THE MINIMAL PARABOLIC EISENSTEIN DISTRIBUTION ON THE DOUBLE COVER OF SL(3) OVER $\mathbb{Q}$.

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ABSTRACT OF THE DISSERTATION

The Minimal Parabolic Eisenstein Distribution on the Double Cover of SL(3) over $\mathbb{Q}$.

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We begin a study of the Fourier Coefficients of a minimal parabolic Eisenstein distribution on the double cover of SL(3) over $\mathbb{Q}$. The central problem in the computation of the Fourier coefficients is a computation of certain exponential sums twisted by the splitting map $s : \Gamma_1(4) \to \{\pm 1\}$, which appear after unfolding the integral defining the Fourier coefficients. In [11], Miller provides a formula for s; unfortunately, this formula does not appear conducive to the computation of the exponential sums; however, while considering a similar computation over number fields containing the 4-th roots of unity, Brubaker-Bump-Friedberg-Hoffstein [5] successfully computed the non-degenerate Fourier coefficients of a minimal parabolic Eisenstein series using a formula for an analog of s in terms of Plücker coordinates. These coordinates are well suited to the computation of the exponential sums and so our main objective is a proof that Miller's formula for s can be written in terms of Plücker coordinates. With this new formula for s we can compute the Fourier coefficients of our Eisenstein distribution. The calculations of these Fourier coefficients will be addressed in a forthcoming work.

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Chapter 1

Introduction

This dissertation begins a study of the Fourier coefficients of the minimal parabolic Eisenstein series on the double cover of SL(3) over $\mathbb{Q}$. This study will be broken into two parts. The first part, which makes up the contents of this dissertation, focuses on studying the splitting map (see Section 3.4). The second part, which will appear in a forthcoming work, utilizes the study of the splitting to execute the Fourier coefficient computation. It is worth mentioning that this Fourier coefficient computation is achieved using the technique of automorphic distributions. Although it is not necessary to use distributions to address this computation, the use of automorphic distributions should be advantageous for certain applications of these results.

To begin, we introduce the metaplectic Eisenstein series and briefly indicate where the splitting enters into the Fourier coefficient computation. Let $\widetilde{G} = \widetilde{\mathrm{SL}}(3,\mathbb{R})$, the double cover of $G = \mathrm{SL}(3,\mathbb{R})$. Let N_R be equal to the group of 3×3 upper triangular unipotent matrices with coefficients in the ring R and let B_R be the group of 3×3 upper triangular matrices with coefficients in the ring R; when $R = \mathbb{R}$ we may omit the subscript. The discrete subgroup $\Gamma_1(4) \subseteq G$ (defined on line (2.2)) can be embedded into $\widetilde{G}$ via the map $S : \Gamma_1(4) \hookrightarrow \widetilde{\mathrm{SL}}(3,\mathbb{R})$, where $S(\gamma) = (\gamma, s(\gamma))$ and $s : \Gamma_1(4) \to \{\pm 1\}$; the map s will be called the splitting map of $\Gamma_1(4)$ and it is introduced in Section 3.4. Let $\Gamma_\infty = \Gamma_1(4) \cap N(\mathbb{Z})$. Finally, $\tilde{\tau}_\lambda$ is a particular element of the principal series representation $\widetilde{V}^{-\infty}_{\lambda,\phi}$ (defined in Subsection 2.1.3).

The metaplectic Eisenstein distribution $\tilde{E}(\tilde{g},\lambda)$ is a vector valued automorphic distribution on $\widetilde{G} = \widetilde{\mathrm{SL}}(3,\mathbb{R})$; for certain complex vectors $\lambda \in \mathbb{C}^3$ the Eisenstein series satisfies the equation

$$\tilde{E}((\tilde{g},\lambda)) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_1(4)} \tilde{\tau}_\lambda(S(\gamma)\tilde{g}).$$

The (m_1, m_2)-th Whittaker distribution is defined to be

$$\int_{N_{\mathbb{Z}}\backslash N_{\mathbb{R}}} \tilde{E}(n\tilde{g}, \lambda)\psi^{-1}_{m_1,m_2}(n)dn, \tag{1.1}$$

where ψ_{m_1,m_2} is a one dimensional character of $N_{\mathbb{Z}}\backslash N_{\mathbb{R}}$. As usual, each Whittaker distribution is most easily computed after the Eisenstein series is broken up into incomplete sums indexed by the Bruhat cells. Let $\tilde{E}_w$ denote the partial Eisenstein series with respect to the Bruhat cell NwB. Specifically,

$$\tilde{E}_w((\tilde{g}, \lambda)) = \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma_1(4)\\ \gamma\in NwB}} \tilde{\tau}_\lambda(S(\gamma)\tilde{g}).$$

Roughly speaking, unfolding the integral (1.1) (with E replaced by E_w) yields a Dirichlet series whose coefficients are exponential sums twisted by the splitting s. When w is the long element of the Weyl group the exponential sums take on the form

$$\sum_{\substack{\gamma\in Nw_\ell B\\ \gamma\in\Gamma_\infty\backslash\Gamma_1(4)/\Gamma_\infty}} s(\gamma)e^{2\pi i(m1\frac{B_1}{A_1}+m2\frac{B_2}{A_2})}. \tag{1.2}$$

Miller [11], using the work of Banks-Levy-Sepanski [1], provides a formula for s, but this formula is not well suited to the computation of (1.2). (However, Miller's formula can be used to execute this computation when w is not the long element.) In [5], Brubaker-Bump-Friedberg-Hoffstein compute the non-degenerate Fourier coefficients of an Eisenstein series on the n-fold cover of SL(3) over a number field K containing the $2n$-th roots of unity; in the case of double covers this means that the base field must contain the 4-th roots of unity so their work does not apply when working over $\mathbb{Q}$. The Brubaker-Bump-Friedberg-Hoffstein formula for s in terms of Plücker coordinates (defined in Section 2.1.2) is convenient when it comes to computing the exponential sums appearing in the Fourier coefficient computation; unfortunately the formula does not hold over $\mathbb{Q}$, the case in which this Eisenstein series can be related to moment problems for quadratic Dirichlet L-series. The main result of Chapter 3 provides a formula for Miller's splitting in terms of Plücker coordinates. The most interesting case, which will be described in the next theorem, occurs when γ is in the big Bruhat cell.

Theorem 1. *Let $\gamma \in \Gamma_1(4)$. Suppose that the Plücker coordinates of γ are given by $(4A_1, 4B_1, C_1, 4A_2, 4B_2, C_2)$, such that $A_1 > 0$, $A_2 \neq 0$, and A_2 is divisible by fewer powers of 2 than A_1 is. Let $D = (A_1, A_2)$, $D_1 = (D, B_1)$, $D_2 = D/D_1$, and let $\epsilon = \left(\frac{-1}{-B_1/D_1}\right)$. Then*

$$s(\gamma) = \left(\frac{-\epsilon}{-A_1A_2}\right)\left(\frac{A_1/D}{A_2/D}\right)\left(\frac{B_1/D_1}{A_1/D}\right)\left(\frac{4B_2/D_2}{\operatorname{sign}(A_2)A_2/D}\right)\left(\frac{D_1}{C_1}\right)\left(\frac{D_2}{C_2}\right). \tag{1.3}$$

(see Section 3.6 for notation)

Although the statement of the previous theorem appears to have restrictive assumptions, the identities of Section 3.5 reduce the general situation to the case of the theorem.

If -1 is assumed to be a square, then this formula essentially matches the the formula of Brubaker-Bump-Friedberg-Hoffstein [5]. Another interesting feature of this formula is the appearance of ϵ. The term involving ϵ will equal 1 when (A_1, A_2) is odd, but need not when (A_1, A_2) is even.

We conclude the introduction with a brief outline of the contents of each chapter. Chapter 2 collects notation and basic computations that will be used throughout the remainder of this dissertation. Most of the material contained in this chapter is fairly standard. Subsection 2.2.2 provides one notable exception. This subsection introduces the Banks-Levy-Sepanski 2-cocycle [1] and collects some basic properties of the 2-cocycle. The choice of a 2-cocycle defines a multiplication on $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ and thus plays an important role in the determination of the splitting s.

In Chapter 3 we study the splitting s. The main result of this section provides a formula for the splitting in terms of Plücker Coordinates as seen in Theorem 1. This result is achieved by studying the effect of certain symmetries on s. Additionally, it is established that s satisfies a type of twisted multiplicativity. The formula for the splitting in terms of Plücker coordinates provides us a means to execute the computation of the Fourier coefficients of the metaplectic Eisenstein series. However, these computations will be addressed in a forthcoming work.

Chapter 2

Notation, Conventions, and Basic Computations

2.1 Notation

2.1.1 SL$(3,\mathbb{R})$ and $\widetilde{\mathrm{SL}}(3,\mathbb{R})$

In this section the notation related to $\mathrm{SL}(3,\mathbb{R})$ and $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ is established. As a set $\widetilde{\mathrm{SL}}(3,\mathbb{R}) \cong \mathrm{SL}(3,\mathbb{R}) \times \{\pm 1\}$. As a topological space $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ is given the nontrivial covering space topology whose existence is guaranteed by the fact that the fundamental group of $\mathrm{SL}(3,\mathbb{R})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. We will indicate how to determine the fundamental group of $\mathrm{SL}(3,\mathbb{R})$ once some notation has been established.

In this dissertation, the group law on $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ is defined using the Banks-Levy-Sepanski 2-cocycle, $\sigma : \mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R}) \to \{\pm 1\}$, constructed in [1] and recalled in Subsection 2.2.2. With the 2-cocycle the group multiplication is defined by

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 \sigma(g_1, g_2)). \tag{2.1}$$

The covering group $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ fits into the exact sequence

$$1 \to \{(1, \pm 1)\} \to \widetilde{\mathrm{SL}}(3,\mathbb{R}) \xrightarrow{\pi} \mathrm{SL}(3,\mathbb{R}) \to 1,$$

where the map π is given by $(g, \epsilon) \mapsto g$ and $(1, \pm 1)$ is contained in the center of $\widetilde{\mathrm{SL}}(3,\mathbb{R})$.

The following list establishes some notation for some subgroups of $\mathrm{SL}(3,\mathbb{R})$ and

$\widetilde{\mathrm{SL}}(3,\mathbb{R})$:

$$\begin{array}{lllllll}
G & = & \mathrm{SL}(3,\mathbb{R}) & , & \widetilde{G} & = & \widetilde{\mathrm{SL}}(3,\mathbb{R}) \\
B & = & \left\{ \begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & \frac{1}{ae} \end{pmatrix} | a, e \neq 0 \right\} & , & \widetilde{B} & = & \left\{ \left(\begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & \frac{1}{ae} \end{pmatrix}, \pm 1 \right) | a, e \neq 0 \right\} \\
B_- & = & \left\{ \begin{pmatrix} a & 0 & 0 \\ d & e & 0 \\ g & h & \frac{1}{ae} \end{pmatrix} | a, e \neq 0 \right\} & , & \widetilde{B}_- & = & \left\{ \left(\begin{pmatrix} a & 0 & 0 \\ d & e & 0 \\ g & h & \frac{1}{ae} \end{pmatrix}, \pm 1 \right) | a, e \neq 0 \right\} \\
N & = & \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\} & , & \widetilde{N} & = & \left\{ \left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, 1 \right) \right\} \\
T & = & \left\{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & \frac{1}{t_1 t_2} \end{pmatrix} | t_i \in \mathbb{R}^\times \right\} & , & \widetilde{T} & = & \left\{ \left(\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & \frac{1}{t_1 t_2} \end{pmatrix}, \pm 1 \right) | t_i \in \mathbb{R}^\times \right\} \\
A & = & \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \frac{1}{ab} \end{pmatrix} | a, b > 0 \right\} & , & \widetilde{A} & = & \left\{ \left(\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \frac{1}{ab} \end{pmatrix}, 1 \right) | a, b > 0 \right\} \\
M & = & \left\{ \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_1\epsilon_2 \end{pmatrix} | \epsilon_1, \epsilon_2 = \pm 1 \right\} & , & \widetilde{M} & = & \left\{ \left(\begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_1\epsilon_2 \end{pmatrix}, \pm 1 \right) | \epsilon_1, \epsilon_2 = \pm 1 \right\} \\
K & = & \mathrm{SO}(3) & , & \widetilde{K} & = & \mathrm{Spin}(3)
\end{array}$$

$$\Gamma = \Gamma_1(4) = \{\gamma \in \mathrm{SL}(3,\mathbb{Z}) | \gamma \equiv \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \pmod 4\} \quad , \quad \Gamma_\infty = N_{\mathbb{R}} \cap \mathrm{SL}(3,\mathbb{Z}). \tag{2.2}$$

Note that with the above notation $\widetilde{B} = \pi^{-1}(B)$, but the analogous relation does not hold for A or N. This is because the Banks-Levy-Sepanski 2-cocycle is defined so that A and N split with respect to $\widetilde{\mathrm{SL}}(3,\mathbb{R})$. To simplify notation $\left(\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \frac{1}{ab} \end{pmatrix}, 1 \right)$ will be written $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \frac{1}{ab} \end{pmatrix}$, and $\left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, 1 \right)$ will be written $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$.

In [11], Miller constructs a map $S : \Gamma_1(4) \hookrightarrow \widetilde{\mathrm{SL}}(3,\mathbb{R})$ such that $S(\gamma) = (\gamma, s(\gamma))$, where $s(\gamma) \in \{\pm 1\}$. This map S is a splitting of $\Gamma_1(4)$ into $\widetilde{SL}(3,\mathbb{R})$ (i.e $\pi \circ S =$id). The definition of the map s, which by abuse of terminology will also be called the splitting, is defined in section 3.4.

The following list establishes some notation for elements of $\mathrm{SL}(3,\mathbb{R})$:

$$\mathrm{t}(a,b,c) = \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}, \quad n(x,y,z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{llll}
w_{\alpha_1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & w_{\alpha_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \\
w_{\alpha_1} w_{\alpha_2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & w_{\alpha_2} w_{\alpha_1} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix},
\end{array}$$

$$\text{and } w_\ell = w_{\alpha_1} w_{\alpha_2} w_{\alpha_1} = w_{\alpha_2} w_{\alpha_1} w_{\alpha_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The last five elements listed above and the identity matrix constitute a complete set of representatives for $W = N(T)/T$, the Weyl group of $\mathrm{SL}(3,\mathbb{R})$.

A few comments are in order. The representatives of the Weyl group listed above are those defined in Section 3 of [1]. These representatives are used in the formula for of the 2-cocycle, which can be found in Section 4 of their previously cited paper. Also note that $w_\ell B w_\ell = B_-$ and $w_\ell N w_\ell = N_-$. Occasionally, if H is a subgroup of B, the notation $H^{op} = w_\ell H w_\ell^{-1}$ may be used.

Let $\mathfrak{a}$ be the Lie algebra of traceless 3×3 diagonal matrices with real entries. For $X \in \mathfrak{sl}(3, \mathbb{R})$, let $\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}$ be the exponential map and let log denote its inverse on A.

The map $K \times A \times N \to \mathrm{SL}(3, \mathbb{R})$, given by $(k, a, n) \mapsto kan$ is a diffeomorphism. This matrix factorization is called the Iwasawa decomposition. Define the maps $\kappa : \mathrm{SL}(3, \mathbb{R}) \to K$, $H : \mathrm{SL}(3, \mathbb{R}) \to \mathfrak{a}$, and $n : \mathrm{SL}(3, \mathbb{R}) \to N$ such that $g \mapsto (\kappa(g), \exp(H(g)), n(g))$ is the inverse of the map $(k, a, n) \mapsto kan$.

The Iwasawa decomposition can be used to prove that $\pi_1(\mathrm{SL}(3, \mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$. The subgroups A and N are contractible spaces and the Iwasawa decomposition provides a diffeomorphism between $\mathrm{SL}(3, \mathbb{R})$ and $K \times A \times N$. Thus $\pi_1(\mathrm{SL}(3, \mathbb{R})) \cong \pi_1(K)$ and it is well known that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z}$.

The definition of the Whittaker distributions involves integration over $\Gamma_\infty \backslash N \cong_{\mathrm{set}} [0, 1]^3$. Specifically, each coset of $\Gamma_\infty \backslash N$ has a unique representative $n(x, y, z)$ such that $0 \le x, y, z < 1$. The invariant measure of N will descend to the quotient $\Gamma_\infty \backslash N$ and is given by Lebesgue measure on $[0, 1]^3$. Thus in terms of coordinates $dn = dxdydz$.

2.1.2 Plücker Coordinates

In this section let $\Gamma = \mathrm{SL}(3, \mathbb{Z})$ and recall the notation of Subsection 2.1.3 and that $w_\ell = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

In the case of $\mathrm{SL}(2, \mathbb{R})$, the Fourier coefficient computation begins by observing that the coset space $\{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) | n \in \mathbb{Z}\} \backslash \mathrm{SL}(2, \mathbb{Z})$ is parameterized by $(c, d) \in \mathbb{Z} \times \mathbb{Z}$ such that $(c, d) = 1$. Specifically the bijection is given by the map $(c, d) \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $a, b \in \mathbb{Z}$ such that $ad - bc = 1$. Similarly, in the case of $\mathrm{SL}(3, \mathbb{R})$ we will be interested in an analogous parameterization for $\Gamma_\infty \backslash \Gamma$. This can be found in [3, Ch 5]. The correspondence is recalled for convenience, but first some notation is introduced.

The **Plücker coordinates** of $\Gamma_\infty\backslash\Gamma$ are defined presently. Given $\left(\begin{smallmatrix} a & b & c \\ d & e & f \\ g & h & i \end{smallmatrix}\right) \in$ $\mathrm{SL}(3,\mathbb{R})$, define six parameters as follows:

$$\begin{aligned} A_1' &= -g, \quad A_2' = -(dh - eg) \\ B_1' &= -h, \quad B_2' = (di - fg) \\ C_1' &= -i, \quad C_2' = -(ei - fh) \end{aligned} \tag{2.3}$$

Note that these parameters satisfy

$$A_1'C_2' + B_1'B_2' + C_1'A_2' = 0,$$

which arises from

$$\det\left(\left(\begin{smallmatrix} g & h & i \\ d & e & f \\ g & h & i \end{smallmatrix}\right)\right) = 0.$$

Now the parameterization of $N\backslash\mathrm{SL}(3,\mathbb{R})$ may be stated.

Theorem 2. *([3, Ch 5]) The map taking* $\left(\begin{smallmatrix} a & b & c \\ d & e & f \\ g & h & i \end{smallmatrix}\right) \in SL(3,\mathbb{R})$ *to* $(A_1', B_1', C_1', A_2', B_2', C_2')$ *defines a bijection between the coset space* $N\backslash SL(3,\mathbb{R})$ *and the set of all* $(A_1', B_1', C_1', A_2', B_2', C_2') \in \mathbb{R}^6$ *such that:* $A_1'C_2' + B_1'B_2' + C_1'A_2' = 0$, *not all of* A_1', B_1', C_1' *equal* 0, *and not all of* A_2', B_2', C_2' *equal* 0. *Furthermore, a coset in* $N\backslash SL(3,\mathbb{R})$ *contains an element of* Γ *if and only if* A_1', B_1', C_1' *are coprime integers and* A_2', B_2', C_2' *are coprime integers.*

Versions of this result hold for other congruence subgroups. In particular, $\Gamma_\infty\backslash\Gamma_1(4)$ can be identified with

$$\left\{ \begin{array}{c} (A_1', B_1', C_1', A_2', B_2', C_2') \in \mathbb{Z}^6 | A_1'C_2' + B_1'B_2' + C_1'A_2' = 0, \\ (A_i', B_i', C_i') = 1,\ A_j' \equiv 0 \pmod 4,\ B_j' \equiv 0 \pmod 4,\ C_j' \equiv -1 \pmod 4 \end{array} \right\}. \tag{2.4}$$

In certain cases it is advantageous to divide out the factors of 4 dividing A_i and B_i. This leads to an equivalent parameterization of $\Gamma_\infty\backslash\Gamma_1(4)$. Let

$$\begin{aligned} A_1' &= 4A_1, \quad A_2' = 4A_2, \\ B_1' &= 4B_1, \quad B_2' = 4B_2, \\ C_1' &= C_1, \quad\ \ C_2' = C_2. \end{aligned} \tag{2.5}$$

Using these modified parameters we have that $\Gamma_\infty\backslash\Gamma_1(4)$ can be identified with

$$\left\{ \begin{array}{c} (4A_1, 4B_1, C_1, 4A_2, 4B_2, C_2) \in \mathbb{Z}^6 | A_1C_2 + 4B_1B_2 + C_1A_2 = 0, \\ (A_i, B_i, C_i) = 1,\ C_j \equiv -1 \pmod 4 \end{array} \right\}. \tag{2.6}$$

2.1.3 Principal Series

The following discussion establishes the preliminaries needed for the definition of the Eisenstein distribution. A complete treatment of automorphic distributions can be found in [13] and [12]. Bate [2] provides an explicit exposition of some of these ideas in the context of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ principal series representations. This dissertation will primarily be concerned with $\mathrm{SL}(3,\mathbb{R})$ and $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ principal series representations and the following discussion introduces notation and addresses some technical points.

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)\,,\ \rho = (1, 0, -1) \in \mathfrak{a}'_{\mathbb{C}}$, and define

$$\begin{aligned} V_\lambda^\infty &= \{f \in C^\infty(\mathrm{SL}(3,\mathbb{R})) | f(gb_-) = \exp((\lambda-\rho)(H(b_-^{-1})))f(g), \text{ for all } b_- \in B_-\}, \\ V_\lambda &= \{f \in L^2_{\mathrm{loc}}(\mathrm{SL}(3,\mathbb{R})) | f(gb_-) = \exp((\lambda-\rho)(H(b_-^{-1})))f(g), \text{ for all } b_- \in B_-\}, \\ V_\lambda^{-\infty} &= \{f \in C^{-\infty}(\mathrm{SL}(3,\mathbb{R})) | f(gb_-) = \exp((\lambda-\rho)(H(b_-^{-1})))f(g), \text{ for all } b_- \in B_-\}. \end{aligned}$$

The group G acts by left translation on each of the three preceding spaces, $\pi(h)(f)(g) = f(h^{-1}g)$. These spaces are called the smooth, locally L^2, and distributional principal series representation spaces, respectively. In the definition of $V_\lambda^{-\infty}$ the equality is understood in the sense of distributions. Note that

$$V_\lambda^\infty \subset V_\lambda \subset V_\lambda^{-\infty}.$$

Let $f_1 \in V_\lambda$ and $f_2 \in V_{-\lambda}$, define the pairing $(\cdot,\cdot)_\lambda : V_{-\lambda} \times V_\lambda \to \mathbb{C}$, by

$$(f_1, f_2)_\lambda = \int_{K/M} f_1(k) f_2(k) dk, \tag{2.7}$$

where $K = \mathrm{SO}(3)$, the integration is taken over a fundamental domain of K/M, and dk is the Haar measure of K. By [10, Theorem 3], if $h \in G$, $f_1 \in V_{-\lambda}$, and $f_2 \in V_\lambda$, then $(\pi(h)f_1, f_2)_\lambda = (f_1, \pi(h^{-1})f)_\lambda$.

For us, distributions will be dual to smooth measures, and thus can be thought of as generalized functions in which the action of the distribution on the measure is given by integration of their product over the full space. Thus, the pairing can be extended to $V_\lambda^{-\infty}$ on the right. Restriction from $V_{-\lambda}$ to its smooth vectors results in a pairing $V_{-\lambda}^\infty \times V_\lambda^{-\infty} \to \mathbb{C}$. Under this pairing $V_\lambda^{-\infty}$ may be identified with the dual of $V_{-\lambda}^\infty$;

this duality is to be understood in the context of topological vector spaces, thus some comments about topology are in order.

