton-Jacobi-Bellman PDE,

**Lemma 4.1.1.** *For any $w \in C_b^{1,2}([t,T]\times\mathbb{R}^n)$, any $0 \leq t \leq \theta \leq T$, and all $u(\cdot) \in \mathcal{U}$ we have:*

$$w(t,x) = \rho^g_{t,\theta}\Big[\int_t^\theta c(s, X_s^{t,x;u}, u_s)\, ds + w(\theta, X_\theta^{t,x;u}) - \bar{\zeta}\Big], \tag{4.8}$$

*where*

$$\bar{\zeta} = \int_t^\theta \Big\{[c^{u_s} + \mathcal{L}^{u_s} w](s, X_s^{t,x;u}) + g\big(s, [\mathcal{D}_x w \cdot \sigma^{u_s}](s, X_s^{t,x;u})\big)\Big\}\, ds. \tag{4.9}$$

We defer the proof to the next section, where a valid approximation scheme is provided. Obviously, the mollified value function $\hat{V}_{h,\epsilon}$ is qualified to Lemma 4.2.4; as a result, if we consider $t \in [0, T-h^2-\varepsilon^2]$, for all $u(\cdot) \in \mathcal{U}$ on $[t, T-h^2-\varepsilon^2]$,

$$\widehat{V}_{h,\varepsilon}(t,x) \;\leq\; \rho^g_{t,T-h^2-\varepsilon^2}\left(\int_t^{T-h^2-\varepsilon^2} c(s, X_s^{t,x,u}, u_s)\, ds + \widehat{V}_{h,\varepsilon}(T-h^2-\varepsilon^2, X^{t,x,u}_{T-h^2-\varepsilon^2}) - \bar{\zeta}\right),$$

with

$$\begin{aligned}\bar{\zeta} &= \int_t^{T-h^2-\varepsilon^2} \Big([c^{u_s} + \mathcal{L}^{u_s}\widehat{V}_{h,\varepsilon}](s,x) + g(s, [\partial_x \widehat{V}_{h,\varepsilon}\cdot\sigma^{u_s}](s,x))\Big)\, ds \\ &\geq -NTe^{NT}\left(\varepsilon + \frac{h}{\varepsilon^2}\right).\end{aligned}$$

If we can obtain $\bar{\zeta} \geq -O(h)$, then by the estimates (4.5)–(4.7) and monotonicity of dynamic risk measure,

$$V_h(t,x) \leq \rho^g_{t,T-h^2-\varepsilon^2}\Big(\int_t^{T-h^2-\varepsilon^2} c(s, X_s^{t,x,u}, u_s)\, ds + V(T-h^2-\varepsilon^2, X^{t,x,u}_{T-h^2-\varepsilon^2})\Big) + O(h^\alpha). \tag{4.10}$$

where $\alpha \in (0,1]$ depending on the choice of $\epsilon$ as a function of $h$.

**Remark 4.1.2.** *It is rather simple to prove on $[T-h^2-\epsilon, T]$, $V^h$ and $V$ are close (via terminal condition).*

By applying dynamic programming equation (3.50) to the right hand side of (4.10), we conclude

$$V_h(t,x) \le \inf_{u(\cdot)\in\mathcal{U}} \rho^g_{t,T-h^2-\varepsilon^2}\Big( \int_t^{T-h^2-\varepsilon^2} c(s, X^{t,x,u}_s, u_s)\, ds \\ + V(T-h^2-\varepsilon^2, X^{t,x,u}_{T-h^2-\varepsilon^2})\Big) + Ne^{NT} h^{\frac{1}{3}} = V(t,x) + Ne^{NT} h^{\frac{1}{3}},$$

as desired.

Unfortunately, $\bar{\zeta}$ can not be bounded below in order of $h$. Specifically, the brilliant idea of N. Krylov is to mollify the whole dynamic programming equation in expectation case, which generates no error because expectation is of linear nature. However, our dynamic risk measure is a convex operator, when being regularized, we can only show

$$\widehat{V}_{h,\varepsilon}(t,x) \le \rho^g_{t,t+h^2}\left[ \int_t^{t+h^2} c(s, X^{t,x;u}_s, \alpha)\, ds + \widehat{V}_{h,\varepsilon}(t+h^2, X^{t,x;\alpha}_{t+h^2}) \right] + O(h). \tag{4.11}$$

After Itô's formulas and other manipulations, it can only end up with $\widetilde{\zeta} = O(1)$. The most fundamental issue is the Brownian motion driving the density process when using the dual representation of dynamic risk measure. Even on a short interval, the error accumulates by the same order as the Brownian motion.

## 4.2 Approximation by $\epsilon$-optimal control

### 4.2.1 Regularity of Risk-averse HJB

Recall the Hamilton-Jacobi-Bellman equation associated with our risk-averse control problem,

$$\min_{u\in U} \Big\{ c(t,x,u) + [\mathcal{L}^u V](t,x) + g\big(t, [\mathcal{D}_x V \cdot \sigma^u](t,x)\big) \Big\} = 0, \tag{4.12}$$

with the boundary condition

$$V(T,x) = \Phi(x), \quad x \in \mathbb{R}^n. \tag{4.13}$$

If we make a slightly stronger assumptions on the coefficients:

**Assumption 4.2.1.** (i) $|b(\cdot,0)| + |\sigma(\cdot,0)|$ *is bounded;*

(ii) *The functions* $b, \sigma \in C_{\mathrm{b}}^1([0,T] \times \mathbb{R}^n)$ *and, for some constant* $C > 0$,

$$|b(t,x_1) - b(t,x_2)| + |\sigma(t,x_1) - \sigma(t,x_2)| \leq C|x_1 - x_2| \quad a.s.$$

$$|b(t,x_1)| + |\sigma(t,x_1)| \leq C|x_1|, \quad a.s..$$

(iii) *The dimension of Brownian motion and the state process coincide, i.e.,* $n = d$, *and*

$$\sigma(t,x)\sigma^\top(t,x) \geq \frac{1}{C}\mathbb{I}, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d.$$

(iv) $g(\cdot, z), \Phi(x) \in C_L(\mathbb{R}^n)$*, where we use a constant C to denote all the Lipschitz constants, and*

$$\sup_{0 \leq t \leq T} |g(t,0)| + |\Phi(0)| \leq C \tag{4.14}$$

**Theorem 4.2.2.** *(Zhang [76], Chap 2, Thm 2.4.1) Under Assumption 5.3.1 and compactness of U,* (4.12)-(4.13) *has a unique classical solution* $V \in C_b^{1,2}([0,T] \times \mathbb{R}^n)$.

In following subsection, we are going to exploit risk-averse Hamilton-Jacobi-Bellman equation (4.12)–(4.13) to design a convergent numerical scheme. And we shall observe, the regularity of solution to the PDE is the cornerstone.

### 4.2.2 Existence of $\epsilon$ – optimal Control

As before, our intention is again to design a piecewise-constant control which facilitates the discrete-time approximation. As suggested by risk-averse dynamic programming equation, for any control $u(\cdot) \in \mathcal{U}$,

$$V(t,x) \leq \rho_{t,\theta}\Big[\int_t^\theta c(s, X_s^{t,x;u}, u_s)\, ds + V(\theta, X_\theta^{t,x;u})\Big],\ 0 \leq t \leq \theta \leq T, \tag{4.15}$$

If we can restrict the class of control to be piecewise-constant, i.e., $u_h \in \mathcal{U}_h$, while being able to control the distance between value function (left hand side of (4.15)) and the term appearing on the right hand side of (4.15) by appropriately choosing $u_h(\cdot)$, then as we enforce the distance to go to 0, a desirable control can be obtained.

This motivates the definition of $\epsilon$-optimal control below.

**Definition 4.2.1.** *If for each $\delta > 0$, there exists a policy $u(\cdot) \in \mathcal{U}$ such that*

$$V(t,x) + \delta \geq \rho_{t,\theta}\Big[\int_t^\theta c(s, X_s^{t,x;u}, u_s)\,ds + V(\theta, X_\theta^{t,x;u})\Big],\ 0 \leq t \leq \theta \leq T, \tag{4.16}$$

*then we call $u(\cdot)$ a $\delta$ – optimal control under risk aversion.*

For piecewise constant control, we also need to introduce the following discretization: partition $[t,T]$ into $M$ equal size subintervals, $I_i = [s_i, s_{i+1})$, for $i = 1, ..., M-1$, *i.e.*, $t = s_0 < s_1 < \cdots < s_M = T$ with $s_{i+1} - s_i = (T-t)/M$. We can further narrow the family of piecewise constant control to be Markovian.

**Definition 4.2.2.** *Let $u = (\bar{u}_1, ..., \bar{u}_{M-1})$, $\bar{u}$ is called a* discrete Markov policy *if $u \in \mathcal{U}$ defines a solution $X$ to controlled diffusion process such that*

$$u_s = \bar{u}_i(X_{s_i}),\ s \in I_i,\ i = 1, ..., M-1. \tag{4.17}$$

Clearly, our intention will be to construct a Markov $\epsilon$–optimal control based on the above discretization. Namely, the following theorem has to be proved

**Theorem 4.2.3.** *Given $V(t,x)$ is the solution to* (3.50)*, under Assumption 3.1.1 , for any $\delta > 0$, a Markovian $\epsilon$-optimal control exists, i.e.,* (4.16) *is valid.*

The proof of the main theorem above relies on Lemma 4.2.4 appearing in Subsection 4.1. Let's restate and give the proof:

**Lemma 4.2.4.** *For any $w \in C_b^{1,2}([t,T]\times\mathbb{R}^n)$, any $0 \leq t \leq \theta \leq T$, and all $u(\cdot) \in \mathcal{U}$ we have:*

$$w(t,x) = \rho^g_{t,\theta}\Big[\int_t^\theta c(s, X_s^{t,x;u}, u_s)\,ds + w(\theta, X_\theta^{t,x;u}) - \bar{\zeta}\Big], \tag{4.18}$$

*where*

$$\bar{\zeta} = \int_t^\theta \Big\{[c^{u_s} + \mathcal{L}^{u_s} w](s, X_s^{t,x;u}) + g\big(s, [\mathcal{D}_x w \cdot \sigma^{u_s}](s, X_s^{t,x;u})\big)\Big\}\,ds. \tag{4.19}$$

*Proof.* For any $u(\cdot) \in \mathcal{U}$, we apply Itô formula to $w(s, X_s^{t,x;u})$:

$$w(\theta, X_\theta^{t,x;u}) - w(t,x) - \int_t^\theta [\mathcal{L}^{u_s} w](s, X_s^{t,x;u})\,ds = \int_t^\theta [\mathcal{D}_x w \cdot \sigma^{u_s}](s, X_s^{t,x;u})\,dW_s. \tag{4.20}$$

Subtraction of $\int_t^\theta g(s, [\mathcal{D}_x w \cdot \sigma^{u_s}](s, X_s^{t,x;u}))\, ds$ from both sides and evaluation of the risk on both sides yields

$$
\begin{aligned}
&\rho_{t,\theta}^g\Big[w(\theta, X_\theta^{s,x;u}) - w(t,x) - \int_t^\theta [\mathcal{L}^{u_s} w](s, X_s^{t,x;u}) + g(s, [\mathcal{D}_x w \cdot \sigma^{u_s}](s, X_s^{t,x;u}))\, ds\Big] \\
&\qquad = \rho_{t,\theta}^g\Big[\int_t^\theta [\mathcal{D}_x w \cdot \sigma^{u_s}](s, X_s^{t,x,u})\, dW_s - \int_t^\theta g(s, [\mathcal{D}_x w \cdot \sigma^{u_s}](s, X_s^{t,x;u}))\, ds\Big].
\end{aligned} \tag{4.21}
$$

The risk measure on the right hand side of (4.21) is the solution of the following backward stochastic differential equation:

$$
\begin{aligned}
Y_t^{t,x;u} = &\int_t^\theta [\mathcal{D}_x w \cdot \sigma^{u_s}](s, X_s^{t,x;u})\, dW_s - \int_t^\theta g(s, [\mathcal{D}_x w \cdot \sigma^{u_s}](s, X_s^{t,x;u}))\, ds \\
&+ \int_t^\theta g(s, Z_s^{t,x;u})\, ds - \int_t^\theta Z_s^{t,x;u}\, dW_s.
\end{aligned} \tag{4.22}
$$

Substitution of $Z_s^{t,x;u} = [\mathcal{D}_x w \cdot \sigma^{u_s}](s, X_s^{t,x;u})$ yields $Y_t^{t,x;u} = 0$. By the uniqueness of the solution of BSDE, the right hand side of (4.21) is zero. Using the translation property on the left hand side of (4.21), we obtain

$$
w(t,x) = \rho_{t,\theta}^g\Big[-\int_t^\theta [\mathcal{L}^{u_s} w](s, X_s^{t,x;u}) + g(s, [\mathcal{D}_x w \cdot \sigma^{u_s}](s, X_s^{t,x;u}))\, ds + w(\theta, X_\theta^{t,x;u})\Big]. \tag{4.23}
$$

This is the same as (4.18). □

Now we can prove Theorem 4.2.3.

