GAMES AND NETWORK FORMATION

by

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A dissertation submitted to the

School of Graduate Studies

Rutgers, The State University of New Jersey

In partial fulfillment of the requirements

For the degree of

Doctor of Philosophy

Graduate Program in Economics

Written under the direction of

Professor Tomas Sjöström

And approved by

New Brunswick, New Jersey

October, 2017

ABSTRACT OF THE DISSERTATION

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This dissertation consists of three studies in communication and information networks. The first chapter analyzes formation of networks when players choose how much time to invest in other players. I assume the information can be transferred using all possible paths in the network and study the model under two different link strength functions. First, under the assumption that the link strength is the arithmetic mean of agents' investment levels, which allows players to form links unilaterally to other players, every player is connected to another either directly or indirectly with no more than two links under any Nash equilibrium. Moreover, the strict Nash equilibrium structure is a star network. Second,under the assumption that the link strength function is Cobb-Douglas in which players have to have bilateral agreement to form links with each other, I show that paired networks in which players are matched in pairs, are Nash equilibria. Moreover, I consider a sequential game in which players choose and announce their investments publicly according to a random ordering. I show that an Assortative Pair Equilibrium, in which players are assortatively matched in pairs according to their information levels, is the only strongly robust Nash equilibrium.

In the second chapter, I consider the model introduced in the first chapter and fully characterize the Nash equilibria and surplus-maximizing outcomes for a three-player game, in order to investigate how equilibrium structures are different from the efficient outcomes and how these structures differ under different link strength functions. At equilibrium, the agents choose to invest all their time with only one agent regardless of the link strength function. More links are formed when the agents are perfect substitutes compared to Cobb-Douglas link strength, in which bilateral agreement is required for link formation. Moreover, the results show that the agents have a tendency to connect to fewer agents with higher investment levels from an efficiency perspective.

In the third chapter, I investigate a model of communication with two agents and a principal, allowing for asymmetric interdependencies between the agents. Each agent has private information on different dimensions of the state of nature. The interdependencies are characterized as action complementarities or substitutabilities between the agents within the same economic environment. I model the communication as cheap talk messages, assuming the information is not verifiable. I look at two decision mechanisms. First, under the centralized decision mechanism, in which agents communicate vertically with the principal and the principal makes the decisions for the agents after observing the reported private information, the communication takes form of a partition equilibrium. Second, under the decentralized mechanism, the agents communicate horizontally with each other via cheap talk and then make the decision for themselves. Under this protocol, I show that when there are strategic interaction between the agents, there are at most two on-the-equilibrium path conditional expectations for each agent. Thus, centralization allows more informative communication compared to decentralization. Moreover, I show that if the agents are strategic complements, it is not possible to have an informative horizontal communication.

Acknowledgements

I would like to express my sincere gratitude to my advisor, Prof. Tomas Sjöström, without whom this work would be simply impossible. I am profoundly thankful for his compassionate supervision, guidance and inspiration on this thesis, for his encouragement, his endless patience and confidence in me. It was a great pleasure to have him as a mentor and a friend, and a blessing to have the chance to benefit from his knowledge, intelligence, wisdom and sense of humor.

I would also like to express my gratitude to my committee members. I have benefited immensely from many discussions and brainstorming sessions with Prof. Colin Campbell, and the guidance and teaching of Prof. Richard McLean. This dissertation would not have been complete without their invaluable comments and encouragement. Their approach to research and teaching will continue to inspire me. I am also grateful to Prof. Kalyan Chatterjee for his comments on my research and generosity in serving on my dissertation committee.

I would like to thank Dorothy Rinaldi and Linda Zullinger for their guidance, care and patience; and Debra Morgan, Donna Ghilino, Paula Seltzer and Janet Budge, Matthew Steiner for their amazing administrative support and many nice conversations.

I am deeply grateful for all my friends I shared great moments and invaluable experiences throughout my entire graduate life at Rutgers University. I am particularly indebted to Evin Açan Millet, Ebru Işgın and Örsan Aytekin for sharing the joy and the pain, motivating and supporting me day and night, and keeping my spirits up during rough times. I feel blessed to have them in my life. I am also deeply grateful for Özlem Tonguç and Selin Zülfikar who were always there for me even miles apart.

A special thanks goes to my husband, Jonathan Jacob Horowitz, for his love and understanding, bringing joy to my life, keeping me focussed during the good days and the bad, and motivating me when things seemed to be insurmountable. He constantly inspires me with his positivity and optimism. I would like to express my deepest gratitude to Maureen and Georg Horowitz, for welcoming me into their family with open arms and made me feel like I am home. I have been very fortunate to have them around.

I would have never reached at this point in my life without the unconditional love and endless support of my parents, Nevin and Duygu Uysal, my sister, Özge Uysal, and my aunts, Nesrin Karaköse and Nurten Demiray whose never-ending prayers, patience, resilience and strong faith in me keep me going. Words cannot express my love and gratitude towards them. At the end of this journey, I know that they are even more excited and happier than me for my accomplishments.

Dedication

To my family

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1 Introduction

This dissertation introduces three studies on communication and information networks. The main motivation behind this dissertation is a desire to answer questions concerning the kinds of network structures that will emerge at equilibrium, and how these structures are different from the efficient ones. The second chapter studies Nash equilibria of a network formation game with weighted link strengths. The third chapter considers the model introduced in the second chapter, and fully characterizes socially optimum outcomes and Nash equilibria of the three player game. Similar to the third chapter, the fourth chapter studies communication between three players-a principal and two agents when the interests of the agents are not aligned.

Following the seminal papers of Jackson and Wolinsky (1996), and Bala and Goyal (2000), the majority of the literature dealing with models of network formation assumes that the agents make binary decisions; they choose whether or not to link to another agent. However, in a wide variety of situations, such as friendships, sharing of information, and trade of goods and services, agents decide not only whom to connect to but also how much to spend on each connection they make. In the second chapter of this dissertation, I analyze the formation of networks when players choose how much time to invest in other players. As opposed to the distance-based utility weighted link formation game of Bloch and Dutta (2009) in which only the shortest or most reliable path is considered, my model assumes the information can be transferred using all possible paths in the network. I assume that each player has an intrinsic value of information to share and one unit of endowment to invest in relationships with others. Once a direct link is formed, the information is transferred both ways with decay. Moreover, indirect links can transmit indirect information. However, the benefits from indirect information transfers are zero when two agents are connected by more than two links.

I study the model assuming two different link strength functions. First, I assume that link strength is the arithmetic mean of agents' investment levels, that is, the agents are perfect substitutes. As a positive investment of an agent is enough for a link to be formed, this specification allows players to form links unilaterally with other players. Therefore, it is reminiscent of the model by Bala and Goyal (2000). Alternatively, I assume that the link strength function is Cobb-Douglas in which players have to have bilateral agreement to form links with each other, which is similar to Jackson and Wolinsky (1996)'s model.

I show that, when the investments are perfect substitutes, every player is connected to another either directly or indirectly with no more than two links under any Nash equilibrium. Moreover, I find that the strict Nash equilibrium structure is a star network, in which there exists a player (the center) such that all other players are connected to the center. On the other hand, using the Cobb-Douglas link strength function, I show that paired networks in which players are matched in pairs, are Nash equilibria. However, I also consider a sequential game in which players choose and announce their strategies publicly according to a random ordering. I show that an Assortative Pair Equilibrium, in which players are assortatively matched in pairs according to their information levels, is the subgame perfect equilibrium of the sequential game for all possible orderings of the players. Therefore, I conclude that the Assortative Pair Equilibrium is the only strongly robust Nash equilibrium.

Unfortunately, a complete characterization of Nash equilibria and strongly efficient outcomes is difficult in network formation problems. In the third chapter of this dissertation, I consider the model introduced in the second chapter, and fully characterize the Nash equilibria and surplus-maximizing outcomes for a three-player game, in order to investigate how equilibrium structures are different from the efficient outcomes and how these structures differ under different link strength functions. At equilibrium, the agents choose to invest all their time with only one agent regardless of the link strength function. More links are formed when the agents are perfect substitutes compared to Cobb-Douglas link strength, in which bilateral agreement is required for link formation. As opposed to the findings of Bloch and Dutta (2009), the results show that the agents have a tendency to connect to fewer agents with higher investment levels from an efficiency perspective when all paths included in the calculation of the indirect benefits from communication.

The model in the previous two chapters assumes that once the agents are connected, the information is shared. However, if the interests of the agents are not aligned, they may strategically withhold information. In the last chapter of this dissertation, I investigate a model of communication with two agents and a principal. I consider a model by Bora (2010) with two agents and a principal and allow for asymmetric interdependencies between the agents. Each agent has private information on different dimensions of the state of nature. The interdependencies are characterized as action complementarities or substitutabilities between different agents within the same economic environment. A typical example of such an environment would be a multi-product firm. Since most of the information held by different departments within the firm is not verifiable, I model communication between the agents as cheap talk messages. I study the model under two different decision mechanisms. The first one is a centralized decision mechanism in which the headquarters makes the production decisions after observing the reported private information of each department. The second one is the decentralized mechanism in which the agents are allowed to communicate with each other via cheap talk and then make the production decision for their departments. I look at the Perfect Bayesian Equilibrium by Fudenberg and Tirole (1991) as the equilibrium concept and focus on the most informative outcome in case of multiple equilibrium outcomes. I show that, under vertical communication protocol, the communication from the agents takes form of a partition equilibrium, in which the state space is partitioned into intervals and agents report the interval which their private information belong to. On the other hand, under horizontal communication, I show that there are at most two on-the-equilibrium path conditional expectations for each agent when there are strategic interaction between the agents, whereas agents fully reveal their private information when there is no strategic interaction. Moreover, I show that if the agents are strategic complements, it is not possible to have an informative horizontal communication. However, if the agents are strategic substitutes, there are parameter configurations that makes the horizontal communication informative. Under these parameter configurations, in which the cost of production is lower compared to the absolute value of the strategic interaction terms, we have a two-partition equilibrium.

2 A Strategic Model of Network Formation with Endogenous Link Strength

2.1 Introduction

Humans are social creatures; social interactions influence our thinking and behavior. Social networks that we belong to influence our opinions and decisions, determining the products we buy, political candidates for whom we vote, the level of education we obtain, and whether our kids get vaccinations. They also play a central role in the transmission of information such as job opportunities and availability of new technologies. To build better models of human behavior, economists cannot ignore the role of social and economic networks. Hence, following in the long tradition of sociology literature, research on networks in economics has grown rapidly over the last two decades.

The structure of a network, i.e., existence of key players, whether everybody is connected or whether groups are segregated, affects the diffusion of information; thus, it's important to gain insight into the network structures likely to emerge and how these structures are related to the ones optimal for a society. Following the seminal papers of Jackson and Wolinsky (1996), and Bala and Goyal (2000), the majority of the literature dealing with models of network formation assumes that the agents make binary decisions; they choose whether or not to link to another agent. However, in a wide variety of situations, such as friendships, sharing of information, and trade of goods and services, agents decide not only whom to connect to but also how much time to spend on each connection they make. Even though treating links as binary quantities helps to overcome computational difficulties, this simplification doesn't allow for analysis of many applications with differing link intensity.

The first work to point out the importance of allowing for richer environments for links strengths in network analysis is that of Granovetter (1973). Granovetter interviewed a hundred people and sent out two hundred questionnaires in the Boston area in the late 1960s to analyze how people find their jobs. The results show that more than half of the people found their jobs through personal contacts. However, the surprising result of the study is that, in most cases, the information on job opportunities came from the people who are not close to the job seeker. Granovetter argues that weak ties play a key role in information transmission within networks because such ties connect people who are dissimilar, and therefore, have nonoverlapping groups of friends.

In this paper, we analyze the formation of networks when players choose how much time to invest in other players. Our analysis is centered around information and friendship networks in which players invest in relationships to exchange information or favours. However, our model is applicable to any situation where players exchange divisible goods. Specifically, we analyze a network formation game in which each player has an intrinsic value of information to share and one unit of endowment to invest in relationships with others. The link strength is a function of investments of the players involved in the link. Once a direct link is formed, the information is transferred both ways with decay. Therefore, an agent's investment decision about a link not only affects his direct benefit from the relationship but also that of the other agent involved in the link. Moreover, we assume that there are benefits from indirect communication.

The extent to which link externalities are accounted for has been limited in the previous literature following the seminal papers by Jackson and Wolinsky (1996), and Bala and Goyal (2000). While calculating the indirect benefits, models using distance-based utility, such as Jackson and Wolinsky's connections model and Bala and Goyal's two-way benefit model, only consider the shortest path between the agents. There is no value added in having multiple paths between agents. However, in wide variety of situations, when the good exchanged in the network is divisible, like information, the good is transferred by using all the paths between the agents. Therefore, our model assumes the information can be transferred using all possible paths in the network. However, in order to add tractability to our analysis, following Brueckner (2006), we assume that the benefits from indirect information transfers are zero when two agents are connected by more than two links. That is, when two agents, i and j, are directly connected, agent i can only obtain indirect information of agent j's direct links via agent j.

Heterogeneity between agents are allowed in terms of their intrinsic value. We assume that individuals are ranked according to their intrinsic value of information. Therefore, the value of a direct link for agent i with agent j depends on three variables: intrinsic value of agent j, the value of the other agents whom agent j is directly connected, and the level of investment of agent j.

One of the challenges of modeling network formation with endogenous link strengths is transforming investments into the link strengths. We study the model under two different link strength functions to model different situations. First, we assume that link strength is the arithmetic mean of agents' investment levels, that is, the agents are perfect substitutes. As a positive investment of an agent is enough for a link to be formed, this specification allows players to form links unilaterally with other players. Therefore, it is reminiscent of the model by Bala and Goyal (2000). Alternatively, in the second case, we assume that the link strength function is Cobb-Douglas in which players have to have bilateral agreement to form links with each other, which is similar to Jackson and Wolinsky's (1996) model.

In order to motivate two different types of link strength functions, we introduce several applications of decentralized information and innovation diffusion systems. The main distinction between the different specifications of the link strength function is the availability of the information and the element of consent to obtain intrinsic information. Additively separable link strength function is more appropriate for the situations when intrinsic information of an agent is available publicly; however, other agents need to invest time in learn about this information. Whereas, Cobb-Douglas is applicable to the situations in which both agents are required to invest in a relationship in order to exchange information. Even though many applications have elements of both, we provide some applications that correspond closely to one compared to the other.

Legitech is one of the examples of a decentralized system of innovation diffusion discussed in E. M. Rogers (1983). Legitech is a computer conferencing system used for exchanging information among legislative staffs of various states. It works like an internet forum: a legislator who is seeking advice can send out a general inquiry on the topic to solicit suggestions over the Legitech computer network to find how other states have responded to this problem. Legislators in other states can respond to the inquiry. The responses can be a specific technical solution or a reference to resources that can supply an answer, such as a reference to a bill originated by a legislator in another state. Other members of the network also have access to answers to others' requests. In terms of the network structure observed, Rogers (1983) notes that certain information sources in Legitech have gained reputation and respect of others on the system for their careful and competent responses to inquiries. Thus, legislators are more likely to follow their advice. In this system, the legislator who is seeking advice mainly bears the cost of obtaining information, since the cost of posting a response, especially the ones consisting of a reference to a source, is low; while the legislator who is seeking advice have to invest in studying the posted reference. Therefore, additively separable link strength function is more appropriate in characterizing this system.

Another application of additively separable link strength function presented in Rogers (1983) is the use of models and on-the-spot conferences in communities for diffusion of innovation regarding health, family planning, and industrial development. A model is defined as a local unit that pioneers in inventing and developing an innovation, in evaluating its results, and in serving as an example for the diffusion of the innovation to other units. The models often have exemplary characteristics of success so that many people want to learn about their techniques. In order to facilitate information transmission, on-the-spot conferences are held at the site of a model. During these meetings, participants observe the innovation in use by a local unit and are able to ask questions about the implementation of the innovation and its effectiveness. The participants, then, decide whether or not to adopt the innovation, and, if they decide to adopt, how to incorporate it into their particular local conditions. Once a participant decides to adopt a technology, the participant not only uses the knowledge of the model but also contributes to it with his experience of the technology. Thus, the flow of benefits is two-way even though only the participant invests in the relationship. Rogers (1983) states that the innovation demonstrated at the exemplary model need not be copied exactly. It is observed that often a great degree of variety can be observed in the forms of an innovation that are actually implemented by local units.

Rogers' (1983) conclusions are also in accordance with the findings of Conley and Udry (2010). They investigate the role of social learning in the diffusion of a new agricultural technology in Ghana using data on farmers' communication patterns. They find that farmers align their inputs according to the information received from the neighbors who were surprisingly successful in previous periods.

In these three applications, the general theme is the emergence of pioneer agents, such as the top contributors in Legitech and the most successful farmers in Ghana. Because of their prestige and success, these pioneer agents gain popularity and influence others' innovation decisions. We show that strict Nash equilibrium of the model using additively separable function is a star network in which there exists a center player such that all other players are connected to. Hence, our results are in accordance with the emergence of the pioneer agents.

Lastly, our motivation for Cobb-Douglas link strength function comes from the book by Blau (1963). Blau (1963) studies consultation patterns among agents working in a federal law enforcement agency. In this law enforcement agency, agents are responsible for the inspection of business establishments and preparing reports on the firms' compliance with the law. As the tasks include complex legal regulations and the reports might lead to legal action against the firms, the agents often need consultation with the other agents. Blau (1963) points out that a consultation is an exchange of values in which both participants gains something by paying a price. The agent seeking advice is able to perform better than he could without receiving any help. By asking for an advice, however, he not only has to spend time explaining his problem to his colleague but also implicitly acknowledge his incompetency to solve a problem. On the other hand, while the consultant has to devote his time to the consultation and disrupt his own work, he gains prestige in return. The final pattern of this social structure is different than what he expects to get: instead of asking advice from a highly competent agent, agents establish partnerships of mutual consultation and less competent agents tend to pair off as partners[1]. In our paper, using the Cobb-Douglas link strength function, we show that paired networks in which players are matched in pairs, are Nash equilibria. However, when we consider a sequential game in which players choose and announce their strategies publicly according to a random ordering,

[1] Latest civil service rating by supervisor is used for competency measure.

we find that an Assortative Pair Equilibrium, in which players are assortatively matched in pairs according to their information levels, is the only subgame perfect equilibrium of the sequential game for all possible orderings of the players. Therefore, we conclude that the Assortative Pair Equilibrium is the only strongly robust Nash equilibrium. Hence, our results are analogous of Blau (1963)'s final pattern, in which the agents are matched in pairs according to their competency levels.

The rest of the chapter proceeds as follows. The next section, Section 2, presents relevant economics literature on network formation games. Section 3 formally introduces the general framework and proves the existence of Nash equilibrium for any continuous, non-decreasing concave link strength function. Section 4 analyzes the case with additively separable link strength function, while Section 5 considers the case with Cobb-Douglas link strength function. Finally, Section 6 concludes.

2.2 Relevant Literature

We start this section with the literature on network formation games with binary link strengths, before introducing the works on weighted link formation games. The literature on network formation games starts with a simultaneous-move game by Myerson (1977), introduced in the context of the formation of communication graphs. In this model, each player simultaneously announces the set of players with whom he would like to be linked. The links are formed if both players involved in the link named each other. However, the main weakness of this model is that it has too many Nash equilibria, including complete network, in which every player is directly linked to other players, and empty network, in which no links are formed. Therefore, this analysis fails to capture the idea that two individuals should communicate and engage in relationship if it is in their mutual benefit.

Later, Aumann and Myerson (1988) propose an extensive form game based on an ordering over all possible links. This model provides the pairs of individuals with the opportunity to communicate and reconsider their decision if the link is not formed in the previous rounds. When a link appears in the ordering, the pair of players involving that link decide on whether or not to form that link knowing the decisions of all pairs coming before them. A decision to form a link is binding and cannot be undone. The game moves through all links initially. If at least one link is formed during the first round, then it starts from the same ordering of links again; however, it moves through only the links that are not formed. Therefore, if a pair of players (i, j) decide not to form a link initially, but some other pair coming after them forms a link, then the pair (i, j) is allowed to reconsider their decision. The game continues until either all links are formed, or there is a round that no new links have formed even though the links that haven't been formed have been reconsidered. Since this is a finite game with perfect information, it always has a subgame perfect equilibrium. However, even in simple settings, it can be very difficult to solve the game by using backward induction. Another shortcoming is that the ordering of the links can have a serious impact on the network structures at the equilibrium.

In Myerson (1977) and Aumann and Myerson (1988), the standard game-theoretic analysis fails to account for the communication and coordination between the agents properly, and provide an insight on why and how the structures at the equilibrium are formed. In order to overcome these issues, Jackson and Wolinsky (1996) introduce a new concept of stability: pairwise stability. A network is defined to be pairwise stable if no pairs of unlinked players both want to form a link, and no player wants to break off a link. This notion of stability is based upon the idea that two players should be able to form a link if it is mutually beneficial to do so. Therefore, formation of a link should involve mutual consent of the individuals to be linked. Jackson and Wolinsky (1996), then, use this new concept of equilibrium to solve two models, connections and co-author models, introduced in the same paper.

The connections model is a simple model of social connections in which links represent social relationships. These relationships offer benefits such as information and favors. However, it is costly to form links. Players directly communicate with those to whom they are linked. There are also benefits from indirect communication from those to whom their adjacent links are connected. The value of the benefit from the indirect links decays with the distance of the relationship. When calculating the indirect benefits, if there are multiple paths between the players, only the shortest path is considered. Jackson and Wolinsky (1996) analyze the model under the assumption that the agents are symmetric, i.e., the cost and the value of a link is the same for all players. They find that a pairwise stable network has at most one non-empty component. They show that a complete network is the unique pairwise stable network in the low cost range; a star, in which there exists a player (the center) such that all other players are connected to, is pairwise stable but not necessarily the unique one in the medium cost range; and each player has either no links or at least two links in any pairwise stable network in the high cost range. Moreover, they characterize strongly efficient networks, the network structures that maximize the total utility of the agents. They show that the unique strongly efficient network is the complete network in the low cost range, a star in the medium cost range, and empty network in the high cost range. In addition, they show the star network is efficient but not pairwise stable for a wide range of parameters[2].

[2] Pairwise stability allows only requires only pairwise incentive compatibility. Therefore, while analyzing for pairwise stability, deviations on a single link is considered at a time. However, there may be situations that coalitions larger than pairs can be formed. There are stronger notions of stability allowing for larger coalitions. For example, Dutta and Mutuswami (1997) study strong stability (coalition-proof Nash equilibrium) with a generalized version of Myerson (1977)'s model to see whether the tension between stability

Furthermore, in the co-author model, players are interpreted as researchers who spend time writing papers. Each link represents a collaboration between a pair of researchers. Each collaboration creates a synergy depending on the time how much they spend together. Moreover, since each player has a fixed amount of time spend on research, the amount of time a player spends on a collaboration decreases with the number of links that the player has. Hence, contrary to connections model in which the indirect communication has benefits, in the co-author model, indirect connections create distractions and result in negative externalities. Jackson and Wolinsky (1996) show that the network consisting of pairs are the strongly efficient networks; and pairwise stable networks can be partitioned into fully intraconnected components, and tend to be over-connected from an efficiency perspective. They conclude that the tension between stability and efficiency arises because the players do not account for the indirect negative effects that their connections bring to their neighbors.

Jackson and Wolinsky (1996) assume that a formation of a link between two agents require mutual consent of the agents. By contrast, Bala and Goyal (2000) weaken this assumption and allow agents to form links with others unilaterally by incurring the cost of the link. This modification in the modeling allows the authors to be able to use Nash equilibrium and its refinements in their analysis. They study both one-way and two-way flow of benefits. In the model with one-way flow, only the player who forms the link benefits from it, while in two-way flow, once the link is formed, both players enjoy the benefits. Moreover, similar to Jackson and Wolinsky (1996), there are also benefits from indirect communication from those to whom their adjacent links are connected. In their benchmark model, the indirect communication is assumed to be frictionless, i.e., without decay. They

and efficiency can be resolved. Similar to Jackson and Wolinksy (1996)'s result that, due to symmetry, the tension between efficiency and stability still remains even if stronger notions of stability is applied, Dutta and Mutuswami (1997) conclude that the conflict between efficiency, stability and symmetry remains.

also analyze the model with decay, in which the value of the benefit from the indirect links decays with the distance of the relationship, and in case of multiple paths, only the shortest path is considered. Moreover, like Jackson and Wolinsky (1996), Bala and Goyal (2000) analyze the model under the assumption that the agents are symmetric.

When there is no decay, Bala and Goyal (2000) show that Nash equilibrium is either the empty network or connected, that is, there exists a path between every pair of players. In the one-way flow model, strict Nash equilibrium structures are the empty network and the wheel network, in which a single directed cycle is formed with each player investing in exactly one link. Moreover, for a large set of parameters, the wheel is also the unique efficient architecture. Whereas, in the two-way flow model, strict Nash equilibrium structures are the empty network and the center-sponsored star network, in which one agent forms all the links. Furthermore, a star is also an efficient network for a class of payoff functions. Where there is decay, strict Nash networks are also connected. However, characterization of the strict Nash and efficient networks becomes difficult as the distances between the agents become relevant in computation of the indirect benefit. By focusing on low levels of decay, they obtain partial results. In one-way flow, the wheel and the star networks are strict Nash; while, in two-way flow model, the star is the unique efficient network and also a strict Nash equilibrium for a wide range of parameters.

There are three papers that drop the assumption of binary link strengths, and therefore, are closely related to our model. In the first one, Bloch and Dutta (2009) analyze a weighted link formation game in which players have fixed endowments to invest in relationships with others. In the baseline model, which is similar to our model with additively separable link strength function, they assume link strength is an additively separable and convex function of individual investments. However, unlike our model, agents use only the path which maximizes the product of link strengths, i.e., the most reliable path. They show that both the stable and efficient network architectures are stars. Moreover, they study the case were the agents' investments are perfect complements. Nonetheless, they could only provide a partial characterization of the stable and efficient structures as the analysis becomes intractable once the indirect benefits are taken into account.

On the other hand, Rogers (2006) considers a weighted link formation game in which all paths between agents are taken into account when calculating the indirect benefits. In this model, each player has an intrinsic positive base utility that would be his payoff in the absence of any network connections and an amount of time to allocate in forming relationships. Players are allowed to be heterogenous with respect to their intrinsic values and their time constraints. In addition to their intrinsic utility, players benefit from other players by interacting with each other. The more time a player spends on a player with higher intrinsic utility, the higher utility he obtains. Specifically, in Rogers' model, the benefit of forming a link is the product of the total value of the other agent and the strength of the relevant link. The total value of each agent is the sum of the benefits from all connections to other agents plus the agent's intrinsic utility. However, once a pair of agents are connected, they obtain each other's intrinsic utility with a weight of the link between them. Thus, this leads to multiple counting of an agent's intrinsic value in his total value. Apart from this, calculating the total value of agents in this way allows Rogers to separate the flow of benefits into "taking" and 'giving" components. That is, in giving model, the link decisions represent the giving of benefits, whereas, in taking model, the link decisions represent the taking of benefits. He finds that with exception of some Nash equilibria in giving model, all stable and efficient networks are identified as interior. Since heterogeneity between agents are allowed, both equilibrium and efficient networks display heterogeneity in link strengths. Moreover, by separating the flow of benefits, Rogers is able to provide new insights with respect to the efficiency of the stable network architectures. Particularly, the source of inefficiency is identified as the giving incentives and the inefficiency is present only when there exists heterogeneity among agents in terms of their budget constraints.

Lastly, another work analyzing heterogenous agents is by Brueckner (2006). Brueckner considers friendship networks concentrating on three player networks. He adopts a stochastic approach to link formation with the probability depending on the noncooperative investment. Even though this alternative approach leads to a simpler mathematical structure, the network architecture at equilibrium cannot be specified. Instead, his analysis focuses on the links which are most likely to form. Moreover, opposed to previous literature, he assumes that benefits are zero when more than two links are involved. Therefore, if agents i and j are connected, then agent i gains from socializing with j's direct friends but receives no benefits from j's indirect friendships. Brueckner shows that individual investment in friendship formation is too low. Moreover, in an asymmetric setting, friendship links involving attractive agent, who has personal magnetism or a broad group of acquaintances, are most likely to form.

2.3 The General Model of Link Formation

In this section, we present the general model of link formation game and prove the existence of the equilibrium before imposing further restrictions to the model. We also introduce the notation and definitions to be used throughout the rest of the paper.

Let $N = \{1, 2, 3..., n\}$ be the set of players. Each player j has information worth r_j and 1 unit of time to allocate across links to others. A strategy for player j's will be denoted

z_j. It consists of his investment levels in the other players:

$$z_j = \{z_{jk}\}_{k \neq j}$$

and must satisfy

$$0 \leq z_{jk} \leq 1$$

for all $k \neq j$ and

$$\sum_{k \neq j} z_{jk} = 1.$$

Let Z_j denote player j's strategy set. A strategy profile consists of a strategy for each player. A strategy profile will be written

$$z = (z_1, z_2, ..., z_n) \in Z \equiv Z_1 \times Z_2 \times ... \times Z_n$$

Let $z_{-j} = (z_1, z_2, ..., z_{j-1}, z_{j+1}, ..., z_n)$ denote the strategies of all the players except j.

Each player can benefit not only from his own information, but also from the information of other players if he is linked to them. The stronger the link is, the greater the share of information is transmitted. However, when information is transmitted along a link, it depreciates by some factor $0 < \beta \leq 1$. Thus, if the direct link between player i and player j has strength $\sigma(z_{ij}, z_{ji})$, then obtaining player j's information via this direct link is worth $\beta\sigma(z_{ij}, z_{ji})r_j$ to player i. Moreover, we assume that links can transmit indirect information. Then, if player j is linked to player k, then player k's information can be indirectly transmitted from k to i via j. Obtaining player k's information via this indirect link is worth $\beta\sigma(z_{ij}, z_{ji})\beta\sigma(z_{jk}, z_{kj})r_k$ to player i. However, following Brueckner (2006), our framework assumes that the benefits from indirect information transfers are zero when more than two links are involved.

We assume that the general link strength function between players i and j

$$\sigma(z_{ij}, z_{ji}) = \sigma_{ij}$$

is continuous, non-decreasing and concave in z_{ij} and z_{ji}. Moreover, we suppose $\sigma(0,0) = 0$ and $\sigma(1,1) = 1$ so that $\sigma_{ij} \in [0,1]$. In the following sections, we will work on with specific link strength functions.

Let $S_j(z)$ denote player j's payoff from strategy profile z. Since we assume information can be transmitted by a chain of no more than two links, then the total amount of information that player j receives from player i, directly and indirectly, is

$$S_{ij}(z) = \left(\beta\sigma_{ij} + \beta^2 \sum_{k \neq i,j} \sigma_{ik}\sigma_{kj} \right) r_i$$

For now, we will assume that player j can obtain less information exclusive to player i than player i. Specifically, we will assume

$$\beta\sigma_{ij} + \beta^2 \sum_{k \neq i,j} \sigma_{ik}\sigma_{kj} < 1 \tag{1}$$

as long as $0 < \beta < 1$. In the next sections, we will prove that (1) holds for each specified link strength function.

Player j's payoff $S_j(z)$ will be his own information plus the total amount of information he receives from others:

$$
\begin{aligned}
S_j(z) &= r_j + \sum_{i \neq j} S_{ij}(z) \\
&= r_j + \sum_{i \neq j} \left(\beta \sigma_{ij} + \beta^2 \sum_{k \neq i,j} \sigma_{ik} \sigma_{kj} \right) r_i
\end{aligned}
$$

We will use Nash equilibrium as the equilibrium concept throughout the paper. A network is Nash stable if there is no profitable deviations by individual agents.

Definition 1 *A strategy profile z is a Nash equilibrium if and only if, for all $j \in N$, it holds that*

$$S_j(z) \geq S_j(z_{-j}, z'_j) \text{ for all } z'_j \in Z_j$$

The timeline for non-cooperative Nash equilibrium for n players is as follows: At time zero, players learn the value of each player's information, i.e., $r_j, j \in N$. Each player has 1 unit of time to invest in communication with other players. At time one, they simultaneously choose how much time to invest in other players. At time two, the players exchange information according to their strategies. Therefore, the optimization problem for each player j is as follows:

$$\text{maximize } S_j(z_j, z_{-j}) = r_j + \sum_{j \neq i} \left(\beta \sigma_{ij} + \beta^2 \sum_{k \neq i,j} \sigma_{ik} \sigma_{kj} \right) r_i \text{ subject to}$$

$$z_j = \{z_{jk}\}_{k \neq j}$$

$$0 \leq z_{jk} \leq 1 \text{ for all } k \in N \setminus \{j\}$$

$$\sum_{k \neq j} z_{jk} = 1$$

Now, we will prove that the general game of link formation has an equilibrium. Notice that the players have infinite strategy sets; therefore, we will use Debreu, Fan, Glicksberg Theorem to prove the existence of the equilibrium.