The map induced by restriction to K defines a vector space isomorphism between $V_{-\lambda}^{\infty}$ and $C^{\infty}(K)$. The family of norms $||\partial^{\alpha}\phi||_u = \sup_{k\in K}\{|\partial^{\alpha}\phi(k)|\}$ define a topology on $C^{\infty}(K)$ which can be transferred to $V_{-\lambda}^{\infty}$ using the previous isomorphism. The dual $V_{\lambda}^{-\infty}$ can be given the strong topology (sometimes called the polar topology) [14, §19]. With respect to these topologies $V_{\lambda}^{-\infty}$ can be identified with the continuous dual of $V_{-\lambda}^{\infty}$. Additionally, V_{λ}^{∞} is dense in $V_{\lambda}^{-\infty}$, and sequential convergence in $V_{\lambda}^{-\infty}$ with respect to the strong topology is equivalent to sequential convergence with respect to the weak topology [14, §34.4].

The pairing just described focuses on the compact model of the principal series representations. The Eisenstein distribution considered in this dissertation will be more amenable to study using the non-compact model of the principal series representation which we describe presently.

Let $w \in W$. As wNB_- is open and dense in $\mathrm{SL}(3,\mathbb{R})$, restriction from $\mathrm{SL}(3,\mathbb{R})$ to wN defines an injection $V_{\lambda}^{\infty} \hookrightarrow C^{\infty}(wN)$ and the pairing is compatible with this injection in the following sense. Let $F : \mathrm{SL}(3,\mathbb{R}) \to \mathbb{C}$, such that $F(gb_-) = e^{2\rho H(b_-)}F(g)$. Then, by a slight modification of Consequence 7 in [9],

$$\int_K F(k)dk = \int_N F(wn)dn. \tag{2.8}$$

Since wNB_-/B_- is a dense open subset of $G/B_- \cong K/M$, the identity captures the idea that removing sets of measure zero will not affect the value of the integral. As $f_1 \in V_\lambda$ and $f_2 \in V_{-\lambda}$ implies that $f_1(gb_-)f_2(gb_-) = \exp(2\rho H(b_-))f_1(g)f_2(g)$, we apply this identity to Equation 2.7 to establish a bridge between the pairings of principal series in the compact and noncompact pictures. Specifically,

$$(f_1, f_2)_\lambda = \frac{1}{8}\int_N f_1(wn)f_2(wn)dn,$$

and so the pairing can be realized as an integration over the non-compact space N.

When dealing with distributions some care must be taken for two reasons. First, restricting distributions to closed submanifolds need not be meaningful. For $f \in V_{\lambda}^{-\infty}$

restriction to wN can be made rigorous. Informally, this is due to f being smooth in the B_- variables and wNB_- being open in $\mathrm{SL}(3,\mathbb{R})$. Second, a distribution cannot necessarily be recovered from its restriction to a dense open subset. This trouble can be remedied with a partition of unity in general, but for the distributions E_w the support will be contained inside $w^{-1}w_\ell NB_-$ for some element of the Weyl group. These two points together justify restriction of $f \in V_\lambda^{-\infty}$ to wN. Again, these points are addressed completely in [13], [12].

Let $\delta_{(0,0,0)}$ be the Dirac delta function at $(0,0,0) \in \mathbb{R}^3$. The vector $\tau \in V_\lambda^{-\infty}$, characterized by

$$\tau\left(w_\ell \left(\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix}\right) b_-\right) = \exp((\lambda-\rho)(H(b_-^{-1})))\delta_{(0,0,0)}(x,y,z),$$

is used to construct an Eisenstein distribution on $\mathrm{SL}(3,\mathbb{R})$. The next proposition collects some of τ's basic properties.

Proposition 3. *Let $\tau \in V_\lambda^{-\infty}$ be as above. Then:*

1. *τ is right N_--invariant.*
2. *$supp(\tau) = w_\ell B_- = Bw_\ell$.*
3. *τ is left N-invariant.*

Proof: The first two properties follow immediately from the definition of τ. For the final claim let $f \in V_{-\lambda}^{\infty}$. By the definition of τ,

$$(f,\tau)_\lambda = \int_N \tau(w_\ell n(x,y,z)) f(w_\ell n(x,y,z))dxdydz = f(w_\ell).$$

On the other hand,

$$(f,\pi(n)\tau)_\lambda = (\pi(n^{-1})f,\tau)_\lambda = f(nw_\ell) = f(w_\ell),$$

where the last equality follows as $w_\ell n w_\ell \in N_-$. Thus, $\pi(n)\tau = \tau$. $\square$

To construct the metaplectic principal series one additional input is needed. Let

$\phi : \widetilde{M} \to \mathrm{SL}(2,\mathbb{C})$ be the representation defined by

$$
\begin{aligned}
\left(\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right),-1\right)^{-1} &\mapsto \left(\begin{smallmatrix}-1&\\&-1\end{smallmatrix}\right)\\
\left(\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right),1\right)^{-1} &\mapsto \left(\begin{smallmatrix}1&\\&1\end{smallmatrix}\right)\\
\left(\left(\begin{smallmatrix}-1&0&0\\0&-1&0\\0&0&1\end{smallmatrix}\right),-1\right)^{-1} &\mapsto \left(\begin{smallmatrix}&-i\\-i&\end{smallmatrix}\right)\\
\left(\left(\begin{smallmatrix}-1&0&0\\0&-1&0\\0&0&1\end{smallmatrix}\right),1\right)^{-1} &\mapsto \left(\begin{smallmatrix}&i\\i&\end{smallmatrix}\right)\\
\left(\left(\begin{smallmatrix}1&0&0\\0&-1&0\\0&0&-1\end{smallmatrix}\right),-1\right)^{-1} &\mapsto \left(\begin{smallmatrix}-i&\\&i\end{smallmatrix}\right)\\
\left(\left(\begin{smallmatrix}1&0&0\\0&-1&0\\0&0&-1\end{smallmatrix}\right),1\right)^{-1} &\mapsto \left(\begin{smallmatrix}i&\\&-i\end{smallmatrix}\right)\\
\left(\left(\begin{smallmatrix}-1&0&0\\0&1&0\\0&0&-1\end{smallmatrix}\right),-1\right)^{-1} &\mapsto \left(\begin{smallmatrix}&-1\\1&\end{smallmatrix}\right)\\
\left(\left(\begin{smallmatrix}-1&0&0\\0&1&0\\0&0&-1\end{smallmatrix}\right),1\right)^{-1} &\mapsto \left(\begin{smallmatrix}&1\\-1&\end{smallmatrix}\right).
\end{aligned}
\tag{2.9}
$$

The kernel of the previous map is trivial and so it follows that $\widetilde{M}$ is isomorphic to

$$\left\{\left(\begin{smallmatrix}\pm1&\\&\pm1\end{smallmatrix}\right),\left(\begin{smallmatrix}\pm i&\\&\mp i\end{smallmatrix}\right),\left(\begin{smallmatrix}&\pm1\\\mp1&\end{smallmatrix}\right),\left(\begin{smallmatrix}&\pm i\\\pm i&\end{smallmatrix}\right)\right\};$$

this group is isomorphic to the quaternion group $Q_8 = \{\pm1, \pm i, \pm j, \pm k\}$ and the representation is the unique two dimensional representation of Q_8. Additionally, $\phi((m,-\epsilon)) = -\phi((m,\epsilon))$, as can be seen from the definition of ϕ.

The ideas considered in the case of $\mathrm{SL}(3,\mathbb{R})$ can be adapted to $\widetilde{\mathrm{SL}}(3,\mathbb{R})$. Let $\lambda, \rho \in \mathfrak{a}'_{\mathbb{C}}$, where $\rho = (1,0,-1)$. Let

$$\widetilde{V}^{\infty}_{\lambda,\phi} = \{f \in C^{\infty}(\widetilde{\mathrm{SL}(3,\mathbb{R})})^2 | f(\tilde{g}\tilde{m}an_-) = \exp((\lambda-\rho)(H(a^{-1})))\phi(\tilde{m}^{-1})f(\tilde{g}),$$
$$\text{for all } \tilde{m}an_- \in \widetilde{M}AN_-\},$$

$$\widetilde{V}_{\lambda,\phi} = \{f \in L^2_{\mathrm{loc}}(\mathrm{SL}(3,\mathbb{R}))^2 | f(\tilde{g}\tilde{m}an_-) = \exp((\lambda-\rho)(H(a^{-1})))\phi(\tilde{m}^{-1})f(g),$$
$$\text{for all } \tilde{m}an_- \in \widetilde{M}AN_-\},$$

$$\widetilde{V}^{-\infty}_{\lambda,\phi} = \{f \in C^{-\infty}(\mathrm{SL}(3,\mathbb{R}))^2 | f(\tilde{g}\tilde{m}an_-) = \exp((\lambda-\rho)(H(a^{-1})))\phi(\tilde{m}^{-1})f(g),$$
$$\text{for all } \tilde{m}an_- \in \widetilde{M}AN_-\}.$$

Note that in this case the vectors in the principal series representations are vector valued.

Define the pairing, $(\cdot,\cdot)_{\lambda,\phi} : \widetilde{V}_{\lambda,\phi} \times \widetilde{V}_{-\lambda,\phi^{-1}} \to \mathbb{C}$, by

$$(f_1, f_2)_{\lambda,\phi} = \int_{\widetilde{K}/\widetilde{M}} f_1(k) f_2(k) d\tilde{k}, \tag{2.10}$$

where the integration is taken over a fundamental domain of $\widetilde{K}/\widetilde{M}$ and $d\tilde{k}$ is the Haar measure of $\widetilde{K} \cong \mathrm{SU}(2) \cong \mathrm{Spin}(3)$, a maximal compact subgroup of $\widetilde{\mathrm{SL}}(3,\mathbb{R})$. Note that $\widetilde{K}$ is the double cover of SO(3). As in the case of $\mathrm{SL}(3,\mathbb{R})$ this pairing can be written as integration in the non-compact model (recall line (2.8)) and (2.10) induces a pairing

$$(\cdot,\cdot)_{\lambda,\phi} : \widetilde{V}^{\infty}_{-\lambda,\phi^{-1}} \times \widetilde{V}^{-\infty}_{\lambda,\phi} \to \mathbb{C}.$$

The element $\tilde{\tau} \in \widetilde{V}^{-\infty}_{\lambda,\phi}$, characterized by

$$\tilde{\tau}\left((w_\ell, 1)\left(\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix}\right)\tilde{m}an_-\right) = \exp((\lambda-\rho)(H(a^{-1})))\phi(\tilde{m}^{-1})\begin{bmatrix} \delta_{(0,0,0)}(x,y,z) \\ 0 \end{bmatrix},$$

will be used to construct a metaplectic Eisenstein distribution on $\widetilde{\mathrm{SL}}(3,\mathbb{R})$.

Proposition 4. *Let $\tilde{\tau} \in \widetilde{V}^{-\infty}_{\lambda,\phi}$ be as above. Then:*

1. *$\tilde{\tau}$ is right $\widetilde{N}_-$-invariant.*

2. *$supp(\tilde{\tau}) = (w_\ell, 1)\widetilde{B}_- = \widetilde{B}(w_\ell, 1)$.*

3. *$\tilde{\tau}$ is left $\widetilde{N}$-invariant.*

Proof: The proof is identical to that of Proposition 3. □

Now we can define the metaplectic Eisenstein distribution as

$$\tilde{E}((g,\epsilon),\lambda) = \sum_{\tilde{\gamma} \in \tilde{\Gamma} \cap \widetilde{N} \backslash \tilde{\Gamma}} \pi(\tilde{\gamma}^{-1})\tilde{\tau}((g,\epsilon)) \tag{2.11}$$

where

$$\pi\left(\left(\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right), 1\right)\right)^{-1}\tilde{\tau}\left(\left(\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix}\right)\tilde{m}an_-\right) = e^{(\lambda-\rho)(a^{-1})}\phi(\tilde{m}^{-1})\left[\begin{smallmatrix} \delta_0(x)\delta_0(y)\delta_0(z) \\ 0 \end{smallmatrix}\right]$$

and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. Standard arguments show that the sum is convergent when the real part of $\lambda_1 - \lambda_2$ and $\lambda_2 - \lambda_3$ is sufficiently large.

To provide some context for the study of the splitting that will follow, we will state a result that identifies the point at which the splitting obstructs further computation. Let

$$\mathbb{S}(A_1, A_2) = \{\gamma \in \Gamma_1(4) | \gamma \text{ has Plücker coordinates of the form } (4A_1, *, *, 4A_2, *, *)\}, \tag{2.12}$$

$$\Sigma(A_1, A_2; m_1, m_2) \stackrel{\text{def}}{=} \sum_{\gamma \in \Gamma_\infty \backslash \mathbb{S}(A_1,A_2)/\Gamma_\infty} s(\gamma) e^{2\pi i (m1 \frac{B_1}{A_1} + m2 \frac{B_2}{A_2})}, \tag{2.13}$$

ane let

$$\psi(\left(\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix}\right)) = \psi_{m_1,m_2}(\left(\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix}\right)) = e^{2\pi i (m_1 x + m_2 y)}. \tag{2.14}$$

Proposition 5. *Let* $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \widetilde{V}^\infty_{-\lambda, \phi^{-1}}$. *Then*

$$\begin{aligned}
(f(\tilde{g}), \int_{N_{\mathbb{Z}} \backslash N_{\mathbb{R}}} & \tilde{E}_{w_\ell}(n\tilde{g}) \psi^{-1}(n) dn)_{\lambda,\phi} \\
&= \Big(\sum_{\substack{A_1>0 \\ A_2>0}} |4A_1|^{-1-\lambda_1+\lambda_2} |4A_2|^{-1-\lambda_2+\lambda_3} \Sigma(A_1, A_2; m_1, -m_2) \\
&\quad - i \sum_{\substack{A_1>0 \\ A_2<0}} |4A_1|^{-1-\lambda_1+\lambda_2} |4A_2|^{-1-\lambda_2+\lambda_3} \Sigma(A_1, A_2; m_1, -m_2) \Big) \\
&\quad \times \left[\int_{\mathbb{R}^3} f_1 \Big(n(x,y,z) \Big(\Big(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix} \Big)_{0}, 1 \Big) \Big) \psi(n(x,y,z)) dx dy dz \right] \\
&\quad + \Big(i \sum_{\substack{A_1<0 \\ A_2>0}} |4A_1|^{-1-\lambda_1+\lambda_2} |4A_2|^{-1-\lambda_2+\lambda_3} \Sigma(A_1, A_2; m_1, -m_2) \\
&\quad + \sum_{\substack{A_1<0 \\ A_2<0}} |4A_1|^{-1-\lambda_1+\lambda_2} |4A_2|^{-1-\lambda_2+\lambda_3} \Sigma(A_1, A_2; m_1, -m_2) \Big) \\
&\quad \times \left[\int_{\mathbb{R}^3} f_1 \Big(n(x,y,z) \Big(\Big(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix} \Big)^{0}, 1 \Big) \Big) \psi(n(x,y,z)) dx dy dz \right]
\end{aligned}$$

This proposition will be proved in a future work.

The computation of the exponential sums $\Sigma(A_1, A_2; m_1, m_2)$ requires an understanding of the splitting map s. The main objective of this dissertation is to provide a method for computing s that will facilitate the computation of the exponential sums.

2.2 Basic Computations

2.2.1 Kronecker Symbol

If $a, b \in \mathbb{R}^\times$, then the **Hilbert Symbol** is defined by

$$(a,b)_{\mathbb{R}} = \begin{cases} 1, & \text{if } a \text{ or } b > 0 \\ -1, & \text{if } a \text{ and } b < 0. \end{cases} \tag{2.15}$$

Let $n \in \mathbb{Z}_{\neq 0}$ with prime factorization $n = \epsilon p_1^{e_1} \dots p_\ell^{e_\ell}$, where $\epsilon = \pm 1$. If $k \in \mathbb{Z}$ the **Kronecker Symbol** is defined by

$$\left(\frac{k}{n}\right) = \left(\frac{k}{\epsilon}\right)\left(\frac{k}{p_1}\right)^{e_1} \cdots \left(\frac{k}{p_\ell}\right)^{e_\ell},$$

where $\left(\frac{k}{p_i}\right)$ is the Legendre symbol when p_i is an odd prime, $\left(\frac{k}{\epsilon}\right) = (k, \epsilon)_{\mathbb{R}}$, and

$$\left(\frac{k}{2}\right) = \begin{cases} 0, & \text{if } k \text{ is even} \\ 1, & \text{if } k \equiv \pm 1 (\text{mod } 8) \\ -1, & \text{if } k \equiv \pm 3 (\text{mod } 8). \end{cases}$$

The formula can be extended to $n = 0$ by

$$\left(\frac{k}{0}\right) = \begin{cases} 1, & \text{if } k = \pm 1 \\ 0, & \text{otherwise.} \end{cases}$$

The following proposition collects some facts about the Kronecker Symbol. These results can be found in [8].

Proposition 6. *(Properties of the Kronecker Symbol) Let $a, b, m, n \in \mathbb{Z}$, $\epsilon = \pm 1$, and let n' and m' denote the odd part of n and m, respectively.*

1. *If $ab \neq 0$, then $\left(\frac{a}{n}\right)\left(\frac{b}{n}\right) = \left(\frac{ab}{n}\right)$.*
2. *If $mn \neq 0$, then $\left(\frac{a}{m}\right)\left(\frac{a}{n}\right) = \left(\frac{a}{mn}\right)$.*
3. *If $n > 0$ and $a \equiv b \, (\text{mod } m)$, where $m = \begin{cases} 4n, & \text{if } n \equiv 2 \, (\text{mod } 4) \\ n, & \text{otherwise} \end{cases}$, then*

$$\left(\frac{a}{n}\right) = \left(\frac{b}{n}\right).$$

4. If $a \not\equiv 3\,(mod\ 4)$ and $m \equiv n\,(mod\ b)$, where $b = \begin{cases} 4|a|, & \text{if } a \equiv 2\,(mod\ 4) \\ |a|, & \text{otherwise} \end{cases}$, *then*

$$(\frac{a}{m}) = (\frac{a}{n}).$$

5.

$$\left(\frac{-1}{n}\right) = (-1)^{\frac{n'-1}{2}} \text{ and } \left(\frac{2}{n}\right) = (-1)^{\frac{(n')^2-1}{8}}.$$

6. (Quadratic Reciprocity) If $gcd(m,n) = 1$, then

$$\left(\frac{m}{n}\right)\left(\frac{n}{m}\right) = (n,m)_{\mathbb{R}}(-1)^{\frac{(m'-1)(n'-1)}{4}}$$

7. $(\frac{(\frac{-1}{m})}{n}) = (-1)^{(\frac{m'-1}{2})(\frac{n'-1}{2})} = ((\frac{-1}{m}),(\frac{-1}{n}))_{\mathbb{R}}$.

2.2.2 2-cocycle

This section begins with a formula for the Banks-Levy-Sepanski 2-cocycle [1] as presented in Miller [11] and then goes on to collect some facts about the 2-cocycle that will be used in subsequent computations.

Recall the Plücker coordinates introduced in (2.1.2) and let $g \in \mathrm{SL}(3,\mathbb{R})$ with Plücker coordinates $(A_1', B_1', C_1', A_2', B_2', C_2')$.

Let

$$X_1(g) = \det(g), \tag{2.16}$$

$$X_2(g) = \begin{cases} -A_2', & \text{if } A_2' \neq 0 \\ B_2', & \text{if } B_2' \neq 0 \text{ and } A_2' = 0 \\ -C_2', & \text{otherwise} \end{cases} \tag{2.17}$$

$$X_3(g) = \begin{cases} -A_1', & \text{if } A_1' \neq 0 \\ -B_1', & \text{if } B_1' \neq 0 \text{ and } A_1' = 0 \\ -C_1', & \text{otherwise} \end{cases} \tag{2.18}$$

$$\Delta(g) = \begin{pmatrix} X_1(g)/X_2(g) & 0 & 0 \\ 0 & X_2(g)/X_3(g) & 0 \\ 0 & 0 & X_3(g) \end{pmatrix} \tag{2.19}$$

It is worth noting that the signs of the Weyl group elements w from Section 2.1.1 are chosen so that $\Delta(w) = e$.

If $g_1, g_2 \in G$ such that $g_1 = naw_1 \dots w_k n'$ is the Bruhat decomposition of g_1, then in Section 4 of [1], Banks-Levy-Sepanski show that the 2-cocycle σ satisfies the formula

$$\sigma(g_1, g_2) = \sigma(a, w_1 \dots w_k n' g_2)\sigma(w_1, w_2 \dots w_k n' g_2) \dots \sigma(w_{k-1}, w_k n' g_2)\sigma(w_k, n' g_2), \tag{2.20}$$

where each factor can be computed using the following rules

$$\begin{aligned} \sigma(a, h) =& \sigma(a, \Delta(h)), \text{ for } a \text{ diagonal}, \\ \sigma(w_\alpha, h) =& \sigma(\Delta(w_\alpha h)\Delta(h), -\Delta(h)), \\ \sigma(t(a_1, a_2, a_3), t(b_1, b_2, b_3)) =& (a_1, b_2)(a_1, b_3)(a_2, b_3). \end{aligned}$$

Some simple identities involving σ are collected in the next lemma.

Lemma 7. *(Banks-Levy-Sepanski [1])*

$$\sigma(n_1 g_1, g_2 n_2) = \sigma(g_1, g_2).$$

$$\sigma(g_1 n, g_2) = \sigma(g_1, n g_2).$$

$$\sigma(n, g) = \sigma(g, n) = 1$$

$$\sigma(g, a) = 1, \text{ when } a \in A.$$

Proof: The first three identities follow from Lemma 4 in Section 3 of [1]. The fourth identity follows from the fact that since we are working over $\mathbb{R}$ the Steinberg symbol may be taken to be the Hilbert symbol defined on line (2.15), and the combination of Theorem 7 a) in [1] and the second to last equation in Section 4 of [1]. $\square$

Chapter 3

The Splitting

This chapter contains a study of the splitting $\Gamma_1(4) \to \widetilde{\mathrm{SL}}(3,\mathbb{R})$, where $\gamma \mapsto (\gamma, s(\gamma))$ and $s : \Gamma_1(4) \to \{\pm 1\}$ is the map defined on line (3.37) in Section 3.4. An understanding of this map is necessary to complete the computation of the Fourier coefficients of the metaplectic Eisenstein distribution. Miller [11] provided a formula for s that involves a non-unique matrix factorization; however, it appears difficult to complete the Fourier coefficient computation using this formula. Thus, this chapter sets out to rewrite Miller's formula in terms of Plücker coordinates which are better suited to this Fourier coefficient computation; this decision is motivated by the success of Brubaker-Bump-Friedberg-Hoffstein's use of these coordinates in [5].

3.1 Block Parameters

Any $\gamma \in \Gamma$ can be written as

$$\gamma = n \begin{pmatrix} a_1 & b_1 & 0 \\ c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 & b_2 \\ 0 & 1 & 0 \\ c_2 & 0 & d_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & b_3 \\ 0 & c_3 & d_3 \end{pmatrix}. \tag{3.1}$$

where $n \in N$ and all of the other factors are in Γ. Note that this factorization is non-unique. The parameters a_i, b_i, c_i, d_i will be called **block parameters**. In [11], Miller proves that $\Gamma_1(4)$ is maximal in $\mathrm{SL}(3,\mathbb{Z})$ such that there exists a splitting homomorphism $\Gamma_1(4) \to \widetilde{\mathrm{SL}}(3,\mathbb{R})$ and provides a formula for this splitting homomorphism using equation (3.1) and the 2-cocycle of Banks-Levy-Sepanski [1].

Theorem 2 (above) shows how Plücker coordinates can be used to provide a description of $\Gamma_\infty \backslash \Gamma$. By relating the block parameters and the Plücker coordinates, this description can be used to identify a unique matrix representative of $\Gamma_\infty \backslash \Gamma$ in terms of the block parameters. Proposition 8 below makes the relationship between block parameters and Plücker coordinates explicit, and Proposition 9 shows how this relationship can be used to pick a unique representative of $\Gamma_\infty\backslash\Gamma$ using the block parameters.