*Proof.* Fix $\alpha > 0$, define a compact cylinder $O_\alpha = [t, T) \times \{x \in \mathbb{R}^n : \|x\| < \alpha\}$. By Theorem 4.2.2, all partial derivatives, $\partial_t V, \partial_x V, \partial_{xx} V$, are uniformly continuous on $\bar{O}_\alpha$, where $\alpha > 0$. Fo any $\beta > 0$, there exists $\gamma > 0$ such that for $|s - s'| < \gamma$ and $|x - x'| < \gamma$,

$$
\Big| \mathcal{G}^u V(s,x) + f(s, x, [\partial_x V \cdot \sigma^u](s,x)) - \mathcal{G}^u V(s', x') - f(s', x', [\partial_x V \cdot \sigma^u](s', x') \Big| < \frac{\beta}{2}, \tag{4.24}
$$

where

$$
\begin{aligned}
\mathcal{G}^u V(s,x) &= \mathcal{L}^u V(s,x) - c^u(s,x), \\
f(s, x, [\partial_x V \sigma^u(s,x)) &= c^u(s,x) + g(s, [\partial_x V \cdot \sigma^u](s,x)),
\end{aligned} \tag{4.25}
$$

for all $(s,x) \in \bar{O}_\alpha$ and $u(\cdot) \in \mathcal{U}$.

We now establish a grid on $\bar{O}_\alpha$: let $\{x \in \mathbb{R}^n : \|x\| < \alpha\} = \cup_i B_i$, for $j = 1, ..., N$, be a disjoint partition with $\{B_j\}_{j=1}^N$ of diameter less than $\gamma/2$. Meanwhile, for the time axis, we make a partition such that the length of each subinterval $I_i = [s_i, s_{i+1}) < \min(\gamma, 1)$. Pick any $x_j \in B_j$ for some $j$, due to Hamilton-Jacobi-Bellman equation, there exists $u_{ij} \in U$ such that

$$\mathcal{G}^{u_{ij}} V(s_i, x_j) + f(s_i, x_j, [\partial_x V \cdot \sigma^{u_{ij}}](s_i, x_j)) < \frac{\beta}{2}. \tag{4.26}$$

It follows from continuity (4.24), for all $s \in I_i$ and $\|x - x_j\| < \gamma$,

$$\mathcal{G}^{u_{ij}} V(s, x) + f(s, x, [\partial_x V \cdot \sigma^{u_{ij}}](s, x)) < \beta. \tag{4.27}$$

The discrete time Markov control policy $u = (\bar{u}_1, ..., \bar{u}_{M-1})$ can be set as follows:

$$\bar{u}_i(x) = \begin{cases} u_{ij}, & \textit{if } x \in B_j, \\ u_0, & \textit{if } (s, x) \in ([t, T] \times \mathbb{R}^n) \setminus \bar{O}_\alpha. \end{cases} \tag{4.28}$$

where $u_0 \in U$ is arbitrary. We can define a control $u(\cdot) \in \mathcal{U}$ for which $X_s^{t,x;u}$ has a solution with $X_t^{t,x;u} = x$ such that

$$u_s = u_{ij}, \ s \in I_i, \ X_{s_i}^{t,x;u} \in B_j. \tag{4.29}$$

Set $\zeta = \int_t^\theta c(s, X_s^{t,x;u}, u_s)\, ds + V(\theta, X_\theta^{t,x;u})$. By Lemma 4.2.4, we have

$$V(t, x) = \rho_{t,\theta}^g(\zeta - \bar{\zeta}). \tag{4.30}$$

The difference between the left hand side and the right hand side corresponds to the solutions of the following BSDEs,

$$Y_t^1 = \zeta + \int_t^\theta g(s, Z_s^1)\, ds - \int_t^\theta Z_s^1 dW_s, \tag{4.31}$$

$$Y_t^2 = \zeta - \bar{\zeta} + \int_t^\theta g(s, Z_s^2)\, ds - \int_t^\theta Z_s^2 dW_s, \tag{4.32}$$

By standard estimates of BSDE,

$$\left\| Y_t^1 - Y_t^2 \right\|^2 \leq C \mathbb{E}_t [\, |\bar{\zeta}|^2 \,] \tag{4.33}$$

for some $C$ which depends only on the Lipschitz constant of the driver $g$.

Define the following event:

$$E = \left\{ \sup_{t \leq s \leq T} \|X_s^{t,x;u}\| \leq \alpha, |X_s^{t,x;u} - X_{s_i}^{t,x;u}| < \frac{\gamma}{2}, \ s \in I_i, \ i = 0, ..., M-1 \right\} \tag{4.34}$$

If $X_{s_i}^{t,x;u} \in B_j$ for some $j$, then $\|X_s^{t,x;u} - X_{s_i}^{t,x;u}\| < \frac{\gamma}{2}$ for every $x_j \in B_j$. Therefore, in the event $E$, we have $\|X_s^{t,x;u} - x_j\| < \gamma$. From (4.27), we obtain,

$$\mathcal{G}^{u_s} V(s, X_s^{t,x;u}) + f(s, [\partial_x V \cdot \sigma^{u_s}](s, X_s^{t,x;u})) < \beta. \tag{4.35}$$

From standard estimates for stochastic differential equations, there exists $C_1 > 0$ and $C_2 > 0$ such that

$$\mathbb{P}_t\Big[\max_{0\le i\le M-1} \max_{s\in I_i} \big\|X_s^{t,x;u} - X_{s_i}^{t,x;u}\big\| \ge \frac{\gamma}{2}\Big] \le \frac{C_1(s_{i+1}-s_i)^2}{\gamma^4} = \frac{C_1(T-t)}{M\gamma^4}, \tag{4.36}$$

and

$$\mathbb{P}_t\Big[\max_{t\le s\le T} \big\|X_s^{t,x;u}\big\| > \alpha\Big] \le \frac{C_2(1+\|x\|)}{\alpha}. \tag{4.37}$$

Therefore,

$$\mathbb{P}_t[E^C] \le \frac{C_1C_2(T-t)(1+\|x\|)}{M\gamma^4\alpha}. \tag{4.38}$$

Since $\theta \le T$,

$$\begin{aligned}
&\mathbb{E}_t\Big[ \Big( \int_t^\theta \mathcal{G}^{u_s} V(s, X_s^{t,x;u}) + f(s, [\partial_x V \cdot \sigma^{u_s}](s, X_s^{t,x;u}))ds\Big)^2 \Big] \\
&= \mathbb{P}_t[E]\mathbb{E}_t\Big[ \Big( \int_t^\theta \mathcal{G}^{u_s} V(s, X_s^{t,x;u}) + f(s, [\partial_x V \cdot \sigma^{u_s}](s, X_s^{t,x;u}))ds\Big)^2 \Big| E \Big] \\
&+ \mathbb{P}_t[E^C]\mathbb{E}_t\Big[ \Big( \int_t^\theta \mathcal{G}^{u_s} V(s, X_s^{t,x;u}) + f(s, [\partial_x V \cdot \sigma^{u_s}](s, X_s^{t,x;u}))ds\Big)^2 \Big| E^C \Big] \\
&\le \beta^2(T-t)^2 + \mathbb{E}_t[\bar{\zeta}^2]\frac{C_1C_2(T-t)(1+\|x\|)}{M\gamma^4\alpha}.
\end{aligned} \tag{4.39}$$

For $\alpha$ and $M$ large enough, the last term can be made smaller than $\delta^2/2C$, if $\beta(T-t) < \frac{\delta}{\sqrt{2C}}$. Then, we obtain from (4.33) the inequality,

$$\rho_{t,\theta}^g[\zeta] - \rho_{t,\theta}^g[\zeta - \bar{\zeta}] \le \delta, \tag{4.40}$$

which implies

$$V(t,x) = \rho_{t,\theta}^g[\zeta - \bar{\zeta}] \ge \rho_{t,\theta}^g[\zeta] - \delta. \tag{4.41}$$

As a result, the control $u(\cdot)$ constructed is a $\delta$ – optimal, as desired. □

# Chapter 5

# A Dual Method For Backward Stochastic Differential Equations with Application to Risk Valuation

## 5.1 Solving Optimal Control Problem in Discrete-Time

In chapter 4, we proposed an approximation of the risk-averse dynamic programming equation by piecewise constant Markov control. To arrive at a purely discrete-time system, it is not sufficient to only ”freeze” the control at the initial point (of a discretized interval) but also the time and state. Without loss of generality, let us focus on policy evaluation. For a finite horizon $\mathcal{T} = [0, T]$, we consider a forward diffusion process of Markovian type:

$$(5.1) \qquad dX_s = b(s, X_s)dt + \sigma(s, X_s)dW_s, \ X_t = x, \quad 0 \le t \le s \le T.$$

where Borel function $b : \mathcal{T} \times \mathbb{R}^n \mapsto \mathbb{R}^n$, $\sigma : \mathcal{T} \times \mathbb{R}^n \mapsto \mathbb{R}^{n\times d}$ are Lipschitz in state and satisfy linear growth condition. Our objective is to apply $g$-dynamic risk measure for the future cost at any time $t$ in $\mathcal{T}$. In particular, we are interested in, for any $(t, x) \in \mathcal{T} \times \mathbb{R}^n$,

$$(5.2) \qquad v(t, X_t) = \rho^g_{t,T}\Big[ \int_t^T c(s, X_s)ds + \Phi(X_T) \Big], \ \textit{where } X_s \textit{ is subject to } (5.1),$$

for terminal function $\Phi$ specified as before and cost function $c$ a measurable mapping from $\mathcal{T} \times \mathbb{R}^n$ to $\mathbb{R}$.

**Remark 5.1.1.** *To recover the controlled diffusion process and original system, one can assume the drift and diffusion functions are parameterized by control $u$ as well as the cost rate $c$. As a result, the value function above is a functional of control $u(\cdot)$, that can be optimized by choosing the best one.*

We partition time horizon $\mathcal{T}$ into $N$ sub-intervals with length $\Delta^N = T/N$ (for notation convenience, we will skip the superscript $N$ if there is no ambiguities), i.e., $t_k = k\Delta^N$ are grid points, for $k = 0, ..., N-1$. Given any initial $(t,x) \in \mathcal{T} \times \mathbb{R}^n$, $\{\widetilde{X}_s\}_{s\in\mathcal{T}}$ proposed below is a continuous approximation of of (5.1),

$$\widetilde{X}_r^{t,x} := x + \int_t^r b\Big(\kappa_t^\Delta(s), \widetilde{X}^{t,x}_{\kappa_t^\Delta(s)}\Big)\, ds + \int_t^r \sigma\Big(\kappa_t^\Delta(s), \widetilde{X}^{t,x}_{\kappa_t^\Delta(s)}\Big)\, dW_s, \quad r \in [t,T]. \tag{5.3}$$

where $\kappa_t^\Delta(s) = t + \Delta[\frac{s-t}{\Delta}]$. We define the value functions corresponding to the $g$-dynamic risk measure applied on the original underlying process (5.1) and approximation process (5.3), respectively,

$$v(t,x) := \rho^g_{t,T}\Big[\int_t^T c(s, X_s)\, ds + \Phi(X_r^{t,x})\Big], \tag{5.4}$$

$$v^\Delta(t,x) := \rho^g_{t,T}\Big[\int_t^{\kappa_t^\Delta(T)} c(\kappa_t^\Delta(s), \widetilde{X}^{t,x}_{\kappa_t^\Delta(s)})\, ds + \Phi(\widetilde{X}^{t,x}_{\kappa^\Delta(T)})\Big]. \tag{5.5}$$

The following theorem shows the closeness between (5.4) and (5.5):

**Theorem 5.1.2.** *For any* $(t,x) \in \mathcal{T} \times \mathbb{R}^n$,

$$\Big| v(t,x) - v^\Delta(t,x) \Big| = O(\Delta^{\frac{1}{2}}). \tag{5.6}$$

*Proof.* Let's set, for $v \in [t,T]$,

$$\bar{b}(v,\eta) := b\Big(\kappa_t^\Delta(v), \eta + \widetilde{X}^{t,x}_{\kappa_t^\Delta(v)} - \widetilde{X}^{t,x}_v\Big),$$
$$\bar{\sigma}(v,\eta) := \sigma\Big(\kappa_t^\Delta(v), \eta + \widetilde{X}^{t,x}_{\kappa_t^\Delta(v)} - \widetilde{X}^{t,x}_v\Big).$$