Theorem 2 *(Debreu, Fan, Glicksberg) Consider a strategic form game* $\left\langle N, (Z_i)_{i\in N}, (S_i)_{i\in N}\right\rangle$ *with infinite strategy sets such that for each* $i \in N$:

1. Z_i *is convex and compact.*

2. $S_i(z_i, z_{-i})$ *is continuous in* z_{-i}*.*

3. $S_i(z_i, z_{-i})$ *is continuous and quasiconcave in* z_i*.*

The game has a pure strategy Nash equilibrium.

Proposition 3 *The general model of link formation game with n-players has a Nash equilibrium.*

Proof. The strategy set Z_i for player i is a simplex defined by

$$z_i = \{z_{ik}\}_{k\neq i}$$

$$0 \leq z_{ik} \leq 1 \text{ for all } k \in N \setminus \{i\}$$

$$\sum_{k\neq i} z_{ik} = 1$$

Thus, each Z_i for all $i \in N$ is convex and compact.

We assume that the general link strength function between players i and j, σ_{ij} is continuous and concave in z_{ij} and z_{ji}.Since the payoff function for player i, $S_i(z_i, z_{-i})$, is linear with respect to σ_{ij}, $S_i(z_i, z_{-i})$ is jointly continuous in both z_i and z_{-i}, and concave in z_i.

Therefore, by Theorem 2, a pure strategy Nash equilibrium exists for the general model of link formation game with n-players. ■

Before we proceed to the analysis, we introduce formal definitions of the concepts to be used in the following chapters.

Definition 4 *A network* (N, σ) *consists of a set of nodes* $N = \{1, 2, ..., n\}$ *and a real-valued n x n matrix,* σ*, where* σ_{ij} *represents the link strengths between the players* i *and* j*. A path in a network* (N, σ) *between players* i *and* j *is a sequence of links* $i_1i_2; i_2i_3; ...i_{K-1}i_K$ *such that* $\sigma_{i_{k-1}i_k} > 0$ *for each* $k \in \{1, 2, ...K\}$ *with* $i_1 = i$ *and* $i_K = j$*, and such that each node in the sequence* $i_1, i_2, ...i_K$ *is distinct.*

Note that, for our model, $\sigma_{ij} = \sigma_{ji}$ for all $i, j \in N$. Therefore, any network in our model is an undirected network and σ is a symmetric matrix.

Definition 5 *A network* (N, σ) *is connected if for each* $i \in N$ *and* $j \in N$ *there exists a path between* i *and* j*.*

Definition 6 *A component of a network* (N, σ) *is a nonempty subnetwork* (N', σ') *such that* $\emptyset \neq N' \subset N$*,* $\sigma' \subset \sigma$ *and*

- (N', σ') *is connected, and*
- *if* $i \in N'$ *and* $\sigma_{ij} > 0$ *for* σ*, then* $j \in N'$ *and* $\sigma_{ij} > 0$ *for* σ'*.*

The following are formal definitions of particular network architectures. These structures are illustrated in the following figure.

Definition 7 *A network* (N, σ) *is an n-player wheel if it consists of* n *directed links and has a single directed cycle that involves* n *players.*

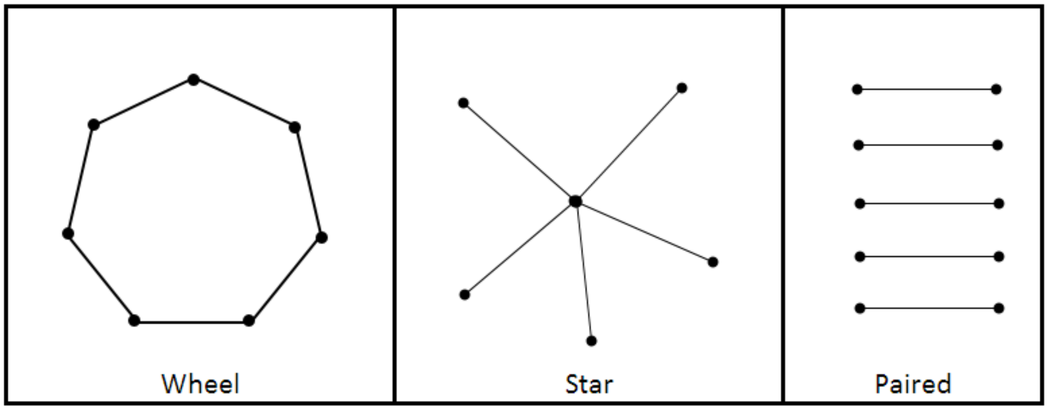


Figure 1: Examples of special network architectures

Definition 8 *A network (N, σ) is a star if there exists a player i (the center of the star) such that all other players connected to the center. That is the player at the center has direct links to $n-1$ players and each of the other players has only one direct link.*

Definition 9 *The player j is ostracized if $z_{ij} = 0$ for all $i \neq j$.*

Definition 10 *A network is said to be paired if for every player i (except one agent ostracized in the case that n is odd), there exists a player j such that*

$$z_{ij} = z_{ji} = 1$$

In the next sections, we will focus on special cases of our model in which we impose restrictions on link strength function. First, we will assume that the strength of the link is an additively separable function. This case allows players form links unilaterally to other players. Then, we will assume that the link strength function is Cobb-Douglas in which players have to have bilateral agreement to form links with each other.

2.4 Additively Separable Link Strength

In the previous section, we assume that the general link strength function between players i and j

$$\sigma(z_{ij}, z_{ji}) = \sigma_{ij}$$

is continuous and concave in z_{ij} and z_{ji}. In this section, we introduce further restriction and assume the link strength function between players i and j is

$$\sigma_{ij} = \frac{1}{2}z_{ij} + \frac{1}{2}z_{ji}$$

In other words, players i and j are perfect substitutes in terms of investment levels.

As σ_{ij} is linear in z_{ij} and z_{ji}, then player j's payoff is also continuous in all actions and concave in his own action s_i. Therefore, we know that there exists a pure strategy Nash equilibrium for the game as the conditions in Theorem 2 holds. Moreover, notice that this link strength function allows players to form links unilaterally to other players. The intuition behind this is that even if player i doesn't want to connect to player j, player j can still obtain some information of player i by spending time on researching on player i.

We will start our analysis with characterization of Nash equilibrium network structures with n-players where investments are perfect substitutes. These results will form the basis for comparison of the network structures under different link strength functions. Moreover, we present a complete characterization of the set of strong Nash equilibrium where the equilibrium network structure is a star.

Let $N = \{1, 2, 3..., n\}$ be the set of players. A strategy for player j's will be denoted z_j.

It consists of his investment levels in the other players:

$$z_j = \{z_{jk}\}_{k \neq j}$$

and must satisfy

$$0 \leq z_{jk} \leq 1$$

for all $k \neq j$ and

$$\sum_{k \neq j} z_{jk} = 1.$$

Let Z_j denote player j's strategy set.

A strategy profile consists of a strategy for each player. A strategy profile will be written

$$z = (z_1, z_2, ..., z_n) \in Z \equiv Z_1 \times Z_2 \times ... \times Z_n$$

Let $S_j(z)$ denote player j's payoff from strategy profile z. If we assume information can be transmitted by a chain of no more than two links, then the total amount of information that player j receives from player i, directly and indirectly, is

$$S_{ij}(z) = \left(\beta \frac{1}{2} \left(z_{ij} + z_{ji} \right) + \beta^2 \sum_{k \neq i,j} \frac{1}{2} \left(z_{ik} + z_{ki} \right) \frac{1}{2} \left(z_{jk} + z_{kj} \right) \right) r_i$$

We have assumed in the previous section that player j can obtain less information exclusive to player i than player i. Specifically, the following inequality holds as long as $0 < \beta < 1$.

$$\beta \sigma_{ij} + \beta^2 \sum_{k \neq i,j} \sigma_{ik} \sigma_{kj} < 1$$

Now, we will show that if $\sigma_{ij} = \frac{1}{2}z_{ij} + \frac{1}{2}z_{ji}$ and $0 < \beta < 1$

$$\begin{aligned} \beta\sigma_{ij} + \beta^2 \sum_{k \neq i,j} \sigma_{ik}\sigma_{kj} &< 1 \\ \beta\frac{1}{2}\left(z_{ij} + z_{ji}\right) + \beta^2 \sum_{k \neq i,j} \frac{1}{2}\left(z_{ik} + z_{ki}\right)\frac{1}{2}\left(z_{jk} + z_{kj}\right) &< 1 \end{aligned}$$

As $0 \leq z_{jk} \leq 1$ for all $k \neq j$,

$$\sigma_{kj} = \frac{1}{2}\left(z_{jk} + z_{kj}\right) \leqslant 1$$

Therefore, we have

$$\begin{aligned} \beta\frac{1}{2}\left(z_{ij} + z_{ji}\right) + \beta^2 \sum_{k \neq i,j} \frac{1}{2}\left(z_{ik} + z_{ki}\right)\frac{1}{2}\left(z_{jk} + z_{kj}\right) &\leqslant \beta\frac{1}{2}\left(z_{ij} + z_{ji}\right) + \beta^2 \sum_{k \neq i,j} \frac{1}{4}\left(z_{ik} + z_{ki}\right) \\ &= \beta\frac{1}{2}\left(z_{ij} + z_{ji}\right) + \beta^2 \sum_{k \neq i,j} \frac{1}{4} z_{ik} + \beta^2 \sum_{k \neq i,j} \frac{1}{4} z_{ki} \\ &= \beta\frac{1}{2}\left(z_{ij} + z_{ji}\right) + \beta^2\frac{1}{4}\left(1 - z_{ij}\right) + \beta^2\frac{1}{4}\left(1 - z_{ji}\right) \\ &= \left(\beta - \frac{\beta^2}{2}\right)\frac{1}{2}\left(z_{ij} + z_{ji}\right) + \frac{\beta^2}{2} \end{aligned}$$

As $\sigma_{ij} \leqslant 1$ and $0 < \beta < 1$, we have

$$\beta\sigma_{ij} + \beta^2 \sum_{k \neq i,j} \sigma_{ik}\sigma_{kj} \leqslant \beta < 1$$

Player j's payoff $S_j(z)$ will be his own information plus the total amount of information he receives from others:

$$S_j(z) = r_j + \sum_{i \neq j} S_{ij}(z)$$

Let $z_{-j} = (z_1, z_2, ..., z_{j-1}, z_{j+1}, ..., z_n)$ denote the strategies of all the players except j. Let (z_{-j}, z'_j) denote the strategy profile when all players except j choose according to z but player j chooses z'_j. A strategy profile z is a Nash equilibrium if and only if, for all $j \in N$, it holds that

$$S_j(z) \geq S_j(z_{-j}, z'_j) \quad \text{for all } z'_j \in Z_j.$$

Let $\Omega(z)$ denote the set of players that some other player invests in under strategy profile z. That is

$$\Omega(z) = \{j : \ z_{ij} > 0 \text{ for some } i\}.$$

Since a player cannot invest in himself $\Omega(z)$ must contain at least two players.

Fix the strategies z_{-j} of all players except player j. Let z_j^a denote the strategy such that $z_{ja} = 1$, i.e., player j invests only in player a. His payoff from this strategy is

$$\begin{aligned}
S_j(z_{-j}, z_j^a) &= r_j + \sum_{i \neq j} \left(\beta \frac{1}{2}\left(z_{ij} + z_{ji}\right) + \beta^2 \sum_{k \neq i,j} \frac{1}{2}\left(z_{ik} + z_{ki}\right) \frac{1}{2}\left(z_{jk} + z_{kj}\right) \right) r_i \\
&= r_j + \left(\beta \frac{1}{2}\left(z_{aj} + 1\right) + \beta^2 \sum_{k \neq a,j} \frac{1}{2}\left(z_{ak} + z_{ka}\right) \frac{1}{2} z_{kj} \right) r_a \\
&\quad + \sum_{i \neq j,a} \left(\beta \frac{1}{2} z_{ij} + \beta^2 \frac{1}{2}\left(z_{ia} + z_{ai}\right) \frac{1}{2}\left(1 + z_{aj}\right) + \beta^2 \sum_{k \neq i,j,a} \frac{1}{2}\left(z_{ik} + z_{ki}\right) \frac{1}{2} z_{kj} \right) r_i \\
&= r_j + \beta \frac{1}{2} r_a + \left(\beta \frac{1}{2} z_{aj} + \beta^2 \sum_{k \neq a,j} \frac{1}{2}\left(z_{ak} + z_{ka}\right) \frac{1}{2} z_{kj} \right) r_a \\
&\quad + \sum_{i \neq j,a} \left(\beta \frac{1}{2} z_{ij} + \beta^2 \frac{1}{2}\left(z_{ia} + z_{ai}\right) \frac{1}{2} z_{aj} + \beta^2 \sum_{k \neq i,j,a} \frac{1}{2}\left(z_{ik} + z_{ki}\right) \frac{1}{2} z_{kj} \right) r_i \\
&\quad + \beta^2 \sum_{i \neq j,a} \frac{1}{2}\left(z_{ia} + z_{ai}\right) \frac{1}{2} r_i \\
&= r_j + \beta \frac{1}{2} r_a + \left(\beta \frac{1}{2} z_{aj} + \beta^2 \sum_{k \neq a,j} \frac{1}{2}\left(z_{ak} + z_{ka}\right) \frac{1}{2} z_{kj} \right) r_a \\
&+ \sum_{i \neq j,a} \left(\beta \frac{1}{2} z_{ij} + \beta^2 \sum_{k \neq i,j} \frac{1}{2}\left(z_{ik} + z_{ki}\right) \frac{1}{2} z_{kj} \right) r_i + \beta^2 \sum_{i \neq j,a} \frac{1}{2}\left(z_{ia} + z_{ai}\right) \frac{1}{2} r_i
\end{aligned}$$

Thus, we have

$$S_j(z_{-j}, z_j^a) = r_j + \beta\frac{1}{2}r_a + \sum_{i \neq j}\left(\beta\frac{1}{2}z_{ij} + \beta^2\sum_{k \neq i,j}\frac{1}{2}(z_{ik} + z_{ki})\frac{1}{2}z_{kj}\right)r_i + \beta^2\sum_{i \neq j,a}\frac{1}{2}(z_{ia} + z_{ai})\frac{1}{2}r_i$$

Similarly,

$$S_j(z_{-j}, z_j^b) = r_j + \beta\frac{1}{2}r_b + \sum_{i \neq j}\left(\beta\frac{1}{2}z_{ij} + \beta^2\sum_{k \neq i,j}\frac{1}{2}(z_{ik} + z_{ki})\frac{1}{2}z_{kj}\right)r_i + \beta^2\frac{1}{2}\sum_{i \neq j,b}(z_{ib} + z_{bi})\frac{1}{2}r_i$$

Therefore, the payoff difference between only investing in player a and only investing in player b is

$$\begin{aligned} & S_j(z_{-j}, z_j^a) - S_j(z_{-j}, z_j^b) && (2) \\ = \quad & \beta\frac{1}{2}(r_a - r_b) + \beta^2\sum_{i \neq j,a}\frac{1}{2}(z_{ia} + z_{ai})\frac{1}{2}r_i - \beta^2\sum_{i \neq j,b}\frac{1}{2}(z_{ib} + z_{bi})\frac{1}{2}r_i && (3) \end{aligned}$$

Lemma 11 *Consider any Nash equilibrium z and let σ denote the implied link-strengths. Suppose $\{a, b\} \subseteq \Omega(z)$, $h \neq b$ and $m \neq a$. Then if $z_{ha} > 0$ and $z_{mb} > 0$ the following inequality must hold:*

$$(\sigma_{ha} - \sigma_{hb})\, r_h + (\sigma_{mb} - \sigma_{ma})\, r_m \leq 0 \tag{4}$$

Proof. Suppose player h invests in $a \in \Omega(z)$, i.e., $z_{ha} > 0$. Then, by the Nash property, we must have

$$S_h(z_{-h}, z_h^a) - S_h(z_{-h}, z_h^b) \geq 0.$$

Suppose some other player, say $m \neq h$, invests in $b \in \Omega(z)$ where $b \neq h$, i.e., $z_{mb} > 0$. By the Nash property, we must have

$$S_m(z_{-m}, z_m^b) - S_m(z_{-m}, z_m^a) \geq 0. \tag{5}$$

Applying (2) we find that

$$\begin{aligned}
& \left[S_h(z_{-h}, z_h^a) - S_h(z_{-h}, z_h^b)\right] - \left[S_m(z_{-m}, z_m^a) - S_m(z_{-m}, z_m^b)\right] \\
= & \left[\beta\frac{1}{2}(r_a - r_b) + \beta^2 \sum_{i \neq h,a} \frac{1}{2}(z_{ia} + z_{ai}) \frac{1}{2} r_i - \beta^2 \sum_{i \neq h,b} \frac{1}{2}(z_{ib} + z_{bi}) \frac{1}{2} r_i\right] \\
& - \left[\beta\frac{1}{2}(r_a - r_b) + \beta^2 \sum_{i \neq m,a} \frac{1}{2}(z_{ia} + z_{ai}) \frac{1}{2} r_i - \beta^2 \sum_{i \neq m,b} \frac{1}{2}(z_{ib} + z_{bi}) \frac{1}{2} r_i\right] \\
= & \beta^2 \frac{1}{2}(z_{ma} + z_{am} - z_{mb} - z_{bm}) \frac{1}{2} r_m + \beta^2 \frac{1}{2}(z_{hb} + z_{bh} - z_{ha} - z_{ah}) \frac{1}{2} r_h
\end{aligned}$$

Therefore,

$$\begin{aligned}
& S_h(z_{-h}, z_h^a) - S_h(z_{-h}, z_h^b) \\
= & \left[S_m(z_{-m}, z_m^a) - S_m(z_{-m}, z_m^b)\right] + \frac{\beta^2}{4}(z_{ma} + z_{am} - z_{mb} - z_{bm}) r_m + \frac{\beta^2}{4}(z_{hb} + z_{bh} - z_{ha} - z_{ah}) r_h \\
\geq & 0
\end{aligned}$$

The term in square brackets is non-positive by (5). Therefore,

$$(z_{ma} + z_{am} - z_{mb} - z_{bm}) r_m + (z_{hb} + z_{bh} - z_{ha} - z_{ah}) r_h \geq 0$$

which implies that (4) holds. ■

Proposition 12 *Every player is connected to every other player either directly or indirectly with no more than two links under any Nash equilibrium z.*

Proof. First, we will prove that the network (N, σ) under the Nash equilibrium z is connected by contradiction. Then, we will show that each player is connected either directly or indirectly with no more than two links.

Suppose that the network (N, σ) under the Nash equilibrium z is not connected. Therefore, there exists a player i and a player j such that there is no path in (N, σ) between i and j. Moreover, since the network (N, σ) is not connected, there would be at least two components of (N, σ); (N', σ') and (N'', σ'') such that $\emptyset \neq N' \subset N$, $\emptyset \neq N'' \subset N$ and $N' \cap N'' = \emptyset$. Without loss of generality, assume that player i is in N' and player j is in N''. Since player i cannot invest in himself, there exists a player a such that $z_{ia} > 0$. Moreover, since (N', σ') is connected, $a \in N'$. Similarly, since player j cannot invest in himself, there exists a player b such that $z_{jb} > 0$ and since (N'', σ'') is connected, $b \in N''$. Since $a \in N'$ and $b \in N''$, we have $i \neq b$ and $j \neq a$. Then, from Lemma 11, as $\{a, b\} \subseteq \Omega(z)$, $i \neq b$ and $j \neq a$, and $z_{ia} > 0$ and $z_{jb} > 0$ the following inequality must hold:

$$(\sigma_{ia} - \sigma_{ib})\, r_i + (\sigma_{jb} - \sigma_{ja})\, r_j \leq 0$$

Since $\sigma_{ia} > 0$ and $\sigma_{jb} > 0$ as $z_{ia} > 0$ and $z_{jb} > 0$, we should have $\sigma_{ib} > 0$ and/or $\sigma_{ja} > 0$. Suppose $\sigma_{ib} > 0$, then there exists a path from player i to j via player b where $\sigma_{ib} > 0$ and $\sigma_{bj} > 0$. This leads to a contradiction with the assumption that the network (N, σ) under the Nash equilibrium z is not connected. Similar argument holds if we have $\sigma_{ja} > 0$ as there exists a path from player i to j via player a where $\sigma_{ia} > 0$ and $\sigma_{aj} > 0$. Therefore, the network (N, σ) under the Nash equilibrium z is connected.

Now, suppose that the shortest link between players i and j is greater than two. Since players i and j cannot invest in themselves, there exists players $a \neq i, j$ and $b \neq a, i, j$ such that $z_{ia} > 0$ and $z_{jb} > 0$. Notice that if $a = b$ or $a = j$ or $b = i$, the shortest link between players i and j becomes less than or equal to two. Then, from Lemma 11, as $\{a, b\} \subseteq \Omega(z)$, $i \neq b$ and $j \neq a$, and $z_{ia} > 0$ and $z_{jb} > 0$ the following inequality must hold:

$$(\sigma_{ia} - \sigma_{ib})\, r_i + (\sigma_{jb} - \sigma_{ja})\, r_j \leq 0$$

Since $\sigma_{ia} > 0$ and $\sigma_{jb} > 0$ as $z_{ia} > 0$ and $z_{jb} > 0$, we should have $\sigma_{ib} > 0$ and/or $\sigma_{ja} > 0$. Suppose $\sigma_{ib} > 0$, then there exists a path from player i to j via player b where $\sigma_{ib} > 0$ and $\sigma_{bj} > 0$. This leads to a contradiction with the assumption that the shortest link between players i and j is greater than two. Suppose $\sigma_{ja} > 0$, then there exists a path from player i to j via player a where $\sigma_{ia} > 0$ and $\sigma_{aj} > 0$. This leads to a contradiction with the assumption that the shortest link between players i and j is greater than two. Therefore, each player i and j should be connected either directly or indirectly with no more than two links. ■

Even though the most distinct feature of our model is the weighted link strength strategies, we will study strict Nash equilibria as a refinement in the next results. For this purpose, we will rewrite player j's payoff as

$$S_j(z) = \frac{\beta}{2} \sum_{i \neq j} z_{ji} \left(r_i + \beta \sum_{k \neq i,j} \sigma_{ik} r_k \right) + \phi\left(z_{-j}\right)$$

where ϕ is a function that doesn't depend on player j's strategy z_j. As player j's objective function $S_j(z)$ is linear in the choice variables, z_{ji} for all $i \neq j$, for any given z_{-j}, player j maximizes his payoff assigning positive link strength only to the players that maximize

$$\pi_{ji} = r_i + \beta \sum_{k \neq i,j} \sigma_{ik} r_k$$

We will denote the unique maximizer of π_{ji} for player j as j^*. If there is a unique maximizer, j^*, then player j should set $z_{jj^*} = 1$.

Lemma 13 *Assume that in a Nash equilibrium z, for distinct players i and j, there exists a unique maximizer j^* of $\pi_{jk}, k \in N \setminus \{j\}$ for player j and there exists a unique maximizer i^* of $\pi_{ik}, k \in N \setminus \{i\}$ for player i. If $j \neq i^*$ and $i \neq j^*$, then $j^* = i^*$.*

Proof. Consider any Nash equilibrium z in which, for distinct players i and j, there exists a unique maximizer i^* of $\pi_{ik}, k \in N \setminus \{i\}$ for player i such that $i^* \neq j$, and player j chooses $z_{ja} = 1$ where $a \notin \{i, i^*\}$. Since $z_{ja} = 1$ and $z_{ia} = 0$, we have

$$\begin{aligned}
\pi_{ja} &= r_a + \beta \sum_{k \neq a,i,j} \sigma_{ak} r_k + \frac{\beta}{2} z_{ai} r_i \\
\pi_{ia} &= r_a + \beta \sum_{k \neq a,i,j} \sigma_{ak} r_k + \frac{\beta}{2} \left(z_{aj} + 1\right) r_j
\end{aligned}$$

Moreover, since $z_{ji^*} = 0$ and $z_{ii^*} = 1$, we have

$$\begin{aligned}
\pi_{ji^*} &= r_{i^*} + \beta \sum_{k \neq i^*,i,j} \sigma_{i^*k} r_k + \frac{\beta}{2} \left(z_{i^*i} + 1\right) r_i \\
\pi_{ii^*} &= r_{i^*} + \beta \sum_{k \neq i^*,i,j} \sigma_{i^*k} r_k + \frac{\beta}{2} z_{i^*j} r_j
\end{aligned}$$

Since player i^* is the unique maximizer of π_{ik} for $k \in N$, we have

$$\begin{aligned}
\pi_{ii^*} &> \pi_{ia} \\
r_{i^*} + \beta \sum_{k \neq i^*,i,j} \sigma_{i^*k} r_k + \frac{\beta}{2} z_{i^*j} r_j &> r_a + \beta \sum_{k \neq a,i,j} \sigma_{ak} r_k + \frac{\beta}{2} \left(z_{aj} + 1\right) r_j \\
r_{i^*} + \beta \sum_{k \neq i^*,i,j} \sigma_{i^*k} r_k - \left(r_a + \beta \sum_{k \neq a,i,j} \sigma_{ak} r_k \right) &> \frac{\beta}{2} \left(z_{aj} + 1 - z_{i^*j}\right) r_j
\end{aligned}$$

As $z_{aj} + 1 - z_{i^*j} \geqslant 0$, we should have

$$r_{i^*} + \beta \sum_{k \neq i^*, i, j} \sigma_{i^*k} r_k - \left(r_a + \beta \sum_{k \neq a, i, j} \sigma_{ak} r_k \right) > 0$$

Then,

$$\pi_{ji^*} - \pi_{ja} = r_{i^*} + \beta \sum_{k \neq i^*, i, j} \sigma_{i^*k} r_k - \left(r_a + \beta \sum_{k \neq a, i, j} \sigma_{ak} r_k \right) + \frac{\beta}{2} \left(z_{i^*i} + 1 - z_{ai} \right) r_i$$

As $z_{i^*i} + 1 - z_{ai} \geqslant 0$, we have

$$\pi_{ji^*} - \pi_{ja} > 0$$

This means player j is not best responding by choosing $z_{ja} = 1$. This means the only candidates for j^* are i and i^*. By the hypothesis in the lemma, $j^* \neq i$. Hence, $j^* = i^*$ ■

Lemma 14 *In a strict Nash equilibrium z, if $j^* = i$ and $i^* = h \neq j$, then $h^* = i$.*

Proof. Choose a strategy profile in which each player $l \neq j$ sets $z_{ll^*} = 1$ with $j^* = i$ and $i^* = h \neq j$. From Lemma 13, as $h \neq i$, we must have $h^* \in \{j, i\}$. Assume $z_{hj} = 1$ and consider the following expressions:

$$\begin{aligned}
\pi_{ji} &= r_i + \beta \sum_{k \neq h, i, j} \sigma_{ik} r_k + \frac{\beta}{2} r_h \\
\pi_{jh} &= r_h + \beta \sum_{k \neq h, i, j} \sigma_{hk} r_k + \frac{\beta}{2} r_i \\
\pi_{ih} &= r_h + \beta \sum_{k \neq h, i, j} \sigma_{hk} r_k + \frac{\beta}{2} r_j \\
\pi_{ij} &= r_j + \beta \sum_{k \neq h, i, j} \sigma_{jk} r_k + \frac{\beta}{2} r_h
\end{aligned}$$

Since $\pi_{ji} - \pi_{jh} > 0$ and $\pi_{ih} - \pi_{ij} > 0$, we have

$$\begin{aligned}
\pi_{ji} - \pi_{ij} &> \pi_{jh} - \pi_{ih} \\
r_i + \beta \sum_{k \neq h,i,j} \sigma_{ik} r_k + \frac{\beta}{2} r_h - \left(r_j + \beta \sum_{k \neq h,i,j} \sigma_{jk} r_k + \frac{\beta}{2} r_h \right) &> \frac{\beta}{2} r_i - \frac{\beta}{2} r_j \\
r_i + \beta \sum_{k \neq h,i,j} \sigma_{ik} r_k + \frac{\beta}{2} r_j &> r_j + \beta \sum_{k \neq h,i,j} \sigma_{jk} r_k + \frac{\beta}{2} r_i \\
\pi_{hi} &> \pi_{hj}
\end{aligned}$$

Therefore, player h is not best responding if he sets $z_{hj} = 1$. Hence, we have $h^* = i$. ■

Proposition 15 *Assume $r_{k-1} > r_k$ for all $k \in N \setminus \{1\}$. There are two sets of strict Nash equilibria of the simultaneous move game, both resulting in a star network. The first set of equilibrium strategies is given by*

$$\begin{aligned}
z_{i1} &= 1 \text{ for all } i \in N \setminus \{1\} \\
z_{12} &= 1
\end{aligned}$$

Moreover, we have

$$S_1 > S_2 > ... > S_N$$

The second set of equilibrium strategies is given by

$$\begin{aligned}
z_{ii^*} &= 1 \text{ for all } i \in N \setminus \{i^*\} \text{ where } i^* \neq 1 \\
z_{i^*1} &= 1
\end{aligned}$$

Moreover, we have

$$S_{i^*} - r_{i^*} > S_1 - r_1$$

$$S_1 > S_2 > S_{i^*-1} > S_{i^*+1} ... > S_N$$

For the second case to be strict Nash equilibrium, we need

$$\begin{aligned} r_{i^*} &> r_1 - \frac{\beta}{1-\beta} \sum_{k \neq 1, i^*, j} \frac{r_k}{2} \text{ where } j = \arg\max_{k \neq 1, i^*} r_k \\ r_{i^*} &> r_2 - \frac{\beta}{1 - \frac{\beta}{2}} \sum_{k \neq 1, i^*, 2} \frac{r_k}{2} \text{ if } r_2 > r_{i^*} \end{aligned}$$

Proof. In a strict Nash equilibrium z, every j has a unique maximizer of $\pi_{ji}, i \in N \setminus \{j\}$ for fixed z_{-j}. Then, player j sets $z_{jj^*} = 1$ in any strict Nash equilibrium. By Lemma 13, we know that for distinct players i and j, if $j \neq i^*$ and $i \neq j^*$, then $j^* = i^*$. This restricts the relationships between two distinct players into two: the players are either directly connected or they are connected to the same player. Moreover, if the players are directly connected, then Lemma 14 eliminates any wheel structure in the equilibrium. Therefore, by Lemma 13 and 14, the only possible strategy profile for a strict Nash equilibrium is the following: There exists a player i^* such that

$$\begin{aligned} z_{ji^*} &= 1 \text{ for all } j \in N \setminus \{i^*\} \\ z_{i^*h} &= 1 \text{ where } h = \arg\max_{k \neq i^*} r_k. \end{aligned}$$

Now, consider the first set of equilibrium strategies given by

$$\begin{aligned} z_{i1} &= 1 \text{ for all } i \in N \setminus \{1\} \\ z_{12} &= 1 \end{aligned}$$

Player 1 is at the center of the network, i.e., $z_{i1} = 1$ for all $i \in N \setminus \{1\}$. Since,

$$\begin{aligned} \pi_{1i} &= r_i \text{ for all } i \neq 1 \\ 2 &= \arg\max_{k\neq 1} r_k \end{aligned}$$

player 1 is strictly best responding by choosing $z_{12} = 1$.

Players $j \in N \setminus \{1, 2\}$ are best responding since

$$r_1 + \beta r_2 + \beta \sum_{k\neq 1,2,j} \frac{r_k}{2} > r_2 + \beta r_1 > r_k + \frac{\beta}{2} r_1 \text{ for all } k \neq 1, 2, j$$

Finally, player 2 is best responding since

$$r_1 + \beta \sum_{k\neq 1,2} \frac{r_k}{2} > r_k + \frac{\beta}{2} r_1$$

for all $k \neq 1, 2$. Under this equilibrium strategies, the surplus for each player is

$$\begin{aligned} S_1 &= r_1 + \beta r_2 + \beta \sum_{k\neq 1,2} \frac{r_k}{2} \\ S_2 &= r_2 + \beta r_1 + \beta^2 \sum_{k\neq 1,2} \frac{r_k}{2} \\ S_n &= r_n + \frac{\beta}{2} r_1 + \beta^2 \sum_{k\neq 1,n} \frac{r_k}{2} \text{ for all } n \in N \setminus \{1, 2\} \end{aligned}$$

Therefore, we have

$$S_1 > S_2 > ... > S_N$$

Now, consider the second set of equilibrium strategies given by

$$\begin{aligned} z_{ii^*} &= 1 \text{ for all } i \in N \setminus \{i^*\} \\ z_{i^*1} &= 1 \end{aligned}$$

Player i^* is at the center of the network, i.e., $z_{ii^*} = 1$ for all $i \in N \setminus \{i^*\}$. Since,

$$\begin{aligned} \pi_{i^*i} &= r_i \text{ for all } i \neq i^* \\ 1 &= \arg\max_{k \neq i^*} r_k \end{aligned}$$

player i^* is strictly best responding by choosing $z_{i^*1} = 1$.

