LAGRANGIAN FLOER THEORY IN SYMPLECTIC FIBRATIONS

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ABSTRACT OF THE DISSERTATION

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Consider a fibration of compact symplectic manifolds $F \to E \to B$ with a compatible symplectic form on E, and an induced fibration of Lagrangian submanifolds $L_F \to L \to L_B$. We develop a Leray-Serre type spectral sequence to compute the Floer cohomology of L in terms of the Floer complex of L_F and L_B when F is symplectically small. Moreover, we write down a formula for the leading order superpotential when F is a Kähler homogeneous space. To solve the transversality and compactness problem, we use the classical approach in addition to the perturbation scheme recently developed by Cieliebak-Mohnke [CM07] and Charest-Woodward [CWb; CWa]. As applications, we find Floer-non-trivial tori in complex flag manifolds and ruled surfaces.

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Chapter 1

Introduction

In topology, the idea of a *fibration* is of central importance. In a certain sense, this is the topological version of a short exact sequence. A fibration provides a natural way of viewing a large space as two smaller ones that are twisted together, or a way of constructing one space from two.

To say something about the topology of a fibration, one typically uses some sort of long exact sequence, or more generally a spectral sequence. This idea was made popular by Leray, Serre, Grothendieck, and others [Ser51; Ler50a; Ler50b; Gro57; Wei94]. For example, to compute the de Rham cohomology of a fiber bundle $F \to E \to B$, where B has a good cover $\mathfrak{U}$, one can use a spectral sequence whose second page is $E_2^{p,q} = H^p(\mathfrak{U}, \mathcal{H}^q)$, the Čech cohomology of the cover, where $\mathcal{H}^q$ is the presheaf $U \mapsto H^q(\pi^{-1}(U))$. The idea goes back to one of Leray [Ler50a; Ler50b], where he develops his spectral sequence to compute sheaf cohomology groups.

We would like to develop a Leray-Serre type spectral sequence in the setting of *pseudo holomorphic curves* and *Lagrangian Floer theory*. Pseudo holomorphic curves were introduced circa 1985 by Gromov [Gro85], and have become a powerful tool in symplectic topology. One of the original applications was in defining a *quantum cup product* on the usual cohomology that allows interactions between cocycles (or their Poincaré duals) that do not "intersect" in the classical situation. Further dynamical applications were considered by Floer and others [Flo88; Flo89].

In this paper, the invariant of interest is *Lagrangian intersection Floer cohomology* [Flo88;

Oh93]. This theory takes as input two transversely intersecting Lagrangians (or often, a single Lagrangian) and in nice cases provides an obstruction to *displacement* by a Hamiltonian isotopy.

Fukaya et. al. (culminating in [FOOO09]) have discovered an underlying algebraic and categorical structure in the information given from Lagrangian intersection theory, called the *Fukaya category* of a symplectic manifold. Through homological mirror symmetry, the derived Fukaya category is expected to be equivalent to the derived category of coherent sheaves in the mirror manifold. Therefore, it seems feasible to try to find some generators for the Fukaya category, or at least some Floer non-trivial Lagrangians.

The Floer cohomology for a single Lagrangian is constructed as follows: Choose a Morse-Smale function on L and form the usual Morse complex $CF(L)$. The Floer differential then counts *quantized* Morse flows–isolated pseudo holomorphic disks $u : (D, \partial D) \to (M, L)$ with boundary markings that map to specified stable/unstable manifolds. Assuming that we have made the right assumptions (L is monotone or weakly unobstructed) and have perturbed the almost complex structure correctly, this differential is well-defined and squares to zero, which gives us a homology theory $HF(L)$.

In this thesis, we will study compact fiber bundles where the base, fiber, and total space are symplectic. We will denote this $F \to E \to B$, where (F, ω_F) and (B, ω_B) are symplectic, and E has an appropriately compatible symplectic form. We suppose further that we have a Lagrangian $L \subset E$ that fibers over a Lagrangian $L_B \subset B$ with Lagrangian fibers $L_F \subset F$, and we proceed to compute the invariant $HF(L)$ from information about $HF(L_B)$ and $HF(L_F)$. The first main result is the derivation of a spectral sequence in the vein of Leray [Ler50a; Ler50b]. Each Morse-Floer configuration $u : (C, \partial C) \to (E, L)$ has a notion of *energy*, which is the symplectic area under pullback $e(u) = \int_C u^*\omega$. The energy of the configurations under the projection $\pi : E \to B$ provides a filtration to induce a spectral sequence.

To provide a better method of computation of this Floer theory, we prove a formula for the *disk potential* of a fibered Lagrangian. A version of the potential function was introduced in [FOOO; FOOO09], and provides a concrete way to show that the Floer cohomology of a Lagrangian torus is isomorphic to the Morse cohomology. Our formula gives the first and second order terms in the disk potential as a computation that takes place in (B, L_B) and (F, L_F). We then show that solving for critical points of this second order potential allows us to say that the particular fibered Lagrangian is Hamiltonian non-displaceable.

Let us now outline the project in further detail. Consider a fiber bundle of smooth compact manifolds $F \to E \to B$ where (F, ω_B) and (B, ω_B) are symplectic, and the symplectic form on E is the weak coupling form:

$$\omega_K = a + K\pi^*\omega_B$$

$$i^*a = \omega_F$$

$$da = 0$$

for large K. The two-form a is what's known as the *minimal coupling form*, and is developed in Guillemin-Lerman-Sternberg [GLS96]. The essence is as follows: Let $TF \oplus H$ be a connection on E. We say that H is Hamiltonian if the holonomy maps are Hamiltonian diffeomorphisms of the fibers. Further suppose that $H = TF^{a\perp}$. Then for $v_1, v_2 \in H$ the assignment $(v_1, v_2) \mapsto a(v_1, v_2)$ is defined as the unique zero average Hamiltonian associated to the vertical component of $[v_1, v_2]$. Since a may degenerate in the horizontal direction, so we choose K large enough so that ω_K is non-degenerate. In section 2, we give more details of this construction.

In order for the Floer theory to work, we have to make some assumptions on the base and fiber. These assumptions involve a crucial quantity assigned to holomorphic curves that is known as the energy. For a symplectic manifold (E, ω), choose an *almost complex* structure J

(where $J^2 = -I$) such that $\omega(\cdot, J\cdot)$ is positive definite. A differentiable map from a Riemann surface $u : (C, \partial C) \to (E, L)$ is said to be J-holomorphic if $J \circ du = du \circ j$, where j is an integrable complex structure on C.

Definition 1. For a map $u : (C, \partial C) \to (E, L)$, the *energy* of u is the symplectic area of C under pullback:

$$e(u) = \int_C u^*\omega$$

Since ω is closed and L is Lagrangian, it follows from Stokes' theorem that this is a homotopy invariant. For curves that are J-holomorphic we have the energy identity [MS04]:

$$\int_C |du|_J^2 d\mathrm{vol}_C = \int_C u^*\omega$$

Thus, for a fixed homology class we get an L^2 bound on the derivative of a J-holomorphic representative, and this is crucial to the compactness results in Floer theory.

We will need L_F to be monotone and B to be rational. Let $\mu(u)$ be the boundary Maslov index associated to a homology class $[u] \in h_2 \circ \pi_2(F, L_F)$, where $h_2 : \pi_2(E, L) \to H_2(E, L)$ is the Hurewicz homomorphism. We say that L_F is *monotone* if there is a $\lambda > 0$ such that $\mu(u) = \lambda \int_C u^*\omega_F$ for any $[u] \in h_2 \circ \pi_2(F, L_F)$. We say that B is rational if ω_B has a non-zero representative in $H^2(B, \mathbb{Q})$. Moreover, we say that L_B is rational if there is an $e \in \mathbb{R}$ such that $h_2 \circ \pi_2(B, L_B) \subset e \cdot \mathbb{Z}$.

Given $L_F \subset F$ and $L_B \subset B$ as above, some natural questions one can ask are:

1. Can we produce a Lagrangian $L \subset E$ as fiber bundle $L_F \to L \to L_B$ given some assumptions on the topology of $F \to E|_{L_B} \to L_B$?

2. If $L \subset E$ of the form $L_F \to L \to L_B$, what can we say about the Floer cohomology of L given that of L_F and L_B?

In this paper, much of the work will culminate to a definitive answer for (2). We will provide an answer for (1) in some special cases.

We set the following definition:

Definition 2. A *Symplectic Mori Fibration* is a fiber bundle of compact symplectic manifolds $(F,\omega_F) \to (E,\omega_K) \xrightarrow{\pi} (B,\omega_B)$, whose transition maps are symplectomorphisms of the fibers, (F,ω_F) is monotone, (B,ω_B) is rational, and $\omega_K = a + K\pi^*\omega_B$ for large K where a is the minimal coupling form associated to a Hamiltonian connection.

Usually, one defines Lagrangian Floer theory over some sort of Novikov ring. In our framework, it is natural to work in a Novikov ring with two formal variables q and r, in which the r-exponent is allowed to be as negative as a multiple of the q-exponent. For $0 < \epsilon < 1$, denote

$$\Lambda^2 := \left\{ \sum_{i,j} c_{ij} q^{\rho_i} r^{\eta_j} | c_{ij} \in \mathbb{C},\ \rho_i, \eta_j \in \mathbb{R},\ \rho_i \geq 0, (1-\epsilon)\rho_i + \eta_j \geq 0 \right.$$
$$\left. \#\{c_{ij} \neq 0, \rho_i + \eta_j \leq N\} < \infty \right\}$$

Let

$$\Lambda_q = \left\{ \sum_{i \geq 0} c_i q^{\rho_i} : c_i \in \mathbb{C},\ \rho_i \in \mathbb{R}_{\geq 0},\ \#\{i : c_i \neq 0, \rho_i \leq N\} < \infty \right\}$$

be the universal Novikov ring with non-negative powers of q and let

$$\Lambda[q] = \left\{ \sum_{i \in \mathbb{N}_0} c_i q^{i\rho} \right\}$$

for some $\rho > 0$. Filtration of the Floer chain complex by base energy induces a spectral sequence, which is our first main theorem:

Theorem 1. *Let $(F,\omega_F) \to (E,\omega_K) \to (B,\omega_B)$ be a symplectic Mori fibration. Suppose we have a fibration of Lagrangians $L_F \to L \to L_B$, with L_F monotone, L_B rational, L Lagrangian with respect to ω_K, and a divisor $D = \pi^{-1}(D_B)$ for a Donaldson hypersurface $[D_B]^{PD} = n[\omega_B]$ of large degree in the base. Choose an a regular, coherent, stabilizing, convergent perturbation datum $(P_\Gamma)_\Gamma$ (for each configuration type Γ). Then there is a spectral sequence $\mathcal{E}_s^*$ that converges* *to $HF^*(L, \Lambda^2)$ whose second page is the Floer cohomology of the family of L_F over L_B. The latter is computed by a spectral sequence with second page*

$$\widetilde{\mathcal{E}}_2 = H^*(L_B, \mathcal{HF}(L_F, \Lambda[r])) \otimes \Lambda[q] \tag{1.1}$$

where the coefficients come from the system that assigns the module $HF(L_{F_p}, \Lambda[r])$ to each critical fiber.

In many cases, it is convenient to use the *disk potential* to compute the Floer cohomology of L. In case the fibers of E are Kähler with a G-invariant integrable complex structure, we use Donaldson's version of the Oka principle to trivialize the bundle u^*E for any holomorphic disk $u : (D, \partial D) \to (B, L_B)$. This allows us to produce J-holomorphic lifts of configurations into (E, L) that are vertically constant (and hence regular!). We use this lifting operation and a small perturbation of almost complex structures to write down a formula for the "second order" potential for L in terms of the leading order potential of L_B and the potential for L_F.

Let $e_v(u) = \int_C u^*a$, $e(\pi \circ u) = \int_C K\pi \circ u^*\omega_B$, and $\mathrm{Hol}_\rho(u)$ be the evaluation at a representation $\rho \in \mathrm{Hom}(\pi_1(L), \Lambda^{2\times})$ on the boundary of a configuration u. Define the 0^{th}-order A_∞ composition map as

$$\mu^0_{L,\rho} = \sum_{\substack{u \in \mathcal{M}(E,L,x)_0 \\ k=1,\ldots,m}} \pm(m!)^{-1}\mathrm{Hol}_\rho(u) q^{e(\pi\circ u)} r^{e_v(u)} x \tag{1.2}$$

where m is the number of marked points on the domain of u. Here we are assuming that the 0-dimensional moduli spaces $\mathcal{M}(E, L, x)_0$ of holomorphic configurations with one output are of expected dimension, which is a large technical obstruction that will be detailed in chapter 4. Let $x \in \mathrm{crit}(f) \subset CF(L, \Lambda^2)$ refer to a generating critical point of a Morse function f.

Definition 3. The *second order potential* for a symplectic Mori fibration is

$$\mathcal{W}^L_0(\rho) = \sum_{\substack{u \in \mathcal{I}_x \\ x \in \mathrm{crit}(f)}} \varepsilon(u)(m!)^{-1}\mathrm{Hol}_\rho(u_i) q^{e(\pi\circ u)} r^{e_v(u)} x$$

where for each x

$$\mathcal{I}_x = \left\{ u \in \mathcal{M}(E, L, x)_0 \,|e(\pi \circ u) = 0 \right\}$$
$$\cup \left\{ u \,|e(u) = \min_{v \in \mathcal{M}(E,L,x)_0} \{e(v) : e(\pi \circ v) \neq 0\} \right\}$$

The number m is the number of marked points on the domain of u_i, and $\varepsilon(u)$ is ± 1 depending on a choice of orientation on the moduli space.

The second order potential counts the holomorphic disks contained in a single fiber, or those with minimal total energy among the homology classes with non-zero base energy. For monotone L_F and large enough K in the weak coupling form, the second order potential becomes the terms of lowest and second lowest total degree in 1.2.

The utility of this definition is the following: We view it as a function $\mathcal{W}_0 : \mathrm{Hom}(\pi_1(L), \Lambda^2) \to CF(L, \Lambda^2)$ and attempt to find critical points. Once one has a non-degenerate critical point ρ, we show that it induces a critical point ξ of μ^0. Finally, we show that at a critical point ξ we have $H(L, \mu^1_\xi) \cong H^{\mathrm{Morse}}(L, \Lambda^2)$. The cohomology $H(L, \mu^1_\xi)$ is by definition the Floer cohomology of L.

To this end, let $W_0^{L_B}(\rho)$ be the terms of minimal energy appearing in $\mu^0_{L_B}$, and $i_{x_M *} : CF(L_F, \Lambda[r]) \to CF(L, \Lambda^2)$ the map induced from inclusion of the fiber above the unique maximum of the Morse function on L_B. In subsection 4.5, we will introduce a lifting operator

$$\mathcal{L} : \mathcal{M}(B, L_B, x)_0 \to \mathcal{M}(E, L, y)_0$$

that lifts unbroken holomorphic configurations in the base to regular holomorphic configurations in the total space with a chosen output y. One can pick y to be the unique maximum of the Morse function restricted to the fiber, and we denote this operator $\mathcal{L}^\wedge$. This operator induces a count of holomorphic disks $\underline{\mathcal{L}}^\wedge \circ W_0^{L_B}$ arising from the count that one does in the base, and induces a function

$$\underline{\mathcal{L}}^{\wedge} \circ W_0^{L_B} : Hom(\pi_1(L), \Lambda^2) \to CF(L, \Lambda^2)$$

Theorem 2. *Let E be a compact symplectic fibration with Kähler fibers, and let $(P_\Gamma)_\Gamma$ be an appropriate choice perturbation data for each configuration type Γ. Then the second order potential for (E, L) decomposes into a sum of a lifted leading order potential and the full potential for the fiber:*

$$\mathcal{W}_0^L(\rho) = \underline{\mathcal{L}}^{\wedge} \circ \mathcal{W}_0^{L_B}(\rho) + i_{x_M*} \circ \mu^0_{L_F, \iota^*\rho} \tag{1.3}$$

As a corollary 4, we show that if L_B has a well defined Floer cohomology (that is, $(\mu^1_{L_B})^2 = 0$ for an undeformed $\mu^1_{L_B}$), then so does L. The details of this theorem and corollaries are carried out precisely in section 5.2.

Before any of the results come about, sections 4.3 4.4 develops the usual transversality and compactness results for the moduli space of J-holomorphic disks. We use a system of domain dependent almost complex structures, as developed in Cieliebak-Mohnke [CM07] and Charest-Woodward [CWb; CWa], to overcome the multiple cover problem in achieving transversality in the base manifold. We summarize the technicalities for rational symplectic manifolds: In order to make use of domain dependent perturbation data on the space of k differentiable, p integrable maps from a disk into E, denoted $\mathrm{Map}(D, E, L)_{k,p}$, one needs the domain to be stable (no non-trivial automorphisms), since when defining the moduli of pseudo-holomorphic curves one identifies domains up to reparameterization. To stabilize our J-holomorphic domain configurations, we use the idea of a stabilizing divisor [CM07] (the existence attributed to [Don96]) that is typically Poincaré dual to some large multiple of the symplectic class. By requiring additional marked points on our configurations to map to the divisor, we obtain stable domains and can therefore use a more refined perturbation system.

The transversality and compactness results in the fibration setting $F \to E \to B$ requires us to balance the aforementioned technique for a rational (B, L_B) with the more classical results for a monotone (F, L_F). The main transversality result requires the use of an *upper triangular* perturbation system (with respect to a symplectic connection $TF \oplus H$) to show that the linearized Cauchy-Riemann operator is surjective in the particular case that a disk is constant along the fibers. One can then apply the classic density argument from [MS04] that uses the regularity for the ajoint of the linearized CR operator. The fact we are using domain-dependent perturbation data for B allows us to choose a section of $T_J\mathcal{J}$ that is only non-zero in a small neighborhood of some point p in the domain, thus bypassing the multiple cover problem inherent in the base manifold. For surjectivity in the fiber, we use the decomposition result for monotone manifolds due to Lazzarini [Laz10]. This removes the need to stabilize components that are horizontally constant, and allows us to use a single almost complex structure for each component that is contained in a fiber. Compactness in this situation is a similar combination of techniques from the rational and monotone cases: basically, we use the divisor in the base to rule out any unstable bubble components under the projection, and the classical type of regularization/dimension count to rule out vertical bubbles. The net result is that the only possibility for an unusual configuration in the limit is the formation of a stable disk component that does not break over critical points and is non-constant in the horizontal direction. Due to the assumption that the minimal Maslov index of L_F is 2, we do get the usual disk bubble connected to a constant disk that cancels in the differential due to the different orderings of the boundary markings.

In order to write down a spectral sequence, we use coefficients from Λ^2, with r appearing as $r^{\int_C u^*a}$ and q appearing as $q^{\int_C K\pi\circ u^*\omega_B}$. Filtering the complex $CF(L, \Lambda^2)$ with respect to q degree induces a spectral sequence similar to the one in [FOOO09] section 6.2. However, the result here is that the second page is the cohomology of the complex $CF(L, \Lambda_{\geq 0}[r])$ with respect to the differential d^0 that counts configurations with no q degree. Morally, the second page contains the Floer theory of the fiber Lagrangian along with the Morse theory of the base.

Once the usual technical results are out of the way, we get a Floer cohomology theory that accepts as input Lagrangian fibrations $L_F \to L \to L_B$ that is about as computable as the theory for the base and fiber. Using this, we find some Floer-non-trivial tori in certain classes of *minimal models*, e.g. $\mathbb{P}^1$ bundles over a Riemann surface; we compute some lower dimensional examples at the end of the paper. The implication of this is further reaching than one would expect, due to a program of Gonzalez-Woodward [GW; WC]. In their program, they use the minimal model program from algebraic geometry to produce Floer-non-trivial generators for the Fukaya category. The starting point is what some refer to as a *Mori* fibration, and at each stage of a *running* of the minimal model program, more generating Lagrangians are created that persist to the beginning of the running, i.e. the original space. Thus, finding Floer-non-trivial Lagrangians in a Mori fibration will (in nice cases) give Floer-non-trivial Lagrangians in the original space. Moreover, the end stage Mori fibration typically has Fano fiber.

In addition to the Mori surfaces exemplified at the end of this paper, the following example of *full flags* has been a toy model for this project.

1.1 Example: Full Flags in $\mathbb{C}^3$

We expose a 3-torus T in the three dimensional complex flag manifold that fibers over the Clifford torus in $\mathbb{P}^2$ that is Floer-non-trivial. It is conjectured that this is the same torus as in [NNU], but viewed from the perspective of our fibration machinery.

Consider the space of nested complex vector spaces $V_1 \subset V_2 \subset \mathbb{C}^3$. We can realize this as a symplectic fiber bundle $\mathbb{P}^1 \to \mathrm{Flag}(\mathbb{C}^3) \to \mathbb{P}^2$, with the both the base and fiber monotone. The type of Lagrangian that we are looking for is of the form $L_F \to L \to L_B$, where L_B and L_F are the so-called Clifford tori in $\mathbb{P}^n$. More generally, L_F is any smooth, simple, closed curve that divides the symplectic area of $\mathbb{P}^1$ into halves. By the Riemann mapping theorem, the Floer cohomology of L_F is isomorphic to that of any equator. Such an L constructed this way *should* be non-displacable, and we describe the construction after some preliminaries.

Holomorphic (but not symplectic) trivializations for Flag($\mathbb{C}^3$) can be realized as follows. Start with a chain of subspaces $V_1 \subset V_2 \subset \mathbb{C}^3$ with $V_1 \in \mathbb{P}^2$ represented as $[z_0, z_1, z_2]$ with $z_0 \neq 0$. Using the reduced row echelon form, there is a unique point in $\mathbb{P}(V_2)$ with first coordinate zero, $[0, w_1, w_2]$. On the open set U_0 of $\mathbb{P}^2$, we get a trivialization

$$\Psi_0 : \mathrm{Flag}(\mathbb{C}^3) \to U_0 \times \mathbb{P}^1$$

$$([z_0, z_1, z_2], V_2) \mapsto ([z_0, z_1, z_2], [w_1, w_2])$$

If $z_1 \neq 0$, then the transition map $U_0 \times \mathbb{P}^1 \to U_1 \times \mathbb{P}^1$ is given by

$$g_{01}([w_1, w_2]) = [-\frac{z_0 w_1}{z_1}, w_2 - \frac{z_2 w_1}{z_1}]$$

which is a well defined element

$$\begin{bmatrix} \frac{-z_0}{z_1} & 0 \\ \frac{-z_2}{z_1} & 1 \end{bmatrix}$$

in $PGL(2)$. A similar transition matrix works for the other trivializations.

Unfortunately, the above *algebraic* viewpoint does not contain any sort of symplectic structure. There is a natural symplectic form that we could use given by viewing Flag($\mathbb{C}^3$) as a coadjoint orbit $U(3)/T$ with

$$\omega_\xi(X, Y) = \xi([X, Y])$$

where X, Y are in $\mathfrak{su}(3)/\{\mathfrak{stab}(\xi)\}$ [Sil01].

Finding a fibered Lagrangian requires a careful argument based on results from Guillemin-Lerman-Sternberg [GLS96]. In $\mathbb{P}^n$, there is a distinguished *Clifford torus*, denoted Cliff($\mathbb{P}^n$) of the form

$$[z_0, \ldots, z_n] : \|z_i\| = \|z_j\| \,\forall i, j$$

that is also realized as the central moment fiber with regard to the action of T^n. It was demonstrated in [Cho04] that this is a monotone, Floer-nontrivial Lagrangian. In $\mathbb{P}^1$, this is simply an equator with respect to a Hamiltonian height function. The main idea is that we want to find a Lagrangian sub-bundle

$$\mathrm{Cliff}(\mathbb{P}^1) \to L \to \mathrm{Cliff}(\mathbb{P}^2)$$

for which we will be able to compute the Floer cohomology.

The relevant result that we will use gives a description of the moment map for a symplectic fibration over a Hamiltonian base manifold that will trivialize the fibration above $\mathrm{Cliff}(\mathbb{P}^2)$. Let $F \to E \to B$ be a symplectic fibration with a compact G-action where the projection is equivariant. Denote ψ as the moment map for the action of G on B. Let Δ be an open set of the moment polytope where the action is free. Given these assumptions, the discussion in [GLS96] section 4.6 leads to the following theorem:

Theorem 3. *[GLS96] Over $U = \psi^{-1}(\Delta)$, there is a symplectic connection Γ such that the moment map for the action on $\pi^{-1}(E)$ with the weak coupling form $a + \pi^*\omega_U$ is $\psi \circ \pi$*

See chapter 4 of [GLS96] for a proof.

In lieu of the ability to change the connection on an open set (see the G-equivariant versions of theorems 6 and 7), this new symplectic structure is not much different from (in fact, isotopic to) the weak coupling form associated the original fiber-wise structure.

We sketch the proof of this theorem, as well as how it ties into our example: The key component involves constructing a space E_W that is a symplectic fibration over the family of reduced spaces W, and one obtains a new symplectic connection (and associated weak coupling form) on $E|_U \to U$ by pulling back the connection from this new space. Moreover, the fibration $E_W \to W$ can be shown to induce a fibration of reduced spaces $(\psi \circ \pi)^{-1}(\alpha)/G \to \psi^{-1}(\alpha)/G$.

In our situation, we take $G = T^2$, $\psi : \mathbb{P}^2 \to \mathfrak{t}^\vee$ to be the associated moment map, and α as the barycenter of the moment polytope for $\mathbb{P}^2$. Thus, the modified connection on $E|_U \to U$ is trivial over $\psi^{-1}(\alpha)$ due to the fact that it is induced from $(\psi \circ \pi)^{-1}(\alpha)/G \to \{point\}$. Therefore, the fibration is symplectically trivial above Cliff$(\mathbb{P}^2)$ with the new connection.

1.1.1 Choosing a Lagrangian and computing Floer cohomology

Once we have the trivialization, we pick a trivial section of Cliff$(\mathbb{P}^1)$, so our Lagrangian is T^3. On the other hand, there may be more ways to pick a non-trivial S^1 bundle over Cliff$(\mathbb{P}^2)$; this is currently under investigation.

To start, we pick a Morse-Smale function on Cliff$(\mathbb{P}^2)$, such as the sum of two height functions $h_1 + h_2$. In the case that the Lagrangian we pick is trivially Cliff$(\mathbb{P}^2) \times$ Cliff$(\mathbb{P}^1)$, we can use the three-way sum of S^1 height functions $h_1 + h_2 + h_3$ as our Morse-Smale function. Alternatively, one can follow a standard recipe when the fibration is non-trivial: Choose a Morse-Smale function on each critical fiber $\pi^{-1}(x_i)$ and extend to the rest of the space using cutoff functions in local trivializations. Explicitly, let ϕ_i : Cliff$(\mathbb{P}^2) \to \mathbb{R}$ be a cutoff function that is 1 in a neighborhood of x_i and 0 outside of some local trivialization $U_i \ni x_i$, with the U_i disjoint. Pick an identification of each critical fiber $\pi^{-1}(x_i)$ with S^1, a height function $g : S^1 \to \mathbb{R}$, and form

$$f(p) = h_1 \circ \pi(p) + h_2 \circ \pi(p) + \sum_{i=0}^{3} \phi_i \circ \pi(p) g(\theta)$$

We will assume that we can perturb this function in a neighborhood near each critical point to make it Morse-Smale and not change the individual critical points.

We use the 2nd order potential from theorem 20 to show that this Lagrangian is Floer-non-trivial. First, we compute the first order potential for $\mathbb{P}^2$: Let the stabilizing divisor be $\cup_{i=1}^{3} \Phi^{-1}(\partial P_i)$ the inverse image of the faces of the moment polytope. For the standard integrable complex structure J_B, take the holomorphic disks

$$u_1(z) = [z, 1, 1],\ u_2(z) = [1, z, 1],\ u_3(z) = [1, 1, z]$$

and suppose that $-h_1 - h_2$ takes its maximum at $x_0 = [1, 1, 1]$. By the classification of disks with boundary in $\mathrm{Cliff}(\mathbb{P}^2)$ and the fact that the integrable complex structure is regular [Cho04], these are the only Maslov index 2 disks (up to reparameterization) through $[1, 1, 1]$. By monotonicity and an index count, x_0 is the only term showing up in $\mu^0_{\mathrm{Cliff}(\mathbb{P}^2)}$. Thus, the leading order potential for the base is

$$\mathcal{W}_0^{\mathrm{Cliff}}(\mathbb{P}^2)(y_1, y_2) = \left(y_1 q^\rho + y_2 q^\rho + \frac{1}{y_1 y_2} q^\rho \right) x_0$$

where $\rho = K \int_D u_i^* \omega_{FS}$. Let x be the unique maximum of f (in the fiber above x_0). The lifted leading order potential 45 is

$$\mathcal{L} \circ \mathcal{W}_0^{\mathrm{Cliff}(\mathbb{P}^2)}(y_1, y_2) = \left(y_1 y_3^k q^\rho r^{e_v(\mathcal{L}u_1)} + y_2 y_3^l q^\rho r^{e_v(\mathcal{L}u_2)} + \frac{1}{y_1 y_2 y_3^j} q^\rho r^{e_v(\mathcal{L}u_3)} \right) x$$

for some integers k, l, and j. The inclusion of the potential for $\mathrm{Cliff}(\mathbb{P}^1)$ in the fiber above x_0 is

$$i_{x_0 *} \circ \mathcal{W}^{\mathrm{Cliff}(\mathbb{P}^1)}(y_3) = \left(y_3 r^{e_v(v_1)} + y_3^{-1} r^{e_v(v_2)} \right) x \tag{1.4}$$

where $v_1(z) = [z, 1]$ resp. $v_2(z) = [1, z]$. Thus, the second order potential for the total Lagrangian is

$$\begin{aligned} \mathcal{W}_0^L(y_1, y_2, y_3) = {} & y_3 r^{e_v(v_1)} + y_3^{-1} r^{e_v(v_2)} \\ & + y_1 y_3^k q^\rho r^{e_v(\mathcal{L}u_1)} + y_2 y_3^l q^\rho r^{e_v(\mathcal{L}u_2)} + \frac{1}{y_1 y_2 y_3^j} q^\rho r^{e_v(\mathcal{L}u_3)} \end{aligned}$$

Let us change basis of $\mathrm{Hom}(\pi_1(L), \Lambda^{2,\times})$ so that the second order potential takes the form

$$\begin{aligned} \mathcal{W}_0^L(y_1, y_2, y_3) = {} & y_3 r^{e_v(v_1)} + y_3^{-1} r^{e_v(v_2)} \\ & + y_1 q^\rho r^{e_v(\mathcal{L}u_1)} + y_2 q^\rho r^{e_v(\mathcal{L}u_2)} + \frac{1}{y_1 y_2 y_3^m} q^\rho r^{e_v(\mathcal{L}u_3)} \end{aligned}$$

for some $m \in \mathbb{Z}$. Since the first part of this expression is symmetric in y_3, we can assume that $m \geq 0$ by a further change of basis.

For the partial derivatives of $\mathcal{W}_0^L$, we have

$$\partial_{y_1}\mathcal{W}_0^L = q^\rho r^{e_v(\mathcal{L}u_1)} - \frac{1}{y_1^2 y_2 y_3^m} q^\rho r^{e_v(\mathcal{L}u_3)} \tag{1.5}$$

$$\partial_{y_2}\mathcal{W}_0^L = q^\rho r^{e_v(\mathcal{L}u_2)} - \frac{1}{y_1 y_2^2 y_3^m} q^\rho r^{e_v(\mathcal{L}u_3)} \tag{1.6}$$

$$\partial_{y_3}\mathcal{W}_0^L = r^{e_v(v_1)} - y_3^{-2} r^{e_v(v_2)} - \frac{m}{y_1 y_2 y_3^{m+1}} q^\rho r^{e_v(\mathcal{L}u_3)} \tag{1.7}$$

The S_3 action on $\mathbb{C}^3$ gives an action on $\mathrm{Flag}(\mathbb{C}^3)$ that permutes the relative homology classes of the lifted disks $\mathcal{L}u_i$. If we assume that the form a is invariant under this action, then we have that $e_v(\mathcal{L}u_1) = e_v(\mathcal{L}u_2) = e_v(\mathcal{L}u_3)$. Hence, setting expressions 1.5 and 1.6 equal to 0 gives three convenient solutions $y_1 = y_2 = y_3^{m/3}$. It remains to solve 1.7, which we sketch. Making the substitutions for y_1, y_2, and setting equal to zero gives us

$$r^{e_v(v_1)} - y_3^{-2} r^{e_v(v_2)} - \frac{m}{y_3^{5m/3+1}} q^\rho r^{e_v(\mathcal{L}u_3)} = 0$$

We normalize via the transformation $y_3 \mapsto y_3^3$

$$r^{e_v(v_1)} - y_3^{-2} r^{e_v(v_2)} - \frac{m}{y_3^{5m+3}} q^\rho r^{e_v(\mathcal{L}u_3)} = 0$$

and clear the denominator

$$y_3^{5m+3} r^{e_v(v_1)} - y_3^{5m+1} r^{e_v(v_2)} - mq^\rho r^{e_v(\mathcal{L}u_3)} = 0$$

Let $\eta = e_v(\mathcal{L}u_3) - e_v(v_2)$. Dividing by the appropriate power of r gives:

$$y^{5m+3} - y_3^{5m+1} - mq^\rho r^\eta = 0 \tag{1.8}$$

While the power of r is negative, we can pick K large enough in the weak coupling form so that $q^\rho r^\eta$ is in the ring Λ^2.

The ring Λ_t becomes a Λ^2-algebra via the homomorphism $q, r \mapsto t$, so let us instead solve the equation

$$y_3^{5m+3} - y_3^{5m+1} - mt^\alpha = 0 \tag{1.9}$$

with $\alpha > 0$.

Equation 1.9 has two unital solutions in the universal Novikov ring Λ_t. Indeed, the reduction mod t has 1 and -1 as solutions. By Hensel's lemma, there are unique solutions $\mathfrak{y}_1 \equiv 1\,(mod\, t)$ and $\mathfrak{y}_2 \equiv -1\,(mod\, t)$ in Λ_t.

