THE INTERPLAY BETWEEN TEACHER QUESTIONING AND STUDENT REASONING

by

MIRIAM GERSTEIN

A dissertation submitted to the

Graduate School-New Brunswick

Rutgers, The State University of New Jersey

In partial fulfillment of the requirements

For the degree of

Doctor of Philosophy

Graduate Program in Education

Written under the direction of

Carolyn Maher

And approved by

________________________

________________________

________________________

________________________

New Brunswick, New Jersey

OCTOBER, 2017
The purpose of the qualitative study is to describe patterns of relationships between teacher questioning and student responses, especially student reasoning. This study is positioned in the longitudinal/cross sectional research study of the development of students’ mathematical thinking and reasoning conducted at Rutgers University that spanned over twenty-five years. Data were analyzed from sessions that were conducted in urban, working class, and suburban settings, from a range of age levels conducted by different researchers and from different content domains. The discourse from classroom settings as well as informal learning environments are examined.

Little research has been conducted on the association between teacher questioning and the production of varied forms of reasoning by students. Hence, the objective was to identify how differing styles of questioning demonstrated by the researchers were associated with the production of students’ reasoning. Representative sessions facilitated by four researchers were examined to investigate researcher questioning and discourse moves and student responses in building proof-like justifications. Results indicated that
certain types of questioning techniques employed by researchers were associated with ways the students formulated their solutions, extended their reasoning, made connections, or otherwise enhanced or refined their solutions. Accompanying the qualitative analysis of the data, are video narratives (VMCAnalytics) that demonstrate the different categories of teacher questioning that were associated with students’ reasoning and their productions of justifications.

By analyzing questioning in various settings, at different age levels, and by multiple researchers, one gains insight into the relationships of teacher questioning and patterns of student reasoning. This study indicates that the extensive use of probing and eliciting questions, as well as questions posed to encourage engagement, engenders a learning environment that is conducive to argumentation, justification, and reasoning. Such targeted questioning can prompt students to articulate their strategies and logic, use evidence to support their conclusions, and justify and give detailed explanation of their thought processes, and serves as a catalyst for students to challenge or support each other’s arguments. Findings, accompanied by the video narratives, are offered to show patterns of effective questioning techniques that may lead students to clarify and formulate their own mathematical thinking and to independently discover mathematical facts and realities.
Acknowledgements

First and foremost, I would like to express my thanks to G-d for all of the wonderful kindness He has shown me throughout my life and for giving me the wherewithal, the words, and the wisdom to complete this work.

I would like to thank my advisor, Carolyn Maher, for her expertise, encouragement, and skillful editing that she has consistently provided throughout my work at Rutgers University. Her wealth of knowledge, and her commitment to her students, helped make the dissertation process as enjoyable as possible. One could not ask for a better doctoral mentor.

My committee members too provided helpful critiques, and insightful ideas that helped me refine my work and guided me throughout the dissertation process. They helped me improve my focus and shared their expertise throughout the entire process. Their helpful criticism and comments were much appreciated.

Several graduate students were most helpful in allowing me to achieve an estimate of reliability. Special thanks to Baila Salb and Esther Winter for their assistance in this regard. Thank you to Dr. Alice Alston for her careful editing of the analytics associated with this dissertation.

Finally, I would like to thank my family for their support and encouragement throughout my studies. My parents have constantly encouraged me to set high goals for myself and to achieve especially in the area of higher education. Without their encouragement and support I would not have undertaken and completed such a monumental task. My husband and children too served as constant source of
encouragement as they cheered me on to complete my dissertation even as they dealt with a wife and mother who was constantly preoccupied with work.
Table of Contents

Abstract ........................................................................................................................................... ii
Acknowledgements ......................................................................................................................... iv
List of Figures .................................................................................................................................. ix
List of Tables ................................................................................................................................. xii

CHAPTER 1 – INTRODUCTION ..................................................................................................... 1
  1.1 Statement of the Problem ........................................................................................................ 1
  1.2 The Longitudinal Study ......................................................................................................... 2
  1.3 Research Questions ............................................................................................................... 5

CHAPTER 2 – REVIEW OF THE LITERATURE ............................................................................. 7
  2.1 Introduction ........................................................................................................................... 7
  2.2 The Role of Teacher Questioning ......................................................................................... 7
    2.2.1 The development of mathematical thinking ................................................................. 7
    2.2.2 Promoting active engagement ...................................................................................... 8
    2.2.3 Shaping productive classroom discourse ................................................................... 9
    2.2.4 Other functions of teacher questioning ...................................................................... 10
  2.3 Developing Good Questions ............................................................................................... 10
    2.3.1 Questioning Techniques ............................................................................................... 10
    2.3.2 Reflecting on Questioning Techniques ......................................................................... 11
  2.4 Wait-time ............................................................................................................................... 13
  2.5 Classification of Questions ................................................................................................... 14
    2.5.1 Showing and Telling, Leading, and Shepherding ......................................................... 14
    2.5.2 General, Specific, Probing, and Leading Questions .................................................... 15
    2.5.3 Pumping, Reflective Toss, and Constructive Challenge ............................................. 16
    2.5.4 Probing, Guiding, and Factual Questions .................................................................... 17
  2.6 The Effect of Questioning ..................................................................................................... 19
    2.6.1 The Effect on Student Responses ............................................................................... 19
    2.6.2 The Effect on Student Achievement ........................................................................... 22
    2.6.3 The Effect on Student Reasoning and Justification .................................................... 28
  2.7 Summary of Findings ............................................................................................................ 33

CHAPTER 3: METHODOLOGY ..................................................................................................... 37
CHAPTER 5: CONCLUSIONS AND IMPLICATIONS

5.1 Introduction

5.2 An Overview of the Findings

5.3 Discussion

5.4 Limitations

5.5 Implications

5.5.1 Implications for Further Research

5.5.2 Implications for Practice

References

Appendix
List of Figures

Figure 3.1. Staircase model of Cuisenaire ® rods .................................................43
Figure 3.2. A train of light green and one red rod alongside a brown rod ...............43
Figure 4.1. A purple and white train alongside a yellow rod ..................................65
Figure 4.2. Two purple rods alongside the blue rod and two yellow rods alongside the blue rod ..........................................................65
Figure 4.3. Staircase model of Cuisenaire rods ......................................................66
Figure 4.4. “Odd” and “Even” rods ...............................................................69
Figure 4.5. Brian’s cases ...............................................................................83
Figure 4.6. Erik’s first arrangement .......................................................................85
Figure 4.7. Erik’s second arrangement ..................................................................85
Figure 4.8. Question types used by T/R 1 during Session 2 .........................................93
Figure 4.9. Question types used by T/R 2 during Session 2 .........................................93
Figure 4.10. Question results during Session 2 ........................................................93
Figure 4.11. Meredith demonstrates that 2/10=1/5 ..................................................94
Figure 4.12. The drawing of the candy bar .............................................................99
Figure 4.13. Researcher Maher records the different number names for half of the candy bar ..................................................100
Figure 4.14. Model with a train of orange and red rods as the unit .............................104
Figure 4.15. 1/2 is one red rod or two white rods longer than 1/3 .............................104
Figure 4.16. Models to support the claim that 1/2 is greater than 1/3 by 1/3 ...............105
Figure 4.17. Model showing two red rods aligned with one purple rod ......................106
Figure 4.18. Jackie and Kelly’s model with dark green rod as 1 ...............................106
Figure 4.19. Erik’s model showing that the white is 1/6 ...........................................106
Figure 4.20. Brian’s model .............................................................................108
Figure 4.21. Erik model to support his upper and lower bound argument ..................109
Figure 4.22. Model showing Michael’s direct reasoning ...........................................109
Figure 4.23. Meredith’s model ..........................................................................110
Figure 4.24. Meredith modifies her model to show the difference to be one sixth …..110
Figure 4.25. Erik’s model to show the difference to be one sixth ..........................110
Figure 4.26. Question types used by T/R 1 during Session 5 ..............................118
Figure 4.27. Question results during Session 5 ....................................................118
Figure 4.28. Alan’s first two models of the same length ....................................124
Figure 4.29. Alan’s third model ............................................................................125
Figure 4.30. Alan’s largest model for comparing two fifths to one half ...............126
Figure 4.31. Question types used by T/R 2 during Session .................................129
Figure 4.32. Question results during Session 10 ..................................................129
Figure 4.33. Question types used by T/R 2 during the interview with Brandon ....153
Figure 4.34. Question results during the interview with Brandon .........................153
Figure 4.35. Question types used by T/R 3 during the Kenilworth sessions ..........185
Figure 4.36. Question results during the Kenilworth sessions .............................185
Figure 4.37. Question types used by T/R 4 during the November 2 IML session ....194
Figure 4.38. Question results during the November 2 IML session .......................195
Figure 4.39. Question types used by T/R 4 during the November 3 IML session ....211
Figure 4.40. Question results during the November 3 IML session .......................212
Figure 5.1. Classification of questions for all researchers .......................................213
Figure 5.2. Sub-codes for the encouraging engagement classification for all researchers ..........................................................................................................................214
Figure 5.3. Sub-codes for the probing classification for all researchers ..............214
Figure 5.4 Classification of question results for all researchers ..........................215
Figure 5.5. Classification of questions for Researcher Carolyn Maher .................218
Figure 5.6. Classification of questions for Researcher Carolyn Maher .................218
Figure 5.7. Sub-codes for the probing classification for Researcher Carolyn Maher ...219
Figure 5.8. Classification of question results for Researcher Carolyn Maher ........219
Figure 5.9. Classification of questions for Researcher Amy Martino ...................221
Figure 5.10. Sub-codes for the encouraging engagement classification for Researcher Amy Martino ..............................................................................................................221
Figure 5.11. Sub-codes for the probing classification for Researcher Amy Martino .....222
Figure 5.12. Classification of question results for Researcher Amy Martino ................222
Figure 5.13. Classification of questions for Researcher Robert B. Davis .................224
Figure 5.14. Sub-codes for the encouraging engagement classification for Researcher Robert B. Davis ..........................................................224
Figure 5.15. Sub-codes for the probing classification for Researcher Robert B. Davis .225
Figure 5.16. Classification of question results for Researcher Robert B. Davis ..........225
Figure 5.17. Classification of questions for Researcher Arthur Powell .....................226
Figure 5.18. Sub-codes for the encouraging engagement classification for Researcher Arthur Powell ................................................................................227
Figure 5.19. Sub-codes for the probing classification for Researcher Arthur Powell.... 227
Figure 5.20 Classification of question results for Researcher Arthur Powell ............228
List of Tables

Table 3.1. Summary of Coding Scheme............................................................... 58
Table 4.1. Classification of teacher questioning and results, Colts Neck Session 2, Part I......66
Table 4.2. Classification of teacher questioning and results, Colts Neck Session 2, Part II.......69
Table 4.3. Classification of teacher questioning and results, Colts Neck Session 2, Part III......73
Table 4.4. Classification of teacher questioning and results, Colts Neck Session 2, Part IV.....74
Table 4.5. Classification of teacher questioning and results, Colts Neck Session 2, Part V......78
Table 4.6. Classification of teacher questioning and results, Colts Neck Session 2, Part VI.....85
Table 4.7. Classification of teacher questioning and results, Colts Neck Session 5, Part I....... 97
Table 4.8. Classification of teacher questioning and results, Colts Neck Session 5, Part II.....100
Table 4.9. Classification of teacher questioning and results, Colts Neck Session 5, Part III.....111
Table 4.10. Classification of teacher questioning and results, Colts Neck Session 10, Part I...122
Table 4.11. Classification of teacher questioning and results, Colts Neck Session 10, Part II...126
Table 4.12. Classification of teacher questioning and results, Colts Neck Interview with Brandon, Part I ..........................................................................................................136
Table 4.13. Classification of teacher questioning and results, Colts Neck Interview with Brandon, Part II ..................................................................................................145
Table 4.14. Classification of teacher questioning and results, Kenilworth session, Part I.......169
Table 4.15. Classification of teacher questioning and results, Kenilworth session, Part II......178
Table 4.16. Classification of teacher questioning and results, IML, Part I.........................191
Table 4.17. Classification of teacher questioning and results, IML, Part II.......................198
Table 4.18. Classification of teacher questioning and results, IML, Part III.....................204
Table 4.19. Classification of teacher questioning and results, IML, Part IV.....................208
CHAPTER 1 – INTRODUCTION

1.1 Statement of the Problem

Research reveals the importance of effective teacher questioning and highlights the role that such questioning plays in the development of students’ mathematical reasoning (Klinzing, Klinzing-Eurich & Tisher, 1985; Martino & Maher, 1999). Questioning can serve as a springboard for further discussion, elaboration of incomplete thought, and a greater conceptual understanding of the mathematical problems at hand. Teacher questioning can give the teacher a glimpse into the mathematical thinking of the students which may not be apparent otherwise. Not only may it help the teacher estimate students’ understanding of the concept being discussed, but it may also provide the teacher with a better understanding of the students’ thought processes, judgment, concerns, or hesitations when coming up with solutions to open-ended mathematical problems. Modeling of good questions may also have a positive effect on the students. They may begin to imitate those forms of questions both when interacting with peers and when working on their own. By asking students for explanations of their work, for further clarification, or by asking students how they would convince their peers of their solutions, teachers may stimulate students to initiate such discussions on their own, or prompt them to formulate cogent justifications and reasoning. Such probing by teachers may also result in students’ posing these questions to themselves, which in turn could help clarify the solutions in their minds before presenting their ideas to others.

The purpose of this case study is to describe patterns of relationships between researcher questioning and student responses and especially student reasoning, as the students worked on open-ended mathematical tasks. My area of research focuses on
classifying teacher researcher questions and the interplay between teacher researcher questioning and student responses and reasoning across different age levels in order to provide insight into the relationships of teacher questioning and patterns of student reasoning. To shape my research, I will examine the literature on teaching and learning mathematics and concentrate specifically on the research concerning the questioning techniques that can help foster an environment conducive to mathematical reasoning and justification.

1.2 The Longitudinal Study

My study is positioned in the longitudinal/cross sectional research study of the development of students’ mathematical thinking and reasoning conducted at Rutgers University. In 1987, researchers from Rutgers University, led by Researcher Carolyn A. Maher, began a longitudinal study in the Kenilworth district in New Jersey (Maher, 2010). This study spanned over twenty-five years as the researchers followed some of the students from first grade to post-college. The researchers also worked with students in New Brunswick, New Jersey as well as in Colts Neck, New Jersey eventually encompassing approximately 80 students from these three districts. The Kenilworth students were studied in grades 1-12, college and post-college. The students from the urban district of New Brunswick were analyzed from grades 4-8 during the summer and again in grade 11. Students from the suburban district of Colts Neck were followed over a number of years in elementary and middle school from grades 2-7 (Francisco & Maher, 2005). In addition to collaboration with schools in the above-mentioned districts, researchers at Rutgers University also implemented an Informal Math Learning (IML)
after-school enrichment program in Plainfield, New Jersey that spanned a three year period.

Interventions were conducted in Kenilworth starting in 1987 at a time when direct instruction and behaviorism directed mathematics teaching. The project started off as professional development intervention. Alice S. Alston and Judith H. Landis worked with the teachers and principal, Fred Rica, to implement activities that would try to help the teachers move away from primarily drill and practice exercises to lessons that would engage the students and invite mathematical understanding. The Rutgers team videoed both teacher workshops as well as classroom sessions. Alice Alston also assisted teachers in their classrooms. The math curriculum was revised to allow for more reasoning and engagement and the researchers introduced the use of manipulatives into math lessons so that the students could build physical representations of their solutions (Maher, 2002, Yankelewitz, 2009).

The longitudinal study began with one class of first graders. First graders at Kenilworth were randomly placed in classes and these students remained together until third grade. When this initial group of 18 students was split after third grade, the school arranged that a focus group consisting of 12 students would continue to work with the researchers. This group of 12 changed during the middle school years as families changed districts and other students joined. There were a few students that were involved in the project throughout. The researchers conducted two 90-minute sessions, as well as one 45 minute session, four to six times each year. The Kenilworth high school was closed during the students’ ninth grade year, but sessions were held in private homes. When the school reopened the next year, the fourteen students continued to be involved by meeting
with researchers after school four to six times for an hour or two over the course of a year. Ten students had participated in the study since grade one (Maher, 2002).

In all, from all of the sessions in this longitudinal/cross sectional study, over 4500 hours of video were collected including videos of problem-solving sessions where the students worked on well-defined, open ended math problems, as well as interviews with individual students, researchers, or small groups. These videos make up the Robert B. Davis Institute for Learning (RBDIL) video collection. The RBDIL also has numerous documents supplementing these videos, including student work, researcher field notes and observations, and many transcripts of videos taken during the class sessions. (Hmelo-Silver et al., 2013).

I have drawn my data largely from sessions at the Harding Elementary School in Kenilworth, New Jersey, sessions at the Conover Road School in Colts Neck, New Jersey, as well as sessions conducted at the Hubbard Middle School in Plainfield, New Jersey, in an attempt to gain insight into the effect of the researcher questions on student reasoning and possible implications for practice. From the sessions conducted at Kenilworth, I focused specifically on sessions conducted by Researcher Robert B. Davis as he introduced the students to the idea of function through an activity called “Guess My Rule.”

In addition to studying the Kenilworth sessions, I have analyzed sessions at the suburban Colts Neck school. Starting in the second grade, students worked on fractions and combinatorics problems in a study that spanned three and half years (1991-1995). I have focused primarily on a sampling of the 50 sessions conducted when the students were in fourth grade (1993-1994). Since there were many sessions where students
worked on fraction problems under the direction of Researchers Carolyn Maher and Amy Martino, I selected a few of the fraction session led by those researchers that include multiple instances of teacher questioning. I have also looked at an interview conducted by Researcher Amy Martino with a fourth grade boy as he works on the Pizza and Towers problems.

I also analyzed sessions from the Informal Math Learning program conducted by Rutgers researchers in Plainfield, New Jersey, focusing my attention specifically on sessions led by Researcher Arthur Powell where he worked with students on the “Guess My Rule” activity.

During the sessions in this cross-sectional, longitudinal study, students worked on open-ended mathematical tasks. The researchers were interested in how the students built mathematical ideas and used various forms of mathematical reasoning when working in an environment where collaboration, thoughtfulness, sharing ideas, building mathematical meaning, and justifying solution were encouraged.

1.3 Research Questions

The present study builds on existent research. Yankelewitz (2009) looked at the different forms of reasoning exhibited by the Colts Neck students during the sessions on fractions. Baldev (2009) analyzed the Informal Math Learning sessions where seventh grade students from low-income urban community built ideas about linear functions and engaged in mathematical reasoning as they worked on the Guess My Rule problem. Giordano (2008) studied the interventions used by Researcher Robert B. Davis as he introduced the concept of linear and quadratic functions through the Guess My Rule activity.
In this study I trace the researchers’ questions during these sessions and the opportunity they afforded for student comprehension and development of mathematical reasoning and justification. I study whether and how their questioning invited students to formulate their solutions, extend their reasoning, make connections, or otherwise enhance or refine their solutions.

The central research question that guided this study is: How does questioning invite student reasoning? This research question includes the following subsidiary questions:

1. What kinds of teacher researcher questions were associated with learning environments where argumentation, justification, and/or reasoning occurred?
2. What evidence is there that teachers/researchers’ questioning provided knowledge of the students’ understanding of the task at hand?
3. What teacher researcher moves triggered students to explain or justify their solution?
   (a) directing attention to components of a developing solution
   (b) directing attention to faulty reasoning
4. What teacher researcher moves challenge students to strengthen their arguments, reorganize their solution, or build more complete or complex arguments and reasoning?
CHAPTER 2 – REVIEW OF THE LITERATURE

2.1 Introduction

Research on teacher questioning in mathematics education has dealt with the importance of questioning (Kazemi and Stipek, 2001; Klinzing, Klinzing-Eurich & Tisher, 1985; Martino & Maher, 1999), the development of good questions (Di Teodoro, Donders, Kemp-Davidson, Robertson & Schuyler, 2011; Moyer and Milewicz, 2002), the classification of questions (Franke et al; Towers, 1998), and the effect of questioning on student achievement, responses, reasoning and justification (Boaler and Staples, 2008; Martino & Maher, 1999; Mueller, Yankelewitz, and Maher, 2014). I have selected to explore a sampling of the relevant literature on these topics in the field of mathematics education since they are highly relevant to my area of proposed research.

2.2 The Role of Teacher Questioning

2.2.1 The development of mathematical thinking

“Questioning is an important part of the teacher’s ability to establish a classroom atmosphere conducive to the development of mathematical thinking” (Burns, 1985, p. 16). By interjecting with timely questions, the teacher can guide the students to formulate more sophisticated reasoning and justifications, revisit a previous question, or help the students link concepts together. By merely being asked for clarification of certain elements of their reasoning, students may be prompted to pay attention to a specific aspect or component of their argument that may assist them in coming up with the proper solution. Through questioning, a teacher can also bring together students with conflicting solutions that may lead to possible changes to a proposed solution.
Burns (1985) posits that it is imperative for teachers to use questioning to foster an atmosphere in their classroom that is conducive to mathematical reasoning. She emphasizes that as important as actual mathematical understanding is for students, it is also essential that they “become powerful mathematical thinkers” (p. 14) and she contends that teacher questioning plays a large role in helping that become a reality. Teacher questioning can help guide students to examine their conjectures and seek ways to justify their solutions both for themselves and to their peers. Often what is lacking in mathematics classrooms is the directing of students’ attention to “deciding on the reasonableness of their solutions, justifying their procedures, verbalizing their processes, [and] reflecting on their thinking” (p. 14). When questioning and reexamining becomes part of the classroom norms, students will feel comfortable sharing their hypotheses, formulating generalizations, explaining their thought processes, and defending their solutions. Questioning can even lead them to “search for - even seek - those insights that rather than converging toward an answer, open up new areas to investigate” (p. 16).

2.2.2 Promoting active engagement.

The IRE (initiate, respond, evaluate) pattern described by Cazden (1986), in which a teacher initiates a question, the students respond, and the teacher in turn evaluates that response, does not allow for the students to be "active agents in their learning" (Hmelo-Silver & Barrows. 2008). Hmelo-Silver and Barrows assert that there are three types of discourse moves that are essential for knowledge building: questioning, statements, and regulatory statements. These discourse moves provide opportunities for the students to be actively engaged in "identifying knowledge problems and collectively
improving their ideas [and it] makes student's thinking visible and open for discussion (p. 52)."

2.2.3 Shaping productive classroom discourse

Kazemi and Stipek (2001) studied videotapes of four lessons on addition of fractions taught in three low-income schools. They classified “high-press exchanges” as consisting of “a mathematical argument, not simply a procedural description” and evidence from the discussion that the students understood “relations among multiple strategies” (p. 59). Kazemi and Stipek found that although all four teachers had implemented inquiry-based learning of mathematics in their classroom by having their students solve open-ended problems in groups, demonstrate their work graphically and numerically, and talk about their strategies to come up with solutions, it was often insufficient to generate authentic mathematical inquiry. They found that teacher questioning played a large role in the level of exchange produced in the classroom. In addition, student errors “provide[d] opportunities to reconceptualize a problem, explore contradictions, and pursue alternative strategies” and their “collaborative work involve[d] individual accountability and reaching consensus through mathematical argumentation” (p. 59).

Cobb, Boufi, McClain and Whitenack (1997) highlight the role of teacher questioning in an article on reflective discourse and its impact on the mathematical development of students. Cobb and his colleagues observed a first grade classroom of eighteen students over the course of a year and conducted videotaped interviews of students where they attempted to assess the students’ development of mathematical strategies in solving tasks. Cobb et al. maintain that the teacher was instrumental in the
mathematical development of her students. In addition to the culture of inquiry that she
created in her classroom and the choice of tasks and activities that she used, Cobb et al.
ote the pivotal role she played in shaping the productive classroom discourse. They
maintain that one of the main ways that a teacher can "proactively support students'
mathematical development is to guide and, as necessary, initiate shifts in the discourse"
(p. 269). A simple "empirical verification" (p. 269) in which the teacher asked the
students how they could be certain that they had come up with all the possible solutions
to the task created an important shift in the discourse, causing the students to "step back
and reorganize what had been done thus far" (p. 269).

2.2.4 Other functions of teacher questioning

Citing Burbules (1993) and King (1999), Hmelo-Silver and Barrows (2008)
delineate many functions of questioning. Questioning can "help with goal setting, guiding
cognitive processing, activating prior knowledge, focusing attention, promoting cognitive
monitoring, and promoting displays of knowledge" (p. 53). In addition, questioning can
be used to obtain information when students display a lack of knowledge. It can be used
to gauge whether students share the same views and understanding, and whether there are
differences of opinion that can be discussed and reconciled. They may also be used as a
means to keep students focused on the task at hand and facilitate productive group
discourse.

2.3 Developing Good Questions

2.3.1 Questioning Techniques

McCullough and Findley (1983) stress that it is important for teachers to come
prepared with clear and concise questions that are both concrete and abstract in nature "in
order to involve all learners in moving toward more critical thinking” (p. 8). Hunter and Russell (1997) emphasize that teachers should first pose questions to the group and then address questions to individuals which can lead to more active involvement by all students. Moreover, modeling good questions can help students learn to ask good question when working with their peers and thereby develop their thinking and reasoning skills.

2.3.2 Reflecting on Questioning Techniques

Although good questioning techniques are vital for expanding student knowledge, Moyer and Milewicz (2002) found that many teachers use a low level of questioning. In a study that examined the questioning strategies used by 48 preservice teachers during audiotaped diagnostic mathematics interviews with children, Moyer and Milewicz found that teachers primarily used questions that involved what they classified as checklisting (asking brief factual questions), instructing rather than assessing, and a limited amount of probing and asking of follow-up questions. Their study demonstrated how vital it is for teachers to reflect on their questioning techniques to determine if they are to maximize the teaching opportunity when interacting with their students and to develop appropriate questioning skills that would elicit student reasoning by guiding the student without divulging the answer.

An article by Di Teodoro, Donders, Kemp-Davidson, Robertson & Schuyler (2011) discusses a collaborative study by four Canadian teachers conducted in grades 2 and 3 in different schools, which analyzed teacher and student questioning during three mathematics tasks. Their goals were to help students gain an understanding of the characteristics of a good question, to encourage high quality questioning to aid in
mathematical understanding, as well as to improve the researchers’ questioning skills. Using a framework for characterizing questions developed by Tienken, Goldberg, and DiRocco (2009), they categorized questions into two groupings, productive and reproductive. Productive questions otherwise termed by Di Teodoro et al. as “deeper” were questions that would “provide students the opportunity to create, analyze or evaluate; these questions are usually open-ended and divergent in nature.” Reproductive questions or “surface questions would include questions that prompt students to imitate, recall, or apply knowledge and information taught by the teacher, through a mimicking process.” Students solved the problems independently but then conducted a “gallery walk” in which they analyzed other students’ work and jotted down questions concerning the others’ solutions. Teachers primarily questioned during the problem solving activity, whereas the students questioned during the “gallery walk” and later class discussions. After categorizing questions as surface, deeper, or unknown, the researchers presented the top ten questions during a subsequent class session and led a discussion about the qualities of the good questions before students embarked on a second task. Results of the study showed that this methodology increased student and teacher confidence in asking questions and, more importantly, that it improved the quality of the questions asked from task to task. These researchers also emphasize how “surface” questions are sometimes very important as they lay the groundwork for “deeper questions.” They conclude that although modeling good questions is important and can help students acquire this vital art, it is also important for students to spend time on learning what constitutes a good question so that they can use this tool when collaborating with others or during class
discussions. Not only did the researchers improve their skills but the students benefitted as well by being part of the process.

2.4 Wait-time

McCullough and Findley (1983) assert that allowing for adequate wait time after posing the question is important to enhance the quality as well as quantity of student responses. Many studies have been conducted experimenting with extending wait-time between questioning and possible effects that it may have (Swift and Gooding, 1983, Tobin, 1980, 1986 DeTure & Miller, 1985). Some researchers (DeTure & Miller, 1985, and Fagan et al., 1981) found that with an increased wait-time the number of teacher questions decreased. Yet although there may have been less questioning, Swift and Gooding (1983) noticed by studying 600 class discussions (40 middle school science classes) that wait-time is crucial when asking questions. They found that if teachers merely waited 2-3 seconds after asking a question and that same amount of time before posing the next question, the quantity as well as quality of student discussions improved. In analyzing data from 20 classes from third to fifth grade, Fagan et al. (1981) also found that the cognitive level of teacher questioning improved in tandem with a small increase in wait time. Studying 20 middle school mathematics and language arts classes, Tobin (1986) obtained similar results. He found that in the mathematics classrooms where wait-time was manipulated, students were challenged with a proportionately greater amount of application questions versus procedural questions than in the control classrooms.
2.5 Classification of Questions

2.5.1 Showing and Telling, Leading, and Shepherding

Aiming to improve teachers’ methods of questioning, British middle school teacher and researcher Julia Towers (1998) sought to classify questioning techniques in order to identify what forms of questioning are effective for promoting student reasoning. She developed a list of themes when analyzing teacher questioning and interventions that she culled from videotapes of her rural British middle school class as well as a case study which she analyzed. In each of her three classes she videotaped a pair of students during three sessions (one pair of sixth graders and two pairs of seventh graders). In a second part of the study she videoed secondary students and their teacher in Vancouver. In this latter strand she focused on only one class during a period of eighteen sessions. In addition to videotaping, she also conducted interviews with teachers and students. Based on the data collected, she categorized teacher questioning into the following fifteen classifications: “showing and telling, leading, shepherding, checking, reinforcing, inviting, clue giving, managing, enculturating, blocking, modeling, praising, rug-pulling, retreating and anticipating” (p. 200). She then clustered these themes under the first three categories, showing and telling, leading, and shepherding on the grounds that these “represent broad teaching styles, rather than strategies” (p. 202). She defined teaching styles as “the broad practices that appear extensively within a particular teacher’s activities.” She found that the teachers in her study used these styles to a greater extent than her other intervention themes. She characterized the remaining themes as teaching strategies which she defined as “usually a brief intervention” and of which a teacher “might have a repertoire of many … upon which to draw” (p. 202). According to Towers,
showing and telling involved providing information without checking for understanding while leading questioning involved frequently asking low level questions. Shepherding was characterized as guiding questions “through subtle nudging, coaxing, and prompting” (p. 30). Although her work included a review of the literature involving teacher intervention and a novel and enlightening categorization of questioning approaches, Towers did not take her investigation to the next vital level of inquiry, which would have been to determine and demonstrate which of the documented interventions were instrumental in eliciting student reasoning.

2.5.2 General, Specific, Probing, and Leading Questions

Franke, et al. (2009) also created coding schemes for teacher questions. They studied three elementary school classrooms (second and third grades) in a large urban school district in Southern California with a 99% minority student body. Using data from videotapes of two sessions for each class, the researchers coded schemes of student and teacher participation. The students worked in small groups on a task, which was then followed by a class discussion before another task was assigned. The researchers focused specifically on follow-up questions posed by the teacher after students had formed some initial response. They were able to identify four types of teacher questioning: “general questions, specific questions, probing sequences of specific questions and leading questions” (p. 383). General questions were not directed at a specific part of a student response or explanation. Specific questions were asked to clarify one aspect of the student response or direct the students’ attention to one part of their explanation. Often teachers would ask a series of these pointed questions, or “probing sequences,” to enable a student to see the error in their statement or to help them enhance their explanations.
Leading questions were used to guide the students to come up with the correct answer or explanation.

These researchers found that teachers frequently asked students for explanations. Although they did so in a variety of ways, these follow-up questions did not always yield further explanations. They found that the posing of a probing sequence of specific questions stimulated students to offer a complete explanation, fill in gaps or correct a prior explanation to a greater extent than the other three kinds of questions. Their data also suggests that in order for teachers to elicit a complete explanation, many specific questions, each aimed at particular details of the students’ explanation, are required. This was true both in cases where the students provided an incorrect initial answer and where they gave an incomplete or ambiguous answer. This questioning technique was found to help the student as well as the teacher and the rest of the class understand the thought process used to come up with the solution, which in turn led students to correct or clarify their reasoning. The technique also enlightened teachers to make better decisions concerning the assignment of additional problems as well as the choice of questions to be used at other opportunities. In this study, the researchers found that leading questions “did not provide opportunities for students to build on their own understanding” (390). They suggest that further research would be useful in this area.

2.5.3 Pumping, Reflective Toss, and Constructive Challenge

Chin (2006) describes different forms of questioning that were effective in eliciting reasoning. Among the questioning techniques she mentions are “pumping” (Hogan & Pressley, 1997), “reflective toss,” (van Zee & Minstrell, 1997) and “constructive challenge.” Pumping places the burden on the student to provide more
information and to further communicate their ideas and would include questions like “What else?” A reflective toss is when a student asks a question or makes a comment, and the teacher refrains from responding but rather throws the question back at the student and possibly hints at ways that the student can come up with an answer on their own. A teacher can also use a “constructive challenge” in response to an incorrect answer instead of directly correcting the student by urging the student to think about and reconsider their answer.

2.5.4 Probing, Guiding, and Factual Questions

Sahin and Kulm (2008) maintain that teachers should be aware of what types of questions they use in their classrooms as well as the motives behind those questions. They analyzed the questions used by a first-year teacher and an experienced teacher using videos of five sessions for each teacher that were taken in two sixth grade Texas public school classes as part of a five-year longitudinal study. Both teachers were teaching the topic of equivalent fractions and conversion between fractions, decimals, and percent. They studied the relationship between the types of questions asked and the juncture during which they were asked. Additionally, they interviewed the teachers to understand their motives behind asking certain questions after showing them short video clips of their class sessions. They categorized teacher questions into the following three categories: probing, guiding, and factual. Probing questions compel students to go beyond remembering facts to build and extend on prior knowledge and encourage deeper thinking. These questions asked students to explain their thinking, apply their knowledge to a specific problem at hand, or to offer justification for their answers. Sahin and Kulm also focused on guiding questions which were asked to help students who were unsure of
how to proceed in a problem, or used to encourage students to remember a certain strategy, or guide them by asking a series of factual questions which would help them scaffold understanding. Guiding questions are thus similar to leading or helping questions. Factual questions that asked the students to recall a fact or state a definition, solution, or procedure for solving a problem, comprised most of the questions asked during the session. This was despite the fact that the lessons were based on reform curricula where the students were given the opportunity to work collaboratively and the teachers attempted to ask many questions rather than lecture exclusively. Sahin and Kulm divided each lesson into four parts, an introduction, a development section, practice and examples and a summary segment. They coded each question asked by the teachers as factual, probing, or guiding and calculated the frequency of those questions per segment of the lesson.

Sahin and Kulm’s (2008) findings were similar to that of Bloom, Englehart, Furst, Hill, and Krathwohl (1956) who found that the majority of the questions posed were factual in nature. They did find that teachers with minimal experience did pose more probing questions than the teacher with a number of years of experience. Although both were using reform based textbooks which offer suggestions of incorporating probing questions into the lesson, Sahin and Kulm posit that the new teacher may have used the book more extensively than the experienced teacher who relied on her traditional teaching methods of assisting students in coming up with solutions and following procedures. They found no correlation between the part of the lesson and the type of question asked, though they did find that one teacher used probing questions often when working with manipulatives since such activity may involve more student interaction and
stimulate student thinking. They also found that probing questions were used with greater consistency during the summary portion of the lesson. Sahin and Kulm suggest that this may be due the fact that teachers have higher expectations of student understanding at the end of the lesson. Yet the large number of probing questions was often the result of asking a number of students similar questions rather than encouraging students to deepen their understanding and form generalizations. The researchers gleaned from the interview data that teachers were cognizant of the motive behind particular questions and what they were trying to elicit from the students. They also found that reflection through watching clips of their sessions was beneficial in that it helped the teachers focus on the advantages of different question types. They conclude that probing and guiding questions are important to foster a classroom environment which supports student conjectures, justifications, and knowledge building.

2.6 The Effect of Questioning

2.6.1 The Effect on Student Responses

In an analysis of seven 45 minute videos in college classrooms primarily comprised of freshman students, Andrews (1980) looked at the different forms of questions used by seven instructors. He classified questions as either “divergent thinking questions” or “convergent questions” (p. 131) based on Guilford’s (1968) classification. Divergent thinking questions can yield many correct answers and are therefore more likely to lead to further discussion. Convergent questions have only one possible solution and therefore student may not participate as much for fear that they will supply the wrong solution. In addition, discussion may be limited by a student providing the answer to the question. He found that questions that yielded the most productive discussions were those
that could be called structured divergent questions. These questions allowed for leeway in responses yet also were sufficiently focused so that the discussion was given adequate direction. He found that although this was consistent across instructors, the richest discussions were found in the classrooms of instructors who consistently utilized such questions. Andrews also studied the wait-time and noted the importance of giving students time to process information. Additionally, he mentions the ill consequences of questions which engendered confusion due to their ambiguity and therefore resulted in prolonged silences from the students. Andrews emphasizes the importance for teachers to capitalize on Bloom’s (1956) taxonomy of questions and to form questions that require the use of analysis, synthesis, and evaluation. He maintains that higher level questions, which allow for analysis, synthesis, or evaluation, will also result in richer discussion than lower level questions that merely expect students to display basic comprehension or memorization. He also stresses that the questions should be stated clearly and that teachers can mention explicitly that many answers are acceptable. He also cautions against the use of questions which he calls “Shotgun” and “Funnel” (p. 150) questions which result in cognitive overload. Andrews coined the term “Shotgun questions” for instances where the instructor asks more than one question together which are inconsistent in that they include different content areas or one question requires deeper analysis while the second requires merely a response with factual information. Instructors may use such questions hoping that at least part of the question will be answered but students may be deterred from responding since they need to first dissect what is being asked of them. Andrews uses the term “Funnel questions” to characterize questions that comprise a series of questions that begin with a broad question but continue with more
structured questions. Here the instructor has a particular response which he/she would like to pull from the students and continuously gets more specific in the queries in order to garner that answer so that the later questions are ultimately more convergent in nature. Andrews stresses that if questions are met with silence by the students, rather than posing a different question, the instructor should evaluate whether there is a need for further clarification or restatement.

Arnold, Atwood, and Rogers (1974) studied the relationship between the level of teacher questions, lapse time, and the responses provided by the students. They studied taped sessions of eleven elementary school teachers in a suburban middle class neighborhood, who had been trained to use questions at higher levels of Bloom’s taxonomy and had been encouraged to allow for an extended wait-time to allow for student responses. Class sessions in science, social studies, math, and language arts were studied with students ranging in age from six to twelve, and a total of one hundred and eighty questions and response segments were analyzed. Three researchers independently classified each question and response using Bloom’s taxonomy. Results from their analysis showed that there was a strong relationship between the question level and the level of response by the students. Although the teachers had been trained to ask questions at all levels of Bloom’s taxonomy, they found that the questions asked with the greatest frequency were at the comprehension level and that the teachers asked very few analysis questions. They attributed this to the fact that teachers may have difficulty forming analysis questions for elementary students. They also found that the time it took to respond to such questions was longer than for the other question types, which they claim indicates the difficulty inherent in such questions.
Cole and Williams (1973) studied audio tapes of lessons conducted by eight teachers with grade levels ranging from second to sixth grade, and found a positive correlation between the level of teacher questioning on the level, length, and syntax of student responses. They concluded that asking higher level questions stimulated higher level responses and that higher level responses were more complex in structure and length than lower level responses.

Dillon (1982) also analyzed the correlation between teacher questioning levels and student responses. Using categories from Bellack, Kliebard, Hyman, and Smith (1966), he classified utterances as defining, interpreting, fact-stating, explaining, opining, or justifying. He also made a distinction between teacher questions and teacher statements. In a sample of 477 teacher-student exchanges, he noted that there was a weak correlation between the level of teacher questions and students responses. He found a stronger correlation between high level teacher statements and high level student responses. He therefore dismisses the assumption that higher level questions will elicit higher level responses since his data indicated that only about half of the time was there any correlation between the two.

2.6.2 The Effect on Student Achievement

In a review of literature on teacher questioning, Gall (1984) states that research has shown that teachers utilizing higher cognitive questioning, which encourages students to do independent thinking, rather than questioning students on facts that had been previously presented to them, had a positive effect on student achievement. Cunningham (1971) and Hunkins (1972) suggest that research has shown that certain questions and strategies are more effective than others for stimulating student thinking and resulting in
greater student achievement. In studies of student achievement in social studies classes, Buggey (1972) and Rogers and Davis (1970) concurred with those findings.

Gall et al. (1978) studied audio tapes of ten 50 minute ecology lessons taught to 336 sixth graders. Two classes in each of six schools were analyzed. They found that asking probing questions so that students could elaborate on their initial response or redirecting questions to other students for elaboration had little effect on student achievement. Similarly, little correlation was found between higher level questions and student achievement. He therefore contested the truism that higher questioning levels impact student achievement.

Winne (1979) also found that there were no significant differences between the effects of high order and low order questions. He selected and reviewed 18 studies conducted on teacher questioning across the spectrum of content knowledge. In these studies there was either a contrast between a group of students being taught with teachers predominantly using fact questions and one being questioned primarily with higher cognitive questions, or a comparison between a period of time in which students’ were being asked mostly fact questions and a later time where they were being asked proportionately more higher cognitive questions. He defines fact questions as questions intended to elicit from the students material that which had been taught to them previously. He deemed such questions as “convergent” or “lower cognitive questions.” “Higher cognitive questions” or “divergent questions” were questions that required the students to “mentally manipulate bits of information previously learned to create an answer, or to support an answer with logically reasoned evidence” (p. 14). The
procedures necessary to answer such questions would be similar to Bloom’s taxonomy of application, analysis, synthesis, and evaluation (Bloom et al. 1956).

Winne (1979) found many methodological flaws within these studies. For example, he asserts that in order for the studies to bear significance, the sessions would have to be replicated albeit with a different teacher or different group of students. But Winne argues that this is nearly impossible even if the teachers followed a prescribed set of questions. Teacher behaviors occurring during the class sessions may have varied across contrasting groups, thereby impacting student achievement as well. For example, one teacher may have corrected students’ misconceptions, whereas the other teacher may have allowed students to challenge the solution. Most importantly, Winne emphasizes that it had not been proven empirically that higher cognitive questions result in the improvement of students’ cognitive processes of manipulating and recalling information. He maintains that testing student achievement some time after the experiment has been conducted (such as with a post-test) is inadequate and the focus should be on the immediate impact of those questions on students’ cognitive processes. Winne divided the studies into three categories, those that showed significant positive results, those that showed negative results, and those that showed insignificant results. 60% showed no significant results and merely 15% showed that higher cognitive questions resulted in greater student achievement.

Redfield and Rousseau (1981) analyzed thirteen of the same studies as Winne and added one additional study. They contest Winne’s findings and attempt to demonstrate that these studies did indeed show that the predominant use of higher cognitive questions in the classroom results in greater student achievement. Their study calculates effect size
to determine the influence of “program monitoring, experimental validity, and level of teacher questioning” (p. 237) and yields an effect size of +.7292. Their results indicate that on average students scored in the 77th percentile after being taught with higher order questioning versus scoring 50th percentile in the absence of such questioning. From data yielded by studies whose validity was closely supervised, they gleaned that greater student achievement can be expected when teachers are trained to use higher cognitive questions.

Samson, Styrkowski, and Weinstein (1987) conducted a quantitative synthesis to analyze the effects of teacher questioning on student achievement. Building on the work of Winne (1979) and Redfield and Rousseau (1981) they conclude that whereas higher cognitive questions indeed positively affect student achievement, the size of that effect is not as great as that posited by Redfield and Rousseau. They assert that although the studies demonstrate only a small median effect, optimal teacher questioning could possibly yield a moderate or large effect but this has yet to be proven. After considering 44 studies, they chose to examine fourteen of these studies, which had sufficient statistical data available, where the independent variable was teacher questioning that could be classified as high or low cognitive level, and where the dependent variable included a measure of student achievement. They found only a small effect with a median effect size of .13 and thereby challenged Redfield and Rousseau’s findings and supported Winne’s assumption that a large effect has yet to be shown.

Brophy (1986) discusses classroom practices which affect student achievement. In a section on teacher questioning, Brophy asserts that students should be able to provide the correct answer to most questions. Certain basic skills will require the teacher to ask
many questions in a drilling fashion that students should be able to answer correctly, but when teachers are attempting to encourage their students to evaluate or apply their knowledge, it may be necessary to ask questions that only a few students can answer or ask questions for which there are multiple correct answers. Citing the conflicting results found by Redfield & Rousseau (1981) and Winne (1979) on the benefit of using higher cognitive questions (at the application, analysis, synthesis, evaluation levels) rather than lower level questioning (at the knowledge and comprehension levels), Brophy notes that even if one can claim that higher level questions do correlate positively to achievement, data from the studies indicates that when the teachers attempted to teach at the higher cognitive levels, a mere 25% of the questions asked were in fact at a higher level.

Brophy emphasizes the importance of lower level questions as a means of leading to higher level questions. He also points out the need for research on question sequences rather than on individual questions since some situations would warrant starting with higher level questions and then probing for details by asking lower level follow-up questions. Other class discussions would benefit from beginning with lower level questions to ensure that students are aware of necessary facts before asking higher level questions that would expect the students to integrate those bits of information. Therefore he maintains that research should focus on the teacher’s objectives, as well as the relevance, quality, and timeliness of the questions asked.

A study by Boaler and Staples (2008) analyzed three mathematics classes comprising 700 students from different schools. Whereas the Railside school students from ethnically diverse backgrounds were being taught with a reform-oriented approach, the other two classes of primarily white students were being taught using more traditional
methods. The researchers studied student achievement as well as attitudes of the students toward mathematics. In addition, they coded teacher practices. One area of interest in this study was their description of the questioning methods utilized by teachers at the Railside school. The researchers classified teacher questioning into three categories, probing, extending, and orienting. Their findings were that the students in the Railside school had larger achievement gains as well as a more positive attitude toward mathematics. At the Railside school, teachers spent only about 4% of the class session presenting lecture material. About 9% of the class session included questioning the students in whole class discussions. During the majority of the session (approximately 72%), the teachers circulated around the room while the students worked on long conceptual problems. They assisted the students and queried them about the work that they were performing. They found that due to careful deliberation and consideration about the questions that they were to ask during each class session, Railside teachers’ questions were more varied than the questions asked of students in the other schools. They found that 62% of the questions were procedural, 17% conceptual, 15% probing, and the remaining 6% were other kinds of questions. The teachers spent time before class comparing and sharing questions that they would ask. Although teacher questioning was not the only focus of this study, it found that it played a large role in student achievement. The researchers emphasized follow-up questions on problems the students solved, and although not explicitly stated in the curriculum, “teachers’ questions significantly shaped the course of implementation” (p. 627) of the tasks. In the traditional algebra classes, the majority of the questions asked were procedural in nature (97% and 99% of the two teachers’ questions respectively). Interestingly, when students at Railside were asked “what does it take to be successful in
mathematics class?’ they named practices such as “asking good questions, rephrasing problems, explaining well, being logical, justifying work, considering answers, and using manipulatives.” When the same question was posed to students in the traditional classes the main responses were that they “needed to concentrate, and pay careful attention (p. 629).”

2.6.3 The Effect on Student Reasoning and Justification

The work of Julia Towers mentioned earlier prompted Amy Martino and Carolyn Maher, researchers in a longitudinal study conducted by faculty at Rutgers University, to analyze how questioning at specific points of a student’s work could serve as a catalyst for development of student reasoning. Martino and Maher found that, in general, students do not usually look to build proofs or justify their solutions. They often believe that their solution itself constitutes sufficient justification. They also found that under natural circumstances, even students working in groups or pairs do not question each other regarding the validity of their solutions. Hence, they posited that it is precisely at the point when students working alone or in groups have come up with a solution or done as much as they could without teacher intervention that teacher questioning becomes essential. Their work therefore focused on this juncture, which they identified as the one at which the interplay between teacher and student is most critical. In their study, they selected a few instances during the longitudinal study where teacher questioning impacted student justification and reasoning. The researchers, teachers, and principals who were to interact with the students in the longitudinal study were cautioned beforehand not to provide any solutions, nor to tell the students whether their solutions were incorrect or not, but merely to listen as the students worked together and to question
them on their work. Martino and Maher’s study described three episodes where teacher questioning facilitated a justification, where questioning offered an opportunity for generalization, and where researcher questioning invited the learner to make connections. Their work demonstrates dramatically how a few simple questions were able to elicit sophisticated reasoning and justification (Martino & Maher, 1999).

Mueller, Yankelewitz, and Maher (2014) used Sahin and Kulm’s (2008) classification of guiding and probing questions and to identify two types of teacher interventions, eliciting an idea and encouraging explanations and justifications. They also looked at teacher moves that were used to make a student’s ideas public. Mueller et al. looked at the activities of a class comprised of 24 Latino and African American students during five, 90 minute sessions as they worked on fractions using Cuisenaire rods. Using video data, they analyzed the effect of teacher moves on student arguments, ideas, and solutions by identifying those three teacher interventions which encouraged students to come up with their own ideas and strategies, explain their thinking, generalize their findings, and make connections. They found that during the first two sessions a large percentage of teacher moves involved making students’ ideas public. During the last three sessions, although making students’ ideas public constituted the greatest percentage of teacher moves, there was a large increase in questions that elicited student ideas. Mueller et al. found that teacher questions or elicit student ideas were often used when the students were working on tasks in small groups that proved to be a challenge for the students such as in a session where the students worked on comparing fractions with different denominators. By asking a series of questions, the teachers tried to encourage students to formulate their ideas and to persist in coming up with a solution. Questions
that encouraged justification and explanation occurred most frequently when students were working on tasks that allowed for different ways of solving the problem. These tasks were often discussed by the whole class and encouraged students to share their differing ideas or to concur with and support another’s arguments and many teacher moves and questions were aimed at helping students make their ideas public. These interventions fostered a classroom environment where students felt comfortable challenging or defending their solutions and allowed them to take responsibility for their learning and assume roles that may have otherwise been left to the teacher. Since the teachers considered all solutions and didn’t correct the students’ assumptions, students gained confidence in their ability to determine the validity of their arguments.

In a book chapter describing the process in which eight year old children build arguments in a supportive environment, Maher (2009) underscores the importance of the researcher questions in eliciting more “articulate explanations both among groups and within groups of students” (p. 130), Maher analyzed two videotaped sessions, each approximately 90 minutes long, where the students worked on a task which consisted of finding all possible four-tall towers that could be constructed by selecting from red and yellow plastic cubes. Encouraged by researcher questions, the students developed more “efficient organizational schemes” which elicited more elaborate arguments as to why they believed they had found all possible patterns. Upon further questioning, some students furnished compelling arguments including arguments by cases, contradiction, and recursive reasoning. Maher emphasizes that it was simply questioning the students and asking for justifications of their solutions that brought about significant proof-like arguments and “elegant verbal justifications” (p. 130). The researcher elicited remarkable
reasoning by young children not by trying to coax them to reason in a certain way but merely by encouraging them to voice their ideas and convince others of their solutions.

In a study conducted by Hmelo-Silver and Barrows (2008), the researchers analyzed five hours of discourse over two problem-based learning (PBL) group meetings and specifically focused on the scaffolding of learning through facilitator questions and the characteristics of the discussion that occurred during those interactive sessions. The sessions were led by expert facilitator Barrows (the second author of the study). In the first session, five second-year medical students were presented with a problem of a patient with pernicious anemia. In the first session, which involved a lot of problem solving discourse, the students attempted to pinpoint what they needed to learn. During the second session they applied their knowledge to the particular problem given to them and reflected on what they had known before approaching the problem and how their thinking had developed through their intellectual analysis and discourse.

Working with videos and transcripts, Hmelo-Silver and Barrows (2008) studied the types of questions asked. They coded the questions using Graesser and Person’s (1994) taxonomy of questions. Graesser and Person (1994) had divided questions into two categories, those that would require a short answer such as “yes” or “no” and those that would necessitate a longer answer consisting of multiple sentences. Short answer questions included verification, disjunctive, concept completion, feature specification, and quantification. Long answer questions included definition, example, comparison, interpretation, causal antecedent, causal consequence, goal orientation, instrumental/procedural, enablement, expectational, and judgmental. Additionally they coded for assertion where the questioner indicates that he/she does not comprehend or is
lacking knowledge or a request/directive where the questioner would like the listener to perform a specific action. Hmelo-Silver and Barrows utilized these categories but also incorporated some additional categories under the heading “task oriented and meta questions” (p. 62). These additional “task oriented and meta” question categories included group dynamics, monitoring, and self-directed learning. They included Graesser and Person’s request/directive classification in this category and they seem to have renamed Graesser and Person's assertion category as "need clarification" (p.62) since the objective of that category seems similar to the one defined by Graesser and Person.

Citing King (1999), Hmelo-Silver and Barrows point out that the first two long answer questions, definition and example, can help lead to “comprehension-oriented discourse,” whereas the others would have a greater impact on knowledge building since they necessitate greater reasoning, clarifying and reorganizing their ideas, and use of inference.

Hmelo-Silver and Barrows (2008) found that the facilitator used many questions from the task-oriented and meta category to direct the students’ focus on their self-directed learning and to aid in monitoring their progress. They found that definition and interpretation questions early on in the session provided the student with the opportunity to see that their understanding of certain topics was limited and highlighted which areas would require additional study. Some questions were effective in eliciting causal reasoning while others served as a catalyst for the students to explain their ideas and helped them problematize. Yet others pressed the students to think more deeply and keep the discussion moving. One particularly lengthy and rich discussion resulted from a facilitator prompt to draw a diagram. The researchers emphasize that questions asked in
an IRE discourse may serve as a tool for evaluation but the same question type when used in a student-centered environment could help the students recognize that their knowledge in a particular area is lacking and prompt them to gain more understanding of that topic. They stress that the open-ended metacognitive questions, which served as scaffolds until the students were able to internalize the concepts, provided a conductive environment for knowledge-building, where students in turn modeled high-level questions, reanalyzed their ideas, refined them and built upon one another’s ideas. Facilitator-led sessions helped generate a collaborative environment of productive knowledge building.

2.7 Summary of Findings

The extant research discusses the importance of teacher questioning in the development of students’ mathematical thinking and highlights many benefits associated with teacher questioning. Questioning can guide students to examine their conjectures and justify their solutions and help them feel comfortable sharing their ideas and arguments with others (Burns, 1985; Burbules, 1993; Hmelo-Silver and Barrows, 2008, & King, 1999). Kazemi and Stipek (2001) found that teacher questioning played a large role in inquiry-based learning and increased the level of discourse and argumentation amongst the students. Hmelo-Silver and Barrows (2008) stress that questioning is necessary to ensure that students are actively engaged in their learning. Cobb et al. (1997) point out that teacher questioning can be used as a tool to create shifts in classroom discourse. Simply asking for empirical verification of a solution helped students evaluate what had been done so far and encouraged them to make an assessment of what they needed to do in order to successfully complete the task. McCullough and Findley (1983) stress the importance of teacher preparation of questions to be used during
the lesson, and Hunter and Russell (1997) provide techniques that increase active engagement through questioning. Moyer and Milewicz (2002) found that preservice teachers used primarily factual questions and emphasized the importance for teachers to reflect on their questioning techniques to ensure that they utilize questioning as a means of guiding students without providing solutions. Di Teodor o et al. (2011) found that having teachers classify their questions as surface questions or deeper question helped to increase their confidence in asking questions and improved the quality of their subsequent questions as well. Swift and Gooding (1983) highlight the benefit of increasing wait-time by 2-3 seconds for improving the quantity and quality of student discussion. Fagen et al. (1981) and Tobin (1986) even found that the cognitive level of teacher questioning improved as well in tandem with a small increase in wait-time.

Several research articles discuss different classification of questions. Towers (1998) classified questions into fifteen categories under three main themes of showing and telling, leading, and shepherding. Franke et al. (2009) classified questions as general, specific, probing, or leading and found that a probing sequence of questions was often most effective in stimulating students to explain their thoughts and solutions fully or to correct a previous argument. They found that leading questions did not encourage students to build understanding. Chin (2006) discusses the merits of pumping, reflective tosses, and constructive challenges in eliciting reasoning. Sahin and Kulm (2008) categorized questions as factual, probing, or guiding and tried to determine whether or not there was a correlation between the question type and the part of the lesson regarding which the question was asked. They found that even though the teachers were attempting to teach in a reform manner, most of the questions asked were factual in nature. They did
not find a correlation between the question type and the segment of the lesson. They also found that teacher reflection after watching videos of the class sessions was beneficial in making teachers cognizant of the benefits of different question types. Andrews (1980) posited that the most productive discussions resulted from structured divergent questions. He advises against the use of “Funnel” or “Shotgun” questions and stresses that questions that require analysis, synthesis, and evaluation, result in richer discussion.

Arnold et al. (1974) found a relationship between the question level and the level of response by the students. Cole and Williams (1973) also found a positive correlation between the two and found that the complexity and structure of the responses were linked as well. Dillon (1980) contests the fact that there is a strong correlation between question level and student responses since his data showed them to be correlated in only half of the instances. Other researchers studied the correlation between the level of teacher questioning and student achievement with conflicting findings (Buggey, 1972; Cunningham, 1971; Hunkins, 1972; Gall, 1978, Redfield and Rosseau, 1981; Rogers and Davis, 1970; Samson et al., 1987; Winne). Brophy (1986) cites these contradictory findings and emphasizes that there are occasions where low level questions are indeed beneficial and therefore research should be focus on the relevance, quality, and timeliness of questions rather than seeking to prove the correlation between the level of teacher questioning and student responses. Boaler and Staples (2008) analyzed the questioning in a reform-oriented classroom and found them to be more varied in nature than those used in traditional classrooms and noted that this affected student attitude toward mathematics as well. Martino and Maher (1999) and Maher (2009) noted that students do not usually justify their solutions without prompting and underscored how a few simple questions
were instrumental in eliciting remarkable reasoning and justification. Mueller et al. (2014) demonstrated how teacher questioning fostered a classroom environment conducive for reasoning and justification. Hmelo-Silver and Barrows (2008) emphasized the need for questioning in a problem-based learning environment and using Graesser and Person’s (1994) classification of questions detailed the varying impact that different question types had on student knowledge building and self-directed learning.

In this review of the literature, I synthesized the research from articles that discuss the importance of questioning, questioning techniques, different classifications of questions, and the effect of questioning on student response, achievement, reasoning, and justification. Although I began by reviewing literature in the area of mathematics education, I found that there were not many studies that dealt with mathematics education exclusively and that there is much to be gleaned from research in other domains such as learning sciences and general education among other content areas. I therefore included important and relevant studies from those areas as well.

This study builds on the literature by identifying instances of teacher questioning that was followed by instances of student reasoning in multiple settings and by multiple teachers/researchers. My study is positioned in the longitudinal/cross sectional research study of the development of students’ mathematical thinking and reasoning conducted at Rutgers University that spanned over twenty-five years. I have taken the vital work of Martino and Maher (1999) further by more comprehensively identifying and studying occasions of researcher questioning and how their questioning was followed by ways the students formulated their solutions, extended their reasoning, made connections, or otherwise enhanced or refined their solutions. Martino and Maher selected a few
instances during the longitudinal study where teacher questioning impacted student justification and reasoning. This study builds on that important work by looking at numerous instances of teacher researcher questioning and the associated student responses, reasoning and justification.

CHAPTER 3: METHODOLOGY

To answer the research questions, I conducted a qualitative study. Due to the fact that this study occurred in a natural setting rather than in a controlled setting, the data lent itself to a qualitative approach. In addition, this study will attempt to portray a detailed view of the topic which can be best carried out using qualitative measures (Creswell, 2006). This study is bound by the number of researchers and sessions to be analyzed.

3.1 Setting

3.1.1 Introduction

My study is positioned in the longitudinal/cross sectional research study of the development of students’ mathematical thinking and reasoning conducted at Rutgers University. The study encompassed data drawn about student mathematical activity from three school districts, Kenilworth, New Jersey, New Brunswick, New Jersey and Colts Neck, New Jersey. This study will examine selected data from the Kenilworth and Colts Neck schools as well as analyze data from the Informal Math Learning after-school program conducted in Plainfield, New Jersey.

3.1.2 Fraction Sessions Led by Researchers Carolyn Maher and Amy Martino at Colts Neck

At the Conover Road School, fraction operations were introduced in the fifth grade. Between September 20 and December 15, 1993 fourth grade students spent
twenty-five sessions exploring fraction ideas. Prior to these sessions the students had merely had an introduction to the idea of a fraction as an operator in third grade. The researchers were interested in how students build fractions as number, construct representations, and ultimately compare fractions and display ideas about fraction equivalence. Of significance in this study as in all parts of the longitudinal study is the fact that researchers did not provide solutions for the problems or indicate whether the students had arrived at the correct solution or not. All educators and researchers were cautioned not to provide any solutions to the students or to tell them if they were right or wrong but were merely allowed to question the students and observe them interacting with their peers. Thus, the students took control of their work and this necessitated the validation of their solution through argument and proof with fellow students and researchers. If a question was not resolved during one session it was usually revisited in a later session.

3.1.2.1 The Colts Neck Sample

The Colts Neck school class consisted of fourteen girls and eleven boys. The main researchers who conducted the sessions were Researchers Carolyn Maher and Amy Martino. The classroom teacher was present for all sessions as well. Some sessions were attended by other members of the Rutgers research team such as Researcher Robert B. Davis. The principal of the school, Judith Landis attended some sessions as well. These educators and researchers occasionally posed questions to the students (Yankelewitz, 2009).

The students were grouped in pairs (with one group of three) and encouraged to discuss their findings with their peers as well as interact with other class members and
participate in group discussions. After students worked with their partner(s), the researcher usually conducted a class discussion where she would discuss the task at hand and give students the opportunity to demonstrate their solutions and explain their reasoning at the overhead projector or blackboard. After they had justified their solutions, the rest of the class was encouraged to voice their opinions and either corroborate or challenge those findings. The students used a variety of manipulatives to illustrate their solutions, such as Cuisenaire rods, ribbons and meter sticks (Yankelewitz, 2009).

3.1.3 Guess My Rule Sessions Led by Researcher Robert B. Davis at Kenilworth

The Harding School is a public K-8 school in Kenilworth, New Jersey. Funded by NSF grants MDR-9053597, REC-9814846, and REC 07071578, Rutgers University collaborated with the Harding School beginning in 1984. Researchers followed a group of students starting in first grade in order to analyze the development of mathematical ideas in children. They continued to follow some of the students during their high school and college years. The study included four strands: 1) Combinatorics and counting 2) Probability 3) Algebra and 4) Precalculus.

Researcher Robert B. Davis, New Jersey Distinguished Professor of Mathematics Education, led a number of sessions at the Harding Public School in Kenilworth where he introduced 12 sixth-graders to algebraic ideas prior to their formal study of algebra. On 9/30/1993 students worked on variables, open sentence, truth set notation, solving linear equations with one variable, solving quadratic equations, solving linear equations with two variables and the Pebbles in a Bag activity which helped students think about numbers with different signs. Davis initially used boxes and triangles in open sentences to represent variables so that the students could learn how to solve for an unknown. He
eventually transitioned from boxes and triangles once the students seemed to have understood the concept of a variable. This session has been analyzed by Spang (2009).

At the end of the session on 9/30/1993, Davis spent approximately ten minutes introducing the Guess My Rule activity. The next session on 10/1/1993 was devoted to the Guess My Rule activity. The sessions on Guess My Rule have been analyzed by Giordano (2008). Mayansky (2007) studied the remaining sessions (10/29/1993, 11/1/1993, 11/12/1993, and 11/15/1993) which were devoted to solving the Towers of Hanoi.

The data for this study are taken from video tapes of sessions that occurred on 9/30/1993 and 10/1/1993 where students worked on the Guess My Rule activity. The students worked with partners, in small groups, and participated in whole class discussions. During these sessions, he encouraged sense-making through questioning to provide opportunity for student understanding of linear and quadratic function ideas.

These classroom sessions were videotaped with three or four cameras. The cameras tried to capture activity at the board and student-researcher interaction. In addition the cameras often focused on groups of students working at their desks as well as on individual work.

3.1.3.1 The Kenilworth Sample

Twelve students participated in this study (Michael, Milin, Jeff, Matt, Stephanie, Ankur, Michelle I, Michelle R, Brian, Romina, Bobby, and Amy Lynn) as part of the longitudinal study. Researcher Robert Davis used effective pedagogy to engage the students. He encouraged the students to make their thinking explicit and public, invited
discourse and conversation between students, and provided students with the opportunity to investigate and use reasoning to come up with solutions to the problems.

3.1.4 Informal Math Learning Program with Researcher Arthur Powell in Plainfield

The Informal Math Learning program (IML), a three year longitudinal study, was started in fall 2003 in the Frank J. Hubbard Middle School in the district of Plainfield, New Jersey. Funded by NSF grant REC - 0309062, researchers from Rutgers University led by Carolyn Maher, Arthur Powell, and Keith Weber, worked with two cohorts of 20 middle-school students in an after-school program. In spring 2004, the program began with a pilot study where seventh grade students in the first cohort worked on Guess My Rule, chosen from Researcher Robert B. Davis's book "Discovery in Mathematics" (1967). The data used in this study comes from the second cohort of seventh graders which worked on similar problems involving linear and quadratic functions during the fall of 2005, finding relationships between two variables. At first the numbers that they were given to work on involved a positive slope but as they successfully completed those tasks the students were introduced to tasks where there was an inverse relationship between the variables with negative slopes and negative y intercepts. This activity as well as the other tasks that the students were given were based on earlier research conducted in the longitudinal study conducted at Kenilworth, Colts Neck, and New Brunswick. In this program though, students had access to and use of the Casio ClassPad 300 educational software. The fall sessions consisted of nine after-school sessions each about 90 minutes long. The dates for those sessions were as follows: 10/27/05, 11/02/05, 11/03/05, 11/16/05, 11/17/06, 12/01/05, 12/07/05, 12/08/05 and 12/15/05. Data from these IML
session were recorded on 53 video CD’s which span eight session. In addition there is a recording of an interview with one of the students (Ariel) where he discusses his solution to the Geese problem. This study focuses on the data that relates to the students working on the Guess My Rule problem during sessions on 11/02/2005 and 11/03/2005.

3.1.4.1 The Plainfield Sample

The Plainfield public schools are comprised of 98% African American and Latino students. During these after-school sessions there were typically 6 to 10 students who attended and worked on the assigned task either alone or with a partner (Baldev, 2009).

3.2 Tasks

At Colts Neck the students worked on fraction activities in pairs and were asked to share their solutions with their classmates. Usually this was one by building models on the overhead projector and explaining and justifying their solutions. The other students were asked if they agreed with the proposed solution, or if they had any problem or alternate solution to offer. During most of the sessions on fractions the students used three-dimensional Cuisenaire ® rods to build their models. Cuisenaire rods are a set of wooden rods consisting of ten rods of different colors and lengths. The shortest rod in the set measures ones centimeter and the longest rod in ten centimeters in length (see figure 3.1 below for an image of the rods). The students also utilized a set of two-dimensional, transparent Cuisenaire rods when presenting their solutions on the overhead projector. Student constructed models using individual rods as well as created models consisting of rods placed end to end which they called trains (see figure 3.2 for an example of a train model). The students also presented their ideas using transparencies on the overhead
projector. In addition, they often recorded their solutions on paper.

*Figure 3.1. Staircase model of Cuisenaire rods*

*Figure 3.2. A train of light green and one red rod alongside a brown rod*

Researchers Amy Martino’s interview with Brandon, a fourth grade student from Colts Neck, involved the Pizza problem and the Towers Problem. Brandon used paper and markers to draw tables for the Pizza Problem. When he worked on reconstructing his Towers he built models utilizing yellow and red Unifix cubes.

At Kenilworth the students, under the direction of Researcher Robert B. Davis worked on a problem called Guess My Rule. The activity Guess My Rule works as follows. One or more student make up a secret rule such as "Whatever number you tell me, I will multiply by three and add four." Other students attempt to guess the rule by providing numbers to the rule makers and analyzing the results that they provide in turn. Alternatively, students are given a table with two columns. One column is labeled with a square and one with a triangle. Using the values in the table, the students attempt to
determine what operations need to be done to the value in the square column to produce the value in the triangle column. This activity was used as a means of introducing the concept of function. According to Alston and Davis (1996) the idea of a function is that there is some systematic procedure so that, whatever number is provided, one knows precisely what to do with it and is able to provide the correct answer. Researcher Robert B. Davis originally introduced this activity in the Madison Project (Davis, 1967, 1980), named after the Madison School in Syracuse, New York. Davis directed the Madison Project from 1965 until the 1970's. Davis used the materials developed in this project to introduce the concept of function to students ranging in age from young children to college-age students.

During the after-school IML program the students worked as well on the Guess My Rule problems. They were given a table of values and asked to find the relationship between the two variables. In the first few problems there was a direct relationship between the variables, and therefore a positive slope. Eventually the tasks included examples where there was an inverse relationship between the variables with negative slopes as well as negative y intercepts. The Plainfield students in the IML program also utilized the Casio ClassPad's graphing feature to help them solve the problems. After working on Guess My Rule students worked on three additional activities: the Ladders problem, the Geese Problem, and the Museum problem.

3.3 Data Collection and Validity

In order to ensure validity I have triangulated the data therefore culled data from the portfolio of videos, transcripts, and researcher notes taken during the sessions as well
student work. By using multiple data sources, the evidence will be supported in many ways (Creswell, 2006).

3.3.1 Video Recordings

Video recordings and their transcripts was chosen to serve as the primary data source for this study due to the fact that it allows for an in-depth analysis of the interactions between the researchers and students as well as interaction between the students. The videos accurately portray the events occurring during class sessions than by field notes or recollection of events after the session. In addition, it affords the researcher the opportunity to observe and analyze the data numerous times which allows for microanalysis of the data. Roschelle (2000) posits that videos "can preserve more aspects of interaction including talking, gesture, eye gaze, manipulatives, and computer displays" (p. 709). Composing a narrative and coding the data is thus enhanced by the added dimensions that video affords and helps ensure the validity of results. Video data was particularly useful in this study as it allowed the researcher to identify instances of teacher questioning and the resulting responses, arguments, and reasoning displayed by students as captured by video. Video also allowed the researcher to use screenshots which served as a necessary component of this work.

In addition to the full videos, screenshots were taken to capture the students’ work which gave the researcher the opportunity to analyze segments of the videos in greater depth. Screenshots were used to capture student work, representations and models. Thus video data and the resulting ability to capture screenshots helped enhance data analysis and informed the research to effectively meet the goals of this study.
3.3.2 Students’ Written Work

Since video data is not without some human and technological biases (Davis, 1989; Hall, 2000; Roschelle, 2000) Pirie (1996) suggests supplementing video recordings with students’ written work in order to allow for a more complete examination of students’ mathematical activity. Therefore another data source used in this study was samples of student work that was collected during the sessions. This work was often recorded during the sessions, or occasionally written after the intervention session had ended. According to Merriam (2014), student work is considered researcher-generated documents since their work has been prepared for the researcher after the study has begun. Student work enhances the researcher’s ability to learn more about the event under analysis. Student work was especially useful in that it provided the researcher with material to supplement the video data and fill in for times where students demonstrated reasoning not only verbally but in written form as well. It aided the researcher in getting a closer look at student representations and lent additional insight into student reasoning.

3.3.3 Field Notes

In addition to video data, transcriptions and samples of student work, I analyzed the researchers’ field notes, that included the tasks assigned to the students, a description of the work done by students at the overhead projector, as well as a recap of the discussions held by the researchers between sessions. Since I am limited in this study to analysis of video data and I was unable to attend the sessions in person, these field notes helped me gain the perspective of those who actively participated in the sessions.
3.4 Data Analysis

Powell, Francisco, and Maher (2003) delineate an analytical model to be used in analyzing video data which "employs a sequence of seven interacting non-linear phases" (413). This model includes: 1. Viewing attentively the video data 2. Describing the video data 3. Identifying critical events 4. Transcribing 5. Coding 6. Constructing storyline 7. Composing narrative. The sections that follow will elaborate on how this was implemented in this study.

3.4.1 Viewing and describing the data

This study focuses specifically on sessions where Researchers Carolyn Maher, Robert B. Davis, Amy Martino, and Arthur Powell led the class session or interacted with students, I first identified videos or clips that involve those researchers and located videos of sessions conducted by these researchers that featured teacher questioning. Specifically, I analyzed sessions led by Researchers Carolyn Maher and Amy Martino where students worked on building fraction ideas as well as an interview conducted by Researcher Amy Martino as a student reconstructs his solutions to the Pizza and Towers problems. In addition, I analyzed sessions led by Researchers Robert B. Davis and Arthur Powell where they worked with students on an activity to introduce the concept of function called Guess My Rule. In many sessions videos were taken using three cameras. These videos captured views of the classroom as a whole, as well as small group discussions.

Powell et al. (2003) encourage researchers to view the video data at first multiple times without using any analytical lens. By viewing the research sessions a number of times the researcher can become familiar with the full session. This preliminary stage will sometimes indicate to the researcher that additional data (such as student work and field
notes) should be looked at as well. Powell et al. suggest that in the second stage the researcher should “map out the video data so that someone reading the descriptions would have an objective idea of the content of the videotapes” (p. 416). During this phase the videos should be described briefly without making any inferences or interpretations. This serves as a time-coded references so that the researcher can easily find a specific event or episode at a later time.

3.4.2 Critical events

Powell et al. (2003) define critical events as an event that "demonstrates a significant or contrasting change from previous understanding, a conceptual leap from earlier understanding" (p. 416). These critical events either support or disaffirm the research hypotheses. The researcher then connects and analyzes these critical events thereby forming a narrative. Powell et al. point out that if a researcher is interested in the impact of teacher intervention on student mathematical thinking, they might deem as important "events that connect teacher interventions and associated student articulation of their thinking" (p. 418).

The present study utilized this definition of critical events. Since the sessions that were studied have numerous instances of teacher questioning and student responses, each session was divided into segments, where a teacher is interacting with one individual student or group of students. There may be instances where such interaction is punctuated by whole class discussions but then the teacher returns to the original student of group of students. I considered such occurrences to be one segment as well. This enabled me to analyze a meaningful give and take between the researcher or teacher and a single individual student or group of students. Since it was not feasible to look at every instance
of teacher questioning in the data, I focused on significant instances of researcher-student interaction that occurred during sessions led by the four researchers. Significant instances was defined as segments of video containing at least two responses from the student. After identifying significant instances of researcher-student interaction, critical events that connected teacher questioning and associated student reasoning and justification were identified.

3.4.3 Transcribing

Powell et al. (2003) underscore the importance of transcribing at least the critical events or any part of the video that relates to and provides evidence to help deal with the research questions. These transcripts are important in order to analyze language employed by teacher researchers and students as well as the flow of ideas. They emphasize though that researchers should not depend solely on transcripts. It is important to watch the corresponding videos carefully in order to pick up on nuances in speech and to observe non-verbal behaviors and communication. The video data for this study has all been transcribed and the transcriptions have been verified. This researcher utilized these transcripts in conjunction with viewing the videos to get a more complete picture of the interaction between researchers and students. All transcripts used in this analysis are included in appendix A.

3.4.4 Coding and Identifying Themes

The researcher identified researcher questioning and coded those questions using coding schemes found in the literature to determine instances and characteristics of teacher questioning and resulting student reasoning. The schemes developed from analysis of the data as well as from the relevant literature. Many classifications of teacher
questioning have been mentioned in the literature, but new codes were added as found appropriate using a grounded theory approach (Glaser & Strauss, 1967) to ensure the inclusion of additional themes that were not included initially in the coding scheme.

After transcribing the data, the researcher coded the data in order to address the research questions. After the initial development of codes, more codes were identified to further refine the classification. The researcher also returned to the literature and culled additional codes to be utilized in classifying the data. At this point an independent researcher reviewed portions of the data in order to verify the coding used. Any differences of opinion regarding coding were discussed by the researchers and the final consensus and understanding of events was incorporated into the final narrative and coding scheme.

3.4.5 Analyzing documents

Examples of student work as well as researcher field notes, and interview data were used to supplement and verify the findings. These documents offered additional insight into the reasoning employed by student. By analyzing the representations that they constructed to explain their solutions and the explanations that they wrote to clarify their solutions the researcher was able to glean information that was not otherwise apparent by simply watching the videos. Researcher field notes also provided verification for the events and vignettes that occurred during the session.

3.4.6 Constructing a Storyline

A storyline is "the result of making sense of the data with particular attention to identified codes" (Powell, et al., 2003 p. 430). To assist in the construction of a storyline once coding was completed, the data was also organized in tables in order to allow for
meaningful analysis and to help the researcher identify relationship between teacher questioning and student reasoning. The tables were useful in assisting the researcher in discovering discourse patterns and teacher questioning types that were especially conducive to eliciting student reasoning.

3.4.7 Composing a Narrative

After constructing a storyline, the data was then strung together in a narrative to describe the sessions and highlight teacher questioning that provided opportunity for student reasoning. Since this study intends to analyze sessions led by different researchers, with each researcher having their own pedagogical style, sessions led by specific researchers were analyzed separately, and differences in pedagogical styles were illustrated through representative RUanalytics. These analytics attempt to capture the interplay that occurred between teacher questioning and student reasoning and argumentation as will be described in the next section.

3.4.8 Creating Analytics

In addition to utilizing some of the components of the analytical model for studying video data as delineated by Powell et al. (2003), this study used the VMC analytic tool to highlight the different categories of teacher questioning that tended to elicit student reasoning and justification. The sections below will briefly describe the Video Mosaic Collaborative (VMC) as well as the VMC Analytic tool that is used to create analytics.

3.4.8.1 The Video Mosaic Collaborative

The Video Mosaic Collaborative (VMC) repository was set up to allow the researchers, teacher educators and teachers as well as others access the Robert B. Davis Institute for Learning
(RBDIL) video collection. The website currently contains 300 videos and clips of elementary and high school students working on tasks across eight content strands; algebra, counting and combinatorics, fractions and rational numbers, geometry, pre-calculus, calculus, and probability (Hmelo-Silver et al., 2013). Together with each clip or video, there is metadata which provides basic information about the video data as well as supplementary information including information about the researchers and students featured on the videos and the mathematical task that the students are working on as well as transcripts, student work, and related publications (Agnew et al., 2010). The VMC allows for ease of access and powerful search tools to enable users view related videos and associated metadata.

3.4.8.2 The VMC Analytic Tool

The VMC Analytic tool which is part of the VMC (located at https://rucore.libraries.rutgers.edu/analytic) is a valuable tool for creating narratives from videos that can be used by teacher educators and researchers to structure a video into meaningful events that can draw user’s attention to specific aspects of a story. Users can created multimedia artifacts by searching for appropriate videos, selecting portions of the video that they would like to use, and annotating these selections. This tool allows multiple events across one or more videos to be combined into a meaningful narrative. The rich descriptions that accompany these events lend insight into the purpose and lessons to be gleaned from the analytic. User’s attentions may be drawn to differing learning styles, teacher moves, or student collaboration that can serve to inform users in their research or classrooms. Users will not have to view the entire video(s) in order to decide whether they are appropriate for use in an intervention or research project. The VMCAAnalytic tool allows multiple users to collaborate on making an analytic together. In addition, it features a threaded discussion component to allow for exchange, comment,
and feedback between users on the general analytic or on specific events included in the analytic. Analytics that are published are shared publicly and since they are repository objects they can be searched similar to other video resources.

The analytics provided in this study lend additional insight into the interplay between teacher questioning and student reasoning. They highlight some of the significant exchanges between researchers and students that demonstrate how a facilitator’s questioning can serve as a springboard for student argumentation, reasoning, and justification.

3.5 The Coding Scheme

Questions were categorized as either encouraging engagement, eliciting, probing, shepherding, factual, divergent, or affective questions. These basic categories were divided into a number of subcategories. Questions were also coded by whether or not they resulted in higher order thinking including student argumentation, justification or reasoning. The following paragraphs elaborate on these coding schemes.

The first category, encouraging engagement, consisted of codes for questions that were used to keep the students engaged and on task and to encourage them to listen attentively to their peers. Included in this category are administrative questions which are used to ascertain that students were engaged and that teacher and student input was clear to all. This is similar to Towers’ (1998) code of managing, which she defines as keeping the students on task and giving instructions. Also included in encouraging engagement are questions that were asked to confirm agreement, questions that were used to check for understanding (Towers 1998), and questioning that was redirected at another student.
In this study, the code eliciting student ideas was used when the teacher took the initiative to elicit student ideas and encouraged students to formulate their own ideas and strategies. Using these questions, researchers/teachers encouraged students to make their thinking public or explicit or to make their explanation visible to other students. Eliciting questions encouraged students to voice their ideas and solutions, explain their thinking and share their thoughts or solutions with others.

The next category of question is probing questions or a probing sequence of specific questions (Franke et al., 2009) that elicited student answers past their initial replies. Further teacher queries were based on student response. These questions were often used to enable students to see the error in their statement or to help them enhance their explanations. They often consisted of a series of more than two related questions about something specific that a student had said and included multiple teacher questions and multiple student responses. This category includes elucidating, redirecting, encouraging critical thinking, and reality-check.