Players $j \in N \setminus \{i^*, 1\}$ are best responding if

$$\begin{aligned} r_{i^*} + \beta r_1 + \beta \sum_{k \neq 1, i^*, j} \frac{r_k}{2} &> r_1 + \beta r_{i^*} \\ \beta \sum_{k \neq 1, i^*, j} \frac{r_k}{2} &> (r_1 - r_{i^*})(1 - \beta) \\ \frac{\beta}{1-\beta} \sum_{k \neq 1, i^*, j} \frac{r_k}{2} &> r_1 - r_{i^*} \\ r_{i^*} &> r_1 - \frac{\beta}{1-\beta} \sum_{k \neq 1, i^*, j} \frac{r_k}{2} \end{aligned}$$

Since this holds for all players $j \in N \setminus \{i^*, 1\}$, we should have

$$r_{i^*} > r_1 - \frac{\beta}{1-\beta} \sum_{k \neq 1, i^*, h} \frac{r_k}{2} \text{ where } h = \arg\max_{k \neq 1, i^*} r_k$$

Finally, when $r_{i^*} < r_2$, player 1 is best responding if

$$\begin{aligned}
r_{i^*} + \frac{\beta}{2} r_2 + \beta \sum_{k \neq 1,2,i^*} \frac{r_k}{2} &> r_2 + \frac{\beta}{2} r_{i^*} \\
\beta \sum_{k \neq 1,2,i^*} \frac{r_k}{2} &> (r_2 - r_{i^*}) \left(1 - \frac{\beta}{2}\right) \\
\frac{\beta}{1 - \frac{\beta}{2}} \sum_{k \neq 1,2,i^*} \frac{r_k}{2} &> r_2 - r_{i^*} \\
r_{i^*} &> r_2 - \frac{\beta}{1 - \frac{\beta}{2}} \sum_{k \neq 1,2,i^*} \frac{r_k}{2}
\end{aligned}$$

Under this equilibrium strategies, the surplus for each player is given by

$$\begin{aligned}
S_1 &= r_1 + \beta r_{i^*} + \beta^2 \sum_{k \neq 1,i^*} \frac{r_k}{2} \\
S_{i^*} &= r_{i^*} + \beta r_1 + \beta \sum_{k \neq 1,i^*} \frac{r_k}{2} \\
S_n &= r_n + \frac{\beta}{2} r_{i^*} + \beta^2 \sum_{k \neq i^*,n} \frac{r_k}{2} \text{ for all } N \setminus \{1, i^*\}
\end{aligned}$$

Therefore, we have

$$S_{i^*} - r_{i^*} > S_1 - r_1$$

and

$$S_1 > S_2 > S_{i^*-1} > S_{i^*+1} ... > S_N$$

■

Even though we know that star networks are the only strict Nash equilibria of the simultaneous move game under additively separable link strength function, there may be Nash equilibrium with different network architecture when $r_{k-1} > r_k$ for all $k \in N \setminus \{1\}$. Consider the following example with four players.

Example 16 *Assume that* $r_1 = 100, r_2 = 99, r_3 = 5, r_4 = 4$ *and* $\beta = \frac{2}{3}$. *As the intrinsic values of players 1 and 2 are very high compared to the other two, under any Nash equilibrium, 1 and 2 would invest all their time to each other. Then, there will be three different Nash equilibrium, in one of which players 3 and 4 allocate their time both on player 1 and 2. The following figure shows the investment levels at the Nash equilibrium. As opposed to the first two architecture, the network structure in the third Nash equilibrium is not a star.*

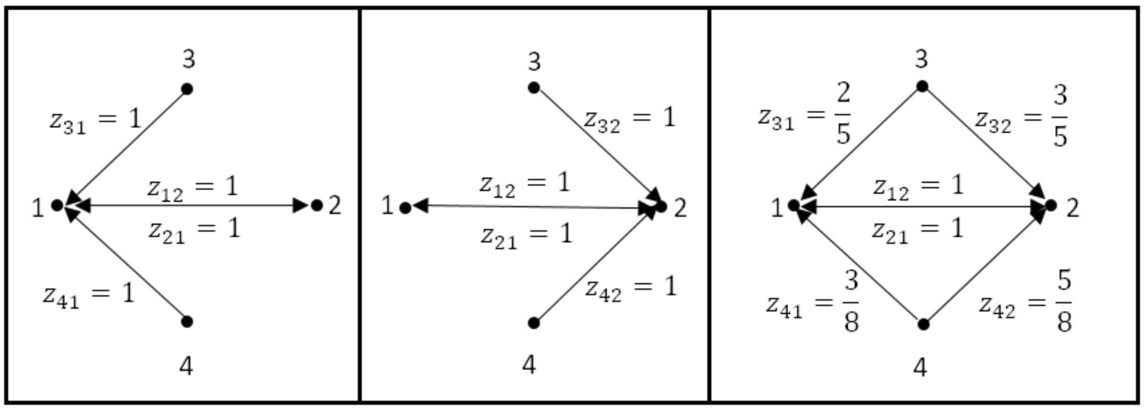


Figure 2: Example of different Nash equilibrium structures

Proposition 17 *Nash equilibrium under additively separable link strength function may not be surplus maximizing outcome.*

Proof. We prove the result by using three player game. Full characterization of Nash equilibrium and surplus maximizing outcome is provided in the next chapter.

Assume that there are only three players and they are ordered in terms of their information so that $r_1 > r_2 > r_3 > 0$. Under these conditions, there is only one Nash equilibrium where $z_{12} = 1, z_{21} = 1, z_{31} = 1$. We will show that a higher level of total surplus could be achieved under some conditions by changing the equilibrium strategy for either player 1 or player 3. The following shows the partial derivative of total surplus with respect to z_{12}

evaluated at $z_{12}=1, z_{21}=1, z_{31}=1$.

$$\left[\frac{\partial S}{\partial z_{12}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = -\frac{1}{4}\beta(r_3(2+\beta)-r_2(2-\beta))$$

If the difference between the intrinsic values of player 2 and 3 is low enough, that is, when

$$\frac{r_2}{r_3} < \frac{2+\beta}{2-\beta}$$

the partial derivative of the total surplus with respect to z_{12} would be negative at Nash Equilibrium. Then, a higher total social surplus could be achieved by decreasing z_{12} by a small amount so that player 1 allocate his time between player 2 and 3. The players' individual surpluses shows that the increase in the total surplus is a result of the increase in player 3's surplus being higher than the total decrease in player 1 and 2's surpluses.

$$\left[\frac{\partial S_1}{\partial z_{12}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = \frac{1}{2}\beta\left(r_2-r_3\right)$$

$$\left[\frac{\partial S_2}{\partial z_{12}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = \frac{1}{4}\beta\left(2r_1-\beta r_3\right)$$

$$\left[\frac{\partial S_3}{\partial z_{12}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = -\frac{1}{4}\beta\left(2r_1+\beta r_2\right)$$

On the other hand, the following shows the partial derivative of total surplus with respect to z_{31} evaluated at $z_{12}=1, z_{21}=1, z_{31}=1$.

$$\left[\frac{\partial S}{\partial z_{31}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = \frac{1}{4}\beta\left(r_1(2-3\beta)-r_2(2-\beta)\right)$$

If the difference between the intrinsic values of player 1 and 2 is low enough, that is, when

$$\frac{r_1}{r_2} < \frac{2-\beta}{2-3\beta}$$

the partial derivative of the total surplus with respect to z_{31} would be negative at Nash Equilibrium. Then, a higher total social surplus could be achieved by decreasing z_{31} by a small amount so that player 3 allocate his time between player 1 and 2. The players' individual surpluses shows that the increase in the total surplus is a result of the increase in player 2's surplus being higher than the total decrease in player 1 and 3's surpluses.

$$\left[\frac{\partial S_1}{\partial z_{31}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = \frac{1}{4}\beta\left(2(1-\beta)r_3 - \beta r_2\right)$$

$$\left[\frac{\partial S_2}{\partial z_{31}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = -\frac{1}{4}\beta\left(2(1-\beta)r_3 + \beta r_1\right)$$

$$\left[\frac{\partial S_3}{\partial z_{31}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = \frac{1}{2}\beta\left(r_1 - r_2\right)(1-\beta)$$

■

2.5 Cobb-Douglas Link Strength

In the general model, we assume that the general link strength function between players i and j

$$\sigma(z_{ij}, z_{ji}) = \sigma_{ij}$$

is continuous and concave in z_{ij} and z_{ji}. In this section, we assume the link strength function between players i and j is

$$\sigma_{ij} = z_{ij} z_{ji}$$

As σ_{ij} is continuous and linear in z_{ij} and z_{ji}, then player j's payoff is also continuous in all actions and concave in his own action s_i. Therefore, we know that there exists a pure strategy Nash equilibrium for the game as the conditions in Theorem 2 holds. Moreover, notice that this link strength function allows players to form links only when there is bilateral agreement between the players as opposed to the additively separable link strength function, in which player could form links unilaterally. The intuition behind this is the players have exclusive information and it can be shared only if they invest time to each other.

Unfortunately, the analysis of Nash equilibria with n-players under Cobb-Douglas link strength function becomes intractable with the inclusion of the indirect benefits into the model. Thus, we will focus only on a refinement to Nash equilibrium. We consider a sequential game with perfect information in which the players announce their strategies according to a random ordering. We show that there is a unique subgame perfect equilibrium of the sequential game which is also Nash equilibrium of the simultaneous move game.

Let $N = \{1, 2, 3..., n\}$ be the set of players. A strategy for player j's will be denoted z_j. It consists of his investment levels in the other players:

$$z_j = \{z_{jk}\}_{k \neq j}$$

and must satisfy

$$0 \leq z_{jk} \leq 1$$

for all $k \neq j$ and

$$\sum_{k \neq j} z_{jk} = 1.$$

Let Z_j denote player j's strategy set. A strategy profile consists of a strategy for each player. A strategy profile will be written

$$z = (z_1, z_2, ..., z_n) \in Z \equiv Z_1 \times Z_2 \times ... \times Z_n$$

Let $S_j(z)$ denote player j's payoff from strategy profile z. The strength of the link between i and j is

$$\sigma(z_{ij}, z_{ji}) = z_{ij} z_{ji}$$

Player j's information is worth r_j. Thus, if the direct link between player i and player j has strength $\sigma_{ij} = \sigma(z_{ij}, z_{ji})$, then obtaining player j's information via this direct link is worth $\beta\sigma(z_{ij}, z_{ji})r_j$ to player i where $0 < \beta < 1$. Moreover, if player j is linked to player k, then player k's information can be indirectly transmitted from k to i via j. Obtaining player k's information via this indirect link is worth $\beta\sigma(z_{ij}, z_{ji})\beta\sigma(z_{jk}, z_{kj})r_k$ to player i.

If we assume information can be transmitted by a chain of no more than two links, then the total amount of information that player j receives from player i, directly and indirectly, is

$$S_{ij}(z) = \left(\beta z_{ij} z_{ji} + \beta^2 \sum_{k \neq i,j} z_{ik} z_{ki} z_{jk} z_{kj} \right) r_i$$

We have assumed that player j can obtain less information exclusive to player i than player i while introducing the general model. Specifically, the following inequality holds as long as $0 < \beta < 1$.

$$\beta\sigma_{ij} + \beta^2 \sum_{k \neq i,j} \sigma_{ik}\sigma_{kj} < 1$$

Now, we will show that if $\sigma_{ij} = \beta z_{ij} z_{ji}$ and $0 < \beta < 1$

$$\begin{aligned} \beta\sigma_{ij} + \beta^2 \sum_{k \neq i,j} \sigma_{ik}\sigma_{kj} &< 1 \\ \beta z_{ij} z_{ji} + \beta^2 \sum_{k \neq i,j} z_{ik} z_{ki} z_{jk} z_{kj} &< 1 \end{aligned}$$

As $0 \leq z_{jk} \leq 1$ for all $k \neq j$,we have

$$\begin{aligned} \beta z_{ij} z_{ji} + \beta^2 \sum_{k \neq i,j} z_{ik} z_{ki} z_{jk} z_{kj} &\leqslant \beta z_{ij} z_{ji} + \beta^2 \sum_{k \neq i,j} z_{ik} z_{jk} \\ &\leqslant \beta z_{ij} z_{ji} + \beta^2 \sum_{k \neq i,j} (1 - z_{ij}) z_{jk} \\ &= \beta z_{ij} z_{ji} + \beta^2 (1 - z_{ij}) \sum_{k \neq i,j} z_{jk} \\ &= \beta z_{ij} z_{ji} + \beta^2 (1 - z_{ij}) (1 - z_{ji}) \\ &< \beta z_{ij} z_{ji} + \beta (1 - z_{ij}) (1 - z_{ji}) \\ &\leqslant \beta z_{ji} + \beta (1 - z_{ji}) \end{aligned}$$

Therefore, if $0 < \beta < 1$, we have

$$\beta\sigma_{ij} + \beta^2 \sum_{k \neq i,j} \sigma_{ik}\sigma_{kj} \leqslant \beta < 1$$

Player j's payoff $S_j(z)$ will be his own information plus the total amount of information he receives from others:

$$\begin{aligned} S_j(z) &= r_j + \sum_{i \neq j} S_{ij}(z) \\ &= r_j + \sum_{i \neq j} \left(\beta z_{ij} z_{ji} + \beta^2 \sum_{k \neq i,j} z_{ik} z_{ki} z_{jk} z_{kj} \right) r_i \end{aligned}$$

Let $z_{-j} = (z_1, z_2, ..., z_{j-1}, z_{j+1}, ..., z_n)$ denote the strategies of all the players except j. Let (z_{-j}, z'_j) denote the strategy profile when all players except j choose according to z but player j chooses z'_j. Let $\Omega(z)$ denote the set of players that some other player invests in under strategy profile z. That is

$$\Omega(z) = \{j : \; z_{ij} > 0 \text{ for some } i\}.$$

Since a player cannot invest in himself $\Omega(z)$ must contain at least two players.

Fix the strategies z_{-j} of all players except player j. Let z_j^a denote the strategy such that $z_{ja} = 1$, i.e., player j invests only in player a. His payoff from this strategy is

$$S_j(z_{-j}, z_j^a) = r_j + \beta z_{ja} r_a + \beta^2 z_{ja} \sum_{k \neq, j, a} z_{ak} z_{ka} r_k$$

Proposition 18 *Assume $r_{k-1} > r_k$ for all $k \in N \setminus \{1\}$. There exists a Nash equilibrium of the simultaneous move game under the following strategies. For every player i, there exists a player j such that*

$$z_{ij} = z_{ji} = 1 \textit{ if } n \textit{ is even}$$

and if n is odd, there will be a player $l \neq 1$ that is ostracized. If n is odd then let

$$z_{lm} = 1 \textit{ where } z_{m1} = z_{1m} = 1$$

Proof. Suppose, we have even number of players. Assume that for every player i, there exists a player j such that

$$z_{ij} = z_{ji} = 1 \text{ if } n \text{ is even}$$

holds except player h, where h is paired with p. Thus, the surplus for player h is

$$S_h(z_{-h}, z_h) = r_h + \beta z_{jp} r_p$$

Notice that this surplus is maximized when $z_{jp} = 1$ and $z_{ji} = 0$ where $i \in N \setminus \{j, p\}$ as

$$S_j = r_j + \beta r_p \geqslant S_j(z_{-j}, z_j^{'}) = r_j + \beta z_{jp} r_p \text{ for all } z_j' \in Z_j$$

and $0 \leqslant z_{jp} \leqslant 1$.This holds for any $j \in N$.

Now, suppose we have odd number of players. Then, there will be a player $l \neq 1$ that is ostracized. Assume that

$$z_{lm} = 1 \text{ where } z_{m1} = z_{1m} = 1$$

Since each player is matched with another player according to the strategy profile above, none of the players spend time with player l. The players whose pair has information less than r_l would want to be paired with the player l. However, since player l's strategy is $z_{lm} = 1$, none of these players have an incentive to deviate. Thus, there is no incentive to deviate for players $i \in N \setminus \{l, m\}$. Moreover, player m has no incentive to deviate as he is paired with player 1. Let's check if player l has an incentive to deviate. As none of the players spend time with player l, the payoff for player l is

$$S_l = r_l$$

Since S_l doesn't depend on z_l, there is no incentive to deviate for player l.Therefore, we have an equilibrium. ■

Corollary 19 *Assume $r_{k-1} > r_k$ for all $k \in N \setminus \{1\}$. There exists a Nash equilibrium of the simultaneous move game where*

$$\begin{aligned} z_{i(i+1)} &= z_{(i+1)i} = 1 \textit{ for all } i \in N \setminus \{n\} \textit{ where } i : odd. \\ z_n &: \textit{free if } n : odd \end{aligned}$$

Therefore, at the equilibrium, we have

$$S_1 > S_2 > ... > S_n$$

This equilibrium will be referred to as "Assortative Pair Equilibrium (APE)".

Proof. Suppose, we have even number of players. Assume that

$$z_{i(i+1)} = z_{(i+1)i} = 1 \text{ for all } i \in N \text{ where } i : odd.$$

holds except player j. Assume that $j : odd$. Thus, the surplus for player j is

$$S_j(z_{-j}, z_j) = r_j + \beta z_{j(j+1)} r_{j+1}$$

Notice that this surplus is maximized when $z_{j(j+1)} = 1$ and $z_{ji} = 0$ where $i \in N \setminus \{j, j+1\}$ as

$$S_j = r_j + \beta r_{j+1} \geqslant S_j(z_{-j}, z_j^{'}) = r_j + \beta z_{j(j+1)} r_{j+1} \text{ for all } z_j^{\prime} \in Z_j$$

and $0 \leqslant z_{j(j+1)} \leqslant 1$.

Assume that $j : even$. Thus, the surplus for player j is

$$S_j(z_{-j}, z_j) = r_j + \beta z_{(j-1)j} r_{j-1}$$

Notice that this surplus is maximized when $z_{j(j+1)} = 1$ and $z_{ji} = 0, i \in N \setminus \{j, j+1\}$ as

$$S_j = r_j + \beta r_{j-1} \geqslant S_j(z_{-j}, z_j^{'}) = r_j + \beta z_{(j-1)j} r_{j-1} \text{ for all } z_j' \in Z_j$$

and $0 \leqslant z_{(j-1)j} \leqslant 1$.This holds for any $j \in N$.

Now, suppose we have odd number of players. Assume that

$$\begin{aligned} z_{i(i+1)} &= z_{(i+1)i} = 1 \text{ for all } i \in N \text{ where } i : odd. \\ z_n &: free \end{aligned}$$

We know from above that there is no incentive to deviate for players $i \in N \setminus \{n\}$ .Let's check if player n has an incentive to deviate. Since each player with odd rank in terms of payoff ordering is matched with the player that are subsequent to themselves according to the strategy profile above, none of the players spend time with player n. This makes the payoff for player n, $S_n = r_n$. Since S_n doesn't depend on z_n, there is no incentive to deviate for player n.

Thus, at equilibrium, we have

$$\begin{aligned} S_i &= r_i + \beta r_{i+1} \text{ if } i : odd \\ S_i &= r_i + \beta r_{i-1} \text{ if } i : even \\ S_n &= r_n \text{ if } n : odd \end{aligned}$$

If $i : odd$, then

$$S_i = r_i + \beta r_{i+1} > S_{i+1} = r_{i+1} + \beta r_i$$

as $0 < \beta < 1$. If $i : even$, then

$$S_i = r_i + \beta r_{i-1} > S_{i+1} = r_{i+1} + \beta r_{i+1}$$

as $r_{k-1} > r_k$ for all $k \in N \setminus \{1\}$ .Thus, we have

$$S_1 > S_2 > ... > S_n$$

■

Notice that when there are even number of players, under APE, each player with odd ranking will be matched with the player that comes after him according to the information ranking. That is, players will be matched in pairs according to their information levels, and they will form links to people who have similar level of information as themselves. On the other hand, if there are odd number of players, the player n who has the least valuable information will not be linked to anyone. Thus, player n will be ostracized and his choice variable z_n will be free.

Consider a sequential game with perfect information. An ordering of the players is chosen randomly at time zero. Then, at time 1, the first player according to the ordering chosen at time zero chooses his links publicly. Then, at time 2, after observing the first player's choices, the second player according to the random ordering chooses his links publicly. Similarly, at time n, after observing the choices of the previous players, player n according to the random ordering chooses his links. After each player choose their links, at time $n+1$, the players exchange information according to their strategies.

Definition 20 *A Nash equilibrium outcome (of the simultaneous move game) is "robust" if there exists SOME ordering of the players, such that the subgame perfect equilibrium of the sequential move game with that ordering, generates that same outcome.*

Definition 21 *A Nash equilibrium outcome (of the simultaneous move game) is "strongly robust" if for ALL possible orderings of the players, the subgame perfect equilibrium of the sequential move game with that ordering, generates that same outcome.*

Proposition 22 *Assuming $r_{k-1} > r_k$ for all $k \in N \setminus \{1\}$, Assortative Pair Equilibrium of the simultaneous move game where*

$$\begin{aligned} z_{i(i+1)} &= z_{(i+1)i} = 1 \textit{ for all } i \in N \setminus \{n\} \;\; \textit{where } i : odd. \\ z_n &: \textit{free if } n : odd \end{aligned}$$

is strongly robust, i.e., it is the only subgame perfect equilibrium of the sequential game for all possible orderings. Therefore, Assortative Pair Equilibrium is the unique equilibrium of the sequential game that is strongly robust.

Proof. Proof will be done in steps.

(Step One) Assume that player j is not ostracized. If $z_{ij} = 0$, then $z_{ji} = 0$ at equilibrium for the simultaneous and sequential move games.

Proof. Assume $z_{ij} = 0$. Since player j is not ostracized, there exists a player h such that $z_{hj} \neq 0$. Moreover, since $\sum_{k \neq j} z_{jk} = 1$, we can write $z_{jh} = 1 - z_{ji} - \sum_{k \neq j,i,h} z_{jk}$. We can substitute this into S_j. If we take the derivative of S_j with respect to z_{ji}, we get

$$\frac{\partial S_j}{\partial z_{ji}} = -\beta z_{hj} r_h - \beta^2 z_{hj} \sum_{k \neq j,h} z_{hk} z_{kh} r_k < 0$$

Thus, $z_{ji} = 0$.

■

(Step Two) If player 2 moves before player 1 and sets $z_{21} = 1$, then player 1 will set $z_{12} = 1$ at equilibrium for the sequential move game.

Proof. Suppose player 2 moves before player 1 in sequential game and sets $z_{21} = 1$. We can substitute $z_{13} = 1 - z_{12} - \sum_{k \neq 1,2,3} z_{1k}$ in player 1's payoff function and take derivative with respect to z_{12}

$$\frac{\partial S_1}{\partial z_{12}} = \beta r_2 - \beta z_{31} r_3 - \beta^2 z_{31} \sum_{k \neq 1,2,3} z_{3k} z_{k3} r_k$$

Since $0 \leqslant z_{k3} \leqslant 1$ and $r_k < r_3$ for $k > 3$, we have $z_{k3} r_k < r_3$. Thus,

$$\frac{\partial S_1}{\partial z_{12}} = \beta r_2 - \beta z_{31} r_3 - \beta^2 z_{31} \sum_{k \neq 1,2,3} z_{3k} z_{k3} r_k > \beta r_2 - \beta z_{31} r_3 - \beta^2 z_{31} r_3 \sum_{k \neq 1,2,3} z_{3k}$$

Notice that $\sum_{k \neq 1,2,3} z_{3k} = 1 - z_{31} - z_{32}$. From Step 1, we also know that $z_{32} = 0$ as $z_{23} = 0$ if player 3 is not ostracized. So, $\sum_{k \neq 1,2,3} z_{3k} = 1 - z_{31}$ and we get

$$\begin{aligned} \frac{\partial S_1}{\partial z_{12}} &> \beta r_2 - \beta z_{31} r_3 - \beta^2 (1 - z_{31}) z_{31} r_3 \\ &> \beta r_2 - \beta z_{31} r_3 - \beta^2 \left(1 - z_{31}\right) r_3 \\ &> \beta r_2 - \beta z_{31} r_3 - \beta \left(1 - z_{31}\right) r_3 \\ &> \beta r_2 - \beta r_3 > 0 \end{aligned}$$

Therefore, player 1 chooses $z_{12} = 1$.

If player 3 is ostracized, then $z_{k3} = 0$ for all $k \neq 1, 2, 3$. Then,

$$\frac{\partial S_1}{\partial z_{12}} = \beta r_2 - \beta z_{31} r_3 - \beta^2 z_{31} \sum_{k \neq 1,2,3} z_{3k} z_{k3} r_k = \beta r_2 - \beta z_{31} r_3 > 0$$

Therefore, player 1 chooses $z_{12} = 1$.

■

(Step Three) If player 2 moves before player 1, then he sets $z_{21} = 1$.

Proof. Fix the strategies z_{-2} of all players except player 2. Let z_2^a denote the strategy such that $z_{2a} = 1$, i.e., player 2 invests only in player a, where $a \neq 1$. His payoff from this strategy is

$$S_2(z_{-2}, z_2^a) = r_2 + \beta z_{a2} r_a + \beta^2 z_{a2} \sum_{k \neq 2,a} z_{ak} z_{ka} r_k$$

Moreover, he knows from the previous step that if he sets $z_{21} = 1$, then player 1 sets $z_{12} = 1$. His payoff from this strategy is

$$S_2(z_{-2}, z_2^1) = r_2 + \beta r_1$$

Notice that since $r_a < r_1$, we have

$$\begin{aligned} S_2(z_{-2}, z_2^a) &= r_2 + \beta z_{a2} r_a + \beta^2 z_{a2} \sum_{k \neq 2,a} z_{ak} z_{ka} r_k < r_2 + \beta z_{a2} r_a + \beta^2 z_{a2} r_1 \sum_{k \neq 2,a} z_{ak} \\ &= r_2 + \beta z_{a2} r_a + \beta^2 \left(1 - z_{a2}\right) z_{a2} r_1 < r_2 + \beta r_1 = S_2(z_{-2}, z_2^1) \end{aligned}$$

Therefore, player 2 cannot get higher payoff by linking to someone rather than player 1. This also implies that player 2 wouldn't use mixed strategies such as $0 < z_{2i} < 1$ for some

$i \in N \setminus \{1\}$. To see this, suppose player 2 invests in $a \in \Omega(z)$, i.e., $z_{2a} > 0$. Then, by the Nash property, we must have

$$S_2(z_{-2}, z_2^a) - S_2(z_{-2}, z_2^b) \geq 0$$

for all $b \in N$. However, we know that

$$S_2(z_{-2}, z_2^a) < S_2(z_{-2}, z_2^1)$$

for $a \in N \setminus \{1\}$. Thus, if player 2 moves before player 1, he sets $z_{21} = 1$.

■

(Step Four) If player 1 moves before player 2 and sets $z_{12} = 1$, then player 2 will set $z_{21} = 1$ at equilibrium for the sequential move game.

Proof. Suppose player 1 moves before player 2 in sequential game and sets $z_{12} = 1$. We can substitute $z_{23} = 1 - z_{21} - \sum_{k \neq 1,2,3} z_{2k}$ in player 2's payoff function and take derivative with respect to z_{21}

$$\frac{\partial S_2}{\partial z_{21}} = \beta r_1 - \beta z_{32} r_3 - \beta^2 z_{32} \sum_{k \neq 1,2,3} z_{3k} z_{k3} r_k$$

Since $0 \leqslant z_{k3} \leqslant 1$ and $r_k < r_3$ for $k > 3$, we have $z_{k3} r_k < r_3$. Thus,

$$\frac{\partial S_2}{\partial z_{21}} = \beta r_1 - \beta z_{32} r_3 - \beta^2 z_{32} \sum_{k \neq 1,2,3} z_{3k} z_{k3} r_k > \beta r_2 - \beta z_{32} r_3 - \beta^2 z_{32} r_3 \sum_{k \neq 1,2,3} z_{3k}$$

Notice that $\sum_{k \neq 1,2,3} z_{3k} = 1 - z_{31} - z_{32}$. From Step One, we also know that $z_{31} = 0$ as

$z_{13} = 0$. So, $\sum_{k \neq 1,2,3} z_{3k} = 1 - z_{31}$ and we get

$$\begin{aligned}
\frac{\partial S_2}{\partial z_{21}} &> \beta r_1 - \beta z_{32} r_3 - \beta^2 (1 - z_{32}) z_{32} r_3 \\
&> \beta r_1 - \beta z_{32} r_3 - \beta^2 \left(1 - z_{32}\right) r_3 \\
&> \beta r_1 - \beta z_{32} r_3 - \beta \left(1 - z_{32}\right) r_3 \\
&> \beta r_1 - \beta r_3 > 0
\end{aligned}$$

Therefore, player 2 chooses $z_{21} = 1$.

■

(Step Five) If player 1 moves before player 2, then he sets $z_{12} = 1$.

Proof. Fix the strategies z_{-1} of all players except player 1. Let z_1^a denote the strategy such that $z_{1a} = 1$, i.e., player 1 invests only in player a, where $a \neq 2$. His payoff from this strategy is

$$S_1(z_{-1}, z_1^a) = r_1 + \beta z_{a1} r_a + \beta^2 z_{a1} \sum_{k \neq 1,a} z_{ak} z_{ka} r_k$$

Moreover, player 1 knows from the previous step that if he sets $z_{12} = 1$, then player 2 sets $z_{21} = 1$. His payoff from this strategy is

$$S_1(z_{-1}, z_1^2) = r_1 + \beta r_2$$

Notice that since $r_a < r_2$, we have

$$\begin{aligned}
S_1(z_{-1}, z_1^a) &= r_1 + \beta z_{a1} r_a + \beta^2 z_{a1} \sum_{k \neq 1,a} z_{ak} z_{ka} r_k < r_1 + \beta z_{a1} r_a + \beta^2 z_{a1} r_2 \sum_{k \neq 2,a} z_{ak} \\
&= r_1 + \beta z_{a1} r_a + \beta^2 \left(1 - z_{a1}\right) z_{a1} r_2 < r_1 + \beta r_2 = S_1(z_{-1}, z_1^2)
\end{aligned}$$

Therefore, player 1 cannot get higher payoff by linking to someone rather than player 2. This also implies that player 1 wouldn't use mixed strategies such as $0 < z_{1i} < 1$ for some $i \in N \setminus \{1\}$. To see this, suppose player 2 invests in $a \in \Omega(z)$, i.e., $z_{1a} > 0$. Then, by the Nash property, we must have

$$S_1(z_{-1}, z_1^a) - S_1(z_{-1}, z_1^b) \geq 0$$

for all $b \in N$. However, we know that

$$S_1(z_{-1}, z_1^a) < S_1(z_{-1}, z_1^2)$$

for $a \in N \setminus \{1\}$. Thus, if player 1 moves before player 2, he sets $z_{12} = 1$.

■

(Step Six) By induction, regardless of the ordering in which players move, we will have

$$\begin{aligned} z_{i(i+1)} &= z_{(i+1)i} = 1 \text{ for all } i \in N \text{ where } i : odd \\ z_n &: \quad free \text{ if } n : odd \end{aligned}$$

at the equilibrium of the sequential move game.

Proof. From the previous steps, we know that regardless of the ordering in which players move in the sequential move game, we will have $z_{12} = z_{21} = 1$. This is due to the fact that these players are the ones with the highest information and cannot get higher payoffs by linking to other players. Since $z_{12} = z_{21} = 1$ at the equilibrium for the sequential move game, all players will have $z_{i1} = z_{i2} = 0$ for all $i \in N \setminus \{1, 2\}$. Then, the sequential move game can be reduced to the game where we only have players $\{3, 4, ..., n\}$ with the same random ordering chosen in time zero not including players 1 and 2. In this case, player 3 and 4 will become the players with the highest level of information. Thus, from Step 1 through Step 5 we know that $z_{34} = z_{43} = 1$ at equilibrium, regardless of the ordering in which players move in the sequential move game. Then, the sequential move game can be reduced to the game where we only have players $\{5, 6, ..., n\}$ with the same random ordering chosen in time zero not including players 1, 2, 3, 4.