By proposition 21, there is a representation $\eta \in \mathrm{Hom}(\pi_1(L), \Lambda_t^\times)$ such that

$$HF(L, \Lambda_t, \eta) \cong H^*(L, \Lambda_t)$$

which says that this Lagrangian is non-displacable via Hamiltonian isotopy and recovers the result from [NNU].

Remark 1. As suggested by Marco Castronovo, one can take the base Lagrangian as one of Vianna's exotic tori [Viab; Viaa]. The computation of the potential for an exotic torus is done in [Viab]. Using this, we believe that the construction and Floer-non-triviality of a Lagrangian which fibers over an exotic torus should be similar to the Clifford case.

1.2 Outline

The paper is divided into six sections and an appendix. In section 2, we follow the literature to lay the necessary groundwork to discuss symplectic fiber bundles.

In section 3, we give a review of Floer theory for *rational* symplectic manifolds, as developed in [CWa; CM07].

In section 4, prove the transversality and compactness results fibered setting.

In section 5, we define Floer theory for a symplectic Mori fibration.

Section 6 is devoted to explicit examples in the case of ruled surfaces.

The appendix is background taken from [CWa].

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Chapter 2

Background: Symplectic fibrations

We would like to unravel what we mean by the definition of a *symplectic Mori fibration* 2, and state some results pertaining to our situation. The idea is that we require the transition maps of our fiber bundle to be symplectomorphisms of the fibers. Then, for any global form a such that $\iota^* a = \omega_F$, we get a connection by taking the symplectic complement $H = TF^{\perp_a}$. If B is also symplectic then the form $a + K\pi^*\omega_B$ is non-degenerate for large K. The main obstruction in this setup is finding such an a that is *closed*. We elaborate.

Following [MS98] chapter 6, we start with fiber bundle with connected total space E with a compact symplectic base (B, ω_B) and fiber (F, ω_F). A *symplectic fibration* is such a space E where the transition maps are symplectomorphisms of the fibers:

$$\Phi_i : \pi^{-1}(U_i) \to U_i \times F$$

$$\Phi_j \circ \Phi_i^{-1} : U_i \cap U_j \times F \to U_i \cap U_j \times F$$

$$(p, q) \mapsto (p, \phi_{ji}(q))$$

where $\phi_{ji} : U_i \cap U_j \to \mathrm{Symp}(F, \omega_F)$ are Čech co-cycles.

Assume that there is a class $[b] \in H^2(E)$ such that $\iota^*[b] = [\omega_{F_p}]$ for all p. Then for large K, a theorem of Thurston (Theorem 6.3 in [MS98]) tells us that there is a symplectic form ω_K on E that represents the class $[b + K\pi^*\omega_B]$ and is compatible with the fibration structure.

Theorem 4 (Thurston, theorem 6.3 from [MS98]). *Let $(F, \omega_F) \to E \to (B, \omega_B)$ be a compact symplectic fibration with connected base. Let ω_{F_p} be the canonical symplectic form on the fiber*

F_p and suppose that there is a class $b \in H^2(E)$ such that

$$\iota_p^* b = [\omega_{F_p}]$$

for some (and hence every) $p \in B$. Then, for every sufficiently large real number $K > 0$, there exists a symplectic form $\omega_K \in \wedge^2(T^\vee E)$ that makes each fiber into a symplectic submanifold and represents the class $b + K[\pi^*\omega_B]$

The existence of the class b is not, a priori, easy. However, one can assume that F is a surface of genus $g \neq 1$.

Lemma 1 (lemma 6.6 from [MS98])**.** *Let $(F, \omega_F) \to E \to (B, \omega_B)$ be a compact symplectic fibration such that the first Chern class $c_1(TF) = \lambda[\omega_F]$ for $\lambda \neq 0$. Then the class $\lambda^{-1}c_1(TM)$ pulls back to $[\omega_F]$*

One then applies Thurston's theorem to get a symplectic representative of $\lambda^{-1}c_1(TM) + K[\pi^*\omega_B]$. Thus, if F is a Riemann surface but not a torus, then E has compatible structure.

From here on, let us denote the two-form representative of the class b from theorem 4 by a. Given that (F_p, a) is fiberwise symplectic, we get a well defined connection by taking the symplectic complement of TF, denoted $H = TF^{\perp_a}$. We will call a connection arising in this way a *symplectic connection*, or equivalently a connection whose parallel transport maps are symplectomorphisms on the fibers. While there may be many (closed) such a that define the same connection H, Guillemin-Lerman-Sternberg [GLS96] and McDuff-Salamon [MS98] give a construction that uses the (assumed) Hamiltonian action of parallel transport.

Theorem 5 ([GLS96],theorem 6.21 from [MS98])**.** *Let H be a symplectic connection on a fibration $F \to E \to B$ with $\dim F = n$. The following are equivalent:*

1. *The holonomy around any contractible loop in B is Hamiltonian.*

2. *There is a unique closed connection form ω_H on E with $i^*\omega_H = \omega_F$ and*

$$\int_F \omega_H^{(n+2)/2} = 0$$

where $\int_F$ is the map from TB that lifts $v_1 \wedge v_2$ and integrates $\iota_{v_1 \wedge v_2} \omega_H^{2n+2}$ over the fiber. The form ω_H is called the minimal coupling form of the symplectic connection H. Any (symplectic) form $\omega_H + K\pi^*\omega_B$ is called a weak coupling form.

The idea is that ω_H is already determined on vertical and verti-zontal components, so it remains to describe it on horizontal components. This is done assigning the value of the zero-average Hamiltonian corresponding to $[v_1^\sharp, v_2^\sharp] - [v_1, v_2]^\sharp$, where the $v_i^\sharp$ are horizontal lifts of base vectors v_i.

One might then ask: if we have two connection forms ω_{H_1} and ω_{H_2}, how are the symplectic forms $\omega_{H_1} + K\pi^*\omega_B$ and $\omega_{H_1} + K\pi^*\omega_B$ related. We have the following result.

Theorem 6 (theorem 1.6.3 from [GLS96])**.** *For two symplectic connections H_i, $i = 1, 2$, the corresponding forms $\omega_{H_i} + K\pi^*\omega_B$ are isotopic for large enough K.*

The hard part is actually finding a Lagrangian in the form $L_F \to L \to L_B$. If we can find such an L, it is not guaranteed to be Lagrangian due to small contributions from the horizontal part of ω_H. However, it seems feasible that we could alter the connection in a neighborhood of L to make it Lagrangian. Precisely, we have

Theorem 7 (theorem 4.6.2 from [GLS96])**.** *Let $A \subset B$ be a compact set, $A \subset U$ an open neighborhood, and H' a symplectic connection for $\pi^{-1}(U)$. Then there is an open subset $A \subset U' \subset U$ and connection H on E such that $H = H'$ over U'.*

In light of theorem 6, nothing is lost if we modify the connection on our candidate Lagrangian and then extend it using theorem 7.

Methods to construct a submanifold $L \subset E$ of the form $L_F \to L \to L_B$ seem to be dependent on the situation. In the case when the ambient base manifold is dimension 2, we do not need to worry about horizontal contributions to ω_H so long as the candidate Lagrangian is parallel to the connection. In particular, we detail some examples of ruled complex surfaces in a later section of this paper.

Chapter 3

Background: Perturbation in rational symplectic manifolds

3.1 Moduli space of treed stable disks

In this section we include the background and results of Charest-Woodward [CWa]. They prove transversality and compactness for *rational*, non-fibered symplectic manifolds and Lagrangians [CWb; CWa], following the techniques of [CM07]. This section is included for completeness and will be adapted for our use in later sections.

A fundamental problem in defining and Floer theory lies in making the right choices of perturbation data to resolve the problems of transversality and compactness. There are a number of methods, including the polyfolds approach and the method of Kuranishi structures. The author chose to use a more geometric approach developed in [CM07; CWb; CWa]. The main idea is to use the existence of a symplectic almost complex divisor that represents the Poincaré dual of (a large multiple) of the symplectic class [Don96] in order to stabilize domains and allow the use of domain dependent almost complex structures. We consider Morse-Floer trees that are stabilized by extra marked points that map to the divisor. We then show that we can choose an appropriate system of perturbation data that regularizes any reasonable configuration, including those with sphere or disk "bubbles". This regularization of bubble configurations allows us to then proof appropriate compactness results (which in turn rules out sphere bubbling).

A *tree* is a planar graph $\Gamma = (\mathrm{Edge}(\Gamma), \mathrm{Vert}(\Gamma))$ with no cycles, that can be decomposed as follows:

1. For nonempty Vert(Γ), Edge(Γ) consists of

    (a) *finite edges* $\mathrm{Edge}_{<\infty}(\Gamma)$ connecting two vertices,

    (b) *semi-infinite edges* Edge_∞ with a single endpoint, or

2. if Vert(Γ) is empty, then Γ has one *infinite edge* and let Edge_∞ denote its two ends.

From $\mathrm{Edge}_\infty(\Gamma)$ we can distinguish one open endpoint as the *root* or the tree, and the other semi-infinite edges being referred to as the *leaves*.

A *metric tree* is a tree with an assignment of length to each finite edge, denoted l : $\mathrm{Edge}_{<\infty}(\Gamma) \to [0, \infty]$. If a finite edge has infinite length, we call that edge broken, and thus we have a *broken metric tree*. We think of this as two metric trees, where the first has a leaf with extremal point ∞_1, that is glued to the extremal point ∞_2 of the root of the second. Finally, a broken metric tree is *stable* if the valence of each vertex is at least 3.

Definition 4. Let D be a collection of disk and sphere domains equipped with an identification to $D \subset \mathbb{C}$ resp. $\mathbb{P}^1$, and a distinguished set of boundary marked points $\{x_i\}$ on the disk boundaries resp. interior marked points $\{z_i\}$ on the disk interiors and sphere components. A *nodal n-marked disk* is the collection of domains together with identifications among the $\{x_i\}$ resp. among the $\{z_i\}$ such that at most two points are identified and the resulting topological space is simply connected. The *boundary* resp. *interior nodes* are the points where two boundary resp. two interior markings are identified. We equip the unidentified boundary markings $\{x_0, \ldots, x_n\}$ with a counter-clockwise cyclic order around the boundary starting with a distinguished root x_0, and the unidentified interior markings with an order $z_1, \ldots, z_n$. A *special point* is a node or marking. A nodal marked disk is *stable* if each sphere component has at least 3 special points, and if each disk component has least three boundary special points or at least one interior special point and one boundary special point.

Denote a nodal n-marked disk as a triple $(S, \underline{x}, \underline{z})$, where S is the surface component together with the nodes, and $\underline{x}$ resp. $\underline{z}$ are the boundary markings resp. the interior markings.

Definition 5. A *treed disk* C is a triple (T, D, o) consisting of

1. a broken metric tree $T = (\Gamma, l)$,
2. a collection $D = (S_v, \underline{x}_v, \underline{z}_v)_{v \in \mathrm{Vert}(\Gamma)}$ of stable marked nodal disks for each vertex v of T, with the number of boundary markings $\underline{x}_v$ equal to the valence of v, and
3. an ordering o of the set of interior markings $\cup_v \underline{z}_v \in \mathrm{int}(D)$, so that we may denote the interior markings $z_1, \ldots, z_m$.

We will be studying J-holomorphic maps from a geometric realization of C, given by replacing the vertices with their corresponding nodal disks by attaching the boundary markings $\underline{x}_v$ to the appropriate edges at v. A treed disk is *stable* if and only if each nodal disk is stable and each vertex in T has valence at least three.

For a nodal marked disk D, there is an associated metric tree $\Gamma(D)$ constructed as follows: Transform each disk/sphere component to a vertex and each node to a length zero edge between the corresponding vertices. Further, replace each boundary marking and interior marking with a semi-infinite edge.

The *combinatorial type* of a treed disk $C = (T, D, o)$ includes the type of tree Γ obtained by gluing $\Gamma(D_v)$ into T such that the root of D_v is closest to the root of T, as well as a labeling of

1. the set of edges $\mathrm{Edge}_{<\infty}(\Gamma)$ of length 0 or ∞, and
2. the set of $\mathrm{Edge}_{<\infty}(\Gamma)$ with finite non-zero length.

The vertices partition into the set

$$\mathrm{Vert}(\Gamma) = \mathrm{Vert}_d(\Gamma) \sqcup \mathrm{Vert}_s(\Gamma)$$

that represent disk resp. sphere domains, and the edges decompose as follows:

$$\begin{aligned}\mathrm{Edge}(\Gamma) = &\mathrm{Edge}_{<\infty,s}(\Gamma) \sqcup \mathrm{Edge}_{<\infty,d}(\Gamma) \sqcup \mathrm{Edge}_{\infty,s} \\ &\sqcup \mathrm{Edge}^{0}_{<\infty}(\Gamma) \sqcup \mathrm{Edge}^{\infty}_{<\infty}(\Gamma) \sqcup \mathrm{Edge}^{(0,\infty)}_{<\infty}(\Gamma) \sqcup \mathrm{Edge}_{\infty}(\Gamma)\end{aligned}$$

that are the spherical nodes, boundary nodes, interior markings, finite edges with zero, infinite, and finite non-zero length, as well as semi-infinite edges.

Moreover, the combinatorial type includes a label of homology classes on the vertices: $[v] \in H_2(X, L)$ if $v \in \mathrm{Vert}_d(\Gamma)$ and $[v] \in H_2(X)$ if $v \in \mathrm{Vert}_s(\Gamma)$.

For each $e \in \mathrm{Edge}_{\infty,s} \cup \mathrm{Edge}_{<\infty,s}(\Gamma)$, let $m(e) \in \mathbb{Z}_{\geq 0}$ be a non-negative integer labeling. This will represent the intersection multiplicity with the divisor.

We encode this data into a moduli space of stable treed disks $\mathfrak{M}^{n,m}$, where n is the number of semi-infinite edges and m the number of interior markings. The connected components of this moduli space can be realized as a product of Stasheff's associahedra, and thus it is a cell complex.

One can stratify $\mathfrak{M}^{n,m}$ by combinatorial type, i.e., for each stable combinatorial type Γ, let $\mathfrak{M}_\Gamma$ be the subset of treed disks of type Γ endowed with the subspace topology. We have a universal treed disk of type $\mathcal{U}_\Gamma \to \mathfrak{M}_\Gamma$ that consists of points (C_m, m), where m is of type Γ and C_m is its geometric realization. $\mathcal{U}_\Gamma$ has the structure of a fiber bundle.

We can view a universal treed disk as a union of two sets: $S_\Gamma \cup T_\Gamma$. The former being the two dimensional part (with boundary) of each fiber, and the later being the one dimensional part. $S_\Gamma \cap T_\Gamma$ is the set of nodes, interior markings, and boundary markings. Given a treed disk C, we can identify nearby disks with C using a local trivialization. This gives us a map for each chart

$$\mathfrak{M}^i_\Gamma \to \mathcal{J}(C) \tag{3.1}$$

where $\mathcal{J}(C)$ are holomorphic structures on the surface part of C.

Definition 6. (Behrend-Manin morphisms of graphs) A *morphism* of graphs $\Upsilon : \Gamma \to \Gamma'$ is a surjective morphism on the set of vertices obtained by combining the following elementary morphisms:

(a) (Cutting edges) Υ *cuts an edge* $e \in \mathrm{Edge}_{<\infty}(\Gamma)$ with infinite length resp. an edge $e \in \mathrm{Edge}_{\infty,s}(\Gamma)$ (spherical node) if the map on vertices is a bijection and

$$\mathrm{Edge}(\Gamma') \cong \mathrm{Edge}(\Gamma) - \{e\} + \{e_+, e_-\}$$

where $e_\pm \in \mathrm{Edge}_\infty(\Gamma')$ are attached to the vertices contained in e. We view Γ' as two disconnected graphs Γ_+, Γ_-.

(b) (Collapsing edges) Υ *collapses an edge* if the map on vertices $\mathrm{Vert}(\Upsilon) : \mathrm{Vert}(\Gamma) \to \mathrm{Vert}(\Gamma')$ is a bijection except the indentification of two vertices in $\mathrm{Vert}(\Gamma)$ that are joined by an edge in $e \in \mathrm{Edge}^0_{<\infty}(\Gamma)$ and

$$\mathrm{Edge}(\Gamma) \cong \mathrm{Edge}(\Gamma') - \{e\}$$

(c) (Making an edge length finite or non-zero) Υ *makes an edge finite or non-zero* if Γ has the same tree as Γ' and the lengths of the edges $\ell(e)$ for $e \in \mathrm{Edge}_{<\infty}(\Gamma')$ are the same except for a single edge e where $\ell(e) = \infty$ resp. 0 and the length $\ell'(e)$ in Γ' is in $(0, \infty)$.

(d) (Forgetting tails)Υ forgets a semi-infinite edge and collapses edges to make the resulting combinatorial type stable. The ordering on $\mathrm{Edge}_{\infty,s}(\Gamma)$ naturally defines one on $\mathrm{Edge}_{\infty,s}(\Gamma')$.

Each of the above operations on graphs corresponds to a map of moduli spaces of stable marked treed disks.

Definition 7. (Morphisms of moduli spaces)

(a) (Cutting edges) Suppose that Γ' is obtained from Γ by cutting an edge of infinite length. There are diffeomorphisms $\overline{\mathfrak{M}}_\Gamma \to \overline{\mathfrak{M}}_{\Gamma'}$ obtained by identifying the two endpoints corresponding to the cut edge and choosing the ordering of the interior markings to be that of Γ.

(b) (Collapsing edges) Suppose that Γ' is obtained from Γ by collapsing an edge. There is an embedding $\overline{\mathfrak{M}}_\Gamma \to \overline{\mathfrak{M}}_{\Gamma'}$ whose image is a 1-codimensional corner of $\overline{\mathfrak{M}}_{\Gamma'}$

(c) (Making an edge finite or non-zero) If Γ' is obtained from Γ by making an edge finite resp. non-zero, then $\overline{\mathfrak{M}}_\Gamma$ embeds in $\overline{\mathfrak{M}}_{\Gamma'}$ as the 1-codimensional corner where e reaches infinite resp. zero length, with trivial normal bundle.

(d) (Forgetting tails) Suppose that Γ' is obtained from Γ by forgetting i-th tail, either in $\mathrm{Edge}_{\infty,s}(\Gamma)$ or $\mathrm{Edge}_\infty(\Gamma)$. Forgetting the i-th marking and collapsing the unstable components and their distance to the stable components (if any) defines a map $\overline{\mathfrak{M}}_\Gamma \to \overline{\mathfrak{M}}_{\Gamma'}$.

We note that all of these maps extend to smooth maps of the corresponding universal treed disks. If Γ is disconnected, say the disjoint union of Γ_1 and Γ_2, then the universal disk is the disjoint union of the pullbacks of the universal disks $\overline{\mathcal{U}}_{\Gamma_1}$ and $\overline{\mathcal{U}}_{\Gamma_2}$.

Orientations can be put on the space of treed disks as follows:

(a) (For a single disk) For $m \geq 1$, we can identify any point in the open stratum of $\overline{\mathfrak{M}}_{n,m}$ with the half space $\mathbb{H} \subset \mathbb{C}$. To be consistent, say we map the root x_0 to ∞, an interior marking z_1 to i and the boundary markings x_i to an $n-1$-tuple of $\mathbb{R} \subset \mathbb{C}$. We then use standard orientations on these spaces.

If $m = 0$, send x_0 to ∞, x_1 to 0, x_2 to 1, and the remaining boundary markings to an ordered tuple of $(1, \infty) \subset \mathbb{R} \subset C$.

(b) (Treed disks with multiple disk components) Given a treed disk in $\overline{\mathcal{U}}_{m,n}$ with an edge of zero length, we can realize it as being in the closure of a higher dimensional stratum by identifying the edge with a node. To obtain an edge of finite non-zero length, we use part (c) from the definition above. That is, the 1-codimensional corner where we have an edge of zero length is also realized as the boundary of the higher dimensional stratum where that edge has finite and non-zero length. Choose orientations on the top dimensional strata that induce the opposite orientations on the aforementioned 1-codimensional corners.

3.2 Treed holomorphic disks

Now that we have the notion of a treed disk, we can begin constructing the moduli of Floer trajectories. The vertices will represent domains for J-holomorphic maps (with boundary in a Lagrangian) while the edge parts will represent flow lines for a domain dependent Morse function.

Fix a metric G on L that extends to a metric on X for which L is totally geodesic. Pick a Morse-Smale function F on L that has a unique maximum x_M. The gradient flow equation is the following initial value problem:

$$\frac{\mathrm{d}\phi_p(t)}{dt} = -\mathrm{grad}_{\phi_p(t)}(F)$$

$$\phi_p(0) = p$$

where X is the gradient vector field of F with respect to g, so the image defines a unique set of points $\phi_p(-\infty, \infty)$ that doesn't depend on $p \in \phi_p(-\infty, \infty)$. Denote the critical set as $\mathcal{I}(L)$, and for $x \in \mathcal{I}(L)$ denote the stable and unstable manifolds of x as

$$W_F^{\pm}(x) = \left\{ \phi_p(t) : \lim_{t \to \mp\infty} \phi_p(t) = x \right\}$$

The difference in sign is a convention that we will follow as though we are considering the stable/unstable manifolds of F, while the gradient flow is morally that of $-F$. The requirement that F is Morse-Smale guarantees that all of these submanifolds intersect transversely, and thus have smooth intersections. The index $I(x)$ is defined to be the dimension of $W_F^-(x) = W_{-F}^+(x)$

An almost complex structure for a X is a fiber-preserving linear map $J : TX \to TX$ such that $J^2 = -I$. J is tamed with respect to ω if $\omega(\cdot, J\cdot)$ is positive definite. Let $\mathcal{J}_\tau(X)$ denote the space of tamed almost complex structures.

Given a disk or sphere domain S equipped with a complex structure j and a taming almost complex structure J on X, the J-holomorphic curve equation for a map $u : S \to X$ is

$$J \circ du = du \circ j$$

that will have boundary in a Lagrangian in the case S is a disk. Achieving transversality for solutions to the gradient flow resp. holomorphic curve equation will involve Morse functions resp. almost complex structures that depend on the domain of the solution. First, we fix sets in the domain on which the perturbation will be non-constant. Let $\overline{\mathcal{S}}_\Gamma \subset \overline{\mathcal{U}}_\Gamma$ be the two-dimensional part of the universal treed disk, and $\overline{\mathcal{T}}_\Gamma \subset \overline{\mathcal{U}}_\Gamma$ be the tree part of the universal treed disk. Fix a compact set

$$\overline{\mathcal{S}}^o_\Gamma \subset \overline{\mathcal{S}}_\Gamma$$

not containing the boundary, nodes, or interior markings, but having open interior in every fiber of the universal disk $\mathcal{U}_\Gamma$. Also fix a compact set

$$\overline{\mathcal{T}}^o_\Gamma \subset \overline{\mathcal{T}}_\Gamma$$

that intersects each universal fiber. Thus, the compliments

$$\overline{\mathcal{S}}_\Gamma - \overline{\mathcal{S}}^o_\Gamma$$

$$\overline{\mathcal{T}}_\Gamma - \overline{\mathcal{T}}^o_\Gamma$$

are neighborhoods of the boundary, interior markings, and nodes resp. neighborhoods of ∞ in each fiber of the universal disk.

Definition 8. (a) (Domain-dependent Morse functions) Let (F, g) be a Morse-Smale pair, and $l > 0$ an integer. A *domain-dependent perturbation* for F of class C^l for type Γ is a

C^l map

$$F_\Gamma : \overline{\mathcal{T}}_\Gamma \times L \to \mathbb{R}$$

equal to F on $\overline{\mathcal{T}}_\Gamma - \overline{\mathcal{T}}^o_\Gamma$.

(b) (Domain-dependent almost complex structure) Let $J \in \mathcal{J}_\tau(X)$ an $l > 0$ an integer. A *domain-dependent almost perturbation for J* of class C^l for type Γ is a C^l class map

$$J_\Gamma : \overline{\mathcal{S}}_\Gamma \to \mathcal{J}_\tau(X)$$

that is equal to J on $\overline{\mathcal{S}}_\Gamma - \overline{\mathcal{S}}^o_\Gamma$.

Definition 9. [CWa](Perturbation Data) A *C^l-perturbation datum for type* Γ is a C^l pair $P_\Gamma = (F_\Gamma, J_\Gamma)$ consisting of a domain-dependent C^l Morse function F_Γ for type Γ and a domain-dependent C^l almost complex structure J_Γ for type Γ. A *perturbation datum* is a family $(P_\Gamma)_{\Gamma \in \gamma}$ for types $\Gamma \in \gamma$.

It will be important to choose a perturbation datum that is compatible with operations on treed disks:

Definition 10. (a) (Cutting edges) Suppose that Γ is a combinatorial type and Γ' is obtained by cutting an edge of infinite length. A perturbation datum on Γ' gives ruse to a perturbation datum for Γ by pushing forward P'_Γ under the map $\overline{\mathcal{U}}'_\Gamma \to \overline{\mathcal{U}}_\Gamma$

(b) (Collapsing edges/making an edge finite or non-zero) Suppose that Γ' is obtained from Γ by collapsing an edge or making an edge finite or non-zero. Any perturbation datum P'_Γ for Γ' induces a datum for Γ by pullback of P'_Γ under $\overline{\mathcal{U}}'_\Gamma \to \overline{\mathcal{U}}_\Gamma$.

(c) (Forgetting tails) Suppose that Γ' is a combinatorial type of stable treed disk obtained from Γ by forgetting a marking. In this case there is a map of universal disks $\overline{\mathcal{U}}_\Gamma \to \overline{\mathcal{U}}'_\Gamma$ given by forgetting the marking and stabilizing. Any perturbation datum P'_Γ induces a datum P_Γ by pullback of P'_Γ

Thus, it makes sense to define a perturbation datum that is compatible with the morphisms on graphs and moduli spaces. We will call this property *coherence*:

Definition 11. A perturbation datum $\underline{P} = (P_\Gamma)$ is *coherent* if it is compatible with the morphisms between different $\mathfrak{M}_\Gamma$ in the sense that

(a) (Cutting edges axiom) If Γ is obtained from Γ' by cutting an edge of infinite length, then $P_{\Gamma'}$ is the pushforward of P_Γ.

(b) (Collapsing edges/making an edge finite or non-zero axiom) If Γ is obtained from Γ' by collapsing an edge or making an edge finite or non-zero, then $P_{\Gamma'}$ is the pullback P_Γ.

(c) (Product axiom) If Γ is the union of types Γ_1, Γ_2 obtained from cutting an edge of Γ', then P_Γ is obtained from P_{Γ_1} and P_{Γ_2} as follows: Let $\pi_k : \overline{\mathfrak{M}}_\Gamma \cong \overline{\mathfrak{M}}_{\Gamma_1} \times \overline{\mathfrak{M}}_{\Gamma_2} \to \overline{\mathfrak{M}}_{\Gamma_k}$ denote the projection onto the kth factor, so that $\overline{\mathcal{U}}_\Gamma$ is the unions of $\pi_1^* \overline{\mathcal{U}}_{\Gamma_1}$ and $\pi_2^* \overline{\mathcal{U}}_{\Gamma_2}$. Then we require that P_Γ is equal to the pullback of P_{Γ_k} on $\pi_k^* \overline{\mathcal{U}}_{\Gamma_k}$

(d) (Ghost-marking independence) If Γ' is obtained from Γ by forgetting markings on components corresponding to vertices with $[v] = 0$, then J_Γ is the pullback of $J_{\Gamma'}$.

Definition 12. Given a perturbation datum P_Γ, a *holomorphic treed disk* in X with boundary in L consists of a treed disk $C = S \cup T$ and a continuous map $u : C \to X$ such that

(a) (Boundary condition) $u(\partial S \cup T) \subset L$.

(b) (Surface equation) On the surface part of S of C the map u is J-holomorphic for the given domain-dependent almost complex structure: if j denotes the complex structure on S, then

$$J_{\Gamma,z} \mathrm{d}u|_S = \mathrm{d}u|_s j.$$

(c) (Tree equation) On the tree part $T \subset C$ the map u is a collection of gradient trajectories:

$$\frac{d}{ds} u|_T = -\mathrm{grad}_{F_{\Gamma,s}}(u|_T)$$

where s is a local coordinate with unit speed so that for every $e \in \mathrm{Edge}_{<\infty}(\Gamma)$, we have $e \cong [0, \ell(e)]$ or $e \cong [0, \infty)$ via s.

A holomorphic treed disk $u : C \to X$ is *stable* iff

(a) Each disk on which u is constant contains at least three special points or at least one interior special point and one other special point.

(b) Each sphere on which u is constant contains at least three special points.

We denote the moduli space of isomorphism classes of connected treed holomorphic disks with n leaves and m interior markings by $\mathcal{M}_{n,m}(L, \underline{P})$. For a connected combinatorial type Γ, $\mathcal{M}_\Gamma(L, P_\Gamma)$ denotes the subset of type Γ.

Denote $\mathcal{I}(L)$ as the set of critical points of F. For a tuple of critical points $\underline{x} = (x_0, \ldots, x_n) \subset \mathcal{I}(L)^n$ let $\mathcal{M}_\Gamma(L, \underline{x}) \subset \mathcal{M}_\Gamma(L)$ denote the subset of isomorphism classes of holomorphic treed disks u that have limits $\lim_{s\to-\infty} u(\phi_{e_i}(s)) = x_i$ for $i \neq 0$ and $\lim_{s\to\infty} u(\phi_{e_0}(s)) = x_0$. The expected dimension of the moduli space is as follows:

$$\begin{aligned} i(\Gamma, \underline{x}) :=& \dim W_F^-(x_0) - \sum_{i=1}^n W_F^-(x_i) + \sum_{i=1}^k I(u_i) + n - 2 - |\mathrm{Edge}^0_{<\infty}(\Gamma)| \\ & - \big(|\mathrm{Edge}_\infty(\Gamma)| - (n+1)/2\big) - 2|\mathrm{Edge}_{<\infty,s}(\Gamma)| - \sum_{e\in \mathrm{Edge}_{\infty,s}} m(e) - \sum_{e\in \mathrm{Edge}_{<\infty,s}} m(e). \end{aligned}$$

We note that $\big(|\mathrm{Edge}_\infty(\Gamma)| - (n+1)/2\big)$ is the number of breakings on Γ.

3.3 Transversality

In order to achieve transversality for the moduli space of stable treed J-holomorphic curves, we need to restrict to a certain class of symplectic manifolds and Lagrangian submanifolds:

Definition 13. (Rationality)

(a) A symplectic manifold (X, ω) is *rational* if the cohomology class $[\omega]$ exists as an element in $H^2(X, \mathbb{Q})$. Equivalently, X is rational if it has a linearization: there is a line bundle $\widetilde{X} \to X$ with a connection whose curvature is $(2\pi k/i)\omega$ for $k \in \mathbb{Z}$.

(b) Let $h_2 : \pi_2(X, L) \to H_2(X, L)$ be the relative Hurewicz morphism. Let $[\omega]^\vee : H_2(X, \mathbb{R}) \to \mathbb{R}$ be the map given by pairing with ω. A Lagrangian $L \subset X$ is *rational* if $[\omega]^\vee \circ h_2(\pi_2(X, L)) = \mathbb{Z} \cdot e$ for some $e > 0$

As we will see later, the rationality assumption will allow the existence of a *stabilizing divisor* to kill any automorphisms of $C \subset \mathcal{U}_\Gamma$ and make sense of domain-dependent data.

Definition 14. (Stabilizing Divisors)

(a) A divisor in X is a closed codimension two symplectic submanifold $D \subset X$. An almost complex structure $J : TX \to TX$ is adapted to a divisor D if D is an almost complex submanifold of (X, J).

(b) A divisor $D \subset X$ is *stabilizing* for a Lagrangian submanifold L if

  (1) $D \subset X - L$, and

  (2) There exists an almost-complex structure $J_D \in (\mathcal{J}, \omega)$ adapted to D such that any J_D holomorphic disk $u : (C, \partial C) \to (X, L)$ with $\omega([u]) > 0$ intersects D in at least one point.

We get the following theorem (from [CWb; CWa; CM07]) as an application of various techniques:

Theorem 8. *[CWa] There exists a divisor $D \subset X$ that is stabilizing for L. Moreover, if L is rational then there exists a divisor $D \subset X$ that is stabilizing for L and such that L is exact in the compliment $(X \setminus D, \omega)$.*

We will need conditions on the interaction between the treed disks and the divisor:

Definition 15. (Adapted stable treed disks) Let (X, L) be a symplectic manifold with Lagrangian L and a codimension two submanifold D disjoint from L. A treed disk $u : C \to X$ with boundary in L is *adapted* to D iff

(a) (Stable domain) The domain C is stable;

(b) (Non-constant spheres) Each component of C that maps entirely to D is constant;

(c) (Markings) Each interior marking z_i maps to D and each component of $u^{-1}(D)$ contains an interior marking.

From here on out, we will use the notation $\mathcal{M}_\Gamma(L, D, P_\Gamma)$ to denote the space of holomorphic tree disks with boundary in L that are adapted to D. We will prove a transversality result for this moduli space, so long as Γ is *uncrowded*. A combinatorial type is called *uncrowded* if each ghost component has at most one interior marking. This condition is necessary to prevent the expected dimension from running away to negative infinity.

For a partial ordering on combinatorial types of treed disks, we say that $\Gamma' \leq \Gamma$ iff Γ is obtained from Γ' by (Collapsing edges/making edge lengths finite or non-zero). Suppose that perturbation data $P_{\Gamma'}$ has been chosen for all $\Gamma' \leq \Gamma$ (i.e. boundary types $\mathcal{U}_{\Gamma'} \subset \overline{\mathcal{U}}_\Gamma$). Denote $\mathcal{P}^l_{\Gamma, \underline{P}_{\Gamma'}}(X, D)$ as the space of perturbation datum $P_\Gamma = (F_\Gamma, J_\Gamma)$ of class C^l equal to the given pair (F, J) on $(\overline{\mathcal{T}}_\Gamma - \overline{\mathcal{T}}^o_\Gamma, \overline{\mathcal{S}}_\Gamma - \overline{\mathcal{S}}^o_\Gamma)$, and such that the restriction of P_Γ to $(\overline{\mathcal{T}}^o_{\Gamma'}, \overline{\mathcal{S}}^o_{\Gamma'})$ is equal to $P_{\Gamma'}$ for each boundary type Γ'. Prescribing this equality guarantees that the resulting collection satisfies the (Collapsing edges/Making edges finite or non-zero) axiom of the coherence condition. Let $\mathcal{P}_\Gamma(X, D)$ be the intersection of the spaces $\mathcal{P}^l_\Gamma(X, D)$ for all $l \geq 0$.