Proposition 8. *If $\gamma \in \Gamma_1(4)$ has block parameters given by*

$$\gamma = n\begin{pmatrix} a_1 & b_1 & 0 \\ c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_2 & 0 & b_2 \\ 0 & 1 & 0 \\ c_2 & 0 & d_2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & b_3 \\ 0 & c_3 & d_3 \end{pmatrix},$$

then γ has Plücker coordinates given by

$$\begin{aligned} A_1' &= -c_2, & B_1' &= -d_2c_3, & C_1' &= -d_2d_3 \\ A_2' &= -(c_1c_3 - d_1c_2a_3), & B_2' &= c_1d_3 - d_1c_2b_3, & C_2' &= -d_1d_2. \end{aligned} \tag{3.2}$$

Proof: Multiplying on the left by n will not change the Plücker coordinates of γ, so suppose that

$$\gamma = \begin{pmatrix} a_1 & b_1 & 0 \\ c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_2 & 0 & b_2 \\ 0 & 1 & 0 \\ c_2 & 0 & d_2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & b_3 \\ 0 & c_3 & d_3 \end{pmatrix}.$$

Multiplying the matrices on the right hand side of this equation leads to the equation

$$\gamma = \begin{pmatrix} a_1a_2 & b_1a_3+a_1b_2c_3 & b_1b_3+a_1b_2d_3 \\ c_1a_2 & d_1a_3+c_1b_2c_3 & d_1b_3+c_1b_2d_3 \\ c_2 & d_2c_3 & d_2d_3 \end{pmatrix}.$$

Now we compute the Plücker coordinates and find that

$$\begin{aligned} A_1' &= -c_2, & B_1' &= -d_2c_3, & C_1' &= -d_2d_3 \\ A_2' &= -(c_1c_3 - d_1c_2a_3), & B_2' &= c_1d_3 - d_1c_2b_3, & C_2' &= -d_1d_2. \end{aligned}$$

□

Note that a_1, b_1, a_2, b_2 do not appear in any of the expressions.

Proposition 9. *The coset of $\Gamma_\infty\backslash\Gamma$ with Plücker coordinates $(A_1', B_1', C_1', A_2', B_2', C_2')$ is represented by the matrix*

$$\begin{pmatrix} a_1 & b_1 & 0 \\ c_1 & \frac{-C_2'}{(B_1',C_1')\epsilon} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_2 & 0 & b_2 \\ 0 & 1 & 0 \\ -A_1' & 0 & (B_1',C_1')\epsilon \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & b_3 \\ 0 & \frac{-B_1'}{(B_1',C_1')\epsilon} & \frac{-C_1'}{(B_1',C_1')\epsilon} \end{pmatrix},$$

where

1. $\epsilon = \left(\frac{-1}{(B_1',C_1')}\right)$.

2. *a_3 is the smallest positive integer satisfying $a_3 \equiv d_3^{-1} \pmod{c_3}$, such that $\frac{d_1c_2a_3 - A_2'}{c_3} \in 4\mathbb{Z}$.*

3. a_i $(i \neq 3)$ *is the smallest positive integer satisfying* $a_i \equiv d_i^{-1}$ *(mod* c_i*).*

4. $c_1 = \frac{d_1 c_2 a_3 - A_2'}{c_3}$.

5. $b_j = \frac{a_j d_j - 1}{c_j}$ *for* $j = 1, 2, 3$.

The appearance of 'smallest' in items two and three is a choice made to ensure uniqueness.

Proof: Let $\gamma \in \Gamma$ with Plücker coordinates $(A_1', B_1', C_1', A_2', B_2', C_2') \in \mathbb{Z}^6$ such that $A_1'C_2' + B_1'B_2' + C_1'A_2' = 0$ and $(A_i', B_i', C_i') = 1$ for $i = 1, 2$. Write

$$\gamma = n \begin{pmatrix} a_1 & b_1 & 0 \\ c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 & b_2 \\ 0 & 1 & 0 \\ c_2 & 0 & d_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & b_3 \\ 0 & c_3 & d_3 \end{pmatrix}.$$

Using line (3.2), the block parameters may be expressed as follows:

$$c_2 = -A_1', \quad d_2 = \left(\frac{-1}{(B_1', C_1')}\right)(B_1', C_1'), \quad c_3 = \frac{-B_1'}{d_2}, \quad d_3 = \frac{-C_1'}{d_2}$$

$$d_1 = \frac{-C_2'}{d_2}, \quad c_1 \frac{-B_1'}{d_2} \equiv -A_2' \,(\mathrm{mod} \frac{A_1'C_2'}{d_2} = d_1 c_2).$$

The equations above do not contain enough information to specify a unique representative in an orbit of Γ with respect to the left action of Γ_∞. As the Plücker coordinates already provide a bijection with $\Gamma_\infty \backslash \Gamma$, by observing how changing the block parameters affect the Plücker coordinates a unique representative can be identified.

Assume the Plücker coordinates $(A_1', B_1', C_1', A_2', B_2', C_2')$ are fixed. This determines $d_1, c_2, d_2, c_3, d_3 \in \mathbb{Z}$ such that $(c_2, d_2) = (c_3, d_3) = 1$, $c_2, c_3 \equiv 0 (\mathrm{mod}\ 4)$, and $d_1, d_2, d_3 \equiv 1 (\mathrm{mod}\ 4)$. Consider the expression for A_2' in line (3.2). As $(c_3, d_1 c_2)$ divides A_2', there are integers c_1 and a_3 that satisfy that equation. Several different integral choices of c_1 and a_3 will result in the same value for A_2'. Take a_3 to be the smallest positive integer satisfying $a_3 \equiv d_3^{-1} (\mathrm{mod}\ c_3)$ and $c_1 \in 4\mathbb{Z}$. Such a pair of integers exist as any γ with the given Plücker coordinates provides a solution to that equation with the desired congruence conditions. This solution can be shifted to ensure that a_3 is positive and minimal subject to the congruence conditions. The relation $(A_2', B_2', C_2') = 1$ implies that $(c_1, d_1) = 1$. Similarly, take a_i $(i \neq 3)$ to be the smallest positive integer satisfying $a_i \equiv d_i^{-1} (\mathrm{mod}\ c_i)$.

Define b_i to be $\frac{a_i d_i - 1}{c_i}$. Finally we must be check that this choice for b_3 is consistent with the expression for B'_2. The equation $A'_1C'_2 + B'_1B'_2 + C'_1A'_2 = 0$ becomes $d_1c_2d_2((a_3d_3 - b_3c_3) - 1) = 0$, when the block parameters are substituted in for the Plücker coordinates. When $c_2 \neq 0$ it follows that the two equations defining b_3 are consistent. When $c_2 = 0$ it follows that B'_2 does not depend on b_3 and so no conflict can arise. □

3.2 Symmetries of Plücker Coordinates

Understanding the effect of certain matrix operations on the space $\Gamma_\infty \backslash \Gamma_1(4)$ will provide a mechanism to reduce the general determination of s, in terms of Plücker coordinates, to that of a more specific case. The reductions are discussed in section 3.7. The current section collects the relevant symmetries.

Proposition 10. *Let $M \in SL(3, \mathbb{R})$ with Plücker coordinates $(4A_1, 4B_1, C_1, 4A_2, 4B_2, C_2)$, $n = n(x, y, z) \in N$, $S_2 = t(1, -1, 1)$, and $S_3 = t(1, 1, -1)$. Then:*

1. *nMn^{-1} has Plücker coordinates*

$$(4A_1, 4B_1 - 4A_1x, C_1 - 4B_1y + 4A_1(xy - z), 4A_2, 4B_2 + 4A_2y, C_2 + 4B_2x + 4A_2z).$$

2. *S_3MS_3 has Plücker coordinates*

$$(-4A_1, -4B_1, C_1, -4A_2, 4B_2, C_2).$$

3. *S_2MS_2 has Plücker coordinates*

$$(4A_1, -4B_1, C_1, 4A_2, -4B_2, C_2).$$

4. *$w_\ell M^{-t} w_\ell$ has Plücker coordinates*

$$(4A_2, -4B_2, C_2, 4A_1, -4B_1, C_1).$$

5. *Let $M \in \Gamma_1(4)$. If D divides (A_1, A_2), $D_1 = (D, B_1)$, $D = D_1D_2$, and $T = t(1, D_2^{-1}, D^{-1})$ then $TMT^{-1} \in SL(3, \mathbb{Z})$ has Plücker coordinates*

$$(4A_1/D, 4B_1/D_1, C_1, 4A_2/D, (4B_2)/D_2, C_2).$$

Furthermore, $TMT^{-1} \in \Gamma_1(4)$ *if and only if* D_2 *divides* B_2*.*

The proof is straightforward matrix algebra and will be omitted. The last symmetry is the most important in terms of the reduction step and we would like to make a few comments about D_2 and its relation to B_2. As D divides (A_1, A_2), $D_1 = (D, B_1)$, $D = D_1 D_2$, and $A_1 C_2 + 4B_1 B_2 + C_1 A_2 = 0$ we see that D_2 divides $4B_2$. Thus it could be the case that D_2 divides $4B_2$ and D_2 does not divide B_2. In this case $4B_2/D_2$ is no longer divisible by 4 and TMT^{-1} can only be shown to live in $SL(3, \mathbb{Z})$. On the other hand, if D_2 divides B_2, then TMT^{-1} will be an element of $\Gamma_1(4)$. Now if D is odd then D_2 divides B_2 and $TMT^{-1} \in \Gamma_1(4)$. We will apply this result when D is odd, but many of the technical aspects of the computation of the splitting in terms of Plücker Coordinates are directly related to this failure of D_2 to divide B_2 in general.

3.3 Some Results on Double Coset Spaces

This section describes some structural features of the set $\Gamma_\infty \backslash \Gamma_1(4) / \Gamma_\infty$. This double coset space is important for two reasons. First, the map $s : \Gamma \to \{\pm 1\}$ is left and right Γ_∞ invariant and so it naturally descends to a map with domain $\Gamma_\infty \backslash \Gamma_1(4) / \Gamma_\infty$. Second, certain subsets of $\Gamma_\infty \backslash \Gamma_1(4) / \Gamma_\infty$ index exponential sums appearing in the computation of the Fourier coefficients of the metaplectic Eisenstein distribution. This section begins with some notation and simple observations. Then an explicit description of the sets indexing the aforementioned exponential sums is provided. Finally, the main result in this section, Proposition 12, is proved. This proposition shows that the sets indexing the exponential sums exhibit a multiplicative structure.

Let

$$\mathbb{S}(A_1, A_2) = \{\gamma \in \Gamma_1(4) | \gamma \text{ has Plücker coordinates of the form } (4A_1, *, *, 4A_2, *, *)\}. \tag{3.3}$$

Note that these sets are left Γ_∞-invariant and that $\Gamma_1(4) = \coprod_{(A_1, A_2) \in \mathbb{Z}^2} \mathbb{S}(A_1, A_2)$. Recall the Plücker coordinates of the coset space $\Gamma_\infty \backslash \Gamma_1(4)$ as described in Theorem 2. By the left Γ_∞-invariance of $\mathbb{S}(A_1, A_2)$, the Plücker coordinates can be passed to the space $\Gamma_\infty \backslash \mathbb{S}(A_1, A_2)$, and $\Gamma_\infty \backslash \Gamma_1(4) = \coprod \Gamma_\infty \backslash \mathbb{S}(A_1, A_2)$. The notation and calculations from section 3.2 provide maps $\mathrm{Ad}(S) : \mathbb{S}(A_1, A_2) \to \mathbb{S}(-A_1, -A_2)$ given by $M \mapsto S_i M S_i^{-1}$, $\phi : \mathbb{S}(A_1, A_2) \to \mathbb{S}(A_2, A_1)$ given by $M \mapsto w_\ell M^{-t} w_\ell$, and $\mathrm{Ad}(n) : \mathbb{S}(A_1, A_2) \to \mathbb{S}(A_1, A_2)$ given by $M \mapsto nMn^{-1}$, where $n \in \Gamma_\infty$. Simple computations show that these maps respect the left and right action of Γ_∞, thus they descend to maps on the double coset spaces.

When $A_1, A_2 \neq 0$, the next proposition establishes that unique representatives of the double coset space $\Gamma_\infty \backslash \mathbb{S}(A_1, A_2)/\Gamma_\infty$ are given by elements of

$$S(A_1, A_2) \stackrel{\text{def}}{=} \{(4A_1, 4B_1, C_1, 4A_2, 4B_2, C_2) \in \mathbb{Z}^6 | A_1C_2 + 4B_1B_2 + C_1A_2 = 0, \\ (A_i, B_i, C_i) = 1,\ C_j \equiv -1 (\text{mod } 4),\ \frac{B_1}{A_1}, \frac{B_2}{A_2}, \frac{C_2}{4A_2} \in [0, 1)\}. \quad (3.4)$$

Proposition 11. *Let $A_1, A_2 \neq 0$ and let $\gamma \in \mathbb{S}(A_1, A_2)$ with Plücker coordinates $(4A_1, 4B_1, C_1, 4A_2, 4B_2, C_2)$. Then there is a unique $n \in \Gamma_\infty$ such that the Plücker coordinates of γn live in $S(A_1, A_2)$. This induces a bijective map*

$$\Gamma_\infty \backslash \mathbb{S}(A_1, A_2)/\Gamma_\infty \to S(A_1, A_2).$$

Proof: The result follows from Theorem 2 describing the Plücker coordinates, the description of the action of $\mathrm{Ad}(n)$ on the coordinates in section 3.2, and the identity

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Specifically, given $M \in \mathbb{S}(A_1, A_2)$ the above factorization of the unipotent matrix shows that there is a unique element $n \in \Gamma_\infty$ such that the Plücker coordinates of Mn are in the set $S(A_1, A_2)$. □

Now we turn to the main result of this section; the sets $\Gamma_\infty \backslash \mathbb{S}(A_1, A_2)/\Gamma_\infty$ exhibit a multiplicative structure. This result is essentially an application of the Chinese Remainder Theorem.

Proposition 12. *Let $A_1, \alpha_1 > 0$, $A_2, \alpha_2 \neq 0$, suppose that $(A_1A_2, \alpha_1\alpha_2) = 1$, A_1, A_2 are odd, and suppose that $A_1\alpha_1 + A_2\alpha_2 \equiv 0$ (mod 4). Let $\mu = (\frac{-1}{-A_1A_2})$. Then*

$$\Gamma_\infty \backslash \mathbb{S}(A_1\alpha_1, A_2\alpha_2)/\Gamma_\infty \cong \Gamma_\infty \backslash \mathbb{S}(A_1, \mu A_2)/\Gamma_\infty \times \Gamma_\infty \backslash \mathbb{S}(\alpha_1, -\mu\alpha_2)/\Gamma_\infty.$$

The bijection is induced by the map

$$(4A_1\alpha_1, 4B_1, C_1, 4A_2\alpha_2, 4B_2, C_2) \mapsto$$

$$((4A_1, 4B_1, \mathcal{C}_1, \mu 4A_2, 4B_2, \gamma C_2),$$
$$(4\alpha_1, 4B_1, \left(\frac{-1}{A_2}\right) A_2 C_1, -\mu 4\alpha_2, -\left(\frac{-1}{A_2}\right)\mu 4B_2, -\mu\left(\frac{-1}{A_2}\right) A_1 C_2)), \quad (3.5)$$

where:

1. $\mathcal{C}_1 = \frac{-A_1\gamma C_2 - 4B_1B_2}{\mu A_2}$.

2. *γ is the smallest positive integer such that $\gamma \equiv 1$ (mod 4) and $\gamma \equiv \alpha_1$ (mod A_2).*

Proof: To establish this bijection, we will construct a map

$$\phi : \Gamma_\infty\backslash\mathbb{S}(A_1\alpha_1, A_2\alpha_2)/\Gamma_\infty \to \Gamma_\infty\backslash\mathbb{S}(A_1, \mu A_2)/\Gamma_\infty \times \Gamma_\infty\backslash\mathbb{S}(\alpha_1, -\mu\alpha_2)/\Gamma_\infty$$

and a map

$$\psi : \Gamma_\infty\backslash\mathbb{S}(A_1, \mu A_2)/\Gamma_\infty \times \Gamma_\infty\backslash\mathbb{S}(\alpha_1, -\mu\alpha_2)/\Gamma_\infty \to \Gamma_\infty\backslash\mathbb{S}(A_1\alpha_1, A_2\alpha_2)/\Gamma_\infty$$

such that ϕ and ψ are inverses. To describe this map and its inverse, we will need an integer $\gamma \in \mathbb{Z}$ such that $\gamma \equiv 1 \pmod 4$ and $\gamma \equiv \alpha_1 \pmod{A_2}$ (A_2 must be odd for this step). To avoid ambiguities we will stipulate that γ is the least positive integer satisfying these conditions. It will be useful to write $\gamma = \alpha_1 + \ell A_2$ for some integer ℓ.

We begin with the construction of ϕ. Let $(4A_1\alpha_1, 4B_1, C_1, 4A_2\alpha_2, 4B_2, C_2) \in S(A_1\alpha_1, A_2\alpha_2)$. Consider

$$((4A_1, 4B_1, \mathcal{C}_1, \mu 4A_2, 4B_2, \gamma C_2),$$
$$(4\alpha_1, 4B_1, \left(\frac{-1}{A_2}\right) A_2 C_1, -\mu 4\alpha_2, -\left(\frac{-1}{A_2}\right)\mu 4B_2, -\mu\left(\frac{-1}{A_2}\right) A_1 C_2)), \quad (3.6)$$

where

$$\mathcal{C}_1 = \frac{-A_1\gamma C_2 - 4B_1B_2}{\mu A_2}.$$

This will be the value of ϕ once we adjust for cosets. For this to be a sensible definition we must show that this pair of 6-tuples resides in $\Gamma_\infty\backslash\mathbb{S}(A_1, \mu A_2)\times\Gamma_\infty\backslash\mathbb{S}(\alpha_1, -\mu\alpha_2)$. At this point is will be a good idea to recall that

$$S(A_1,A_2)=\{(4A_1,4B_1,C_1,4A_2,4B_2,C_2)\in\mathbb{Z}^6|A_1C_2+4B_1B_2+C_1A_2=0,\\ (A_i,B_i,C_i)=1,\ C_j\equiv -1(\text{mod } 4),\ \frac{B_1}{A_1},\frac{B_2}{A_2},\frac{C_2}{4A_2}\in[0,1)\},\quad (3.7)$$

and the bijection of Proposition 11. First we will show that

$(4A_1,4B_1,\mathcal{C}_1,\mu 4A_2,4B_2,\gamma C_2)\in\Gamma_\infty\backslash\mathbb{S}(A_1,\mu A_2)$. This involves showing the following six conditions are satisfied:

1. $A_1\gamma C_2+4B_1B_2+\mathcal{C}_1\mu A_2=0$
2. $\mathcal{C}_1\in\mathbb{Z}$
3. $\mathcal{C}_1\equiv -1 \pmod 4$
4. $\gamma C_2\equiv -1 \pmod 4$
5. $(A_1,B_1,\mathcal{C}_1)=1$
6. $(\mu A_2,B_2,\gamma C_2)=1$

First we show that $A_1\gamma C_2+4B_1B_2+\mathcal{C}_1\mu A_2=0$. The definition of $\mathcal{C}_1$ was made so that this equation would hold.

Second we show that $\mathcal{C}_1\in\mathbb{Z}$. Recall that $\gamma=\alpha_1+\ell A_2$. Additionally, we will use the relation $A_1\alpha_1C_2+4B_1B_2+A_2\alpha_2C_1=0$ which is a consequence of the assumption that $(4A_1\alpha_1,4B_1,C_1,4A_2\alpha_2,4B_2,C_2)\in S(A_1\alpha_1,A_2\alpha_2)$. Thus

$$\begin{aligned}\mathcal{C}_1 &= \tfrac{-A_1\gamma C_2-4B_1B_2}{\mu A_2} && (\text{def of } \mathcal{C}_1)\\ &= \mu\alpha_2\tfrac{-A_1\alpha_1C_2-4B_1B_2}{A_2\alpha_2}-\mu\ell A_1C_2 && (\gamma=\alpha_1+\ell A_2)\\ &= \mu\alpha_2C_1-\mu\ell A_1C_2. && (A_1\alpha_1C_2+4B_1B_2+A_2\alpha_2C_1=0)\end{aligned}\quad(3.8)$$

Therefore $\mathcal{C}_1$ is an integer. For later reference we record the identity

$$\mathcal{C}_1=\mu\alpha_2C_1-\mu\ell A_1C_2.\quad(3.9)$$

Third we show that $\mathcal{C}_1\equiv -1 \pmod 4$. Since $C_2\equiv -1 \pmod 4$,$\gamma\equiv 1 \pmod 4$, and $\mu=(\frac{-1}{-A_1A_2})$, it follows that

$$\mathcal{C}_1=\frac{-A_1\gamma C_2-4B_1B_2}{\mu A_2}$$

$$\equiv -A_1(1)(-1)\mu(A_2)^{-1} \equiv -1 \,(\mathrm{mod}\ 4),$$

as desired.

Fourth we show that $\gamma C_2 \equiv -1$ mod 4. This follows as $\gamma \equiv 1 \pmod 4$ and $C_2 \equiv -1 \pmod 4$.

Fifth we show that $(A_1, B_1, \mathcal{C}_1) = 1$. Recall that $\mathcal{C}_1 = \mu\alpha_2 C_1 - \mu\ell A_1 C_2$. This implies that $(A_1, B_1, \mathcal{C}_1) = (A_1, B_1, \mu\alpha_2 C_1)$. We can show that $(A_1, B_1, \mu\alpha_2 C_1) = 1$ as $(A_1, \alpha_2) = 1$ and $(A_1\alpha_1, B_1, C_1) = 1$.

Sixth we will show that $(\mu A_2, B_2, \gamma C_2) = 1$. This follows as $(A_2\alpha_2, B_2, C_2) = 1$, $\gamma \equiv \alpha_1 \pmod{A_2}$, and $(A_2, \alpha_1) = 1$.

Having verified the six conditions we can conclude that $(4A_1, 4B_1, \mathcal{C}_1, \mu 4A_2, 4B_2, \gamma C_2) \in \Gamma_\infty \backslash \mathbb{S}(A_1, \mu A_2)$.

Now we will show that $(4\alpha_1, 4B_1, \left(\frac{-1}{A_2}\right) A_2 C_1, -\mu 4\alpha_2, -\left(\frac{-1}{A_2}\right)\mu 4B_2, -\mu\left(\frac{-1}{A_2}\right) A_1 C_2)) \in \Gamma_\infty \backslash \mathbb{S}(\alpha_1, -\mu\alpha_2)$. This involves showing the following five conditions are satisfied:

1. $\alpha_1(-\mu\left(\frac{-1}{A_2}\right) A_1 C_2) + 4(B_1)(-\mu\left(\frac{-1}{A_2}\right) B_2) + (-\mu\alpha_2)(\left(\frac{-1}{A_2}\right) A_2 C_1) = 0$
2. $\left(\frac{-1}{A_2}\right) A_2 C_1 \equiv -1 \pmod 4$
3. $-\mu\left(\frac{-1}{A_2}\right) A_1 C_2 \equiv -1 \pmod 4$
4. $(\alpha_1, B_1, \left(\frac{-1}{A_2}\right) A_2 C_1) = 1$
5. $(-\mu\alpha_2, -\mu\left(\frac{-1}{A_2}\right) B_2, -\mu\left(\frac{-1}{A_2}\right) A_1 C_2) = 1$

The first condition follows as

$$\alpha_1(-\mu\left(\frac{-1}{A_2}\right) A_1 C_2) + 4(B_1)(-\mu\left(\frac{-1}{A_2}\right) B_2) + (-\mu\alpha_2)(\left(\frac{-1}{A_2}\right) A_2 C_1)$$
$$= -\mu\left(\frac{-1}{A_2}\right)(A_1\alpha_1 C_2 + 4B_1 B_2 + A_2\alpha_2 C_1) = 0.$$

The last equality follows as $(4A_1\alpha_1, 4B_1, C_1, 4A_2\alpha_2, 4B_2, C_2) \in S(A_1\alpha_1, A_2\alpha_2)$.