Equation (5.3) can be rewritten as:

$$\widetilde{X}_r^{t,x} = x + \int_t^r \bar{b}\,(s, \widetilde{X}_s^{t,x})\, ds + \int_t^r \bar{\sigma}\,(s, \widetilde{X}_s^{t,x})\, dW_s.$$

The following estimates is well known (see, for example, chap.2, [77]):

$$\mathbb{E}\Big[\sup_{t\le r\le T} \Big| X_r^{t,x} - \widetilde{X}_r^{t,x} \Big|^2\Big] = O(\Delta). \tag{5.7}$$

Since $V^\Delta(t,x)$ and $V(t,x)$ are the solutions of BSDE (5.8) and (5.9) below, respectively,

$$y_t^1 = \Phi(\widetilde{X}_r^{t,x}) + \int_t^r \Big(g(s, z_s^1) + c(\kappa_t^\Delta(s), \widetilde{X}^{t,x}_{\kappa_t^\Delta(s)})\Big)\, ds - \int_t^r z_s^1\, dW_s, \tag{5.8}$$

$$(5.9)\qquad y_t^2 = \Phi(X_r^{t,x}) + \int_t^r \Big( g(s, z_s^2)ds + c(s, X_s^{t,x}) \Big) ds - \int_t^r z_s^2\, dW_s,$$

standard estimates from BSDE implies (see, chap. 3, [59])

$$(5.10)\qquad \Big|\, y_t^1 - y_t^2 \,\Big|^2 = \mathcal{O}(\Delta).$$

The proof is completed. □

The above theorem leads us to the recursion form:

$$(5.11)\qquad \begin{aligned} v^k(\widetilde{X}_k) := v^\Delta(k\Delta, \widetilde{X}_{k\Delta}) &= \rho^g_{k,k+1}\Big[ c(k\Delta, \widetilde{X}_{k\Delta})\Delta + v^{k+1}(\widetilde{X}_{(k+1)\Delta}) \Big] \\ &= c\,(\,k\Delta, \widetilde{X}_{k\Delta}\,)\,\Delta + \rho^g_{k,k+1}\,[\, v^{k+1}(\widetilde{X}_{(k+1)\Delta})\,] \end{aligned}$$

where $\rho^g_{k,k+1}[\cdot] := \rho^g_{k\Delta,(k+1)\Delta}[\cdot]$ for $k = 1, ..., N-1$. If $\rho^g_{k,k+1}[\cdot]$ is conditional expectation, i.e., $\mathbb{E}_{k\Delta}[\cdot]$, the evaluation is straightforward provided the distribution of the next step value function $v^{k+1}(\widetilde{X}_{(k+1)\Delta})$. However, due to the non-linearity of our evaluation, the risk evaluation present above requires solving FBSDEs system on a short interval, $[k\Delta, (k+1)\Delta]$. To be explicit, the following FBSDEs has to be solved,

$$(5.12)\qquad Y_t = Y_{t+\Delta} + \int_t^{t+\Delta} g(s, Z_s)\, ds - \int_t^{t+\Delta} Z_s\, dW_s, \quad \textit{for } t = k\Delta, \textit{ where } k \in \{0, ..., N-1\},$$

where $Y_{t+\Delta} = v^{k+1}(\widetilde{X}_{(k+1)\Delta})$. In this situation, solving (5.12) analytically is hopeless, and we need to resort to a numerical method. And the proposed numerical method should have the property that the errors accumulated on each interval will not explode in the end. On the other hand, as finding optimal control is the the target of the risk-averse control problem, the numerical algorithm for risk evaluation should be done in an efficient manner.

Our main effort in this chapter is to develop an algorithm for our risk evaluation which can be applied to more general backward stochastic differential equation with convex and positive-homogeneous driver. We, first, take advantage of dual representation of the backward stochastic differential equation to convert risk evaluation to a stochastic control problem where the control is the *Radon-Nikodym derivative process*. Then, by exploring optimality conditions, we show that piecewise constant density (control) provides a close approximation on the short interval. Last, the backward induction extends the approximation to a finite time horizon, while keeping the error of order higher than discretization step. A financial application in risk management is given at the end to present some numerical results and sensitivity analysis.

## 5.2 Deficiency of Euler's Method for Risk Evaluation

Before jumping into our approximation scheme of risk evaluation, we review one of widely used numerical solution to backward stochastic differential equation, which is proposed by Zhang [76], among others [49, 31, 20, 10]. We raise the issue that such approximation scheme does not preserve monotonicity and time consistency of risk evaluation on discretized interval.

Consider the interval $[t, t+\Delta]$. Given $X_t = x$, the discretized SDE has the form:

$$\widetilde{X}^{t,x}_{t+\Delta}(t,x) = x + b(t,x)\,\Delta + \sigma(t,x)\,\Delta W, \;\; \textit{where}\; \Delta W = W_{t+\Delta} - W_t. \tag{5.13}$$

Since $b$, $\sigma$ and the terminal function $v^{\Delta}(t+\Delta,\cdot)$ are known, we can re-express the random variable that is going to be evaluated by the risk measure as a functional of the initial state $x$ and $\mathcal{F}_T$-measurable random variable $\Delta W$,

$$\xi^{\Delta,x}(\Delta W) := v\,(t+\Delta, \widetilde{X}^{t,x}_{t+\Delta}) \tag{5.14}$$

which is *Lipschitz* in $\Delta W$. We apply *Euler's method* to BSDE associated with g-evaluation. Following [76], the numerical recipe reads:

$$\begin{aligned} \widetilde{Y}_{t+\Delta} &= \xi^{\Delta,x}(\Delta W), \\ \widetilde{Z}_t &= \frac{1}{\Delta}\mathbb{E}_t[\widetilde{Y}_{t+\Delta}\Delta W], \\ Y_t &= \mathbb{E}_t[\widetilde{Y}_{t+\Delta}] + g\,(\,t,\,\widetilde{Z}_t\,)\,\Delta \end{aligned} \tag{5.15}$$

To explain, we first compute the terminal data of BSDE, $\widetilde{Y}_{t+\Delta}$ based on the specific functional form, then multiply it with increment of Brownian motion, after taking expectation and re-scaling, the approximated $Z$ is obtained; last, the current evaluation $\widetilde{Y}_t$ is a linear combination of conditional expected value of $\widetilde{Y}_{t+\Delta}$ and driver implemented on computed $\widetilde{Z}_t$.

For continuous approximation of $(\widetilde{Y}_\cdot, \widetilde{Z}_\cdot)$, define

$$\widetilde{Y}^{\Delta}_r := \widetilde{Y}_{t+\Delta} - g\,(\,t,\,\widetilde{Z}_t\,)(r-t) + \int_t^r \widetilde{Z}_s\,dW_s, \quad t < r \le t+\Delta \tag{5.16}$$

**Lemma 5.2.1.** *The following identity holds:*

$$\widetilde{Z}_t\Delta = \mathbb{E}_t\Big[\int_t^{t+\Delta} \widetilde{Z}_s\,ds\Big] \tag{5.17}$$

*Proof.* Since $\xi^{\Delta,x} \in L^2(\Omega, \mathcal{F}_{t+\Delta}, \mathbb{P})$, there is a sequence of $(\epsilon^k)_k$ of random variables in $\mathbb{D}^{1,2}$ converging to $\xi^{\Delta,x}$ in $L^2(\Omega, \mathcal{F}_{t+\Delta}, \mathbb{P})$.[1] Then, it follows from the *Clark-Ocone formula* that, for all $k$,

$$\epsilon^k = \mathbb{E}_t[\,\epsilon^k\,] + \int_t^{t+\Delta} \zeta_s^k \, dW_s, \quad \textit{where } \zeta_s^k := \mathbb{E}_s[\,\mathcal{D}_s \epsilon^k\,], \quad t \le s \le t+\Delta.$$

It can be computed that,

$$\begin{aligned} \widetilde{Z}_t \Delta &= \mathbb{E}_t[\,\xi^{\Delta,x} \Delta W\,] \\ &= \lim_{k\to\infty} \mathbb{E}_t\Big[\,\epsilon^k \Delta W\,\Big] \\ &= \lim_{k\to\infty} \mathbb{E}_t\Big[\int_t^{t+\pi} \mathcal{D}_s \epsilon^k ds\Big] \\ &= \lim_{k\to\infty} \mathbb{E}_t\Big[\int_t^{t+\Delta} \zeta_s^k \, ds\Big] \end{aligned} \tag{5.18}$$

by the *Malliavin integration by parts formula* and the tower property of conditional expectations. We then estimate,

$$\begin{aligned} \left|\mathbb{E}_t\Big[\int_t^{t+\pi} (\,\zeta_s^k - \widetilde{Z}_s\,)\, ds\Big]\right| &\le \left|\mathbb{E}_t\Big[\int_t^{t+\Delta} (\,\zeta_s^k - \widetilde{Z}_s)\Big]\right|^{\frac{1}{2}} \\ &= \left|\mathbb{E}_t\Big[\,\epsilon^k - \mathbb{E}_t[\,\epsilon^k\,] - (\,\xi^{\Delta,x} - \mathbb{E}_t[\,\xi^{\Delta,x}\,])\,\Big]^2\right|^{\frac{1}{2}} \\ &\le 2\left|\mathbb{E}_t\Big[\,\xi^{\Delta,x} - \epsilon^k\,\Big]^2\right|^{\frac{1}{2}} \end{aligned}$$

Since $\epsilon^k$ converges to $\xi^{\Delta,x}$ in $L^2(\Omega, \mathcal{F}_{t+\Delta}, \mathbb{P})$, the last inequality together with (5.18) provide the required result. □

With the help of lemma above, the following results can be proved (see Bouchard and Touzi [10]), which shows the algorithm (5.15) has error of order $\sqrt{\Delta}$.

**Lemma 5.2.2.** *Define*

$$\bar{Z}_t := \frac{1}{\Delta}\mathbb{E}_t\Big[\int_t^{t+\Delta} \widetilde{Z}_s \, ds\Big] \tag{5.19}$$

*Then,*

$$\limsup_{\Delta\to 0} \Delta^{-1}\Big\{\sup_{t\le s<t+\Delta} \mathbb{E}_t\Big[\big|Y_s - Y_t\big|^2\Big] + \mathbb{E}_t\Big[\int_t^{t+\Delta} \big|\widetilde{Z}_s - \bar{Z}_t\big|^2 \, ds\Big]\Big\} < +\infty \tag{5.20}$$

[1] $\mathbb{D}^{1,2}$ is the Hilbert space making $\epsilon$ and its *Malliavin derivative* bounded in square integrable sense(see formal definition in the last section of this chapter).

**Theorem 5.2.3.** *For* Euler method *with approximation solution* $(\widetilde{Y}_\cdot, \widetilde{Z}_\cdot)$ *and original solution* $(Y_\cdot, Z_\cdot)$,

$$\limsup_{\Delta\to 0}\left\{\sup_{t\le s\le t+\Delta}\mathbb{E}_t\left[\left|\widetilde{Y}_s - Y_s\right|^2\right] + \mathbb{E}_t\left[\int_t^{t+\Delta}\left|\widetilde{Z}_s - Z_s\right|^2 ds\right]\right\} \le K\Delta$$

Let us investigate the risk measure induced by *Euler's approximation* (5.15). For terminal value $\xi^{\Delta,x}(\Delta W)$, that is, we set

$$\widetilde{\rho}^g_{t,t+\Delta}\left[\xi^{\Delta,x}(\Delta W)\right] := \mathbb{E}_t\left[\xi^{\Delta,x}(\Delta W)\right] + g\left(\frac{1}{\Delta}\mathbb{E}_t\left[\xi^{\Delta,x}(\Delta W)\Delta W\right]\right)\Delta \tag{5.21}$$

It can be shown easily that $\widetilde{\rho}^g_{t,t+\pi}[\cdot]$ satisfies:

(i) **Normalization property:** $\xi \equiv 0$ implies $\widetilde{\rho}^g_{t,t+\Delta}[\xi] \equiv 0$;

(ii) **Translation invariance:** for any $\eta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})$,

$$\begin{aligned}\widetilde{\rho}^g_{t,t+\Delta}\left[\xi^{\Delta,x}(\Delta W) + \eta\right] &= \mathbb{E}_t\left[\xi^{\Delta,x}(\Delta W)\right] + \eta + g\left(\frac{1}{\Delta}\mathbb{E}_t\left[\xi^{\Delta,x}(\Delta W)\Delta W + \eta\,\Delta W\right]\right)\Delta \\ &= \mathbb{E}_t\left[\xi^{\Delta,x}(\Delta W)\right] + \eta + g\left(\frac{1}{\Delta}\mathbb{E}_t\left[\xi^{\Delta,x}(\Delta W)\Delta W\right]\right)\Delta \\ &= \widetilde{\rho}^g_{t,t+\Delta}\left[\xi^{\Delta,x}(\Delta W)\right] + \eta.\end{aligned} \tag{5.22}$$

(iii) **Convexity:** for any $\xi^{1,\Delta,x}(\Delta W), \xi^{2,\Delta,x}(\Delta W) \in L^2(\Omega, \mathcal{F}_{t+\Delta}, \mathbb{P})$ and $\lambda \in L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$, by convexity of driver $g$,

$$\begin{aligned}& g\left(t, \frac{1}{\Delta}\mathbb{E}_t\left[\lambda\,\xi^{1,\Delta,x}(\Delta W)\Delta W + (1-\lambda)\,\xi^{2,\Delta,x}(\Delta W)\Delta W\right]\right) \\ \le{}& \lambda\, g\left(t, \frac{1}{\Delta}\mathbb{E}_t\left[\xi^{1,\Delta,x}(\Delta W)\Delta W\right]\right) + (1-\lambda)\, g\left(\frac{1}{\Delta}\mathbb{E}_t\left[\xi^{2,\Delta,x}(\Delta W)\Delta W\right]\right)\end{aligned}$$

thus the convexity follows immediately.