Theorem 9. *[CWa]* (Transversality) *Suppose that* Γ *is an uncrowded type of stable treed marked disk of expected dimension* $i(\Gamma, \underline{x}) \leq 1$. *Suppose regular coherent perturbation data for types of stable treed marked disks* Γ' *with* $\Gamma' \leq \Gamma$ *are given. Then there exists a comeager subset* $\mathcal{P}^{\mathrm{reg}}_\Gamma(X, D) \subset \mathcal{P}_\Gamma(X, D)$ *of* regular perturbation data *for type* Γ *compatible with the previously chosen perturbation data such that if* $P_\Gamma \subset \mathcal{P}^{\mathrm{reg}}_\Gamma(X, D)$ *then*

1. (Smoothness on each stratum) *The moduli space* $\mathcal{M}_\Gamma(L, D, P_\Gamma)$ *of adapted stable treed disks of type* Γ *is a smooth manifold of expected dimension.*

2. (Tubular neighborhoods) *If* Γ *is obtained from* Γ' *by collapsing an edge of* $\mathrm{Edge}_{<\infty,d}(\Gamma')$ *of making an edge finite or non-zero or by gluing* Γ' *at a breaking, then the stratum* $\mathcal{M}_{\Gamma'}(L, D, P_{\Gamma'})$ *has a tubular neighborhood in* $\overline{\mathcal{M}}_\Gamma(L, D, P_\Gamma)$.

3. (Orientations) *There exist orientations on* $\mathcal{M}_\Gamma(L, D, P_\Gamma)$ *compatible with the morphisms*

(Cutting an edge) *and* (Collapsing an edge/Making an edge finite or non-zero) *in the following sense:*

(a) If Γ *is obtained from* Γ' *by* (Cutting an edge) *then the isomorphism* $\mathcal{M}_{\Gamma'}(L, D, P_{\Gamma'}) \to \mathcal{M}_{\Gamma}(L, D, P_{\Gamma})$ *is orientation preserving.*

(b) If Γ *is obtained from* Γ' *by* (Collapsing an edge) *or* (Making an edge finite or non-zero) *then the inclusion* $\mathcal{M}_{\Gamma'}(L, D, P_{\Gamma'}) \to \overline{\mathcal{M}}_{\Gamma}(L, D, P_{\Gamma})$ *has orientation (from the decomposition*

$$T\mathcal{M}_{\Gamma}(L, D)|_{\mathcal{M}_{\Gamma'}(L,D,P_{\Gamma'})} \cong \mathbb{R} \oplus T\mathcal{M}_{\Gamma'}(L, D, P_{\Gamma'})$$

and the outward normal orientation on the first factor) given by a universal sign depending only on Γ, Γ'.

Proof. See [CWa] $\square$

3.4 Compactness

We wish to have compactness of the 0 and 1 dimensional components of the moduli space $\overline{\mathcal{M}}_{\Gamma}(L, D)$ satisfying a certain energy bound, and a natural realization of the boundary $\partial\overline{\mathcal{M}}_{\Gamma}(L, D)$ as a concatenation of treed disks in $\overline{\mathcal{M}}_{\Gamma'}(L, D)$ for $\Gamma' < \Gamma$. We record the following theory from [CWa]:

Definition 16. For $e > 0$, we say that an almost complex structure $J_D \in \mathcal{J}_\tau(X, D)$ is *e-stabilized* by a divisor D iff

(a) (Non-constant spheres) D contains no non-constant J_D-holomorphic spheres of energy less than e.

(b) (Sufficient intersections) each non-constant J_D-holomorphic sphere $u : S^2 \to X$ resp. J_D-holomorphic disk $u : (D, \partial D) \to (X, L)$ with energy less than e has at least three resp. one intersection points with the divisor D. That is, $u^{-1}(D)$ has order at least three resp. one.

Definition 17. A divisor D with Poincaré dual $[D]^\wedge = k[\omega]$ for some $k \in \mathbb{N}$ has *sufficiently large degree* for an almost complex structure J_D iff

- $([D]^\wedge, \alpha) \geq 2(c_1(X), \alpha) + \dim(X) + 1$ for all $\alpha \in H_2(X, \mathbb{Z})$ representing non-constant J_D-holomorphic spheres.
- $([D]^\wedge, \beta) \geq 1$ for all $\beta \in H_2(X, L, \mathbb{Z})$ representing non-constant J_D-holomorphic disks.

Given $J \in \mathcal{J}_\tau(X, \omega)$ denote by $\mathcal{J}_\tau(X, D, J, \theta)$ as the space of tamed almost complex structures $J_D \in \mathcal{J}_\tau(X, \omega)$ such that $\|J_D - J\| < \theta$ (in the sense of [[CM07],lemma 8.3] or 4.1) and J_D preserves TD. We need the following lemma.

Lemma 2. *[CWa] For θ sufficiently small, suppose that D has sufficiently large degree for an almost complex structure θ-close to J. For each energy $e > 0$, there exists an open and dense subset $\mathcal{J}^*(X, D, J, \theta, e) \subset \mathcal{J}_\tau(X, D, J, \theta)$ such that if $J_D \in \mathcal{J}^*(X, D, J, \theta, e)$, then J_D is e-stabilized by D. Similarly, if $D = (D^t)$ is a family of divisors for J^t, then for each energy $e > 0$, there exists a dense and open subset $\mathcal{J}^*(X, D^t, J^t, \theta, e)$ in the space of time-dependent tamed almost complex structures $\mathcal{J}^*(X, D^t, J^t, \theta)$ such that if $J_{D^t} \in \mathcal{J}^*(X, D^t, J^t, \theta, e)$, then J_{D^t} is e-stabilized for all t.*

Let Γ be a type of stable treed disk, and let $\Gamma_1, \ldots, \Gamma_l$ be the components formed by deleting boundary nodes of positive length, and $\overline{U}_{\Gamma_1}, \ldots, \overline{U}_{\Gamma_l}$ the corresponding decomposition of the universal curve. In case L is rational and exact in the complement of D, any stable treed disk with domain of type Γ and transverse intersections with the divisor has energy at most

$$n(\Gamma_i, k) := \frac{n(\Gamma_i)}{k}$$

on the component $\overline{U}_{\Gamma_i}$, where $n(\Gamma_i)$ is the number of markings on $\overline{U}_{\Gamma_i}$ with D.

Let $J_D \in \mathcal{J}_\tau(X, D, J, \theta)$ be an almost complex structure that is stabilized for all energies, (e.g., something in the intersection of $J_D \in \mathcal{J}^*(X, D, J, \theta, e)$ for all energies). For each energy e, there is a contractible open neighborhood $\mathcal{J}^{**}(X, D, J_D, \theta, e)$ of J_D in $J_D \in \mathcal{J}^*(X, D, J, \theta, e)$ that is e-stabilized.

Definition 18. A perturbation datum $P_\Gamma = (F_\Gamma, J_\Gamma)$ for a type of stable treed disk Γ is *stabilized* by D if J_Γ takes values in $\mathcal{J}^*(X, D, J, \theta, n(\Gamma_i, k))$ on $\overline{U}_{\Gamma_i}$

Theorem 10. (Compactness for fixed type)*[CWa] For any collection $\underline{P} = (P_\Gamma)$ of coherent, regular, stabilized perturbation data and any uncrowded type Γ of expected dimension at most one, the moduli space $\overline{\mathcal{M}}_\Gamma(L, D)$ of adapted stable treed marked disks of type Γ is compact and the closure of $\mathcal{M}_\Gamma(L, D)$ contains only configurations with disk bubbling.*

Proof. See [CWa]. $\square$

Chapter 4

Transversality and compactness in the fibered setting

We would like to use parts of the previous scheme to help us achieve transversality for the moduli space of curves into symplectic Mori fibrations. To recall:

Definition 19. A *symplectic Mori fibration* is a fiber bundle of symplectic manifolds $(F, \omega_F) \to (E, \omega_K) \xrightarrow{\pi} (B, \omega_B)$, where (F, ω_F) is monotone, (B, ω_B) is rational, and $\omega_K = a + K\pi^*\omega_B$ for large K with $\iota^* a = \omega_F$.

Definition 20. A *fibered Lagrangian* is a Lagrangian in a symplectic Mori fibration $L \subset E$ such that there are Lagrangians $L_F \subset F$ and $L_B \subset B$ and π induces a fiber bundle $L_F \to L \to L_B$

In general, the Floer cohomology of L_B may not be defined due to bubbling. However, the usual transversality and compactness should still hold for L if we combine the technical results for L_F and L_B. On the other hand, our primary interest is in $L \subset E$ which, a priori, is neither monotone nor part of a rational symplectic manifold, so we take care in this section to make sure that the usual results hold. Even if the pair (E, L) were rational, we would like a nice way to compute invariants in the fibered case.

Summarily, we pull back the divisor from the base to stabilize Floer trajectories that intersect fibers transversely, and use the usual monotone results for pseudo holomorphic curves that lie completely in a fiber.

4.1 Divisors

This is an expository section on the existence of a Donaldson hypersurface that is stabilizing for a given Lagrangian. To start, we repeat the definition of a weakly stabilizing divisor.

Definition 21. [CWa]

(a) A *symplectic divisor* in B is a closed codimension two symplectic submanifold $D \subset B$. An almost complex structure $J : TB \to TB$ is adapted to a divisor D if D is an almost complex submanifold of (B, J).

(b) A divisor $D \subset B$ is *stabilizing* for a Lagrangian submanifold L if

  (1) $D \subset B - L$, and

  (2) There exists an almost complex structure J_D adapted to D such that any J_D-holomorphic disk $u : (C, \partial C) \to (B, L)$ with $\omega(u) > 0$ intersects D in at least one point.

(c) An almost complex structure is *adapted* to D if D is an almost complex submanifold of B.

The existence of this is highly non-trivial. In the case of a smooth projective variety Bertini's theorem tells us that there are plenty of smooth hypersurfaces $D \subset B$ [Har97]. Picking one that does not intersect L requires a further analysis as in [AGM01].

To find at least one symplectic divisor $D \subset B \setminus L$ in the general case, we appeal to the fact that B is rational. Let K be an integer such that $K[\omega] \in H^2(M, \mathbb{Z})$. Then there is a complex line bundle $\tilde{B} \to B$ such that $c_1(\tilde{B}) = K[\omega]$. Since $0 = K[\omega|_L] \in H^2(L, \mathbb{Z})$, there is some power t so that $\tilde{B}^{\otimes t}|_L$ is topologically trivial over L. Thus, choose a section l of $\tilde{B}^{\otimes t}$ that is non-vanishing on L, and take a small smooth perturbation so that that l intersects the zero section transversly, so that $l^{-1}(0)$ is smooth. [CWa]

For a given J, we would like to know if we can find a symplectic divisor that also stabilizes L and such that J is adapted to D. While this may seem like a lot to ask, one can use the techniques of Donaldson [Don96] and Auroux-Gayet-Mohsen [AGM01] to find an approximately J-holomorphic submanifold. For a symplectic divisor D, and an ω-compatible J, let us define the Kähler angle of D with respect to J as

$$\Theta_D(J) : D \to [0, \pi], \qquad x \mapsto \cos^{-1}\left(\frac{\omega_x^k}{k!\Omega_{T_x D}}\right) \tag{4.1}$$

where $\Omega_{T_x D}$ is the volume form induced from the metric $\omega(\cdot, J\cdot)$ and an orientation (see section 8 of [CM07]). One says that a symplectic divisor is θ-approximately holomorphic for J if its Kähler angle is $\Theta_D(J) \leq \theta$ for all $x \in D$. Let $t_0 = |\mathrm{Tor}(H(L))|$. We say that a symplectic divisor is of degree d if $[D]^{PD} = d[\omega]$. We have the following lemma:

Lemma 3 (lemma 4.17 from [CWb])**.** *Let B be rational and $J \in \mathcal{J}(\omega)$. There exists an integer $k_m > 0$ such that for every $\theta > 0$ there is an integer $k_\theta > 0$ such that for every $k > k_\theta$ there exists a θ-approximately holomorphic symplectic divisor D of degree $t_0 k_m k$ that is stabilizing for L.*

Remark 2. More is true if we are to assume that there is a line bundle with connection $\tilde{B} \to B$ that is covariant constant when restricted to L (or L is *rational* in the sense of [CWb]). In this case, L becomes exact in the compliment of $(B \setminus D, \omega)$, and so the symplectic area of any $[u] \in \pi_2(B, L)$ is proportional to the intersection number with the divisor.

A major result from [CWb][CM07] is that we can actually find an almost complex symplectic divisor that is stabilizing for L:

Lemma 4 (lemma 4.18 from [CWb])**.** *Let (B, ω) be a rational and compact symplectic manifold. Then there exists divisors $D_d \subset B$ of arbitrarily large degree d with adapted almost complex structures J_{D_d} such that the pair (D_d, J_{D_d}) stabilize L.*

4.1.1 Divisors in the fibered setting

To be able to use the perturbation scheme from section 3, we pick a divisor in B and take its inverse image under π to get a divisor in E.

For a compatible almost complex structure J_B on (B, ω_B), there exists a natural almost complex structure $\pi^* J_B$ on the connection bundle $H = TF^{\perp a}$ of E. We will denote this a.c.s. J_B by abuse of notation. An almost complex structure on the sub-bundle H is called *basic* if it is the pullback of some a.c. structure from the base.

We will achieve transversality by using almost complex structures of the form

$$J_{ut} = \begin{bmatrix} J_F & K \\ 0 & J_B \end{bmatrix}$$

where the block decomposition is with respect to the connection $TF \oplus H$ on E.

Thus, for a stabilizing pair (D_B, J_{D_B}) for L_B, the pair $(\pi^{-1}(D_B), J_D)$ with

$$J_D = \begin{bmatrix} J_F & K \\ 0 & \pi^* J_{D_B} \end{bmatrix} \tag{4.2}$$

forms an almost complex submanifold for any particular choice of J_F and K. This almost complex submanifold is stabilizing for L with respect to J_D-holomorphic disks that have positive area when projected to the base.

Definition 22. 1. A divisor D is stabilizing for L if it is the inverse image of a stabilizing divisor D_B for L_B in sense of definition 21 :

There exists an almost-complex structure $J_{D_B} \in (\mathcal{J}, \omega_B)$ adapted to D_B such that any J_{D_B} holomorphic disk $u : (C, \partial C) \to (B, L_B)$ with $\omega_B([u]) > 0$ intersects D_B in at least one point.

2. We label an adapted a.c.s. as in 4.2 J_D.

4.2 Perturbation Data

4.2.1 Adapted Morse functions and pseudo-gradients

Part of the input data requires the choice of a Morse-Smale function and a Riemannian metric on L. It will be important later on that we choose the function so that it descends to a datum on B. We can construct a Morse function on L by the following recipe: take Morse functions b resp. g on L_B resp. L_F. Take trivializations $\{(U_i, \Psi_i)\}$ with the U_i small neighborhoods of the critical points $\{x_i\}$ for b. Let ϕ be a sum of bump functions equal to $\epsilon << 1$ in a neighborhood of each x_i and 0 outside U_i. The function $f = \pi^* b + \epsilon \pi^* \phi g$ is a Morse function for L with the property that its restriction to fibers near critical points of b is also Morse.

To ensure that critical points only occur in critical fibers of f, the ϵ can be made small enough so that the derivative of the bump function doesn't contribute significantly to the horizontal component of the flow. The Morse function can then be perturbed in small neighborhoods outside of critical points to a Morse-Smale function.

An approach to Morse theory that is more adapted to the fibration setting is that of a pseudo-gradient with details carried out in [[Hut] section 6.3]. The connection H on E induces a connection $TL \cap H \oplus TF$ on L (this particular connection is merely convenient and can be chosen in many ways). Let us choose a Riemannian metric G_B on L_B and let $X_b := \mathrm{grad}_{G_B} b \in TL_B$. Assume that G_B is given by the Euclidean metric in neighborhood of critical points of b. Then X_b has a horizontal lift to the connection on TL. Next, choose a metric G_F on L_F and denote $X_g := \mathrm{grad}_{G_F} g$, with G_F the pullback of the Euclidean metric in a neighborhood of critical points of g. We will show that

$$X_g \oplus X_b \in TL_F \oplus TL \cap H$$

has the property of a *pseudo-gradient* for the Morse function f with respect to the metric $G_F \oplus G_B$. By this, we mean the following:

Definition 23. Let f be a Morse function on a Riemannian manifold (M, G). A *pseudo-gradient* for f is a vector field X such that $X_p = 0$ for $p \in \mathrm{crit}(f)$ and

1. $G(\mathrm{grad}_G(f), X) \leq 0$ and equality holds only at critical points of f
2. In a Morse chart for f centered at $p \in \mathrm{Crit}(f)$, $\mathrm{grad}_e f = X$ where e is the standard Euclidean metric.

We check that our construction satisfies the pseudo-gradient property: Let $G := G_F \oplus G_B$. From the construction of f, we have

$$\mathrm{grad}_G f = \phi \cdot \mathrm{grad}_{G_F} g \oplus (\mathrm{grad}_{G_B} b + g \cdot \mathrm{grad}_{G_B} \phi)$$

By the Morse lemma and the local Euclidean property of the metric G_B, there are coordinates centered at $p \in \mathrm{Crit}(b)$ so that G_B is Euclidean and

$$b(x) = \pm x_1^2 \pm \cdots \pm x_n^2.$$

In this chart, we choose ϕ to be a bump function that is radial, with

$$\|g\||\partial_i \phi| < 2|x_i|$$

and such that $d\phi$ has annular support. It is clear that $G(\mathrm{grad}_{G_B} b + g \cdot \mathrm{grad}_{G_B} \phi, X_b) \leq 0$ in these coordinates (with equality as in property 1). It is also clear that $G(\mathrm{grad}_{G_B} b + g \cdot \mathrm{grad}_{G_B} \phi, X_b) < 0$ outside of these coordinates where ϕ vanishes. Moreover, we have that $G(\phi \cdot \mathrm{grad}_{G_F} g, X_g) \leq 0$ with equality only at critical points of g and outside the support of ϕ. The locality property 2 follows from the Euclidean-near-critical-points property of $G_F \oplus G_B$ and the Morse lemma for b and g.

Morse theory for pseudo-gradients is spelled out in chapter 2 of [AD14]. In particular, the Smale condition can be achieved by perturbing the pseudo gradient $X := X_g \oplus X_b$ in finitely many neighborhoods outside of critical points.

A particular nicety of the pseudo-gradient approach is the following: any flow line γ for X on L projects to a flow line $\pi \circ \gamma$ for X_B in L_B. This feature makes it so that Morse-Floer trajectories behave well under the projection $\pi : E \to B$.

Finally, we can define the type of perturbation data that we will be using: Fix a Morse function f and pseudo gradient $X_g \oplus X_b$ as in the above connection. In regards to the connection $TF \oplus H$ on E, let

$$\mathcal{J}^l_{ut}(E, \omega_K) = \left\{ J_{ut,\tau} = \begin{bmatrix} J_{TF} & J_H \\ 0 & J_B \end{bmatrix} | J^2_{ut} = -I, \; J_{ut} \in \mathcal{J}^l_\tau(E, \omega_K) \right\}$$

be the space of taming upper triangular almost complex structures for (E, ω_K). Let $TL = TL_F \oplus TL_H$ be a connection on L.

Definition 24. An *M-type perturbation datum* for $(F \to E \to B, \omega)$ of type Γ, denoted $P_\Gamma(E, D)$, is a map $\mathcal{U}_\Gamma \to \mathcal{J}^l_{ut}(E, \omega_K) \times \mathrm{Vect}^l(TF) \oplus \mathrm{Vect}^l(H_L)$ where the first factor has J_B resp. J_H equal to J_D resp. 0 in a neighborhood of the interior markings, spherical nodes, and on the boundary component of each disk. The vector field factor is required to be equal to X in a neighborhood of ∞. Denote the set of all perturbation data for type Γ by $\mathcal{P}_\Gamma(E, D)$.

Definition 25. A perturbation datum for a collection of combinatorial types γ is a family $(P_\Gamma)_{\Gamma \in \gamma}$

We modify our definition of treed holomorphic disk 12 to be a flow on edges:

Definition 26. A *treed holomorphic disk* with respect to the pseudo-gradient X_f for f satisfies the properties of 12 with the following instead of the (Tree equation):

(c)' (Tree equation') On the tree part $T \subset C$ the map u is a collection of flows:

$$\frac{d}{ds} u|_T(p) = X_f(u(p))$$

We will also refers to the above as a *pearly Morse trajectory* or a *Morse-Floer trajectory.*

From here on, we use the terms "Morse flow" or "Morse trajectory" to describe a flow of any pseudo-gradient that is compatible with our Morse function.

4.2.2 Block upper triangular almost complex structures

For an even dimensional real vector space V with $V = X \oplus Y$, the space $\mathcal{J}_{ut}(V)$ can be viewed as a (trivial) vector bundle $\mathcal{K} \to \mathcal{J}_{ut} \to \mathcal{J}_X \times \mathcal{J}_Y$, where the base are the bundles of a.c. structures on X resp. Y respectively. Suppose that $\dim X = 2m$ resp $\dim Y = 2n$ with almost complex structure J resp. K. The matrices L that make $\begin{bmatrix} J & L \\ 0 & K \end{bmatrix}$ into an almost complex structure on V structure satisfy the linear relation $JL + LK = 0$. For J_0 resp. K_0 in normal form $\begin{bmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{bmatrix}$, this is the set of matrices $\begin{bmatrix} A & B \\ B & -A \end{bmatrix}$ where A and B are $m \times n$. The set of almost complex structures on X resp. Y are given by the homogeneous space $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$ resp. $GL(2m, \mathbb{R})/GL(m, \mathbb{C})$. For $J = CJ_0C^{-1}$, $K = DK_0D^{-1}$, the fiber at (J, K) is given by matrices of the above form conjugated by C, D^{-1}.

For a contractible open neighborhood U of (J, K), choose a section of the bundle $GL(2n, \mathbb{R}) \times GL(2m, \mathbb{R}) \to \mathcal{J}_X \times \mathcal{J}_Y$. This gives a smooth choice of coset representatives $([A], [B]) \mapsto (s_1(A), s_2(B)) \in A \cdot GL(n, \mathbb{C}) \times B \cdot GL(m, \mathbb{C})$. We define a local trivialization of $\mathcal{J}^l_{ut}(V)$ on U by

$$([A], [B], L) \mapsto ([A], [B], s_1^{-1}(A) \cdot L \cdot s_2(B)).$$

Transition maps for a choice of section $([A], [B]) \mapsto (t_1(A), t_2(B))$ over an intersecting V are given by

$$L \mapsto t_1^{-1}(A) s_1(A) \cdot L \cdot s_2^{-1}(B) t_2(B).$$

Now let us choose a symplectic form $\omega_k = a + k\omega$ on V with $V = X \oplus Y$, $a|_X$ is non-degenerate, $Y = X^{a\perp}$ and ω is a symplectic form on Y with $k >> 1$. We would like to pick an open set of $\mathcal{J}_{ut,\tau}(V, \omega_k)$ (that we will also denote $\mathcal{J}_{ut,\tau}(V, \omega_k)$ by abuse of notation) such that there is a well defined projection to $\mathcal{J}_\tau(X, a) \times \mathcal{J}_\tau(Y, \omega)$: Indeed, write $v = x \oplus y \in X \oplus Y$. Then we can write down the fiber of this projection as the open set of L such that

$$a(x, Jx) + a(y, Ky) + k\omega(y, Ky) > a(x, Ly)$$

for all $x \oplus y$. Let us see that this is open: By Cauchy-Schwarz, $|a(x, Ly)| \leq \|x\|_a \cdot \|JLy\|_a$. Thus we need to choose L such that

$$\|x\|_a^2 - \|x\|_a \|JLy\|_a + a(y, Ky) + k\omega(y, Ky) > 0$$

We notice that $a(y, Ky) + k\omega(y, Ky)$ is positive definite for large enough k. Viewing this as quadratic in $\|x\|_a$, it is sufficient to only consider L's such that

$$\|JLy\|_a^2 < 4(a(y, Ky) + k\omega(y, Ky)) \tag{4.3}$$

for all y in a compact set. This is equivalent to choosing an L whose $(a + k\omega, a)$ norm is sufficiently small.

The above argument establishes $\mathcal{J}^l_{ut,\tau}(V, \omega)$ as a fiber bundle over $\mathcal{J}_{X,\tau}(\omega) \times \mathcal{J}_{Y,\tau}(\omega)$, whose fiber is an open ball in a linear space.

In general we will consider an open set of the space $\mathcal{J}^l_{ut,\tau}(E, \omega_K)$ (by abuse of notation, also denoted $\mathcal{J}^l_{ut,\tau}(E, \omega_K)$), that is a Banach manifold and can be realized as a Banach bundle $\mathcal{J}_{\mathcal{A}} \to \mathcal{J}^l_{ut,\tau}(E, \omega) \to \mathcal{J}^l_{TF,\tau} \times \mathcal{J}^l_{B,\tau}$, where the fiber at a point (J_F, J_B) is the space of sections over E such that $J_F J_H + J_H J_B = 0$ and the supremum norm of J_H with respect to (J_B, J_F) is sufficiently small. Thus the tangent space to a point

$$\begin{bmatrix} J_F & J_A \\ 0 & J_B \end{bmatrix}$$

is given by the set of matrices $\begin{bmatrix} W_F & W_A \\ 0 & W_B \end{bmatrix}$ where the W_F resp. W_B anti-commute with their respective almost complex structure and W_A satisfies $J_F W_A + W_A J_B = 0$.

It should be noted that the space of such upper triangular structures that are adapted the the divisor is still a Banach bundle, as this only imposes a constraint on the base part of the structure.

4.2.3 Coherence and π-stability

The type of requirements that we need for stability are slightly more delicate than those in the case from section 3.

As above, the *combinatorial type* Γ of a treed disk will contain the following information:

1. the set of vertices, edges, edges lengths, and node/marking type where edges meet vertices,
2. the homotopy class that each vertex is required to represent as a domain for a map u,
3. the tangency of each interior marking to the divisor $\pi^{-1}(D_B)$ along the connection H, and
4. a binary marking that dictates how each vertex as a disk/sphere domain behaves with respect to π (see below).

Definition 27. A *binary marking* of a combinatorial type Γ is a subset of the vertices and edges, denoted $m\text{Vert}(\Gamma)$ and $m\text{Edge}(\Gamma)$, for which any map $u : C \to E$ is required to map the domain for $mv \in m\text{Vert}(\Gamma)$ resp. $me \in m\text{Edge}(\Gamma)$ to a constant under π. The set of unmarked vertices and edges will be denoted $u\text{Vert}(\Gamma)$ resp. $u\text{Edge}(\Gamma)$.

Let $(\Gamma, \overline{x})$ be a combinatorial type and let $[v_i]$ denote the homology class of the vertex v_i. Let S_i denote the disk/sphere domain corresponding to the vertex v_i.

Definition 28. The combinatorial type $(\pi_*\Gamma, \pi(\overline{x}))$ is the combinatorial type of the underlying metric tree of Γ along with the labeling $\pi_*[v_i]$ and boundary markings $\pi(x_i)$.

Definition 29. The π-stabilization map $\Gamma \mapsto \Upsilon(\Gamma)$ is defined on combinatorial types by forgetting any unstable vertex v_i for which $[v_i] = 0$ and identifying edges as follows:

1. If v_i has one incoming edge e_i and one outgoing edge f_i that is closer to the root, then $\text{Vert}(\Upsilon(\Gamma)) = \text{Vert}(\Gamma) - \{v_i\}$ and we identify the edges e_i and f_i: $\text{Edge}(\Upsilon(\Gamma)) = \text{Edge}(\Gamma)/\{f_i \sim e_i\}$ where $\ell(\Upsilon(f_i)) = \ell(\Upsilon(e_i)) = \ell(e_i) + \ell(f_i)$

2. If v_i has one outgoing edge e_i and no incoming edge, then $\Upsilon(\Gamma)$ has vertices $\text{Vert}(\Upsilon(\Gamma)) = \text{Vert}(\Gamma) - \{v_i\}$ and edges as follows:

   (a) If $\ell(e_i) = 0$, then $\text{Edge}(\Upsilon(\Gamma)) = \text{Edge}(\Gamma) - \{e_i\}$

   (b) If $\ell(e_i) > 0$, then $\text{Edge}(\Upsilon(\Gamma)) = \text{Edge}(\Gamma)$ and set $\ell(\Upsilon(e_i)) = \ell(e_i)$

It follows that $\Upsilon(\pi_*\Gamma)$ forgets (unstable) marked vertices and identifies the adjacent edges. This is precisely the combinatorial type of $\pi \circ u$ if u is of type Γ.

Definition 30. A combinatorial type is called *π-stable* if $\Upsilon(\pi_*\Gamma)$ is stable.

Definition 31. A *coherent* M-type perturbation datum $(P_\Gamma)_\Gamma$ for π-stable types is one with the following properties:

1. $J_{\Gamma,TF}$ is constant on each surface component of the universal treed disk $\mathcal{U}_\Gamma$

2. On domains corresponding to marked vertices, all perturbation data is constant and $J_H = 0$ on the corresponding surface component.

3. If Γ' is obtained from Γ by forgetting a marked vertex and stabilizing the domain, then the perturbation data P_Γ agrees with the pullback of $P_{\Gamma'}$ under the natural map of universal disks.

4. The collection $\{(J_{\Gamma,H}, J_{\Gamma,B}, f_\Gamma)\}_\Gamma$ obeys the axioms for a coherent perturbation system from the rational case (11).

4.3 Transversality

This section establishes the smoothness of moduli spaces whose expected dimension is 0 or 1.

Definition 32. A Floer trajectory $u : C \to E$ based on a π-stable combinatorial type is called *π-adapted* to D if $\pi \circ u$ is adapted to D_B in sense of definition 15:

(a) (Stable domain) The geometric realization of $\Upsilon(\pi_*\Gamma)$ is a stable domain;

(b) (Non-constant spheres) Each component of C that maps entirely to D is constant;

(c) (Markings) Each interior marking z_i maps to D and each component of $u^{-1}(D)$ contains an interior marking.

Denote by $\mathcal{M}_\Gamma(L, D, P)$ the moduli space of type Γ Floer trajectories with boundary in L that are π-adapted to D with respect to some M-type perturbation datum P, and for a tuple $(x_0, \dots, x_n)$, by $\mathcal{M}_\Gamma(L, D, P, \overline{x})$ the ones that limit to x_0 along the root and $(x_1, \dots, x_n)$ along the leaves, arranged in counterclockwise order.

The expected dimension of the stratum $\mathcal{M}_\Gamma(E, D, P, \overline{x})$ is

$$\begin{aligned}\iota(\Gamma, \overline{x}) :=& \dim W_X^+(x_0) - \sum_{i=1}^{n} \dim W_X^+(x_i) + \sum_{i=1}^{m} I(u_i) + n - 2 - |\mathrm{Edge}^0_{<\infty}(\Gamma)| \\ &- \left(|\mathrm{Edge}_\infty(\Gamma)| - (n+1)\right)/2 - 2|\mathrm{Edge}_{<\infty,s}(\Gamma)| - |\mathrm{Edge}_{\infty,s}(\Gamma)| \\ &- \sum_{e \in \mathrm{Edge}_{\infty,s}} m(e) - \sum_{e \in \mathrm{Edge}_{<\infty,s}} m(e).\end{aligned}$$

Let $\overline{\mathcal{S}}_\Gamma \subset \overline{\mathcal{U}}_\Gamma$ be the two-dimensional part of the universal treed disk, and $\overline{\mathcal{T}}_\Gamma \subset \overline{\mathcal{U}}_\Gamma$ be the tree part of the universal treed disk. Fix a compact set

$$\overline{\mathcal{S}}^o_\Gamma \subset \overline{\mathcal{S}}_\Gamma$$

not containing the boundary, nodes, or interior markings, but having open interior in every fiber of the universal disk $\mathcal{U}_\Gamma$. Also fix a compact set

$$\overline{\mathcal{T}}^o_\Gamma \subset \overline{\mathcal{T}}_\Gamma$$

having non-trivial intersection with each universal fiber. Thus, the complements

$$\overline{\mathcal{S}}_\Gamma - \overline{\mathcal{S}}^o_\Gamma$$

$$\overline{\mathcal{T}}_\Gamma - \overline{\mathcal{T}}^o_\Gamma$$

are neighborhoods of the boundary, interior markings, and nodes resp. neighborhoods of ∞ in each fiber of the universal disk. We require that the perturbation data vanish in these neighborhoods. In addition, we only consider types Γ that are uncrowded.

We say that a type $\Gamma' \leq \Gamma$ iff Γ is obtained from Γ' by (collapsing edges/making an edge length finite or non-zero).