Continuing in the same manner, by induction, we can conclude that at the equilibrium of the sequential move game, regardless of the ordering in which players move, we have

$$z_{i(i+1)} = z_{(i+1)i} = 1 \text{ for all } i \in N \text{ where } i : odd$$

Moreover, if n is odd, then all the players except player n is matched with another player. Thus, his payoff is $S_n = r_n$, making his payoff independent of his choice variable z_n. Therefore, we have $z_n : free$ if n is odd at the subgame perfect equilibrium. ■ ■

Proposition 23 *Nash equilibrium under Cobb-Douglas link strength function may not be surplus maximizing outcome.*

Proof. Assume that there are only three players and they are ordered in terms of their information so that $r_1 > r_2 > r_3 > 0$. From Proposition 22, only the Assortatively Pair Equilibrium is strongly robust. Under this equilibrium, player 1 and 2 spend all their time with each other. Therefore, player 3 is obstrasized. Assume that player 3 sets $z_{31} = 1$. We will show that a higher level of total surplus could be achieved under some conditions by changing player 1's equilibrium strategy. The following shows the partial derivative of total surplus with respect to z_{12} evaluated at $z_{12} = 1, z_{21} = 1, z_{31} = 1$.

$$\left[\frac{\partial S}{\partial z_{12}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = -\beta\left(r_3 - r_2 + \beta r_2 + \beta r_3\right)$$

If the difference between the intrinsic values of player 2 and 3 is low enough, that is, when

$$\frac{r_2}{r_3} < \frac{1+\beta}{1-\beta}$$

the partial derivative of the total surplus with respect to z_{12} would be negative at the Assortative Pair Equilibrium with $z_{12} = z_{21} = z_{31} = 1$. Then, a higher total social surplus could be achieved by decreasing z_{12} by a small amount so that player 1 has a link with both player 1 and 2. The players' individual surpluses shows that the increase in the total surplus is a result of the increase in player 3's surplus being higher than the total decrease in player 1 and 2's surpluses.

$$\left[\frac{\partial S_1}{\partial z_{12}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = \beta\left(r_2 - r_3\right)$$

$$\left[\frac{\partial S_2}{\partial z_{12}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = \beta\left(r_1 - \beta r_3\right)$$

$$\left[\frac{\partial S_3}{\partial z_{12}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = -\beta\left(r_1 + \beta r_2\right)$$

■

2.6 Conclusion

In this chapter, we analyze the formation of networks when players choose how much time to invest in other players. This is one of the few papers on weighted link formation that includes all possible paths in the network in calculation of indirect benefits. We assume that each player has an intrinsic value of information to share and one unit of endowment to invest in relationships with others. Once a direct link is formed, the information is transferred both ways with decay. Moreover, indirect links can transmit indirect information. However, the benefits from indirect information transfers are zero when two agents are connected by more than two links.

We study the model under two different link strength functions. First, we assume that the link strength is the arithmetic mean of agents' investment levels, i.e., the agents are perfect substitutes. As a positive investment of an agent is enough for a link to be formed, this specification allows players to form links unilaterally with others. We show that, when the investments are perfect substitutes, every player is connected to another either directly or indirectly with no more than two links under any Nash equilibrium. Moreover, we find that the strict Nash equilibrium structure is a star network in which all players are connected to the one with the highest total value of information.

Alternatively, we assume that the link strength function is Cobb-Douglas. Since a link between a pair of players is formed only when each of them invests in the relationship, players have to have bilateral agreement to form links with each other. Under the Cobb-Douglas link strength function, we show that paired networks, in which players are matched in pairs, are Nash equilibria. However, we also consider a sequential game in which players choose and announce their strategies publicly according to a random ordering. We show that an Assortative Pair Equilibrium, in which players are assortatively matched in pairs according to their information levels, is the only subgame perfect equilibrium of the sequential game for all possible orderings of the players. Therefore, we conclude that the Assortative Pair Equilibrium is the only strongly robust Nash equilibrium. Lastly, for both link strength functions, Nash equilibria may not be a surplus-maximizing outcome.

Equilibrium network structures vary with the link strength function. The main distinction between the different specifications of the function is the availability of the information and the element of consent to exchange information. Even though many applications have elements of both, additively separable link strength function is more appropriate for the situations when intrinsic information of an agent is available publicly; whereas, Cobb-Douglas is applicable to the situations in which both agents are required to invest in a relationship in order to exchange information. Our results are consistent with the real life applications. Particularly, pioneer agents emerge in the applications that one-sided investment is sufficient for forming a link. These situations are reminiscent to additively separable case, in which strict Nash equilibrium is a star network. Moreover, collaboration network discussed in Blau (1963) requires two-sided investments. Blau (1963) reports that the agents establish partnerships of mutual consultation and less competent agents tend to pair of as partners. This network architecture is akin to the Assortative Pair Equilibrium of Cobb-Douglas case.

Future work can proceed in a number of interesting directions. In this work, we assume that agents differ only in their intrinsic value of information. However, the agents can exhibit asymmetries in terms of endowment levels, that is, some agents may have more time to invest in relationships with others. It would be interesting to examine the effect of this additional availability on the decisions of agents. Another line of asymmetry is the coefficient for decay. In the real world, people are heterogeneous in term of their communication skills. Thus, the effectiveness of the communication may differ accordingly. This situation could be examined by allowing for different levels of decay.

Another line for future research is to weaken the assumption of the benefits from indirect information transfers are zero when two agents are connected by more than two links.

Weakening this assumption increases the benefits from indirect communication. Therefore, especially in the additively separable case, we may observe different equilibrium architectures. However, we should note that weakening this assumption may result in computational difficulties.

3 Nash-Stable and Strongly Efficient Networks in Three-Player Weighted Link Formation Game: Full Characterization

3.1 Introduction

In the second chapter, we analyze the formation of networks when players choose how much time to invest in other players. This is one of the few papers on weighted link formation that includes all possible paths in the network in calculation of indirect benefits. Unfortunately, a complete characterization of Nash equilibria and strongly efficient outcomes, the network structures that maximize the total utility of the agents, is difficult in network formation problems, as the analysis becomes intractable once the indirect benefits are taken into account. In this chapter, we consider the model introduced in the second chapter and fully characterize the Nash equilibria and surplus-maximizing outcomes for a three-player game, in order to investigate how equilibrium structures are different from the efficient outcomes and how these structures differ under different link strength functions.

As discussed in the second chapter, there exists a general tension between stability and efficiency[3]. In their seminal paper, Jackson and Wolinsky (1996) analyze the connections and co-author model. In the connections model, individuals benefit from indirect connections. Therefore, players would prefer to have their neighbors have more connections rather than fewer. For the connections model, Jackson and Wolinsky (1996) show that for a wide variety of parameters, surplus-maximizing networks are not stable. They also conclude that the tension between stability and efficiency arises because the players do not account for the indirect benefits that their connections bring to their neighbors. That is, when forming

[3] We will provide a brief discussion of the tension between efficiency and stability here. For more detailed discussion of the papers, please refer to the second chapter.

a link, a player only pays attention to his own payoff and does not consider whether the link would increase the payoff of others.

Additionally, in the co-author model, indirect connections result in negative externalities as they create distractions. Therefore, individuals prefer to connect with players with less connections. By analyzing the co-author model, Jackson and Wolinsky (1996) show that stable networks are over-connected. The inefficiency of the stable networks are caused by the fact that a player only pays attention to his own benefit when forming a link and does not consider the harm of this additional link to his partners. In short, in both models, Jackson and Wolinsky (1996) show that private and social incentives are not aligned and self-centered incentives may lead to inefficient networks.

On the other hand, Bala and Goyal (2000) weaken the assumption of Jackson and Wolinsky (1996) that a formation of a link between two agents require mutual consent of the agents. They allow agents to form links with others unilaterally by incurring the cost of the link and analyze the model under both one-way and two-way flow of benefits. In the model with one-way flow, only the player who forms the link benefits from it, while in two-way flow, once the link is formed, both players enjoy the benefits. Bala and Goyal (2000) find that the efficient and stable networks coincide only when the cost of forming a link is very high or very low in the model with two-way flow. On the contrary, in the one-way flow model, they show that for large set of parameters, the unique efficient network is the wheel network, in which a single directed cycle is formed with each player investing in exactly one link, which is also a strict Nash equilibrium structure.

One of the papers that is closely related to our paper is Bloch and Dutta (2009), which drops the assumption of binary link strengths. They analyze a weighted link formation game in which players have fixed endowments to invest into communicating with others.

However, unlike our model, only the most reliable path, the path that maximizes the product of link strengths, is used to calculate the indirect benefits. They show that under additively separable link strength function, the stable and efficient network architectures are stars.

Lastly, Rogers (2006) studies a weighted link formation game by separating the flow of benefits into "taking" and 'giving" components. Similar to model, all paths between the players are taken into account when calculating the indirect benefits. He finds that all stable and efficient networks are identified as interior with exception of some Nash equilibria in giving model. By separating the flow of benefits, Rogers identifies the source of inefficiency as the giving incentives. He also concludes that the inefficiency is present only when there exists heterogeneity among agents in terms of their budget constraints.

Our results show that for additively separable link strength function, the unique Nash equilibrium is a star network in which all players invest all their time in one player with the highest value of information except themselves. However, depending on the information levels of the players and the efficiency of the communication, the surplus-maximizing outcome structure can be a triangle, in which all agents are connected with each other, or a star, in which player 2 and player 3 do not invest time with each other. Therefore, unlike the results of Bloch and Dutta (2009), Nash equilibrium coincides with the surplus-maximizing outcome only when the differences between the agents 'information levels are relatively high and/or the efficiency of the communication is relatively low when all paths are included in the calculation of the indirect benefits.

Alternatively, under Cobb-Douglas link strength function, we show that under Nash equilibrium player 1 is matched with either player 2 or player 3, and the other player is ostracized. However, unlike the results of Rogers (2006), the is no interior solution for surplus-maximizing outcome. Depending on the information levels of players 2 and 3, and the efficiency of the communication, the surplus-maximizing outcome structure can be a star, in which player 2 and player 3 invest all their time with player 1, or a pair, in which players 1 and 2 are matched and player 3 is ostracized. For the both link strength functions, the results show that the agents have a tendency to connect to fewer agents with higher investment levels from an efficiency perspective.

The rest of the chapter proceeds as follows. The next section, Section 2, analyzes the three player game with additively separable link strength function. Section 3 analyzes the three player game with Cobb-Douglas link strength function. Finally, Section 4 discusses the results and concludes.

3.2 Three Player Game with Additively Separable Link Strength Function

3.2.1 Nash Equilibrium

In this section, we fully characterize the set of Nash equilibrium under additively separable link strength with three players.

Assume the players are ordered in terms of their information so that $r_1 > r_2 > r_3 > 0$. That is, player 1 has the most information and player 3 has the least. Each player has 1 unit of time that he can invest in relationships (links) with the other players. Let z_{ij} the amount of time player i invests in the link to j. Since player i has one unit of time to allocate, $z_{ij} + z_{ik} = 1$. The strength of the link between i and j is

$$\sigma(z_{ij}, z_{ji}) = \frac{1}{2} z_{ij} + \frac{1}{2} z_{ji}$$

Notice $\sigma(1,1) = 1$, so that $\sigma(z_{ij}, z_{ji}) \leq 1$ always holds.

The strategy set for player i consists of the time allocated to other players and is denoted z_i. The strategy set for each player must satisfy the following properties:

$$z_i = \{z_{ik}\}_{k \neq i}$$

$$0 \leq z_{ik} \leq 1 \text{ for all } k \neq i$$

$$\sum_{k \neq i} z_{ik} = 1 \text{ for each player } i$$

Each player can benefit not only from his own information, but also from the information of other players if he is linked to them. The stronger the link is, the greater the share of information is transmitted. However, when information is transmitted along a link, it depreciates by some factor $0 < \beta \leq 1$. Thus, if the direct link between player i and player j has strength $\sigma(z_{ij}, z_{ji})$, then obtaining player j's information via this direct link is worth $\beta\sigma(z_{ij}, z_{ji})r_j$ to player i. Moreover, we assume that links can transmit indirect information. Then, if player j is linked to player k, then player k's information can be indirectly transmitted from k to i via j. Obtaining player k's information via this indirect link is worth $\beta\sigma(z_{ij}, z_{ji})\beta\sigma(z_{jk}, z_{kj})r_k$ to player i. The surplus for player i consists of the direct and indirect information obtained by communicating with other players and is denoted as S_i. So, the surplus for player i can be calculated to be

$$S_i = r_i + \beta\frac{z_{ik} + z_{ki}}{2}(r_k + \beta\frac{z_{kj} + z_{jk}}{2}r_j) + \beta\frac{z_{ij} + z_{ji}}{2}(r_j + \beta\frac{z_{kj} + z_{jk}}{2}r_k) \text{ for } i \neq k \neq j$$

Since there are only three players,

$$z_{ik} + z_{ij} = 1 \text{ for all } i \neq k \neq j.$$

$$z_{ij} = 1 - z_{ik}$$

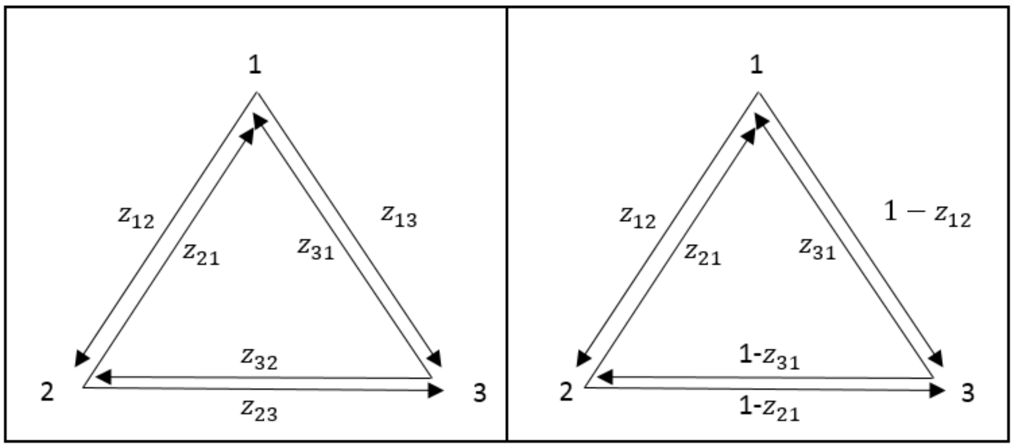


Figure 3: Substitution of the link strengths under additively separable link strength function

We can rewrite S_i by using the figure above:

$$\begin{aligned} S_i &= r_i + \beta\frac{z_{ik} + z_{ki}}{2}(r_k + \beta\frac{z_{kj} + z_{jk}}{2}r_j) + \beta\frac{z_{ij} + z_{ji}}{2}(r_j + \beta\frac{z_{kj} + z_{jk}}{2}r_k) \\ &= r_i + \beta\frac{z_{ik} + z_{ki}}{2}(r_k + \beta\frac{(1 - z_{ki}) + (1 - z_{ji})}{2}r_j) + \beta\frac{(1 - z_{ik}) + z_{ji}}{2}(r_j + \beta\frac{(1 - z_{ki}) + (1 - z_{ji})}{2}r_k) \\ &= r_i + \beta\frac{z_{ik} + z_{ki}}{2}(r_k + \beta\frac{2 - z_{ki} - z_{ji}}{2}r_j) + \beta\frac{1 - z_{ik} + z_{ji}}{2}(r_j + \beta\frac{2 - z_{ki} - z_{ji}}{2}r_k) \end{aligned}$$

So, player í's problem is to maximize his own surplus by choosing z_{ik} subject to $0 \leq z_{ik} \leq 1$ and $z_{ij} = 1 - z_{ik}$.

Proposition 24 *The Nash equilibrium strategies for non-cooperative game for* $0 < \beta \leq 1$

are

$$z_{12} = 1, z_{13} = 0$$

$$z_{21} = 1, z_{23} = 0$$

$$z_{31} = 1, z_{32} = 0$$

So, the link strengths between the players are

$$\sigma_{12} = 1$$

$$\sigma_{13} = \frac{1}{2}$$

$$\sigma_{23} = 0$$

and the surplus for each player is

$$\begin{aligned} S_1 &= r_1 + \beta r_2 + \frac{\beta}{2} r_3 \\ S_2 &= r_2 + \beta r_1 + \frac{\beta^2}{2} r_3 \\ S_3 &= r_3 + \frac{\beta}{2} r_1 + \frac{\beta^2}{2} r_2 \end{aligned}$$

where $S_1 > S_2 > S_3$.

Proof. The proof is in the appendix. ■

3.2.2 Surplus-Maximizing Outcome

In this section, we fully characterize surplus-maximizing link structure, in which the total utility of the agents is maximized, under additively separable link strength with three

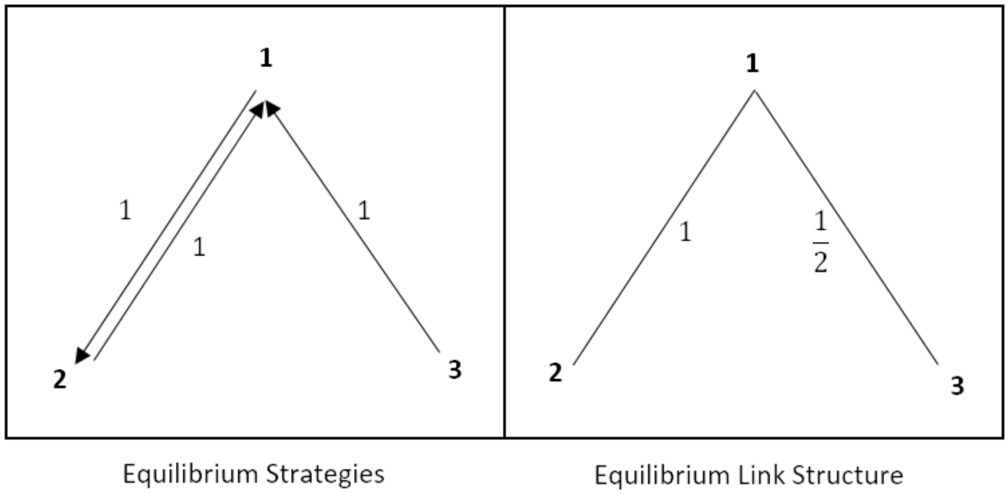


Figure 4: Nash equilibrium strategies and link structure under additively separable link strength function

players. We again assume the players are ordered so $r_1 > r_2 > r_3 > 0$ holds.

The surplus for player i consists of the direct and indirect information obtained by communicating with other players and is denoted as S_i and defined as

$$S_i = r_i + \beta\frac{z_{ik} + z_{ki}}{2}(r_k + \beta\frac{z_{kj} + z_{jk}}{2}r_j) + \beta\frac{z_{ij} + z_{ji}}{2}(r_j + \beta\frac{z_{kj} + z_{jk}}{2}r_k) \text{ for } i \neq k \neq j.$$

The total surplus is calculated by

$$S = S_1 + S_2 + S_3$$

Hence, the social planner's problem is to maximize the social surplus by choosing the link strengths z_1, z_2, z_3 subject to

$$z_i = \{z_{ik}\}_{k \neq i}$$

$$0 \leq z_{ik} \leq 1 \text{ for all } k \neq i$$

$$\sum_{k \neq i} z_{ik} = 1 \text{ for each player } i$$

From the point of view of social surplus, it is only the link strengths σ_{ij} that matter, not the z_{ij}. However, the z_{ij} matter for the overall resource constraint: because $z_{ij} + z_{ik} = 1$ we obtain the following constraint on the link strengths:

$$\sigma_{12} + \sigma_{23} + \sigma_{13} = \frac{3}{2}$$

We claim, the planner can maximize surplus in two steps. First, choose the link strengths $\sigma_{12}, \sigma_{23}, \sigma_{13}$ to maximize surplus subject to $0 \leq \sigma_{ij} \leq 1$ and $\sigma_{12} + \sigma_{23} + \sigma_{13} = \frac{3}{2}$. Second, allocate the individual links z_{ij} so everything adds up correctly.

Lemma 25 *Suppose we have σ_{ij} such that $0 \leq \sigma_{ij} \leq 1$ and $\sigma_{12} + \sigma_{23} + \sigma_{13} = \frac{3}{2}$. Then we can always find z_{ij} such that $0 \leq z_{ij} \leq 1$ and $z_{ij} + z_{ik} = 1$ and which satisfy*

$$\sigma_{12} = \frac{1}{2} z_{12} + \frac{1}{2} z_{21} \tag{6}$$

$$\sigma_{23} = \frac{1}{2} z_{23} + \frac{1}{2} z_{32} \tag{7}$$

$$\sigma_{13} = \frac{1}{2} z_{13} + \frac{1}{2} z_{31} \tag{8}$$

Proof. The proof is in the appendix. ■

From now on, then, for the sake of social surplus, we can forget the z_{ij}. The total surplus can be calculated to be

$$\begin{aligned} S &= r_1 \left(1+\beta(\sigma_{12}+\sigma_{13})\left(1+\beta\sigma_{23}\right)\right) + r_2 \left(1+\beta(\sigma_{12}+\sigma_{23})\left(1+\beta\sigma_{13}\right)\right) \\ &\quad + r_3 \left(1+\beta(\sigma_{13}+\sigma_{23})\left(1+\beta\sigma_{12}\right)\right) \end{aligned}$$

The social planner's problem is to maximize the social surplus by choosing the link strengths $\sigma_{12}, \sigma_{23}, \sigma_{13}$ subject to $0 \leq \sigma_{ij} \leq 1$ and $\sigma_{12}+\sigma_{23}+\sigma_{13} = \frac{3}{2}$. Notice that since $\sigma_{23} = \frac{3}{2} - (\sigma_{12}+\sigma_{13})$, we can rewrite the total surplus as

$$\begin{aligned} S &= r_1 \left(1+\beta(\sigma_{12}+\sigma_{13})\left(1+\beta(\frac{3}{2}-\sigma_{12}-\sigma_{13})\right)\right) + r_2 \left(1+\beta(\frac{3}{2}-\sigma_{13})\left(1+\beta\sigma_{13}\right)\right) \\ &\quad + r_3 \left(1+\beta(\frac{3}{2}-\sigma_{12})\left(1+\beta\sigma_{12}\right)\right) \end{aligned}$$

Proposition 26 *The social planner's problem has a unique solution.*

Proof. The proof is in the appendix. ■

Note that since the objective function is strictly concave and the constraints are linear, any local maximum of the social surplus will be a global maximum. Therefore, while solving for the social planner's problem, it is sufficient to look at necessary conditions. In other words, Kuhn-Tucker conditions are sufficient for finding global maxima.

Proposition 27 *For any socially optimal solution, $\sigma_{12} > \sigma_{13} > \sigma_{23}$ holds.*

Proof. *The proof is in the appendix.* ■

Proposition 28 *The solution to the social planner's problem is given in the following figure.*

Proof. The proof is in the appendix. ■

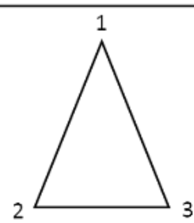
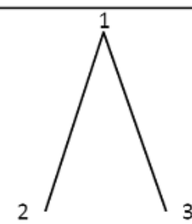
	$\frac{r_2}{r_1} > \frac{2-3\beta}{2-\beta}$ $(\sigma_{23} > 0)$	$\frac{r_2}{r_1} \leq \frac{2-3\beta}{2-\beta}$ $(\sigma_{23} = 0)$
$\frac{r_2}{r_3} \geq \frac{2+\beta}{2-\beta}$ $(\sigma_{12} = 1)$	$\sigma_{12} = 1$ $\sigma_{13} = \left(\frac{r_1(2-\beta) + r_2(3\beta-2)}{4\beta(r_1 + r_2)}\right)$ $\sigma_{23} = \left(\frac{r_1(3\beta-2) + r_2(2-\beta)}{4\beta(r_1 + r_2)}\right)$	$\sigma_{12} = 1$ $\sigma_{13} = \frac{1}{2}$ $\sigma_{23} = 0$
$\frac{r_2}{r_3} < \frac{2+\beta}{2-\beta}$ $(\sigma_{12} < 1)$	$\sigma_{12} = \left(\frac{4r_1r_2 + r_3(r_1 + r_2)(3\beta-2)}{4\beta(r_1r_2 + r_1r_3 + r_2r_3)}\right)$ $\sigma_{13} = \left(\frac{4r_1r_3 + r_2(r_1 + r_3)(3\beta-2)}{4\beta(r_1r_2 + r_1r_3 + r_2r_3)}\right)$ $\sigma_{23} = \left(\frac{4r_2r_3 + r_1(r_2 + r_3)(3\beta-2)}{4\beta(r_1r_2 + r_1r_3 + r_2r_3)}\right)$	$\sigma_{12} = \left(\frac{r_2(2+3\beta) - r_3(2-3\beta)}{4\beta(r_2 + r_3)}\right)$ $\sigma_{13} = \left(\frac{r_2(3\beta-2) + r_3(2+3\beta)}{4\beta(r_2 + r_3)}\right)$ $\sigma_{23} = 0$
Optimal Link Structure		

Figure 5: Surplus-maximizing link structure under additively separable link strength function

Proposition 29 *For any interior socially optimum outcome, and any* $r_i > 0, r_j > 0, r_k > 0$ *and* $0 < \beta \leq 1$, *the strength of the link* σ_{ij} *is increasing in* r_i *and* r_j *and decreasing in* r_k.

Proof. The proof is in the appendix. ■

3.3 Three Player Game with Cobb-Douglas Link Strength Function

3.3.1 Nash Equilibrium

In this section, we fully characterize the set of Nash equilibrium under Cobb-Douglas link strength with three players.

Assume the players are ordered in terms of their information so that $r_1 > r_2 > r_3 > 0$.

That is, player 1 has the most information and player 3 has the least. Each player has 1 unit of time that he can invest in relationships (links) with the other players. Let z_{ij} the amount of time player i invests in the link to j. Since player i has one unit of time to allocate, $z_{ij} + z_{ik} = 1$. The strength of the link between i and j is

$$\sigma(z_{ij}, z_{ji}) = z_{ij} z_{ji}$$

Notice $\sigma(1,1) = 1$, so that $\sigma(z_{ij}, z_{ji}) \leq 1$ always holds. Moreover, $\sigma(z_{ij}, 0) = 0$ without loss of generality. Therefore, there should be bilateral agreement between the players in order to be linked with each other.

The strategy set for player i consists of the time allocated to other players and is denoted z_i. The strategy set for each player must satisfy the following properties:

$$z_i = \{z_{ik}\}_{k \neq i}$$

$$0 \leq z_{ik} \leq 1 \text{ for all } k \neq i$$

$$\sum_{k \neq i} z_{ik} = 1 \text{ for each player } i$$

Each player can benefit not only from his own information, but also from the information of other players if he is linked to them. The stronger the link is, the greater the share of information is transmitted. However, when information is transmitted along a link, it depreciates by some factor $0 < \beta \leq 1$. Thus, if the direct link between player i and player j has strength $\sigma(z_{ij}, z_{ji})$, then obtaining player j's information via this direct link is worth $\beta\sigma(z_{ij}, z_{ji})r_j$ to player i. Moreover, we assume that links can transmit indirect information. Then, if player j is linked to player k, then player k's information can be indirectly transmitted from k to i via j. Obtaining player k's information via this indirect link is worth $\beta\sigma(z_{ij}, z_{ji})\beta\sigma(z_{jk}, z_{kj})r_k$ to player i. The surplus for player i consists of the direct and indirect information obtained by communicating with other players and is denoted as S_i. So, the surplus for player i can be calculated to be

$$S_i = r_i + \beta z_{ik}z_{ki}(r_k + \beta z_{kj}z_{jk}r_j) + \beta z_{ij}z_{ji}(r_j + \beta z_{kj}z_{jk}r_k) \text{ for } i \neq k \neq j.$$

Since there are only three players,

$$z_{ik} + z_{ij} = 1 \text{ for all } i \neq k \neq j.$$

$$z_{ij} = 1 - z_{ik}$$

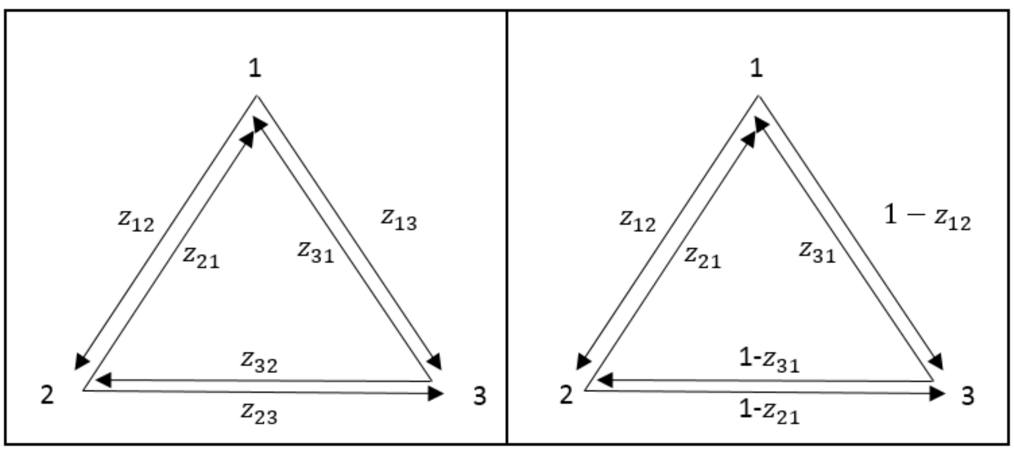


Figure 6: Substitution of the link strengths under Cobb-Douglas link strength function

So we can rewrite S_i by using the figure above:

$$S_i = r_i + \beta z_{ik}z_{ki}(r_k + \beta z_{kj}z_{jk}r_j) + \beta z_{ij}z_{ji}(r_j + \beta z_{kj}z_{jk}r_k)$$

$$= r_i + \beta z_{ik}z_{ki}(r_k + \beta(1 - z_{ki})(1 - z_{ji})r_j) + \beta(1 - z_{ik})z_{ji}(r_j + \beta(1 - z_{ki})(1 - z_{ji})r_k)$$

So, player i's problem is to maximize his own surplus by choosing z_{ik} subject to $0 \leq z_{ik} \leq 1$ and $z_{ij} = 1-\ z_{ik}$.

Proposition 30 *There are two sets of Nash equilibrium strategies for non-cooperative game for* $0 < \beta < 1$*. The first set of strategies is given by*

$$z_{12} = 1, z_{13} = 0$$

$$z_{21} = 1, z_{23} = 0$$

$$z_{31} = free, z_{32} = 1 - z_{31}$$

So, the link strengths between the players are

$$\sigma_{12} = 1$$

$$\sigma_{13} = 0$$

$$\sigma_{23} = 0$$

and the surplus for each player is

$$S_1 = r_1 + \beta r_2$$

$$S_2 = r_2 + \beta r_1$$

$$S_3 = r_3$$

where $S_1 > S_2 > S_3$*. The second one is*

$$z_{12} = 0, z_{13} = 1$$

$$z_{21} = 0, z_{23} = 1$$

$$z_{31} = 1, z_{32} = 0$$

So, the link strengths between the players are

$$\sigma_{12} = 0$$

$$\sigma_{13} = 1$$

$$\sigma_{23} = 0$$

and the surplus for each player is

$$S_1 = r_1 + \beta r_3$$

$$S_2 = r_2$$

$$S_3 = r_3 + \beta r_1$$

where $S_1 > S_3 > S_2$*. Moreover, the social surplus in the first set of Nash equilibria is greater than the second one.*

Proof. The proof is in the appendix. ■

Notice that the social surplus in the second set of Nash equilibria is lower than the first one. However, we know from the previous chapter that only the first set of Nash equilibria is strongly robust.

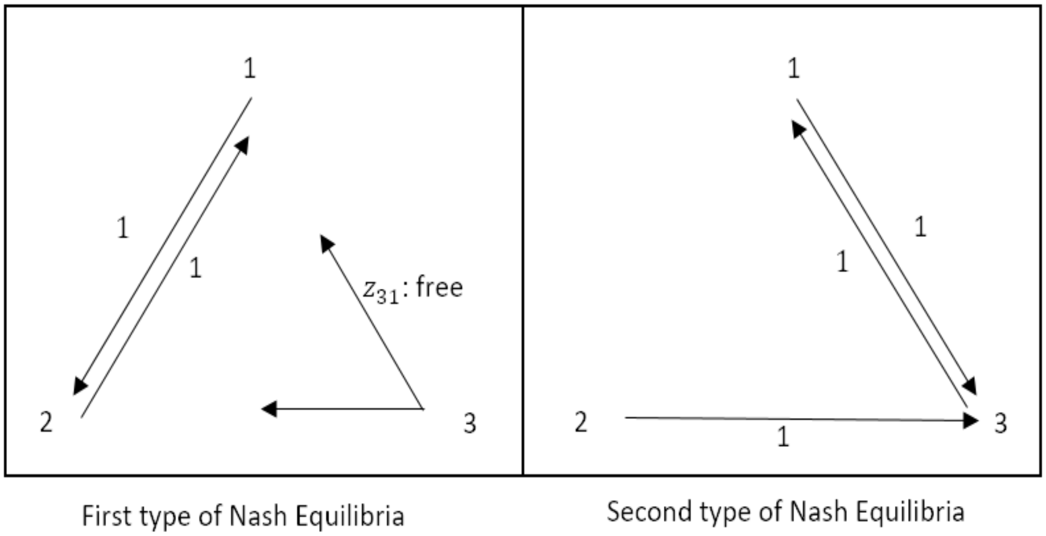


Figure 7: Nash equilibria under Cobb-Douglas link strength

3.3.2 Surplus-Maximizing Outcome

In this section, we fully characterize surplus-maximizing link structure, in which the total utility of the agents is maximized, under Cobb-Douglas link strength with three players. We again assume the players are ordered so $r_1 > r_2 > r_3 > 0$ holds.