However, **monotonicity** cannot hold in general because of the extra term – increment of the Brownian motion. Even worse, such approximating dynamic risk measure $\widetilde{\rho}^g_{t,t+\Delta}[\cdot]$ is not **time consistent**. Because of these, although the numerical scheme can work with controllable errors, the evaluation after discretization loses its interpretation as a risk measure, which may lead to instability and inefficiency. In the following, we develop an alternative numerical method to efficiently compute risk associated with the underlying dynamic. The approximating risk measure designed satisfies all properties of a dynamic risk measure.

## 5.3 Dual Method of Risk Evaluation

From now on, we shall focus on the new method to solve risk evaluation problems (in general, BSDE with convex and positive-homogeneous driver). The notation will be slightly different from the previous discussion, but should not affect reading.

### 5.3.1 Initial Set-up

Given a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by $d$-dimensional Brownian motion $\{W_t\}_{t\in[0,T]}$, we consider the following stochastic differential equation:

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t,\ \ X_0 = x, \quad t \in [0, T], \tag{5.23}$$

with measurable $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, and $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^d$. Also, the following assumptions about the drift and volatility terms are made.

**Assumption 5.3.1.** (i) $|b(\cdot, 0)| + |\sigma(\cdot, 0)|$ *are bounded;*

(ii) *The functions* $b, \sigma \in C^1_{\mathrm{b}}([0, T] \times \mathbb{R}^n)$*, the constant* $C > 0$ *denotes the Lipschitz constants*

$$|b(t, x_1) - b(t, x_2)| + |\sigma(t, x_1) - \sigma(t, x_2)| \leq C|x_1 - x_2| \quad a.s.$$

$$|b(t, x_1)| + |\sigma(t, x_1)| \leq C|x_1|, \quad a.s..$$

(iii) *The dimension of the Brownian motion and the state process coincide, i.e.,* $n = d$*, and*

$$\sigma(t, x)\sigma^\top(t, x) \geq \frac{1}{C}\mathbb{I}, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d.$$

Our intention is to evaluate risk of a terminal cost generated by the forward process (5.23):

$$\rho_{0,T}[\Phi(X_T)], \tag{5.24}$$

where $\Phi \in C_L(\mathbb{R}^n)$ is bounded, and $\{\rho_{s,t}\}_{0\leq s\leq t\leq T}$ is a dynamic risk measure consistent with the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. We refer the reader to [59] for a comprehensive discussion on risk measurement and filtration-consistent evaluations.

As discussed in Chapter 2, a special role in the dynamic risk theory is played by *g-evaluations* which are defined by one-dimensional backward stochastic differential equations of the following form:

$$-dY_t = g(t, Y_t, Z_t)\, dt - Z_t\, dW_t, \quad Y_T = \Phi(X_T),\ \ t \in [0, T], \tag{5.25}$$

with $\rho^g_{t,T}[\Phi(X_T)]$ defined to be equal to $Y_t$. The driver $g$ is jointly Lipschitz in $(y, z)$, and the process $g(\cdot, 0, 0)$ bounded.

The evaluation of risk is equivalent to the solution of a *decoupled* forward–backward system of stochastic differential equations (5.23)–(5.25). An important virtue of this system is its *Markov property*:

$$\rho_{t,T}[\Phi(X_T)] = v(t, X_t), \tag{5.26}$$

where $v : [0, T] \times \mathbb{R}^n \to \mathbb{R}$. We have

$$v(t, x) = \rho^x_{t,T}[\Phi(X^{t,x}_T)], \quad (t, x) \in [0, T] \times \mathbb{R}^n, \tag{5.27}$$

where $\{X^{t,x}_s\}$ is the solution of the system (5.23) restarted at time $t$ from state $x$:

$$dX^{t,x}_s = b(s, X^{t,x}_s)\, ds + \sigma(s, X^{t,x}_s)\, dW_s, \quad s \in [t, T], \quad X^{t,x}_t = x, \tag{5.28}$$

and $\rho^x_{t,T}[\Phi(X^{t,x}_T)]$ is the (deterministic) value of $Y^{t,x}_t$ in the backward equation (5.25) with terminal condition $\Phi(X^{t,x}_T)$.

Numerical methods for solving forward equations are very well understood (see, e.g., [39]). We focus, therefore, on the backward equation (5.25). So far, a limited number of results are available for this purpose. The most prominent is the Euler method with functional regression (see, e.g., [20, 10, 76]). Our intention is to show that for drivers satisfying additional coherence conditions, a much more effective method can be developed, which exploits time-consistency, duality theory for risk measures, and the maximum principle in stochastic control.

### 5.3.2 Stochastic Maximum Principle

In this section, we decipher the optimality conditions of the stochastic control problem (2.39)–(2.40). Since only the process $\{\Gamma_{t,s}\}_{s\in[t,T]}$ is controlled, the analysis is rather standard. For completeness, we repeat some important steps here.

Suppose $\hat{\mu}$ is the optimal control; then, for any $\mu \in \mathcal{A}$ and $0 \le \alpha \le 1$, we can form a perturbed control function

$$\mu^\alpha = \hat{\mu} + \alpha(\mu - \hat{\mu}).$$

It is still an element of $\mathcal{A}$, due to the convexity of the sets $A_s$. The processes $\hat{\Gamma}$, $\Gamma$ and $\Gamma^\alpha$ are the state processes under the controls $\hat{\mu}$, $\mu$, and $\mu_\alpha$, respectively.

We linearize the state equation (2.40) about $\hat{\Gamma}$ to get, for $s \in [t, T]$,

$$d\eta_s^\mu = [\hat{\mu}_s \eta_s^\mu + \hat{\Gamma}_{t,s}(\mu_s - \hat{\mu}_s)]\, dW_s, \quad \eta_t^\mu = 0. \tag{5.29}$$

It is evident that this equation has a unique strong solution. Denote

$$h_s^\alpha = \frac{1}{\alpha}[\Gamma_{t,s}^\alpha - \hat{\Gamma}_{t,s}] - \eta_s^\mu, \quad s \in [0, T].$$

The following result justifies the usefulness of the linearized equation (5.29).

**Lemma 5.3.2.**

$$\lim_{\alpha \to 0} \sup_{0 \le s \le T} \|h_s^\alpha\|^2 = 0. \tag{5.30}$$

*Proof.* We first prove that

$$\lim_{\alpha \to 0} \sup_{t \le s \le T} \|\Gamma_{t,s}^\alpha - \hat{\Gamma}_{t,s}\|^2 = 0. \tag{5.31}$$

We have

$$d(\Gamma_{t,s}^\alpha - \hat{\Gamma}_{t,s}) = (\mu_s^\alpha \Gamma_{t,s}^\alpha - \hat{\mu}_s \hat{\Gamma}_{t,s})\, dW_s = ((\mu_s^\alpha - \hat{\mu}_s)\hat{\Gamma}_{t,s} + \mu_s^\alpha(\Gamma_{t,s}^\alpha - \hat{\Gamma}_{t,s}))\, dW_s. \tag{5.32}$$

By Itô isometry,

$$\begin{aligned}
\|\Gamma_{t,r}^\alpha - \hat{\Gamma}_{t,r}\|^2 &= \int_t^r \|(\mu_s^\alpha - \hat{\mu}_s)\hat{\Gamma}_{t,s} + \mu_s^\alpha(\Gamma_{t,s}^\alpha - \hat{\Gamma}_{t,s})\|^2\, ds \\
&\le 2\int_t^r \|(\mu_s^\alpha - \hat{\mu}_s)\hat{\Gamma}_{t,s}\|^2\, ds + 2\int_t^r \|\mu_s^\alpha(\Gamma_{t,s}^\alpha - \hat{\Gamma}_{t,s})\|^2\, ds \\
&\le 2\int_t^r \|(\mu_s^\alpha - \hat{\mu}_s)\hat{\Gamma}_{t,s}\|^2\, ds + K\int_t^r \|\Gamma_{t,s}^\alpha - \hat{\Gamma}_{t,s}\|^2\, ds,
\end{aligned}$$

where $K$ is a constant. Since the first integral on the right hand side converges to 0, as $\alpha \to 0$, the Gronwall inequality yields (5.31).

We can now prove (5.30). Combining (5.32) and (5.29), we obtain the stochastic differential equation for $h^\alpha$:

$$\begin{aligned}
dh_s^\alpha &= \left\{\frac{1}{\alpha}[(\hat{\mu}_s + \alpha(\mu_s - \hat{\mu}_s))\Gamma_{t,s}^\alpha - \hat{\mu}_s\hat{\Gamma}_{t,s}] - \hat{\mu}_s\eta_s^\mu - \hat{\Gamma}_{t,s}(\mu_s - \hat{\mu}_s)\right\} dW_s \\
&= \left\{\frac{1}{\alpha}\hat{\mu}_s[\Gamma_{t,s}^\alpha - \hat{\Gamma}_{t,s}] + (\mu_s - \hat{\mu}_s)[\Gamma_{t,s}^\alpha - \hat{\Gamma}_{t,s}] - \hat{\mu}_s\eta_s^\mu\right\} dW_s \\
&= \left\{\hat{\mu}_s h_s^\alpha + (\mu_s - \hat{\mu}_s)[\Gamma_{t,s}^\alpha - \hat{\Gamma}_{t,s}]\right\} dW_s.
\end{aligned}$$

Since the processes $\{\hat{\mu}_s\}$ and $\{\mu_s\}$ are bounded, Itô isometry yields again

$$\|h_r^\alpha\|^2 \le K \int_t^r \|h_s^\alpha\|^2 \, ds + K \int_t^r \|\Gamma_{t,s}^\alpha - \hat{\Gamma}_{t,s}\|^2 \, ds,$$

where $K$ is constant. By the Gronwall inequality, using (5.31), we get the desired result. □

The convergence result above directly leads to the following variational inequality.

**Lemma 5.3.3.** *For any $\mu \in \mathcal{A}$ we have*

$$\mathbb{E}[\, \xi_T \eta_T^\mu \,] \le 0. \tag{5.33}$$

*Proof.* Since $\hat{\mu}$ is the optimal control,

$$\mathbb{E}[\, \xi_T ( \Gamma_{t,T}^\alpha - \hat{\Gamma}_{t,T} \,) \,] \le 0.$$

Lemma 5.3.2 leads to

$$\lim_{\alpha \to 0} \mathbb{E}\Big[ \xi_T \frac{1}{\alpha} (\Gamma_{t,T}^\alpha - \hat{\Gamma}_{t,T}) \Big] = \mathbb{E}[\xi_T \eta_T^\mu] \le 0,$$

as required. □

We now express the expected value in (5.33) as an integral, to obtain a pointwise variational inequality (the maximum principle). To this end, we introduce the following backward stochastic differential equation (the *adjoint equation*):

$$dp_s = -k_s \hat{\mu}_s \, ds + k_s \, dW_s, \;\; p_T = \xi_T, \;\; s \in [t, T], \tag{5.34}$$

with $\xi_T = \Phi(X_T^{t,x})$. By construction, $\mathbb{E}[\, \xi_T \eta_T^\mu \,] = \mathbb{E}[\, \hat{p}_T \eta_T^\mu \,]$. Applying the Itô formula to the product process $p_s \eta_s^\mu$, we obtain

$$d(p_s \eta_s^\mu) = \Big( k_s \eta_s^\mu + \hat{p}_s [\hat{\mu}_s \eta_s^\mu + \hat{\Gamma}_{t,s}(\mu_s - \hat{\mu}_s)] \Big) \, dW_s + k_s \hat{\Gamma}_{t,s}(\mu_s - \hat{\mu}_s) \, ds.$$

If follows that

$$\mathbb{E}[\xi_T \eta_T^\mu] = \mathbb{E}\Big[ \int_t^T k_s \hat{\Gamma}_{t,s}(\mu_s - \hat{\mu}_s) ds \Big]. \tag{5.35}$$

We can summarize our derivations in the following version of the maximum principle. We define the Hamiltonian $H : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$:

$$H(\gamma, \nu, \kappa) = k\gamma \cdot \nu.$$

**Theorem 5.3.4.** *For almost all $s \in [t, T]$, with probability 1,*

$$H(\hat{\Gamma}_{t,s}, \hat{\mu}_s, k_s) = \max_{\nu \in A_s} H(\hat{\Gamma}_{t,s}, \nu, k_s).$$

*Proof.* For any $\mu \in \mathcal{A}$, we define the set

$$\mathcal{G} = \{ (\omega, s) \in \Omega \times [t, T] : k_s \hat{\Gamma}_{t,s}(\mu_s - \hat{\mu}_s) > 0 \}.$$

We construct a new control $\mu^* \in \mathcal{A}$:

$$\mu^*_s = \begin{cases} \mu_s, & (\omega, s) \in \mathcal{G}, \\ \hat{\mu}_s, & \text{otherwise.} \end{cases}$$

The measurability and adaptedness of $\mu^*$ can be easily verified. It follows from (5.33) and (5.35) that

$$\mathbb{E}\Big[ \int_t^T k_s \hat{\Gamma}_{t,s}(\mu^*_s - \hat{\mu}_s) ds \Big] \le 0.$$

By the construction of $\mu^*$,

$$\iint_{\mathcal{G}} k_s \hat{\Gamma}_{t,s}(\mu^*_s - \hat{\mu}_s)\, ds\, \mathbb{P}(d\omega) \le 0.$$

Since the integrand is positive on $\mathcal{G}$, the product measure of $\mathcal{G}$ must be zero. □

### 5.3.3 Regularity of the Integrand in the Adjoint Equation

We also make a stronger assumption about the drift and diffusion terms of the forward system, and about the terminal cost function.