Theorem 11 (Tranversality). *Suppose Γ is an uncrowded combinatorial of expected dimension $\iota(\Gamma, \overline{x}) \leq 1$. Suppose that a coherent system of M-type perturbation data has been chosen for all types $\Gamma' \leq \Gamma$. Then there is a comeager subset of* M-type perturbation datum $\mathcal{P}_\Gamma^{\mathrm{reg}}(E, D) \subset \mathcal{P}_\Gamma(E, D)$*, that is coherently compatible with the previously chosen data, such that the following hold:*

1. *The moduli space $\mathcal{M}_\Gamma(L, D, P)$ for $P \in \mathcal{P}_\Gamma^{reg}$ is a smooth manifold of expected dimension.*
2. *The (orientations) and (tubular neighborhoods) statements from theorem 9 hold.*

Proof. The proof follows some of the ideas in [CWa] in addition to making special choices of perturbation data for the fiber and upper triangular part. If C is a nodal disk of type Γ, for $p \geq 2$ and $k > 2/p$ let $\mathrm{Map}^0(C, E, L)_{k,p}$ denote the space of (continuous) maps from C to E with boundary and edge components in L that are of the class $W^{k,p}$ on each disk, sphere, and edge. We have the following standard result:

Lemma 5. $\mathrm{Map}^0(C, E, L)_{k,p}$ *is a C^q Banach manifold, $q < k - n/p$, with local charts centered at u given by the product space of vector fields that agree at disk nodes and interior markings:*

$$\bigoplus_{(v,e) \in Vert(\Gamma) \oplus Edge(\Gamma)} W^{k,p}(C, u_v^* TE, u_{v,\partial C}^* TL) \oplus_{\mathrm{Edge}_d} W^{k,p}(C, u_e^* TL)$$

where the chart into Map^0 *is given by geodesic exponentiation with respect to some metric on E that makes L and D totally geodesic.*

Let $\mathrm{Map}_\Gamma^0(C, E, L)_{k,p} \subset \mathrm{Map}^0(C, E, L)_{k,p}$ denote the submanifold of maps whose spheres and disks map to the labeled homology classes that have the prescribed tangencies to the divisor, and whose marked vertices are constant with respect to π.

In general, the space $\mathrm{Map}^0_\Gamma(C,E,L)_{k,p}$ is a C^q Banach submanifold where $q < k - n/p - \max_e m(e)$. Following Dragnev [Dra04], the universal space is constructed as follows. Given a trivialization of the universal disk $C \in U^i_\Gamma \to \mathfrak{M}^i_\Gamma$,we get a map $m \mapsto j(m) \in \mathcal{J}(S)$ obtained by identifying nearby curves with C. Consider the space

$$\mathcal{B}^i_{k,p,\Gamma,l} := \mathfrak{M}^i_\Gamma \oplus \mathrm{Map}^0_\Gamma(C,E,L)_{k,p} \oplus \mathcal{P}^l_\Gamma(E,D). \tag{4.4}$$

Over this Banach manifold we get a vector bundle $\mathcal{E}^i_{k,p,\Gamma,l}$ given by

$$(\mathcal{E}^i_{k,p,\Gamma,l})_{m,u,J,F} \subset \bigoplus_{v,e\in\Gamma} \Lambda^{0,1}_{j,J,\Gamma}(C, u_v^*(TF \oplus H))_{k-1,p} \tag{4.5}$$

$$\oplus \Lambda^1(C, u_e^* TL)_{k-1,p} \tag{4.6}$$

where $\Lambda^{0,1}_{j,J,\Gamma}(C, u_v^*(TF\oplus H))_{k-1,p}$ resp. $\Lambda^1(C, u_e^*TL)_{k-1,p}$ denote the space of sections of $(0,1)$-forms resp 1-forms, and $(\mathcal{E}^i_{k,p,\Gamma,l})_{m,u,J,F}$ is the subspace of sections that vanish to order $m(e)-1$ at the node or marking corresponding to e. Local trivializations of this bundle are given by parallel transport along geodesics in E via the associated Hermitian connection in the fibers. For the transition maps to be C^q, we need the l in $\mathcal{J}^l_{ut,\tau}$ large enough so that $q < l - k$.
There is a C^q section $\overline{\partial} : \mathcal{B}^i_{k,p,\Gamma,l} \to \mathcal{E}^i_{k,p,\Gamma,l}$ via

$$(m,u,J,F) \mapsto (\overline{\partial}_{j(m),J} u_S, (\frac{d}{d\mathrm{s}} - X) u_T) \tag{4.7}$$

with

$$\overline{\partial}_{j(m),J} u_S := \mathrm{d}u_S + J \circ \mathrm{d}u_S \circ j(m) \tag{4.8}$$

The a.c. structure J depends on $(m,p) \in \mathfrak{M}^i_\Gamma \oplus C$. The *local universal moduli space* is defined to be

$$\mathcal{M}^{univ,i}_\Gamma(E,L,D) := \overline{\partial}^{-1} \mathcal{B}^i_{k,p,\Gamma,l} \tag{4.9}$$

where $\mathcal{B}^i_{k,p,\Gamma,l}$ is identified with the zero section.

With respect to the variable in $\mathrm{Map}^0_\Gamma(C,E,L)_{k,p}$, the linearization of the Cauchy Riemann operator $\overline{\partial}_{j(m),J}$ is

$$D_{u,J,j}(\xi) = \triangledown\xi + J \circ \triangledown\xi \circ j - J(u)(\triangledown_\xi J)\partial_{j(m),J} u_s \tag{4.10}$$

We also have zeroth order terms coming from the domain dependent data: The differential of $\overline{\partial}_{j(m),J}$ at a J_Γ holomorphic map with respect to the variable in $\mathcal{P}_\Gamma(E, D)$ is given by

$$T_{J_\Gamma}\mathcal{P}_\Gamma \to \Lambda^{0,1}(C, u^*TE)_{k-1,p}, \qquad K \mapsto K \circ \mathrm{d}u_S \circ j \tag{4.11}$$

The surjectivity argument for linearized $\overline{\partial}$ is divided into multiple cases: given a component u_v of a Floer trajectory, the component can either be constant in the horizontal direction, the vertical direction, both, or neither. Notably, we have the splittings of the domain of the linearized Cauchy-Riemann operator:

$$\begin{aligned} D_u(K) : W^{k,p}(S, u^*TF, u^*_{\partial S}TF \cap TL) \oplus W^{k,p}(S, u^*H, u^*_{\partial S}H \cap TL) \\ \to \Lambda^{0,1}_{j,J_\Gamma}(S, u^*TF \oplus H)_{k-1,p} \end{aligned} \tag{4.12}$$

While the range does not split in such a manner (unless $J_H \equiv 0$), we have the nice feature of additional freedom in the choice of perturbation data. Now, supposing that u is J-holomorphic, D_u restricts to a map

$$D_u : W^{k,p}(S, u^*TF, u^*_{\partial S}TF \cap TL) \to \Lambda^{0,1}_{j,J_\Gamma}(S, u^*TF)_{k-1,p} \tag{4.13}$$

By construction, any J-holomorphic disk/sphere u gives rise to a J_B holomorphic disk/sphere $\pi \circ u$. We use this fact in each of the following 3 cases:

Case 1: *u is only constant in the horizontal direction.*

In this case, the domain corresponds to a marked vertex of Γ. In the vertical direction, we have that $du_F \circ j = J_{TF} du_F$ since the horizontal differential vanishes. Thus, u is a J_{TF}-holomorphic curve in the monotone manifold F_p (with boundary conditions in $F_p \cap L$ in the disk case). First, assume that u is simple. In this case, we use the standard argument from [MS04] to get surjectivity for the restriction of the linearized operator in (4.13).

Now suppose u is a disk component but not simple. Then by decomposition results due to [Laz10], we have that u represents a sum of elements of $H_2(E, L)$. If $\dim F \geq 3$, we must have that $I(u) = mI(\tilde{u})$, where $\tilde{u} \circ p = u$ for simple J-holomorphic $\tilde{u}$ and holomorphic covering map p. Replacing u with $\tilde{u}$ in the configuration Γ gives a simple configuration $\widetilde{\Gamma}$, which can be made regular by the above paragraph. Since $I(\tilde{u}) \geq 2$ and $\iota(\widetilde{\Gamma}, \overline{x}) \geq 0$, we must have had that $\iota(\Gamma, \overline{x}) \geq 2$, which is a contradiction. The case when $\dim F = 2$ is similar, see [Cha].

If u is a non-constant and nowhere injective sphere component attached to a configuration $\widetilde{\Gamma}$, then we must have that $u = \tilde{u} \circ p$ for a degree $d > 1$ branched covering map p. From this, we get that $2c_1(A_u) = 2dc_1(A_{\tilde{u}}) > 0$ since u is non-constant and F is monotone. The configuration Γ with u replaced by $\tilde{u}$ is regular by the above paragraph, and so it has expected dimension. This gives us that Γ with the map u must be of index ≥ 2, which goes against the assumption.

The above argument tells us that

$$D_u^{TF} : W^{k,p}(S, u^*TF, u^*TF \cap TL) \to \Lambda^{0,1}_{j,J_\Gamma}(S, u^*TF)_{k-1,p}$$

is surjective. To see that

$$\begin{aligned} D_u : W^{k,p}(S, u^*TF, u^*TF \cap TL) \oplus W^{k,p}(S, u^*H, u^*H \cap TL) \\ \to \Lambda^{0,1}_{j,J,\Gamma}(S, u^*TF \oplus H)_{k-1,p} \end{aligned}$$

is surjective, it suffices to check surjectivity after composing with the projection

$$\Lambda^{0,1}_{j,J,\Gamma}(S, u^*TF \oplus H)_{k-1,p} \xrightarrow{d\pi_*} \Lambda^{0,1}_{j,J_B}(S, \pi \circ u^*TB)_{k-1,p}$$

since $\Lambda^{0,1}_{j,J,\Gamma}(S, u^*TF)_{k-1,p} = \ker d\pi_*$. We have the following commutative diagram

$$
\begin{tikzcd}
W^{k,p}(S, u^*TE, u^*TL) \arrow[dd, two heads, "\pi_{k,p}"] \arrow[dr, "D_u"] & \\
& \Lambda^{0,1}_{j,J_\Gamma}(S, u^*TF \oplus H)_{k-1,p} \arrow[d, two heads, "d\pi_*"] \\
W^{k,p}(S, \pi \circ u^*TB, \pi \circ u^*TL_B) \arrow[r, "D_{\pi\circ u}"] & \Lambda^{0,1}_{j,J_B}(S, \pi \circ u^*TB)_{k-1,p}
\end{tikzcd}
$$

where $D_{\pi\circ u}$ is surjective by a doubling argument for constant disks; see the argument in theorem 15.

Case 2: u is only constant in the vertical direction.

To get surjectivity onto the first summand we leverage the upper triangular part of the a.c. structure. First consider the case when u has no tangencies to the divisor. Following the type of argument in [MS04], we prove that the image of the linearized map is dense in $\Lambda^{0,1}(S, u^*TF)_{k-1,p}$. Suppose that the image is not dense. Since this is a Fredholm operator, the image is closed. By the Hahn-Banach theorem, there is an non-zero element $\eta \in \Lambda^{0,1}(S, u^*TF)_{k-1,q}$ such that

$$\int_C \langle D_u^{TF}\xi + K \circ du_H \circ j, \eta \rangle = 0 \tag{4.14}$$

for every $\xi \in W^{k,p}(S, u^*TF)$ and K with $J_F K + K J_B = 0$. Thus, we have the following identities:

$$\int_C \langle D_u^{TF}\xi, \eta \rangle = 0 \tag{4.15}$$

$$\int_C \langle K \circ du_H \circ j, \eta \rangle = 0 \tag{4.16}$$

It follows [MS04] that η is a solution the Cauchy-Riemann type equation

$$D_u^{TF*}\eta = 0$$

where D_u^{TF*} is the formal adjoint. Thus, η is of class $(k-1, q)$, and it follows that $\eta \neq 0$ on a dense subset of S.

Lemma 6. *Let* $0 \neq \eta \in Y$ *and* $0 \neq \xi \in X$ *with corresponding a.c. structures* J_Y *resp.* J_X*. Then there is a* K *with* $J_Y K J_X = K$ *such* $K\xi = \eta$

Proof. This requires us to find a complex anti-linear K such that $K\xi = \eta$, which is straightforward. See [MS04].

$\square$

Pick a point p where $du_H \neq 0 \neq \eta$ that is contained in the complement of $\overline{\mathcal{U}}_\Gamma^{\text{thin}}$. Then there is a $K_0 \in T_{J_{u(p)}}\mathcal{J}$ such that $\langle K_0 \circ du_{H,p} \circ j, \eta(p)\rangle > 0$. From the perturbation data $J_\Gamma : C \to \mathcal{J}_{ut}(\omega, D)$, we construct a section $K_\Gamma : C \to T_{J_\Gamma}\mathcal{J}_{ut}$ such that $K_\Gamma(p, u(p)) = K_0$ and K_Γ is supported in a sufficiently small neighborhood $U \times V$ with u injective on U and $\langle K_\Gamma(x, u(x)) \circ du_{H,x} \circ j, \eta(x)\rangle > 0$ whenever $K_\Gamma(x, u(x)) \neq 0$. We must then have that

$$\int_C \langle K \circ du_H \circ j, \eta\rangle > 0$$

which is a contradiction. Therefore, the linearized operator must be surjective onto the TF part of the summand in this case.

When there are tangencies to the divisor, the above method in combination with Lemma 6.6 from [CM07] gives surjectivity.

To check surjectivity onto the compliment of $\Lambda^{0,1}(S, u^*TF)_{k-1,q}$, we again use the fact that $\Lambda^{0,1}(S, u^*TF)_{k-1,q} = \ker d\pi_*$ and that the diagram from case 1 commutes. It is then enough to check the surjectivity of

$$W^{k,p}(S, \pi \circ u^*TB, \pi \circ u^*TL_B) \xrightarrow{D_{\pi\circ u}} \Lambda^{0,1}_{j,J_B}(S, \pi \circ u^*TB)_{k-1,p}$$

wherefore $\pi \circ u$ is adapted to D_B after forgetting constant and unstable surface components, so surjectivity follows from the work of [CWa].

Case 3: $du_H, du_F \neq 0$

This is a combination of cases 1 and 2 by working with the splitting $\Lambda^{0,1}_{j,J,\Gamma}(S, u^*TF \oplus H)_{k-1,p} \cong \ker d\pi_* \oplus \Lambda^{0,1}_{j,J_B}(S, \pi \circ u^*TB)_{k-1,p}$

Surjectivity on the edges is a matter of a standard argument: The linearization of the operator $\frac{d}{ds} - X$ at a solution u_T is

$$(V, X_\Gamma) \mapsto \nabla_s V - X_\Gamma u_T$$

where $(V, X_\Gamma) \in u_T^* TL \times \mathrm{Vect}_c^l(\overline{\mathcal{T}}_\circ \times L, \mathbb{R})$. If $x \in u_T^* TL$ is in the cokernel this linearization, then we have

$$\int_T \langle \frac{d}{ds} V, x \rangle = 0$$

$$\int_T \langle X_\Gamma u_T, x \rangle = 0$$

for all $V \in u_T^* TL$ and $X_\Gamma \in \mathrm{Vect}_c^l(\overline{\mathcal{T}}_\circ \times L, \mathbb{R})$. Thus, if $x \neq 0$, choose X_Γ so that the pairing $\langle X_\Gamma u_T, v \rangle$ is non-trivial and positive. This gives a contradiction.

By the implicit function theorem, this universal moduli space is a C^q Banach manifold. The general theory of real Cauchy-Riemann operators [MS04] tells us that the linearization $D_u + K \circ du \circ j$ is Fredholm, so has finite dimensional kernel. We now consider the restriction of the projection $\Pi : \mathcal{B}^i_{k,p,\Gamma,l} \to \mathcal{P}^l_\Gamma(E, D)$ to the universal moduli space. The kernel and cokernel of this projection are isomorphic the kernel and cokernel of the operator D_u, respectively. Thus, Π is a Fredholm operator with the same index as D_u. Let $\mathcal{M}_d^{univ,i}$ be the component of the universal space on which Π has Fredholm index d. By the Sard-Smale theorem, for q large enough, the set of regular values of Π, $\mathcal{P}_\Gamma^{l,reg}(E, D)_{d,i}$, is comeager. Let

$$\mathcal{P}_\Gamma^{l,reg}(E, D)_d = \bigcap_i \mathcal{P}_\Gamma^{l,reg}(E, D)_{d,i}$$

Then this is also a comeager set. An argument due to Taubes (see [MS04]) shows that the set of smooth regular perturbation datum

$$\mathcal{P}_\Gamma^{reg}(E, D)_d = \bigcap_l \mathcal{P}_\Gamma^{l,reg}(E, D)_d$$

is also comeager. For $P_\Gamma = (J_\Gamma, f_\Gamma)$ in the set of smooth regular data, notate $\mathcal{M}^i_\Gamma(E, L, D, P_\Gamma)$ as the space of P_Γ trajectories in the trivialization i, a C^q manifold of dimension d. By elliptic regularity, every element of $\mathcal{M}^i_\Gamma(E, L, D, P_\Gamma)$ is smooth. Using the transition maps for the universal curve of Γ, we get maps $g_{ij} : \mathcal{M}^i_\Gamma \cap \mathcal{M}^j_\Gamma \to \mathcal{M}^i_\Gamma \cap \mathcal{M}^j_\Gamma$ that serve as transition maps for the space

$$\mathcal{M}_\Gamma(E, L, D, P_\Gamma) = \bigcup_i \mathcal{M}^i_\Gamma(E, L, D, P_\Gamma)$$

Since each piece $\mathcal{M}^i_\Gamma(P_\Gamma)$ and the moduli space of treed disks is Hausdorff and second countable and the moduli space of treed disks is, it follows that $\mathcal{M}_\Gamma(P_\Gamma)$ is Hausdorff and second countable.

The gluing argument that produces the tubular neighborhood of $\mathcal{M}'_\Gamma(E, L, D, P)$ in $\mathcal{M}_\Gamma(E, L, D, P)$ is the same as in [CWb; CWa]. The matter of assigning compatible orientations is also similar.

□

4.4 Compactness

The main goal of this section is to establish the boundary strata of the compactified moduli space $\overline{\mathcal{M}}_\Gamma(L, D, P_\Gamma)$ as a product of moduli $\mathcal{M}_{\Gamma'}(L, D, P_{\Gamma'}) \times \mathcal{M}_{\Gamma''}(L, D, P_{\Gamma''})$ of sub types when we have a coherent perturbation datum. We use the existence of a divisor D_B and an appropriate choice of perturbation datum to rule out sphere bubbling in the base, and then complete the result with well known facts about compactness in monotone symplectic manifolds.

Definition 33. For a divisor $D = \pi^{-1}(D_B)$, we say that an adapted (upper triangular) a.c. structure J with basic lower block diagonal J_{D_B} is *e-stabilized* by D if J_{D_B} is e-stabilized by D_B as in definition 16:

(a) (Non-constant spheres) D_B contains no non-constant J_{D_B}-holomorphic spheres of energy less than e.

(b) (Sufficient intersections) each non-constant J_{D_B}-holomorphic sphere $u : S^2 \to B$ resp. J_{D_B}-holomorphic disk $u : (D, \partial D) \to (B, L_B)$ with energy less than e has at least three

three resp. one intersection points resp. point with the divisor D_B. That is, $u^{-1}(D_B)$ has order at least three resp. one.

.

Definition 34. We say that D is of large enough degree for an adapted J if D_B is for J_{D_B} as in definition 17:

1. $([D_B]^\wedge, \alpha) \geq 2(c_1(B), \alpha) + dim(B) + 1$ for all $\alpha \in H_2(B, \mathbb{Z})$ representing non-constant J_{D_B}-holomorphic spheres.

2. $([D_B]^\wedge, \beta) \geq 1$ for all $\beta \in H_2(B, L_B, \mathbb{Z})$ representing non-constant J_{D_B}-holomorphic disks.

A similar result holds as in Lemma 2 for a dense open set that is e stabilizing. Indeed, suppose we have a basic a.c. structure J_{D_B} for which D_B is of sufficiently large degree and is θ-close to J_B. There is an open, dense set $\mathcal{J}^*_\tau(B, D_B, J_B, \theta, e) \subset \mathcal{J}_\tau(B, D_B, J_B, \theta)$ given by Lemma 2. To get a collection of upper triangular e-stabilizing a.c. structures on E, we take the inverse image of this set under the projection $\pi : \mathcal{J}_{ut,\tau} \to \mathcal{J}_{B,\tau}$. We shall denote the (dense, open) set obtained in this manner $\mathcal{J}^*_\tau(E, D, J_B, \theta, e)$.

For a π-stable combinatorial type Γ, let $\Gamma_1, \ldots, \Gamma_l$ be the decomposition obtained by cutting at boundary disk nodes of positive finite length. Let $\overline{U}_{\Gamma_1}, \ldots, \overline{U}_{\Gamma_l}$ be the corresponding decomposition of the universal curve. In the case where L_B is rational, any stable treed holomorphic disk projected to B with domain of unmarked type Γ_i and transverse intersections with the divisor has energy at most

$$n(\Gamma_i, k) := \frac{n(\Gamma_i)}{C(k)} \tag{4.17}$$

on the component $\overline{U}_{\Gamma_i}$, where $n(\Gamma_i)$ is the number of markings on $\overline{U}_{\Gamma_i}$ and $C(k)$ is an increasing linear function of k arising in the construction of D_B in [CM07].

Definition 35. A perturbation datum $P_\Gamma(E, D) = (F_\Gamma, J_\Gamma)$ for a type of stable treed disk Γ is *stabilized* by D if J_Γ takes values in $\pi^{-1}\mathcal{J}^*_\tau(B, D_B, J_B, \theta, n(\Gamma_i, k))$ on $\overline{U}_{\Gamma_i}$

If the sub types Γ_i are only obtained by deleting positive finite length nodes, then we must have

$$\iota(\Gamma, \overline{x}) = \sum_{i=1}^{l} \iota(\Gamma_i)$$

Moreover, if $\iota(\Gamma, \overline{x}) \leq 1$, then regularity can be achieved for Γ by theorem 11, so each of the sub types Γ_i are also regular and therefore of positive index. On the other hand, if the configuration Γ has an edge of length 0, then we can obtain a configuration of index $\iota(\Gamma) + 1$ by either deforming the node into a single disk or making the edge length non-zero by (tubular neighborhood) part of theorem 11. Therefore, the limit of configurations that leave a length 0 edge unchanged will not contribute to the A_∞ algebra, so we will not pay heed to this case.

Theorem 12. *For any collection $P = (P_\Gamma)$ of coherent, regular, stabilized M-type perturbation data and any uncrowded type Γ of expected dimension at most one, the moduli space $\overline{\mathcal{M}}_\Gamma(L, D, P_\Gamma)$ of π-adapted stable treed marked disks of type Γ is compact and the closure of $\mathcal{M}_\Gamma(L, D, P_\Gamma)$ only contains configurations with horizontally non-constant disk bubbles.*

Proof. We check sequential compactness. Let Γ be a connected, uncrowded, π-stable combinatorial type, and let $u_\nu : C_\nu \to E$ be a sequence of J_Γ-holomorphic maps. As in the above discussion, decompose Γ into regular sub types $\Gamma_1, \ldots, \Gamma_l$ of index 0 or 1 that contain only marked or unmarked vertices.

Case 1: Γ_i is an unmarked partial sub type.

Since we are on an unmarked subtype, the π-adapted Floer trajectories are actually *adapted* to D in the sense of [CWa]. The sequence $u^\nu : C_i^\nu \to E$ has a subsequence that converges in the Gromov topology to $u : C_i' \to E$ for a possibly unstable curve class $[C_i']$ of combinatorial type Γ_i^∞. Since $\pi(u^\nu) \mapsto \pi(u)$, the fact that u is π-adapted follows from [CWa]. We include the argument here for completeness' sake.

Since $J_\Gamma = J_D \in \mathcal{J}_\tau^*(B, D_B, J_B, \theta, n(\Gamma_i, k))$ over D, D_B contains no $\pi_* J_D$-holomorphic spheres from $\pi(u)$. Thus, the (non-constant spheres) property.

Any unstable disk component u_j in the limit would be J_D-holomorphic. Unless it is constant, $\pi \circ u_j$ would be J_{D_B}-holomorphic and have at least one intersection with D_B by the stabilizing property of D_B. Thus, unstable disk components can only occur in the vertical direction.

Similarly, suppose we have a non-constant unstable sphere component u_j. Then $\pi \circ u_j$ has energy at most $n(\Gamma_i, k)$ since it is the limit of types with this energy bound. Since $J_\Gamma = J_D$ on $\pi \circ u_j$, there must be at least three intersection points with D_B on this component, unless $\pi \circ u_j$ is constant. Thus, unstable sphere components only occur in the vertical direction.

Therefore, for an unmarked sub-type, the only unstable component that we can pick up is constant in the horizontal direction (a marked vertex). We argue that this cannot occur: If we have a vertical sphere bubble v from an unmarked disk, then it must have positive energy and hence positive Chern number by the monotone property of the fiber. The limiting configuration is regular by appropriate choice of coherent perturbation data, and by the coherence condition we also have regularity for the configuration without the sphere bubble, so both are of non-negative expected dimension. This tells us that a vertical sphere bubble is a codimension 2 phenomenon. However, the index of Γ_i is at most 1, and since the index of the limit is at most the limit of the indices, we arrive at a contradiction.

We would like to rule out vertical disk bubbles, which the rest of case 1 is devoted to. Suppose that in the limit, we get a vertical disk bubble $u_1 : C_1 \to (F, L_F)$ off of an unmarked sub type $u_2 : C_2 \to (E, L)$. Let us first consider the case that u_2 is non-constant: The index of the limiting configuration Γ_i^∞ is either 0 or 1, and so u lies in a smooth moduli space by 11. The minimal Maslov index for a J_F holomorphic disk with boundary in L_F is 2 by assumption, so the map u_2 lies in a moduli space of type $\widehat{\Gamma_i^\infty}$ that is smooth of expected dimension by theorem 11 since $\iota(\widetilde{\Gamma_i^\infty}) \leq \iota(\Gamma_i^\infty) \leq 1$. However, this tells us that u_2 is a codimension 2 configuration, which is a contradiction.

Next, suppose that u_1 is constant on surface components (so a Morse flow tree). First suppose that the total limiting configuration has only one input (so u_1 has two inputs). Then there are algebraic cancellations in the output of μ^1 given by considering the differing orientations on the moduli spaces corresponding to the different orderings of the marked points on u_1.

In the case where we have more than one input, we see that u_1 must be equal to a critical point x_0. The disk bubble u_2 must be part of a non-constant J_F-holomorphic configuration $\underline{u_2}$ with a single output to x_0. By monotonicity of L_F and the assumption on the minimal Maslov number, $I(u_2) \geq 2$. Moreover, $\underline{u_2}$ is regular by the coherence assumptions on the perturbation datum. In order for the total configuration to be isolated, we must have that that $\dim W_f^+(x_0) = 0$. Therefore, all of the inputs of the original sequence of configurations are x_0. Since the Maslov index is preserved under taking limits, this shows that the original combinatorial type has an expected dimension of at least 2: This is a contradiction.

Case 2: Γ_i is a marked sub type.

By construction, the pseudo-gradient X restricted to any critical fiber is X_g, a pseudo-gradient for g. Thus, for a connected marked sub type mapping to a critical fiber, we are considering Morse-Floer trajectories on a monotone Lagrangian $L_{F_b} \subset F_b$. Away from the critical fibers, the flow lines intersect the fibers transversely, so the only marked configurations contained in non-critical fibers are nodal-disks with zero length edges.

The index formula that gives us that the dimension of the open strata for an admissible set of critical points $(x_0, \ldots, x_n)$, after modding out by isomorphism, is:

$$\begin{aligned}\iota(\Gamma, \overline{x}) :=& \dim W_X^+(x_0) - \sum_{i=1}^{n} \dim W_X^+(x_i) + \sum_{i=1}^{n} I(u_i) + n - 2 \\ & - |\mathrm{Edge}^0_{<\infty}(\Gamma)| - |\mathrm{Edge}_\infty(\Gamma) - (n+1)|/2 - 2|\mathrm{Edge}_{<\infty,s}(\Gamma)|\end{aligned}$$

where $I(u_i)$ is either the Maslov index of u_i or $2c(A_i)$ with A_i as the spherical homology class of u_i. For a fixed energy, Gromov compactness gives us a subsequence that Gromov-Floer converges to a limiting treed holomorphic trajectory u of the same energy, that is again contained in a single fiber. The total Maslov/Chern class $\sum_i I(u_i)$ is preserved under limits (see [Oh93]), and this configuration is of expected dimension ≤ 1. Thus it can be made regular by the transversality argument above. First assume that the limiting configuration Γ contains a non-constant sphere bubble. Because of the spherical node and the fact that $c_1(A_i) = \lambda E(A_i)$, this configuration is of codimension at least 2, giving negative expected dimension. This is a contradiction.

The case against a vertical disk bubble is the same as at the end of Case 1.

□

4.5 The case of a Kähler fiber

When the fibers are Kähler, we prove a version of the Oka principle to lift holomorphic curves in the base to the total space. The Kähler structure need not be regular, so we describe a correspondence between disks/spheres for a fiber wise integrable J_I and disks for a small perturbation of J_I. Later, this correspondence will help us compute the associated superpotential for a fibered Lagrangian and subsequently some Floer cohomology groups.

We set blanket assumptions for this section. Let G be a compact lie group and $G_{\mathbb{C}}$ its complexification.

Definition 36. 1. A *symplectic Kähler fibration* is a symplectic Mori fibration

$$(F, \omega_F) \to (E, \omega_K) \to (B, \omega_B)$$

with structure group G such that (F, ω_F) is a Hamiltonian G space. We assume that the action of G extends to an action of $G_{\mathbb{C}}$ for which there exists a $G_{\mathbb{C}}$-invariant integrable ω_F-compatible complex structure J_G.

2. A fibered Lagrangian in this situation is a sub-bundle $L_F \to L \to L_B$ of Lagrangians such that L_B is rational and L_F is monotone G-invariant. We assume that the transition functions are Čech co-cycles in G, so that E has an associated principle G-bundle.

An example of a symplectic Kähler fibration a Hirzebruch surface, with structure group S^1.

The flag manifold 1.1 is not a good example, as the structure group can only be reduced to $SU(2)$, so there is no invariant Lagrangian $L_F \subset \mathbb{P}^1$. However, we will see that results of this section can still be applied, due to the fact that $SU(2)$ is simply connected.

On the other hand, any bundle of the form $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}_m) \to \mathbb{P}^2$ is an example of a symplectic Kähler fibration, for which one can find fibered Lagrangians.

Let $TF \oplus H$ be a symplectic connection on (E, ω_K), and denote J^G as an almost complex structure that tames ω_K of the form

$$J^G = \begin{bmatrix} J_G & 0 \\ 0 & J_B \end{bmatrix}$$

where the block decomposition is with respect to the decomposition $TF \oplus H$, and J_B is part of a coherent, domain dependent perturbation datum for (B, L_B, D_B). Denote

$$J_\Gamma = \begin{bmatrix} J_F & K \\ 0 & J_B \end{bmatrix}$$

as part of a domain-dependent coherent perturbation datum for E given in theorem 9.

Let $u : D \to B$ be a J_B holomorphic map. The pullback of the fiberwise Kähler form a induces a connection $TF \oplus H_D$ on u^*E, where $H_D = TF^{\perp u^* a}$ such that the projection $u^*\pi$ is holomorphic with respect to the structure

$$\begin{bmatrix} J_G & 0 \\ 0 & j \end{bmatrix} \tag{4.18}$$

on u^*E and j on D.

Let P_{u^*E} be the associated principle G-bundle to u^*E. Since parallel transport on u^*E is G valued, H_D defines a connection on P_{u^*E}. We also denote this connection H_D.

The space of diffeomorphisms of P_{u^*E} covering the identity form an infinite dimensional Lie group, called the group of gauge transformations $\mathcal{G}(P_{u^*E})$. Let $\phi \in \mathcal{G}(P_{u^*E})$, and let g_p be the unique element in G such that $\phi(p) = p \cdot g_p$. We require that ϕ preserve the G-structure of P_{u^*E}: $\phi(p \cdot g) = \phi(p) \cdot g$. From this we get

$$p \cdot g \cdot g_{p \cdot g} = p \cdot g_p \cdot g$$

so that $g_{p \cdot g} = g^{-1} g_p g$. Thus $\mathcal{G}(P_{u^*E}) \cong C^\infty(P_{u^*E}, G)^G$, where the superscript denotes the equivariance $f(p \cdot g) = g^{-1} f(p) g$. When the domain of u is a disk, we have that $P_{u^*E} \cong D \times G$ as a smooth G-bundle, so the group of gauge transformations becomes $C^\infty(D, G)$.

In a similar manner, we denote the complexified gauge group as $\mathcal{G}_{\mathbb{C}} := C^\infty(D, G_{\mathbb{C}})$.

Given a $\mathcal{G} \in \mathcal{G}_{\mathbb{C}}(P_{u^*E})$ we can obtain a different connection via $\mathcal{G}(H_D)$. Thus, the complex gauge transformations act on the space of connections of P_{u^*E}. Moreover, we can consider holomorphic gauge transformations, that we define as elements in $\mathrm{Hol}(D, G_{\mathbb{C}})$.

We will expose the following result:

Theorem 13. *There is a holomorphic gauge transformation $\mathcal{G} \in Hol(D, G_{\mathbb{C}})$ that is G-valued along ∂D such that $\mathcal{G}(H_D)$ is a flat connection on P_{u^*E}. Thus, $u^*E \cong D \times F$ as a (j, J_G) holomorphic fiber bundle via an isomorphism that preserves Lagrangian boundary conditions.*

We will see that this theorem is a corollary of a result by Donaldson, which is an h-principle for complex manifolds with boundary:

Theorem 14. *[Don92] Let $V \to Z$ be a holomorphic vector bundle over a complex manifold* *with boundary. Let f be a metric over the boundary. Then there is a unique Hermitian metric h satisfying:*

1) $h|_{\partial Z} = f$

2) $i\Lambda F_h = 0$

Item (2) is known as the *Hermitian Yang-Mills* equation. F_h is the curvature of the connection associated to h, and $\Lambda : \mathrm{Map}^{(1,1)} \to \Omega^0$ is the Kähler component in the decomposition $\Omega^{(1,1)} \otimes V \cong \Omega_0^{(1,1)} \otimes V + \Omega^0\omega \otimes V$.

When Z is complex dimension one, the Hermitian Yang-Mills equation says that the Chern connection induced by h is flat. Moreover, when $Z = (D, \partial D)$, the existence of a flat connection allows trivialization via parallel transport.