Elucidating questions were questions used to clarify a student response. The teacher used these questions to query a student about what he or she meant and to allow the student to elaborate on what he or she had said.

Redirecting questions asked the students to consider analyzing the question further and think about the implications of the response and how it would relate to other solutions or statements. This type of questioning invites a student to “fold back,” an element of Pirie and Kieren’s theory for the dynamical growth of mathematical understanding and its associated model (Kieren, Pirie, & Gordon Calvert, 1999; Martin & Pirie, 2003; Pirie & Kieren, 1994). Folding back is considered by some to be the “central
key construct” of the model (Martin, 2008). Pirie and Kieren (1991) define folding back as

A person functioning at an outer level of understanding when challenged may invoke or fold back to inner, perhaps more specific local or intuitive understandings. This returned to inner level activity is not the same as the original activity at that level. It is now stimulated and guided by outer level knowing. The metaphor of folding back is intended to carry with it notions of superimposing one’s current understanding on an earlier understanding, and the idea that understanding is somehow ‘thicker’ when inner levels are revisited. This folding back allows for the reconstruction and elaboration of inner level understanding to support and lead to new outer level understanding. (p. 172)

Questions were also included in the classification of probing if they were used to encourage critical thinking or augment analytical cognizance. This sub-code was used when the teacher attempted to increase the students’ critical awareness about what their response had been and what their underlying assumptions may have been in making a particular statement or offering a particular answer and thereby make them cognitively aware of the reasons behind their thinking. This category would also be used if the teacher used questioning to ask the students whether they had completely answered the question at hand. Questions were also coded if they were used as a calculated shifting of the discussion to ideas or objects that were at odds with a previous discussion or conclusion to provide opportunity for students to step back and reevaluate their thinking. This researcher refers to such questions as reality-check. Towers (1998) uses the term rug-pulling.

Another classification used was shepherding, also referred to in the literature as leading or guiding (Franke et al., 2009; Towers, 1998). Franke et al. provide the difference between probing and leading. They write that probing question or a series of pointed questions are often used to enable students to see the error in their statement or to
help them enhance their explanation. Leading questions are used to guide the students to come up with the correct answer or explanation. Towers (1998) classifies leading questions as “an extended stream of interventions aimed at directing the student towards a specific answer or position, often involving step-by-step explanations.” This classification is similar to Towers’ classification called shepherding, which she defines as “an extended stream of interventions directing a student towards understanding through subtle nudging, coaxing, and prompting.” She elaborates on this by delineating additional sub-codes, including inviting and clue-giving. Inviting is defined as “suggesting a new and potentially fruitful avenue of exploration” which is “more open ended than clue-giving” (p. 201). Clue giving differs from inviting in that it is a “deliberate attempt to point the student to the correct answer or the preferred route” (p. 201).

The next classification used was factual questions where the teacher attempted to elicit facts, rules, calculations or other knowledge that had been taught previously. Such questions are often prefaced with “how much”, “what is”, “when can you”, and similar phrases. This category included questions that asked students to recall information or questions that required a simple yes or no response.

The next category of questions was divergent questions or open-ended questions (Guilford, 1968). Such questions could be answered in more than one way or had no definite correct or incorrect answers. These involved creative and/or conceptual reasoning and were used to invite discussion.

Questions were classified as affective questions if they prompted students to express an opinion, individual mindset, ideal, preference or emotion.
In addition to the above classifications, the data was also analyzed and coded to determine which question types were associated with higher order thinking or mathematical reasoning, justification or argumentation (Martino & Maher, 1999). Questions were coded a second time to show which questions invited students to work out solutions rather than memorize them or necessitated noting similarities among different facts and relationships and formulating meaningful rules from these findings. These types of questions prompted students to give more detailed explanations of their strategies, generalize solutions, and/or make connections. Specifically, I looked for questioning which facilitated awareness of solutions presented by other students or invited argumentation, whereby the questioning served as an opportunity for students to provide reasons for or against something. In addition, questions were coded if they facilitated justification where students used evidence or data to support a conclusion. The data was further analyzed to see whether the researcher invited inductive or deductive student reasoning whereby they made inferences by either formulating a rule based on a set of facts or testing a rule on the basis of specific phenomena or facts. Questioning was coded as promoting reasoning if it was associated with student mathematical reasoning that included direct reasoning, indirect reasoning, reasoning by cases, generic reasoning, and upper and lower bounds or other forms of reasoning. The data were also coded to determine if the questioning invited students to form generalizations, make connections, or to find relationships among concepts and things, thus promoting the use of analogical reasoning. Questioning was also coded as higher order questioning if it fostered assessment and application of knowledge.
The table below contains the list of codes used to classify the data (encouraged engagement, eliciting, probing, factual, divergent/open-ended, and affective) as well codes that were applied if the questioning facilitated higher order thinking (facilitated awareness of solutions presented by other students, promoted argumentation, facilitated justification or promoted reasoning, offered opportunity for generalization, invited learners to make connections/promoted analogy, promoted assessment, or promoted use/application). The table also delineates numerous sub codes for each classification. Definitions and examples are provided for select codes to give insight into nuances and differences between codes.

Table 3.1.

<table>
<thead>
<tr>
<th>Code</th>
<th>Definition</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>ENCOURAGING ENGAGEMENT</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| 1. Administrative             | Questions that ascertain that students are engaged and that teacher and student input is clear to all. | “Can we proceed?”
|                               |                                                                           | “Does anyone need help with understanding that?”
|                               |                                                                           | “Could you clarify that?”
|                               |                                                                           | “Does anyone have any questions about this formula/conclusion/concept?” |
| 2. Confirming agreement       |                                                                           | “Do you all agree with Meredith’s solution?”
|                               |                                                                           | “Are you all convinced?”
<p>|                               |                                                                           | “Do the rest of you agree?”                                           |
| 3. Checking for understanding | Attempting to ascertain whether or not a student understood the teacher’s or student’s explanation or discourse. | “Do you understand why it is two fourths?”                              |
| 4. Shifting questioning to other students |                                                                           | “Alan, do you see anything wrong with Sarah’s response?”                |</p>
<table>
<thead>
<tr>
<th>5. Encouraging students to listen</th>
<th>Encouraging students to listen to their peer or to the researcher</th>
<th>“Did everyone hear what James said?”</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>ELICITING</strong></td>
<td>Encouraging students to formulate their own ideas and strategies. Encouraging students to make thinking public/explicit/ Encouraging students to make explanation visible to other students Encouraging students to explain their thinking and share their thoughts or solutions with others.</td>
<td>“What do you think David?”</td>
</tr>
<tr>
<td><strong>PROBING QUESTIONS OR PROBING SEQUENCE OF SPECIFIC QUESTIONS</strong></td>
<td>Questions that elicit student answers past their initial replies. Further teacher queries are based on student response. These questions were often used to enable a student to see the error in their statement or to help them enhance their explanations. Often consisted of a series of more than two related questions about something specific that a student said and included multiple teacher questions and multiple student responses.</td>
<td>Teacher: Does someone have a different number name? Danielle: I thought it would be two fourths. Teacher: You thought two fourths? How did you think that? Danielle: Because if she got a half, then the top two rows is a half, and then that’s two fourths. Teacher: Why are they two fourths? Can you help the class understand that? Danielle: Because there’s four rows. Teacher: Because there are four rows and we if talk about the first two rows that’s two fourths?</td>
</tr>
<tr>
<td>1. Elucidating</td>
<td>Questions used to clarify a statement of solution presented by a student</td>
<td>“Can you use other words to tell me what you mean?” “Can you give me specifics about that statement?”</td>
</tr>
<tr>
<td>2. Redirecting/Folding back</td>
<td>Questions that asked the students to consider analyzing the question further and think about the</td>
<td>“If what you are saying is correct, what bearing does that have on …?” How does your response compare</td>
</tr>
</tbody>
</table>
| **3. Encouraging critical thinking/ Augmenting analytical cognizance** | Attempting to increase the students’ critical awareness about what their response had been and what their underlying assumptions may have been in making a particular statement or offering a particular answer and thereby make them cognitively aware of the reasons behind their thinking. Using questioning to ask the students whether they had completely answered the question at hand. | “What are you taking for granted?”
“Why do you think so?”
“Is there any other reason you think so?” |
|---|---|---|
| **SHEPHERDING (LEADING/GUIDING)** | The teacher guides students toward particular answers or explanations and provides opportunities for students to respond | Teacher: Kimberly, if you drive at 60 miles an hour how long will it take to get from City A to City B if A and B are 360 miles from each other?”
Kimberly: “I don’t know.”
Teacher: “If you drive at 60 miles per hour how long will it take you to get from New York to Newark, which are 60 miles from each other?”
Kimberly: “One hour.”
Teacher: “How many 60 miles are there in 360 miles?”
Kimberly: “I see, it would take 6 hours to get from City A to City B.” |
| **1. Inviting** | Question serves to provide a new avenue of exploration. Not as specific as clue-giving | “What other rod could we use to help us solve this problem?” |
| **2. Clue-giving** | Question is used to deliberately point the students to the correct answer or means of arriving at the solution. | “How about if we tried the blue rod?” |
| **3. Reality check (rug-pulling):** | Calculated moving of discussion to ideas or questions | }
objects that are at odds with previous discussion or conclusion in order to compel the student to step back and reevaluate his or her thinking.

**FACTUAL QUESTIONS**

Questions that elicit facts, rules, calculations or other knowledge that had been taught previously. Such questions are often prefaced with “how much”, “what is”, “when can you”, and similar phrases.

<table>
<thead>
<tr>
<th>1. Recall</th>
<th>“Do you remember the formula we used yesterday to calculate the area of a triangle?”</th>
</tr>
</thead>
<tbody>
<tr>
<td>2. Questions requiring yes or no answers</td>
<td>“Are these two lines equal in length?”</td>
</tr>
</tbody>
</table>

**DIVERGENT QUESTIONS/OPEN-ENDED QUESTIONS**

Questions that can be answered in more than one way or have no definite correct or incorrect answers. These involve creative and/or conceptual reasoning and can be used to promote discussion.

| Questions which prompt students to express an opinion, individual mindset, ideal, preference or emotion. | “What is your opinion about this?” |
| “How would you feel if that happened?” | “What is your attitude toward…?” |

**QUESTIONS THAT ELICITED HIGHER ORDER THINKING OR ENCOURAGED ARGUMENTATION/EXPLANATION/JUSTIFICATION/REASONING**

| 1. Facilitated Awareness of Solutions Presented by Other Students | Encouraged students to provide reasons for or against something. |
| 2. Promoted Argumentation | |

**AFFECTIVE QUESTIONS**

Questions which prompt students to express an opinion, individual mindset, ideal, preference or emotion.
3. Facilitated Justification or 
Promoted Reasoning

Encouraged students to use evidence or data to support a conclusion
Encouraged students to make inferences by either formulating a rule based on a set of facts or testing a rule by on the basis of specific phenomena or facts (inductive or deductive). Promoted mathematical reasoning which includes direct reasoning, indirect reasoning, reasoning by cases, generic reasoning, and upper and lower bound arguments.

4. Offered Opportunity for 
Generalization

Encouraged students to make a statement about a larger set of object based on observations on a smaller set that is contained in the larger one.

5. Invited Learners to Make 
Connections/Promoted Analogy

Encouraged students to find resemblance, contrast, identity or other relationships among concepts and things.

6. Promoted Assessment

Encouraged students to weigh, rank, rate, estimate, or determine resulting from the matching of concepts or things to accepted rules or realities.

7. Promoted Use/Application

Encouraged students to apply ideas or rules acquired in a learning situation in a different setting or scenario.

As noted above, many of the codes were drawn from the literature while others were added using a grounded theory approach. Teacher questioning was classified by the types of questions posed and the whether the questioning resulted in student argumentation, justification, or reasoning. The narrative details the exchanges between the researchers and students, describes the juncture at which the question was posed and the student’s original reasoning before the teacher intervention. The tables embedded in the narrative display student reasoning after the intervention to see how it was changed by the intervention.

3.6 Validity

Throughout the process of data collection and analysis, care was taken to ensure the validity of results. The data was triangulated through the use of multiple data sources, video, transcript, student work, and researcher notes. In addition, transcripts of the
sessions were verified by an independent researcher. During the coding stage as well, data was coded by the researcher and verified by an independent researcher. Any differences in coding was discussed between researchers until a consensus was reached regarding the most fitting code to be applied. The researcher also attempted to portray the sessions through a descriptive narrative to enable the reader to come to arrive at their own conclusion which would hopefully be similar to the conclusions drawn by the researcher.

CHAPTER 4: RESULTS

4.1 Introduction

Different forms of questions asked by researchers during the various studies are examined as they facilitated sessions where students worked on open-ended tasks. Attention is given to the interplay between research questions and the evolution of student reasoning. Narratives for each session are provided with attention to the interplay of researcher questioning and student reasoning. For student conjectures, the ensuing forms of reasoning are presented. Screenshots of models built by the students or samples of student work are illustrated along with the narrative to provide detail as needed. When it was judged helpful in understanding the narrative, student work is displayed. Other relevant student work appears in the appendices.

The tables presented for each narrative include the teacher questions, the classification of that teacher questioning, as well as the associated student response, reasoning and justification. In addition histograms are provided based on these tables to show how many questions were asked according to the main coding scheme as well
histograms to show the number of instances of student argumentation, justification, and reasoning.

4.2 Analysis of teacher questioning by Researchers Carolyn Maher and Amy Martino

In this section the varied questioning techniques employed by Researcher Carolyn Maher (identified as T/R 1) and Researcher Amy Martino (identified as (T/R 2) during sessions devoted to fraction exploration at the Colts Neck School were identified. The questioning used by Researcher Amy Martino in an interview conducted with Brandon, a fourth grade student from the Colts Neck school is also analyzed. During this interview Researcher Martino observes Brandon as he reconstructs work that he had done in previous class session

4.2.1 Colts Neck Sept 21, 1993 (Session 2) with Researchers Carolyn Maher and Amy Martino

At the start of the session 2 on fractions, on Sept 21, 1993, T/R 1 asked the students to recap what they had done day before for a visiting guest (Tom Purdy). Jessica and Michael gave a brief summary and Erik provided an example. He said for instance, if they gave the blue rod the number name one, they would want to use the blocks to determine what would be half. T/R 1 asked Erik for the solution to that example and the students all tried to find a rod that they could call one half. The students were not able to find such a rod. One student suggested that the yellow rod could be called one half but other students disagreed with that suggestion. As the students tried to find a rod that fit those specifications, T/R 1 asked them to solve another problem. “But suppose I wanted to call the yellow rod, I wanted to give it a number name one half. Can you tell me what I would have to call one?” (line 2.0.29). Brian picked up two yellow rods and one orange
rod and stated that the two yellow rods were equal in length to the orange rod. T/R 1 said that she was still concerned about the problem that Erik had created. Erik said that he did not think there was a rod that could be called one half of the blue rod. T/R 1 asked David to tell the class what he thought about the problem. David concurred with Erik and when T/R 1 questioned him further and asked him to explain his reasoning at the overhead projector, David put a purple rod next to a yellow rod on the overhead projector (Figure 4.1). Then he added a white rod to the purple rod to create a train. He explained that usually the shorter rod is “only one block apart.” (line 2.0.58). Next he put two yellow rods together and placed a blue rod next to the yellow train. He then placed two purple rods together and placed a blue rod next to that train (Figure 4.2) He explained his models by saying that “[I]f you have two yellows, it would be too tall and if you have two purples, that would be too short” (line 2.0.58). He then set up all ten rods in a staircase fashion in order of length (Figure 4.3) and stated that since there is no other rod between the yellow and purple “there’s no way you can do that” (line 2.0.60). Here David clearly showed by using an upper and lower bounds argument that there was no rod half the length of the blue rod.

*Figure 4.1. A purple and white train alongside a yellow rod*
Figure 4.2. Two purple rods alongside the blue rod and two yellow rods alongside the blue rod

Figure 4.3. Staircase model of Cuisenaire rods

Table 4.1

<table>
<thead>
<tr>
<th>Line</th>
<th>Researcher</th>
<th>Researcher question</th>
<th>Question type</th>
<th>Question result</th>
<th>Student response/reasoning/justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0.48</td>
<td>T/R 1:</td>
<td>David, what do you think? What does David think? I can’t hear you, David. Hold on.</td>
<td>Eliciting Encouraging engagement - encouraging students to listen</td>
<td>Promoted argumentation</td>
<td>David said that there is no rod that could be called a half</td>
</tr>
<tr>
<td>2.0.50</td>
<td>T/R 1:</td>
<td>Why, David? Slowly and loud.</td>
<td>Probing – elucidating Probing – encouraging critical thinking</td>
<td>Promoted reasoning</td>
<td>David said that two yellow rods would be too big and two purples are too small</td>
</tr>
<tr>
<td>2.0.52</td>
<td>T/R 1:</td>
<td>What about something between purple and yellow?</td>
<td>probing – elucidating</td>
<td>Facilitated justification and promoted reasoning</td>
<td>David answered that there is no other rod</td>
</tr>
<tr>
<td>2.0.54</td>
<td>T/R 1:</td>
<td>Why not?</td>
<td>Probing - elucidating</td>
<td>Facilitated justification and promoted reasoning</td>
<td></td>
</tr>
<tr>
<td>2.0.55</td>
<td>T/R 1:</td>
<td>Show us what you have there, David. Why do you think there isn’t any? Cause I think you built it to show us. Can you show us your yellow and your purple?</td>
<td>Probing – promoting critical thinking</td>
<td>Facilitated justification and promoted reasoning</td>
<td></td>
</tr>
<tr>
<td>2.0.57</td>
<td>T/R 1:</td>
<td>David, why don’t you come up here and explain your reasoning. ...What’s, what’s your reasoning?</td>
<td>Eliciting - encouraging student to make thinking explicit</td>
<td>Facilitated justification and promoted reasoning</td>
<td>David demonstrated at the OHP that two yellows would be too long and two purples would be too short. He then built a staircase model to demonstrate that there are no other rods that could be called one half of the blue rod. David clearly showed by using an upper and lower bounds argument that there was no rod half the length of the blue rod.</td>
</tr>
<tr>
<td>2.0.61</td>
<td>T/R 1:</td>
<td>Are you all convinced?</td>
<td>Confirming agreement</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Classification of teacher questioning and result, Colts Neck Session 2 – Part I
Jessica disagreed with David and said that she believed there was a solution to the problem since three green rods were equal in length to the blue rod. David refuted this by saying that Erik was looking for a half, meaning that since Jessica was referring to rods that would be given the number name thirds in this example, it could not be the correct solution.

Erik then offered another solution. He claimed that the yellow and purple rods could be called halves even though they were not equal. T/R 1 requested that Erik repeat what he said. Adding to what he said before, Erik explained that perhaps the yellow rod could be given the number name three quarters and the purple rod could be given the number name one quarter. Again Erik was incorrect in his assumption that one half simply meant one part of a whole comprised of two parts rather than the mathematical definition of one half. Rather than responding to that faulty argument, T/R 1 asked David what he thought. David responded that he thought “you would need the same” (line 2.0.68) meaning that he thought that the two halves would have to be equal. T/R 1 used a compelling example to question Erik on his faulty definition of one half. She asked him suppose the rods were bricks of gold instead, would he consider it a fair split if she kept the yellow rod and gave Erik the purple rod. Erik held onto his earlier claim and said he thought it was fair. T/R 1 then addressed this question to the class and the students all said together that they did not believe it was a fair distribution. Kimberly said that it would not be fair because the person who receive the yellow rod would not get as much since that rod is smaller.

Erik then changed his argument slightly but again used a faulty definition of one half. He said that one could call part of it one quarter and the other part three quarters. He
explained that “it just wouldn’t be halves… But it would still look like you’re dividing it into halves” (line 2.0.77). Erik still defended his position but modified his argument slightly by saying that it would only look like a half but would actually be quarters.

Brian interjected and repeated Jessica’s suggestion that it could be divided into thirds. David once again responded to the initial question and asked them to find halves not thirds. Next Alan and Jessica both countered Erik’s argument. Alan stressed that both halves would have to be equal and Jessica agreed and said that Erik’s model could not be showing halves because “those two aren’t both even halves (2.0.86)

T/R 1 then posed the following question to Erik: “Can you divide things in halves and have them different sizes?” (line 2.0.91) Erik responded that

Well, see. This isn’t exactly dividing into halves. But I’m still using two blocks, but not… I’m dividing it in half still using two blocks, but one block is bigger than the other block. So it’s like using three quarters and one quarter, but you’re only using two blocks so it’s almost like dividing it in half.

(line 2.0.92)

Here Erik modified his argument due to the researcher’s questioning but he still did not explain clearly that in order to be one half the two rods would have to be equal.

Andrew then argued with Erik and said that Erik says he wants a half but if he uses the purple and yellow rods then they could not be halves. T/R 1 asked Alan to review the points that had been raised in the discussion. Alan explained that it is not possible to divide it into halves since one would have to use rods of different sizes but that it could be divided into thirds. David then demonstrated to the class how if one built a staircase model of the rods, then one could divide those rods into “even” or “odd” rods. (Figure 4.4) He explained that all of the rods that he classified as “even” were double the length of another rod, whereas the “odd” rods were not twice the length of another rod. He then
explained that since the blue rod was an “odd” rod one could therefore not find a rod that could be called one half of the blue rod.

Figure 4.4. “Odd” and “Even” rods

T/R 1 then questioned Erik on the explanation that he had provided earlier. She asked Erik if he had meant that he had come up with two rods that were equal in length to the blue rod but were not equal in length. Erik conceded that the two rods could not be given the number name one half since in order to be considered half the two rods would have to be equal in length.

Table 4.2

Classification of teacher questioning and results, Colts Neck Session 2 – Part II

<table>
<thead>
<tr>
<th>Line</th>
<th>Researcher</th>
<th>Researcher question</th>
<th>Question type</th>
<th>Question result</th>
<th>Student response/reasoning/justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0.65</td>
<td>T/R 1:</td>
<td>Did you all hear what Erik said? Erik, do you want to say that one more time?</td>
<td>Encouraging engagement - encouraging students to listen</td>
<td>Erik had suggested that a purple and yellow rod could be considered halves of the blue rod. Erik repeated his faulty reasoning but added that perhaps the yellow rod could be given the number name three quarters and the purple rod could be given the number name one quarter.</td>
<td></td>
</tr>
<tr>
<td>2.0.67</td>
<td>T/R 1:</td>
<td>What do you think, David?</td>
<td>Encouraging engagement - shifting questioning to another student Eliciting</td>
<td>Promoted reasoning</td>
<td>David said he believes that the two halves would have to be equal</td>
</tr>
<tr>
<td>2.0.69</td>
<td>T/R 1:</td>
<td>You think you would need the same?</td>
<td>Probing - elucidating</td>
<td>Promoted reasoning</td>
<td>David said yes – they would need to be equal</td>
</tr>
<tr>
<td>2.0.72</td>
<td>T/R 1:</td>
<td>You don’t need the same? In other words, I</td>
<td>Probing - encouraging critical thinking</td>
<td>Promoted reasoning</td>
<td>Erik held onto his earlier claim and said he thought it was fair</td>
</tr>
</tbody>
</table>
could call this a half [the yellow rod] and I can call this a half [the purple rod]. Suppose this is a brick of gold and we’re going to share it, Erik. And I’m going to take the yellow half and you get the purple half. Fair?

<table>
<thead>
<tr>
<th>Time</th>
<th>Transcript</th>
<th>Encouragement</th>
<th>Argumentation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Kimberly disagreed with Erik and states that is would not be fair for one person to receive the purple which is smaller than the yellow. Erik then changed his argument slightly but again used a faulty definition of one half. He said that one could call part of it one quarter and the other part three quarters</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>Transcript</th>
<th>Encouragement</th>
<th>Argumentation</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.078</td>
<td>T/R 1: What do you think, Brian?</td>
<td>Encouraging engagement - shifting questioning to other students</td>
<td>Promoted reasoning</td>
</tr>
<tr>
<td></td>
<td>Brian said that the blue rod could be split into thirds</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>Transcript</th>
<th>Encouragement</th>
<th>Argumentation</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.080</td>
<td>T/R 1: Is that, is that the question?</td>
<td>Probing - elucidating</td>
<td>Promoted argumentation and justification</td>
</tr>
<tr>
<td></td>
<td>David clarified that they are looking for halves. Alan added that in order to be considered halves the two rods would have to be equal in length.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>Transcript</th>
<th>Encouragement</th>
<th>Argumentation</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.089</td>
<td>T/R 1: (Addressing the question to Erik) What do you think of that?</td>
<td>Encouraging engagement - shifting questioning to another student</td>
<td>Promoted justification</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>Transcript</th>
<th>Encouragement</th>
<th>Argumentation</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.091</td>
<td>T/R 1: Can you divide things in halves and have them different sizes? I think that’s what Jessica is asking and Alan and David.</td>
<td>Probing - Encouraging critical thinking</td>
<td>Promoted justification</td>
</tr>
<tr>
<td></td>
<td>Erik modified his argument due to the researcher’s questioning but he still did not explain clearly that in order to be one half the two rods would have to be equal.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>Transcript</th>
<th>Encouragement</th>
<th>Argumentation</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.093</td>
<td>T/R 1: Andrew? What do you think about that, Andrew?</td>
<td>Encouraging engagement - shifting questioning to another student</td>
<td>Promoting argumentation and reasoning</td>
</tr>
<tr>
<td></td>
<td>Andrew argued with Erik and said that Erik says he wants a half but if he uses the purple and yellow rods then they could not be halves</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>Transcript</th>
<th>Encouragement</th>
<th>Argumentation</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.095</td>
<td>T/R 1: It seems to me we have some differences here, don’t we? Um. How many of you agree with Erik? [no hands are raised, children giggle] How many of you disagree with</td>
<td>Confirming agreement</td>
<td></td>
</tr>
<tr>
<td>Time</td>
<td>Role</td>
<td>Text</td>
<td>Probing/Encouraging</td>
</tr>
<tr>
<td>-------</td>
<td>-------</td>
<td>-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------</td>
<td>------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>2.0.95</td>
<td>T/R</td>
<td>Erik? [all hands are raised, more giggling].</td>
<td>Alan explained that it is not possible to divide it into halves since one would have to use rods of different sizes but that it could be divided into thirds</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Hm, okay, what’s the issue, do you think, here in the disagreement? Can somebody summarize the issue? Alan, do you want to try again?</td>
<td></td>
</tr>
<tr>
<td>2.0.98</td>
<td>T/R 1</td>
<td>Because I think, go ahead, David. What do you think?</td>
<td>David demonstrated to the class how if one built a staircase model of the rods, then one could divide those rods into “even” or “odd” rods. He explained that all of the rods that he classified as “even” were double the length of another rod, whereas the “odd” rods were not twice the length of another rod. He then explained that since the blue rod was an “odd” rod one could therefore not find a rod that could be called one half of the blue rod.</td>
</tr>
<tr>
<td>2.0.100</td>
<td>T/R 1</td>
<td>Okay, let me see, I think that we have. Maybe, Erik, the way we can resolve this is, I don’t think I’m hearing you say, Erik, that you want to call yellow one half and purple one half. I don’t, I don’t hear you say that. You’re not saying that, are you?</td>
<td>Erik agreed that the yellow and purple rods cannot be considered halves of the blue rod.</td>
</tr>
<tr>
<td>2.0.102</td>
<td>T/R 1</td>
<td>You’re saying that you agree with the rest of the class that if you call something one half of something… They have to be the same size.</td>
<td>Erik agreed</td>
</tr>
<tr>
<td>2.0.108</td>
<td>T/R 1</td>
<td>You are in essence answering a different question, maybe?</td>
<td>Erik agreed</td>
</tr>
</tbody>
</table>
Next T/R 1 posed another task to the class. She asked “If you were designing a new set of rods and you wanted to call the blue rod one, can you tell me what that new rod might look like so that you would be able to call it a half?”

Erik and Alan worked together to find a solution to this task. Erik at first said that there was no answer to the problem because a rod with the length of nine could not be divided equally. Alan explained to him that the problem was “if you could” (line 2.0.122), but Erik continued and pointing to the orange rod he said “if this is ten, then this (the blue rod) is nine. It’s impossible to divide this evenly” (line 2.0.125). Alan once again explained to him that they were trying to come up with a rod that did not exist in the set, but they could create a new rod with another color by dividing the blue rod into two equal part. He stated that such a rod would be four and a half. Erik persisted and said that there was no such rod. To prove his point he lined up nine white rods next to the blue rod. Counting four white rods from one side and five white rods from the other side, he demonstrated that the blue rod, which was equal in length to nine white rods, could not be divided equally.

During this time T/R 2 discussed the task with Meredith and Sarah. She asked them to explain the problem and how they would go about coming up with a solution. Meredith explained to T/R 2 that since the orange rod is ten and the blue rod is nine, then half of the blue rod would be four and a half (centimeters). She demonstrated that a train of a purple rod and half of a white rod would be equal to four and a half (centimeters).

Table 4.3

Classification of teacher questioning and results, Colts Neck Session 2 – Part III
<table>
<thead>
<tr>
<th>Line</th>
<th>Transcript</th>
<th>Action</th>
<th>Justification/Reasoning</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0.162</td>
<td>T/R 2: You two seem very quiet over here. What do you two think about this? What would the new rod you design be like? Do you understand the question? Sarah, do you understand the question? [Sarah nods her head] What are you being asked to do?</td>
<td>Eliciting</td>
<td>Encouraging engagement-checking for understanding</td>
<td>Sarah and Meredith argued with each other if they know how to do the problem. Meredith explained the task at hand.</td>
</tr>
<tr>
<td>2.0.173</td>
<td>T/R 2: That’s uneven, ok, so you think if we were, if we were able to go into a workshop with wood and build a new rod, what would that rod look like? If we could go and build and make a rod any way we’d like, what would that rod look like? How would you describe it? Would you describe it in terms of these other rods here?</td>
<td>Probing – encouraging critical thinking</td>
<td>Promoted justification and reasoning</td>
<td>Meredith explained that the orange rod is equivalent to 10 (cm) and the blue rod is 9 (cm) and therefore the new rod would have to be 4 ½ (cm).</td>
</tr>
<tr>
<td>2.0.175</td>
<td>T/R 2: Alright. Why is that?</td>
<td>Probing - elucidating</td>
<td>Promoted reasoning and justification</td>
<td>Meredith explained that half of 9 would equal 4 ½. She demonstrated that a train of a purple rod and half of a white rod would be equal to four and a half (centimeters).</td>
</tr>
</tbody>
</table>

T/R 1 called everyone’s attention and asked Jackie, Beth, and Graham to share their solution with the class. Jackie together with Graham explained that the new rod would be comprised of a purple rod and half of a white rod. They explained that the set of rods would now include a rod half of the length of the white rod, making it the smallest rod in the set. Michael and David stressed that once a new rod is created, more rods would have to be added to the set so that there would be rods half the length of the new rod while Brian explained that “No matter what there’ll always be something that won’t equal something, like… If you cut these little ones in half, then there wouldn’t be something for the little ones to make a half out of them” (lines 2.0.204-2.0.206).

T/R 1 asked the class if anyone had another idea for designing a new rod that could be given the number name one half when the blue rod was called one. James came...
up to the overhead and explained to the class that by using one light green rod and half of another light green rod it would be equal in length to half of the blue rod.

Table 4.4

Classification of teacher questioning and results, Colts Neck Session 2 – Part IV

<table>
<thead>
<tr>
<th>Line</th>
<th>Researcher</th>
<th>Researcher question</th>
<th>Question type</th>
<th>Question result</th>
<th>Student conjecture, response, reasoning, or justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0.185</td>
<td>T/R 1:</td>
<td>Could you speak nice and loud? Cause I’m a student back here and I can’t hear you. Do you want to try and talk really loud?</td>
<td>Encouraging engagement - administrative -</td>
<td>Promoted reasoning and justification</td>
<td>Jackie reiterated that they thought the new rod would be comprised of a purple rod and half of a white rod. They decided that the color name for the new rod would be light pink. Graham added that the smallest rod in the set would now be half of a white rod.</td>
</tr>
<tr>
<td>2.0.189</td>
<td>T/R 1:</td>
<td>Did you all hear what they said? No, they, Kimberly didn’t hear you, dear.</td>
<td>Encouraging engagement - administrative</td>
<td></td>
<td>Jackie explained their model again.</td>
</tr>
<tr>
<td>2.0.193</td>
<td>T/R 1:</td>
<td>And you said something else, what would your smallest rod be?</td>
<td>Probing – elucidating</td>
<td></td>
<td>Jackie repeated that the smallest rod would now be half of a white rod.</td>
</tr>
<tr>
<td>2.0.195</td>
<td>T/R 1:</td>
<td>What are you going to call that? You’re the designers. What are you going, it’s not going to be white, what do you think? You want to help them out? You could have other consultants to this design. Why don’t you call on someone for help and consulting? Graham? Beth?</td>
<td>Probing – elucidating - encouraging students to listen</td>
<td>Promoted reasoning and justification</td>
<td>Graham explained that they would have to cut the smallest rod in half (he indicated the clear one would need to be divided which is the OHP version of the white rod).</td>
</tr>
<tr>
<td>2.0.198</td>
<td>T/R 1:</td>
<td>I see some hands up. Why don’t you see if...?</td>
<td></td>
<td></td>
<td>Michael interjected that if one made a new rod one would need to create a whole new set in order to make a half of the new rod.</td>
</tr>
<tr>
<td>2.0.200</td>
<td>T/R 1:</td>
<td>What do you think, Graham? What do the rest of you think? Do you think there would have to be a whole new set? There are some other people who have opinions. Why don’t you go, who’s going to, why don’t you take it, Jackie? You call on people, okay?</td>
<td>Eliciting</td>
<td>Promoted reasoning</td>
<td>Brian said that “No matter what there’ll always be something that won’t be equal to something.”</td>
</tr>
<tr>
<td>2.0.205</td>
<td>T/R 1:</td>
<td>Can you say a little more about that, Brian? Nice</td>
<td>Probing-elucidating</td>
<td>Promoted reasoning</td>
<td>Brian reiterated that there wouldn’t be a rod</td>
</tr>
<tr>
<td>Time</td>
<td>Speaker</td>
<td>Action</td>
<td>Description</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>---------</td>
<td>--------</td>
<td>-------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0.208</td>
<td>T/R 1:</td>
<td>Did you all hear what Brian said? That’s, Brian, I, we didn’t hear back here. Kimberly and I are trying hard. Can you turn around and say it nice and loud?</td>
<td>Encouraging engagement – administrative, encouraging students to listen</td>
<td>Brian repeated his claim again.</td>
<td></td>
</tr>
<tr>
<td>2.0.210</td>
<td>T/R 1:</td>
<td>What do you think about that? David, you had your hand up. Do you have a different point?</td>
<td>Encouraging engagement – confirming agreement, shifting questioning to other students</td>
<td>Promoted reasoning</td>
<td></td>
</tr>
<tr>
<td>2.0.216</td>
<td>T/R 1:</td>
<td>What do you think, Jacquelyn?</td>
<td>Encouraging engagement - shifting questioning to other students</td>
<td>Jackie agreed with Michael.</td>
<td></td>
</tr>
<tr>
<td>2.0.218</td>
<td>T/R 1:</td>
<td>It’s an interesting question, isn’t it? It’s an interesting question. So in other words, when you designed a solution, you’re telling me, for the problem where you’re making now a pink rod, is that what you’re calling it?</td>
<td>Probing - elucidating</td>
<td>Jackie agreed.</td>
<td></td>
</tr>
<tr>
<td>2.0.220</td>
<td>T/R 1:</td>
<td>You’re creating a pink rod. And as I understand it, the pink rod is made up of purple and half a white. Is that what you said? Um. You solved the problem of having a rod that you can call one half when you call the blue rod one, right? But then, as some of you pointed out, then your smallest rod is then, with this new design, your smallest rod is then,</td>
<td>Probing - elucidating</td>
<td>Meredith filled in that it would be half of the white rod.</td>
<td></td>
</tr>
<tr>
<td>2.0.224</td>
<td>T/R 1:</td>
<td>White rod, right? And what are you going to call that? Let’s give that a name. Let’s give that a name. Can you give that a name? It’s not white any more. It’s half of white. What color name shall we give it?</td>
<td>Encouraging engagement - shifting questioning to other students, Eliciting</td>
<td>Jackie offered that a rod half the length of the white rod should be called light blue.</td>
<td></td>
</tr>
<tr>
<td>2.0.226</td>
<td>T/R 1:</td>
<td>Did anybody have another way to make the argument? James?</td>
<td>Encouraging engagement – shifting questioning to other students, Eliciting</td>
<td>Promoted reasoning and justification</td>
<td></td>
</tr>
<tr>
<td>2.0.229</td>
<td>T/R 1:</td>
<td>Is that okay? That’s another way, huh? Does anybody have another way? …Do you think there’s still another way?</td>
<td>Encouraging engagement - confirming agreement, Eliciting</td>
<td>James at the overhead projector demonstrated that one and half light green rods could also be considered half of the blue rod and that new rod could be given the color name light blue.</td>
<td></td>
</tr>
<tr>
<td>2.0.233</td>
<td>T/R 1:</td>
<td>What do you think, Dr. Martino? You want to</td>
<td>Encouraging engagement</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
After completing that task, T/R 2 began a new set of tasks with the class. She began by putting an orange and yellow rod on the overhead projector, and asked the students, “If we call the orange ‘two’, what can we say about yellow?” The students worked briefly in groups on this task. Danielle, working with Gregory, explained her solution to him. Placing two yellow rods next to an orange rod, she explained that since two yellow rods were equal in length to the orange rod each yellow rod would be called one. Gregory disagreed with her and said that “when the orange is one, we went like a half down” (line 2.0.240). Listening to this exchange, the classroom teacher asked Gregory if the orange rod was called one in the task at hand. Gregory realized his argument was faulty and conceded that the yellow rod could then be given the number name one. Meredith too explained to her partner Sarah that the yellow rod would be called one since the orange rod had been given the number name two. Brian as well demonstrated this solution in front of the class on the overhead projector.

T/R 2 then changed the task and asked “What if I change the name of the orange to ‘six’... what number name would I call the yellow? Kimberly offered that the yellow rod would then be called five. When questioned about her solution Kimberly explained Look here [pointing to Brian’s model] before you said that [the orange rod] would equal two, and then Brian said that [yellow rod] would equal one. So now you’re saying that that [orange rod] equals six, so I figured that if that equaled one before [yellow rod] it would equal five now” (line 2.0.270). T/R 2 asked the class what they thought about Kimberly’s argument. Alan disagreed with her solution and explained that in the previous
task the orange rod was called two and the yellow rod was given the number name one.
He explained that now that orange rod was called six then the yellow rod which is half
the length of the orange rod would need to be called three. Jessica then reiterated Alan’s
argument. T/R 1 then interjected and said that she was curious why Kimberly thought the
yellow rod would be called five. Kimberly explained that she thought that since when the
orange was two the yellow was one, then here when the orange is six the yellow would be
called five. T/R 1 repeated Kimberly’s argument while demonstrating with rods at the
overhead. She asked, “So you're saying if this [yellow rod] is five and this [yellow rod] is
five, this [orange rod] is six?” (line 2.0.280). Kimberly then admitted that she had made a
mistake and T/R 1 encouraged her to explain her thinking. Kimberly acknowledged that
she had forgotten that although the sum of one and one is two, five and five does not
equal six. T/R 1 then asked her what the orange rod would be called if the yellow rod was
given the number name five to ensure that she fully understood the solution. Kimberly
explained that in that case it would be called ten.

Table 4.5

*Classification of teacher questioning and results, Colts Neck Session 2 – Part V*

<table>
<thead>
<tr>
<th>Line</th>
<th>Researcher</th>
<th>Researcher question</th>
<th>Question type</th>
<th>Question result</th>
<th>Student conjecture, response, reasoning, or justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0.234</td>
<td>T/R 2:</td>
<td>Alright. Let’s try something a little different now. Ok. Now, if we call the orange “two”, what can we say about yellow? Think about it for a minute, and you want to talk to your partner?</td>
<td>Encouraging engagement – administrative</td>
<td>The students worked on the task.</td>
<td></td>
</tr>
<tr>
<td>2.0.251</td>
<td>T/R 2:</td>
<td>Who haven’t we heard from? Let’s see. Brian, what do you think, now when we call this, we give this the number name two, the orange, what number name are we going to give</td>
<td>Eliciting</td>
<td>Brian answered one.</td>
<td></td>
</tr>
<tr>
<td>Time</td>
<td>T/R 2:</td>
<td>Prompt</td>
<td>Response</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>--------</td>
<td>--------</td>
<td>----------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0.253</td>
<td>Why one? You want to come up here.</td>
<td>Probing - elucidating</td>
<td>Brian explained the yellow rod would be called one since the orange rod had been given the number name two.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0.255</td>
<td>Okay, so you’d consider each of these [yellow rods] a one, is that what you’re saying?</td>
<td>Probing - elucidating</td>
<td>Brian said no but explained again that the two yellow rods would be equal in length to the orange rod and therefore would be called one.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0.257</td>
<td>O.k. So the number name you’re giving yellow then was what?</td>
<td>Probing - elucidating</td>
<td>Brian answered one.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0.259</td>
<td>Okay, alright, one. What do you think about that? Does anyone want to come- Who agrees with that, that you give the yellow the number name one? Ok. Does anybody disagree with that? I heard, I heard some</td>
<td>Encouraging engagement – confirming agreement</td>
<td>Erik said he had another number name.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0.261</td>
<td>Ok, what would you call it, Erik?</td>
<td>Eliciting</td>
<td>Erik said that if the orange one had the number name one then the yellow rod would be called one half.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0.267</td>
<td>Let me ask you another question then, I'm going to ask this to everybody, too. What if I change the name of the orange to six? What would I call the yellow-what number name would I call the yellow? Let’s see, uh, somebody I haven’t had a chance to talk with, James, is your hand up? Kimberly?</td>
<td>Eliciting</td>
<td>Kimberly answered that the yellow rod would be called five.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0.269</td>
<td>Five. That’s interesting. Can you come up and tell us about that?</td>
<td>Probing - elucidating</td>
<td>Kimberly presented a faulty argument saying that since in the previous example when orange had been given the number name two, the yellow rod was called one, here when orange is six, the yellow rod would be called five.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0.373</td>
<td>That’s interesting. What do you think about that, some of these other folks? Did you all hear Kimberly’s argument here? She’s saying when you call this one, the number name two, the orange, that the yellows were each one, ok, they had the number name of one. She’s saying, so if I call this six now, she’d call that five. What do you</td>
<td>Encouraging engagement – confirming agreement</td>
<td>Alan disagreed with her solution and explained that in the previous task the orange rod was called two and the yellow rod was given the number name one. He explained that now that orange rod was called six then the yellow rod which is half the length of the orange rod would</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
think? [Meredith and others shake their heads negatively.] Ok, I see some people are shaking their heads and I want to hear why. Uh, let’s see. Alan?

| 2.0.273 | T/R 2: | Okay. So we have another argument. What do you all think about Alan’s argument? He’s calling this [yellow rod] three, the number name three when I call this [orange rod] the number name six? Meredith? | Encouraging engagement – confirming agreement Eliciting | Meredith’s response is inaudible. |
| 2.0.278 | T/R 1: | I’m curious how Kimberly thought of five? Can you help me understand why you think five? | Probing – elucidating | Promoted reasoning |
| 2.0.280 | T/R 1: | The yellow is five... That’s where I am confused. So you’re saying if this [yellow rod] is five and this [yellow rod] is five, this [orange rod] is six? | Probing - elucidating | Kimberly admitted that she had made a mistake. |
| 2.0.282 | T/R 1: | You didn’t mean that? What did you mean, Kimberly? | Probing - elucidating | Kimberly said she made a mistake since one plus one equals two but five plus five does not equal six. |
| 2.0.287 | T/R 1: | If you want this to be five, what would you have to call the orange? | Probing - elucidating | Kimberly answered ten. |
| 2.0.289 | T/R 1: | You’d have to call orange ten. Do you agree with that? | Encouraging engagement – confirming agreement | Many students answered yes. |

The last task during the session was posed by T/R 2: *I’m going to call the orange and light green together one…Can you find a rod that has the number name one half?*

The students worked on this task for a few minutes in groups and then discussed their solutions in a whole class discussion. Brian built a train consisting of an orange and light green and put alongside it two dark green rods with a white rod between the dark green.

He explained his solution to T/R 2 who was circulating around the classroom saying that
there was no rod that could be called one half of the orange and light green train since the train was equal to thirteen and “thirteen is an odd number” (line 2.0.301). When questioned further, Brian explained that since thirteen is an odd number the train could not be divided into two equal parts. He reasoned that the only way to divide it would be to cut a white rod in half and put one part on each side together with the dark green rod. Brian said he had an additional solution and demonstrated to T/R 2 that if he put four light green rods with a white rod in the middle, two light green rods and a half of the white rod could be considered half the length of the orange and light green train.

    T/R 2 then discussed the task with Jessica and Laura. Jessica and Laura had built a model similar to Brian’s first model, where they had placed a two dark green rods and one white rod beneath the orange and light green train. They had placed the white rod though at the end of the train of two dark green rods. Jessica explained to the researcher that they needed to create a new rod since no rod existed in the set that could be considered half the length of the orange and light green train. When T/R 2 asked Jessica what the new rod would look like, Jessica mistakenly said that the new rod would consist of a dark green rod and a white rod. Laura then explained that it would be the dark green rod and half of the white rod. Jessica agreed and showed T/R 2 what such a model would look like. Meredith, working with Sarah, also built a model similar to Brian’s and Jessica’s model after trying to find a rod that would be considered “six” (line 2.0.329). When T/R 2 approached her desk she provided similar reasoning to the other students and said that half of the white rod and a green rod would be equal to half the length of the orange and light green train.
In the whole class discussion, Andrew constructed the model that had been created by Brian, Jessica, and Meredith and explained that the dark green rod and half of the white rod would be equal to half of the orange and light green train. T/R 2 asked the class to explain why there was no rod in the set that could be called half of the train. Meredith at first offered a partially faulty explanation. She explained: “Well, because you want to have seven and six, seven, but there are no rods that are really seven, and you need it to be thirteen.” T/R 2 asked Meredith to explain her thinking and Meredith continued, “Well, take two greens and a white… And there's no blocks that have half on them, and for the uneven numbers, for the odd numbers you need a half, because you can’t make it without it.” Although Meredith was correct in her argument that the length of the train was odd, she mistakenly said that she would need a rod that was seven white rods long.

T/R 2 asked Brian what Meredith meant by “odd” since he too had used the term earlier. Brian did not answer that the question but rather demonstrated how he had attempted to find all of the ways that the rods could be combined to provide a solution to the task. In figure 4.5 one can see the different models that Brian constructed.

Figure 4.5. Brian’s cases
Meredith then showed the class another solution to the problem on the overhead projector. This solution had been built by Brian earlier but had not been presented to the class. Together with Sarah, Meredith built a train consisting of a yellow, light green, and yellow rod. Meredith explained that if the light green rod was cut in half then “one and a half” could be added to each yellow rod and those together could be called one half (line 2.0.374). Meredith’s explanation was unclear since she did not clarify what she meant by “one and a half.” T/R 2 asked the class if they understood what Meredith meant and Graham offered that Meredith meant that if one split the light green rod in half and allocated each piece to one side then one could create a train that could be given the number name one half.

T/R 2 then asked Meredith what the orange and light green train would be called if the light green rod was three. Meredith answered that the train would be called thirteen. T/R 2 asked for further clarification by saying, “You were thinking of the whole length of the train as being thirteen of what? Thirteen blues, thirteen oranges, thirteen what?” (lines 2.0.281–2.0.283). Meredith explained that she mean thirteen yellows. She said that if the light green rod was cut and painted yellow it would equal the length of the train and be equal to thirteen. T/R 2 then turned to the class for help in understanding Meredith’s reasoning. Erik offered to clarify and explained that the train was equal to thirteen white rods. As he provided this clarification, Meredith built added white rods to her model on the overhead.

Erik offered to provide one more solution before the close of the session. At the overhead projector he built a model consisting of a train of two light green rods and seven white rods. He then explained:
“I figured you could take two light greens and put them there. And then after that I just took all these, the clear ones, and I figured, well, I put down seven. And I figured that they are all equal, and if you have these two you would have three and then you could take one and put it on that and so it would be four, five, you would have three, four, and then four, five, six, six, and then seven.”

Erik indicated that he was dividing the white rods into two groups, with each group containing three or four rods. T/R2 asked him to clarify and he explained that the first group would equal the length of six white rods and the second group would be equal in length to seven white rods (Figure 4.6). T/R 2 asked him what he would do with the extra white rod in the second group. Erik changed his model and put three white rods in the place of one green rod. He then divided the rods into three groups by putting a light green rod together with a white rod in two of the groups and a light green rod with two white rods in the third group (Figure 4.7). T/R 2 continued to question him and asked what he would do with the fourth white rod. She suggested that if they went back to the original model (of two light green rods and seven white rods), they could split the extra white rod in half. Erik concurred and said that half of the white rod could be placed in each of the two groups.

Figure 4.6. Erik’s first arrangement

Figure 4.7. Erik’s second arrangement

Table 4.6
<table>
<thead>
<tr>
<th>Line</th>
<th>Researcher</th>
<th>Researcher question</th>
<th>Question type</th>
<th>Question result</th>
<th>Student conjecture, response, reasoning, or justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0.291</td>
<td>T/R 2:</td>
<td>Suppose we made a train, ok, I'm going to take Erik's idea from earlier, and I'm going to call the orange and [light] green together, one. You like that? I'm calling this one. The orange and green train together, one. Now I didn't work this problem, but, I'm curious, can you find a rod that has the number name one half?</td>
<td>Encouraging engagement - administrative</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0.300</td>
<td>T/R 2:</td>
<td>There was no rod that worked perfectly when you take two of them?</td>
<td>Probing - elucidating</td>
<td>Promoted reasoning</td>
<td>Brian responded that there was no rod since the train was 13 (cm) in length and thirteen is an odd number</td>
</tr>
<tr>
<td>2.0.302</td>
<td>T/R 2</td>
<td>What does that have to do with it?</td>
<td>Probing - elucidating</td>
<td>Promoted reasoning</td>
<td>Brian responded that one cannot divide thirteen evenly unless he would divide a white rod as they had done during a previous task.</td>
</tr>
<tr>
<td>2.0.306</td>
<td>T/R 2:</td>
<td>Okay, so we are going to develop a new rod again. We'd have to go back to the workshop and make a new rod. Ok, This is wonderful. What do you think, Erin? Do you agree with what Brian said? Have you checked and made sure that there aren't any pairs of the same that would fit here?</td>
<td>Encouraging engagement – confirming agreement</td>
<td></td>
<td>Erin nodded affirmatively.</td>
</tr>
<tr>
<td>2.0.314</td>
<td>T/R 2:</td>
<td>Ok, so what would one half of this green and orange train be? How much of this, in other words? Can you show me?</td>
<td>Probing - elucidating</td>
<td></td>
<td>Jessica stacked one green rod on top of the other. She put a green and white rod on the bottom and green rod on top.</td>
</tr>
<tr>
<td>2.0.316</td>
<td>T/R 2:</td>
<td>So what do you think, Laura? Do you know what I’m asking her? I want to be able to see the one half in my head.</td>
<td>Eliciting</td>
<td>Promoted reasoning</td>
<td>Jessica held up a train of green and white rods and said it would be called one half.</td>
</tr>
<tr>
<td>2.0.320</td>
<td>T/R 2:</td>
<td>What do you think, Laura?</td>
<td>Eliciting</td>
<td></td>
<td>Laura agreed with Jessica that a green rod together with half of a white rod would equal one half.</td>
</tr>
<tr>
<td>2.0.362</td>
<td>T/R 2:</td>
<td>Ok, is there somebody who would like to share a solution with me to show me one half of this train which has an orange and a green? To show me something that is one half as long as this orange and green train. Ok, let’s see, uh, Andrew? Could you come up and show? If you worked with Mark the two of you can come up and show us?</td>
<td>Eliciting</td>
<td>Promoted reasoning</td>
<td>Andrew said that if they took a green rod and half of a white rod it would be half as long as the orange and green train.</td>
</tr>
<tr>
<td>2.0.365</td>
<td>T/R 2:</td>
<td>Ok, do you all follow</td>
<td>Encouraging</td>
<td>Promoted</td>
<td>Meredith began to explain that</td>
</tr>
</tbody>
</table>
what Andrew said here? Erik, did you have a comment on that? You had something different. First of all, what do you think of this? Does this work? Looks like we're into inventing our own rods again, right? Making up a rod, a new rod here. Why do you think that works? I mean, why do you think that that works? You have any ideas? Meredith?

<p>| 2.0.367 | T/R 2: | Ok, can you say a little bit more about that? | Probing – elucidating | Promoted reasoning and justification | Meredith said to take two green rods and a white rod because for odd numbers there wouldn’t be a rod equal to half so one would need to use the white rod. |
| 2.0.371 | T/R 2: | Ok, Brian said something like that too, about the numbers being odd. Brian, what did you want to add? | Eliciting | Promoted reasoning | Brian said that by splitting the white one in half, they could make a half just as they had done in a previous task. |
| 2.0.373 | T/R 2: | Does anybody else have anything they want to add to this before I begin a new problem? | Encouraging engagement - administrative | Promoted reasoning and justification | Meredith built a train consisting of a yellow, light green, and yellow rod and said that one could add one and a half to one yellow and one and a half to the other yellow. |
| 2.0.375 | T/R 2: | What do you mean, one and a half? Does anyone know what Meredith means? I don't want you to tell me yet. Does anyone knows what Meredith means by adding one and a half to the yellow on each side? Where did she get one and a half from? I see a couple of hands, let’s see. Graham | Probing – elucidating | Promoted reasoning and justification | Graham answered from the light green rod. |
| 2.0.377 | T/R 2: | The light green, ok. How does this become one and a half? What piece of it [the train] becomes one and a half? I don’t understand. | Probing – elucidating | Promoted reasoning and justification | Graham held up the light green rod and explained that they could split it in the middle and put one and a half on each side. |
| 2.0.379 | T/R 2: | Oh, okay, all right. So if I cut that [Light green rod] down the middle, I see, okay. Well, if we’re calling this light green three, what are you calling this train with the light green and the orange together? | Probing – elucidating | Promoted reasoning and justification | Meredith answered thirteen. |
| 2.0.381 | T/R 2: | You were thinking of the whole length of the train as being thirteen of what? | Probing – elucidating | Promoted reasoning and justification | Meredith reiterated thirteen. |
| 2.0.385 | T/R 2: | Thirteen yellows? | Probing – elucidating | Promoted reasoning | Meredith answered if you turn the light green rod into yellow by cutting the light green rod in half and painting each piece yellow it would be equal thirteen and be equal in length to the orange and green. |</p>
<table>
<thead>
<tr>
<th>Time</th>
<th>T/R 2:</th>
<th>Dialogue</th>
<th>Type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0.389</td>
<td></td>
<td>Do you understand my question, though? She keeps saying thirteen for the train that I made with the orange and the green. I don’t understand where she’s getting the number thirteen from. Why thirteen?</td>
<td>Probing - elucidating</td>
<td>Promoted reasoning and justification Erik answered that the length of the train is thirteen and one could demonstrate that by lining up white rods along the train.</td>
</tr>
<tr>
<td>2.0.394</td>
<td></td>
<td>Oh! So then what you’re saying is if you line up the little white cubes along the, uh, the train with the orange and the green there’d be thirteen of them?</td>
<td>Probing - elucidating</td>
<td>Erik answered yes</td>
</tr>
<tr>
<td>2.0.396</td>
<td></td>
<td>Do we have a minute to do another one, or do we have to clean up.</td>
<td>Encouraging engagement - administrative</td>
<td>Erik showed T/R 2 another possible solution and built a model consisting of a train of two light green rods and seven white rods.</td>
</tr>
<tr>
<td>2.0.398</td>
<td></td>
<td>Ok, alright. So you figured then that you can put, have seven on each of our halves?</td>
<td>Probing - elucidating</td>
<td>Promoted reasoning              Erik explained that that the first group would equal the length of six white rods and the second group would be equal in length to seven white rods</td>
</tr>
<tr>
<td>2.0.402</td>
<td></td>
<td>Ok, so, in other words, one of these could go here with this group, one of these goes here, back and forth like this. Ok, what happens to this guy, though? [pointing to the white rod to the far right] How can I be fair in making my two halves the same size? What could I do?</td>
<td>Probing – elucidating</td>
<td>Erik changed his model and put three white rods in the place of one green rod. He then divided the rods into three groups by putting a light green rod together with a white rod in two of the groups and a light green rod with two white rods in the third group</td>
</tr>
<tr>
<td>2.0.404</td>
<td></td>
<td>Uh huh, oh, but I have one for this guy, and one for this guy, one for this guy, and what about this guy? [She points to the remaining white rod on the far right.]</td>
<td>Probing - elucidating</td>
<td></td>
</tr>
<tr>
<td>2.0.406</td>
<td></td>
<td>I think we ran into the same problem, didn’t we? Would you agree that if we went back to this model, Erik, where we had these [She rearranges the rods] and we were divvying them up. Would you agree that maybe I could take this one [white rod] and saw it in half, if I had a saw?</td>
<td>Probing – elucidating</td>
<td>Erik concurred</td>
</tr>
<tr>
<td>2.0.408</td>
<td></td>
<td>And then what could I do with it, if I sawed it in half?</td>
<td>Probing - elucidating</td>
<td>Promoted reasoning              Erik said that half of the white rod could be placed in each of the two groups.</td>
</tr>
<tr>
<td>2.0.410</td>
<td></td>
<td>Ok, ok, I think we are almost out of time, aren’t we? We probably need to clean up. Unless we have a minute? Ok, yes, ok, what do you have to share with us? Brian?</td>
<td>Encouraging engagement - administrative</td>
<td>Promoted reasoning and justification Brian showed his model of a train consisting of a purple, yellow and purple rod and explained that the yellow could be cut in half.</td>
</tr>
</tbody>
</table>
4.2.1.1 Questioning techniques used in the Sep 21, 1993 session at Colts Neck

In this session, questioning is used in a variety of ways. In the beginning of the session, instead of the researcher reviewing what had been done in a previous session, as is usually done in ordinary classroom sessions, T/R 1 asked the students to recap and describe what they had accomplished in the previous session. In response, Erik provided an example to explain that in the previous session the class had been told that a rod had the number name one and they would have to find a rod that would be called one half. His choice of a rod for this example, the blue rod, led to a spirited class discussion where the students debated whether there could be a rod one half the length of the blue rod. T/R 1 allowed the students to lead the discussion by asking students to share their thinking, and demonstrate their solutions at the overhead projector. T/R 1 used questions such as “David, what do you think? What does David think?” (line 2.0.48) to elicit student responses and to encourage students to listen to their peers. When David conjectured that there was no rod in the set that could be given the number name one half when the blue rod was given the number name one, T/R 1 used probing questions to elucidate the issue at hand and allow David to formulate a cogent argument and clarify and justify his solution for the class. T/R 1 used questions such as “What about something between purple and yellow?” (line 2.0.52) to facilitate justification and reasoning. In order to help the students fully understand David’s argument, T/R 1 asked David to come up to the overhead and explain his reasoning to the class. This elicited
justification and reasoning where David built a staircase model and used an upper and lower bound argument to justify his conjecture that there was no rod that could be called half of the blue rod. T/R 1 confirmed the class’s agreement by asking “Are you all convinced?” (line 2.0.61). When Erik presented a faulty argument where he claimed that two different length rods could be considered halves, T/R 1 encouraged all students to listen to his argument and then encouraged student engagement by shifting questioning to other students to elicit counterarguments. Erik did not change his position, and T/R 1 used probing questions which invited him to change his argument. She compared the rod to a brick of gold and asked Erik if he thought it would be fair if she would get the larger “half.” Although Erik held onto his earlier claim, when T/R 1 confirmed agreement from the rest of the class, it elicited argumentation, where other students presented counterarguments. T/R 1 used probing questions to gain clarification of their arguments and encourage critical thinking. T/R 1 asked students for additional input to elaborate on other student responses to help the students and researchers fully understand the solution being presented. During these exchanges, questioning served to elicit argumentation, reasoning and justification numerous times. In instances where students presented correct arguments as well as in cases where they offered faulty arguments, T/R 1 often used a sequence of probing questions (Franke et al, 2009) which helped the students explain their solution more completely or revise their faulty explanation. When students provided faulty explanations, rather than correcting them, the researcher also allowed other students the opportunity to share their thoughts and present their arguments. Although Erik persisted in his faulty reasoning, T/R 1 repeatedly returned to his explanation until he realized on his own that his reasoning was faulty. By using compelling examples and
by asking Erik questions regarding his explanation, Erik at first modified his reasoning slightly due to researcher questioning until later on in the session the researcher was able to elicit correct reasoning and justification. By using a probing sequence of specific questions in an interchange that included multiple teacher questions and multiple student responses, which helped clarify students’ arguments and encourage critical thinking, Erik gained clarity and the solution was clarified for the rest of the class as well. Throughout the exchange with Erik, T/R 1 punctuated the conversation with questions to confirm agreement or disagreement from the rest of the class. After a number of arguments had been presented, the researcher asked a student to review the different points that had been discussed. Alan was able to recap clearly the different forms of reasoning that had been presented earlier.

In the next part of the session, while students worked on the task at first in pairs, T/R 2 asked students to explain the problem and how they would go about coming up with a solution. Then during the whole class discussion, T/R 1 encouraged a number of students to present their solutions in front of the class. When Kimberly presented a faulty argument, which was countered by other students, T/R 1 used probing questions to gain clarification regarding Kimberly’s solution and encouraged critical thinking. Kimberly was then able to explain why she had initially come up with her faulty argument and how she realized that her solution was erroneous. T/R 1 then used additional questioning to ensure that Kimberly was able to apply that knowledge to another example, which indicated that she fully understood the solution.

In the latter part of the session, T/R 2 circulated around the classroom and questioned students on their models, thereby eliciting reasoning and allowing students to
make their thinking explicit. Students presented many forms of reasoning, including direct, indirect, and recursive reasoning and upper and lower bound arguments. Then in a whole class discussion she used many probing questions to clarify student responses. By asking students to attempt to explain other students’ reasoning, she was able to lead the students to the correct answer while at the same time clarifying the concepts for the entire class. Figures 4.8 – 4.10 show the question types and associated student reasoning that occurred during this session.

![Question Type](image)

*Figure 4.8. Question types used by T/R 1 during Session 2*
Figure 4.9. Question types used by T/R 2 during Session 2

Figure 4.10. Question results during Session 2
4.2.1.2 Analytics for the Sep 21, 1993 session at Colts Neck

The analytic “Task Design Prompts Fourth Grade Students to Use Multiple Forms of Reasoning” (Winter, 2015a) portrays the researcher question and student arguments during the first part of Session 2 where the students worked on the first task. “Fourth Graders Design a New Rod” (Gerstein, 2015b) describes the next part of the session where students work on the task of designing a new rod that could be called one half of the blue rod.

The analytic that presents the questions asked by T/R 2 and the associated student argumentation, reasoning and justification during the second part the session is entitled “Fourth Graders Reason by Cases as They Explore Fraction Ideas” (Winter, 2015b).

In addition, Winter (2015c) in an analytic entitled “Fourth graders build towards proportional reasoning,” depicts the events during this session where Kimberly erroneously uses additive reasoning and Alan challenges her solution. Researcher probing and questioning invited Kimberly to explain her misconception and revise her solution with the use of proportional reasoning.

See also “Establishing norms and creating a mathematical community” (Gerstein, 2015a) which portrays teacher questioning during the preceding Session 1 at Colts Neck and demonstrates how the researchers thereby created an atmosphere of mathematical inquiry. This analytic illustrates the importance of establishing socio-mathematical norms (Steencken, 2001; Yackel and Cobb, 1996) that establish what criteria establish the validity of a mathematical solution in the mathematics classroom. In this classroom, a norm was immediately established that clearly formed justifications that were to be agreed upon by all members of the mathematical community. In addition, the analytic
shows how a mathematical community was created by promoting group interaction and encouraging students to share their ideas and challenge one another. This analytic demonstrates the importance of collaboration and the ways that students can learn by constructing their own problems in an attempt to challenge their peers.

“The Interplay between Teacher Questioning and Student Reasoning: Utilizing Probing Questions to Elicit Argumentation, Justification, and Reasoning” (Gerstein, 2017a) focuses on the questions posed by Researcher Maher during this session and the associated student argumentation, justification, and reasoning.

4.2.2 Colts Neck Sept 29, 1993 (Session 5) with Researchers Carolyn Maher and Amy Martino

At the beginning of Session 5 on Fractions, T/R 1 informed the class that they would be continuing a discussion they had started in a previous session. She placed an overhead on the projector and asked the students to read the problem written there and to tell her if they remembered discussing this problem previously. The problem that she put on the overhead projector read “Is $1/5=2/10$?” Meredith offered to show what they had worked on at the overhead projector. At first she built a model consisting of an orange rod and two yellow rod and said that the orange rod could be given the number name one and the yellow rods would be given the number name one half. T/R 1 tried to get Meredith to complete her explanation and therefore asked her to clarify how her model answered the question. Meredith then incorrectly gave the yellow rods the number name two tenths and showed how five red rods were equal in length to the two yellow rods. T/R 1 again tried to get her to correct her explanation and therefore asked her to once again explain her argument to the class. Meredith fixed her model by using ten white
rods. She gave the white rods the number name one tenth and explained that two white rods could be called two tenths. She then moved the five red rods to demonstrate that the red rods were called one fifth. She also placed two white rods under a red rod on the side of the overhead projector to emphasize that they were equal in length and stated that two tenths equals one fifth (figure 4.11). T/R 1 then used questioning to make sure that everyone understood and agreed with Meredith.

![Figure 4.11. Meredith demonstrates that 2/10=1/5](image)

Brian offered to go to the overhead and using Meredith’s model, he reiterated Meredith’s argument. Erik too concurred with Meredith and stated that since the two white rods were called two tenths and one red rod was called one fifth, then the two are equal.

**Table 4.7**

*Classification of teacher questioning and results, Colts Neck Session 5 – Part I*

<table>
<thead>
<tr>
<th>Line</th>
<th>Speaker</th>
<th>Researcher question</th>
<th>Question type</th>
<th>Question result</th>
<th>Student conjecture, response, reasoning, or justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0.3</td>
<td>T/R 1:</td>
<td>Do you know what that question is? Do you remember that? Can you read that statement? What does that say, Michael?</td>
<td>Encouraging engagement - administrative</td>
<td>Michael read the task. <em>Is 1/5 = 2/10?</em></td>
<td></td>
</tr>
<tr>
<td>5.0.6</td>
<td>T/R 1:</td>
<td>How many of you agree that is what that says? That’s my question. Is one fifth equal to two tenths? [quiet in the room]</td>
<td>Encouraging engagement - confirming agreement</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.0.7</td>
<td>T/R 1:</td>
<td>Do you remember talking about any of those ideas at all? What do you remember about that, Meredith?</td>
<td>Eliciting</td>
<td>Promoted reasoning and justification</td>
<td>Meredith built a model consisting of an orange rod and two yellow rod and</td>
</tr>
</tbody>
</table>
Meredith responded one half

Meredith incorrectly gave the yellow rods the number name two tenths and showed how five red rods were equal in length to the two yellow rods.

Meredith fixed her model by using ten white rods. She gave the white rods the number name one tenth and explained that two white rods could be called two tenths. She then moved the five red rods to demonstrate that the red rods were called one fifth. She also placed two white rods under a red rod on the side of the overhead projector to emphasize that they were equal in length and stated that two tenths equals one fifth.

Meredith answers that they are equal

Brian answered that he agreed with Meredith.

Brian reiterated what Meredith had said.

Erik too concurred with Meredith and stated that since the two white rods were called two tenths and one red rod was called one fifth, then the two are equal.

Thank you gentlemen. [Brian and Erik return to their seats.]
T/R 1 then reminded the students about the candy bar story that they had talked about in an earlier session. After showing the students a drawing of the candy bar (Figure 4.12), she asked the students to see if they could come up with alternative number names for half of the candy bar. The students worked for a few minutes and then discussed their solutions in the ensuing whole class discussion.

Figure 4.12. The drawing of the candy bar

Jackie said that another number name for half of the candy bar could be six twelfths. Danielle offered another solution of two fourths and explained that if she took the top two rows it would be two fourths. T/R 1 asked her to explain how she found two fourths in order to make her thinking explicit and Danielle said “Because there’s four rows.” T/R 1 asked the class if they agreed with Danielle’s solution and since some students seemed to indicate that they did not agree, T/R 1 asked Brian to explain. Brian’s explanation was not clear and therefore Danielle clarified her solution for the class by
stating, “If there’s four fourths, and half of four is two, so two fourths would be a half.”
Brian then offered another number name for half of the candy bar. He explained that
there were six pairs of pieces in the candy bar and therefore three sixths could be another
alternate number name. T/R 1 summed up the different solutions offered by writing on
the overhead “1/2 = 6/12 = 2/4 = 3/6” (Figure 4.13). She then asked the students if it also
made sense that one fifth was equal to two tenths and the students agreed that that was
correct.

Figure 4.13. Researcher Maher records the different number names for half of the candy
bar

Table 4.8
Classification of teacher questioning and results, Colts Neck Session 5 – Part II

<table>
<thead>
<tr>
<th>Line</th>
<th>Speaker</th>
<th>Researcher question</th>
<th>Question type</th>
<th>Question result</th>
<th>Student conjecture, response, reasoning, or justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0.36</td>
<td>T/R 1:</td>
<td>Do you remember the candy bar? Remember. Did you get any? [Students say yeah.] This was the little candy bar, it looks something like this. If you excuse my sketch. Do you remember it looks like this. It was broken up into three columns and then four rows. Do you remember that? Remember I was giving half of this little candy bar to Dr. Martino ……? Remember how appreciative she was? Remember that? We said we were giving her half the candy bar, didn’t we? Right? Okay? Could someone have told me another name, another number name for how much of the candy bar I gave her? Do you understand my question? If this is</td>
<td>Encouraging engagement - administrative, checking for understanding, Eliciting</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
my candy bar and I gave her that much, right? One number name we said was one half; didn’t we? Can someone think of another name that very exactly tells me how much of the candy bar I gave Dr. Martino? And if you think you know, why don’t you discuss it with your neighbor and see if you agree.

| 5.0.37 | T/R 1: | Eliciting | Promoted reasoning | Jackie offered the number name six twelfths and explained that Dr. Martino received 6 pieces out of a total of 12 pieces. |
|--------|--------|------------|------------------|

<table>
<thead>
<tr>
<th>5.0.43-5.0.45</th>
<th>T/R 1:</th>
<th>Encouraging engagement - encouraging students to listen, confirming agreement, Eliciting</th>
<th>Many students agreed with Jackie and Danielle offered the number name two fourths.</th>
</tr>
</thead>
</table>

| 5.0.47 | T/R 1: | Probing - elucidating | Promoted reasoning | Danielle explained that the top two rows would be considered half and therefore could be given the number name two fourths. |
|--------|--------|------------------------|------------------|

<table>
<thead>
<tr>
<th>5.0.49</th>
<th>T/R 1:</th>
<th>Encouraging engagement - confirming agreement, Eliciting</th>
<th>Danielle explained that the top two rows would be considered two fourths.</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>5.0.51</th>
<th>T/R 1:</th>
<th>Probing - elucidating</th>
<th>Danielle explained that there are four rows.</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>5.0.53</th>
<th>T/R 1:</th>
<th>Encouraging engagement - confirming agreement</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>5.0.54</th>
<th>T/R 1:</th>
<th>Encouraging engagement - checking for understanding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>T/R</td>
<td>Role</td>
</tr>
<tr>
<td>-------</td>
<td>-----</td>
<td>------</td>
</tr>
<tr>
<td>5.0.56</td>
<td>T/R 1:</td>
<td>Eliciting</td>
</tr>
<tr>
<td>5.0.58</td>
<td>T/R 1:</td>
<td>Probing - elucidating</td>
</tr>
<tr>
<td>5.0.60</td>
<td>T/R 1:</td>
<td>Encouraging engagement - shifting questioning to another student, Eliciting</td>
</tr>
<tr>
<td>5.0.63</td>
<td>T/R 1:</td>
<td>Probing - elucidating</td>
</tr>
<tr>
<td>5.0.65</td>
<td>T/R 1:</td>
<td>Probing - elucidating</td>
</tr>
<tr>
<td>5.0.67</td>
<td>T/R 1:</td>
<td>Encouraging engagement - confirming agreement</td>
</tr>
<tr>
<td>5.0.69</td>
<td>T/R 1:</td>
<td>Encouraging engagement - checking for understanding, confirming agreement, Probing - encouraging critical thinking</td>
</tr>
<tr>
<td>5.0.71</td>
<td>T/R 1:</td>
<td>Encouraging engagement - confirming agreement, Factual - recall</td>
</tr>
<tr>
<td>5.0.75</td>
<td>T/R 1:</td>
<td>Factual - recall</td>
</tr>
<tr>
<td>5.0.77</td>
<td>T/R 1:</td>
<td>Factual - recall</td>
</tr>
</tbody>
</table>
During the next part of the session the students discussed a problem that they had worked on during the last session. T/R 1 asked the students about the problem they had been working on during that session. Michael and Andrew reminded everyone that they had been working on the task: “Which is bigger ½ or 1/3?” T/R 1 asked the class which one they had found to be bigger, and Laura answered that one half was bigger. T/R 1 requested that Laura and her partner Jessica show their solution on the overhead projector. Jessica and Laura built a model of an orange and red train and then lined up two dark green rods and three purple rods (Figure 4.14). They gave the orange and red train the number name one, they called the dark green rod one half, and the purple rod they called one third. Jessica explained that one could see from their model that one half was bigger than one third. T/R 1 asked the class if they agreed with Jessica. Audra said that she agreed and explained that the model showed that one half is greater than one third. T/R 1 then asked Jessica, Laura, and Audra, how much bigger one half is than one third. Jessica showed that the difference between one half and one third was a red rod. T/R 1 asked them if they could give a number name to the red rod. Audra and Jessica each built a second model with a purple and red train, and a dark green rod. Then they lined up two red rods next to this model and said that the number name would be one third (Figures 4.15 and 4.16). Since the girls were using a different model they answered the question incorrectly.
Figure 4.14. Model with a train of orange and red rods as the unit

Figure 4.15. 1/2 is one red rod or two white rods longer than 1/3

Figure 4.16. Models to support the claim that 1/2 is greater than 1/3 by 1/3

T/R 1 asked the class once again if they agreed with the argument that the girls had presented. Kelly said that she agreed with what they presented and went up to the overhead. There she built a train of two light green rods. Then she put a light green rod next to the red rod and stated that the light green rod had the number name one half and the red rod had the number name one third. Kelly did not present a complete explanation.

T/R 1 asked Brian what he thought about the arguments that had been presented. Before presenting his argument Brian first reiterated what he believed to be the first arguments. He asked for confirmation that the girls had said that the difference between one half and one third was one third. Brian then stated that he did not agree with the girls’ solution. He believed the difference to be one sixth. He reasoned and demonstrated correctly that the red rod was half of the purple rod (Figure 4.17) and therefore would be
called one sixth and that the difference between one half and one third is one sixth. He showed that the purple rod had been given the number name one third and that two red rods were equal in length to the purple rod. Each of the red rods would be called one sixth and both together would be called two sixths which would be the same as one third.

Figure 4.17. Model showing two red rods aligned with one purple rod

Jessica objected to Kelly and Jackie’s argument saying that their argument was different than the original argument. Jackie repeated their argument using a more complete model consisting of a dark green rod, two light green rods, and three red rods (Figure 4.18). She called the dark green one, the reds one third and the light green one half. She then showed that the difference between one half and one third was one white rod. Jessica one again objected to Jackie’s explanation saying that “I think they are making a different size candy bar” (line 4.0.146). T/R 1 asked if one was allowed to make such a change. Jessica incorrectly explained that it was not allowed because one half of the larger model would be a different size than one half of the smaller model.

Figure 4.18. Jackie and Kelly’s model with dark green rod as 1
T/R 1 asked Jackie to once again to restate her argument. Jackie repeated her argument and explained once again that one half was larger than one third “by one which is the white one” (line 5.0.162). T/R 1 clarified for the class that Jackie had indeed used a different model than Jessica and Audra. She asked them if they had also claimed that the difference between the two fractions was one. When they admitted that they had, T/R 1 asked them if that meant that the white rod was called one. They responded in the affirmative and T/R 1 asked if they were calling both the white rod and the green rod one. Jackie, realizing the fallacy in her argument, laughed and said no.

Erik offered to clarify what Jackie and Kelly meant by their models and arguments. He explained that they probably meant that the dark green rod was one, and since one could line up six rods next to the dark green rod, the white rods would be called one sixth (Figure 4.19). He said “I think that’s what they’re trying to say but they just, they’re just not saying it. I think they just, they want to call it one sixth (line 5.0.178). Jackie and Kelly agreed that the difference was one sixth. T/R 1 asked them why the white rod could not be called one and Jackie and Kelly answered that it could not be one since the dark green rod was called one.

*Figure 4.19. Erik’s model showing that the white is 1/6*
Erik then interjected and said that the white rods could be called one if the dark green rod was called six, and the light green rod would be called three. T/R 1 asked him if this could be done within the same problem and Erik admitted that it could not.

T/R 1 asked Jessica to once again explain her first model. Jessica explained incorrectly again that the difference between one half and one third was a red rod which would be called one third. Brian once again countered this by saying that if one third is cut in half, each part would be called one sixth. Placing a red rod on the purple rod in Jessica’s model, he explained that the red had to be called one sixth and not one third since two thirds was larger than one half by one sixth (Figure 4.20).

Figure 4.20. Brian’s model

Erik together with some interjections from Brian demonstrated sophisticated reasoning to show that the difference between one half and one third is one sixth. He said that the answer could not be one third, and going to the overhead he added that the dark green is the one half and the purple rods are thirds. Using an upper and lower bound argument, he asked: “How could one half be bigger than the thirds by one third? Because, and you have the half and the thirds together that the half is almost as big as two thirds, but yet the two thirds aren’t exactly, are not exactly, the green, the dark green is not, the dark green is not exactly as big as two, two thirds but, two thirds, it’s the, but it’s
far enough so that the two thirds are not bigger than it by one third” (line 5.0.209) He also argued that the rod that was one half is equal to a “third and a half” (line 5.0.211 and Figure 4.21).

*Figure 4.21. Erik model to support his upper and lower bound argument*

Michael then added his thoughts and lining up rods as he spoke said that since six red rods were equal in length to the train of orange and red rods it would be called one sixth (Figure 4.22). Brian and Erik both agreed with Michael’s argument.

*Figure 4.22. Model showing Michael’s direct reasoning*

T/R 1 then asked Meredith to share her understanding with the class. Meredith explained that the difference between one third and one half could not be one third. She demonstrated this by lining up two purple rods next to a dark green rod (Figure 4.23) and explained that since the two purple rods which are each called thirds extended past the dark green which is the half the difference could not be a third. An intercom interrupted
her so T/R I asked her to repeat herself. Meredith removed one of the purple rods and showed that the difference between the dark green rod and one purple rod was a red rod and stated that the difference is one sixth. (Figure 4.24).

![Figure 4.23. Meredith’s model](image)

**Figure 4.23.** Meredith’s model

Erik repeated his previous argument and said that since one half is bigger than a third but it’s not as big as two thirds, it therefore could not be a third bigger. He reiterated once again that the difference would be one sixth. He also place a red rod next to the dark green rod to demonstrate that two thirds was larger than one half by a red rod, or one sixth (Figure 4.25).

![Figure 4.25. Erik’s model to show the difference to be one sixth](image)

**Figure 4.25.** Erik’s model to show the difference to be one sixth
T/R 1 encouraged the students to write about one of the arguments that had been given and to state whether or not they agreed with the argument. Jessica said that she now agreed with Brian and Erik saying that the red rod was one sixth and not one third. After her comment, T/R 1 brought the session to a close.