**Assumption 5.3.5.** *The functions $b, \sigma, \Phi \in C^2_b([0, T] \times \mathbb{R}^n)$, and*

$$|\sigma(s, x) - \sigma(t, x)| \le C|s - t|^{\frac{1}{2}}$$

*for all $s, t \in [0, T]$ and all $x \in \mathbb{R}^n$.*

Consider the forward–backward system (5.28) and (5.34). The key to our further estimates is the following regularity result about the integrand $\{k_t\}$ in the adjoint equation (5.34).

**Lemma 5.3.6.** *A constant $C$ exists, such that for all $0 \le t < s \le T$, and all $x \in \mathbb{R}^n$,*

$$\|k_s - k_t\| \le C|s - t|^{\frac{1}{2}}. \tag{5.36}$$

*Proof.* The quasilinear parabolic partial differential equation corresponding to the forward–backward system (5.28)–(5.34) has the following form (see, *e.g.* [50, sec. 8.2]),

$$u_t(t,x) + u_x(t,x)b(t,x) + \frac{1}{2}\mathrm{tr}\big(u_{xx}(t,x)\sigma(t,x)\sigma^T(t,x)\big) + u_x(t,x)\sigma(t,x)\hat{\mu}_t = 0, \tag{5.37}$$

with the boundary condition $u(T,x) \equiv \Phi(x)$. Due to the linearity of the driver of (5.34), the terms with $u_x$ can be collapsed. Then the equation (5.37) is the Feynman-Kac equation for

$$u(s,\tilde{X}_s^{t,x}) = \mathbb{E}\big[\Phi(\tilde{X}_T^{t,x})\big|\mathcal{F}_s\big], \quad s \in [t,T], \tag{5.38}$$

where

$$d\tilde{X}_s^{t,x} = [b(s,\tilde{X}_s^{t,x}) + \sigma(s,\tilde{X}_s^{t,x})\hat{\mu}_s]\,ds + \sigma(s,\tilde{X}_s^{t,x})\,dW_s, \quad s \in [t,T], \quad \tilde{X}_s^{t,x} = x.$$

Ma and Yong [50] consider it on page 195 in formula (1.12). Under Assumption 5.3.5, the equation (5.37) has a classical solution $u(\cdot,\cdot)$, and then the process

$$k_s = u_x(s,\tilde{X}_s^{t,x})\,\sigma(s,\tilde{X}_s^{t,x}), \quad s \in [t,T], \tag{5.39}$$

is the solution of the adjoint equation (5.34). By [50, Prop. 8.1.1], a process $H \in \mathcal{H}^{2,n\times n}[t,T]$ exists, such that the process $G_s = u_x(s,\tilde{X}_s^{t,x})$ satisfies the following $n$-dimensional BSDE:

$$G_t = \Phi_x(\tilde{X}_T^{t,x}) + \int_t^T \Big([b_x(s,\tilde{X}_s^{t,x}) + \sigma_x(s,\tilde{X}_s^{t,x})\hat{\mu}_s]G_s + \sigma_x(s,\tilde{X}_s^{t,x})H_s\Big)\,ds - \int_t^T H_s\,dW_s. \tag{5.40}$$

We obtain the following estimate:

$$\begin{aligned} \|k_s - k_t\| &= \|u_x(s,\tilde{X}_s^{t,x})\,\sigma(s,\tilde{X}_s^{t,x}) - u_x(t,x)\,\sigma(t,x)\| \\ &\le \|u_x(s,\tilde{X}_s^{t,x})\,\sigma(s,\tilde{X}_s^{t,x}) - u_x(s,\tilde{X}_s^{t,x})\,\sigma(t,x) + u_x(s,\tilde{X}_s^{t,x})\,\sigma(t,x) - u_x(t,x)\,\sigma(t,x)\| \\ &\le \|u_x(s,\tilde{X}_s^{t,x})\|\,\|\sigma(s,\tilde{X}_s^{t,x}) - \sigma(t,x)\| + \|\sigma(t,x)\|\,\|G_s - G_t\|. \end{aligned} \tag{5.41}$$

The first term on the right hand side of (5.41) can be bounded with the help of Assumption 5.3.5:

$$\begin{aligned} \|\sigma(s,\tilde{X}_s^{t,x}) - \sigma(t,x)\| &\le \|\sigma(s,\tilde{X}_s^{t,x}) - \sigma(s,x)\| + \|\sigma(s,x) - \sigma(t,x)\| \\ &\le C_1|s-t|^{\frac{1}{2}} + C_2\|\tilde{X}_s - x\| \le C_3|s-t|^{\frac{1}{2}}, \end{aligned}$$

where $C_1$, $C_2$, and $C_3$ are some universal constants. It follows from (5.40) that

$$G_r - G_t = -\int_t^r \Big([b_x(s,\tilde{X}_s) + \sigma_x(s,\tilde{X}_s)\hat{\mu}_s]G_s + \sigma_x(s,\tilde{X}_s)H_s\Big)\,ds + \int_t^r H_s\,dW_s.$$

Therefore, the second term on the right hand side of (5.41) can be bounded as $\|G_s - G_t\|^2 \leq C_4|s-t|$. Integrating these estimates into (5.41), we obtain (5.36) with a universal constant $C$. □

### 5.3.4 Error Estimates for Constant Controls on Small Intervals

To reduce an infinite dimensional control problem to a finite dimensional vector optimization, we partition the interval $[0,T]$ into $N$ short pieces of length $\Delta = T/N$, and develop a scheme for evaluating the risk measure (5.24) by using constant dual controls on each piece. We denote $t_i = i\Delta$, for $i = 0, 1, \ldots, N$.

For simplicity, in addition to Assumption 2.2.11, we assume that the driver $g$ does not depend on time, and thus all sets $A_t = \partial g(0)$ are the same. We denote them with the symbol $A$; as we shall see later on this is not a major restriction.

If the system's state at time $t_i$ is $x$, then the value of the risk measure (5.27) is then the optimal value of problem (2.39). By dynamic programming,

$$v(t_i, x) = \rho^x_{t_i,t_{i+1}}[v(t_{i+1}, X^{t_i,x}_{t_{i+1}})].$$

The risk measure $\rho^x_{t_i,t_{i+1}}[\,\cdot\,]$ is defined by problem (2.39), with terminal time $t_{i+1}$ and the function $\Phi(\cdot)$ replaced by $v(t_{i+1},\,\cdot\,)$. Equivalently, it is equal to $Y^{t_i,x}_{t_i}$, in the corresponding forward–backward system on the interval $[t_i, t_{i+1}]$:

$$dX^{t_i,x}_s = b(s, X^{t_i,x}_s)\,ds + \sigma(s, X^{t_i,x}_s)\,dW_s, \quad X^{t_i,x}_{t_i} = x, \tag{5.42}$$

$$-dY^{t_i,x}_s = g(Z^{t_i,x}_s)\,ds - Z^{t_i,x}_s\,dW_s, \quad Y^{t_i,x}_{t_{i+1}} = v(t_{i+1}, X^{t_i,x}_{t_{i+1}}). \tag{5.43}$$

Under Assumption 5.3.5, the function $v(\cdot,\cdot)$ is the classical solution of the associated Hamilton–Jacobi–Bellman equation:

$$v_t(t,x) + v_x(t,x)b(t,x) + \frac{1}{2}\operatorname{tr}\big(v_{xx}(t,x)\sigma(t,x)\sigma^T(t,x)\big) + g\big(v_x(t,x)\sigma(t,x)\big) = 0, \tag{5.44}$$

with the terminal condition $v(T,x) = \Phi(x)$.

Suppose we use a constant control in the interval $[t_i, t_{i+1}]$:

$$\mu_s := \hat{\mu}_{t_i} = \operatorname*{arg\,max}_{\nu\in A}\ k_{t_i}\nu, \ \ \forall s \in [t_i, t_{i+1}], \tag{5.45}$$

where $(p,k)$ solve the adjoint equation corresponding to (5.34):

$$dp_s = -k_s\hat{\mu}_s\,ds + k_s\,dW_s, \quad s \in [t_i, t_{i+1}], \quad p_{t_{i+1}} = v(t_{i+1}, X^{t_i,x}_{t_{i+1}}). \tag{5.46}$$

We still use $\hat{\Gamma}$ to denote the state evolution under the optimal control, while $\Gamma$ is the process under control $\mu$ defined in (5.45). It is well-known that the value function $v(\cdot,\cdot)$ of the system (5.42)–(5.43) is in $C^2_{\mathrm{b}}([0,T]\times\mathbb{R}^n)$; see, for example, [76, Thm. 2.4.1]. Therefore, the bounds developed in section 5.3.3 remain valid for the processes $(p,k)$ in (5.46).

Our objective is to show that a constant $C$ exists, independent of $x$, $N$, and $i$, such that the approximation error on the $i$th interval can be bounded as follows:

$$0 \le \mathbb{E}[v(t_{i+1}, X^{t_i,x}_{t_{i+1}})(\hat{\Gamma}_{t_i,t_{i+1}} - \Gamma_{t_i,t_{i+1}})] \le C\Delta^{\frac{3}{2}}. \tag{5.47}$$

The fact that we do not know $k_{t_i}$ will not be essential; later, we shall generate even better constant controls by discrete-time dynamic programming.

We can now derive some useful estimates for the constant control function (5.45).