Sketch of proof for 14. We study the evolution equation

$$h^{-1}\frac{\partial h}{\partial t} = -2i\Lambda F_h, \quad h|_{\partial Z} = f \tag{4.19}$$

starting with some arbitrary smooth extension h_0 of f. The key facts to establish a limiting solution to this heat flow are short/long time existence, the observation that the highest order term of $-2i\Lambda F_h$ is the laplacian, and the following lemma:

Lemma 7. *[Don92]Suppose $\theta \geq 0$ is a sub-solution to the heat equation on $Z \times [0, \infty)$, e.g. $\dfrac{\partial \theta}{\partial t} + \Delta\theta \leq 0$. If $\theta \equiv 0$ on ∂Z for all time, then θ decays exponentially*

$$\sup \theta(z, t) \leq C\mathrm{e}^{-\mu t}$$

where μ depends only on Z, and C on the initial value of θ.

One then observes that for a solution to 4.19, the quantity $\varepsilon = \|i\Lambda F_h\|_h^2$ is non-negative sub-solution to the homogeneous heat flow.

On the other hand, consider the bundle of metrics $\mathcal{H} \to V$, with transition functions given by some reduction of structure group on V and metric . The quantity $i\Lambda F_h$ is the time derivative of h in this bundle, and the quantity $\sqrt{\varepsilon}$ is the velocity of the family $h_t(z)$. By the above exponential bound, $h_t(z)$ has finite length as a path in the fiber $\mathcal{H}_z$. By completeness of $\mathcal{H}_z$, there is a limiting metric in each fiber. Then by [Don92], there is a subsequence of h_n that converges in C^∞ to h_∞ that by (4.19) is a solution to the Yang-Mills problem. □

Proof of 13 following 14. Since J_G is $G_{\mathbb{C}}$ invariant and G acts by symplectomorphisms on F, we get an induced representation $\rho : G \to \mathrm{Symp}(V, \omega_F) \cap GL(V, J_G)$ for some symplectic vector space V. We form the associated vector bundle $\mathcal{V} := u^*E \times_\rho V$ that has an integrable complex structure and Hermitian metric arising from $(\omega_F, J_G) := h_0$. Applying Donaldson's result 13 we get a flat Hermitian metric h_∞ that agrees with h_0 on ∂D. The following is immediate:

1. There is a complex gauge transformation $\mathcal{G} \in \mathrm{Hol}(D, G_{\mathbb{C}})$ so that $\mathcal{G}^* h_0 = h_\infty$.

2. By $G_{\mathbb{C}}$ invariance, $\mathcal{G}^* J_G = J_G$, so J_G is compatible with h_∞.

3. Since $h_0 = h_\infty$ on ∂D, so $\mathcal{G}$ is constant there.

4. $(\mathcal{V}, h_\infty, J_G)$ has a flat Chern connection, so it can be trivialized by parallel transport. This induces a further isomorphism

$$\Phi : \mathcal{V} \xrightarrow{\cong} D \times V \tag{4.20}$$

   which corresponds to an element $\mathcal{G}_\Phi \in C^\infty(D, G)$ since the connection is G-valued.

Applying the complex gauge transformation $\mathcal{G}_\Phi \circ \mathcal{G}$ to u^*E we have an isomorphism of (j, J_G) complex manifolds

$$u^*E \xrightarrow{\cong} (D \times F, \partial D \times L_F) \tag{4.21}$$

that preserves the sub-bundle $(u|_{\partial D})^* L$ by the G-invariance of L_F.

□

Corollary 1. *For a symplectic Kähler fibration and any J^G-holomorphic disk $u : (D, \partial D) \to (E, L)$, the bundle $(\pi \circ u)^* E$ is holomorphically trivial. Thus u can be written as $\pi \circ u \times \tilde{u}$:*

$(D,\partial D) \to (B, L_B) \times (F, L_F)$, where $\tilde{u}$ is a J_G-holomorphic section of the bundle $\pi \circ u^*E$ with boundary values in L_F.

Remark 3. In the case of $\mathrm{Flag}(\mathbb{C}^3)$, the structure group is $SU(2)$. Since $SU(2)$ is simply connected, the boundary values of the complex gauge transformation $\mathcal{G}_\Phi \circ \mathcal{G}$ can be extended to a gauge transformation $\mathcal{G}_\mathbb{R} \in C^\infty(D, G)$. The resulting bundle $\mathcal{G}_\mathbb{R}^{-1} \circ \mathcal{G}_\Phi \circ \mathcal{G}$ has a flat connection (since $C^\infty(D, G)$ preserves curvature), so it is trivial. The result of Theorem 13 and the above corollary continue to hold in this situation.

4.5.1 Existence of regular lifts

Given a regular J_B-holomorphic disk, we show that we can find a regular lift using the results of the previous section.

It follows from theorem 13 that any section $\tilde{u} : (D, \partial D) \to D \times (F, L_F)$ will induce a J^G-holomorphic section of the bundle $\widehat{u} : D \to u^*E$ with L_F boundary values. In particular, we can choose a $p \in L_F$ and take the constant section $\tilde{u}_p(z) = (z, p)$. We will prove a theorem below that such a section is regular in the sense that the linearized $\overline{\partial}$ is surjective. However, we would like to characterize these "vertically constant" sections based on their covariant derivatives. To this end, let H be a connection on $E \to B$, and for $Y \in TB$ let $\Phi_{t,Y}$ be the associated flow of the horizontal lift of Y to H.

Definition 37. The *covariant derivative* of a section s with respect to Y and H is

$$\nabla_Y^H s := \frac{d}{dt}|_0 \Phi_{-t,Y}(s_{\Phi_{t,Y}})$$

This is going to be a section of the bundle $s^*TF \to B$. It is a general fact that $\nabla_Y s(p)$ only depends on the value of Y at the particular point p, see [DK90] for instance.

For a smooth map $u : D \to B$, we can define a covariant derivative on u^*E by $\nabla_Y^{u^*H} X := u^* \nabla_{du(Y)} X$. We note again that the dependence in Y is only at a single point $z \in D$, so we need not define $du(Y)$ everywhere.

If $\mathcal{G}$ is an element of the complex gauge group of E, then the covariant derivative of $\mathcal{G}(H)$ is given by $\nabla^{\mathcal{G}(H)}s = \mathcal{G}\nabla^H(\mathcal{G}^{-1}s)$ [DK90]. Therefore, if s is an H covariant constant section in the sense that

$$\nabla^H s \equiv 0$$

then $\mathcal{G}s$ is a $\mathcal{G}(H)$ covariant constant section. This shows that the following definition is independent of complex gauge transformation:

Definition 38. A J^G-holomorphic map u is *vertically constant* with respect to H if it induces a covariant-constant section of $\pi \circ u^*E$.

When $\pi \circ u^*E \cong D \times F$ and TD is the trivial connection, we have that $\pi_{TF} \circ du = \nabla^{TD}u$, so the covariant derivative agrees with the derivative of a section in the vertical direction.

The uniqueness statement in Donaldson's result gives us a uniqueness of vertically constant lifts:

Lemma 8. *Let $u : D \to (B, L_B)$ be a J_B-holomorphic disk with a distinguished boundary value $u(x_0)$. Given a point $p \in \pi^{-1}(u(x_0)) \cap L_F$ there is a unique vertically constant $\begin{bmatrix} J_G & 0 \\ 0 & J_B \end{bmatrix}$-holomorphic lift $\hat{u}$ with $\hat{u}(x_0) = p$*

Proof. For $u^*E \cong D \times F$ the statement is clear. Therefore, it suffices to show that the gauge transformation in theorem 13 is unique.

Let $\hat{u} : D \to D \times F$ be the section $\hat{u}(z) = (z, p)$ and let $\mathcal{G}$ be a trivializing (complex) gauge transformation from theorem 13, given by first applying the unique complex gauge transformation $\mathcal{G}_{\mathbb{C}}$ from 14 that is the identity on the boundary, followed by the real gauge transformation $\mathcal{G}^p_{\mathbb{R}}$ that trivializes the bundle via parallel transport from the point p. Then $\mathcal{G}\hat{u}$ is a vertically constant section of u^*E with $\mathcal{G}\hat{u}(x_0) = p$. If $\mathcal{G}^{p'}_{\mathbb{R}}$ is a trivializing gauge transformation corresponding to parallel transport from a point p', then there is an element $g \in G$ so that $\mathcal{G}_{\mathbb{R}} = g \cdot \mathcal{G}'_{\mathbb{R}}$

since the connection is flat. Either g fixes $p \in L_F$ or it does not. Since the gauge transformations commute with the G action, we have that $\hat{u}$ is unique up to $\mathrm{stab}_G(p)$, but any choice of coset representative will give the same section since $\hat{u}$ is constant.

□

Theorem 15. *Let $u : C \to B$ be a regular J_B-holomorphic type with one output $u(x_0) = q \in L_B$, no sphere components, and no broken edges. Then for any $p \in \pi^{-1}(q) \cap L_F$, there is a unique lift $\hat{u} : C \to E$ with $\hat{u}(x_0) = p$ that is vertically constant on disk components and J^G-holomorphic*

Proof. First, we construct a lift by matching a chain of boundary conditions, and then we prove transversality. Let u_0 be the restriction of u to the disk component D_0 closest to the output, let E_{01} denote the edge between D_0 and some adjacent component D_1, and let u_1 denote the restriction of u to D_1 that meets E_{01} at x_2. By lemma 8, there is a unique vertically constant lift $\hat{u}_0$ with $\hat{u}_0(x_0) = p$. Let x_1 be the boundary point corresponding to the output on D_0. The projection of the flow of X starting at $\hat{u}(x_1)$ agrees with the flow of X_b, so flowing X for time $\ell(E_{01})$ lands at a point $p_1 \in \pi^{-1}(u(x_2))$. Take the unique lift of $\hat{u}_1$ of u_1 with $\hat{u}(x_2) = p_1$, and continue in this fashion until a lift of u is constructed on every disk component. This gives a J^G-holomorphic Floer trajectory $\hat{u} : C \to E$ with boundary in L.

The linearized operator at this particular $\hat{u}$ is actually surjective, and this is due to the facts that we are using a block diagonal almost complex structure and that $\hat{u}$ is vertically constant. Indeed, let us focus on a single disk component. The range splits in the expression 4.12 as

$$\begin{aligned} D_{\hat{u}}(K) : W^{k,p}(D, \hat{u}^*TF, \hat{u}^*_{\partial S}TF \cap TL) \oplus W^{k,p}(D, \hat{u}^*H, \hat{u}^*_{\partial S}H \cap TL) \\ \to \Lambda^{0,1}_{j,J_G}(D, \hat{u}^*TF)_{k-1,p} \oplus \Lambda^{0,1}_{j,J_B}(D, u^*H)_{k-1,p} \end{aligned}$$

so it suffices to show surjectivity onto each summand; We achieve surjectivity onto $\Lambda^{0,1}_{j,J_G}(S, u^*TF)_{k-1,p}$ by the usual doubling trick for constant maps (outlined below), and surjectivity onto the second summand by the argument for theorem 11 following [CWa].

To show surjectivity onto the first summand, it is enough to consider the properties of $D_{\tilde{u}}(K) : W^{k,p}(D, \tilde{u}^*TF, \tilde{u}^*_{\partial S}TF \cap TL) \to \Lambda^{0,1}_{j,J_G}(S, \tilde{u}^*TF)$. Since $\tilde{u}$ is constant, it extends to a map $\tilde{u} : \mathbb{P}^1 \to F$. The map

$$\Lambda^{0,1}_{j,J_G}(\mathbb{P}^1, TF)_\infty \to \Lambda^{0,1}_{j,J_G}(D, TF)_\infty$$

is surjective by a Schwartz reflection principle, so for the smooth case it suffices to show that the linearized operator is surjective onto $\Lambda^{0,1}_{j,J_G}(\mathbb{P}^1, TF)_\infty$. Since J_G is integrable, $D_{\tilde{u}}$ is precisely the Dolbeault operator for this vector bundle, and the cokernel is the Dolbeault cohomology group $H^{0,1}(\mathbb{P}^1, \tilde{u}^*TF)$. On the other hand, $\tilde{u}^*TF$ splits as a direct sum of holomorphic line bundles with respect to J_G, all of which are trivial, so it suffices to consider $H^{0,1}(\mathbb{P}^1, \mathbb{C}) \cong H^{1,0}(\mathbb{P}^1, \mathbb{C}^\vee) \cong \mathbb{C}$. But the linearized operator at $\tilde{u}$ is not identically 0, so it must be surjective. Thus $D_{\tilde{u}}$ is surjective when we consider smooth η. In the $(k-1, p)$ case, we use the usual elliptic bootstrapping argument of the adjoint operator $D^\vee_{\tilde{u}}$.

□

Contained in the the proof of this theorem is the fact that we can achieve transversality for vertically constant J^G-holomorphic disks:

Corollary 2. *Let $u : D \to (B, L_B)$ be a J_B-holomorphic disk at which the linearized $\bar{\partial}_{J_B}$ operator is surjective. Then the linearized $\bar{\partial}_{J^G}$ operator is surjective at any vertically constant J^G-holomorphic lift of u.*

Remark 4. By the uniqueness content in Donaldson's result 14, the vertically constant lift that we get is unique once we prescribe a constant boundary condition in L_F. We will refer to this unique lift as the *Donaldson lift* through $p \in L_F$.

For completeness, we record the following result: Let $J_\Gamma = \begin{bmatrix} J_F & K \\ 0 & J_B \end{bmatrix}$ be a regular (domain-dependent) almost complex structure from theorem 11, and $u_\Gamma : S \to E$ a regular configuration of index 0 without sphere components. We would like to see that $\pi \circ u_\Gamma$ is regular and thus lies in a moduli of expected dimension.

Lemma 9. *Let $u_\Gamma : U_\Gamma \to E$ be a regular J_Γ-holomorphic configuration for smooth J_Γ. Then $\pi \circ u_\Gamma$ is a regular J_B-holomorphic configuration.*

Proof. The fact that $\pi \circ u$ is J_B holomorphic is clear. Furthermore, by the choice of pseudo-gradient perturbation data and the discussion in section 4.2.1, we have that $\pi \circ u$ is a Morse flow on edges. It remains to check transversality.

For $u_\Gamma|_D =: u$ restricted to a single disk or sphere component, we need to show that the linearized operator

$$D_{\pi\circ u} : W^{k,p}(S, \pi \circ u^*TB, \pi \circ u^*TL_B) \to \Lambda^{0,1}_{j,J_B}(S, \pi \circ u^*TB)_{k-1,p}$$

is surjective for $k > 2/p$ with $p \geq 2$. By the regularity assumption on u we have that

$$\begin{aligned} D_u : W^{k,p}(S, u^*TF, u^*_{\partial S}TF \cap TL) \oplus W^{k,p}(S, u^*H, u^*_{\partial S}H \cap TL) \\ \to \Lambda^{0,1}_{j,J_\Gamma}(S, u^*TF \oplus H)_{k-1,p} \end{aligned}$$

is surjective, where H is the symplectic connection. The projection $d\pi : u^*TF \oplus H \to u^*H$ is equivariant with respect to the almost complex structures (J_Γ, J_B), and so it induces a map

$$d\pi_* : \Lambda^{0,1}_{j,J_\Gamma}(S, u^*TF \oplus H)_{k-1,p} \to \Lambda^{0,1}_{j,J_\Gamma}(S, u^*H)_{k-1,p}$$

via $\eta \mapsto d\pi \circ \eta$. To see that this projection is surjective, we use the isomorphism

$$\begin{aligned} \Phi_{J_\Gamma,J^G} : \Lambda^{0,1}_{j,J_\Gamma}(S, u^*TF \oplus H)_{k-1,p} &\to \Lambda^{0,1}_{j,J^G}(S, u^*TF \oplus H)_{k-1,p} \\ \eta &\mapsto \frac{1}{2}(\eta + J^G \circ \eta \circ j) \end{aligned}$$

Wherefore $\Lambda^{0,1}_{j,J^G}(S, u^*TF \oplus H)_{k-1,p}$ now splits as

$$\begin{aligned} &\Lambda^{0,1}_{j,J^G}(S, u^*TF)_{k-1,p} \oplus \Lambda^{0,1}_{j,J^G}(S, u^*H)_{k-1,p} \\ \cong\ &\Lambda^{0,1}_{j,J^G}(S, u^*TF)_{k-1,p} \oplus \Lambda^{0,1}_{j,J^G}(S, \pi \circ u^*TB)_{k-1,p} \end{aligned}$$

One checks that the following diagram commutes:

$$\begin{tikzcd}
\Lambda^{0,1}_{j,J_\Gamma}(S, u^*TF \oplus H)_{k-1,p} \arrow[dd, "d\pi_*"] \arrow[dr, hook, two heads, "\Phi_{J_\Gamma, J^G}"] & \\
& \Lambda^{0,1}_{j,J^G}(S, u^*TF)_{k-1,p} \oplus \Lambda^{0,1}_{j,J^G}(S, \pi \circ u^*TB)_{k-1,p} \arrow[dl, two heads, "\pi_2"] \\
\Lambda^{0,1}_{j,J_B}(S, \pi \circ u^*TB)_{k-1,p} &
\end{tikzcd}$$

It commutes due to the fact that both are spaces of anti-holomorphic forms with J_B in the lower diagonal block. Thus, we see that $d\pi$ is surjective. To see that $D_{\pi \circ u}$ is surjective, we just need to check that the next diagram commutes:

$$\begin{tikzcd}
W^{k,p}(S, u^*TE, u^*TL) \arrow[dd, two heads, "\pi_{k,p}"] \arrow[dr, two heads, "D_u"] & \\
& \Lambda^{0,1}_{j,J_\Gamma}(S, u^*TF \oplus H)_{k-1,p} \arrow[d, two heads, "d\pi_*"] \\
W^{k,p}(S, \pi \circ u^*TB, \pi \circ u^*TL_B) \arrow[r, "D_{\pi \circ u}"] & \Lambda^{0,1}_{j,J_B}(S, \pi \circ u^*TB)_{k-1,p}
\end{tikzcd}$$

□

4.5.2 Perturbation of vertically constant lifts

In lieu of the regularity statement of theorem 15 and corollary 2, it may seem that we can just compute vertically constant J-holomorphic disks with J^G. However, a perturbation datum (P_Γ) containing J^G may not satisfy the coherence axioms 31, causing compactness to fail. Therefore, we want to make sure that the J^G-holomorphic disks can be related to some J_Γ-holomorphic disks where J_Γ is part of a coherent datum. We accomplish this relation by using perturbations that are contained in a small enough neighborhood of J^G.

Definition 39. For a combinatorial type Γ with $\iota(\Gamma, \bar{x}) \leq 1$, and two regular almost complex structures J_i $i = 0, 1$, define a *regular smooth homotopy* $J_t \in \mathcal{J}_\Gamma^{reg}(J_0, J_1) \subset \mathcal{J}_\Gamma(J_0, J_1)$ as

$$J_t : [0,1] \to \mathcal{J}^{\infty}_{ut,\tau}$$

$$J_0 = J_0$$

$$J_1 = J_1$$

such that the linearized operator

$$D_{u,J_t} + \frac{\partial}{\partial t} J_t \circ du \circ j$$

is surjective at each $t \in [0,1]$ and $u \in \mathcal{M}_\Gamma(L, J_t)$.

We use the following adaption of a theorem by McDuff-Salamon:

Theorem 16. *[MS04] For a combinatorial type Γ with $\iota(\Gamma, \bar{x}) \leq 1$ and a regular J_B, let J_i, $i = 0, 1$ be regular upper triangular almost complex structures for type $(\Gamma, \bar{x})$ such that $\iota(\Gamma, \bar{x}) \leq 1$, and let $\mathcal{J}_{\Gamma, J_B}(J_0, J_1)$ be the Banach manifold of smooth upper triangular homotopies that are constantly equal to J_B in the lower diagonal block. Then there is a Baire set of smooth homotopies $\mathcal{J}^{reg}_{\Gamma, J_B}(J_0, J_1) \subset \mathcal{J}_{\Gamma, J_B}(J_0, J_1)$ such that if $J_t \in \mathcal{J}^{reg}_{\Gamma, J_B}(J_0, J_1)$, then there is a parameterized moduli space $\mathcal{W}_{\Gamma, J_B}(J_t)$ that is a smooth oriented manifold with boundary*

$$\partial \mathcal{W}_\Gamma(J_t) = \mathcal{M}_\Gamma(L, J_0)^- \sqcup \mathcal{M}_\Gamma(L, J_1)$$

so that these two moduli spaces are oriented (compact) cobordant.

Proof. The proof is the same as an argument of theorem 11. The projection $\mathcal{M}^{univ}_\Gamma(L, J_t) \to \mathcal{J}_{\Gamma, J_B}(J_0, J_1)$ has the same Fredholm index and cokernel dimension as the linearized $\bar{\partial}$ operator, and the points where the projection is surjective are precisely the regular homotopies. One then uses the Sard-Smale theorem to find a Baire set where the projection is submersive. □

Let $0 < \epsilon < \epsilon_0 < 1$. The above theorem tells us that $\mathcal{M}_\Gamma(L, J_0)$ and $\mathcal{M}_\Gamma(L, J_\epsilon)$ compact cobordant, and thus are diffeomorphic for $\epsilon << 1$. Indeed, any regular homotopy induces a cobordism $\mathcal{W}^\vee_\Gamma(J_t) \to [0,1]$ that is a submersion at 0. Since the property of being a submersion is an open condition, this must be a submersion in a neighborhood $[0, \epsilon)$.

Since our choice of smooth homotopy can lie in a Baire set, we may assume that there is some $\epsilon \in [0, \epsilon_0)$ such that J_ϵ is regular for (Γ, y_0). Indeed, let us identify an open neighborhood $B_\epsilon(J_0)$ of J_0 with an open neighborhood of 0 in $T_{J_0}\mathcal{J}^\infty_{ut,\tau}$. Then the image of the map $\mathcal{J}^{reg}_{\Gamma,J_B}(J_0, J_1) \times [0, \epsilon_0) \to T_{J_0}\mathcal{J}^\infty_{ut,\tau}$ given by

$$(J_t, s) \mapsto J_s$$

forms a Baire set of a neighborhood of J_0. The intersection of this Baire set with the set of regular perturbation data $\mathcal{J}^{reg}_\Gamma(E, L) \cap B_\epsilon(J_0)$ must also be Baire.

With this in mind, let us only consider perturbations J_Γ in theorem 11 such that J_F resp. K are C^∞ close to J_G resp. 0.

For a connected combinatorial type Γ for B, let us restrict to the case when $\iota(\Gamma, x_0) = 0$ with a single output, and such that Γ has no sphere components or breakings at critical points. Let $u \in \mathcal{M}_\Gamma(B, L_B, J_B, x_0)$ with domain C, and for simplicity let us assume that the evaluation at the marked point z_0 on the boundary of C has $ev_u(z_0) = x_0$. For any lift y_0 of x_0, Donaldson's heat flow gives us a unique vertically constant lift v of u with $ev_v(z_0) = y_0$. For the purposes of this section, we repeat the index formula:

$$0 = \iota(\Gamma, x_0) = \sum_{i=1}^{m} I(u_i) - 2 - |\mathrm{Edge}^0_{<\infty}(\Gamma)| - |\mathrm{Edge}_{\infty,s}(\Gamma)|$$
$$- \sum_{e \in \mathrm{Edge}_{\infty,s}} m(e)$$

where $|\mathrm{Edge}^0_{<\infty}(\Gamma)|$ resp. $|\mathrm{Edge}_{\infty,s}(\Gamma)|$ is the number of disk nodes resp. number of interior markings that intersect the divisor. Let $(\pi^*\Gamma, y_0)$ be the combinatorial type of the Donaldson lift through y_0. This contains the following information:

1. The combinatorial type of the underlying metric tree $T = (\mathrm{Vert}(\Gamma), \mathrm{Edge}(\Gamma))$

2. The labelings $D : \mathrm{Vert}(\Gamma) \to \pi_2(E, L)$ of the relative homotopy classes

3. The enumeration $m : \mathrm{Edge}_{\infty,s}(\Gamma) \to \mathbb{Z}_{\geq 0}$

Let J_B be a regular, domain dependent perturbation data for the type Γ in (B, L_B) as in theorem 9, and let

$$J_{K,J_F} = \begin{bmatrix} J_F & K \\ 0 & J_B \end{bmatrix}$$

reference a taming almost complex structure on $(E, a + N\pi^*\omega_B)$ for which K is also domain dependent. Then, by theorems 1112, there is a Baire set of these for which the moduli space

$$\mathcal{M}_{\pi^*\Gamma}(L, J_{K,J_F}, y_0)$$

is smooth and compact of dimension 0.

Crucially, we want to see J^G is regular for configurations of type $\pi^*\Gamma$, so that we can use theorem 16 to construct a parameterized moduli space.

Lemma 10. *J^G is a regular almost complex structure for the moduli space $\mathcal{M}_{\pi^*\Gamma}(L, J^G, y_0)$.*

Proof. By corollary 2, the linearized operator is surjective at any vertically constant lift, so we show that $u \in \mathcal{M}_{\pi^*\Gamma}(L, J^G, y_0)$ is necessarily vertically constant.

Suppose that u is not vertically constant. The combinatorial type $\pi^*\Gamma$ has the same geometric realization as that of the regular configuration Γ, so it has no sphere components. Without lack of generality, let us assume that $C = D$ is a single disk component. Once we trivialize $\pi \circ u^*E \cong D \times F$ following theorem 13, we get an induced J_G-holomorphic map $v : D \to (F, L_F)$ with $\int_C u^*\omega_F > 0$. By the monotonicity of L_F,

$$\lambda \int_D u^*\omega_F = \mu(v^*TF, v^*TL_F) > 0$$

where $\mu(\cdot, \cdot)$ is the boundary Maslov index. It follows from the axioms of the Maslov index that $\mu(u^*TE, u^*TL) = \mu(v^*TF, v^*TL_F) + \mu(\pi \circ u^*TB, \pi \circ u^*TL_B) > \mu(\pi \circ u^*TB, \pi \circ u^*TL_B)$.

But for a Donaldson lift w of $\pi \circ u$, we have $\mu(w^*TE, w^*TL) = \mu(\pi \circ u^*TB, \pi \circ u^*TL_F)$, so u is not in the same relative homology class as a Donaldson lift. This is a contradiction on the homology class $[\pi^*\Gamma]$. $\square$

We will need a corollary of this argument later:

Corollary 3. *Let $\mu(\cdot,\cdot)$ be the boundary Maslov index, and u a J^G-holomorphic disk with Lagrangian boundary. Then $\mu(u^*TF, u^*TL_F) \geq 0$ and $\mu(u^*TF, u^*TL_F) = 0$ if and only if u is vertically constant.*

Proof. Take v as in the previous proof. Since the gauge transformation $\mathcal{G}$ is an isomorphism that preserves L_F, we have that $\mu(u^*TF, u^*TL_F) = \mu(v^*TF, v^*TL_F) \geq 0$.

For the second statement, we have that $v \equiv 0$ iff $\mu(v^*TF, v^*TL_F) = 0$ by monotonicity. Since the property of *vertically constant* is independent of gauge, we have that u is vertically constant iff $v \equiv 0$.

$\square$

For a regular upper triangular datum close to J^G that is part of coherent system $\mathcal{P} = (P_\Gamma)_\Gamma$ from theorem 9, we apply theorem 16 to pick a homotopy $J_t \in \mathcal{J}^{reg}_{\Gamma, J_B}(J^G, J_{K,J_F})$ that gives us a parameterized moduli space of dimension 1:

$$\mathcal{W}_{\pi^*\Gamma}(J_t)$$

$$\partial \mathcal{W}_\Gamma(J_t) = \mathcal{M}_{\pi^*\Gamma}(E, L, J^G)^- \sqcup \mathcal{M}_{\pi^*\Gamma}(E, L, J_{K,J_F})$$

The map $p : \mathcal{W}_{\pi^*\Gamma} \to [0,1]$, $(u, J_t) \mapsto t$ has a surjective derivative at $p^{-1}(0)$ by the definition of cobordism, so it must be a submersion in a neighborhood $p^{-1}([0, \epsilon_0))$, which shows that $p^{-1}(0)$ and $p^{-1}(\epsilon)$ are diffeomorphic for $\epsilon < \epsilon_0$. Thus in theorem 9, we choose perturbation coherent data for type $\pi^*\Gamma$ that lies along some regular homotopy and is close enough to J^G.

With this, we can finally define a *lifting operator*:

Definition 40. Choose a combinatorial type (Γ, x_0) for (B, L_B) of expected dimension 0, and let $\pi^*\Gamma$ be the unique combinatorial type of a Donaldson lift through y_0. For a coherent, regular, upper triangular perturbation datum $(P_\Gamma)_\Gamma$ that lies in a small enough neighborhood of J^G for any type $(\pi^*\Gamma, y_0)$, and a choice of regular homotopy $J_t \in \mathcal{J}^{reg}_{\Gamma, J_B}(J^G, J_{\pi^*\Gamma})$, the *lifting operator* is defined as the map between moduli spaces

$$\mathcal{L}^{\Gamma}_{J_t, y_0} : \mathcal{M}_\Gamma(B, L_B, J_B, x_0) \to \mathcal{M}_{\pi^*\Gamma}(E, L, J_{\pi^*\Gamma}, y_0) \tag{4.22}$$

as the map that factors through

$$\mathcal{M}_\Gamma(B, L_B, J_B, x_0) \to \mathcal{M}_{\pi^*\Gamma}(E, L, J^G, y_0) \to \mathcal{M}_{\pi^*\Gamma}(E, L, J_{\pi^*\Gamma}, y_0)$$

as first taking the unique J^G-holomorphic Donaldson lift $\hat{u}$ with output y_0, and then applying the isotopy $(\hat{u}, J^G) \mapsto (\hat{u}_\epsilon, J_\epsilon)$ along $\mathcal{W}_{\pi^*\Gamma}(J_t)$ with $J_\epsilon = J_{\pi^*\Gamma}$

The lifting operator does depend on our choice of homotopy, but only up to a permutation of the points in the resulting moduli space $\mathcal{M}_{\pi^*\Gamma}(E, L, J_{\pi^*\Gamma}, y_0)$. The discussion in this section shows that the operator is well-defined.

We will also reference the following

Definition 41. The *unperturbed lifting operator* $\mathcal{L}^{\Gamma}_{J^G, y_0}$ is the map

$$\mathcal{M}_\Gamma(B, L_B, J_B, x_0) \to \mathcal{M}_{\pi^*\Gamma}(E, L, J^G, y_0)$$

from 4.22

We will use both of these operators to compute the leading order potential in section 5.2.

Chapter 5

Floer invariants for symplectic fibrations

5.1 Leray-Serre for Floer Cohomology

We derive a spectral sequence that converges to the Floer cohomology of L a la Leray-Serre. Our coefficient ring is a modification of the universal Novikov ring in two variables:

$$\Lambda^2 := \left\{ \sum_{i,j} c_{ij} q^{\rho_i} r^{\eta_j} | c_{ij} \in \mathbb{C},\ \rho_i \geq 0, (1-\epsilon)\rho_i + \eta_j \geq 0 \right.$$
$$\left. \#\{c_{ij} \neq 0, \rho_i + \eta_j \leq N\} < \infty \right\}$$

for a fixed $0 < \epsilon < 1$ that we will specify in later lemma 11. Choose a brane structure on the Lagrangian L (as in the appendix) and let $\mathrm{Hol}_\rho(u)$ be the evaluation of u with respect to a chosen rank one local system $\rho : \pi_1(L) \to (\Lambda^2)^\times$.

The symplectic form on E is the weak coupling form $\omega_K = a + K\pi^*\omega_B$ for $K >> 1$. We define the *vertical symplectic area* of a J-holomorphic configuration u as

$$e_v(u) = \int_C u^* a$$

This is a topological invariant, although it may not be positive due to horizontal contributions. To avoid mentioning "K" too many times, denote $e(\pi \circ u) = \int_C K(\pi \circ u)^* \omega_B$.

Label the critical points in L by x^i_j, where j enumerates the critical points y_j such that $\pi(x^i_j) = y_j$, and define the *Floer chain complex* of a fibered Lagrangian as:

$$CF(L, \Lambda^2) := \bigoplus_{x^i_j \in \mathrm{Crit}(f)} \Lambda^2 \langle x^i_j \rangle$$

For a taming, coherent, stabilizing (as in the appendix) perturbation datum $\mathcal{P} = (X_\Gamma, J_\Gamma)_\Gamma$, define the A_∞ maps as:

$$\mu^n(x_1, \dots, x_n) =$$

$$\sum_{x_j^i, [u] \in \overline{\mathcal{M}}_\Gamma(L, D, \underline{x})_0} (-1)^\lozenge (\sigma(u)!)^{-1} \mathrm{Hol}_\rho(u) r^{e_v(u)} q^{e(\pi \circ u)} \varepsilon(u) x_j^i \quad (5.1)$$

where $\lozenge = \sum_{i=1}^n i|x_i|$ and $\sigma(u)$ is the number of interior markings on Γ_u. $\varepsilon(u)$ is ± 1 depending on the orientation of the moduli space that contains $[u]$ 11. By the following lemma, this sum is well defined for appropriate choice of ϵ

Lemma 11. *$\mu^n(x_1, \dots, x_n) \in CF(L, \Lambda^2)$ for ϵ small enough.*

Proof. The transversality 11 and compactness 12 theorems tells us that for a fixed energy E, there are finitely many J-holomorphic configurations with $\int_C u^* \omega_K \leq e$. Thus, the output satisfies the criterion $\#\{i, j : c_{ij} \neq 0, e(\pi \circ u) + e_v(u) \leq N\} < \infty$.

For $\epsilon << 1$, the form $\omega_{K(1-\epsilon)}$ is also a non-degenerate symplectic form. If $\mathcal{P}^{reg}(\omega_K)$ is the Baire set of perturbation data that tames ω_K from theorems 1112, we have that

$$\mathcal{P}_\Gamma^{reg}(\omega_{K(1-\epsilon)}) \subset \mathcal{P}_\Gamma^{reg}(\omega_K)$$

as an open subset by the discussion in subsection 4.2.2 (specifically, inequality 4.3). Therefore, one can either chose perturbation from a slightly smaller set, or take K_1 so that $K_1(1-\epsilon) \geq K$ and consider the A_∞ maps for L in the symplectic manifold (E, ω_{K_1}). $\square$

In the next section, we will prove that these maps actually satisfy the A_∞-algebra axioms.