Table 4.9
Classification of teacher questioning and results, Colts Neck Session 5 – Part III

<table>
<thead>
<tr>
<th>Line</th>
<th>Speaker</th>
<th>Researcher question</th>
<th>Question type</th>
<th>Question result</th>
<th>Student conjecture, response, reasoning, or justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0.95</td>
<td>T/R 1:</td>
<td>How many of you worked out which is bigger? One half or one third? [several hands go up] How many of you think they are the same? [all hands go down] How many of you think one is bigger? [several hands go up again] Which is bigger? One half or one third. Laura.</td>
<td>Encouraging engagement - administrative, Eliciting</td>
<td>Laura said one half is bigger</td>
<td></td>
</tr>
<tr>
<td>5.0.97</td>
<td>T/R 1:</td>
<td>You say one half is bigger. What do the rest of you think? Do you think one half is bigger? [several students provide affirmation] Do you think you can convince Dr. Davis that that’s the case? [several hands go up] Can you convince Dr. Davis that one half is bigger than one third? By the way, do you know how much bigger? How many of you think you know how much bigger it is? Ok, that’s the second question.</td>
<td>Encouraging engagement - confirming agreement Eliciting</td>
<td>Jessica and Laura built a model to the overhead and Jessica explained that one could see from their model that one half was bigger than one third.</td>
<td></td>
</tr>
<tr>
<td>5.0.99</td>
<td>T/R 1:</td>
<td>Can you tell me what number name you’re calling the orange and the red rod?</td>
<td>Probing - elucidating</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.0.101</td>
<td>T/R 1</td>
<td>You’re calling the orange and red rod one? Can you say that again, what number names gave to each of those rods so I can hear from back here?</td>
<td>Encouraging engagement - encouraging students to listen</td>
<td>Jessica and Laura explained that they called the dark green rod one half, and the purple rod they called one third.</td>
<td></td>
</tr>
<tr>
<td>5.0.106</td>
<td>T/R 1:</td>
<td>What do the rest of you think? What do you think? Audra what do you think of what… the two young ladies built up there?</td>
<td>Encouraging engagement - confirming agreement, shifting questioning to another student</td>
<td>Audra agreed with Jessica and Laura</td>
<td></td>
</tr>
<tr>
<td>5.0.110</td>
<td>T/R 1:</td>
<td>How many of you agree with the argument that a half is bigger than a third with the argument that was made here? Ok, did you figure out how much bigger?</td>
<td>Encouraging engagement - confirming agreement, Eliciting</td>
<td>Audra and Jessica built a second model with a purple and red train and a</td>
<td></td>
</tr>
</tbody>
</table>
Then they lined up two red rods next to this dark green rod. Then they lined up two red rods next to this model and said that the number name would be one third. Since the girls were using a different model they answered the question incorrectly.

<table>
<thead>
<tr>
<th>Time</th>
<th>Speaker</th>
<th>Text</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0.118 T/R 1</td>
<td>Okay so these young ladies have proved that one half is bigger than one third and it’s—one half is one third bigger than one third. What do you think? That one half is one third bigger than one third. What do you think about what they just proved? Now you were all watching their argument up there and have they convinced you? I don’t know if they have convinced Dr. Davis. Um, but I am wondering if they have convinced you? Kelly. What do you think, do you agree with this?</td>
<td>Encouraging engagement - confirming agreement, Probing - encouraging critical thinking</td>
</tr>
<tr>
<td>5.0.122 T/R 1</td>
<td>Brian you are making a face, what do you think? Do you agree with them?</td>
<td>Kelly agreed with their solution. She built a model but did not present a complete argument to explain her thinking.</td>
</tr>
<tr>
<td>5.0.128 T/R 1</td>
<td>Let me make sure I understand this. You’re calling this one, right? And you’re calling this one third,</td>
<td>Brian disagreed with the girls’ argument</td>
</tr>
<tr>
<td>5.0.132 T/R 1</td>
<td>Right? And you’re calling this one half? And you’re saying one half is bigger than one third by one third? [Girls agreeing with her as she demonstrates what they said]</td>
<td>The girls agreed with T/R 1</td>
</tr>
<tr>
<td>5.0.134 T/R 1</td>
<td>Ok, I would like all of you—How many of you agree? How many of you disagree? Now if you disagree you have to say why you disagree because they are saying that one third is smaller than a half, one half is bigger than a third, it’s one third bigger than a third, that’s what they are saying. Now either you have to agree, or disagree or not know. How many of you aren’t sure? [several hands go up] A few of you aren’t sure, but some of you disagree. And if you disagree we have to say what’s wrong with their argument. There must be something wrong with their argument if you disagree. Or maybe their argument is right because I’m very confused. Brian what do you think?</td>
<td>Encouraging engagement - confirming agreement, Probing - encouraging critical thinking Eliciting</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Promoted reasoning and justification</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Brian reasoned and demonstrated correctly that the red rod was half of the purple rod and therefore would be called one sixth and that the difference between one half and one third is one sixth. He showed that the purple rod had been given the number name one third and that two red rods were equal in length to the purple rod. Each of the red rods would be called one sixth and both together would be called two sixths which would be the same as one third.</td>
</tr>
<tr>
<td>5.0.140</td>
<td>T/R 1:</td>
<td>So you’re giving a red the number name one sixth and I understand the young ladies up at the overhead are giving red the number name one third and can red have the number name one third and one sixth at the same time? That’s my question.</td>
</tr>
<tr>
<td>5.0.147</td>
<td>T/R 1:</td>
<td>Is that allowed?</td>
</tr>
<tr>
<td>5.0.149</td>
<td>T/R 1:</td>
<td>Why not? What’s wrong with that? In what way it is not fair?</td>
</tr>
<tr>
<td>5.0.151</td>
<td>T/R 1:</td>
<td>Ok so what do you ladies think? Are you making different size candy bars? What are calling the candy bar when you started the problem? What was one? What did you call one if you’re thinking of candy bars when you began the problem?</td>
</tr>
<tr>
<td>5.0.153</td>
<td>T/R 1:</td>
<td>Is that what you built when you went up there, you said the dark green is one? Is that what you said?</td>
</tr>
<tr>
<td>5.0.155</td>
<td>T/R 1:</td>
<td>Ok then use the—okay if your calling dark green one then I want to hear your argument which is bigger a half or a third and by how much?</td>
</tr>
<tr>
<td>5.0.157</td>
<td>T/R 1:</td>
<td>Okay you think a half is bigger than one third and you’re calling the dark green one? Did you change your mind?</td>
</tr>
</tbody>
</table>
| 5.0.161 | T/R 1: | Can you show me why that’s a half? | Probing - elucidating | Promoted reasoning and justification | Jackie once again explained her argument while demonstrating with rods that three red rods and two light green rods both equaled the dark green. She explained that the difference between the light green rod which
they gave the number name one half if larger than the red rods which they called one third by one white rod.

<table>
<thead>
<tr>
<th>Time</th>
<th>T/R 1:</th>
<th>Probing - elucidating</th>
<th>Jackie said that the light green is one bigger than the red rod.</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0.163-65</td>
<td>Ok, I see that you switched what you made, um, your model, uh, but you showed me that one half is still bigger than a third and you still believe that. But what number name did you give to white? You said it was a white rod bigger but I didn’t hear what number name you gave to white. I thought I heard you say it’s one bigger… Did you say that?</td>
<td></td>
<td>Jackie answered one.</td>
</tr>
<tr>
<td>5.0.169</td>
<td>And what number name are you calling the white?</td>
<td>Probing - elucidating</td>
<td>Jackie answered one.</td>
</tr>
<tr>
<td>5.0.171-73</td>
<td>You all agree with that?</td>
<td>Encouraging engagement - confirming agreement</td>
<td>Jackie, realizing the fallacy in her argument, laughed and said no</td>
</tr>
<tr>
<td>5.0.175</td>
<td>What I thought I heard you say. [asking the class] You hear my question? Is everybody hearing my question? You said you called the light green one, you said you called the red one third, and you said you called the light green one half. Right? And now the white one, right… [puts the white and red together next to the light green] The white one which tells you how much bigger it is, you said you’re calling it one. So you’re calling this one and this one [pointing to the white and dark green].</td>
<td>Encouraging engagement - checking for understanding</td>
<td>Erik called out that he thought he knew what Jackie meant.</td>
</tr>
<tr>
<td>5.0.177</td>
<td>Erik, what do they mean? I’m so confused.</td>
<td>Eliciting</td>
<td>He explained that they probably meant that the dark green rod was one, and since one could line up six rods next to the dark green rod, the white rods would be called one sixth.</td>
</tr>
<tr>
<td>5.0.183</td>
<td>Is that what you meant to say?</td>
<td>Encouraging engagement - confirming agreement</td>
<td>Jackie and Kellie responded in the affirmative</td>
</tr>
<tr>
<td>5.0.185</td>
<td>So you’re saying then you all agree, that’s what, you all really wanted to call the little white one, one sixth and not one? When you call the light green one? So I’m a little concerned now? Are you agreeing with Brian or disagreeing with Brian that the number name that you would give for how much bigger one half is than one third? Is how much? One half is how much bigger than one third?</td>
<td>Encouraging engagement - confirming agreement, checking for understanding</td>
<td>Jackie and Kellie answered one sixth.</td>
</tr>
<tr>
<td>5.0.187</td>
<td>T/R 1:</td>
<td>Is it one or one sixth?</td>
<td>Probing - elucidating</td>
</tr>
<tr>
<td>5.0.189</td>
<td>T/R 1:</td>
<td>You’re sure it’s one sixth?</td>
<td>Encouraging engagement - confirming agreement</td>
</tr>
<tr>
<td>5.0.191</td>
<td>T/R 1:</td>
<td>Why can’t it be one?</td>
<td>Encouraging engagement - checking for understanding</td>
</tr>
<tr>
<td>5.0.193</td>
<td>T/R 1:</td>
<td>The dark green is one? I understand when-</td>
<td></td>
</tr>
<tr>
<td>5.0.197</td>
<td>T/R 1:</td>
<td>But can you do that in the same problem?</td>
<td>Probing - encouraging critical thinking</td>
</tr>
<tr>
<td>5.0.199</td>
<td>T/R 1:</td>
<td>I still want to go back to the problem Brian was helping them with the problem up there, I still wonder if we can solve this one because you started with this other one and you said that the orange and red [together] are one, right? Isn’t that what you said?</td>
<td>Encouraging engagement - administrative</td>
</tr>
<tr>
<td>5.0.201</td>
<td>T/R 1:</td>
<td>Right, and you said it’s bigger by the red, right? And the question was, what number name do you give to the red? Now if you really understood what mistake you made here maybe you’ll figure out what mistake you made up there.</td>
<td>Eliciting</td>
</tr>
<tr>
<td>5.0.203</td>
<td>T/R 1:</td>
<td>Well how can you build a model and say that one half is bigger than a third by a sixth and build another model that says one half is bigger than a third by a third? How is that possible? I am so confused. Brian, it’s just his face tells me that he is so unhappy with that. Do you believe that Brian? They’re still telling me that one half is bigger by—one half is bigger than one third by one third. Can anyone tell me what’s going on here? I am so confused.</td>
<td>Probing - encouraging critical thinking, Encouraging engagement - confirming agreement, Eliciting</td>
</tr>
</tbody>
</table>
| 5.0.208 | T/R 1: | You think it can be a third and a half? How many think they could be a third and a half? How many of you don’t think it could be a third and a sixth? How many of you disagree? | Encouraging engagement - confirming agreement | Erik with some interjections from Brian demonstrated sophisticated reasoning using an upper and lower bound argument to show that the
difference between one half and one third is one sixth. Meredith too reasoned why the difference had to be one sixth and justified her arguments by demonstrating with the rods at the overhead.

4.2.2.1 Summary of questioning in Colts Neck session Sept 29

In this session, questioning was used in many different ways. When Meredith began to give an incorrect explanation, questioning invited her to get her back on track. Questioning was also used to make sure that other students agreed with a proposed solution. T/R 1 asked her to “tell the class what you’re thinking, let’s see what they think” (line 5.0.19). When asked to repeat her explanation, Meredith realized her mistake and fixed her original model. Questioning in this session was also used as an effective tool to create a student-centered classroom where T/R 1 confirmed agreement from the other students and questions provided the opportunity for argumentation, justification, and reasoning. Rather than the researcher reviewing what had been done in a previous session, T/R 1 encouraged the students to remind everyone what work they had done and to demonstrate it in front of the class. She used questions such as “What do you think?” (line 5.0.43), “Can you help the class understand that?” (line 5.0.51), “If you disagree, why do you disagree” (line 5.0.53), and “Who wants to pretend to be a teacher or explainer, if you agree?” (line 5.0.56) to confirm agreement, elicit student reasoning, check for understanding, and encourage critical thinking. In this part of the session, the students came up with many different number names for half of the candy bar and justified their responses in front of the classroom.
In the next part of the session, where the students discussed the task “which is bigger \( \frac{1}{2} \) or \( \frac{1}{3} \)?,” T/R 1 asked students to present their solution in front of the class. This led to an animated discussion that elicited a variety of reasoning. T/R 1 repeatedly asked students to share their thoughts and whether they agreed with the proposed solution. A number of students presented faulty or incomplete arguments. T/R 1 questioned their reasoning and asked other students to make their thinking explicit. Once again T/R 1 used questions such as “How many of you disagree?... And if you disagree we have to say what’s wrong with their argument” (line 5.0.134) and similar questions to allow for argumentation and provide the opportunity for critical thinking. T/R 1 also used many probing/elucidating questions so that the students presenting their faulty argument could clarify and explain their model. After this sequence of probing questions and the input of a few other students, Jessica realized the fallacy in her argument. Another student presented a complete and correct justification for his model and proposed solution. The discussion eventually resulted in students offering complex reasoning, including upper and lower bound arguments, as well as many instances of direct and indirect reasoning. Figures 4.26 – 4.27 show the question types and associated student reasoning that occurred during this session.
Figure 4.26. Question types used by T/R 1 during Session 5

Figure 4.27. Question results during Session 5
4.2.2.2 Analytics that highlight teacher questioning for session 5 at Colts Neck

Gerstein (2015) portrays the teacher questioning and the resultant student reasoning when the students revisit the task “Is 1/5 = 2/10?” during session 5 in an analytic “Fourth Graders Analyses of Equivalence: 1/5 or 2/10?” In addition to the teacher questioning and student reasoning showcased in this analytic, this analytic also shows how revisiting a task provided an opportunity for students to strengthen their understanding of the topic. Maher and Martino (1996) contend that when students are given sufficient time to work on a problem and are given the opportunity to discuss their solutions with their peers, they often express differences of opinion. These conflicts are not always resolved immediately; sometimes pushed off to a later class session, and possibly deferred over an extended period of time. This allows students to think about the problem and upon revisiting the identical or similar problem at a later date, students are able to build on their ideas and solidify their understanding. As noted by Francisco and Maher (2005), revisiting a concept in a related problem “helps students build rich and durable forms of mathematical understanding of mathematical concepts” (p. 371).

Molloy (2015) in her analytic “Analysis on Student Collaboration and Comparing Fractions,” also focuses on Session 5, where the students grapple with the tasks “Is 2/10 = 1/5 and “Which is greater ½ or 1/3 and by how much.” She depicts the whole class debates where students argue about the solutions presented and provide the opportunity for their fellow students to change their misconceptions. Once again teacher questioning is highlighted in this analytic. Expert teacher questioning invited the students to make their thinking explicit, encouraged confirmation or disagreement, and provided the opportunity for students to clarify and refine their solutions.
The following analytic also portrays the teacher questioning used during the latter part of Session 5: *Comparing 1/2 and 1/3: Confusion about the Unit* (Van Ness & Alston, 2015b). This analytic describes the spirited discussion of the comparison task “Which is greater ½ or 1/3 and by how much?” that had started in earlier sessions. For additional analytics that capture sessions where the students dealt with this task see “Which is Larger, 1/2 or 1/3? An Introduction to Comparing Unit” (Van Ness & Alston, 2015a), which describes Session 2, where the students begin their exploration of this task. See also “Comparing Models and Justifying the Choice of Unit” (Van Ness & Alston, 2015c) that describes events during Sessions 4 and 6 where students revisit and reconsider the candy bar metaphor and discuss the importance of the unit when working with fractions.

The part of the session where the students discuss the task, “Which is bigger ½ or 1/3?”, is depicted in the analytic “The interplay between Researcher Carolyn Maher’s questions and fourth grade student reasoning while exploring fraction comparisons” (Gerstein, 2017d).

4.2.3 Colts Neck October 8, 1993 (session 10) with Researcher Amy Martino

At the beginning of session 10 on fractions, T/R 2 discussed with the students the work they had done in the previous session. Andrew and Jessica told the class that they had tried to construct a large model using trains for thirds and fourths. T/R 2 asked the students if they had written down their models and Andrew answered that he had done so. T.R 2 then asked the students how many models they thought they would be able to construct. Erik suggested that there would be a lot of possible models that could be built. He said,
Well, because see, what me, Alan and I figured, is if you start with one rod, and you can divide one rod that's a large number into thirds and fourths, then you just count down by two, because we think that even numbers you can divide into fourths and thirds, but odd numbers you can't, so it was like, if we started with the orange rod… you could probably divide it into thirds and fourths. And then just go down two and then just keep going down until whatever number you get and then you'll just keep going down and you should be able to.

(line 10.2.23)

Erik tried to explain that if you start with a big rod that could be divided into thirds and quarters and then find a rod that is two centimeters shorter you should be able to divide it as well. Erik’s reasoning was incorrect here as this would only be true for a limited number of rods and would not be true for all the rods that he indicated.

Alan interrupted Erik and used an argument that he had stated in the previous session. He said that if they used four oranges they would not be able to find thirds unless they created a new rod by putting two rods together.

T/R 2 asked the class if they agreed with Erik’s argument. Erik then changed his argument and said that most even rods could be used to create models that showed thirds and quarters. Michael said that he did not think that all even rods could be divided into thirds and fourths. T/R 2 asked Erik what he meant by the term even rod. Erik explained that if one lines up whites next to it and makes sure that all of the white rods are “real tight” and then see if they can be divided into half. David commented that he had shown the class in a previous session that the white rod would be called one, the next rod in length would be the red rod which would be considered even, the light green rods which would be considered three would be odd and so on.

T/R 2 told the class that students who had not completed recording their models from the previous session that showed the difference between two thirds and three fourths
could continue working on the that task, while those students who had already completed the task could start working on a new task.

Table 4.10

*Classification of teacher questioning and results, Colts Neck Session 10 – Part I*

<table>
<thead>
<tr>
<th>Line</th>
<th>Researcher</th>
<th>Researcher question</th>
<th>Question type</th>
<th>Question result</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.2.2</td>
<td>T/R 2:</td>
<td>Ok, so you started to make, you started to use trains [yeah] to make your number one and then you were able to work from there. What else did you find out? That's interesting. What problem were you working on? Somebody tell me what the problem was that you were working on here? Amy.</td>
<td>Encouraging engagement - administrative? Eliciting</td>
<td>Amy said the task was two thirds or three fourths</td>
</tr>
<tr>
<td>10.2.4</td>
<td>T/R 2:</td>
<td>Okay, two thirds or three fourths, and what did Dr. Maher want you to do with this? What did she ask you to do, yesterday? I'm trying to build this in my own mind as to what she asked you. Uh, let's see, Jessica.</td>
<td>Probing - elucidating</td>
<td>Jessica said they were trying to build bigger models</td>
</tr>
<tr>
<td>10.2.6</td>
<td>T/R 2:</td>
<td>Ok, so they were making like bigger models? Erik, did you want to add to that?</td>
<td>Probing - elucidating, Encouraging engagement - shifting questioning to another student</td>
<td>Erik responded that they were trying to make two models</td>
</tr>
<tr>
<td>10.2.10</td>
<td>T/R 2:</td>
<td>And that's basically what she said? Did she ask you to think any more about this or write about it or- Meredith?</td>
<td>Factual - recall</td>
<td>Erik said they were told to write about it and Meredith adds that they also drew their models on paper</td>
</tr>
<tr>
<td>10.2.15</td>
<td>T/R 2:</td>
<td>Ok. What I was wondering is, I heard that, I heard that yesterday there were some, some people came up with some really big models. Were you able to draw those?</td>
<td>Encouraging engagement - administrative</td>
<td>Erik said they couldn’t draw them because they were too big but Andrew said he did draw his model.</td>
</tr>
<tr>
<td>10.2.18</td>
<td>T/R 2:</td>
<td>Ahh, neat. Ok, so Andrew drew one. What he did was he taped some paper together. That's very clever. So maybe we can actually record some of those bigger ones today for people who did that, uh, what I'm wondering is, now that you've had a chance to build some models, and you came up with some different ones, from what I'm hearing, how many models do you think it's possible to build for</td>
<td>Eliciting</td>
<td>Erik said it would be possible to build a lot of models.</td>
</tr>
<tr>
<td>Time</td>
<td>T/R 2:</td>
<td>Prompts and Techniques</td>
<td>Notes</td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>--------------------------------------------</td>
<td>----------------------------------------------------------------------------------------</td>
<td>------------------------------------------------------------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>10.2.22</td>
<td>Comparing those two fractions?</td>
<td>A lot, you want to say more about why that's so?</td>
<td>Erik tried to explain that if you start with a big rod that could be divided into thirds and quarters and then find a rod that is two centimeters shorter you should be able to divide it as well. Erik’s reasoning was incorrect here as this would only be true for a limited number of rods and would not be true for all the rods that he indicated. Alan added that if they used four oranges they would not be able to find thirds unless they created a new rod by putting two rods together.</td>
<td></td>
</tr>
<tr>
<td>10.2.29</td>
<td>Ok, what do you think about this theory that, uh, that Erik and Alan have about the even numbers? They said that they think they can divide even numbers into thirds and fourths, and uh, Erik said that</td>
<td>Eliciting</td>
<td>Michael said that not all even numbers worked.</td>
<td></td>
</tr>
<tr>
<td>10.2.33</td>
<td>Ok, that's interesting. Does anyone else have any other ideas about, about, the number of models you could build? You know some people here saying a whole lot. What do you think? Do you agree with that? Do you disagree with that? If you agree with it, why do you agree with it? Michael, do you want to say something?</td>
<td>Eliciting, Encouraging engagement - confirming agreement, shifting questioning to another student</td>
<td>Erik modified his conjecture that they can divide most even numbers into thirds and fourths.</td>
<td></td>
</tr>
<tr>
<td>10.2.41</td>
<td>What do you mean by an even number rod?</td>
<td>Probing - elucidating</td>
<td>Erik explained that if one lines up whites next to it and makes sure that all of the white rods are “real tight” and then see if they can be divided into half.</td>
<td></td>
</tr>
<tr>
<td>10.2.45</td>
<td>Ok, ok, David, did you want to add, you want to add to that? Or you want to comment on that?</td>
<td>Eliciting, Encouraging engagement - shifting questioning to another student</td>
<td>David commented that he had shown the class in a previous session that the white rod would be called one, the next rod in length would be the red rod which would be considered even, the light green rods which would be considered three would be odd and so on.</td>
<td></td>
</tr>
<tr>
<td>10.2.49</td>
<td>Because of the number of whites you could put alongside of it to show? Is that why you're saying that, in other words? Why you're giving it a name two because you can use two</td>
<td>Probing - elucidating</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Alan worked alone at his desk on the new task that had been provided by T/R 2. The task was stated as follows: Compare one half or two fifths. Which is larger, and by how much?

T/R 2 came over to Alan’s desk to look at his work. Alan showed her the two models that he had built. The first model consisted of an orange rod, five red rods, two yellow rods, and ten white rods. He explained to T/R 2 that the one half is larger than one fifth. He demonstrated that the difference between them was one white rod which could be called one tenth (Figure 4.28) He showed T/R 2 his second model where he had used a red and brown train instead of the orange rod in his first model.

![Figure 4.28. Alan’s first two models of the same length](image)

T/R 2 asked Alan if he could build a model with a different length to demonstrate that the difference between one half and two fifths is one tenth. Alan built a model using two orange rods, five purple rods, two additional orange rods, and ten red rods (Figure 4.29). He explained once again to T/R 2 that the difference between one half (the number name given here to the orange rods) and two fifths (the number name given to the purple rods) was again one tenth which in this model was represented by one red rod.
T/R 2 asked Alan how he had built his model so quickly and specifically why he had chosen the two orange rods to be given the number name one. Alan explained that he had built his second model based on his first model. His first model consisted of one orange rod, five red rods, two yellow rods, and ten white rods. He had simply doubled the amount of orange rods and was thereby able to come up with same solution. He further extended his solution by predicting how he could easily build a third model based on the previous ones. He explained as follows:

Because up here, I knew that this was ten, and two tens would be twenty, and I knew that that would work, so it takes two of those to complete it using a double ten. So one of those [points to red rods] filled in the gap. Probably if you used another one [takes a third orange and gestures to show that a fourth orange rod would be placed along with the first three] another two, you could fill in that with more purples and using more reds, too.

(lines 10.2.72-10.2.73)

T/R 2 asked him to try to build a third model to see whether or not that prediction would be true. A few minutes later she returned to his desk and asked him about his next model. Alan showed her another model that he had constructed and explained that he had doubled the length of his previous model (Figure 4.30). Purple rods would now be given the number name tenths, brown rods would be called fifths and now the red rods would be given the number name twentieths. He explained to T/R 2 that he would now need forty white rods. These white rods would be given the number name fortieths. Alan continued his argument. He put five blue rods on his model and said,

You can't make the model any bigger than this, you would have to use one blue. It wouldn't be the exact size. So you can't make a model any bigger than this,
without making a train, making all these uneven. So basically, this is the only model you can make that's even without using trains, like this one here, that would make all of these unequal.

(line 10.2.154)

Alan said that he would need another four orange rods to be combined with the other orange rods that had been given the number name one. He implied that he could not build a large model that would be based on his current models since five blue rods would not be equal in length to the train of eight orange rods.

Figure 4.30. Alan’s largest model for comparing two fifths to one half

Table 4.11

Classification of teacher questioning and results, Colts Neck Session 10 – Part II

<table>
<thead>
<tr>
<th>Line</th>
<th>Researcher</th>
<th>Researcher question</th>
<th>Question type</th>
<th>Question result</th>
<th>Student conjecture, response, reasoning, or justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.2.62</td>
<td>T/R 2:</td>
<td>Ok, do you think now that you can, I mean you're working on this one I can see, do you think you can come up with one that's a different total length than, one's that's different from these two, these two have the same length?</td>
<td>Eliciting Probing- encouraging critical thinking</td>
<td>Alan said that he has begun to work on such a model</td>
<td></td>
</tr>
<tr>
<td>10.2.66</td>
<td>T/R 2:</td>
<td>Does that one work?</td>
<td>Probing - elucidating</td>
<td>Alan responded in the affirmative</td>
<td></td>
</tr>
<tr>
<td>10.2.68</td>
<td>T/R 2:</td>
<td>A working model here? Tell me about that one.</td>
<td></td>
<td>Alan explained his model</td>
<td></td>
</tr>
<tr>
<td>10.2.72</td>
<td>T/R 2:</td>
<td>Can I ask you a question now? Why did you choose the two oranges to be one? You seemed to come up with that pretty quickly.</td>
<td>Probing - elucidating</td>
<td>Alan using recursive reasoning explained that he had built his second model based on his first model. His first model consisted of one orange rod, five red rods, two yellow rods, and ten white rods. He</td>
<td></td>
</tr>
</tbody>
</table>
had simply doubled the amount of orange rods and was thereby able to come up with same solution. He further extended his solution by predicting how he could easily build a third model based on the previous one.

<table>
<thead>
<tr>
<th>Time</th>
<th>T/R 2:</th>
<th>Eliciting</th>
<th>Alan said he will try to build a bigger model and builds a model double the length of this previous model.</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.2.76</td>
<td>Ok, so you did ten, you called it ten and twenty because of the little white ones. That's an interesting theory, could you kind of test that one out for me and, see, see if you could build a bigger model?</td>
<td>Eliciting</td>
<td>Alan said he will try to build a bigger model and builds a model double the length of this previous model.</td>
</tr>
<tr>
<td>10.2.157</td>
<td>Oh my goodness, can you imagine the size of that? If you want- if you wanted to make a train, though, where you were adding a different color rod on the end of this train of four oranges, do you think you could come up with other models?</td>
<td>Probing - encouraging critical thinking</td>
<td>Alan said that he would need another four orange rods to be combined with the other orange rods that had been given the number name one. He implied that he could not build a large model that would be based on his current models since five blue rods would not be equal the length to the train of eight orange rods.</td>
</tr>
<tr>
<td>10.2.159</td>
<td>Ok, could you, I almost hate to ask this but could you, we have a couple of minutes left, could you try to trace this so we don’t lose what you did here? Uh, or maybe you can draw a sketch of it, okay?</td>
<td>Encouraging engagement - administrative</td>
<td></td>
</tr>
</tbody>
</table>
about the number of models you could build?” and “What do you think? Do you agree with that? Do you disagree with that? If you agree with it, why do you agree with it?”

In the next part of the session, where students worked on building models at their desks, T/R 2 questioning provided the opportunity for sophisticated reasoning. She asked Alan to construct different models to show the difference between one half and two fifths. When she noticed that Alan had constructed additional models in quick succession, she questioned him on his method of creating new models. T/R 2’s questioning invited Alan to reason recursively and he proceeded to extend his solution and predict how he could create additional models to represent the difference between one half and two fifths. Figures 4.31 and 4.32 show the question types and associated student reasoning that occurred during this session.

![Figure 4.31. Question types used by T/R 2 during Session 10](image-url)
Figure 4.32. Question results during Session 10

4.2.3.2 Analytics for Session 10 at Colts Neck

The analytic “Extending the Doubling Conjecture” (Uptegrove, 2016) highlights the researcher questioning and associated student reasoning from the last part of the session where Alan built and explained his models to T/R 2.

Salb (2015) also showcases the teacher questioning and Alan’s recursive reasoning shown during that part of the session where he works on a task comparing two fifths and one half. Alan not only builds multiple models, but also predicts the form of larger models as he is questioned by T/R 2.

Winter (2015d) uses the events of this session as well in an analytic “The development of upper and lower bound arguments while comparing fractions” to show how Alan responded to T/R 2’s questioning by using an upper and lower bound argument. He had created a train rod model of four orange rods that represented one to
compare the fractions one half and two fifths. He showed that two orange rods would be called one half and that the brown rods were called fifths. He then used an upper bound argument to explain that the model could not be bigger than four orange rods if you want to call a single rod one fifth and a train of rods that are the same one half.

“The Interplay between Teacher Questioning and Student Reasoning: Utilizing Probing Questions to Elicit Argumentation, Justification, and Reasoning” (Gerstein, 2017a) focuses in part on the questions posed by Researcher Martino as she discussed with Alan his various models that he built during this session and the associated student justification, and reasoning.

4.2.4 Researcher Amy Martino and Brandon working with Pizza and Towers at Colts Neck, April 5, 1993

Researcher Amy Martino conducted a skillful task-based interview with Brandon, a student at the Colts Neck school on April 5th 1993. Brandon, a 10-year-old 4th grade boy, explained how he had solved two problems in earlier class sessions. In the first class session, Brandon had worked with a partner on the following task: Your group has two colors of Unifix Cubes. Work together and make as many different towers four cubes high as is possible when selecting from two colors. The students were asked to solve the problem and convince others that their solution was complete and that they had not recorded duplicates. In the second session, students, again working with partners, were given the following problem: A local pizza shop has asked us to help design a form to keep track of certain pizza choices. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms and pepperoni. How many different choices for pizza does a customer have? Find a way to
convince each other that you have accounted for all possibilities. In both session the 
students had been encouraged to convince themselves as well as their peers that their 
solution was indeed complete.

During a session on March 11, 1993, Brandon created a code for representing the 
presence or absence of specific toppings for the pizza problem by using a 1 to indicate the 
presence of a particular topping or a 0 to mark the absence of a topping. Using this 
system, Brandon created a chart and organized his data to account for all of the possible 
pizza combinations by delineating all cases of zero toppings, one topping, two toppings, 
three toppings and four toppings.

T/R 2 began the interview on April 5th, 1993 by showing Brandon the work that 
he had done during that earlier session. She explained to Brandon that researchers from 
Rutgers were very much interested in the work that he had performed and asked him to 
help her understand it so that she could explain it to them. T/R 2 pointed to the zeroes 
and ones that filled Brandon's page and asked him how those numbers had to do with 
pizzas. Brandon offered to redo his work so that she could understand. As Brandon began 
to work, T/R 2 asked him if he remembered what the problem was. Brandon answered in 
the affirmative and explained that they were trying to see how many different ways could 
be devised to create a pizza with four toppings. He elaborated that the four possibilities 
were mushroom, sausage, pepper, and pepperoni. He represented the four toppings by 
writing P for pepper, M for mushroom, S for sausage, and peponi for pepperoni. He 
began to reconstruct his chart but realized right away that he was doing it differently than 
the way he had ultimately solved the problem, so he took a new sheet of paper and started 
over. He explained to T/R 2 that "since they're in order" (line 14) instead of "skipping"
around he would try to do it in order. Brandon said that at first he started to go "one, two, three, four" but that he would now do it in an easier fashion. T/R 2 encouraged him to explain how he was now going to change his method of finding all possible toppings. Brandon explained that first one could have nothing on the pizza. He indicated this by writing zeroes under the four column headings (pepper, mushroom, sausage and pepperoni). As he continued to draw a chart for T/R 2, he delineated the other options. He indicated the presence of an item by using a "1" and the absence of a topping by writing a "0." He explained that the second option would be pepper with nothing else. The third pizza could have mushroom with nothing else. The next one could have sausage with nothing else and finally pepperoni with nothing else. Then he continued by listing two toppings per pizza. He started with pepper and mushroom and nothing else, then pepper and sausage and nothing else and pepper and pepperoni and nothing else. Then Brandon stated that since we’re all done with pepperoni (Brandon meant pepper) you could have mushroom and sausage with nothing else. T/R 2 interjected and asked him to make his thinking explicit and explain how he knew he could go on to mushrooms at this point. Brandon asked "Why didn't I use pepper anymore?" (line 23). Brandon explained that because he had already used pepper if he wrote mushroom and pepper it would be a duplicate. Brandon then went on to explain "so each time you go 3, 2, 1" (line 26). T/R 2 asked him to clarify what he meant by 3, 2, 1. He explained that with the first topping there are three other toppings that could be used together with it. Then when moving on to the next topping there are only two. T/R 2 then asked Brandon to clarify again why when he moved to mushrooms with sausage and pepperoni but not with pepper. Brandon once again explained his thinking by saying "I already got pepperoni-mushroom,
pepperoni-sausage. That’d be the same thing. It’s just like saying you have an airplane and a car, saying you got a car and an airplane. It’s still the same thing" (line 30). T/R 2 said that she now understood. She then asked Brandon to explain what zeroes and ones represented. Brandon explained that zero meant that there was nothing. He added that he was not sure where he got the idea to represent it with ones and zeroes. He said that "It just popped into my head" (line 34). Continuing his chart he said that there could be mushroom with pepperoni, and that all that was left was sausage and pepperoni. He added that he couldn’t do pepperoni alone because "that'd be the same as up there" (line 34). R/2 asked him to repeat that last part of his explanation and Brandon explained once more that he had already put in three zeroes in the first three columns and a one in the last column and he indicated the line where he had written that. T/R 2 said that she now understood but asked "How will we know when we're done?" (line 39). Brandon explained that "We’ll run out of ways" (line 40) and said that he would explain how he would know when they finish all of the possibilities. He moved on to "threes" (line 40) and said that the pizza could have pepper, mushroom, and sausage and nothing else. Then continuing on to line 13 of his chart, Brandon said that there could be a pepper, mushroom, no sausage and pepperoni. Then he admitted as he wrote line 14, "This is where it gets really tricky." (line 44). On line 14 he indicated that there would be pepper, no mushroom, sausage, and pepperoni. On line 15, he started to put a pepper, mushroom and no sausage but corrected himself and instead put no pepper, mushroom, sausage and pepperoni.

Then putting one's in all columns, he said they could then have a pizza with all of the toppings. At that point Brandon exclaimed that he believed that he had doubles since
he should only have 15 ways. T/R 2 suggested that they go back to look at work that he had done previously. As Brandon checked his work, T/R 2 asked how he would know that all of the line items were different from each other. T/R 2 pointed to one line of his old chart and asked him how he would know that there was no duplicate of that line. Brandon said he knew because he drew lines, and indicating that all of the lines from 16 back to 12 were correct, he said that if he goes back up to line 12 he could stop there because that would be "twos" (line 60) and he wouldn't have put a three into the twos group. T/R 2 asked "these are in groups… Then you're thinking about these in groups?" (line 61). Brandon answered in the affirmative and explained that there was a one group where all of them have one topping. In the twos group each has two toppings and in the threes group they all have three toppings. The final group has all toppings. Brandon said that he liked his old chart better and that the one that he just made was confusing. T/R 2 asked him to show her the different groups on the new chart. Using a different colored marker, Brandon drew lines to show the demarcation between groups. He showed her the ones group. T/R 2 asked him about line 1 (which was written as 0, 0, 0, 0). Brandon explained that line 1 would be a separate group - the zero group. T/R 2 interrupted him as he continued to show her the groups and asked "Do we have all the ones we could possibly have in the ones group?" (line 81). Brandon answered that he had all of them and T/R 2 encouraged him to explain. Brandon reviewed his work and emphasized that if he would add any other choice, there would be a duplicate. Brandon said that the twos groups "is like the most" (line 87). T/R 2 asked him what he meant by that and Brandon explained that there were more choices in that group. Brandon was going to continue on the threes group, but T/R 2 asked him again "Can you convince me that there is, there
aren't any more in the twos group. That there aren't seven or eight" (line 91). Brandon explained that by looking to see if there are ones in the same column you could figure out if there is a duplicate. T/R 2 asked him to clarify how he knew there were no more with mushroom. Brandon explained that "each time you get less. If you start off with pepperoni you got three choices because there’s mushroom, sausage, pepperoni. Get to mushroom, you only got two choices because pepperoni you already used with mushroom" (line 94). T/R 2 indicated she understood and Brandon continued with the threes group. Brandon said that in that group the combinations would be pepper, mushroom, sausage; pepper mushroom pepperoni; pepper sausage pepperoni. T/R 2 asked him if those were all the choices and Brandon continued that they would then start with mushrooms and could have mushroom, sausage and pepperoni. He said that those were all the choices. Brandon said "I know you're going to ask why” (line 100) so he explained that no matter what combination he made there would be an identical one in his chart. He ended by saying that "So then your only choice left is having an all pizza with everything" (line 100). T/R 2 asked him what he would call the final group and Brandon offered "The total" (line 102). T/R 2 reminded him that he had called one group the zeroes, and the next one the ones. Brandon continued and said "You had two toppings three toppings, four toppings" (line 104).

Table 4.12

Classification of teacher questioning and results, Colts Neck Interview with Brandon – Part I

<table>
<thead>
<tr>
<th>Line</th>
<th>Researcher question</th>
<th>Question type</th>
<th>Question result</th>
<th>Student conjecture, response, reasoning, or justification</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Line</th>
<th>T/R 2</th>
<th>Elicitation/Probing</th>
<th>Analysis/Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>T/R 2</td>
<td>Okay? Do you want to</td>
<td>Brandon offered</td>
</tr>
<tr>
<td></td>
<td></td>
<td>tell me what you</td>
<td>to redo his work</td>
</tr>
<tr>
<td></td>
<td></td>
<td>were doing here and</td>
<td>so that T/R 2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>how this, how these</td>
<td>could understand.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>turn out to be</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>pizzas – these zeros</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>and ones? Let me</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>line that up for</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>you. [Brandon:</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Okay.] Okay.</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>T/R 2</td>
<td>How are you going to</td>
<td>Brandon</td>
</tr>
<tr>
<td></td>
<td></td>
<td>change this now?</td>
<td>constructed a</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>chart beginning</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>with a blank</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>pizza. He</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>indicated the</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>presence of an</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>item by using a</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>“1” and the</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>absence of a</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>topping by</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>writing a “0.”</td>
</tr>
<tr>
<td>21</td>
<td>T/R 2</td>
<td>How did you know,</td>
<td>Brandon</td>
</tr>
<tr>
<td></td>
<td></td>
<td>how did you know to</td>
<td>explained how he</td>
</tr>
<tr>
<td></td>
<td></td>
<td>go to mushroom</td>
<td>had delineated</td>
</tr>
<tr>
<td></td>
<td></td>
<td>now? I’m, I’m</td>
<td>all the pizzas</td>
</tr>
<tr>
<td></td>
<td></td>
<td>interested in that.</td>
<td>with two</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Okay.</td>
<td>toppings that</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>contained</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>peppers. He</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>explained that</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>each time</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>“you go 3, 2, 1”</td>
</tr>
<tr>
<td>27</td>
<td>T/R 2</td>
<td>What do you mean 3,</td>
<td>Brandon</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2, 1?</td>
<td>explained how</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>in each line</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>there were</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>successively</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>less choices.</td>
</tr>
<tr>
<td>29</td>
<td>T/R 2</td>
<td>Okay, but I don’t,</td>
<td>He explained</td>
</tr>
<tr>
<td></td>
<td></td>
<td>what I don’t</td>
<td>that it would be</td>
</tr>
<tr>
<td></td>
<td></td>
<td>understand is why</td>
<td>a duplicate.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>when you move to</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>mushrooms, why you</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>can put it with</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>sausage and you can</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>put it with</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>pepperoni, but you</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>can’t put it with</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>peppers?</td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>T/R 2</td>
<td>I see. Okay. I</td>
<td>Brandon explained</td>
</tr>
<tr>
<td></td>
<td></td>
<td>understand why you</td>
<td>that a zero</td>
</tr>
<tr>
<td></td>
<td></td>
<td>did that now. What</td>
<td>means nothing</td>
</tr>
<tr>
<td></td>
<td></td>
<td>do these zeros and</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>ones mean? Like</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>what does the zero</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>represent here?</td>
<td></td>
</tr>
<tr>
<td>39</td>
<td>T/R 2</td>
<td>You missed a couple</td>
<td>Brandon explained</td>
</tr>
<tr>
<td></td>
<td></td>
<td>here. How will we</td>
<td>that they</td>
</tr>
<tr>
<td></td>
<td></td>
<td>know when we're</td>
<td>would “run out</td>
</tr>
<tr>
<td></td>
<td></td>
<td>done?</td>
<td>of ways” (line</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>10) and</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>continued</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>constructing</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>his chart and</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>explaining it.</td>
</tr>
<tr>
<td>61</td>
<td>T/R 2</td>
<td>Okay, yeah, these</td>
<td>Brandon answered</td>
</tr>
<tr>
<td></td>
<td></td>
<td>are in groups? Then</td>
<td>in the</td>
</tr>
<tr>
<td></td>
<td></td>
<td>you’re thinking</td>
<td>affirmative and</td>
</tr>
<tr>
<td></td>
<td></td>
<td>about these in</td>
<td>explained that</td>
</tr>
<tr>
<td></td>
<td></td>
<td>groups?</td>
<td>there was a one</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>group where all</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>of them have one</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>topping. In the</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>twos group each</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>had two</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>toppings and in</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>the threes group</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>they all had</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>three toppings.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>The final group</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>had all toppings.</td>
</tr>
<tr>
<td>73</td>
<td>T/R 2</td>
<td>Can you show me what,</td>
<td>Using a different</td>
</tr>
<tr>
<td></td>
<td></td>
<td>can you show, you</td>
<td>colored marker</td>
</tr>
<tr>
<td></td>
<td></td>
<td>have them in groups</td>
<td>Brandon drew</td>
</tr>
<tr>
<td></td>
<td></td>
<td>here, can you show</td>
<td>lines to show</td>
</tr>
<tr>
<td></td>
<td></td>
<td>me what those</td>
<td>the demarcation</td>
</tr>
<tr>
<td></td>
<td></td>
<td>groups are on here?</td>
<td>between groups.</td>
</tr>
<tr>
<td>75</td>
<td>T/R 2</td>
<td>Okay. And what group</td>
<td>Brandon explained</td>
</tr>
<tr>
<td></td>
<td></td>
<td>is that?</td>
<td>it would be</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>the ones group.</td>
</tr>
<tr>
<td>77</td>
<td>T/R 2</td>
<td>Okay. And what does</td>
<td>He explained it</td>
</tr>
<tr>
<td></td>
<td></td>
<td>that mean? The ones</td>
<td>would have only</td>
</tr>
<tr>
<td></td>
<td></td>
<td>group?</td>
<td>one topping.</td>
</tr>
<tr>
<td>79</td>
<td>T/R 2</td>
<td>Okay. What about this</td>
<td>He explained</td>
</tr>
<tr>
<td></td>
<td></td>
<td>one right up here?</td>
<td>the first line</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>was the zero</td>
</tr>
<tr>
<td>81</td>
<td>T/R 2</td>
<td>Do we have, can I, I</td>
<td>Brandon reviewed</td>
</tr>
<tr>
<td></td>
<td></td>
<td>just want to stop</td>
<td>his work and</td>
</tr>
<tr>
<td></td>
<td></td>
<td>you for a second. Do</td>
<td>emphasized that</td>
</tr>
<tr>
<td></td>
<td></td>
<td>we have all the</td>
<td>if he would add</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ones that we could</td>
<td>any other choice,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>possibly have in the</td>
<td>there would be</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ones group?</td>
<td>a duplicate.</td>
</tr>
<tr>
<td>89</td>
<td>T/R 2</td>
<td>What do you mean the</td>
<td>Brandon explained</td>
</tr>
<tr>
<td></td>
<td></td>
<td>most?</td>
<td>why the twos</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>group would have</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>the most</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>choices.</td>
</tr>
</tbody>
</table>
T/R 2 then asked Brandon "Does this problem with the pizzas remind you of any other problem we've done this year?" (line 109). This critical question served as a springboard for Brandon to make the connection between the pizza and the towers problem. Brandon asked T/R 2 if she meant that they were similar in the way in which he did the problem. Brandon said that it reminded him of the blocks (towers) because he did it in order and made groups similar to what he did with the pizza. T/R 2 prompted him by saying "now we built towers that were…" (line 113) and Brandon remembered that they had created opposites. T/R 2 asked him if he remembered how many towers he had made. Brandon said he thought he could remember what he had done and started to write out tower combinations on paper using shaded and non-shaded squares. He came up with six towers. T/R 2 asked him if he thought that was it and Brandon said yes. T/R 2 offered him some Unifix cubes to work with. Brandon created towers by putting together two reds and two yellows. He made the following combinations: red-red-yellow-yellow,
Brandon called that one group. Then he made red-yellow-red-yellow and yellow-red-yellow-red and called that "another" group (line 122). Then he combined four yellows and four reds and called it "Another group. Another pair" (line 122). Brandon counted it up "Okay, 2, 4, 6," and then realized that he could make more than six combinations. He pieced together red-yellow-yellow-red and yellow-red-red-yellow. Counting them again he said "Eight. I think that’s all" (line 122). T/R 2 asked him "Is there a way you could be sure you have them all?" (line 123). Brandon responded that he was unsure. T/R 2 said that when they worked with pizzas he had been quite sure that he had found all of the possible topping combinations. Brandon said that he had been sure since he had created a graph. T/R 2 asked him if there was a way that they could make a graph about towers. Brandon said that he could make a graph about anything. Brandon started drawing representations of the cubes by drawing a shaded square to indicate red and a non-shaded square to indicate yellow. T/R 2 said she was confused how she would keep track of the different combinations and asked again if it was possible to make a graph. Brandon said it would be "too confusing" and then remembered that when he had worked on it together with his partner they tried to make as many combinations as possible until they realized that they had run out of possibilities. T/R 2 said she would like to bring it to a higher level. Brandon suddenly realized that he had missed some combination. T/R 2 offered him some additional blocks and he created the following combinations: yellow-red-yellow-yellow and red-yellow-red-red. T/R 2 commented that he created that combination very quickly and Brandon explained that he was making pairs of opposites by replacing red with yellow and vice versa. Brandon then put together a yellow-red-red-red and then a red-yellow-yellow-yellow. Brandon then
rearranged his towers and said you could start from the bottom and work your way up. T/R 2 asked him what he meant by "working your way up" and Brandon elaborated by saying that if he started with one red and three yellows the next combination could be a yellow-red-yellow-yellow and then a yellow-yellow-yellow-red. T/R 2 acknowledged that Brandon was moving the red cube each time and said she thought that was a good idea. Brandon continued forming combinations as he created towers consisting of yellow-red-yellow-yellow and yellow-yellow-red-yellow. He said "It's kind of like stairs" (line 154). Brandon then said "it’s kind of like the pizza problem. You start off with maybe group like this one would be the ones group" (line 158).

Brandon mentioned that he had not done it this way when working with the towers problem but he had just noticed that that he could do it this way. Brandon then pulled out all of the towers that he had constructed that had one cube of one color and the other three cubes of the alternate color. This group consisted of a tower made up of yellow-yellow-yellow-red/red-red-red-yellow, yellow-yellow-red-yellow/red-red-yellow-red, yellow-red-red-red/yellow-yellow-yellow-yellow, yellow-red-yellow-yellow/red-yellow-red-red lined up. He then chose towers that would be considered twos. Brandon lined the following towers in front of himself: yellow-yellow-red-red-red-yellow and red-yellow-yellow-red/yellow-red-red-yellow. He then added to this group red-yellow-red-yellow/yellow-red-red-yellow. T/R 2 asked him how many he had in the twos group. Brandon counted six towers and T/R 2 commented that that was interesting. Brandon mentioned that there were more ways to rearrange the ones in the ones group than in the twos group. T/R 2 asked him whether the towers in the twos group had two yellows or two reds. Brandon explained that there were two of each color and that they would have
to be like that to be in the twos group. Brandon stated that there would be no threes group. T/R 2 questioned him on that and Brandon elaborated that the threes group would be the same as the ones group. T/R 2 asked him if it would be possible to call the ones group a threes group. Brandon said that either name would be ok. T/R 2 asked him to focus on the colors and look once more at the groups. She asked him to pick a color and Brandon chose yellow. T/R 2 asked him to focus on yellow and show her which towers would be included in the ones group. Brandon held up a tower consisting of red-yellow-red-red. He named it a ones yellow tower or a threes red tower. T/R 2 asked him if there are other towers consisting of one yellow and the remaining cubes red. Brandon selected red-red-red-yellow. Brandon then pulled out a red-red-yellow-red and stated that "that's all the one yellows" (line 214). He quickly corrected himself and said "No, there's one more yellow. There must be one more." As he said that, he found another tower consisting of yellow-red-red-red. T/R 2 asked him how he had known that there must be one more. Brandon said he realized he needed to even the groups out. He then added, pointing to the ones yellows towers which he put in order by creating a staircase with the yellow cubes, "And also, it’s like the pizza problem. You work your way down. Like pepperoni, mushroom, sausage, and pepper” (line 218). T/R 2 asked him what a zero would look like if they were focusing on yellow. In response, Brandon knocked down all the towers and moved them over. "It would be nothing" (line 230). T/R 2 asked “A zero, a zero tower, if we’re looking at yellow, would be nothing?” When Brandon confirmed his faulty answer, T/R 2 asked him what would be considered a ones. Brandon picked up the towers and pulled out towers that would be considered ones: yellow-red-red-red, red-yellow-red-red, red-red-yellow-red, and red-red-red-yellow. He said the rest of the rods
would be threes and fours. T/R 2 said "Tell me again how this is like the pizzas. Brandon responded: "Well, you have one pepperoni. That’d be like, one pepperoni is like, since we’re looking at yellow, the yellow would be one and the reds would be zeroes. You could have one pepper, like it shows here, and right there, I got then it’s like stairs, you were, if I draw a line down…” (line 238). Brandon leaned over and took a marker and he continued "Then you’d go across, draw a line down here. Go across, draw a line down here. Go across, draw a line down there. Go across so you would have like one, one, one, one. Sort of like here you have one pepperoni, one mushroom, one sausage, and one pepper" (line 242). As he spoke he drew a line horizontally under line 2 of his chart and then down to line three, four and five in a staircase fashion to highlight the ones he had put under the column pepper, mushroom, sausage, and pepperoni on those lines respectively. T/R 2 asked "Is what you’re saying to me then like a yellow cube here is like a number one when in your chart?" (line 243). Brandon replied that if they were focusing on red then red would be a number one. T/R 2 tells him that they should continue with yellow and asks him what would come next. She took the towers consisting of yellow-red-red-red, red-yellow-red-red/, red-red-yellow-red/red-red-red-yellow saying that those were ones. Brandon compared a twos group to a pizza that would have two toppings such as pepperoni and mushroom. T/R 2 helped him find towers that would go into the twos group. He laid down the towers that he could compare with his chart of pizza toppings. He put a yellow-yellow-red-red on top of line 6. Then he put a yellow-red-yellow-red tower on top of line 7 and explained to T/R 2 how it would correspond to the one-zero-one-zero on line 7. Next he showed T/R 2 a yellow-red-red-yellow which would correspond to line 8 (one-zero-zero-one). He put down a red-yellow-yellow-red to
correspond to the line 9. Continuing on to his second page he lined up the remaining towers consisting of red-yellow-red-yellow and red-red-yellow-yellow to correspond to lines 10 and 11 respectively. T/R 2 asked him they were done with the twos group. Brandon said yes- "Look at the chart” (line 262). He then started on threes. T/R 2 asked him whether they were still focusing on yellows. Brandon picked up the towers consisting of yellow-yellow-yellow-red, yellow yellow-red-yellow, yellow-red-yellow-yellow, and a red-yellow-yellow-yellow, and said that those would be threes. He showed T/R 2 how they corresponded to lines 12, 13, 14 and 15 respectively. Brandon picked up the tower consisting of 4 yellows and said "Now, if we’re just focusing on yellow, this then would be the pizza with everything” (line 266). T/R 2 asked him whether they were missing any. Brandon answered no and T/R 2 showed him a tower of red and said "We have this guy left" (line 269). Asking him what name they could give to that tower, Brandon answered that "this would be the zero guy…. Finally found out what the red would be. Red zero guy” (line 272).

T/R 2 said she didn’t need Brandon to do it but that theoretically they could have focused on red rather than on yellow and worked it out the same way. Brandon said that they could do that and demonstrated by transposing the ones towers with the threes tower. He pointed to the towers with one red and three yellows and said that those would be the ones. He said the twos group would remain the same and after being asked by T/R 2 about the group that consisted of three reds and one yellow tower, he said that it would be the threes group. T/R 2 asked him if they would now have to change the number names of the red and yellow and Brandon said the reds would now be called one and the
yellows zero. T/R 2 asked him if he was convinced that he had found all of the towers and pizzas. Brandon said yes and he counted the towers for a total of 16.

Brandon offered that they find the various combinations for the pizza toppings by working with opposites as well. T/R 2 asked how that would work but said that she didn't expect him to put together another whole chart again to show the possible pizza choices sorted by opposites. He wrote on the chart pepperoni, mushroom, no sausage, and no pepperoni. No pepper, no mushroom, sausage, pepperoni and said "Two opposites" He picked up two towers red-red-yellow-yellow/yellow-yellow-red-red and demonstrated how it corresponded to 1-1-0-0/ 0-0-1-1. T/R 2 then asked: "Oh. Can I ask you what you think now? Which do you think, which argument do you think is more convincing: grouping them by zero, one, two, three, and four, those four categories, which is five categories, [Brandon: Or by opposites.] or by opposites? Which convinces you that you have them all? Which way?" (line 295). Brandon said it depends on which way a person prefers. And T/R 2 said her concern was “how do I not know there’s another one out there in space somewhere that I haven’t thought of with an opposite?”

Brandon selected towers r-y-r-r and its opposite y-r-y-y. Then he chose y-y-y-r and its opposite r-r-r-y. He selected y-y-r-y and r-r-y-r and another pair y-r-r-r and r-y-y-y. He arranged them so that the towers with one yellows would be in a staircase fashion. He put together pairs of towers with two reds and two yellows as well: yellow-red-red-yellow/red-yellow-yellow-red, red-red-yellow-yellow/yellow-yellow-red-red and red-yellow-red-yellow/yellow-red-yellow-red. T/R 2 said she was still troubled how they would be certain they found all combinations by using opposites. He had shown her when constructing the pizza chart that there could be no other combination but was still
uncertain about using this method for towers. Brandon said that if one kept trying to make towers one would not come up with additional towers: "No matter what you try to do, you’ll always end up with one of those" (line 304). T/R 2 ended off this part of the interview by praising Brandon for how he represented the towers on the chart and matched zeroes and ones to red and yellow cubes. She said "That’s really neat the way you thought of that. You know I can, I can see that. And you’ve convinced me with this picture that you found all the ones and the zeroes. And it’s very nice… You’re a very good thinker, you know that?" (line 305-307).

Table 4.13

*Classification of teacher questioning and results, Colts Neck Interview with Brandon – Part II*

<table>
<thead>
<tr>
<th>Line</th>
<th>Researcher</th>
<th>Researcher question</th>
<th>Question type</th>
<th>Question result</th>
<th>Student conjecture, response, reasoning, or justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>105-109</td>
<td>T/R 2</td>
<td>You call it four toppings. Sure. This is very, very fascinating, Brandon. I’m really, really understanding what you did here now. Can I ask you a question? Now, we’ve done some different problems when I’ve been in this year with Mrs. Zalee. Does this problem with the pizzas remind you of any other problems we’ve done this year?</td>
<td>Probing - redirecting</td>
<td>Invited learner to make connections</td>
<td>This question served as a springboard for Brandon to make a connection with the towers problem and ultimately show how the pizza problem and towers problem are isomorphic. Brandon said it reminded him of the towers problem since there too he did it in groups. Brandon started to draw diagrams on paper of possible towers.</td>
</tr>
<tr>
<td>121</td>
<td>T/R 2</td>
<td>That’s interesting. Could you show me how those looked like with blocks? [Brandon: Okay.] I’ll give you some blocks. You could sit down. I just want to get these for you.</td>
<td>Eliciting</td>
<td></td>
<td>Brandon built eight towers.</td>
</tr>
<tr>
<td>123</td>
<td>T/R 2</td>
<td>We talked about this before. Is there a way you could be sure you</td>
<td>Probing - encouraging</td>
<td></td>
<td>Brandon said he was</td>
</tr>
<tr>
<td>Page</td>
<td>T/R</td>
<td>Text</td>
<td>Critical Thinking</td>
<td>Unsure</td>
<td></td>
</tr>
<tr>
<td>------</td>
<td>-----</td>
<td>------</td>
<td>-------------------</td>
<td>--------</td>
<td></td>
</tr>
<tr>
<td>125</td>
<td>T/R</td>
<td>Like when you did pizzas, you really seemed sure that you had them all. Don’t you?</td>
<td>Probing – encouraging critical thinking</td>
<td>Brandon explained he knew he had all pizza possibilities since he had constructed a graph.</td>
<td></td>
</tr>
<tr>
<td>127</td>
<td>T/R</td>
<td>Is there a way to do that with towers?</td>
<td>Probing – encouraging critical thinking</td>
<td>Brandon answered that one could make a graph of everything including towers.</td>
<td></td>
</tr>
<tr>
<td>129</td>
<td>T/R</td>
<td>Now, that would be interesting. Could you, do you think you could do it with towers? How would we make a graph with towers? What would that look like?</td>
<td>Probing - encouraging critical thinking</td>
<td>Brandon once again started to draw towers on the paper</td>
<td></td>
</tr>
<tr>
<td>131</td>
<td>T/R</td>
<td>Now, what I’m confused about is how can I, how am I going to keep track of this? Like here you showed me very nicely how I could keep track, but how am I going to know that I’ve, I’ve thought them all up if they’re…</td>
<td>Probing - encouraging critical thinking</td>
<td>Brandon at first said it was too hard to do a graph since he would need to draw blocks.</td>
<td></td>
</tr>
<tr>
<td>135</td>
<td>T/R</td>
<td>Would you, would you have to draw the blocks to do this? Could you…</td>
<td>Probing - elucidating</td>
<td>Brandon remembered other combinations of towers and started constructing them and therefore did not immediately respond to this question.</td>
<td></td>
</tr>
<tr>
<td>151</td>
<td>T/R</td>
<td>What do you mean by working your way up?</td>
<td>Probing - elucidating</td>
<td>Promoted reasoning and justification</td>
<td>Brandon explained how he had constructed different towers by working in a staircase fashion</td>
</tr>
<tr>
<td>169</td>
<td>T/R</td>
<td>How many, how many towers do you have in the twos group here?</td>
<td>Probing - elucidating</td>
<td></td>
<td></td>
</tr>
<tr>
<td>185</td>
<td>T/R</td>
<td>Oh, I see. Now you had, I just wanna, I just wanna finish what you had here. You had this guy and you had this guy also. [Creates 4 yellows and 4 reds] Okay, and they were in a group by themselves. Okay, so you said that these all had two. Two yellows or two reds?</td>
<td>Probing - elucidating</td>
<td>Brandon said they had two of each color and added that there would only be a ones group and not a threes group</td>
<td></td>
</tr>
<tr>
<td>187</td>
<td>T/R</td>
<td>Why not?</td>
<td>Probing - encouraging critical thinking</td>
<td>Brandon explained that he thought the ones group would be the same as the threes group</td>
<td></td>
</tr>
<tr>
<td>189</td>
<td>T/R</td>
<td>Can I ask you now, if, could we call, if I, if I wanted to, could I call this all a threes group?</td>
<td>Probing - elucidating</td>
<td>Brandon said the towers indicated could be called the one or the</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td></td>
<td></td>
</tr>
<tr>
<td>191</td>
<td>T/R 2</td>
<td>Okay, Why?</td>
<td>Probing - elucidating</td>
<td>Brandon explained how it had one block of one color and three blocks of the opposite color.</td>
<td></td>
</tr>
<tr>
<td>195</td>
<td>T/R 2</td>
<td>Now if I asked you to focus on the colors, and to take another look at your groups. Say we focused on, which color do you want to focus on? Red or yellow?</td>
<td>Probing – encouraging critical thinking</td>
<td>Brandon focused on yellow</td>
<td></td>
</tr>
<tr>
<td>199</td>
<td>T/R 2</td>
<td>Okay, if I asked you to focus on a particular color like yellow, okay, and then I asked you to tell me what the ones towers were, could you do that?</td>
<td>Probing encouraging critical thinking</td>
<td>Brandon showed towers that would be considered a ones yellow</td>
<td></td>
</tr>
<tr>
<td>205</td>
<td>T/R 2</td>
<td>Sure would, wouldn’t you? If were looking at these 8 towers here, and we’re looking at yellow [Brandon: Oh like almost yellow], and we’re looking for ones, what would, what would be a ones tower here?</td>
<td>Probing - encouraging critical thinking</td>
<td>Brandon identified a tower that could be considered a ones tower</td>
<td></td>
</tr>
<tr>
<td>211</td>
<td>T/R 2</td>
<td>Okay, what else would be a, a one yellow here?</td>
<td>Eliciting</td>
<td>Brandon found another</td>
<td></td>
</tr>
<tr>
<td>213</td>
<td>T/R 2</td>
<td>You, can you, that’s interesting. Can you pull the rest of them out that would be?</td>
<td>Encouraging engagement - administrative</td>
<td>Brandon found one more tower and exclaimed that there had to be one more one yellow tower</td>
<td></td>
</tr>
<tr>
<td>215</td>
<td>T/R 2</td>
<td>Why? Why did you say there must be one more?</td>
<td>Probing - elucidating</td>
<td>Brandon indicated how the yellow blocks in the ones yellow towers form steps and explained how he was working his way down similar to the way he had worked in the pizza problem</td>
<td></td>
</tr>
<tr>
<td>225</td>
<td>T/R 2</td>
<td>What would the zero one look like if we’re looking at yellow?</td>
<td>Eliciting</td>
<td>Brandon at first removed all blocks</td>
<td></td>
</tr>
<tr>
<td>231</td>
<td>T/R 2</td>
<td>A zero, a zero tower, if we’re looking at yellow, would be nothing?</td>
<td>Probing - elucidating</td>
<td>He stated that if one were focusing on yellow there would be nothing in the zero group.</td>
<td></td>
</tr>
<tr>
<td>233</td>
<td>T/R 2</td>
<td>Well, what was a ones yellow?</td>
<td>Probing - elucidating</td>
<td>Brandon showed which towers would be included in the ones, twos, threes and fours group.</td>
<td></td>
</tr>
<tr>
<td>Line</td>
<td>T/R 2</td>
<td>Text</td>
<td>Probing</td>
<td>Promoted reason and justification and invited the learner to make connections</td>
<td>Brandon showed how towers are similar to the ones and zeroes on his chart – how the ones group is in a step-like fashion as well.</td>
</tr>
<tr>
<td>------</td>
<td>-------</td>
<td>--------------------------------</td>
<td>----------</td>
<td>-----------------------------------------------------------------</td>
<td>----------------------------------------------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>237</td>
<td>You go so fast for me, Brandon. Tell me again how this is like the pizzas.</td>
<td>Probing - elucidating</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>243</td>
<td>Is what you’re saying to me then like a yellow cube here is like a number one when in your chart?</td>
<td>Probing - elucidating</td>
<td></td>
<td></td>
<td>Brandon said that if they were focusing on red then red would be the number one.</td>
</tr>
<tr>
<td>245</td>
<td>Well, let’s continue with yellow. This is interesting. I think this is really neat. Now, what would come next with what we have here if we want to reorganize? You said these would be like the one, yellows.</td>
<td>Eliciting, Probing - encouraging critical thinking</td>
<td>Promoted reasoning and justification</td>
<td>Brandon reorganized his towers by first organizing the ones with one yellow and the remaining cubes red.</td>
<td></td>
</tr>
<tr>
<td>247</td>
<td>Where were all our twos towers? There’s, is this one? How many were there? [Brandon: Here, I’m gonna lie these down.] How many were there? I forgot.</td>
<td>Encouraging engagement - administrative</td>
<td></td>
<td>Brandon laid down the towers and indicated how towers corresponded to zeros and ones on his pizza chart.</td>
<td></td>
</tr>
<tr>
<td>253</td>
<td>Yeah, where would the tower be that would look like this pizza?</td>
<td>Probing - encouraging critical thinking</td>
<td>Invited learner to make connections</td>
<td>Brandon picked out the appropriate tower to correspond to the pizza to which T/R 2 was pointing. He continued by showing many other towers and how they would correspond to his pizza chart.</td>
<td></td>
</tr>
<tr>
<td>261</td>
<td>Fascinating. Okay. And then are we out of them? And are we out of towers with two?</td>
<td>Eliciting</td>
<td>Promoted reasoning and justification</td>
<td>Brandon said that on the pizza chart there were no more options in the two’s group and that likewise there were no more tower combinations.</td>
<td></td>
</tr>
<tr>
<td>263</td>
<td>Can we, are we still sticking with yellow as the color we’re focusing on?</td>
<td>Probing - elucidating</td>
<td></td>
<td>Brandon organized towers of three yellow cubes and one red cubes according to the corresponding lines on the pizza chart.</td>
<td></td>
</tr>
<tr>
<td>267</td>
<td>Oh. I see. Okay. And are we missing any?</td>
<td>Probing - encouraging critical thinking</td>
<td></td>
<td>Brandon said no.</td>
<td></td>
</tr>
<tr>
<td>269</td>
<td>You know what I’m wondering? We have this guy left. [Picks up 4 reds] Right? [Brandon: Yeah, because we’re not focusing on him.] Because he’s the opposite of</td>
<td></td>
<td></td>
<td>Brandon says that the all red tower would be the “zero guy.”</td>
<td></td>
</tr>
<tr>
<td>Time</td>
<td>T/R</td>
<td>Text</td>
<td>Probing</td>
<td>Reasoning</td>
<td>Justification</td>
</tr>
<tr>
<td>-------</td>
<td>------</td>
<td>------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------</td>
<td>---------</td>
<td>-----------</td>
<td>---------------</td>
</tr>
<tr>
<td>275</td>
<td>T/R 2</td>
<td>Okay. Could we have done it the, I just wanna ask you. You don’t have to do it, but could we have done it the other way around? Could we have just focused on red and, and gotten it to work the same way? Branden transposed the ones towers with the threes towers and explained which towers would be included in the ones, twos, and threes group.</td>
<td>Probing</td>
<td></td>
<td></td>
</tr>
<tr>
<td>277</td>
<td>T/R 2</td>
<td>And, and what would these be? What would these be?</td>
<td>Probing</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
is what you’re saying, how do I know, how do I know in terms of opposites that that’s all of them?

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>309</td>
<td>T/R 2</td>
<td>Colin thought of the graph?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Probing – elucidating</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Brandon explained that it was his partner Colin’s idea to make a chart. Once they decided to make a chart they split up and worked independently and then compared answers.</td>
</tr>
</tbody>
</table>

4.2.4.1 *Questioning techniques used in the interview with Brandon at Colts Neck*

Since this was a one-on-one task-based and lengthy interview, there were numerous questions asked by T/R 2. With the encouragement of T/R 2, Brandon reconstructed the chart that he had developed when solving the pizza problem in an earlier class session. During the first part of the interview, T/R 2 asked many questions to allow Brandon to share his thought process and reasoning in creating the table. T/R 2 used many probing/elucidating questions to clarify Brandon's organization of the data and to provide the opportunity for him to explain how he was certain that he had come up with all possibilities. Then T/R 2 segued into the towers problem that Brandon has also had the opportunity to tackle in a previous session by asking Brandon if the pizza problem was similar in some way to another task that he had worked on. She asked him “Does this problem with the pizzas remind you of any other problems we’ve done this year?” (line 109). This question encouraged Brandon to fold back to previous understanding. Brandon independently connected it to the towers problem that he had worked on four months earlier using Unifix cubes. Brandon recalled that they had worked on the Towers problem as well and he reconstructed towers by finding opposites as he had done in an earlier class and quickly rebuilt his solution set to that problem. When T/R
2 questioned if this was the best way to ensure that he had found all of the possible tower combinations, Brandon paused for a minute and then reorganized his towers and compared it to the pizza problem. He created five groups, towers with no yellow cubes, towers with one yellow cube, towers with two yellow cubes, towers with three yellow cubes, and towers with four yellow cubes. When T/R 2 questioned him further about his organization, Brandon showed T/R 2 how each tower corresponded to another line on his pizza chart by placing a Unifix tower on the corresponding pizza line. He was able to quickly and correctly match each of his sixteen towers to another line item, thus applying his argument by cases that he had developed when working on the pizza problem to the towers problem. Through careful questioning, T/R 2 built on Brandon’s initial proof by cases and provided the opportunity for magnificent reasoning where Brandon made the connection between the pizza and towers problems, compared the two solutions, and realized and proved how they are equivalent in structure and isomorphic. Thus, T/R's questioning invited Brandon to make connections and provided the opportunity for Brandon to express convincing justification for the two problems.

During the segment of the interview where Brandon was creating groups of towers by cases, T/R 2 employed the use of clue giving to guide Brandon to sort the cubes into the five cases. Brandon initially conjectured that there would only be a “ones group” but not a “threes group.” Rather than indicating that he was wrong in that assumption T/R 2 guided Brandon in the correct direction through questioning, thus enabling him to organize the towers into five groups. T/R 2 asked him to focus on a specific color by saying, “Now if I asked you to focus on the colors, and to take another look at your groups. Say we focused on, which color do you want to focus on? Red or
yellow? (line 195) and “Okay, if I asked you to focus on a particular color like yellow, okay, and then I asked you to tell me what the ones towers were, could you do that? (line 199). Capitalizing on Brandon’s thinking out loud, T/R 2 invited him to provide justification for his assumptions. After identifying towers that could be included in the ones group, Brandon exclaimed that there had to be one more tower that could be included in that group. T/R 2 questioned how he knew that and Brandon organized his towers in a staircase fashion and provided a justification for his assumption. Brandon at first did not identify a tower that could be classified as a “zero tower” and upon questioning asserted that no tower would fit into that category. Once again T/R 2 guided him by simply asking him “Well, what was a ones yellow?” thereby allowing Brandon to explain what towers had been included in the ones group as well as the twos, threes, and fours group, and ultimately show which tower would be considered a zero tower. By asking him “You don’t have to do it, but could we have done it the other way around? Could we have just focused on red and, and gotten it to work the same way?” (line 275), T/R 2 was able to ascertain how Brandon had organized the data by cases. Brandon reorganized the towers by transposing the ones tower with the threes tower and he explained once again what towers would be included in each group if they were focusing on red. Figures 4.33 and 4.34 show the question types and associated student reasoning that occurred during this session.
Figure 4.33. Question types used by T/R 2 during the interview with Brandon

Figure 4.34. Question results during the interview with Brandon
4.2.4.2 Analytics for the interview with Brandon at Colts Neck

Hmelo-Silver (2011) created an analytic, “Using video to teach about transfer: The case of Brandon,” to teach her students about analogical transfer. In this analytic one sees Brandon making the connection between the pizza and towers problems.

In addition, a video from the The Private Universe Project in Mathematics, by the The Annenberg Foundation and Science Media Group of the Harvard Smithsonian Astrophysical Observatory which includes an interview with Researcher Carolyn Maher, highlights major events and discussions during Researcher Amy Martino’s interview with Brandon. This video can be viewed at http://dx.doi.org/doi:10.7282/T3VX0FRD (Maher, Alston, Dann & Steencken, 2000).

“The Interplay between Researcher Martino’s Questioning and Brandon’s Reasoning: Utilizing Probing Questions to Invite Students to Make Connections” (Gerstein, 2017b) focuses on the questions posed by Researcher Martino during her interview with Brandon and the associated student justification and reasoning.

4.3 Analysis of teacher questioning by Researcher Robert B. Davis

In this section I will look at sessions led by Researcher Robert B. Davis as he introduced sixth grade students to algebraic ideas using the Guess My Rule activity. Before starting the Guess My Rule activity, the students had worked on other algebra ideas such as variables, open sentence and truth sets (see Spang, 2009). The students worked on Guess My Rule during sessions conducted on 9/30/1993 and 10/1/1993. During the 9/30/1993 session, the students first worked on Pebbles in the Bag problems, Finding Truth Set problems and then during the last ten minutes of class began Guess My Rule problems. On 10/1/1993 the students continued working on Guess My Rule
problems using a worksheet provide by Researcher Davis. The following examples appeared on the worksheet.

<table>
<thead>
<tr>
<th></th>
<th>□</th>
<th>Δ</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>11</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>□</th>
<th>Δ</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>13</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>□</th>
<th>Δ</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>□</th>
<th>Δ</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>27</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>37</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>47</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>57</td>
</tr>
<tr>
<td></td>
<td>( \Delta )</td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>18</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>28</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>38</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>48</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>( \Delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>( \Delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>( \Delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>( \Delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>
4.3.1 Kenilworth session, September 30, 1993 with Researcher Robert B. Davis

T/R 3 began this part of the session by giving the students a handout and asked the students what they thought they would be doing. The students read the title of the handout and responded that they would be working on Guess My Rule. T/R 3 then encouraged the students to pay attention to Milin and Milin explained that they would be guessing what would go into the equation. T/R 3 agreed and, pointing to the whiteboard where he had written in the first part of the session an equation and a truth set for it, he elaborated on what Milin had said. He told the students that previously they had been given an equation and had been asked to come up with the truth set. Now he would be providing them with the truth set and they would have to come up with the equation. T/R 3 told the students that when they believed they had come up with a solution, they should write it down and show it to him.

The first problem on the handout said:

<table>
<thead>
<tr>
<th></th>
<th>□</th>
<th>△</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>11</td>
<td></td>
</tr>
</tbody>
</table>
Milin wrote on the handout (□ X 2) +1= Δ next to the problem. He told T/R 3 that he knew the equation for the first example since it was the same as the one they had done on the board. T/R 3 told him to continue on to the next example. Meanwhile Matt and Stephanie started to work on the problem. Matt noticed a pattern and said "I have it" (C43). Stephanie asked him to explain his solution to her and as he pointed to the numbers under the box and the triangle he said "Look. Plus one, plus two, plus three…” (C48). Matt turned to T/R 3 and told him that he had found a solution. T/R 3 asked him to write it down. Matt asked him if he could just tell him the equation. T/R 3 responded that it is difficult to "say an equation" (C68). Matt went over to T/R 3 and told him his solution. T/R 3 acknowledged Matt's idea by saying, "Okay, that's a good idea" (C88).

Michelle R., Romina and Brian also worked on problem one. Michelle and Romina both noticed that the difference in numbers under the triangle column was two. T/R 3 asked "Has someone got an equation to show me?” Brian and the other students in his group went over to T/R 3 and Brian explained their solution. T/R 3 asked him "That's certainly an interesting idea…What are you going to show me ultimately?” (C136-C138), thereby encouraging him to write an equation. Michelle reasoned that the difference between the first row was one, the second row two, and in each successive row the difference increased by one.

T/R 3 called the entire group together. He told them that he would like to make sure that they all agreed on what they were trying to do. He reminded them that in the previous activity they had been working on truth sets. He emphasized that with that activity they had been given the equation and had worked out the truth set by working out
pairs of numbers that would make it true. He asked the students what had changed in the current activity. One student offered that now they were making the equation. T/R 3 reiterated that in the current activity he was giving them the numbers that would make it true, the truth set, and he would like the students to come up with the equation. He mentioned that they did not have enough time during the current session to write the equation but that they would return to it the next day.

As T/R 3 wrapped up the session Romina told T/R 3 that she had come up with a solution. Pointing to problem 1 she told T/R 3 that the difference between the values in the triangle column was two. T/R 3 told her "That's a very good idea and that's going to be very useful" (C193). Michelle added that in problem three the difference between successive triangles was three. T/R 3 complimented her by saying "That's very nice" (C199).

<table>
<thead>
<tr>
<th>Line</th>
<th>Speaker</th>
<th>Researcher question</th>
<th>Question type</th>
<th>Question result</th>
<th>Student conjecture, response, reasoning, or justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>C121-12</td>
<td>T/R 3</td>
<td>That's certainly an interesting idea. But now what is it that you're going to… We won't have time to do it today. What are you going to show me ultimately?</td>
<td>Probing - encouraging critical thinking</td>
<td>Promoted justification</td>
<td>Brian went back to his desk and continued to work on the problem, eventually coming up with an equation.</td>
</tr>
</tbody>
</table>

**4.3.1.1 Questioning techniques used in Sept 30, 1993 at Kenilworth**

During this very short segment of Guess My Rule, T/R 3 asked Brian when he came up to show him his solution, "That's certainly an interesting idea…What are you going to show me ultimately?" (C121-124), thereby encouraging him to write an equation. This prompted Brian to give more thought to the problem and ultimately come up with an equation.
4.3.2 Kenilworth session, October 1, 1993 with Researcher Robert B. Davis

T/R 3 began the class session with another discussion about secrets. He asked the students what scientists do. Jeff responded that they discover and invent. T/R 3 asked what kind of problems they hoped scientists would solve. Students offered different responses such as curing certain diseases such as AIDS and inventing solar powered cars.

T/R 3 explained to the students that there were secrets in science that scientists were still seeking to discover such as the cure for numerous diseases. He asked the students for names of scientists. Jeff and another student offered Thomas Edison, Alexander Graham Bell, and Einstein. T/R 3 said that Einstein would probably be the first scientist to come to mind. T/R 3 elaborated that scientists who discover important things are proud to have their name associated with the discovery and like to get credit for it. He said that there are secrets but asked the students if scientists sometimes share their information. Jeff responded "Sometimes" (line A25) and T/R 3 concurred with him and said "They have to, they have to actually because in the long-run…no single person could do it all by themselves, so they have to share" (line A26). T/R 3 continued and explained to the students that they should try to find out secrets "that's sort of fun and that's what you do in science and in mathematics" (line A38) but that they should share them as well. He told them that they could keep the secret to themselves for a short while and give others the opportunity to discover the secret, but then at some point they should share it with others.

Next T/R 3 reviewed what the students had worked on during the previous session.
He reminded them about the equation "box times box minus something times box plus something equals zero" and asked "Do you remember what you were trying to find some numbers. What did those numbers do?" (line A38). Michelle answered that the numbers replaced the boxes or triangles and T/R 3 added that by doing so, they made a true statement. He complimented the students on their work and said that many students had found the secret. T/R 3 reminded them that they had worked on two problems that had seemed impossible, that they had not finished them, and that they would not be finishing those examples during the current session either.

T/R 3 then showed them the equation box times two plus one equals triangle. He asked Stephanie what they had done with that. Stephanie answered that they had to put a number in the box and a number in the triangle so that the equation would be true. T/R 3 said that her explanation was correct and asked her if they put a zero into the box what number would they put into the triangle. Stephanie answered one. T/R 3 asked Michelle R. if she remembered what she had written on her paper. Michelle responded that she did not remember and T/R 3 suggested that she take her paper and write it on the board. Michelle wrote \(\Box \times 2 + 1 = \triangle\). T/R 3 asked her whether she saw where she was missing one parenthesis. Michelle closed the parenthesis so that the equation read \((\Box \times 2) + 1 = \triangle\). Then she placed a zero in the box and a one in the triangle. T/R 3 asked the students if they agree that that was what they had been doing. Then he asked Michelle what they had done next. Michelle said they had tried to find a secret to it with a pattern. T/R 3 said that some students had found an interesting secret which might be a secret they should share with others but Ankur said that they shouldn't share it. Jeff interjected that he thought they should share the secret and T/R 3 said they would share it but not right away. He
then reminded them that the next task that they had worked on involved a table he had provided for them and asked them what they had been supposed to do with them. Romina replied that they had tried to find the equation. T/R 3 agreed with that and started to hand back student worksheets. When everyone had a paper he told the students to discuss problem 2 on the worksheet with their neighbor.

Ankur immediately told T/R 3 that he had finished problem number two and T/R 3 asked him to come up and show him what he had done. Ankur had written on his paper next to problem two:

\[
\begin{array}{c|c}
\square & \Delta \\
0 & 5 \\
1 & 7 \\
2 & 9 \\
3 & 11 \\
4 & 13 \\
\end{array}
\]

\((0 \times 2) + 5 = \Delta\)

T/R 3 suggested that he write it on the board so that it could be recorded by the camera and Ankur wrote \(\square\) time two plus five. Ankur explained that the next row in the table would work as well. T/R 3 agreed that they would all work and asked Ankur to work on additional problems.