**Lemma 5.3.7.** *A constant $C$ exists, such that for all $x$, $N$ and $i$*

$$\mathbb{E}\Big[v(t_{i+1}, X^{t_i,x}_{t_{i+1}})\int_{t_i}^{t_{i+1}}(\hat{\mu}_s - \mu_s)\hat{\Gamma}_{t_i,s}\,dW_s\Big] \le C\Delta^{\frac{3}{2}}. \tag{5.48}$$

*Proof.* From (5.46) we get:

$$v(t_{i+1}, X^{t_i,x}_{t_{i+1}}) = p_{t_i} - \int_{t_i}^{t_{i+1}} k_s\hat{\mu}_s\,ds + \int_{t_i}^{t_{i+1}} k_s\,dW_s. \tag{5.49}$$

Then the left hand side of (5.48) can be written as follows:

$$\begin{aligned}
&\mathbb{E}\Big[\Big(p_{t_i} - \int_{t_i}^{t_{i+1}} k_t\hat{\mu}_t\,dt + \int_{t_i}^{t_{i+1}} k_t\,dW_t\Big)\int_{t_i}^{t_{i+1}}(\hat{\mu}_s - \mu_s)\hat{\Gamma}_{t_i,s}\,dW_s\Big]\\
&\quad = -\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} k_t\hat{\mu}_t\,dt\int_{t_i}^{t_{i+1}}(\hat{\mu}_s - \mu_s)\hat{\Gamma}_{t_i,s}\,dW_s\Big]\\
&\qquad + \mathbb{E}\Big[\int_{t_i}^{t_{i+1}} k_t\,dW_t\int_{t_i}^{t_{i+1}}(\hat{\mu}_s - \mu_s)\hat{\Gamma}_{t_i,s}\,dW_s\Big].
\end{aligned} \tag{5.50}$$

The first term on the right hand side of (5.50) can be bounded by the Cauchy-Schwarz inequality and the Itô isometry:

$$\begin{aligned}
&-\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} k_t\hat{\mu}_t\,dt\int_{t_i}^{t_{i+1}}(\hat{\mu}_s - \mu_s)\hat{\Gamma}_{t_i,s}\,dW_s\Big]\\
&\le \Big(\mathbb{E}\Big[\Big(\int_{t_i}^{t_{i+1}} k_s\hat{\mu}_s\,ds\Big)^2\Big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}\Big[\Big(\int_{t_i}^{t_{i+1}}(\hat{\mu}_s - \mu_s)\hat{\Gamma}_{t_i,s}\,dW_s\Big)^2\Big]\Big)^{\frac{1}{2}}\\
&\le C_1\Delta\Big(\int_{t_i}^{t_{i+1}}\mathbb{E}\Big[(\hat{\mu}_s - \mu_s)^2\hat{\Gamma}^2_{t_i,s}\Big]\,ds\Big)^{\frac{1}{2}} \le C_1C_2\Delta^{\frac{3}{2}},
\end{aligned}$$

where $C_1$ and $C_2$ are some constants. The second term on the right hand side of (5.50) can be evaluated as follows:

$$
\begin{aligned}
&\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} k_t\, dW_t \int_{t_i}^{t_{i+1}} (\hat{\mu}_s - \mu_s)\hat{\Gamma}_{t_i,s}\, dW_s\Big] = \mathbb{E}\Big[\int_{t_i}^{t_{i+1}} k_s(\hat{\mu}_s - \mu_s)\hat{\Gamma}_{t_i,s}\, ds\Big]\\
&\quad = \mathbb{E}\Big[\int_{t_i}^{t_{i+1}} (k_s\hat{\mu}_s\hat{\Gamma}_{t_i,s} - k_{t_i}\hat{\mu}_{t_i})\, ds\Big] + \mathbb{E}\Big[\Big(\int_{t_i}^{t_{i+1}} (k_{t_i} - k_s\hat{\Gamma}_s)\hat{\mu}_{t_i}\, ds\Big]\\
&\quad = \mathbb{E}\Big[\Big(\int_{t_i}^{t_{i+1}} (\sigma_A(k_s\hat{\Gamma}_{t_i,s}) - \sigma_A(k_{t_i}))\, ds\Big] + \mathbb{E}\Big[\Big(\int_{t_i}^{t_{i+1}} (k_{t_i} - k_s\hat{\Gamma}_{t_i,s})\hat{\mu}_{t_i}\, ds\Big].
\end{aligned}
$$

Here, $\sigma_A(z) = \max_{\mu\in A}\langle z, \mu\rangle$ is the support function of the set $A$.

By Lemma 2.2.16, a constant $C_3$ exists, such that $\|\hat{\Gamma}_{t_i,s} - 1\|^2 \le C_3|s - t_i|$. Since the support function is Lipschitz continuous, we can write the following estimate (again, $C_4$ is a sufficiently large constant)

$$
\begin{aligned}
&\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} k_t\, dW_t \int_{t_i}^{t_{i+1}} (\hat{\mu}_s - \mu_s)\hat{\Gamma}_{t_i,s}\, dW_s\Big]\\
&\le C_4 \int_{t_i}^{t_{i+1}} \mathbb{E}[|k_s\hat{\Gamma}_{t_i,s} - k_{t_i}|]\, ds \le C_4 \int_{t_i}^{t_{i+1}} \mathbb{E}[|k_s(\hat{\Gamma}_{t_i,s} - 1)| + |k_s - k_{t_i}|]\, ds\\
&\le C_4\left(\int_{t_i}^{t_{i+1}} \|\hat{\Gamma}_{t_i,s} - 1\|^2\, ds\right)^{\frac{1}{2}}\left(\int_{t_i}^{t_{i+1}} \|k_s\|^2\, ds\right)^{\frac{1}{2}} + C_4\left(\int_{t_i}^{t_{i+1}} \|k_s - k_{t_i}\|^2\, ds\right)^{\frac{1}{2}}\\
&\le C\Delta^{\frac{3}{2}},
\end{aligned}
$$

where $C$ is a sufficiently large constant. In the last step we used Lemma 5.3.6. □

We also have the estimate below:

**Lemma 5.3.8.** *A constant $C$ exists, such that for all $x$, $N$, and $i$*

$$
\mathbb{E}\Big[v(t_{i+1}, X_{t_{i+1}}^{t_i,x}) \int_{t_i}^{t_{i+1}} (\hat{\Gamma}_{t_i,s} - \Gamma_{t_i,s})\mu_s\, dW_s\Big] \le C\Delta^{\frac{3}{2}}. \tag{5.51}
$$

*Proof.* We proceed as in the proof of the previous lemma. We use (5.49) and express the left hand side of (5.51) as follows:

$$
\begin{aligned}
&\mathbb{E}\Big[v(t_{i+1}, X_{t_{i+1}}^{t_i,x}) \int_{t_i}^{t_{i+1}} (\hat{\Gamma}_{t_i,s} - \Gamma_{t_i,s})\mu_s\, dW_s\Big]\\
&\quad = -\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} k_t\hat{\mu}_t\, dt \int_{t_i}^{t_{i+1}} (\hat{\Gamma}_{t_i,s} - \Gamma_{t_i,s})\mu_s\, dW_s\Big]\\
&\qquad + \mathbb{E}\Big[\int_{t_i}^{t_{i+1}} k_t\, dW_t \int_{t_i}^{t_{i+1}} (\hat{\Gamma}_{t_i,s} - \Gamma_{t_i,s})\mu_s\, dW_s\Big].
\end{aligned} \tag{5.52}
$$

The first term on the right hand side of (5.52) can be dealt with by the Cauchy-Schwarz inequality and Itô isometry, exactly as before:

$$\begin{aligned}&\left|\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} k_s\hat{\mu}_s\, ds \int_{t_i}^{t_{i+1}} (\hat{\Gamma}_{t_i,s} - \Gamma_{t_i,s})\mu_s\, dW_s\Big]\right| \\ &\leq \left(\mathbb{E}\Big[\Big(\int_{t_i}^{t_{i+1}} k_s\hat{\mu}_s\, ds\Big)^2\Big]\right)^{\frac{1}{2}} \left(\int_{t_i}^{t_{i+1}} \mathbb{E}[(\hat{\Gamma}_{t_i,s} - \Gamma_{t_i,s})^2|\mu_s|^2]\, ds\right)^{\frac{1}{2}} \leq C_1\Delta^{\frac{3}{2}}.\end{aligned}$$

To estimate the second term, consider two controlled state processes:

$$\begin{aligned}\hat{\Gamma}_{t_i,t} &= 1 + \int_{t_i}^{t} \hat{\mu}_s\hat{\Gamma}_{t_i,s}\, dW_s, \\ \Gamma_{t_i,t} &= 1 + \int_{t_i}^{t} \mu_s\Gamma_{t_i,s}\, dW_s.\end{aligned}$$

Taking the difference yields,

$$\hat{\Gamma}_{t_i,t} - \Gamma_{t_i,t} = \int_{t_i}^{t} (\hat{\mu}_s - \mu_s)\hat{\Gamma}_{t_i,s}\, dW_s + \int_{t_i}^{t} (\hat{\Gamma}_{t_i,s} - \Gamma_{t_i,s})\mu_s\, dW_s. \tag{5.53}$$

By Itô isometry,

$$\mathbb{E}[(\hat{\Gamma}_{t_i,t} - \Gamma_{t_i,t})^2] \leq C_2|t - t_i|.$$

Thus, we can write the bound:

$$\begin{aligned}&\left|\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} k_t\, dW_t \int_{t_i}^{t_{i+1}} (\hat{\Gamma}_{t_i,s} - \Gamma_{t_i,s})\mu_s\, dW_s\Big]\right| \\ &\quad = \left|\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} k_s(\hat{\Gamma}_{t_i,s} - \Gamma_{t_i,s})\mu_s\, ds\Big]\right| \\ &\quad \leq C_1\left(\int_{t_i}^{t_{i+1}} \mathbb{E}[|k_s|^2]\, ds\right)^{\frac{1}{2}} \left(\int_{t_i}^{t_{i+1}} \mathbb{E}[(\hat{\Gamma}_{t_i,s} - \Gamma_{t_i,s})^2]\, ds\right)^{\frac{1}{2}} \leq C\Delta^{\frac{3}{2}},\end{aligned}$$

where $C$ is a sufficiently large constant. □

We can now compare the value of the functional (2.39) with the value achieved by a constant control $\mu$.

**Theorem 5.3.9.** *Suppose Assumptions 5.3.1, 2.2.11, and 5.3.5 are satisfied. Then a constant $C$ exists, independent of $x$, $N$ and $i$, such that inequality* (5.47) *holds.*

*Proof.* Using (5.53), we obtain

$$\begin{aligned}\mathbb{E}[v(t_{i+1}, X_{t_{i+1}}^{t_i,x})(\hat{\Gamma}_{t_i,t_{i+1}} - \Gamma_{t_i,t_{i+1}})] \\ = \mathbb{E}\left[v(t_{i+1}, X_{t_{i+1}}^{t_i,x}) \int_{t_i}^{t_{i+1}} (\hat{\mu}_s - \mu_s)\hat{\Gamma}_{t_i,s}\, dW_s\right] \\ + \mathbb{E}\left[v(t_{i+1}, X_{t_{i+1}}^{t_i,x}) \int_{t_i}^{t_{i+1}} (\hat{\Gamma}_{t_i,s} - \Gamma_{t_i,s})\mu_s\, dW_s\right].\end{aligned}$$

Combining the estimates from Lemmas 5.3.7 and 5.3.8, we obtain the postulated result. □

An even smaller error than (5.47) can be achieved by choosing the *best* constant control in the interval $[t_i, t_{i+1}]$. For a constant $\mu_t \equiv v$, where $v \in A$, the dual state equation (2.40) has a closed-form solution, the exponential martingale:

$$\Gamma_{t_i,t} = \exp\Big(v(W_t - W_{t_i}) - \frac{t - t_i}{2}|v|^2\Big).$$

It follows that an $O(\Delta^{\frac{3}{2}})$ approximation of the risk measure can be obtained by solving the following simple vector optimization problem:

$$\tilde{\rho}^x_{t_i,t_{i+1}}[v(t_{i+1}, X_{t_{i+1}}^{t_i,x})] := \max_{v\in A} \mathbb{E}\Big[v(t_{i+1}, X_{t_{i+1}}^{t_i,x}) \exp\Big(v(W_{t_{i+1}} - W_{t_i}) - \frac{\Delta}{2}|v|^2\Big)\Big]. \tag{5.54}$$

Opposite to (5.45), we do not need to know $k_{t_i}$ to solve this problem.

By Theorem 5.3.9,

$$v(t_i, x) - \tilde{\rho}^x_{t_i,t_{i+1}}[v(t_{i+1}, X_{t_{i+1}}^{t_i,x})] \le C\Delta^{\frac{3}{2}}. \tag{5.55}$$

By construction, the approximating measure of risk $\tilde{\rho}^x_{t_i,t_{i+1}}[\cdot]$ is coherent and satisfies all properties (i)-(iii) of Theorem 2.2.12.

### 5.3.5 Discrete-Time Approximations by Dynamic Programming

The time-consistency of dynamic risk measure leads to the nested form below:

$$\rho_{0,T}[\,\Phi(X_T)\,] = \rho_{t_0,t_1}\Big[\rho_{t_1,t_2}\Big[\ldots\rho_{t_{N-2},t_{N-1}}\Big[\rho_{t_{N-1},t_N}[\Phi(X_T)]\Big]\ldots\Big]\Big]. \tag{5.56}$$

By using optimal constant dual controls on each interval $[t_i, t_{i+1})$, we may approximate this composition by dynamic programming. For $i = N$ we define $\tilde{v}_N(x) = \Phi(x)$. Then, for $i = N-1, N-2, \ldots, 0$, and for $x \in \mathbb{R}^n$, we restart the diffusion (5.23) from $x$ at time $t_i$ as in (5.42).

Having obtained $X_{t_{i+1}}^{t_i,x}$, we can calculate the approximate risk measure (5.54) on the interval $[t_i, t_{i+1}]$:

(5.57) $$\tilde{v}_i(x) = \tilde{\rho}^x_{t_i,t_{i+1}}[\tilde{v}_{i+1}(X_{t_{i+1}}^{t_i,x})] = \max_{\nu\in A} \mathbb{E}\Big[\tilde{v}_{i+1}(X_{t_{i+1}}^{t_i,x}) \exp\Big(\nu(W_{t_{i+1}} - W_{t_i}) - \frac{\Delta}{2}|\nu|^2\Big)\Big].$$

**Theorem 5.3.10.** *Suppose Assumptions 5.3.1, 2.2.11, and 5.3.5 are satisfied. Then a constant $C$ exists, such that for all $N$ and $x$ we have:*

(5.58) $$v(t_i, x) - \tilde{v}_i(x) \le C(N-i)\Delta^{\frac{3}{2}}, \quad i = 0, 1, \ldots, N.$$

*In particular,* $v(0, x) - \tilde{v}_0(x) \le CT\Delta^{\frac{1}{2}}$.