Let us filter the Floer chain complex by q degree: $\mathcal{F}_q^k CF(L)$ is generated by critical points with coefficients from Λ^2 of minimal degree $\geq k\lambda$ in the q variable for some fixed λ that we will specify later.

For this section, let us assume that we have a solution b to the Maurer-Cartan equation. In section 5.2, we will provide sufficient conditions on L_F and L_B for which we can find some natural solution to this. Let $h_2 : \pi_2(E,L) \to H_2(E,L)$ be the relative Hurewicz morphism. From the definition of a rational Lagrangian, the image of the energies $[\omega_B] \circ h_2(\pi_2(B,L_B))$ is discrete. This allows us to use a smaller Novikov ring:

$$\Lambda_d^2 := \left\{ \sum_{i,j} c_{ij} q^{i\rho} r^{\eta_j} | c_{ij} \in \mathbb{C},\, i \in \mathbb{Z}_{\geq 0} \geq 0, (1-\epsilon)i\rho + \eta_j \geq 0 \right.$$
$$\left. \#\{i,j : c_{ij} \neq 0, i\rho + \eta_j \leq N\} < \infty \right\}$$

Where $\rho \geq 0$ is the energy quantization for $(B, K\omega_B)$. Let us pick a solution b to the Maurer-Cartan equation for the A_∞ algebra $CF(L,\Lambda^2)$. Then μ_b^1 respects the filtration by q.

Let Λ_r be the subring of Λ^2

$$\Lambda_r := \left\{ \sum_i c_i r^{\eta_i} | c_i \in \mathbb{C},\, \eta_j \in \mathbb{R}_{\geq 0},\, \#\{i : c_i \neq 0, \eta_j \leq N\} < \infty \right\}$$

and similarly for Λ_q. Define Λ_t as the analogous ring

$$\Lambda_t := \left\{ \sum_i c_i t^{\eta_i} | c_i \in \mathbb{C},\, \eta_j \in \mathbb{R}_{\geq 0},\, \#\{i : c_i \neq 0, \eta_j \leq N\} < \infty \right\}$$

for a formal variable t. Define the *Floer cohomology* of L with respect to this rank one local system, Maurer-Cartan solution, brane structure, and coherent perturbation datum to be

$$HF^*(L,\Lambda^2) := H^*(\mu_b^1)$$

In section 5.3, we will show that this is independent of choices and is an obstruction to displacement by constructing a natural map to $HF(L,\Lambda(t))$.

Filtration by q-degree with step size less than ρ leads us to the following result:

Theorem 17. *Let $F \to E \to B$ be a symplectic Mori fibration along which we have a fibration of Lagrangians $L_F \to L \to L_B$ with L_B rational, and a divisor $D = \pi^{-1}(D_B)$ for a stabilizing divisor D_B of large enough degree in the base. Choose a regular, coherent, stabilizing, convergent, M-type perturbation datum $(P_\Gamma)_\Gamma$ and a solution b to the Maurer-Cartan equation. Then there is* *a spectral sequence $\mathcal{E}_s^*$ that converges to $HF^*(L, \Lambda^2)$ whose second page is the Floer cohomology of the family of L_F over L_B. The latter is computed by a spectral sequence with second page*

$$\widetilde{\mathcal{E}}_2^* = H^*(L_B, \mathcal{HF}(L_F, \Lambda_r)) \otimes gr(\mathcal{F}_q \Lambda_q) \tag{5.2}$$

where the coefficients come from a system that assigns the module $HF(L_{F_p}, \Lambda_r)$ to each critical fiber.

Proof. We pick the filtration step $\lambda > 0$ so that $0 < \lambda < \rho$, or arbirarily if $\rho = 0$. This induces a the spectral sequence $\mathcal{E}_s^*$ with the differential $\delta = \mu_b^1$. We will first show that the criteria from *The Complete Convergence Theorem* from [Wei94] section 5.5 are satisfied. These include showing that the filtration is *exhaustive* and *complete*, and showing that the spectral sequence is *regular*. We suppress some notation by setting $CF(L) := CF(L, \Lambda^2)$.

The filtration is exhaustive if $CF(L, \Lambda^2) = \cup_{k \geq 0} \mathcal{F}_q^k CF(L)$, which is clear in this situation.

The filtration is complete if

$$\varprojlim CF(L)/\mathcal{F}_q^k CF(L) = CF(L)$$

For simplicity, let us first assume that the rank of $CF(L)$ over Λ^2 is one, or equivalently we show that the filtration on Λ^2 is complete. Here, the inverse system is given by the projection $\pi_{kl} : CF(L)/\mathcal{F}_q^k CF(L) \to CF(L)/\mathcal{F}_q^l CF(L)$

$$\sum_{\rho_i \leq k} f_i(r) q^{\rho_i} \mapsto \sum_{\rho_i \leq l} f_i(r) q^{\rho_i}$$

by forgetting the terms of q-degree $\geq l$, where $f_i(r) = \sum_j c_{ij} r^{\eta_j} \in \Lambda(r)$ with $\eta_j \geq -(1-\epsilon)\rho_i$ and $\lim_{i \to \infty} \rho_i = \infty$. The inverse limit is constructed as

$$\varprojlim CF(L)/\mathcal{F}_q^k CF(L) =$$
$$\left\{ \Big(f_i(q,r) \Big) \in \prod_{i=0}^{\infty} CF(L)/\mathcal{F}_q^k CF(L) : \pi_{jk}(f_k(q,r)) = f_j(q,r) \; \forall j \leq k \right\}$$

where each $f_k(q,r) = \sum_{\rho_i \le k} f_i(r) q^{\rho_i}$ and $(1-\epsilon)\rho_i + \eta_j \ge 0$. Surely we have an inclusion

$$CF(L) \subset \varprojlim CF(L)/\mathcal{F}_q^k CF(L)$$

given by collecting all of the degree ρ_i terms:

$$\sum_{i,j} c_{ij} q^{\rho_i} r^{\eta_i} = \sum_i f_i(r) q^{\rho_i} \mapsto \Big(\sum_{\rho_i \le k} f_i(r) q^{\rho_i} \Big)$$

The reverse map is given by

$$\left(\sum_{\rho_i \le k} f_i(r) q^{\rho_i} \right) \mapsto \sum_{i=0}^{\infty} f_i(r) q^{\rho_i}$$

and we want to know that this converges in Λ^2. In other words, if $f_i(r) = \sum_{j=0}^{\infty} c_{ij} r^{\eta_j}$, we want to know that $\#\{c_{ij} \ne 0 : \rho_i + \eta_j \le N\} < \infty$. Given that $\eta_j \ge -(1-\epsilon)\rho_i$, we have that $\rho_i + \eta_j \ge \epsilon \rho_i$, which goes to ∞ as $i \to \infty$. It follows that the sum converges.

When $\mathrm{rank}(CF(L)) \ge 2$ use the fact that the filtration and inverse system projections commute with the direct sum decomposition, so that the inverse limit is the direct sum of the inverse limits. I.e.

$$\begin{aligned} \varprojlim CF(L)/\mathcal{F}_q^k CF(L) &\cong \varprojlim \bigoplus_{i=1}^{n} \Lambda^2/\mathcal{F}_q^k \Lambda^2 \langle x_i \rangle \\ &\cong \bigoplus_{i=1}^{n} \varprojlim \Lambda^2/\mathcal{F}_q^k \Lambda^2 \langle x_i \rangle \end{aligned}$$

Thus, the filtration is complete.

Next, we nee to show that the spectral sequence is regular, i.e. that $\delta_s = 0$ for $s >> 1$. We imitate the idea behind theorem 6.3.28 in [FOOO09]. Essentially, the proposition we need is the following

Proposition 1. *There exists a $c > 0$ such that*

$$\delta(CF(L)) \cap \mathcal{F}_q^k CF(L) \subset \delta(\mathcal{F}_q^{k-c} CF(L))$$

that doesn't depend on k.

Proof of proposition. We find a *standard generating set*, which is inspired from the notion of a *standard basis* in [Definition 6.3.1, [FOOO09]]. Let $\{v_i\}_{i=1}^m$ be a generating set for for Δ over Λ^2 (for example, given by $\delta(x_j^i)$). Denote $F(v_i) = k$ as the lowest power of q appearing in v_i, and let $\sigma(v_i) \in CF(L, \Lambda(r))$ the be the coefficient of the lowest power of q. For example, the element $v = q^\rho \sum_{j \geq 0} c_{1j} r^{\eta_j} y_j$ has $F(v) = \rho$ and $\sigma(v) = \sum_{j \geq 0} c_{1j} r^{\eta_j} y_j$.

Let M be a finitely generated Λ^2-module and suppose $\mathcal{S} = \{v_1, \ldots, v_k\}$ is a generating set for M. $\mathcal{S}$ is said to be a *standard generating set* for M if (after reordering) there exists a sequence $(\rho^j) \in (0, \infty]^k$ so that

$$F(v_i) \leq F(v_{i+1})$$

$$r^{\rho^j} \sigma(v_j) \notin \Lambda_r \cdot \{\sigma(v_i)\}_{i=1}^{j-1}$$

Lemma 12. *A standard generating set for finitely generated M exists.*

Proof. Let $\{v_1, \ldots, v_k\}$ be a generating set ordered so that $F(v_i) \leq F(v_{i+1})$. Denote $F(v_j) = \rho_j$ and denote

$$\mathcal{S}_j = \Lambda_r \cdot \{\sigma(v_1), \ldots, \sigma(v_j)\}.$$

If $k = 1$, then there is nothing to show. If $\sigma(v_j) \notin \mathcal{S}_{j-1}$, then there is also nothing to show. Assume that $r^{\rho^j} \in \mathcal{S}_{j-1}$ for

$$0 \leq \rho^j \leq (1 - \epsilon)\min_{1 \leq i \leq j-1} F(v_j) - F(v_i).$$

Then there are elements $a_{ij} \in \Lambda_r$ so that

$$r^{\rho^j} \sigma(v_j) = \sum_{i=1}^{j-1} a_{ij} \sigma(v_i)$$

Hence,

$$v_j^1 = v_j - \sum_{i=1}^{j-1} a_{ij} r^{\rho^j} q^{\rho_j - \rho_i} v_i$$

is such that $F(v_j^1) > F(v_j)$ and

$$\Lambda^2 \cdot \{v_1, \ldots, v_j\} = \Lambda^2 \cdot \{v_1, \ldots, v_{j-1}, v_j^1\}$$

We then replace v_j with v_j^1, reorder $\{v_j^1, v_{j+1}, \ldots, v_k\}$ by increasing q-valuation, and continue. If it happens that $r^{\rho_i^j}\sigma(v_j^i) \in \mathcal{S}_{l_i}$ for all i and some non-decreasing sequence l_i and decreasing ρ_i^j, then $v_j^i \in \Lambda^2\{v_1, \ldots, \widehat{v_j}, \ldots, v_k\}$ for all i and we can disregard v_j from the original generating set. □

By applying a change of coordinates to $CF(L)$, the existence of a standard generating set for $\delta(CF(L))$ essentially says that the matrix for δ is square upper triangular with columns that obey the q filtration. Roughly, we can start with the generating set $\delta(x_i)$ and reorder as decribed by the algorithm in the proof of Lemma 12. Notably, if $(s_{i(j)})$ is the matrix for δ in these coordinates, we have that $s_{i(j)} = 0$ for $i > j$ and for each j:

$$\min_i F(s_{i(j-1)}) \leq \min_i F(s_{i(j)}).$$

Moreover, any element in the range is expressible as a linear combination of the columns

$$v = \sum a_i v_i,$$

and the expression is unique provided the relative difference in r-valuation of $\sigma(a_i)$ is not too large. For such a generating set, we can set

$$c = \max\ F(s_{i(j)}).$$

□

Let

$$Z_s^k = \left\{x \in \mathcal{F}^k CF(L) \mid \mu_b^1(x) \in \mathcal{F}^{k+s-1} CF(L)\right\} + \mathcal{F}^{k+1} CF(L) \tag{5.3}$$

$$B_s^k = \left\{\mu_b^1(\mathcal{F}^{k-s+2} CF(L)) \cap \mathcal{F}^k CF(L)\right\} + \mathcal{F}^{k+1} CF(L) \tag{5.4}$$

$$\mathcal{E}_s^k = Z_s^k / B_s^k \tag{5.5}$$

and let $r > r_0$ with $r_0 - 1 - c \geq 1$ and $\chi \in Z_r^k$. Then $\mu_b^1(\chi) \in CF(L) \cap \mathcal{F}^{k+r-1} CF(L)$, so by our proposition, $\mu_b^1(\chi) \in \mu_b^1(\mathcal{F}^{k+r-1-c} CF(L)) \subset \mu_b^1(\mathcal{F}^{k+1} CF(L))$ since $k + r - 1 - c >$

$k + r_0 - 1 - c > k + 1$. The differential induces a map

$$\mu_b^1 : Z_r^k \to \mathcal{F}^{k+r-1}CF(L) \to e_r^{k+r-1}$$

that we must show is 0. Indeed, $B^{k+r-1} \supset \mu_b^1(\mathcal{F}^{k+1})$, so $\mu_b^1(\chi) = 0 \in \mathcal{E}_r^{k+r-1}$.

Now, we apply the complete convergence theorem from [Wei94]. Since the filtration of this cohomology spectral sequence is bounded below, it converges to $H^*(\mu_b^1)$.

Next, we calculate the second page. By definition, we have

$$\mathcal{E}_1^* \cong CF(L, \mathbb{C}) \otimes_{\mathbb{C}} gr_*(\mathcal{F}\Lambda^2)$$

where

$$gr_*(\mathcal{F}\Lambda^2) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{F}_q^n \Lambda^2 / \mathcal{F}_q^{n+1} \Lambda^2$$

is the associated graded ring of Λ^2. The differential δ_1 on $\mathcal{E}_1^*$ is induced from the Floer differential on $CF(L)$, and counts pearly Morse trajectories whose pearls project to constants.

To compute $\mathcal{E}_2^*$ via a spectral sequence, we follow Leray-Serre for Morse cohomology of a fibration. Let $\mathfrak{F}CF(L, \Lambda^2)$ be the filtration of $(CF(L, \Lambda^2), \delta_1)$ by degree of $\pi(x_i)$ as a critical point of b. This is a well-defined filtration of cell-complexes, since any Morse-Floer configuration that is horizontally constant on disk component projects to a Morse flow line between critical points in the base. The latter only exists when the output critical point has index more than the index of the input.

The base index filtration is bounded, so by the *classical convergence theorem* from [Wei94], we get a spectral sequence $\tilde{\mathcal{E}}_r$ that converges to $\mathcal{E}_2$.

It remains to compute the second page of the spectral sequence $\widetilde{\mathcal{E}}_r$:

1. The $0th$ page is the associated graded complex with differential d_0 that counts pearly Morse trajectories between critical points of the same base index whose pearls project to

constants. By the construction of the pseudo-gradient, flow lines project to flow lines. Since there are no flow lines in L_B between critical points of the same index, d_0 counts pearly Morse trajectories contained exclusively in critical fibers. Therefore,

$$\widetilde{\mathcal{E}}_1 = \bigoplus_{y \in \mathrm{Crit}(b)} HF(L_{F_y}, \Lambda_r)$$

2. d_1 counts trajectories u on $\mathcal{E}_2$, for which we know that $\pi \circ u$ is constant on surface components. Moreover, u must form a trajectory between critical point classes $[x_{out}]$ and $[x_{in}]$ such that $W_b^+(\pi(x_{out})) - W_b^+(\pi(x_{in}))$ is at most 1 (hence, exactly one when d_1 is induced on $\ker(d_0)/\mathrm{im}(d_0)$). Following our convention, $c[x_{out}]$ is a summand of $d_1[x_{in}]$.

To compute $\widetilde{\mathcal{E}}_2$ we need to understand exactly what trajectories are in item 2. We show that these must be zero energy Morse flows in L by virtue that they lie in a 0-dimensional moduli space and project to Morse flow lines between relative index 1 critical points.

For a contradiction, suppose that $[u] \in \mathcal{M}_\Gamma(L, D, x_{out}, x_{in})$ has a non-trivial pearl with image in $\pi^{-1}(p)$ which is counted by d_1 on $\widetilde{\mathcal{E}}_1$. Let $u|_S$ denote the restriction of a representative to the pearl component, with $u(y_{out}) \in W_X^+(x_{out})$ and $u(y_{in}) \in W_X^-(x_{in})$. Since the almost complex structure is the same in each fiber, $u|_S$ is a regular J_F-holomorphic configuration. Say $u(y_{out}) \in W_{X_g}^+(z_{out})$ and $u(y_{in}) \in W_{X_g}^-(z_{out})$ in L_F. Let $(\Gamma_F, z_{out}, z_{in})$ be the combinatorial type of $u|_S$. Since the total configuration u is isolated, we must have that $\iota(\Gamma_F, z_{out}, z_{in}) = 0$. Flowing X starting at z_{out} gives an critical point x'_{out} in the same fiber as x_{out}. Since index of a critical point is an orientable diffeomorphism-invariant, $\dim W_{X_g}^+(z_{out}) = \dim W_{X_g}^+(x'_{out})$. Similarly, there is a critical point x'_{in} for X_g (and X) in the same fiber as x_{in}. We have

$$\begin{aligned}&\dim W_X^+(x_{out}) - \dim W_X^+(x_{in}) = \\ &\quad \dim W_{X_b}^+(\pi(x_{out})) + \dim W_{X_g}^+(x'_{out}) - \dim W_{X_b}^+(\pi(x_{in})) - \dim W_{X_g}^+(x'_{in})\end{aligned}$$

But since we are on $\tilde{\mathcal{E}}_1$, we know that

$$\dim W_{X_b}^+(\pi(x_{out})) - W_{X_b}^+(\pi(x_{in})) = 1$$

which tells us that $\iota(\Gamma, x_{out}, x_{in}) = \iota(\Gamma_F, z_{out}, z_{in})) + 1$. This is a contradiction on the the definition of the Floer differential.

□

Remark 5. As an alternative to using propostion 1, one can simply use the existance of such a $c' > 0$ for the total energy filtration, given by [Proposition 6.3.9 [FOOO09]]. If such a c' is sharpened by a holomorphic configuration u, then by a computation in the next subsection:

$$\epsilon K \int_C u^* \omega_B \leq \int_C u^*(a + K\omega_B) \leq c'$$

where the left hand side holds for any holomorphic u and $K >> \epsilon$. Thus, one can take $c = \dfrac{c'}{\epsilon}$.

5.1.1 The spectral sequence in the setting of the usual Novikov ring

In order to provide a link to the invariants in the literature, we define the A_∞ algebra of a fibered Lagrangian with coefficients in the universal Novikov ring Λ_t.

Definition 42. For $e(u) = \int_S u^* \omega_K$, define the one variable A_∞-maps as

$$\nu^n(x_1, \ldots, x_n) = \sum_{x_j^i, [u] \in \overline{\mathcal{M}}_\Gamma(L, D, \underline{x})_0} (-1)^\diamondsuit (\sigma(u)!)^{-1} \mathrm{Hol}_\rho(u) t^{e(u)} \epsilon(u) x_j^i \quad (5.6)$$

In theorem 19, we will prove that the A_∞-maps 5.1 satisfy the axioms of an A_∞-algebra composition, and the proof is the same for 5.6.

Choose a regular, coherent, convergent perturbation datum $(P_\Gamma) = (J_\Gamma, F_\Gamma)$ (which can be the same choice as in theorem 17). Assume that L is weakly unobstructed, i.e. there is a solution to the Maurer-Cartan equation b so that

$$(\nu_b^1)^2 = 0$$

(see appendix 7) we define the Floer cohomology of L with coefficients in Λ_t to be

$$HF(L, \Lambda_t) := H(CF(L, \Lambda_t), \nu_b^1)$$

Following [Oh], we can compute the cohomology via a spectral sequence, whose first page is the Morse cohomology: By energy quantization, there is a minimal number e_0 so that for any (P_Γ)-holomorphic configuration u with boundary in L, we have

$$e_0 \leq e(u)$$

Therefore, let us filter the complex $CF(L, \Lambda_t) =: CF(L)$ by t-degree:

$$\mathcal{F}_t CF(L, \Lambda_t) : CF(L) \supset t^\rho CF(L) \supset t^{2\rho} CF(L) \supset \dots \tag{5.7}$$

for $\rho = e_0$. By positivity of energy, the differential ν_b^1 preservers this filtration. Denote the induced spectral sequence $\mathcal{G}_s$. The convergence of this spectral sequence is another application of the Complete Convergence Theorem [Wei94] as in the proof of theorem 17. However, this case is simpler, as exhaustiveness and completeness follow easily, and regularity follows from lemma 6.3.2 in [FOOO09]. We have that

$$\mathcal{G}_\infty \cong gr_*(\mathcal{F}_t HF(L, \Lambda_t)) \tag{5.8}$$

Details can be found in [FOOO09].

Altering the filtration step

The choice of ρ above was arranged so that

$$\mathcal{G}_2 \cong gr_*(\mathcal{F}_t H^{Morse}(L, \Lambda_t))$$

However, we are free to choose ρ so that the second page provides a different type of information:

Theorem 18. *Let $F \to E \to B$, $L_F \to L \to L_B$, a divisor $D = \pi^{-1}(D_B)$, and $(P_\Gamma)_\Gamma$ be as in theorem 17, and b a solution to the Maurer-Cartan equation for ν^n. Then there is a spectral* *sequence $\mathcal{B}_s^*$ that converges to $HF^*(L, \Lambda_t)$ whose second page is the Floer cohomology of the family of L_F over L_B. The latter is computed by a spectral sequence with second page*

$$\widetilde{\mathcal{B}}_2^* = H^*(L_B, \mathcal{HF}(L_F, \Lambda_t)) \otimes gr(\mathcal{F}_t\Lambda_t) \tag{5.9}$$

Sketch of proof. For a particular Morse function g on L_F, let $\Sigma_{max,F,g}$ be an upper bound on the energy of disks appearing in the Floer differential for (F, L_F). This can be finite due to the fact that (F, L_F) is a monotone pair that is also compact.

For $\epsilon << 1$ and K large enough, the coupling form $a + (1-\epsilon)K\pi^*\omega_B$ is positive definite, so we get the inequality

$$a + K\pi^*\omega_B \geq \epsilon K\pi^*\omega_B$$

If Σ_B is the energy quantization for B, choose K large enough so that

$$\epsilon K\Sigma_B > \Sigma_{max,F}$$

and choose ρ so that

$$\epsilon K\Sigma_B > \rho > \Sigma_{max,F}$$

Considering the spectral sequence induced by the filtration 5.7 with step size ρ, the second page is the cohomology of the complex

$$(CF(L, \Lambda_t), \delta_0) \tag{5.10}$$

where δ_0 counts pearly Morse trajectories that have 0 energy when projected to B. Filtration of 5.10 by base Morse index induces another spectral sequence that converges to the cohomology as in theorem 17. □

5.2 A_∞-algebras and disk potentials

The *disk potential*, introduced in the physics literature and mathematically in [FOOO09], is a powerful tool that is used to compute the Lagrangian Floer cohomology in toric manifolds. In this section, we prove that we can associate an A_∞ algebra to a fibered Lagrangian and derive a relationship between the potential for the base and that of the total space.

For a symplectic Mori fibration and a fibered Lagrangian L, define its A_∞-*algebra* as the family

$$A(L) = (CF(L, \Lambda^2), \mu^n)$$

with

$$\mu^n(x_1, \dots, x_n) = \sum_{x_0, [u] \in \overline{\mathcal{M}}_\Gamma(L,D,\underline{x})_0} (-1)^\diamondsuit (\sigma(u)!)^{-1} \mathrm{Hol}_L(u) r^{e_v(u)} q^{e(\pi \circ u)} \varepsilon(u) x_0 \tag{5.11}$$

This sum is well defined by our transversality and compactness results for fibrations. We will prove, up to signs/orientations, that these products satisfy the A_∞ axioms

$$0 = \sum_{\substack{n,m \geq 0 \\ n+m \leq d}} (-1)^{n + \sum_{i=1}^n |a_i|} \mu^{d-m+1}(a_1, \dots, a_n, \mu^m(a_{n+1}, \dots, a_{n+m}), a_{n+m+1}, \dots, a_d)$$

Theorem 19. *For a coherent, regular, stabilizing M-type perturbation system, the products in 5.11 satisfy the axioms of a A_∞-algebra.*

Proof. For bounded energy, theorems 11 and 12 say that the compactification of the 1-dimensional component of the moduli space $\mathcal{M}(L, D, \underline{x}, e(u) \leq k)_1$ is a compact 1-manifold with boundary. Thus, with proper orientations:

$$0 = \sum_{\Gamma \in \mathfrak{M}_{m,n}} \sum_{[u] \in \partial \overline{\mathcal{M}}_\Gamma(L,D,\underline{x},e(u) \leq k)_1} \frac{1}{m!} \mathrm{Hol}_L(u) r^{e_v(u)} q^{e(\pi \circ u)} \epsilon(u) \tag{5.12}$$

where we divide by $m!$ to signify that there are $m!$ different orderings of interior markings for a given configuration. More importantly, each boundary combinatorial type is obtained by gluing two types Γ_1, Γ_2 along a broken edge that is a root for Γ_1 resp. leaf for Γ_2. Since our perturbation is coherent with respect to cutting an edge, we have that

$$\partial\overline{\mathcal{M}}_\Gamma(L,D,\underline{x})_1 \cong \bigcup_{y,\Gamma_1,\Gamma_2} \overline{\mathcal{M}}_{\Gamma_2}(L,D,x_0,\dots,x_{i-1},y,x_{i+1+k}\dots x_n)_0$$

$$\times\,\overline{\mathcal{M}}_{\Gamma_1}(L,D,y,x_i,\dots,x_{i+k})_0$$

Thus, for each boundary $[u]=[u_1]\times[u_2]$, we have that

$$\varepsilon(u)=\varepsilon(u_1)\varepsilon(u_2)$$

$$\mathrm{Hol}_L(u)=\mathrm{Hol}_L(u_1)\mathrm{Hol}_L(u_2)$$

$$e_v(u)=e_v(u_1)+e_v(u_2)$$

$$e(\pi\circ u)=e(\pi\circ u_1)+e(\pi\circ u_2)$$

Let $m_i=\sigma(u_i)$. Then for each $[u_1]\times[u_2]$ of combinatorial type $\Gamma_1\times\Gamma_2$, there are $m_1!m_2!$ ways to order the interior markings. These observations give us the formula

$$0=\sum_{y,\Gamma_1,\Gamma_2}\sum_{\substack{[u_1]\in\overline{\mathcal{M}}_{\Gamma_1}(L,D,y,x_i,\dots,x_{i+k})_0\\ [u_2]\in\overline{\mathcal{M}}_{\Gamma_2}(L,D,x_0,\dots,x_{i-1},y,x_{i+1+k},\dots x_n)_0}}\frac{1}{m_1!m_2!}$$

$$\mathrm{Hol}_L(u_1)\mathrm{Hol}_L(u_2)r^{e_v(u_1)}r^{e_v(u_2)}q^{e(\pi\circ u_1)}q^{e(\pi\circ u_2)}\epsilon(u_1)\epsilon(u_2) \quad (5.13)$$

This is the n^{th} A_∞ relation up to signs. $\square$

5.2.1 The disk potential

The 0^{th} order multiplication map is of particular interest. In the literature this is sometimes referred to as the *superpotential* or the *disk potential*. One can often compute the non-displaceability of a Lagrangian by simply finding critical points of μ^0. We prove a formula that expresses the leading order part of μ^0 as a sum of terms coming from the base and fiber.

Definition 43. The *second order potential* for a symplectic Mori fibration is

$$\mathcal{W}_0^L(\rho)=\sum_{\substack{u\in\mathcal{I}_x\\ x\in\mathrm{crit}(f)}}\varepsilon(u_i)(m!)^{-1}\mathrm{Hol}_\rho(u)q^{e(\pi\circ u)}r^{e_v(u)}x$$

where for each x

$$\mathcal{I}_x = \left\{ u \in \mathcal{M}(E, L, x)_0 \,|e(\pi \circ u) = 0 \right\} \\ \cup \left\{ u \,|e(u) = \min_{v \in \mathcal{M}(E,L,x)_0} \{e(v) : e(\pi \circ v) \neq 0\} \right\}$$

This potential captures any isolated holomorphic disks counted by μ^0_L that are contained in a single fiber, along with the holomorphic disks of minimal energy in the total space that have non-zero base energy. By theorem 12, the minimal energies are obtained for L, since there are only finitely many isolated holomorphic configurations bounded by a finite energy. In our framework, it seems unreasonable to expect much information from the *leading order potential* as defined in the literature [FOOO]. For instance, we can take K in the weak coupling form to be so large that the only terms of minimal energy are contained in a single fiber. Using this same line of reasoning, let x_F be the unique maximum of $g|_{L_F}$. Since L_F is monotone and has minimal Maslov number 2, we have that

$$\mu^0_{L_F}(\theta) = \sum_{u \in \mathcal{M}(F, L_F, x_F)} \mathrm{Hol}_\theta(u) r^\kappa x_F$$

where the sum is finite and κ is a multiple of 2 corresponding to the energy of Maslov index 2 disks. It follows that the only terms with $q = 0$ in μ^0 appear as coefficients of x_M, the unique maximum of f.

As in the proof of theorem 18, let $\epsilon << 1$ be such that $\omega_{K(1-\epsilon)}$ is also tamed by (J_Γ), and take K large enough so that

$$\epsilon K \Sigma_B > \kappa$$

where Σ_B is the energy quantization constant for (B, L_B, ω_B). Then the terms in 43 with $e(\pi \circ u) \neq 0$ are precisely the terms of 2nd lowest degree appearing in μ^0_L as a coefficient of the maximum x_M and the terms of lowest degree for any other critical point.

Define the leading order potential for L_B to be the minimal energy terms appearing in $\mu^0_{L_B}$:

Definition 44. For generating critical points $x \in CF(L_B, \Lambda_q)$, the *leading order potential* for a Lagrangian in a rational symplectic manifold as in section 3 is

$$\mathcal{W}_0^{L_B}(\theta) = \sum_{\substack{x,\, u \in \mathcal{M}(B, L_B, x)_0 \\ e(u) = \min\{e(v) :\, v \in \mathcal{M}(B, L_B, x)_0\}}} (\sigma(u)!)^{-1} \mathrm{Hol}_\theta(u) \varepsilon(u) q^{e(u)} x \tag{5.14}$$

The main theorem of this section is that we can express the second order potential as the sum of the leading order potentials for the base and fiber. In order to get a representation for $\pi_1(L_B)$ from one for $\pi_1(L)$, we use the lifting operator 4.22 from section 4.5. Given a representation $\rho \in \mathrm{Hom}(\pi_1(L), (\Lambda^2)^\times)$, define a representation

$$u \mapsto \mathrm{Hol}_\rho(\mathcal{L}^\Gamma_{J_t, y_0} u)$$

for elements in $u \in \mathcal{M}_{\pi^*\Gamma}(B, L_B, J_B, x)$.

Enumerate the critical fibers F_i, and let y_i be the unique maximum of $g|_{L_{F_i}}$ in the fiber above x_i. Using the lifting operator from section 4.5, we perform a transformation on the leading order base potential:

Definition 45. The *lifted leading order potential* for $L_F \to L \to L_B$ is

$$\mathcal{L}_{J_t} \circ \mathcal{W}_0^{L_B}(\rho) =$$

$$\sum_{\substack{i,\, u \in \mathcal{M}(B, L_B, x_i)_0 \\ e(u) = \min\{e(v) :\, v \in \mathcal{M}(B, L_B, x_i)_0\}}} (\sigma(u)!)^{-1} \mathrm{Hol}_\rho(\mathcal{L}^\Gamma_{y_i} u) \varepsilon(\mathcal{L}^\Gamma_{y_i} u) q^{e(u)} r^{e_v(\mathcal{L}^\Gamma u)} y_i$$

that takes each configuration $u \in \mathcal{M}(B, L_B, x_i)$ and computes its Donaldson lift through y_i. We remark that there may be cancellations between configurations in 5.14 that may not occur in the lifted potential. We interpret the lifted potential as a formula in its own right and not as something dependent on $W_0^{L_B}(\theta)$ in an algebraic sense.

By theorem 16 and the discussion thereafter, the choice of regular homotopy J_t induces a bijection on moduli spaces $\mathcal{M}_{\pi^*\Gamma}(E, L, y_i, J_i)_0$ for $i = 0, 1$. Therefore it does not matter which regular homotopy we pick, or if we pick one at all so long as we know that there exists one.

If x_M is the unique maximum for b on L_B, let $i_{x_M *} : CM(L_F, \mathbb{C}) \to CM(L, \mathbb{C})$ be the map induced from the inclusion $i_{x_M} : L_F \to L$ of L_F as the fiber of x_M. Let $\iota^* : \mathrm{Hom}(\pi_1(L), (\Lambda^2)^\times) \to \mathrm{Hom}(\pi_1(L_F), (\Lambda^2)^\times)$ be the map on representations induced from $\iota_{x_M *}$.

Theorem 20. *Let E be a compact symplectic Kähler fibration. There is an open neighborhood of $\mathcal{U}_G$ of J_G resp. $\mathcal{U}_K$ of 0 such that if*

$$J^\Gamma_{ut} = \begin{bmatrix} J^\Gamma_F & K^\Gamma \\ 0 & J^\Gamma_B \end{bmatrix}$$

is choice of regular, coherent, convergent, smooth perturbation datum with $K^\Gamma \in \mathcal{U}_K$ and $J^\Gamma_B \in \mathcal{U}_G$, then there is a homotopy J_t with $J_0 = J^G$ and $J_1 = J^\Gamma_{ut}$ that is regular for vertically constant types Γ such that the second order potential for (E, L) decomposes into a sum of the lifted leading order potential and the full potential for the fiber:

$$\mathcal{W}^L_0(\rho) = \mathcal{L} \circ \mathcal{W}^{L_B}_0(\rho) + i_{x_M *} \circ \mu^0_{L_F, \iota^* \rho} \tag{5.15}$$

First, we state a sequence of lemmas to make the main proof go smoothly. The primary focus will be on configurations which are horizontally non-constant and of minimal energy. Let Γ be an unmarked, π-stable combinatorial type of a non-zero energy J_Γ-holomorphic configuration $u : C_\Gamma \to (E, L)$ through $W^+_X(y)$ appearing in definition 43, and $\pi_*\Gamma$ the combinatorial type obtained by changing the labeling of homology classes to $\pi_*[v_i]$. Since Γ is unmarked, we have that $\pi_*[v_i] = 0$ if and only if $[v_i] = 0$: in the notation of subsection 4.2.3 , $\Upsilon(\pi_*\Gamma) = \pi_*\Gamma$.