Michelle and Ankur came up a few minutes later with another completed problem. T/R 3 complimented them on their solution by saying "That's clever" (line A104) and once again T/R 3 suggested that they show it in front of the camera. Ankur had written on his paper next to problem 3:
T/R 3 then asked for volunteers who had completed problem number two to show everyone what they had done. He then asked “is it okay if we give away secrets or is it too early to do that?” (A122). As one student suggested that they wait, T/R 3 agreed to do so. Michael told T/R 3 that he wanted the other students to divulge their secrets right away. T/R 3 informed him that others had asked to hold off with it. Michael asked if he was the only one who did not know the secret and T/R 3 assured him that he was not.

T/R 3 asked the students what problems they had done so far. Matt informed T/R 3 that he had found the secret and after listening to him, T/R 3 cautioned him to be careful because there may have been more than one secret that he needed to consider. When Michael showed T/R 3 his work, T/R 3 told him that it's very nice and told him to see if Milin agreed with him.

T/R 3 asked students to come and explain their solution in front of the camera. Michelle I and Ankur showed their work. Michelle had written for problems two and three:

\[
\begin{array}{c|c}
\hline
0 & 5 \\
1 & 7 \\
\hline
\end{array}
\]

\[
(0 * 3) + 1 = D
\]
Michelle pointed to the five and seven under the triangle in problem two and explained that the difference between the five and seven is two and one would have to add two to arrive at the next number in the triangle column.

Ankur added that they had found the difference and then "on the second one we just figured it out" (line A213). T/R 3 told them that they had done nice work and pointing to problem 3 he said "you did that work on this problem down here" (line A213). Michelle explained that in problem three the difference between the first two numbers in the triangle column was three and hence they had come up with x3 in their equation. T/R 3 complimented them by saying “that's very nice, that's a very important idea. Thank you” (line A216).

Amy-Lynn and Bobby also showed their work in front of the camera. Pointing to problem 2, Amy-Lynn explained that in problem 2 they had used the five "as a plus number."

The following was written on Amy-Lynn’s paper:
2.  

<table>
<thead>
<tr>
<th>□</th>
<th>△</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
</tr>
</tbody>
</table>

3.  

<table>
<thead>
<tr>
<th>□</th>
<th>△</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
</tr>
</tbody>
</table>

Pointing to problem 3 she said that they had used the one (the first number under the triangle column) "as a plus number" (line A218). T/R 3 said "Okay, now you have another number, how did you find that other number?" Pointing to the equations that she had written, he asked "You used the two here and the three here - how did you find that?" (line A219). Bobby explained that if "you minus that [the seven] from the first number and that's how you get the times sign" (line A224). T/R 3 tried to clarify and asked "okay, you're saying you subtracted the five from the seven. Bobby said "the seven from the five", and T/R 3 answered that he thought if you subtract seven from five you get negative two. Bobby realized his error and corrected himself by saying “Oh, okay, then five from seven" (line A228).

T/R 3 complimented them on their work by saying, "That’s very nice, those are very important ideas" (line A229).
T/R 3 mentioned that many students had come up with the secret but there were specific questions that were causing them difficulty.

<table>
<thead>
<tr>
<th></th>
<th>□</th>
<th>△</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>26</td>
</tr>
</tbody>
</table>

Romina and Brian came up to show their work on problem number six. Romina showed T/R 3 that the difference between the numbers in the triangle column was 1,3,5,7. In between those numbers is two. T/R 3 asked for clarification by asking "where is it that it goes by two, can you show that?" (line A251). Romina explained what she had said once more and T/R 3 told them that "Now you need to figure out what to do with that but it's a wonderful idea. Thank you."

T/R 3 asked the students if anyone else had found the equation for number six. Michelle I and Ankur came up to show and explain their solution for problem six in front of the camera. Ankur began talking and said that "Whatever the first number is, equals the second number" (line A269). Ankur meant by this that the number in the box column had to be multiplied by itself. He continued by saying "So we put two in the second one and always goes here, so if this is three, three goes here and plus one" (line A271). Ankur added that they needed to add one each time.

Michelle tried to clarify and showed T/R 3 that in line four of the table if three is multiplied by three and they add one that would give the value in the triangle column of
ten. T/R 3 commented that "you haven't quite really found the formula" (line A274). Michelle responded that she believed the secret was that the number in the box always goes next to it, meaning that the number is multiplied by itself. T/R 3 encouraged her to write it by asking "Can you think of a way to write that? ... That's a really neat idea, that’s a really neat idea" (line A277). Ankur asked if they had discovered the secret and T/R 3 answered "If you can find a way to write it you've really got if figured out" (A279).

T/R 3 called the class together and asked Mike to come up to the board to show how he had arrived at a solution. He told him to do one of the first five because "that's what people feel happiest about" (line A280). Mike chose to demonstrate his solution for problem two. He explained that there is always a pattern in the triangle column. He wrote on the whiteboard box x 2 + but did not fill in the missing number. He then continued with a lengthy but faulty explanation of how he would come up with the other number.

“Zero can go into five, five times, (T/R 3 started to correct him but Mike continued) "Yeah whatever, so we leave that one out. One goes into seven, seven times, leave that one out. The two does not go into nine, so take one out to make it eight, so we’re going to have to have plus one. So this two goes into eight. Two times four is eight, plus one is nine, so that’s nine. The eleven, take one away and it’s ten, but it doesn’t go into ten. So you take one away so you’re going to have to take one away from eight, so it’s going to be seven” (line A288). T/R 3 suggested that others may have an idea that is easier than what Mike was proposing.

Matt offered that the missing number is five since it was the number in the first four. T/R 3 clarified that he was adding five since he was taking the number from the pair zero five. T/R 3 made sure that everyone agreed with Matt's solution by asking "Okay, is everybody happy with that, you all know that?" (line A296).
A student asked for them to demonstrate number six that had been troubling many students. Ankur stated that he had an answer for number six and that he thought he had a way to write it. T/R 3 at first suggested that he demonstrate his solution in front of the class but then changed his mind and suggested to Ankur to speak in front of the camera. Reading his solution from his paper Ankur explained that "the two numbers in the brackets are always the same and the number after the bracket is always one" (line A310). T/R 3 asked him if he could figure out how to write it "using the box and triangle method of writing?" (line A313). Ankur was unsure of what T/R 3 meant so T/R 3 tried to clarify by saying that instead of using words he should try to use boxes and triangles. He mentioned that Michelle had almost said the equation when they had approached her earlier. T/R 3 suggested that they try to work on it further.

T/R 3 called for everyone's attention and emphasized that the students should be trying to write their solution using the box and triangle notation just as Mike had done when he had demonstrated his solution on the whiteboard. He said that many of the students "have got some very clever things, but you are writing them in words" (line A325). He asked them to take their ideas and try to write them using boxes and triangles. Ankur again told T/R 3 his solution for problem 6. He said "the number that's here, you see will always go here" (line A330). T/R 3 asked him if he could think of a way of writing that so that others would know that the number "that goes here has to go here too" (line A331). Ankur asked him if he could show him an equation with a "box here and a box here" (line A334) and T/R 3 asked him “you want to go and do that?” (line A335). Amy Lynn and Bobby showed T/R 3 their solution by the camera. Amy Lynn had written on the worksheet:
Amy-Lynn explained their solution and T/R 3 told them that their solution looks like it would work and asked them if they could write it with the box and triangle notation. Ankur and Michelle I. showed him their equation where they had written \((\Box x \Box + 1 = \Delta)\). T/R 3 asked Michelle if she agreed with what Ankur had written. When she responded that she did, T/R 3 complimented their work saying "That is elegant. That is great" (line A354).

T/R 3 asked Michelle and Ankur to share part of the secret without revealing everything. When they came up Michelle asked T/R 3 if he'd like them to say what they had written without telling everyone the "code." T/R 3 told them quietly that they should tell everyone that the number outside of the box had to be the same as the number inside of the box. Ankur protested that by saying that it would give the whole secret away but T/R 3 and Michelle disagreed. T/R 3 told the class that Michelle and Ankur were trying to decide how much of the secret to publish.

Michelle wrote on the board: \((\Box x _) + 1 = \Delta\). Michelle explained that the number after the parenthesis would always be one and the number and the number in the parenthesis would always be the same as the number in the box.

Table 4.14

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>((0 x 1) + 1 = 1)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>((1 x 1) + 1 = 2)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>((2 x 2) + 1 = 5)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>((3 x 3) + 1 = 10)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>17</td>
<td>((4 x 4) + 1 = 17)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>26</td>
<td>((5 x 5) + 1 = 26)</td>
<td></td>
</tr>
</tbody>
</table>
## Classification of teacher questioning and results, Kenilworth session Part I

<table>
<thead>
<tr>
<th>Line</th>
<th>Speaker</th>
<th>Researcher question</th>
<th>Question type</th>
<th>Question result</th>
<th>Student conjecture, response, reasoning, or justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>A277</td>
<td>T/R 3</td>
<td>You suppose there would be a way to write that. Can you think of a way to write that? The number that goes in the box is also the number that is next to it. How could you write that? That’s a really neat idea, that’s a really neat idea.</td>
<td>Probing - encouraging critical thinking</td>
<td>Promoted justification</td>
<td>Ankur and Michelle attempted to write the formula. They came back a few minutes later and told T/R 3 that they had written “The numbers in the bracket are always the same. The number after the bracket is always one.”</td>
</tr>
<tr>
<td>A313</td>
<td>T/R 3</td>
<td>Okay, now you certainly managed to write it, could you write it using the box and triangle method of writing?</td>
<td>Probing - encouraging critical thinking</td>
<td>Promoted justification</td>
<td>Ankur asked for clarification and then went to work on the problem some more together with Michelle. They came up later and once again tried to explain their solution to T/R 3.</td>
</tr>
<tr>
<td>A331</td>
<td>T/R 3</td>
<td>Yeah, that’s certainly right, okay, um, can you think of a way to write it so that we’ll know that the number that goes here has to go here too.</td>
<td>Probing - encouraging critical thinking</td>
<td>Promoted justification</td>
<td>Ankur and Michelle worked again to try to come up with the formula. They came back and showed their solution to T/R 3. They had written: ((\square x \square) + 1 = \Delta) and Ankur explained it to T/R 3.</td>
</tr>
<tr>
<td>A351</td>
<td>T/R 3</td>
<td>Yup, do you agree?</td>
<td>Encouraging engagement - confirming agreement</td>
<td>Michelle agreed with Ankur.</td>
<td></td>
</tr>
<tr>
<td>A360</td>
<td>T/R 3</td>
<td>How many people have got number six?</td>
<td>Encouraging engagement - administrative</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A366</td>
<td>T/R 3</td>
<td>Do you want to come and write it and tell it to the camera?</td>
<td>Encouraging engagement - administrative</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A374</td>
<td>T/R 3</td>
<td>Michelle, without saying what you and Ankur have done, can you tell people what you said in the original? Do you remember what you said?</td>
<td>eliciting</td>
<td>Michelle and Ankur discussed how much of the secret should be revealed. Then Michelle wrote on the board: ((\square x \square) + 1 = \Delta). Michelle explained that the number after the bracket is always one and the number</td>
<td></td>
</tr>
<tr>
<td>A389</td>
<td>T/R 3</td>
<td>I think everybody is well agreed on that. Isn’t that true a lot of people have decided on</td>
<td>Encouraging engagement - Confirming</td>
<td>Brian answered that they got that as well.</td>
<td></td>
</tr>
</tbody>
</table>
Brian showed T/R 3 his work. He had written \((\Box \times \_ ) + 1 = \Delta\) but explained to T/R 3 that the code would be "square times square plus one equals triangle" (line A434). T/R 3 pointed out that he had not written it that way and Brian agreed that he had left out the second square. T/R 3 asked Brian and Romina to write it down correctly.

Michelle asked T/R 3 if she could share the code with the class. T/R 3 told the class, "let me say the people with the secret would like to publish it now, when scientists really discovered something they do what they call publishing, they send it to a journal and it gets printed and everybody reads it"(line A450). Michelle asked the class: "If the number here is going to be the same as the number here, what shape do you think that is going to be?" (line A432) One student offered that it would be a square. Michelle told everyone that box times box plus one equals triangle was the code.
T/R 3 asked everyone to try to solve number seven. Bobby told T/R 3 that he and Amy-Lynn had already completed number seven.

Bobby explained that zero times zero plus five equals five and one times one plus five equals six. T/R 3 asked them how they would write that with box and triangle notation. Amy-Lynn suggested "Box times box plus five equals triangle" (line A479). Jeff came up to T/R 3 and showed him the same solution.

Michelle R. and Jeff came up to T/R 3 to show him their equation for problem eight:

Michelle R. had written on her paper next to problem 8:

\[ 0 \times 0 - 0 = 0 \]
1 x 1 - 1 = 0
2 x 2 - 2 = 2
3 x 3 - 3 = 6

T/R 3 was surprised that they had solved another problem so quickly. Jeff and Michelle told T/R 3 that the equation was "box times box minus box equals triangle" (line A500). T/R 3 complimented them by saying "that's wonderful" (line A501).

Amy-Lynn and Bobby came up to explain their solution for number nine to T/R 3.

<table>
<thead>
<tr>
<th>9.</th>
<th>□</th>
<th>△</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>½</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4 ½</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>12½</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>18</td>
</tr>
</tbody>
</table>

Amy Lynn explained:

Okay, we had the zero times the zero and then plus zero and we got zero. Then we added one, so it’s one times zero plus a half and we got a half. And then we did zero one two times zero plus two equals two. And then we did one two three, three times zero plus four and a half equals four and a half. And then you took four times zero plus eight equals eight. And then five times zero plus twelve and a half equals twelve and a half. Six times zero plus eighteen equals eighteen. (line A510)

T/R 3 suggested that they continue working on the problem and see if they could find an easier way to solve it. He asked them "Where do you suppose those halves are coming from, what do you suppose is making those halves in the problem?" (line A511). Amy-Lynn answered "the whole" and T/R 3 once again encouraged them to think about the problem some more.
Michelle, Romina, Ankur and Brian came up to explain their solution for problem number eight. Ankur explained that they divided the triangle by the square. $\Delta \div \square$

He said, "so you just divide the triangle by the square, then we wrote it like a code. So, six divided by three is two. So you divide twenty by five and that’s four, thirty by six that’s five, twelve by four that’s three" (line A523). Here Ankur tried to explain the pattern that they had found.

<table>
<thead>
<tr>
<th>□</th>
<th>△</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
</tr>
</tbody>
</table>

$0 \times 0 - 0 = 0$
$1 \times 1 - 1 = 0$
$2 \times 2 - 2 = 2$
$3 \times 3 - 3 = 6$

T/R 3 tried to understand the connection between the written representation and the pattern so he asked them what they would get by dividing the box number by the triangle. Ankur responded that they would get the answer. T/R 3 tried to clarify so he asked "What answer? So here I would take the six and divide it by three" (line A530). Brian said that the answer would be two. T/R 3 asked him "What's the two?" (line A532). Brian tried to explain but had difficulty expressing the pattern he saw in written form. T/R 3 encouraged them to try form a valid equation with an equal sign to explain the
pattern he had seen. T/R 3 acknowledged that he did see an interesting pattern. He said "Let’s see what we do here, let’s try it. When I had here, zero divided by one is zero, and I had two divided by two that was one, and six divided by three that was two. Oh, I see something interesting is happening" (line A538). Brian added that they noticed a pattern that it "goes up, like one, two, three, four, five, and you multiply six times five plus zero and you get that number” (line A539). Although he tried to express the formula verbally he still did not know how to translate it into a written equation. T/R 3 encouraged them to continue working on it so that they could come up with a valid equation.

T/R 3 then addressed the whole class and explained that a standard formula is one that uses a pattern to arrive at the triangle value. He stressed that they were supposed to write the equation in a way that the triangle value is the number derived from the operations performed on the box value.

Okay, there is one thing that I would like to talk to you about, can I get a place to sit here. Notice there are different kinds of secrets, different people are making up, but this kind of thing which is called a formula, it’s what mathematicians call a formula, that formula lets you if I tell you the number in the box, that lets you find what the number in the triangle is. Okay, now some of you have some very interesting secrets, I’m not saying don’t use it, but some of you use something that depends on knowing what the number in the triangle is, but you see what we got here doesn’t. [On board is (□ x □) + 1 = Δ]. Okay, it only depends on the number in the box; if I tell you the number in the box then you can find the number in the triangle. So we’re particularly looking for formulas like this where you don’t need to know the number in the triangle, all you need is to put the number in the box and that will tell you the number in the triangle. (line A551)

Bobby and Amy-Lynn told T/R 3 they had a secret about number eight. He explained that the difference between the triangle values increased by two. First the difference was zero, then two, then four etc. T/R 3 wrote the table for problem eight and
asked them to share this idea with the rest of the class. Bobby wrote his idea on the board as he explained how he found a pattern in the difference between the triangle values.

<table>
<thead>
<tr>
<th>O</th>
<th>△</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
</tr>
</tbody>
</table>

T/R 3 asked Jeff if he too had something to show everyone. Jeff conjectured that when the triangle column begins with two zeroes the equation would be box times box minus a number equals the triangle. T/R 3 asked the others if anyone had a formula for the problem and added that Jeff and the others that he was working with had come up with a formula. Brian, Michelle, Ankur, and Romina came up to explain their equation to the class. Brian explained,

“Like, divide two into two and you get one, if you multiply two times one plus zero you get two and that’s what you’re supposed to get because two’s in the triangle. And the triangle’s supposed to be at the end. Divide six times three and you get two…. Awesome, and that’s one and then you take the one and the two out of there and the two times one, I have to move down, and then you just plus zero cause that’s the number you have up there to start with and then it equals that
As Brian spoke he wrote on board: \((2 \times 1) + 0\) (the two is in a box). T/R 3 responded to this incorrect argument by saying that he's not sure everyone could follow that argument, and that although they had a nice idea they did not actually arrive at the correct formula. Although Brian did not use the triangle in his faulty formula, once again T/R 3 stressed to the class that in order to meet the requirement for a formula they would have to find a formula that performs an operation on the box value in order arrive at the triangle value.

T/R 3 gave the students one last problem which he said could be called problem ten.

<table>
<thead>
<tr>
<th>□</th>
<th>△</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>31</td>
</tr>
<tr>
<td>6</td>
<td>63</td>
</tr>
<tr>
<td>7</td>
<td>127</td>
</tr>
</tbody>
</table>

He told them that it is a difficult problem and that they would not have time to solve it during the current class session but the students would hopefully be able to solve it in a few weeks’ time.

Brian, Romina, Michelle, and Ankur came up and showed T/R 3 what they had noticed about this last example. Brian explained to T/R 3 that the difference between the triangle values was doubling; between zero and one there was a difference of one, between one and three, there was a difference of two, between three and seven, there was a difference
of four. T/R 3 complimented them on that idea by saying "Oh, ah, that's really clever" (line A594) and suggested that they give it some more thought.

T/R 3 told the class that they had run out of time but he did want to discuss one last thing about secrets. Milin exclaimed that they shouldn't keep secrets. T/R 3 asked the class for their opinion about keeping secret and what bad things could happen as a result of keeping secrets. The students animatedly argued with each other for a moment about the value of keeping secrets. T/R 3 interjected and said "I think we do need to keep thinking about it, we do need to find a way to this so that everybody’s comfortable with it, but there is also a case to be made for keeping secrets because what I’ve said sometimes to people is suppose Michael and I went to the gym and Michael did a lot of weight lifting and I watched him. Who gets stronger? He would. What’s that got to do with secrets?" (line A604). Romina answered that if T/R 3 would want to get strong he would have to do the weight lifting himself. T/R 3 agreed and ended the session by saying "Cause if you want to be good at figuring things out you better practice figuring things out" (line A607).

Table 4.15
Classification of teacher questioning and results, Kenilworth session Part II

<table>
<thead>
<tr>
<th>Line</th>
<th>Speaker</th>
<th>Researcher question</th>
<th>Question type</th>
<th>Question result</th>
</tr>
</thead>
<tbody>
<tr>
<td>A433</td>
<td>T/R 3</td>
<td>See if you can, you helped them get it, can you come and explain it to the camera?</td>
<td>Encouraging engagement - administrative</td>
<td>Brain explained that the code would be “square times square plus one equals triangle.”</td>
</tr>
<tr>
<td>A437</td>
<td>T/R 3</td>
<td>That’s very nice why does that work? Because the rule says</td>
<td>Probing - elucidating</td>
<td>T/R 3 provided an answer to his own</td>
</tr>
</tbody>
</table>
whatever you write in one square you have to write the same in the other square.

<table>
<thead>
<tr>
<th>Page</th>
<th>A445</th>
<th>T/R 3</th>
<th>Yeah, why don’t you say that to the camera?</th>
<th>Bobby responded that he already had the opportunity to say it to the camera.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>aA450</td>
<td>T/R 3</td>
<td>Okay, let me say the people with the secret would like to publish it now, when scientists really discovered something they do what they call publishing, they send it to a journal and it gets printed and everybody reads it. You can erase it. Are you ready for them to publish this is that alright? Okay, could we get it quiet please? So they say they’re going to tell you their discovery now.</td>
<td>Encouraging engagement – encouraging students to listen</td>
</tr>
<tr>
<td></td>
<td>A471</td>
<td>T/R 3</td>
<td>Okay, does everybody understand that?</td>
<td>Encouraging engagement – checking for understanding</td>
</tr>
<tr>
<td></td>
<td>A478</td>
<td>T/R 3</td>
<td>So how would you write that with box and triangle notation?</td>
<td>Probing – encouraging critical thinking</td>
</tr>
<tr>
<td></td>
<td>A490</td>
<td>T/R 3</td>
<td>Can you write that here with the boxes and triangles?</td>
<td>Probing – encouraging critical thinking</td>
</tr>
<tr>
<td></td>
<td>A507</td>
<td>T/R 3</td>
<td>Okay, that’s an interesting idea, Amy-Lynn and Bobby would you come and say your thing to the camera? Why don’t you talk this time?</td>
<td>Encouraging engagement - administrative</td>
</tr>
<tr>
<td></td>
<td>A509</td>
<td>T/R3</td>
<td>Okay, are you going to explain it?</td>
<td>Probing - elucidating</td>
</tr>
</tbody>
</table>
three, three times zero plus four and a half equals four and a half. And then you took four times zero plus eight equals eight. And then five times zero plus twelve and a half equals twelve and a half. Six times zero plus eighteen equals eighteen

<table>
<thead>
<tr>
<th>Time</th>
<th>T/R</th>
<th>Speech</th>
<th>Probing – encouraging critical thinking</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>A511</td>
<td>T/R 3</td>
<td>I think it would be worth thinking about that some more and see if you can find an easier way to deal with all of that. Where do you suppose those halves are coming from, what do you suppose is making those halves in the problem?</td>
<td></td>
<td>Amy-Lynn suggested that the half comes from the whole.</td>
</tr>
<tr>
<td>A526</td>
<td>T/R 3</td>
<td>Now why don’t you take that triangle number and divide it by the box number what do I get? What do I get at that point?</td>
<td></td>
<td>Ankur responded that you get the answer</td>
</tr>
<tr>
<td>A528</td>
<td>T/R 3</td>
<td>What answer?</td>
<td>Probing - elucidating</td>
<td>Brian said that the answer would be two.</td>
</tr>
<tr>
<td>A532</td>
<td>T/R 3</td>
<td>What’s the two?</td>
<td>Probing - elucidating</td>
<td>Brian tried to explain but had difficulty expressing the pattern he saw in written form. T/R 3 encouraged them to try form a valid equation with an equal sign to explain the pattern he had seen. T/R 3 acknowledged that he did see an interesting pattern.</td>
</tr>
<tr>
<td>A551</td>
<td>T/R 3</td>
<td>So we’re particularly looking for formulas like this where you don’t need to know the number in the triangle, all you need is to put the number in the box and that will tell you the number in the triangle. Okay, um, I think we really have come close to being out of time would anybody like to say anything about this?</td>
<td>Eliciting</td>
<td>Bobby offered that he had a secret.</td>
</tr>
<tr>
<td>A553</td>
<td>T/R 3</td>
<td>You’ve got a secret about the whole thing, okay, well we got time to do a couple more things. You want to say it to the camera or to everybody</td>
<td>Encouraging engagement - administrative</td>
<td>Bobby said he had a secret for number eight and he would like to say it in front of the camera.</td>
</tr>
<tr>
<td>A555</td>
<td>T/R 3</td>
<td>Who’s talking?</td>
<td>Encouraging engagement - administrative</td>
<td></td>
</tr>
<tr>
<td>A560</td>
<td>T/R 3</td>
<td>Okay, they have an interesting idea that maybe you’ve all thought of, but it’s worth making sure you know it. Would you explain it to everybody?</td>
<td>Eliciting Promoted reasoning</td>
<td>Bobby explained that the difference between the triangle values increased by two</td>
</tr>
<tr>
<td>A563</td>
<td>T/R 3</td>
<td>O.k., I think that’s worth thinking about, Jeffery did you have something to show?</td>
<td>Eliciting Promoted reasoning</td>
<td>Jeff asked if he should show everyone in the class.</td>
</tr>
<tr>
<td>A565-56 6</td>
<td>T/R 3</td>
<td>It’s up to you are you ready to show everyone? Okay, anybody else coming to help, or are you doing it by yourself?</td>
<td>Encouraging engagement - administrative</td>
<td>Jeff conjectured that when the triangle column begins with two zeroes the equation would be box times box minus a number equals the triangle</td>
</tr>
<tr>
<td>A568</td>
<td>T/R 3</td>
<td>Okay, how many people got this, some people have the formula for this and I think Jeff actually had the formula for this. Does anyone else have the formula for this?</td>
<td>Eliciting Promoted reasoning (faulty)</td>
<td>Brian tried to explain his group’s idea by saying, “Like, divide two into two and you get one, if you multiply two times one plus zero you get two and that’s what you’re supposed to get because two’s in the triangle. And the triangle’s supposed to be at the end. Divide six times three and you get two…. Awesome, and that’s one and then you take the one and the two out of there and the two times one, I have to move down, and then you just plus zero cause that’s the number you have up there to start with and then it equals that two. This number should always be in the triangle at the end.</td>
</tr>
<tr>
<td>A586</td>
<td>T/R 3</td>
<td>Do you want to say it to the camera or do</td>
<td>Eliciting Promoted reasoning (faulty)</td>
<td>Brian responded that he wanted to share his</td>
</tr>
</tbody>
</table>
you want to say it to everybody.

**solution in front of the camera.** Brian explained to T/R 3 that the difference between the triangle values was doubling: between zero and one there was a difference of one, between one and three, there was a difference of two, between three and seven, there was a difference of four.

<table>
<thead>
<tr>
<th>Time</th>
<th>T/R 3</th>
<th>Probing - elucidating</th>
<th>Brian answered “between zero and one is one.”</th>
</tr>
</thead>
<tbody>
<tr>
<td>A590</td>
<td>Well, now I’m not sure. What number am I looking at here?</td>
<td>Probing - elucidating</td>
<td>Brian answered “You add one and one and you get two which is the difference between there.”</td>
</tr>
<tr>
<td>A592</td>
<td>So what am I going to do add that?</td>
<td>Probing - elucidating</td>
<td></td>
</tr>
<tr>
<td>A597</td>
<td>Okay, we really have run out of time, I think, anybody have anything you want to say as our last parting remark before we… parting remark. Anybody want to say anything? What? Could I sit there for a second? Can we talk for one second about keeping secrets?</td>
<td>Eliciting</td>
<td>Milin exclaimed, “we shouldn’t keep secrets.”</td>
</tr>
<tr>
<td>A599</td>
<td>You shouldn’t keep secrets? Well, there are two sides to it? What’s the bad thing about keeping secrets?</td>
<td>Probing – encouraging critical thinking</td>
<td>Jeff offered that other people wouldn’t find out.</td>
</tr>
<tr>
<td>A604</td>
<td>Let me turn it around and argue the other case a little bit. I think we do need to keep thinking about it, we do need to find a way to this so that everybody’s comfortable with it, but there is also a case to be made for keeping secrets because what I’ve said sometimes to people is suppose Michael and I went to the gym and Michael did a lot of...</td>
<td>Eliciting</td>
<td>Romina answered “He’s doing the lifting. If you want to get stronger you got to figure it out” and T/R 3 reiterated that thought by saying “Cause if you want to be good at figuring things out you better practice figuring things out.”</td>
</tr>
</tbody>
</table>
4.3.2.1 Questioning techniques used in the Oct 1, 1993 session at Kenilworth

During the second session on Guess My Rule, T/R 3 used questioning and created a learning environment conducive to critical thinking and justification. He asked Ankur and Michelle, “You suppose there would be a way to write that. Can you think of a way to write that? The number that goes in the box is also the number that is next to it. How could you write that? That’s a really neat idea, that’s a really neat idea” (line A277). Ankur and Michelle continued to work on the problem and later tried to justify their solution once again but still without having written a formula. T/R 2 once again provided the opportunity for critical thinking by asking them, “Okay, now you certainly managed to write it, could you write it using the box and triangle method of writing?” (line A313). After spending some more time on that task, they again approached T/R 3 with an explanation but had not yet come up with a formal equation. T/R 3 once again encouraged them to think of a way to write their ideas in equation form by asking, “Yeah, that’s certainly right, okay, um, can you think of a way to write it so that we’ll know that the number that goes here has to go here too” (line A331). Ankur and Michelle were finally successful in coming up with the equation $(\square \times \square) + 1 = \Delta$.

T/R 3 repeatedly asked students to come share their solutions in front of the camera or by addressing the whole class. When he felt that one of the students was consistently speaking on behalf of his/her partner, he tried to encourage the quieter of the two to explain his/her work as well. T/R 3 also used questioning to elucidate students’
representations and solutions. This helped students clarify their ideas for themselves and perfect their solutions and equations. When Amy-Lynn and Bobby showed T/R 3 their work he said “I think it would be worth thinking about that some more and see if you can find an easier way to deal with all of that. Where do you suppose those halves are coming from, what do you suppose is making those halves in the problem?” When Amy-Lynn could not provide a satisfactory answer, T/R 3 encouraged them to continue thinking about the problem. When Brian and his group also had difficulty with one of the problems, T/R 3 made sure to emphasize that they are “particularly looking for formulas like this where you don’t need to know the number in the triangle, all you need is to put the number in the box and that will tell you the number in the triangle” (A551). T/R 3 ended off the session by asking the students to talk about keeping secrets. When Milin expressed his dislike of secrets, T/R 3 led a discussion about the importance of occasionally keeping secrets and provided an apt analogy. He said “suppose Michael and I went to the gym and Michael did a lot of weight lifting and I watched him. Who gets stronger? He would. What’s that got to do with secrets?” This allowed the students to think for themselves how the idea of someone watching another lift weights was analogous and relevant to the idea of secrets. Figures 4.35 and 4.36 show the question types and associated student reasoning that occurred during this session.
**Figure 4.35.** Question types used by T/R 3 during the Kenilworth sessions

**Figure 4.36.** Question results during the Kenilworth sessions
4.3.2.2 Analytics for the sessions at Kenilworth

The analytic entitled “Using Questioning to Promote Conceptual Understanding: Robert B. Davis Introduces Algebra Ideas to Sixth Graders” (Grey, 2015) portrays the various questions asked by T/R 3 during the Kenilworth sessions on Guess My Rule to clarify and encourage students to refine their ideas and think more deeply about the tasks. In addition, “Student Perseverance in Discovering Patterns: Guess My Rule with Robert B. Davis” (Makovec, 2015) depicts the students working on the Guess My Rule tasks. This analytic features pairs of students showing T/R 3 their work. It also contains a few events showing Michelle and Ankur as they try to find the equation for the particularly challenging function table in Problem 6, which is the first function that is not linear. T/R 3 uses questions in this analytic to provide the opportunity for students to clarify and understand the students’ work and to encourage them to try to find the correct equation.

“Early algebra learning and the interplay between Researcher Robert B. Davis’ questioning and 6th graders' reasoning” (Gerstein, 2017e) portrays the researcher questions and associated student reasoning during the Guess My Rule session.

4.4 Analysis of teacher questioning by Researcher Arthur Powell

In this section I will examine the various questions asked by Researcher Arthur Powell (identified as T/R 4) as he worked with students in the Informal Math Learning Program (IML) on Guess My Rule problems. The students worked on the Guess My Rule activity on November 2 and November 3, 1993.
4.4.1 Informal Math Learning Program (IML), Nov 2, 2005, with Researcher Arthur Powell in Plainfield, New Jersey

T/R 4 started off the session saying "We're going to play a game. And the game we're going to play is called…" (line 1). Ariel interjected and said Guess My Rule. Brandon said that someone had told them that they would be playing that game. T/R 4 explained that in order to play the game, he would think of some rule and Dawud interjected and said that they would have to try and guess it. T/R 4 said that that was correct but he asked "do you know how you're going to try and guess the rule?" (line 11). Brandon answered that they would be working with numbers and T/R 4 elaborated and said that he would be giving them a number and he would tell them what the rule does to the number. He explained that he wouldn't be telling them the rule but he would be giving them the result. T/R 4 said that since there were five students present he would choose the number five. He said that his rule did something to the number five and resulted in thirteen. Ariel and Dawud both said that T/R 4 just added eight to get from five to thirteen. T/R 4 said "Don't say what you think the rule is" (line 22). Dawud asked how they were supposed to know. T/R 4 reiterated that the first number was five and asked what the result was and Brandon said thirteen. T/R 4 said that now it was the students turn to choose a number and he would tell them what the rule will do to that number. He cautioned them that he did not want them to guess the rule yet. Dawud chose the number three. T/R 4 asked the students what they thought the rule would do to three. James answered that it would "make it into eleven" (line 31). T/R 4 said that the rule would not make eleven, it would take the number three and make it seven. T/R 4 asked what results they had gotten so far and Brandon said five and thirteen. T/R 4 wrote it for them under
the columns square and triangle. Under the square column he wrote five and three and under the triangle thirteen and seven. T/R 4 asked them to give him another number. Dawud offered six. T/R 4 asked them what they think the rule would do to six. Dawud conjectured that six would go to twenty four. T/R 4 asked the students if they all agreed and Brandon said he thought it would go to ten. Yonny offered fifteen and Ariel said "It depends on the number, that's how much you add on (line 61). Yonny suggested eight and Brandon said he thought it would be ten. T/R 4 asked the students if they were ready for the number and he wrote down 16. Ariel tried to explain by saying "Look, cause 5 was at an eight, then you added two more and for like number 3 it was four. And if it was four, you would add a six. And for five you would add eight" (line 67). T/R 4 tried to understand what Ariel was saying and asked holding up the chart for the class to see "Ariel thinks that the rule is, any number that I put here, under this column [the square column] is that right? He says that any number I put in this column, you do what, Ariel?" (line 68). Ariel explained that the rule was that one needed to add two to what was done in the previous number. He explained that by five they added eight and for six they added ten, and by three they added four. T/R 4 suggested that they try another number and asked James to choose a number. James chose eight. T/R 4 asked what they thought the rule would do to eight. Brandon answered eighteen and T/R 4 suggested that they think carefully and look at all the numbers. He walked around showing the students the paper. Ariel called out "Oh, I got it, I know I got it... It’s twenty-two!" Brandon disagreed and once again repeated his solution of eighteen. T/R 4 asked Ariel how he got twenty-two. James answered "Add two" (line 84). T/R 4 said he would write the numbers and drew a square and a triangle on top of the two columns. Ariel said "I know, like, a way to
represent it too. Like, the square is the numbers going in, like, the triangle, it goes into like a say a factory and it comes out the triangle number” (line 89). T/R 4 asked everyone to listen to what Ariel had said. Ariel repeated his explanation. The students argued for a short while each offering different numbers. Ariel exclaimed "Look at the five, look at the five. It’s twenty-two!" (line 113). T/R 4 wrote the number twenty two under the triangle heading next to the eight. T/R 4 asked James to choose a "box number" (line 121). Other students whispered to him to choose four so he chose four. T/R 4 wrote four down on the overhead chart under the square column]. Dawud said the answer would be twenty four, Brandon thought it would be eight, and Ariel said it would be ten. Ariel explained his solution saying: "Cause, you’re going to add six. Cause, for five you added um… Cause for the five you had thirteen, and then the six you got eighteen, you just added um ‘what’s your name’ to this one and ‘what’s your name’ to that one” (line 133 and 137). T/R 4 wrote 10 on the overhead in the triangle column, corresponding to the four entry in the square column. Yonny said the answer was ten and everyone shouted out ten. T/R 4 then told Ariel not to say the answer the next time but to wait until everyone else got it. He then wrote a zero under the square column. Brandon and Dawud told T/R 4 to write zero in the corresponding triangle column as well. Yonny disagreed and said to put a two. T/R 4 wrote a negative two on the overhead. T/R 4 asked the students if they had guessed his rule yet. Ariel said he did but then said "No, I didn't" (line 161). He continued "Zero done messed up my whole thing. Look, this is what I thought: So, for four you added six, for five you added eight, for six you added ten, I mean…. twelve. Wait, yeah yeah yeah" (line 164). T/R 4 suggested that they try to do another problem.
T/R 4 allowed Yonny to write the number one on the overhead under the square column and a 15 under the corresponding triangle column. Brandon said he thought the rule was "By fifteen (line 177).” Ariel suggested that they put a two into the table and Brandon reiterated that request saying "Do it in a row" (line 183). Yonny complied and put a 2 under the square column and a 25 in the triangle column. Brandon asked him to continue by putting a three. T/R 4 said, "Okay, hold on. Now before you write down 3, what do you think it’s going to be?" (line 189). Yonny put a three in the square column and 35 in the triangle column. Brandon suggested that it was going by tens but then corrected himself that it could not be by tens because they started off with 15. Ariel suggested that Yonny was just putting a 5 at the end of each number. The students continue to shout out suggestions. T/R 4 asked them to think of the rule that would produce a result for four. Christian said he knew the rule and said, "The rule is, like, basically you got the same thing. You get, you doing the same numbers, like: 1, 1 2, 2 3,3 and you just adding five to the same numbers. You put forty, I think the next number you going to put is forty-five" (line 209). Yonny said that forty five was the correct result for four. T/R 4 said "The rule is you’re using some operation to get the number. Okay? You got to think, what is the operation that he is using to get the number" (line 214). Brandon and Christian call out "Times ten" multiple times. T/R 4 asks them if that works. Yonny hinted to them that times ten was only half of the operation. Brandon changed the rule to times ten divided by five and Christian echoed that answer. T/R 4 suggested that they try that suggestion and see if it works. Christian corrected himself and said that it would be times ten plus five. T/R 4 asked Yonny if Christian had found the rule and asked Yonny to state the rule so that everyone could hear. Yonny repeated the rule for everyone. T/R 4
asked Christian to make a rule and then told the class that Chris would give them one number and show them the result. Christian put in ten under the square column and a twenty in the triangle column. Christian put two under the first column. T/R 4 asked the students what they thought the answer would be. Brandon offered four as the solution. Christian wrote a four under the triangle column. All the students shouted out various answers. Christian changed the answer on the overhead to twelve. Brandon said the answer was plus ten, which was the correct solution. T/R 4 concluded that this was the last question they would be doing together as a class. Next they would work in pairs on some more Guess My Rule problems.

Table 4.16

Classification of teacher questioning and results, IML Part I

<table>
<thead>
<tr>
<th>Line</th>
<th>Speaker</th>
<th>Researcher question</th>
<th>Question type</th>
<th>Question result</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>T/R 4</td>
<td>You know the game? Guess my rule?</td>
<td>Encouraging</td>
<td>Brandon said they were told they would be playing that game</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>engagement</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>T/R 4</td>
<td>He told you that you were going to play the game or that you had worked on that game already?</td>
<td>Probing</td>
<td>Brandon said that he had told them.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>elucidating</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>T/R 4</td>
<td>He told you which?</td>
<td>Probing</td>
<td>Brandon responded that they had been told they would be playing Guess My Rule.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>elucidating</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>T/R 4</td>
<td>You’re going to try and guess it, but do you know how you’re going to try and guess the rule?</td>
<td>Encouraging</td>
<td>Brandon offered that they would be working with numbers</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>engagement</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>checking for understanding</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>T/R 4</td>
<td>Okay, ready for it? My rule takes the number...</td>
<td>Encouraging</td>
<td>Brandon exclaimed that he shouldn’t tell them the rule.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>engagement</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>administrative</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>T/R 4</td>
<td>You ready... Okay, here’s my rule, you ready for it?...</td>
<td>Encouraging</td>
<td>Brandon answered five.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>engagement</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>administrative</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>T/R 4</td>
<td>So, the first number was what?</td>
<td>Encouraging</td>
<td>Brandon said thirteen</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>engagement</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>checking for understanding</td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>T/R 4</td>
<td>And it gave you?</td>
<td>Encouraging</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>engagement</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>checking for understanding</td>
<td></td>
</tr>
<tr>
<td>Time</td>
<td>T/R 4</td>
<td>Statement</td>
<td>Type</td>
<td>Notes</td>
</tr>
<tr>
<td>------</td>
<td>-------</td>
<td>-----------</td>
<td>------</td>
<td>-------</td>
</tr>
<tr>
<td>30</td>
<td>T/R 4</td>
<td>So, Dawud said three, and my number... my, what do you think my number is going to do to three?</td>
<td>Eliciting</td>
<td>James conjectured that it would make it eleven.</td>
</tr>
<tr>
<td>32</td>
<td>T/R 4</td>
<td>You think it’s going to make it into eleven? Nope, my rule takes the number three and makes it seven.</td>
<td>Encouraging engagement - Administrative</td>
<td>Ariel asked how they got thirteen from five.</td>
</tr>
<tr>
<td>42</td>
<td>T/R 4</td>
<td>Okay, Dawud said... What number did Dawud just say?</td>
<td>Probing - elucidating</td>
<td>Dawud repeated six. Yonny offered a number as well.</td>
</tr>
<tr>
<td>46</td>
<td>T/R 4</td>
<td>What do you think my rule is going to do to six?</td>
<td>Eliciting</td>
<td>Brandon said “I know!”</td>
</tr>
<tr>
<td>49</td>
<td>T/R 4</td>
<td>Tell me what you think, if we have six, what do you think is going to happen to six?</td>
<td>Eliciting</td>
<td>Dawud said that six would become twenty-four</td>
</tr>
<tr>
<td>51</td>
<td>T/R 4</td>
<td>Does everyone agree?</td>
<td>Encouraging engagement - confirming agreement</td>
<td>Brandon disagreed and conjectured that it would become ten.</td>
</tr>
<tr>
<td>53</td>
<td>T/R 4</td>
<td>You think six is going to go to ten?</td>
<td>Probing - elucidating</td>
<td>Dawud defended his answer by saying that they “we made five with eight, we made seven with four. So six, it will be twenty-four.”</td>
</tr>
<tr>
<td>56</td>
<td>T/R 4</td>
<td>Does anyone else have a guess?</td>
<td>Encouraging engagement - shifting questioning to another student</td>
<td>Brandon answered ten.</td>
</tr>
<tr>
<td>59</td>
<td>T/R 4</td>
<td>Hold on, he says it’s going to be more than thirteen? Why does he think that?</td>
<td>Probing - elucidating</td>
<td>Yonny said the answer is fifteen and Ariel tried to answer the researchers question by saying “it depends on the number, that’s how much you add on.” Yonny added that he believed the answer is eight.</td>
</tr>
<tr>
<td>64</td>
<td>T/R 4</td>
<td>Alright, is everybody ready? I’m going to write down the number. [Writes down sixteen]</td>
<td>Encouraging engagement - administrative</td>
<td>Ariel exclaims that he knew it would be ten and explained the pattern that it (the y) was increasing by two.</td>
</tr>
<tr>
<td>68</td>
<td>T/R 4</td>
<td>Ariel thinks that the rule is, any number that I put here [holds up the chart to the class], under this column [the square column] is that right? He says that any number I put in this column, you do what, Ariel?</td>
<td>Probing - elucidating</td>
<td>Ariel explained his reasoning once more that he believed it increased by two in the triangle column.</td>
</tr>
<tr>
<td>70</td>
<td>T/R 4</td>
<td>Yea? Let’s try one other number. Let’s try one other number, James, would you like to pick a number to try?</td>
<td>Encouraging engagement - confirming agreement, Eliciting</td>
<td>James chose eight.</td>
</tr>
<tr>
<td>72</td>
<td>T/R 4</td>
<td>Alright, James says eight. What number do you think my rule is going to give back for eight?</td>
<td>Eliciting</td>
<td>Brandon offered eighteen and Ariel said twenty-two.</td>
</tr>
</tbody>
</table>
4.4.1.1 Questioning techniques used in the November 2, 1993 session at Plainfield

T/R 4 used a variety of questions to engage the students and to encourage them to listen to their peers and think critically. When a student offered a solution, T/R 4 asked “Why does he think that?” (line 59) to encourage the other students to attend to the reasoning of another student. He punctuated the sessions with questions such as “Does everyone agree?” to allow for greater participation from the students. When Ariel seemed to have come up with a correct solution, T/R 4 encouraged the students to listen to him. Ariel also offered an apt analogy for the idea of function and T/R 4 used questioning to
ensure that the other students had paid attention to his explanation. He also encouraged the students to think about how Ariel had come up with his solution by questioning the students “So how’s Ariel doing this?” (line 141). Figures 4.37 and 4.38 show the question types and associated student reasoning that occurred during this session.

**Figure 4.37.** Question types used by T/R 4 during the November 2 IML session
4.4.1.2 Analytics for the November 2, 1993 session at Plainfield

Pierce’s (2014) analytic, “Ariel Constructing Linear Equations for Guess My Rule and the Ladder Problem” contains events depicting T/R 4 and the IML students as T/R 4 introduces Guess My Rule before the students break up to work in pairs on the tasks. This analytic clearly shows the researcher questioning and subsequent student responses as they begin to learn about the idea of function.

“Researcher Powell Introduces Functions in an Informal Math Learning Environment: The Interplay Between Teacher Questioning and Student Reasoning” (Gerstein, 2017c) focuses in part on the questions posed by Researcher Powell as he introduced Guess My Rule and the associated student justification, and reasoning.

**4.4.2 Informal Math Learning Program (IML), Nov 3, 2005, with Researcher Arthur Powell in Plainfield, New Jersey**

During the second session on Guess My Rule the students were given a number of problems. Brandon worked with Yonny on problem 1.

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>
Looking at problem 1 on the worksheet, Yonny said "Plus one, plus two, plus three, plus four, see I got it. I am too smart" (line 769). Yonny noticed a pattern and created a rule. He wrote on the side of the Guess My Rule Problem 1 sheet: "The rule is that when you add you add by one more." Yonny realized that the value added to the X value to produce the corresponding Y value increased by one. Brandon wrote on the side of his worksheet: The Rule is +1+2+3 and it keeps going to six. T/R 4 asked Brandon and Yonny what they had accomplished so far and questioned how they came up with the rule. Brandon explained that you have to do "Plus one, plus two, plus three, plus four, plus five, plus six, plus seven…" (line 783). T/R 4 asked them what they would get if he gave them the number six. Brandon answered that it would be thirteen. T/R 4 asked him to elaborate and Brandon explained, "Look, OK, look, it’s one, zero plus one equals one. Zero plus, I mean one plus two equals three, three plus two equals five, three plus four equals seven, four plus five equals nine, and five plus six equals eleven and six plus seven equals, what you got here, equals thirteen" (line 796). Next T/R 4 asked Brandon and Yonny what the value of Y would be if X = 20. Yonny extended the chart for problem 1 adding 6-10 under the x column.

He put in the following:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>11</td>
</tr>
</tbody>
</table>

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>13</td>
</tr>
<tr>
<td>7</td>
<td>15</td>
</tr>
<tr>
<td>8</td>
<td>17</td>
</tr>
</tbody>
</table>
When Yonny got to X = 10 on the table he told Brandon that it "just doubles by two" (line 809). T/R 4 asked what he was referring to when he said that it doubled by two. T/R 4 asked him to clarify and Yonny explained his thinking by saying "Like one plus two plus three plus yeah. Like one plus two [equals] three, plus two [equals] five" (line 831). Since the y was going from one to three to five Yonny considered that to be doubling by two. T/R 4 asked "So you call that doubling by two?" (line 832) and Yonny answered in the affirmative. Brandon and Yonny continued to work on the problem. Brandon said "And twenty would be forty-one. Forty-One. Forty-One. Forty-One. The total would be forty-one" (line 835). Yonny said "He got it" (line 132) and did not continue extending the table. T/R 4 asked him how he came up with the number and Brandon explained that "On this side, on this side the number goes up by two. So I skipped by two all the way to twenty" (line 838). R/4 told them that he would like to give them another problem to work on but before he did that he wanted to ask them "What would Y be if X is one hundred?" (line 845), to which Yonny replied, "Oh you can't make us do that…" (line 847). T/R 4 asked them if there was another way to come up with the solution without having to go through all the numbers until 100. Yonny at first answered no but soon thereafter changed his mind and provided an answer. He said "Well, I think it could be like forty-one times five" (line 855) He explained his solution using multiplicative reasoning, saying, "Forty-one is twenty and twenty is a factor of a hundred. So, it multiplies by five to get on it. So, I just multiplied forty-one by five" (lines 857). T/R 4 asked them to think about the problem and that he would return
shortly. When he returned a short while later, he asked them if they had come up with a solution. Yonny said the answer is two hundred and five. Brandon explained with reasoning similar to what Yonny had said before: "Forty one times five, because twenty is a factor of one hundred. Twenty forty-one, so forty-one times five, cause twenty times five equals a hundred, so we just took the five, from the twenty, so we took the forty one and multiplied it by five" (line 888).

Table 4.17

Classification of teacher questioning and results, IML Part II

<table>
<thead>
<tr>
<th>Line</th>
<th>Speaker</th>
<th>Researcher question</th>
<th>Question type</th>
<th>Question result</th>
<th>Student conjecture, response, reasoning, or justification</th>
</tr>
</thead>
</table>
| 760  | T/R 4   | You think you got it? | Eliciting     | Promoted reasoning | Brandon jokingly says yes, then no, then yes. Yonny explained that he found the pattern by saying “plus one, plus two, plus three, plus four…” [Brandon writes on the side of the Guess My Rule Problem 1 sheet: The Rule is +1+2+3 and it keeps going to six.]
| 778  | T/R 4   | So would you guys tell me what you have so far? | Eliciting     |               | Brandon said they finished.
| 782  | T/R 4   | Show me. How did you come up with the rule? | Probing – elucidating |               | Brandon explained that they had found the pattern “plus one, plus two…”
| 787  | T/R 4   | Suppose if I gave you the number six, what would it be? | Probing – encouraging critical thinking |               | Brandon answered that the result would be thirteen
| 789  | T/R 4   | How did you get that? | Probing - elucidating |               | Brandon said that you would have to add six plus seven
| 795  | T/R 4   | Six plus seven? | Probing – elucidating | Promoted reasoning and justification | Brandon explained that by adding one to the zero in the x column, it equaled one. If one added two to the one in the x column it equaled three. Adding three in the next row would equal five, adding four to five would equal nine, five and six equals eleven and six plus seven equals
| T/R 4 | Probing – encouraging critical thinking | Yonny responded that he was not sure and Brandon started to do some calculations on his paper.

800 | Eliciting | Yonny at first said that twenty was too big of a number to work with and that Brandon will work on the problem but then says that he will work on it together with Brandon. Yonny extended his chart until x= 10. Before filling in the y column he said it "just doubles by two."

804 | Probing – encouraging critical thinking | Brandon explained that they increased the y column by skipping by two “all the way to twenty.”

826 | Encouraging engagement - administrative | Yonny at first said that twenty was too big of a number to work with and that Brandon will work on the problem but then says that he will work on it together with Brandon. Yonny extended his chart until x= 10. Before filling in the y column he said it "just doubles by two."

828 | Probing - elucidating | Brandon explained that they increased the y column by skipping by two “all the way to twenty.”

830 | Probing - elucidating | Yonny explained that the in the y column went from one to three to five it was doubling by two.

832 | Probing - elucidating | Yonny answered in the affirmative. They continued working on the problem until they determined that the answer to 20 would be 41.

837 | Probing – elucidating | Brandon explained that they increased the y column by skipping by two “all the way to twenty.”

845 | Probing – encouraging critical thinking | Yonny exclaimed that he can’t make them work the problem all the way to 100.

849 | Probing – encouraging critical thinking | Brandon said they could multiply 100 by two.

851 | Probing – encouraging critical thinking | Yonny answered no.

853 | Probing – encouraging critical thinking | Brandon responded that he would have to work on it all day. Yonny said that he believed the solution would be forty-one times five.

859 | Probing – encouraging critical thinking | Brandon responded that he would have to work on it all day. Yonny said that he believed the solution would be forty-one times five.

872 | Eliciting | Brandon answered that they did not come up with an answer.

881 | Encouraging engagement - administrative | Yonny answered no.
Brandon responded that the answer would be "Forty one times five, because twenty is a factor of one hundred." He then explained that since twenty times five equals 100 they could multiply forty-one by five.

T/R 4 told Yonny and Brandon that he would like to give them another problem.

He handed them a worksheet with Problem 2 for Guess My Rule. The problem stated:

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
</tr>
</tbody>
</table>

Yonny quickly responds "I got the rule already" (line 901). When T/R 4 came over to question them about their solution, Brandon explained to him that "Alright the rule is, it's going up by one on the X side and it's going up by two on the Y side. Easy" (line 911). T/R 4 asked him how he would determine a number for Y if he gave him a number for X, for example if X were seven. Brandon answered that the y would be seventeen. T/R 4 asked him if he was sure and Brandon replied that he was certain that the answer was seventeen. T/R 4 asked them if they remembered what he had asked them by the previous problem. Brandon said to find it for one hundred and T/R 4 said that they should first try twenty. Brandon asked Yonny for assistance in writing it out and T/R 4 asked him if there was a better way to find the rule rather than writing it all out. He suggested
that they look at the chart carefully to determine whether they could come up with the rule. Brandon started to work on the problem. He wrote:

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td>7</td>
<td>17</td>
</tr>
<tr>
<td>8</td>
<td>29</td>
</tr>
<tr>
<td>9</td>
<td>21</td>
</tr>
</tbody>
</table>

Then he crossed out what he had written after x= 5 and y= 15 and rewrote the chart as follows:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>17</td>
</tr>
<tr>
<td>7</td>
<td>19</td>
</tr>
</tbody>
</table>

Brandon had already written, "The rule is the numbers are going up by 1 on the x axis and by two on the y axis.” Brandon once again asked Yonny to assist him in filling in his table. Yonny asked him if he found the answer yet and Brandon said he came up with forty five. T/R 1 asked them how they came up with forty five. Brandon explained by showing T/R 4 what they had done in Problem 1 for Guess My Rule:

Because, OK, look here. No here, let me tell you. Let me tell you. Look here, ten, ten, look in the other ones we did ten, but we did all the way it but I found out that if you add ten by ten would equal twenty right. And then you would do the twenty times two that would equal, I mean the ten times two, this ten, I mean like this ten by two, would equal to the twenty, and then the twenty by
the answer of ten, twenty five, would give you forty five, because it worked in this one. So I thought it would work in this one too.

(line 964)

T/R 4 asked him how it would work in the current exercise. Brandon pointed to the problem 1 table and said “ten plus ten equals twenty, alright. That’s how you get this one [points to X = 20 in the Guess My Rule Problem 1 table] the X-axis. So the X-axis, then twenty plus twenty one equals. That’s how I got it” (line 968). Then he writes the following on the problem 1 worksheet:

\begin{align*}
10 \\
+10 \\
20 & \text{ X-axis} \\
+21 \\
41 & \text{ Y-axis}
\end{align*}

T/R 4 asked him what he would get for the second problem. Brandon wrote the following on the problem 2 worksheet.

\begin{align*}
10 \\
+10 \\
20 \\
+25 \\
45
\end{align*}

T/R 4 asked him where he got the 25. Brandon explained that he got it from "the answer of ten. From the Y-axis with that equals up, I mean, with the one that matches up with the ten" (line 985). T/R 4 asked them to try and see if they would come up with the same answer by doing it by their initial method of putting down all the numbers in their table up to x = twenty. Brandon started to write the following:

\begin{align*}
6 & \quad 17 & \quad 41 \\
19 & \quad 43
\end{align*}
Brandon said "Oh, forty-one, forty-one. See it’s the same answer as I told you. Yeah, we smart. Not dumb" (line 1002) Brandon goes to T/R 4 and tells him that they have come up with the same results.

Table 4.18

*Classification of teacher questioning and results, IML Part III*

<table>
<thead>
<tr>
<th>Line</th>
<th>Speaker</th>
<th>Researcher question</th>
<th>Question type</th>
<th>Question result</th>
<th>Student conjecture, response, reasoning, or justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>908</td>
<td>T/R 4</td>
<td>How’d you guys do?</td>
<td>Eliciting</td>
<td>Brandon answered that they completed the problem.</td>
<td></td>
</tr>
<tr>
<td>910</td>
<td>T/R 4</td>
<td>Oh yeah, what did you guys come up with?</td>
<td>Eliciting</td>
<td>Promoted reasoning</td>
<td>Brandon explained that in the y column the numbers were increasing by one and in the y column they increased by two.</td>
</tr>
<tr>
<td>912</td>
<td>T/R 4</td>
<td>OK, and so if I give you a number for X. How are you going to tell me what number Y is going to be? For example, if X is seven.</td>
<td>Probing – encouraging critical thinking</td>
<td>Promoted reasoning</td>
<td>Brandon answered that seven (in the x column) would be seventeen (in the y column).</td>
</tr>
<tr>
<td>916</td>
<td>T/R 4</td>
<td>You sure?</td>
<td></td>
<td></td>
<td>Brandon said he would check again and then said he was sure of his solution.</td>
</tr>
<tr>
<td>920</td>
<td>T/R 4</td>
<td>Now, remember the question I asked you about the last rule you</td>
<td>Factual - recall</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td></td>
<td></td>
</tr>
<tr>
<td>922</td>
<td>T/R 4</td>
<td>What happens…</td>
<td>Brandon interjected that T/R 4 would like to them to come up with the solution to 100.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>924</td>
<td>T/R 4</td>
<td>Well before we get to a hundred, what about twenty?</td>
<td>Brandon asked Yonny for assistance in writing it out.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>927</td>
<td>T/R 4</td>
<td>You got to write it all out, huh?</td>
<td>Brandon answered in the affirmative.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>929</td>
<td>T/R 4</td>
<td>I wonder if you can find a different way of getting your rule, so that you don’t have to write it all out.</td>
<td>Brandon begins to extend the table. He decides that the answer is forty five.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>930</td>
<td>T/R 4</td>
<td>Why don’t you take a look at your numbers in the table, and see whether or not you come up with another of getting it? OK?</td>
<td>Yonny began to respond but Brandon interjected and explained that he added ten and ten to get twenty. Then he added twenty five which equaled forty five.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>962</td>
<td>T/R 4</td>
<td>OK, so you say it’s forty-five, how did you come up with it?</td>
<td>Brandon explained that in the first problem he had added ten and ten to get twenty and then added twenty one. He justified his answer by writing the following on his paper:</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>10 +10 20 X-axis +21 41 Y-axis</td>
<td></td>
<td></td>
</tr>
<tr>
<td>965</td>
<td>T/R 4</td>
<td>So it worked here, show me how it worked here?</td>
<td>Brandon explained that in the first problem he had added ten and ten to get twenty and then added twenty one. He justified his answer by writing the following on his paper:</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>10 +10 20 +25 45</td>
<td></td>
<td></td>
</tr>
<tr>
<td>976</td>
<td>T/R 4</td>
<td>Hmm…so for the other guess my rule problem what did you think you got that?</td>
<td>Brandon repeats his answer for problem two and writes on his paper:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>984</td>
<td>T/R 4</td>
<td>Hmm…and where did the twenty five come from?</td>
<td>Brandon explained that he got the twenty five from number in the y column that corresponded with x=10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>986</td>
<td>T/R 4</td>
<td>Umm, suppose we do it your other way and figure</td>
<td>Brandon asked if they should “go all the way”</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
out what twenty would be? administrative

up." He then extended his chart until y=43 and then went to tell T/R 4 that he had indeed come up with the same solution.

T/R 4 gives them a third example to work on.

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
</tr>
</tbody>
</table>

Brandon looked at the sheet and says "Ahh, this is easy! By one on the X axis, by three on the Y axis. Uh, I got skills" (line 1017). He turned and shouted to T/R 4 that he had finished. T/R 4 said that he would ask them again to find the result if x = 20. Brandon extended the problem 3 table as follows.

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
</tr>
<tr>
<td>6</td>
<td>19</td>
</tr>
<tr>
<td>7</td>
<td>22</td>
</tr>
<tr>
<td>8</td>
<td>25</td>
</tr>
<tr>
<td>9</td>
<td>28</td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

Yonny said that the next number was going to be twenty nine. Brandon corrected him and said "It’s not going to be twenty-nine. It’s got to go up by three" (line 1038).