The surplus for player i consists of the direct and indirect information obtained by communicating with other players and is denoted as S_i and defined as

$$S_i = r_i + \beta z_{ik} z_{ki}(r_k + \beta z_{kj} z_{jk} r_j) + \beta z_{ij} z_{ji}(r_j + \beta z_{kj} z_{jk} r_k) \text{ for } i \neq k \neq j.$$

The total surplus is calculated by

$$S = S_1 + S_2 + S_3$$

Hence, the social planner's problem is to maximize the social surplus

$$\begin{aligned} S &= r_1(1+\beta(z_{12}z_{21}+(1-z_{12})z_{31})(1+\beta(1-z_{21})(1-z_{31}))) \\ &+ r_2(1+\beta(z_{12}z_{21}+(1-z_{21})(1-z_{31}))(1+\beta(1-z_{12})z_{31})) \\ &+ r_3(1+\beta((1-z_{12})z_{31}+(1-z_{21})(1-z_{31}))(1+\beta z_{12}z_{21})) \end{aligned}$$

by choosing the link strengths z_1, z_2, z_3 subject to

$$z_i = \{z_{ik}\}_{k\neq i}$$

$$0 \leq z_{ik} \leq 1 \text{ for all } k \neq i$$

$$\sum_{k\neq i} z_{ik} = 1 \text{ for each player } i$$

Proposition 31 *The social planner's problem has a solution.*

Proof. The proof is in the appendix. ■

Let us write the Lagrangian function as follows.

$$\begin{aligned} \mathcal{L} &= r_1(1+\beta(z_{12}z_{21}+(1-z_{12})z_{31})(1+\beta(1-z_{21})(1-z_{31}))) \\ &+ r_2(1+\beta(z_{12}z_{21}+(1-z_{21})(1-z_{31}))(1+\beta(1-z_{12})z_{31})) \\ &+ r_3(1+\beta((1-z_{12})z_{31}+(1-z_{21})(1-z_{31}))(1+\beta z_{12}z_{21})) \\ &+ \lambda_1(1-z_{12}) + \lambda_2(1-z_{21}) + \lambda_3(1-z_{31}) \end{aligned}$$

The solutions of the following system of equations will be the critical points of the Lagrangian function, $\mathcal{L}$:

$$\frac{\partial \mathcal{L}}{\partial z_{12}} = r_1\beta(z_{21} - z_{31})(1 + \beta(1 - z_{21})(1 - z_{31}))$$
$$+ r_2\beta(z_{21}(1 + \beta(1 - z_{12})z_{31}) - \beta z_{31}(z_{12}z_{21} + (1 - z_{21})(1 - z_{31}))$$
$$+ r_3\beta(-z_{31}(1 + \beta z_{12}z_{21}) + \beta z_{21}((1 - z_{12})z_{31} + (1 - z_{21})(1 - z_{31})) - \lambda_1$$

$$\frac{\partial \mathcal{L}}{\partial z_{12}} \leq 0$$
$$\frac{\partial \mathcal{L}}{\partial z_{12}} z_{12} = 0$$

$$\frac{\partial \mathcal{L}}{\partial z_{21}} = r_1\beta(z_{12}(1 + \beta(1 - z_{21})(1 - z_{31})) - \beta(1 - z_{31})(z_{12}z_{21} - (1 - z_{12})z_{31})$$
$$+ r_2\beta(z_{12} - (1 - z_{31}))(1 + \beta(1 - z_{12})z_{31})$$
$$+ r_3\beta(-(1 - z_{31})(1 + \beta z_{12}z_{21}) + \beta z_{12}((1 - z_{12})z_{31} + (1 - z_{21})(1 - z_{31}))) - \lambda_2$$

$$\frac{\partial \mathcal{L}}{\partial z_{21}} \leq 0$$
$$\frac{\partial \mathcal{L}}{\partial z_{21}} z_{21} = 0$$

$$\frac{\partial \mathcal{L}}{\partial z_{31}} = r_1\beta((1 - z_{12})(1 + \beta(1 - z_{21})(1 - z_{31})) - \beta(1 - z_{21})(z_{12}z_{21} + (1 - z_{12})z_{31}))$$
$$+ r_2\beta(-(1 - z_{21})(1 + \beta(1 - z_{12})z_{31}) + \beta(1 - z_{12})(z_{12}z_{21} + (1 - z_{12})(1 - z_{31})))$$
$$+ r_3\beta(z_{21} - z_{12})(1 + \beta z_{12}z_{21})) - \lambda_3$$

$$\frac{\partial \mathcal{L}}{\partial z_{31}} \leq 0$$

$$\frac{\partial \mathcal{L}}{\partial z_{31}} z_{31} = 0$$

$$0 \leqslant z_{12} \leqslant 1$$

$$0 \leqslant z_{21} \leqslant 1$$

$$0 \leqslant z_{31} \leqslant 1$$

$$0 \leq \lambda_2$$

$$0 \leq \lambda_3$$

$$0 \leq \lambda_4$$

Since we know that the solution to the social planner's problem exists and we have linear inequality constraints, the solutions to the system will satisfy the Kuhn-Tucker first order conditions for the social planner's problem. Thus, the solution to the social planner's problem will be among the critical points of $\mathcal{L}$. Before solving for the critical points of $\mathcal{L}$, we will restrict the set of possible solutions with the Lemma 32 and 33.

Lemma 32 *For socially optimum outcome, if $z_{ij} = 0$, then we should have $z_{ji} = 0$.*

Proof. The proof is in the appendix. ■

The previous lemma restricts the possible link strengths into 7 cases which is summarized in the following figure. The next lemma further restricts the possible link strengths into 3 cases.

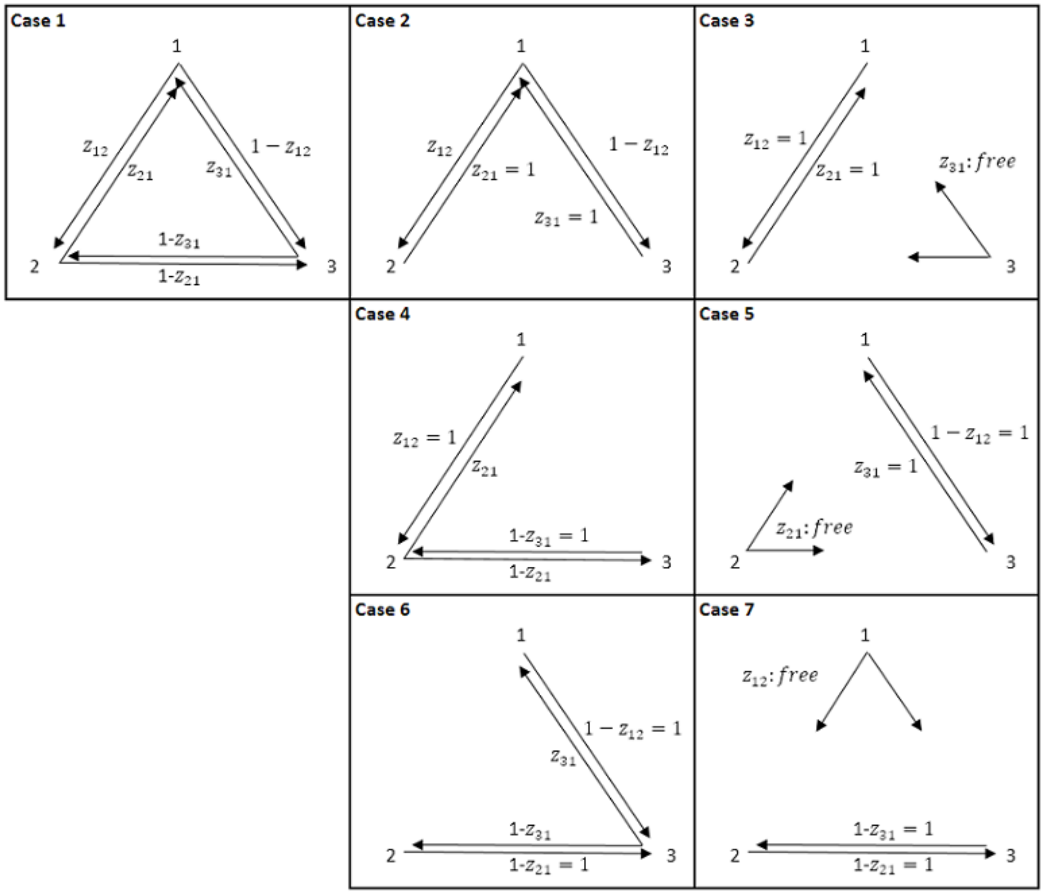


Figure 8: Possible cases for socially optimum outcome under Cobb-Douglas link strength function

Lemma 33 *There are only 3 possible outcomes for socially optimum:*

interior solution where $0 < z_{12} < 1$, $0 < z_{21} < 1$, $0 < z_{31} < 1$,

only two links are formed where $0 < z_{12} < 1$, $z_{21} = z_{31} = 1$,

and only one link is formed where $z_{12} = z_{21} = 1, 0 \leq z_{31} \leq 1$.

Proof. The proof is in the appendix. ■

Proposition 34 *For any socially optimal solution,* $\sigma_{12} > \sigma_{13} \geqslant \sigma_{23}$ *holds.*

Proof. The proof is in the appendix. ■

Proposition 35 *For any interior solution, we have* $\frac{1}{2} < z_{12} < 1$, $\frac{1}{2} < z_{21} < 1$ *and* $\frac{1}{2} < z_{31} < 1$ *at the optimum.*

Proof. The proof is in the appendix. ■

Proposition 36 *When* $0 < \beta \leqslant 1$*, the solution for the social planner's problem is*

- *For* $\frac{r_2}{r_3} < \frac{1+\beta}{1-\beta}$*, the optimal solution is given by*

$$z_{12} = \frac{1}{2} + \frac{r_2 - r_3}{2\beta(r_2 + r_3)}$$

$$z_{21} = 1$$

$$z_{31} = 1$$

  So, the link strengths between the players are

$$\sigma_{12} = \frac{1}{2} + \frac{r_2 - r_3}{2\beta(r_2 + r_3)}$$

$$\sigma_{13} = \frac{1}{2} - \frac{r_2 - r_3}{2\beta(r_2 + r_3)}$$

$$\sigma_{23} = 0$$

- *For* $\frac{r_2}{r_3} \geqslant \frac{1+\beta}{1-\beta}$*, the optimal solution is given by*

$$z_{12} = 1$$

$$z_{21} = 1$$

$$z_{31} : free$$

Proof. The proof is in the appendix. ■

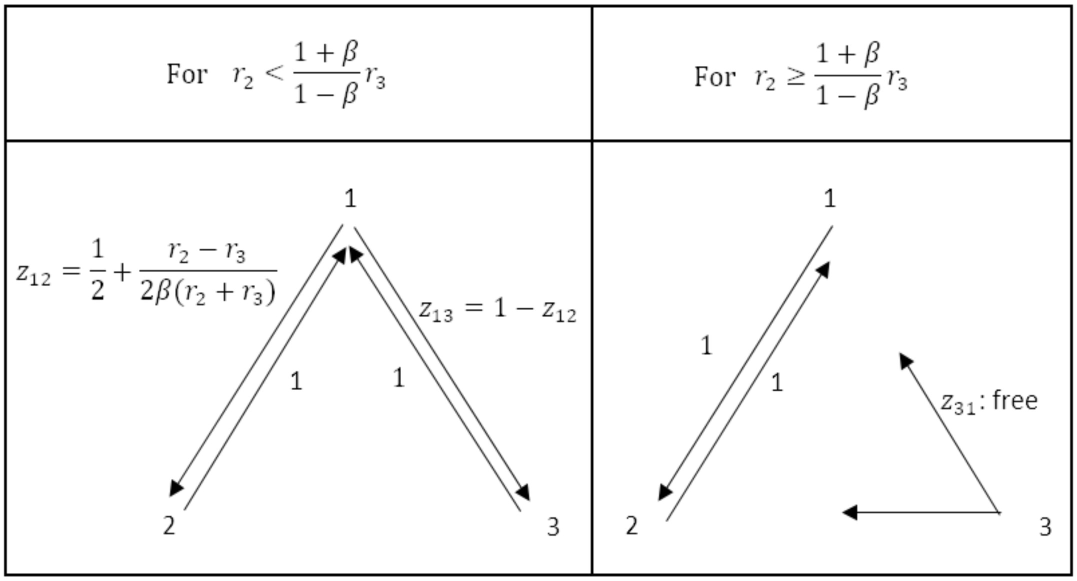


Figure 9: Surplus-maximizing link structure under Cobb-Douglas link strength function

3.4 Conclusion

In this chapter, we analyze the weighted link formation game introduced in the second chapter, and fully characterize Nash equilibrium and socially optimum outcome for the game with three players. Similar to the second chapter, all possible paths are included in the network in calculation of indirect benefits. We assume that each player has an intrinsic value of information to share and one unit of endowment to invest in relationships with others. Once a direct link is formed, the information is transferred both ways with decay. Moreover, indirect links can transmit indirect information with a decay.

We study the model under two different link strength functions. First, we assume that the link strength is the arithmetic mean of agents' investment levels, i.e., the agents are perfect substitutes. As a positive investment of an agent is enough for a link to be formed, this specification allows players to form links unilaterally with others. We show that, when the investments are perfect substitutes, the unique Nash equilibrium is a star network in which all players invest all their time in one player with the highest value of information except themselves. On the other hand, depending on the agents' information levels and the efficiency of the communication, β, the surplus-maximizing outcome structure can be a triangle, in which all agents are connected with each other, or a star, in which player 2 and player 3 do not invest time with each other. Nash equilibrium coincides with the surplus-maximizing outcome only when the differences between the agents' information levels are relatively high and/or the efficiency of the communication, β, is relatively low. Thus, when players maximize their own surplus, they tend to over-invest in the relationship with the player who has the highest information level except themselves.

Alternatively, we assume that the link strength function is Cobb-Douglas. Since a link between a pair of players is formed only when each of them invests in the relationship, players have to have bilateral agreement to form links with each other. Under the Cobb-Douglas link strength function, we show that there are two different Nash equilibria, in which players are matched in pairs. In the first type of Nash equilibrium, player 1 and 2 invest all their time with each other, where player 3 is ostracized. In the second type of Nash equilibrium, similar to the n-player case, there is a coordination problem between players 1 and 2. Therefore, under this equilibrium, players 1 and 3 are matched in pairs and player 2 is ostracized, resulting in a lower total surplus compared to the first type of Nash equilibrium.

On the other hand, for Cobb-Douglas link strength function, depending on the information levels of players 2 and 3, and the efficiency of the communication, β, the surplus-maximizing outcome structure can be a star, in which player 2 and player 3 invest all their time with player 1, or a pair, in which players 1 and 2 are matched and player 3 is ostracized. The first type of Nash equilibrium coincides with the surplus-maximizing outcome only when the differences between the information levels of players 2 and 3 are relatively high and/or the efficiency of the communication, β, is relatively low. Therefore, similar to the additively separable link strength function, when players maximize their own surplus, they tend to over-invest in the relationship with the player who has the highest information level except themselves.

In conclusion, equilibrium and efficient network structures vary with the link strength function. At equilibrium, the agents choose to invest all their time with only one agent regardless of the link strength function. More links are formed when the agents are perfect substitutes compared to Cobb-Douglas link strength, in which bilateral agreement is required for link formation. For both link strength functions, Nash equilibria may not be a surplus-maximizing outcome. As opposed to the findings of Jackson and Wolinsky (1996), Bala and Goyal (2000) and Bloch and Dutta (2009), the results show that the agents have a tendency to connect to fewer agents with higher investment levels from an efficiency perspective.

Future work on weighted link formation games can proceed in a number of interesting directions. As discussed in the previous chapter, one line for future research is to weaken the assumption of the benefits from indirect information transfers are zero when two agents are connected by more than two links. Weakening this assumption increases the benefits from indirect communication. Even though this assumption is not restrictive in the three-player game, it adds tractability to the analysis of n-player game. To examine the effects of this assumption, future work should be done by weakening this assumption starting with a four-player game.

Another line discussed previously is introducing asymmetries between agents, such as asymmetries in terms of the agents' endowment levels and communication skills. However, adding additional asymmetries may result in computational difficulties. Therefore, we suggest that future work on the effects of these additional asymmetries should start with the analysis of the three-player game.

4 Strategic Information Flows in Multi-Agent Environments

4.1 Introduction

The model analyzed in the previous chapters assumes that once the agents are connected, the information is shared. However, if the interests of the agents are not aligned, they may strategically withhold information. In this chapter, we study strategic information flows within economic environments characterized by interdependencies among agents. A typical example of such an environment would be a multi-product firm. Specifically, we investigate a model of communication with two agents and a principal, allowing for asymmetric interdependencies between the agents. Each agent has private information on different dimensions of the state of nature. The interdependencies are characterized as action complementarities or substitutabilities between different agents within the same economic environment.

Since most of the information held by different departments within the firm is not verifiable, we model communication between the agents as cheap talk messages. In the canonical cheap talk model, defined in their seminal paper by Crawford and Sobel (1982), there is an informed agent on the state of the world who sends costless messages to a decision maker (principal). The best action from the perspective of the principal and the agent depends on the agent's information. However, the agent is biased, i.e., his best action is different than the principal's best action. Moreover, the agent's information is not contractible. Under these conditions, Crawford and Sobel (1982) show that communication is informative to a certain extent when the agent's bias is not large.

Similar to Crawford and Sobel (1982), we will assume that the decisions made within an organization are complex; and hence, they are not contractible. As the organization lacks commitment, the only formal authority will be the allocation of the decision rights. By using cheap talk model, we look at two different communication protocols between uninformed principal and two agents who are privately informed about an independent aspect of the state of the world. Under the first protocol, which we will refer to as vertical communication, we have a centralized decision mechanism in which the principal (the headquarters) makes the production decisions for each agent (department) after observing the reported private messages of the agents. Under the second protocol, which we will refer to as horizontal communication, we have a decentralized mechanism in which the agents are allowed to communicate with each other via cheap talk and then make the production decision for their departments.

We will investigate how action interdependencies between the agents, i.e., action complementarities or substitutabilities, affect the communication outcome. Actions of the agents are strategic complements if an agent's payoff to increase his own action is increasing in the level of other agent's action. On the contrary, actions of the agents are strategic substitutes if an agent's payoff to increase his own action is decreasing in the level of other agent's action. Our model is an extension of a paper by Bora (2010). Bora (2010) examines the effect of competition on the internal organization of a multi-divisional firm and he characterize decentralized and centralized equilibrium. However, his analysis is symmetric in the sense that agents have the same coefficient for action interdependencies. In this paper, we investigate how different coefficients of agents for interdependencies contribute to the informativeness of the communication in these two mechanisms.

We assume that each agent maximizes his own payoff function under both communication protocols, whereas, the principal's payoff function, which is included in the analysis during vertical communication, is the sum of the payoff functions of the agents. Therefore, the principal is only concerned about efficiency. Moreover, we assume the payoff functions of the agents satisfy the increasing differences property. That is, an agent's payoff to increase his action is higher when his private information signals that he has a higher type.

We look at the Perfect Bayesian Equilibrium by Fudenberg and Tirole (1991) as the equilibrium concept and focus on the most informative outcome in case of multiple equilibrium outcomes. Under vertical communication protocol, the communication from the agents takes form of a partition equilibrium. That is, the state space is partitioned into intervals and agents report the interval which their private information belong to.

On the other hand, under horizontal communication, if there is no strategic interaction between agents, then agents fully reveal their private information. However, when there are strategic interaction between the agents, there are at most two on-the-equilibrium path conditional expectations for each agent. If the agents are strategic complements, both agents would like the other agent to think that he has the highest type. Hence, it is not possible to have an informative horizontal communication under strategic complementarities. On the contrary, if the agents are strategic substitutes, there are parameter configurations in which the cost of production is lower compared to the absolute value of the strategic interaction terms making the horizontal communication informative. Under these parameters, we have a two-partition equilibrium. However, when the cost of production is higher and the absolute value of the strategic interaction terms are lower, then the horizontal communication, again, becomes uninformative.

The rest of the chapter proceeds as follows. The next section, Section 2, presents relevant economics literature on strategic information transmission. Section 3 formally introduces the model. Section 4 analyzes vertical communication between the principal and the agents while Section 5 analyzes horizontal communication between the agents. Finally, Section 6 concludes.

4.2 Relevant Literature

This chapter relates to four main strands of literature: organizational economics, literature on experts, pre-play communication in incomplete information games, and economics of networks.

There are variety of works focus on communication and decision-making within organizations. Dessein (2002) studies efficiency trade-off between choosing delegation and centralization of the decision rights by using a setup similar to Crawford and Sobel (1982). He shows that delegation is preferred if and only if the divergence of the preferences between the principal and the agent is small. Alonso and Matouschek (2008) look at an alternative decision mechanism in which the principal delegates authority but holds the right to constrain the action set from which the agent chooses. Similar to Dessein (2002), they show that internal delegation is optimal when the agent's preferences are sufficiently aligned. Unlike our model, these models only include univariate state of the nature and one decision variable. However, in this paper, we are interested in the case of multidimensional state of nature with two actions to be taken.

Alonso (2008) studies allocation of decision rights in a firm which takes two-dimensional decision with unidimensional uncertainty. He concludes that full delegation is optimal when actions are substitutes, and the principal needs to make the decision for one of the actions if when there are complementarities. Moreover, Alonso et al. (2008a) investigates a firm that sells a single product in different markets with privately informed agents on their own local conditions. Due to the arbitrage, the firm has to set a single price. Under these conditions, they conclude that decentralization is optimal when the agents have sufficiently different local conditions.

More closely related papers, Alonso et al. (2008) and Rantakari (2008) are interested in the trade-off between adaptation and coordination in a multi-divisional firm. Alonso et al. (2008) show that an increased need for coordination improves horizontal communication but worsens vertical communication. Therefore, they conclude that delegation can dominate centralization even when coordination is extremely important relative to adaptation. Moreover, like our model, they show that centralization is more informative than decentralization. Rantakari (2008) extends the analysis by allowing asymmetric divisions in size. Moreover, in addition to vertical and horizontal communication structures, he also looks at the decision mechanism in which one division makes the production decisions for both. He shows that when the incentive conflicts between the divisions are small, centralization is always dominated by decentralization. However, our paper differs from these two articles in two ways. First, these papers only look at specific form of complementarities between the agents, whereas we consider both action substitutabilities and complementarities. Second, they treat coordination as an exogenous variable, whereas, in our model, the need for coordination arises endogenously from the interactions within the firm. Later, Rantakari (2011) includes the importance of coordination as a choice variable. In his model, decision makers can obtain information with a cost. He shows that if the cost of information decreases, then the preference for decentralization increases.

As previously mentioned, in this paper, we use the model introduced by Bora (2010) and extend the analysis to asymmetric agents with different coefficients for interdependencies. He shows that the quality of communication varies with the interactions between the agents. Particularly, he finds that the extend of informative communication is very limited under decentralization. Moreover, he shows that if the agents are strategic complements and positive externalities, then there is no information horizontal communication.

In another closely related article, Alonso, Dessein, Matouschek (2015) examine the effect of competition on the internal organization of a multi-divisional firm. They characterize decentralized and centralized equilibrium under a set of different communication and decision making mechanisms. They show that even if agents have superior information about local conditions, and their incentive conflicts are negligible, a centralized organization can be better at adapting to local information than a decentralized one. Therefore, they conclude that an increase in product market competition that makes adaptation more important can favor centralization rather than decentralization.

The vertical communication structure is similar to a situation in which the principal makes a decision by consulting two experts about different dimensions of uncertainty. Thus, the second strand of literature related to our paper is literature on experts. There are various papers analyzing the quality of communication between an informed principal and a single informed but biased agent. Spector (2000) studies cheap-talk games introduced by Crawford and Sobel (1982) and complements their results by showing that, when the speaker's and the receiver's preferences are close, the most informative equilibrium converges toward full information transmission. Ottoviani (2000) formulates a model of advice study financial retail industry. His model consists of an informed agent (financial adviser) transmitting information to an uninformed principal (investor) with uncertain degree of strategic sophistication. He studies incentives for truthful information disclosure and information acquisition, and the role of explicit monetary transfers. Krishna and Morgan (2008), later, analyze monetary transfers in detail, whereas Ottoviani and Squintani (2006) analyze the case with lower degree of sophistication in detail.

On the other hand, Krishna and Morgan (2001) analyze a model of expertise with two perfectly informed but biased experts who observe the same unidimensional state of nature. They show that full revelation is not possible. Moreover, they show that it is only beneficial to consult both experts when they are biased in the opposite direction. Battaglini (2002) extends the model of expertise into a multidimensional setting. He finds that, contrary to the unidimensional case, if there is more than one sender, full revelation of information in all states of nature is generically possible, even when the conflict of interest is arbitrarily large. Battaglini (2004) study policy advice by several biased experts with noisy private information. Contrary to the previous findings, he finds that full revelation of information is never possible. Our model differs from these papers in the literature of experts. First, we assume that the bias of the agents are endogenous and depend on the interdependencies between the agents, whereas these papers treat them as exogenous. Second, we assume each agent only observes one dimension of the state of nature, while, in these papers, the experts observe all dimensions of the uncertainty, either perfectly or imperfectly.

The third strand of literature related to our paper is on pre-play communication in incomplete information games as the horizontal communication protocol can be considered as one. One of the papers closely related to ours is Baliga and Morris (2002). They use a two-person, finite type and finite action game with one-sided incomplete information to analyze the role of cheap talk. They show that when there are strategic complementarities and positive externalities, then there is no communication at the equilibrium in the cheap talk game. They also show that no communication result is maintained in a game with two-sided uncertainty as long as only one player is allowed to talk. However, if both agents are allowed to talk at the cheap talk stage of the game, then they show that there are non-monotonic equilibria in which lowest and the highest types send the same signal, while the medium type send a different message. This is different from our result that when there are strategic complements, an informative horizontal communication is not possible.

Baliga and Sjostrom (2004) extend the two-sided incomplete information game to a continuum of types by analyzing a situation in which two players simultaneously decide whether or not to acquire new weapons in an arms race game. Each player's type is independently drawn from a continuous distribution and determines the player's propensity to arm. They show that if there is no pre-play communication, then an arms race takes place. However, if the players are allowed to communicate via cheap talk, then the probability of an arms race takes place is significantly lower. Later, Baliga and Sjostrom (2012) extend the model in Baliga and Sjostrom (2004) by introducing another player, an extremist. They study a game of conflict with incomplete information in which two players choose hawkish or dovish actions. An extremist, who can be a hawk or a dove, attempts to manipulate decision-making by sending a public message. They show that if actions are strategic complements, a hawkish extremist sends a public message which triggers hawkish behavior from both players, increasing the likelihood of conflict, and reducing welfare. If actions are strategic substitutes, a dovish extremist sends a public message causing one player to become more dovish and the other more hawkish. Additionally, they show that if actions are strategic substitutes, then a hawkish extremist is unable to manipulate decision making, whereas, if actions are strategic complements, then a dovish extremist is unable to manipulate decision making.

The final strand of literature related to our paper combines networks and strategic communication. Calvo-Armengol and de Marti (2009) study a game in which all agents share a common decision problem and coordination is required between individual actions. Agents have private information and they share their own private information through pairwise communication. They fully characterize the decision functions and the equilibrium payoffs given a communication structure. Moreover, they find that adding communication channels is not always beneficial as it may result in miscoordination. In their model, agents are assumed to have non-conflicting objectives and efficient networks are characterized under physical communication constraints. On the contrary, Hagenbach and Koessler (2010) study the equilibrium communication networks that arise under strategic communication constraints. They study strategic endogenous communication in a network game with biased agents by using cheap talk. In their model, agents would like to choose an action to coordinate their actions with others and be close to a common state of nature. However, each agent's ideal proximity to that state varies. They show that, in equilibrium, pre-play communication depends on the conflict of interest between these agents, and the number and preferences of the other agents with whom they communicate. Additionally, they show that central agents in terms of preference have a tendency to communicate more and to have a greater impact on decisions. In a similar paper, Galeotti et al. (2013) study a multi-player communication model with privately informed decision makers who have different preferences about the actions they take. Players communicate via cheap talk to influence each others' actions in their favor. They show that welfare at equilibrium depends on the number of truthful messages sent and how evenly they are distributed across decision makers.

In contrast to Hagenbach and Koessler (2010) and Galeotti et al. (2013), Calvo-Armengol et al. (2015) look at strategic communication in a network game with costly and verifiable information. In their model, each agent cares about both adaptation and coordination. They study directed payoff interactions among agents with local knowledge. Similar to our model in the second chapter, they also allow for various communication intensity among pairs of agents. They fully characterize information and influence flows. They also show that when the coordination motive becomes more important, the influence of an agent on all his peers approximates to his directed payoff interactions.

4.3 The Model

There are three players in our model: agent 1 and 2, and a principal. The payoff function of agent $i \in \{1, 2\}$ is given by

$$U_i(\theta_i, q_i, q_j) = \theta_i q_i - \alpha q_i^2 + \beta_i q_i q_j,$$

where $j \neq i$, q_i is the action of agent i and θ_i is his private information. The principal's payoff function is the sum of the payoffs of the agents, i.e.,

$$U_P(\theta_1, \theta_2, q_1, q_2) = U_1(\theta_1, q_1, q_2) + U_2(\theta_2, q_1, q_2).$$

We assume that $q_i \in \mathbb{R}$, $i = 1, 2$, and θ_1 and θ_2 are drawn independently from two uniform distributions on $[0, 1]$ interval. We will investigate the problem under the restriction that $\alpha > 0$. In addition, we assume that $|2\alpha| > |\beta_1 + \beta_2|$, so that the principal's payoff function is strictly concave in q_1 and q_2.

As a result of these assumptions, we get the following results: First, the payoff function of agent i is strictly concave in q_i, so that given θ_i, q_j, it has a unique maximum in q_i. Second, U_i has increasing differences in (θ_i, q_i), i.e., $U_{12}^i > 0$. This implies that given q_j, the maximizer q_i is an increasing function of θ_i. Finally, when $\beta_i > 0$, U_i has strategic complementarities in (q_i, q_j), and the maximizer q_i is an increasing function of q_j. On the other hand, when $\beta_i < 0$, U_i has strategic substitutabilities in (q_i, q_j), and the maximizer q_i is an decreasing function of q_j.

We study the model under two communication structures. First, we look at the vertical communication game in which the agents send costless messages to the principal simultaneously, and the principal chooses both q_1 and q_2. Second, we look at the horizontal communication game in which the agents send costless messages to each other simultaneously, and after observing the messages, each agent chooses their own action q_i.

4.4 Vertical Communication

There are three stages in the vertical communication game. At the initial stage, nature independently chooses θ_1 and θ_2. In the second stage, agent i observes her private information θ_i and chooses m_i from a set of feasible signals $\mathcal{M}_i = [0, 1]$. In the last stage, the principal observes (m_1, m_2) and chooses $(q_1, q_2) \in \mathbb{R}^2$. We denote the strategy of the agent i as $\mu_i : \Theta_i \to \mathcal{M}_i$ and the strategy set of the principal as $y : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathbb{R}^2$. An assessment is given by (μ_1, μ_2, y, P), where $P(\cdot|m_1, m_2)$ is the density of the principal's beliefs conditional on (m_1, m_2).

We use Perfect Bayesian Equilibrium (Fudenberg and Tirole (1991)) as the solution concept. The following three conditions characterize an equilibrium of the game:

- The principal's beliefs over (θ_1, θ_2) conditional on observing (m_1, m_2) are formed using Bayes' rule whenever possible.

- Given $P(\cdot|m_1, m_2)$, the principal chooses (q_1, q_2) to maximize $E[U_P(\theta_1, \theta_2, q_1, q_2)|m_1, m_2]$, i.e.,

$$y(m_1, m_2) = (q_1, q_2) = \arg\max_{\bar{q}_1, \bar{q}_2} \int_{(\theta_1, \theta_2) \in (\Theta_1 \times \Theta_2)} U_P(\theta_1, \theta_2, \bar{q}_1, \bar{q}_2) dP(\theta_1, \theta_2 | m_1, m_2).$$

- Given y and μ_j, agent i chooses m_i to maximize $E[U_i(\theta_i, a_1, a_2)]$, i.e.,

$$\mu_i(\theta_i) = m_i = \arg \int_0^1 U_i(\theta_i, y(\bar{m}_i, \mu_j(\theta_j))) d\theta_j.$$

At the last stage of the game, after observing the message profile (m_1, m_2), the principal solves the following problem:

$$\max_{(q_1,q_2)} \int_{(\theta_1,\theta_2)\in(\Theta_1\times\Theta_2)} [\theta_1 q_1 + \theta_2 q_2 + (\beta_1 + \beta_2) q_1 q_2 - \alpha(q_1^2 + q_2^2)] dP(\theta_1, \theta_2 | m_1, m_2)$$

or more simply,

$$\max_{(q_1,q_2)} E[\theta_1 q_1 + \theta_2 q_2 + (\beta_1 + \beta_2) q_1 q_2 - \alpha(q_1^2 + q_2^2) | m_1, m_2].$$

Since the messages are chosen independently, $E[\theta_i | m_i, m_j] = E[\theta_i | m_i]$. Thus, the problem reduces to:

$$\max_{(q_1,q_2)} q_1 E[\theta_1 | m_1] + q_2 E[\theta_2 | m_2] + (\beta_1 + \beta_2) q_1 q_2 - \alpha(q_1^2 + q_2^2).$$

From the first order conditions, the solution to the problem is:

$$y(m_1, m_2) = (q_1(E[\theta_1|m_1], E[\theta_2|m_2]), q_2(E[\theta_1|m_1], E[\theta_2|m_2]))$$

where

$$q_i(E[\theta_1|m_1], E[\theta_2|m_2]) = \frac{2\alpha E[\theta_i|m_i] + (\beta_1 + \beta_2) E[\theta_j|m_j]}{4\alpha^2 - (\beta_1 + \beta_2)^2}$$

Now let us define

$$a = \frac{2\alpha}{4\alpha^2 - (\beta_1 + \beta_2)^2} \quad \text{and} \quad b = \frac{\beta_1 + \beta_2}{4\alpha^2 - (\beta_1 + \beta_2)^2} \tag{9}$$

Then,

$$q_i(E[\theta_1|m_1], E[\theta_2|m_2]) = aE[\theta_i|m_i] + bE[\theta_j|m_j] \tag{10}$$

Next, after introducing the necessary definitions, we will characterize the equilibrium of the vertical communication game.