Proposition 2. $\iota(\pi_*\Gamma, \pi(y)) \leq \iota(\Gamma, y)$.

Proof of proposition. The quantity ι is topological, so this follows from the fact that π is surjective and that all of the input data (divisor, pseudo-gradient, almost complex structure) are compatible with π. Specifically, flow lines of X project to flow lines of X_b

Lemma 13. *For an interior marked point $z_0 \in C$ for u, the degree of tangency of $\pi \circ u$ at z_0 is the same as that of u at z_0.*

Proof. This essentially follows from the definition of tangency degree as the order of vanishing along the normal bundle to the divisor. Let $u(z) = (u_D(z), u_{N_D}(z))$ in coordinates centered at $u(z_0)$ and adapted to $D \oplus N_D$ where N_D is the normal bundle to D. Further, choose a neighborhood of z_0 in which u is J_D-holomorphic. The degree of tangency of u at z_0 is defined as

$$m_u(z_0) = j : \min_{j \geq 0} d_{z_0}^{(j+1)} u_{N_D} \neq 0$$

Since $\pi(N_D) = N_{D_B}$ and $\pi \circ u = (\pi \circ u_D(z), u_{N_D}(z))$ in coordinates adapted to D and π, it is clear that $m_{\pi \circ u}(z_0) = m_u(z_0)$ □

It follows from this discussion and corollary 3 that $\iota(\pi_* \Gamma, \pi(x)) \leq \iota(\Gamma, x)$.

□

The one caveat of the above proposition is that $\iota(\pi_* \Gamma, x)$ could be negative if there any unstable disk components D_i such that $\pi \circ u|_{D_i}$ is constant. This does not happen if Γ is an unmarked type.

Proof of theorem. The outline of the proof is as follows: First, we pick a configuration corresponding to a term on the left hand side of 5.15, and show that it is either horizontally constant, or that it must be a lift of a configuration through x to a configuration through the unique maximum in L_x. Then, we show that we can realize a term in 45 as a term on the left hand side of 5.15.

Suppose that we have a J_B^Γ-holomorphic configuration u with one output x in the base of index 0 and minimal non-zero energy $e(u) = e_x$. We first record a few things about the combinatorial type of u

1. Γ does not have any sphere components once an appropriate divisor and perturbation datum have been chosen.

2. The index formula is

$$\iota(\Gamma, x) = \dim W^+_{X_b}(x) + \sum_{i=1}^{m} I(u_i) - 2 - |\mathrm{Edge}^0_{<\infty}(\Gamma)|$$
$$- |\mathrm{Edge}_{\infty,s}(\Gamma)| - \sum_{e \in \mathrm{Edge}_{\infty,s}} m(e) = 0$$

where the u_i are the individual disk components for u;

We can also assume that Γ is an unbroken trajectory, by the definition of the moduli space and the nature of the bubbling phenomenon. We apply the lifting operator 4.22 to obtain a regular, $J_{ut}^{\pi^*\Gamma}$-holomorphic configuration $\mathcal{L}_{J_t,y}u : C \to E$ through any generator $y \in CF(L_x, g)$ of the Morse chain complex of the fiber L_x with Morse function g. This configuration is vertically constant before perturbation, and since perturbation is a homotopy equivalence of maps, we have that $I(u) = I(\mathcal{L}_{J_t,y}u)$. Moreover, the lifting operation preserves divisor intersection multiplicity by the argument in 13 , and perturbing in the direction of J_Γ can be arranged so that the lower diagonal block of J_t is constantly J_B^Γ: Thus $\pi \circ \mathcal{L}_{J_t,y}u = u$. It follows from these observations that the lifted and perturbed configuration will be index 0 precisely when $\dim W^+_X(y) = \dim W^+_{X_b}(x)$, which occurs only when y is the unique maximum of $g|_{L_{F_x}}$. Hence, a configuration in the lifted leading order potential is realized as a configuration listed on the left hand side of 5.15.

Going in the opposite direction, suppose that u is an index 0 configuration into (E, L) with some output $y \in CF(L, \Lambda^2)$ corresponding to a term appearing on the left hand side of 5.15. By the property that $e(u)$ is minimal, we have that either u has no marked components or is comprised entirely of marked components. Let us take the latter (easier) case first: u defines an index 0 J_F-holomorphic configuration in the fiber $\pi^{-1}(x)$ with output $i^*y \in CF(L_x, g)$ in the Morse chain complex of $g|_{L_x}$. Using u, we can construct an equivalent J_F-holomorphic configuration u_w in the critical fiber $\pi^{-1}(w)$ with output $\phi(y)$, where ϕ is an appropriate symplectomorphism. Any configuration u_w must have positive index by regularity, so it must be that x is the unique maximum on L_B. Hence, the configuration $u_x = u$ is realized in the potential for the fiber on the right hand side of 5.15.

Finally, take the case that the combinatorial type Γ of u consists purely of unmarked components. Projecting to the base yields a regular configuration by lemma 9 whose combinatorial type is $(\Upsilon(\pi_*\Gamma), \pi(y))$ in the notation of subsection 4.2.3 . Since Γ has no marked components, we have that $\Upsilon(\pi_*\Gamma) = \pi_*\Gamma$. By proposition 5.2.1 and the regularity/stability of $\pi \circ u$, it follows that $\iota(\pi_*\Gamma, \pi(y)) = 0$.

Through a sequence of lemmas, we deduce the possibility for y and u, after which the theorem will follow:

Proposition 3. *y must be the unique maximum of $g|_{L_x}$, and u is in the image of the lifting operator.*

Proof. We derive a lower bound on the index of the configuration u, and show that it must be > 0 unless y is the unique maximum on L_x.

Let $\mathcal{L}_{J^G,y}$ be the unperturbed lifting operator 41 through $y \in L_x$. We compare the indices of the configurations $u^G := \mathcal{L}_{J^G,y}(\pi \circ u)$ and u. The index of u^G is 0 precisely when $W^+_X(y) = W^+_{X_b}(\pi(y))$, and the same holds true for the perturbed lifting operator $\mathcal{L}_{J_t,y}$ (so precisely when y is the maximum of $g|_{L_x}$). Let J_t be a regular homotopy (as in section 4.5) such that $J_0 = J^G$ and $J_1 = J^\Gamma_{ut}$, along which the lower diagonal block $(J_t)_B = J_B$ is constant. The parameterized moduli space $\mathcal{W}_\Gamma(E, L, \{J_t\}_{0<t\leq 1}, y)$ for $t \neq 0$ is cut out transversely for small perturbation data by theorem 16 . Thus, we can pick a sequence of maps in $u_i \in \mathcal{W}_\Gamma(E, L, \{J_t\}_{0<t\leq 1}, y)$ that are J_{δ_i}-holomorphic such that $\pi \circ u_i = \pi \circ u$ and J_{δ_i} is regular for u_i. By Gromov compactness, the limit v is a J^G-holomorphic configuration, possibly with

sphere components. Since $\pi \circ u_i = \pi \circ u$, we must have that $\pi \circ v = \pi \circ u$, and any sphere/disk bubbles are contained in the fibers.

On one hand, u^G is vertically constant, so $\mu(u^{G*}TF, u^{G*}TL_F) = 0$ by corollary 3. On the other hand, we have the following: $\mu(u^*TF, u^*TL_F) = \mu(v^*TF, v^*TL_F)$ since v is homotopic to u, $\mu(v^*TF, v^*TL_F) \geq 0$ by corollary 3, and $m_u(z_i) = m_{\mathcal{L}\pi\circ u}(z_i)$ by lemma 13 . Therefore, $0 \leq \iota(u^G, y) = \iota(u, y) - \mu(u^*TF, u^*TL_F) + \mu(u^{G*}TF, u^{G*}TL_F)$, so $\iota(u^G, w) \leq \iota(u, w)$. This inequality tells us that the only index 0 configuration that is possible is when y is the unique maximum of $g|_{L_x}$ and v is vertically constant. Therefore, v must be in the image of the unperturbed lifting operator and u is in the image of the lifting operator.

□

This concludes the proof of theorem 20. □

An important consequence of this formula lies in the role of unique maximum $x_M \in CF(L, \Lambda^2)$. This element often behaves like a unit in the A_∞ algebra for L, (see 7[CWa]), and generally it can be shown that if μ_L^0 is a multiple of x_M, then $(\mu_L^1)^2 = 0$ and the Floer cohomology of L is defined. The following is an immediate corollary of 20:

Corollary 4. *If $\mu^0_{L_B,\theta}$ is a multiple of the unique maximum of b on L_B, then $\mu^0_{L,\rho}$ is a multiple of the unique maximum of an appropriate Morse function $g + \pi^* b$ on L.*

To actually see that x_M is a strict unit for $A(L)$, we require more delicate assumptions on the coherence axioms for perturbation data, and the introduction of weighted edges. This is done in detail in [CWa].

Aside from units, the main point of potential is to allow us to calculate Floer cohomology in simple way. The general theorem that one looks to prove is: if we have a critical point of this potential at a particular representation, then the differential at the higher pages of the Morse-to-Floer spectral sequence vanishes and the Floer cohomology is isomorphic to the Morse cohomology. This is detailed in the next subsection.

5.2.2 The weak divisor equation

Moduli spaces defined via divisorial perturbations satisfy a weak form of the *divisor equation* from Gromov-Witten theory. With this principle one can use the cup product to count the number of holomorphic disks in a certain moduli, which allows us to get a relation between μ^1 and the derivative of μ^0.

First, we want to know that the exponential function makes sense in Λ^2: In order for the exponential sum to converge, we have to introduce a natural topology as in [Fuk].

Definition 46. Let

$$\mathbf{x}_k = \sum_{i,j} c_{ij}^k q^{\rho_i} r^{\eta_j} \in \Lambda^2, \quad \mathbf{x} = \sum_{i,j} c_{ij} q^{\rho_i} r^{\eta_j}$$

be in Λ^2. We say that $\mathbf{x}_k$ converges to $\mathbf{x}$ if each c_{ij}^k converges to c_{ij} in $\mathbb{C}$

Lemma 14. *For $p \in \Lambda^2$, the sequence $\sum_{n=0}^N \frac{p^n}{n!}$ converges in the above topology.*

Proof. Define

$$\exp(p) = \sum_{n=0}^{\infty} \frac{p^n}{n!}$$

If this is well defined in Λ^2, then it is clear that this is the limit of the sequence. Let $p = \sum_{i,j\geq 0} c_{ij} q^{\rho_i} r^{\eta_j} \in \Lambda^2$ with ρ_0 and η_0 such that $\rho_0 + \eta_0 = \min_{i,j} \rho_i + \eta_j$. Then $\exp(p) = 1 + \sum_{n=0}^{\infty} c_{00}^n q^{n\rho_0} r^{n\eta_0}/n! +$ higher degree terms. From $\rho_0 + \eta_0 \geq 0$ and $(1-\epsilon)\rho_0 + \eta_0 \geq 0$, we have that $\rho_0 + \eta_0 = 0$ if and only if $\rho_0 = 0$ and $\eta_0 = 0$, so we take two cases. First let us assume that $\rho_0 + \eta_0 > 0$. Then $n\rho_0 + n\eta_0 \to \infty$, so there are finitely many non-zero coefficient below any bounded degree.

If $p = c_{00} +$ higher degree terms with $c_{00} \neq 0$, then

$$\exp(p) = 1 + \sum_{n=1}^{\infty} c_{00}^n/n! + \sum_{n=1}^{\infty} c_{00}^{n-1} q^{\rho_1} r^{\eta_1}/n! + \cdots$$

It is also clear that $\exp(p) \in \Lambda^2$ in this case. $\square$

It follows that we can endow the space of representations $\mathrm{Hom}(\pi_1(L), \Lambda^{2\times}) \cong H^1(L, \Lambda^{2\times})$ with the structure of a smooth manifold using exp. Indeed, the exponential allows us to form the tensor product

$$H^1(L, \Lambda^2) \otimes_{\Lambda^2} \Lambda^{2\times}$$

If we assume that $H^1(L, \Lambda^2)$ is free, then $H^1(L, \Lambda^2) \otimes_{\Lambda^2} \Lambda^{2\times} \cong H^1(L, \Lambda^{2\times})$, and we have a surjective map given by $H^1(L, \Lambda^2) \to H^1(L, \Lambda^2) \otimes_{\Lambda^2} \Lambda^{2\times}$ that is locally an isomorphism. By the universal coefficients and Hurewicz homomorphism, $H^1(L, \Lambda^{2\times}) \cong \mathrm{Hom}(H_1(L), \Lambda^{2\times}) \cong \mathrm{Hom}(\pi_1(L), \Lambda^{2\times})$. This gives us a tangent space at each point, and it only remains to check that the transition maps are smooth.

From the above discussion, both μ_L^0 and the second order potential define smooth maps $H^1(L, \Lambda^{2\times}) \to CF(L, \Lambda^2)$. Let $\gamma = \sum_{i=1}^k x^k$ be a Morse 1-cocycle and let $W^+(\gamma) = \bigcup W_f^+(x^k)$ be the homology class of the union of the stable manifolds of the critical points appearing in the expression of γ (so this is a codimension 1 cycle). We have the following straightforward computation from [CWa]:

$$\begin{aligned}\mu_\rho^1(\gamma) &= \sum_{u \in \mathcal{M}(L,D,y)_0} \mathrm{Hol}_\rho(u)[\partial u, W^+(\gamma)]\varepsilon(u) q^{e(\pi \circ u)} r^{e_v(u)} y \\ &= \partial_\gamma \mu_\rho^0\end{aligned}$$

Where the perturbations for the moduli space $\mathcal{M}(L, D, y)_0$ are given by pullback under the map that forgets an incoming edge. Thus if ρ is a critical point for μ_ρ^0, then the Floer differential vanishes on Morse 1-cocycles.

In practice, it is much easier to search for critical points of ν^0 5.16. As in the discussion in section 5.3 of this chapter, we have a ring homomorphism

$$\mathfrak{f} : \Lambda^2 \to \Lambda_t$$

given by

$$\sum_{i,j\geq 0} c_{ij} q^{\alpha_i} r^{\eta_j} \mapsto \sum_{i,j\geq 0} c_{ij} t^{\alpha_i+\eta_j}$$

that makes Λ_t into a Λ^2-algebra. We have

$$HF(L,\Lambda^2)\otimes_{\Lambda^2}\Lambda_t \cong HF(L,\Lambda_t)$$

and the weak divisor equation

$$\nu^1_\rho(\gamma) = \partial_\gamma \nu^0_\rho$$

holds for Morse 1-cocycles. Let

$$\mathcal{W}^L_{0,t} := \mathcal{W}^L_0 \otimes 1 \in CF(L,\Lambda_t)$$

denote the *single variable second order potential* for L. The following theorem is a powerful tool introduced in [FOOO09][FOOO]:

Theorem 21. *Let* 0 *be a solution to the Maurer Cartan equation for* $(CF(L),\mu^n)$ *(and hence for* $(CF(L),\nu^n)$*). Suppose that for a* $\rho \in Hom(\pi_1(L),\Lambda_t^\times)$*,* $D_\rho \mathcal{W}^L_{0,t} = 0$ *and the Hessian* $D^2_\rho \mathcal{W}^L_{t,0}$ *is surjective. Suppose further that* $H^*(L,\Lambda_t)$ *is generated by degree one elements via the cup product. Then there is a representation* $\tau \in Hom(\pi_1(L),\Lambda_t^\times)$ *so that*

$$HF(L,\Lambda_t,\tau)\cong H^{Morse}(L,\Lambda_t)$$

Proof. Let $\zeta_x = \min_{v\in\overline{\mathcal{M}}(L,D,x)_0} e(v)$, and define

$$\mathcal{B}_2(\theta) = \sum_{\substack{x\in\mathrm{Crit}(f)\\ u\in\mathcal{J}_x}} (-1)^{\diamondsuit}(\sigma(u)!)^{-1}\mathrm{Hol}_\theta(u) t^{e(u)}\varepsilon(u)x$$

where $\mathcal{J}_x \subset \overline{\mathcal{M}}(L,D,x)_0$ with

$$\mathcal{J}_x := \left\{ u : e(u) = \zeta_x \right\}$$

if $x \neq x_M$ the maximum of f, and

$$\mathcal{J}_x := \left\{ u : e(u) = \zeta_x \right\} \cup \left\{ u : e(u) = \min_{\substack{v\in\overline{\mathcal{M}}(L,D,x)_0\\ e(v)\neq\zeta_x}} e(v) \right\}$$

if $x = x_M$. These minima are attained due to the compactness result for configurations with finite energy 12. In order to relate $\mathcal{B}_2$ to $\mathcal{W}_{0,t}$, we choose K in the weak coupling form to be large enough: As in the proof of theorem 18, let K be such that

$$\epsilon K \Sigma_B > 2\lambda$$

where λ is the monotonicity constant for L_F and Σ_B is energy quantization for (B, L_B). As in the discussion after definition 43, we have

$$\mathcal{B}_2 = \mathcal{W}_{0,t}$$

for this particular K.

$\mathcal{B}_2(\theta)$ is the first and second order terms of ν^0_θ. Therefore, we can apply the induction argument from the proof of the strongly non-degenerate case of Theorem 10.4 [FOOO] to produce a critical point ξ of ν^0. By the weak divisor equation we have that the Floer differential ν^1 vanishes on Morse 1-cocycles.

Finally, we use an induction on cohomological degree and symplectic area, similar to Lemma 13.1 of [FOOO09] and Proposition 2.31 of [CWa], to argue that ν^1 vanishes on all Morse cocycles.

Lemma 15. $\nu^1(\gamma) = 0$ *for any Morse cocycle on the first page of the Morse-to-Floer spectral sequence 5.8.*

Proof. By the A_∞ relations,

$$\nu^1(\nu^2(\gamma_1, \gamma_2)) = \pm\nu^2(\nu^1(\gamma_1), \gamma_2)) \pm \nu^2(\gamma_1, \nu^1(\gamma_2))$$
$$\pm\nu^3(\gamma_1, \gamma_2, \nu^0) \pm \nu^3(\gamma_1, \nu^0, \gamma_2) \pm \nu^3(\nu^0, \gamma_1, \gamma_2)$$

We remark that there are no ν^3 terms due to the assumption that ν^0 is a strict unit. Let ν^n_β be the sum of terms in ν^n_β containing t^β for $\beta \geq 0$. We have

$$\sum_{\beta_1+\beta_2=\beta} \nu^1_{\beta_1}(\nu^2_{\beta_2}(\gamma_1, \gamma_2)) = \sum_{\beta_1+\beta_2=\beta} \pm\nu^2_{\beta_1}(\nu^1_{\beta_2}(\gamma_1), \gamma_2)) + \sum_{\beta_1+\beta_2=\beta} \pm\nu^2_{\beta_1}(\gamma_1, \nu^1_{\beta_2}(\gamma_2))$$

Move any terms involving $\beta_2 \neq 0$ on the right hand side to the other side:

$$\nu^1_\beta(\gamma_1 \cup \gamma_2) = \sum_{\beta_1+\beta_2=\beta} \pm\nu^2_{\beta_1}(\nu^1_{\beta_2}(\gamma_1), \gamma_2)) + \sum_{\beta_1+\beta_2=\beta} \pm\nu^2_{\beta_1}(\gamma_1, \nu^1_{\beta_2}(\gamma_2))$$
$$- \sum_{\beta_1+\beta_2=\beta,\, \beta_2>0} \nu^1_{\beta_1}(\nu^2_{\beta_2}(\gamma_1, \gamma_2))$$

Using this expression, we proceed by induction on cohomological degree and energy. For $d_i \in \mathbb{N}_0$ and $\alpha_i \in \mathbb{R}_{\geq 0}$, say that $(\alpha_1, d_1) \leq (\alpha_2, d_2)$ if $\alpha_1 < \alpha_2$ or $\alpha_1 = \alpha_2$ and $d_1 < d_2$. The base step $(0, d)$ is given, since the computation takes place on the second page of the Morse-to-Floer spectral sequence. The above expression immediately implies the induction step, since the first two terms on the right hand side vanish by the induction hypothesis, since $(\beta_2, \deg(\gamma_i)) \leq (\beta, \deg(\gamma_1 \cup \gamma_2))$. The last term on the right hand side also vanishes by the induction hypothesis since $\beta_1 < \beta$. This proves the lemma. $\square$

It follows from the lemma that the Morse-to-Floer spectral sequence collapses after the first page, and

$$HF(L, \Lambda_t) \cong H^{Morse}(L, \Lambda_t)$$

$\square$

5.3 Invariance

One wants to know that the Floer cohomology defined before theorem 17 is independent of choices of M-type perturbation datum and Maurer-Cartan solution. While a Hamiltonian isotopy of L may destroy the fibration structure, we want to know if our definition of fibered Floer cohomology coincides with any standard definition.

5.3.1 Choices of M-type perturbation data and pullback divisors

To establish the invariance of the Fukaya algebra (within choices that are compatible with the fibration), one can follow the proof of invariance in Charest-Woodward [CWa]. We summarize their result for rational $L_B \subset B$:

For two divisors D_{B_1} and D_{B_2} of the same degree and two stabilizing perturbation datum $\mathcal{P}^0$ $\mathcal{P}^1$, one defines a theory of *quilted*-$\mathcal{P}^{01}$-holomorphic treed disks that are $\mathcal{P}^0$ resp. $\mathcal{P}^1$ at the root resp. leaves and are $\mathcal{P}_t^{01}$-holomorphic in between for some some path between $\mathcal{P}^0$ and $\mathcal{P}^1$. The full result is:

Theorem 22 ([CWa] Corollary 3.12)**.** *For any stabilizing divisors D^0 and $D^1 \subset B \setminus L_B$, and any convergent, coherent, regular, stabilized perturbation systems $\underline{\mathcal{P}}_1$ and $\underline{\mathcal{P}}_2$, the Fukaya algebras $CF(L, \underline{\mathcal{P}}_1)$ and $CF(L, \underline{\mathcal{P}}_2)$ are convergent homotopy equivalent.*

Pick a time parameterization for each quilted type that takes 0 on the root, 1 on the leaves, and only depends on the edge distance from the single quilted component. We assume that the two divisors we pick are built from homotopic sections of the same line bundle. Given an energy e, lemma 2 guarantees the existence of a path (or even an open dense set) of a.c structures J_{D^t} such that D_t contains no J_{D^t}-holomorphic spheres. We then take a time dependent perturbation system $\mathcal{P}_t^{01}$ that takes values in the open, dense set guaranteed by lemma 2 and is equal J_{D^t} on the thin part of the domain. Then, transversality and compactness follow for quilted P_t^{01} treed disks, and we can define a *perturbation morphism* P^{01} from $\mathcal{P}^0$ to $\mathcal{P}^1$ on products by taking the isolated $\mathcal{P}_t^{01}$ trajectories. This, in turn defines an A_∞ morphism between the A_∞ algebras $CF(L, \mathcal{P}^0, D^0)$ and $CF(L, \mathcal{P}^1, D^1)$. To show that the composition of the two perturbation morphisms $P^{10} \circ P^{01}$ is homotopic to the identity, one develops a similar theory with *twice-quilted* treed disks.

When the divisors D^0 and D^1 are not of the same degree, one follows [[CM07], Theorem 8.1] to find divisors $D^{0'}$ and $D^{1'}$ built from homotopic sections of the same line bundle that are ϵ-transverse to D^0 resp. D^1. One uses a theory of holomorphic configurations that are adapted to both D^0 and $D^{0'}$ to get a perturbation morphism (and similarly for D^1 resp. $D^{1'}$) as in the proof of [[CWa] Theorem 3.11]. Finally, we apply the previous argument to the data adapted to the divisors $D^{0'}$ and $D^{1'}$ to get a perturbation morphsim with homotopy inverse between data adapted to these two divisors.

The fibered situation

The divisor $\pi^{-1}(D_B)$ used in our Floer theory is not stabilizing, but the Floer theory is well behaved with respect to $\pi^{-1}(D_B)$ as chapter 4 shows. Thus, we follow the same quilted construction as [[CWa] Section 3] to prove invariance of divisor $\pi^{-1}(D_B)$ and M-type perturbation system. For two divisors D_B^0 and D_B^1 built from homotopic sections of a line bundle and two M-type perturbation datum $\mathcal{P}^0$ resp. $\mathcal{P}^1$ for π-stable types, we choose a path of coherent M-type perturbation data $\mathcal{P}_t^{01}$ and define a theory of quilted $\mathcal{P}^{01}$-holomorphic π-stable types that are π-adapted to $\pi^{-1}(D_B^0)$ near the root and to $\pi^{-1}(D_B^1)$ near the leaves. This theory defines an A_∞ "perturbation" morphism P^{01} where one checks that $P^{01} \circ P^{10}$ is homotopic to the identity.

When the divisors D_B^i are of different degrees, the story is the same as in the rational case [[CWa] Theorem 3.11]. Given a homotopic divisors of high degrees $D_B^{i'}$ that intersects D_B^i ϵ-transversely, the divisors $\pi^{-1}(D_B^i)$ and $\pi^{-1}(D_B^{i'})$ also intersect ϵ-transversely. Therefore, one uses a quilted theory of types that are π-adapted to both $\pi^{-1}(D_B^0)$ and $\pi^{-1}(D_B^{0'})$ to get a perturbation morphism, and similarly for $i = 1$.

Agreement with rational case

In case the pair (E, L) is rational, we would like to see that our definition of the Fukaya algebra agrees with that of [CWa]. We begin with the ring homomorphism

$$\mathfrak{f} : \Lambda^2 \to \Lambda(t)$$

$$\sum_{i,j} c_{ij} q^{\rho_i} r^{\eta_i} \mapsto \sum_{i,j} c_{ij} t^{\rho_i} t^{\eta_j}$$

where $\Lambda(t)$ is the universal Novikov field. This algebra homomorphism is well defined by the definition of Λ^2: specifically, the requirement that $\#\{c_{ij} \neq 0 : \rho_i + \eta_j \leq N\} < \infty$ tells us that $\sum_{i,j\geq 0} c_{ij} t^{\rho_i+\eta_j}$ converges in $\Lambda(t)$. Therefore, $\Lambda(t)$ is a Λ^2 module and we can form the base change by extending $\mathfrak{f}$ linearly to a map between Λ^2-modules

$$\underline{\mathfrak{f}} : CF(L, \Lambda^2) \to CF(L, \Lambda(t))$$

As in the proof of proposition 21, define the single variable A_∞-maps on $CF(L, \Lambda(t))$ as

$$\nu^n(x_1, \dots, x_n) = \sum_{x_0, [u] \in \overline{\mathcal{M}}_\Gamma(L, D, \underline{x})_0} (-1)^\diamondsuit (\sigma(u)!)^{-1} \mathrm{Hol}_\rho(u) t^{e(u)} \varepsilon(u) x_0 \quad (5.16)$$

then $\underline{\mathfrak{f}}$ defines an A_∞-morphism

$$\nu_b^n(\underline{\mathfrak{f}} x_1, \dots, \underline{\mathfrak{f}} x_n) = \underline{\mathfrak{f}} \mu_b^n(x_1, \dots, x_n)$$

In particular, we get a chain map at the $n = 1$ level that induces an isomorphism

$$CF(L, \Lambda^2) \otimes_{\Lambda^2} \Lambda(t) \cong CF(L, \Lambda(t))$$

as chain complexes. Since $\Lambda(t)$ is a field, we have the identity

$$HF(L, \Lambda^2) \otimes_{\Lambda^2} \Lambda(t) \cong HF(L, \Lambda(t))$$

It follows that tensoring the result of 17 with $\Lambda(t)$ gives something that we expect to be a Hamiltonian isotopy invariant.

We want to see that this definition coincides with Charest-Woodward's definition of the Fukaya algebra [CWa]. We sketch the idea as follows. The divisor $\pi^{-1}(D_B)$ is not stabilizing for L, but as in [Theorem 8.1 [CM07]], we take a Hermitian line bundle $X \to E$ with a unitary connection of curvature $i\omega$, and construct a section s_k of $X^{\times k}$ whose intersection with the zero section is ϵ-transverse to $\pi^{-1}(D_B)$. The set $D := s_k^{-1}(0)$ is a smooth codimension 2 sub-manifold such that

1. $[D]^{PD} = k[\omega]$ for large k,

2. D is ϵ-transverse to $\pi^{-1}(D_B)$, and

3. by an extension of lemma 2 and ϵ-transversality, there is a perturbation datum (P'_Γ) that is arbitrarily close to (P_Γ) and agrees with (P_Γ) on $\pi^{-1}(D_B)$ and makes D into a stabilizing divisor for L.

We then construct a theory of quilted (P'_Γ)-holomorphic configurations that are π-adapted to D_B up to the quilted component and adapted to D from the quilted component and onward. The compactness and transversality of these types is expected to hold, and it is further expected that we get a perturbation morphism $P^{01} : CF(L, P_\Gamma, \pi^{-1}(D_B)) \to CF(L, P'_\Gamma, D)$. When we consider the reverse map $P^{10} : CF(L, P'_\Gamma, D) \to CF(L, P_\Gamma, \pi^{-1}(D_B))$, it is expected that an adaption of Charest-Woodward's [CWa] results will show that these two morphisms provide a homotopy equivalence of A_∞-algebras. The details of this section are to be carried out in the shortened version of this thesis.

Chapter 6

Applications

6.1 Flags

The demonstration of a Floer-non-trivial three torus in $\text{Flag}(\mathbb{C}^3)$ inspires a search of non-displaceable tori in higher dimensional partial or full flag manifolds using our framework. We give a construction of a fibered Lagrangian in a type of partial flag manifold, and compute the disk potential in an easy case.

Let F_n^k be the manifold of partial flags $V_1 \subset \cdots \subset V_k \subset \mathbb{C}^n$ where $\dim_{\mathbb{C}} V_i = i$. Identify this homogeneous $U(n)$ space with the coadjoint orbit $G \cdot \xi$ equipped with the $U(n)$-equivariant KKS form $\omega_\xi(X,Y) = \langle \xi, [X,Y] \rangle$. Technically we need small fibers for the Floer theory to work, so we won't try to keep track of the symplectic structure too much. The idea is this: assume that we have chosen $L_k \subset F_n^k$ with $HF(L_k, \Lambda(t)) \neq 0$, and proceed to construct L_{k+1}.

1. Base step: We illustrate the step from $k = 1$ to $k = 2$ to derive a model for the higher order cases. $F_n^1 = \mathbb{P}^n$, so let us take $L_1 = \text{Cliff}\mathbb{P}^n$. we that a fibration $\pi_2 : F_n^2 \to \mathbb{P}^n$ with fiber $\mathbb{P}^{n-1}$, so by theorem 3, there is a T^{n-1}-invariant open neighborhood $\mathcal{U}_1$ of $\text{Cliff}\mathbb{P}^n$ and a symplectic connection on F_n^2 such that the moment map for the action of $T^{n-1} \subset U(n)/U(1)$ on $\pi_2^{-1}(\mathcal{U}_1)$ is $\Phi \circ \pi_2$, where $\Phi : \mathbb{P}^n \to \mathfrak{t}^{n-1\vee}$ is the moment map for $\mathbb{P}^n$. At the level set of $\Phi \circ \pi_2$ which lies above $\text{Cliff}\mathbb{P}^n$, we have a symplectic trivialization $\Phi \circ \pi_2^{-1}(0) \cong \text{Cliff}\mathbb{P}^n \times \mathbb{P}^{n-1}$ by an argument similar to that in example 1.1. Thus, pick the Lagrangian $L_2 = \text{Cliff}\mathbb{P}^n \times \text{Cliff}\mathbb{P}^{n-1} \subset F_n^2$. If one chooses, one may deform back to the original symplectic connection.

We calculate the second order potential for L_2 using theorem 20 and show that it has a critical point. Since it is a product torus, let y_i be the evaluation of the local system ρ on the first n factors, and z_j be the evaluation on the last $n-1$ factors. The (leading order) potential for Cliff$\mathbb{P}^n$ is

$$\mathcal{W}_0^{\mathrm{Cliff}\mathbb{P}^n}(\theta) = \sum_{i=1}^{n} y_i q^{e(u_i)} + \frac{1}{y_1 \cdots y_n} q^{e(u_{n+1})} \tag{6.1}$$

where $e(u_i) = \frac{1}{n+1}$ [FOOO]. We leave out the unique maximum since it is known that Cliff$\mathbb{P}^n$ is unobstructed. The lifted potential of Cliff$\mathbb{P}^n$ in F_n^2 is

$$\mathcal{L} \circ \mathcal{W}_0^{\mathrm{Cliff}\mathbb{P}^n}(\rho) = \sum_{i=1}^{n} y_i z^{\nu_i} q^{e(u_i)} + \frac{1}{y_1 \cdots y_n z^{\nu_1} \cdots z^{\nu_n}} q^{e(u_{n+1})} \tag{6.2}$$

where ν_i is the multindex $z^{\nu_i} := z_1^{\nu_{i1}} \cdots z_n^{\nu_{in}}$ given by evaluation of $\mathcal{L}u_i$ on the local system ρ. By theorem 20 the second order potential of L_2 is

$$\begin{aligned}\mathcal{W}_0^{L_2}(\rho) = &\sum_{i=1}^{n} y_i z^{\nu_i} q^{e(u_i)} + \frac{1}{y_1 \cdots y_n z^{\nu_1} \cdots z^{\nu_n}} q^{e(u_{n+1})}\\ &+ \sum_{i=1}^{n-1} z_i r^{e_v(w_i)} + \frac{1}{z_1 \cdots z_{n-1}} r^{e_v(w_n)}\end{aligned}$$

we have the partial derivative with respect to the y-variables

$$\partial_{y_i} \mathcal{W}_0^{L_2} = q^{e(u_i)} + \frac{-1}{y_i \cdot y_1 \cdots y_n z^{\nu_1} \cdots z^{\nu_n}} q^{e(u_{n+1})} \tag{6.3}$$

and with respect to the z variables

$$\begin{aligned}\partial_{z_j} \mathcal{W}_0^{L_2} = &\sum_{i=1}^{n} \nu_{ij} y_i z^{\nu_i - \mathfrak{e}_j} q^{e(u_i)} + \frac{-\sum_{i=1}^{n} \nu_{ij}}{y_1 \cdots y_n z^{\nu + \mathfrak{e}_j}} q^{e(u_{n+1})}\\ &+ r^{e_v(w_j)} + \frac{-1}{z_j \cdot z_1 \cdots z_{n-1}} r^{e_v(w_n)}\end{aligned}$$

where $\mathfrak{e}_j = (0, \dots, 0, 1, 0, \dots, 0)$ is the j^{th} standard multindex and $\nu = \sum_i \nu_i$. Solutions to 6.3 are given by $y_1 = \cdots = y_n = e^{2\pi i k/(n+1)}$ and $z_1 = \cdots = z_{n-1} = e^{2\pi i \ell/|\nu|}$ with $|\nu|$ denoting the sum of the indices. Solutions to $\partial_{z_j} \mathcal{W}_0^{L_2} = 0$ only seem possible if $z_j \equiv 1\ \forall j$ (unless we have more information about the lifted representation) and $y_1 = \cdots = y_n = e^{2\pi i k/(n+1)}$.