Second we show that $\left(\frac{-1}{A_2}\right) A_2 C_1 \equiv -1 \pmod 4$. This follows as $\left(\frac{-1}{A_2}\right) A_2 \equiv 1 \pmod 4$ and $C_1 \equiv -1 \pmod 4$.

Third we will show that $-\mu\left(\frac{-1}{A_2}\right)A_1C_2 \equiv -1 \pmod 4$. This follows as $-\mu\left(\frac{-1}{A_2}\right)A_1 = \left(\frac{-1}{A_1}\right)A_1 \equiv 1 \pmod 4$ and $C_2 \equiv -1 \pmod 4$.

Fourth we will show that $(\alpha_1, B_1, \left(\frac{-1}{A_2}\right)A_2C_1) = 1$. To see this note that $(A_1\alpha_1, B_1, C_1) = 1$ and $(A_2, \alpha_1) = 1$.

Fifth we will show that $(-\mu\alpha_2, -\mu\left(\frac{-1}{A_2}\right)B_2, -\mu\left(\frac{-1}{A_2}\right)A_1C_2) = 1$. To see this note that $(A_2\alpha_2, B_2, C_2) = 1$ and $(A_1, \alpha_2) = 1$.

Having verified the five conditions we can conclude that

$$(4\alpha_1, 4B_1, \left(\frac{-1}{A_2}\right)A_2C_1, -\mu 4\alpha_2, -\mu\left(\frac{-1}{A_2}\right)4B_2, -\mu\left(\frac{-1}{A_2}\right)A_1C_2) \in \Gamma_\infty\backslash\mathbb{S}(\alpha_1, -\mu\alpha_2).$$

Putting everything together we find that the pair of 6-tuples

$$\begin{aligned}&((4A_1, 4B_1, \mathcal{C}_1, \mu 4A_2, 4B_2, \gamma C_2),\\ &\qquad (4\alpha_1, 4B_1, \left(\frac{-1}{A_2}\right)A_2C_1, -\mu 4\alpha_2, -\left(\frac{-1}{A_2}\right)\mu 4B_2, -\mu\left(\frac{-1}{A_2}\right)A_1C_2)),\end{aligned} \tag{3.10}$$

is an element of $\Gamma_\infty\backslash\mathbb{S}(A_1, \mu A_2) \times \Gamma_\infty\backslash\mathbb{S}(\alpha_1, -\mu\alpha_2)$. Now we can apply the canonical map from $\Gamma_\infty\backslash\mathbb{S}(\cdot,\cdot)$ to $\Gamma_\infty\backslash\mathbb{S}(\cdot,\cdot)/\Gamma_\infty$. Let

$$\phi : \Gamma_\infty\backslash\mathbb{S}(A_1\alpha_1, A_2\alpha_2)/\Gamma_\infty \to \Gamma_\infty\backslash\mathbb{S}(A_1, \mu A_2)/\Gamma_\infty \times \Gamma_\infty\backslash\mathbb{S}(\alpha_1, -\mu\alpha_2)/\Gamma_\infty$$

be the map constructed above.

To establish the bijection we construct the inverse map ψ. Consider

$$((4A_1, 4b_1, c_1, \mu 4A_2, 4b_2, c_2), (4\alpha_1, 4\beta_1, \gamma_1, -\mu 4\alpha_2, 4\beta_2, \gamma_2)) \in S(A_1, \mu A_2) \times S(\alpha_1, -\mu\alpha_2).$$

Using the bijection between $S(\cdot,\cdot)$ and $\Gamma_\infty\backslash\mathbb{S}(\cdot,\cdot)/\Gamma_\infty$ described in Proposition 11, we will identify this element of $S(A_1, \mu A_2) \times S(\alpha_1, -\mu\alpha_2)$ with its corresponding element in $\Gamma_\infty\backslash\mathbb{S}(A_1, \mu A_2)/\Gamma_\infty \times \Gamma_\infty\backslash S(\alpha_1, -\mu\alpha_2)/\Gamma_\infty$. Let

$$\begin{aligned}&\psi(((4A_1, 4b_1, c_1, \mu 4A_2, 4b_2, c_2), (4\alpha_1, 4\beta_1, \gamma_1, -\mu 4\alpha_2, 4\beta_2, \gamma_2))\\ &\qquad = (4A_1\alpha_1, 4B_1, C_1, 4A_2\alpha_2, 4B_2, C_2).\end{aligned} \tag{3.11}$$

We define B_i, C_i presently. Define B_1 to be the integer such that $B_1 \equiv b_1 \pmod{A_1}$, $B_1 \equiv \beta_1 \pmod{\alpha_1}$, and such that $B_1/A_1\alpha_1 \in [0, 1)$. Define B_2 to be the integer such that $B_2 \equiv b_2 \pmod{A_2}$, $B_2 \equiv -\mu\left(\frac{-1}{A_2}\right)\beta_2 \pmod{\alpha_2}$, and such that $B_2/A_2\alpha_2 \in [0,1)$. The definition of C_2 will be a bit more involved.

For the definition of C_2 we begin by rewriting the congruence conditions of the previous paragraph as equalities; specifically, $b_1 = B_1 - xA_1$, $b_2 = B_2 + y\mu A_2$, $\beta_1 = B_1 - x'\alpha_1$, and $\beta_2 = -\mu\left(\frac{-1}{A_2}\right)B_2 + y'(-\mu)\alpha_2$, where x, y, x', and $y' \in \mathbb{Z}$. Let C_2 be defined to be the unique integer satisfying $C_2 \equiv \gamma^{-1}(c_2 - 4B_2x) \pmod{4A_2}$, $C_2 \equiv -\mu\left(\frac{-1}{A_2}\right)A_1^{-1}(\gamma_2 - 4(-\mu\left(\frac{-1}{A_2}\right))B_2x') \pmod{4\alpha_2}$, and $C_2/4A_2\alpha_2 \in [0,1)$. For later we record two more equalities; let $z, z' \in \mathbb{Z}$ such that $c_2 = \gamma C_2 + 4B_2x - 4\mu A_2z$ and $\gamma_2 = -\mu\left(\frac{-1}{A_2}\right)A_1C_2 - 4\mu\left(\frac{-1}{A_2}\right)B_2x' + 4\mu\alpha_2z'$. Finally define $C_1 = \frac{-A_1\alpha_1C_2 - 4B_1B_2}{A_2\alpha_2}$.

Let us take a moment to describe the heuristic that suggests the definition of C_2. If ψ is to be the inverse of ϕ, based on the right action of Γ_∞, we expect that $c_2 = \gamma C_2 + 4B_2x - 4\mu A_2z$, where $\gamma \in \mathbb{Z}$ is defined as in the statement of the proposition and z is some integer. Thus $C_2 \equiv \gamma^{-1}(c_2 - 4B_2x) \pmod{4A_2}$. Similarly, we expect that $\gamma_2 = -\mu\left(\frac{-1}{A_2}\right)A_1C_2 - 4\mu\left(\frac{-1}{A_2}\right)B_2x' + 4\mu\alpha_2z'$. Thus $C_2 \equiv -\mu\left(\frac{-1}{A_2}\right)A_1^{-1}(\gamma_2 - 4(-\mu\left(\frac{-1}{A_2}\right))B_2x') \pmod{4\alpha_2}$.

Now we return to the proof. We must show that $(4A_1\alpha_1, 4B_1, C_1, 4A_2\alpha_2, 4B_2, C_2) \in S(A_1\alpha_1, A_2\alpha_2)$. This involves showing that the following six conditions are satisfied:

1. $A_1\alpha_1C_2 + 4B_1B_2 + A_2\alpha_2C_1 = 0$
2. $C_2 \equiv -1 \pmod 4$
3. $(A_2\alpha_2, B_2, C_2) = 1$
4. $C_1 \in \mathbb{Z}$
5. $C_1 \equiv -1 \pmod 4$
6. $(A_1\alpha_1, B_1, C_1) = 1$

First we show that $A_1\alpha_1C_2 + 4B_1B_2 + A_2\alpha_2C_1 = 0$. The definition of C_1 was picked so that this equality would be satisfied.

Second we show that $C_2 \equiv -1 \pmod 4$. By the definition of C_2, we have $C_2 \equiv \gamma^{-1}(c_2 - 4B_2x) \equiv \gamma^{-1}c_2 \equiv -1 \pmod 4$, as desired.

Third we show that $(A_2\alpha_2, B_2, C_2) = 1$ as

$$(A_2\alpha_2, B_2, C_2) = (A_2, B_2, C_2)(\alpha_2, B_2, C_2) = (A_2, b_2, c_2)(\alpha_2, \beta_2, \gamma_2) = 1.$$

Fourth we will show that $C_1 = \frac{-A_1\alpha_1 C_2 - 4B_1B_2}{A_2\alpha_2}$ is an integer. Since $C_2 \equiv \gamma^{-1}(c_2 - 4B_2x) \pmod{4A_2}$, $B_1 = b_1 + xA_1$, and $B_2 \equiv b_2 \pmod{A_2}$ it follows that

$$A_1\alpha_1 C_2 + 4B_1B_2 \equiv A_1\alpha_1(\gamma^{-1}(c_2 - 4b_2x)) + 4(b_1 + xA_1)b_2)) \pmod{A_2}. \tag{3.12}$$

From the congruence $\gamma \equiv \alpha_1 (\text{mod } A_2)$ we see that

$$(3.12) \equiv (A_1c_2 + 4b_1b_2) \pmod{A_2}. \tag{3.13}$$

We can complete this calculation by observing that since $(4A_1, 4b_1, c_1, \mu 4A_2, 4b_2, c_2) \in S(A_1, \mu A_2)$ it follows that $A_1c_2 + 4b_1b_2 + \mu A_2 c_1 = 0$ so

$$(3.13) \equiv 0 \pmod{A_2}. \tag{3.14}$$

Now we will show that $-A_1\alpha_1C_2 - 4B_1B_2$ is divisible by α_2. We will use the following identities: $C_2 \equiv -\mu\left(\frac{-1}{A_2}\right)A_1^{-1}(\gamma_2 - 4(-\mu\left(\frac{-1}{A_2}\right))B_2x') \pmod{4\alpha_2}$, $\beta_1 = B_1 - x'\alpha_1$, and $\beta_2 = -\mu\left(\frac{-1}{A_2}\right)B_2 + y'(-\mu)\alpha_2$. The first and third identities can be combined to show that

$$\begin{aligned} C_2 \equiv& -\mu\left(\frac{-1}{A_2}\right)A_1^{-1}(\gamma_2 - 4(-\mu\left(\frac{-1}{A_2}\right))B_2x') \\ \equiv& -\mu\left(\frac{-1}{A_2}\right)A_1^{-1}(\gamma_2 - 4\beta_2x') \pmod{\alpha_2}. \end{aligned}$$

Now if we consider $A_1\alpha_1C_2 + 4B_1B_2$, we can rewrite this quantity using the equations for B_1 and B_2, and the new expression for C_2 to get

$$A_1\alpha_1C_2 + 4B_1B_2 \equiv -\mu\left(\frac{-1}{A_2}\right)(\alpha_1(\gamma_2 - 4\beta_2x') + 4(\beta_1 + x'\alpha_1)\beta_2) \pmod{\alpha_2}. \tag{3.15}$$

The terms involving the product $\alpha_1\beta_2$ cancel and we are left with

$$(3.15) \equiv -\mu\left(\frac{-1}{A_2}\right)(\alpha_1\gamma_2 + 4\beta_1\beta_2) \pmod{\alpha_2}. \tag{3.16}$$

Now $(4\alpha_1, 4\beta_1, \gamma_1, -\mu 4\alpha_2, 4\beta_2, \gamma_2) \in S(\alpha_1, -\mu\alpha_2)$ implies that $\alpha_1\gamma_2 + 4\beta_1\beta_2 - \mu\alpha_2\gamma_1 = 0$ so

$$(3.16) \equiv 0 \pmod{\alpha_2}. \tag{3.17}$$

As $(A_2, \alpha_2) = 1$, we can put the two divisibility results from lines (3.14) and (3.17) together to see that C_1 is an integer.

Unfortunately, to continue we will require two additional identities involving C_1. The first is

$$C_1 = \frac{\gamma_1 + 4B_1 y' + 4\alpha_1(-x'y' - z')}{\left(\frac{-1}{A_2}\right) A_2}. \tag{3.18}$$

The second is

$$C_1 = \frac{c_1 + B_1 y - A_1(xy + z) - \mu\ell A_1 C_2}{\mu\alpha_2}. \tag{3.19}$$

We postpone the proof of these identities for the moment.

Fifth we show that $C_1 \equiv -1 \pmod 4$. As $\gamma_1 \equiv -1 \pmod 4$ and A_2 is odd this result follows by considering the identity $C_1 = \frac{\gamma_1 + 4B_1 y' + 4\alpha_1(-x'y' - z')}{\left(\frac{-1}{A_2}\right) A_2}$.

Sixth we show that $(A_1\alpha_1, B_1, C_1) = 1$. We begin by observing that $(A_1\alpha_1, B_1, C_1) = (A_1, B_1, C_1)(\alpha_1, B_1, C_1)$. To see that $(\alpha_1, B_1, C_1) = 1$ recall that $B_1 = \beta_1 + x'\alpha_1$, $C_1 = \frac{\gamma_1 + 4B_1 y' + 4\alpha_1(-x'y' - z')}{\left(\frac{-1}{A_2}\right) A_2}$, and $(A_2, \alpha_1) = 1$. These results imply that

$$(\alpha_1, B_1, C_1) = (\alpha_1, \beta_1 + x'\alpha_1, \frac{\gamma_1 + 4B_1 y' + 4\alpha_1(-x'y' - z')}{\left(\frac{-1}{A_2}\right) A_2}) = (\alpha_1, \beta_1, \gamma_1) = 1.$$

It remains to show that $(A_1, B_1, C_1) = 1$. By the second identity for C_1 we have $C_1 = \frac{c_1 + B_1 y - A_1(xy - z) - \mu\ell A_1 C_2}{\mu\alpha_2}$. As $(A_1, \alpha_2) = 1$ we see that $(A_1, B_1, C_1) = (A_1, B_1, c_1 + B_1 y) = (A_1, B_1, c_1)$. As $b_1 = B_1 - xA_1$ we see that $(A_1, B_1, c_1) = (A_1, b_1, c_1)$. Since $(4A_1, 4b_1, c_1, \mu 4A_2, 4b_2, c_2) \in S(A_1, \mu A_2)$, and $(A_1, b_1, c_1) = 1$ and so $(A_1, B_1, C_1) = 1$.

Next we will prove the two equalities involving C_1. First we address equation (3.18),

$$\frac{-A_1\alpha_1 C_2 - 4B_1 B_2}{A_2\alpha_2} = \frac{\gamma_1 + 4B_1 y' + 4\alpha_1(-x'y' - z')}{\left(\frac{-1}{A_2}\right) A_2}. \tag{3.20}$$

Since $\beta_1 = B_1 - x'\alpha_1$,

$$A_2\alpha_2 \mathrm{RHS}((3.20)) = \left(\frac{-1}{A_2}\right) \alpha_2(\gamma_1 + 4B_1 y' + 4\alpha_1(-x'y' - z')) \tag{3.21}$$

$$= \left(\frac{-1}{A_2}\right) (\gamma_1\alpha_2 + 4\beta_1\alpha_2 y' - 4\alpha_1\alpha_2 z'). \tag{3.22}$$

Before addressing the left hand side of the equality, recall that

$$-\mu \left(\frac{-1}{A_2}\right) A_1 C_2 = \gamma_2 + 4\mu \left(\frac{-1}{A_2}\right) B_2 x' - 4\mu\alpha_2 z'.$$

Thus,

$$\begin{aligned} A_2\alpha_2\text{LHS}((3.20)) =& -A_1\alpha_1C_2 - 4B_1B_2 \\ =&\alpha_1(\mu\left(\frac{-1}{A_2}\right)\gamma_2 + 4B_2x' - \left(\frac{-1}{A_2}\right)4\alpha_2z') - 4B_1B_2 \\ =&\alpha_1(\mu\left(\frac{-1}{A_2}\right)\gamma_2 - \left(\frac{-1}{A_2}\right)4\alpha_2z') + 4B_2(\alpha_1x' - B_1). \end{aligned} \tag{3.23}$$

As $\beta_1 = B_1 - x'\alpha_1$ we have

$$\begin{aligned} (3.23) =&\alpha_1(\mu\left(\frac{-1}{A_2}\right)\gamma_2 - \left(\frac{-1}{A_2}\right)4\alpha_2z') - 4B_2\beta_1 \\ =&\alpha_1\mu\left(\frac{-1}{A_2}\right)\gamma_2 - \alpha_1\left(\frac{-1}{A_2}\right)4\alpha_2z' - 4B_2\beta_1. \end{aligned} \tag{3.24}$$

Next we can replace B_2 using the equality $\beta_2 = -\mu\left(\frac{-1}{A_2}\right)B_2 + y'(-\mu)\alpha_2$ to get

$$\begin{aligned} (3.24) =&\alpha_1\mu\left(\frac{-1}{A_2}\right)\gamma_2 - 4\left(\frac{-1}{A_2}\right)\alpha_1\alpha_2z' - 4(-\mu\left(\frac{-1}{A_2}\right)\beta_2 - \left(\frac{-1}{A_2}\right)y'\alpha_2)\beta_1 \\ =&\mu\left(\frac{-1}{A_2}\right)(\alpha_1\gamma_2 + 4\beta_1\beta_2) - 4\left(\frac{-1}{A_2}\right)\alpha_1\alpha_2z' + 4\left(\frac{-1}{A_2}\right)y'\alpha_2\beta_1. \end{aligned} \tag{3.25}$$

Next we apply the identity $\alpha_1\gamma_2 + 4\beta_1\beta_2 - \mu\alpha_2\gamma_1 = 0$ to see that (3.22) is equal to (3.25). Thus

$$\frac{-A_1\alpha_1C_2 - 4B_1B_2}{A_2\alpha_2} = \frac{\gamma_1 + 4B_1y' + 4\alpha_1(-x'y' - z')}{\left(\frac{-1}{A_2}\right)A_2},$$

as desired.

Now we address equation (3.19),

$$\frac{-A_1\alpha_1C_2 - 4B_1B_2}{A_2\alpha_2} = \frac{c_1 + B_1y - A_1(xy + z) - \mu\ell A_1C_2}{\mu\alpha_2}.$$

Recall that $\gamma = \alpha_1 + \ell A_2$. From this expression for γ it follows that

$$\mu\alpha_2C_1 = \frac{-A_1\alpha_1C_2 - 4B_1B_2}{\mu A_2} = \frac{-A_1\gamma C_2 - 4B_1B_2}{\mu A_2} - \mu\ell A_1C_2. \tag{3.26}$$

Now as $\gamma C_2 = c_2 - 4B_2x + 4\mu A_2z$ it follows that

$$A_1\gamma C_2 + 4B_1B_2 + \mu A_2c_1 = A_1(c_2 - 4B_2x + 4\mu A_2z) + 4B_1B_2 + \mu A_2c_1 \tag{3.27}$$

$$= A_1c_2 + 4B_1B_2 + \mu A_2c_1 - 4A_1B_2x + 4\mu A_1A_2z. \tag{3.28}$$

Now by using the equations $b_1 = B_1 - xA_1$, $b_2 = B_2 + y\mu A_2$, and $A_1c_2 + 4b_1b_2 + c_1\mu A_2 = 0$ it follows that

$$\begin{aligned}(3.28) =& A_1c_2 + 4(b_1 + xA_1)(b_2 - y\mu A_2) + c_1\mu A_2 - 4A_1(b_2 - y\mu A_2)x + 4\mu A_1A_2z \\ =& (A_1c_2 + 4b_1b_2 + c_1\mu A_2) - 4b_1y\mu A_2 + 4\mu A_1A_2z \\ =& -4b_1y\mu A_2 + 4\mu A_1A_2z \end{aligned} \tag{3.29}$$

After another application of $b_1 = B_1 - xA_1$ we get

$$(3.29) = -4\mu B_1A_2y + 4\mu A_1A_2(xy + z). \tag{3.30}$$

Thus we have shown that

$$A_1\gamma C_2 + 4B_1B_2 + \mu A_2c_1 = -4\mu B_1A_2y + 4\mu A_1A_2(xy + z). \tag{3.31}$$

By combining lines (3.26) and (3.31) it follows that

$$\mu\alpha_2(\frac{-A_1\alpha_1C_2 - 4B_1B_2}{A_2\alpha_2}) = c_1 + 4B_1y - 4A_1(xy + z) - \mu\ell A_1C_2. \tag{3.32}$$

Note that the lack of symmetry between the two expressions for C_1, which can be found on lines (3.18) and (3.19), is the consequence of the appearance of γ in the following congruence $C_2 \equiv \gamma^{-1}(c_2 - 4B_2x) \pmod{4A_2}$, and the absence of γ in the congruence $C_2 \equiv -\mu\left(\frac{-1}{A_2}\right)A_1^{-1}(\gamma_2 - 4(-\mu\left(\frac{-1}{A_2}\right))B_2x') \pmod{4\alpha_2}$

Finally we must show that the maps ϕ and ψ are inverses. We begin by computing $\psi \circ \phi$. Let $(4A_1\alpha_1, 4B_1, C_1, 4A_2\alpha_2, 4B_2, C_2) \in S(A_1\alpha_1, A_2\alpha_2)$. The image of this point under the map ϕ is

$$\begin{aligned}&((4A_1, 4B_1, \mathcal{C}_1, \mu 4A_2, 4B_2, \gamma C_2), \\ &\quad (4\alpha_1, 4B_1, \left(\frac{-1}{A_2}\right)A_2C_1, -\mu 4\alpha_2, -\left(\frac{-1}{A_2}\right)\mu 4B_2, -\mu\left(\frac{-1}{A_2}\right)A_1C_2)),\end{aligned} \tag{3.33}$$

where

$$\mathcal{C}_1 = \frac{-A_1\gamma C_2 - 4B_1B_2}{\mu A_2}.$$

Now let $x, x', y, y', z, z' \in \mathbb{Z}$ such that

$$((4A_1, 4B_1 - 4A_1x, \mathcal{C}_1 - 4B_1y + 4A_1(xy - z), \mu 4A_2, 4B_2 + \mu 4A_2y, \gamma C_2 + 4B_2x + \mu 4A_2z),$$

$$(4\alpha_1, 4B_1 - 4\alpha_1 x', \left(\frac{-1}{A_2}\right) A_2 C_1 - 4B_1 y' + 4\alpha_1(x'y' - z'),$$
$$-\mu 4\alpha_2, -\left(\frac{-1}{A_2}\right) \mu 4B_2 - \mu 4\alpha_2 y', -\mu \left(\frac{-1}{A_2}\right) A_1 C_2 - \left(\frac{-1}{A_2}\right) \mu 4B_2 x' - \mu 4\alpha_2 z'))$$

is an element of $S(A_1, \mu A_2) \times S(\alpha_1, -\mu\alpha_2)$. Now the image of this element under the map ψ is

$$(4A_1\alpha_1, 4B_1^*, C_1^*, 4A_2\alpha_2, 4B_2^*, C_2^*),$$

where the definitions of B_1^* ,C_1^*, B_2^*, and C_2^* can be found in the two paragraphs following line (3.11). We claim that there are integers x^*, y^*, z^* such that right multiplication of $(4A_1\alpha_1, 4B_1^*, C_1^*, 4A_2\alpha_2, 4B_2^*, C_2^*)$ by $n(x^*, y^*, z^*)$ will equal $(4A_1\alpha_1, 4B_1, C_1, 4A_2\alpha_2, 4B_2, C_2)$.