Definition 37 *Let (μ_1, μ_2, y, P) be a Perfect Bayesian Equilibrium. An action profile (q_1, q_2) is on-the-equilibrium path if there exists a type profile (θ_1, θ_2) who chooses (m_1, m_2) and $y(m_1, m_2) = (q_1, q_2)$. More formally, (q_1, q_2) is on-the-equilibrium path if there exists θ_1, θ_2 such that $(q_1, q_2) = y(\mu_1(\theta_1), \mu_2(\theta_2))$. Similarly, we say that $t_i \in [0, 1]$, $i = 1, 2$, is an on-the-equilibrium path expectation if there exists a type θ_i' such that $t_i = E[\theta_i|\mu_i(\theta_i')]$.*

Lets start with calculating the expected payoffs of the agents reporting their true types. The expected payoff of agent 1 of type θ_1 to sending a message that would induce the conditional expectation equal to x is calculated by

$$\begin{aligned} E[U_1(\theta_1, x)|m_2] &= \int_0^1 [\theta_1 q_1(x, E[\theta_2|\mu_2(\theta_2')]) - \alpha q_1(x, E[\theta_2|\mu_2(\theta_2')])^2 \\ &+ \beta_1 q_1(x, E[\theta_2|\mu_2(\theta_2')]) q_2(x, E[\theta_2|\mu_2(\theta_2')])] d\theta_2'. \end{aligned}$$

The marginal change in the expected payoff agent 1 is calculated by:

$$\frac{\partial}{\partial x} E[U_1(\theta_1, x)|m_2] = \frac{\partial}{\partial x} E[\theta_1 q_1(x, m_2) - \alpha q_1(x, m_2)^2 + \beta_1 q_1(x, m_2) q_2(x, m_2)] \tag{11}$$

Since agent 1 does not observe the type of the second agent, when he is sending the message, he uses the following expectation on agent 2's type

$$\int_0^1 E[\theta_2|\mu_2(\theta_2')]d\theta_2' = E[\theta_2] = 1/2 \tag{12}$$

As $q_i(E[\theta_1|m_1], E[\theta_2|m_2])$'s defined in 10 after substituting 12 into 11 and taking the partial derivative with respect to x, we get the change in the expected payoff of agent 1 with type θ_1 as

$$\begin{aligned}
\frac{\partial}{\partial x}E[U_1(\theta_1,x)|m_2] &= \frac{\partial}{\partial x}[\theta_1(ax+\frac{b}{2}) - \alpha(ax+\frac{b}{2})^2 + \beta_1(ax+\frac{b}{2})(\frac{a}{2}+bx)] \\
&= \theta_1 a - 2\alpha a(ax+\frac{b}{2}) + \beta_1(a(\frac{a}{2}+bx) + (ax+\frac{b}{2})b) \\
&= \theta_1 a + x(2ab\beta_1 - \alpha a^2) + \beta_1\frac{a^2+b^2}{2} - \alpha ab
\end{aligned}$$

If we substitute for 9 and simplify, we get

$$\begin{aligned}
\frac{\partial}{\partial x}E[U_1(\theta_1,x)|m_2] &= \frac{2\alpha}{(4\alpha^2-(\beta_1+\beta_2)^2)}\theta_1 + \frac{2\alpha(2\beta_1(\beta_1+\beta_2)-4\alpha^2)}{(4\alpha^2-(\beta_1+\beta_2)^2)^2}x \\
&\quad + \frac{\beta_1(\beta_1+\beta_2)^2 - 4\alpha^2\beta_2}{2(4\alpha^2-(\beta_1+\beta_2)^2)^2}
\end{aligned}$$

Therefore, agent 1 has incentives to lie about his type depending on β_1, β_2 and α. However, as it is shown by Crawford and Sobel (1982), partially informative equilibrium is still possible. Under this equilibrium, state-space is partitioned so that any message m_i reveals only the interval that θ_i belongs to. As it is shown in the following lemma, different types of each agent will form an interval in terms of their strategies in the equilibrium.

Lemma 38 *If $\underline{\theta}_i$ and $\overline{\theta}_i$ prefer to send m_i to the principal, then any $\theta_i^0 \in [\underline{\theta}_i, \overline{\theta}_i]$ also prefers to send m_i.*

Proof. Let $E[\theta_i|m_i]$ be the conditional expectation of the principal to type θ_i upon receiving the message m_i and m_i' be another signal such that $E[\theta_i|m_i'] < E[\theta_i|m_i]$. For simplicity, denote $q_1(m_i, m_j) = q_1$, $q_2(m_i, m_j) = q_2$ and $q_1(m_i', m_j) = q_1'$, $q_2(m_i', m_j) = q_2'$. Since $\alpha > 0$, $q_1' < q_1$ for every m_j. Note that for any m_j,

$$[U_i(\theta_i^0, q_1, q_2) - U_i(\theta_i^0, q_1', q_2')] = [U_i(\underline{\theta}_i, q_1, q_2) - U_i(\underline{\theta}_i, q_1', q_2')] + \underbrace{(\theta_i^0 - \underline{\theta}_i)(q_1 - q_1')}_{>0}. \tag{13}$$

Integration of the equation 13 with respect to θ_j yields that the expected payoff of type θ_i^0 to choosing m_i is strictly higher than the payoff to choosing m_i'.

Similarly, for any m_j,

$$[U_i(\theta_i^0, q_1, q_2) - U_i(\theta_i^0, q_1', q_2')] = [U_i(\overline{\theta}_i, q_1, q_2) - U_i(\overline{\theta}_i, q_1', q_2')] + \underbrace{(\theta_i^0 - \overline{\theta}_i)(q_1 - q_1')}_{>0}. \tag{14}$$

Integration of the equation 14 with respect to θ_j also yields that the expected payoff of type θ_i^0 to choosing m_i is strictly higher than the payoff to choosing m_i'. Thus, agent i with type θ_i^0 will never find it optimal to choose a message that would yield a conditional expectation lower than $E[\theta_i|m_i]$. Similar argument holds for agent i with type θ_i^0 and any signal m_i'' be another signal such that $E[\theta_i|m_i'] > E[\theta_i|m_i]$. Thus, agent i with type θ_i^0 will never find it optimal to choose a message that would yield a conditional expectation higher than $E[\theta_i|m_i]$. ■

In the next proposition, we summarize the equilibrium conditions for the vertical communication game. For simplicity, we will focus on the pure reporting strategies, as, without loss of generality, all the other equilibria are economically equivalent to the pure reporting equilibria defined in Proposition 39. Moreover, we will not provide a restriction on beliefs for messages that are the out-of-equlibrium path. This is because we can support the equilibrium provided in 39 with different the out-of-equlibrium beliefs.

Proposition 39 *The equilibrium of the vertical communication game is characterized by as follows where* $0 = k_{i,0} < k_{i,1} < \cdots < k_{i,n} < ... < 1$ *for all* $i \in \{1, 2\}, n \in \mathbb{N}$

$$\mu_i(\theta) = \frac{k_{i,n} + k_{i,n+1}}{2} \text{ if } \theta_i \in [k_{i,n}, k_{i,n+1}) \tag{15}$$

$$P\left(\theta_1, \theta_2 | \frac{k_{1,n} + k_{1,n+1}}{2}, \frac{k_{2,m} + k_{2,m+1}}{2}\right) = \begin{cases} \frac{1}{(k_{1,n+1} - k_{1,n})(k_{2,m+1} - k_{2,m})} \\ \text{if } (\theta_1, \theta_2) \in [k_{1,n}, k_{1,n+1}] \times [k_{2,m}, k_{2,m+1}] \\ 0 \text{ otherwise} \end{cases} \tag{16}$$

$$\begin{aligned} q_1\left(\frac{k_{1,n} + k_{1,n+1}}{2}, \frac{k_{2,m} + k_{2,m+1}}{2}\right) &= \frac{2\alpha(k_{1,n} + k_{1,n+1}) + (\beta_1 + \beta_2)(k_{2,m} + k_{2,m+1})}{2(4\alpha^2 - (\beta_1 + \beta_2)^2)}, \\ q_2\left(\frac{k_{1,n} + k_{1,n+1}}{2}, \frac{k_{2,m} + k_{2,m+1}}{2}\right) &= \frac{2\alpha(k_{2,m} + k_{2,m+1}) + (\beta_1 + \beta_2)(k_{1,n} + k_{1,n+1})}{2(4\alpha^2 - (\beta_1 + \beta_2)^2)}. \end{aligned} \tag{17}$$

$$k_{i,n+1} = \frac{2\alpha^2 - 2\beta_i(\beta_1 + \beta_2)}{2\alpha^2 - \beta_i(\beta_1 + \beta_2)} k_{i,n} - k_{i,n-1} - \frac{\beta_i(\beta_1 + \beta_2)^2 - 4\alpha^2\beta_j}{\alpha(2\beta_i(\beta_1 + \beta_2) - 4\alpha^2)} \text{ for every } n = \mathbb{N} \tag{18}$$

Proof. Given Lemma 38, we have a partition equilibria, in which type space is partitioned into intervals such that types in the same interval report the same message. Specifically, there exists a partition $\{k_{i,n}\}_{n\in\mathbb{N}}$ such that $0 = k_{i,0} < k_{i,1} < \cdots < k_{i,n} < ... < 1$ and if $\theta, \theta' \in (k_{i,n}, k_{i,n+1})$, then we have $\mu_i(\theta) = \mu_i(\theta')$.

If reporting strategies of agents are given as in 15, then 16 and 17 follow. Thus, we need to determine the partition $\{k_{i,n}\}_{n\in\mathbb{N}}$ so that each type of agent 1 and 2 is best responding. Since U^i is continuous in θ_i, for each $k_{i,n}$, each agent is indifferent between sending a lower message $\frac{k_{i,n}-1+k_{i,n}}{2}$ and a higher message$\frac{k_{i,n}+k_{i,n+1}}{2}$. By following the partition equilibrium logic, we find the incentive-compatible partitions by identifying the types that each agent is indifferent between sending a lower message and a higher message, i.e.,

$$EU_i(\theta_i^M, q_i(E[\theta_i|m_i^L], .), q_i(E[\theta_i|m_i^L], .)) = EU_i(\theta_i^M, q_i(E[\theta_i|m_i^H], .), q_i(E[\theta_i|m_i^H], .)) \quad (19)$$

The family of incentive-compatible partitions is obtained by solving this indifference condition, 19,:

$$k_{i,n+1} = C_{i,1}k_{i,n} - k_{i,n-1} - C_{i,2}$$

where

$$C_{i,1} = \frac{2\alpha^2 - 2\beta_i(\beta_1 + \beta_2)}{2\alpha^2 - \beta_i(\beta_1 + \beta_2)} \quad \text{and} \quad C_{i,2} = \frac{\beta_i(\beta_1 + \beta_2)^2 - 4\alpha^2\beta_j}{\alpha(2\beta_i(\beta_1 + \beta_2) - 4\alpha^2)}$$

■

4.5 Horizontal Communication

4.5.1 Autarchy

Under autarchy, the game is played between agents who cannot communicate. After observing their own type θ_i but not θ_j, each agent i simultaneously choose q_i. We will adopt the Bayesian equilibrium as the equilibrium concept and consider only symmetric strategies.

Proposition 40 *The symmetric equilibrium of the autarchy game is given by $f_i : \Theta_i \rightarrow \mathbb{R}$ such that*

$$f_i(\theta_i) = \frac{\theta_i}{2\alpha} + \frac{\beta_i(2\alpha + \beta_j)}{4\alpha(4\alpha^2 - \beta_i\beta_j)}$$

where agent i of type θ_i chooses an action according to f_i defined above and $i \neq j$, $i, j \in \{1, 2\}$.

Proof. Suppose that agent j plays according to f_j. Given the strategy of agent j, the objective function for agent i is:

$$\max_{q_i} \int_0^1 [\theta_i q_i - \alpha q_i^2 + \beta q_i f_j(\theta_j)] d\theta_j$$

Since the objective function is strictly concave in q_i, the first order condition

$$q_i = \frac{\theta_i}{2\alpha} + \frac{\beta_i}{2\alpha} \int_0^1 f_j(\theta_j) d\theta_j$$

is necessary and sufficient for a maximum. Therefore, in equilibrium

$$q_i = f_i(\theta_i) = \frac{\theta_i}{2\alpha} + \frac{\beta_i}{2\alpha} \int_0^1 f_j(\theta_j) d\theta_j.$$

This equation is a Fredholm integral equation of the second kind with the simplest degenerate kernel[4] whose solution is given by:

$$f_i(\theta_i) = \frac{\theta_i}{2\alpha} + \frac{\beta_i(2\alpha + \beta_j)}{4\alpha(4\alpha^2 - \beta_i \beta_j)}.$$

■

4.5.2 Horizontal Communication

There are three stages in the horizontal communication game. At the initial stage, nature independently chooses θ_1 and θ_2. In the second stage, after observing their private information, agent i independently and simultaneously sends a message m_i to the other agent from a set of feasible signals $M_i = [0, 1]$. In the last stage, after observing the messages,

[4] See Polyanin and Manzhirov (1998).

each agent independently chooses q_i. As in the vertical communication game, we focus on the Perfect Bayesian Equilibria of the game.

Let m_1, m_2 be sent in equilibrium. If agent i with θ_i sends m_i, the optimal action, q_i, solves

$$\max_{\bar{q}_i} E[\theta_i \bar{q}_i - \alpha \bar{q}_i^2 + \beta_i \bar{q}_i q_j | m_j]$$

The solution is

$$q_i = \frac{\theta_i}{2\alpha} + \frac{\beta_i}{2\alpha} E[q_j | m_j]$$

Now if we plug q_j back into the previous equation, we get

$$q_i = \frac{\theta_i}{2\alpha} + \frac{\beta_i}{2\alpha} E[\frac{\theta_j}{2\alpha} + \frac{\beta_j}{2\alpha} E[q_i | m_i] | m_j]$$

Since m_i and m_j are chosen independently, the previous equation reduces to

$$q_i = \frac{\theta_i}{2\alpha} + \frac{\beta_i}{4\alpha^2} E[\theta_j | m_j] + \frac{\beta_i \beta_j}{4\alpha^2} E[q_i | m_i] \tag{20}$$

If we take expectations conditional on m_i, we obtain from the previous equation

$$E[q_i | m_i] = \frac{2\alpha}{4\alpha^2 - \beta_i \beta_j} E[\theta_i | m_i] + \frac{\beta_i}{4\alpha^2 - \beta_i \beta_j} E[\theta_j | m_j] \tag{21}$$

Therefore, combining 20 and 21, we get:

$$q_i = \frac{\theta_i}{2\alpha} + \frac{\beta_i \beta_j}{2\alpha(4\alpha^2 - \beta_i \beta_j)} E[\theta_i | m_i] + \frac{\beta_i}{4\alpha^2 - \beta_i \beta_j} E[\theta_j | m_j]. \tag{22}$$

Proposition 41 *If $\beta_1 \neq 0$ and $\beta_2 \neq 0$, there are at most two on-the-equilibrium path conditional expectations for each agent.*

Proof. Let θ and $\theta' = \theta + \varepsilon$ be two conditional expectations associated with m_1 and m_1' on agent 1's signaling strategy, i.e., $E[\theta_1|m_1] = \theta$ and $E[\theta_1|m_1'] = \theta'$. For the agent 1 of type θ_1, the difference in the utility between sending m_1' and m_1 is

$$\underbrace{\frac{\beta_1\beta_2}{2\alpha(4\alpha^2-\beta_1\beta_2)^2}}_{K_1}\varepsilon[\theta_1\underbrace{(4\alpha^2-\beta_1\beta_2)}_{K_2}+\beta_1\beta_2\theta+\underbrace{\alpha\beta_1}_{K_3}+\frac{\beta_1\beta_2\varepsilon}{2}] \tag{23}$$

or more simply,

$$K_1\varepsilon[\theta_1K_2+\beta_1\beta_2\theta+K_3+\frac{\beta_1\beta_2\varepsilon}{2}] \tag{24}$$

Let us assume for a contradiction that there exist three on-the-equilibrium path expectations θ, $\theta' = \theta + \varepsilon_1$, and $\theta'' = \theta' + \varepsilon_2$ with $\varepsilon_1, \varepsilon_2 > 0$. Since θ' is on-the-equilibrium path, there exists an agent 1 with type θ_1^* who prefers θ' to θ, i.e.

$$K_1\varepsilon_1[\theta_1^*K_2+\beta_1\beta_2\theta+K_3+\frac{\beta_1\beta_2\varepsilon_1}{2}] \geq 0. \tag{25}$$

The difference in utility for this agent between sending a signal that would yield the conditional expectation $\theta'' = \theta' + \varepsilon_2$ and the signal that yields θ' equals to

$$K_1\varepsilon_2[\theta_1^*K_2+\beta_1\beta_2(\theta+\varepsilon_1)+K_3+\frac{\beta_1\beta_2\varepsilon_2}{2}] \tag{26}$$

We can rewrite 26 as:

$$K_1\varepsilon_2[\theta_1^*K_2+\beta_1\beta_2\theta+K_3+\frac{\beta_1\beta_2\varepsilon_1}{2}]+K_1\varepsilon_2[\frac{\beta_1\beta_2}{2}(\varepsilon_1+\varepsilon_2)] \tag{27}$$

As the expression in 25 is greater than or equal to zero, the first part of the expression in 27 is also greater than or equal to zero. In addition, we can rewrite the second part of the expression as:

$$\varepsilon_2[\frac{(\beta_1\beta_2)^2}{4\alpha(4\alpha^2-\beta_1\beta_2)^2}(\varepsilon_1+\varepsilon_2)]. \tag{28}$$

Since $\beta_1 \neq 0$ and $\beta_2 \neq 0$, the second part of the expression is strictly greater than zero. Thus, the expression in 27 is strictly greater than zero. This means that the agent 1 with type θ_1^* would strictly prefer to induce the expectation θ'' to θ'. This contradicts with the assumption that θ' is induced. Therefore, there can be at most two on-the-equilibrium conditional expectations for the agent 1.

The same argument applies to the agent 2 with type θ_2^*. Thus, when $\beta_1 \neq 0$ or $\beta_2 \neq 0$, there can be at most two on-the-equilibrium conditional expectations for agent 1 and agent 2. ■

Proposition 42 *If $\beta_1 = 0$ or $\beta_2 = 0$, there exists a fully revealing equilibrium.*

Proof. When $\beta_1 = 0$ or $\beta_2 = 0$, the expression in (4.12) equals to zero. Thus, each type of each agent is indifferent between sending any of the signals. Therefore, the signaling strategy defined by $\mu_i(\theta_i) = m_i = \theta_i$ for $i = 1, 2$ is part of an equilibrium. ■

It is concluded from Proposition 41 that the most informative equilibrium under $\beta_i \neq 0$ will take form of two-partition equilibrium. Let us characterize the conditions when a two-partition equilibrium exists. Under two-partition equilibrium with symmetric strategies, there exists a type $k_i \in (0,1)$ for each agent i such that agent i is indifferent between sending a low signal inducing a conditional expectation $\frac{k_i}{2}$ and a high signal inducing a conditional expectation $\frac{1+k_i}{2}$. By substituting the restrictions $\theta_1 = k_1$, $\theta = \frac{k_1}{2}$ and $\varepsilon = \frac{1}{2}$ into 23, the following condition is obtained

$$k_1 = -\frac{\beta_1(4\alpha+\beta_2)}{8(4\alpha^2-\beta_1\beta_2)} \in (0,1)$$

Similar argument holds for agent 2 with $\theta_2 = k_2$, $\theta = \frac{k_2}{2}$ and $\varepsilon = \frac{1}{2}$. So,

$$k_2 = -\frac{\beta_2(4\alpha + \beta_1)}{8(4\alpha^2 - \beta_1\beta_2)} \in (0, 1)$$

Lets start with calculating the expected payoffs of the agents reporting their true types. The expected payoff of agent 1 of type θ_1 to sending a message that would induce the conditional expectation equal to x is calculated by

$$\begin{aligned} E[U_1(\theta_1, x)|m_2] &= \int_0^1 [\theta_1 q_1(x, E[\theta_2|\mu_2(\theta_2')]) - \alpha q_1(x, E[\theta_2|\mu_2(\theta_2')]))^2 \\ &+ \beta_1 q_1(x, E[\theta_2|\mu_2(\theta_2')])q_2(x, E[\theta_2|\mu_2(\theta_2')])]d\theta_2'. \end{aligned}$$

The marginal change in the expected payoff agent 1 is calculated by:

$$\frac{\partial}{\partial x} E[U_1(\theta_1, x)|m_2] = \frac{\partial}{\partial x} E[\theta_1 q_1(x, m_2) - \alpha q_1(x, m_2)^2 + \beta_1 q_1(x, m_2) q_2(x, m_2)] \tag{29}$$

Since agent 1 does not observe the type of the second agent, when he is sending the message, he uses the following expectation on agent 2's type

$$\int_0^1 E[\theta_2|\mu_2(\theta_2')]d\theta_2' = E[\theta_2] = 1/2 \tag{30}$$

As $q_i(E[\theta_1|m_1], E[\theta_2|m_2])$'s defined in 22 after substituting 30 into 29 and taking the partial derivative with respect to x, we get the change in the expected payoff of agent 1 with type θ_1 as

$$\begin{aligned}
\frac{\partial}{\partial x}E[U_1(\theta_1,x)|m_2] &= \frac{\partial}{\partial x}[\theta_1(\frac{\theta_1}{2\alpha}+\frac{\beta_1\beta_2}{2\alpha(4\alpha^2-\beta_1\beta_2)}x+\frac{\beta_1}{2\,(4\alpha^2-\beta_1\beta_2)}) \\
&\quad -\alpha(\frac{\theta_1}{2\alpha}+\frac{\beta_1\beta_2}{2\alpha(4\alpha^2-\beta_1\beta_2)}x+\frac{\beta_1}{2\,(4\alpha^2-\beta_1\beta_2)})^2 \\
&\quad +\beta_1(\frac{\theta_1}{2\alpha}+\frac{\beta_1\beta_2}{2\alpha(4\alpha^2-\beta_1\beta_2)}x+\frac{\beta_1}{2\,(4\alpha^2-\beta_1\beta_2)})(\frac{\alpha+\beta_2 x}{(4\alpha^2-\beta_1\beta_2)})] \\
&= \frac{\theta_1\beta_1\beta_2}{2\alpha(4\alpha^2-\beta_1\beta_2)}-\frac{\beta_1\beta_2}{(4\alpha^2-\beta_1\beta_2)}(\frac{\theta_1}{2\alpha}+\frac{\beta_1\beta_2 x}{2\alpha(4\alpha^2-\beta_1\beta_2)}+\frac{\beta_1}{2\,(4\alpha^2-\beta_1\beta_2)}) \\
&\quad +\beta_1(\frac{\alpha+\beta_2}{(4\alpha^2-\beta_1\beta_2)})(\frac{\theta_1}{2\alpha}+\frac{\beta_1\beta_2}{2\alpha(4\alpha^2-\beta_1\beta_2)}x+\frac{\beta_1}{2\,(4\alpha^2-\beta_1\beta_2)})+ \\
&\quad +\beta_1\frac{\beta_1\beta_2}{2\alpha(4\alpha^2-\beta_1\beta_2)}\frac{\alpha+\beta_2 x}{(4\alpha^2-\beta_1\beta_2)}) \\
&= \underbrace{\frac{(\beta_1\beta_2)^2}{2\alpha(4\alpha^2-\beta_1\beta_2)^2}}_{H_1}x+\underbrace{\frac{\beta_1\beta_2}{2\alpha(4\alpha^2-\beta_1\beta_2)}}_{H_2}\theta_1+\underbrace{\frac{\beta_1^2\beta_2}{2\,(4\alpha^2-\beta_1\beta_2)}}_{H_3}
\end{aligned} \tag{31}$$

Similarly, the expected payoff of agent 2 of type θ_2 to sending a message that would induce the conditional expectation equal to x is calculated by

$$\begin{aligned}
E[U_2(\theta_2,x)|m_1] &= \int_0^1[\theta_2 q_2(x,E[\theta_1|\mu_1(\theta_1')])-\alpha q_2(x,E[\theta_1|\mu_1(\theta_1')])^2 \\
&+ \beta_2 q_1(x,E[\theta_1|\mu_1(\theta_1')])q_2(x,E[\theta_1|\mu_1(\theta_1')])]d\theta_1'.
\end{aligned}$$

The marginal change in the expected payoff agent 1 is calculated by:

$$\begin{aligned}
\frac{\partial}{\partial x}E[U_2(\theta_2,x)|m_1] &= \frac{\partial}{\partial x}E[\theta_2 q_2(x,m_1)-\alpha q_2(x,m_1)^2+\beta_2 q_1(x,m_1)q_2(x,m_1)] \\
&= \underbrace{\frac{(\beta_1\beta_2)^2}{2\alpha(4\alpha^2-\beta_1\beta_2)^2}}_{H_1}x+\underbrace{\frac{\beta_1\beta_2}{2\alpha(4\alpha^2-\beta_1\beta_2)}}_{H_2}\theta_2+\underbrace{\frac{\beta_1\beta_2^2}{2\,(4\alpha^2-\beta_1\beta_2)}}_{H_4}
\end{aligned} \tag{32}$$

As $\frac{(\beta_1\beta_2)^2}{2\alpha(4\alpha^2-\beta_1\beta_2)^2}>0$ when $\beta_i\neq 0$, the payoff function $E[U_i(.)]$ is convex in x for both agents. Therefore, the payoffs of agent i is minimized at $E[U_i(.)]=0$. In the next figure, we will analyze the informativeness of the horizontal communication by using the equations 31 and 32. This figure shows three different outcomes associated with different parameter configurations. The lines labeled as l_i represent the situation in which the equation 31 equals to zero. That is,

$$l_i : x = -\frac{H_2}{H_1}\theta_1 - \frac{H_3}{H_1}$$

Notice that, the lines l_i shows the expectations that yields the minimum utility for agent 1, as $E[U_i(.)]$ is convex in x. Similar argument can be made for agent 2 by using the equation 32. In that case, the lines l_i represent the situation in which the equation 32 equals to zero. That is,

$$l_i : x = -\frac{H_2}{H_1}\theta_1 - \frac{H_4}{H_1}$$

Before analyzing the different outcomes under different parameter configurations, lets take a closer look at H_i's under $\beta_i \neq 0$. We know that for all parameter values, $H_1 > 0$. Now, we assume that β_i's have the same signs, i.e., we have either $\beta_i < 0$ or $\beta_i > 0$. Thus, $H_2 = \frac{\beta_1\beta_2}{2\alpha(4\alpha^2-\beta_1\beta_2)} > 0$. To see this, we need to look at the term $(4\alpha^2 - \beta_1\beta_2)$ at the denominator as we know $\alpha > 0$ and $\beta_1\beta_2 > 0$. Since $|2\alpha| > |\beta_1 + \beta_2|$, we have

$$4\alpha^2 > (\beta_1 + \beta_2)^2 = \beta_1^2 + 2\beta_1\beta_2 + \beta_2^2$$

If we subtract $\beta_1\beta_2$ from both sides, we obtain

$$4\alpha^2 - \beta_1\beta_2 > \beta_1^2 + \beta_1\beta_2 + \beta_2^2 > 0$$

Thus, $H_2 > 0$. On the other hand, the sign of H_3 and H_4 depends on β_2 and β_1 respectively, as $(4\alpha^2 - \beta_1\beta_2) > 0$. If $\beta_2 > 0$, then $H_3 > 0$, and if $\beta_2 < 0$, then $H_3 < 0$. Similarly, if

$\beta_1 > 0$, then $H_4 > 0$, if $\beta_1 > 0$, then $H_4 > 0$.

Lastly, lets take a look at the lines l_i. For agent 1, we have

$$
\begin{aligned}
l_i^1 \quad &: \quad x = -\frac{H_2}{H_1}\theta_1 - \frac{H_3}{H_1} \\
x \quad &= \quad -\underbrace{\frac{4\alpha^2 - \beta_1\beta_2}{\beta_1\beta_2}}_{K_1}\theta_1 - \underbrace{\frac{\alpha\left(4\alpha^2 - \beta_1\beta_2\right)}{\beta_2}}_{K_2}
\end{aligned}
$$

Additionally, for agent 2, we have

$$
\begin{aligned}
l_i^2 \quad &: \quad x = -\frac{H_2}{H_1}\theta_1 - \frac{H_4}{H_1} \\
x \quad &= \quad -\underbrace{\frac{4\alpha^2 - \beta_1\beta_2}{\beta_1\beta_2}}_{K_1}\theta_1 - \underbrace{\frac{\alpha\left(4\alpha^2 - \beta_1\beta_2\right)}{\beta_1}}_{K_3}
\end{aligned}
$$

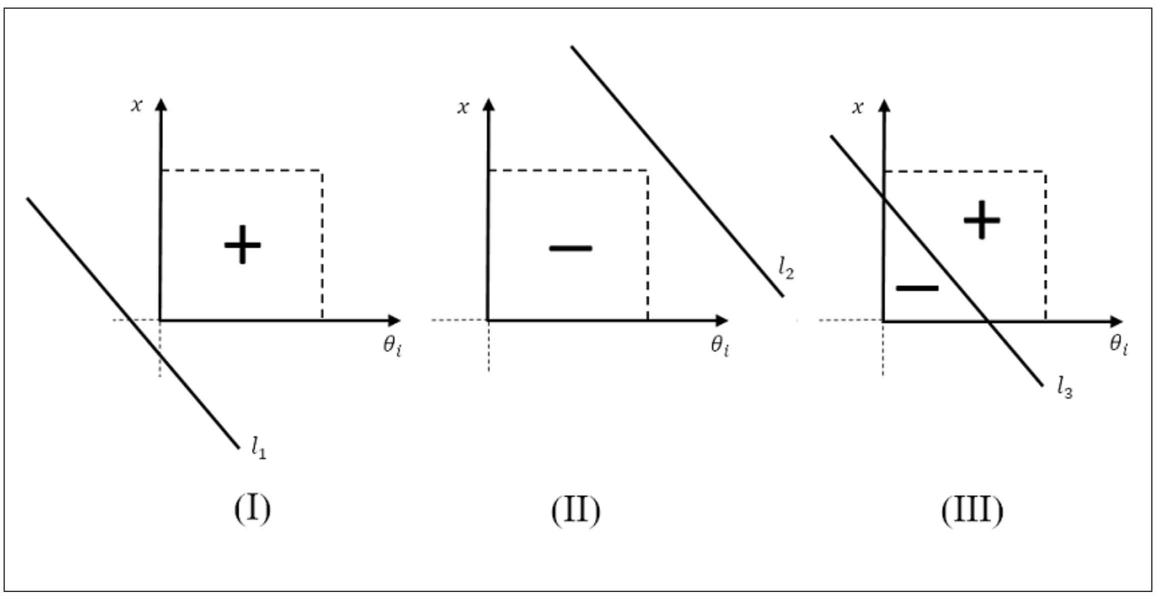


Figure 10: Informativeness of horizontal communication

In the first part of the figure, (I), l_1 is such that, for every θ_i, the marginal utility at every x is positive. Thus, every type of agent i would want agent j to believe that agent i

has the highest type. Therefore, for the case (I), informative communication is not possible.

In the second part of the figure, (II), l_2 is such that, for every θ_i, the marginal utility at every x is negative. Thus, every type of agent i would want agent j to believe that agent i has the lowest type. Therefore, similar to the case (I), informative communication is not possible for the case (II).

Notice that as informative communication is not possible for the cases (I) and (II), the agents will choose their actions according to the case under autarchy.

On the other hand, in the third part of the figure, (III), the line l_3 partitions the (θ_i, x) space into two. In the next figure, we will discuss the result of Proposition 41 by taking a closer look at the third case.