*Proof.* The result follows by backward induction. It is obviously true for $i = N$. If it is true for $i + 1$, we can easily verify it for $i$. By the translation property of $\tilde{\rho}^x_{t_i,t_{i+1}}[\,\cdot\,]$ and (5.55) we obtain:

$$\begin{aligned} v(t_i, x) - \tilde{v}_i(x) &= v(t_i, x) - \tilde{\rho}^x_{t_i,t_{i+1}}[\tilde{v}_{i+1}(X_{t_{i+1}}^{t_i,x})] \\ &\le v(t_i, x) - \tilde{\rho}^x_{t_i,t_{i+1}}[v(t_{i+1}, X_{t_{i+1}}^{t_i,x})] + C\Delta^{\frac{3}{2}} \le C(N-i)\Delta^{\frac{3}{2}}, \end{aligned}$$

as required. □

In practice, the forward process (5.28) is simulated in an approximate way, for example, by *Euler's method*:

(5.59) $$\tilde{X}_{t_{i+1}}^{t_i,x} = x + b(t_i, x)\,\Delta + \sigma(t_i, x)\,\Delta W,\ \Delta W \sim N(0, \sqrt{\Delta}\mathbb{I}).$$

It is well known that for small $\Delta$, the error of this Euler scheme is $O(\Delta^{\frac{1}{2}})$. Since $\tilde{X}_{t_{i+1}}^{t_i,x}$ is a normal random vector, streamlined calculation of the risk measure is possible. Denoting by $\mathcal{N}$ a standard normal random vector with independent components, we can simplify the calculation of the risk measure in (5.57) as follows:

(5.60) $$\tilde{v}_i(x) \approx \max_{\nu\in A} \mathbb{E}\Big[\tilde{v}_{i+1}(x + b(t_i, x)\Delta + \sigma(t_i, x)\mathcal{N}) \exp\Big(\Delta^{\frac{1}{2}}\nu\mathcal{N} - \frac{\Delta}{2}|\nu|^2\Big)\Big].$$

Observe that the same normal random vector $\mathcal{N}$ is used in both terms of this expression.

**Remark 5.3.11.** *Our earlier assumption of time-homogeneity of $g$ is barely a restriction after discretization, because $g$ can be piecewise $\alpha$-Hölder continuous between the grid points. As long as the risk aversion does not change abruptly, the numerical method developed can be easily adapted to the case of a time-dependent driver.*

## 5.4 Application to Financial Risk Management

### 5.4.1 Introduction and Motivation

After a credit crunch, the management of risk is an increasingly important function of any financial institutions, e.g., major investment banks. The primary goal is to make sure to have sufficient capital reserves against potential loss in the future. Such risk management is divided into two stages: *scenario generation* and *portfolio re-pricing*.

Scenario generation refers to simulation of sample paths over a given time horizon. This is also called the *outer stage*, where *Monte Carlo simulation* is usually performed to generate paths governed by stochastic differential equation. Repricing of a portfolio amounts to the computation of the portfolio value at the risk time horizon, given a particular scenario of risk factor. The portfolio can consist of derivative securities with nonlinear payoffs that, in conjunction with financial models, require Monte Carlo simulation for this *inner stage* as well (see figure 5.1). Thus, in real world application, the risk measurement requires calculation of a two-level nested Monte Carlo simulation. Lastly, the risk evaluation is done by risk measure $\rho$, a functional that maps future random exposure to a real number. Examples of risk measure can be *value at risk*, *conditional value at risk*, *probability of loss*, e.t.c.. Such structure leads to challenging computational task. Especially, the inner step simulation has to be done for each scenarios generated in the first stage. Much researches have been devoted to address the computation issue; to name a few, Gordy and Juneja [32], Lee and Glynn [45], Lesnevski et al [46, 47] and Rockafellar and Uryasev [65].

To fix the notation, on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let us consider the problem of measuring the risk of a portfolio of assets at the risk horizon $t = \tau$, while standing at time 0. We denote the current wealth, i.e., net present value of portfolio, by $\mathcal{F}_0$ measurable random variable $X_0$(known quantity). At time $\tau$, the value of the portfolio is then a $\mathcal{F}_\tau$-measurable random variable $X_\tau$. In almost all real world applications, we shall assume a probabilistic model for the evolution of uncertainty between times 0 and $\tau$. For example, a stochastic differential equation run in between is a qualified model. Suppose the outcome space $\Omega$ is a set of possible future scenarios, each of which incorporates sufficient information so as to determine all assets prices at the risk horizon. Then, in each scenario $\omega \in \Omega$, the portfolio has value $X_\tau(\omega)$. The

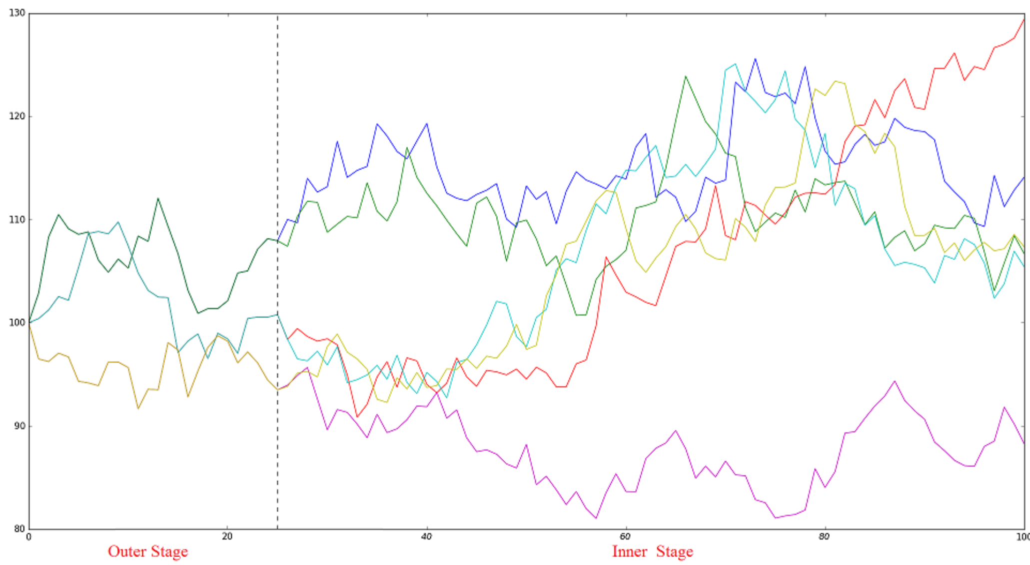


Figure 5.1: Two Stage Simulation

*mark-to-market (MTM)* loss of this portfolio at time $\tau$ in scenario $\hat{\omega} \in \Omega$ is given by

$$L(\hat{\omega}) = X_0 - X_\tau(\hat{\omega}). \tag{5.61}$$

The usual risk measurement is static in the sense that it evaluates the risk of exposure at risk horizon only at current time 0. For instance, if we use the probability of loss model, the approximation of risk evaluation $\mathbb{E}[\mathbf{1}_{\{L \geq c\}}]$ is simply

$$\rho[L] \approx \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\{L(\omega_i) \geq c\}}, \tag{5.62}$$

where $N$ is the number of scenarios in the outer stage and $c$ is certain pre-determined threshold. However, such an evaluation can be only executed at the initial time; if one wants to check the risk at the intermediate point of the risk horizon, the whole model has to be re-run. Given the computational efforts of nested simulation, it can be very burdensome. In addition, repeated simulations can cause inconsistency of risk evaluation, which is also undesired.

In our work, we use dynamic risk measure and its approximation algorithm proposed in Theorem 5.56, to measure the risk associated with the portfolio dynamically. In this way, the risk can be monitored continuously and consistently, in other words, for any time instant $t$ within the risk horizon, the evolution of risk can be traced.

### 5.4.2 Example – Single Put Option

To better illustrate dynamic risk evaluation, let us consider a specific example of a portfolio consisting of a long position in a single put vanilla option, which expires in $T$ years and has strike price $K$. The underlying stock, say $ABC$, follows the geometric Brownian motion with the initial price $S_0$, mean $\mu$ and volatility $\sigma$, under real world probability measure $\mathbb{P}$, its dynamics is given by the following SDE:

$$\frac{dS_t}{S_t} = bdt + \sigma dW_t, \quad t \in [0, T] \tag{5.63}$$

Here, $W_t$ is $\mathbb{P}$-Brownian motion. Let us also set the flat interest rate level as $r$; therefore, under risk-neutral pricing framework, we have stock dynamics,

$$\frac{dS_t}{S_t} = rdt + \sigma d\widetilde{W}_t \tag{5.64}$$

where $\widetilde{W}_t$ is a $\mathbb{Q}$-Brownian motion. With these specifications, the initial value of the put can be easily calculated by plugging into the *Black-Scholes (BS)* formula. It yields

$$\begin{aligned}\mathcal{P}(0, S_0) &:= BS(0, S_0, \sigma, K, T) \\ &= S_0 N(d_+(T, S_0)) - KD(0, T)N(d_-(T, S_0)),\end{aligned}$$

where $N(\cdot)$ stands for the cumulative distribution function of normal distribution and

$$\begin{aligned}d_+(\tau, x) &= \frac{1}{\sigma\sqrt{\tau}}[\ln\frac{x}{K} + (r + \frac{1}{2}\sigma^2)\tau], \\ d_-(\tau, x) &= \frac{1}{\sigma\sqrt{\tau}}[\ln\frac{x}{K} + (r - \frac{1}{2}\sigma^2)\tau].\end{aligned}$$

Let us fix a risk horizon $\tau$, and denote the price of at the risk horizon as $S_\tau(\omega)$. Then, the exposure (or, MTM) at time $\tau$ is the difference of the initial put price $\mathcal{P}(0, S_0)$ and the risk-neutral price of the option at time $\tau$, i.e.,

$$\Phi(S_\tau(\omega)) := \mathcal{P}(0, S_0) - \mathbb{E}^{\mathbb{Q}}[\,(S_T - K)^+ \,\big|\, S_\tau(\omega)] \tag{5.65}$$

Here, to get $S_\tau(\omega)$, we have to simulate the path of stock under real-world measure, i.e., according to (5.63). Then, to compute the right hand side, we only need to work out the second term; again, it can be computed analytically by Black-Scholes formula. It is well known that, in this case, the loss function $\Phi(\cdot)$ is Lipschitz with respect to state. We are in the situation to apply the dynamic risk measure,

$$\rho^g_{t,\tau}[\,\Phi(S_\tau)\,] := y_t, \;\; where\; y_t = \Phi(S_\tau) + \int_t^\tau g(s, Z_s)ds - \int_t^\tau Z_s dW_s, \;\; t \in [0, \tau], \tag{5.66}$$

which enables us to view the risk at any time $t$ before the risk horizon.

As for implementation details, instead of using Monte Carlo simulation, we use a tree for the outer stage to generate the evolution of states. This is because, when evaluating risk, by using our specific algorithm (5.15), we can reduce functional optimization to vector optimization. However, since backward induction has to be implemented, the state space also needs to be discretized, which makes tree structure appealing.

### 5.4.3 Numerical Experiments

We now present the numerical results based on the following data:

$$K = 95,\;\; T = 0.75,\;\; S_0 = 100,\;\; \mu = 0.08,\;\; \sigma = 0.2,\;\; r = 0.03,\;\; \tau = 0.2.$$

For risk evaluation, we specify the generator to be:

$$g(z) = \gamma \| \max\{z\mathcal{N}, 0\}\|_p,\;\; \mathcal{N} \sim N(0, 1),$$

where the parameters $\gamma > 0$ and $p \geq 1$ model risk aversion. The corresponding set of ambiguity is then:

$$A = \partial g(0) = \{\, l \in \mathbb{R}^n_+ : |l|_q \leq \gamma k\,\},$$

with $1/p + 1/q = 1$, and

$$k = \begin{cases} \frac{1}{\sqrt{2}}(2m(2m-1)\cdots(m+1))^{\frac{1}{2m}}, & \text{if } p = 2m, \\ (2^m\sqrt{2}\pi m!)^{\frac{1}{2m+1}}, & \text{if } p = 2m+1. \end{cases}$$

Fix $p = 2$, at time 0, given $\mathcal{F}_\tau$-measurable loss $\Phi(\cdot)$ in (5.65). Table 5.1 and Figure 5.2 summarize the valuation when varying the step size and risk tolerance $\gamma$. We can observe convergence of the numerical method, as the step size decreases, uniformly over the whole range of $\gamma$.