Let us begin with B_1^* and B_2^*. We know that

$$B_1 \equiv B_1 - A_1 x \equiv B_1^* \pmod{A_1}$$

and

$$-\left(\frac{-1}{A_2}\right) \mu B_2 \equiv -\left(\frac{-1}{A_2}\right) \mu B_2 - \mu A_2 y' \equiv -\left(\frac{-1}{A_2}\right) \mu B_2^* \pmod{A_2}.$$

Considering the analogous congruences modulo α_1 and α_2 leads to the relations

$$B_1 \equiv B_1^* \pmod{\alpha_1} \qquad \text{and} \qquad B_2 \equiv B_2^* \pmod{\alpha_2}.$$

Now $B_i/(A_i\alpha_i) \in [0, 1)$ and $B_i^*/(A_i\alpha_i) \in [0, 1)$. Therefore, by the Chinese Remainder Theorem $B_1 = B_1^*$ and $B_2 = B_2^*$.

Similarly,

$$\begin{aligned} C_2^* \equiv &\gamma^{-1}((\gamma C_2 + 4B_2 x + \mu 4A_2 z) - 4B_2 x) \\ \equiv &C_2 \pmod{4A_2} \end{aligned}$$

and

$$\begin{aligned} C_2^* \equiv &-\mu \left(\frac{-1}{A_2}\right) A_1^{-1}((-\mu \left(\frac{-1}{A_2}\right) A_1 C_2 - \mu \left(\frac{-1}{A_2}\right) 4B_2 x' - \mu 4\alpha_2 z') + \mu \left(\frac{-1}{A_2}\right) 4B_2 x') \\ \equiv &C_2 \pmod{4\alpha_2}. \end{aligned}$$

Again $C_2/(4A_2\alpha_2) \in [0,1)$ and $C_2^*/(4A_2\alpha_2) \in [0,1)$, so the Chinese Remainder Theorem implies that $C_2^* = C_2$. From this we can conclude that $C_1^* = C_1$ as well and thus $\psi \circ \phi = \text{id}$.

The last thing that we must check is that $\phi \circ \psi = \text{id}$. Consider

$$((4A_1, 4b_1, c_1, \mu 4A_2, 4b_2, c_2), (4\alpha_1, 4\beta_1, \gamma_1, -\mu 4\alpha_2, 4\beta_2, \gamma_2)) \in S(A_1, \mu A_2) \times S(\alpha_1, -\mu\alpha_2).$$

Let

$$\begin{aligned} &\psi(((4A_1, 4b_1, c_1, \mu 4A_2, 4b_2, c_2), (4\alpha_1, 4\beta_1, \gamma_1, -\mu 4\alpha_2, 4\beta_2, \gamma_2)) \\ &\qquad = (4A_1\alpha_1, 4B_1, C_1, 4A_2\alpha_2, 4B_2, C_2), \end{aligned} \tag{3.34}$$

where B_1, C_1, B_2, C_2 are defined as in the two paragraphs following line (3.11). Let the image of this element under the map ϕ be

$$\begin{gathered} ((4A_1, 4B_1 - 4A_1x, \mathcal{C}_1 - 4B_1y + 4A_1(xy - z), \mu 4A_2, 4B_2 + \mu 4A_2 y, \gamma C_2 + 4B_2 x + \mu 4A_2 z), \\ (4\alpha_1, 4B_1 - 4\alpha_1 x', \left(\frac{-1}{A_2}\right) A_2 C_1 - 4B_1 y' + 4\alpha_1(x'y' - z'), \\ -\mu 4\alpha_2, -\left(\frac{-1}{A_2}\right)\mu 4B_2 + \mu\alpha_2 y', -\mu\left(\frac{-1}{A_2}\right) A_1 C_2 - \left(\frac{-1}{A_2}\right)\mu 4B_2 x' - \mu 4\alpha_2 z')), \end{gathered}$$

where $x, x', y, y', z, z' \in \mathbb{Z}$ such that this pair of 6-tuples is an element of $S(A_1, \mu A_2) \times S(\alpha_1, -\mu\alpha_2)$.

Now

$$B_1 \equiv b_1 \pmod{A_1} \qquad \text{and} \qquad B_1 \equiv B_1 - A_1 x \pmod{A_1}.$$

Furthermore, $b_1/A_1 \in [0,1)$ and $(B_1 - A_1x)/A_1 \in [0,1)$, thus $b_1 = B_1 - A_1 x$.

Similarly,

$$B_1 \equiv \beta_1 \pmod{\alpha_1} \qquad \text{and} \qquad B_1 \equiv B_1 - \alpha_1 x' \pmod{\alpha_1}.$$

Furthermore, $\beta_1/\alpha_1 \in [0,1)$, and $(B_1 - \alpha_1 x')/\alpha_1 \in [0,1)$ thus $\beta_1 = B_1 - \alpha_1 x'$.

We may consider the analogous comparisons using B_2. In particular,

$$B_2 \equiv b_2 \pmod{A_2} \qquad \text{and} \qquad B_2 \equiv B_2 + \mu A_2 y \pmod{A_2}.$$

Since $b_2/(\mu A_2) \in [0,1)$, and $(B_2 + \mu A_2 y)/(\mu A_2) \in [0,1)$ it follows that $b_2 = B_2 + \mu A_2 y$.

Next we have

$$B_2 \equiv -\mu \left(\frac{-1}{A_2}\right) \beta_2 \,(\text{mod } \alpha_2) \quad \text{and} \quad -\mu \left(\frac{-1}{A_2}\right) B_2 \equiv -\mu \left(\frac{-1}{A_2}\right) B_2 + \mu\alpha_2 y' \,(\text{mod } \alpha_2).$$

Since $\beta_2/(-\mu\alpha_2) \in [0,1)$, and $(-\mu \left(\frac{-1}{A_2}\right) B_2 + \mu\alpha_2 y')/(-\mu\alpha_2) \in [0,1)$, it follows that $\beta_2 = -\mu \left(\frac{-1}{A_2}\right) B_2 + \alpha_2 y'$.

Now we will show that $\gamma C_2 + 4B_2 x + \mu 4A_2 z = c_2$ and $-\mu \left(\frac{-1}{A_2}\right) A_1 C_2 \mu - \left(\frac{-1}{A_2}\right) B_2 x' - \mu\alpha_2 z' = \gamma_2$. In the first case note that

$$C_2 \equiv \gamma^{-1}(c_2 - 4B_2 x)(\text{mod } 4A_2) \tag{3.35}$$

and

$$C_2 \equiv \gamma^{-1}((\gamma C_2 + 4B_2 + \mu 4A_2 z) - 4B_2 x)(\text{mod } 4A_2). \tag{3.36}$$

Since $c_2/(\mu 4A_2) \in [0,1)$ and $(\gamma C_2 + 4B_2 + \mu 4A_2 z)/(\mu 4A_2) \in [0,1)$ it follows that

$$\gamma C_2 + B_2 x + \mu A_2 z = c_2.$$

Similarly,

$$C_2 \equiv -\mu \left(\frac{-1}{A_2}\right) A_1^{-1}(\gamma_2 + 4\mu \left(\frac{-1}{A_2}\right) B_2 x')(\text{mod } 4\alpha_2)$$

and

$$C_2 \equiv -\mu \left(\frac{-1}{A_2}\right) A_1^{-1}((-\mu \left(\frac{-1}{A_2}\right) A_1 C_2 - \left(\frac{-1}{A_2}\right) \mu 4B_2 x' - \mu 4\alpha_2 z') + 4\mu \left(\frac{-1}{A_2}\right) B_2 x')(\text{mod } 4\alpha_2).$$

As $\gamma_2/(-\mu 4\alpha_2) \in [0,1)$ and $(-\mu \left(\frac{-1}{A_2}\right) A_1 C_2 - \left(\frac{-1}{A_2}\right) \mu 4B_2 x' - \mu 4\alpha_2 z')/(-\mu 4\alpha_2) \in [0,1)$ it follows that $\gamma_2 = -\mu \left(\frac{-1}{A_2}\right) A_1 C_2 - \left(\frac{-1}{A_2}\right) \mu 4B_2 x' - \mu 4\alpha_2 z'$.

From this we can conclude that $c_1 = \mathcal{C}_1 - 4B_1 y + 4A_1(xy - z)$ and $\gamma_1 = \left(\frac{-1}{A_2}\right) A_2 C_1 - 4B_1 y' + 4\alpha_1(x'y' - z')$. Finally we see that $\phi \circ \psi = \text{id}$. $\square$

3.4 The Splitting in Terms of Block Parameters

This brief section recalls the formula for the splitting in terms of block parameters, as shown in [11]. Suppose that $\gamma \in \Gamma$ and

$$\gamma = n \begin{pmatrix} a_1 & b_1 & 0 \\ c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 & b_2 \\ 0 & 1 & 0 \\ c_2 & 0 & d_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & b_3 \\ 0 & c_3 & d_3 \end{pmatrix},$$

where each $\left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right) \in \Gamma_1(4) \subseteq \mathrm{SL}(2,\mathbb{Z})$. Then

$$s(\gamma) = \left(\frac{c_1}{d_1}\right)\left(\frac{c_2}{d_2}\right)\left(\frac{c_3}{d_3}\right) s_{NA}(\gamma), \tag{3.37}$$

where

$$s_{NA}(\gamma) = \begin{cases} (c_1, d_1) & , \quad c_1, c_2 \neq 0,\ c_3 = 0 \\ (c_1c_2(c_1c_3 - d_1c_2a_3), c_1a_3)(a_3, c_2c_3) & , \quad c_1, c_2, c_3, c_1c_3 - d_1c_2a_3 \neq 0 \\ (c_2a_3, c_1a_3)(a_3, c_2c_3) & , \quad c_1, c_2, c_3 \neq 0,\ c_1c_3 - d_1c_2a_3 = 0 \\ (a_3, c_2c_3) & , \quad c_1 = 0,\ c_2, c_3 \neq 0 \\ 1 & , \quad \text{otherwise.} \end{cases} \tag{3.38}$$

A few remarks are in order. First, note that this formula shows that s is left Γ_∞-invariant. Second, note that s is not a group homomorphism, but rather that $s(\gamma_1\gamma_2) = s(\gamma_1)s(\gamma_2)\sigma(\gamma_1, \gamma_2)$. Finally recall that $4A_2 = -(c_1c_3 - d_1c_2a_3)$.

3.5 Symmetries of the Splitting

This section describes how the map s is affected by some of the symmetries described in section 3.2. These symmetries arise from automorphisms of $\Gamma_1(4)$ that preserve Γ_∞, and thus induce an action on the space of double cosets, $\Gamma_\infty\backslash\Gamma_1(4)/\Gamma_\infty$.

3.5.1 Conjugation

The simplest automorphisms that preserve Γ_∞ are given by conjugation by elements of Γ_∞. This action will leave the splitting unchanged.

Proposition 13. *Let $\gamma \in \Gamma_1(4)$ and $n \in \Gamma_\infty$. Then $s(n\gamma n^{-1}) = s(\gamma)$.*

Proof: Consider $s(n\gamma n^{-1})$. From 'The Guide for the Metaplexed' [11] it follows that s is left Γ_∞ invariant. Thus it is sufficient to consider $s(\gamma n)$ where $\gamma \in \Gamma_1(4)$ and $n \in \Gamma_\infty$. Now by the definition of the splitting $s(\gamma n) = s(\gamma)s(n)\sigma(\gamma, n)$. By results from 'The Guide for the Metaplexed' [11] and Banks-Levy-Sepanski [1], $s(n) = 1$ and $\sigma(\gamma, n) = 1$. Therefore $s(\gamma n) = s(\gamma)$ and it follows that $s(n\gamma n^{-1}) = s(\gamma)$. □

In addition to the left Γ_∞-invariance of s, the proof of this result shows that s is right Γ_∞-invariant as well; therefore, s is well defined on the double coset space $\Gamma_\infty\backslash\Gamma_1(4)/\Gamma_\infty$. Next consider conjugation by the elements $S_2 = \mathrm{t}(1,-1,1)$ and $S_3 = \mathrm{t}(1,1,-1)$ defined in Section 3.2. The results included in the next Proposition are not exhaustive, but rather only include symmetries that will be used in the sequel.

Proposition:

$$s(S_3\gamma S_3) = \begin{cases} (c_1a_3,-1)s(\gamma) & , c_1,c_2,c_3 \neq 0, A_2 = 0 \\ s(\gamma) & , \text{ otherwise} \end{cases}$$

If $A_1, A_2 \neq 0$. Then

$$s(S_2\gamma S_2) = -\mathrm{sign}(A_1A_2)s(\gamma).$$

Proof: Let $S = S_3$. If $\gamma = n\gamma_1\gamma_2\gamma_3$, then

$$\begin{aligned} S\gamma_1S &= \gamma_1, \\ S\gamma_2S &= \begin{pmatrix} a_2 & 0 & -b_2 \\ 0 & 1 & 0 \\ -c_2 & 0 & d_2 \end{pmatrix}, \\ S\gamma_3S &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & -b_3 \\ 0 & -c_3 & d_3 \end{pmatrix}. \end{aligned}$$

From the formula for s, contained in section 3.4, it follows that

$$s(S\gamma S) = \begin{cases} (c_1a_3,-1)s(\gamma) & , c_1,c_2,c_3 \neq 0, A_2 = 0 \\ s(\gamma) & , \text{ otherwise} \end{cases}.$$

Now let $S = S_2$. If $\gamma = n\gamma_1\gamma_2\gamma_3$, then

$$\begin{aligned} S\gamma_1S &= \begin{pmatrix} a_1 & -b_1 & 0 \\ -c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ S\gamma_2S &= \gamma_2, \\ S\gamma_3S &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & -b_3 \\ 0 & -c_3 & d_3 \end{pmatrix}. \end{aligned}$$

From the formula for s, contained in section 3.4, it follows that

$$s(S\gamma S) = \begin{cases} \mathrm{sign}(d_1)s(\gamma) & , c_1,c_2 \neq 0, c_3 = 0 \\ -\mathrm{sign}(a_3)\,\mathrm{sign}(c_1c_2(-A_2))\,\mathrm{sign}(c_1a_3)s(\gamma) & , c_1,c_2,c_3,A_2 \neq 0 \\ \mathrm{sign}(a_3)s(\gamma) & , c_1 = 0, c_2,c_3 \neq 0 \\ s(\gamma) & , \text{ otherwise} \end{cases}$$

$$= \begin{cases} \operatorname{sign}(d_1)s(\gamma) & , c_1, c_2 \neq 0, c_3 = 0 \\ -\operatorname{sign}(A_1A_2)s(\gamma) & , c_1, c_2, c_3, A_2 \neq 0 \\ \operatorname{sign}(a_3)s(\gamma) & , c_1 = 0, c_2, c_3 \neq 0 \\ s(\gamma) & , \text{ otherwise} \end{cases} .$$

To finish the proof we will show that $\operatorname{sign}(d_1) = -\operatorname{sign}(A_1A_2)$ when $c_1, c_2 \neq 0, c_3 = 0$, and $\operatorname{sign}(a_3) = -\operatorname{sign}(A_1A_2)$ when $c_1 = 0, c_2, c_3 \neq 0$. Let us begin with the case where $c_1, c_2 \neq 0, c_3 = 0$.

In this case, by (3.2) we know that $c_3 = 0$ implies that $4A_2 = d_1(-A_1)a_3$ and $a_3 = 1$. Thus we see that $\operatorname{sign}(d_1) = -\operatorname{sign}(A_1A_2)$.

In the second case, by 3.2 $c_1 = 0$ implies that $4A_1 = d_1c_2a_3$ and $d_1 = 1$. Thus we see that $\operatorname{sign}(a_3) = -\operatorname{sign}(A_1A_2)$.

□

3.5.2 Cartan Involution Composed with the Long Element

Proposition: Let $\gamma \in \Gamma_1(4)$ with Plücker coordinates $(4A_1, 4B_1, C_1, 4A_2, 4B_2, C_2)$. Consider the involution $\phi : \gamma \mapsto w_\ell \gamma^{-t} w_\ell^{-1}$. When A_1 and A_2 are not equal to 0

$$s(\phi(\gamma)) = (-A_1, -A_2)s(\gamma).$$

When $A_1, B_2 \neq 0$ and $A_2 = 0$,

$$s(\phi(\gamma)) = (-A_1, B_2)s(\gamma).$$

A similar result holds on the other cells, but these identities will not be needed.

Proof: By the formula in Section 3.4, if $\gamma = n\gamma_1\gamma_2\gamma_3$, then

$$s(\gamma) = s(\gamma_1)s(\gamma_2)s(\gamma_3)\sigma(\gamma_1, \gamma_2\gamma_3)\sigma(\gamma_2, \gamma_3).$$

Similarly,

$$s(\phi(\gamma)) = s(\phi(\gamma_1))s(\phi(\gamma_2))s(\phi(\gamma_3))\sigma(\phi(\gamma_1), \phi(\gamma_2)\phi(\gamma_3))\sigma(\phi(\gamma_2), \phi(\gamma_3))$$

(Note that $\phi(\Gamma_\infty) = \Gamma_\infty$ is needed in this computation). Direct calculation shows that

$$\begin{aligned}
\phi(\gamma_1) &= \phi\left(\begin{pmatrix} a_1 & b_1 & 0 \\ c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_1 & b_1 \\ 0 & c_1 & d_1 \end{pmatrix}, \\
\phi(\gamma_2) &= \phi\left(\begin{pmatrix} a_2 & 0 & b_2 \\ 0 & 1 & 0 \\ c_2 & 0 & d_2 \end{pmatrix}\right) &= \begin{pmatrix} a_2 & 0 & -b_2 \\ 0 & 1 & 0 \\ -c_2 & 0 & d_2 \end{pmatrix}, \\
\phi(\gamma_3) &= \phi\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & b_3 \\ 0 & c_3 & d_3 \end{pmatrix}\right) &= \begin{pmatrix} a_3 & b_3 & 0 \\ c_3 & d_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

These computations show that $s(\phi(\gamma_1)) = s(\gamma_1)$, $s(\phi(\gamma_3)) = s(\gamma_3)$, and $s(\phi(\gamma_2)) = \left(\frac{-c_2}{d_2}\right) = \left(\frac{c_2}{d_2}\right) = s(\gamma_2)$. The second to last equality follows as $d_2 \equiv 1$ (4).

It remains to compute $\sigma(\gamma_1, \gamma_2\gamma_3)\sigma(\gamma_2, \gamma_3)\sigma(\phi(\gamma_1), \phi(\gamma_2)\phi(\gamma_3))\sigma(\phi(\gamma_2), \phi(\gamma_3))$. First note that the computation of $\sigma(\gamma_2, \gamma_3)$ can be found in Proposition 4.2 in [11]. Specifically, Miller proves that

$$\sigma(\gamma_2, \gamma_3) = \begin{cases} (a_3, c_2c_3), & c_2, c_3 \neq 0 \\ 1, & \text{otherwise.} \end{cases} \tag{3.39}$$

Next we compute $\sigma(\phi(\gamma_2), \phi(\gamma_3))$. If $c_3 = 0$, then $\gamma_3 \in N$ and $\sigma(\phi(\gamma_2), \phi(\gamma_3)) = 1$. Assume that $c_2 \neq 0$ and $c_3 \neq 0$.

$$\begin{aligned}
\sigma(\phi(\gamma_2), \phi(\gamma_3)) =& \sigma\left(\begin{pmatrix} a_2 & 0 & -b_2 \\ 0 & 1 & 0 \\ -c_2 & 0 & d_2 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 & 0 \\ c_3 & d_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \\
=& \sigma\left(\begin{pmatrix} & & c_2^{-1} \\ & 1 & \\ -c_2 & & d_2 \end{pmatrix}, \begin{pmatrix} a_3 & & \\ c_3 & a_3^{-1} & \\ & & 1 \end{pmatrix}\right) \\
=& \sigma\left(\begin{pmatrix} c_2^{-1} & & \\ & -1 & \\ & & -c_2 \end{pmatrix} w_{\alpha_1}w_{\alpha_2}w_{\alpha_1}, \begin{pmatrix} 1 & & \frac{-d_2}{c_2} \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} a_3 & & \\ c_3 & a_3^{-1} & \\ & & 1 \end{pmatrix}\right) \\
=& \sigma\left(\begin{pmatrix} c_2^{-1} & & \\ & -1 & \\ & & -c_2 \end{pmatrix} w_{\alpha_1}w_{\alpha_2}w_{\alpha_1}, \begin{pmatrix} a_3 & & \\ c_3 & a_3^{-1} & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}\right) \\
=& \sigma\left(\begin{pmatrix} c_2^{-1} & & \\ & -1 & \\ & & -c_2 \end{pmatrix} w_{\alpha_1}w_{\alpha_2}w_{\alpha_1}, \begin{pmatrix} a_3 & & \\ c_3 & a_3^{-1} & \\ & & 1 \end{pmatrix}\right)
\end{aligned} \tag{3.40}$$

These five equalities follow from applications of the identity

$\sigma(n_1g_1n_2, g_2n_3) = \sigma(g_1, n_2g_2)$ and basic matrix algebra. The next three equalities follow from the definition of σ.

$$\begin{aligned}
(3.40) =& \sigma\left(\begin{pmatrix} c_2^{-1} & & \\ & -1 & \\ & & -c_2 \end{pmatrix}, w_{\alpha_1}w_{\alpha_2}w_{\alpha_1}\begin{pmatrix} a_3 & & \\ c_3 & a_3^{-1} & \\ & & 1 \end{pmatrix}\right) \sigma\left(w_{\alpha_1}, w_{\alpha_2}w_{\alpha_1}\begin{pmatrix} a_3 & & \\ c_3 & a_3^{-1} & \\ & & 1 \end{pmatrix}\right) \\
&\times \sigma\left(w_{\alpha_2}, w_{\alpha_1}\begin{pmatrix} a_3 & & \\ c_3 & a_3^{-1} & \\ & & 1 \end{pmatrix}\right) \sigma\left(w_{\alpha_1}, \begin{pmatrix} a_3 & & \\ c_3 & a_3^{-1} & \\ & & 1 \end{pmatrix}\right) \\
=& \sigma\left(\begin{pmatrix} c_2^{-1} & & \\ & -1 & \\ & & -c_2 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & a_3^{-1} & \\ & & a_3 \end{pmatrix}\right) \sigma\left(\begin{pmatrix} 1 & & \\ & a_3^{-1} & \\ & & a_3 \end{pmatrix}\begin{pmatrix} a_3^{-1} & & \\ & 1 & \\ & & a_3 \end{pmatrix}, \begin{pmatrix} -a_3^{-1} & & \\ & -1 & \\ & & -a_3 \end{pmatrix}\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \sigma\left(\begin{pmatrix} a_3^{-1} & & \\ & 1 & \\ & & a_3 \end{pmatrix}\begin{pmatrix} a_3^{-1} & & \\ & a_3 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} -a_3^{-1} & & \\ & -a_3 & \\ & & -1 \end{pmatrix}\right)\\
&\times \sigma\left(\begin{pmatrix} a_3^{-1} & & \\ & a_3 & \\ & & 1 \end{pmatrix}\begin{pmatrix} c_3^{-1} & & \\ & c_3 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} -c_3^{-1} & & \\ & -c_3 & \\ & & -1 \end{pmatrix}\right)\\
=&(c_2,a_3)^2(-1,a_3)^2(a_3,-a_3)^2(a_3,-1)(a_3c_3,-c_3)(a_3c_3,-1)^2 \qquad (3.41)
\end{aligned}
$$

The final steps follow from properties of the Hilbert Symbol.

$$(3.41) = (a_3,-1)(a_3c_3,-c_3) = (a_3,-1)(a_3,-c_3)(c_3,-c_3) = (a_3,c_3)$$

If either c_2 or c_3 is equal to 0, then $\sigma(\phi(\gamma_2),\phi(\gamma_3)) = 1$. Thus

$$\sigma(\phi(\gamma_2),\phi(\gamma_3))\sigma(\gamma_2,\gamma_3) = \begin{cases} (a_3,c_2), & c_2,c_3\neq 0 \\ 1, & \text{otherwise.} \end{cases}$$