Brandon continued to extend his table:

```
10  31
```

Brandon wrote on the side of the paper:

```
10  +10
20  +31
51
"Ten plus ten is twenty, plus thirty-one is fifty-one. Finished, and we got twenty. I wonder if they noticed" (line 1047) He stood up and went over to T/R 4. T/R 4 asked him how he had come up with the result. Brandon explained:
So far what we think, that we know is that, alright, on the X-axis it goes up by one. On the Y-axis it goes up by three. And then we, to get twenty, we take the ten and we did what we did the last time, we did ten plus ten equals twenty plus the number of ten, that matches up by ten, and then we add by thirty one, and so it was twenty plus thirty one which equals fifty one. And we got it, so um the number that twenty matches up with is…

T/R 4 said that he didn’t know if he believed that the answer was fifty one. Yonny offered that the answer was really sixty one. T/R 4 asked Yonny to clarify for Brandon why he believed the answer to be sixty one. Yonny explained that he thought that it should be double thirty one and therefore should be sixty one. He said “Because in the previous ones, it said like twenty is divisible by four, it went up to forty so the answer was kind of like the answer was kind of like double, but now the answer came up to thirty one, so I think it should be double of that, to get sixty-one” (line 1056).

T/R 4 suggested that they try to figure out whether the result for twenty would be fifty one or sixty-one. Brandon extended his table as follows:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>28</td>
</tr>
<tr>
<td>10</td>
<td>31</td>
</tr>
<tr>
<td>11</td>
<td>34</td>
</tr>
<tr>
<td>12</td>
<td>37</td>
</tr>
<tr>
<td>13</td>
<td>40</td>
</tr>
<tr>
<td>14</td>
<td>43</td>
</tr>
<tr>
<td>15</td>
<td>46</td>
</tr>
<tr>
<td>16</td>
<td>49</td>
</tr>
<tr>
<td>17</td>
<td>52</td>
</tr>
<tr>
<td>18</td>
<td>55</td>
</tr>
<tr>
<td>19</td>
<td>58</td>
</tr>
<tr>
<td>20</td>
<td>61</td>
</tr>
</tbody>
</table>
Brandon realized that the result should have been sixty one rather than fifty one and went over to tell T/R 4.

T/R 4 asked everyone to write their names and the date on their worksheets and said they would continue the next day by discussing the work that they had done that day. Another researcher collected their work and with that the session came to a close.

Table 4.19
Classification of teacher questioning and results, IML Part IV

<table>
<thead>
<tr>
<th>Line</th>
<th>Speaker</th>
<th>Researcher question</th>
<th>Question type</th>
<th>Question result</th>
<th>Student conjecture, response, reasoning, or justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1020</td>
<td>T/R 4</td>
<td>Now, I’m going to ask you the same question for twenty.</td>
<td>Probing – encouraging critical thinking</td>
<td>Promotes reasoning</td>
<td>Brandon complains about having to solve for twenty but then works on coming up with the answer. He extended the table until x=20, y = 31. He then wrote: 10 +10 20 +31 51</td>
</tr>
<tr>
<td>1049</td>
<td>T/R 4</td>
<td>OK, so show me how did you come up with it?</td>
<td>Probing - elucidating</td>
<td>Promoted reasoning and justification</td>
<td>Brandon explained that he used the same method that he had employed with the first two problems. He added ten and ten and then added thirty one to get fifty one.</td>
</tr>
<tr>
<td>1051</td>
<td>T/R 4</td>
<td>Fifty-one? I don’t know whether I believe that…</td>
<td>Probing – elucidating</td>
<td>Promoted argumentation</td>
<td>Yonny said he thought the answer was sixty one.</td>
</tr>
<tr>
<td>1055</td>
<td>T/R 4</td>
<td>Why do you think it’s sixty-one?</td>
<td>Probing - elucidating</td>
<td>Promoted reasoning and justification</td>
<td>Yonny explained that he thought that it should be double thirty one since in the</td>
</tr>
</tbody>
</table>
4.4.2.1 Questioning techniques used in the November 3, 1993 session at Plainfield

In this session, T/R 4 questioned Brandon and Yonny as they worked on some Guess My Rule problems. He used probing questions to provide the opportunity for them to explain and clarify their work. His elucidating questions included questions such as “How did you get that?” (line 789) and “help me understand what happens here?” (line 830). He also encouraged them to think more deeply about the tasks and to encourage them to come up with a formula by asking them “Umm, and suppose I give you twenty? What do you think it will be?” (line 797) and “What would Y be if X is one hundred?” (line 845). After presenting the latter challenging question he asked them “So, is there another way to think about this rule?” (line 851) to once again encourage them to come up with a formula instead of merely extending the table to answer his questions. He then left them for a few minutes to give them time to think about the question. When he returned, he continued to question them, thereby encouraging them to think critically about their work to come up with a formula. He asked them “You got to write it all out,
huh? I wonder if you can find a different way of getting your rule, so that you don’t have to write it all out.” (line 927 and 929). He again gave them some time to think about the problem by saying “Why don’t you take a look at your numbers in the table, and see whether or not you come up with another way of getting it? OK?” (line 930). Upon his return, he used a variety of probing question to encourage the students to explain their thinking and clarify how they had arrived at their solutions. Figures 4.39 and 4.40 show the question types and associated student reasoning that occurred during this session.

Figure 4.39. Question types used by T/R 4 during the November 3 IML session
Figure 4.40. Question results during the November 3 IML session

4.4.2.2 Analytics for the November 3, 1993 session at Plainfield

“Learning about functions in the Informal Math Learning program and the interplay between teacher questioning and student reasoning” focuses in part on the questions posed by Researcher Powell and the associated student justification, and reasoning that occurred during the IML sessions.

CHAPTER 5: CONCLUSIONS AND IMPLICATIONS

5.1 Introduction

In this section, an overview of the findings will be provided. Patterns that were noticed in the data will be described. This will include a discussion of the similarities and differences in teacher questioning that were observed among the four researchers. The findings as well as limitations and implications of the study will be presented.

5.2 An Overview of the Findings

Numerous questions were asked during the sessions in Colts Neck, Kenilworth, and Plainfield. In many instances questions were asked in clusters before students were
given the opportunity to respond. Since these questions were asked at one time they are counted as one question yet were often classified in more than one way. In the three sessions led by Researchers Carolyn Maher (T/R 1) and Amy Martino (T/R 2) at Colts Neck, 174 instances of teacher questioning were identified. Researcher Amy Martino asked 60 questions during her interview with Brandon. 38 teacher questions were identified during the sessions led by Researcher Robert B. Davis at Kenilworth. Researcher Arthur Powell posed a total of 76 questions during the IML sessions at Plainfield while he conducted a whole class discussion and worked with Brandon and Yonny on some tasks. Figures 5.1-5.4 shows the number of questions that fell into each category.

![Figure 5.1. Classification of questions for all researchers](image)

*Figure 5.1. Classification of questions for all researchers*
Figure 5.2. – Sub-codes for the encouraging engagement classification for all researchers

Figure 5.3. Sub-codes for the probing classification for all researchers
5.3 Discussion

Researchers Carolyn Maher and Amy Martino used numerous questions during the sessions at Colts Neck to encourage student engagement and to give the students an opportunity to think deeply about the problems and independently without being given solutions or led to the correct answers. The majority of the questions asked fell into the categories of encouraging engagement, eliciting, and probing. Many questions were asked to encourage students to listen to one another and have them confirm or disagree with a fellow classmate’s ideas or solutions. Numerous questions were also asked to elicit student ideas and encourage students to make their thinking explicit. Probing questions were asked most frequently. These included questions to elucidate student responses, encourage critical thinking, or assist students in becoming aware about what their response had been and what their underlying assumptions may have been when presenting their ideas or offering a solution. A mere few factual questions were asked.
during these sessions and no divergent or affective questions were identified. The masterful questions asked by Researchers Carolyn Maher and Amy Martino served to invite argumentation and provided the opportunity for students to express justification and reasoning. The classroom thus became a laboratory of ideas set forth by the students themselves and actively challenged by their peers until they arrived at solutions that were acceptable to all. The researchers also attempted to include all students in the discussion, by asking for confirmation and by eliciting student ideas.

Researcher Carolyn Maher used many questions to keep the students engaged, on task, and to encourage them to listen attentively to their peers. She often used questioning to confirm agreement and disagreement. These questions were often followed by cautioning the students that if they did not agree with the conjectures of another student they would have to support their counterargument and make clear in what regard they were disagreeing and why. During whole class discussion she often redirected questioning to another student so that he/she could support the initial student’s claim, state an argument, or try to clarify the intent of the initial claim. Occasionally Researcher Maher used questioning to check for understanding.

Researcher Maher also used questioning to elicit student ideas, thereby encouraging students to formulate their own ideas and strategies. She used such questions to encourage students to formulate their own ideas and strategies and encouraged students to make their thinking public or explicit or to make their explanation visible to other students. These eliciting questions encouraged students to voice their ideas and solutions, explain their thinking, and share their thoughts or solutions with others.
The questions most frequently posed by Researcher Maher fell into the category of probing questions or a probing sequence of specific questions that elicited student answers past their initial responses. These questions were often used to provide the opportunity for students to see the error in their statement or to help them enhance their explanations. These elucidating questions and questions that encouraged critical thinking seemed to be a very effective form of questioning as they were associated with many instances of argumentation, justification, and reasoning. When Researcher Maher queried a student about what he or she meant she allowed for the student to elaborate, clarify, or for other students to propose alternate solutions. She also used probing questions to augment analytical cognizance. Here Researcher Maher attempted to increase the students’ critical awareness about what their response had been and what their underlying assumptions may have been in making a particular statement or offering a particular answer and thereby make them cognitively aware of the reasons behind their thinking.

Researcher Maher did not use shepherding questions to guide the student to come up with the correct response, thus allowing students to come to conclusions on their own and to refine their solutions without being directed towards a specific answer or position.

Factual questions, divergent and effective questions were rarely used during sessions led by Researcher Maher. Her questioning mainly served to create a mathematical community where group interaction was promoted and students were encouraged to share their ideas and challenge one another. The varied forms of reasoning that students expressed as well as the spirited argumentation and justification displayed during these sessions testify to the effectiveness of Researcher Maher’s questioning techniques in creating a learning environment where students were given the opportunity
for higher order thinking. See figures 5.5-5.8 for a summary of questioning techniques employed by Researcher Maher and for data regarding the associated student responses.

**Figure 5.5. Classification of questions for Researcher Carolyn Maher**

**Figure 5.6. – Sub-codes for the encouraging engagement classification for Researcher Carolyn Maher**
**Figure 5.7.** Sub-codes for the probing classification for Researcher Carolyn Maher

**Figure 5.8.** Classification of question results for Researcher Carolyn Maher
Researcher Amy Martino utilized many of the same questioning techniques discussed above as she worked with the Colts Neck students together with Researcher Maher. Through gentle probing she was able to steer the students to think more deeply about the tasks, increase collaborative discourse, and clarify and refine their solutions. Like Researcher Maher, she used numerous probing questions which served to elucidate, encourage critical thinking, and augment analytical cognizance. She also peppered class discussions with invitations for other student to confirm agreement or disagreement and tried to pull non-participating students into the discussion by asking them to share their thoughts or encouraging them to share alternate solutions. These questions provided opportunity student argumentation, justification, and reasoning.

During Researcher Martino’s interview with Brandon she posed many similar questions to the ones that she had asked during the class sessions. Many of these questions were eliciting and probing questions. These questions were asked to provide the opportunity for Brandon to express his thought process in writing his chart and building towers. She utilized probing questions to provide him with the opportunity to additional patterns in his data without actually giving away solutions. Here, in addition to providing the opportunity for justification and reasoning multiple times, her questioning tactics invited Brandon to make connections between his solutions to the two tasks that on the surface appeared different but were structurally similar, and ultimately enabled him to discover an isomorphism. Figures 5.9-5.12 shows the number of questions that fell into each category.
Figure 5.9. Classification of questions for Researcher Amy Martino

Figure 5.10. – Sub-codes for the encouraging engagement classification for Researcher Amy Martino
Figure 5.1. Sub-codes for the probing classification for Researcher Amy Martino

Figure 5.12. Classification of question results for Researcher Amy Martino
Researcher Robert B. Davis was an expert educator who used a variety of techniques to keep the students engaged and to elicit reasoning. His student-centered classroom was a noisy, lively place where students worked excitedly to come up with solutions to the task at hand. According to Spang (2009), Davis’s pedagogy included “‘setting the stage’, revisiting an earlier idea, using ‘secrets’, whispering answers into the microphone, providing positive reinforcement, soliciting agreement, engaging students from their seat to the board, withholding information from the students that they were expected to discover, and asking questions to probe their thinking.” Spang asserts that the students eventually modeled Davis’s pedagogy and asked each other questions rather than revealing rules. He gave the students a sense of control by asking them permission to change an answer. He probed the students’ thinking and provided a lot of positive reinforcement. As Spang writes: “Davis provided a friendly safe environment for the students to actively learn and build algebraic ideas.” Using the idea of a "secret" proved to be a motivating factor that encouraged students to continue working on the task until they had successfully discovered the secret and it also fostered interaction between students as they shared and discussed their secrets.

Researcher Davis surprisingly did not ask as many questions as Drs. Maher and Martino. He asked only a few probing questions during the sessions under analysis. Although he did not ask as many questions during the sessions as the other researchers, the few questions that he did ask created a learning environment conducive to student justification and reasoning and provided the opportunity for students to refine their solution. Through the means of this questioning and other teaching strategies together with engaging tasks, he created an atmosphere in the classroom that was conducive to
student self-directed learning, intrinsic motivation, and mathematical investigation and reasoning. Figures 5.13-5.16 illustrate the types of questions asked by Researcher Davis as well as the associated student responses.

**Figure 5.13.** Classification of questions for Researcher Robert B. Davis

**Figure 5.14.** – Sub-codes for the encouraging engagement classification for Researcher Robert B. Davis
**Figure 5.15.** Sub-codes for the probing classification for Researcher Robert B. Davis

**Figure 5.16.** Classification of question results for Researcher Robert B. Davis
Researcher Arthur Powell used questioning during the class discussion to encourage engagement, to encourage students to listen to their peers and to try to understand their proposed solutions. When working with Yonny and Brandon on tasks, he used questioning to try to keep their attention focused on work and to encourage them to think more deeply about the tasks. He repeatedly asked them to work with numbers such as twenty as a way of trying to get them to come up with a formula. Although during these sessions Brandon and Yonny did not arrive at the formula that Researcher Powell had hoped they would find, nevertheless, his questioning promoted argumentation, justification, and reasoning. The figures below (Figures 5.17-5.20) illustrate the types of questions asked as well as the associated student reasoning.

![Question Type Chart]

*Figure 5.17. Classification of questions for Researcher Arthur Powell*
Figure 5.18. – Sub-codes for the encouraging engagement classification for Researcher Arthur Powell

Figure 5.19. Sub-codes for the probing classification for Researcher Arthur Powell
As evident from the discussion above, the researchers in this study were able to estimate the students’ understanding of the tasks at hand and used questioning to provide opportunity for students to explain and justify their solutions. The researchers used probing questions extensively in order to direct students’ attention to components of developing solutions or to faulty reasoning. These questions were successful in creating a learning medium where students thought critically about their work and the solutions of their fellow classmates. In addition, many questions classified as encouraging engagement helped students engage in discussion, focus their attention on the work and solutions of their peers, check for understanding, and confirm agreement or disagreement among the students. Questions classified as encouraging engagement, eliciting, and probing provided opportunity for student interaction and collaboration. These questions sparked spirited discussions, as students tried to justify and defend their solutions, voiced their objections to their peers’ ideas, and carefully explained and clarified their ideas.
Eliciting and probing questions were effective in challenging students to strengthen their arguments, reorganize their solution, or build more complete or complex arguments and reasoning. Data from this study indicate that questions that encouraged engagement and invited student ideas as well as probing questions helped create a rich learning environment for student argumentation, justification, and reasoning on numerous occasions. The students in these classrooms felt compelled to justify and explain their solutions and to give detailed explanation of their reasoning, and they were encouraged by the researchers to voice their disagreement or provide support for another’s arguments. All this supported a learning environment where students worked out solutions on their own rather than memorize formulas or rules, independently noticed similarities and relationships, and formulated meaningful rules from their findings.

Questioning invited students to give detailed explanations of their strategies and served as a catalyst for students to use evidence to support their conclusions. This in turn resulted in student argumentation, where explanations and solutions were challenged and fellow students offered their own ideas. Thus, student discourse was replete with argumentation, justification, and reasoning. In total, 101 instances of such student responses were noted for argumentation, justification and reasoning. In five instances students made connections, thereby indicating that they had grasped the concepts and were not merely applying formulas or spitting back information that they had been taught.

As some of the studies discussed in the review of the literature indicated, teachers use many factual questions even when attempting to teach in a reform manner. The researchers in this study utilized factual questions in only a few instances. They primarily
used questions to encourage engagement, elicit student ideas, and probe for further clarification or to encourage critical thinking.

5.4 Limitations

This study did not attempt to compare researcher questioning styles, but merely noted patterns that were found in the data. The researchers worked with different class sizes, different student populations, and in some cases dissimilar tasks and had different levels of rapport with the students. Thus, this study did not attempt to ascertain which researcher utilized the most effective questioning techniques. As was evident from the data, each researcher created a student-centered classroom and invited numerous instances of argumentation, justification and reasoning.

In addition, due to the qualitative nature of this work, one cannot generalize findings. Since this research can be classified as a case study, one would need to replicate it in order to confirm its validity (Yin, 2003). Also, additional question classifications could have been utilized and other sub-codes could have been added which would have given the data additional nuance and lent themselves to further analysis and interpretation.

5.5 Implications

Questioning is an important pedagogical tool that can be utilized by mathematics teachers to provide opportunity for argumentation, justification, reasoning and other higher order thinking. The findings from this study can help other researchers identify patterns in teacher questioning that appeared to have been effective in eliciting higher order thinking. Mathematics educators can study the learning environment created by the researchers and implement some of the questioning techniques that provided the
opportunity for collaboration and invited mathematical argumentation and discourse as well as sophisticated justification and reasoning.

The analytics referenced in this dissertation are intended to be a vivid accompaniment to the narratives presented in this study, so that one can observe the actual questions and resulting argumentation that played out during the sessions and gain additional insight into the interplay between teacher questioning and student reasoning. These analytics showcase some of the significant exchanges between researchers and students that demonstrate how a facilitator’s questioning can serve as a tool to foster a learning environment conducive for student argumentation, reasoning, and justification.

5.5.1 Implications for Further Research

It may be enlightening to study the juncture at which effective questions were asked in order to identify the types of questions that were vital for helping students deepen their understanding. Such study may shed light on what questions can be posed in response to a student’s meaningful explanation or incomplete reasoning in order to help the student perfect or deepen his or her understanding or what can be asked in response to expressions of faulty reasoning. By analyzing the data with the addition of these “timing” codes, a new layer of understanding can be gleaned from the data. These “timing” codes would indicate whether probing questions were in response to an incomplete argument or faulty argument or to the student having difficulty expressing the solution or approaching the problem. It would help mathematics educators identify the questions that were particularly effective in assisting a student who was having difficulty with a concept or unsure of how to arrive at a solution.
In addition, research can be done to compare the age level of students with the questions asked. Questioning techniques that appeared to be effective at eliciting justification and reasoning may have different results when implemented at a different grade level.

Researcher questioning in this study was associated with many instances of student argumentation, justification and reasoning. In light of this, one may speculate that these questions may have triggered certain behaviors. Future studies can be designed to study whether deliberate attention to certain questions result in specific student behaviors.

In addition, research can be done to see what teacher researcher moves were associated with students’ interacting and/or collaborating. One can also study the wait-time and analyze whether increased wait-time was important to enhance the quality as well as quantity of student responses.

5.5.2 Implications for Practice

Teachers, teacher educators, and researchers can all benefit by studying the data presented in this study. In addition to looking closely at teacher questioning and the associated student response, they can study how mathematical norms were established in these classrooms that made students feel comfortable arguing, refuting, and justifying solutions as they worked together with their partner and as they joined in class discussions. The nature and structure of the tasks can also be studied (see Lo, 2010) so that well-structured tasks can be implemented in the classroom, which would help assure that mathematics learning not merely consist of memorization and application of formulas and procedures. The correct environment, coupled with well-structured tasks
and effective questioning can be replicated in order to invite higher order thinking, justification and reasoning.

As Researcher Maher so eloquently says on the PUP Math video featuring Brandon discovering an isomorphism:

Brandon had an opportunity to think deeply about a problem. And he had an opportunity to talk to someone about his ideas. I think we have to remember - We see Brandon and we all so impressed with what he did. And what he did was very impressive. But at that time, the schools grouped students according to math ability. They don't do that anymore. This was many years ago. And Brandon was in the lowest group. And when later we went to the teachers with what we found, with our interview of Brandon, and we said, "Look. Look at this! This is just absolutely brilliant. This is wonderful; this is amazing!" And they hadn't seen anything like that, they told us. Well, I think we don't see these things because we don't give students an opportunity to show us their thinking. I think the world is full of Brandons. We just don't take the time to find them and to listen to them. We don't have mechanisms to pull them out. I think they're all over.

Through careful planning and effective questioning we can pull out all the Brandons. We can help all students tap into their latent mathematical abilities and guide them towards increased collaboration, rich mathematical discourse, argumentation, justification, reasoning, and ultimately joy in learning and discovering mathematics.
References


Dillon J. T (1983). Teaching and the art of questioning, Bloomington Ind, Phi Delta Kappa (Fastback No. 194).


Appendix A

Transcripts

Colts Neck Sept 21, 1993 Session 2, Sept. 21, 1993 (Front, Side, and OHP) ............. 2
Colts Neck Session 5, Sept. 29, 1993 (Front, Side, and OHP) .................................. 23
Colts Neck, Session 10, Oct. 8, 1993 (Front) .............................................................. 41
Colts Neck April 5, 1993, Interview with Brandon ...................................................... 50
Kenilworth Sept 30, 1993 (Session 1 on Guess My Rule) ........................................... 82
Kenilworth Oct 1, 1993 (Session 2 on Guess My Rule) .............................................. 88
Plainfield, IML Sessions November 2-3, 2005 ......................................................... 112
Colts Neck Session 2, Sept. 21, 1993 (Front, Side, and OHP)

Line | Time  | Speaker | Transcript
--- | --- | --- | ---
2.0.2 | | Jessica: | Um. We did activities with rods and we um had to see like which rods were bigger and we had to…um, we did math problems with them.
2.0.3 | | T/R 1: | Okay. Somebody want to sum up a little bit more? Michael?
2.0.4 | | Michael: | Um. We, well, we, what we did is, we called one ‘one’ and then we had to decide the littler one, what it would be called, one thirds, one fourth, or a half of that, the bigger block.
2.0.5 | | T/R 1: | You know, I don’t want to embarrass Mr. Purdy, but you have to go very slow for him. He often needs an example.
2.0.6 | | Tom: | I need to see it.
2.0.7 | | T/R 1: | He needs to see it. He just… that’s the way he learns. Can you help him a little better then, Michael? Maybe make up and example for him or somebody? We really need to help him out…Erik, you want to help while Michael is thinking of something else?
2.0.8 | | Erik: | Well, let’s [He picks up a blue rod.] If we said that the blue rod would be one whole, um, we’d figure out what, we’d take all the blocks and try and figure out what would be half of it. [He holds the purple rod next to the blue rod.] And let’s, I figured that the purple block would be half of it. So, well, no, not exactly, but…
2.0.9 | | T/R 1: | Mr. Purdy goes through the same thing, Erik.
2.0.10 | | Erik: | But if we call this one whole [holding up the blue rod], we’d figure out which block would be one half of it
2.0.11 | | Tom: | Uh huh.
2.0.12 | | Erik: | And which block would equal up the two blocks of… these two blocks of it, that would equal up to one of these we’d call that one half of the whole block. So, that’s basically what we did.
2.0.13 | | T/R 1: | You’re not going to help solve it for him? [to Tom]
2.0.14 | | Tom: | I was going to say, did you find it? Or
2.0.15 | | Erik: | Oh, oh.
2.0.16 | | Tom: | I mean I don’t be- I mean, you’re making me believe you can’t do it.
2.0.17 | | Erik: | Well, yeah we did, but
2.0.18 | | Tom: | You’re making me believe maybe you can’t do it.
2.0.19 | | Erik: | No. We did find it. I just can’t remember which one it was. [He holds up two dark green rods, end to end, next to the blue rod and discards them when he sees that two dark green rods are not equal in length to the blue rod.] I think it was
the... [He measures two yellow rods, end to end, to the blue rod.]

2.0.20 T/R 1: Maybe some of you can help Erik out.
2.0.21 Erik: I think it was the dark green.
2.0.22 Tom: You’re saying the blue one is one [Children are working with the rods.]

2.0.23 S Student: Try the yellow.
2.0.24 S Alan: The little green one was the thirds. The yellow was the half.
No. The yellow is the halves of the orange one.
2.0.25 S Erik: I don't think there is one.
2.0.26 Tom: I think you picked a good one.
2.0.27 T/R 1: Erik, Erik. Suppose I wanted, suppose I wanted to call the yellow one “one half”. Suppose I wanted to do that.
2.0.28 Student: Found it! [Erik turns quickly as this is said – off camera view].
2.0.29 T/R 1: But suppose I wanted to call the yellow rod, I wanted to give it a number name one half. Can you tell me what I would have to call one?

2.0.30 F Meredith: Oh, oh!
2.0.31 T/R 1: I think you need to get your rods and build it for me.
2.0.32 F Meredith: Oh.
2.0.33 T/R 1: If I wanted to call the yellow one half, can you show me
2.0.34 S Alan: Easy
2.0.35 T/R 1: What would I have to call one?
2.0.36 S Alan: It’s orange. It’s easy. See? The orange one. [Erik is working with his rods.]

2.0.37 F Meredith: Oh! Yes I can, yes I can, yes I can. Oh!
2.0.38 F T/R 1: Brian, you want to tell Mr. Purdy?
2.0.39 0:08:38 Brian2: Well, these two blocks equal up to this one whole. [He holds up two yellow rods in his left hand and an orange rod in his right hand.]

2.0.40 Tom: Those two blocks equal up to one whole. So how much is each one? Each one of the yellows?
2.0.41 Brian2: One half.
2.0.42 0:08:57 T/R 1: So you are going to call the yellow one half? I’m still worried about Erik’s problem. Erik wants to call this one [She holds up a blue rod] and Erik is trying to call something one half. Don’t you want to help Erik out?

2.0.43 F Meredith: No.
2.0.44 S Erik: I don’t think there is one.
2.0.45 S Alan: A little green makes a third out of that. Look I can do it
2.0.46 T/R 1: If you call the dark blue one “one”
2.0.47 Alan: One, two, three.
2.0.49 David: I don’t think that you can do
2.0.50 T/R 1: Why, David? Slowly and loud.

2.0.51 0:09:25 David: I don’t think that you can do that because if you put two yellows that’d be too big, but then if you put two purples that’s uh, that’s uh, that’d be too short and

2.0.52 T/R 1: What about something between purple and yellow?

2.0.53 David: I don’t think there is anything.

2.0.54 T/R 1: Why not? [David pauses.]

2.0.55 Show us what you have there, David. Why do you think there isn’t any? Cause I think you built it to show us. Can you show us your yellow and your purple?

2.0.56 David: Well, I was thinking. Cause there’s usually, the tall one… [inaudible]

2.0.57 T/R 1: David, why don’t you come up here and explain your reasoning. David doesn’t think it’s possible because Mr. Purdy said, “Well, maybe it’s not possible.” So let’s, let’s see. Let’s help him out a little. Here’s the two yellows and here’s the two purples. What’s, what’s your reasoning? Let’s listen to what David has to say.

2.0.58 0:10:20 David: [F - Meredith builds some erect models on her desk as David explains] [He comes to the overhead and puts a blue rod onto it. He places a yellow rod and a purple rod, end to end, with one white rod - Figure O-10-33] All right. You see usually, um, they are only one, with the shorter one, only one block apart [Figure O-11-01]. Like that and so these, but then if you have for the blues, like if you have two yellows, it would be too tall and if you have two purples [He puts two yellow rods, end to end, next to the blue rod and then two purples next to another blue rod - Figure F-11-56]

2.0.59 T/R 1: Do you need another purple? Here

2.0.60 David: That’d be too short and then there’s really nothing in between ‘cause if you do [He builds a ‘staircase’ of rods, beginning with the longest, orange rod, then places blue, etc. until he reaches the shortest rod, the white one.] And then here [between the yellow and the purple rods], there’s nothing in between, right here, so there’s no way that you can do that.

2.0.61 T/R 1: Are you all convinced? Jessica? Jessica has a question for you, David.

2.0.62 0:12:00 Jessica: But if you put three greens to it you could

2.0.63 0:12:04 David: Yeah, but Erik said, Erik wants the half. [inaudible]…’cause I figured that out, too.

2.0.64 0:12:11 Erik: I think you could do it, but they’re… See, I figure if you take a yellow and a purple it’s equal [to the length of the blue rod - Figure S-12-24]. They’re not exactly the same, but they’re both halves. Because the purple would be half of this even though the yellow is bigger because if you put the purple on the bottom and the yellow on top it’s equal, so they’re both
halves, but only one’s bigger than the other. So it equals up to the same thing.

2.0.65 T/R 1: Did you all hear what Erik said? Erik, do you want to say that one more time? How many heard what Erik said? How many would like to hear it a second time? Ok, Erik, would you say that one more time to David and the rest of us?

2.0.66 0:12:54 Erik: If this would be one whole [the blue rod], you could take the yellow to be and you could call it one half [holding a yellow rod next to the blue]. But if you took another yellow it would be too big. But if you took a purple with the yellow, and put it on top of yellow, it equalled to the blue. So, the purple would be a half and the yellow would be a half, except that the yellow would just be one bigger than the other. Or maybe you could call this three quarters [holding the yellow rod] and you could call this one quarter [holding the purple rod]. And, but it would still equal up to the whole. [F - As Erik speaks, Jessica models the blue rod and the yellow and purple train at her desk]

2.0.67 T/R 1: What do you think, David?

2.0.68 David: I didn’t think of that. [Erik chuckles. David places a yellow and a purple rod end to end, next to a blue rod.] Like that. Cause I was thinking that, um, that you would need the same.

2.0.69 T/R 1: You think you would need the same?

2.0.70 David: Yeah, but that might

2.0.71 Erik: You don’t really.

2.0.72 T/R 1: You don’t need the same? In other words, I could call this a half [the yellow rod] and I can call this a half [the purple rod]. Suppose this is a brick of gold and we’re going to share it, Erik. And I’m going to take the yellow half and you get the purple half. Fair?

2.0.73 Erik: Yeah.


2.0.75 Erik: Well, well I mean-

2.0.76 Kimberly: Yes, cause the pink is, the purple is smaller than the yellow and the person who got the purple wouldn’t have as much.

2.0.77 Erik: Yeah, but you could call this three quarters and this one quarter and it would still be equal up to the whole. Then it, just wouldn’t be halves, it would be quarters. But it would still look like you’re dividing it into halves, but you’re really dividing into quarters.

2.0.78 14:07 F T/R 1: What do you think, Brian?

2.0.79 Brian: Well, you could, you could use say, if there, if there was three people – you could at least split it into thirds, you could at least split it into thirds.
T/R 1: Is that, is that the question?
David: Well, no. It’s not. You see we’re trying to do it in halves.
T/R 1: [To Brian] We’re trying to work on halves.
Brian: Oh.
T/R 1: Okay. Alan.
Alan: When you’re dividing things into halves, both halves have to be equal – in order to be considered a half.
Jessica: [inaudible] this isn’t a half. Those two aren’t both even halves.
T/R 1: Erik?
Erik: Yeah?
T/R 1: What do you think of that?
Erik: Can you divide things in halves and have them different sizes? I think that’s what Jessica is asking and Alan and David.
Erik: Well, see. This isn’t exactly dividing into halves. But I’m still using two blocks, but not… I’m dividing it in half still using two blocks, but one block is bigger than the other block. So it’s like using three quarters and one quarter, but you’re only using two blocks so it’s almost like dividing it in half.
T/R 1: Andrew? What do you think about that, Andrew?
Andrew: Well if he’s saying, he’s saying that he wants a half, but if he puts that, a purple and a yellow, he won’t have a half. He would have three quarters and one quarter. And he wants a half.
T/R 1: It seems to me we have some differences here, don’t we? Um. How many of you agree with Erik? [no hands are raised, children giggle] How many of you disagree with Erik? [all hands are raised, more giggling]. Hm, okay, what’s the issue, do you think, here in the disagreement? Can somebody summarize the issue? Alan, do you want to try again?
Alan: Um. You can’t, if you’re div, you can’t divide that into halves, because you’d have to use rods that are of different sizes, but you could divide it into thirds using rods that are the same size which, which is the light green rods.
Erik: But I didn’t want thirds.
T/R 1: [inaudible] can be very helpful to Mr. Purdy. Because I think, go ahead, David. What do you think?
David: [at OHP, pointing to the rods on the OHP] I think that some of these that you can’t do like this would be odd. [David moves the white rod to one side.] this could be even. [David begins a new group with the red rod.] This would be odd. [He moves the light green rod next to the white rod.] Be
even. [He moves the purple rod next to the red rod. Continuing in this manner, he moves the yellow, black and blue rods next to the white and light green rods. He moves the dark green, brown and orange rods next to the red and purple rods.] This, be, you see, then when you get up to here, blue would be odd, but like with brown, you could take these two [He places two purple rods next to the brown rod.] and put them together and that would be even. Take the orange, put the yellows, with the orange and that would be even [He does this as he is speaking - Figure O-17-20].

2.0.100 T/R 1: Okay, let me see, I think that we have. Maybe, Erik, the way we can resolve this is, I don’t think I’m hearing you say, Erik, that you want to call yellow one half and purple one half. I don’t, I don’t hear you say that. You’re not saying that, are you?

2.0.101 Erik: No [agreeing that he is not saying that].

2.0.102 T/R 1: You’re saying that you agree with the rest of the class that if you call something one half of something

2.0.103 Erik: Yeah

2.0.104 T/R 1: They have to be the same size.

2.0.105 Erik: Yeah, yeah.

2.0.106 T/R 1: Right?

2.0.107 Erik: Yeah.

2.0.108 T/R 1: You are in essence answering a different question, maybe?

2.0.109 Erik: Yeah.

2.0.110 T/R 1: Where you were saying, “Well, if I call this one, there are other rods that make up one and maybe they’re not the same size.” I think you’re very generous, Erik. Not as generous as Beth and Kimberly. And if we’re talking about bricks of gold, letting me have the larger one if we’re sharing one half. I, I really appreciate your generosity. I know Mr. Purdy wouldn’t be so generous. Is that right, Mr. Purdy?

2.0.111 Tom: That’s right.

2.0.112 T/R 1: That’s right. But I do appreciate your generosity, so we’ll have to talk later about some, some sharing. Um. We could go into business together, Erik. But I think that what we’re saying from this is the point that David is making and Alan and some of you have expressed very nicely, that if we are calling a rod one half, okay, if we call a rod one half, of, let’s say, a rod that we called one, was given a number rod one, there are two conditions that have to be satisfied. Can you tell me what those conditions are? And I think one more time as a summary because you’re saying that purple could not be considered one half because one of the conditions isn’t met, right? I mean, they’re both [two purple rods] the same size.
2.0.113 David: Um, hm. But they don’t, um, if you put like that [He puts two purple rods together.], they don’t, uh, they’re not as big as the blue.

2.0.114 T/R 1: Do you agree? Do you all see the second condition that’s not met? See the space in here? Or if you can put them like this, see that space? And I think that David has made another very powerful, interesting argument that I’d like you to think about. He claims that that’s missing, right? And that there couldn’t be another rod in between to do it, right? That’s interesting, now, you know, suppose you had to manufacture these rods and make another color. Okay? Here we have a purple rod that’s too small, right? To qualify to be a half. Do you agree the purple’s too small? And here we have a yellow one, right? That’s too big, right? To qualify, do you see that? If you were designing a new set of rods and you wanted to call the blue rod one, okay? Can you tell me what that new rod might look like so that you would be able to call it a half? [pause] Do you understand my question? We have rods here with ten, we have ten colors, don’t we? You told me that yesterday.

2.0.115 Students: Yeah.

2.0.116 0:20:29 T/R 1: Right? And you all told me that if I wanted to call blue one in terms of the box you have, right? You can’t find a rod that you could give a number name one half. Isn’t that what you all told me? [mumbles of agreement] That’s a problem. Because, um, there’s another school that wants to have rods where they want to call blue one and have another rod that they can give a number name one half. Okay? Now can you tell me what the design of that rod might begin to look like? Why don’t you talk to your neighbor and think about that problem? Do you understand my problem?

2.0.117 Students: Yeah.

2.0.118 0:21:06 T/R 1: We know it can’t be purple and we know it can’t be yellow. What do you think, David? You’ve convinced me that’s there’s nothing in between. [T/R 1 and David confer at the OHP. Their conversation cannot be heard.]

2.0.119 SIDE VIEW

2.0.120 [Erik and Alan as partners begin immediately:]

2.0.121 0:21:10 Erik: It can’t be anything ’cause you can’t divide nine equally. You see if this is

2.0.122 Alan: If you could

2.0.123 Erik: No you can’t. This is ten.

2.0.124 Alan: If you could make a rod.

2.0.125 Erik: If this is ten [the orange rod], then this [the blue rod] is nine. It’s impossible to divide this evenly.
2.0.126  Alan: Different rods. You might be able to, like if you divide a blue rod in half you could that that length and make a new color and that would equal up to halves. Which would mean it would be like [noise]

2.0.127  Erik: It’s impossible. You can’t divide it in half. You can’t divide it in half, Alan.

2.0.128  Alan: Right, you could divide it in half if you had [inaudible] parts. You could divide it in half, but having equal parts, but you couldn’t have equal numbers.

2.0.129  Erik: [inaudible]

2.0.130  Alan: If you cut this [the blue rod] down the middle, it would be four and a half, [inaudible] the same length.

2.0.131  Erik: Four and a half. You can’t make a rod that’s four and a half.

2.0.132  Alan: Um, hm. So you can’t divide into anything.

2.0.133  Erik: Except thirds.

2.0.134  Alan: Except thirds. Or, or singles.

2.0.135  0:22:12 Erik: You can’t divide it into halves. “Cause I put this up here and there are nine of these and one, two, three, four, five. One, two, three, four [pause] four, one two, there four five. One, two, three, four. One, two, three, four. One, two, three, four, five [He is counting the two groups of white rods next to the blue rod - Figure S-22-24].

2.0.136  Alan: Over here you have thirds.

2.0.137  Erik: You can divide it into thirds, but you can’t divide it into halves.

2.0.138  Alan: You can divide it into thirds. You can divide it into ninths.

2.0.139  Erik: But you can’t divide it into halves.

2.0.140  Alan: You can’t divide it into anything else but thirds and ninths.

2.0.141  Erik: Exactly, you’re right.

2.0.142  Alan: Just thirds or ninths. That’s all you can do. That’s productive reasoning.

2.0.143  Erik: What?

2.0.144  Alan: Productive reasoning. So there can be only thirds and ninths. And the umm singular rods. And you can’t divide it into halves.

2.0.145  Erik: Exactly. It’s impossible to divide it in halves.

2.0.146  Alan: That can’t be done.

2.0.147  Erik: It’s impossible, Alan. You can’t divide it into halves.

2.0.148  Alan: It’s been proven.

2.0.149  Erik: Exactly. [noise]

2.0.150  0:23:30 Alan: Mind handing some over my way? All right um, what I’m going to do right now is make out of everything, I’m going to halve or third every color,

2.0.151  Erik: You can’t halve every color, you can third every color.

2.0.152  Alan: [singing] I can third every color. I can halve every color.

2.0.153  Erik: Except blue.
2.0.154  Alan: You can third.
2.0.155  Erik: You can third. You can third.
2.0.156  Alan: And ninth.
2.0.157  Erik: and ninth.
2.0.158  Alan: Now black.
2.0.159  **FRONT VIEW**
2.0.160  [Brian G. and Jacqueline built staircases independently.]
2.0.161  [Sarah and Meredith sat quietly; T/R 2 approached the partners.]
2.0.162  T/R 2: You two seem very quiet over here. What do you two think about this? What would the new rod you design be like? Do you understand the question? Sarah, do you understand the question? [Sarah nods her head] What are you being asked to do?
2.0.163  Sarah: I know. [to Meredith] Let’s see if you do.
2.0.164  Meredith: You don’t know.
2.0.165  Sarah: Yes I do. Can you do it?
2.0.166  Meredith: Let’s see you do it.
2.0.167  T/R 2: My am I getting the feeling neither of you are going to be able to tell me. [Sarah and Meredith laugh]
2.0.168  Sarah: We aren’t good talkers
2.0.169  T/R 2: Oh, yes you are.
2.0.170  Meredith: She’s asking us to find a rod that will make up a whole, that will make blue. Find one that will fit.
2.0.171  T/R 2: Ok, so,
2.0.172  Sarah: Inaudible
2.0.173  T/R 2: That’s uneven, ok, so you think if we were, if we were able to go into a workshop with wood and build a new rod, what would that rod look like? If we could go and build and make a rod any way we’d like, what would that rod look like? How would you describe it? Would you describe it in terms of these other rods here?
2.0.174  Meredith: This [orange] is ten, this [blue] is nine. And if you split this in half, it would have to be four and a half and four and a half.
2.0.175  T/R 2: Alright. Why is that?
2.0.176  Meredith: Because half of nine is four and a half.
2.0.177  T/R 2: Ok. So…
2.0.178  Meredith: And then, and you would have to make one, say, you had four, and then you make, this is a four and this is a four [purple rods]. You’d have to have a half on this four and a half on this four. [model is not in camera view] [T/R 1 starts talking, T/R 2 and Meredith speak but inaudible]
2.0.179  **WHOLE CLASS**
2.0.180  0:24:10 T/R 1: Okay, I’d like us, if you don’t mind, if we can stop for a minute and I’m going to ask Beth, Graham, and Jackie to
come up and pose their solutions. I heard a few of your solutions. I know David has a solution I heard already, up front. I’d like to hear some other possible solutions. You can clear off there [the OHP] what you don’t need.

2.0.181 F T/R 2: What do you think, Sarah? [Sarah nods yes] I’m anxious I want to make sure you share that.

2.0.182 F Meredith: You don’t need the brown, all you need is

2.0.183 Beth: I got it, right here.

2.0.184 0:25:00 Jackie: [Places purple then white then purple rods in a line on OHP - Figure O-27-44]. We thought that to make a new rod we would make, um, we would cut this white one in half and attach it

2.0.185 T/R 1: Could you speak nice and loud? Cause I’m a student back here and I can’t hear you. Do you want to try and talk really loud?

2.0.186 0:25:44 Jackie: We thought of, to cut the white one in half and add it to one rod [purple] and then add it to the other rod [purple]. And we thought the color would be light pink.

2.0.187 Graham: [To Jackie] And the smallest one would be a half ‘cause it was the white one.

2.0.188 Jackie: And the smallest one would be a half ‘cause it was the white one.

2.0.189 T/R 1: Did you all hear what they said? No, they, Kimberly didn’t hear you, dear.

2.0.190 Jackie: We thought to cut the white one

2.0.191 T/R 1: You can come in front and talk nice and loud, I know you can Graham.

2.0.192 Jackie: We could cut the white one in half and add it to the purple rod and add one one half to one purple rod and the other to the other one and we thought that we could call the color light pink.

2.0.193 T/R 1: And you said something else, what would your smallest rod be?

2.0.194 Jackie: Oh, yeah. Our smallest rod would be half of the white one.

2.0.195 T/R 1: What are you going to call that? [some giggling] You’re the designers. What are you going, it’s not going to be white, what do you think? You want to help them out? You could have other consultants to this design. Why don’t you call on someone for help and consulting? Graham? Beth?

2.0.196 Graham: We cut the clear one [the OHP version of the white rod] in half to like make this. Then you’d, then you would have to cut like a reg- a regular one in half to be your smallest one [F - model on Brian’s desk - staircase, with the top filled in by white rods, Figure F-26-18]

2.0.197 F Meredith: [whispers to Sarah] They took my answer.

2.0.198 T/R 1: I see some hands up. Why don’t you see if…?
Michael: If you’re going to make a new rod, then you’d have to make a whole new set because there’d have to be a half of that rod, too.

Michael: If you cut these little ones in half, then there wouldn’t be something for the little ones to make a half out of them.

Brian: No matter what, there’ll always be something that won’t be equal to something, like

[laughter]

David: Well, before I told you. I thought that, uh, to cut it in half, too, but then I realized that, uh, that you would have to make a whole set.

Meredith: Well, you could just, if you do that then you’d have to cut the ones that are separate, the little blocks into halves, all of them, so then you could make it equal.

Jacquelyn: Um, it, I agree with Michael. ‘Cause if you do that, um, it changes the whole pattern ‘cause this has a set in pattern to it and the whole thing would change.

Jackie: You’re creating a pink rod. And as I understand it, the pink rod is made up of purple and half a white. Is that what you said? Um. You solved the problem of having a rod that you
can call one half when you call the blue rod one, right? But then, as some of you pointed out, then your smallest rod is then, with this new design, your smallest rod is then,

2.0.221 Meredith: Half
2.0.222 T/R 1: Half of the
2.0.223 Meredith: White
2.0.224 T/R 1: White rod, right? And what are you going to call that? Let’s give that a name. Let’s give that a name. Can you give that a name? It’s not white any more. It’s half of white. What color name shall we give it?

2.0.225 Jackie: Light blue.
2.0.226 T/R 1: Pardon? Light blue? Okay, so your smallest rod is going to be light blue. But I heard some other people say, like Brian in particular, and some others, Meredith, that, okay, you’ve solved that problem, but you could expect new problems. Yeah. That’s interesting. Well, that’s something to think about. You did a really nice job. Did anybody have another way to make the argument? James? [James goes to OHP]

2.0.227 F [Sarah has a model of P-W-P built on her desk.]
2.0.228 0:30:43 James: Well, I thought that if you had a blue rod as one, you could take light green, imagine there are two others here. Then you could split the middle one in half and you could call that a light blue rod.

2.0.229 T/R 1: Is that okay? That’s another way, huh? Does anybody have another way? …Do you think there’s still another way?
2.0.230 Meredith: Mmm hmm.
2.0.231 T/R 1: Do you think there are other ways than this? That’s really good? Now, Mr. Purdy, did that help you?
2.0.232 Tom: Yes.
2.0.233 T/R 1: That’s great. Ok, well that was very very helpful. What do you think, Dr. Martino? You want to give them some more problems? [Dr. Martino says something] You want to do some more? She’s going to challenge you with some other ones. Um, uh oh, here we go, Mr. Purdy pay attention, because she’s going to really challenge you.

2.0.234 T/R 2: Alright. Let’s try something a little different now. Ok. Now, if we call the orange “two”, what can we say about yellow? Think about it for a minute, and you want to talk to your partner?

2.0.235 SIDE VIEW
2.0.236 [Erik and Alan discuss, but conversation inaudible.]
2.0.237 0:32:54 [CT asked Danielle to explain. She said “one”. CT noticed that Danielle’s partner, Gregory, was frowning.]
2.0.238 CT: He’s not convinced; he’s a little shaky. Can you explain to him why you’d call it one?
Danielle: Because if this is two [yellow rods next to an orange rod]. Then these [yellow rods] should be one. Because see if these are two, there's two of them. We call that [orange rod] two. We call these one [yellow rods] because this [orange rod] is called two.

Gregory: When the orange is one, we went like a half down

CT: [To Gregory] Is this [orange rod] still one?

Gregory: No, it's two.

CT: So then

Gregory: Yeah, one.

FRONT VIEW

Meredith: You’re using all the yellows

[Kimberly raises her hand. She has built a model of two yellow rods under the orange rod. Sarah raises her hand. Meredith returns to her desk]

Sarah: I have them. I only have two! [Meredith goes to back of room].

Meredith: Oh! She called orange two. One half? Two? Then this would have to be one.

WHOLE CLASS

T/R 2: [Class called together by T/R 2] Ok. I’m anxious to hear some answers to this, hear what people have come up with. I hear, I hear a couple of different things here and I think that’s something- let’s see if we can get some answers up here and discuss them. Uh, let’s see. Who haven't we heard from? Let’s see. Brian, what do you think, now when we call this, we give this the number name two, the orange, what number name are we going to give to yellow?

Brian: One.

T/R 2: Why one? You want to come up here. You can come up here and show us. [Brian goes to overhead.]

Brian: You would put two yellows together and it would be the same size as that, and even if and that’s like having, so if this [orange rod], is considered a two. Then those two [yellow rods] would be considered like a regular orange, so it would be considered a one.

T/R 2: Okay, so you’d consider each of these [yellow rods] a one, is that what you’re saying?

Brian: No, that like together they would equal the same as that [orange] so it would be a one.

T/R 2: O.k. So the number name you’re giving yellow then was what?

Brian: One, one.

T/R 2: Okay, alright, one. What do you think about that? Does anyone want to come- Who agrees with that, that you give
the yellow the number name one? Ok. Does anybody disagree with that? I heard, I heard some-

2.0.260 0:34:53 Erik: I have another name. You can call it another name.
2.0.261 T/R 2: Ok, what would you call it, Erik?
2.0.262 Erik: Well, see, do you have to call the orange two?
2.0.263 T/R 2: Well, I've arbitrarily picked that I'm calling the orange two.
2.0.264 Erik: Well you could call it one, and if you call it one, then two yellows would be a half. If that would be considered, if the orange would be considered two, then you'd call those [yellow rods] one. But if you can call it [orange] one, you could call those [yellows] halves.

2.0.265 TR/ 2: That's interesting, what if I call the orange…uh
2.0.266 Brian: [at overhead]. There might be other ways. You can split them, you can maybe split it into thirds, and call that a one but we don’t have enough thirds-
2.0.267 0:35:38 T/R 2: Okay, yeah, you probably could….Let me ask you another question then, I'm going to ask this to everybody, too. What if I change the name of the orange to six. What would I call the yellow- what number name would I call the yellow? Let’s see, uh, somebody I haven’t had a chance to talk with, James, is your hand up? Kimberly?

2.0.268 Kimberly: Five.
2.0.269 T/R 2: Five. That’s interesting. Can you come up and tell us about that? [Kim goes to the overhead.]
2.0.270 Kimberly: Look here[ pointing to Brian’s model] before you said that [the orange rod] would equal two, and then Brian said that [yellow rod] would equal one. So now you’re saying that that [orange rod] equals six, so I figured that if that equaled one before [yellow rod] it would equal five now. [F - Sarah has built a model on her desk 36:10]

2.0.373 0:36:54 T/R 2: That’s interesting. What do you think about that, some of these other folks? Did you all hear Kimberly's argument here? She's saying when you call this one, the number name two, that the yellows were each one, ok, they had the number name of one. She's saying, so if I call this six now, she’d call that five. What do you think? [Meredith and others shake their heads negatively.] Ok, I see some people are shaking their heads and I want to hear why. Uh, let’s see. Alan?

2.0.272 36:25 F Alan: [Goes to the overhead] You said that the orange rod was six. And before you said that this was two and this [yellow rod] was one. So now if you’re calling this [orange rod] six, and half of six would be three. So that’s

2.0.273 T/R 2: Okay. So we have another argument. What do you all think about Alan’s argument? He's calling this [yellow rod] three,
the number name three when I call this [orange rod] the number name six? Meredith?

2.0.274 Student: Yeah.
2.0.275 Meredith: [inaudible]
2.0.276 T/R2: You agree with that? Jessica?
2.0.277 Jessica I agree with him because like half of six is three so that would
2.0.278 T/R 1: I’m curious how Kimberley thought of five? Can you help me understand why you think five?
2.0.279 Kimberly: Well, before you said that was two, the orange was two, and the yellow was one. So now you're saying it's six, so the yellow could be five.
2.0.280 T/R 1: The yellow is five... That’s where I am confused. So you're saying if this [yellow rod] is five and this [yellow rod] is five, this [orange rod] is six?
2.0.281 Kimberly: Ok, I made a mistake, I-
2.0.282 T/R 1: You didn't mean that? What did you mean, Kimberly?
2.0.283 Kimberly: Well, I made the mistake. I figured it out now.
2.0.284 T/R 1: Tell me what you were thinking. I'm curious about what you were thinking.
2.0.285 T/R 2: That's what I want to know.
2.0.286 Kimberly: I made the mistake thinking from before, I forgot that adding one and one is two, but five and five isn’t six, so, I made that mistake.
2.0.287 T/R 1: If you want this to be five, what would you have to call the orange?
2.0.288 Kimberly: Ten.
2.0.289 T/R 1: You’d have to call orange ten. Do you agree with that? [students: Yeah] What a class! You're going to have trouble stumping them, Dr. Martino.
2.0.290 0:38:33 T/R 2: I know, this is tough! Okay, let’s try another one. Umm, okay if we call [long pause]let's see …
2.0.291 T/R 1: I think we're going to have to consult to give you a problem hard enough. You're just getting too good for us and… how about this one? Suppose we made a train, ok, I'm going to take Erik’s idea from earlier, and I'm going to call the orange and [light] green together, one. You like that? I'm calling this one. The orange and green train together, one. Now I didn't work this problem, but, I’m curious; can you find a rod that has the number name one half?
2.0.292 SIDE VIEW
2.0.293 0:39:33 Alan: Erik, look! Erik, look, this is the biggest I can find, you see?
2.0.294 Erik: I’m trying to figure it out.
2.0.295 Alan: You'd have to have this
2.0.296 Erik: I’m trying to figure it out.
2.0.297 Alan: There’s no way to call something half [in this train].
Erik: How do you know?

Brian: [Brian G and Erin as partners. Brian raised his hand and T/R 2 joined them. Brian built the following model: O-LG train with G-W-G train directly beneath.] It's like the other problem we had. You split that [white rod] in half and then put one side on one side [green rod] and then take the other half on the other side [of the other green rod]. It's like what we did last time.

T/R 2: There was no rod that worked perfectly when you take two of them?

Brian: No, because ten and three equals thirteen and thirteen is an odd number.

T/R 2: What does that have to do with it?

Brian: Well, with thirteen, you can’t split thirteen in half equally. Except you take a one and you split it in half and you put one side on, you put one half on one side and you put the other half on the other side, like what we did last time.

T/R 2: Oh, that’s interesting.

Brian: And you would change the color.

T/R 2: Okay, so we are going to develop a new rod again. We'd have to go back to the workshop and make a new rod. Ok. This is wonderful. What do you think, Erin? Do you agree with what Brian said? Have you checked and made sure that there aren't any pairs of the same that would fit here? [Erin nods affirmatively.]

Brian: You could probably do it another way. That's what James did and I thought it would probably work again. Maybe it would work, it would probably work. When he was using the blue with the nine, he was using these others [light green rods], so I thought [He places four light green rods (12cm) under the train of orange and light green (13 cm).] No, no. Oh yeah you could do this like we just did. [He places one white rod between the light green rods. His train is LG-LG-W-LG-LG] Yeah, I think so, yeah.

T/R 2: Okay, so show me where one half would be. One half of that [orange and light green] train.

Brian: Well, right there [he points to the white rod] would be the half of one.

T/R 2: Down the center there.

Brian: Yeah.

T/R 2: Nice thinking, Brian. Let me see what some other people have come up with.

[T/R 2 left Brian and Erin and joined Laura and Jessica. T/R 2 questioned the girls about Jessica’s model, a train of G-G-W beneath the train of O-LG] …for one half. So you'd had
to like invent a new rod. So like, here's the dark greens. And you'd have to, that doesn't work, so you'd have to put a white.

2.0.314 T/R 2: Ok, so what would one half of this green and orange train be? How much of this, in other words? Can you show me?

2.0.315 Jessica: [Stacking one green rod on top of the other: G then G-W on bottom]. Well

2.0.316 T/R 2: So what do you think, Laura? Do you know what I’m asking her? I want to be able to see the one half in my head.

2.0.317 Jessica: This [holding up a train of green and white rods] would be one half.

2.0.318 T/R 2: Okay, that [green rod] and the white?

2.0.319 Jessica: Well, it’s sort of in thirds, but if you, if you like say if this [orange and light green train] was one, then this [green-green-white train] would be two. And you have to like pretend that this [G-G-W train] was one whole right here.

2.0.320 T/R 2: What do you think, Laura?

2.0.321 Laura: Well, I think that like one of these and half of this one [white] would be half.

2.0.322 T/R 2: Okay, so

2.0.323 Jessica: Yeah, half of the white.

2.0.324 T/R 2: Okay, so if I imagine that I had a saw, a small saw and I could cut that [white rod] in half, then you'd take a green and the half of that.

2.0.325 Jessica: Yeah, so then half of this and this would be one half and half that would be the other half, and that would be one half.

2.0.326 FRONT VIEW

2.0.327 Meredith: It’s thirteen. Do we have a seven? What’s seven?

2.0.328 Sarah: Green doesn’t work.

2.0.329 Meredith: Oh, ok. This is easy. One, we need a six. Six

2.0.330 Sarah: Blue [picks up blue rod, puts it down, inaudible]

2.0.331 Meredith: Green, no, purple, I think purple.

2.0.332 40:18 F Sarah: No purple’s right here [pointing to her staircase, Figure F-40-18]. One, two three four five six [pointing to brown rod].

2.0.333 Meredith: It’s not a six, see watch. This is a ten, right? Ten, nine eight. This is eight, two less than this [holding a brown rod]. That’s one less

2.0.334 Sarah: I know that.

2.0.335 Meredith: Yellow

2.0.336 Sarah: No, yellow won’t work.

2.0.337 Meredith: Yeah? Oh, yeah, um.

2.0.338 Sarah: Oh

2.0.339 Meredith: Oh yeah, it will.

2.0.340 Sarah: No it won’t

2.0.341 Meredith: Give me back the yellow

2.0.342 Sarah: No it won’t!
2.0.343 41:08  Meredith: I’ll show you. I’ll prove it to you! Yellow. Watch. [She puts Y-W-Y under the train of O-LG - Figure F-41-08. Sarah chuckles. Meredith dismantles her model.] What’s highest, the next highest after yellow?

2.0.344  Sarah: No no no, you took my thing apart.

2.0.345  Meredith: It’s not that, it was purple, then red, this

2.0.346  Sarah: No green.

2.0.347  Meredith: This

2.0.348  Sarah: No we already tried that.

2.0.349  Meredith: Dark green.

2.0.350  Sarah: Dark green doesn’t work.

2.0.351  Meredith: Who said it doesn’t. Watch.

2.0.352  Sarah: I tried it.

2.0.353 41:53 F  Meredith: Yes it does, remember halves? [Meredith’s model: DG-W-DG. Figure F-41-53]

2.0.354  Sarah: Yes, I do. [Both girls raise their hands.]

2.0.355  Meredith: Oh, oh! It works. [T/R 2 joins them.]

2.0.356  BOTH VIEWS

2.0.357 0:43:27  T/R 2: Can I join you? Have you come up with an idea?

2.0.358 0:43:33  Meredith: Yeah, since, I didn’t end up doing halves with the whites, I took two greens and I put a white in the middle, and if I cut the white in the middle in half, then you would have six and a half and six and a half.

2.0.359  Sarah: And cut that up.

2.0.360  Meredith: And then you would have it.

2.0.361  T/R 2: Okay, so you’re telling me, I see what you’re doing. Ok. You’re going to cut this right down the middle to give me a half.

2.0.362  [To the class] Ok. I want to hear from some people now. I hear some wonderful thinking here. This was not an easy one. Ok, is there somebody who would like to share a solution with me to show me one half of this train which has an orange and a green? To show me something that is one half as long as this orange and green train. Ok, let’s see, uh, Andrew? Could you come up and show? If you worked with Mark the two of you can come up and show us? I’d like everybody to watch what they do because if you have a different way of thinking about it, I’d like to see that also.

2.0.363 0:44:41  Andrew: Well we thought that if we had to find one half of that, we took two dark greens and the white one. And we said if we split the white one in half, then it would be half, because if you put the white one there, it would equal up the the train that you made. [Andrew places a train of green, white, and green on OHP next to orange and green - Figure O-46-42]

2.0.364  Meredith: [whispering] Quick, give me the yellow, I need the yellow. I have a solution, I have a solution.
T/R 2: Ok, do you all follow what Andrew said here? Erik, did you have a comment on that? You had something different. First of all, what do you think of this? Does this work? Looks like we're into inventing our own rods again, right? Making up a rod, a new rod here. Why do you think that works? I mean, why do you think that that works? You have any ideas? Meredith?

Meredith: Well, because you want to have seven and six, seven, but there are no rods that are really seven, and you need it to be thirteen. So, those two blocks and half of that would equal up to it, and it would help-

T/R 2: Ok, can you say a little bit more about that?

Meredith: Well, take the two greens and take a white. And you do that.

T/R 2: So you're showing the two greens and the white that are up here. Ok, it's just like our picture up here

Meredith: And there's no blocks that have half on them, and for the uneven numbers, for the odd numbers you need a half, because you can't make it without it.

T/R 2: Ok, Brian said something like that too, about the numbers being odd. Brian, what did you want to add?

Brian: Well, like what we did last time with, when Mrs. Maher was talking about, about if we split the gold equally, what you could do is, well, I thought of a lot of ways. So like, once I have the white cube right in the middle, you split that in half, right in the middle. That's what we did last time.

T/R 2: Great! Ok, well you came up with several different ways. I see one of the ways that Brian has is, he used light greens, all light greens and one white, right? Ok. That would be another way to do it, wouldn't it? That's really very nice. [Brian built 5 models for the orange and light green train: G-W-G, LG-LG-W-LG-LG, R-R-R-W-R-R-R, P-Y-P - Figure S-47-12] Ok you two can take a seat. Does anybody else have anything they want to add to this before I begin a new problem? [Sarah and Meredith raise their hands] Ok, let's see, Sarah? [Sarah and Meredith go to the overhead.] You can take these off if you want to. Oh, you need another light green, you know I don’t think we have any for the overhead, so maybe we could just use one of the regular - why don’t we try those, ok? [talk about the rod looking black on the overhead] Well, we can pretend that it’s a light green, can’t we? Ok, go ahead.

Meredith: [She builds a train of Y-LG-Y - Figure O-48-31] If that's a light green, then you could just make a yellow and add one and a half to the yellow and one and a half to the other yellow.

T/R 2: What do you mean, one and a half? Does anyone know what Meredith means? I don't want you to tell me yet. Does
anyone knows what Meredith means by adding one and a half to the yellow on each side? Where did she get one and a half from? I see a couple of hands, let’s see. Graham.

2.0.376 Graham: The light green
2.0.377 T/R 2: The light green, ok. How does this become one and a half? What piece of it [the train] becomes one and a half? I don’t understand.
2.0.378 Graham: You like split it in the middle, and it would be one and a half on each side. [He holds up the light green rod and shows cutting through the middle of it.]

2.0.379 0:49:08 T/R 2: Oh, okay, all right. So if I cut that [Light green rod] down the middle, I see, okay. Well, if we’re calling this light green three, what are you calling this train with the light green and the orange together?

2.0.380 Meredith: Well, the yellow is I think the yellow’s um, I think yellow is about five long, and the green in the middle [Counting cm in the train] Ten [two yellow rods], eleven, twelve, and then thirteen [for the light green rod], thirteen yellows.

2.0.381 T/R 2: You were thinking of the whole length of the train as being thirteen of what?

2.0.382 Meredith: Thirteen
2.0.383 T/R 2: Thirteen blues, thirteen oranges, thirteen what?
2.0.384 Meredith: Thirteen yellows.
2.0.385 T/R 2: Thirteen yellows?
2.0.386 Meredith: If you turn the light green into yellows.
2.0.387 T/R 2: I don’t understand.

2.0.388 Meredith: Well, if you cut that [light green rod] in the middle and then you just paint the light green of each piece yellow and you’re making it thirteen and it will be equal to the train.

2.0.389 T/R 2: Do you understand my question, though? She keeps saying thirteen for the train that I made with the orange and the green. I don’t understand where she’s getting the number thirteen from. Why thirteen?

2.0.390 Erik: Wait, she's getting thirteen from the number of the whole train?

2.0.391 T/R 2: Well she keeps saying that the length of this is thirteen.
2.0.392 Erik: Yeah, I know, I know where she's getting it.

2.0.393 0:50:47 Erik: Well, see, If you take one of the orange rods and take all these little things [white rods] and you put it up to it, it will equal ten. And then if you do the same thing with the light green rod, it'll equal three. And if you have ten and three it's thirteen [As Erik speaks, Meredith lines three white rods on top of the green rod - Figure O-51-14].

2.0.394 T/R 2: Oh! So then what you're saying is if you line up the little white cubes along the, uh, the train with the orange and the
green there'd be thirteen of them? I understand, ok, I understand what you're saying, that's wonderful.

2.0.395  Erik: Yeah, thirteen
2.0.396  T/R 2: Do we have a minute to do another one, or do we have to clean up.

2.0.397 0:51:45  Erik: I have another solution. [Some talk by T/R 2… He goes to the overhead and puts two light green rods under the orange and light green train. He adds seven white rods to the right of the light green rods - Figure O-53-00]. I figured you could take two light greens and put them there. And then after that I just took all these, the clear ones [white rods]; and I figured, well, I put down seven. And I figured that they all equal, and if you have these two you would have three and then you could take one and put it on that and so it would be four, five, you would have three, four [He motions that he is adding one W to the LG , one W to the other LG, etc.], and then four, five, five, six, six, and then seven.

2.0.398  T/R 2: Ok, alright. So you figured then that you can put, have seven on each of our halves?
2.0.399  Erik: Yeah, of the halves, and then like you’re making a new rod.
2.0.400  T/R 2: So there’d be seven and seven? What do you think about that? He’s saying

2.0.401  Erik: Yeah, well no, well, I mean, not seven and seven, seven and six. It’s an odd number of white, theclears, so it wouldn’t be seven one would be seven and one would be six

2.0.402  T/R 2: Ok, so, in other words, one of these could go here with this group, one of these goes here, back and forth like this. Ok, what happens to this guy, though? [pointing to the white rod to the far right] How can I be fair in making my two halves the same size? What could I do?

2.0.403  Erik: What you could do, I think what you could do is, hmm, you could take this [white]. And you could replace those two, those three with a light green, yeah, one of the light greens like that. [He moves three whites and places a light green in his model.]

2.0.404  T/R 2: Uh huh, oh, but I have one for this guy, and one for this guy, one for this guy, and what about this guy? [She points to the remaining W on the far right.]

2.0.405  Erik: Oh, what this guy would go
2.0.406  T/R 2: I think we ran into the same problem, didn’t we? Would you agree that if we went back to this model, Erik, where we had these [She rearranges the rods] and we were divvying them up. Would you agree that maybe I could take this one [white rod] and saw it in half, if I had a saw?

2.0.407  Erik: Yeah.
2.0.408  T/R 2: And then what could I do with it, if I sawed it in half?
2.0.409  Erik: Then you could put half here and half here [pointing to the two columns of rods]

2.0.410  T/R 2: Ok, ok, I think we are almost out of time, aren't we? We probably need to clean up. Unless we have a minute? Ok, yes, ok, what do you have to share with us? Brian?

2.0.411 54:55  F Brian 2: [Brian, directly in front of T/R2, raises his hand and shows his model of P-Y-P - Figure F-54-55] If you take two purples and you put them on the sides and you put yellow in the middle and you cut it in half, then there'd be, then it would be equal.

2.0.412  T/R 2: Ok, that's sort of like the solutions that other people were talking about, but you used different colors in order to show that, right? That's really nice. Ok I think we're going to clean up for today, and one thing that we're hoping that you all do, hopefully we're coming back on Friday, but what we're hoping you'll all do ... is write about what we worked on the past few days, etc.

2.0.413 0:56:42  S End of class.
I wasn’t here earlier this week. And I know that Dr. Martino shared so much with me but I, I still was just so interested in some of the questions you were asking. And, I thought maybe we would start with the one that I understand there was some discussion about and see if we could think about that together. Do you know what that question is? Do you remember that? Can you read that statement? What does that say, Michael?

5.0.4  
*Task 1 is 1/5 = 2/10?*

5.0.5 08:35  
Michael: It says, is one fifth equal to two tenths? [Figure S-8-31]

5.0.6 8:40  
T/R 1: How many of you agree that is what that says? That’s my question. Is one fifth equal to two tenths? [quiet in the room]

5.0.7 8:48  
T/R 1: Do you remember talking about any of those ideas at all? What do you remember about that, Meredith?

5.0.8 8:54  
Meredith: Um?

5.0.9 8:55  
T/R 1: You want to tell us?

5.0.10 8:56  
Meredith: We had used our Cuisenaire Rods.

5.0.11 8:59  
T/R 1: Come up and show us what the issue is. Because, Dr. Davis also wasn’t here.

[Meredith moves to take position at the overhead to explain].

5.0.12 9:11  
Meredith: Say this is called one. [She positions an orange rod on the screen.] Then you … took this …. [Then she positions what looks to be two yellow rods of equal length under the orange rod. Both rods together are equivalent to the length of the orange rod.] This would be half of the orange rod [pointing at one yellow rod] … and this would be un …. called the one [pointing at the orange rod] and these would be one [pointing at the orange rod] and these would be called the two halves [pointing at the yellow rods - Figure O-11-08].

5.0.13 9:34  
T/R 1: Ok, so what name would you give the yellow rod?

5.0.14 9:37  
Meredith: A half

5.0.15  
T/R 1: One half. Ok.

5.0.16 9:42  
Meredith: And, this would be called one [pointing at the orange rod]

5.0.17 9:45  
T/R 1: And that would be called one. Can you tell me what that has to do with my question? Is one fifth equal to two tenths?

5.0.18 9:49  
Meredith: Um. Two tenths. You call this two tenth. This is usually a tenth [she places two yellow rods on the projector] and then you put this up and you call this two tenth and then you take this and you put it up to them [she places five red rods near the yellow rods] and it’s five- they equal up to it [Figure O-12-02].

5.0.19 10:22  
T/R 1: Can you say that so the class hears you and see what they think? I am not- did you all hear what Meredith said? I am not- I didn’t hear either. But maybe if you can tell the class what your thinking, let’s see what they think.
Meredith: Well if you have the orange rod and you put it up to the yellow rod, the yellow rods, it’s two halves and that would be called two tenths. Uh, yeah. Two tenths… wait two tenths would be …… two ones … [she places 2 white rods on the screen] That is what two tenths would be, ‘because these ones are tenths [pointing at the white rods and looking at T/R 1 - Figure O-12-52].

T/R 1: Why don’t you tell the class and-

Meredith: These ones are tenths [pointing at the white rods] ‘cause this orange rod is one and when you put the ones up to it there’s ten of them and two of them would equal two tenths. And this [she moves the red rods directly above the orange rod] is one fifth to the orange rod. And if you take one of them [she moves one red rod above the two white rods] it’s equal to two tenths [Figure O-13-21].

T/R 1: So what is your conclusion if I ask you the question, is two tenths equal to one fifth?

Meredith: Yes.

T/R 1: You think it is. Ok, let’s have some discussion; Thank you, Meredith. Let’s see what other people think. Do we have some other discussion about that? Brian?

[Brian comes to the overhead.]

T/R 1: By the way, how many of you agree with what Meredith said? How many aren’t sure? How many of you disagree? Ok, so we have some not sures and agree, so let’s see Brian which category are you in? Do you agree or are you not sure?

Brian: I agree.

T/R 1: Ok, you want to explain why you agree?

Brian: Well, I agree because it’s just like having one of these reds being a whole and one of these [a white rod] being a half. So it’s just like saying, it’s just like saying, two halves equal a whole. It’s the same as being two tenths equal one fifth.

T/R 1: What do you think? Erik is making a face. Erik do you want to, do you want to say what you think out to Brian? [Erik goes to the overhead.]

Erik: Well, I kind of agree with Meredith, because if you take the orange rod, it takes ten of the white rods to equal up to the orange and five of the red ones. Then if you take two of these [points to white rods] which is two tenths and this [red rod] is one fifth. Because, well, it’s, it takes five of them to equal up to the orange rod and, and if you put two tenths next to it they equal up to each other and you’ll have one fifth and one tenth- and two tenths together and they both equal up to the same amount.
Brian: [whispering] Like what I said kind of. [Erik and Brian chuckle]

T/R 1: Ok, anybody else? Any other discussion? Thank you gentlemen. [Brian and Erik return to their seats.]

T/R 1: Any other discussion? Anybody else have something to say about that? [No response] How many of you believe that one tenth-two tenths and one fifth represent the same length? Do you think it would be okay to give them the same number name? How many of you think it would be ok? We can call this two tenths or we can call this one fifth. How many of you agree with that? [several hands go up] Yeah. Makes sense doesn’t it?

Task 2: What other number names can we give to one half of a candy bar?

T/R 1: Do you remember the candy bar? Remember. Did you get any? [Students say yeah.] This was the little candy bar, it looks something like this. If you excuse my sketch. Do you remember it looks like this. It was broken up into three columns and then four rows. Do you remember that? Remember I was giving half of this little candy bar to Dr. Martino ……..? Remember how appreciative she was? Remember that? We said we were giving her half the candy bar, didn’t we? Right? Okay? Could someone have told me another name, another number name for how much of the candy bar I gave her? Do you understand my question? If this is my candy bar and I gave her that much, right? One number name we said was one half, didn’t we? Can someone think of another name that very exactly tells me how much of the candy bar I gave Dr. Martino? And if you think you know, why don’t you discuss it with your neighbor and see if you agree. [discussion] Ok, so discuss it. You have to be able to prove it and say why. [Groups working together - Figure O-17-08].

T/R 1: Ok, Are you ready to share your ideas? I heard a couple of different number names. How many of you think you have another number name that tells me how much of the candy bar Dr. Martino got? We already agree that one number name was one half. Right? That represented how much of the candy bar. How many of you think you have a different number name? Ok, Jackie. [Jackie stands]

Jackie: Um, well we thought it was six twelfths, because-

T/R 1: Jackie, I am sorry I can’t hear, it’s so loud here

Jackie: Um, we thought is was, um, six twelfths because there are um twelve pieces in all and she got six, and six makes half.

T/R 1: What do you think?

Erik: I have another one.
T/R 1: What do you think? Did you all hear, uh, what Jackie said? How many of your heard what Jackie said? Raise your hand if you heard what she said. [several hands go up] How many of you agree with what Jackie said? [several hands go up] That another number name you said Jackie, was…[asking the student to repeat]

Jackie: Six—

T/R 1: --Six twelfths. How many of you agree with that? [several hands go up] Does anyone disagree with that? [all hands go down quickly] Does someone have a different number name? I heard some other number names as I walked around. [Danielle raises her hand] Danielle?

Danielle: Um, I thought, um, it would be, um, two fourths.

T/R 1: You thought two fourths? How did you think that?

Danielle: Because if she got a half, then the top two rows, um, is a half, and then that’s two fourths.

T/R 1: Then we can think of this as two fourths. What do you think about that? How many of you think that’s another number name? Two fourths? Some people aren’t sure. Danielle why don’t you come up and show them what you’re thinking? I am not sure if people were following you. [Danielle walks up to the projector] Danielle thinks that another number name for how much of the candy bar we gave Dr. Martino is two fourths so we’re interested in knowing how Danielle was thinking about that.

Danielle: Well, this row and this row is a half [points to top two rows together] and then these are two fourths. [pointing to the top two rows again]

T/R 1: Why are they two fourths? Can you help the class understand that?

Danielle: Because there’s four rows.

T/R 1: Because there are four rows and we if talk about the first two rows that’s two fourths? What do you think? How many of you think that’s another number name for how much of the candy bar we gave her. How many of you agree with that? How many of you disagree? If you disagree, why do you disagree? [Kelly and Mark seem confused.]

Mark? Not sure why. [Mark shrugs his shoulders] Kelly was your hand up for disagree. No. Is it that you don’t see that it’s two fourths? You don’t see that it’s two fourths? Can someone help Kelly and Mark then, they don’t disagree, but they don’t see it. Brian?

Brian: I just found out another way that a half can be… [block noises and mumblings cut off the last part of what he said] [Brian says he found another number name and is asked to wait until the answer of 2/4 is explained. He tries to explain.]

T/R 1:
You have another way that a half would be. Ok, remember that and hold on to that. I would like someone to help the people who don’t quite see that that’s two fourths. Can some one try to explain that? Who wants to pretend to be a teacher or explainer, if you agree? Somebody want to give it a try? Nobody wants to try? Brian?

Brian: If I agree? Ok. [Brian proceeds to the front of the room to use the screen] Well, I agree on two fourths because there’s, because three times, because four times three equals twelve and if you split it in half there’d be three fourths,[he has turned the grid around]. Well, well, I agree because there are four thirds on there but, but when there’s a half there are only, there’s three fourths instead of four thirds.

Wait. Hold up. Where are your four thirds?

Well…well, well when you said … to split it in half [he covers six of the twelve blocks] like that there are only two fourths left over, yeah.. there are only two fourths when it’s a whole there are four thirds … I can’t explain it too well…

I am not sure I understand what you are saying. Can some else help? Danielle do you want to go up there and ….? Danielle:No, I just have something to say …. Cause, If there’s four fourths, um, and half of four is two, so two fourths would be a half.

But I thought about it the other way around.

What do you mean by the other way around, um, Brian?

Brian: Well, instead of, well she went two fourths like going across, like cause those are fourths and that’s a fourth and there are two of them. I have another way. It’s, I have three, I think three sixths.

Three sixths?

Because well, because a whole, there’s a whole, if there’s [looks at overhead and counts with fingers] yeah it’s three sixths. Because there are six of them, there, there, there I found groups of sixths. There’s one sixth, another sixth, another sixth, another sixth, another sixth, another sixth and if you split them in half, there’s three of them on top

That’s very interesting. What do you think about that? Brian found another way. That’s very interesting, Brian. So, he is saying that these two little pieces of candy, and these two, and these two and these two, and these two and these two, I don’t know if that is what you said Brian, but something like that.

Yeah.

You can talk about taking this candy bar and sharing it in six pieces where we would have two wedges to be one piece. Do you follow that? That’s neat. And so you are saying that,
if you got three of those, right? One of the two, one of the two, one of the two, then you still get a half of candy bar. So, what do you think? You’re saying that you think that three sixths is another name for one half? What do you think? Ok, let’s write this down. Thank you very much Brian, that’s interesting.

5.0.70 24:12 T/R 1: So what do we have here? The first question, does one half equal two tenths? Do you believe that? No, I am sorry. We said one fifth equals two tenths. When we went to our candy bar we said one half equaled? One half of the candy bar. This [one fifth equals two tenths] was with the rods.

5.0.71 24:49 T/R 1: One half equals six twelfths. Do you agree to that? Six twelfths? What else? What’s another number name? Brian?

5.0.72 25:05 Brian: We did two fourths?
5.0.73 25:06 T/R 1: Nice and loud.
5.0.74 25:07 Brian: We did two fourths? [louder]
5.0.75 25:08 T/R 1: Two fourths….Do we have another number name for half of the candy bar? Jessica?

5.0.76 00:25:18 Jessica: No, I was going to say one half.

5.0.77 25:22 T/R 1: We have one half. Are these all the ones we had? Danielle.

5.0.78 25:25 Danielle: There are six twelfths

5.0.79 25:26 T/R 1: Six twelfths, I think we have. We said six twelfths, two fourths, we said one half, Brian?

5.0.80 25:35 Brian: Three sixths

5.0.81 25:36 T/R 1: Brian says three sixths. Remember that? Is that correct? Three sixths. How did we get three sixths? Do you remember? [Brian shakes his head yes. Figure O-27-07]

5.0.82 25:46 T/R 1: Any others? What do you think about that? Let’s go back to the original question here. You agree then that if we call this one and we call the red rod what number name?

[mumblings of one fifth from the students]

5.0.83 26:10 T/R 1: One fifth and we call the two whites together what number name did we give it? The two whites together.

5.0.84 26:16 Brian: Two tenths…

5.0.85 26:17 T/R 1: Two tenths. I can also give it the number name…..? What else can I call the two white ones besides two tenths? What did we decide? [Quiet] What other number name can I give the white ones besides two tenths, if I call the orange rod one. Michael? [Figure O-28-29]

5.0.86 26:48 Michael: One fifth
5.0.87 26:49 T/R 1: I can call it one fifth. Alright. And what other number name can I call the red one.

5.0.88 26:55 Michael: Two tenths
5.0.89 26:56 T/R 1: Two tenths, is that right? [Michael: Mmm hmmm]

5.0.90 Task 3- Which is bigger one half or one third and by how much?
5.0.91 00:27:00 T/R 1: Very interesting. Ok. Do you have any comments or questions about this? I see some beautiful things you are making. Some beautiful pieces of architecture. Ok, well maybe we will leave this go. Tell me what you did the last time Dr. Martino was here. What was the problem you were working on? [Michael raises his hand] Anybody want to tell me and tell Dr. Davis? You were working on a problem I think in class together I think you were in groups, weren’t you? Do you all want to think for a moment and maybe discuss with your partner to help you remember what you were working on? [Michael’s hand is still up] Michael?

5.0.92 27:44 Michael: We were working on the candy bar problem. Like, with like which is bigger a half or one third and we were using candy bars to show that.

5.0.93 27:55 T/R 1: Ok, so you were working on which is bigger, one half or one third. Andrew?

5.0.94 00:28:00 Andrew: Yeah, we were working on, we had to write about um and we had to do an example on it, and um to see if which is bigger, one half or one third.

5.0.95 28:11 T/R 1: How many of you worked out which is bigger? One half or one third? [several hands go up] How many of you think they are the same? [all hands go down] How many of you think one is bigger? [several hands go up again] Which is bigger? One half or one third. Laura.

5.0.96 00:28:31 Laura: One half.

5.0.97 29:32 T/R 1: You say one half is bigger. What do the rest of you think? Do you think one half is bigger? [several students provide affirmation] Do you think you can convince Dr. Davis that that’s the case? [several hands go up] Can you convince Dr. Davis that one half is bigger than one third. By the way, do you know how much bigger? How many of you think you know how much bigger it is? Ok, that’s the second question. Ok, I really would like someone to come up. Jessica maybe and Laura can come up to the overhead and show Dr. Davis how you decided which is bigger. And see if you can convince us of your result. [Jessica and Laura come to overhead - Figure O-31-38].

5.0.98 00:29:32 Jessica: Well, um, one third would be just this piece here [she points to the purple rod] and one half of that would be [she sets up two dark green rods] and one half would be this [one dark green rod] and one third is bigger than one half cause this [purple rod] would be one third and then this bigger piece [dark green rod] would be one half of that. And-

5.0.99 30:18 T/R 1: Can you tell me what number name you’re calling the orange and the red rod?

5.0.100 30:22 Jessica Um, one.
You’re calling the orange and red rod one? Can you say that again, what number names gave to each of those rods so I can hear from back here?

Jessica [whispers to Laura] You say. Um, this would be, this, we’re counting this as one whole [orange and red train] and I think this [dark green rod] has two and this [purple rod] has, wait, um. Um [giggling], um I can’t we called it, yeah, [Laura helps her out] I think this one was one-

Laura: That was one third

Jessica: this was one third, and this was one half.

Laura: One half.