If we vary the underlying asset volatility, as well as strike price of the contract, we can construct the *risk surface*. As Table 5.2 and Figure 5.3 show, the risk is plotted against different combinations of volatility $\sigma$ and strike price $K$.

Table 5.1: Risk Valuation Convergence Table

| step size | $\gamma = 0.1$ | $\gamma = 0.3$ | $\gamma = 0.4$ | $\gamma = 0.6$ | $\gamma = 0.8$ | $\gamma = 1.0$ |
|---|---|---|---|---|---|---|
| 0.4 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 0.2 | 0.03407 | 0.23907 | 0.34080 | 0.54207 | 0.73957 | 0.93217 |
| 0.1 | 0.06371 | 0.37300 | 0.52628 | 0.82895 | 1.12492 | 1.41239 |
| 0.08 | 0.07086 | 0.40174 | 0.56573 | 0.88956 | 1.20628 | 1.51403 |
| 0.05 | 0.08261 | 0.44695 | 0.62757 | 0.98446 | 1.33392 | 1.67410 |
| 0.04 | 0.08687 | 0.46282 | 0.64924 | 1.01771 | 1.37878 | 1.73064 |
| 0.02 | 0.09622 | 0.49671 | 0.69544 | 1.08872 | 1.47500 | 1.85268 |
| 0.01 | 0.10165 | 0.51579 | 0.72141 | 1.12877 | 1.52971 | 1.92284 |
| 0.008 | 0.10287 | 0.51998 | 0.72712 | 1.13760 | 1.54184 | 1.93852 |
| 0.005 | 0.10485 | 0.52674 | 0.73632 | 1.15187 | 1.56152 | 1.96407 |
| 0.004 | 0.10557 | 0.52919 | 0.73965 | 1.15705 | 1.56869 | 1.97344 |
| 0.002 | 0.10720 | 0.53465 | 0.74709 | 1.16864 | 1.58483 | 1.99465 |
| 0.001 | 0.10822 | 0.53798 | 0.75163 | 1.17574 | 1.59479 | 2.00786 |
| 0.0008 | 0.10845 | 0.53876 | 0.75269 | 1.17740 | 1.59713 | 2.01099 |
| 0.0005 | 0.10886 | 0.54007 | 0.75447 | 1.18021 | 1.60109 | 2.01630 |
| 0.0004 | 0.10901 | 0.54057 | 0.75515 | 1.18127 | 1.60260 | 2.01833 |

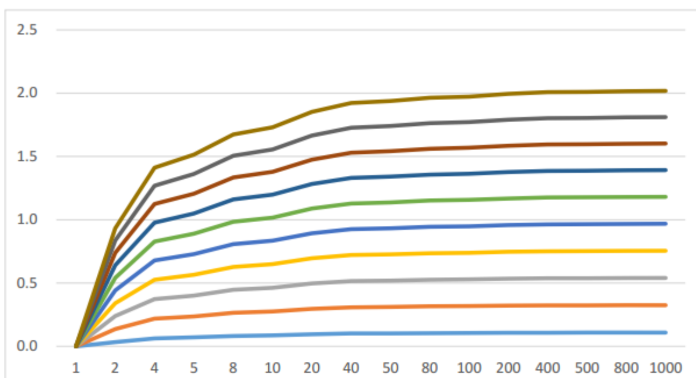


Figure 5.2: Convergence Graph: low to high, from $\gamma = 0.1$ to $\gamma = 1.0$, the horizontal axis stands for different number of steps while the vertical axis is the measurement of risk.

Table 5.2: Risk Surface Table

| $K, \sigma$ | 0.1 | 0.3 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|---|---|---|---|---|---|---|---|---|
| 70 | -0.0002 | -0.1479 | -0.0901 | 0.0566 | 0.2612 | 0.5101 | 0.7918 | 1.0962 |
| 80 | -0.0041 | -0.0188 | 0.2444 | 0.4645 | 0.7290 | 1.0279 | 1.3517 | 1.6918 |
| 90 | 0.0737 | 0.3661 | 0.7508 | 1.0114 | 1.3099 | 1.6379 | 1.9869 | 2.3489 |
| 100 | 0.7941 | 1.0211 | 1.4004 | 1.6691 | 1.9782 | 2.3177 | 2.6780 | 3.0506 |
| 110 | 2.4089 | 1.8858 | 2.1561 | 2.4081 | 2.7100 | 3.0475 | 3.4087 | 3.7833 |
| 120 | 3.9179 | 2.8641 | 2.9802 | 3.2007 | 3.4842 | 3.8108 | 4.1655 | 4.5361 |
| 130 | 4.6752 | 3.8600 | 3.8387 | 4.0233 | 4.2831 | 4.5938 | 4.9376 | 5.3003 |

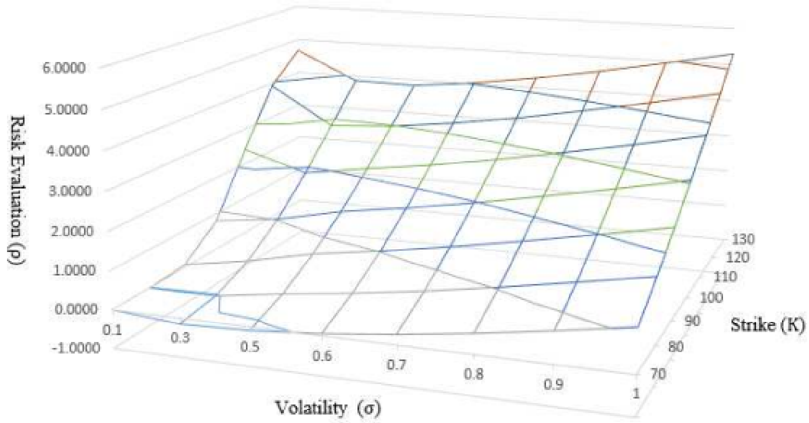


Figure 5.3: The dependence of risk on the strike price and volatility.

As we can observe, if the stock *ABC* becomes volatile, the risk of the portfolio should increase, because volatility implies uncertainty. Moreover, since the current stock price is 100, as the strike price increases, the risk also goes up, which indicates being engaged in out of money trade is riskier than at-the-money or in-the-money. Thus, the risk surface constructed coincides with intuition, which validates the risk evaluation approximation.

# Chapter 6

# Conclusion

In this thesis, we discussed optimal control of a dynamic system with risk-aversion. In particular, we investigated the diffusion setting, i.e., with dynamics given by a Markovian stochastic differential equation. The challenges arise due to the non-linearity of the evaluation, which is successfully overcome by adopting a backward stochastic differential equation driven by Brownian motion. The special structure of a dynamic risk measure corresponds to a family of BSDE with convex (or coherent, resp.) driver. The dynamic risk measure inherits all basic properties from static risk measure that enables derivation of risk-averse dynamic programming equation whose connection with Hamilton-Jacobi-Bellman (HJB) equation can also be obtained with an extra term capturing the form and degree of risk aversion.

As in the case of optimal control under expectation, there are multiple approaches to finding solution to the original problem. Our work focuses on the Markov-Chain Approximation (MCA). We raise the issue that the integral regularization method proposed in N. Krylov[41] cannot be adapted to our case for the non-linearity of risk-averse dynamic programming equation. Having realized Hamilton-Jacobi-Bellman equation is essential for the approximation scheme, we extensively use the regularity of the PDE to put forward an $\epsilon$-optimal control approximation. Essentially, we construct a piecewise constant Markov control that is $\epsilon$-optimal. In this case, we can make the original value function and one under $\epsilon$-optimal control arbitrarily close. Nevertheless, in this case, we lose track of the convergence rate.

Risk measure is a nonlinear operator, especially, in the continuous time setting, it amounts to solving a BSDE. After discretization, the policy evaluation requires, on each interval, finding solutions of a forward-backward stochastic differential equation system (FBSDEs). An analytical approach is not feasible (otherwise, we would directly solving controlled FBSDE in the first place); we aimed at an efficient numerical solution to FBDSE on each short interval.

The classical algorithm developed by J. Zhang [76](among many others) is heavily based on Malliavin calculus; it provides fast computation but loses the fundamental properties of being monotonic and time-consistent when adapted to risk evaluation. We invented an approximation through dual representation that is Malliavin calculus free, which converts risk evaluation to a stochastic optimal control problem of special form. With the same idea as above, we replace the functional type of *Radon-Nikodym derivative* by a real-valued vector; as a result, the optimization problem is a simple Euclidean space optimization that can be solved by any non-linear optimizer. After the replication to all sub-intervals, the whole policy evaluation can be solved by successive backward induction. We also provide a useful application to financial risk management, where we measure the risk (exposure at risk horizon) of a financial derivative that is marked to the market before maturity. The real world practice is to use a static risk measure, e.g., value at risk, average value at risk. Since the nature of our evaluation is dynamic, it enables us to trace the risk as time goes on. In addition, the controlled system developed allows us not only to evaluate the risk but also provides advice to control risk.

# Notation

Here is the list of notations that are frequently used in this work:

- $\mathbb{E}_t[\,\cdot\,] := \mathbb{E}[\,\cdot\,|\,\mathcal{F}_t]$.
- $\mathcal{P}^m[t,T]$: the set of $\mathbb{R}^m$-valued adapted processes on $\Omega \times [t,T]$.
- $L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^m)$: the set of $\mathbb{R}^m$-valued $\mathcal{F}_t$-measurable random variables $\xi$ such that $\|\xi\|^2 := \mathbb{E}[\,|\xi|^2\,] < \infty$; for $m = 1$, we write it $L^2(\Omega, \mathcal{F}, \mathbb{P})$.
- $\mathcal{S}^{2,m}[t,T]$: the set of elements $Y \in \mathcal{P}^m[t,T]$ such that $\|Y\|^2_{\mathcal{S}^{2,m}[t,T]} := \mathbb{E}[\,\sup_{t\le s\le T} |Y_s|^2\,] < \infty$; for $m = 1$, we write it $\mathcal{S}^2[t,T]$.
- $\mathcal{H}^{2,m}[t,T]$: the set of elements $Y \in \mathcal{P}^m[t,T]$, such that $\|Y\|^2_{\mathcal{H}^{2,m}[t,T]} := \mathbb{E}[\,\int_t^T |Y_s|^2\,ds] < \infty$; for $m = 1$ we write it $\mathcal{H}^2[t,T]$.[1]
- $\mathcal{M}^2[0,T]$: the product space defined as $\mathcal{M}^2[0,T] = \mathcal{S}^2[0,T] \times \mathcal{H}^2[0,T]$.
- $C^{i,j}([t,T]\times\mathbb{R}^m)$ the space of functions $f : [t,T]\times\mathbb{R}^m \to \mathbb{R}$, which are $i$-th differentiable with respect to the first argument and $j$-th differentiable with respect to the second argument, with all these derivatives continuous with respect to both arguments.
- $C_{\mathrm{b}}^{i,j}([t,T]\times\mathbb{R}^m)$ the space of functions $f \in C^{i,j}([t,T]\times\mathbb{R}^m)$ with all derivatives bounded and continuous with respect to both arguments.
- $C^\infty(B)$: the space of functions $f : B \mapsto \mathbb{R}$ that are infinitely continuously differentiable with respect to all arguments and have compact support on $B \subset \mathbb{R}^n$.
- $C_L(B)$: the space of Lipschitz continuous functions $f : B \to \mathbb{R}$.
- For Borel measurable function $w : [0,T]\times\mathbb{R}^n$, the seminorm $\left\| w(t,x) \right\|_{2,1}$ is defined as

[1] When the norm is clear from the context, the subscripts are skipped.

follows:

$$\begin{aligned}\left\|w\right\|_{2,1} = \sup_{(t,x)}\left|w(t,x)\right| + \sup_{(t,x)}\left\|\mathcal{D}_x w(t,x)\right\| + \sup_{(t,x)}\left\|\mathcal{D}^2_{xx} w(t,x)\right\| + \sup_{(t,x)}\left|\partial_t w(t,x)\right| \\ + \sup_{(t,x),(s,y)} \frac{\left\|\mathcal{D}^2_{xx} w(t,x) - \mathcal{D}^2_{xx} w(s,y)\right\|}{|t-s| + |x-y|} + \sup_{(t,x),(s,y)} \frac{\left|\partial_t w(t,x) - \partial_t w(s,y)\right|}{|t-s| + |x-y|}.\end{aligned}$$

In the formula above, we use $\mathcal{D}_x$ and $\mathcal{D}^2_{xx}$ to denote the gradient and the Hessian matrix, and the supremum is always over $(t,x), (s,y) \in [0, T-\varepsilon^2] \times \mathbb{R}^n$.

# Curriculum Vitae

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### *Education*

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