However, if $c_3 = 0$ it follows that $a_3 = 1$ so if fact we have

$$\sigma(\phi(\gamma_2),\phi(\gamma_3))\sigma(\gamma_2,\gamma_3) = (a_3,c_2) \qquad (3.42)$$

It remains to consider $\sigma(\gamma_1,\gamma_2\gamma_3)$ and $\sigma(\phi(\gamma_1),\phi(\gamma_2)\phi(\gamma_3))$. In Proposition 4.2 in [11], Miller proves that

$$\sigma(\gamma_1,\gamma_2\gamma_3) = \begin{cases} (c_1c_2(-A_2),c_1a_3), & c_1,c_2,A_2\neq 0 \\ (c_2a_3,c_1a_3), & c_1,c_2\neq 0, A_2 = 0 \\ 1, & \text{otherwise.} \end{cases}$$

Now we will compute $\sigma(\phi(\gamma_1),\phi(\gamma_2)\phi(\gamma_3))$. Note that if $c_1 = 0$, then this 2-cocycle is equal to 1. Thus assume that $c_1 \neq 0$. As $a_j \equiv 1$ (4), thus $a_j \neq 0$. In the following computation let

$$
\begin{aligned}
\alpha &= \text{the first nonzero quantity among } c_2, \tfrac{-A_2}{c_1a_2}, \tfrac{1}{a_2a_3},\\
\beta &= \text{the first nonzero quantity among} \tfrac{-A_2}{c_1}, \tfrac{1}{a_3},\\
\delta &= \text{the first nonzero quantity among} - c_2a_3, \tfrac{1}{a_2},\\
h &= \begin{pmatrix} a_2a_3 & & \\ \frac{-A_2}{c_1} & a_3^{-1} & \\ -c_2a_3 & & a_2^{-1} \end{pmatrix}.
\end{aligned}
$$

Then using the identity $\sigma(n_1g_1n_2,g_2n_3) = \sigma(g_1,n_2g_2)$ and matrix algebra it follows that

$$\sigma(\phi(\gamma_1),\phi(\gamma_2)\phi(\gamma_3)) = \sigma\left(\begin{pmatrix} 1 & & \\ & a_1 & b_1 \\ & c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & & -b_2 \\ & 1 & \\ -c_2 & & d_2 \end{pmatrix}\begin{pmatrix} a_3 & b_3 & \\ c_3 & d_3 & \\ & & 1 \end{pmatrix}\right)$$

$$
\begin{aligned}
=&\sigma\left(\begin{pmatrix} 1 & & \\ & & -c_1^{-1} \\ & c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2a_3 & a_2b_3 & -b_2 \\ c_3 & d_3 & \\ -c_2a_3 & -c_2b_3 & d_2 \end{pmatrix}\right) \\
=&\sigma\left(\begin{pmatrix} 1 & & \\ & & -c_1^{-1} \\ & c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2a_3 & & \\ c_3 & a_3^{-1} & \\ -c_2a_3 & & a_2^{-1} \end{pmatrix}\begin{pmatrix} 1 & \frac{b_3}{a_3} & \frac{-b_2}{a_2a_3} \\ & 1 & \frac{b_2c_3}{a_2} \\ & & 1 \end{pmatrix}\right) \\
=&\sigma\left(\begin{pmatrix} 1 & & \\ & & -c_1^{-1} \\ & c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2a_3 & & \\ c_3 & a_3^{-1} & \\ -c_2a_3 & & a_2^{-1} \end{pmatrix}\right). \qquad (3.43)
\end{aligned}
$$

Now factor the matrix in the first entry using the $\mathrm{SL}(2,\mathbb{R})$ Bruhat decomposition on the lower right 2×2 block and apply the identity $\sigma(n_1g_1n_2, g_2n_3) = \sigma(g_1, n_2g_2)$ to see that

$$
\begin{aligned}
(3.43) =&\sigma\left(\begin{pmatrix} 1 & & \\ & c_1^{-1} & \\ & & c_1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \frac{d_1}{c_1} \\ & & 1 \end{pmatrix}, \begin{pmatrix} a_2a_3 & & \\ c_3 & a_3^{-1} & \\ -c_2a_3 & & a_2^{-1} \end{pmatrix}\right) \\
=&\sigma\left(\begin{pmatrix} 1 & & \\ & c_1^{-1} & \\ & & c_1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \frac{d_1}{c_1} \\ & & 1 \end{pmatrix}\begin{pmatrix} a_2a_3 & & \\ c_3 & a_3^{-1} & \\ -c_2a_3 & & a_2^{-1} \end{pmatrix}\right) \\
=&\sigma\left(\begin{pmatrix} 1 & & \\ & c_1^{-1} & \\ & & c_1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}, \begin{pmatrix} a_2a_3 & & \\ \frac{-A_2}{c_1} & a_3^{-1} & \frac{d_1}{c_1a_2} \\ -c_2a_3 & & a_2^{-1} \end{pmatrix}\right) \\
=&\sigma\left(\begin{pmatrix} 1 & & \\ & c_1^{-1} & \\ & & c_1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}, \begin{pmatrix} a_2a_3 & & \\ \frac{-A_2}{c_1} & a_3^{-1} & \\ -c_2a_3 & & a_2^{-1} \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & * \\ & & 1 \end{pmatrix}\right) \\
=&\sigma\left(\begin{pmatrix} 1 & & \\ & c_1^{-1} & \\ & & c_1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}, \begin{pmatrix} a_2a_3 & & \\ \frac{-A_2}{c_1} & a_3^{-1} & \\ -c_2a_3 & & a_2^{-1} \end{pmatrix}\right). \qquad (3.44)
\end{aligned}
$$

Recall the definition of Δ from subsection 2.2.2. Now the definition of σ shows that

$$
\begin{aligned}
(3.44) =&\sigma\left(\begin{pmatrix} 1 & & \\ & c_1^{-1} & \\ & & c_1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}\begin{pmatrix} a_2a_3 & & \\ \frac{-A_2}{c_1} & a_3^{-1} & \\ -c_2a_3 & & a_2^{-1} \end{pmatrix}\right) \\
&\times\sigma\left(\begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}, \begin{pmatrix} a_2a_3 & & \\ \frac{-A_2}{c_1} & a_3^{-1} & \\ -c_2a_3 & & a_2^{-1} \end{pmatrix}\right) \\
=&\sigma\left(\begin{pmatrix} 1 & & \\ & c_1^{-1} & \\ & & c_1 \end{pmatrix}, w_{\alpha_2}h\right)\sigma(w_{\alpha_2}, h) \\
=&\sigma\left(\begin{pmatrix} 1 & & \\ & c_1^{-1} & \\ & & c_1 \end{pmatrix}, \Delta(w_{\alpha_2}h)\right)\sigma(\Delta(w_{\alpha_2}h)\Delta(h), -\Delta(h)) \\
=&\sigma\left(\begin{pmatrix} 1 & & \\ & c_1^{-1} & \\ & & c_1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\alpha} & & \\ & \frac{\alpha}{\beta} & \\ & & \beta \end{pmatrix}\right) \\
&\times\sigma\left(\begin{pmatrix} \frac{1}{\alpha} & & \\ & \frac{\alpha}{\beta} & \\ & & \beta \end{pmatrix}\begin{pmatrix} \frac{1}{\alpha} & & \\ & \frac{\alpha}{\delta} & \\ & & \delta \end{pmatrix}, \begin{pmatrix} \frac{-1}{\alpha} & & \\ & \frac{-\alpha}{\delta} & \\ & & -\delta \end{pmatrix}\right) \\
=&(c_1,\beta)(\beta\delta,-\delta). \qquad (3.45)
\end{aligned}
$$

The last two equalities follow from properties of the Hilbert Symbol.

$$
(3.45) = (c_1,\beta)(\beta,-\delta)
$$

$$=(-c_1\delta, \beta)$$

Thus,

$$\sigma(\phi(\gamma_1), \phi(\gamma_2)\phi(\gamma_3))\sigma(\gamma_1, \gamma_2\gamma_3) = (\alpha, \beta c_1\delta). \tag{3.46}$$

If we combine lines (3.42) and (3.46) and use the fact that $A_1 \neq 0$ we get

$$\sigma(\phi(\gamma_2), \phi(\gamma_3))\sigma(\gamma_2, \gamma_3)\sigma(\phi(\gamma_1), \phi(\gamma_2)\phi(\gamma_3))\sigma(\gamma_1, \gamma_2\gamma_3)$$
$$= (c_2, a_3)(c_2, \beta c_1(-c_2a_3)) = (c_2, \beta c_1). \tag{3.47}$$

Remember that this equation is only valid if $c_1 \neq 0$. If $c_1 = 0$, then as mentioned above $\sigma(\phi(\gamma_1), \phi(\gamma_2)\phi(\gamma_3))\sigma(\gamma_1, \gamma_2\gamma_3) = 1$ and so we get

$$\sigma(\phi(\gamma_2), \phi(\gamma_3))\sigma(\gamma_2, \gamma_3)\sigma(\phi(\gamma_1), \phi(\gamma_2)\phi(\gamma_3))\sigma(\gamma_1, \gamma_2\gamma_3) = (c_2, a_3). \tag{3.48}$$

If $A_1, A_2, c_1 \neq 0$, then by line (3.47) we get

$$(a_3, -A_2)(-A_1, -A_1A_2a_3) = (-A_1, -A_2).$$

If $A_1, A_2 \neq 0, c_1 = 0$, then $d_1 = 1$ and $A_2 = -A_1a_3$. In this case line (3.48) gives

$$(c_2, a_3) = (-A_1, -A_1A_2) = (-A_1, -A_2).$$

If $A_2 = 0$, Then $c_1, c_3 \neq 0$ and $c_1c_3 = d_1c_2a_3$. If additionally, $A_1, B_2 \neq 0$, then by line (3.47) we have

$$\sigma(\phi(\gamma_1), \phi(\gamma_2)\phi(\gamma_3))\sigma(\gamma_1, \gamma_2\gamma_3) = (c_2, a_3c_1) = (c_2, c_2c_3d_1) = (c_2, -c_3d_1)$$

$$= (-A_1, -C_2B_1) = (-A_1, -C_2(4B_1B_2)B_2) = (-A_1, C_2^2A_1B_2) = (-A_1, B_2).$$

□

3.6 The Splitting in Terms of Plücker Coordinates

Theorem 14. *Let* $\gamma \in \Gamma_1(4)$ *with Plücker coordinates* $(4A_1, 4B_1, C_1, 4A_2, 4B_2, C_2)$ *such that* $A_1 > 0$*, and* $A_2/(A_1, A_2) \equiv 1\,(mod\ 2)$*. Let* $D = (A_1, A_2)$*,* $D_1 = (D, B_1)$*,* $D_2 = D/D_1$*, and let* $\epsilon = \left(\frac{-1}{-B_1/D_1}\right)$*. Then* $s(\gamma) =$

$$\left(\frac{-\epsilon}{-A_1A_2}\right)\left(\frac{A_1/D}{A_2/D}\right)\left(\frac{B_1/D_1}{A_1/D}\right)\left(\frac{4B_2/D_2}{sign(A_2)A_2/D}\right)\left(\frac{D_1}{C_1}\right)\left(\frac{D_2}{C_2}\right). \tag{3.49}$$

The proof of this theorem occupies the next two sections. Section 3.7 contains a reduction step that reduces the computation to the case where D is a power of 2. In Section 3.8 a nice representative of the double coset $\Gamma_\infty \gamma \Gamma_\infty$ is identified and the computation is executed using this representative.

A few remarks are in order regarding the formula for the splitting. First, this formula can be used to derive the value of the splitting on any input using the symmetries of Section 3.5. This point will be discussed more throughly in Section 3.8. Second, if -1 is assumed to be a square then this formula essentially reduces to the formula of Brubaker-Bump-Friedberg-Hoffstein [5]. One notable difference is that the roles of A_i and C_i are swapped. Third, $\left(\frac{-\epsilon}{-A_1A_2}\right) = 1$ when A_1 and A_2 are odd. This follows as $A_1 \equiv -A_2 \pmod 4$; this congruences is a consequence of the equation $A_1C_2 + 4B_1B_2 + C_1A_2 = 0$ and the congruences $C_i \equiv -1 \pmod 4$.

3.7 The Splitting: The Reduction

Proposition 15. *Let $\gamma \in \Gamma_1(4)$ with Plücker Coordinates $(4A_1, 4B_1, C_1, 4A_2, 4B_2, C_2)$, such that $A_1, A_2 \neq 0$. Suppose that D divides (A_1, A_2). Let $D_1 = (D, B_1)$ and let $D_2 = D/D_1$. Suppose that D_2 divides B_2. Let $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & D_2^{-1} & 0 \\ 0 & 0 & D^{-1} \end{pmatrix}$. Then $S\gamma S^{-1} \in \Gamma_1(4)$ with Plücker Coordinates $(4A_1/D, 4B_1/D_1, C_1, 4A_2/D, 4B_2/D_2, C_2)$ and*

$$s(\gamma) = s(S\gamma S^{-1}) \left(\frac{D_1}{C_1}\right)\left(\frac{D_2}{C_2}\right). \tag{3.50}$$

Proof: As the Plücker coordinates satisfy the previously mentioned divisibility conditions, γ is of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ D_2a_{21} & a_{22} & a_{23} \\ Da_{31} & D_1a_{32} & a_{33} \end{pmatrix}.$$

If γ is factored into blocks, then

$$\gamma = n \begin{pmatrix} a_1 & b_1 & 0 \\ D_2c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 & b_2 \\ 0 & 1 & 0 \\ Dc_2 & 0 & d_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & b_3 \\ 0 & D_1c_3 & d_3 \end{pmatrix}.$$

Recall the definition of $\mathbb{S}(A_1, A_2)$ from line (3.3). Using Proposition 10 we see that $\gamma \in \mathbb{S}(A_1D, A_2D)$ is mapped into $\mathbb{S}(A_1, A_2)$ via conjugation by the matrix

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & D_2^{-1} & 0 \\ 0 & 0 & D^{-1} \end{pmatrix}.$$

Explicitly,

$$S\gamma_1 S^{-1} = \begin{pmatrix} a_1 & D_2 b_1 & 0 \\ c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$S\gamma_2 S^{-1} = \begin{pmatrix} a_2 & 0 & D b_2 \\ 0 & 1 & 0 \\ c_2 & 0 & d_2 \end{pmatrix},$$
$$S\gamma_3 S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & D_1 b_3 \\ 0 & c_3 & d_3 \end{pmatrix}.$$

Using Miller's formula for the splitting as described in Section 3.4,

$$\begin{aligned} s(S\gamma S^{-1}) =& s(S\gamma_1 S^{-1}) s(S\gamma_2 S^{-1}) s(S\gamma_3 S^{-1}) \\ & \times \sigma(S\gamma_1 S^{-1}, S\gamma_2\gamma_3 S^{-1}) \sigma(S\gamma_2 S^{-1}, S\gamma_3 S^{-1}) \\ =& \left(\frac{c_1}{d_1}\right)\left(\frac{c_2}{d_2}\right)\left(\frac{c_3}{d_3}\right) \sigma(S\gamma_1 S^{-1}, S\gamma_2\gamma_3 S^{-1}) \sigma(S\gamma_2 S^{-1}, S\gamma_3 S^{-1}) \\ =& \left(\frac{c_1 D_2}{d_1}\right)\left(\frac{c_2 D}{d_2}\right)\left(\frac{c_3 D_1}{d_3}\right)\left(\frac{D_2}{d_1}\right)\left(\frac{D}{d_2}\right)\left(\frac{D_1}{d_3}\right) \\ & \times \sigma(S\gamma_1 S^{-1}, S\gamma_2\gamma_3 S^{-1}) \sigma(S\gamma_2 S^{-1}, S\gamma_3 S^{-1}) \\ =& s(\gamma)\left(\frac{D_2}{-C_2/d_2}\right)\left(\frac{D}{d_2}\right)\left(\frac{D_1}{-C_1/d_2}\right) \\ =& s(\gamma)\left(\frac{D_1}{-C_1}\right)\left(\frac{D_2}{-C_2}\right). \end{aligned}$$

The fourth equality follows from formula (3.38) paired with the fact that conjugation by S only scales the block parameters by positive numbers. Thus the nonarithmetic factor remains unchanged and it follows that

$$s(\gamma) = s(S\gamma S^{-1})\left(\frac{D_1}{C_1}\right)\left(\frac{D_2}{C_2}\right).$$

□

3.8 The Splitting: The Reduced Case

This section contains the computation of the splitting when $(A_1, A_2) = 2^{\ell}$. By combining this formula with the reduction step, the transformation of the formula for the splitting in terms of block parameters into a formula in terms of Plücker coordinates will be complete.

If the reduction step could be applied with $D = (A_1, A_2)$, the computation would be more straightforward. Unfortunately, (A_1, A_2) is not always a valid choice in the reduction step as conjugation by the prescribed matrix, S, may result in a matrix that does not reside in $\Gamma_1(4)$. Only the prime 2 can cause this trouble (This is proved in a remark after Proposition 10.), so the reduction step of Section 3.7 can be used to remove the odd part of the GCD of A_1 and A_2. So suppose $D = (A_1, A_2) = 2^\ell$. By applying the symmetries of Section 3.2 we can impose the additional constraints $A_1 > 0$, and $A_2/2^\ell \equiv 1 \pmod 2$; the calculations of Section 3.5 describe the effect of these symmetries on the value of the splitting. Specifically, the Cartan involution composed with conjugation by the long element can be used to swap A_1 and A_2; thus, if A_2 is more even than A_1 they can be swapped; if the new A_1 is negative, then conjugation by $t(1, 1, -1)$ will flip the sign of A_1. Thus the computation in the general case can be reduced to that of the special case just described. The evaluation of the splitting in this special case is the content of the following proposition.

Proposition 16. *Let* $\gamma \in \Gamma_1(4)$ *with Plücker coordinates* $(4A_1, 4B_1, C_1, 4A_2, 4B_2, C_2)$ *such that* $A_1 > 0$*, and* $A_2/(A_1, A_2) \equiv 1\,(mod\ 2)$*. Let* $D = (A_1, A_2) = 2^\ell$*,* $D_1 = (D, B_1)$*,* $D_2 = D/D_1$*, and let* $\epsilon = \left(\frac{-1}{-B_1/D_1}\right)$*. Then* $s(\gamma) =$

$$\left(\frac{-\epsilon}{-A_1A_2}\right)\left(\frac{A_1/D}{A_2/D}\right)\left(\frac{B_1/D_1}{A_1/D}\right)\left(\frac{4B_2/D_2}{sign(A_2)A_2/D}\right)\left(\frac{D_1}{C_1}\right)\left(\frac{D_2}{C_2}\right). \tag{3.51}$$

Proof: Let $\gamma \in \Gamma_1(4)$ have Plücker Coordinates $(4A_1, 4B_1, C_1, 4A_2, 4B_2, C_2)$. Let $B_j = 2^{\delta_j} b_j$, where $j = 1, 2$, and $(2, b_j) = 1$. Let $\epsilon = \left(\frac{-1}{-b_1}\right) = \left(\frac{-1}{-B_1}\right)$.

The splitting and the extra conditions on A_1 and A_2 are right Γ_∞-invariant. Thus multiplication on the right by an element of Γ_∞ can be used to reduce the computation to that of a representative with still more favorable properties. If $x, z \in \mathbb{Z}$, then $\gamma n(2^{\delta_1+2}x, 0, z)$ has Plucker Coordinates $(4A_1, 4\mathcal{B}_1, \mathcal{C}_1, 4A_2, 4\mathcal{B}_2, \mathcal{C}_2)$ where

$$\mathcal{B}_1 = B_1 - (2^{\delta_1+2}x)A_1, \tag{3.52}$$

$$\mathcal{B}_2 = B_2,, \tag{3.53}$$

and

$$\mathcal{C}_2 = C_2 - 4B_2(2^{\delta_1+2}x) - 4A_2z. \tag{3.54}$$

The factor of 2^{δ_1+2} in front of x ensures that $\mathrm{ord}_2(\mathcal{B}_1) = \mathrm{ord}_2(B_1)$ and $\mathcal{B}_1 \equiv B_1 (\mathrm{mod}\ 4)$. Now we will specify certain congruences conditions (limiting the possible choices for x and z) that will aid in our computation of the splitting. Using the Chinese Remainder Theorem, choose $x \in \mathbb{Z}$ subject to the conditions

$$\begin{aligned} \mathcal{B}_1 &\equiv B_1 \pmod{2^{\delta_1+2}A_1}, \\ \mathcal{B}_1 &\equiv 1 \pmod{A_2/2^{\ell}}, \\ \mathcal{B}_1 &< 0. \end{aligned}$$

Choose $z \in \mathbb{Z}$ subject to the conditions

$$\begin{aligned} \mathcal{C}_2 &\equiv 1 \pmod{\mathcal{B}_1/2^{\delta_1}}, \\ \mathcal{C}_2 &\equiv C_2 - 4B_2 2^{\delta_1+2}x \pmod{4A_2}, \\ \mathcal{C}_2 &< 0. \end{aligned}$$

The Chinese Remainder Theorem can be applied to the above system of congruences because $\mathcal{B}_1 \equiv 1 \pmod{A_2/2^{\ell}}$. This more suitable representative will be used to compute the splitting.

By Proposition 9, $\gamma n(2^{\delta_1+2}x, 0, z)$ can be factored so that it has block parameters

$$\begin{array}{lllllll} c_2 &=& -4A_1 &,\ d_2 &=& \pm(\mathcal{B}_1, \mathcal{C}_1) &= \pm(\mathcal{B}_1, \frac{A_1\mathcal{C}_2+4\mathcal{B}_1\mathcal{B}_2}{-A_2}), \\ c_3 &=& -4\mathcal{B}_1/d_2 &,\ d_3 &=& -\mathcal{C}_1/d_2 &= \frac{A_1\mathcal{C}_2+4\mathcal{B}_1\mathcal{B}_2}{A_2 d_2}, \\ d_1 &=& -\mathcal{C}_2/d_2 &,\ \frac{-\mathcal{B}_1}{d_2}c_1 &=& -A_2 + \frac{-\mathcal{C}_2}{d_2}(-A_1)(a_3), & \end{array} \tag{3.55}$$

where $a_3 > 0$. Note that $(\mathcal{B}_1, A_2/2^{\ell}) = (\mathcal{B}_1, \mathcal{C}_2) = 1$. Therefore $d_2 = 1$ as $(A_1, \mathcal{B}_1, \mathcal{C}_1) = 1$.

By equation (3.38),

$$s_{\mathrm{NA}}(\gamma) = \begin{cases} (c_1(-A_1)(-A_2), c_1) & , c_1 \neq 0 \\ 1 & , c_1 = 0, \end{cases}$$

as $c_2, c_3 \neq 0$ and $a_3 > 0$. However, $(c_1(-A_1)(-A_2), c_1) = (-A_1A_2, c_1)$. If $A_1A_2 < 0$, this is equal to 1. If $A_1A_2 > 0$, this is equal to $\mathrm{sign}(c_1)$. However, we have $A_1A_2 > 0$, $\mathcal{B}_1 < 0$, $\mathcal{C}_2 < 0$, and $a_3 > 0$. Additionally, in these new coordinates $-\mathcal{B}_1 c_1 = -A_2 + (-\mathcal{C}_2)(-A_1)a_3$ as $d_2 = 1$. Thus $\mathrm{sign}(c_1) < 0$.

Recall that $A_1 > 0$. Thus if $A_2 < 0$, then $s_{NA}(\gamma) = 1$. If $A_2 > 0$, then $s_{NA}(\gamma) = -1$ when $c_1 \neq 0$ and $s_{NA}(\gamma) = 1$ when $c_1 = 0$. However, $c_1 = 0$ implies that $A_2 < 0$. Thus in either case $s_{\mathrm{NA}}(\gamma) = -\mathrm{sign}(A_2)$.