By proposition 21, there is a local system ξ such that $HF(L_2, \Lambda(t), \xi) \cong H^{Morse}(L_2, \Lambda(t)) \neq 0$.

2. Induction step: Assume that we have constructed a Lagrangian $L_k \subset F_n^k$ with $HF(L_k, \Lambda(t)) \neq 0$. Viewing $\pi : F_n^k \to \mathbb{P}^n$ as a symplectic fibration we assume that L_k fibers over Cliff$\mathbb{P}^n$. Apply theorem 3 to get a symplectic connection on F_n^k such that the moment map for the action of T^{n-1} on the open set $\pi^{-1}(\mathcal{U}_1)$ is $\Phi \circ \pi$. Now consider the symplectic fibration $\pi_{k+1} : F_n^{k+1} \to F_n^k$ and apply theorem 3 to once again deform the symplectic form so that the moment map for the T^{n-1} action on $\pi_{k+1}^{-1}(\pi^{-1}(\mathcal{U}_1))$ is $\Phi \circ \pi \circ \pi_{k+1}$. Then, $\pi_{k+1}^{-1}[\pi^{-1}(\text{Cliff}\mathbb{P}^n)] \cong \mathbb{P}^{n-k+1} \times \pi^{-1}(\text{Cliff}\mathbb{P}^n)$ as a symplectic fibration, so choose $L_{k+1} \cong \text{Cliff}\mathbb{P}^{n-k+1} \times L_k$.

The computation of the disk potential and the critical points involves many different indices and multindices in this case, but is morally similar to the base case. We omit the calculation.

6.2 Projective ruled surfaces

There are some low dimension applications that naturally show up in the Gonzalez-Woodward symplectic minimal model program [WC; GW]. In dimension 4, a typical end stage of running of the minimal model program is a so called *ruled surface*, or a holomorphic $\mathbb{P}^1$ bundle over a Riemann surface. These occur in the classification of surfaces due to Enriques-Kodaira [BHPV04], which we review in this section. Then, we show that one can construct a fibered Lagrangian torus that is Floer-non-trivial.

In the classification of projective surfaces [BHPV04], there is the case where no exterior powers of the cannonical line bundle admit holomorphic sections. More precisely, let E be a projective surface, and $K_E = T^*E \wedge T^*E$ be the *canonical line bundle.* We have the object $H^0(E, K_E)$ that is the vector space of holomorphic sections. Form the sequence of integers $P_i(E) = \dim H^0(E, K_E^{\otimes i})$. If $P_i(E) = 0$ for all positive integers i, then the *Kodaira dimension* of E is said to be $-\infty$ (This is in contrast to the other possible cases when $P_i(E)$ has asymptotics like i^k for $k \geq 0$). Surfaces with Kodaira dimension equal to $-\infty$ are known as the ruled surfaces, where E fibers as a $\mathbb{P}^1$ bundle over a Riemann surface B. For the complete classification, see [BHPV04].

Basic cohomology theory gives us that any algebraic $\mathbb{P}^n$ bundle with structure group $PGL(n+1, \mathbb{C})$ over a Riemann surface B is actually the projectivization of a vector bundle. This follows from the long exact sequence of sheaf cohomology groups

$$\to H^1(B, \mathcal{GL}(n, \mathbb{C})) \to H^1(B, \mathcal{PGL}(n-1, \mathbb{C})) \to H^2(B, \mathcal{O}_B)$$

Since B is a curve, $H^2(B, \mathcal{O}_B) = 0$ by Grothendieck vanishing. Thus, $H^1(B, \mathcal{GL}(n, \mathbb{C})) \to H^1(B, \mathcal{PGL}(n, \mathbb{C}))$ is surjective, so that every algebraic projective bundle lifts to an algebraic vector bundle.

6.2.1 Ruled surfaces with $\mathbb{P}^1$ base

We compute the disk potential for a Lagrangian torus in the Hirzebruch surface $\mathbb{F}_n$ that may not be a level set of the moment map.

Let $E \to \mathbb{P}^1$ be a rank two complex vector bundle. A result of Grothendieck [BHPV04], say that this splits as a sum of complex line bundles $\mathcal{O}_n \oplus \mathcal{O}_m$. One can then projectivize $\mathbb{P}(E) \to \mathbb{P}^1$. Since tensoring before projectivizing has no effect, we can normalize to get $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}_n) \to \mathbb{P}^1$, which we define as the n^{th} Hirzebruch surface $\mathbb{F}_n$.

Take Cliff$\mathbb{P}^1 := L_B$. Appealing to a deformation lemma in [Sei08]16, one can deform the connection above Cliff$\mathbb{P}^1$ so that the holonomy is trivial, and pick a section of Cliff$\mathbb{P}^1 := L_F$ in above L_B. We then deform back to the original symplectic form to obtain a Lagrangian torus $L \subset \mathbb{F}_n$ that fibers as Cliff$\mathbb{P}^1 \to L \to$ Cliff$\mathbb{P}^1$.

For a representation $\rho \in \mathrm{Hom}(\pi_1(L), \Lambda^\times)$ let y_1,y_2 denote the values on the canonical generators. We take the weak coupling form $\omega = a + K\pi^*\omega_B$ with $\iota^* a = \omega_F$, and a regular perturbation datum that is close to the integrable structure. Let u_i $i = 1, 2$ denote the Maslov index 2 disks in $\mathbb{P}^1$ with a single marked boundary point and boundary contained in L_B, and v_i the holomorphic disks of the same type with boundary in L_F.

Following section 4.5, we will take the vertically constant $J^G = \begin{bmatrix} J_G & 0 \\ 0 & J_B \end{bmatrix}$-holomorphic lifts and then perturb. The evaluation of the representation on the lift of u_1 is y_1 if we were to consider the fibration as trivial over the hemisphere described by u_1. This determines the representation evaluated on a vertically constant lift of u_2, which is $y_1^{-1}y_2^{-n}$ since the transition map between the two hemispheres of the base $\mathbb{P}^1$ is $[z_1, z_2] \mapsto [z_1, e^{2\pi i n\theta} z_2]$. We then perturb these lifts to $J = \begin{bmatrix} J_F & K \\ 0 & J_B \end{bmatrix}$-holomorphic disks, denoted $\mathcal{L}u_i$, where this J is part of a coherent perturbation datum. The perturbation process is a homotopy equivalence, so the evaluation on the local system does not change after perturbing. In order to avoid cluttered notation, we let $e(u) := e(\pi \circ u) = \int \pi \circ u^*\omega_B$. From the formula 5.15 we have

$$\mathcal{W}_L^0(\rho) = y_2 r^{e_v(v_1)} + y_2^{-1} r^{e_v(v_2)} + y_1 q^{e(u_1)} r^{e_v(\mathcal{L}u_1)} + y_1^{-1} y_2^{-n} q^{e(u_2)} r^{e_v(\mathcal{L}u_2)}$$

The partial derivatives are as follows:

$$\partial_{y_1} \mathcal{W}_L^0(\rho) = q^{e(u_1)} r^{e_v(\mathcal{L}u_1)} - y_1^{-2} y_2^{-n} q^{e(u_2)} r^{e_v(\mathcal{L}u_2)} \tag{6.4}$$

$$\partial_{y_2} \mathcal{W}_L^0(\rho) = r^{e_v(v_1)} - y_2^{-2} r^{e(v_2)} - n y_1^{-1} y_2^{-n-1} q^{e(u_2)} r^{e_v(\mathcal{L}u_2)} \tag{6.5}$$

The base Lagrangian is a Clifford torus, so we have that $e(u_1) = e(u_2)$. If we assume thusly that $e_v(\mathcal{L}u_1) = e_v(\mathcal{L}u_2)$, then the zero locus of 6.4 contains the locus $y_1^2 = y_2^{-n}$. Substituting into 6.5 gives the equation

$$r^{e_v(v_1)} - y_2^{-2} r^{e(v_2)} - n y_2^{\frac{-n-2}{2}} q^{e(u_2)} = 0$$

Viewing this expression under the transformation $y_2 \mapsto y_2^2$ gives a degree $n+2$ polynomial

$$r^{e_v(v_1)} - y_2^{-4} r^{e_v(v_2)} - n y_2^{-n-2} q^{e(u_2)} r^{e_v(\mathcal{L}u_2)} = 0 \tag{6.6}$$

Since the fiber Lagrangian is a Clifford torus, we can assume that $r^{e_v(v_1)} = r^{e_v(v_2)}$. Dividing out by this variable gives a simpler equation:

$$1 - y_2^{-4} - n y_2^{-n-2} q^{e(u_2)} r^{e_v(\mathcal{L}u_2) - e_v(v_1)} = 0 \tag{6.7}$$

Assuming $n > 2$, we can consider 6.7 as a monic equation

$$y_2^{n+2} - y_2^{n-2} - n q^{\rho} r^{\eta} = 0 \tag{6.8}$$

with $\rho = \int K u_2^* \omega_B$ and $\eta = e_v(\mathcal{L}u_2) - e_v(v_1)$. While η may be negative, we can take $K >> 1$ in the weak coupling form so that $q^{\rho} r^{\eta}$ actually lies in our ring Λ^2.

By Hensel's lemma, there are four unital solutions to 6.8 in Λ^2 (see discussion in example 1.1). It follows from proposition 21 that there is a representation $\tau \in \mathrm{Hom}(\pi_1(L), \Lambda_t^{\times})$ so that

$$HF(L, \Lambda_t, \tau) \cong H^{Morse}(L, \Lambda_t)$$

6.2.2 Base genus ≥ 2

We give a description of a Lagrangian in a ruled surface E with base genus ≥ 2, and show that is has non-trivial Floer cohomology whenever the topology allows.

Let B be a Riemann surface of genus ≥ 2. Considering Lagrangians as simple closed curves, we say that an embedded Lagrangian is *balanced* if it is null-homologous and

$$\frac{\mathrm{Area}(B_+)}{\chi(B_+)} = \frac{\mathrm{Area}(B_-)}{\chi(B_-)} \tag{6.9}$$

whenever L_B divides B into two Riemann surfaces with boundary. The notion of balanced is really a monotonicity condition of sorts, and allows one to construct the Lagrangian intersection theory. It has been observed by Seidel, Efimov and others [Efi12; Sei11] that these curves generate the Fukaya category.

Since L_B is not contractible, we have that $\pi_2(B, L_B) = 1$, so there are no non-trivial disks with boundary in L_B. Therefore, the Floer cohomology is isomorphic to the classical Morse cohomology with Novikov coefficients (which one can see from the spectral sequence in [FOOO09]), and this balanced Lagrangian is non-displaceable

Now let V be a rank 2 vector bundle over B and $\mathbb{P}(V) \to E \to B$ be its projectivization. Let us pick a Lagrangian with $L_F \subset \mathbb{P}^1$ dividing the symplectic area of $\mathbb{P}^1$ into two equal parts. As in [FOOO], one can compute the disk potential for L_F, and the requirement that it is balanced gives us critical points at the representations ± 1, hence $HF(L_F, \Lambda[t]) \cong H^*(L_F, \Lambda[t])$.

Let us denote our ruled surface as $\Sigma_{g,V}$, with base genus g and ruling V. To find a subbundle $L_F \to L \to L_B$ in $\Sigma_{g,V}$, we deform the connection and use parallel transport to flow out a torus. Let L_B be balanced, simple and closed as above with a parameterization γ_B, and let ω define a connection on $\Sigma_{g,V}$ by $TF \oplus H$ with $H = TF^{\omega\perp}$. Then, parallel transport along γ_B gives maps

$$\phi_s : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(s))$$

that are Hamiltonian diffeomorphisms since the $\mathbb{P}^1$ fiber is simply connected. Following Seidel [Sei08] section 15, we then deform the symplectic form (and horizontal splitting) by $d\alpha$ where α is a vanishes on TF to prescribe different parallel transport maps. This is made precise in the following lemma:

Lemma 16. *[Sei08] Let γ be a path in B and let ψ_s be a Hamiltonian isotopy of $F_{\gamma(0)}$ starting with $\psi_0 = Id$. Then there is a deformation of the fibration along γ that extends to all of E such* *that the parallel transport maps along γ satisfy*

$$\phi = \phi_s \circ \psi_s$$

In particular, when γ is a loop as in this case, we can deform the structure so that holonomy around the loop is the identity. Thus, pick a simple closed curve $L_F \subset F$ and look at the image of its transport along γ. This gives us a Lagrangian torus in the deformed manifold.

Proof of lemma. The idea is as follows: Let α be a 1-form that vanishes on TF and in a neighborhood of $\pi^{-1}(\gamma)$. Then $\omega + d\alpha = \omega$ when restricted to TF, and so is non-degenerate. Let $Y^\sharp = (X, Y)$ (in the ω splitting) be a horizontal lift (in the $\omega + d\alpha$ splitting) of a vector field Y. Then, $\mathcal{L}_{Y^\sharp}\alpha = 0$ on TF since it is the pullback of a base form, and vertically we have

$$\begin{aligned} 0 &= \iota_{Y^\sharp}(\omega + d\alpha) \\ &= \iota_X \omega + \iota_{Y^\sharp} d\alpha \\ &= \iota_X \omega - d\iota_Y \alpha \end{aligned}$$

which says that parallel transport in the $\omega + d\alpha$ splitting is infinitesimally the Hamiltonian flow of $-\iota_Y\alpha$. Thus, for a Hamiltonian isotopy ψ_s with associated time-dependent Hamiltonian H_s let α be any 1-form that vanishes on TF such that

$$\iota_Y \alpha|_{\pi^{-1}(\gamma(s))} = H_s$$

and vanishes outside of a neighborhood of $\pi^{-1}(\gamma)$. Parallel transport with respect to the α splitting will then be prescribed by $\phi_s \circ \psi_s$. The desired deformation is then

$$\omega + K\pi^*\omega_B \mapsto \omega + td\alpha + K\pi^*\omega_B$$

for K large enough. □

The deformation only changes the symplectic form by an exact form. Thus, an application of Moser's theorem gives us a symplectic isotopy back to the original symplectic structure, which in turn gives a Lagrangian.

We use the spectral sequence to compute $HF(L, \Lambda^2)$. According 17, the second page is the cohomology of the Morse chain complex of L_B with coefficients in the local system $\mathcal{HF}(L_F, \Lambda_r)$. The filtration is with respect to the base energy, but the differential induced on any of the higher pages does not include any q terms. Therefore, the sequence collapses after the second page, and we have that the Floer cohomology of L is isomorphic to the homology of the complex $CF(L)$ with differential δ_0 that counts isolated Floer trajectories in each fiber in addition to zero-energy Morse configurations in the base:

$$gr_*(HF(L, \Lambda^2)) \cong \mathcal{E}_2((CF(L), \Lambda^2), \delta_0, \mathcal{F}_q)$$

The second page of the Floer fibration spectral sequence can be computed via the usual Leray-Serre spectral sequence of a fibration with vertical differential given by δ_F^{Floer} and horizontal differential given by δ_B^{Morse}

$$\mathcal{E}_2(CF(L, \Lambda^2), \delta_0, \mathcal{F}_q) \cong \mathcal{E}_\infty^{LS}(C(L), \delta_B^{\text{Morse}} \pm \delta_F^{\text{Floer}}, \mathcal{F}_{bd})$$

where the filtration $\mathcal{F}_{bd}$ is given by base degree, i.e. $\deg \pi(x_i)$ for x_i a critical point on the total space. The second page of this is given as follows

$$\mathcal{E}_2^{LS} \cong H^*(L_B, \mathcal{HF}(L_F, \Lambda_r)) \otimes gr(\mathcal{F}_q \Lambda_q)$$

as in theorem 17.

A modification of the proof of theorem 17 will show us that the higher differentials on $\mathcal{E}_r^{LS}$ count only Morse flow lines in the total space. Briefly, the point in the base at which a vertical disk component occurs induces a free parameter in the moduli space, so it is not an isolated configuration. The proof of the existence of this free parameter is slightly more complicated that in the proof of 17, as a trajectory between critical points of a higher base index difference no longer projects to a reparameterization of a Morse trajectory.

Assuming the above discussion holds, the Floer cohomology of L Lagrangian only depends on the topology and the action of $\pi_1(S^1)$ on the Floer cohomology of the fibers. In particular, if L is a product Lagrangian and the action is trivial, then we have

$$\mathcal{E}_\infty^{LS} \cong \mathcal{E}_2^{LS} \cong H^*(L_B) \otimes HF(L_F, \Lambda_r) \otimes gr(\mathcal{F}_q \Lambda_q)$$

so that

$$HF(L, \Lambda^2) \cong H^*(L_B, \Lambda_q) \otimes H(L_F, \Lambda_r)$$

which shows that L is non-displaceable.

Chapter 7

Appendix

For completeness, we include some aspects of the A_∞-algebra and Maurer-Cartan equation for a rational Lagrangian in a rational symplectic manifold. This section is taken from [CWa].

7.1 A_∞ algebras and composition maps

We define the necessary algebraic notions to consider Fukaya algebras of Lagrangians. Define the *universal Novikov field* of formal power series:

$$\Lambda = \left\{ \sum_i c_i q^{\rho_i} | \quad c_i \in \mathbb{C}, \rho_i \in \mathbb{R}, \quad \rho_i \to \infty \right\} \tag{7.1}$$

The subalgebra of only non-negative powers will be denoted $\Lambda_{\geq 0}$ (similarly $\Lambda_{>0}$).

The axioms for an A_∞ algebra are as follows. Let A be a $\mathbb{Z}_g$-graded vector space and let

$$\mu^d : A^{\otimes d} \to A[2-d]$$

be multilinear maps. (A, μ^d) is said to be an A_∞ *algebra* if the composition maps satisfy the following relations:

$$0 = \sum_{n,m \geq 0 \, n+m \leq d} (-1)^{n + \sum_{i=1}^n |a_i|} \mu^{d-m+1}(a_1, \ldots, a_n,$$

$$\mu^m(a_{n+1}, \ldots, a_{n+m}), a_{n+m+1}, \ldots, a_d)$$

We will also need the notion of an A_∞ morphism between two algebras. Let A_0 and A_1 be two A_∞ algebras.

Definition 47. An A_∞ morphism from A_0 to A_1 is a collection of maps

$$\mathcal{F}^d : A_0^{\otimes d} \to A_1[1-d], \; d \geq 0$$

such that the following equation holds:

$$\sum_{i+j\leq d}(-1)^{i+\sum_{j=1}^{i}|a_j|}\mathcal{F}^{d-j+1}(a_1,\ldots,a_i,\mu^j_{A_0}(a_{i+1},\ldots,a_{i+j}),a_{i+j+1},\ldots,a_d) =$$

$$\sum_{i_1+\ldots i_m=d}\mu^m_{A_1}(\mathcal{F}^{i_1}(a_1,\ldots,a_{i_1},\ldots,\mathcal{F}^{i_m}(a_{i_1+\cdots+i_{m-1}+1},\ldots,a_d))$$

In order to properly define the Fukaya algebra for a Lagrangian, we require that the Lagrangian have additional structure, called a *brane structure*. Let E be a symplectic manifold and $\mathrm{Lag}(E)$ the fiber bundle bundle whose fiber at p is the Grassmanian of Lagrangian subspaces of T_pE. For an even integer g, a *Maslov cover* is a g-fold cover $\mathrm{Lag}^g(E) \to \mathrm{Lag}(E)$ such that the induced two-fold cover $\mathrm{Lag}^g(E)/\mathbb{Z}_{g/2} \to \mathrm{Lag}(E)$ is the oriented double cover. A Lagrangian submanifold is *admissible* if it is compact and oriented (we assume connectedness for now).

A *grading* on L is a lift of the canonical map

$$L \to \mathrm{Lag}(X), \quad l \mapsto T_lL$$

to $\mathrm{Lag}^g(X)$. A *relative spin structure* for L is a lift of the transition maps $\psi_{\alpha\beta}$ for TL to Spin satisfying the cocycle condition

$$\psi_{\alpha\beta}\psi^{-1}_{\alpha\gamma}\psi_{\beta\gamma} = i^*\varepsilon_{\alpha\beta\gamma}$$

where $\varepsilon_{\alpha\beta\gamma}$ is a 2-cycle on E. Let

$$\Lambda^\times = \{c_0 + \Sigma_{i>0}c_iq^{\rho_i} \subset \Lambda_{\geq 0}|c_0 \neq 0\}$$

be the subgroup of formal power series with invertible leading coefficient. A *rank one local system* (with values in $\Lambda^\times$) is a representation $\pi_1(E) \to \Lambda^\times$. A *brane structure* for a compact oriented (connected) Lagrangian L consists of the following data:

1. A Maslov cover $\mathrm{Lag}^g(E) \to$ with a grading,
2. A rank one local system with values in $\Lambda^\times$ and
3. A relative spin structure with the given 2-cycles.

An *admissible Lagrangian brane* is an admissible Lagrangian submanifold equipped with a brane structure. For such an object, the space of Floer cochains is defined as

$$CF(L) = \bigoplus_{d \in \mathbb{Z}_g} CF_d(L), \quad CF_d(L) = \bigoplus_{x \in \hat{\mathcal{I}}_d(L)} \Lambda\langle x \rangle$$

Given a Lagrangian brane L, we denote by $\mathrm{Hol}_L(u) \in \mathbb{C}^\times$ the evaluation of the local system on the homotopy class of loops defined by going around the boundary of the treed disk once. We denote by $\sigma([u])$ the number of interior markings of $[u] \in \overline{\mathcal{M}}_\Gamma(L, D, \underline{x})$.

Definition 48. For regular stabilizing coherent perturbation data (P_Γ) define the composition maps

$$\mu^n : CF(L)^{\otimes n} \to CF(L)$$

on critical points by the following equation:

$$\mu^n(x_1, \ldots, x_n) = \sum_{x_0, [u] \in \overline{\mathcal{M}}_\Gamma(L, D, \underline{x})_0} (-1)^{\lozenge} (\sigma([u])!)^{-1} \mathrm{Hol}_L(u) q^{e([u])} \epsilon([u]) x_0 \tag{7.2}$$

where $\lozenge = \sum_{i=1}^n i|x_i|$.

So far, we have neglected to mention anything about units. In fact, everything that has been recorded so far can be done to incorporate a *strict unit*.

Definition 49. Let A be an A_∞ algebra. A *strict unit* for A is an element e_A such that

$$\mu^2(e_A, a) = a = (-1)^{|a|} \mu^2(a, e_a)$$

$$\mu^n(\ldots, e_A, \ldots) = 0, \ n \neq 2$$

An A_∞-algebra is called *strictly unital* if it equipped with a strict unit.

One obtains such a thing by replacing the unique maximum x with 3 copies such that

$$i(x_M^\bullet) = i(x_M^\circ) = 0, \ i(x_M^\Delta) = -1$$

The notion of a treed holomorphic disk, morphisms of moduli spaces, and a coherent perturbation system can be modified to incorporate these three additional copies. See [CWa] for the full details.

Let $\widehat{CF}(L)$ be the chain complex with this additional structure

We have the following theorem.

Theorem 23 (A_∞ relations)**.** *[CWa] Let $\mathcal{P}$ be a coherent,stabilizing, regular perturbation datum. Then $(\widehat{CF}(L), \{\mu^n\}_n)$ is A_∞ algebra with strict unit. The subcomplex $CF(L)$ is an A_∞-algebra without unit.*

Sketch of proof. For an admissible tuple $(x_0, \dots, x_n)$, components of the moduli space $\overline{\mathcal{M}}(L, D, \underline{x})_1$ are compact manifolds with (possibly overlapping) boundary. Thus they obey the following relation:

$$0 = \sum_{\Gamma \in \mathfrak{M}_{n,m}} \sum_{[u] \in \partial \overline{\mathcal{M}}_\Gamma (L,D,\underline{x})_1} (\sigma(u))^{-1} \varepsilon(u) q^{e(u)} \mathrm{Hol}(u) \tag{7.3}$$

When Γ is a type without weights, then the boundary points of the moduli space are types with a (one additional) breaking, equivalent to the union of types Γ_1 and Γ_2 with n_1 resp. $n - n_1 - 1$ leaves. By the (product axiom),

$$\partial \overline{\mathcal{M}}(L, D, \underline{x})_1 = \bigcup_{y, \Gamma_1, \Gamma_2} \mathcal{M}_{\Gamma_1}(L, D, x_0, \dots, x_{i-1}, y, x_{i+n_2}, \dots, x_n) \times \mathcal{M}_{\Gamma_2}(L, D, y, x_i, \dots, x_{i+n_2-1}) \tag{7.4}$$

Say $\sigma([u]) = m$ Since there are m choose m_1, m_2 ways of distributing the interior markings to the two component graphs,

$$0 = \sum_{\substack{i, m_1+m_2=m \\ [u_1] \in \mathcal{M}_{\Gamma_1}(L,D,x_0,\dots,x_{i-1},y,x_{i+n_2},\dots,x_n)_0 \\ [u_2] \in \mathcal{M}_{\Gamma_2}(L,D,y,x_i,\dots,x_{i+n_2-1})_0}} (m!)^{(-1)} \binom{m}{m_1} q^{e(u_1)+e(u_2)} \varepsilon(u_1)\varepsilon(u_2) \mathrm{Hol}_L(u_1) \mathrm{Hol}(u_2) \tag{7.5}$$

This is the A_∞ relation up to signs, and it now remains to show that the signs arising from the orientations are consistent with those of the A_∞ relations. We refer the reader to [CWa].

$\square$

Next, we include the necessary statements to find a perturbation system so that the resulting A_∞ algebra is convergent:

Definition 50. A perturbation system $\underline{P} = (P_\Gamma)$ is *convergent* if for each energy bound E, there exists a constant $C(E)$ such that for any Γ and any treed J_Γ-holomorphic disk $u : C \to X$ of type Γ, the total Maslov index $I(u) := \sum I(u_i)$ satisfies

$$(e(u) < e) \Rightarrow (I(u) < c(E)). \tag{7.6}$$

Lemma 17. *[CWa] Any convergent, coherent, regular, stabilizing perturbation system $\underline{P} = (P_\Gamma)$ defines a convergent Fukaya algebra $\widehat{CF}(L, \underline{P})$.*

Proposition 4. *[CWa] There exist convergent, coherent, regular, stabilizing perturbations $\underline{P} = (P_\Gamma)$.*

See [CWa] for the proof.

Bibliography

[AD14] M. Audin and M. Damian. *Morse Theory and Floer Homology.* London: Springer-Verlag, 2014.

[AGM01] D. Auroux, D. Gayet, and J.-P. Mohsen. "Symplectic hypersurfaces in the complement of an isotropic submanifold". In: *Math. Ann.* 321 (2001), pp. 739–754.

[BHPV04] W.P. Barth, K. Hulek, C.A.M. Peters, and A. Van de Ven. *Compact Complex Surfaces.* Berlin: Springer-Verlag, 2004.

[Cha] F. Charest. *Source spaces and perturbations for cluster complexes.* arXiv: `1212.2923 [math.SG]`.

[CWa] F. Charest and C. Woodward. *Floer cohomology and flips.* arXiv: `1508.01573v3 [math.SG]`.

[CWb] F. Charest and C. Woodward. *Floer trajectories and stabilizing divisors.* arXiv: `1401.0150 [math.SG]`.

[Cho04] C.-H. Cho. "Holomorphic disks, spin structures, and the Floer cohomology of the Clifford torus". In: *Interneational mathematics research notices* 35 (2004), pp. 1803–1843.

[CM07] K. Cieliebak and K. Mohnke. "Symplectic hypersurfaces and transversality in Gromov-Witten theory". In: *J. Symplectic Geom.* 3.5 (2007), pp. 281–356.

[Don92] S.K. Donaldson. "Boundary problems for Yang-Mills fields". In: *Journal of geometry and physics* 8 (1992), pp. 89–122.

[Don96] S.K. Donaldson. "Symplectic submanifolds and almost-complex geometry". In: *J. Symplectic Geom.* 44 (1996), pp. 666–705.

[DK90] S.K. Donaldson and Kronheimer. *The Geometry of Four-Manifolds.* New York: Oxford University Press, 1990.

[Dra04] D.L. Dragnev. "Fredholm theory and transversality for noncompact pseudoholomorphic maps in symplectizations". In: *Comm. Pure Appl. Math.* 57.6 (2004), pp. 726–763.

[Efi12] A. Efimov. "Homological mirror symmetry for curves of higher genus". In: *Advances in Mathematics* 230.1 (June 2012), pp. 493–530.

[Flo88] A. Floer. "Morse thoery for lagrangian intersections". In: *J. Diff. Geo.* 28 (1988), pp. 513–547.

[Flo89] A. Floer. "Symplectic fixed points and holomorphic spheres". In: *Comm. Pure Appl. Math.* 41 (1989), pp. 775–813.

[Fuk] K. Fukaya. *Cyclic symmetry and adic convergence in Lagrangian Floer theory.* arXiv: `0907.4219 [math.dg]`.

[FOOO09] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. *Lagrangian intersection Floer theory: anamoly and obstruction.,* volume 46 of *AMS/IP Studies in Advanced Mathematics.* Providence, RI: American Mathematical Society, 2009.

[FOOO] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. *Lagrangian Floer theory on compact toric manifolds I.* arXiv: `0802.1703v3 [math.SG]`.

[GW] E. Gonzaléz and C. Woodward. *Quantum cohomology and toric minimal model programs.* arXiv: `1207.3253 [math.AG]`.

[Gro85] M. Gromov. "Psuedo holomorphic curves in symplectic manifolds". In: *Invent. Math.* 82 (1985), pp. 307–347.

[Gro57] A. Grothendieck. "L'anneau spectral et l'anneau fibré d'homologie d'un espace localement compact et d'une application continue". In: *Tohoku Math. J.* 9.2 (1957), pp. 119–221.

[GLS96] V. Guillemin, E. Lerman, and S. Sternberg. *Symplectic fibrations and multiplicity diagrams*. Cambridge: Cambridge University Press, 1996.

[Har97] R. Hartshorne. *Algebraic Geometry*. New York, NY: Springer-Verlag, 1997.

[Hut] M. Hutchings. *Floer homology of families I*. arXiv: `0308115 [math.sg]`.

[Laz10] L. Lazzarini. "Relative frames on J-holomorphic curves". In: *J. Fix. P. Th. Appl.* 151 (2010), pp. 177–242.

[Ler50a] J. Leray. "L'anneau spectral et l'anneau fibré d'homologie d'un espace localement compact et d'une application continue". In: *J. Math. Pures Appl.* 29 (1950), pp. 1–139.

[Ler50b] J. Leray. "L'homologie d'un espace fibré dont la fibre est connexe". In: *J. Math. Pures Appl.* 29 (1950), pp. 169–213.

[MS98] D. McDuff and D. Salamon. *Introduction to symplectic topology*. Oxford: Clarendon Press, 1998.

[MS04] D. McDuff and D. Salamon. *J-holomorphic curves and symplectic topology*. Providence, RI: American Mathematical Society, 2004.

[NNU] T. Nishinou, Y. Nohara, and K. Ueda. *Toric degenerations of Gelfand-Cetlin systems and potential functions*. arXiv: `0810.3470 [math.sg]`.

[Oh93] Y.-G. Oh. "Floer cohomology of Lagrangian intersections and psuedo-holomorphic disks I". In: *Comm on Pure and Applied Mathematics* 46 (1993), pp. 949–994.

[Oh] Y.-G. Oh. "Floer cohomology, spectral sequences, and the Maslov class of Lagrangian embeddings". In: *International Mathematics Research Notices* 1996.15 (), pp. 305–346.

[Sei08] P. Seidel. *Fukaya categories and Picard-Lefschetz theory*. Zürich: European Mathematical Society (EMS), 2008.

[Sei11] P. Seidel. "Homological mirror symmetry for the genus 2 curve". In: *Journal of Algebraic Geometry* 20.4 (2011), pp. 727–769.

[Ser51] J.P. Serre. "Homologie singuliére des espaces fibrés". In: *Applications, Ann. of Math.* 54 (1951).

[Sil01] A.-C. da Silva. *Lectures on Symplectic Geometry.* Berlin-Heidelberg: Springer-Verlag lecture notes in mathematics, 2001.

[Viaa] R. Vianna. *Infinitely many exotic monotone Lagrangian tori in* $\mathbb{CP}^2$. arXiv: `1409.2850 [math.sg]`.

[Viab] R. Vianna. *On exotic Lagrangian tori in* $\mathbb{CP}^2$. arXiv: `1305.7512 [math.sg]`.

[Wei94] C.-A. Weibel. *An Introduction to Homological Algebra (Cambridge Studies in Advanced Mathematics 38).* Cambridge, UK: Cambridge University Press, 1994.

[WC] C. Woodward and F. Charest. *Floer cohomology and flips.* arXiv: `1508.01573v3 [math.SG]`.