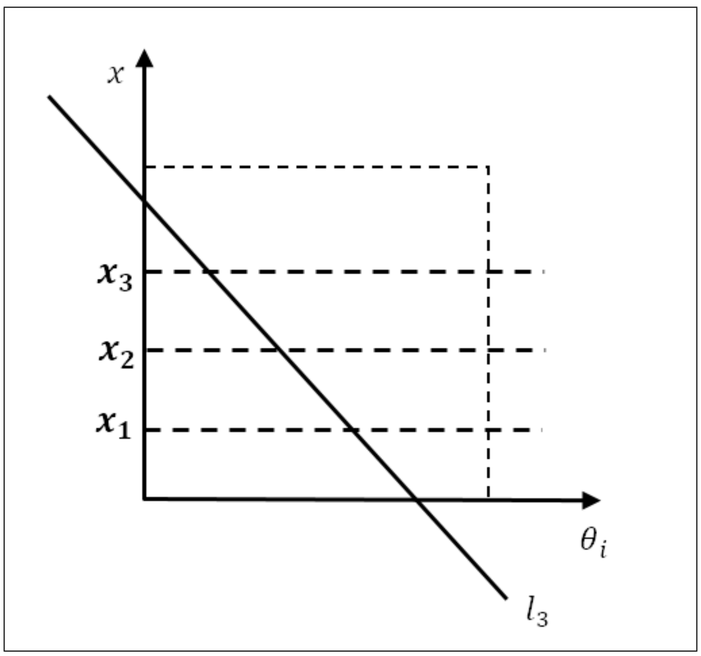


Figure 11: Horizontal Communication with two partitions

Lets assume that we have three expectations, x_1, x_2 and $x_{3.}$ induced at equilibrium. Then, none of the types would send a message yielding the expectation x_2. This is because, depending on the agent's type, sending a message to induce the highest or the lowest type would be strictly preferred to sending a message to induce x_2. Therefore, there will be a cutoff type for this case such that the types smaller than this cutoff type prefer lower expectations as marginal utility is negative, and the types greater than this cutoff type prefer higher expectations as marginal utility is positive. Therefore, we have a two-partition equilibrium for this case.

Now, lets take a look at the horizontal communication under strategic complementarities and substitutabilities. We start with strategic complementarities by assuming $\beta_i > 0$. Then, we have $H_1, H_2, H_3, H_4 > 0$ and $K_1, K_2, K_3 > 0$. This means that we have the case (I) depicted in the Figure 10 under strategic complementarities. As we discussed earlier, both agents would like the other agent to think that he has the highest type as the marginal utility of the agent increases with x. Hence, it is not possible to have an informative horizontal communication under strategic complementarities.

On the other hand, if we assume strategic substitutabilities, $\beta_i < 0$, then we have $H_1, H_2 > 0$, $H_3, H_4 < 0$ and $K_1 > 0$, $K_2, K_3 < 0$. Then, depending on the parameters α and β_i's, we may have informative horizontal communication. If the parameters are such that K_2 and K_3 are high in absolute value, then we have the case (II) depicted in Figure 10. This case happens for the parameter configurations with relatively higher α compared to $|\beta_i|$'s making K_2 and K_3 are high in absolute value. As discussed both agents would like the other agent to think that he has the lowest type as the marginal utility of the agent decreases with x. Hence, it is not possible to have an informative horizontal communication if K_2 and K_3 are high in absolute value.

However, for parameter configurations making K_2 and K_3 are low in absolute value, we have informative horizontal communication. Under these parameters, we have the case (III) depicted in Figure 10. The parameter configurations with relatively lower α compared to $|\beta_i|$'s make K_2 and K_3 are low in absolute value. If that is the case, then we have a two-partition equilibrium making the horizontal communication informative.

4.6 Conclusion

In this chapter, we study strategic information flows and communication in organizations. Specifically, we look at two communication protocols, vertical and horizontal communication, between uninformed principal and two agents who are privately informed about an independent aspect of the state of the world. In addition, we allow for asymmetric interdependencies between the agents. We show that strategic interdependencies between agents in the form on complementarities and substitutabilities lead informed agents to distort their information in communications.

Under the vertical communication protocol, there is a centralized decision mechanism in which the principal makes the production decisions for the agents after observing the private messages of the agents. We show that, similar to Crawford and Sobel (1982), Rantakari (2008), Alonso et al. (2008) and Bora (2010), the Perfect Bayesian Equilibrium of the vertical communication protocol takes form of a partition equilibrium, in which the state space is partitioned into intervals and the agents report the interval their private information belong to.

Under the horizontal communication protocol, there is a decentralized decision mechanism in which the agents communicate with each other via cheap talk then make the production decisions for their own departments. We show that the agents fully reveal their private information when there is no strategic interaction between them. However, when there are strategic interactions, the Perfect Bayesian Equilibrium has at most two on-the-equilibrium path conditional expectations for each agent. If the agents are strategic complements, then an informative horizontal communication is not possible as each agent want the other one to think that he has the highest type. On the other hand, when agents are strategic substitutes, there exists parameter configurations in which the cost parameter α is lower compared to $|\beta_i|$'s such that the equilibrium takes form of a two-partition equilibrium.

There can be multiple lines for future research. In this chapter, we study asymmetric interdependencies between two agents. It would be interesting to look at the communication between more than two agents. Moreover, we only study two communication protocols. Other communication structures can also be analyzed. For example, in an alternative communication structure, agents can communicate via cheap talk and then one of the agents makes the production decision for both divisions. Furthermore, more rounds of communications can be added to the model. Additionally, the model can be analyzed by using sequential communication protocols.

Appendix

In this appendix, we provide formal proofs of the propositions in Chapter 3.

Proof of Proposition 24. If we take the derivative of player i's surplus with respect to his choice variable, we get

$$\begin{aligned}
\frac{\partial S_i}{\partial z_{ik}} &= \frac{\beta}{2}r_k + \frac{\beta^2}{2}(\frac{2 - z_{ki} - z_{ji}}{2})r_j - \frac{\beta}{2}r_j - \frac{\beta^2}{2}(\frac{2 - z_{ki} - z_{ji}}{2})r_k \\
&= \frac{\beta}{2}(r_k - r_j) - \frac{\beta^2}{2}(\frac{2 - z_{ki} - z_{ji}}{2})(r_k - r_j) \\
&= (r_k - r_j)(\frac{\beta}{2} - \frac{\beta^2}{2}(\frac{2 - z_{ki} - z_{ji}}{2})) \\
&= \frac{\beta}{2}(r_k - r_j)(1 - \beta(\frac{2 - z_{ki} - z_{ji}}{2}))
\end{aligned}$$

Notice that $\frac{2 - z_{ki} - z_{ji}}{2}$ is the link strength between player k and player j. So, if $0 < \beta < 1$, then

$$1 - \beta(\frac{2 - z_{ki} - z_{ji}}{2}) > 0$$

Therefore, if $(r_k - r_j) > 0$, then $\frac{\partial S_i}{\partial z_{ik}} > 0$, making the surplus of player i increasing in z_{ik}. So, player i will invest all of his time with player k.Similarly, if $(r_k - r_j) < 0$, then $\frac{\partial S_i}{\partial z_{ik}} < 0$, making the surplus of player i decreasing in z_{ik}. So, player i will invest all of his time with player j.

Since $r_2 > r_3$, player 1 will choose $z_{12} = 1$, since $r_1 > r_3$, player 2 will choose $z_{21} = 1$ and since $r_1 > r_2$, player 3 will choose $z_{31} = 1$, making the link strengths between the players

$$\begin{aligned}
\sigma_{12} &= \frac{z_{12} + z_{21}}{2} = 1 \\
\sigma_{13} &= \frac{z_{13} + z_{31}}{2} = \frac{1}{2} \\
\sigma_{23} &= \frac{z_{23} + z_{32}}{2} = 0
\end{aligned}$$

■

Proof of Lemma 25. Without loss of generality, let σ_{23} be the weakest link and σ_{12} the strongest. Then $\sigma_{23} \leq 1/2 \leq \sigma_{12}$. Let $z_{32} = 2\sigma_{23}$ and $z_{23} = 0$, so that (7) holds. Then let $z_{31} = 1 - z_{32}$, and let z_{13} be such that (8) holds, i.e.,

$$\sigma_{13} = \frac{1}{2}z_{13} + \frac{1}{2}(1 - z_{32}) = \frac{1}{2}z_{13} + \frac{1}{2}(1 - 2\sigma_{23})$$

This means

$$z_{13} = 2(\sigma_{13} + \sigma_{23}) - 1 \tag{33}$$

Since $1/2 \leq \sigma_{12} \leq 1$ and $\sigma_{12} + \sigma_{23} + \sigma_{13} = \frac{3}{2}$, we have

$$\sigma_{13} + \sigma_{23} + 1 \geq \sigma_{13} + \sigma_{23} + \sigma_{12} = 3/2 \geq \sigma_{13} + \sigma_{23} + \frac{1}{2}$$

and so

$$\frac{1}{2} \leq \sigma_{13} + \sigma_{23} \leq 1$$

This means (33) lies between 0 and 1. Finally, let $z_{12} = 1 - z_{13}$ and let z_{21} be such that (6) holds, i.e.,

$$\sigma_{12} = \frac{1}{2}z_{12} + \frac{1}{2}z_{21} = \sigma_{12} = \frac{1}{2}\left(1 - z_{13}\right) + \frac{1}{2}z_{21} = \frac{1}{2}\left(1 - \left(2(\sigma_{13} + \sigma_{23}) - 1\right)\right) + \frac{1}{2}z_{21}$$

so that

$$z_{21} = 2\left(\sigma_{12} + \sigma_{13} + \sigma_{23}\right) - 2 = 1$$

We have found all the z_{ij} we are looking for.

■

Proof of Proposition 26. The social planner's problem is equivalent to maximizing

$$\begin{aligned} S &= r_1\left(1+\beta(\sigma_{12}+\sigma_{13})\left(1+\beta(\frac{3}{2}-\sigma_{12}-\sigma_{13})\right)\right)+r_2\left(1+\beta(\frac{3}{2}-\sigma_{13})\left(1+\beta\sigma_{13}\right)\right) \\ &\quad +r_3\left(1+\beta(\frac{3}{2}-\sigma_{12})\left(1+\beta\sigma_{12}\right)\right) \end{aligned}$$

by choosing the link strengths σ_{12}, σ_{13} subject to $0 \leq \sigma_{12} \leq 1$ and $0 \leq \sigma_{13} \leq 1$

The Hessian matrix is given by

$$H = \begin{bmatrix} -2\beta^2\left(r_1+r_3\right) & -2\beta^2 r_1 \\ -2\beta^2 r_1 & -2\beta^2\left(r_1+r_2\right) \end{bmatrix}$$

As $-2\beta^2\left(r_1+r_3\right) < 0$ and $\det\left(H\right) = 4\beta^2\left(r_1r_2+r_1r_3+r_2r_3\right) > 0$, the Hessian matrix is negative-definite. Thus, the objective function is strictly concave in the decision variables. We know that the social planner's objective function S is continuous function over a compact set defined by $0 \leq \sigma_{12} \leq 1$ and $0 \leq \sigma_{13} \leq 1$. Thus, the solution to the problem exists. Hence, the solution is unique. ■

Proof of Proposition 27. Let $\sigma^*_{12}, \sigma^*_{13}, \sigma^*_{23}$ be the socially optimal link structure for given $r_1 > r_2 > r_3 > 0$ and $0 < \beta < 1$. So, for all $\epsilon > 0$

$$\begin{aligned} S(\sigma^*_{12}, \sigma^*_{13}, \sigma^*_{23}) - S(\sigma^*_{12}, \sigma^*_{13}+\epsilon, \sigma^*_{23}-\epsilon) &\geqslant 0 \\ \beta\epsilon\left(r_2-r_1\right)(1-\beta\sigma^*_{12})+\beta^2\epsilon(r_1+r_2)\left(\sigma^*_{13}-\sigma^*_{23}-\epsilon\right) &\geqslant 0 \\ \beta(r_1+r_2)(\sigma^*_{13}-\sigma^*_{23}-\epsilon) &\geqslant (r_1-r_2)(1-\beta\sigma^*_{12}) \\ \sigma^*_{13}-\sigma^*_{23}-\epsilon &\geqslant \frac{(r_1-r_2)(1-\beta\sigma^*_{12})}{\beta(r_1+r_2)} \\ \sigma^*_{13}-\sigma^*_{23} &\geqslant \frac{(r_1-r_2)(1-\beta\sigma^*_{12})}{\beta(r_1+r_2)} - \epsilon \end{aligned}$$

Since $\sigma_{12}^* \leqslant 1$ and $\beta < 1$, we have $(1 - \beta\sigma_{12}^*) > 0$. Thus,

$$\frac{(r_1 - r_2)(1 - \beta\sigma_{12}^*)}{\beta(r_1 + r_2)} > 0$$

since $r_1 > r_2$. So,

$$\begin{aligned} \sigma_{13}^* - \sigma_{23}^* - \epsilon &\geqslant \frac{(r_1 - r_2)(1 - \beta\sigma_{12}^*)}{\beta(r_1 + r_2)} \\ \sigma_{13}^* - \sigma_{23}^* &\geqslant \frac{(r_1 - r_2)(1 - \beta\sigma_{12}^*)}{\beta(r_1 + r_2)} - \epsilon \end{aligned}$$

Let $0 < \epsilon < \frac{(r_1 - r_2)(1 - \beta\sigma_{12}^*)}{\beta(r_1 + r_2)}$. Then,

$$\begin{aligned} \sigma_{13}^* - \sigma_{23}^* &\geqslant \frac{(r_1 - r_2)(1 - \beta\sigma_{12}^*)}{\beta(r_1 + r_2)} - \epsilon > 0 \\ \sigma_{13}^* &> \sigma_{23}^* \end{aligned}$$

Similarly, let $\sigma_{12}^*, \sigma_{13}^*, \sigma_{23}^*$ be the socially optimal link structure for given $r_1 > r_2 > r_3 > 0$ and $0 < \beta < 1$. So, for all $\epsilon > 0$

$$\begin{aligned} S(\sigma_{12}^*, \sigma_{13}^*, \sigma_{23}^*) - S(\sigma_{12}^* + \epsilon, \sigma_{13}^* - \epsilon, \sigma_{23}^*) &\geqslant 0 \\ \beta\epsilon\left(r_3 - r_2)(1 - \beta\sigma_{23}^*) + \beta^2\epsilon(r_2 + r_3)\right)(\sigma_{12}^* - \sigma_{13}^* - \epsilon) &\geqslant 0 \\ \beta(r_2 + r_3)(\sigma_{12}^* - \sigma_{13}^* - \epsilon) &\geqslant (r_2 - r_3)(1 - \beta\sigma_{23}^*) \\ \sigma_{12}^* - \sigma_{13}^* - \epsilon &\geqslant \frac{(r_2 - r_3)(1 - \beta\sigma_{23}^*)}{\beta(r_2 + r_3)} \\ \sigma_{12}^* - \sigma_{13}^* &\geqslant \frac{(r_2 - r_3)(1 - \beta\sigma_{23}^*)}{\beta(r_2 + r_3)} - \epsilon \end{aligned}$$

Since $\sigma_{23}^* \leqslant 1$ and $\beta < 1$, we have $(1 - \beta\sigma_{23}^*) > 0$. Thus,

$$\frac{(r_2 - r_3)(1 - \beta\sigma_{23}^*)}{\beta(r_2 + r_3)} > 0$$

since $r_1 > r_2$. So,

$$\begin{aligned}
\sigma_{12}^* - \sigma_{13}^* - \epsilon &\geqslant \frac{(r_2 - r_3)(1 - \beta\sigma_{23}^*)}{\beta(r_2 + r_3)} \\
\sigma_{12}^* - \sigma_{13}^* &\geqslant \frac{(r_2 - r_3)(1 - \beta\sigma_{23}^*)}{\beta(r_2 + r_3)} - \epsilon
\end{aligned}$$

Let $0 < \epsilon < \frac{(r_2 - r_3)(1 - \beta\sigma_{23}^*)}{\beta(r_2 + r_3)}$. Then,

$$\begin{aligned}
\sigma_{12}^* - \sigma_{13}^* &\geqslant \frac{(r_2 - r_3)(1 - \beta\sigma_{23}^*)}{\beta(r_2 + r_3)} - \epsilon > 0 \\
\sigma_{12}^* &> \sigma_{13}^*
\end{aligned}$$

■

Proof of Proposition 28. As we have pointed it out previously, Kuhn-Tucker conditions will be sufficient for global maxima.

Let us write the Lagrangian function as follows.

$$\begin{aligned}
\mathcal{L} = {} & r_1 \left(1 + \beta(\sigma_{12} + \sigma_{13}) \left(1 + \beta\sigma_{23}\right)\right) + r_2 \left(1 + \beta(\sigma_{12} + \sigma_{23}) \left(1 + \beta\sigma_{13}\right)\right) \\
& + r_3 \left(1 + \beta(\sigma_{13} + \sigma_{23}) \left(1 + \beta\sigma_{12}\right)\right) \\
& + \lambda_1(\frac{3}{2} - \sigma_{12} - \sigma_{13} - \sigma_{23}) + \lambda_2(1 - \sigma_{12}) + \lambda_3(1 - \sigma_{13}) + \lambda_4(1 - \sigma_{23})
\end{aligned}$$

Thus, the solutions of the maximization of the social welfare problem is given by the following system of equations:

$$\frac{\partial \mathcal{L}}{\partial \sigma_{12}} = r_1\beta\left(1+\beta\sigma_{23}\right) + r_2\beta\left(1+\beta\sigma_{13}\right) + r_3\beta^2(\sigma_{13}+\sigma_{23}) - \lambda_1 - \lambda_2 \leq 0$$

$$\frac{\partial \mathcal{L}}{\partial \sigma_{12}}\sigma_{12} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \sigma_{13}} = r_1\beta\left(1+\beta\sigma_{23}\right) + r_2\beta^2\left(\sigma_{12}+\sigma_{23}\right) + r_3\beta(1+\beta\sigma_{12}) - \lambda_1 - \lambda_3 \leq 0$$

$$\frac{\partial \mathcal{L}}{\partial \sigma_{13}}\sigma_{13} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \sigma_{23}} = r_1\beta^2\left(\sigma_{12}+\sigma_{13}\right) + r_2\beta\left(1+\beta\sigma_{13}\right) + r_3\beta(1+\beta\sigma_{12}) - \lambda_1 - \lambda_4 \leq 0$$

$$\frac{\partial \mathcal{L}}{\partial \sigma_{23}}\sigma_{23} = 0$$

$$\sigma_{12}+\sigma_{23}+\sigma_{13} = \frac{3}{2}$$

$$0 \leqslant \sigma_{12} \leqslant 1$$

$$0 \leqslant \sigma_{13} \leqslant 1$$

$$0 \leqslant \sigma_{23} \leqslant 1$$

$$0 \leq \lambda_2$$

$$0 \leq \lambda_3$$

$$0 \leq \lambda_4$$

Since $r_1 > r_2 > r_3 > 0$ and by Proposition 27, the solution is characterized by four cases.

Case 1: Interior Solution: $0 < \sigma_{23} < \sigma_{13} < \sigma_{12} < 1$

The restrictions on the system of equations are:

$$\lambda_2 = \lambda_3 = \lambda_4 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \sigma_{12}} = \frac{\partial \mathcal{L}}{\partial \sigma_{13}} = \frac{\partial \mathcal{L}}{\partial \sigma_{23}} = 0$$

So, the solution is

$$\sigma_{12} = \frac{4r_1r_2 + r_3(r_1+r_2)(3\beta-2)}{4\beta(r_1r_2+r_1r_3+r_2r_3)}$$

$$\sigma_{13} = \frac{4r_1r_3 + r_2(r_1+r_3)(3\beta-2)}{4\beta(r_1r_2+r_1r_3+r_2r_3)}$$

$$\sigma_{23} = \frac{4r_2r_3 + r_1(r_2+r_3)(3\beta-2)}{4\beta(r_1r_2+r_1r_3+r_2r_3)}$$

Notice that, since $r_1 > r_2 > r_3$, we have $\sigma_{23} < \sigma_{13} < \sigma_{12}$. So, we only need to check $0 < \sigma_{23}$ and $\sigma_{12} < 1$.

For $0 < \sigma_{23}$, we need

$$\frac{r_1(r_2+r_3)}{r_2r_3} < \frac{4}{2-3\beta} \text{ if } \beta < \frac{2}{3}$$

For $\sigma_{12} < 1$, we need

$$\frac{r_3(r_1+r_2)}{r_1r_2} > \frac{4(1-\beta)}{2+\beta}$$

Now, let's assume

$$\frac{r_2}{r_1} > \frac{2-3\beta}{2-\beta}$$

and

$$\frac{r_2}{r_3} < \frac{2+\beta}{2-\beta}$$

hold. Then, we have

$$\frac{r_1}{r_2} < \frac{2-\beta}{2-3\beta}$$

and

$$\frac{r_2+r_3}{r_3} < \frac{4}{2-\beta}$$

.So for $\beta < \frac{2}{3}$ we have

$$\frac{r_1}{r_2}\frac{r_2+r_3}{r_3} < \frac{2-\beta}{2-3\beta}\frac{4}{2-\beta}$$
$$\frac{r_1(r_2+r_3)}{r_2r_3} < \frac{4}{2-3\beta}$$

Moreover, when

$$\frac{r_2}{r_1} > \frac{2-3\beta}{2-\beta}$$

and

$$\frac{r_2}{r_3} < \frac{2+\beta}{2-\beta}$$

hold, we have

$$\frac{r_1+r_2}{r_1} > \frac{4-4\beta}{2-\beta}$$

and

$$\frac{r_3}{r_2} > \frac{2-\beta}{2+\beta}$$

Then, as $0 < \beta < 1$, we obtain

$$\frac{r_1+r_2}{r_1}\frac{r_3}{r_2} > \frac{4-4\beta}{2-\beta}\frac{2-\beta}{2+\beta}$$
$$\frac{r_3(r_1+r_2)}{r_1r_2} > \frac{4(1-\beta)}{2+\beta}$$

Case 2: $0 < \sigma_{23} < \sigma_{13} < \sigma_{12} = 1$

The restrictions on the system of equations are:

$$\lambda_3 = \lambda_4 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \sigma_{12}} = \frac{\partial \mathcal{L}}{\partial \sigma_{13}} = \frac{\partial \mathcal{L}}{\partial \sigma_{23}} = 0$$

$$\sigma_{23} + \sigma_{13} = \frac{1}{2}$$

So, the solution is

$$\sigma_{12} = 1$$

$$\sigma_{13} = \frac{r_1(2-\beta) + r_2(3\beta - 2)}{4\beta(r_1 + r_2)}$$

$$\sigma_{23} = \frac{r_1(3\beta-2) + r_2(2 - \beta)}{4\beta(r_1 + r_2)}$$

Again, notice that $\sigma_{23} < \sigma_{13} < \sigma_{12} = 1$ for $r_1 > r_2 > r_3$.

For $0 < \sigma_{23}$ and $\sigma_{13} < 1$, we need

$$\frac{r_2}{r_1} > \frac{2-3\beta}{2-\beta} \text{ if } \beta < \frac{2}{3}$$

For $\lambda_2 = r_2(1 + 2\beta\sigma_{13} - \frac{3\beta}{2}) - r_3(1 + \frac{\beta}{2}) \geqslant 0$, we need

$$\frac{r_3(r_1 + r_2)}{r_1 r_2} \leqslant \frac{4(1-\beta)}{2+\beta}$$

Notice that

$$\frac{r_1}{r_2} < \frac{2-\beta}{2-3\beta} \iff \frac{r_1}{r_1 + r_2} < \frac{2-\beta}{4(1-\beta)}$$

Thus, from the previous two equations, we obtain

$$\frac{r_3(r_1+r_2)}{r_1r_2} \leqslant \frac{4(1-\beta)}{2+\beta}$$
$$\frac{r_3(r_1+r_2)}{r_1r_2}\frac{r_1}{r_1+r_2} \leqslant \frac{4(1-\beta)}{2+\beta}\frac{2-\beta}{4(1-\beta)}$$
$$\frac{r_3}{r_2} \leqslant \frac{2-\beta}{2+\beta}$$
$$\frac{r_2}{r_3} \geqslant \frac{2+\beta}{2-\beta}$$

Case 3: $0 = \sigma_{23}, \sigma_{13} = \frac{1}{2}, \sigma_{12} = 1$

The restrictions on the system of equations are:

$$\lambda_3 = \lambda_4 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \sigma_{12}} = \frac{\partial \mathcal{L}}{\partial \sigma_{13}} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \sigma_{23}} \leqslant 0$$

So, the solution is

$$\sigma_{12} = 1$$

$$\sigma_{13} = \frac{1}{2}$$

$$\sigma_{23} = 0$$

For $\lambda_2 = r_2\beta(1-\frac{\beta}{2}) - r_3\beta(1+\frac{\beta}{2}) \geqslant 0$, we need

$$\frac{r_2}{r_3} \geqslant \frac{2+\beta}{2-\beta}$$

For $\frac{\partial \mathcal{L}}{\partial \sigma_{23}} \leqslant 0$, we need

$$\frac{r_2}{r_1} \leqslant \frac{2-3\beta}{2-\beta}$$

Notice that

$$\frac{r_2}{r_3} \geqslant \frac{2+\beta}{2-\beta} \iff \frac{r_2+r_3}{r_3} \geqslant \frac{4}{2-\beta}$$

From $\frac{r_2}{r_1} \leqslant \frac{2-3\beta}{2-\beta}$, $\frac{r_2+r_3}{r_3} \geqslant \frac{4}{2-\beta}$ and $\frac{r_2+r_3}{r_3} > 0$, we have

$$\begin{gathered}\frac{r_1}{r_2} \geqslant \frac{2-\beta}{2-3\beta} \iff \\ \frac{r_1(r_2+r_3)}{r_2 r_3} \geqslant \frac{2-\beta}{2-3\beta}\frac{r_2+r_3}{r_3} \geqslant \frac{2-\beta}{2-3\beta}\frac{4}{2-\beta} = \frac{4}{2-3\beta} \text{ if } \beta < \frac{2}{3} \\ \frac{r_1(r_2+r_3)}{r_2 r_3} > 0 > \frac{4}{2-3\beta} \text{ if } \beta > \frac{2}{3}\end{gathered}$$

Thus, we get

$$\frac{r_1(r_2+r_3)}{r_2 r_3} \geqslant \frac{4}{2-3\beta}$$

Moreover,

$$\frac{r_2}{r_1} \leqslant \frac{2-3\beta}{2-\beta} \iff \frac{r_1}{r_1+r_2} \geqslant \frac{2-\beta}{4(1-\beta)}$$

Thus, from $\frac{r_1}{r_1+r_2} \geqslant \frac{2-\beta}{4(1-\beta)}$ and $\frac{r_2}{r_3} \geqslant \frac{2+\beta}{2-\beta}$, we have

$$\begin{gathered}\frac{r_1}{r_1+r_2}\frac{r_2}{r_3} \leqslant \frac{2-\beta}{4(1-\beta)}\frac{2+\beta}{2-\beta} \\ \frac{r_1 r_2}{r_3(r_1+r_2)} \leqslant \frac{2+\beta}{4(1-\beta)} \\ \frac{r_3(r_1+r_2)}{r_1 r_2} \leqslant \frac{4(1-\beta)}{2+\beta}\end{gathered}$$

Case 4: $0 = \sigma_{23}, 0 < \sigma_{13} < \sigma_{12} < 1$

The restrictions on the system of equations are:

$$\lambda_2 = \lambda_3 = \lambda_4 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \sigma_{12}} = \frac{\partial \mathcal{L}}{\partial \sigma_{13}} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \sigma_{23}} \leqslant 0$$

So, the solution is

$$\sigma_{12} = \frac{r_2(2+3\beta) - r_3(2-3\beta)}{4\beta(r_2+r_3)}$$

$$\sigma_{13} = \frac{r_2(3\beta-2) + r_3(2+3\beta)}{4\beta(r_2+r_3)}$$

$$\sigma_{23} = 0$$

For $\sigma_{13} = \frac{3}{2} - \sigma_{12}$ and $\sigma_{12} < 1$, we have $\frac{1}{2} < \sigma_{13}$.Thus, we need

$$\frac{r_2}{r_3} < \frac{2+\beta}{2-\beta}$$

For $\frac{\partial \mathcal{L}}{\partial \sigma_{23}} \leqslant 0$, we need

$$\frac{r_1(r_2+r_3)}{r_2 r_3} \geqslant \frac{4}{2-3\beta}$$

Note that we have

$$\frac{r_2}{r_3} < \frac{2+\beta}{2-\beta}$$
$$\frac{r_2+r_3}{r_3} < \frac{4}{2-\beta}$$

Then,

$$\frac{4}{2-3\beta} \leqslant \frac{r_1(r_2+r_3)}{r_2 r_3} \leqslant \frac{r_1}{r_2}\frac{(r_2+r_3)}{r_3} < \frac{r_1}{r_2}\frac{4}{2-\beta}$$
$$\frac{4}{2-3\beta} < \frac{r_1}{r_2}\frac{4}{2-\beta}$$
$$\frac{r_2}{r_1} > \frac{2-3\beta}{2-\beta}$$

■

Proof of Proposition 29. Since

$$\frac{\partial\sigma_{ij}}{\partial r_i} = \frac{{r_j}^2 r_k(6-3\beta)}{4\beta(r_1r_2+r_1r_3+r_2r_3)^2} > 0$$

$$\frac{\partial\sigma_{ij}}{\partial r_j} = \frac{{r_i}^2 r_k(6-3\beta)}{4\beta(r_1r_2+r_1r_3+r_2r_3)^2} > 0$$

and

$$\frac{\partial\sigma_{ij}}{\partial r_k} = \frac{(r_i+r_j)r_ir_j(3\beta-6)}{4\beta(r_1r_2+r_1r_3+r_2r_3)^2} < 0$$

the strength of the link σ_{ij} is increasing in r_i and r_j and decreasing in r_k. ■

Proof of Proposition 30. If we take the derivative of player 1's surplus with respect to his choice variable, we get

$$\frac{\partial S_1}{\partial z_{12}} = r_2(z_{21} - \beta(1-z_{21})(1-z_{31})z_{31}) - r_3(z_{31} - \beta(1-z_{21})(1-z_{31})z_{21})$$

Similarly ,

$$\frac{\partial S_2}{\partial z_{21}} = r_1(z_{12} - \beta(1 - z_{12})z_{31}(1 - z_{31})) - r_3((1 - z_{31}) - \beta(1 - z_{12})z_{31}z_{12})$$
$$\frac{\partial S_3}{\partial z_{31}} = r_1((1 - z_{12}) - \beta z_{12}z_{21}(1 - z_{21})) - r_2((1 - z_{21}) - \beta z_{12}z_{21}(1 - z_{12})$$

Notice that if

$$r_2(z_{21} - \beta(1 - z_{21})(1 - z_{31})z_{31}) - r_3(z_{31} - \beta(1 - z_{21})(1 - z_{31})z_{21}) > 0$$

then $\frac{\partial S_1}{\partial z_{12}} > 0$ and $z_{12} = 1$.Thus,

$$\frac{\partial S_2}{\partial z_{21}} = r_1(z_{12} - \beta(1 - z_{12})z_{31}(1 - z_{31})) - r_3((1 - z_{31}) - \beta(1 - z_{12})z_{31}z_{12})$$
$$= r_1 - r_3(1 - z_{31}) > 0$$

making $z_{21} = 1$. Moreover, if $z_{12} = 1$ and $z_{21} = 1$ then

$$\frac{\partial S_3}{\partial z_{31}} = r_1((1 - z_{12}) - \beta z_{12}z_{21}(1 - z_{21})) - r_2((1 - z_{21}) - \beta z_{12}z_{21}(1 - z_{12}) = 0$$

So, z_{31} can take any value between 0 and 1. That is $z_{31} : free$. Lastly, we have to check

$$\frac{\partial S_1}{\partial z_{12}} = r_2(z_{21} - \beta(1 - z_{21})(1 - z_{31})z_{31}) - r_3(z_{31} - \beta(1 - z_{21})(1 - z_{31})z_{21}) = r_2 - r_3z_{31} > 0$$

under $z_{12} = 1$, $z_{21} = 1$ and $z_{31} : free$. This is the first set of Nash equilibria.

On the other hand, if

$$r_2(z_{21} - \beta(1 - z_{21})(1 - z_{31})z_{31}) - r_3(z_{31} - \beta(1 - z_{21})(1 - z_{31})z_{21}) < 0$$

then $\frac{\partial S_1}{\partial z_{12}} < 0$ and $z_{12} = 0$.Thus,

$$\frac{\partial S_2}{\partial z_{21}} = r_1(z_{12} - \beta(1 - z_{12})z_{31}(1 - z_{31})) - r_3((1 - z_{31}) - \beta(1 - z_{12})z_{31}z_{12})$$

$$= -r_1\beta(1 - z_{12})z_{31}(1 - z_{31}) - r_3(1 - z_{31}) < 0$$

making $z_{21} = 0$. Moreover, if $z_{12} = 0$ and $z_{21} = 0$ then

$$\frac{\partial S_3}{\partial z_{31}} = r_1((1 - z_{12}) - \beta z_{12}z_{21}(1 - z_{21})) - r_2((1 - z_{21}) - \beta z_{12}z_{21}(1 - z_{12})$$

$$= r_1 - r_2 > 0$$

So, $z_{31} = 1$.