The computation of the arithmetic part remains. The arithmetic part of the splitting is given by $\left(\frac{-A_1}{d_2}\right)\left(\frac{-\mathcal{B}_1/d_2}{-\mathcal{C}_1/d_2}\right)\left(\frac{c_1}{-\mathcal{C}_2/d_2}\right)$. The computation begins by noting that $(\mathcal{B}_1/2^{\delta_1}, A_2) = 1$, and $d_2 = 1$. Thus,

$$\begin{aligned}&\left(\frac{-A_1}{d_2}\right)\left(\frac{-\mathcal{B}_1/d_2}{-\mathcal{C}_1/d_2}\right)\left(\frac{c_1}{-\mathcal{C}_2/d_2}\right)\\ &= \left(\frac{\epsilon 2^{\delta_1}}{-\mathcal{C}_1}\right)\left(\frac{-\epsilon\mathcal{B}_1/2^{\delta_1}}{\mathrm{sign}(A_2)[(A_1/2^{\ell})\mathcal{C}_2 + 4(\mathcal{B}_1\mathcal{B}_2/2^{\ell})]}\right)\left(\frac{-\epsilon\mathcal{B}_1/2^{\delta_1}}{\mathrm{sign}(A_2)A_2/2^{\ell}}\right)\left(\frac{c_1}{-\mathcal{C}_2}\right). \quad (3.56)\end{aligned}$$

The decision to introduce $\epsilon = \left(\frac{-1}{-B_1}\right)$ in the previous line will allow us to apply the statement in Proposition 6 having to do with periodicity in the bottom entry in a future part of this computation. In addition, using the equation involving c_1 in line (3.55) leads to

$$\begin{aligned}(3.56) = &\left(\frac{\epsilon 2^{\delta_1}}{-\mathcal{C}_1}\right)\left(\frac{-\epsilon\mathcal{B}_1/2^{\delta_1}}{\mathrm{sign}(A_2)(A_1/2^{\ell})\mathcal{C}_2}\right)\left(\frac{-\epsilon\mathcal{B}_1/2^{\delta_1}}{\mathrm{sign}(A_2)A_2/2^{\ell}}\right)\\ &\times\left(\frac{-\mathcal{B}_1}{-\mathcal{C}_2}\right)\left(\frac{-A_2 + (-\mathcal{C}_2)(-A_1)(a_3)}{-\mathcal{C}_2}\right). \quad (3.57)\end{aligned}$$

The next equality follows as $-\mathcal{C}_1 \equiv 1 \pmod 4$ and $-\mathcal{C}_2 > 0$.

$$\begin{aligned}(3.57) = &\left(\frac{2^{\delta_1}}{-\mathcal{C}_1}\right)\left(\frac{-\epsilon\mathcal{B}_1/2^{\delta_1}}{\mathrm{sign}(A_2)(A_1/2^{\ell})\mathcal{C}_2}\right)\left(\frac{-\epsilon\mathcal{B}_1/2^{\delta_1}}{\mathrm{sign}(A_2)A_2/2^{\ell}}\right)\left(\frac{-\mathcal{B}_1}{-\mathcal{C}_2}\right)\\ &\times\left(\frac{\mathrm{sign}(A_2)A_2/2^{\ell}}{-\mathcal{C}_2}\right)\left(\frac{2^{\ell}}{-\mathcal{C}_2}\right). \quad (3.58)\end{aligned}$$

The next step follows from quadratic reciprocity.

$$\begin{aligned}(3.58) = &\left(\frac{2^{\delta_1}}{\mathcal{C}_1}\right)\left(\frac{-\epsilon\mathcal{B}_1/2^{\delta_1}}{\mathrm{sign}(A_2)(A_1/2^{\ell})\mathcal{C}_2}\right)\left(\frac{-\epsilon\mathcal{B}_1/2^{\delta_1}}{\mathrm{sign}(A_2)A_2/2^{\ell}}\right)\left(\frac{-\mathcal{B}_1}{-\mathcal{C}_2}\right)\\ &\left(\frac{-\mathcal{C}_2}{\mathrm{sign}(A_2)A_2/2^{\ell}}\right)\left(\frac{2^{\ell}}{\mathcal{C}_2}\right). \quad (3.59)\end{aligned}$$

Now replace the $\mathcal{C}_2$ appearing in the top entry using the equation $A_1\mathcal{C}_2+4\mathcal{B}_1\mathcal{B}_2+\mathcal{C}_1A_2 = 0$ and use periodicity in the top entry to see that

$$(3.59) = \left(\frac{2^{\delta_1}}{\mathcal{C}_1}\right)\left(\frac{-\epsilon\mathcal{B}_1/2^{\delta_1}}{\mathrm{sign}(A_2)(A_1/2^{\ell})\mathcal{C}_2}\right)\left(\frac{-\epsilon\mathcal{B}_1/2^{\delta_1}}{\mathrm{sign}(A_2)A_2/2^{\ell}}\right)\left(\frac{-\mathcal{B}_1}{-\mathcal{C}_2}\right)$$

$$\times\left(\frac{A_1/2^\ell}{\operatorname{sign}(A_2)A_2/2^\ell}\right)\left(\frac{4\mathcal{B}_1\mathcal{B}_2/2^\ell}{\operatorname{sign}(A_2)A_2/2^\ell}\right)\left(\frac{2^\ell}{\mathcal{C}_2}\right). \quad (3.60)$$

Next rearrange the terms and use that $\mathcal{B}_1 < 0$ and $A_1 > 0$ to get

$$(3.60) = \left(\frac{A_1/2^\ell}{A_2/2^\ell}\right)\left(\frac{-\epsilon\mathcal{B}_1/2^{\delta_1}}{\operatorname{sign}(A_2)(A_1/2^\ell)}\right)\left(\frac{(4\mathcal{B}_1\mathcal{B}_2/2^\ell)(-\epsilon\mathcal{B}_1/2^{\delta_1})}{\operatorname{sign}(A_2)A_2/2^\ell}\right)\left(\frac{-\mathcal{B}_1/2^{\delta_1}}{\mathcal{C}_2}\right)$$
$$\times\left(\frac{-\epsilon\mathcal{B}_1/2^{\delta_1}}{\mathcal{C}_2}\right)\left(\frac{2^{\delta_1}}{\mathcal{C}_1}\right)\left(\frac{2^{l+\delta_1}}{\mathcal{C}_2}\right). \quad (3.61)$$

Next consider the terms involving $\mathcal{C}_2$ to get

$$(3.61) = \left(\frac{A_1/2^\ell}{A_2/2^\ell}\right)\left(\frac{-\epsilon\mathcal{B}_1/2^{\delta_1}}{\operatorname{sign}(A_2)(A_1/2^\ell)}\right)\left(\frac{(4\mathcal{B}_1\mathcal{B}_2/2^\ell)(-\epsilon\mathcal{B}_1/2^{\delta_1})}{\operatorname{sign}(A_2)A_2/2^\ell}\right)$$
$$\times\left(\frac{(-\epsilon\mathcal{B}_1/2^{\delta_1})(-\mathcal{B}_1/2^{\delta_1})}{\mathcal{C}_2}\right)\left(\frac{2^{\delta_1}}{\mathcal{C}_1}\right)\left(\frac{2^{l+\delta_1}}{\mathcal{C}_2}\right)$$
$$= \left(\frac{A_1/2^\ell}{A_2/2^\ell}\right)\left(\frac{-\epsilon\mathcal{B}_1/2^{\delta_1}}{\operatorname{sign}(A_2)(A_1/2^\ell)}\right)\left(\frac{(4\mathcal{B}_1\mathcal{B}_2/2^\ell)(-\epsilon\mathcal{B}_1/2^{\delta_1})}{\operatorname{sign}(A_2)A_2/2^\ell}\right)\left(\frac{2^{\delta_1}}{\mathcal{C}_1}\right)\left(\frac{\epsilon 2^{l+\delta_1}}{\mathcal{C}_2}\right). \quad (3.62)$$

The remaining equalities follow from basic properties of the Kronecker Symbol.

$$(3.62) = \left(\frac{A_1/2^\ell}{A_2/2^\ell}\right)\left(\frac{-\epsilon\mathcal{B}_1/2^{\delta_1}}{\operatorname{sign}(A_2)(A_1/2^\ell)}\right)\left(\frac{-\epsilon 2^{\delta_1+2-l}\mathcal{B}_2}{\operatorname{sign}(A_2)A_2/2^\ell}\right)\left(\frac{2^{\delta_1}}{\mathcal{C}_1}\right)\left(\frac{\epsilon 2^{l+\delta_1}}{\mathcal{C}_2}\right)$$
$$= \left(\frac{\epsilon}{\operatorname{sign}(A_2)\mathcal{C}_2}\right)\left(\frac{A_1/2^\ell}{A_2/2^\ell}\right)\left(\frac{-\epsilon\mathcal{B}_1/2^{\delta_1}}{A_1/2^\ell}\right)\left(\frac{-\epsilon 2^{\delta_1+2-l}\mathcal{B}_2}{\operatorname{sign}(A_2)A_2/2^\ell}\right)\left(\frac{2^{\delta_1}}{\mathcal{C}_1}\right)\left(\frac{2^{l+\delta_1}}{\mathcal{C}_2}\right)$$
$$= \left(\frac{\epsilon}{\operatorname{sign}(A_2)\mathcal{C}_2}\right)\left(\frac{A_1/2^\ell}{A_2/2^\ell}\right)\left(\frac{-\epsilon\mathcal{B}_1/2^{\delta_1}}{A_1/2^\ell}\right)\left(\frac{-\epsilon\mathcal{B}_2/2^{\delta_2}}{\operatorname{sign}(A_2)A_2/2^\ell}\right)$$
$$\times\left(\frac{2^{\delta_1}}{\mathcal{C}_1}\right)\left(\frac{2^{l+\delta_1}}{\mathcal{C}_2}\right)\left(\frac{2^{\delta_1+\delta_2+2-l}}{\operatorname{sign}(A_2)A_2/2^\ell}\right) \quad (3.63)$$

The next line shows that the formula remains the same when the modified Plücker coordinates are switched back to the original Plücker coordinates using the equations of lines (3.52), (3.53), and (3.54). The switch for B_1 and B_2 is direct, but it appears that some care is needed when dealing with C_1 and C_2. The case when $\ell > 0$ is straightforward as $\mathcal{C}_i \equiv C_i (\text{mod } 8)$ for $i = 1, 2$. Unfortunately, this need not be true when $\ell = 0$.

Supposing that $\ell = 0$ there will be two cases to consider: $\delta_1 = 0$ and $\delta_1 \neq 0$. If $\delta_1 = 0$ both exponents will be 0 and there is nothing to show. The last case to consider is when $\ell = 0$ and $\delta_1 > 0$. In this case $A_1C_2 + C_1A_2 \equiv 0 (\text{mod } 8)$ and $A_1\mathcal{C}_2 + \mathcal{C}_1A_2 \equiv 0 (\text{mod } 8)$. As A_1, A_2 are odd, $C_1C_2 \equiv \mathcal{C}_1\mathcal{C}_2 (\text{mod } 8)$. Thus $\left(\frac{2^{\delta_1}}{C_1}\right)\left(\frac{2^{\delta_1}}{C_2}\right) = \left(\frac{2^{\delta_1}}{\mathcal{C}_1}\right)\left(\frac{2^{\delta_1}}{\mathcal{C}_2}\right)$. Therefore,

$$(3.63) = \left(\frac{\epsilon}{\text{sign}(A_2)C_2}\right)\left(\frac{A_1/2^\ell}{A_2/2^\ell}\right)\left(\frac{-\epsilon B_1/2^{\delta_1}}{A_1/2^\ell}\right)\left(\frac{-\epsilon B_2/2^{\delta_2}}{\text{sign}(A_2)A_2/2^\ell}\right) \times \left(\frac{2^{\delta_1}}{C_1}\right)\left(\frac{2^{l+\delta_1}}{C_2}\right)\left(\frac{2^{\delta_1+\delta_2+2-l}}{\text{sign}(A_2)A_2/2^\ell}\right). \quad (3.64)$$

Now the computation will bifurcate. Case 1 will consist of $\delta_1 \leq l$ and Case 2 will consist of $\delta_1 > l$.

Case 1: If $\delta_1 \leq l$ then $D_1 = (2^\ell, B_1) = 2^{\delta_1}$ and $D_2 = 2^{l-\delta_1}$. Thus,

$$\left(\frac{-B_1/2^{\delta_1}}{A_1/2^\ell}\right) = \left(\frac{-B_1/D_1}{A_1/2^\ell}\right),$$

$$\left(\frac{-\epsilon B_2/2^{\delta_2}}{\text{sign}(A_2)A_2/2^\ell}\right)\left(\frac{2^{\delta_1+\delta_2+2-l}}{\text{sign}(A_2)A_2/2^\ell}\right) = \left(\frac{-\epsilon 4B_2/D_2}{\text{sign}(A_2)A_2/2^\ell}\right),$$

$$\left(\frac{2^{\delta_1}}{C_1}\right)\left(\frac{2^{l+\delta_1}}{C_2}\right) = \left(\frac{2^{\delta_1}}{C_1}\right)\left(\frac{2^{l-\delta_1}}{C_2}\right) = \left(\frac{D_1}{C_1}\right)\left(\frac{D_2}{C_2}\right).$$

Case 2: If $\delta_1 > l$ then $D_1 = (2^\ell, B_1) = 2^\ell$ and $D_2 = 1$. Thus,

$$\left(\frac{-B_1/2^{\delta_1}}{A_1/2^\ell}\right) = \left(\frac{-B_1/D_1}{A_1/2^\ell}\right)\left(\frac{2^{\delta_1-l}}{A_1/2^\ell}\right),$$

$$\left(\frac{-\epsilon B_2/2^{\delta_2}}{\text{sign}(A_2)A_2/2^\ell}\right)\left(\frac{2^{\delta_1+\delta_2+2-l}}{\text{sign}(A_2)A_2/2^\ell}\right) = \left(\frac{-\epsilon 4B_2/D_2}{\text{sign}(A_2)A_2/2^\ell}\right)\left(\frac{2^{\delta_1-l}}{\text{sign}(A_2)A_2/2^\ell}\right),$$

$$\left(\frac{2^{\delta_1}}{C_1}\right)\left(\frac{2^{l+\delta_1}}{C_2}\right) = \left(\frac{2^{\delta_1}}{C_1}\right)\left(\frac{2^{l-\delta_1}}{C_2}\right)$$
$$= \left(\frac{D_1}{C_1}\right)\left(\frac{D_2}{C_2}\right)\left(\frac{2^{\delta_1-l}}{C_1}\right)\left(\frac{2^{\delta_1-l}}{C_2}\right).$$

Finally we must show that

$$\left(\frac{2^{\delta_1-l}}{A_1/2^\ell}\right)\left(\frac{2^{\delta_1-l}}{-A_2/2^\ell}\right)\left(\frac{2^{\delta_1-l}}{C_1}\right)\left(\frac{2^{\delta_1-l}}{C_2}\right) = 1. \quad (3.65)$$

To see this note that $\delta_1 > l$ implies that $\delta_1 > 0$. Thus $(A_1/2^\ell)C_2 + C_1(A_2/2^\ell) \equiv 0 \pmod 8$ or equivalently, $(A_1/2^\ell)C_2 \equiv -C_1(A_2/2^\ell) \pmod 8$. Thus equation (3.65) holds and Case 2 is complete.

In either case the arithmetic part of the splitting is given by

$$s_A(\gamma) = \left(\frac{\epsilon}{\text{sign}(A_2)C_2}\right)\left(\frac{A_1/D}{A_2/D}\right)\left(\frac{-\epsilon B_1/D_1}{A_1/D}\right)\left(\frac{-\epsilon 4B_2/D_2}{\text{sign}(A_2)A_2/D}\right)\left(\frac{D_1}{C_1}\right)\left(\frac{D_2}{C_2}\right).$$

After incorporating the non-arithmetic factor and groupings the ϵ terms and the -1 terms together the formula simplifies to

$$s(\gamma) = \left(\frac{-\epsilon}{-A_1A_2}\right)\left(\frac{A_1/D}{A_2/D}\right)\left(\frac{B_1/D_1}{A_1/D}\right)\left(\frac{4B_2/D_2}{\text{sign}(A_2)A_2/D}\right)\left(\frac{D_1}{C_1}\right)\left(\frac{D_2}{C_2}\right).$$

□

At this point we can combine Proposition 15 and Proposition 16 to prove Theorem 14.

Proof of Theorem 14: Let $\gamma \in \Gamma_1(4)$ with Plücker coordinates $(4A_1, 4B_1, C_1, 4A_2, 4B_2, C_2)$ such that $A_1 > 0$, and $A_2/(A_1, A_2) \equiv 1 \pmod 2$. Let $D = (A_1, A_2)$, $D_1 = (D, B_1)$, $D_2 = D/D_1$, and let $\epsilon = \left(\frac{-1}{-B_1/D_1}\right)$. Let D', D_1', and D_2' denote the odd part of D, D_1, and D_2 respectively. Let $S = t(1, (D_2')^{-1}, (D')^{-1})$. Since $S\gamma S^{-1}$ has Plücker coordinates $(4A_1/D', 4B_1/(D_1)', C_1, 4A_2/D', 4B_2/(D_2)', C_2)$, Proposition 15 shows that $s(\gamma) = s(S\gamma S^{-1})\left(\frac{D_1'}{C_1}\right)\left(\frac{D_2'}{C_2}\right)$. Let $\epsilon' = \left(\frac{-1}{-B_1/(D_1')}\right)$ and note that $\epsilon' = \epsilon$. Since $(A_1/D', A_2/D')$ is even we can apply Proposition 16 to evaluate $s(S\gamma S^{-1})$. In particular, we have

$$\begin{aligned} s(S\gamma S^{-1}) &= \left(\frac{-\epsilon'}{-(A_1/D')(A_2/(D')}\right)\left(\frac{(A_1/D')/(D/D')}{(A_2/D')/(D/D')}\right)\left(\frac{(B_1/D_1')/(D_1/D_1')}{(A_1/D')/(D/D')}\right) \\ &\qquad \times \left(\frac{4(B_2/D_2')/(D_2/D_2')}{\operatorname{sign}(A_2/D')(A_2/D')/(D/D')}\right)\left(\frac{(D_1/D_1')}{C_1}\right)\left(\frac{(D_2/D_2')}{C_2}\right) \\ &= \left(\frac{-\epsilon'}{-A_1A_2}\right)\left(\frac{A_1/D}{A_2/D}\right)\left(\frac{B_1/D_1}{A_1/D}\right) \\ &\qquad \times \left(\frac{4B_2/D_2}{\operatorname{sign}(A_2)A_2/D}\right)\left(\frac{(D_1/D_1')}{C_1}\right)\left(\frac{(D_2/D_2')}{C_2}\right). \end{aligned}$$

As $s(\gamma) = s(S\gamma S^{-1})\left(\frac{D_1'}{C_1}\right)\left(\frac{D_2'}{C_2}\right)$ and $\epsilon = \epsilon'$ we have

$$s(\gamma) = \left(\frac{-\epsilon}{-A_1A_2}\right)\left(\frac{A_1/D}{A_2/D}\right)\left(\frac{B_1/D_1}{A_1/D}\right)\left(\frac{4B_2/D_2}{\operatorname{sign}(A_2)A_2/D}\right)\left(\frac{D_1}{C_1}\right)\left(\frac{D_2}{C_2}\right).$$

□

A few remarks are in order. First, remember that this formula only holds when $A_1 > 0$ and $\operatorname{ord}_2\left(\frac{A_1}{A_2}\right) \geq 0$. The other cases may be described using symmetries of s as indicated at the beginning of this section. Second, $\epsilon = \left(\frac{-1}{B_1}\right)$ during the computation in the reduced case. Thus, $\epsilon = \left(\frac{-1}{-B_1/D_1}\right)$ in the general case and not $\left(\frac{-1}{B_1}\right)$, which is only true in the reduced case. Third, when (A_1, A_2) is odd, then both A_1 and A_2 must be odd since $A_1 + A_2 \equiv 0 \pmod 4$. So it follows that $\left(\frac{\epsilon}{-A_1A_2}\right) = 1$. Therefore, ϵ only influences the value of the splitting when (A_1, A_2) is even.

At this point, we would like to reiterate why we set out to prove Theorem 14. Recall that

$$\Sigma(A_1, A_2; m_1, m_2) = \sum_{\gamma \in \Gamma_\infty \backslash \mathbb{S}(A_1,A_2)/\Gamma_\infty} s(\gamma) e^{2\pi i (m1 \frac{B_1}{A_1} + m2 \frac{B_2}{A_2})}.$$

This was first introduced on line (2.13). The Fourier coefficients of the metaplectic Eisenstein series are built out of certain Dirichlet series, and when $A_1, A_2 \neq 0$, the exponential sums, $\Sigma(A_1, A_2; m_1, m_2)$, appear as coefficients of some of these Dirichlet series. Theorem 14 allows us to write $\Sigma(A_1, A_2; m_1, m_2)$ in the following explicit form. Recall line (3.4),

$$S(A_1, A_2) = \{(4A_1, 4B_1, C_1, 4A_2, 4B_2, C_2) \in \mathbb{Z}^6 | A_1 C_2 + 4B_1 B_2 + C_1 A_2 = 0, \\ (A_i, B_i, C_i) = 1,\ C_j \equiv -1 (\text{mod } 4),\ \frac{B_1}{A_1}, \frac{B_2}{A_2}, \frac{C_2}{4A_2} \in [0, 1)\}. \quad (3.66)$$

We have seen in Proposition 11 that $S(A_1, A_2)$ is in bijection with $\Gamma_\infty \backslash \mathbb{S}(A_1, A_2)/\Gamma_\infty$, so when A_1 and A_2 satisfy the hypothesis of Theorem 14 it follows that

$$\Sigma(A_1, A_2; m_1, m_2) = \sum_{S(A_1,A_2)} \left(\frac{-\epsilon}{-A_1 A_2}\right) \left(\frac{A_1/D}{A_2/D}\right) \left(\frac{B_1/D_1}{A_1/D}\right) \left(\frac{4B_2/D_2}{\text{sign}(A_2) A_2/D}\right) \\ \times \left(\frac{D_1}{C_1}\right) \left(\frac{D_2}{C_2}\right) e^{2\pi i (m1 \frac{B_1}{A_1} + m2 \frac{B_2}{A_2})}. \quad (3.67)$$

This expression is still a bit unwieldy, but the result of the next section shows that it is enough to consider the case were $A_1 = p^k$, and $A_2 = \pm p^\ell$. With this final reduction the exponential sum $\Sigma(A_1, A_2; m_1, m_2)$ can be computed explicitly; this computation will be the content of a forthcoming paper.

3.9 Twisted Multiplicativity

Proposition 17. *Let $A_1, \alpha_1 \in \mathbb{Z}_{>0}$, $A_2, \alpha_2 \in \mathbb{Z}$ such that A_1, A_2 are odd, $(A_1 A_2, \alpha_1 \alpha_2) = 1$, $A_1 \alpha_1 + A_2 \alpha_2 \equiv 0$ (mod 4), and $\frac{\alpha_2}{(\alpha_1, \alpha_2)} \equiv 1$ (mod 2). Let $\mu = \left(\frac{-1}{-A_1 A_2}\right)$. Then with respect to the map from Proposition 12, $\phi : S(A_1 \alpha_1, A_2 \alpha_2) \to S(A_1, \mu A_2) \times S(\alpha_1, -\mu \alpha_2)$, the following holds:*

$$s(\gamma) = s(\pi_1(\phi(\gamma))) s(\pi_2(\phi(\gamma))) \left(\frac{\alpha_2}{(\frac{-1}{A_1}) A_1}\right) \left(\frac{\alpha_1}{A_2}\right), \quad (3.68)$$

where π_i is the projection onto the i-th factor.