What do the rest of you think? What do you think? Audra what do you think of what… the two young ladies built up there?

Audra: I agree because-

Want to speak to the class [asking her to go up front]

Audra: [comes to overhead] I agree because if you saw what the, um half, was here and then you saw what, no, what the half was here and then you saw what the third was there, and you saw that the half was bigger than the third.

How many of you agree with the argument that a half is bigger than a third with the argument that was made here? Ok, did you figure out how much bigger?

It’s two, two [places two white rods next to purple rod]

It’s a red bigger, but [Figure S-32-28]

You’re saying it’s a red rod bigger or two white ones but that’s what I see you have built there, but I would like you to tell me what number name you have for how much bigger it is.

Um, wait, it’s one third bigger, I think [organizing the dark green and red blocks together].

Jessica: I think it’s one third bigger too because if you put the red to the green

You’d see that there’s three

You need three and if you put the purple one to it also and then it takes one third of them. [Showing the purple differs from the dark green by one red block - Figure O-34-49]

Okay so these young ladies have proved that one half is bigger than one third and it’s- one half is one third bigger than one third. What do you think? That one half is one third bigger than one third. What do you think about what they just proved? Now you were all watching their argument up there and have they convinced you? [Child in front row has his hand up] I don’t know if they have convinced Dr. Davis. Um, but I am wondering if they have
convinced you? Kelly. What do you think, do you agree with this? [Kelly stands up, and comes to the front of class]

Kelly: Um, yes.

T/R 1: You agree with them.

Kelly: Well, if you have, um, a red, if you hold, um, well, if you have, um, well we used these and we went like, and then we like held reds up and we showed that um, that um, one half is bigger by, because this part is smaller, and this is supposed to be one, one third so that’s how we did it [Figure O-36-23].

T/R 1: Brian you are making a face, what do you think? Do you agree with them?

Brian: Not really.

T/R 1: Brian doesn’t agree with you

Audra or Jessica I think that’s like changing the problem because we are using the dark greens and she [Kelly] is using the light greens.

T/R 1: Oh, hmm.

Jessica: If you take this and it has three thirds [Brian builds the model on his desk]

T/R 1: Let me make sure I understand this. You’re calling this one, right? And you’re calling this one third,

Jessica: And we’re calling this

T/R 1: And you’re calling this one third

Jessica: One third

T/R 1: Right? And you’re calling this one half? And you’re saying one half is bigger than one third by one third? [Girls agreeing with her as she demonstrates what they said]

Jessica and Audra: Yeah, you can put three of these, three reds up to one green and then it would take one, one third of the red to make um, to go there, like.

T/R 1: Ok, I would like all of you—How many of you agree? How many of you disagree? Now if you disagree you have to say why you disagree because they are saying that one third is smaller than a half, one half is bigger than a third, it’s one third bigger than a third, that’s what they are saying. Now either you have to agree, or disagree or not know. How many of you aren’t sure? [several hands go up] A few of you aren’t sure, but some of you disagree. And if you disagree we have to say what’s wrong with their argument. There must be something wrong with their argument if you disagree. Or maybe their argument is right because I’m very confused. Brian what do you think?

Brian: Well, when they said one third is bigger than one half by one third. I think they said, is that what they said? Well, I don’t really agree, because well if you split, if you
split one of the thirds in half which would make [counting the blocks], which would make a sixth. I think it’s a sixth bigger. Like, well, [holds his rods], um should I go up there?

5.0.136 37:11 T/R 1: Sure, ladies can you make a little space here for Brian. Maybe you need to have a little conference here, we have some disagreement.

5.0.137 37:21 Brian: [He goes to the overhead.] Well, see for um, when they said it was one half bigger, if you split a third in half it’d make a sixth, like one, two, three, four, five six. Like, like pretending they were, like pretending they were split in half. If you split one of these in half and you have three of them up there they’d make, they’d make six and any way, and when you split them in half right in the middle over there it’s kind of like that, it’s kind of like this, there was this was, that was the one third [points to a purple rod] and that was the one half [points to the dark green rod] on the bottom and so it’s just like this and the red I’m pretending is like, is like, is a half of one of the purples and you see when I split it in half it’s, it’s one sixth and, and it equals, and it equals up to a green [Figure O-39-51].

5.0.138 00:38:30 T/R 1: I’m hearing you say Brian that the number name for red is one sixth and the reason why is—

5.0.139 38:32 Brian: Well, I mean a red, I’m considering a red one sixth [Dr Maher: yeah] because two of these [red rods] equals, see they’re two, they’re two sixths, two halves of one purple and the purple is a third and the half of one third is sixth, there’s sixths [Figure O-40-19].

5.0.140 .00:38:57 T/R 1: So you’re giving a red the number name one sixth and I understand the young ladies up at the overhead are giving red the number name one third and can red have the number name one third and one sixth at the same time? That’s my question.

5.0.141 Brian: Well, what I mean is-

5.0.142 00:39:13 T/R 1: I heard what you said Brian, I just wish everyone would listen here because your going to have to decide and write about this in a few minutes and you’re going to have to decide of the arguments which you agree with; Brian’s argument or the argument of the other people. And you need to know the arguments of both people so you can write about them and tell me which do you believe and why. And if you don’t believe an argument you have to tell why you don’t believe it, and if you believe an argument you have to be able to prove it. So we have two different arguments at the table and it’s very important that you listen so you understand what the arguments are. [camera pans to Graham who appears to have made the Eiffel Tower with his block, this
may be because he is bored or does not understand what is going on]

5.0.143 00:39:50 Jessica: Kelly and Jackie have something else that like goes with this like—

5.0.144 39:55 T/R 1: Ok we will hear Kelly and Jackie and we will hear Brian’s again. Brian said it and I know some of you heard it, I heard it. But I would like you all to listen to these arguments.

5.0.145 00:40:02 Jackie: Well, we would call this dark green one and the reds one third and the light green one half, and we thought the, we thought one third was bigger by one of these white things. [Her model is using 6 cm as the unit.]

5.0.146 40:21 Jessica: Oh, I think they’re making a different size candy bar

5.0.147 40:25 T/R 1: Is that allowed?

5.0.148 40:28 Jessica: Um, no.

5.0.149 40:30 T/R 1: Why not? What’s wrong with that? In what way it is not fair?

5.0.150 00:40:33 Jessica: Because if you give someone half of this one [12cm?] and then one half of that one [6cm?] and this is bigger than [takes a light green and dark green rod in hand].

5.0.151 40:50 T/R 1: Ok so what do you ladies think? Are you making different size candy bars? What are calling the candy bar when you started the problem? What was one? What did you call one if you’re thinking of candy bars when you began the problem?

5.0.152 00:41:00 One of the girls: The dark green…

5.0.153 41:02 T/R 1: Is that what you built when you went up there, you said the dark green is one? Is that what you said?

5.0.154 41:10:06 One of the girls: Yeah…[the girls look at each other in agreement]

5.0.155 41:10 T/R 1: Ok then use the—okay if your calling dark green one then I want to hear your argument which is bigger a half or a third and by how much?

5.0.156 00:41:14 Jackie: Okay, we think that a half is bigger than the third.

5.0.157 41:18 T/R 1: Okay you think a half is bigger than one third and you’re calling the dark green one? Did you change your mind?

5.0.158 41:23 Jackie: Yeah, and we think light green is a half [of the 6 cm model].

5.0.159 41:25 T/R 1: Well show me your argument now and tell me which is bigger a half and a third and by how much?

5.0.160 41:31 Jackie: Okay, this is, this is a half [light green] and the red is a third.

5.0.161 41:35 T/R 1: Can you show me why that’s a half?

5.0.162 00:41:36 Jackie: Because if you put these all together they equal up to the one…[Showing that three reds, two light greens both equal the dark green which is one] and we think the light green which is a half is bigger than the red by, by one which is this white one. [Showing the difference between red and LG is a white. Figure O-43-42]

5.0.163 T/R 1: Ok, I see that you switched what you made, um, your model, uh, but you showed me that one half is still bigger than a
third and you still believe that. But what number name did you give to white? You said it was a white rod bigger but I didn’t hear what number name you gave to white. I thought I heard you say it’s one bigger

5.0.164 Jackie: Yeah.
5.0.165 T/R 1: Did you say that?
5.0.166 Jackie: Yeah, the green, the light green is one bigger than the red. And the red is one bigger, the light green is one bigger
5.0.167 T/R 1: And what number name are you calling the white?
5.0.168 Jackie: One
5.0.169 T/R 1: You all agree with that?
5.0.170 Jackie: Actually, I used this to um, to tell that the light green is one white bigger.
5.0.171 T/R 1: Ok, and the number name you are giving to the white you’re saying is one,
5.0.172 Jackie: Yeah.
5.0.173 T/R 1: you called the green one and your calling the white one?
5.0.174 Jackie: No. [giggling]
5.0.175 T/R 1: That’s what I thought I heard you say. [asking the class] You hear my question? Is everybody hearing my question. You said you called the light green one, you said you called the red one third, and you said you called the light green one half. Right? And now the white one, right… [puts the white and red together next to the light green] The white one which tells you how much bigger it is, you said you’re calling it one. So your calling this one and this one [pointing to the white and dark green].
5.0.176 Erik: [from his seat] I think I know what they mean.
5.0.177 T/R 1: Erik, what do they mean I’m so confused.
5.0.178 Erik: [walks to the overhead] I think they mean that they want to call this, the dark green one, one whole, and they want to call this, yeah, like you line all the whites up to it which I think should be six and they want to call it one sixth. I think that’s what they’re trying to say but they just, they’re just not saying it. I think they just, they want to call it one sixth [Figure O-45-54].
5.0.179 T/R 1: I don’t see six of them up there.
5.0.180 Erik: Well however many are up there that what they are trying to say.
5.0.181 Jessica: Yeah because I think they meant
5.0.182 Erik: I think you wanted to say not one whole but one sixth [Figure O-46-07].
5.0.183 T/R 1: Is that what you meant to say?
5.0.184 Girls: Yeah.
5.0.185 T/R 1: So you’re saying then you all agree, that’s what, you all really wanted to call the little white one, one sixth and not
one? When you call the light green one? So I’m a little concerned now? Are you agreeing with Brian or disagreeing with Brian that the number name that you would give for how much bigger one half is than one third? Is how much? One half is how much bigger than one third?

5.0.186 44:45  Girls:  Um, one, one sixth.
5.0.187 44:46  T/R 1:  Is it one or one sixth?
5.0.188 44:48  Girls:  One sixth.
5.0.189 44:49  T/R 1:  You’re sure it’s one sixth?
5.0.190 44:50  Girls:  Yea.
5.0.191 44:51  T/R 1:  Why can’t it one?
5.0.192 44:52  Girls:  Because that’s be um, the dark green.
5.0.193  T/R 1:  The dark green is one? I understand when-
5.0.194 00:45:01  Erik:  But I think you can call it one because you can make the dark green bigger—
5.0.195 45:05  T/R 1:  But they didn’t, they called the dark green one, Erik —
5.0.196 45:06  Erik:  [continuing]: You could call, you can call the dark green one six, the dark green rod six and then you could call the light green rod three—
5.0.197 45:15  T/R 1:  But can you do that in the same problem?
5.0.198 45:18  Erik:  No [slumps down in his chair]
5.0.199 45:20  T/R 1:  Yeah you can’t change the rules in the problem, now I want to go back—[maybe recognizing Erik ]that’s very very nice and Erik that was very helpful to me and to the folks up there but I still want to go back to the problem Brian was helping them with the problem up there, I still wonder if we can solve this one because you started with this other one and you said that the orange and red [together] are one, right? Isn’t that what you said?
5.0.200 45:43  Jessica:  This is one whole, and then this is one third and this is one half. [pointing to the three different rod lengths]
5.0.201 45:45  T/R 1:  Right, and you said it’s bigger by the red, right? And the question was, what number name do you give to the red? Now if you really understood what mistake you made here maybe you’ll figure out what mistake you made up there.

[girls whisper to each other]
5.0.202 00:45:58  Jessica:  Well, we and we, um, named, well, three reds equal up to um, one greens and then you put the purple next to it and you need one more red, you need a red to go next to the purple, so it would be one third.

5.0.203 00:46:34  T/R 1:  Well how can you build a model and say that one half is bigger than a third by a sixth and build another model that says one half is bigger than a third by a third? How is that possible? I am so confused. Brian, it’s just his face tells me that he is so unhappy with that. Do you believe that Brian? They’re still telling me that one half is bigger by—
one half is bigger than one third by one third. Can anyone tell me what’s going on here? I am so confused.

5.0.204  00:47:16  Brian:   I don’t- I still don’t think so, well, because, well, well, see like I said before when you split the ahh, when you split the thirds in half and they make sixths, it’s still like [He goes to the overhead.]

5.0.205  47:48  T/R 1:   So Brian is giving the red rod a different number name, he’s not calling it a third he’s calling it a sixth. They don’t believe that though, they still want to call it a third. Someone has to-

5.0.206  48:02  Brian:   See, well, because when you put it right there you see that, you see that there’s one of these, if you put one of these on top of it you might see that, that it’s that much that, that red, that red is that much bigger than one of the halves because one of these reds I’m calling is, is a sixth and anyway a half of one of these, a half of one of the thirds. But when you put it on top of one of the thirds it’s that much bigger than one of the halves [Figure O-50-08].

5.0.207  00:48:51  Jessica:   Well, I think they might both be answers.

5.0.208  48:54  T/R 1:   You think it can be a third and a half? How many think they could be a third and a half? How many of you don’t think it could be a third and a sixth? How many of you disagree?

5.0.209  49:10  Erik:   I don’t think you can have an answer of a third because if you have one half [he goes to the overhead] and if you take the one half which would be the dark green, you have the one half and then these [purple rods] are the thirds. How could one half be bigger than the thirds by one third? Because, and you have the half and the thirds together that the half is almost as big as two thirds, but yet the two thirds aren’t exactly, are not exactly, the green, the dark green is not, the dark green is not exactly as big as two, two thirds but, two thirds, it’s the, but it’s far enough so that the two thirds are not bigger than it by one third [Figure O-51-53].

5.0.210  00:50:15  Brian:   I kind of agree with Erik. I think now I disagree with them [referring to the girls].

5.0.211  50:19  Erik:   I don’t really think that if you have this [a purple rod] that you could have one third bigger than it [Brian - yeah] because it’s got to be one third and probably a third and a half.

5.0.212  50:30  Brian:   Yeah, he’s right.

5.0.213  50:31  Erik:   It couldn’t be, it couldn’t be exactly a third.

5.0.214  50:34  Brian:   Cause one third bigger, this would be one third bigger like that to the end over there [Figure O-52-40]. That would actually be like this [showing with the dark green and purple pieces], this would really be one third bigger and there’s still some left over and there’s still about [Figure O-52-51]

5.0.215  00:50:56  Erik:   A half left over.
5.0.216 51:02  Brian: Yeah, there’s still, there’s still one more, there’s still one more piece left, like about a sixth left [Figure O-53-04].

5.0.217 51:05  Erik: Cause it’s like if you have, if you have the like dark green and it doesn’t exactly equal up to, it doesn’t exactly equal up. It’s less than two thirds but it’s more than one third. It’s just about one third and a half. So it couldn’t be exactly a third bigger than it and it couldn’t be exactly two thirds or it couldn’t be exactly one third bigger. It had to be one third and a half.

5.0.218 51:37  T/R 1: Michael wanted to say something for a long time and has been very patient.

5.0.219 00:51:39  Michael:

BREAK
In side
VIDEO

Umm, I think it should be called one sixth because [he goes to the overhead] because if you put six reds up to one orange [arranges six reds under the orange w/ red rod train] with a red then it would equal, there would be, there would be, it would be the same size just, so it would be called one sixth because reds like that [Figure O-54-17].

5.0.220  Brian: Yeah, I agree with Michael and Erik

5.0.221  T/R 1: So, so Brian, Michael is offering another way of thinking about that red as being one sixth. You thought about the red as being one sixth to make a half of a third and Michael is saying that red is one sixth

5.0.222  Erik: Yeah, Michael is right because it takes three sixths to equal one half, and if-

5.0.223  T/R 1: I see Meredith is wanting to say something.

5.0.224  Meredith: I agree with Erik, Michael and Brian because if you do call that a sixth, a sixth, and if you put the dark green and two thirds, you said it was, you said it was, um, they said that it’s a third bigger, if you did a third bigger, this is called a third and then you put it there, you see negative, [Figure O-55-18, interrupted by intercom. Meredith placed a red rod next to the dark green rod - Figure O-55-27]

5.0.225  T/R 1: I’m sorry, Meredith, could you start again

5.0.226  Meredith: You said it was one third bigger, that can’t be true because one third bigger

5.0.227  Erik and Brian: Yeah

5.0.228  Brian: It’s about one sixth less. So it can’t be a third bigger.

5.0.229  Erik: And also, like

5.0.230  Meredith: So it’s one sixth bigger [Figure O-55-55]

5.0.231 52:43  Erik: And also yeah, and also, I think because if you have the light green, the light green, it’s not bigger than, it’s not bigger than
the, it’s not bigger than the umm third, it’s not bigger than two thirds. It’s bigger than one third, but it’s not as big as two thirds so it’s less than two thirds but more than one third. So it can’t be a third bigger. And if you have that to make it two thirds large, there has to be a sixth [Figure O-56-29].

5.0.232 53:19 T/R 1: Well that is really something, uh I think --
5.0.233 53:23 Michael: It’s sort of like one sixth in both cases.
5.0.234 53:27 T/R 1: Well you find that you are consistent, you do get one sixth when you use both models. I am really interested I, uh, hearing about what all of you are thinking about these arguments [To the children at the overhead] You can sit down now. Thank you very very much, that was very very helpful. What I am going to ask you to do in the next class, Mrs. Phillips, is if they can have some time tomorrow where they could have their rods, or the next chance you get to… so maybe the substitute would let them do that. If they could have their rods, and I’d like you to write about that there are two arguments here. Right? There are two models that were built and I would like you to try to tell me what you believe were the arguments and if you had to persuade somebody who was confused, if you could try to think what the confusion was, or you can tell me what all the things you learned about this model. You had some wonderful arguments that you gave up here, and there were so many ideas that I know that it’s hard to catch all the ideas to listen, but maybe if you had the rods and you talked to each and you worked together maybe that would be helpful. What do you think Jessica?

5.0.235 00:54:37 Jessica: Well, I think now that I agree with Brian and Erik.
5.0.236 54:42 T/R 1: You now agree with Brian and Erik? Why did you change your mind Jessica?
5.0.237 54:43 Jessica: Because I, I saw that um, it wasn’t the same as um, it can’t, it couldn’t be one third.
5.0.238 54:52 T/R 1: Why couldn’t it be one third?
5.0.239 54:53 Jessica: Because you’d, it, you’d have to add um, a red and that would be one sixth.
5.0.240 54:59 T/R 1: Okay now that’s very interesting, if you could write about that, and why you changed your mind when they gave their arguments, that would be really interesting I would really like reading when you write about these things, you do so nicely. And what you also might want to do is if you didn’t understand peoples’ arguments because you know maybe sometimes when someone’s talking up here it’s hard to catch it all, maybe you can get them aside and talk to them privately. And say you know I don’t really quite understand what you were doing, can you help me. I’m sure Jessica and
Laura would be very happy to show you their models and show you the way they were first thinking about it. And maybe how their thinking changed, right, and why? I think that would be very interesting. I think we need to talk about this problem some more. It’s a very important problem. What do you think? How many of you have enjoyed working this problem? [several hands go up] How many of you have found it very hard work? [A couple hands stay up] Some of you haven’t? Dr. Davis what do you think?

5.0.241 55:58 Dr. Davis: Um, I think that this is one of the most interesting discussions in mathematics I have ever heard. You people were sensational. I’m gonna be curious about how it all comes out when you write about it, to see if you really all agree…But you’re doing some very good mathematics. I’m very impressed.

5.0.242 56:16 T/R 1: The thinking in this class is absolutely wonderful and I’m just so impressed at your thinking and the way you’re writing about your ideas. So I just can’t wait—Is it possible Mrs. Phillips that they could write or do you have something else planned for tomorrow.

5.0.243 56:29 CT: Umm, yes, tomorrow your question is, uhh do you believe that one half is larger than one third and by how much and if you didn’t understand the uhh explanation you will have the Cuisenaire rods in front of you and you will work it out. You’ll have time to talk to your neighbors about it and you will write a discussion that I will love to see, and that I know that all our Rutgers friends will love to see. You’re being given the paper now so that you have it. You write in what everybody?

5.0.244 57:01 Student: Pen.

5.0.245 57:03 CT: Pen, pen, you know where the pens are, alright. This is not homework but this is paper ready for you to go tomorrow, your notebook paper. It’ll be the first thing on the agenda. Any questions? Yes, sir.

5.0.246 57:18 Brian: Could we do it for homework? Start on it for homework if we would like too.

5.0.247 57:23 CT: Alright, alright, if you fully understand, surely.

5.0.248 57:26 T/R 1: And then you could help other people understand your thinking and be available – [cuts out] Thank you very very much, I can’t wait to come back on Friday. See you then, thank you.

5.0.249 57:30: End of class
41

Colts Neck, Session 10, Oct. 8, 1993 (Front)

<table>
<thead>
<tr>
<th>Line</th>
<th>Time</th>
<th>Speaker</th>
<th>Transcript</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.2.1</td>
<td>00:16</td>
<td>Andrew:</td>
<td>So, everybody started doing that and like combining, um, two like browns and saying that was fourth of the…</td>
</tr>
<tr>
<td>10.2.2</td>
<td>00:27</td>
<td>T/R 2:</td>
<td>Ok, so you started to make, you started to use trains [yeah] to make your number one and then you were able to work from there. What else did you find out? That's interesting. What problem were you working on? Somebody tell me what the problem was that you were working on here? Amy.</td>
</tr>
<tr>
<td>10.2.3</td>
<td>00:44</td>
<td>Amy:</td>
<td>Um, which was larger, two thirds or three fourths?</td>
</tr>
<tr>
<td>10.2.4</td>
<td>00:48</td>
<td>T/R 2:</td>
<td>Okay, two thirds or three fourths, and what did Dr. Maher want you to do with this? What did she ask you to do, yesterday? I'm trying to build this in my own mind as to what she asked you. Uh, let’s see, Jessica.</td>
</tr>
<tr>
<td>10.2.5</td>
<td>01:01</td>
<td>Jessica:</td>
<td>Um, well, actually, we just really tried to bring more problems up than we did last time, and like some people were making big ones, and like, combining like, two browns and like making that equal one.</td>
</tr>
<tr>
<td>10.2.6</td>
<td>01:18</td>
<td>T/R 2:</td>
<td>Ok, so they were making like bigger models? Erik, did you want to add to that?</td>
</tr>
<tr>
<td>10.2.7</td>
<td>01:22</td>
<td>Erik:</td>
<td>Well, we also had to, I think she gave us, like the question that Amy said, that she gave us that question and we had to make two models to explain it.</td>
</tr>
<tr>
<td>10.2.8</td>
<td>01:30</td>
<td>T/R 2:</td>
<td>Mmm hmm.</td>
</tr>
<tr>
<td>10.2.9</td>
<td>01:33</td>
<td>Erik:</td>
<td>That's basically all she said.</td>
</tr>
<tr>
<td>10.2.10</td>
<td>01:51</td>
<td>T/R 2:</td>
<td>And that's basically what she said? Did she ask you to think any more about this or write about it or- Meredith?</td>
</tr>
<tr>
<td>10.2.11</td>
<td>01:42</td>
<td>Erik:</td>
<td>Yeah she told us to write about it.</td>
</tr>
<tr>
<td>10.2.12</td>
<td>01:42</td>
<td>Meredith:</td>
<td>Well, she told us to um, like, draw our models on a piece of paper.</td>
</tr>
<tr>
<td>10.2.13</td>
<td>01:47</td>
<td>T/R 2:</td>
<td>Ok, ok, has everybody here had an opportunity to do that?</td>
</tr>
<tr>
<td>10.2.14</td>
<td>01:51</td>
<td>Erik and other students</td>
<td>Yeah.</td>
</tr>
<tr>
<td>10.2.15</td>
<td>02:03</td>
<td>T/R 2:</td>
<td>Ok. What I was wondering is, I heard that, I heard that yesterday there were some, some people came up with some really big models. Were you able to draw those?</td>
</tr>
<tr>
<td>10.2.16</td>
<td>02:03</td>
<td>Erik:</td>
<td>Well, we didn't draw 'em, we couldn't draw 'em, too big.</td>
</tr>
<tr>
<td>10.2.17</td>
<td>02:07</td>
<td>Andrew:</td>
<td>Ahh, neat. Ok, so Andrew drew one. What he did was he taped some paper together. That's very clever. So maybe we can actually record some of those bigger ones today for people who did that, uh, what I'm wondering is, now that you've had a chance to build some models, and you came up with some different ones, from what I'm hearing, how many models do you think it's possible to build for comparing those two fractions?</td>
</tr>
</tbody>
</table>
Erik: Comparing what, two thirds and three fourths?

T/R 2: Yeah.

Erik: A lot.

T/R 2: A lot, you want to say more about why that's so?

Erik: Well, because see, what me, Alan and I figured, is if you start with one rod, and you can divide one rod that's a large number into thirds and fourths, then you just count down by two, because we think that even numbers you can divide into fourths and thirds, but odd numbers you can't, so it was like, if we started with the orange rod, we could prob-, you could probably divide it into thirds and fourths. And then just go down two and then just keep going down until whatever number you get and then you'll just keep going down and you should be able to Faulty Conjecture

Alan: We also realized that the bigger, like if you put three of these together, that if you put four, you couldn't third that unless you made a new rod using two others to be bigger than the orange. U/L

T/R 2: Oh.

Alan: So the big- so if the mod- the bigger the model or, this is the biggest model you can get without having not being able to third it. The bigger the model, then you can't third it.

T/R 2: Oh.

Alan: Like four oranges you can't third it without making a new rod. But three oranges you could call that a whole and have three more oranges as the thirds.

T/R 2: Ok, what do you think about this theory that, uh, that Erik and Alan have about the even numbers? They said that they think they can divide even numbers into thirds and fourths, and uh, Erik said that

T/R 2: Mmm hmm.

Michael: Um, I agree with Erik and Alan, um, we, we, um, we came up basically with the same thing, but we also found out that some of the ev- some of the even numbers didn't work modifies conjecture

T/R 2: Mmm hmm.

Michael: You couldn’t divide them into thirds and fourths.
One of the even numbers we found that you could divide into thirds and fourths is the dark green rod. Faulty.

Oh, that one works, so that's an even number one that works.

Mmm hmmm.

What do you mean by an even number rod?

Well, a rod that if you put all of the whites up to it.

Mmm hmmm

All the whites real tight, and you determine if you can divide it in half.

Ok, ok, David, did you want to add, you want to add to that? Or you want to comment on that?

I want to comment that an even rod, is, before, when I got up there, maybe about like a week ago, um, I said that like the white would be one, the reds would be two, so the reds are even, and then the light greens are three,

They're odd, and then the purple is even.

Because of the number of whites you could put alongside of it to show? Is that why you're saying that, in other words? Why you're giving it a name two because you can use two whites to show it? Ok that's interesting. Well, what I'd like to do is, is there, first of all, is there anyone here who had not had an opportunity to record their models [hands raised]. Ok, there are some people who haven't. I'd like to give those people an opportunity to record, I think it was two or three models? Ok, if you have done that, I want to give you something else to try, ok? And if we have time then the other people who finish recording can try it as well, but what I'd like to do is, if your models were big

Yeah

We've got tape and we've got paper, we've got blank white paper up here, we've got notebook paper, um, you could, you could tape some sheets together like Andrew's done. I think that's a really neat idea. Ok? I want to come over and talk to Andrew about that. Um, for those of you who have, for those of you who need to do this now, who need to work on your models, please raise your hand, and um, you've got rods, and if you need markers, I have markers here, I'll come around with, and paper. Ok? The rest of you, I want to have you work on something else. [students gather to receive materials]. Ok, you need paper? Ok, is there anybody in here who has had a chance to record all their models and would like to begin thinking about something else? Ok, I'll be around. [camera focuses on David, Meredith, and Erik on the floor, then moves to James, Jacqueline, and Amy. T/R 2 is overheard posing the new problem to a student in the
background]. Which is bigger, one half or two fifths, and by how much? Decide which is bigger, and by how much, ok? And please, when you get a model, call me over.

10.2.52 11:32  Amy: [James builds a twenty-four cm model using orange, blue, and yellow] What are we doing? Which one?

10.2.53 11:34  James: Any one you want. After we do... after we both after we do both of these models we'll do another problem, k?

10.2.54  Jacquelyn: What's next?

10.2.55 12:26  James: I don't know.

10.2.56 12:32  Amy: An orange and a purple, two oranges and a purple.

10.2.57 13:59  Alan: [camera moves to Alan, who has built a model of an orange, five red rods, two yellow rods, and ten white rods. Figure F-12-47] One that has a lot of reds. [gets another box of rods, works, T/R 2 joins.] I'm trying to work on my third model.

10.2.58 16:54  T/R 2: You already got two!

10.2.59  Alan: [Discussing model with one orange rod, five red rods, two yellow rods, and ten whites] the one is the orange, there are the fifths, there are the halves, and there are the tenths. Ok, so, um, if you took out two of those [two red rods] that would be two fifths, and that would be a half [a yellow]. And one of those would fill in the gap [white rod], so it would be one tenth, so it would be a half is bigger than two fifths by one tenth [Figure F-17-22].

10.2.60 17:21  T/R 2: Neat, ok.

10.2.61  Alan: And then down here it's basically the same size as the orange, I just made a train of a brown and a red, [inaudible, Alan's second model is a brown and red train, two yellow rods, and five red - Figure F-17-36]

10.2.62 17:29  T/R 2: Ok, do you think now that you can, I mean you're working on this one I can see, do you think you can come up with one that's a different total length than, one's that's different from these two, these two have the same length?

10.2.63 17:44  Alan: I'm working on that one down here.

10.2.64  T/R 2: You're working on that one, it looks like you're working on a bigger model here.

10.2.65  Alan: Mmm hmmm. [Alan's third model is two blue rods, four purple rods, but he dismantles it and builds one with two trains of two orange rods, five purple rods, and ten red rods]

10.2.66 19:00  T/R 2: Does that one work?

10.2.67  Alan: Yup.

10.2.68  T/R 2: A working model here? Tell me about that one.

10.2.69  Alan: [Figure F-19-01] Ok, the two oranges make the whole.

10.2.70 19:07  T/R 2: So we're calling this, the two oranges together one. Ok.

10.2.71  Alan: And these, the five purples are the fifths, and the two oranges again are just the halves, now down here, the reds are the tenths. And again if you remove that [two purples and an
orange] it would take one of those [red] to fill in the gap, so it's bigger by, one half is bigger than two fifths by one tenth.

10.2.72 19:35 T/R 2: Can I ask you a question now? Why did you choose the two oranges to be one? You seemed to come up with that pretty quickly.

10.2.73 19:43 Alan: Because up here, I knew that this was ten, and two tens would be twenty, and I knew that that would work, so it takes two of those to complete it using a double ten. So one of those [red rods] filled in the gap. Probably if you used another one [takes a third orange and gestures to show that a fourth orange rod would be placed along with the first three] another two, you could fill in that with more purples and using more reds, too.

10.2.74 20:07 T/R 2: Interesting!
10.2.75 Alan: And it could make more.
10.2.76 20:09 T/R 2: Ok, so you did ten, you called it ten and twenty because of the little white ones. That's an interesting theory, could you kind of test that one out for me and, see, see if you could build a bigger model?

10.2.77 20:22 Alan: I'm trying to build a bigger model.
10.2.78 T/R 2: If you need more stuff, I've got more rods
10.2.79 20:27 Alan: I can use from these models [dismantles first model]
10.2.80 T/R 2: Ok. I've also got more rods up there. Ok, I'm going to come back, [camera moves to the floor]

10.2.81 20:47 David: You my best friend
10.2.82 Erik: Before he com- oh he's here.
10.2.83 21:02 V1: Erik, can you tell me about this?
10.2.84 Erik: Well,
10.2.85 David: We were working on this thing, cuz before she made um one that had one orange one blue and one black
10.2.86 Meredith: We told you about it yesterday!
10.2.87 21:12 V1: Well, we're gonna make sure we got it down.
10.2.88 David: And then I thought that maybe if we double it. Well, first, first we had the reds and they were one twelfth
10.2.89 21:24 Erik: Purples
10.2.90 David: [Figure F-21-35] No, we had the reds were the one twelfths, and we had the whites there were one twenty-fourths. So then, um, so then I thought if we doubled it, then the purples would be uh, one twelfth, and then, then I thought the greens might be kind of half of it, so, well, maybe not half but the greens were one seventeenth.

10.2.91 21:53 Erik: Sixteenth.
10.2.92 David: Oh, sixteenths, then, um, so then the reds are one twenty-fourths and the whites are one forty-eighths. And, you can't really make anything like halves, or like one-thirds.
10.2.93 22:12 Meredith: You would need a new model, maybe. If you put ten up to it, it won’t do it.
10.2.94 V1: Ok, so, now hold on a minute.
10.2.95 Erik: Well, we did this before, we did this a couple of days ago, and sixteenths fit. But it’s a little off.
10.2.96 22:26 V1: You said those greens are what?
10.2.97 Erik: They’re sixteenths.
10.2.98 22:30 V1: But I’m not convinced about that end part.
10.2.99 Erik: Yeah, I know, we had it yesterday.
10.2.100 David: I’m not sure if they’re lined up right though.
10.2.101 22:52 Erik: Couldn’t we do the browns? Could we do the browns as opposed to blacks?
10.2.102 V1: I don’t know this looks a little different than yesterday.
10.2.103 Erik: Can we do the browns as opposed to the blacks? We can have the dark greens
10.2.104 Meredith: We were going to do the red.
10.2.105 23:21 T/R 2: How’s this big model coming?
10.2.106 Erik: Not too good
10.2.107 David: Not too good.
10.2.108 Erik: We had it, we had it better yesterday.
10.2.109 T/R 2: What happened?
10.2.110 Erik: It was fine yesterday but now it doesn’t work.
10.2.111 Meredith: Oh I see what’s wrong!
10.2.112 23:31 T/R 2: What do you think’s messing things up?
10.2.113 Meredith: [Figure F-23-30] It needs a one.
10.2.114 Erik: But can’t we, can’t we, can’t we trade in one of the blacks for a brown?
10.2.115 23:49 David: But then that wouldn’t fit.
10.2.116 Erik: Yeah it would.
10.2.117 23:51 David: It would mess everything up though, Erik. The purples wouldn’t fit, the greens wouldn’t fit, the whites would fit, but maybe not the reds.
10.2.118 24:02 Meredith: No, if we trade it for a... no let’s trade it for a blue [Figure F-24-36].
10.2.119 24:08 T/R 2: Oh, I see, you’re calling, I see, you’re calling one of those, that top train, with the oranges and the blues and the blacks?
10.2.120 24:16 Meredith: Because then if you put another green here.
10.2.121 24:19 Erik: Oh, yeah! But,
10.2.122 Erika and David: What about the purple?
10.2.123 Meredith: Just take the purples out, you don’t need the purple.
10.2.124 David: Well, then that’s going to mess everything up, Meredith.
10.2.125 Erik: Then what will be the twelfths? No yeah, then what would be the twelfths?
10.2.126 Meredith: We don’t need the twelfths!
10.2.127 Erik: Yeah we do.
10.2.128 David: Because that’s the whole thing.
Erik: That's the whole question. That's the whole answer. It's either three twenty-
Meredith: Well, where's the two thirds?
Erik: Well, we don't really know.
Meredith: [laughing] But the question is which is bigger, two thirds or three fourths.
David: Well, Erik, um, remember, fourths, if green was one twelfth then that would be it, but like I said before that I thought that well we don't really need the greens.
Erik: Wait wait wait wait wait. This isn't the model we did before. The model we did before I believe was three oranges and like something else
David: No it wasn't, cause I remember your original model was an orange a blue and a black, and then I thought if we doubled it.
Erik: What if we did just, an, two oranges, two blues, one black and one blue. That one's not totally messing it up.
Meredith: Except the purples
Erik: Purple we could figure out-
Meredith: Wait! Wait. I've an idea. Take away this, put on this [an orange instead of a black - Figure F-25-49]
Erik: Oh no
Meredith: And then put a one there. Then you could put one here, it would fit better.
Erik: Then put a red [some inaudible conversation]
David: Do you really need the green?
Erik: No, not really.
David: So should we just take it out?
Erik: Yeah, cuz I mean it's giving us too big of a problem, and we don't need it. I don't know why we put it on.
David: I just did that because I thought the green ones were the twelfths.
Erik: [Figure F-27-48] Yeah, I know. This is a- oh let's measure it! It is approximately, fifty-three. No, it's fifty-two. No it's fifty-two and a half.
Meredith: No it isn't. Watch. It needs to be equal.
David: Erik, it starts like that.
Erik: No it doesn't start at one, it starts at zero. [take away meter sticks, mess it up, fix it] Yes. [start putting reds on model]
Meredith: Another one, another one, another one, another one.
T/R 2: Ok, Alan, now you tried to make it with four orange. Tell me about this model and tell me just what you told me before.
Alan: [Figure F-30-58] Ok, originally I had two oranges, and that was, that would only use twenty of the whites, but if you added another two of them on it would be forty of the whites. So the whites down here are the fortieths. And the purples
would take five to use for the two, and another five over here, so that would be the tenths. And now, these, if you put two oranges together, the two oranges each would be the halves. These would be the twentieths [reds], and the browns would be the fifths. Now there should be, I think nineteen more on here to complete the fortieths. You can't make the model any bigger than this, you would have to use one blue. It wouldn't be the exact size. [places five blue rods - Figure F-31-18]. So you can't make a model any bigger than this, without making a train, making all these uneven. So basically, this is the only model you can make that's even without using trains, like this one here, that would make all of these unequal.

10.2.155 31:41 T/R 2: So, if I wanted to continue my train with oranges, you're saying, I would have trouble showing

10.2.156 Alan: Another four, no, another four oranges to fit five more of the browns on, so it would be a yard long probably.

10.2.157 31:57 T/R 2: Oh my goodness, can you imagine the size of that? If you wan- if you wanted to make a train, though, where you were adding a different color rod on the end of this train of four oranges, do you think you could come up with other models?

10.2.158 32:09 Alan: Well, it could be, but this is the, basically the only equal model using, you know, tenths twentieths, fortieths. ... for the whole.

10.2.159 32:21 T/R 2: That's interesting, that's really interesting Ok, could you, I almost hate to ask this but could you, we have a couple of minutes left, could you try to trace this so we don't lose what you did here? Uh, or maybe you can draw a sketch of it, okay? And just label a sketch of it. That would be easier for you, let me get you some paper. [leaves to get paper] Alan, you know what would be really, you know what would be helpful to me, I want to make sure that you get the information down [inaudible]. What you used, in other words, oranges to make the ones, and purples to make the tenths, can you write that?

10.2.160 33:20 Erik: [camera moves. Erik, David and Meredith a building a really long model.] We can use every single rod possible. You could take off the purples because we don't need them. We're making this gigantic-

10.2.161 33:47 V1: Ok, but, uh, Erik, is this going to help you solve the problem?

10.2.162 33:52 Erik: Yeah.

10.2.163 V1: How?

10.2.164 Erik: We're making it big.

10.2.165 V1: How do you think this will help you solve the problem? Are you going to let that equal your whole?

10.2.166 Erik: Uh huh.
That's going to be a pretty big whole.

They have a big whole down there.

Do you think this will help you solve the problem that we want you to solve?

I don't think it will.

David doesn't think so.

Neither do I, really.

But we just want to do it anyway.

Put it this way, you know you probably could but it's gonna take you a while and we're running out of time. So can you show me what you got with the model that you had on here before? [Erik destroys the long model]. Cause we're gonna, we're gonna have to clean up in a couple of minutes, ok? So

Uh oh

So what do you have here?

Well,

So how's it coming over here?

Well, let's see.

[Figure F-36-17] Well, we have, as the whole we have two oranges, two blues and two blacks, because David said that Meredith made an original model that was one orange, one blue and when black, and then-

One orange, One blue, and one black, and then, well, she had um, the reds were one twelfth and then the whites were one twenty-fourth, and then

We did, we doubled

Put that down, we don't need that, alright?

We doubled two oranges two greens and two blacks

Instead of one orange one blue and one black.

The purples would be the twelfths, the reds would be the um twenty-fourths

I'm not convinced about this, okay?

Yeah, wait a minute.

Wait, you say how much are the purples?

Alright, here wait a minute. Alright, Meredith made this model with one orange, one blue and one black,

Yeah, and it had thirds and it had fourths.

And then the, so then reds-

So how much are the purples?

The purples are one twelfth.

I'm not sure, I don't see that.

Well, see, one two three four five six seven eight nine ten eleven twelve, thirteen! We made a mistake.

There's thirteen of them. What does that tell you?

They're one thirteenths.

Thirteenths.
David: Because you see yesterday

Erik: Yesterday this whole thing came out perfect.

David: Yesterday we had one twelfth, the greens were one sixteenth, and now they're one seventeenth.

Meredith: Hah ha ha ha ha you were wrong.

V1: Well, what, how could that be?

Erik: Why do we need oranges on top?

CT: Kindly if you have work with your name on it that you want to share with Dr. Martino, give it to her please.

V1: I don't know. What do you, what do you guys think what do you think happened? Because, you know, I see thirteen things here.

Meredith: I don't think they know how to count.

Erik: I think Meredith sabotaged it. [inaudible, laughter]

David: Well, I think I think, yesterday, maybe it was three blues

Erik: No it was smaller.

V1: It looks pretty, well, let's get it - this is the model you guys just had, right?

Meredith: No, we had one that was straighter.

V1: Ok, well, let's even out the ends. Okay? Now that looks pretty straight to me. Okay, now, these are all even, but I see, yeah there's thirteen, aren't there?

Meredith: We don't need twelfths.

Erik: That's the whole point!

Meredith: What's the point of twelfths? The point is two thirds and three fourths.

Erik: The answer is one twelfth.

David: Meredith, I made this thing to show that when you double it. To show that when you double it. The reds were one twelfths, now the reds aren’t one twelfths, now the reds are, uh,

Erik: So you're trying to show that with different models, thirds, that they're twelfths, that if the numbers will change, no they're changing size but they don't change in answer!

V1: Ok, guys, you gotta start putting the stuff away. I'm afraid we need a little bit more work on that model. End of class

Colts Neck April 5, 1993, Interview with Brandon
<table>
<thead>
<tr>
<th>Line</th>
<th>Time</th>
<th>Speaker</th>
<th>Transcript</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>R3</td>
<td>Brandon</td>
<td>Assuming that I have got it with me. Yes, I do. It got a little chopped off in the photocopier, but you can help me understand what it meant. Like the bottom I think did. Okay, let’s see. Now this was on…[Brandon lines up the two sheets of paper]. Yeah we need some tape or something there. Okay. And, you can feel free to add to this. Alls I was I was telling you as we were walking over. The people at Rutgers were very interested with what you were doing here and in order for me to be able to explain to them what you did, I need to understand it.</td>
</tr>
<tr>
<td>2</td>
<td>Brandon</td>
<td>Brandon</td>
<td>Okay.</td>
</tr>
<tr>
<td>3</td>
<td>R3</td>
<td>Brandon</td>
<td>Okay? Do you want to tell me what you were doing here and how this, how these turn out to be pizzas – these zeroes and ones? Let me line that up for you. [Brandon: Okay.] Okay.</td>
</tr>
<tr>
<td>4</td>
<td>Brandon</td>
<td>Brandon</td>
<td>I’ll do it over again so then you can see how I did it.</td>
</tr>
<tr>
<td>5</td>
<td>R3</td>
<td>Brandon</td>
<td>Oh, that would be neat. I would love that. [Brendon uncaps and recaps markers.] Are any of the working today?</td>
</tr>
<tr>
<td>6</td>
<td>Brandon</td>
<td>Brandon</td>
<td>Well, these two have no points.</td>
</tr>
<tr>
<td>7</td>
<td>R3</td>
<td>Brandon</td>
<td>Okay. Look at that. They got all mushed down. Okay.</td>
</tr>
<tr>
<td>8</td>
<td>Brandon</td>
<td>Brandon</td>
<td>Well, since there are three toppings, four toppings that is…</td>
</tr>
<tr>
<td>9</td>
<td>R3</td>
<td>Brandon</td>
<td>Do you remember what the problem was?</td>
</tr>
<tr>
<td>10</td>
<td>Brandon</td>
<td>Brandon</td>
<td>Yeah, I remember how many toppings, how many different ways could you make toppings with four</td>
</tr>
</tbody>
</table>
toppings. Pepperoni pizza and all that stuff.

11  R3  Yeah, there was pepperoni…

12  Brandon  Mushroom, sausage, and pepper, pepper. Pepper, mushroom sausage, pepperoni.

13  R3  Yeah, so we selected from those four toppings.

14  Brandon  Okay, first you could have a plain cheese pizza with nothing on it. Then since they’re in order, instead of going put checking off and doing one sausage and skip around doing all the pepperoni, pepper, mushroom, it would be easier to go in order. So, first I do pepper with blank. Nothing else. That would be a way too. And you can have a pepper and sausage with nothing else. Then you can have peppers, pepper, sausage, mushroom. Well, this kind of isn’t actually the way I did it. I went like one, one had plain of that plain, and that plain, that plain, that and so on. It was the wrong way.

15  R3  Yeah, I remember you did. I think you did that when you started the last time, too, in class.

16  Brandon  First, I started going like one, two, three, four. But…

17  R3  Right. One, one with one, and then a pizza with two.

18  Brandon  It’s going to be easier going this way.

19  R3  How are you going to change this now?

20  Brandon  I’m just going to change the way I do it. Okay. A blank. Nothing on the pizza. Then you can have one pepper on the pizza with nothing else. One mushroom on the pizza with nothing else. Then you can have a couple of sausages on the pizza with nothing else. Maybe a couple pepperonis. And then if you don’t want any of that, you can start getting fancy and going to twos. So have a pepperoni and mushroom and nothing else. Then, a pepperoni and sausage with nothing else. Pepper and pepperoni and nothing else. And so on. Then since we’re all done with pepperoni, you could have a mushroom and sausage with nothing else.

21  R3  How did you know, how did you know to go to mushroom now? I’m, I’m interested in that. Okay.

22  Brandon  Why didn’t I use pepper anymore?
R3: Yeah, yeah. Why didn’t you use pepper anymore?

Brandon: Because I all ready used pepper there. It’s all ready mushroom and pepper, and if I did mushroom, put one down for mushroom and then pepper, that’d be the same thing.

R3: Oh, I see.

Brandon: So, that’ll be like a zero. So, each time you go 3, 2, 1.

R3: What do you mean 3, 2, 1?

Brandon: First, you could use all the toppings, then you could only, then since you used all ready that, you can’t use, it’s, that’s one less. So, you could only use those three. Then use all that, all those. You get three choices with, with the first one. And with the second one, with mushrooms, you only get two choices because there’s only sausage and pepperoni. And then with sausage, you could only do pepperoni.

R3: Okay, but I don’t understand is why when you move to mushrooms, why you can put it with sausage and you can put it with pepperoni, but you can’t put it with peppers?

Brandon: Because that’d be the same thing. Because if I do that and put a one there. Right there. I all ready got pepperoni-mushroom, pepperoni-sausage. That’d be the same thing. It’s just like saying you have an airplane and a car, saying you got a car and an airplane. It’s still the same thing.

R3: I see. Okay. I understand why you did that now. What do these zeroes and ones mean? Like what does the zero represent here?

Brandon: You have nothing on that. That’s nothing. I don’t know why I chose to use zeroes and ones.

R3: I was going to ask you about that where you got this idea from.

Brandon: I don’t know how I got it. It just popped into my head. So then you could have a mushroom and the one pepperoni and since then you got, you already used sausage and pepperoni, you could, so all you have left is, then you can’t do anything, then all you could do is a sausage and pepperoni. And now, you can’t just do, put one by pepperoni because then that’d be the same as up there.
Say that again, that last part.

Since I, if I do that and go put three zeroes and a one right there because like that. That’d be the same right there.

I see. Okay, so that would be the same as the pepperoni pizza with nothing else on it.

Then…

You missed a couple here. How will we know when we’re done?

We’ll run out of all ways. I’ll show, I could, you could tell when you run out. Now we could go to threes if you really like this stuff and if you’re really rich. One pepperoni, a mushroom and sausage, nothing else. Anyway, I noticed also, the numbers, each time you go up into like another group of like if you go from twos to threes, it’s like the numbers get higher sort of.

What do you mean by the numbers get higher?

Well, like you, when you look at two, well, that, I couldn’t say that. But, sometimes when you just do those - a pepperoni and mushroom and nothing else. It’s like 1,100. Then you do pepperoni, mushroom and sausage and it’s 1,110.

I see what you mean by the numbers getting higher.

It’s kind of confusing doing it that way. So then 13, you could have, 13 you could have a pepperoni, a mushroom, no sausage, and pepperoni. This is where it gets really tricky. For 14, you could have, since a pepperoni, no mushroom, a sausage, and a pepperoni. For 15, you could have a pepperoni, a mushroom, no sausage. Oh I got that. Okay. That’s one I got already. Okay for 15 you may be able to have no pepperoni, a mushroom, and sausage and pepperoni. And 16 would be a pizza with everything.

And, what would that look like?

Oh man, I think I got doubles here somewhere. I have no idea.

Oh, let’s see.

Because that you should only have 15 ways.
Can we go back and check that? You’re looking back at this right?

Yeah, that. This one’s a right easy one. Okay.

We check this.

We have a blank pizza, correct. Correct. Correct. Correct. Correct. All these are correct.

Are they all different from each other?

Yeah because I checked them with coloring(?). Oh here are the threes. This is where I get, got mixed up. A pepperoni, mushroom, and sausage. And a pepperoni sausage, a pepper sausage and pepperoni then you could have a pepper mushroom and pepperoni. Still different. For 15 you could have a pepperoni, mushroom, sausage and pepperoni and pepper.

And what’s this down here?

Well, that would be 16 because we got doubles somewhere. We got doubles. That one was a double. That one would be exactly a mushroom, sausage, pepperoni, sausage, pepperoni, so we cross that one out.

Okay. But what about that one?

That one. Those three?

How can we figure out is this one a duplicate?

No. That one? I thought we draw, drew a line. A mushroom sausage and pepperoni. Don’t have it. [R3: Okay.] We need straight ones. [R3: Okay.] And then you could only get up to 12 that’s where you’re going to stop because then it’s all twos. And I wouldn’t have a three and a two in the twos group like...

Okay, yeah, these are in groups? [Brandon: Yeah, like first there’s like…] Then you’re thinking about these in groups?

Yeah, like first there’s a one group you would only have one topping. The twos group you only have two toppings. The threes group you only have three toppings. And the final group you have all toppings.

I see. So, you might if you in fact you were reorganizing this you might move this one. [Brandon: Yeah, that one’s…] Where would you move it to?
Brandon: Well, if a mushroom, pepperoni, sausage I’d move that right there between 14 and 15.

R3: Okay, why would you move it there?

Brandon: Because it goes after that you have pepperoni-mushroom-sausage, pepperoni mushroom and pepper, and if I move it up one, that would be incorrect because you have still you would have a you would start off with a mushroom and you’re still working with pepperoni, pepper so it’d probably go here because then you go into another different way, different group.

R3: I see. So is this one we did okay? Does this one work too?

Brandon: No, that one’s kind of confusing for me. That one wouldn’t work that well. That’s like the best one.

R3: This is the best one?

Brandon: Yeah, that’s a good, good working one.

R3: Why is that one confusing?

Brandon: Because I kind of like got confused during the middle.

R3: Can you show me what, can you show, you have them in groups here, can you show me what those groups are on here?

Brandon: Okay. You could go. There’s one group. And, let me use a different color.

R3: Okay. And what group is that?

Brandon: Okay. Here. The ones group.

R3: Okay. And what does that mean? The ones group?

Brandon: You only have one topping in that group.

R3: Okay. What about this one right up here?

Brandon: That would be a totally separate group. The zero group. Nothing. Now, you go into the ones group…

R3: Do we have, can I, I just want to stop you for a second. Do we have all the ones that we could possibly have in the ones group?

Brandon: Yeah.

R3: How do you know that?
Brandon: Because I went down one pepperoni, no mushroom, no sausage, no pepperoni. One mushroom, no pepper, sausage and pepperoni. One sausage, no pepperoni, mushroom and pepper. Then one pepperoni with nothing else.

R3: Interesting. Okay.

Brandon: Because if I did, because if I did that again right there that would be the same right there. [R3: Okay.] Because no matter where I put it, it would have a same in that same group. Like a one there then a one there.

R3: I see. Okay.

Brandon: Then you could have a twos group which would go about. A twos groups is like the most.

R3: What do you mean the most?

Brandon: You get the most out of two because you get more, you get more choices than one, and, and, you get more choices. There’s like you could have more on. You get there’s more different choices like pepperoni-mushroom, pepperoni-sausage, pepper-pepperoni, and that so on so like the two group is like the biggest. Then you got the threes group…

R3: Can we slow down again? This is really very interesting, but I’m wondering now you have one, two, three, four, five, six in the twos group. Can you convince me that there is, there aren’t anymore in the twos group. That there aren’t seven or eight.

Brandon: You go pepper-mushroom. That’s one. Pepper-sausage. That’s two. Pepper-pepperoni. Three. Then you can’t do anymore because you already used sausage once and mushrooms once in to tell that you are a and to see that you made duplicate look over there in one because if you just look there you’ll see another one. But, if you see a zero there, that means it’s not a duplicate because you got nothing there. So if you there’s a one-one, then that’d be the same as there. Then you get into mushrooms. Mushroom-sausage. Mushroom-pepperoni. No more because then you would see another one down there in that same group with the mushroom and the sausage.

R3: How come, how come there are no more thought with the mushroom?

Brandon: Because each time you get less. If you start off with pepperoni, you got three choices because there’s
mushroom, sausage, pepperoni. Get two mushroom, you only got two choices because pepperoni you already used with mushroom.

R3 I see.

Brandon Then the same with sausage. Only you do pepperoni. Then if I put pepperoni by itself that’d be the same as up there.

R3 Okay. I’m following.

Brandon So then you get after three, you go peppers-mushroom-sausage, pepper-mushroom-pepperoni, then pepper, pepper-sausage-and pepperoni.

R3 Is that all?

Brandon Then you could do, start with mushroom. Mushroom-sausage-and pepperoni and that’s all for the threes group. Because, I know you’re going to ask why, because since if you do that that if you put that one there, it would still be the same, it would be the same as that. And since you’re doing threes and there’s only have three left, you could only have all three. All ones and no zero. And no pepper because you already used that. And no matter where you put another one, it would be the same as any one of those threes up there. So then your only choice left is having an all pizza with everything.

R3 Interesting. And what are we calling this group?

Brandon The all. I don’t know what I’d call that. The total.

R3 Okay. The total? We call these the zeroes, the one toppings, right?

Brandon You had two toppings, three toppings, four toppings.

R3 You call it four toppings. Sure. This is very, very fascinating, Brandon. I’m really, really understanding what you did here now. Can I ask you a question?

Brandon Yeah. What?

R3 Now, we’ve done some different problems when I’ve been in this year with Mrs. Zalee (?).

Brandon Yeah with the blocks and all that.

R3 Does this problem with the pizzas remind you of any other problems we’ve done this year?

Brandon Like what do you mean? Like the way I did it?
In any way, does it remind you of any of the problems that we’ve done? [Brandon: Does it?] It could be in the way you’ve done them.

Brandon

Oh yeah, it kind of a little reminds me of the blocks ’cause you would because my partner and I, whoever that was, I think it was Colin, did it in order. Like one, like you would do yellow, red, yellow, red then switch it around. Do the opposite red, yellow, red, yellow. It’s kind of what we do here. Just do it in groups. Like that’s what we did with the blocks with my partner.

R3

Now we built towers that were…

Brandon

Yeah. Opposite. How many ways could we make towers? How many ways could we make pizzas? The same problem.

R3

The same kind of thing. Do you remember how many towers there were? [Brandon: There was…] That goes way back.

Brandon

I think I could remember.

R3

Would you like, would you like these? Or would you like these?

Brandon

[Writes out tower combinations] Oh wait. I think I could do it. We had an all tower. Two an all tower. Shaded. Non-shaded. We had another tower. We were working with fours, right? [R3: Yes.] Then that then you would have the opposite of that which that would be two. That would be a group of four. And six because you would have, maybe you could have and maybe let’s say umm one and one then you could maybe have over there, One and one. Yeah, there was six towers. Six towers I remember.

R3

And that was, that was all the towers you could make?

Brandon

Yeah, six towers.

R3

That’s interesting. Could you show me how those looked like with blocks? [Brandon: Okay.] I’ll give you some blocks. You could sit down. I just want to get these for you.

Brandon

[Puts together blocks] Two yellows two reds. One and put red on the bottom yellow on top. One group. [Combines two reds and two yellows. Then makes red-yellow-red-yellow and yellow-red-yellow-red] Another group. [Combines four yellows and four reds] Another group. Another pair. Okay, 2, 4, 6, and
now let’s try something, oh wait, there’s more than 6. [Pieces red-yellow-yellow-red and yellow-red-red-yellow] Another group. Okay, I think that’s all. Two, four, six, eight. Eight. I think that’s all.

123 20:33 R3 We talked about this before. Is there a way you could be sure you have them all?

124 Brandon I don’t, I kind of don’t not know.

125 R3 Like when you did pizzas, you really seemed sure that you had them all. Don’t you?

126 Brandon Yeah, yeah, because that was on a graph.

127 R3 Is there a way to do that with towers?

128 Brandon Yeah, you could do a graph with anything. Towers.

129 R3 Now, that would be interesting. Could you, do you think you could do it with towers? How would we make a graph with towers? What would that look like?

130 Brandon Shaded equals red. Not shaded equals yellow. Okay. You could have a shade. Okay, it would be either like a shaded like I was doing before a shaded, all shaded and then not shaded. Just a plain old tower. A red, one with red yellow like that one would be with that, that would be that. That would be one group. [R3: Okay. What connects?] You could have…then you could maybe have shaded, not shaded and shaded, red-yellow-red-yellow. Then you could have yellow-red-yellow-red.

131 R3 Okay. I’m confused…you know what I’m confused about, Brandon? [Brandon: What?] Now, what I’m confused about is how can I, how am I going to keep track of this? Like here you showed me very nicely how I could keep track, but how am I going to know that I’ve, I’ve thought them all up if they’re…

132 Brandon I know. Well, it’s harder to do it with the graph because you can’t do like…

133 R3 Could we do a graph for this?

134 Brandon You can’t do like, you can’t do yellows and reds because that would be too confusing because then you would have to draw the blocks. And if you just go like blank that, it would be too confusing, so you have to draw the blocks. Because here with blocks, that’s one group. You could do two blocks.
Would you, would you have to draw the blocks to do this? Could you…

Oh wait. Now I remember how we found it out. We kept trying different ways. We tried, and since there was a, you could only use four blocks, we kept trying every single way we could think of. That’d be the same as that. Whatever we tried, we still got the same.

What I’m wondering about now. I want to take this a step higher than we talked about it in December.

Brandon

Oh wait, wait, wait. There’s some more.

Ah. There is more. Okay.

I knew there was more.

You can see I wasn’t convinced you had them all. Yeah. You want some more blocks?

I knew there had to be some more. Okay. What was that? You could do two yellows. Two red, a yellow, and a red. [Created yellow-red-yellow-yellow and red-yellow-red-red] That would be another group. Here we go, here we go.

Well, you can think these all up in your head very quickly.

Well, all you gotta do is make one that you could make the opposite of. All I had to do is make this one tower. Then you, I, it instantly came in my head. I could put red there, red there, and red there. And so then, you could make total opposites like that. [R3: I see. So these are all opposites.] Like where there was a yellow, you could put a red. Where there was a red, you would put a yellow.

So these are all pairs of opposites?

Yeah. You could have a yellow-red, a yellow.

I’ll line these up so you could see them.

Yellow-red-red-red.

Oh, I’ll get you more blocks. You don’t have to keep jumping up. Looks like you need some more yellows while you’re over there.

This should be way more than I need. Okay, then you could have those. Then since, once you’ve seen one group, you could make then you could make just the
same with doing opposites. Another group. But actually, it would be easier when I did this. Wait a minute, same. Okay, that group’s gone. Okay, so you could start off with a bottom then work your way up. You could work your way up like this.

151  
25:25  
R3  
What do you mean by working your way up?

152  
Brandon  
Well, once you’ve, when you get to one of one color like one yellow and all reds or one red all yellow, you can start with the bottom and move like for the red group, you could start with one red three yellows, and go a yellow-red-yellow-yellow, yellow-yellow-red-yellow, and [R3: You’re moving the red.] yellow-yellow-yellow-red.

153  
R3  
You’re moving the red. Okay, so you’re moving it up one each time. [Brandon: Yeah, then maybe you could have...] That’s a good idea.

154  
Brandon  
[Makes yellow-red-yellow-yellow and yellow-yellow-red-yellow] Then you can do that. That’d be another one. It’s kind of like stairs. Then you could have all yellows and one red. Now you just switch it around and do the opposite. That one would go with what was that? That?

155  
R3  
These guys?

156  
Brandon  
Yeah, I made, then you could make one of these. I would go…

157  
R3  
So you already have that first one. I see.

158  
Brandon  
Yeah, I didn’t notice that. [R3: That’s interesting.] You could do that one. Two reds-yellow-red. Instead of two yellows-red-yellow, two reds-yellow-red. That’d be another pair. Then you could do maybe all reds and then a yellow on top. And it’s kind of like the pizza problem. You start off with maybe group like this one would be the ones group.

159  
R3  
Let’s see what you’re talking about.

160  
Brandon  
Oh, now. I see this now. This is like the ones group. You only have one of the opposite color in there. This isn’t how I did it, but I, but I just noticed it.

161  
R3  
This is fascinating to me.

162  
I  
Brandon  
I just noticed it. Then you would have, that would be the ones group. You only have one in there.

163  
R3  
I guess you got this pair. [Brandon: And that pair. And that pair.] And this pair.
And that pair. That pair would be the ones group.

Okay. These are the ones then.

[Has yellow-yellow-red/yellow-red-yellow-red-yellow, yellow-yellow-red-yellow/red-red-yellow-red, yellow-red-red/red-yellow-yellow-yellow, yellow-red-yellow-yellow-red-yellow-yellow lined up.] Then you, then in pizza, this would be like the whole group. [Holds red-red-red-yellow-yellow-yellow] All groups. Save that for last. One group. Now you have the two groups. [Holds red-yellow-yellow-red-yellow-red-yellow] The twos. You have a red, you have two of the opposite colors in there. Same with here. That’s one group. Two of these.

These are the ones.

Here are the twos group. [Has yellow-yellow-red/yellow-red-yellow-yellow and red-yellow-yellow-red/red-red-yellow-yellow lined up in front of him] You have two of the opposite color in there. These would be like plain pizzas. Plain pizzas or all pizzas. Oh wait. These would be in the twos groups. [Adds red-yellow-yellow-yellow-red-yellow-red] They’ve got two of the opposite color.

How many, how many towers do you have in the twos group here?

Pairs or each separate?

Separate towers.

Two, four, six. Six.

Six of them. Now, that’s interesting. Isn’t it?

Eight. Eight and oh yeah. Eight and ones. Each time you go to a higher level of towers like you have most of ones because you could do one there, one there, one there and one there, and that would have a ton of those. But with the twos you could only have, you must have two in each so that takes away from the other, the opposite color so that you can’t pile them as high. What I’m trying to say is like with one, with the ones you could do like that [makes yellow-red-red-red], but then since you have the others when you try to do it, you can’t pile them as high with the opposite
color. And if you tried to do that, you couldn’t do that. You couldn’t pile it as high as the opposite color.

175  R3  I’m not following.

176  Brandon  I know. I’m not following myself either. Okay.

177  R3  I’m very interested in what you did here. The way you said it’s like groups?

178  Brandon  I know. Yeah, what you do is then you could have one, you have one of the opposite color in there that means there would be more ways I’m going to say opposite because there are more yellows of the opposite. I’m going to say opposite because if I was using green and yellow, I won’t say red and yellow. So then you would have an opposite which would make it more but here you go, you use…

179  R3  Aren’t there opposites over here?

180  Brandon  Yeah, but here, over here, you must use two of each. Two reds and two yellows. If you use two reds and one yellow, it’d be three. Two reds and three yellows it’d be five. The same for yellow, the same would go for yellow. So you’d have to have less, because you would have to use two of the opposite. So you could have these. Like that right there. And since, if we’re working with fives, there’d have many more. But since we aren’t not working with fives, you would only be able to do it like this. There must be at least two of the opposite color for it to go into that group or go into the ones group.

181  R3  Now, can I ask you a question? This is very interesting.

182  Brandon  Because if you go like this, [makes yellow-yellow-red-yellow] [R3: Maybe you can help me.] ones groups. For the ones groups, you’d have to have at least two, two colors. Like…

183  31:02  R3  I think I’m starting to understand. Like there’s. Is this what you’re saying to me: that there’s more ways to rearrange something like this [holds yellow-red-yellow-red] than there is to rearrange something like this [holds red-yellow-red-yellow].

184  Brandon  Yeah, yeah, because this, you could start with the bottom and put one up each step. That would be a lot. But this, if you do it, you would have to go like. This would be a good example [holding yellow-red-yellow-red]. Put these two there [pointing to red], then you could move them one step higher. You
moved them another step higher, they would look like this. How would I do that? You would start from this, the low part, move it up to one higher part. Now if you moved to a higher part, it would look like this [creates red-yellow-red-yellow-yellow]. [R3: Yeah.] But that’s five so you can’t. [R3: I see.] But if you take off one, then it’d still be the same, no matter which you take off. The top same. The bottom same. [R3: Okay now you had…] The bottom, take that off, it’s the same as that. Take off the top, it’s the same as that.

185 R3 Oh, I see. Now you had, I just wanna, I just wanna finish what you had here. You had this guy and you had this guy also. [Creates 4 yellows and 4 reds] Okay, and they were in a group by themselves. Okay, so you said that these all had two. Two yellows or two reds?

186 Brandon Two. They had, they had two of each color. They must to be in the twos groups. If they had three of each color and one of the opposite, they would be in ones. And you won’t have any threes group because that would be the same because…

187 R3 Why not?

188 Brandon Because for threes group, a three group, you have three yellows one red. That would be the same. A three group is like that group because three opposite color so that would be a three group would be the same as one group.

189 R3 Can I ask you now, if, could we call, if I, if I wanted to, could I call this all a threes group?

190 Brandon Yeah. You could call it a one or three group.

191 R3 Okay. Why?

192 Brandon Because you could call it a three group because it has three opposite colors. Three colors of one and one opposite. Or you could call it a ones group because it has one opposite color. You could call it three or ones. It doesn’t matter which.

193 R3 What if I asked you to focus now on the color in these towers?

194 Brandon What do you mean? Oh just use color?

195 R3 Now if I asked you to focus on the colors, and to take another look at your groups. Say we focused on, which color do you want to focus on? Red or yellow?
Brandon: I don’t care.

R3: Your favorite?

Brandon: Let’s do yellow.

R3: Okay, if I asked you to focus on a particular color like yellow, okay, and then I asked you to tell me what the ones towers were, could you do that?

Brandon: What the ones towers with yellow would be? Only if you used yellow?

R3: We’re looking at yellow.

Brandon: Oh, so there’s no red? If you only used yellow?

R3: No, what I’m saying is if we’re putting our focus on the yellow because there is red and yellow in all of these, right?

Brandon: Yeah, because if you used just yellow, you would use only that tower [holds 4 yellow tower].

R3: Sure would, wouldn’t you? If we’re looking at these 8 towers here, and we’re looking at yellow [Brandon: Oh like almost yellow], and we’re looking for ones, what would, what would be a ones tower in?

Brandon: A ones yellow tower?

R3: Yeah.

Brandon: [Holding red-yellow-red-red] That would be a ones yellow tower. And that would be a threes red tower.

R3: Okay, so it’s a one yellow and three red.

Brandon: Yeah.

R3: Okay, what else would be a, a one yellow here?

Brandon: [Selects red-red-red-yellow] One yellow.

R3: You, can you, that’s interesting. Can you pull the rest of them out that would be?

Brandon: One yellow. And that’s all the one yellows. No, there’s one more one yellow. There must be one more.

R3: Why? Why did you say there must be one more?

Brandon: Because, because before I only saw three and there’s 5 up there. So three, five take away three is two. So there must be, so you could put another one in that
group which would even it out.

R3: Oh okay. So you’re evening the groups out?

Brandon: Anyway, I just looked at the tops because I saw no yellows bottom pieces, right there. [R3: I see.] And also, it’s like the pizza problem. You work your way down. [Points to ones yellow towers] Like pepperoni, mushroom, sausage, and pepper.

R3: Wait, do that again for me. [Brandon: Yeah.] This is, this one here.

Brandon: Yeah, because we’re kind of like see how this sheet says, how I…

R3: Let’s look at the ones. Where was that?

Brandon: Here’s that sheet. How I…

R3: Is this it? This was your last one.

Brandon: Yeah, that’s page one. It’s kind of like that. You start with zero. You have, you could…

R3: What would the zero one look like if we’re looking at yellow?

Brandon: Zero one?

R3: Zero yellow.

Brandon: [Takes away blocks] Blank.

R3: Now, I don’t get what you just did.

Brandon: It would be nothing. A zero one’s tower would be…

R3: A zero, a zero tower, if we’re looking at yellow, would be nothing?

Brandon: Yeah, it would be nothing.

R3: Well, what was a ones yellow?

Brandon: A ones yellow. If I could find it.

R3: You showed me one yellow.

Brandon: Yeah. A ones yellow tower. Here’s a twos. One’s red. One yellow. Now just look at the top of the groups. Ah here we go. And all these are different groups. All the rest are threes. All the rest are fours.

R3: You go so fast for me, Brandon. Tell me again how this is like the pizzas.
Well, you have one pepperoni. That’d be like, one pepperoni is like, since we’re looking at yellow, the yellow would be one and the reds would be zeroes. You could have one pepper, like it shows here, and right there, I got then it’s like stairs, you were, if I draw a line down...

You need a pen? Let’s get these out of the way.

If I draw a line down here like this, it’d go like sort of look like stairs.

Then you’d go across, draw a line down here. Go across, draw a line down here. Go across, draw a line down there. Go across so you would have like one, one, one, one. Sort of like here you have one pepperoni, one mushroom, one sausage, and one pepper.

Is what you’re saying to me then like a yellow cube here is like a number one when in your chart?

If we’re focusing on red, then red would be a number one.

Well, let’s continue with yellow. This is interesting. I think this is really neat. Now, what would come next with what we have here if we want to reorganize? You said these would be like the one, yellows. [Takes y-r-r-r/y-r-y-r/r-r-y-y/r-r-r-r]

These would be the ones group. [R3: Now, what about…] Now you would start with the two yellow group. [R3: Okay.] But since, but since we’re working with opposites, the two yellows group, you could only use one ‘cause that would be just the same. Two pepper, no it wouldn’t. Then you would have maybe a pepperoni, mushroom…

Where were all our twos towers? There’s, is this one? How many were there? [Brandon: Here, I’m gonna lie these down.] How many were there? I forgot.

There was…here’s a two. If you lie ‘em down and face them sideways, it would, okay like you’re focusing on yellow, you put yellow right there [puts blocks down on pizza number chart], that’d, if these would stretch them out far enough like put down for that you would have yellow-yellow-yellow, nothing. Oh, oh, that’s page two.

Let’s go back to page one.
Page one is always going everywhere.

Page one keeps flying south.

Okay. First, you would have like one yellow-yellow-red-red. Same here. Because if you lie 'em down, stand them up, it’d be harder to have to stand up the paper. So, yellow-yellow, one red. [R3: Now, I understand.] That would be a two. Then you could have...

Yeah, where would the tower be that would look like this pizza?

Right here. Right here you would have a yellow stand for one. So it would have a yellow one, red zero, yellow one, red zero. [R3: I see.] That would be another one.

So this would come next.

Yeah. Now, you could have…

We have other ones.

Yeah, right here. Now right here, you could have a yellow, yellow-red-red-yellow. Yellow-red-red-yellow.

[Holding yellow-red-red-yellow] So what would this pizza look like? This one?

That would be pepperoni and pepper. Pepper and pepperoni. But now if we’re doing it in opposite groups, this would be. Wait. Let’s do this one. [Holding yellow-red-yellow-red] Now, we’d do, that’d be yellow-yellow-yellow. Wait a minute, There’s, I need that one. [Holds red-yellow-yellow-red] Then you’d do like red is zero. So you have nothing there. That’s correct. Yellow, yellow, yellow, yellow correct. Red zero. Correct. So that’d be another one. Now, you could do, you use this for that on page two. Now, page two you would have this one. [Holding red-yellow-red-yellow] Red zero, yellow one, red zero, yellow one. That go, that’s another goodie. [R3: Okay.] Now, all you would have left was two reds two yellows. Two reds two yellows.

Fascinating. Okay. And then are we out of them? And are we out of towers with two?

Yeah, because there would be no more. Look at the chart. Now, now, where is the threes? [R3: Okay, now we…]But now, if we do threes…
Can we, are we still sticking with yellow as the color we’re focusing on?

If we’re focusing on playing yellow, then you would swipe out, then these would be the ones for yellow.

These are the twos.

Yeah. [Holding y-r-y-y and y-y-y-r] This would be the threes for yellow because then you would have. 1-2-3. 1-2-3-blank. That’s in for threes. Then you would have a two, a two zero one. One one zero one. [Pointing to y-r-y-y] One one zero one. And that’s in again. Then you would have one zero one one. [Pointing to y-r-y-y] One zero one one. That’s in. Now you could have a zero one one one. [Holding r-y-y-y] Zero one one one. That’s in. Now, if we’re just focusing on yellow, this then would be the pizza with everything.

Oh. I see. Okay. And are we missing any?

You know what I’m wondering? We have this guy left. [Picks up 4 reds] Right? [Brandon: Yeah, because we’re not focusing on him.] Because he’s the opposite of this guy.

Yeah, we’re not focusing on red.

If we had to call him a name…

This would be the zero. Oh yeah, since the red would stand for zero, this would be the zero guy.

This is neat. This is really neat, Brandon.

Finally found out what the red would be. Red zero guy.

Okay. Could we have done it the, I just wanna ask you. You don’t have to do it, but could we have done it the other way around? Could we have just focused on red and, and gotten it to work the same way?

The same way. Just, just look like this. [Transposes ones standing towers with threes standing towers] Here is the ones group. Twos group. [R3: One red. Okay.] But the twos group would be the same. And then all you do…

And, and what would these be? What would these be?
Brandon: That would be the threes group. Just switch, then just switch those around. Same thing.

R3: Neat. Now, would, would we be changing the number names for red and yellow? In other words, when we did this...

Brandon: Now the reds would be one and the yellow would be zero.

R3: This is really nice. Are you convinced that you’ve found all the towers and all the pizzas?

Brandon: Yeah. Yeah. All the towers. All the pizzas.

R3: They both come out to how many?

Brandon: 16. 2-4-6-8-10-12-14-16.

R3: You convinced of this now?

Brandon: Yeah.

R3: Yeah? This is really nice.

Brandon: Oh, and you could also do the opposites on the pepperonis and sausage thing.

R3: Oh. How would that work?

Brandon: It, it would be easy. But it would be a little more. It would be a little harder.

R3: Yeah, but you could just show me how to start. You don’t have to do the whole thing now.

Brandon: I won’t. I’m not going to do the whole thing.

R3: Because I don’t want to make you nuts here.

Brandon: Okay. So, you could have a pepperoni, a mushroom, no sausage, no pepperoni. No pepper, no mushroom, sausage, pepperoni. Two opposites. [R3: Interesting.] And that would be just like these two. Nope. If we’re focusing on yellow, it would be sort of like [R3: Yeah, which ones would it look like?] 1-1-0-0. These two. [Selects r-r-y-y/y-y-r-r] [R3: If we’re focusing on…] Yeah, if we’re focusing on yellow, it would be 1-1-0-0. [Holds y-y-r-r] 1-1-0-0. 0-0-1-1. [Holds r-r-y-y] 0-0-1-1.

R3: Oh. Can I ask you what you think now? Which do you think, which argument do you think is more convincing: grouping them by zero, one, two, three, and four, those four categories, which is five categories, [Brandon: Or by opposites.] or by
opposites? Which convinces you that you have them all? Which way?

Brandon: Tough.

R3: Because they’re are both ways of going about them, right?

Brandon: It depends on what you like. If you’re better with opposites, do opposites. If you’re better with grouping or playing by categories, do categories.

R3: Interesting. I guess my one concern with opposites is that, how do I not know there’s another one out there in space somewhere that I haven’t thought of with an opposite? You know what I’m saying?

Brandon: We’re only using four blocks, so you could only start out with four, you could, well, to put, let’s put it, let’s make it easier. You could only have two, you could only use four blocks, so you, there, there, there wouldn’t be that many opposites. The, an opposite would be like take that and that. [Selects r-y-r/y-r-y-y] One opposite. Another opposite would be [Selects y-y-y-r/r-r-y-r] that and that. Another opposite would be [Selects y-y-r-y/r-r-r-y] this. This. And this. This. [Selects y-r-r-r/y-y-y-y] Those were two opposites and to prove that those are all the opposites to ones, you could, you would make stairs, different colors, multi-colored stairs. Y-r-y-r. Then you could have those. All yellows, all the yellows would go up, in stairs, like stairs. The three, the one group for yellows or three groups for red would look like that. And the same would go for here. And now, since we’re only using four, any more would be in the twos or threes, twos or threes groups.

R3: Okay. Now the twos, I can see that for these making a staircase, [Brandon: Yeah, those were easy.] but the twos, [Brandon: Are hard.] thinking about that in opposites, and knowing that I in fact know them all, I find that difficult to do.

Brandon: Okay. You have two of those. [Selects y-r-r-y/r-y-y-r] And since you’re only using four, you’ve got two of those. Two of these. [Selects r-r-y-y/y-y-r-r] Two of these. [Selects r-y-r-y/y-r-y-r] And that’s all for the twos group. 2-4-6.

R3: But, how do you know that though? I mean, I know you showed it to me with your pizza drawings, [Brandon: I know.] but if I’m doing it in terms of opposites now, which is what you’re saying, how do I know, how do I know in terms of opposites that that’s
all of them?

Brandon: Find them. You can’t do anymore if you try, whatever you try, it comes out one of those. Because since they’re opposites, if, if we never had that one, and we looked hard enough, you could see that you have y-r-r-y, but you don’t have r-y-y-r. No matter what you try to do, you’ll always end up with one of those. Like for that, whatever you do, you try to do one, you will, you will either end up with like y-y-r-r, r-r-y-y. Anyway, it’s just like, it’s just like turning them upside down. You start a y-y-r-r like that would turn upside down would become that. That’s sort of, and no matter what you try, you would always end up with one of those.

R3: I’m fascinated by what you showed me with how you can lay this like this. In fact, look, it fits. And, and how this, let’s see this one, [Selects r-y-y-r] number 10 here, would be this one, right, because we call the ones yellow. That’s really neat the way you thought of that. You know I can, I can see that. And you’ve convinced me with this picture that you found all the ones and the zeroes. And it’s very nice.

Brandon: And all the others.

R3: You’re a very good thinker, you know that?

Brandon: It was Colin’s idea for the graph. I was, I was thinking about what to do because I didn’t know how to divide. Once Colin got…

R3: Colin thought of the graph?

Brandon: Yeah, so then we, so then I, we started doing, working separately. We split up then we went and came back together and compared answers.

R3: That’s really a good strategy. Can I ask you a question? I’m just curious. What do your mom and dad do for a living?

Brandon: My dad makes pool liners. My mom just works at home.

R3: That’s hard work, Brandon. Take my word for it. That’s really hard work. That’s what my mom does, too. Okay. I want to ask you one more question before I let you go. [Brandon: What?] Have you thought at all about the towers of different sizes anymore? Remember, we started, we started to talk about that. We talked about towers that were three high and five high. We were talking about different
heights.

Brandon: Oh three high. Those would be, there would be about 6, less than 6 ways.

R3: How do you know that?

Brandon: Maybe. 6. There would be 6 ways.

R3: Can I ask you a question?

Brandon: I’m going to make a theory there is 6.

R3: Okay. Let’s test, let’s test that theory. Can we test it?

Brandon: [No.] Well, you could do whatever way you like. I almost wish I could see you test it with your graph. Can you make a graph to show that for threes?

Brandon: I don’t think so. It’d be hard. When you’re using only, when you’re using only colors and cubes, and, and, it’s hard to make a graph. More likely it’d be easier to make the cubes than the graph, and make like, but you can’t say like yellow red. It’d be easier if you called them something like maybe pepper, peppers, tomato, carrot, and tomato.

R3: You know what I’m thinking?

Brandon: No, zucchini and tomato.

R3: Which, kind of like what you showed me here where this kind of went with this? [yeah] Okay. Could you do that for threes if you had a key that kind of told [oh yeah] you that zero and one were whatever color they are?

Brandon: Yeah, but even if I do threes, I could, a three would never fit on that chart no matter what I do.

R3: Why not?

Brandon: Because since we’re using four different toppings, and since we’re only stacking them in three stories high…

R3: Well, could we make our chart look a little different?

Brandon: Yeah, if we take off one. It would be wrong then because then you would have to make it less. There would be many opposites. There would be many sames. So you could have a yellow-yellow, a yellow-red-yellow, or a red-yellow-red. That’s one pair. And since you’re only working with three, if you have my favorite, then it’s 6.
You said 6.

So then since you can’t do it anymore, you could only do. Then you could do a red. Yup. It’s 6 alright. The answer is 6 already. You could tell. You could tell it’s 6.

How? I couldn’t tell.

You could tell it’s 6 already just by making that one more. Actually, you could tell it’s 6 by just looking at that. By just in your head. Since there’s only 3, you could draw it actually too.

Let’s get another piece of paper.

No, there’s enough room on the back of this. Don’t waste paper. So you could do just like shaded would be red. 1-2-3. Shaded-shaded-shaded. Oh wait. No, it would be more than 6, I think. Then you could have blank-blank-blank. Same as always. Maybe shaded-blank-shaded. Or this, this one would go with that one.

Okay. These are opposites then?

Yeah, then that then you could have blank-shaded-blank. [R3: Okay.] Another one would be, this would be group of two. Four. That would be 4. Here’s 6. You could have one blank, blank. Blank-blank-blank one-one. Then you could have blank-blank-shaded. Shaded-shaded-blank. Six. Oh, the answer would be 8.

Okay, so wait, now these stand for shadeds?
[Brandon: Yeah.] These things that look like little squiggles? [Brandon: Those are reds.] And these are blanks so they’re zeroes? [Brandon: Yeah.] Can you fix these so I can see that too because I’m looking at two different systems here? This is good. Interesting. Okay, now how do you know you have them all?

Well, I’ll make the ones into cubes. These two would go with that would be that. [Combines y-r-y/r-y-r] This would be this.

Okay, that’s a pair of opposites. I see that.

Now, where is that other one? Then you could have those two, which would be together. [Combines y-y/y/r-r-r] Then maybe, where’s that other pair of threes I had? Okay. This, and you could have a red and two yellows. Then a yellow and two reds. Okay, now after
that, oh, I confused myself kind of. Now, now it’s right. Then you could have those two. You already have 6 groups. It should be 8 because I forgot about, without counting these [holding r-r-r/y-y-y]. If you didn’t count it, it would be 6, but those are colors, so they count. Then you could work like from bottom up just like over there. You start with a bottom red and yellow, you work your way up to those two. And you could only do these, it depends on what color you’re looking at. That would be one then you could have two reds and a yellow. And that would be all.

341 55:57 R3 What do you mean by work your way up? I think I see it, but I want to be sure.

342 Brandon You start with two yellows and a red. A red, a yellow, and a red on the bottom. [Okay] You can hook those on and make that one group. Just for one, pretend those are not there. You start off with those wherever you see the main opposites, then you would have like another one right there. Those would be the two main opposites. Then you could work, then you would go to the highest, and those would be the two main opposites. [R3: I see. Okay] Take out all the yellow, yellow ones and they’d look like that.

343 R3 You’re going to make the…

344 Brandon Staircase.

345 R3 Staircase.

346 Brandon And when you do it, the staircase should only have three stairs for it to be right.

347 R3 Why?

348 Brandon Because if, to tell if you have one of the same, it would look like this. Then you say, “Oh I got another one.” Put it somewhere. The stair goes down. You have, you walk up a stair. Walk across two stairs, and go up. Fine, switch around. You go up three stairs, down one. But there you would have to crawl, jump to a stair, walk up and go down, [R3: I see.] so it would make no sense. So that, then, you would take that away, and the same, is true for the red.

349 R3 So you got these, these, and these. [Points to r-r-y/r-y-r/y-y-y-r; y-y-r/y-r-y/r-y-y; r-r/-r/y-y-y] And, you sort of showed me with this picture here, sort of like what you did on this chart here, right? [R3: Yeah] Is there a way, I guess what I’m wondering is now, could we
apply this chart you made, which I think is a very interesting nice strategy to towers of different heights or pizzas with different amounts of toppings?

Brandon: Yeah. Easy.

R3: How, how would that work? Like I’m wondering…

Brandon: Okay. Now, let’s focus on red now. I’m going to focus on red. Right there. 1-1-0-0. 1-1-0-0. Okay. But if you use the yellow, this one [r-y-r], it wouldn’t be right. The first three would be right. You would have red 1-1-0-0. 1-1, what happened to the other? Take out, you would have to…