Lastly, we have to check

$$\frac{\partial S_1}{\partial z_{12}} = r_2(z_{21} - \beta(1 - z_{21})(1 - z_{31})z_{31}) - r_3(z_{31} - \beta(1 - z_{21})(1 - z_{31})z_{21}) = -r_3 < 0$$

under $z_{12} = 0$, $z_{21} = 0$ and $z_{31} = 1$. This is the second set of Nash equilibria. ■

Proof of Proposition 31. Since the social planner maximizes the objective function over the set

$$z_i = \{z_{ik}\}_{k \neq i}$$

$$0 \leq z_{ik} \leq 1 \text{ for all } k \in N \setminus \{i\}$$

$$\sum_{k \neq i} z_{ik} = 1$$

which is compact and the objective function S is a continuous over this compact set, the problem has a solution. ■

Proof of Lemma 32. As Kuhn-Tucker conditions are necessary for optimality, the following holds for all socially optimal outcome.

$$\frac{\partial \mathcal{L}}{\partial z_{ji}} = r_i\beta((1-z_{ik})(1+\beta(1-z_{ki})(1-z_{ji})) - \beta(1-z_{ki})(z_{ik}z_{ki}+(1-z_{ik})z_{ji}))$$
$$+ r_k\beta(-(1-z_{ki})(1+\beta(1-z_{ik})z_{ji}) + \beta(1-z_{ik})(z_{ik}z_{ki}+(1-z_{ik})(1-z_{ji})))$$
$$+ r_j\beta(z_{ki}-z_{ik})(1+\beta z_{ik}z_{ki})) - \lambda_j$$

Lets assume $z_{ij}=0$. This means $z_{ik}=1$. For any $0 \leqslant z_{ki} \leqslant 1$,

$$\frac{\partial \mathcal{L}}{\partial z_{ji}} = -\beta(1-z_{ki})(z_{ik}z_{ki}+(1-z_{ik})z_{ji})) - r_k\beta(1-z_{ki}) - r_j\beta(1-z_{ki})(1+\beta z_{ki})) - \lambda_j$$
$$= -(\beta(1-z_{ki})(z_{ik}z_{ki}+(1-z_{ik})z_{ji})) + r_k\beta(1-z_{ki}) + r_j\beta(1-z_{ki})(1+\beta z_{ki})) + \lambda_j) \leqslant 0$$

as each term in the brackets is positive. Thus, we should have $z_{ji}=0$. ■

Proof of Lemma 33. Let's assume that Case 4 where $z_{12}=1$, $0 < z^*_{21} < 1$, $z_{31}=0$ is the socially optimum outcome.

$$S(1, z^*_{21}, 0) = r_1(1+\beta z^*_{21}(1+\beta(1-z^*_{21}))) + r_2(1+\beta) + r_3(1+\beta(1-z^*_{21})(1+\beta z^*_{21}))$$

$$S(z^*_{21}, 1, 1) = r_1(1+\beta) + r_2(1+\beta z^*_{21}(1+\beta(1-z^*_{21}))) + r_3(1+\beta(1-z^*_{21})(1+\beta z^*_{21}))$$

$$S(1, z^*_{21}, 0) - S(z^*_{21}, 1, 1) = -\beta\left(r_1-r_2\right)\left(1-\beta z^*_{21}\right)\left(1-z^*_{21}\right) \leqslant 0$$
$$S(1, z^*_{21}, 0) \leqslant S(z^*_{21}, 1, 1)$$

Thus, Case 4 cannot be socially optimal as it is dominated by $z_{12}=z^*_{21}$, $z_{21}=1$, $z^*_{31}=1$

for $r_1 \geqslant r_2$.

Let's assume that Case 5 where $z_{12} = 0$, $z_{21} : free$, $z_{31} = 0$ is the socially optimum outcome.

$$S(0, z_{21}, 1) = r_1(1+\beta) + r_2 + r_3(1+\beta)$$

$$S(1, 1, z_{31}) = r_1(1+\beta) + r_2(1+\beta) + r_3$$

$$S(0, z_{21}, 1) - S(1, 1, z_{31}) = -\beta\,(r_2 - r_3) \leqslant 0$$

$$S(0, z_{21}, 1) \leqslant S(1, 1, z_{31})$$

Thus, Case 5 cannot be socially optimal as it is dominated by $z_{12} = 1,\;\; z_{21} = 1,\; z_{31} : free$ for $r_2 \geqslant r_3$.

Let's assume that Case 6 where $z_{12} = 0,\;\; z_{21} = 0,\;\; 0 < \;\; z^*_{31} < 1$ is the socially optimum outcome.

$$S(0, 0, z^*_{31}) = r_1(1+\beta z^*_{31}(1+\beta(1-z^*_{31}))) + r_2(1+\beta(1-z^*_{31})(1+\beta z^*_{31})) + r_3(1+\beta)$$

$$S(1, z^*_{31}, 0) = r_1(1+\beta z^*_{31}(1+\beta(1-z^*_{31}))) + r_2(1+\beta) + r_3(1+\beta(1-z^*_{31})(1+\beta z^*_{31}))$$

$$S(0, 0, z^*_{31}) - S(1, z^*_{31}, 0) = -\beta z^*_{31}\,(r_2 - r_3)\,(\beta z^*_{31} - \beta + 1) \leqslant 0$$

$$S(0, 0, z^*_{31}) \leqslant S(1, z^*_{31}, 0)$$

Thus, Case 6 cannot be socially optimal as it is dominated by $z_{12} = 1,\;\; z_{21} = z^*_{31},\; z^*_{31} = 0$ for $r_2 \geqslant r_3$.

Let's assume that Case 7 where $z_{12}: free$, $z_{21}=0,\ z_{31}=0$ is the socially optimum outcome.

$$S(z_{12},0,0) = r_1 + r_2(1+\beta) + r_3(1+\beta)$$

$$S(1,1,z_{31}) = r_1(1+\beta) + r_2(1+\beta) + r_3$$

$$S(z_{12},0,0) - S(1,1,z_{31}) = -\beta\,(r_1 - r_3) \leqslant 0$$

$$S(z_{12},0,0) \leqslant S(1,1,z_{31})$$

Thus, Case 7 cannot be socially optimal as it is dominated by $z_{12}=1,\ \ z_{21}=1$, $z_{31}: free$ for $r_2 \geqslant r_3$.

■

Proof of Proposition 34.

We can rewrite the social surplus in the following way:

$$S = r_1 + r_2 + r_3 + \beta\,(r_1+r_2)\,(z_{12}z_{21} + \beta(1-z_{12})z_{31}(1-z_{21})(1-z_{31}))$$
$$+\,\beta\,(r_1+r_3)\,((1-z_{12})z_{31} + \beta z_{12}z_{21}(1-z_{21})(1-z_{31}))$$
$$+\,\beta\,(r_2+r_3)\,((1-z_{21})(1-z_{31}) + \beta(1-z_{12})z_{12}z_{21}z_{31})$$

Since $r_1+r_2 > r_1+r_3 > r_2+r_3$, the coefficients of these variables should also be ranked accordingly at the surplus-maximizing outcome. Otherwise, social surplus can be increased by reallocation of links strengths. Thus, we should have

$$z_{12}z_{21} + \beta(1-z_{12})z_{31}(1-z_{21})(1-z_{31}) > (1-z_{12})z_{31} + \beta z_{12}z_{21}(1-z_{21})(1-z_{31}) \quad (35)$$

and

$$(1-z_{12})z_{31}+\beta z_{12}z_{21}(1-z_{21})(1-z_{31}) > (1-z_{21})(1-z_{31})+\beta(1-z_{12})z_{12}z_{21}z_{31} \tag{36}$$

From 35, we have

$$\begin{aligned}
z_{12}z_{21}+\beta(1-z_{12})z_{31}(1-z_{21})(1-z_{31}) &> (1-z_{12})z_{31}+\beta z_{12}z_{21}(1-z_{21})(1-z_{31}) \\
z_{12}z_{21}-\beta z_{12}z_{21}(1-z_{21})(1-z_{31}) &> (1-z_{12})z_{31}-\beta(1-z_{12})z_{31}(1-z_{21})(1-z_{31}) \\
z_{12}z_{21}\left(1-\beta(1-z_{21})(1-z_{31})\right) &> (1-z_{12})z_{31}\left(1-\beta(1-z_{21})(1-z_{31})\right)
\end{aligned}$$

As $(1-\beta(1-z_{21})(1-z_{31}))>0$ at any surplus-maximizing outcome by Proposition 33, we have

$$z_{12}z_{21}=\sigma_{12}>(1-z_{12})z_{31}=\sigma_{31} \tag{37}$$

From 36, we have

$$\begin{aligned}
(1-z_{12})z_{31}+\beta z_{12}z_{21}(1-z_{21})(1-z_{31}) &> (1-z_{21})(1-z_{31})+\beta(1-z_{12})z_{12}z_{21}z_{31} \\
(1-z_{12})z_{31}-\beta(1-z_{12})z_{12}z_{21}z_{31} &> (1-z_{21})(1-z_{31})-\beta z_{12}z_{21}(1-z_{21})(1-z_{31}) \\
(1-z_{12})z_{31}\left(1-\beta z_{12}z_{21}\right) &> (1-z_{21})(1-z_{31})\left(1-\beta z_{12}z_{21}\right)
\end{aligned}$$

As $1-\beta z_{12}z_{21}>0$ at any surplus-maximizing outcome except Case 3, where $\sigma_{31}=\sigma_{23}=0$, by Proposition 33, we have

$$(1-z_{12})z_{31}=\sigma_{31}>(1-z_{21})(1-z_{31})=\sigma_{23} \tag{38}$$

Therefore, by 37 and 38, we have

$$\sigma_{12} > \sigma_{31} \geq \sigma_{23}$$

■

Proof of Proposition 35. Using the first first-order condition 42a, we will get $z_{12} > \frac{1}{2}$ if the following holds:

$$0 < \frac{(r_1+r_2)(z_{21}-\beta(1-z_{21})(1-z_{31})z_{31})}{2\beta z_{21}z_{31}(r_2+r_3)} + \frac{(r_1+r_3)(-z_{31}+\beta(1-z_{21})(1-z_{31})z_{21})}{2\beta z_{21}z_{31}(r_2+r_3)}$$

As $2\beta z_{21}z_{31}(r_2+r_3) > 0$, we need

$$\begin{aligned} 0 &< (r_1+r_2)(z_{21}-\beta(1-z_{21})(1-z_{31})z_{31}) + (r_1+r_3)(-z_{31}+\beta(1-z_{21})(1-z_{31})z_{21}) \\ 0 &< (z_{21}-z_{31})\, r_1\,(1+\beta(1-z_{21})(1-z_{31})) + z_{21}\,(r_2+r_3\beta(1-z_{21})(1-z_{31})) \qquad (39) \\ &\quad -z_{31}\,(r_3+r_2\beta(1-z_{21})(1-z_{31})) \end{aligned}$$

Notice that as $r_2 > r_3$,we have

$$r_2 + r_3\beta(1-z_{21})(1-z_{31}) > r_3 + r_2\beta(1-z_{21})(1-z_{31})$$

Therefore, if $z_{21} \geq z_{31}$, the inequality in 39 holds and $z_{12} > \frac{1}{2}$.

Moreover, we have the following inequality 37 from the proof of Proposition 34:

$$z_{12}z_{21} > (1-z_{12})z_{31}$$

Thus, if $z_{21} < z_{31}$, then we should have

$$\begin{aligned} z_{12} &> (1 - z_{12}) \\ z_{12} &> \frac{1}{2} \end{aligned}$$

Hence, we have $z_{12} > \frac{1}{2}$ for any $0 < z_{21} < 1$ and $0 < z_{31} < 1$.

Similarly, using the second first-order condition 42b, we will get $z_{21} > \frac{1}{2}$ if the following holds:

$$0 < \frac{(r_1 + r_2)(z_{12} - \beta(1 - z_{31})(1 - z_{12})z_{31})}{2\beta z_{12}(1 - z_{31})(r_1 + r_3)} + \frac{(r_2 + r_3)(-(1 - z_{31}) + \beta z_{12}(1 - z_{12})z_{31})}{2\beta z_{12}(1 - z_{31})(r_1 + r_3)}$$

As $2\beta z_{12}(1 - z_{31})(r_1 + r_3) > 0$, we need

$$\begin{aligned} 0 &< (r_1 + r_2)(z_{12} - \beta(1 - z_{31})(1 - z_{12})z_{31}) + (r_2 + r_3)(-(1 - z_{31}) + \beta z_{12}(1 - z_{12})z_{31}) \\ 0 &< (z_{12} - (1 - z_{31}))\, r_2\, (1 + \beta(1 - z_{12})z_{31}) + z_{12}\, (r_1 + r_3\beta(1 - z_{12})z_{31}) \qquad (40) \\ &\quad -(1 - z_{31})\, (r_3 + r_1\beta(1 - z_{12})z_{31}) \end{aligned}$$

Notice that as $r_2 > r_3$,we have

$$r_1 + r_3\beta(1 - z_{12})z_{31} > r_3 + r_1\beta(1 - z_{12})z_{31}$$

Therefore, if $z_{12} \geq 1 - z_{31}$, the inequality in 40 holds and $z_{21} > \frac{1}{2}$.

Moreover, we obtain the following inequality 38 from the proof of Proposition 34:

$$z_{12}z_{21} > (1 - z_{21})(1 - z_{31})$$

Thus, if $z_{12} < 1 - z_{31}$, then we should have

$$\begin{aligned} z_{21} &> (1 - z_{21}) \\ z_{21} &> \frac{1}{2} \end{aligned}$$

Hence, we have $z_{21} > \frac{1}{2}$ for any $0 < z_{12} < 1$ and $0 < z_{31} < 1$.

Lastly, using the third first-order condition 42c, we will get $z_{31} > \frac{1}{2}$ if the following holds:

$$0 < \frac{(r_1 + r_3)(1 - z_{12} - \beta(1 - z_{21})z_{12}z_{21})}{2\beta(1 - z_{12})(1 - z_{21})(r_1 + r_2)} + \frac{(r_2 + r_3)(-(1 - z_{21}) + \beta(1 - z_{12})z_{12}z_{21})}{2\beta(1 - z_{12})(1 - z_{21})(r_1 + r_2)}$$

As $2\beta(1 - z_{12})(1 - z_{21})(r_1 + r_2) > 0$, we need

$$\begin{aligned} 0 &< (r_1 + r_3)(1 - z_{12} - \beta(1 - z_{21})z_{12}z_{21}) + (r_2 + r_3)(-(1 - z_{21}) + \beta(1 - z_{12})z_{12}z_{21}) \\ 0 &< ((1 - z_{12}) - (1 - z_{21}))\, r_3\, (1 + \beta z_{12}z_{21}) + (1 - z_{12})\,(r_1 + r_2\beta z_{12}z_{31}) \qquad (41) \\ &\quad -(1 - z_{21})\,(r_2 + r_1\beta z_{12}z_{21}) \end{aligned}$$

Notice that as $r_1 > r_2$,we have

$$r_1 + r_2\beta z_{12}z_{31} > r_2 + r_1\beta z_{12}z_{31}$$

Therefore, if $1 - z_{12} \geq 1 - z_{21}$, the inequality in 41 holds and $z_{31} > \frac{1}{2}$.

Moreover, we have the following inequality by combining 37 and 38 from the proof of Proposition 34:

$$(1 - z_{12})z_{31} > (1 - z_{21})(1 - z_{31})$$

Thus, if $1 - z_{12} < 1 - z_{21}$, then we should have

$$
\begin{aligned}
z_{31} &> (1 - z_{31}) \\
z_{31} &> \frac{1}{2}
\end{aligned}
$$

Hence, we have $z_{31} > \frac{1}{2}$ for any $0 < z_{12} < 1$ and $0 < z_{21} < 1$.

Therefore, we conclude that for any interior solution, we must have $\frac{1}{2} < z_{12} < 1$, $\frac{1}{2} < z_{21} < 1$, $\frac{1}{2} < z_{31} < 1$. ■

Proof of Proposition 36. By Lemma 32 and 33, we know that there are only 3 possible solutions for the social planner's problem.

Case 1: Interior Solution: $0 < z_{12} < 1$, $0 < z_{21} < 1$, $0 < z_{31} < 1$

The restrictions on the system of equations are:

$$
\lambda_1 = \lambda_2 = \lambda_3 = 0
$$

$$
\frac{\partial \mathcal{L}}{\partial z_{12}} = \frac{\partial \mathcal{L}}{\partial z_{21}} = \frac{\partial \mathcal{L}}{\partial z_{31}} = 0
$$

From the partial derivatives, we get:

$$
z_{12} = \frac{1}{2} + \frac{(r_1 + r_2)(z_{21} - \beta(1 - z_{21})(1 - z_{31})z_{31})}{2\beta z_{21} z_{31}(r_2 + r_3)} + \frac{(r_1 + r_3)(-z_{31} + \beta(1 - z_{21})(1 - z_{31})z_{21})}{2\beta z_{21} z_{31}(r_2 + r_3)} \tag{42a}
$$

$$
z_{21} = \frac{1}{2} + \frac{(r_1 + r_2)(z_{12} - \beta(1 - z_{31})(1 - z_{12})z_{31})}{2\beta z_{12}(1 - z_{31})(r_1 + r_3)} + \frac{(r_2 + r_3)(-(1 - z_{31}) + \beta z_{12}(1 - z_{12})z_{31})}{2\beta z_{12}(1 - z_{31})(r_1 + r_3)} \tag{42b}
$$

$$
z_{31} = \frac{1}{2} + \frac{(r_1 + r_3)(1 - z_{12} - \beta(1 - z_{21})z_{12}z_{21})}{2\beta(1 - z_{12})(1 - z_{21})(r_1 + r_2)} + \frac{(r_2 + r_3)(-(1 - z_{21}) + \beta(1 - z_{12})z_{12}z_{21})}{2\beta(1 - z_{12})(1 - z_{21})(r_1 + r_2)} \tag{42c}
$$

There is no interior solution for the social planner's problem. To see this, lets take a look at the case where $r_1 = r_2 = r_3$ and $\beta = 1$. Then, under these restrictions, the interior solution to the social planner's problem is

$$z_{12} = z_{21} = z_{31} = \frac{1}{2}$$

Under these investment levels, the total surplus is

$$S(z_{12}, z_{21}, z_{31}) = S(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = \frac{21}{4} r_1$$

On the other hand, if player 2 and player 3 choose to invest all their time in player 1, and player 1 allocates his time equally between player 2 and player 3, we have

$$z_{12} = \frac{1}{2}, z_{21} = z_{31} = 1$$

Under this corner solution, the total surplus is

$$S(z_{12}, z_{21}, z_{31}) = S(\frac{1}{2}, 1, 1) = \frac{11}{2} r_1$$

Therefore, we have

$$S(\frac{1}{2}, 1, 1) > S(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$$

for $r_1 = r_2 = r_3$ and $\beta = 1$, and the interior solution cannot be optimal.

Notice that under $\beta = 1$, there is no decay in indirect communication. As β decreases, the benefits from indirect communication decreases. Hence, the social planner has less incentive to allocate investment to indirect communication.

Lets take a look at how first-order conditions vary with changes in the differences between the players' information levels r_i's. In the following graph, the first order conditions 42a are plotted for different values for r_1, r_2, r_3 where β is held constant at 1. We know from Proposition 35 that for any interior solution, we have $\frac{1}{2} < z_{12} < 1$, $\frac{1}{2} < z_{21} < 1$ and $\frac{1}{2} < z_{31} < 1$. Thus, the graph restricted to this area. The first order conditions 42a, 42b and 42c for $r_1 = r_2 = r_3 = 1$ and $\beta = 1$ are plotted by using blue, red and green respectively. The first order conditions 42a, 42b and 42c for $r_1 = 2, r_2 = 1.5, r_3 = 1$ and $\beta = 1$ are plotted by using yellow, purple and pink respectively. As the differences between r_1, r_2 and r_3 increases, the first order condition 42a shifts upwards from blue to yellow, and the first order condition 42b shifts right from green to purple. On the other hand, as the differences between r_1, r_2 and r_3 increases, the first order condition 42c shifts down from red to pink. This shift makes 42c move away from the intersection of 42a and 42b. Therefore, 42a, 42b and 42c do not intersect in the region of $\frac{1}{2} < z_{12} < 1$, $\frac{1}{2} < z_{21} < 1$ and $\frac{1}{2} < z_{31} < 1$. Therefore, there is no interior solution to the social planner's problem.

Case 2: $0 < z_{12} < 1$, $z_{21} = z_{31} = 1$.

The restrictions on the system of equations are:

$$\lambda_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial z_{12}} = \frac{\partial \mathcal{L}}{\partial z_{21}} = \frac{\partial \mathcal{L}}{\partial z_{31}} = 0$$

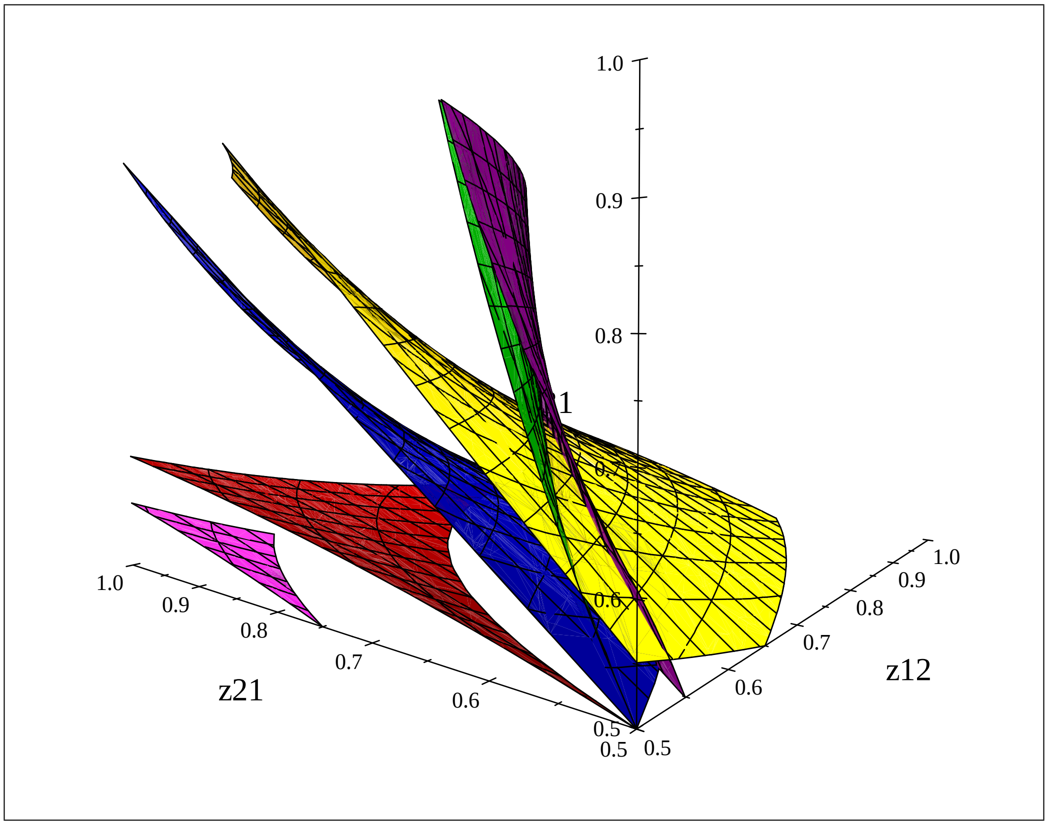


Figure 12: First order conditions under different r_1, r_2 and r_3 levels

For $z_{21} = z_{31} = 1$, we get

$$\frac{\partial \mathcal{L}}{\partial z_{12}} = r_2(1 + \beta(1 - z_{12}) - \beta z_{12}) + r_3(-1 - \beta z_{12} + \beta(1 - z_{12})) = 0$$

$$r_2(1 + \beta) - r_3(1 - \beta) = 2\beta(r_2 + r_3)z_{12}$$

$$z_{12} = \frac{1}{2} + \frac{r_2 - r_3}{2\beta(r_2 + r_3)}$$

As $r_2 > r_3$, $z_{12} > 0$. So, for $z_{12} < 1$, we need to have

$$z_{12} < 1$$

$$\frac{1}{2} + \frac{r_2 - r_3}{2\beta(r_2 + r_3)} < 1$$

$$r_2(1-\beta) < r_3(1+\beta)$$

$$\frac{r_2}{r_3} < \frac{1+\beta}{1-\beta}$$

We need to check $\lambda_2 \geqslant 0$ and $\lambda_3 \geqslant 0$.

$$\frac{\partial \mathcal{L}}{\partial z_{21}} = r_1\beta z_{12} + r_2\beta z_{12}(1 + \beta(1 - z_{12})) + r_3\beta^2 z_{12}(1 - z_{12}) - \lambda_2 = 0$$

$$\lambda_2 = r_1\beta z_{12} + r_2\beta z_{12}(1 + \beta(1 - z_{12})) + r_3\beta^2 z_{12}(1 - z_{12}) > 0$$

for all $0 < z_{12} < 1$.

$$\frac{\partial \mathcal{L}}{\partial z_{31}} = r_1\beta(1 - z_{12}) + r_2\beta^2(1 - z_{12})z_{12} + r_3\beta(1 - z_{12})(1 + \beta) - \lambda_3 = 0$$

$$\lambda_3 = r_1\beta(1 - z_{12}) + r_2\beta^2(1 - z_{12})z_{12} + r_3\beta(1 - z_{12})(1 + \beta) > 0$$

for all $0 < z_{12} < 1$.

Case 3: $z_{12} = z_{21} = 1$, z_{31} : *free.*

The restrictions on the system of equations are:

$$\lambda_3 = 0$$

$$\frac{\partial \mathcal{L}}{\partial z_{12}} = \frac{\partial \mathcal{L}}{\partial z_{21}} = 0$$

For $z_{12}=1$, $z_{21}=1$, $z_{31}=0$ to be equilibrium, we should have $\lambda_1 \geqslant 0$.

$$\frac{\partial \mathcal{L}}{\partial z_{12}} = r_1\beta + r_2\beta(1-\beta) - \lambda_1 = 0$$

, then we have

$$\lambda_1 = \beta r_1 + \beta r_2(1-\beta) > 0$$

as $\beta < 1$. Moreover, we must have $\lambda_2 \geqslant 0$

$$\frac{\partial \mathcal{L}}{\partial z_{21}} = r_1\beta(1+\beta-2\beta) + r_3\beta(-1+\beta-2\beta) - \lambda_2 = 0$$

$$\lambda_2 = r_1\beta(1-\beta) - r_3\beta(1+\beta) \geqslant 0$$

$$r_1\beta(1-\beta) \geqslant r_3\beta(1+\beta)$$

$$\frac{r_1}{r_3} \geqslant \frac{1+\beta}{1-\beta}$$

The following figure summaries conditions where Case 2 and Case 3 are optimum. Notice that the condition for Case 2, $\frac{r_2}{r_3} < \frac{1+\beta}{1-\beta}$, holds in Region 1 and Region 2, whereas the condition for Case 3, $\frac{r_1}{r_3} \geqslant \frac{1+\beta}{1-\beta}$, holds in Region 2 and Region 3. Thus, we have to check which set of strategy provides higher total social surplus in Region 2. Assume $\frac{r_1}{r_3} \geqslant \frac{1+\beta}{1-\beta}$ and $\frac{r_2}{r_3} < \frac{1+\beta}{1-\beta}$ so that the set of strategies is in Region 2. Then, the social surplus under the strategy $z_{12} = \frac{1}{2} + \frac{r_2 - r_3}{2\beta(r_2+r_3)}$, $z_{21} = z_{31} = 1$(Case 2) is

$$S(\frac{1}{2}+\frac{r_2-r_3}{2\beta(r_2+r_3)},1,1) = r_1(1+\beta)+r_2(1+\beta z_{12}(1+\beta(1-z_{12})))+r_3(1+\beta(1-z_{12})(1+\beta z_{12}))$$

whereas the social surplus under the strategy $z_{12} = z_{21} = 1$, $z_{31} : free$ (Case 3) is

$$S(1,1,z_{31}) = r_1(1+\beta) + r_2(1+\beta) + r_3$$

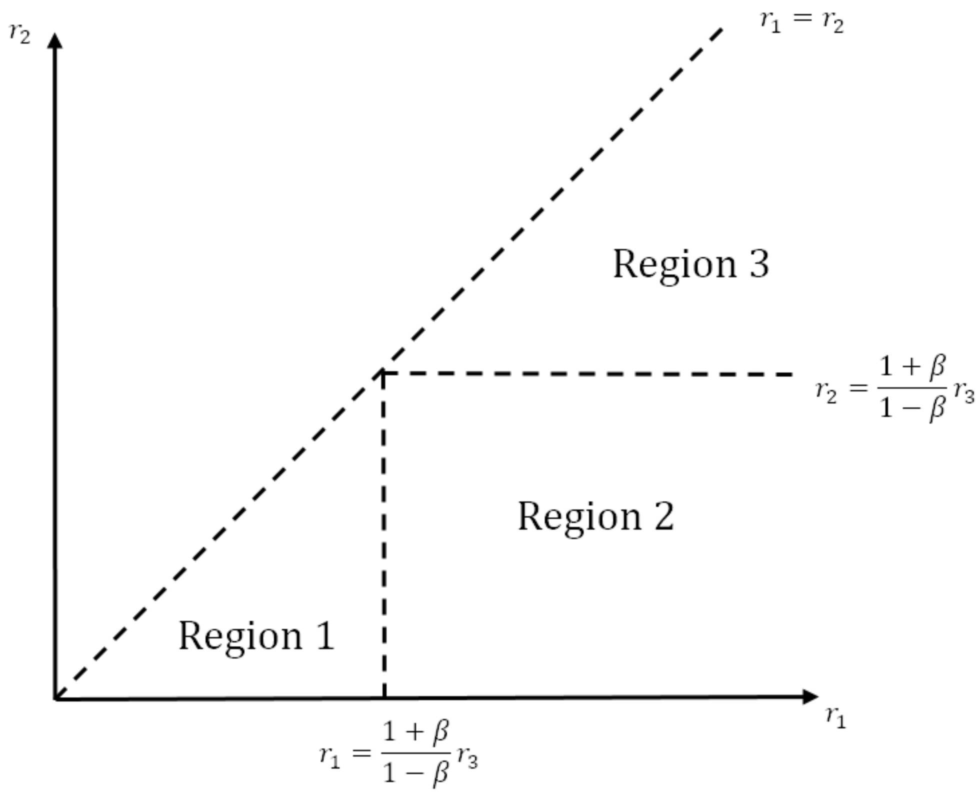


Figure 13: Nash equilibrium for different r_1 and r_2 levels

Thus,

$$\begin{aligned} S(1,1,z_{31}) - S(\frac{1}{2} + \frac{r_2 - r_3}{2\beta(r_2+r_3)}, 1, 1) &= r_2\beta(1 - z_{12}(1 + \beta(1 - z_{12}))) - r_3\beta(1 - z_{12})(1 + \beta z_{12}) \\ &= \beta(1 - z_{12})(r_2 - r_3 - \beta z_{12}(r_2 + r_3)) \\ &= \beta(\frac{1}{2} - \frac{r_2 - r_3}{2\beta(r_2+r_3)})(r_2 - r_3 - \beta(\frac{1}{2} + \frac{r_2 - r_3}{2\beta(r_2+r_3)})(r_2 + r_3)) \\ &= -\frac{1}{4\,(r_2 + r_3)}\,(r_3 - r_2 + \beta r_2 + \beta r_3)^2 < 0 \\ S(1,1,z_{31}) &< S(\frac{1}{2} + \frac{r_2 - r_3}{2\beta(r_2+r_3)}, 1, 1) \end{aligned}$$

if $\frac{r_1}{r_3} \geqslant \frac{1+\beta}{1-\beta}$ and $\frac{r_2}{r_3} < \frac{1+\beta}{1-\beta}$. Therefore, Case 2 where $z_{12} = \frac{1}{2} + \frac{r_2 - r_3}{2\beta(r_2+r_3)}$, $z_{21} = z_{31} = 1$ is optimal in Region 2. As a result, the optimal solutions are divided into two groups:

- For $\frac{r_2}{r_3} < \frac{1+\beta}{1-\beta}$, the optimal solution is given by $z_{12} = \frac{1}{2} + \frac{r_2 - r_3}{2\beta(r_2+r_3)}$, $z_{21} = z_{31} = 1$.
- For $\frac{r_2}{r_3} \geqslant \frac{1+\beta}{1-\beta}$, the optimal solution is given by $z_{12} = z_{21} = 1$, $z_{31} : free$

$\blacksquare$

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