Note that the visible asymmetry in the formula is a result of the asymmetry contained in the hypotheses. In particular, since $\alpha_1 > 0$, $\left(\frac{\alpha_1}{\pm A_2}\right) = \left(\frac{\alpha_1}{A_2}\right)$.

Proof: Recall the map

$$(A_1\alpha_1, B_1, C_1, A_2\alpha_2, B_2, C_2)$$

$$\overset{\phi}{\mapsto} ((A_1, B_1, \mathcal{C}_1, \mu A_2, B_2, \gamma C_2),$$

$$(\alpha_1, B_1, \left(\frac{-1}{A_2}\right) A_2C_1, -\mu\alpha_2, -\left(\frac{-1}{A_1}\right) B_2, \left(\frac{-1}{A_1}\right) A_1C_2)).$$

Let $D = (A_1\alpha_1, A_2\alpha_2)$, $d = (A_1, A_2)$, and $\delta = (\alpha_1, \alpha_2)$. Then $D = (A_1\alpha_1, A_2\alpha_2) = (A_1, A_2)(\alpha_1, \alpha_2) = d\delta$, as $(A_i, \alpha_j) = 1$. Similarly, define $D_1 = (D, B_1)$, $d_1 = (d, B_1)$, and $\delta_1 = (\delta, B_1)$. Again $D_1 = d_1\delta_1$. Let $D_2 = \frac{D}{D_1}$, $d_2 = \frac{d}{d_1}$, and $\delta_2 = \frac{\delta}{\delta_2}$. Note that $D_2 = d_2\delta_2$. Let $\epsilon = \left(\frac{-1}{-B_1/D_1}\right)$, $\epsilon_1 = \left(\frac{-1}{-B_1/d_1}\right)$, and $\epsilon_2 = \left(\frac{-1}{-B_1/\delta_1}\right)$.

Begin with the formula of the splitting described in Theorem 14,

$$\begin{aligned} s(\gamma) &= \left(\frac{-\epsilon}{-A_1\alpha_1A_2\alpha_2}\right)\left(\frac{A_1\alpha_1/D}{A_2\alpha_2/D}\right)\left(\frac{B_1/D_1}{A_1\alpha_1/D}\right) \\ &\qquad \times \left(\frac{4B_2/D_2}{\text{sign}(A_2\alpha_2)A_2\alpha_2/D}\right)\left(\frac{D_1}{C_1}\right)\left(\frac{D_2}{C_2}\right) \\ &= \left(\frac{-\epsilon}{\alpha_1\mu\alpha_2}\right)\left(\frac{A_1\alpha_1/D}{A_2\alpha_2/D}\right)\left(\frac{B_1/D_1}{A_1\alpha_1/D}\right) \\ &\qquad \times \left(\frac{4B_2/D_2}{\text{sign}(A_2\alpha_2)A_2\alpha_2/D}\right)\left(\frac{D_1}{C_1}\right)\left(\frac{D_2}{C_2}\right). \end{aligned}$$

The second equality above follows as $A_1 + \mu A_2 \equiv 0 \pmod 4$.

$$\begin{aligned} s(\pi_1(\phi(\gamma))) &= \left(\frac{-\epsilon_1}{-A_1\mu A_2}\right)\left(\frac{A_1/d}{\mu A_2/d}\right)\left(\frac{B_1/d_1}{A_1/d}\right) \\ &\quad \times \left(\frac{4B_2/d_2}{\text{sign}(\mu A_2)\mu A_2/d}\right)\left(\frac{d_1}{\mathcal{C}_1}\right)\left(\frac{d_2}{\gamma C_2}\right) \\ &= \left(\frac{A_1/d}{\mu A_2/d}\right)\left(\frac{B_1/d_1}{A_1/d}\right)\left(\frac{4B_2/d_2}{\text{sign}(\mu A_2)\mu A_2/d}\right)\left(\frac{d_1}{\mathcal{C}_1}\right)\left(\frac{d_2}{\gamma C_2}\right). \end{aligned}$$

Again the second equality above follows as $A_1 + \mu A_2 \equiv 0 \pmod 4$.

$$s(\pi_2(\phi(\gamma)))$$

$$= \left(\frac{-\epsilon_2}{-\alpha_1(-\mu\alpha_2)}\right)\left(\frac{\alpha_1/\delta}{-\mu\alpha_2/\delta}\right)\left(\frac{B_1/\delta_1}{\alpha_1/\delta}\right)$$

$$\times \left(\frac{-4\left(\frac{-1}{A_2}\right)\mu B_2/\delta_2}{\text{sign}(\alpha_2)\alpha_2/\delta}\right)\left(\frac{\delta_1}{A_2C_1}\right)\left(\frac{\delta_2}{A_1C_2}\right)$$

Next we consider the product of the analogous terms in the formula of each splitting. First consider the terms involving ϵ.

$$\left(\frac{-\epsilon}{\alpha_1\mu\alpha_2}\right)\left(\frac{-\epsilon_2}{-\alpha_1(-\mu\alpha_2)}\right)=\left(\frac{\epsilon\epsilon_2}{\alpha_1\mu\alpha_2}\right)=\left(\frac{\left(\frac{-1}{d_1}\right)}{\alpha_1\mu\alpha_2}\right). \tag{3.69}$$

Consider the terms involving A_1, A_2, α_1, or α_2.

$$\begin{aligned}
&\left(\frac{A_1\alpha_1/D}{A_2\alpha_2/D}\right)\left(\frac{A_1/d}{\mu A_2/d}\right)\left(\frac{\alpha_1/\delta}{-\mu\alpha_2/\delta}\right)\\
&=\left(\frac{(A_1/d)(\alpha_1/\delta)}{(A_2/d)(\alpha_2/\delta)}\right)\left(\frac{A_1/d}{\mu A_2/d}\right)\left(\frac{\alpha_1/\delta}{-\mu\alpha_2/\delta}\right)\\
&=\left(\frac{(A_1/d)}{(\alpha_2/\delta)}\right)\left(\frac{(A_1/d)}{(A_2/d)}\right)\left(\frac{(\alpha_1/\delta)}{(\alpha_2/\delta)}\right)\left(\frac{(\alpha_1/\delta)}{(A_2/d)}\right)\left(\frac{A_1/d}{\mu A_2/d}\right)\left(\frac{\alpha_1/\delta}{-\mu\alpha_2/\delta}\right)\\
&=\left(\frac{A_1/d}{(\alpha_2/\delta)}\right)\left(\frac{\alpha_1/\delta}{A_2/d}\right)\left(\frac{A_1/d}{\mu}\right)\left(\frac{\alpha_1/\delta}{-\mu}\right)\\
&=\left(\frac{A_1/d}{(\alpha_2/\delta)}\right)\left(\frac{\alpha_1/\delta}{A_2/d}\right).
\end{aligned} \tag{3.70}$$

The last equality above follows as $A_1, \alpha_1 > 0$. Next consider the terms involving B_1.

$$\begin{aligned}
&\left(\frac{B_1/D_1}{A_1\alpha_1/D}\right)\left(\frac{B_1/d_1}{A_1/d}\right)\left(\frac{B_1/\delta_1}{\alpha_1/\delta}\right)\\
&=\left(\frac{B_1/(d_1\delta_1)}{A_1/d}\right)\left(\frac{B_1/(d_1\delta_1)}{\alpha_1/\delta}\right)\left(\frac{B_1/d_1}{A_1/d}\right)\left(\frac{B_1/\delta_1}{\alpha_1/\delta}\right)\\
&=\left(\frac{\delta_1}{A_1/d}\right)\left(\frac{d_1}{\alpha_1/\delta}\right).
\end{aligned} \tag{3.71}$$

Now we consider the terms involving B_2.

$$\begin{aligned}
&\left(\frac{4B_2/D_2}{\operatorname{sign}(A_2\alpha_2)A_2\alpha_2/D}\right)\left(\frac{4B_2/d_2}{\operatorname{sign}(\mu A_2)\mu A_2/d}\right)\left(\frac{-4\left(\frac{-1}{A_2}\right)\mu B_2/\delta_2}{\operatorname{sign}(\alpha_2)\alpha_2/\delta}\right)\\
&=\left(\frac{4B_2/(d_2\delta_2)}{\operatorname{sign}(A_2)A_2/d}\right)\left(\frac{4B_2/(d_2\delta_2)}{\operatorname{sign}(\alpha_2)\alpha_2/\delta}\right)\left(\frac{4B_2/d_2}{\operatorname{sign}(A_2)A_2/d}\right)\left(\frac{-4\left(\frac{-1}{A_2}\right)\mu B_2/\delta_2}{\operatorname{sign}(\alpha_2)\alpha_2/\delta}\right)\\
&=\left(\frac{\delta_2}{A_2/d}\right)\left(\frac{-\mu\left(\frac{-1}{A_2}\right)d_2}{\operatorname{sign}(\alpha_2)\alpha_2/\delta}\right).
\end{aligned} \tag{3.72}$$

Next consider the terms involving C_1 or $\mathcal{C}_1$.

$$\begin{aligned}
&\left(\frac{D_1}{C_1}\right)\left(\frac{d_1}{\mathcal{C}_1}\right)\left(\frac{\delta_1}{A_2C_1}\right)\\
&=\left(\frac{d_1}{C_1}\right)\left(\frac{d_1}{\mathcal{C}_1}\right)\left(\frac{\delta_1}{A_2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{C_1\mathcal{C}_1}{d_1}\right)\left(\frac{\delta_1}{A_2}\right) \\
&= \left(\frac{C_1(\mu C_1\alpha_2)}{d_1}\right)\left(\frac{\delta_1}{A_2}\right) \\
&= \left(\frac{\mu\alpha_2}{d_1}\right)\left(\frac{\delta_1}{A_2}\right). \qquad (3.73)
\end{aligned}
$$

The second equality above follows from an application of Quadratic Reciprocity and noting that $C_1\mathcal{C}_1 \equiv 1 \pmod 4$ and $d_1 > 0$. The third equality above follows from the identity for $\mathcal{C}_1$ on line (3.9). Finally consider the terms involving C_2.

$$
\begin{aligned}
&\left(\frac{D_2}{C_2}\right)\left(\frac{d_2}{\gamma C_2}\right)\left(\frac{\delta_2}{A_1C_2}\right) \\
&= \left(\frac{d_2}{\gamma}\right)\left(\frac{\delta_2}{A_1}\right) \\
&= \left(\frac{\gamma}{d_2}\right)\left(\frac{\delta_2}{A_1}\right) \\
&= \left(\frac{\alpha_1}{d_2}\right)\left(\frac{\delta_2}{A_1}\right). \qquad (3.74)
\end{aligned}
$$

The third equality above follows from an application of Quadratic Reciprocity and noting that $\gamma \equiv 1 \pmod 4$ and $d_2 > 0$. The last equality follows as $\gamma \equiv \alpha_1 \pmod{A_2}$.

Now the task is to simplify the product of lines (3.69)-(3.74). This quantity is given by

$$
\left(\frac{\left(\frac{-1}{d_1}\right)}{\alpha_1\mu\alpha_2}\right)\left(\frac{A_1/d}{(\alpha_2/\delta)}\right)\left(\frac{\alpha_1/\delta}{A_2/d}\right)\left(\frac{\delta_1}{A_1/d}\right)\left(\frac{d_1}{\alpha_1/\delta}\right)\left(\frac{\delta_2}{A_2/d}\right)
$$
$$
\times\left(\frac{-\mu\left(\frac{-1}{A_2}\right)d_2}{\mathrm{sign}(\alpha_2)\alpha_2/\delta}\right)\left(\frac{\mu\alpha_2}{d_1}\right)\left(\frac{\delta_1}{A_2}\right)\left(\frac{\alpha_1}{d_2}\right)\left(\frac{\delta_2}{A_1}\right).
$$

The terms will be rearranged and then simplified. The equalities between some lines will involve multiple steps.

$$
\left(\frac{\left(\frac{-1}{d_1}\right)}{\alpha_1\mu\alpha_2}\right)\left(\frac{A_1/d}{\alpha_2/\delta}\right)\left(\frac{\alpha_1/\delta}{A_2/d}\right)\left(\frac{\delta_2}{A_2/d}\right)\left(\frac{\delta_1}{A_1/d}\right)\left(\frac{\delta_1}{A_2}\right)
$$
$$
\times\left(\frac{d_1}{\alpha_1/\delta}\right)\left(\frac{\mu\alpha_2}{d_1}\right)\left(\frac{\delta_2}{A_1}\right)\left(\frac{\left(\frac{-1}{A_1}\right)d_2}{\mathrm{sign}(\alpha_2)\alpha_2/\delta}\right)\left(\frac{\alpha_1}{d_2}\right)
$$

$$
\begin{aligned}
&= \left(\frac{\left(\frac{-1}{d_1}\right)}{\alpha_1\mu\alpha_2}\right)\left(\frac{A_1/d_1}{\alpha_2/\delta}\right)\left(\frac{\alpha_1/\delta_1}{A_2/d}\right)\left(\frac{\delta_1}{A_1}\right)\left(\frac{\delta_1}{A_2/d}\right)\left(\frac{\delta_2}{A_1}\right)\\
&\qquad\qquad \times \left(\frac{d_1}{\alpha_1/\delta}\right)\left(\frac{\left(\frac{-1}{A_1}\right)}{\alpha_2/\delta}\right)\left(\frac{\mu\alpha_2}{d_1}\right)\left(\frac{\alpha_1}{d_2}\right)\left(\frac{\left(\frac{-1}{A_1}\right)d_2}{\mathrm{sign}(\alpha_2)}\right)\\
&= \left(\frac{\left(\frac{-1}{d_1}\right)}{\alpha_1\mu\alpha_2}\right)\left(\frac{A_1/d_1}{\alpha_2/\delta}\right)\left(\frac{\alpha_1}{A_2/d}\right)\left(\frac{\delta}{A_1}\right)\\
&\qquad\qquad \times \left(\frac{d_1}{\left(\frac{-1}{\alpha_1/\delta}\right)\alpha_1/\delta}\right)\left(\frac{\left(\frac{-1}{A_1}\right)}{\alpha_2/\delta}\right)\left(\frac{\mu\alpha_2}{d_1}\right)\left(\frac{\alpha_1}{d_2}\right)\left(\left(\frac{-1}{A_1}\right), \mathrm{sign}(\alpha_2)\right)\\
&= \left(\frac{\left(\frac{-1}{d_1}\right)}{\alpha_1\mu\alpha_2}\right)\left(\frac{A_1/d_1}{\alpha_2/\delta}\right)\left(\left(\frac{-1}{A_1}\right), \mathrm{sign}(\alpha_2)\right)\left(\frac{\alpha_1}{A_2/d_1}\right)\\
&\qquad\qquad \times \left(\frac{\delta}{A_1}\right)\left(\frac{\left(\frac{-1}{\alpha_1/\delta}\right)\alpha_1/\delta}{d_1}\right)\left(\frac{\left(\frac{-1}{A_1}\right)}{\alpha_2/\delta}\right)\left(\frac{\mu\alpha_2}{d_1}\right). \qquad (3.75)
\end{aligned}
$$

The third equality combines two terms with α_1 in the top position and uses quadratic reciprocity. The next chain of equalities follows from basic properties of the Kronecker Symbol.

$$
\begin{aligned}
(3.75) &= \left(\frac{\left(\frac{-1}{d_1}\right)}{\alpha_1\mu\alpha_2}\right)\left(\frac{A_1/d_1}{\alpha_2/\delta}\right)\left(\left(\frac{-1}{A_1}\right), \alpha_2\right)\left(\frac{\alpha_1}{A_2/d_1}\right)\left(\frac{\delta}{A_1}\right)\\
&\qquad\qquad \times \left(\frac{\left(\frac{-1}{\alpha_1/\delta}\right)\alpha_1}{d_1}\right)\left(\frac{\left(\frac{-1}{A_1}\right)}{\alpha_2/\delta}\right)\left(\frac{\mu\alpha_2/\delta}{d_1}\right)\\
&= \left(\frac{\left(\frac{-1}{d_1}\right)}{\alpha_1\mu\alpha_2}\right)\left(\frac{\left(\frac{-1}{A_1/d_1}\right)A_1/d_1}{\alpha_2/\delta}\right)\left(\left(\frac{-1}{A_1}\right), \alpha_2\right)\left(\frac{\alpha_1}{A_2}\right)\left(\frac{\delta}{A_1}\right)\\
&\qquad\qquad \times \left(\frac{\left(\frac{-1}{\alpha_1/\delta}\right)}{d_1}\right)\left(\frac{\left(\frac{-1}{d_1}\right)}{\alpha_2/\delta}\right)\left(\frac{\mu\alpha_2/\delta}{d_1}\right)\\
&= \left(\frac{\left(\frac{-1}{d_1}\right)}{\alpha_1\mu\alpha_2}\right)\left(\frac{\alpha_2/\delta}{\left(\frac{-1}{A_1/d_1}\right)A_1/d_1}\right)\left(\left(\frac{-1}{d_1}\right), \alpha_2\right)\left(\frac{\alpha_1}{A_2}\right)\left(\frac{\delta}{A_1}\right)\\
&\qquad\qquad \times \left(\frac{\left(\frac{-1}{\alpha_1/\delta}\right)}{d_1}\right)\left(\frac{\left(\frac{-1}{d_1}\right)}{\alpha_2/\delta}\right)\left(\frac{\mu\alpha_2/\delta}{d_1}\right)
\end{aligned}
$$

$$\begin{aligned}
&= \left(\frac{\left(\frac{-1}{d_1}\right)}{\alpha_1\mu\alpha_2}\right)\left(\frac{\alpha_2/\delta}{\left(\frac{-1}{A_1/d_1}\right)A_1}\right)\left(\left(\frac{-1}{d_1}\right),\alpha_2\right)\left(\frac{\alpha_1}{A_2}\right)\left(\frac{\delta}{A_1}\right) \\
&\qquad\qquad \times \left(\frac{\left(\frac{-1}{\alpha_1/\delta}\right)}{d_1}\right)\left(\frac{\left(\frac{-1}{d_1}\right)}{\alpha_2/\delta}\right)\left(\frac{\mu}{d_1}\right) \\
&= \left(\frac{\left(\frac{-1}{d_1}\right)}{\alpha_1\mu\alpha_2}\right)\left(\frac{\alpha_2}{\left(\frac{-1}{A_1}\right)A_1}\right)\left(\left(\frac{-1}{d_1}\right),\alpha_2\right)^2\left(\frac{\alpha_1}{A_2}\right)\left(\frac{\left(\frac{-1}{\alpha_1/\delta}\right)}{d_1}\right) \\
&\qquad\qquad \times \left(\frac{\left(\frac{-1}{d_1}\right)}{\alpha_2/\delta}\right)\left(\frac{\mu}{d_1}\right). \qquad (3.76)
\end{aligned}$$

Now we rearrange the terms and apply the identity $\left(\frac{\left(\frac{-1}{a}\right)}{b}\right) = \left(\left(\frac{-1}{a}\right),\left(\frac{-1}{b}\right)\right)$ to get

$$(3.76) = \left(\frac{\alpha_2}{\left(\frac{-1}{A_1}\right)A_1}\right)\left(\frac{\alpha_1}{A_2}\right)\left(\frac{\left(\frac{-1}{d_1}\right)}{\alpha_1\mu\alpha_2}\right)\left(\frac{\left(\frac{-1}{\mu\alpha_1/\delta}\right)}{d_1}\right)\left(\frac{\left(\frac{-1}{\alpha_2/\delta}\right)}{d_1}\right). \qquad (3.77)$$

Finally, another application of the identity $\left(\frac{\left(\frac{-1}{a}\right)}{b}\right) = \left(\left(\frac{-1}{a}\right),\left(\frac{-1}{b}\right)\right)$ completes the computation.

$$\begin{aligned}
(3.77) &= \left(\frac{\alpha_2}{\left(\frac{-1}{A_1}\right)A_1}\right)\left(\frac{\alpha_1}{A_2}\right)\left(\frac{\left(\frac{-1}{d_1}\right)}{\alpha_1\mu\alpha_2}\right)\left(\frac{\left(\frac{-1}{\mu\alpha_1\alpha_2}\right)}{d_1}\right) \\
&= \left(\frac{\alpha_2}{\left(\frac{-1}{A_1}\right)A_1}\right)\left(\frac{\alpha_1}{A_2}\right).
\end{aligned}$$

□

3.10 The Splitting on Other Cells

The content of Section 3.6 is a description of the splitting map on the big cell. However, the computation of the degenerate Fourier coefficients requires an understanding of the splitting on the other cells as well. This section collects the description of s on the smaller Bruhat cells. Recall that

$$\begin{aligned}
w_{\alpha_1} &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
w_{\alpha_2} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.
\end{aligned}$$

Proposition 18. *Let* $\gamma \in \Gamma_1(4)$ *with Plücker Coordinates* $(A_1, B_1, C_1, A_2, B_2, C_2)$. *Then:*

Cell	$(A_1, B_1, C_1, A_2, B_2, C_2)$	$s(\gamma)$
B	$(0, 0, -1, 0, 0, -1)$	*1*
$Bw_{\alpha_1}B$	$(0, 0, -1, 0, B_2, C_2)$	$\left(\frac{B_2}{-C_2}\right)$
$Bw_{\alpha_2}B$	$(0, B_1, C_1, 0, 0, -1)$	$\left(\frac{-B_1}{-C_1}\right)$
$Bw_{\alpha_1}w_{\alpha_2}B$	$(0, B_1, C_1, A_2, B_2, C_2)$	$\left(\frac{A_2/B_1}{-C_2}\right)\left(\frac{-B_1}{-C_1}\right)$
$Bw_{\alpha_2}w_{\alpha_1}B$	$(A_1, B_1, C_1, 0, B_2, C_2)$	$(-A_1, B_2)\left(\frac{-A_1/B_2}{-C_1}\right)\left(\frac{B_2}{-C_2}\right)$
$Bw_\ell B$	$(A_1, B_1, C_1, A_2, B_2, C_2)$	*Equation (3.51)*

Proof:

Case 1: B

$\Gamma_1(4) \cap B$ consists of upper triangular unipotent matrices with integer coefficients. Thus the splitting is trivial on this cell.

Case 2: $Bw_{\alpha_1}B$

In this cell $A_1 = B_1 = A_2 = 0$ and $C_1, B_2 \neq 0$. Thus by Proposition 8

$$c_2 = c_3 = 0$$
$$d_2 = d_3 = 1$$
$$c_1 = B_2$$
$$d_1 = -C_2.$$

Thus

$$\gamma = n \begin{pmatrix} a_1 & b_1 & 0 \\ B_2 & -C_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} n',$$

where $n, n' \in \Gamma_\infty$. By the left and right Γ_∞-invariance of s, assume that

$$\gamma = \begin{pmatrix} a_1 & b_1 & 0 \\ B_2 & -C_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As $c_2 = 0$ the nonarithmetic factor will be equal to 1, by equation (3.38). Thus by equation (3.37)

$$s(\gamma) = \left(\frac{B_2}{-C_2}\right).$$

Case 3: $Bw_{\alpha_2}B$

A similar analysis shows that

$$s(\gamma) = \left(\frac{-B_1}{-C_1}\right).$$

Case 4: $Bw_{\alpha_1}w_{\alpha_2}B$

Similarly,

$$s(\gamma) = \left(\frac{A_2/B_1}{-C_2}\right)\left(\frac{-B_1}{-C_1}\right).$$

Case 5: $Bw_{\alpha_2}w_{\alpha_1}B$

This case appears to be less straightforward and will utilize the symmetry of s with respect to the Cartan involution composed with the conjugation by the long element. Recall the computations of section 3.5. It was shown that $s(\phi(\gamma)) = (-A_1, B_2)s(\gamma)$. By combining this identity with the result of case 4 it follows that

$$s(\gamma) = (-A_1, B_2)\left(\frac{-A_1/B_2}{-C_1}\right)\left(\frac{B_2}{-C_2}\right).$$

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