R3: Could we take off a column maybe and then look at it?

Brandon: You would have, it would be, you would have a lot of opposites.

R3: I see what you’re saying.

Brandon: Because they have opposites. [R3: Like what happened to the…] Like now you have a two, now you have groups of, twos become, threes become twos, groups of twos become ones. Sort of. Some of the groups. Wait. Okay. You would have opposites like here’s one group. Let me just see, let me just find that twos group. There’s the ones group. The twos group. Take that off, you would have in this group, where’s the other sheet with that? Alright. Here’s how we have lots of opposites. I mean lots of same. Okay. Now, take that off. You would have. Now, here’s a threes group. Fine, you take off one. Now, you only have a pepperoni, pepper and sausage. Pepper-sausage. Take one, just take off one column. That screws the whole thing up. Pepper-sausage. Pepper-sausage.

R3: I see. I see duplicates.

Brandon: Yeah, for all. Oh, take off one. 1-2-3. 1-2-3.

R3: How many duplicates do you think there are going to be, now that we took off a topping here? [Brandon: A lot.] Is every tower here, is everything here now, this number going to have a duplicate, do you think?

Brandon: Yeah. About. Yeah. Every one probably. In the twos group 1-2. Threes group, that’d be here. 1-2. 1-2. Every one has a duplicate. We’ll call them. take off two columns, then everyone, then you would, you
would have way less. You would have 1. You would have lots of duplicates.

That’s a good question. How many, how many possibilities would there be if we just had the two columns with our zeroes and ones? You’re right, we’re going to have a lot of duplicates.

That’s a good question. How many, how many possibilities would there be if we just had the two columns with our zeroes and ones? You’re right, we’re going to have a lot of duplicates.

Brandon Only two.

What would they be?

Pepperoni and. No. Only one. Only pepperoni with mushroom.

You sure?

Wait. There’d be 3. There’d be 3. Pepperoni no mushroom. That’d, that would be one. Pepperoni-mushroom. That would be two. No pepperoni-no mushroom that would be three. And that’s all you could do. [R3: I see]. Right here you have a number 10. Like 1-0, 1-0. And there’d be more 1-0s. One, another 1-0. And now for, and now you’ve got 11s. 1-1. 1-1. Wait, wait. You’d have 1. 1-0. 0-1. 0-1. Oh now, you would have 3, you would have 3 duplicates. Three of each one would have 2 more. Two others the same.

Two others the same?

Yeah, if you, wait if you use 3, each one would have only one other the same.

So, if I’m looking at 0-0, [Brandon: Yeah.] how many times do I, will 0-0 appear?

One. One 0-0. Two 0-0. And, there’s one more somewhere. Three 0-0.

So there’s one…

Two, three.

Two. What about this one?

That’s….oh yeah, that would be four. 0-0. 0-0. 0-0. 0-0.

Do you agree with that? [Yeah] Did I count right? Interesting. Okay, so let’s take this back to towers now. Okay, we did pizzas with three toppings, two, three, four toppings. Now, you’re telling me about pizzas with two toppings. Okay? If we go back to towers, selecting from two colors, red and yellow
again, how many towers do you think there would be that would be two cubes tall?

Brandon: Two cubes? There would only be three?

R3: What does it look like?

Brandon: No, make that 4, make it 4. [Creates r-r/y-y/y-r/y-r-y]

That’s all.

R3: Okay. Now, let’s go back to pizzas when we have two toppings. Let me take another sheet here.

Brandon: What color do you want to focus on?

R3: I’m going to call, I’m going to say we’re gonna have mushroom and [Brandon: Pepperoni.] pepperoni as our choice. Okay? Okay, I’ll just write “m” for mushroom. Okay. Now those are the two we’re selecting from, right? [Brandon: Yeah.] How does this match up to towers now? We made. You said that there were three pizzas with two toppings, right? What would those be again? Could you show me with?

Brandon: With just these two, you would have 1-0. 0-0-1.

R3: And what would, what would, where would the?

Brandon: Oh, no, wait. We gotta to start over again. First, I want to, first I have to know, are we going to do it by how, you could have like just one pepperoni, how many pieces of that could you have on like you could have on just one pepperoni on a slice, two pepperonis on a slice, three pepperonis on a slice.

R3: Does that matter?

Brandon: Yes, it does a lot. Okay. Let’s do mushroom and anchovies. You could have one mushroom, zero anchovies. Two mushrooms, zero anchovies. Three mushrooms, zero, zero anchovies. Four mushrooms, zero anchovies. [R3: I see what you’re saying.] You could go as, you could go up to a billion.

R3: Is that what we were doing though before when we were looking at these?

Brandon: No, no before when we were doing the pizza problem, you would just do one.

R3: What did the zero and one mean? I…

Brandon: Like one could mean, yeah, you have it. Zero, zero would mean no. That’s nothing.
R3  I see. If we go back to that system, of it, of what you called one and zero. You said one meant it had it. Zero meant it didn’t have it. Then how many pizzas could we make selecting from anchovies and mushrooms?

Brandon  Oh, without depending on how many… [R3: Yeah.] Just with that, it would have one mushroom-zero anchovies. Zero mushrooms-one anchovy. And I think…

R3  So there were how many?

Brandon  Four.

R3  Four. Does that match up to what we did over here with these?

Brandon  Yeah. Yeah. Let’s focus on it now. Red, red zero-zero. Yellow one-one. 0-0. 0-1. 1-0. 1-0. And there would be 0-0. 0-0. Yeah.

R3  That is really fascinating. Do you think that’s always going to work? Like does it matter how many toppings we have? Could we do something like this?

Brandon  Yeah, but if we’re going to do it, like how I said earlier, how much, how many toppings you’re going to have on one pizza, like, we’re going to have four pieces, four mushrooms on a slice, you would have to tell us what the top number of [R3: Sure.] pieces you could have on a pizza or else you would go onto a billion.

R3  Oh sure. See, I guess, I guess, when I was thinking about the pizzas in my head, [Brandon: Yeah.] I was just thinking that we we’re kind of throwing the stuff on the pie. Have you ever seen those machines like at Domino’s Pizza where it comes out of, like it throws it on the pie, and I wasn’t worrying because some pieces may get four slices of pepper, but some may only get one or two you know because it’s kind of scattered by the machine. So, I was just thinking about whether it has some on the pie or it doesn’t have some on the pie. Not worrying about the individual pieces.

Brandon  Well, if it was like, if it was being handmade, pulling a certain amount each, you would. Someone would have to tell such top number to stop at. And if it was 10, and you’re only using mushrooms-and anchovies, there would be 20. You could have 20 in total.
You think so?

Yeah. Ten. You could only go up to 10 slices, 10 pieces on each pizza. 10 x 2 because there is only two of those.

Two toppings. And what would the 20 mean here?

That in total, in total you could have 20, 20 on each, if, card if like on each card you could have like 20. It would be 20 if you wanted the total amount. The biggest. You would have 10 mushrooms. 10 anchovies.

On the pie?

But. Mainly, if it were medium there’s 8. You would have to do 10x8. And 8 on each slice, there would be 10. 10x8. 80. 80 pieces in total.

Oh. Okay, so you’ve told me number of pieces and this is the amount of toppings.

Yeah, if you only ordered one pizza with everything, the top amount would be 20.

The top amount of what?

How many pieces of each you could get in total, since you could get 10 mushrooms in total and 10 anchovies together. [Ten mushrooms and 10 anchovies around the pie.] And since you’re only using two, that would be 2. And times it by two.

Is this on a slice or a pie? Ten of each? Is that what you’re...

Oh that would, with that. [I’m imagining this pizza.] This would, this would be just one piece. Because, because mainly a medium pizza would be 8 pieces, so you would have to do 10x8.

I see. To tell me, to tell me how much mushroom is on the pie or mushroom and anchovy?

If, if you want a pizza with everything, it would be, you would have 20 thingamabobs on each.

I see. Okay. I understand what you’re saying. Did you think this was kind of interesting doing this? [Yeah.] Connecting this up? This was really fun for me. Would you come back and talk to me some other time? [Okay.] I’d love that.

What time is it? I’m a little worried.
It’s late. It’s getting to be late. It’s quarter to eleven.

Quarter to eleven? Oh. That’s okay. [R3: Is that okay?] Yeah. It doesn’t.

Where are you supposed to be now?

We’re just coming out of reading.

Okay. Can I take you back then? If you’re concerned about the time.

No, I may, I may have missed my music lesson, but that’s okay.

Is that okay? Are you going to get in trouble?

It’s okay. No, it’s okay.

Do I need to talk to anybody for you? Or…

I could tell Mr. Franco during band practice.

Okay. I’m sorry you missed your lesson. You know, I didn’t want you to do that, but this was just so fascinating to me. You could leave those, Brandon. I’ll fix those later. I think I probably should get you back to your class. I, I’m, I’m glad that you said that we could come back and talk with you though if we, you know, if we would like because I bet there, after people see this tape from Rutgers, that they’re going to have more questions for me. I’m going to need to talk to you again, I think.

There you go.

Kenilworth Sept 30, 1993 (Session 1 on Guess My Rule)

Camera View: Combined
Date of filming: 9/30/93
Harding public school, Kenilworth NJ, Guess my Rule (GMR) problem
Transcribed by: Patricia Giordano
Date of transcription: 11/2006
Verified by: Dina Honigwachs
Date of verification: 7/2007
Format revised by: Patricia Giordano
Date of revision: 8/2007
**Whole class discussion**

<table>
<thead>
<tr>
<th>Time</th>
<th>Speaker</th>
<th>Transcription</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RBD</td>
<td>OK, can somebody explain what it is we’re doing?</td>
</tr>
<tr>
<td>Students</td>
<td></td>
<td></td>
</tr>
<tr>
<td>RBD</td>
<td></td>
<td>Can you listen? Can you listen to Milin please?</td>
</tr>
<tr>
<td>Milin</td>
<td></td>
<td>We’re guessing, um, what’s going to be the equation.</td>
</tr>
<tr>
<td>RBD</td>
<td></td>
<td>Exactly right. This time we’ve reversed it, right? A minute ago we had the</td>
</tr>
<tr>
<td></td>
<td></td>
<td>equation and we worked out the truth set. Now I’m going to tell you, on these</td>
</tr>
<tr>
<td></td>
<td></td>
<td>papers here, I tell you the truth set and I want you to tell me the equation.</td>
</tr>
<tr>
<td>Students</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ankur</td>
<td></td>
<td>Like that? [points to what RBD has written from earlier with box and triangle]</td>
</tr>
<tr>
<td>RBD</td>
<td></td>
<td>Yeah. That’s right. You’ve got it? If you’ve got it write it down, and come</td>
</tr>
<tr>
<td></td>
<td></td>
<td>show it to me, would you? If you figure out what the equation is, write it</td>
</tr>
<tr>
<td></td>
<td></td>
<td>down and you can show me.</td>
</tr>
<tr>
<td>RBD</td>
<td></td>
<td>If you figure it out write it down. [Students begin talking amongst themselves. RBD sits but does not speak for a minute.] If you’ve figured out the equation, write it down and come show it to me.</td>
</tr>
</tbody>
</table>

***At this point, discussion occurs among students sitting near one another. The groups are shown together. The discussions were happening simultaneously.***

<table>
<thead>
<tr>
<th>Milin</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Milin</td>
<td></td>
<td>[Writes (□ × 2)+1= △ next to first problem] The first one I know. It’s the same as the one on the board.</td>
</tr>
<tr>
<td>RBD</td>
<td></td>
<td>Oh, OK, the next one.</td>
</tr>
</tbody>
</table>

**Ankur, Michelle R**

<table>
<thead>
<tr>
<th>Ankur</th>
<th></th>
<th>[Ankur goes up to RBD with his paper with an equation on it. RBD looks at paper.]</th>
</tr>
</thead>
<tbody>
<tr>
<td>RBD</td>
<td></td>
<td>I think if you try some of these other values. Let’s try that one.</td>
</tr>
<tr>
<td>Ankur</td>
<td></td>
<td>Inaudible</td>
</tr>
<tr>
<td>RBD</td>
<td></td>
<td>It’s reasonable. It’s a sensible approach. If we try it. Let’s see if we</td>
</tr>
<tr>
<td></td>
<td></td>
<td>try it. If we put zero in the box, so that’s zero plus 5 equals 5. So that</td>
</tr>
<tr>
<td></td>
<td></td>
<td>works. Now try it with one and two. If I put one in I get one times four is</td>
</tr>
<tr>
<td></td>
<td></td>
<td>four, plus five is...</td>
</tr>
<tr>
<td>Ankur</td>
<td></td>
<td>[Nods head, sits back down. Matt goes up to talk to RBD, followed by Brian,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Romina, Amy Lynn, Michelle I, and Bobby.]</td>
</tr>
</tbody>
</table>

**Matt, Stephanie**
Stephanie: How interesting.
Matt: I have it! I got it. I got it. I got it.
Stephanie: How interesting. Matt, we’re partners. Help. Help, and if you know the secret, tell me. [Stephanie leans over and looks over Matt’s shoulder and they talk to each other. They are pointing at the paper.]
Matt: Look. Plus one, plus two, plus three, plus four, plus five, plus six. [He is pointing to the first problem on the ditto. As he says “plus one, plus two…” he points to the numbers under the box and triangle.

\[
\begin{array}{c|c|c}
\hline
\square & \Delta & \text{he says} \\
\hline
0 & 1 & \text{plus one} \\
1 & 3 & \text{plus two} \\
2 & 5 & \text{plus three} \\
3 & 7 & \text{plus four} \\
4 & 9 & \text{plus five} \\
5 & 11 & \text{plus six} \\
\hline
\end{array}
\]
Stephanie: So what does this mean?
Matt: What’s it mean? One, two, three, four, five, six, get it? [He is pointing to the chart again.]
Stephanie: OK, one, three, five, seven
Matt: This one starts with five, and six is eleven. This one starts with three, and four is seven. [He continues but it is inaudible.]
Matt: [looking towards RBD] I found out one of them.
RBD: If you figured out the equation write it down.
Matt: Can I just tell you?
RBD: It’s kind of hard to say an equation.
Matt: It’s sort of; I sort of found how you could make them add up.
Stephanie: Wait, here’s what I don’t get though.
Student: One, three, five, seven, one, three, five, seven, right?
Stephanie: It’s almost like, a pattern almost, because like, the two can go in the one and the three. There’s a three here for some unknown reason. [Points to box value 3] Then the four can go in between the three and the five, and the five can go in between…See, that’s what doesn’t make sense. It repeats. But then it, like, two can fit between the three and the one. Then it repeats again. Then comes the four, and then it repeats once. [She is drawing short diagonal segments between the numbers in the chart from a number under the box to a number under the triangle.] Same thing. [Points to second example.] You know what I mean? Now, like this one…[Points to third example.] This one [points to first example] it like repeats, it goes, then it repeats, goes, repeats, goes, repeats. Oh, goodness.
Matt: [He gets out of his seat and goes to talk to RBD. He is pointing to his paper.] Plus one, plus two, plus three, plus four, plus five. [Inaudible.]
OK that’s a good idea.

[He returns to his seat. Brian, Romina, Amy Lynn, Michelle I, and Bobby talk to RBD.]

Michelle I, Romina, Brian, Amy Lynn, Bobby

Voice 1  Even, odd, even, odd.
Voice 2  It can’t be.
Voice 1  Even, odd.
Voice 2  Where?
Bobby  I know the secret.
Michelle I  [Michelle I has gotten out of her seat and is standing between Romina and Brian.] One, three, five, seven, nine, twelve, thirteen, fifteen, and so on and so on.
Romina  Yeah and five…
Michelle I  [Inaudible.]
Romina  No, thirteen.
Michelle I  [Inaudible.]
Romina  Yeah, that’s what I thought but then he… [indicating Brian] Because it goes evens, evens, and evens. [She points to problem #1 then problem #2 then problem #3 as she says this.] Yeah, that’s a five, that’s a three, that’s a seven.
Bobby  Just plug those zeros, one, two, three, four, five.
Amy Lynn  Odd, odd, odd. [She points to problem 2.]
Brian  Oohh. [Covers mouth while making noise.]
Bobby  Oh, I get it. [Reaches over and points to paper.]
Brian  [Makes noise du-ur.]
Michelle I  Say that again?
Brian  One, two three, four, five, six. [He points to the pairs in problem #1 as he says each number.] Six, and six, five, five, four, four. [He points to problems 1, 2, 3.]
Romina  Oh duh, we knew that much.
Bobby  Five, six, seven, eight, nine.
Michelle I  I think it’s this. You just go down the number line here and you just skip two.
Romina  Skip two. [Same time as Michelle I.] One, two, three, four, five, six. [Pointing to box, triangle pairs in problem 1.]
Brian  I got one! I got one. I got one.
Romina  Could you run that past…
Brian  [Inaudible. Pointing at paper.] Oh yeah, oh yeah.
Michelle I  Yeah, but does it work for the rest? There’s five and there’s six, there’s five, there’s six,…
Brian  Seven, eight, nine. Oh yeah, we gotta show him.
RBD  Has somebody got an equation to show me? You have? Come show me. Come show me the equation.
Brian
He wants me to show you. Excuse me. I’m coming through with an answer. [This group of five students gets up and goes to RBD.

RBD
[Brian is pointing to his paper. He is speaking to RBD but it is inaudible.] That’s certainly an interesting idea. But now what is it that you’re going to...We won’t have time to do it today. What are you going to show me ultimately?

Bobby
I got an idea.

RBD
You’ve got an idea?

Bobby
[Points to the white board chart.] The times number, that you put there, you add it to the first number unless it’s zero, and then you get your answer.

RBD
That’s certainly what you do.

Michelle I
[Points to the white board chart to the 0,1 pair] See there’s one in between there, then two, then three, then four. [points to each successive pair]

Whole class discussion

RBD
Wait, wait. Can everybody sit down for a second? I want to make sure we agree on what we’re trying to do here. What did we do the first time? I gave you an equation, right? [Pauses. Students quiet down.] The first time I gave you an equation and what did we do? We worked out numbers, pairs of numbers that would make it true, right? Now what are we doing? I’ve changed it. What are we doing now?

Student
Making equations

RBD
Now I’m telling you the numbers that would make it true, I’m telling you the pairs of numbers that would make it true and you’re going to tell me the equation. We probably won’t have time for that today. We’re really out of time. [Discusses returning next day to do more of this.] Let’s make sure everybody remembers. Would somebody say, what was the secret for this kind of equation? Let me tell you what mathematicians call this. This is known as a quadratic equation. What’s the secret that you have for this?

Ankur
The two numbers, when you add them they give you the number on the left and when you multiply them they give you the number on the right.

RBD
Thank-you.

Michelle I, Romina

Michelle I
[Michelle I and Romina are talking very quietly while RBD is wrapping up.] Brian, your thing doesn’t work for here. [Points to problem on second page. Others turn page. She turns back to first page.] And it doesn’t work for here either. [Points to problem 3.] Look…
Romina Seven… [She is looking at second page.]
Michelle I Look, look, go back to the first page. Number three. It’s one, three
Romina It skips.
Michelle I Let’s see if it works here. [Turn page. Now pointing at problem4.] It works. It works.
Romina Thank-you. Thank-you very much.
Michelle I [Looking towards RBD.] We got a different answer because our other one didn’t work.
Romina I got the secret. [Romina and Michelle I get out of seats to go to talk to RBD.]
Michelle I On this first side it just like goes down the number line.
Romina [Pointing to problem #1] See there’s one and three which is two, two, adding two.
RBD That’s a very good idea and that’s going to be very useful.
Michelle I And then here (pointing to problem #3) the way I see it, four minus one is three, which is three in between, and there’s three in between each of them.
RBD That’s very nice.
Michelle I And it keeps on going for every problem.
RBD That’s really very nice.
Michelle I It’s right. It’s right.
Kenilworth Oct 1, 1993 (Session 2 on Guess My Rule)

Date of filming 10/1/93
Guess my Rule (GMR) problem
Transcribed by Poroshat Shakoor
Date of transcription 1/2/2007
Format revised by Patricia Giordano
Date of revision 6/2007
Verified by Dina Honigwachs
Date of verification 7/2007

CD 1 of 2

<table>
<thead>
<tr>
<th>Line</th>
<th>Time</th>
<th>CD Line</th>
<th>Speaker</th>
<th>Transcription</th>
</tr>
</thead>
<tbody>
<tr>
<td>00 00 00</td>
<td>00 06 00</td>
<td>Ann 1</td>
<td>RBD</td>
<td>Okay I guess we’re ready to start. Hi.</td>
</tr>
<tr>
<td>Ann 2</td>
<td>Student</td>
<td>Hello.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ann 3</td>
<td>RBD</td>
<td>Um, I wanted to talk a little bit more about that question of secrets because I think that’s an interesting part of what we do. Um, what sort of things do scientists do?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ann 4</td>
<td>Jeff</td>
<td>Discover advance.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ann 5</td>
<td>RBD</td>
<td>Yeah, that’s right. Um, are there any problems you hope scientists will solve in the next few years?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ann 6</td>
<td>Jeff</td>
<td>Cure certain diseases.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ann 7</td>
<td>AmyLynn</td>
<td>AIDS.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ann 8</td>
<td>Teacher</td>
<td>Yeah, that would be on the top of my list. Yeah I think that’s right.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ann 9</td>
<td>Student</td>
<td>Solar powered cars.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ann 10</td>
<td>Student2</td>
<td>Yeah, like solar powered cars.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ann 11</td>
<td>RBD</td>
<td>Now, now there’s a sense the sort of a two sided ah thing about ah secrets. When people, obviously since you don’t know, people don’t know how to deal with certain kinds of cancer and things like that, there are secrets, it’s not because somebody is keeping the secret, it is because nobody knows, right, and people are trying to find out what it is. Um, now, I think it’s clear that the people who make some of these discoveries are very proud of it and they like their name attached to it. Do you know the names of any scientists?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ann 12</td>
<td>Jeff</td>
<td>Do I think the people now or famous people?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ann 13</td>
<td>RBD</td>
<td>Famous people.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ann 14</td>
<td>Student</td>
<td>Or people dead.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ann 15</td>
<td>Jeff</td>
<td>Thomas Edison</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ann 16</td>
<td>Student</td>
<td>Alex Graham Bell.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ann 17</td>
<td>RBD</td>
<td>Alexander Graham Bell. What did he do?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ann 18</td>
<td>Student</td>
<td>He invented the phone.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ann 19</td>
<td>RBD</td>
<td>Yeah.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ann 20</td>
<td>Student</td>
<td>Thomas Edison.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ann 21</td>
<td>Student</td>
<td>Einstein, I guess</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Einstein would probably be the first thing that would occur to me. He’s a scientist. Okay…okay. So you have some idea of that and people really are very proud the theory of relativity, which is the thing Einstein thought about and worked out. Yeah, I’m sure he’s very proud that it’s attached to his name. And so there’s a sense in which people um… clearly there are secrets because nobody knows what to do until somebody figures it out or finds it. Uh, and then when they do they like to get credit for it. So, I mean that’s the sense in which okay there are secrets…but do scientists also share this information? Yeah, I think that Jeff got it probably exactly right, sometimes. They have to, they have to actually because in the long-run…uh in the long-run no single person could do it all by themselves, so they have to share. Matt.

I was wondering what does the mean anyway, e equals m c squared? Well that’s interesting… How do you use that? Energy equals… Yeah that’s right e stands for energy do you know what the m stands for? m c squared. What does the m stand for? E equals m c squared. Mechanical. We could look it up. Well, we could talk about that, but probably not today. Why don’t you see if you can find out and we can talk about that, but I think maybe I propose not to do that today? Maybe a dictionary.

So the main thing is uh we need we need to do both of these things. I think that you do want to try to find secrets occasionally because that’s, that’s sort of fun and that’s what you do in science and, and in mathematics, but we also want to share them too, and so we have to work out which we’re doing. Maybe the first time you find a secret you keep it a secret for little bit so other people can think about it too and see if they can find it. And then at some point, probably, we want to share it. Um, okay, now there was really a sort of neat thing that happened last time… oh what were we working on? Remember, we were doing equations that were box times box minus something times box plus something equals zero. What were we taught to do? Do you remember what you were trying to find some numbers what did those numbers do? The numbers replaced like the empty boxes or triangles. And they made a true statement didn’t they… when you did it… said it was equal to zero and that was true. Okay, and we did quite a few of those and you got to be quite good at that I think. And various people found the
secret and I guess by now everybody knows what it and
we didn’t quite agree whether it’s was one or two
secrets, most people say it’s two, but I think some
people here like you persuaded us it’s one. Um, what’s
the secret to that?

Ann 41 Milin It’s one big secret.
Ann 42 RBD It’s one big secret? Matt.
Ann 43 Matt That the… the two multiple… the two num the numbers
have to like when you add them up it has to equal… it
has to equal the number to the, to the left and be
multiples of the number to the right.
Ann 44 RBD Well you might not really mean multiples, when you
multiply them…
Ann 45 Matt Yeah, be able to multiply them…
Ann 46 RBD Yeah, yeah right when you multiply them they give you
the number on the right and that’s certainly right. Okay
and I think that everybody was good at that. And then
we started working on, well maybe before I leave that…
uh those two equations on the bottom came up because
uh, uh, Milin actually proposed one of them and then
somebody proposed the other one. Jeff, what was
special about them.

Ann 47 Jeff Cause, there were two prime numbers in it so it was like
impossible…
Ann 48 Student no.
Ann 49 Jeff …or you had to go into decimals or whatever.
Ann 50 RBD okay, we left that hanging a little bit and I think I’m
going to leave it hanging again today, but it’s a very
interesting problem and it certainly looks like it might
be impossible doesn’t it? And we might have to use
some other kinds of numbers or something. Okay… um
now then we started working on the sort of thing that’s
on the top up there. Um, we started with that equation
box times two plus one equals triangle. Right, and what
did we do then, Stephanie what did we do?

00 05 09-00 11 09

Ann 51 Stephanie Well, we had to put a number in the box and a number
in the triangle so that the equation was true.
Ann 52 Teacher Exactly what we were doing, and when we did that if
we put zero in that box what number did we put in the
triangle?
Ann 53 Stephanie One. And we made that table there, right. Okay, and
now then we, in fact actually um Michelle where…yeah
um I’m sorry
Michelle R. Uh, you remember what you wrote on your
paper.
Ann 54 RBD No.
Ann 55 Michelle R You want to take it and maybe write it here so that
everybody can see it. Here, just stand there. Well a
couple of them anyhow. [Michele goes to write on the
board]
Ann 56 RBD Well, you suppose you can get it if you wrote small do
you suppose you could get it up by the table the way
you did it on your paper?
Ann 57 RBD Up here?
Yeah, cause that was sort of neat the way you did that.

And you left out one parenthesis; do you see where you left it out?

Oh. [Michelle R closes the parenthesis \((x^2) + 1 = \Delta\) and places a zero in the box and one in the triangle.] Should I do more?

Well that’s probably enough, but she went down and did that, and you agree that that’s what we were doing?

Yeah.

Now, what did we do then? We, then we turned the problem around and did something different. Michael what’d we do then? Michelle?

We tried to find a secret to it with a pattern like how the numbers...

Okay, and some of you did find a very interesting secret and it might be an appropriate one to share, um no, Ankur says that we shouldn’t do that.

Yes we should.

Well, okay, well we won’t we won’t do it just now we will sooner or later. We will sooner or later okay, uh, but we started, we started turning the problem around didn’t we and for the other problems I gave you the table. Here, here I gave you the equation and we made the table, right, but now in the other problems, I gave you the table and what are you supposed to do?

Find the equation.

Yeah, find the equation. Uh, and now for the second problem, let me pass this back to you. This is Stephanie’s, uh the other ones, uh Michelle’s, that’s Ankur’s, that’s Amy-Lynn’s, that’s yours, that’s yours, and who’s is this?

That’s mine.

That’s oh yeah, okay good. Ah, some people didn’t get one.

Whose is that?

I got a really neat one here.

That’s his. That’s his. Well okay, let me, uh, let’s see what we can do here.

Okay, who didn’t get one back? Anybody didn’t get one back? Okay, that’s that. I’m sorry…who’s is that?

I have no idea, this ain’t the one we’re doing right now

Yeah, but it’s the one we’re about to do isn’t it?

No, that’s the one we did before.

Oh I see okay, okay.

Whose is this?

I need the one you have there.

This thing?

I need the one that you have in your folder.

Well we can pass out some new ones. Who doesn’t have one now? Jeff you don’t have one.

Whose is this?

And, okay, Michael you don’t have one. Okay, why
don’t you put your names on them right away so you make sure they get back to the right people? Ah, why don’t you talk to your neighbors and see what you can do with problem two. We know about problem one. So, problem two, you’ve got the table and you’re trying to find the equation, just what Romina told us.

Ann 88    Ankur
I’m finished.

Ann 89    RBD
You’re finished? On all of them or number two?

Ann 90    Ankur
Just number two

Ann 91    RBD
Number two. Okay, come show me would you? [Ankur walks up to the board.] I’ll tell you what I’d like you to do. Let’s stand over here and put it down so the camera can see it. And let’s see; let me see if I can read it too. Okay, well, so in general it would be box times two, right.

Ann 92    Ankur
Right.

Ann 93    RBD
Okay, is that right it…?

Ann 94    Ankur
Yeah.

Ann 95    RBD
It would be box times two plus five okay.

Ann 96    Ankur
Then one and seven would work too and…

Ann 97    RBD
Yeah they all work. Okay, um, okay, why don’t you go and do the third one, see if you can do a few more of them

Ann 98    Student
Plus four. And whatever you get…

Ann 99    RBD
Yeah, if anybody needs more paper here is some paper.

Ann 100   Student
Yeah okay

Ann 101   Student
Four times four plus five…

Ann 102   Student
Say it again Matt.

Ann 103   Student
Yeah I see it.

Ann 104   RBD
[Ankur and Michelle I walk up to teacher with a completed problem]
That’s clever would you put it over there and let the camera get a look at it.

Ann 105   Student
One times one plus five…these would have to be all the same equation.

Ann 106   Student
you figure these go up by two every time and these just go up by one. Maybe you could put a box over here.

Ann 107   RBD
(To Michelle I and Ankur) maybe you could say a word or two about how you did it.

Ann 108   Michelle I
All right. The three is here because there is three between each number and the one is with that one and it would just equal one.

Ann 109   RBD
Huh, that’s really neat. That’s very nice. See if you could do some work with it.

Ann 110   RBD
I couldn’t hear what you said.

Ann 111   Matt
This plus one, two, three, four, five, six, seven, eight. Those go up to six and it’s five, six, seven, eight, nine.

Ann 112   RBD
Yeah that’s true isn’t it?

Ann 113   Matt
And then this one is all prime numbers, one, three, five, seven

Ann 114   RBD
Uh.

Ann 115   Matt
It sort of like it goes in a certain…

Ann 116   RBD
Now wait, I don’t see the one, three, five, and seven, I see one, four, seven, ten

Ann 117   Matt
But, plus one, plus three, five
Oh, oh, I see, oh, oh that’s neat.

I think I found the plus one.

Plus this times three plus five.

I had the plus one.

How many, how many people have got problem two done? You got three. Okay, I need somebody to come and do two, Amy-Lynn would you do three? You got three, would you do three? I need somebody to do two.

Uh, okay, Michelle would you do two? Show everybody how you did two. Okay we need to talk about that, is it okay if we give away secrets or is it too early to do that?

Could we tell it later on?

Wait.

We should give the secret right now

We should give the secrets right now, but some other people are saying it’s too early.

Why… am I the only one who doesn’t know it?

I doubt that. Michael, are you working with Milin on it?

Yeah, but he won’t tell me the secret.

He won’t tell you the secret.

Naughty, naughty.

Why should he Mike?

Well, because they’re working on it together, how can they work together

Is that negative two or just two?

That’s negative two.

Okay, gotcha.

Oh my god it does it does.

No duh, that’s why I’m writing it down.

Do you have to write the numbers in there too?

I’m not writing them

We’re starting at something, all we have to do is find the connection and…

We have to do is find the connection and…

It’s pretty easy.

No, wait look I have something…

Do you know the secrets?

Yeah, yeah, Milin is trying to make sure that Michael figures it out too, which is a good thing to do.

[Stephanie writes under number two,

\[4 \quad 2 + 5 = 13 \text{(in a triangle)}\]

Stephanie writes under number three,

\[0 \quad 2 + 1 = 1 \text{(in a triangle)}\]

Could I, could I get some idea how we’re coming along Here?

Very good.

Um, what’s the, how many problems have you people done?

Well, we’re on the second one and we thought we could just fly through.

Oh okay and uh Bobby and Amy-Lynn you’re on what number four?
Ann 152  Amy-Lynn  four
Ann 153  RBD  You’re on number four and Ankur and Michele you’re on what number?
Ann 154  Ankur  Six.
Ann 155  Matt  I think we found the secret.
Ann 156  RBD  You found the secret great.
Ann 157  Student  Hold on what did you write?
Ann 158  RBD  You want to be careful there might be more than one secret that you might want to think about.
Ann 159  Brian  It doesn’t work.
Ann 160  Romina  It almost works, zero times three, zero plus one. How come it doesn’t work?
Ann 161  Brian  I know it still doesn’t work it’s still zero plus three
Ann 162  Romina  Zero times three plus one equals one.
Ann 163  RBD  Michael that’s very nice.
Ann 164  Brian  Oh, okay one okay.
Ann 165  RBD  See if Milin agrees with you.
Ann 166  Student  Six plus five is eleven.
Ann 167  Student  Yup I got this one.
Ann 168  Ankur  The difference between that one and that one.
Ann 169  RBD  That’s right.
Ann 170  Ankur  And then you add the next odd numbers.
Ann 171  RBD  Right, okay that’s certainly and that’s a good idea, but there might be some other things you might want to think about. You’re doing very nicely.
Ann 172  Michelle I  Thank you.
Ann 173  Ankur  Are these right?
Ann 174  RBD  You know they’re right, but you want to make sure you write where the box is. I guess you’re saying the box goes here because it wouldn’t always be zero, it might be a one or a two, or whatever you put in its place.
Ann 175  Michelle I  Got it.
Ann 176  Stephanie  Look at that. If you put the top number up here in this place, it works like…
Ann 177  RBD  That’s right and it’s a good thing to do. At some point, at some point, I want you to say that, let me find out some way we can get this thing up for the camera. Linda, how can Stephanie say this to the camera so we get both the picture and the audio?
Ann 178  Linda  Um, up through there.
Ann 179  RBD  Well, but I don’t want everybody to see it.
Ann 180  Linda  Well, I was just working on that one
Ann 181  RBD  Do you want me to let her use this?
Ann 182  Linda  Yeah, that’d be fine.
Ann 183  RBD  It’s up to you which ever camera you like. You’ve got to see it too. Okay, wait one second, we’re trying to figure out how to get, how to make it possible for you to talk to the camera.
Ann 184  Stephanie: Jeff, don’t help them, Jeff.
Ann 185  Jeff  Okay.
Ann 186  Student  You’ve got the secret?
Ann 187  RBD  Which camera?
Ann 188  Student  You’ve finally got the secret
Ann 189  RBD  Okay, here you can hold this.
Ann 190  Stephanie  Okay.
And now we’re trying to get the camera on this. He’ll tell us when you can start talking.

Well, I wanted some people to have a chance to talk to the camera and so they can tell the camera the secret without telling everybody the secret we don’t want to ruin it.

I think what we’ll do is turn it this way

Is that a minus two?

No, it’s a plus one.

Stephanie’s trying to explain this and we wanted to get it on the camera and the audio at the same time.

Maybe you could let us know what it is and we could fix it?

Okay, the secret is that the first number in the triangle row, if you put that in this place right before the equal sign it’ll work all the time, so you just have to take the first, the first number in the triangle row and put it before the equal sign.

Thank you. We’ll take the paper back. Who else has a secret you’re ready to tell to the camera, you’re not going to tell the whole world, but you’re going to tell the camera?

You told your secret to the camera you think, you might come and do it again.

Oh, Amy-Lynn come do it.

Oh, I get it now.

This, the front page?

Which page, the front or the second one?

It’s up to you, whatever you would like, whatever you want to say.

I’ll put mine on the second page.

You better do one at a time cause it’s hard for them to get both of them on the camera. Who’s talking?

We’re both talking.

So hold up the mike where it will pick up your voice.

Um, for here you subtracted like seven minus five, and you got two, and that’s what you added on for each next thing.

The difference for each between each two numbers we put that in the first column, in the first, here right here for number two and on the second one we just figured it out that.

That was very nice. You did that work also on this problem down here.

yeah, because see four minus one is three, and then that’s how we got the three there

Okay, that’s very nice, that’s a very important idea. Thank you.

Okay, Amy-Lynn. You and Bobby are going to say what you did. Let’s set it down and Roger’s going to get it on the camera here. And I can change these around if
you want me to. All right. Now here’s the microphone for whoever’s talking.

Ann 218 Amy- Lynn I don’t know. All right, well we, the five here we use it as a plus number. Over here, like the one over here, we use it as a plus number. And then the seven we used it as a plus number.

Ann 219 RBD Okay, now you have another number, how did you find that other number? You used the two here and the three here how did you find that?

Ann 220 Bobby Minus the seven for five and the plus six.

Ann 221 RBD Can you show me? Point.

Ann 222 Bobby See seven, if you add the five, you take it away from there and one times two

Ann 223 RBD So you’re saying this seven here, is that correct?

Ann 224 Bobby If you minus that from the first number and that’s how you get the times sign.

Ann 225 RBD Okay, you’re saying you subtracted the five from the seven.

Ann 226 Bobby The seven from the five.

Ann 227 RBD You subtract seven from five and I think you get negative two.

Ann 228 Bobby Oh, okay, then five from seven.

Ann 229 RBD Okay, that’s very nice. That’s very nice, those are very important ideas.

Ann 230 Student You got number six, you got number six.

Ann 231 RBD Anybody else have a secret you want to say to the camera?

Ann 232 Student No we know it.

Ann 233 RBD You know it, you know it all no point in telling everybody he knows it. A lot of people are saying they know the secret, but they’re stuck on a certain problem that’s giving them difficulties.

Ann 234 Michael I’m almost done.

Ann 235 RBD Okay, is anybody problem six anybody got six?

Ann 236 Brian We know what it is, but we can’t put it inside the thing.

Ann 237 Romina Yeah we know it.

Ann 238 Brian It keeps going up by two.

Ann 239 Romina No what’s in between goes.

Ann 240 Brian In between like one, three, five.

Ann 241 Romina He means this doesn’t go up by two what’s in between it goes up by two.

00 25 00 00 31 00

Ann 242 RBD Oh, way hey can you come up, let’s erase this and come and show us, okay?

Ann 243 Romina Come on Brian.

Ann 244 RBD Maybe, maybe we’ll go to the camera, maybe we’ll go to the camera so they can still think about it, okay. Yeah, yeah good.

Ann 245 RBD Now this is the microphone here.

Ann 246 Romina Hold on he has to come up

Ann 247 RBD Well, you’ve got two microphones it won’t hurt. Now what you want to do is put it down so that Michael can get his camera set. You got to try and stay out of his way.

Ann 248 Romina What should I say what I wrote for number six?

Ann 249 RBD Yeah, yeah.

Ann 250 Romina Well, I think for six that, like, numbers aren’t really, like
what’s in between, well what’s in between the numbers is two. So like, so one, what’s between one and two is… well, what I mean is what’s between two and five is three and what’s between five and ten is five. Then if, when I do all that between one like the numbers in between it goes by two.

Ann 251  RBD  Okay, where is it that it goes by two, can you show that? Make sure the camera can see it.
Ann 252  Romina  well, one, between one and three is two, between three and five is two, between five and seven is two, and between seven and nine is two.
Ann 253  RBD  Okay, that’s what I need. Now you need to figure out what to do with that, but it’s a wonderful idea. Thank you.
Ann 254  Student  Is there an answer for six?
Ann 255  RBD  I’m sorry.
Ann 256  Student  Is there an answer for six?
Ann 257  RBD  Has anybody figured out the equation for six? Has anybody figured out the equation for six?
Ann 258  Michael  We keep getting those stupid fours in the way.
Ann 259  Student  Negative number.
Ann 260  RBD  Matt have you completed that? Okay, come and say that to the camera why don’t ya.
Ann 261  Michael  This one’s ten plus ten that’s twenty.
Ann 262  Matt  I didn’t figure it out, but it has something to do with the prime numbers, but…
Ann 263  RBD  Okay, whoever’s going to talk needs the mike, who’s talking?
Ann 264  Michelle I  We’re both talking.
Ann 265  RBD  You’re both talking.
Ann 266  Ankur  He’s listening.
Ann 267  RBD  You got to hold the mike.
Ann 268  Michelle I  See, um, this is how we did it like, like you talk.
Ann 269  Ankur  Whatever the first number is equals the second number.
Ann 270  Michelle I  Whatever number in the box is here
Ann 271  Ankur  So we put two in the second one and always one goes here so if this is three, three goes here and plus one.
Ann 272  Michelle I  See, and if it works here, three times three would be nine and then the one there would be ten.
Ann 273  Ankur  So if it’s four.
Ann 274  RBD  But, you haven’t quite really found the formula, really.
Ann 275  Ankur  So if it’s four, we get four and four that equals eight, sixteen and then plus one.
Ann 276  Michelle I  I think that the secret is that the number in the box always goes, always goes next to it…
Ann 277  RBD  You suppose there would be a way to write that. Can you think of a way to write that? The number that goes in the box is also the number that is next to it. How could you write that? That’s a really neat idea, that’s a really neat idea.
Ann 278  Ankur  Is that the secret?
Ann 279  RBD  If you can find a way to write it you’ve really got it figured out.
Ann 280  RBD  Um, okay I guess I’d like to one on the board so that everybody gets to see one. Okay, could we get everybody to think about one problem, the same
problem for a minute, and let’s do one of the first five I think that’s what people feel the happiest about. Who is going to come and explain one? Who has not had a chance to talk to the camera? Yeah Mike, why don’t you come up and explain one right up here and explain it. What number?

Ann 282 RBD
It’s your choice, not one, but two or three or four or five.

Ann 283 Mike
I’ll do number two.

Ann 284 RBD
Uh, okay you see where the…

Ann 285 Mike

For, whenever you have a number under the triangle there has to be a pattern, see five and seven, the difference is two. Five plus two is seven. Seven plus two is nine, nine plus two is eleven, apparently it’s two, you see here. Okay, since two, and of the first number, which is going to be box times two equals, no, plus something…

Ann 286 Mike

Zero can go into five, five times,

Ann 287 RBD
Well,

Ann 288 Mike
Yeah whatever, so we leave that one out. One goes into seven, seven times, leave that one out. The two does not go into nine, so take one out to make it eight, so we’re going to have to ad plus one. So this two goes into eight. Two times four is eight, plus one is nine, so that’s nine. The eleven, take one away and it’s ten, but it doesn’t go into ten. So you take one away so you’re going to have to take one away from eight, so it’s going to be seven.

Ann 289 RBD
Well, I think some people have an idea that’s a lot easier than that. Uh, could everybody hear what Mike was saying? He said he’s going to start with box times two and I take it everybody agrees with that, isn’t that right, you saw where he got the two. So, now you want to say where you get the five from.

Ann 290 RBD
Um, now he has a sort of complicated thing he’s doing here, but I’m wondering if there might not be something easier. Matt.

Ann 291 Matt
It is box times two, but it is plus five. What’s the first number on the right?

Ann 292 RBD
So he says…

Ann 293 Matt
Box times two would be fine, but then you add five.
Yeah Matt, you see where he says he’s getting the five from the pair zero five, he says that’s the number you want to take. You want to write that?

So plus five. It’ll work.

[Mike writes
\[
\begin{array}{ccc}
\square & \Delta \\
0 & 5 & (\square \times 2) + 5 = \Delta \\
1 & 7 & 2 \\
2 & 9 & 2 & 8 & +1 \\
3 & 11 & 2 & 7 \\
4 & 13 & & & \\
\end{array}
\]
]

It’ll work. Okay, is everybody happy with that you all know that?

You know that’s a very important set of ideas in mathematics

What about number six.

I think we have something for six, but…

You think you’ve got something for six.

I think we found out how to write it.

You found out how to write it, okay, well come and show people.

Can we do this like…?

Now, wait, maybe you don’t want to do that, you’re right

Aw, can we just show you?

Yeah let’s just show the camera, huh, let’s just show the camera.

Can we show you first?

Yeah you could. I’ll be right back okay.

I’m not saying anything. Okay, the number in the, the two numbers in the brackets are always the same and the number after the bracket is always one.

That’s certainly, right, uh, do you suppose you could why don’t you, let me get out of the way, here, so the camera could get some shots and maybe you should set that down so the camera can get that. And maybe you could explain it. When she says you can talk you can talk.

The two numbers in the brackets are always the same; the number after the bracket is always one.

Okay, now you certainly managed to write it, could you write it using the box and triangle method of writing?

What do you mean?

Well, you know what we did here we used the box and triangle instead of using words we used the box and triangle. See if you can do that.

Like two times two just write the problem?

Well, write down the equation with the box and triangle in it. Yeah, Michelle almost said it when you were down here before.

I did?

Yes you did. Why don’t you go and work on it some more you’ve almost got it.
Ann 320  RBD  You’ve got something you want to tell the camera?
Ann 321  Brian  Yeah.
Ann 322  RBD  Well, come on up, tell the camera. Okay I better give you a microphone here. Here’s a mike.
Ann 323  Brian  Thanks. I was arguing with her… because if you do this zero times two plus one you get one and that’s that so move it up there. Then, one times two plus one is three and that’s that. And then two times two plus one is five. And three times two plus one is seven and there’s that and four times two plus one is nine, and there it is. And five times two plus one equals eleven. So it’s just like this, move that up there.

Ann 324  RBD  I think that’s a good idea, um can you try, it wouldn’t hurt to think about that a little bit more cause I think you may find something that’s simpler and find a way to write all that with the boxes and triangles.
Ann 325  RBD  Can I draw attention to something that Michael did, I’m not sure that everybody caught it and it was a very good idea. Um, and it is that he has written this formula using this box and triangle notation, okay. He used the box and triangle notation to write the formula. Now, uh, some of you have got some very clever things, but you are writing them in words. So the question is can you go and do the sort of thing Mike did and write the box and triangle. Take some of these other ideas, but write them, have you done it?
Ann 326  Ankur?  Yeah.
Ann 327  RBD  Okay, come and tell the camera. Well, at some point we will we need to talk about…
Ann 328  Michelle I  You’ve got to try and figure it out first though before we tell you.
Ann 329  RBD  Yeah, I think it’s a good idea to try and figure it out yourself.
Ann 330  Ankur  Okay, the number here, the number that’s here, you see will always go here and always in this place will go the one and the answer goes here which is four the answer is seventeen.
Ann 331  RBD  Yeah, that’s certainly right, okay, um, can you think of a way to write it so that we’ll know that the number that goes here has to go here too.
Ann 332  Ankur  Like we did on the back?
Ann 333  RBD  No, with symbols and boxes and triangles we…
Ann 334  Ankur  Can we do it like this? The box here and the box here.
Ann 335  RBD  Yeah, you want to go and do that?
Ann 336  Ankur  Yeah.
Ann 337  Student  I just have to write one more thing.
Ann 338  RBD  So I guess the hard problem that people are working on is number six, isn’t it?
Ann 339  RBD  Um, yeah, you got six? Did you say it to the camera yet? Why don’t you come do that? Now which one are you talking about? You’re talking about this one right here.
Ann 340  Amy-Lynn  Okay, well, it’s a zero times one and then it’s like the plus one and then it goes from like the one times one cause it’s just zero, one two, three, four, five. And then it goes one, one two. You see there.
Ann 341  RBD  It could be zero.
Yeah it could be zero. Zero, one, two, three, four, five. And then you just keep the plus one and you go.

All right, can you think of a way of writing that so, it looks like it certainly works. Can you write that with that box and triangle notation? See if you can?

Okay.

What do you mean by box and triangle notation?

Well, you’re not going to write exactly the same thing because this was the answer for a different question. There is some method that goes boxes and triangles to write it.

This is the same pattern as six.

Let me see, uh, Michelle and Ankur were you about to, oh you’re working on it okay. You’ve got it okay, let’s get it down.

It’s the wrong way he’s got it backwards.

When the box is here the number that goes here that goes in the next box would be legal. So these two have to be the same.

Yup, do you agree?

Yeah.

And this is always one and the triangle is the answer.

That is elegant. That is great.

Yes.

Is that what you were thinking?

Yeah, that’s what I did, but I didn’t invent it somebody told me it.

We invented it.

You reinvented it.

How many people have got number six?

Is there a way to get number six?

What? Uh, Ankur and Michelle think they have.

We’ve got a secret.

You have?
the camera?

Ann2 367  Romina  [asks the RBD] So the equation can change?
Ann2 368  RBD  Well it’s going to have to be different from this; it wouldn’t be that it’ll be a different kind of equation.
Ann2 369  Jeff  The only difference between this and that is…
Ann2 370  Brian  Can you change the numbers in the triangle? Please say yes
Ann2 371  RBD  Well, there might actually be ways to do that to.
Ann2 372  Brian  We did that and it works the way I have it.
Ann2 373  Michael  But, the difference between one and four is three, the difference between four and seven is three, the difference between seven and ten is three, that means that’s going to be the first number, three, cause box times three. But, the pattern in this is one three five seven and nine so how we going to…
Ann2 374  RBD  I think Mike is making a point I’d like to pursue. I think maybe it would be ok try sharing a little bit of the secret without telling everybody everything.
[Michelle raises her hand] Michelle, without saying what you and Ankur have done, can you tell people what you said in the original? Do you remember what you said?
Ann2 375  Ankur  We’re thinking what we should say.
Ann2 376  RBD  No wait, say it to everybody, come and do it here on the front board. Now, don’t tell them your nice method of writing it, okay? Because remember, you said something before.
Ann2 377  Michelle  Just tell them what we got for the written answer not like the code?
Ann2 378  RBD  Do you remember what you said to me before you invented the nice way of writing it
Ann2 379  RBD  Yeah, right here. The number outside the box had to be the same as the number inside the box
Ann2 380  Ankur  But that will give it away.
Ann2 381  RBD  Oh, no it won’t
Ann2 382  Michelle  Well not exactly we didn’t get the code right after we did that.
Ann2 383  RBD  Okay, they’re discussing how much of the secret they’re prepared to publish at this stage, well a little bit I guess.
Ann2 384  RBD  I think that’s known as espionage. Okay it would be worth listening because this is a very interesting idea and they’re trying to be careful, they’re still giving you a chance to invent it, but they’re going to tell you something worth thinking about.
Ann2 385  Jeff  They don’t want to give us a chance. they just want to be smart and keep it from us
Ann2 386  Michelle  [Michelle writes on the board, ]
( □ x _ ) + 1 = Δ
Well, this is what we got, but you have to write it in the code.
Ann2 387  RBD  Well, can we get it quiet please, Jeffery and everybody can we get it quiet please.
Ann2 388  Michelle  The number you add after the bracket is always one.
Ann2 389  RBD  I think everybody is well agreed on that. Isn’t that true a lot of people have decided on that.
Ann2 390  Brian  Yeah, we got that.
Ann2 391  Michelle and Ankur  and the number that is here is always the number that’s in the box cause if you put zero here, zero times zero is zero, plus one equals one.
Ann2 392  Brian and  What about three?
Romina: Yeah, but figure out the rest yourself. Jeff the number that goes here always goes here. You have to figure out what the code is.

Ankur: Yeah, but figure out the rest yourself. Jeff the number that goes here always goes here. You have to figure out what the code is.

RBD: Well, yeah. Do you want to come and show it to the camera? Do you want to show it to me? Just say it. Okay, say it. Okay, wait a second can we get it quiet please because I want to hear what Michael is saying.

Michael: For number three the difference between one and four is three, the difference between four and seven is three too and the difference between seven and ten is three so that is going to be the first number but on number 6, the difference between one and two is one and the difference between two and five is three so which one goes in the box?

RBD: Yeah, why don’t you come and write that on the white board so everybody can see what you’re doing. What you just said.

Ankur: Which ever number goes here…

Michael: Yeah, but the problem is…

Ankur: Listen Mike, this is zero and it goes in the box so it has to go here too.

Michael: No, but we’re trying to find out what goes in the second.

Ankur: In this, in this?

Michael: No, no.

Michelle: [Writes: 1 x 1 + 1 = 2] If you put a one in here, one times one is one plus one is two.

Michael: What goes here?

Ankur: Whatever goes in the box goes here, if this is ten then ten goes here, if this is five then five goes here.

Michelle: Whatever’s in the box goes next.

Ankur: And one always goes here.

Michelle: It works every time.

RBD: Let me keep track of where we stand because I’m getting a little confused.

Brian: We have to write a code? How we got it or something?

RBD: No, wait, do the same thing that Michael did originally.

RBD: Do you want to say it to the camera?

Jeff: No I don’t want to say it. This is going to be the number in the square multiplied by itself plus one always is going to give you the answer.

RBD: Yeah, what have you written here, that’s very interesting.

Jeff: Zero times zero plus one equals one.

RBD: But you did something it’s neat I think you need to tell that to the camera. Do you want to do it?

Jeff: Not really.

RBD: Sure you don’t? Let me give you the microphone.

Jeff: Uh, I don’t want to.

RBD: But it’s a great idea.

Jeff: But what am I going to say though.

RBD: Well, you can say here this is a good idea.

Jeff: That’s easy.

RBD: Well later on if you want to, but let’s say it to the camera right now. Here’s the microphone, let me get out of the way.

Jeff: How am I going to do this though?

RBD: Well, here’s what you can do.

Jeff: Can I draw it on this?

RBD: Well, you could.
yeah I’ll just draw it on this and show it as an example. When am I going?

we can wait. okay good. so um, Jeff is going to tell you about this problem right here.

well what it is, no. Can I start? Okay. The number over in the box is always going to multiply by itself in the parenthesis and then you’re going to add it by one and you’ll get the number that’s supposed to be in the triangle. The same thing with this one here, one times one is one plus one equals two. Two times two is four plus one equals five and it just works for the rest of the six. Is that it?

yeah, good.

see if you can, you helped them get it, can you come and explain it to the camera?

yes. okay, you can talk this time. well, you see that number is the same as that number the code would be square times square plus one equals triangle.

yeah, that’s neat but you didn’t write it that way.

I didn’t put the square.

That’s very nice why does that work? Because the rule says whatever you write in one square you have to write the same in the other square.

That’s the same as that.

That’s very nice, okay, why don’t you write it on this paper now.

you could write it.

oh, come on.

just write the code?

yeah. put your names on it too so we know who did it. Thank you very much.

when you minus this one from this one you get the multiplying number.

yeah, why don’t you say that to the camera?

I already did.

No. it doesn’t work for all of them.

Could we tell everybody the code?

you want to.

Okay, let me say the people with the secret would like to publish it now, when scientists really discovered something they do what they call publishing, they send it to a journal and it gets printed and everybody reads it. You can erase it. Are you ready for them to publish this is that alright? Okay, could we get it quiet please? So they say they’re going to tell you their discovery now.

you have the box time that plus one is the triangle. [She writes on the board \[ (x \times x) + 1 = \Delta \]]

someone pick a number that will go like here.

Okay, Ankur says tell him a number and he’ll tell you how it works.

eighty six.

eighty six is too high.

They want eighty six we’ll give them eighty six.

We don’t care just show us how to do it. They’re just going to do eighty six to make us mad.

Now what’s eighty six times eighty six, you said eighty-six
now figure that out.

Ann2 459  Student  You said you could solve. Do it.
Ann2 460  Ankur  We can if we had a piece of paper.
Ann2 461  Michelle  If you have the number here and the number here is going to be the same as the number here what do you think that is going to be?

Ann2 462  RBD  Okay, this is really the key point so it would be very important to listen carefully cause they’re really telling you the secret.
Ann2 463  Michelle  If the number here is going to be the same as the number here, what shape do you think that is going to be?
Ann2 464  Student  A square.
Ann2 465  Michelle  That’s it that’s the whole thing that’s the code. That’s the code.

Ann2 466  Ankur  Isn’t that easy?
Ann2 467  Jeff?  That’s the code - square times square plus one equals triangle?
Ann2 468  RBD  That’s what you had too isn’t it?
Ann2 469  Michelle  I told you.
Ann2 470  Jeff  That isn’t very difficult. If we knew what it was we just didn’t put it down how it was supposed to be.
Ann2 471  RBD  Okay, does everybody understand that? Okay, let’s see if anybody can do number seven.

Ann2 472  Bobby  We’re already done with it.
Ann2 473  RBD  You’ve got seven already, come say it to the camera.
Ann2 474  Jeff  We have to work on seven now.
Ann2 475  Bobby  It’s like zero times zero plus five equals, then, and we just kept on moving the numbers up to one times one, two times two, three times three, four…

Ann2 476  RBD  Say that first one again.
Ann2 477  Bobby  Well zero times zero plus five equals five. One times one plus five equals six. Two times two plus five equals nine.

Ann2 478  RBD  So how would you write that with box and triangle notation?
Ann2 479  Amy-Lynn  Box times box plus five equals triangle.
Ann2 480  RBD  Okay that’s very nice thank you very much. Why don’t you put your names down so we know who did it okay?

Ann2 481  Student  I want to sit with Ankur next time we come.
Ann2 482  Bobby  The first and last or just the first?
Ann2 483  RBD  It’s up to you. Whichever you prefer.
Ann2 484  Romina  Brian you are confusing me so much.
Ann2 485  RBD  O.K. Thank you very much.
Ann2 486  RBD  That’s a good idea. Here’s the microphone.
Ann2 487  Jeff  It’s the same problem

Ann2 488  RBD  We want to get this so the camera can see what you’re doing.
Ann2 489  Jeff  It was the same problem, the same code as the one before, box times box but instead of plus one it’s plus five equals the number, equals triangle.

Ann2 490  RBD  Can you write that here with the boxes and triangles?
Ann2 491  Jeff  So it would be like box times box plus five equals triangle. It’s easy
Ann2 492  RBD  That’s nice.
Ann2 493  Jeff  Thank you.
Ann2 494  RBD  Oh Jeff, would you come and write your name on it too?
Ann2 495  Student  Don’t talk so loud.
Michelle R. walks up to RBD

You got another one, you got eight good heavens.

Student
We got eight too.

Jeff
Okay, she has everything written down. Put it so the camera can see it.

Jeff and Michelle R
It’s box times box minus box equals triangle.

RBD
That’s wonderful.

Jeff
Good.

RBD
I didn’t think you’d get that one.

Jeff
And it works every time. That was easy, once you get the basic one.

RBD
You’ve got nine.

Amy-Lynn
Number nine.

RBD
Okay, are you going to explain it?

Amy-Lynn
Okay, we had the zero times the zero and then plus zero and we got zero. Then we added one, so it’s one times zero plus a half and we got a half. And then we did zero one two times zero plus two equals two. And then we did one two three, three times zero plus four and a half equals four and a half. And then you took four times zero plus eight equals eight. And then five times zero plus twelve and a half equals twelve and a half. Six times zero plus eighteen equals eighteen.

RBD
I think it would be worth thinking about that some more and see if you can find an easier way to deal with all of that. Where do you suppose those halves are coming from, what do you suppose is making those halves in the problem?

Amy-Lynn
Uh, the whole.

RBD
Yeah, I think so. Okay, keep thinking about it because I think you can do more.

RBD
Okay, all four of you are coming to explain this right?

Jeff
What number are you explaining?

Michelle
Number eight. Romina you have the best hand-writing.

RBD
You need to show Roger which one you want him to be
Michelle: This one right here.
Romina: Oh, I have to talk. You can talk this time.
Michelle: Fine, I’ll talk I don’t care, we’ll both talk it doesn’t make a difference. Okay, I don’t know how to explain it.
Ankur: You divide the triangle by the square, so you just divide the triangle by the square, then we wrote it like a code. So, six divided by three is two.
Michelle: Tone it down some?
Ankur: Oh, her ears are hurting. So you divide twenty by five and that’s four, thirty by six that’s five, twelve by four that’s three
Brian: But do this, you multiply six times five plus zero because zero was the first number up in the triangle.
Ankur: So on the board like its square times square it’s square times, like when you divide twelve by four that number goes in the second place.
RBD: Now why don’t you take that triangle number and divide it by the box number what do I get? What do I get at that point?
Ankur: You get the answer.
RBD: What answer?
Brian: You see that’s.
RBD: So here, I would take the six and divide it by three.
Brian: Get two.
RBD: What’s the two?
Brian: You see three goes into six two times, and if you set it up like this three times two plus zero, you get six and that’s that number, I didn’t put the triangle in.
RBD: Okay, that’s an interesting idea.
Brian: We have it, it’s triangle divided by square. But then we have to put times something
RBD: You tell me you get something, why don’t you say equals and tell me what you get.
Brian: Divided it’s not plus.
RBD: Let’s see what we do here, let’s try it. When I had here, zero divided by one is zero, and I had two divided by two that was one, and six divided by three that was two. Oh, I see something interesting is happening.
Brian: It goes up, like one, two, three, four, five, and you multiply six times five plus zero and you get that number.
RBD: Okay, that’s an interesting idea. I think that if you keep thinking about it you might find some other ways.
Brian: Let’s find out number nine and then go back to it.
RBD: That’s a very good idea.
RBD: You want to come and tell the camera, no, okay.
Michelle: Could I go get a quick drink?
Jeff: This is the one for nine.
RBD: Who’s doing the talking, you doing the talking?
Stephanie: Michelle figured it out.
RBD: Okay, then Michelle should do the talking. Let’s try and get it where Roger can see it.
Michelle: You multiply the one times one and then subtract a half of one. And then you multiply two times two and then subtract half of two times two and then you get your answer, which is triangle. Three times three is nine and minus four and a half and that’s your answer four and a half. It keeps going on.
Okay, we do need to talk about some of these I think. Let me talk to everybody okay? Well, I think you know what you said is right. Okay, I won’t tell them.

Okay, there is one thing that I would like to talk to you about, can I get a place to sit here. Notice there are different kinds of secrets, different people are making up, but this kind of thing which is called a formula, it’s what mathematicians call a formula, that formula let’s you if I tell you the number in the box, that let’s you find what the number in the triangle is. Okay, now some of you have some very interesting secrets, I’m not saying don’t use it, but some of you use something that depends on knowing what the number in the triangle is, but you see what we got here doesn’t. [On board is ( □ x □ ) + 1 = Δ]

Okay, it only depends on the number in the box; if I tell you the number in the box then you can find the number in the triangle. So we’re particularly looking for formulas like this where you don’t need to know the number in the triangle, all you need is to put the number in the box and that will tell you the number in the triangle. Okay, um, I think we really have come close to being out of time would anybody like to say anything about this?

We’ve got a secret about the whole thing.

You’ve got a secret about the whole thing, okay, well we got time to do a couple more things. You want to say it to the camera or to everybody?

To the camera. It’s just for number eight.

Who’s talking?

He can.

Well, like here it’s zero and then it goes plus two then plus four then plus six then plus eight then plus ten. Then more, plus twelve then plus fourteen.

You wouldn’t be willing to tell that to everybody.

Yeah, why not? It works for this one.

But it might be an interesting idea. Let me write the table to show this

[Writes on board.]

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
</tr>
</tbody>
</table>

Okay, they have an interesting idea that maybe you’ve all thought of, but it’s worth making sure you know it. Would
you explain it to everybody?  
It starts at…you explain it I’m not good at explaining it.

Ann2 561 Amy-Lynn

Ann2 562 Bobby

Here it’s zero and then this is two then four, then six, eight, then ten and if you keep going down it would be twelve, fourteen if it went on.

[Bobby writes on board]

<table>
<thead>
<tr>
<th>□</th>
<th>△</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
</tr>
</tbody>
</table>

O.k., I think that’s worth thinking about, Jeffery did you have something to show?

Ann2 563 RBD

Should I show everyone?

Ann2 564 Jeff

It’s up to you are you ready to show everyone?

Ann2 565 RBD

Okay, anybody else coming to help, or are you doing it by yourself?

Ann2 566 RBD

I’ll do it by myself. Basically when you start with two zeros you’re going to subtract, you know how when it’s like box times box plus one equals triangle, but if it’s there’s two zeros starting out it’s going to be box times box minus a number equals triangle. We don’t really know, but we tried different numbers and it works out every time. Yes you do, if there’s double zeros on the thing, you minus. Yeah we did we subtracted on eight and subtracted on nine too. No I’m not going to show you my paper.

Ann2 567 Jeff

Okay, how many people got this, some people have the formula for this and I think Jeff actually had the formula for this. Does anyone else have the formula for this? Jeff, yeah, you’re working with Stephanie, and Michelle, and Mike is that right? Matt too, so the four of you. Okay, but they’re carefully not showing you their formula, but they did tell you one interesting thing to think about.

Ann2 568 RBD

We’re ready to explain what we got.

Ann2 569 Michelle

Okay.

Ann2 570 RBD

Okay, everybody goes up. Should I write I don’t think I should write. I don’t know what to write. Should I just this out. Well, see, don’t erase the whole thing. Okay, you just divide basically to get the answer.

[Brian is referring to
Romina writes:

\[ \begin{array}{c|c|c}
\hline
\square & \Delta \\
\hline
0 & 0 & > 0 \\
1 & 0 & > 2 \\
2 & 2 & > 4 \\
3 & 6 & > 6 \\
4 & 12 & > 8 \\
5 & 20 & > 10 \\
6 & 30 & \\
\hline
\end{array} \]

Like, divide two into two and you get one, if you multiply two times one plus zero you get two and that’s what you’re supposed to get because two’s in the triangle. And the triangle’s supposed to be at the end. Divide six times three and you get two.

Ann2 572  RBD

I think you mean divide six by three and you get two.

Ann2 573  Brian

You do. Awesome, and that’s one and then you take the one and the two out of there and the two times one, I have to move down, and then you just plus zero cause that’s the number you have up there to start with and then it equals that two. This number should always be in the triangle at the end.

Wrote on board: \((2 \times 1) + 0\)

Ann2 574  RBD

Okay, I’m not sure that everybody could follow that or not it’s a nice idea, but it’s not quite this formula. What did they, they depend on using the number in the triangle right, and when you really want a formula you want something where if you didn’t know the number in the triangle, you could do something with the number in the box to get it. This is a nifty idea, and you could use it, but it does something different.

Ann2 575  RBD

Okay. I’d show you one last problem. Now, we wouldn’t solve it today it’s a very hard problem, no the one that I’m about to show you.

Ann2 576  Student

Is that number ten.

Ann2 577  RBD

Yeah, you could put it as number ten it’s a good idea.

[ writes on board]

\[ \begin{array}{c|c|c}
\hline
\square & \Delta \\
\hline
0 & 0 \\
1 & 1 \\
2 & 3 \\
3 & 7 \\
4 & 15 \\
5 & 31 \\
\hline
\end{array} \]
Now we’re certainly not going to solve that today, but I’ll bet you you are going to solve it in the next few weeks.

I’m not sure when we come back I’ll have to look it up. It really is a hard problem. But it is a very interesting problem. Good heavens.

when we have to multiply box times box could it be box plus box

Could be. Could be all kinds of things.

I got it. You’re the one who started it.

Are you ready to explain that to the camera?

Well, we have to make sure it works, sixteen and sixteen is thirty two. It does work, but sixty-three.

What about us?

Do you want to say it to the camera or do you want to say it
to everybody.

Camera. Okay.

Okay, that’s the wrong thing this is better. That’s the right one. Maybe you’ve got it all written. Okay, here’s the microphone.

I explain okay. Well, see, one is the difference between zero and one. And if you add one plus one it equals two which is the difference there. And two plus two is four which is the difference between there. Four plus four is eight which is the difference between there. And you add sixteen and sixteen which is thirty two the difference between there and so on.

See the numbers, you add that number with itself and you get the answer which is the difference between the next numbers.

Well, now I’m not sure. What number am I looking at here?

Between zero and one is one.

So what am I going to do add that?

You add one and one and you get two which is the difference between there.

Oh, ah, that’s really clever.

If you add the one and two you get three. And so on you do it for everything.

Okay, you may want to think about that some more, but that’s a very clever idea.

Okay, we really have run out of time, I think, anybody have anything you want to say as our last parting remark before we… parting remark. Anybody want to say anything? What? Could I sit there for a second? Can we talk for one second about keeping secrets?

We shouldn’t keep secrets.

You shouldn’t keep secrets? Well, there are two sides to it?

What’s the bad thing about keeping secrets?

Other people wouldn’t find out.

If another person finds out and says it, as long as you…

As long as you eventually tell it’s all right.
Could you keep it for a couple weeks and see how many people die before I give it out.

Let me turn it around and argue the other case a little bit. I think we do need to keep thinking about it, we do need to find a way to this so that everybody's comfortable with it, but there is also a case to be made for keeping secrets because what I've said sometimes to people is suppose Michael and I went to the gym and Michael did a lot of weight lifting and I watched him who gets stronger? He would. What's that got to do with secrets?

Nothing. Oh, you would have to do it yourself.

He's doing the lifting. If you want to get stronger you got to figure it out.

But maybe if you want to make sure of things before you do it.

We really are out of time we have to stop.

Tell us the secret or else I'm going to kill myself.

I think that would be going way too far.

Michelle, think medicine what if you found a cure but you don't know how to use it?

Thank you very much I'll see you next time. Thank you.
am going to think in my mind of some rule...

And then you got to guess it.

You're going to try and guess it, but do you know how you're going to try and guess the rule?

Yea, we're going to be working with numbers.

You're going to be working with numbers, that's right. And I'm going to, I'll mention a number and I'm going to give you a number and I'm going to tell you what my rule does to that number. That is, I'm not going to tell you the rule, but I'll tell you the result. Okay, ready for it? My rule takes the number...

You can't tell us what the rule is!

No, no. I'm not going to tell you the rule. I'm going to tell you what my rule does. Okay.

Oh.

You ready? Let's see, there are five of you here. Okay, here's my rule, you ready for it? Since there are five of you, the first number that my rule is going to work on, is the number five. And what my rule does to number five, it does some things to it, and what comes, the result... You ready for the result? Thirteen.

Is it... Oh, I know what to do.

Oh, oh. Hold on.

... He just added eight.

Just add the eight.

Don't, Don't say what you think the rule is.

Then how are we supposed to know...

So, the first number was what?

Five, and then...

And it gave you?

Thirteen.

Okay. So, now your job is to give me a number and I will tell you what my rule will do to that number. Alright, but I don’t want you to try and guess the rule just yet.

Three! Three! Three.

So, Dawud said three, and my number... my, what do you think my number is going to do to three?

It’s going to make it into eleven.

You think it’s going to make it into eleven? Nope, my rule takes the number three and makes it seven.

So, how do you get thirteen from the five? Oh, I get it...

Well, you have to guess my rule.
I get it! But, like for every...

...I know your rule!

Hold on. Okay, so what are the two results we have so far.

Alright, five and thirteen.

Five to the thirteen.Alright, I’m going to write this down for you. So far... [Makes a chart divided into square and triangle entries, with square numbers of five and three, and triangle numbers of thirteen and seven, respectively] So you can mention another number and I’ll tell you...

I know they both, they’re both higher...

Six.

Okay, Dawud said... What number did Dawud just say?

Six.

Wait, I want to say a number, he said one. Can I say four?

Don’t say a number, guess the rule!

Excuse me. One person at a time. Dawud said six. What do you think my rule is going to do to six?

I know!

Look, Look, Look. When you had...

Tell me what you think, if we have six, what do you think is going to happen to six?

It’s going to be twenty-four, then thirty-six.

Does everyone agree?

I think six is going to go to ten.

You think six is going to go to ten?

Cause, look. We made five, we made five with eight, we made seven with four. So six, it will be twenty-four.

Mmhmm. Mmhmm.

Does anyone else have a guess?

I say ten, I say ten.

It’s going to be twenty-four, then thirty-six.

Hold on, he says it’s going to be more than thirteen? Why does he think that?

Fifteen.

Cause, cause the first one is five and like I think that you keep on adding on, cause the number... Like three you added it on four, for five, its matching the five you added on eight. It depends on the number, that’s how much you add on.

It’s eight. It’s going to be eight.

I think it’s going to be ten.
Alright, is everybody ready? I’m going to write down the number. [Writes down sixteen]

I knew it was ten!

What are you talking about, ‘it’s ten’?

Look, cause 5 was at an eight, then you added two more and for like number 3 it was four. And if it was four, you would add a six. And for five you would add eight.

Ariel thinks that the rule is, any number that I put here [holds up the chart to the class], under this column [the square column] is that right? He says that any number I put in this column, you do what, Ariel?

Well, like, it would depending on that number you would add two on to what you did to the last one, right. Like, cause five you added eight, and for six you added ten. So, and for three you added four. Wait, yea, four. So it would be, eight then four, then ten. So, like five is just one number lower than six and it was eight. And since the six is ten, so I figured it would go by each number, add two.

Yea? Let’s try one other number. Let’s try one other number. James, would you like to pick a number to try?

Eight.

Alright, James says eight. What number do you think my rule is going to give back for eight?

Eight? Umm... Hold on.

Eighteen!

Nah.

Think carefully. Look at all the numbers. [Walks around with the paper]

Could I see that paper?

I’ll hold on to this.

Oh, I got it, I know I got it... It’s twenty-two!

Eighteen!

Twenty-two.

Twenty-two.

How did you get twenty-two, Ariel?

Add two.

What you talking about, ‘add two’? Oh, I found a dollar.

It would be funny if I got nine. No, it’s seventeen.

It would be funny if you were wrong. I think it’s twenty-two.

Alright, I’m going to put on the overhead the
numbers that we already have. Okay, can everyone see what I’m writing?

89 6:08 Ariel I know, like, a way to represent it too. Like, the square is the numbers going in, like, the triangle, it goes into like a say a factory and it comes out the triangle number.

90 6:18 R1 Alright, would everyone hold on a second? I’d like everyone to hear what Ariel just said.

91 6:23 Yonny Let Ariel talk.

92 6:25 R1 Okay, Ariel, go ahead.

93 6:26 Ariel Alright, so like, the square could be the number you’re putting in and it can say like, it can go to like the factory, or something like that and it come out the number in the triangle, triangle number. Square number and triangle number.

94 6:40 R1 Okay, so did you hear what Ariel said?

95 6:43 Yonny Yup, he’s mad smart.

96 6:44 Brandon No, he talks too fast.

97 6:45 R1 Dawud, could you tell us what Ariel said?

98 6:47 Dawud Umm, I didn’t hear him.

99 6:50 Brandon He talks too fast.

100 6:51 Dawud Can you repeat that again, please? Can you repeat that?

101 6:53 Ariel The square could be like a type a number, and then when it goes into, say, like a factory or something like that. It would come out the triangle type of number.

102 7:03 Brandon So, square is a number and triangle is a factor? Is that what you trying to say?

103 7:07 Yonny Look, look. It’s like the square, is like the bigger kind of shape, and then the like triangle goes into the square to make...

104 7:17 Dawud It keep on multiplying by four.

105 7:19 R1 Is that what you think?

106 7:20 Dawud Look, that’s eight, four, then twelve. So, that’s eight, four, twelve...

107 7:23 Brandon You mean adding? You mean adding? You mean factors of four?

108 7:29 Dawud Yea.

109 7:30 Ariel No, it’s not!

110 7:31 Brandon He means factors of four.

111 7:35 R1 Alright, here goes the number.

112 7:36 Dawud So, that’s that’s going to be eighteen. No it’s going to add up by eighteen. No, it’s going to be sixteen.

113 7:39 Ariel Look at the five, look at the five. It’s twenty-two!

114 7:44 Yonny It’s twenty-two!
7:48 Ariel It’s twenty-two.
7:49 James Twenty-four. Twenty-four.
7:52 Ariel Twenty-two!

[All the students are shouting out numerical answers and then R1 writes down the answer to 8 on the overhead]

7:52 Ariel Told you it’s twenty two!
7:54 James How is it twenty-two?
7:57 R1 Okay, I am going to ask Christian to give us a number. Christian, give us a box number.

[Students whisper to Christian to pick four]

8:09 Christian Mmmmm, four?
8:20 Brandon I know what that is!
8:21 Ariel Four? That’s easy, that’s...
8:22 Dawud It’s going to be twenty-four. It’s going to be twenty-four! Twenty-four! Twenty-four!
8:24 Brandon Eight! Eight!
8:27 Ariel Ten. It’s ten.
8:29 Brandon Thank you. [nods to Ariel]
8:31 Yonny Oh, yeah it’s ten, yeah.
8:33 Dawud Twenty-four.
8:34 Ariel Cause, you’re going to add six. Cause, for five you added um..
8:37 Dawud But, how...but how are you going to go
8:39 Brandon Wait, so it’s minus another two...
8:40 Dawud ...minus four, six
8:42 Ariel Cause for the five you had thirteen, and then the six you got eighteen, you just added um ‘what’s your name’ to this one and ‘what’s your name’ to that one.

[R1 writes ten on the overhead in the triangle column, corresponding to the four entry in the square column]

8:47 Yonny The number is ten.

[Everyone shouts out ten]

8:53 R1 So how’s Ariel doing this? [students are chatting amongst themselves] Well, let’s see on the next, on the next go-around. Excuse me. Ariel? I want on the next go-around, for you to be quiet. Don’t tell us what the [unclear] You, you think of it and we’re going to see if everyone else gets it. Okay? [R1 writes a 0 under the square column]


[Some students shout out answers and others chat]
Brandon: No, no, no, no. Don’t put zero.
Christian: Come on, Brandon.
Brandon: You be quiet.
Dawud: It’s zero.
Yonny: No, two.
Brandon: No, it’s not. It’s two.
Jay: My, my rule says... [Writes down negative two on the overhead under the triangle column, corresponding to the zero entry in the square column]
Dawud: It might be four.
[Both Yonny and Brandon shout out that it’s two]
Yonny: Negative two.
Brandon: I told you it was something two. Ohhhhh. I told you it was something two. Oh, that relates to our [unclear]
R: Have you guessed, have you guessed my rule yet?
Brandon: Yeah, the rule is by two.
R: What do you think the rule is? [Gesturing towards Ariel]
Ariel: I think from zero... No, I didn’t.
Brandon: I got your rule.
Dawud: Two, two, eight, eight.
Ariel: Zero done messed up my whole thing. Look, this is what I thought: So, for four you added six, for five you added eight, for six you added ten, I mean... twelve. Wait, yeah yeahyeah.
[Dawud is frustrated that Ariel cut him off]
Dawud: Look, can I guess? It’s eight, and then it goes, then it go twelve. Then it go... um, I think it’s supposed to be by adding four, then add eight, then it add four, and then it add eight.
Yonny: I think that the total you get, it adds by two
R: Okay, well, let’s try another.

Clip 2 22-23
R: Let’s try another one, okay? Alright, is everybody ready?
Yonny: Okay, pick a number.
Brandon: Oh, oh! Me, me, me! One!
R: Okay, so does everyone see that when there is a one... [Yonny has written a square/triangle chart on the overhead with 1 in the square column and a 15
Brandon: Oh, I know your rule already.
Yonny: What?
Brandon: By fifteen.
Yonny: Nope.
Brandon: Oh, darn it! Sike, I don’t know.
Yonny: Do another one.
Christian: I don’t know, I don’t know.
Ariel: Two! Two! Two!
Brandon: Two, yeah let’s do two! Let’s do it in a row. Do it in a row.
[Yonny writes a 2 in the square column and a 25 in the triangle column]
Ariel: We get forty-five? Oh, two, yeah yeah yeah, two.
Brandon: Oh, go three! Go three! Go three! But, why your by so big, my dude? Oh I almost got, I think I get his rule.
Dawud: Nah, go three.
Brandon: No, go four! I mean, go to three, and then switch it up. This is something like five.
[Everyone is talking over each other]
James: No, it’s not by ten. It can’t be by ten because he started off with 15.
Brandon: I think it’s going to be by ten, I meant, by ten.
Ariel: By ten.
Brandon: No, it’s not by ten. It can’t be by ten because he started off with 15.
Ariel: Yeah, that’s true, yup.
Brandon: So, I think it is...
Dawud: It’s fifteen...Let me think a little while...
Brandon: I think it’s, I think it’s... [unclear]
Dawud: No, no, no. Go four, go four, go four. Go four.
Christian: Stop drawing it so fast. You’re doing it too fast.
Dawud: Forty-five, forty-five.
Brandon: Oh, I know what’s next. You’re just, you’re just putting a five at the end of each number.
James: No, it’s fourteen, then it’s thirteen, then it’s twelve.
[Everyone is talking over each other]
Christian: It’s fourteen, then it’s twelve. I mean, two to twenty-two, then it’s...
Brandon: I know the rule! I know the rule, now. It’s by five.
R1: Okay. Everyone, hold up, hold on. Think of a rule that will produce what you think the number for four [unclear] okay?
Brandon: It’s by five. It’s by five. It’s by fifteen. I think the
rule...

Oh! I know the rule! [jumping up and down]

Sit down, sit down.

I know the rule! The rule is, like, basically you got the same thing. You get, you doing the same numbers, like: 1, 1 2, 2 3, 3 and you just adding five to the same numbers. You put forty, I think the next number you going to put is forty-five.

Yes.

See? Exactly, I’m too smart.

Okay, but the, his rule... One, one second, hang on.

Oh, snap! I get it... Yea, I know the rule now.

The rule is you’re using some operation to get the number. Okay? You got to think, what is the operation that he is using to get the number.

He’s just putting a five at the end of the number.

I know, basically, you put a five at the end of each number.

Times ten!

Times ten!

[Both Christian and Brandon say, ‘times ten’ multiple times to R1 and Yonny]

Okay, so does that work?

Times ten.

That’s half of it.

Times five.

Does that work?

Times ten divided by five!

Times ten divided by five.

No.

Times ten, times ten... times five.

Okay, try it out. See whether or not your rule... see whether it works.

Times ten plus five.

No.

Yea, bud, yea! Give me that marker.

Times ten what?

Yup, did he get it?

Yea!

Times ten what? Plus five?

Okay, okay. Hold on, hold on, hold on. What, what...

Yea! [Brandon congratulates Christian]

Tell him what, tell them what the rule was.

Times ten plus five.

That was easy!
242 2:45  Christian It’s five, ten...
243 2:45  Yonny It’s times ten plus five.
244 2:47  Brandon Oh, that was easy! I should have know that...
245 2:48  R1 Could you have a seat? [saying this to Yonny]
        Come, we’re going to make a rule. [gesturing to Christian] Okay, we’re going to do one more before
        we have you work on some of these problems. Okay, let’s see if you get them.
246 2:55  Christian I say whoever is talking [unclear] he’s just stalling, yo.
247 2:56  Brandon Oh, I want to do one more.
248 3:00  R1 Do you want to fill out the chart? [talking to Christian]
249 3:00  Brandon Alright, alright. Let’s go, Chris! Let’s go, Chris!
250 3:04  R1 Okay, Chris is going to give you one number and he is going to show you what you get. Chris?
251 3:10  Christian Alright, I’m about to put the number, guys.
252 3:12  James But, you don’t get one.
253 3:13  Brandon You said if he... No, erase that, put one.
254 3:15  Yonny Put a triangle two...[clip was cut off]
255 3:16  Brandon You go to put...[clip was cut off]
256 3:17  R1 Alright, does everyone see what happens to ten?
257 3:20  Brandon Yea, times two.
258 3:22  R1 So, put down another number. What number do you want to try now?
259 3:24  Brandon Oh, two, two, two!
260 3:25  Ariel Two, two.
261 3:27  Brandon Hey, hold on, wait. Be quiet, I think I know this.
        [Christian writes 20 first and then corrects it, so there is a 2 under the square column after 10]
262 3:30  Brandon No, two! Not 20! 2! Not 20.
263 3:34  R1 Okay, don’t put the answer yet. [speaking to Christian]
264 3:35  Yonny It’s ten.
265 3:36  R1 What do you think the answer is going to be?
266 3:37  Brandon Four! Four! Four! Four!
267 3:38  Dawud Oh, let him finish. Go, go, go! [talking to Christian who is writing down the answer for 2]
        [Christian writes under the triangle column 4, the answer for 2]
268 3:42  Ariel It’s got to be easy, cause he finished to quick.
        [Everyone is shouting out an answer]
269 3:44  Brandon Times, times four! Times, times two! Times two!
270 3:45  Dawud Ten, times ten, times two!
271 3:49  Brandon I said it before ya’ll.
like...

276 3:53 Ariel Times two.
277 3:54 Dawud Times likes four...
278 3:56 Yonny Times two take away two. I mean, times
279 3:58 Brandon No, it’s times two.
280 4:00 Dawud Times two times plus two.
281 4:01 James Ooooooo.
282 4:02 Brandon Times two, times two. I said that when they had the ten.
283 4:08 Dawud Hahahaha, what is that? That’s twelve?
284     [Christian changes the two under the triangle column to a twelve]
285 4:11 Brandon Why you switch it up, man? Oh, I know it anyways.
286 4:14 Christian What is it?
287 4:15 Brandon Plus ten!
288 4:15 Dawud Plus ten!
289     [The answer is ‘plus ten’ and Brandon goes up to the overhead. The video cuts to another clip.]
290 4:20 R1 I said this was going to be the last one and then we’re going to work in pairs on some other ‘Guess My Rule’ problems. And then we’ll hand it out to you on paper, okay?

Clip 3 22-23
291 00:00 James I just noticed one plus one is two, So you couldn’t get that so I thought about one times two, and that will be two and then add one, and that’s three.
292 00:11 G4 And then that’s how you figured out this rule.
293 01:24 [Ariel and James are working on the Guess My Rule Problem 1:

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>11</td>
</tr>
</tbody>
</table>
294 00:13 G4 And then you verified and said it was all right. Do you want another problem?
295 00:15 James Whatever.
296 00:17 G4 Alright, here you go. Let’s try this one.
297 [G4 gives the Guess My Rule Problem 2 sheet:

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
</tr>
</tbody>
</table>
Five plus two. Oh, this is easy. It’s the same thing, just ... wait a minute, yeah, it is the same thing, no, no it’s not, yeah, yeah. It’s the same thing.

What do you mean by same thing?

It’s zero times two plus five is five.
[There is a lot of background noise]

Zero times two plus five is five. One times two is two plus five is is seven. Then two times two, wait a minute, I mean two plus seven …

Three times five is eight, I mean, fifteen, three times five is fifteen …

I didn’t say five, I said plus five. Its times two...

Plus five. Three times two is six.

So one times two is two plus five is seven and then two times two is four plus five is nine, three times three is six, I mean three times... two, two, two, two [taps repeatedly with his finger in between the columns], let me write.

Times two plus five [writes ×2 + 5 on the side of the table]

I got this big O mark.

Zero times two plus five equals five [writes ×2 + 5 in between the two columns]

[pointing his finger on the values in the table] And then look one times two is two and two plus five is seven. Then two times two is four and plus five is nine. Three times three, times two, two, two, is six plus five is eleven, then four times two is eight plus five is thirteen.

[looks at G4] This is easy. Done. Next.

[laughs] How do you compare the first one and the second one when you were telling me it is the same thing?

[shuffling the papers] Because it is both times two. You are just changing the adding. Next [taps his pen on the table and looks at G4]

Times something plus something. Now we can take a break.

Do you think that there’s another rule that we can have? Or do you think that is the only rule we can get?

[James has written ‘×2 + 5’ in the Guess My Rule Problem 2 sheet. He has also written 25 in the X
column and 55 in the Y column.]

332 02:14 Ariel That’s the only one we can get.
333 02:15 James Times, no, times four plus no...
334 [Ariel and James are working on the problem]
335 02:32 Ariel Oh, I see a pattern. I can’t even do that, it’s too hard. I see a pattern though. It is going to be straight up. I’m done.
336 02:43 James Ohhhh. I see it. It’s add by five, Zero plus five. No, first is five, you add five right here. Then you add six, then you add seven, then you add eight, then nine, and then right here you add twenty-five, thirty.
337 03:07 Ariel Hold on. How are we doing this again?
338 03:10 [James has written in between the columns:
339 03:16 Ariel Because zero plus five is five and then one plus six is seven, and then two plus seven is nine, then three plus eight is eleven, and four plus nine is thirteen [looks at G4]
340 03:16 James And what about for twenty-five?
341 03:33 James And then twenty-five plus thirty is fifty-five. Why thirty?
342 03:37 James Because twenty-five plus thirty is fifty-five. Well, I just put that because you asked me what’s twenty-five plus …
343 03:44 James I was just curious to see because I see here you said I’m going to add five here, then I’m going to add six here, then I’m going to add seven here, how you know when you get down to twenty-five, you are going to add thirty there?
344 03:55 James Ten, eleven, twelve, then thirteen, and then fourteen, and then fifteen [looks at G4] and then for twenty-five …
345 04:03 James And then you are telling me that you get to thirty?
346 04:05 James [nods]
347 04:06 James What if I wanted to know the value for eighty-two?
348 04:11 James You mean we can write eighty-two right here?
349 04:13 James Hmm, hmm.
350 04:15 James Then you will try to [mumbles and looks at the
G4 04:31  Do you want Ariel to help you? Do you want to explain to him what you have found?
Ariel 04:34  No, I don’t remember.
G4 04:36  OK, then how about if I ask you for eighty-two, X is eighty-two?
Ariel 04:40  X is eighty-two? And then you would just....
James 04:43  Oh X is eighty-two? You said that that was eighty-two? [James gestures to the paper]
Ariel 04:46  Oh, that’s easy.
James 04:47  Oh, I’m sorry, I’m sorry. If X was eighty-two.
[There is some inaudible talk between James and G4 and there is a lot of background noise.]
James 05:14  Multiply eighty-two by two and then add five to the answer and then I get one hundred and sixty nine.
[James has written:
82
×2
164
+5
169]
G4 05:26  OK, so this was using this equation. [inaudible].
James 05:27  Yeah.
G4 05:28  Ok, and then you told me you found another way of finding the numbers?
James 05:30  Yeah, and then you notice that eighty-two plus …
G1 05:37  [to Ariel] Can you explain to R2 what you have found here?
Ariel 05:40  Oh, for like every number you add another one for like zero is five, since you’re going to one, now you add another is six, going to two you add another is seven, going to three you add another is eight, going to four you add another is nine.
R2 05:56  OK. And so how would you express that rule?
Ariel 06:00  Huh? Well, the first rule that we came up with was times two plus five. Because zero times two plus five is five. One times two is two plus five is seven and then we can up with …
R2 06:12  Does it work for the others?
Ariel 06:14  Yeah, it worked for all of them... and then we came up with this one.
R2 06:17  Can you show me?
Ariel: [writes $0 \times 2 + 5 = 5$] Zero times two plus five equals five.

Ariel: [writes $1 \times 2 + 5 = 7$] One times two plus five equals seven.

Ariel: Wait a minute. Yeah, yeah.

Ariel: [writes $2 \times 2 + 5 = 9$] Two times two plus five equals nine.

Ariel: [writes $3 \times 2 + 5 = 11$] Then its three times two plus five equals eleven. And the pattern is five plus two is seven plus two is nine plus two is eleven plus two is thirteen.

R2: [to Ariel] Do you and James agree on that?

Ariel: Yeah.

R2: Have you guys talked about it?

Ariel: Yeah.

R2: OK, would you like another challenge?

Ariel: Yeah. We have already done two.

R2: Ah, you have done two of them. Let me give you a third one. [goes away from the table]

James: It’s adding by all odd numbers.

G4: Tell me what you mean by its add by odd numbers.

James: Zero plus one is one plus three is four. This plus five is seven... Yeah, and then three plus seven is ten …

G4: OK, so what would be if we did zero, one, two, three, what if we had four? What would be our $Y$ if I give you four for $X$?

James: Thirteen.

G4: Could you write it down on your chart?

James: [thinking for a few seconds] I don’t know. See, you be giving me hard stuff.

G4: [James is writing something.] I don’t know. See, you be giving me hard stuff.

James: I’m trying to challenge you.

James: I don’t want to.

G4: But you’re too smart not to be challenged.

[James shakes his head.]
OK, so what you think we can do?

[James mumbles some numbers.]

Oh, fifty-five.

How did you come up with that?

[James mumbles inaudibly and is writing.]

The camera now focuses on Ariel.]

[to Ariel] What was that word? Can you explain me this?

I already figured this one out.

But can you first explain me this what you got and then you can tell me that one.

[pointing to the Guess My Rule Problem 2 table] Oh, I found that I was like, I came up with the first rule we got was times two plus five, zero times two plus five is five, one times two plus five is seven, and so on and so on and then we came up with...

So, how you got this? What is this? How you got this?

One plus six? Oh, that is for this rule. [pointing to what he has written on the right side of the table]. The other rule that I came up with, that James came up with, was that you add for every like number you add on one to add to the number. So for zero, you’re adding five that equals five. For one you add on one and it equals six, so one plus six is seven.

Hmm, hmm.

Then for the two it would be plus seven is nine.

Hmm, hmm.

Three plus eight is eleven, four plus nine is thirteen, and five plus ten is fifteen. And the pattern I also see is that it keeps on adding two, we have five plus two is seven, seven plus two is nine, plus two is eleven, plus two is thirteen, plus two is fifteen [looks at G1] OK, then what is this here? [pointing to what he has written on the right side of the table]

And here it continues, six, seven, eight, nine, ten, eleven, twelve, and here it will continue fifteen plus two seventeen, plus two is nineteen, plus two is twenty-one, plus two is twenty-three, plus two is twenty-five, plus two is twenty-seven, plus two is twenty-nine.

OK. And how you figure out this rule?

This one? [has Guess My Rule Problem 3] It’s easy. This is times … Oh, man, I forgot. Oh, yeah, times three plus one [writes on the side of the table ×3 + 1]

Times three plus one. Done. Because 0 times 3 plus
1 equals 1. And 1 times 3 is 3 plus 1 is 4. 2 times 3 is 6 plus 1 is 7. And 3 times 3 is 9 plus 1 is 10.

[Ariel writes as he talks. He has written $\times 3 + 1$ in between the first three pairs of values in the table.]

What will you get for four?

What you will get for four?

Four? That’s easy.

[Ariel extends the Guess My Rule Problem 3 table as follows:]

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
</tr>
<tr>
<td>6</td>
<td>19</td>
</tr>
</tbody>
</table>

I got to add the odd number. Because you know zero plus one... zero plus one is one. One plus three is...

[James is working on the problem separately]

[inaudible] It's just adding three. You find the answer easily. 1 plus 3 is 4. 4 plus 3 is 7. 7 plus 3 is 10. You just keep on adding three and you get all of your answers.

I got to add the odd number. Because you know zero plus one... zero plus one is one. One plus three is...

It’s just adding three. You find the answer easily. 1 plus 3 is 4. 4 plus 3 is 7. 7 plus 3 is 10. You just keep on adding three and you get all of your answers.

[Ariel extends the table by writing the next pair as 7 and 21.]

So, Ariel let me ask you, what would you get for twenty-five? Because James got what he got for twenty-five. I want to see if it is the same.

I just got one thing...

You are trying to find for twenty-five? Ariel?

[Ariel keeps writing something in his sheet.]

Seventy six.

Seventy six?

Yeah.

Seventy-six for what?

For, what’s the name, what was the one again?
129

Ariel: Yeah, twenty-five.

James: It’s not no seventy-six, it’s eighty!

Ariel: First, ten times two [writes $10 \times 2$, he has extended the table till 10 in the X column and 30 in the Y column] the thing of it is thirty [pointing to 30 in the Y column and changes the 10 in $10 \times 2$] so then it would be 30 times 2 is 60 then 5 is 16, plus 16 is 76.

James: Ten plus five is fifteen.

Ariel: What happened?

James: Ten plus five is fifteen!

Ariel: OK.

G1: Why did you do times two, can you tell me?

Ariel: Oh, because she said twenty-five, so the thing for ten is thirty and ten times two is twenty, so thirty times two is sixty, that will be twenty, and then for the five is sixteen.

James: What? This is what I got. I got eighty.

James: [pointing to something that he has written] Five times five is twenty-five. Five times eleven is fifty-five. Five times sixteen is eighty because I just found that out. So I got eighty.

[Ariel is writing on his paper and mumbling some numbers]

James: Ten times eight is eighty.

Ariel: Huh?

[Ariel is writing in his paper and James is mumbling some numbers.]

James: … and then thirty times eight is …

Ariel: Two hundred and forty.

[Ariel is extending the table as follows:

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
</tr>
<tr>
<td>6</td>
<td>19</td>
</tr>
<tr>
<td>7</td>
<td>21</td>
</tr>
</tbody>
</table>
Ariel: Seventy-five. Seventy-five.
G4: [to Ariel] So Ariel what do we get here? If X equals twenty-five, then what does Y equal?
Ariel: Huh?
G4: If X equals twenty-five, so Y equals?
Ariel: Seventy-five.
G4: [to James] And James you said when X equals twenty-five, what does Y equal?
James: Eighty.
G4: Eighty, so Ariel got seventy-five and you got eighty [to James]. So what do you guys think?
Ariel: [looking at his work] Oh, wait a minute. It is seventy-six. If you follow my rule, [pointing to what he has written on paper] twenty-five times … oh, wait a minute [putting his forefingers on his temples] so I must have messed up somewhere because twenty-five times three is seventy-five plus one is seventy-six like I got up here [looks at G4].
G4: [to James] What do you think James?
James: [looking at his work in the paper] I got eighty. I don’t see how he got it? Well, I’m going to go by my rule. [mumbles something to G4]
Ariel: Just do the rule on the number. Twenty-five times three plus one … but, wait a minute. Do you even have the same rule as me?
James [mumbles and shakes his head]

Ariel Exactly! You have a different rule. That’s why!

James [writing in his paper] But my rule worked. Twenty-five …

Ariel [shuffling his papers] Like this one had two different rules. It could have more than one rule.

Ariel makes some funny noises, adjusts his clothing and finally becomes conscious of the camera. James keeps writing in his paper and is mumbling inaudibly as he is writing.]

James [to Ariel] Seventy-six.

Ariel Seventy-six? Yeahhh!

G4 [to James] And you explain what your rule is?

James Mine is every number odd add in order like plus one, plus three, plus five.

G4 What are you adding that plus one, plus three, plus five to?

James To X?

Ariel Hmm? [looking closely at James’s paper] What do you say?

James [pointing to the table in his paper] I start off with one, then add the next number by three, then add the next number by five, then seven, then nine, add odd number in that order.

Ariel [looking at James’ paper and moving his head vigorously]

G4 [to Ariel] Does that make sense to you?

Ariel Yeah!

Clip 6 22-23

G4 OK, so what if I gave you eighty-four for X?

James Huh?

G4 What if I gave you eighty-four for X?

Ariel Let’s see. [pointing his finger to G4] I got that on lock down.

Ariel starts extending the table writing 26 in the X column and then stops.]

Ariel Wait a minute. You said eighty-four, right? That’s
easy.

[James writes in his paper:
84
x3
252
+ 1 = 253]

568 Ariel Two fifty three.
569 G4 [to James] What do you think?
570 James Two fifty three for what?
571 Ariel I used my rule.
572 James Eighty-four?
573 Ariel I just used my rule. Eighty-four times three is two hundred and fifty-two plus one is two hundred and fifty-three.
574 G4 [to James] What do you think?
575 James [turning his head towards G4] Eighty-four? Why eighty-four?
576 G4 I don’t know. I picked up any number.
577 James Stop giving us these big numbers.
578 Ariel [laughs]
579 James What if I give a thousand two hundred and fifty?
580 G4 A thousand two hundred and fifty?
581 James [shaking his head] I don’t. I don’t want to do that. I don’t want to do that.
582 Ariel [starts writing in his paper and mumbles]
583 Ariel has written in his paper:
1,250
x3
3,750
+ 1
3,751]
584 Ariel Three thousand seven hundred and fifty-one.
585 G4 My goodness, what do you think about this James?
586 James What?
587 Ariel One two fifty times three is three seven fifty plus one is three seven fifty one.
588 Ariel [pointing something in his paper to G4 who is coming around to his side] I know that there is some pattern… between this one and this one. I don’t know how to say it.
589 G4 Between which one and which one?
590 Ariel I know twenty-five times three is seventy-five, there is something out there, I just don’t know how to explain it.
591 G4 What about twenty-five?
Ariel: It is seventy-six. I forgot to write down. [writes something in the paper]

G4: Alright, so what are you noticing with the numbers?

Ariel: Huh?

G4: What are you noticing with the numbers?

Ariel: I just noticed that twenty-five times three is seventy-five that’s how I solved this real quick.

G4: Do you think there is anything that is same with what you are doing and what James came up with?

Ariel: Uh, uhh, ahhh, I don’t know. Oh, wait a minute. Uh, hmm, maybe. I don’t know. Because I’m adding three, that’s an odd number, he is adding odd numbers, so might be the same. [shrugs]

G4: So might be the same.

Ariel: I’m doing odd and he is doing odds, so … [shrugs again] same thing.

G4: So odd numbers are the common [stops talking and steps back and waves to someone in the background]

Ariel: [to G4] Have you guys got more problems?

G4: I think you should make up your own.

G4: [bending a little towards Ariel] would you like to make up your own so that maybe we could give it to the rest of the class?

Clip 1

G2: [in the background] You can take five minutes to write down what you have found yesterday and will get together and share our rules and ways of finding the rules.

Ariel: [handing some papers to James] There you go.

G2: [coming near Ariel] How many rules did you guys do?

Ariel: Three.

G2: [going around to James] James, Ariel is writing this number three, can you write the other?

G2: [talking to James] [inaudible]

G2: [James takes a blank transparency. Ariel continues to write on the transparency.]

G2: [Ariel has drawn the following table on the transparency:]

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>
[James is creating a table for a Guess My Rule problem. He has written the following table on paper:

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>18</td>
</tr>
<tr>
<td>4</td>
<td>21</td>
</tr>
<tr>
<td>5</td>
<td>24</td>
</tr>
</tbody>
</table>

James [showing to Ariel the table that he has drawn] Can you find out this rule? Guess my rule.

Ariel [inaudible] I will get it.

Ariel [Ariel takes the writing pad with the paper from James.] [looking closely at the paper and pointing with his pen] That’s a nine?

James [continuing to look at his computer screen] Huh?

Ariel That’s a nine?

James Yeah, right here.

Ariel [continuing to look closely at the paper] This is pretty easy.

G1 comes near to Ariel’s table.]
Ariel: Got it.

James: What is it?

Ariel: It is times two … [pauses] no, it is times three plus six.

James smiles. Ariel gives back the writing pad and the paper to James.

Ariel: [taking the transparency on which he has written the table] What this? I have written what I did yesterday? Are we supposed to write all the rules?

G1: [pointing to something in the table that Ariel has written on the transparency] How you got this?

Ariel: Four times three is twelve.

G1: Can you explain that?

Ariel: [inaudible] [pointing with his pen and mumbling some numbers]

G1: OK, let me understand what you did. Four times …

Ariel: Three.

G1: Three. What is that?

Ariel: Plus one is thirteen.

G1: [pointing to something in the transparency] How did you get this?

Ariel: Six times three is eighteen plus one is nineteen.

G1: Can you explain me this?

Ariel: Oh, seven … oh, I messed up. No wonder, that’s why. Yeah, I messed up. This is twenty-two.

Ariel: And this is supposed to be twenty-five. Wait a minute. Yeah, that is supposed to be twenty-five.

James: [pointing to the table that James has created] So what’s the rule for this one?

James: About the rule?

G4: Hmm, hmm.

James: X you plus three and then times …

G4: Wait, what you do first to the X? You add three and then you multiply by three? Can you show if that works for these values?

James: I want to write down. [starts writing on another paper]

G4: Yeah, you can use one of that.

James: Zero plus three is three times three is nine.

[James has written: $0 + 3 = 3 \times 3 = 9$]
James: One plus three is four times three is twelve. Two plus three equals five times three is eighteen.

[James has written:
\[1 + 3 = 4 \times 3 = 12\]

\[2 + 3 = 5 \times 3 = 18\]
]

G4: Five times three? What is five times three?

James: Fifteen. [corrects the 18 to 15]

G4: Let's see. Is that what they have here? So is this the table given to you? To guess the rule or you started with the rule.

James: [mumbling some numbers] Yeah, this is right.

G4: So this is the rule that you invented?

James: Yeah.

G4: Oh. So is this something that they asked you yesterday or you just thought of?

James: I just thought of.

G4: Oh, very nice.

G4: [pointing to what James had written] Can you think of a way of writing this rule in a more general form? If you want to tell someone without giving specific examples how would you tell what the rule is?

James: Plus three times three.

[James has written: \(X + 3 \times 3 = Y\).]

G4: X is the number that you start with?

James: Yeah.

G4: OK.

James: X plus three times three equals Y.

G4: So if I see it written this way, then you think I would do first I would add three and then I would multiply the result by three?

James: Yeah.

G4: Is this, how about this? [writes ‘5 + 2 \times 3’] Five plus two times three. Just like this. How would you work on this? What would you do?

James: Why is it two?

G4: Huh?

James: Why is it two?

G4: Oh, this is a different rule, I just gave an example. What would you do for that?

James: [writes: \(= 21\).]
You say it is twenty-one, how did you go about it?

Five plus two is seven, seven times three is twenty-one.

OK, so you did the operations in this order [pointing her pen from left to right] the order in which they were written? OK.

But conventionally, do you think this is the way how everybody interprets? Let’s ask him (Ariel).

So is that another way you could do? So [inaudible]. Let’s check with him, if he interprets the way.

How about you? [pointing to Ariel] If I gave you this to preform, how would you go about it? To find the answer for this expression?

Twenty-one. Five plus two is seven times three is twenty-one.

[looking at T1] So he [Ariel] interpreted the same way as he [James] did.

Two times three is six and that is five, so I would say the answer is eleven.

Oh, I know what you do. You put this in parentheses [puts parentheses (5 + 2) × 3.]

Why you put parentheses?

That shows that you should do that first [pointing with his pen to the parentheses] before you do anything else. That’s what our teacher did.

So you need that to avoid confusion here? You think we should put the parentheses here?

[Ariel nods.]

[to James] Do you agree with him?

[James nods.]

Ok, so then going back to your rule here [pointing to the paper] would you leave it this way or what would you do, would you do anything else to it? Would you leave this rule the same way?

[pointing to the paper] This rule?

No, this. This is your rule, right? This is yours.

He said [inaudible]

OK, just because people will not form the equation [inaudible]

[Yonny and Brandon are working on the Guess My Rule Problem 1:]

X

Y
Plus five, plus six … I finished it, I finished it.

Brandon: Yeah. No, no I don’t think. I know. [jokes] Sike nah, I think I got it.

Brandon: No, Yonny is slow. No, you are adding it by one! And then by, when you add it by one, and then you add another one by two or one? Come on, Yonny.

Brandon: [talking to Yonny] I finished the problem already, OK? Without your help.

Yonny: Ok, thank you.

Brandon: It’s by one, by one, by one.

Brandon: No you got to add one, then add one, then add another one.

Yonny: [pointing to the Guess My Rule Problem 1 sheet] No, so it would be like plus one, plus two, plus three, plus four, see I got it. I am too smart.

Brandon: [joins Yonny] Plus five, plus six. See how smart. I am smart.

Yonny: I just said that. He has no proof. [jokes around and laughs]

Brandon: Shut up! [laughs]

Brandon: I’m so smart

[Brandon writes on the side of the Guess My Rule Problem 1 sheet: The Rule is +1+2+3 and it keeps going to six.]

Brandon: Done!

[Yonny writes on the side of the Guess My Rule Problem 1 sheet: The rules is that when you add you add by one more.]

Brandon: Finished!

R1: So would you guys tell me what you have so far?

Brandon: Yea, we finished.

R1: [to Yonny] Which one did you do?

Yonny: Everything.

R1: Show me. How did you come up with the rule?

Brandon: It was easy. Just looked at it, you know because. I just looked at it. Plus one, plus two, plus three, plus
four, plus five, plus six, plus seven… I mean plus six.

You just have to add another one to…

To everything.

Plus one, plus two, plus three, and so on, etcetera …

Suppose if I gave you the number six, what would it be?

That’ll be uhmm… six to umm… six to thirteen.

How did you get that?

Because, umm, because you have to, umm… when you got to here, plus six, you have to add six plus seven.

Wait, I am not sure if I understand.

You see five, eleven? yeah five plus six equals eleven.

OK

And then so you have to add six plus seven because you have to add, umm, yeah, six plus seven because you have to add to get thirteen.

Six plus seven?

Look, OK, look, its one, zero plus one equals one. Zero plus, I mean one plus two equals three, three plus two equals five, three plus four equals seven, four plus five equals nine, and five plus six equals eleven and six plus seven equals, what you got here, equals thirteen.

Umm, and suppose I give you twenty? What do you think it will be?

Umm, I am not sure.

[Brandon starts writing, while Yonny is busy doing something on the computer.]

[to Brandon] You got it? OK, I want you guys to work on that one.

OK.

Yonny? [He is trying to get Yonny’s attention and getting him to start working]

Yeah? What’d you say?

Twenty, suppose box is twenty, in this case, suppose X is twenty.

Umm… that’s too big of a number. Brandon can handle it. Sike, nah, I’ll do it though. We’ll do it, Brandon. [whistles and starts working on the problem.]

Fifteen … [whistles again]. Man, why do you have to say twenty? Couldn’t he have said…

Eleven.

I know, right. [thinks and whistles again]
Yonny: [to Brandon] I know, I know Rule 2. Like, right here it doubles by 2.

Brandon: Yeah that’s the rule I was about to say.

Yonny: No you didn’t.

Brandon: Thereason that I wasn’t about to say that was because it don’t work right here [points with his pen to where X = 6 and Y = 13 in Yonny’s table. Yonny continues to whistle.]

The camera focuses on what Yonny is writing.

Yonny has written:

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>11</td>
</tr>
<tr>
<td>6</td>
<td>13</td>
</tr>
<tr>
<td>7</td>
<td>15</td>
</tr>
<tr>
<td>8</td>
<td>17</td>
</tr>
<tr>
<td>9</td>
<td>19</td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

R1: Can you repeat what you just said? I didn’t quite hear you.

Yonny: I said this doubles by two.

R1: What doubles by two?

Yonny: The Y.

R1: For example, help me understand what happens here?

Yonny: Like one plus two plus three plus yeah. Like one plus two [equals] three, plus two [equals] five.

R1: So you call that doubling by two?

Yonny: Yah.

[Both Yonny and Brandon continue to work on the problem.]

Brandon: [mumbles while working] I got it! And twenty would be forty-one. Forty-One. Forty-One. Forty-One. The total would be forty-one.

Yonny: He got it. [stops working, putting the pen and paper aside.]

R1: Suppose I ask you, if give you a big number. How did you get twenty?

Brandon: On this side, on this side, the number goes up by two. So, I skipped by two all the way to twenty.

R1: Hmm… I see

Brandon: Look, look you see. Now you don’t.
R1: Yonny...

Brandon: Yes.

R1: I want to give you guys another problem. But before I give you the other problem, I want to ask you a question.

Yonny: Yes.

R1: What would Y be if X is one hundred?  
Yonny: What?

Yonny: Oh you can’t make us do that…

Brandon: Oh if X is one hundred, what would Y be? I don’t feel like going up to one hundred.

R1: So is there another way to get there?

Brandon: Yeah, well hundred times two.

R1: So, is there another way to think about this rule?

Yonny: [yawns] No.

R1: So if I ask you about the hundred, what’s the problem? You don’t have to work out?

Brandon: I have to work it all out all day.

Yonny: Well, I think it could be like forty-one times five.

Brandon: Yeah well you have to do something.

Yonny: But like forty-one is twenty and twenty is a factor of a hundred. So, it multiplies by five to get on it. So, I just multiplied forty-one by five.

Brandon: And what would you get, stupid?

R1: So could you guys think about that for a while?

Yonny: What?

R1: I want you to think about that.

Yonny: Ahh, we can think about it.

Brandon: Yes sir. Think about it.

R1: And I will be back.

Yonny: I am thinking.

Brandon: Think about it. Take a sec.

[Brandon starts looking at Yonny’s screen.]

Brandon: Come on bro, think of the dag on question.

Yonny: What are you talking bro…You about to play games. [starts laughing]

Brandon: What are you talking about? There is nothing on my screen. Do you see it? Do you see me clicked on anything? Do you see anything blue around here? Around anything? Nothing.

[Brandon starts looking at Yonny’s screen.]

R1: So guys, did you come up with anything?

Brandon: No, I don’t know it.

Yonny: I’m not sure.

Brandon: Yonny wasn’t working on it.

Yonny: What are you talking about?
Brandon: I started thinking.
Yonny: You are such an unpopular person. You are so whack.
Brandon: It don’t wanna... It don’t wanna go slow down. Oh, here it is.
Yonny: He’s kind of slow in the brain.
R1: You want someone to help you?
Brandon: Maybe.
R1: How’d you get that?
Brandon: Forty one times five, because twenty is a factor of one hundred. Twenty forty-one, so forty-one times five, cause twenty times five equals a hundred, so we just took the five, from the twenty, so we took the forty one and multiplied it by five.

Alright well, I want to give you another problem.
No, come on [speaks softly]
[R1 gives to Yonny and Brandon the Guess My Rule Problem 2 sheet:

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
</tr>
</tbody>
</table>

Brandon: Oh, I thought you were giving us a whole another problem. Come on, man!
Yonny: OK, I got the rule already.
Brandon: Yah, I did too. Man, how are you going to leave us, man?
Yonny: It’s the same rule as this one.
Brandon: Yah we solved it, because we are the only one that’s on it.
Brandon: Alright the rule is, it’s going up by one on the X side and it’s going up by two on the Y side. Easy.
Yonny: I’m done.
[Brandon returns and starts playing video games on the computer. Brandon is still working on the problem. They both get up and go somewhere.]
How’d you guys do?

Brandon

Finished.

Oh yeah, what did you guys come up with?

Brandon

That, on this side, on the X axis, it’s going up by one and on the Y axis it’s going up by two.

OK, and so if I give you a number for X. How are you going to tell me what number Y is going to be?

By adding it to this.

For example, if X is seven.

Uh huh, seven [mumbles] Seventeen.

You sure?

No, let me check again. You said seven, right?

Umm.

[mumbles some numbers and then says] Yah, I’m sure. Yah, I’m sure.

Now, remember the question I asked you about the last rule you came up with?

What?

What happens…

For one hundred?

Well before we get to a hundred, what about twenty?

Yonny get over here.

I got to write it.

You got to write it all out, huh?

Yah.

I wonder if you can find a different way of getting your rule, so that you don’t have to write it all out?

Why don’t you take a look at your numbers in the table, and see whether or not you come up with another of getting it? OK?

OK.

[Brandon starts working on the problem.]

I think it’s going to be … [pauses] never mind. Hold on.

Oh, no nono, I think I messed up. I skipped five, that’s it.

[The camera focuses on what Brandon is writing. He has written the following:]

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td>7</td>
<td>17</td>
</tr>
<tr>
<td>8</td>
<td>29</td>
</tr>
<tr>
<td>9</td>
<td>21</td>
</tr>
</tbody>
</table>
Brandon cancels everything he wrote after the pair \(X = 5\) and \(Y = 15\).]

Yonny Are you writing?
[Brandon writes on the side:

Brandon has already written: The rule is the numbers are going up by 1 on the \(X\) axis and up by two on the \(Y\) axis.]

Brandon Come on bro, help me fix this. I don’t know this, he said to find twenty. You do it, I don’t feel like doing it.

Brandon writes on the side:

Brandon has already written: The rule is the numbers are going up by 1 on the \(X\) axis and up by two on the \(Y\) axis.]

Yonny Are you done?
Brandon No.

Yonny Because we have one more problem for you.
Brandon I mean, yeah, I’m done, but you’re trying to make us do twenty again.

Yonny Found it?
Brandon No.

Yonny It’s forty-three. No, do the next one. Do the next one.
Brandon It is forty-three.

Yonny No, do the next one, do the next one.
Brandon No it is forty-five.

Yonny OK, forty-five. Write it down.
Brandon Got it. It’s forty-five. Yeah.

Brandon writes the following in his Guess My Rule
Problem 1 sheet:

10
+10
20 X-axis
+21
41
Y-axis]

10 20 X-axis
+25
45]

Hmm…so for the other guess my rule problem what did you think you got that?

Brandon Ten plus ten equals twenty. Plus twenty five equals forty-five.

[Brandon writes the following in his Guess My Rule Problem 2 sheet:

10
+10
20
+25
45]

Hmm…and where did the twenty five come from?

Brandon From the answer of ten. From the Y-axis with that equals up, I mean, with the one that matches up with the ten.

R1 Umm, suppose we do it your other way and figure out what twenty would be?

Brandon Oh go all the way up?

R1 See whether or not…if you get the same result.

[R1 leaves Brandon. Yonny asks Brandon something about the game and they start talking about the game. Then Brandon starts writing the following:

6 17
19
43
21
23
25
27
29
31
33
35
37
39

Dag, I don’t think it’s going to be the same answer. Oh, forty-one, forty-one. See it’s the same answer as I told you. Yeah, we smart. Not dumb.
[Yonny plays on the computer while Brandon goes to R1.]

Brandon [in the background] We got the same results.

R1 [in the background] You came up with the same results? Alright did I give you guys, that was the second problem, right? I have to get a third one for you. OK. We have one more.

R1 [coming near to Yonny’s seat and trying to get Yonny’s attention] Yonny?

Brandon Well, I was the only one doing it.

R1 Well, it’s OK; you guys work on the third one and see how you come up with the new one. OK?

Clip 3 - 26

G6 hands them the Guess My Rule Problem 3 sheet:

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
</tr>
</tbody>
</table>

Brandon I got skills. [repeats a few times and looks at the Guess My Rule Problem 3 sheet] Ahh, this is easy! By one on the X axis, by three on the Y axis. Uhh, I got skills.

Brandon [to Yonny] Didn’t I tell you I got skills? [says aloud twice]

Brandon [turns around and shouts to R1 who is in the background] Finished! Told you we got skills like that.

R1 Now, I’m going to ask you the same question for twenty.

Brandon Oh, man, I hate twenty. Why does twenty have to be twenty? [to Yonny] Don’t you hate twenty? Why does it have to be twenty? [starts writing something on the paper.]

Brandon writes the following on the Guess My Rule Problem 3 sheet:

The rule is the number go up by one on the X-axis and by three on the Y-axis.

Brandon laughs and talks with other students who are in the background and then extends the Guess
My Rule Problem 3 table as follows:

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
</tr>
<tr>
<td>6</td>
<td>19</td>
</tr>
<tr>
<td>7</td>
<td>22</td>
</tr>
<tr>
<td>8</td>
<td>25</td>
</tr>
<tr>
<td>9</td>
<td>28</td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

2025 X Y

Twenty-nine, so it is going to be forty-nine for the next one. Twenty-nine.

Brandon It’s not going to be twenty-nine. It’s got to go up by three.

Brandon writes on the side:

10 + 10 = 20
20 + 31 = 51

Ten plus ten is twenty, plus thirty-one is fifty-one. Finished, and we got twenty. I wonder if they noticed. [stands up and goes to R1]

Brandon Excuse me. Finished and we did the twenty.

R1 OK, so show me how did you come up with it?

Brandon So far what we think, that we know is that. Alright, on the X-axis it goes up by one. On the Y-axis it goes up by three. And then we, to get twenty, we take the ten and we did what we did the last time, we did ten plus ten equals twenty plus the number of ten, that matches up by ten, and then we add by thirty one, and so it was twenty plus thirty one which equals fifty one. And we got it, so um the number that twenty matches up with is…

R1 Fifty-one? I don’t know whether I believe that…

Yonny Well I think I believe it’s sixty-one.

Brandon It’s sixty-one?

Yonny Well I think it’s sixty-one.

R1 Why do you think it’s sixty-one?

Yonny Because in the previous ones, it said like twenty is divisible by four, it went up to forty so the answer
was kind of like the answer was kind of like double, but now the answer came up to thirty one, so I think it should be double of that, to get sixty-one.

Brandon

Oh, alright, OK.

Yonny

Sixty-one.

Brandon

OK now, now since I did this, means you got to go up to twenty. I’m not going to...

R1

Yonny, So why don’t you guys figure out what twenty is going to be? Whether it is going to be fifty-one or sixty-one?

Brandon continues to extend the table as follows:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>28</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>31</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>34</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>37</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>40</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>43</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>46</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>49</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>52</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>55</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>58</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>61</td>
<td></td>
</tr>
</tbody>
</table>

R1

Do you think it’s going to be fifty-one?

Brandon shakes his head no

R1

No? Why do you say that? Why do you say that?

Yonny

Like at 18 it’s going to be fifty-one.

R1

Why do you say you don’t you think it’s going to be fifty-one?

Brandon

Cause, it’s going up by three this time not by um...

Brandon

Nope, it’s not fifty-one, it’s probably sixty-one, like you said.

Yonny

[Yonny yawns] I’m tired.

Brandon

Nope, it’s not fifty-one.

R1

Yonny, did you put today’s date and your name on each one of those?

Brandon

Yup, it’s sixty-one brother. We still got it right, since you said it sixty-one. We got it right now, shut up.

Brandon goes up to R1.

R1

[in the background] OK, can I have everyone’s attention? We are at the end of today’s session and I know that each group
worked very hard, this is very good. We have more problems, but we are not going to have that opportunity today, but we start tomorrow’s session by talking about what we did today. Please write your name and today’s date and we will continue tomorrow. OK.

1089
1090 [G6 collects their papers. G6 and Brandon have a discussion about his work while R1 is speaking to the class]

1091
1092 5:31 Brandon Ten plus ten equals twenty.
1093 G6 Right. And then why you add thirty-one here?
1094 Brandon Because that matches up with ten.
1095 G6 And why you choose this one? [pointing to a number on the paper]
1096 Brandon Because it’s the best one that gets up to twenty, quicker than any other one, instead of by five.
1097 G6 So, why you don’t choose five?
1098 Brandon Because, it take too long. It takes too long.
1099 5:53 G6 It takes longer. So this one is easier. Exactly.

1100 Clip 4 - 26

1101 0:00 R1 We have 1 to 3.
1102 0:02 Christian Yeah, yeah.
1103 0:03 R1 We said we was going to use multiplication right?
1104 0:04 Christian Yeah, yeah.
1105 0:07 R1 Christian, what I need you to do is drop your paper. Come over here (beckoning Christian to come closer to Dawud and work together).
1106 0:10 Christian Here, you want me to be like this so Dawud could be like you know, be like... (playful banter).
1107 0:12 Dawud Yeah, yeah, yeah.
1108 0:13 R1 Yes, so let’s look at this here. Let’s look at this here. I like the fact that you’re going over. You said 1, 2, 3. Maybe we should try to use multiplication (pointing to the X side of the table).
1109 0:26 Christian Mike check, one two one two (Christian and Dawud laugh).
1110 0:28  R1  How can we utilize multiplication here? [Christian and Dawud are chatting and laughing] You said multiplication (speaking to Christian). Now if you said multiplication, come on.

1111 0:40  Christian  I don’t know this man! I’m tired.

1112 0:43  R1  Let’s think about this. Let’s just look at one to three. How can you use multiplication?

1113 0:45  Christian  My melon is dry (referencing his brain). [Dawud and Christian laugh]

1114 0:47  R1  Let’s look from one to three. How can you use multiplication? From one to three. [R1 tries to get their attention]

1115  Christian and Dawud are chatting

1116 0:56  R1  Listen, gentlemen. First and foremost, we are focus as much as we can.

1117 1:01  Christian  But we just come out of school. We are tired.

1118 1:03  R1  Let’s look at from one... I understand that! (to Christian)

1119 1:05  Dawud  Go eat a cookie.

1120 1:07  R1  That’s why we’re only looking at this one, from one to three. How can we... What... Give me an equation or number sentence that I could go from one to three. [Christian and Dawud grumble about working] We said it wasn’t two, right? Guys, we said it wasn’t plus two because if it was plus two then this would follow. Let’s try another, from one to three. From one to three.

1121 1:25  Christian  We leave in ten minutes don’t we?

1122 1:27  R1  That’s the reason why. Let’s do one to three. And Mrs. Morris has got to go and because I really, I just want to find this problem.

1123 1:34  Christian  No, you don’t.

1124 1:35  R1  Yes, I do. From one to three, what can we do? We did plus two, and it’s not here. Plus two is not going to work for us. So, let’s look at the next one. Because plus two is our only way. So, let’s look at another one, we said multiplication.


1126 1:51  Dawud  Hahaha, he stuck his finger

1127 1:53  R1  OK, Dawud. While he is getting his fingers out, let’s look at this one. How about multiplication? [Waits for Dawud to respond] And Christian you’re the one. [R1 pulls Christian’s finger out of the opening] You’re OK now. OK, so let’s look. We did addition and it’s not it. So we use multiplication. Cause you
said, didn’t you say that we could use multiplication, right? Christian! Did you or did you not say we can use multiplication?

1128 2:16 Christian Yes, I did.
1129 2:17 R1 Thank you. Dawud, sit on your fingers.
1130 2:20 Christian Sit on your fingers!
1131 2:22 R1 OK, let’s look at this here. From one to the... So, what, I need another. Let’s go. Not I need, we all need, we are all sharing.

1132 2:29 Christian Yeah, go ahead. Do your thing, Dawud.
1133 2:32 R1 Come on, Dawud.
1134 2:33 Dawud I’m sitting on my hands, so I don’t do it.
1135 2:35 R1 How about? So, what can we do with the two? We can’t add it, what else could we do?
1136 2:38 Dawud Uhhh. You hear this? Do you hear this?
1137 2:42 R1 [R1 decides to put the current problem aside] Let’s try. We’ll go to another one, then. We’ll come back to that, OK? Let’s try this one.

1138 2:49 Dawud It’s the same one.
1139 2:50 R1 It’s not the same one. It’s a little different. Christian, look at this. We changed.
1140 2:53 Dawud Five, six... Five, six, seven, eight, nine. Woot, I’m done.
1141 2:53 Christian Hold up, I got to get some pinballs. You need to see this. Y’all looking at my pinball game?
1142 3:00 R1 Three times one is three. And you got plus three.
1143 3:07 Dawud That equals six. But when you plus it, it just goes up one.
1144 3:14 R1 So, let’s change it. You want to revise it?
1145 3:16 Dawud Yeah, 3 times 1 plus 4 equals 7. So, when, you keep this the same, but you just add the pluses right here.
1146 3:24 R1 But, aren’t these two different rules?
1147 3:27 Dawud But, you just add up one. You just go up one.
1148 3:28 R1 So, we got to change, we got to think about it. Do me a favor? Write your name and today’s date.

1149 3:32 Dawud Well, you just add up one. You just go up one.

1150 **Clip 1 - 27**

1151 X  Y
    0  1
    1  3
    2  5
    3  7
    4  9
I think my rule gunna be add one. You just add one, you just add one every time you go up.

What like this? (Pointing to number in X column.)

Yeah like go one then go two, then it go three, then it go four, then it go five, then it go six. You just adding one.

So the X’s are adding one.

Yeah

What about the Y’s?

They Y’s, they just are the same like, when you put one it equals over here. Like when we put two it comes right there (points to X axis)

Okay

And then we go three, it counts right here. Say the number supposed to add with, and it goes to the opposite side.

Okay I’m not sure I understand, I’ll read what you write down.

When add the number like 0+1=1 goes to the x’s

When add the number like zero plus one, one does to the x’s. So if you

Have zero plus one (overhead announcement) One plus one get two? I see, do you have a rule? About how I could get the, if I gave you an x how you would get the y? Remember when we did that? The guess my rule.

That when you get the x, you add one. Then you add one it goes to the x’s side, then you add that number, which comes to the x’s side. So basically when you add the number it’s going to go to the x side.

Okay, what about the y side? Do you have any rule for the y side?

Yeah, it goes by two. Like one, two

Three, oh three plus two is five, five plus two is seven, like that?

mhhmm

Lets see, maybe you should write that too. So you have two rules.

When add like numbers like 0+1=1 goes to the x’s the y’s side add two. Read this. For the y side, when you add two.

What about this, suppose I told you I had something like this (y=x+x+1)
What do you think of that rule?

Dawud

y equals xx. Which y would be to (inaudible) plus one.

R2

Do you think that rule is true?

Dawud

It can if you rearrange the numbers.

R2

How do you mean?

Dawud

Like x equals y plus x and then oh! I think this

R2

You know the reason why I said I thought this was true? Because if x is zero, I get zero plus zero plus zero plus one equals one.

Dawud

Yeah, you would get that.

R2

Would that work for all of them?

Dawud

No. Because one, look. If you go in order this side. One, two, three, four, five.

R2

Okay, what about one, three? One plus one plus one is three, so it works f for this one right?

Dawud

Mhmm.

R2

Do you think it works for all of them?

3:10 Dawud

Yeah, like if you add it with some of the numbers. Like for this one four, two, two plus one, plus one. Equals four. So when you use this number (points to x) you got to change it. Like for this one (points to number four) this one would be five. Three plus one plus one equals five. For this one (points to number five) three plus two plus one equals six.

R2

Okay

Dawud

Three plus one plus one equals five.

R3

Three plus

Dawud

one

R3

No, if you mean to say this x is three right?

Dawud

Uuhh.

R3

Then what should be here? (points to second x)

Dawud

One.

R3

But this is also x, it has to be the same value correct?

Dawud

No.

R3

I mean if you already have three then the other should be three right?

R3

What ever number we put here (points to first x) we should also put here (points to second x)

Dawud

Two plus two plus one equals five.

R3

So is it five?

Dawud

Yeah

4:17 R3

Then can you explain me this?

Dawud

Huh?

R3

Will you explain the last (inaudible)
Dawud mumbles inaudible numbers to himself

Okay let me explain lets see how this works for this.

Like look at this.

Dawud X don’t have to be both.

R3 Zero plus zero plus one is one. See how you have one. Let's do this one (points to numbers one and three) x will be one and y will be three, so x is what here? Okay x is two and y is five.

Dawud Two plus two plus one is five.

R3 Five, very good. Lets check this, three plus three plus one.

Dawud Four plus four plus one equals nine.

R3 Is that correct?

Dawud Yupp

R3 And what about this? (points to five and eleven)

Dawud Five plus five plus one equals eleven.

R3 Okay, now let me ask you something. If I give you some x values like six, seven, eight, nine, ten. Can you find out the y values? And write it down.

Dawud Writes: 13,15,17,19,21

R3 Okay, can you explain to me how you go this?

Dawud Cause one, three, five, seven, nine, eleven. Two. So you add two. Thirteen, fifteen, seventeen, nineteen, twenty-one.

R3 Okay, but then we were checking, we were checking it differently right? We were using this here right? (points to equation y=x+x+1) So what was this then?

Dawud Six. See I can explain myself. Six plus six equals twelve, plus one equals thirteen. For fifteen you could put seven plus seven plus one for fifteen. (inaudible) for twenty ten plus ten plus one equals twenty-one.

R3 Okay, in case I give you a number lets say, 100.

James and Ariel, come here. I have got another challenge for you

Ariel Oh, that’s easy. I got it, I got it. No, I don’t. [Ariel works on the problem] Ohhhh I got it. It goes… add one. [Ariel writes something on his paper] I got it, but I don’t know how to explain it. It’s add one.
Negative one goes to one, makes that. You add in one, wait a minute, no, no, oh yeah. Negative one is zero. This is one, this two, this is three, this is four (noting the difference between the y values). 0,1,2,3,4. [Ariel writes these numbers in between the y values] I’m done. Wait, um Miss? I’m done.

What’s the rule?

It’s going to keep on adding one to the y-axis thing.

It’s what? What is the rule?

I’m done.

You’re done already?

Okay, write it down and explain why it works. And I’ll be right there.

Oh, god. Lalalalala. [Ariel begins to sing to himself, while writing his solution on the paper] The next one would be eleven.

When you did this, what did you mean by doing this?

Cause it goes from negative one to zero, and then one, and then two, and then three, and then four. Cause…

So negative one goes to zero?

Yeah, cause here… No, that’s wrong. I don’t know that. But, I know this one. It goes one plus zero is one, two plus one is three, three plus two is five, four plus… it keeps on adding one to the y. Got it.

Add one to the y?

Yeah, add one to the y. No, like, to the number. For like, it goes, for every number it goes up like this is plus two.

So, I say zero. What do you say?

Negative one.

I say one, what are you going to say?

One, (inaudible).

Two, what do you say?

Three.

I say three, what do you say?

Five.

Five. So what is the rule?

And I noticed a pattern, I noticed the pattern too. It go, plus two, plus two, plus two. What? [Ariel gets distracted by another person calling him]

So, I get two to the number that I give you?
Ariel to the number that you give me, like depending on the number. Like with starting out with one you add zero. Two, you add one. Three, you add two. Four, you add three. Five, you add four. [Ariel says this while pointing at his paper] I’m right, you’re wrong. What is you doing, James? You haven’t even solved it all.

I already know it.

No, you don’t.

Plus five times one, I mean, plus five minus one.

The rule is plus four minus one. No, it’s not. It’s plus three minus one. No, it’s not. I got it. I don’t really know how to say it, like how I did the other ones. I know it’s add one to the y. [Ariel points to the y column of the chart]

Negative one… Why did you write this zero next to one? [R1 points to the y value of 1]

Cause, it went in like… I see it as a factory, this line. [Ariel points to the column line dividing the x and y columns] It goes in an x number, it comes out a y number. I came out still 1 so it must be it didn’t do anything to it, so its 0.

Okay, so in this case, to this number, you didn’t add anything. Okay.

Yeah. So, 2 went in and came out 3, is 1. 3 went in and its 2, cause it came out 5. It’s adding, I just added these and I saw that that’s how its going. Like here… [Ariel begins to write the differences between the y values for his chart that weren’t filled out before] This is: 5, 6, 7, 8, 9. There you go.

And here, its zero? What did you get for zero? [R1 points to Ariel’s paper]

That, I still haven’t figured out. I think it’s zero plus negative one. Negative one.

Zero plus negative one.

Yeah, it’s negative one. That’s what I think, I’m not really sure about that.

Okay, so you said that the rule is?

Add one to the y. I mean to the number, for like for how the number goes up you add one to what you’re adding to. So, basically to the y.

Add one, the rule would be add one to the y.

To the y. So, for every number… [Ariel is writing this on his paper]

What is the number that you are given first?
Ariel: Oh, yeah. The funny thing is that I solved it right away. Ask him. I solved it like in two seconds.

Yonny: Yo, how the heck he be playing video games, when I want to play video games? But she said that we can’t play no video games?

R1: So, what if I say to you, for example eleven, what would you come up with?

Ariel: Eleven?

R1: Yeah.

Ariel: It would be twenty-one.

R1: Twenty-one? Why?

Ariel: Eleven, twenty-one. Because if you follow this it would be 10 [Ariel gestures to his chart with the differences of the y values filled in]

R1: Oh, okay. Maybe R2 could give you a challenge problem.

Ariel: Okay.

R2: I’m not sure they are done with that one yet.

R1: Oh, okay.

Ariel: I’m done.

R1: What is it?

Ariel: You add one to the y. It’s a pattern. For every number you add one to what you’re going to come out with.

R3: If I say twenty, what would the y be in that case? If I say twenty.

Ariel: (Ariel mumbles some numbers while calculating for twenty) It’s thirty-eight.

R1: I don’t think that, I don’t think that’s right.

Ariel: What? You want me to prove it? The ten is half of twenty, that’d be nine. Nine times two is eighteen. It would be adding the eighteen.

R1: Oh, that’s good, that’s good. I like that. So, oh! You would be adding the eighteen.

Ariel: It would be thirty-eight.

R1: Why are you adding twenty to eighteen?

Ariel: Huh? I’m not, No… Yeah, I’m adding eighteen to twenty. Cause of this pattern I saw that...

R1: It’s going up by two, right?

R3: Yeah, it goes…

R1: Shouldn’t you be adding the nineteen?

R3: But here it’s not (inaudible) to get one, its like if you
add zero…

1:07 Ariel Exactly! Nineteen times two is thirty-eight.

1:07 R3 One plus zero, one. Two plus one, three. Three plus two, five.

1:08 R1 Four plus three, seven.

1:09 R3 So he’s not seeing the difference this time.

1:10 R1 Could you give me a formula? Like they had before?

1:11 Ariel That’s the hard part. Well, that’s like add one…

1:12 R3 Well, what’s the answer for twenty? How much is for twenty?

1:13 Ariel Huh?

1:14 R3 For twenty is how much?

1:15 Ariel For twenty it’s thirty-eight.

1:16 R3 Are you sure?

1:17 Ariel Yes, cause ten times two is twenty. So you just got to multiply the things you did for this. And nine times two is eighteen and nineteen times two is thirty-eight and you get your answer.

1:18 R3 Really?

1:19 Ariel Yeah, you get your answer.

1:20 R3 If you go on, would you get thirty-eight?

1:21 Ariel Yeah. I would do it right now. [Begins to calculate all the values up to twenty on his paper in a table format. Video cuts to him at x=12] (inaudible) It’s 23, 25, (saying and writing the y values) 27, 29, 31, 33. I’m getting closer. Thirty… No, it’s, this is 35. I’m getting closer… Wait a minute. How the crap am I wrong? I did something wrong. I did something wrong cause I got to be right. I did something wrong, I think. Yeah, I did something wrong, I did something wrong.

2:53 R3 Is one of those answers wrong?

1:24 Ariel I think.

1:25 R3 The ones that you just got? So, how did you get this thirty-eight again? You said ten, nineteen times two? How big is ten, twenty is two times ten.

1:26 Ariel Yeah, nineteen times two is thirty-eight.

1:27 R3 So that’s why you think that the number that goes with twenty is thirty-eight. Okay, but if you do this…

1:28 Ariel I don’t know why! [Ariel is trying to figure out where he made his mistake] Wait a minute, if you’re coming from… I know, I just had it in my head. Okay, so if it’s thirty-nine then it must have added
twenty. So it’s doing two, yeah! Exactly, cause it keeps on adding two, two, two, two (etc.). And ten times two equals twenty and here it was nineteen and it added twenty. This plus twenty equals 39, which is that right there (pointing to the paper).

1325  4:02    R3
1326  Ariel  Ten, and you’re going to have to go ten more numbers to get to twenty. Cause ten plus ten is twenty. So I did ten times two, cause it keeps on going up by two, to get you twenty. Then you add twenty to that nineteen and it get you thirty-nine.

1327  R3
1328  Ariel  I don’t understand how you got the nineteen. Okay, this nineteen here (pointing at the paper)?

1329  R3
1330  Ariel  Yeah, this nineteen there (pointing at the ten on the paper). Cause it came out of the thing, nineteen.

1331  4:58    R3
1332  Clip 3- 28-29
1333
1334  0:00    R1
1335  Ariel  So the rule, When I ask you for...
1336  R1  Oh, Y+2, y +2 for every number.
1337  Ariel  Y+ two?
1338  R1  Yeah,
1339  Ariel  But what is the question that I have? okay
1340  R1  Y keeps on adding two to itself. Y keeps on adding two to itself. Y started off as negative one but when it added two to itself it became one.
1341  Ariel  When I ask for each on this table do I have to ask for Y value or the x value?
1342  R1  Y value, Y started out as negative one and negative one plus two is one.
work) you have to add two to your y value. Right?

1343  Ariel  Um hum, yeah
1344  R1    So if y is negative one..
1345  Ariel  Plus two is one and that plus two is three and.
1346  R1    Negative one plus two is one, so what does negative one get? What corresponds to negative one?
1347  Ariel  So negative one to add two to it, first of all you got to bring it to positive zero, plus one is one.
1348  R1    According to your table you that's negative one, that's.. what corresponds to negative one?
1349  Ariel  Two.
1350  R1    Two? According to the table?
1351  Ariel  Oh, zero, zero zero.
1352  R1    So is your rule correct?
1353  Ariel  Yeah wait a minute it's..
1354  R1    I have to give you a value correspondence to the y value that I give you right?
1355  Ariel  Yeah, like this I got it too.
1356  R1    If I say negative one according to your rule what is the x value that corresponds to negative one?
1357  Ariel  Zero.
1358  R1    According to your rule?
1359  Ariel  No, you have to switch the rule, like here you have to do one minus zero (pointing to paper) equals one. Three minus one equals two. Five minus two equals three. Seven minus three equals four. Nine minus four equals five. Eleven minus five equals six. Thirteen minus six equals seven. Fifteen minus seven equals eight. And that's another way to prove my rule.
1360  R1    So that would be another rule?
1361 02:20  Ariel  Yeah, that would be another rule if it were to be y first and then x second. For subtracting one.
1362  R1    So there are like two kind of rules?
1363  Ariel  Yeah.
1364  R1    Two different rules?
Ariel: Yeah, like if this switching and (inaudible)

R1: Ok, so for...

Ariel: Like this is adding and y x would be subtracting.

R1: This is adding?

Ariel: Yeah.

R1: Adding two when I give you the value of y.

Ariel: Y it would be subtracting two.

R1: Okay so.

Ariel: This is, if you were to put y first it would be minus two like the number would go in (inaudible).

R1: Ok so what is (points to paper) this useful for? What is it used for (pointing to rule on student work).

Ariel: Cuz y, it started out as negative one and if you add two to that it's one and if you add two to one that it's three then you add two to three that its five.

R1: So y is negative one so you okay so the value of the value that corresponds to negative one is one?

Ariel: Wait what?

R1: According to this rule.

Ariel: According to this rule zero, zero came out negative one. It's subtracting one.

R1: Okay let me see

Ariel: It's two it's like this because for this the rule does.

R1: Okay for this when I say y is negative one do I get this number or this number.

Ariel: Y is negative one plus this rule yeah this number, yeah yeah exactly.

R1: So I get this number. When y is this I get this number (going down the y column) when I say y is three I get this number. And what are these numbers for (pointing to x column)?

Ariel: These are the numbers that x come out as.

R1: So is there another rule for these?

Ariel: Well if you reverse it, negative one..actually yeah yeah yeah, negative one, but the funny thing is to get to negative one to one don't you have to
subtract from zero? If you subtract from zero it would be zero minus one. But to get to this you have to add one, add two but this is subtracting if you switch it over. To get this you have to subtract but for these you have to add.

R1 Oh so how can I deal with a rule like that? When will I know I have to subtract.

Ariel Oh I got it zero equals subtraction and any number above zero equals adding.

Ariel Cuz dealing with zero will be subtracting and a number higher than zero you are adding.

R1 Okay so you're getting a rule to write to write the numbers on this column right. In this case we're working with the y. Is there any way that I can predict where is the entry that corresponds with four?

Ariel If you know all of these up here yes.

R1 Do I have to know all of them?

Ariel No not all of them.

R1 If I say one thousand five, if x is one thousand five what is the next number?

Ariel The next number?

R1 What is the number that corresponds to one thousand and five?

Ariel Inaudible

R1 So you see that I’m asking you, to get the number the x number, (inaudible).

R1 This is a very interesting

R1 You think you already have a rule?

R1 If this is your rule y plus two.

Ariel No my rule is if you're dealing with zero you're subtracting higher than one you're adding

R1 So if your dealing with zero you’re subtracting

Ariel If your dealing with zero you're subtracting one, but if you're dealing with one and up you're adding two.

R1 So if I’m dealing with zero I have to subtract one..

Ariel One. If you're dealing with a number higher than zero you have to add two.
Ariel: You have to add two to this column (pointing to x column), you have to add two to this column.

R1: What value will correspond to negative one?

Ariel: Negative one? One?

R1: According to your rule

R1: What x would be to y? To negative one? I'm giving a value for x, I'm saying x is negative one. According to your rule what will be y?

Ariel: Zero.

R1: Why?

Ariel: No wait a minute, negative two.

R1: Negative two? Do you see any pattern here? Negative one, one, three, five, seven

Ariel: You're adding two.

R1: If I have negative one here? You have what? Negative two. Do you still have the same pattern.

Ariel: If you have negative one then you'll have to go to zero, you're going to be subtracting, making it negative two.

R1: (inaudible) So what happens to your pattern in the table?

Ariel: It starts going up, by one.

R1: You mean increasing or decreasing.

Ariel: Increasing, as it goes down it is increasing.

R1: By how much is it increasing.

Ariel: By two,

R1: But before according to your rule negative one would be negative two. Is it still increasing by two.

Ariel: No you're increasing by one it would go negative two, negative one.

R1: This is very interesting, have you seen other tables like this right? What happens here?

Ariel: This is minus eight

R1: This is negative eight, negative five, negative two, one, four, seven. Is this increasing or decreasing?
1432  Ariel  it's
1433  R1  The y value, increasing or decreasing
1434  Ariel  The y value, is increasing
1435  R1  Are increasing right, by how much is each value increasing?
1436  Ariel  Three.
1437  R1  Three, okay and here to go from this from this entry to this entry it increase by how much (pointing down y column).
1438  Ariel  From here to here (pointing from x column to y column) zero minus eight is that, one minus..
1439  Ariel  I got it already it's going to go eight, six, four.
1440  R1  Why?
1441  Ariel  Cuz zero minus eight is negative eight, one minus six is negative five, and two minus four is negative two, so it's gonna go eight, six, four then it would go hold on, eight six four, two, three, so it comes up to three, what? Cuz here it's subtracted two and it came out to this, here they do nothing it comes out to same, and here they added two it came out to seven. So it would be..
1442  R1  So you add, subtract
1443  Ariel  You subtract, till it be the same then you add two. No but wait if you follow the rule, you see that the next thing will be minus two six minus two will be four. Seven you would leave the same as seven.
1444  R1  Why? So this values are increasing, decreasing or staying the same (pointing to x column)
1445  Ariel  So it's subtracts, stays the same and adds
1446  R1  When do I know that I have to subtract and when I have to add.
1447  Ariel  Well you'll see depending on the, you know you're subtracting when the number comes out smaller, and you see the pattern it subtracts, it stays the same, adds two. So then you'd go – subtracts, stays the same, add two. Subtract, stays the same, add two. Subtract, stays the same, add two.
1448  R1  Negative eight, from negative eight, the y goes from negative eight to negative five. As x increases the negative eight goes to negative five. Negative
five goes to negative two. Negative two goes to one.

Ariel: It's adding three.
R1: Adding three right,
Ariel: Adding three, adding three, adding three, adding three, so this should be like ten this should be thirteen like I wrote, and this should be sixteen like I wrote.

Ariel: Ok eight times two is sixteen. Seven plus six is..
R1: Where is the eight, where is the rule?
Ariel: I see the y value is increasing, by three.
R1: Increasing by three, so can you find any rule such that I just give you a value for x and you say y value, you say the corresponding value for y.

Ariel: Inaudible
R1: Yeah like that something like twenty two, can you come up with a corresponding value for twenty two, (inaudible) without completing the table.
Ariel: Haa no that's hard. Well umm well, oh hold up twenty two is increasing by three...it might be forty two. Might be forty two, let me complete the table. (works on table) no that's wrong, cuz if two is negative two.. (inaudible)

Ariel: You're finding the value for what?
Ariel: Twenty two, its fifty eight.
R1: Fifty eight? Yeah that's true, fifty eight

R1: What if I say thirty? Thirty, write thirty. Thirty, and I say that the y value is a eighty-two. Do you think I'm right?
Ariel: Eighty-two, I think...
R1: Now for twenty-six, how much would that be?
Ariel: Twenty-six would be seventy.

R1: Twenty-six would be how much? It’s right here (pointing to paper) Okay for twenty-seven, for twenty-eight, for twenty-nine, for thirty, thirty-two (Ariel is counting along). How could I predict that without doing all this process?
Ariel: Umm... I have no idea...
R1: If x is zero you get negative eight
Ariel: What I noticed was half of twenty, to get twenty-two it was fifty-eight half of twenty-two is eleven. Oh yeah, what’s twenty-five. And twenty-five times three is fifty.
R1: Twenty Five
Ariel: Yeah, fifty times three.
R1: Twenty-five times three, how much is twenty-five times three?
Ariel: Seventy-five, I mean seventy-five. So (inaudible)
R1: Twenty-five times what?
Ariel: Two, twenty-five times two equals fifty. Cuz eleven times two is twenty-two is fifty and then if you add eight you get that. And thirty-eight, half of thirty-eight is, I mean thirty is fifteen and fifteen and thirty-seven. Thirty-seven times two is... Ok... Seventy-four plus eight. I think times three is eight. I mean times two plus eight.
R1: What times two plus eight?
Ariel: Umm.. like
R1: What number was it?
Ariel: I did for the twenty-two and for the thirty.
R1: Okay, for thirty you found it was what times two?
Ariel: Fifteen times two.
R1: Fifteen?
Ariel: No, thirty-seven times two.
R1: For thirty?
Ariel: Yeah for thirty. Cause half of thirty is fifteen and what I got for fifteen was thirty-seven. And thirty-seven times two...
R1: Why do you take half of thirty?
Ariel: Cause that times two gets you thirty.
R1: Ah, because you want to relate it to those numbers. So, half of fifteen, half of thirty is fifteen so it’s thirty-seven.
Ariel: So two thirty-seven’s will be seventy-four plus eight is eighty-two.
R1: Thirty-seven plus?
Ariel: Eight, I mean, thirty-seven times, I mean, plus thirty-seven is seventy-four plus eight is eighty-two. And for twenty-two...
R1: So in that case to get thirty, we would do two times thirty-seven plus eight
Multiply the y value by two and then add eight.

Ok let’s see here. Zero. What is the corresponding value for zero?

Negative eight

Negative eight. X is going to increase by one. When x is one what is the corresponding value?

Umm, negative five

Negative five, more or less than the amount that corresponded to zero?

More

How much more?

Negative eight plus three

So, it means that it is increasing right? So to zero, it’s negative eight. To one, it’s negative five. So it means three more, right? To two is how much?

Negative two.

How much more than the number corresponding to zero?

Negative five plus…

No here, to two is negative two.

It’s six more.

Six more. To this one it was how much more?

Three.

To this number is?

Six.

Six more. To three it will be?

Nine, wait I thought you meant for this. So for three it will be minus two.

So for that how much more is that than the amount corresponding to zero?

Nine.

Nine, so do you see any pattern?

Yea, three, six, nine, so then the next one will be twelve. Because four. Yeah twelve.

Twelve? So for four how many times… how much did it increase?

It increased by twelve because negative eight plus twelve is four.

So four increased by twelve right? This increased by twelve. When it was four it increased by twelve?

No

No? Oh yeah but if you okay if we refer to zero. We see when we watch one it increased by three, when you watch two it increased by six, when you watch…
compare it to zero. When you watch three, it increase (inaudible) when you watch four it increased to twelve. So how can you get to twelve using four as a factor?

1524 Ariel Four times three, times three plus eight.
1525 R1 Times three plus eight. Write it down. Okay, so for zero that would be what? Times three, that would be?
1526 Ariel Zero times three… zero plus eight (following along with what R1 is saying)
1527 R1 Zero plus eight, that would be? Eight not negative eight. So is it plus eight?
1528 Ariel No minus eight.
1529 R1 Write it minus eight. Okay so try it again.
1530 Ariel So it’s times three minus negative eight
1531 R1 Minus eight or minus negative eight
1532 Ariel Minus eight.
1533 R1 What about one?
1534 Ariel One times three is three minus eight is negative five
1535 R1 What about the next one?
1536 Ariel Two times three is six minus eight is negative two.
1537 R1 How do you get y?
1538 Ariel How do you get y? Multiplying by three and subtracting eight.
1539 R1 So y equals y equals…
1540 Ariel Y equals times three minus eight
1541 R1 Is that the rule? Can you make sure that this is the rule
1542 Ariel Cause, five times three is fifteen minus eight is seven. Six times three is eighteen minus eight is ten. Seven times three is twenty-one minus eight is thirteen.
1543 R1 Yeah, you got it.
1544 Clip 6
28-29
1545 00:00 R1 Now according to what you did here, is this correct?
1546 Ariel This is wrong, this is all wrong, this is hmm.. Okay, this is negative plus minus zero times… No, no zero times one equals zero minus one wait, cause one times one equals one, two times. One times zero plus, no, one times zero minus one… No, no, no, no, its not minus, cause, wait a minute… cause two times, two times zero plus one equals three.
1547 R1 Two times zero?
1548 Ariel Plus one equals three.
Plus one? Two times zero, how much is two times zero?

Two.

Two times zero?

Oh, zero. Plus three. Oh yeah, yeah I got it. You subtract one from the x, you subtract one from the x because zero minus one equals negative one. Wait a minute. One minus one is one, this is adding. But, but I mean for this like here it was, if you subtract three from two you get one. You subtract one from one zero, subtract negative one from zero it’s zero. Subtract five from three, I mean, subtract three from five it’s two, subtract four from seven is three subtract five from nine it’s four and it keeps going and going. It’s seven, six, five, four…. It’s higher, one higher, one higher, one higher…

So what do I subtract?

You subtract, you subtract wait a minute, you subtract to know like how much your adding from the x from the y. Subtract the x from the y.

So if you say that I subtract the x value from the y, in that case I would have to give you two numbers already. The idea is that if I give you one number you can get the other number in this row, in this, I mean, this column.

Hold on zero minus…

Look, look at what you did before. If I say two, according to the rule you that you gave me, I say two times three is six minus eight, minus two. Now I say, six times three, eighteen. Minus eight, ten. Do you see that I found the ten? You’re doing a very wonderful job coming with all this numbers, which is true, what you said here is true. Five minus three is two, seven minus four is three, it’s correct. But, how can I find a way to predict the value of y just giving you a value for x. That’s what you have to.

(Ariel is thinking) Oh I think I got it, I got it, I got it, times two minus one.

Write it down. Minus one. Lets’ see if it is true

Zero times two zero minus one is negative one. One times two is two minus one is one. Two times two is four minus one is three. Three times two is six minus one is five. Four times two is eight minus one is seven.
Researcher: So you’re at this point now what does this tell you to do?
Ramses: To go this way
Researcher: y
Ramses: Oh y, oh okay, So it’ll be right there
Researcher: You got it! So You went over one and up three
Ramses: Where will the next one to then 2, 5
Researcher: Ok, Remember, wait, we don’t put a dot until we’re done with the whole thing
Ramses: Y, five times will go right here.
Researcher: You went over 2 first then....
Ramses: So it will go right here.
Researcher: 0 1 2, this is where you started. And now do your fives.
Ramses: One, two, three, four, five. Right here.
Researcher: Got it.
Ramses: So three will go right here
Researcher: Is that three? Where’s zero
Ramses: Zero right here
Researcher: Now count over. No, no, no, zero.
Ramses: Oh. Zero, one, two, three that’s three
Ramses: And seven times one.
Researcher: No, zero.
Ramses: Zero, one, two, three, four, five, six, seven.
Researcher: You got it, now try the next one
Ramses: 4 right here. Zero, one, two, three, four, and y going up 9 times 0 1 2 3 4 5 6 7 8 9
Researcher: Excellent
Ramses: One, two...

2:01
Ramses: Oh wait
Ramses: I mean 0 1 2 3 4 5 going up 11 times so 0 1 2 3 4 5 6 7 8 9 10 11
Researcher: You got it. Ok, what do you notice about those points
Ramses: They’re all going in a diagonally pattern
Researcher: Very good, so another thing the students were doing when they were working on this yesterday. Actually none of them did that but you’re one step ahead like I said. But the other thing they did what kind of pattern or equation or rule could we write that explains the relationship between these numbers. So what can we do to x to get to y as an answer. So we’re looking for a rule that works for every pair, every x,y pair that we have here. So for example I might look at this first pair and say my rule is x+1 = y because 0+1=1 then I need to check and see if it works here. x+1 is what?
Ramses: x+1 is
Researcher: In this case x is 1 so x+1 is going to be 2 not 3, so that rule didn’t work. So we’re trying to find a rule that works for every one of these pairs. So what is it that I can do to x to get y as an answer?
We want that same rule to work for every pair of numbers that we have here. That’s what everybody was working on yesterday. So that’s the next thing I want you to think about.

Ramses

Alright

Researcher

What kind of rule can you think of that works for every pair of numbers there?

Ramses

Alright

Researcher

Ok, want to just think about it for a few minutes?

Ramses

Yes.
Researcher: So, how do I get from point A to point B, and from B to C and C to D. How, how’s the relationship? Because point A represents this ordered set, B represents this ordered set, C like all these points represent A, B, C, D, E, & F and how is there a pattern between x and y as that line goes?

Ramses: Because

Researcher: What would the rule be if I put if I told you, you’re graphing those points if I asked you, if x was ten where would I put y.

Ramses: X + 10, You would um...

Researcher: We can make it smaller or we can arrow over. But on this graph or this graph or lets even start uhh think about ten but even more close to home what about six?

Ramses: Alright. Well if it’s six: zero, one, two, three, four, five, six. Y would start right here and then you would count up. Any number underneath y.

Researcher: So before you knew it because they gave it to you, but they don’t give you that one. What do you think it would be, based on your information here what do you think y would be if x was six.

Ramses: Oh, I see what you’re saying. Y would be thirteen.

Researcher: Why is that?

Ramses: Because all these numbers are odd numbers and these so the next number will be an odd number. Because they keep skipping. 1 then they skip 2, 3 they skip 4, 5 they skip 6, so they'll probably skip 10 and it’ll be 12 no, no it’ll be 13. I got mixed up on that one

Researcher: So that being the case, what happens when I don’t know the number before? So you can tell me that after six it would be seven after thirteen what would it be?

Ramses: After six it would be thirteen.

Researcher: What about for seven, if x was seven?

Ramses: It would be fifteen.

Researcher: Ummhumok, that’s, that’s good that you’re seeing this trend, we could keep graphing points all day. But remember a line, points form lines and they can go forever in either direction. So if I wanted to know another point, for instance, not in line with that, like fifteen. How would I do that without having to count like you’re counting?

Ramses: You could just start, you don’t have to count it you can just start where you left of at. You left of at five start from five and count five, six, seven, eight, nine,
So you’re solution is to start at a larger number?

So if you were to graph it, wait hold on no not yet, we have some graph paper here James, my friend.

Oh yeah yeah yeah, I could graph it. Hold on, wait a minute, yeah yeah yeah I know how to do this. This is one right here. One two three four five six seven eight nine ten eleven. So there’s one two three four five six seven eight nine ten eleven twelve thirteen easy.

I don’t want to do this

You don’t want to graph? Tell me, if I gave you twenty-five for x what would be your y?

Wait a minute, you can’t do that. Oh wait a minute, don’t this got to be zero? This got to be zero right? Don’t that first dot go to be zero?

Which one

This one right here (points pen to first dot on first row above the axis), this the zero then the the one, then the two, you know?

You mean to say this one?

Yeah

What’s this?

Cause zero five, there ain’t no zero line.

Okay well what is this point?

This

Yes

Nah, I did this point up here. Zero five. But I think shouldn’t this be like zero

Where is zero five, can you show me?

Right here (points to point that he marked (0,5) on his page).

And what is this point?

That’s one

And you’re saying this is zero (pointing to the point Ariel has marked)?

Yeah

Now what is this?

Five. That’s zero x zero y.

That is x zero.

Exactly

Can you point down there?

So I think I should do it like this, zero one two thee four.
So you mean to say that this is zero?

R2: You mean to say this is zero?
Ariel: Yeah. That’s how we do it in class.
R2: Okay Ariel, can you tell me what is this then? If this is zero what is this?
Ariel: Zero.
R2: You mean to say this is zero?
Ariel: No, I mean this point.
Ariel: Which one?
R2: This line, the line that are points
Ariel: Oh, I don’t know. I think that it might be zero.

Clip 10

You guys take about five minutes to write down what you found the other day- yesterday, on your transparencies and then we’ll get together up here and share our rules, and also our ways for finding rules together.

Brandon: This is the problem number one, and this is the rule. “The rule is that the x axis goes up by one and the y axis goes up by two.
R2: What do you mean by x axis?
Brandon: This side, the x side.
R2: This is the x side? Okay, and what do you mean by this y axis?
Brandon: It’s the y side.
R2: Okay, do you know how to graph this valueinaudible.
Brandon: Written Graph Below
two five, three seven, four nine, five eleven. So Brandon and Yonnie, wanna tell us what you did?
Brandon: Alright, the x axis goes up by one, for example zero, one, two, three, four, five, y’all get it now? Y’all get it?
R1: Tell us about the y axis
Brandon: About the y axis? It goes up by two.
Ariel: Just say times two plus one.
Brandon: By two, by one, three, five, seven, nine, eleven. Do y’all get it?
James: Yeah
Brandon: No you don’t.
R1: Is that the rule you guys have stated there?
Brandon: Yes.
R2: Do you guys agree about that? Anybody with any questions?
Brandon said it was wrong. Then we did graph and it was true.

See I told you I was right.

No I did it wrong because it’s upside down.

So can you explain what you got in the graph there?

We got...I did do it wrong I think

Brandon said it was wrong. Then we did graph and it was true.

See I told you I was right.

No I did it wrong because it’s upside down.

So can you explain what you got in the graph there?

We got...I did do it wrong I think

Brandon said it was wrong. Then we did graph and it was true.

See I told you I was right.

No I did it wrong because it’s upside down.

So can you explain what you got in the graph there?

We got...I did do it wrong I think

Brandon said it was wrong. Then we did graph and it was true.

See I told you I was right.

No I did it wrong because it’s upside down.

So can you explain what you got in the graph there?

We got...I did do it wrong I think

Brandon said it was wrong. Then we did graph and it was true.

See I told you I was right.

No I did it wrong because it’s upside down.

So can you explain what you got in the graph there?

We got...I did do it wrong I think

Brandon said it was wrong. Then we did graph and it was true.

See I told you I was right.

No I did it wrong because it’s upside down.

So can you explain what you got in the graph there?

We got...I did do it wrong I think

Brandon said it was wrong. Then we did graph and it was true.

See I told you I was right.

No I did it wrong because it’s upside down.

So can you explain what you got in the graph there?

We got...I did do it wrong I think

Brandon said it was wrong. Then we did graph and it was true.

See I told you I was right.

No I did it wrong because it’s upside down.

So can you explain what you got in the graph there?

We got...I did do it wrong I think
your pen to go over the dot so I know which one you’re talking about. So what do you guys think about Brandon and Yonnie’s rule? It describes what’s happening in the table there, but can you describe it differently though?

Ariel: They could have just said times two plus one.

R1: Guess his rule yall

Ariel: Which rule is this?

R1: Before we do that, I’d be happy to do that. But will you explain to us what was different about your rule with number 1?

Ariel: I ain’t do no rule, did number three.

R1: You did number three? Why were you interrupting Brandon’s?

Ariel: Because that’s how I did number one.

R1: Okay, so lets...(cuts out and returns to both boys at projector)

Ariel: (writing out equation 5x2=10+1=11) One equals eleven. There you go. Five times two plus one equals this, eleven.

R1: Does that would for all of those?

Ariel: Yeah, zero times two is zero, plus one is one. One times two is two plus one is three. Two times two is four plus one is five. Three times two is six plus one is 7. And four times two is 8 plus one is nine.

R1: Okay, how could we write that rule in general? How would you write that out?

Ariel: Oo. There you go, times two plus one.

R1: We are taking there about x and y

Brandon: That’s that rule

Ariel: We are taking there about x and y

Brandon: I made that too big

R1: Cuts to another clip of Ariel writing on board.

R1: What do you guys think of the rule they got there?

Class: I think it’s right

R1: How is that different for what you did Brandon?

Brandon: The reason why its different is because they multiplied instead of adding what we did. It’s the
same thing, but they just multiplied it like its two
times...excuse me. It’s two times zero, which equals
zero of course everybody should know. And then add
one.

1717 2:03 R1 Alright, I need everybody to give Ariel and James
your attention because they have a new guess my rule
they would like you to try. Alright, everybody look
up front there. You might even want to grab a scrap
sheet of paper on your table and copy this down.

1718 Brandon I already got it.
1719 James What is it?
1720 Brandon Times two plus three
1721 James Wrong
1722 Brandon But you could do it! Times two plus three
1723 James Zero times two is zero, plus three that’s three.
(Repeats himself)
1724 Brandon I said plus one.
1725 Ariel Can I tell them? Hold on give them a chance, before
they give up, cause they will.
1726 R1 Our goal is to come up with some sort of a rule, a
relationship between the x and the y. So what
Brandon showed us was a great example of rule. X
times two plus one. But that does not work for this set
of data. What I’d like for everybody to spend the next
few minutes doing excuse me guys.

1727 Ariel Can I give them a clue?
1728 R1 No, no clues. This is a guess my rule that everyone
has to work on next. I would like you to come up with
a rule for this, go ahead and draw yourself a graph for
it if you’d like.
1729 Brandon I think I got it
1730 R1 This is a rule I want you to try, if you think you got it
write it down on a sheet of paper for me and be
prepared to show me that it works. James and Ariel
come here I’ve got another challenge for you.
1731 Ariel Oh, it easy.
1732 James Talking to Brandon about his problem. Zero plus one,
plus three?
1733 Brandon No, not like that. I know how the rules go. Plus one
on the X axis and plus three on the y axis.
1734 Ariel James come on
1735 James to Brandon: no.
1736 Brandon That what it look like
1737 James: No, too bad
1738
1739 Clip
This is a guess my rule that everybody has to work on next. I would like you to come up with a rule for this, go ahead and draw yourself a graph for that even if you’d like.

Brandon: I think I got it

R1: This is a rule I want you to try, if you think you got it write it down on a sheet of paper for me and be prepared to show me that it works. James and Ariel come here I’ve got another challenge for you.

Ariel: Oh, it easy.

James: *Talking to Brandon about his problem.* Zero plus one, plus three?

Brandon: No, not like that. I know how the rules go. Plus one on the X axis and plus three on the y axis.

Ariel: James come on

James: to Brandon: no.

Brandon: That what it look like

James: No, too bad

Brandon: inaudible Plus one on the x axis, plus three on the y axis ain’t that true? Can’t that be true? Yo pay attention to me while I’m speaking to you.

R2: Is it a challenge? Or is it easy to do?

Brandon: James and them cheated. They talk about it not plus one plus three. Look, can’t this be true though Mrs. Patrick. Plus one on the x axis. Plus one on the x axis, plus three on the y axis. Can’t that be true? Tell me.

R2: Is that what you’re noticing? Is that the trend you are noticing?

Brandon: That’s what I’m noticing, but they said that’s not it, so I’m trying to find out what it is.

R2: So you’re noticing it’s going plus one this way (pointing to x axis) on this side is that what you’re saying?

Brandon: Yeah

R2: And what’s going on, on this side?

Brandon: Yeah, plus three.

R2: So what’s the relationship between this and this?

Brandon: What you mean?

R2: Like if you found a relationship going down on both sides, what do you think the relationship between this column and that column? Is there any trend there?

Brandon: So then I got to find for what? The eighteen?

R2: I’m not telling you what to find I’m just asking you since you found a trend this way found a trend the
other way what do you think about that way.

1764 R3 I mean the trends you found are correct is it going down by one, this is going up by three.

1765 R4 Okay I have a question, didn’t that rule help you. Suppose x was six

1766 Brandon Yeah then twenty-seven.

1767 R4 Yes how we get twenty-seven.

1768 Brandon Excuse me?

1769 R4 How we get twenty-seven?

1770 Brandon Because twenty-four plus three equals twenty-seven.

1771 R4 Okay, and if x is hundred.

1772 Brandon I don’t know

1773 R4 So do you think your rule should work for all values of x? you think so?

1774 Brandon uh, excuse me?

1775 R4 I mean if x is six then you say y is twenty-seven right?

1776 Brandon Yeah

1777 R4 Okay and if x is seven?

1778 Brandon It would be thirty

1779 R4 And if x is twenty?

1780 Brandon I don’t know. Can’t go that high! I can’t go that high until I got to work my way up there.

1781 R2 Okay, but I mean Dr. Weber is absolutely correct, you are finding a trend there going down. But do you think that there’s any trends…

1782 R3 It’s not that you’re wrong, it’s just that you’re not really doing what is being asked. Does that make sense?

1783 Brandon mhmhm

1784 R3 We’re not looking for the trends going down like this, or the trends going down like that. What we’re looking for is sort of an equation. If I know what x is, you’ve got to tell me what y is. In and out. Yeah, so if I tell you x is fifteen, how can I find what y is?

1785 Brandon I don’t know

1786 3:36 R3 Maybe we can look at the last rule they found

1787 Brandon It said multiply by two, add by one.

1788 R3 That is what they did last time but would that work here?

1789 Brandon I was gunna say look, they say multiply by two add by one then they only talking bout that one side. I don’t get it

1790 R3 Oh, can we, maybe we can bring up what they had last time. Lets take a look at this. They multiplied by two and add by one that always works right?
Brandon: That’s only for this side though.
R3: No, no no no. What do you think this means? (Points to equation x^2+1) Let’s look at this one. Two, five. They multiply x by two. What’s two times two?
Brandon: Four.
R3: And then they added one, what’s four plus one?
Brandon: Oh right! I get it!
R3: Wait does that work for this one? Three plus three.
Brandon: Uh huh.
R3: Plus one.
Brandon: Yeah, that works.
R3: Five plus five, plus one.
R2: So now if we told you one hundred what would you get? Or if we told you twenty that was the first one.
R3: Yeah what would twenty?
R2: Yeah good.
R3: Two hundred and one. No wait…
R2: For one hundred it would be.
R3: Yeah good.
R2: What about twenty?
Brandon: Ummm, fourty-one.
R3: Good good!
R2: So now you understand that you need to find a rule that works without you having to keep.
R3: Maybe you could try that for this one?
R2: Because yours works if you know the one before. If you don’t know the one before…
Brandon: (doing work on paper)
R3: Thinking about this one now?
Brandon: Mhmm.
R1: How you doing Brandon?
Brandon: Bad.
R1: What’s the problem?
Brandon: I don’t know the problem.
R1: Are you staring at this one up here still?
Brandon: Yeah, trying to find the answer.
R1: Tell me what you notice so far.
Brandon: That you have to multiply by something on the sides. You have to multiply it from the x side and then add it on the rest of them. Something like that.
R1: Okay. Have you tried anything yet?
Brandon: Yeah.
R1: What are you trying?
Brandon: Multiplying by three.
R1: Okay. Did you find something that worked there?
Brandon: Yeah, so far.
R1: What did you do?
Brandon: I did you multiply three by one and add nine. And the three by zero and add nine. No, that doesn’t work.

R1: Which one doesn’t work?

Brandon: The three by zero.

R1: What’s three times zero?

Brandon: Zero and then add nine.

R1: That works.

Brandon: Oh, I’m thinking I had to get twelve. Okay if that works, two by three, six. Six plus nine is wait, yeah, no. Fifteen or fourteen? No, it’s fifteen. And that works. Um, three times three equals nine, equals eighteen. Four times three equals twelve plus nine, yes that works too.

R1: I think you’re on to something here.

Brandon: And then five times three equals fifteen, yup, I found the answer.

R1: Okay so how would you tell me what to do?

Brandon: That you would have to multiply the x side by three and then add on the y side by nine. Do you get it?

R1: No I don’t get it, try it again.

Brandon: Okay, on the x side see how it has zero.

R1: Yes

Brandon: You have to multiply zero by three

R1: Okay, and I get zero

Brandon: Okay, then you have to add nine. See in the y side its nine.

R1: Oh so what I did to the x side I add nine?

Brandon: Yes

R1: Okay, you said add it to the y side before so I got confused. So for example, tell me if I am doing this right. So for five, you say I’d do five times three uh huh

Brandon: And that’s fifteen

R1: Yeah

Brandon: And then do I add nine to fifteen or do I add nine to twenty-four?

R1: Add nine to fifteen

Brandon: Add nine to fifteen, okay. And that equals twenty-four so that works. And you said it worked for all of them?

R1: Yes

Brandon: How would we write that rule, how would we write it as a statement then? You’re gunna write out that rule to tell every body else that your rule works. What would you write?

Brandon: Multiply the x side
Why don’t you do that, actually write it on there.

Brandon WRITES: Multiply the x axis by three and add nine to your answer for the y axis number.

It’s interesting, does this always work? Can you explain to me how that works with like four here?

Four times three equals twelve, plus nine equals twenty-one.

Oh that always works

Yeah, it did when I did it.

That’s really interesting. That’s good. Could you tell me if that if x is ten?

It is ten times three equals thirty, plus nine, thirty-nine.

That’s always works

Yeah, it did when I did it.

That’s really interesting. That’s good. Could you tell me if that if x is ten?

Multiply the x axis by three and add nine. Add nine to your answer for the y axis.

What do you mean by x axis?

Axis. Oh I would need one of those graph papers. Graph sheets, this here it is (gets a graphing paper transparency).

Okay

These, write the dots. I mean the x axis. See the x axis right here?

Okay, and where it that on your table over there? I’m just wondering where you’re looking at the table. And then when I asked you what x axis was you show me your graph. So what does x axis mean in your table? You said multiply the x axis by three. So what are you telling me to multiply by three?

Multiply the numbers that are under x

Okay

Or on the left side.

Okay, and over here they would fall and represent those on this axis is that what you’re telling me?

Yes on the bottom and then up and down is on the y.

This is what I want you thinking about okay? That’s what you’re noticing there, we’re looking for a rule that actually tells me what I can do with this number to get this number as the answer. And that rule has to
work with every pair of numbers here. That's what we're asking you guys to find out. So like on that last one we said \( x \) times two plus one. We can take this number, multiply it by two, add one and that's where that answer came from. That’s what they were saying on this one.

1884 Yonnie Ariel come here.
1885 R1 He's working on a different problem there.
1886 Yonnie He gunna tell me the answer
1887 R1 He’s not going to tell you the answer, he is going to see if you can figure it out. Okay? But this is what was different about this one. You guys are talking about what happens in this direction, which is a good description because that’s important. But what we’re asking you to find is a rule of what tells us what to do with this number in order to get this as the answer. So for this particular one it was \( x \) times two plus one.

1888 Yonnie This times two plus one.
1889 R1 Right this ones times two plus one, right. And it works for all of these.
1890 Yonnie Yeah
1891 R1 So what rule would work for all of these? That you would always do to this \( x \) number to get this as the answer. So that’s what you’re trying to find. Okay? You can talk to Brandon about it up there if you’d like

1892 Yonnie Wait.
1893 R1 I’ll be back to check on you in a little
1894 Yonnie I found it
1895 R1 One times three equals three, where’d twelve come from?
1896 Yonnie It’s times three plus nine, I’m done.
1897 R1 Times three plus nine?
1898 Yonnie Yeah
1899 R1 Okay, prove to me that it always works.
1900 Yonnie Oh god this teachers insane
1901 R1 Insane to know if it always works.
1902 Yonnie \textit{Mumbles work out loud} three times three plus nine
1903 R1 Okay how do we write that rule in then? What times three? Or what is that? Is that plus three? Is that what you were doing here?
1904 Yonnie Oh, there you go I’m done. Ariel I finished your problem.
1905 R1 Alright that’s good. It’s an idea, except you have to write it in another way. What does this mean? When I see something written like that with three different
pairs it tells me that I’m going to graph these sets of numbers. How would you write this as an equation? What are you multiplying by three? What over here and you multiplying?

<table>
<thead>
<tr>
<th>Time</th>
<th>Name</th>
<th>Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>1906</td>
<td>Yonnie</td>
<td>x? Oh wait</td>
</tr>
<tr>
<td>1907</td>
<td>R1</td>
<td>Okay.</td>
</tr>
<tr>
<td>1908</td>
<td>Yonnie</td>
<td>I mean n</td>
</tr>
<tr>
<td>1909</td>
<td>R1</td>
<td>Or in this case what are these numbers (Pointing to the x column)</td>
</tr>
<tr>
<td>1910</td>
<td>Yonnie</td>
<td>x times three plus nine equals y.</td>
</tr>
<tr>
<td>1911</td>
<td>R1</td>
<td>Can you write it for me? I’ve got another one for you to try.</td>
</tr>
<tr>
<td>1912</td>
<td>Yonnie</td>
<td>I’m done!</td>
</tr>
</tbody>
</table>